# Quantification in Description Logics of Typicality

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submitted to the Examining Committee.

*"Either this is madness or it is Hell."*
*"It is neither," calmly replied the voice of the Sphere, "it is Knowledge; it is Three Dimensions: open your eye once again and try to look steadily."*

---

*Edwin Abbott, Flatland: A Romance of Many Dimensions*

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The idea that man is inherently social and does not accomplish anything alone is old. It's in the lines of John Donne's poem, *no man is an island*. More recently, an Austrian bodybuilder put it more eloquently than I could, and I quote him:

> "I didn't make it that far on my own. I mean, to accept that credit or that medal, would discount every single person that has helped me get here today, that gave me advice, that made an effort, that lifted me up when I fell. And it gives the wrong impression that we can do it all alone. None of us can. The whole concept of the self-made man or woman is a myth."

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# Resumo

Igor de Camargo e Souza Câmara. **Quantificação em lógicas de descrição de tipicalidade**. Tese (Doutorado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2023.

Esta tese explora a intersecção das Lógicas de Descrição (DLs) com a teoria dos protótipos no contexto da representação de conhecimento e raciocínio. DLs são formalismos largamente usados para representação de conhecimento e servem como a espinha dorsal da Web Semântica. Esta tese argumenta que incorporar a teoria dos protótipos às DLs pode ser um incremento desejável e pavimenta o caminho para a introdução de raciocínio não-monotônico. Essa ampliação expandiria o raciocínio baseado em DLs para incluir regularidades que são tipicamente, mas nem sempre, verificadas. Por exemplo, ela poderia incluir conhecimento como *pássaros tipicamente voam*, uma regularidade que é verdadeira em quase todos os casos, mas admite exceções. Essa forma de raciocínio é crucial para a inferência inspirada nos processos cognitivos humanos e também para lidar com problemas como o raciocínio sob informação incompleta.

A abordagem desta tese se insere na tradição que combina DLs e tipicalidade através do raciocínio *derrotável* (defeasible), em particular, pela adoção de *inclusões derrotáveis de conceitos* (DCIs). Raciocínio baseado em materialização é uma das técnicas mais proeminentes dessa tradução. Essa técnica se resume a redução de inferências derrotáveis enriquecendo o lado esquerdo das inclusões com conceitos que representam axiomas derrotáveis. Esses conceitos são chamados a materialização dos axiomas que eles representam. Semânticas distintas baseadas em materialização são caracterizadas pelas técnicas que usam para selecionar os conjuntos de axiomas que serão materializados com um dado conceito.

Embora as semânticas baseadas em materialização sejam inegavelmente bem sucedidas, elas possuem sérias limitações. Em particular, elas possuem uma natureza proposicional e, portanto, não podem estender informação derrotável através de quantificadores, um problema conhecido como *negligência de quantificadores*. Portanto, *pássaros tipicamente voam* e *pardais são pássaros* permitem a conclusão de que *pardais tipicamente voam*, No entanto, não é possível concluir de *gatos comem pássaros* que *gatos tipicamente comem animais voadores*.

A tese se apoia nos recém introduzidos modelos de tipicalidade para abordar essas limitações e definir um maquinário semântico que melhore semânticas já existentes e inclua nelas propriedades de primeira-ordem. Ela expande o já estabelecido framework semântico de modelos de tipicalidade para a lógica $\mathcal{EL}_\bot$, uma semântica parametrizada em forças (strenghts) e coberturas (coverages) com seis variações cobrindo semânticas baseadas em materialização já existentes. Adicionalmente, a tese propõe um novo framework para a lógica $\mathcal{ELI}_\bot$, que inclui uma semântica proposicional, equivalente às semânticas baseadas em materialização, e uma semântica aninhada que resolve o problema da negligência de quantificadores.

**Palavras-chave:** Lógicas de Descrição. Tipicalidade. Raciocínio derrotável. Negligência de Quantificadores.

# Abstract


Igor de Camargo e Souza Câmara. **Quantification in Description Logics of Typicality**. Thesis (Doctorate). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2023.

This thesis delves into the intersection of Description Logics (DLs) and prototype theory in the context of knowledge representation and reasoning. DLs are formalisms widely used in knowledge representation and serve as the backbone of the Semantic Web. The thesis proposes that integrating aspects from prototype theory into DLs would be a desirable upgrade, enabling the introduction of nonmonotonic reasoning. This augmentation would expand reasoning based on DLs to include regularities that are typically verified but not always true. For instance, it would include knowledge such as *birds typically fly*, which is generally but not always true. Inferences of this kind are fundamental for modeling human-inspired reasoning and tackling problems like reasoning under incomplete information.

The approach taken in this thesis follows the tradition of combining DLs and typicality through defeasible reasoning by using defeasible concept inclusions (DCIs). Materialization-based semantics is one of the most successful techniques for dealing with defeasible knowledge in DLs. This technique reduces checking defeasible entailments such as concept subsumption and instance checking to an enriched classical query, in which concepts representing defeasible axioms are added to the left-hand side of the inclusion. These concepts are called the *materialization* of the axioms they represent. Distinct materialization-based semantics are characterized by their techniques to select the axioms to materialize with any given concept.

Although materialization-based semantics are undeniably successful, they suffer from some serious drawbacks. In particular, they share a propositional nature and, therefore, cannot extend defeasible information through quantifiers, a problem known as *quantification neglect*. Hence, *birds typically fly* and *robins are birds* allow concluding that *robins typically fly*. However, it is impossible to conclude from *cats eat birds* that *cats typically eat flying animals*.

The thesis builds on the recently-introduced typicality models to address these limitations to define a semantical framework that improves existing semantics and includes first-order properties. It expands the existing framework for typicality models for the logic $\mathcal{EL}_{\bot}$, which is a semantics parametrized along strengths and coverages with six variations covering existing semantics. Additionally, the thesis proposes a new framework for the logic $\mathcal{ELI}_{\bot}$, which includes a propositional semantics equivalent to materialization-based reasoning and a nested semantics that solves quantification neglect for existing materialization-based semantics.



**Keywords:** Description Logics. Typicality. Defeasible Reasoning. Quantification neglect.

# List of Abbreviations

| | |
|---|---|
| ABox | Assertional box |
| DBox | Defeasible terminological box |
| DKB | Defeasible knowledge base |
| DL | Description Logic |
| KB | Knowledge base |
| KLM | The hierarchy of NMR systems proposed in [KLM90] |
| NMR | Nonmonotonic Reasoning |
| OWL | Web-ontology Language |
| QN | Quantification Neglect |
| TBox | Terminological box |
| TI | Typicality Interpretation |
| TM | Typicality model |

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# Chapter 1

# Introduction

Description logics (DLs) are a family of knowledge representation formalisms that are used to represent and reason about knowledge in a structured and logical manner. The primary role of DLs in knowledge representation and reasoning is to provide a formal and expressive language for representing and reasoning about concepts and their relationships. DLs are widely used in various applications, such as computational ontologies through the Web Ontology Language (OWL) profiles.

The philosophical basis for concept representation in FOL and, by extension, in DLs is the *classical theory of concepts*. Despite the name, the classical theory of concepts is not a single, well-defined theory. Historically, several related theories put together under the umbrella term "classical theory" described the inner working of concepts. Those theories, which date back to Aristotle, envisage concepts as sets of individuals defined by a list of features, which can be combined compositionally to form more complex ones. Over the last century, this theory received critiques from the empirical cognitive sciences, especially in the work of Eleanor Rosch and colleagues, and philosophy, exemplified by the work of the philosopher Ludwig Wittgenstein.

Wittgenstein developed the idea of a *family resemblance* [Wit09]. According to it, instances of a concept share some properties. However, there is no single list that each instance *must* have, an effect that mirrors the resemblance among family members: some have a nose with certain properties, others share the eye color, and so forth. Each belongs to the family, but there is no single list of properties instantiated exactly by all.

During the 1970s, this idea was put forward by several empirical works in the cognitive sciences, defying the reign of classical theory. Among those works, the one defended by Eleanor Rosch and her collaborators, published in a series of papers from the 1970s onwards, had a considerable impact. According to her *prototype theory*, concept membership is not equivalent to the binary set membership of standard set theory but is better characterized in terms of a *gradient*. Hence, individuals are more or less representative of a given concept.

> "Although logic may treat categories as though membership is all or none, natural languages possess linguistic mechanisms for coding and coping with gradients of category membership." [Ros78, p. 199]

Furthermore, the idea of having a single list of necessary and sufficient conditions characterizing concepts is also abandoned in favor of non-essential features not considered in a classical-inspired theory. Typicality effects are seen in every domain of human conceptualization, from color categorization to the classification of human-made objects.

The borders of concepts are another point of disagreement between the two traditions. While, according to classical theory, borders are crisp, prototype theory and some related theories postulate fuzzy boundaries. This stance is backed up by empirical experiments in which human subjects usually agree on the classification of those objects closer to the prototype – or in the central region of the conceptual space, in Gärdenfors' (2000) conceptual spaces' terminology – but may disagree, or take up more time to make their judgment, in outlier instances.

A final topic of divergence is compositionality. Classical theory and Description Logics also present concepts as compositional entities. Therefore, newer concepts can be built from more basic ones with the help of combining constructors. "Flying animal" denotes the intersection of animals with flying objects. However, compositionality has been heavily criticized. Problematic cases include, for example, concepts such as *stone lion* – which is not an actual lion made of stone but probably a statue of a lion.

The table below summarizes the main disagreements between the classical and prototype theories: There are several reasons to port features from prototype theory to DLs.

| | **Classical Theory** | **Prototype Theory** |
|---|---|---|
| Concepts are characterized by | Necessary and sufficient conditions | Shared properties and similarity to optimal members |
| Membership | All or nothing | Graded |
| Borders | Crisp | Fuzzy |
| Compositionality | Yes | No |

**Table 1.1:** *comparison between classical and prototype theory*

There are many valuable inferences achievable only through typicality-inspired nonmonotonic reasoning. Suppose a doctor encounters a patient with fever, cough, and loss of smell in 2021. They may conclude that the patient suffers from COVID-19, even though he has not confirmed the presence of the SARS-CoV-2 virus via a PCR sample analysis. The somewhat rushed conclusion can provide valuable guidance to treatment while they wait for more accurate information. Later, when new information comes out, the doctor may keep the conclusion if the presence of the virus is confirmed or nonmonotonically withdraw it if that is not the case. The capacity for drawing conclusions from incomplete information is central to many human endeavors.

Another motivation is dealing with noise in the data, making knowledge representation more robust and error resistant. With concepts described by strict inclusions, a single non-coping entity may threaten a knowledge base by spreading inconsistency. When some of the information is not *obligatory*, but *expected*, entities that do not conform can be handled as unusual or atypical instances, not paradoxical ones.

Finally, there may be value in modeling human reasoning in itself. Drawing human-like conclusions from information can create more robust AI systems and, at the same time, shed light on the mechanisms of human reasoning.

The task of combining DLs and typicality has been investigated by more than a decade. There are several approaches, such as circumscribed KBs [BLW06], [BLW09], [Bon+15a], modal-like typicality operators [Gio+07], [Gio+08], [Gio+09], [Gio+13], [Var18] defeasible concept inclusions [CS10], [CS12], [Cas+14a], [BV18] and defeasible and probabilistic constructors [Poz17], [Poz18]. Materialization-based reasoning combined with defeasible inclusions is a very influential framework for handling typicality. This reasoning paradigm employs classical inference to compute defeasible (i.e. typical) inclusions between concepts, denoted by $C \sqsubseteq\!\!\!\!\sim D$. The procedure is to define new semantics in which a knowledge base satisfies certain defeasible inclusions iff classical semantics satisfy an enriched version of the same inclusion. More specifically, if we want to check whether $\mathcal{K} \models_{\mathsf{mat},} C \sqsubseteq\!\!\!\!\sim D$, we check $\mathcal{K} \models C \sqcap \overline{\mathcal{U}} \sqsubseteq D$, where $\overline{\mathcal{U}}$ is a special concept whose extension matches the elements that satisfy a set of DCIs $\mathcal{U} = \{E_1 \sqsubseteq\!\!\!\!\sim F_1, \ldots, E_n \sqsubseteq\!\!\!\!\sim F_n\}$. The concept $\overline{\mathcal{U}}$ is called the *materialization* of $\mathcal{U}$.

Several different materialization-based semantics exist, such as rational, relevant, lexicographic, and skeptical [CS10], [CS12], [Cas+14a], [GG20]. They differ in their criteria for selecting the DCIs to enrich the concept on the left-hand side. Part of the success of materialization-based semantics is due to its reliance on classical reasoning, a design choice that enables efficient and effective inference through well-optimized reasoners. Moreover, materialization-based systems are based on closures defined alongside the KLM hierarchy for nonmonotonic reasoning, which is a very influential ordered set of features for nonmonotonic systems [KLM90].

A major issue with combining KLM postulates with DL is that the postulates are defined over a propositional setting. Therefore, they do not cover DL's first-order components. More specifically, they do not say anything about the behavior of concepts nested within quantifiers. Overall, materialization-based reasoning for DLs suffers from the same ailment. Materialization-based defeasible semantics enable defeasible inferences such as

- birds typically fly;
- robins are birds;
- **therefore** robins typically fly.

However, when concepts occur nested within quantifiers, the semantical framework cannot reach them with defeasible information. Therefore, it does *not* draw inferences such as

- birds typically fly;
- cats eat birds;
- **therefore** cats typically eat flying animals.

This limitation is known as *quantification neglect* and was noted independently by [Bon+15a], [PT17a], and [Bon19].

Recently, a semantics based on a special class of models – typicality models – was developed for the lightweight defeasible DL $\mathcal{EL}_\bot$ to solve quantification neglect. Typicality model semantics presents a general framework capable of modeling several materialization-based semantics. It is parametrized along two features: strength and coverage. Strength relates to the materialization-based procedure on the background of the semantics, while coverage deals with the spread of defeasible information. There are two possibilities for coverage:

- **propositional**, for a semantic paradigm that does not extend defeasible information through quantifiers, and
- **nested**, for semantics in which defeasible information traverses arbitrarily long chains of quantifiers.

We present strengths corresponding to major materialization-based semantics, such as rational materialization-based reasoning proposed originally in [CS10]. A semantics based on typicality models with rational strength and propositional coverage coincides perfectly with materialization-based rational reasoning.

Typicality models deal with quantification neglect by a procedural upgrade that travels from propositional to nested coverage. The final result is a defeasible semantics that addresses one of the major drawbacks of materialization-based reasoning for defeasible DLs, enabling more sophisticated reasoning with typicalities.

This dissertation's main goal is to contribute to improving DLs of typicality through semantics based on typicality models. To this end, we continue investigating unsolved problems in the framework for $\mathcal{EL}_\bot$ and push typicality models to more expressive semantics.

For $\mathcal{EL}_\bot$, we introduce a new strength corresponding to the materialization-based lexicographic semantics. Besides, we fully compare all the six inference relations based on typicality models for $\mathcal{EL}_\bot$, which was not done to this date. This comparison sheds light on major challenges in reasoning defeasibly in a first-order setting.

Intending to extend typicality models to DLs in the Horn fragment of FOL, we present a comprehensive study for the logic $\mathcal{ELI}_\bot$. The addition of inverse roles is known to greatly impact the expressivity and complexity of the logic by allowing a restricted form of value restrictions (i.e. universal quantification). Our preliminary study shows that adapting typicality models for more expressive frameworks is possible, but it is a challenging task. To retrace the path from propositional to nested reasoning, we introduced heavy technical machinery to account for the increased expressivity. The result is a working defeasible semantics overcoming quantification neglect over some reasoning tasks such as defeasible subsumption and defeasible instance checking.

The dissertation is structured as follows:

- Chapter 2 introduces description logics, including a more detailed coverage of $\mathcal{EL}_\bot$ and $\mathcal{ELI}_\bot$ and some fundamental results on them.
- Chapter 3 introduces nonmonotonic reasoning through the KLM framework, which is crucial to our approach by being the foundation of many of the DLs of typicality,

including materialization-based systems in general.

- Chapter 4 gives an extensive and detailed overview of the current research in the area. It covers several different techniques spread over different DLs. We compare strengths and pinpoint shortcomings of the most relevant systems examined. To the best of our knowledge, no published survey covers this topic. Therefore, this is the first contribution of this dissertation. The content is formatted as an independent paper and will be submitted for publication.

The second part of the dissertation is concerned with typicality models.

- Chapter 5 presents a very brief overview of typicality models, discussing the main intuitions and commonalities between the two systems examined afterward.
- Chapter 6 presents results on typicality models for the DL $\mathcal{EL}_\bot$. Some of the results were previously established by Pensel and Turhan (2017), (2018), (2018), and Pensel (2019). Besides the differences in presentation for some results, we introduce two novelties to this framework which are the main contributions of this chapter. First, we introduce a new strength based on the lexicographic closure, defining propositional and nested lexicographic reasoning. Then, we present a broad comparison between all six semantics based on typicality models for $\mathcal{EL}_\bot$ and three materialization-based semantics. Finally, we sketch a technique for extending the semantics to encompass instance checking for the rational strength.
- Chapter 7 concentrates the major contributions of this dissertation. It presents a new framework for typicality models for the logic $\mathcal{ELI}_\bot$. This framework includes new definitions of domain and satisfaction and an independent upgrade procedure to lift propositional to nested reasoning. The upgrade procedure for $\mathcal{EL}_\bot$ relies on several properties absent for $\mathcal{ELI}_\bot$. Hence the new procedure is only inspired by the original one. We also consider an adaptation of the instance checking extension from $\mathcal{EL}_\bot$ that covers reasoning of rational strength in both coverages.
  - Some of the results of this chapter have been presented in [CT22b], [CT22a], and [CT23].
- Finally, chapter 8 discusses some issues common to both typicality models frameworks covered in the dissertation. This discussion highlights the challenges of reasoning defeasibly in a first-order setting and points to some possible drawbacks of the typicality model's solution.

# Part I

# Background

# Chapter 2

# Description Logics

DESCRIPTION LOGICS (DLs) are a family of logical-based knowledge representation formalisms. They are decidable fragments of first-order logic that cover unary predicates (named *concepts*) and binary predicates (named *roles*). In that sense, they extend the expressivity of propositional logic without sacrificing decidability. DLs are tailored to specific use cases. Therefore, reflecting the trade-off between expressiveness and tractability, they vary in their computational complexity. They go from lightweight formalisms, such as $\mathcal{EL}$, to very expressive ones, such as $\mathcal{SROIQ}$. Because DLs have a first-order nature, the language of DLs usually includes quantifiers.

A wide range of real-life applications justifies the study of DLs. The most notable one is their role as the theoretical backbone of computational ontologies through the web-ontology language (OWL). Beyond the Semantic Web, ontologies are one of the most popular ways of representing knowledge. Successful examples include ontologies in the biomedical domain, such as gene ontology (GO) [Ash+00] or GALEN medical terminology [Rec+95]. Other interesting applications include ontology-based data access (OBDA), where ontologies are used to query databases, and ontology-based data integration (OBDI), where ontologies are employed to integrate heterogeneous databases covering shared knowledge domains.

## 2.1 Syntax

DLs represent knowledge from a universe of discourse by characterizing its elements. The language to describe the behavior of the elements is composed of concepts (i.e., unary predicates), which refer to sets of elements, and roles (i.e., binary predicates), which refer to pairs of elements. More concretely, concepts characterize things such as *birds*, *pet*, *orphan*, and *central nervous system*. Roles, on the other hand, cover relationships such as *parentOf*, *eats*, *is part of*, and *regulates*. Concepts and roles are built from basic building blocks called *names*. Formally, two disjoint sets – $\mathsf{N_C}$, for concepts, and $\mathsf{N_R}$, for roles – define the basic building blocks. Complex concepts arise from the combination of names and roles by *constructors* connected by a well-defined syntax. Definition 2.1 specifies the grammar for the DL $\mathcal{ALC}$, which is the starting point for many DLs.

**Definition 2.1** ($\mathcal{ALC}$ concepts)**.** *Let* $\mathsf{N_C}$ *and* $\mathsf{N_R}$ *be two sets s.t.* $\mathsf{N_C} \cap \mathsf{N_R} = \emptyset$*,* $A \in \mathsf{N_C}$ *and* $r \in \mathsf{N_R}$*. Let* $\top$ *and* $\bot$ *be two constants. An* $\mathcal{ALC}$ *concept* $C$ *is given by:*

$$C, D \coloneqq \top \mid \bot \mid A \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \exists r.C \mid \forall r.C$$

The constructors $\neg$, $\sqcap$, $\sqcup$, $\exists$, $\forall$ are called negation, conjunction, disjunction, existential restriction and value restriction.

DLs also enable reference to specific elements of the universe of discourse through *named individuals.* A set of names, denoted by $\mathsf{N_I}$, is introduced to this end. As before, this set is disjoint w.r.t. $\mathsf{N_C}$ and $\mathsf{N_R}$.

Now, two important notions related to concepts will be introduced – *subconcepts* and *quantified concepts.* The former allows identifying the construction blocs that make any given concept, while the latter pinpoints concepts within the scope of some quantifier.

**Definition 2.2** (Subconcepts & quantified concepts)**.** *Let* $C$ *be a concept. Then, the set of subconcepts of* $C$*, denoted by* $\mathsf{Sub}(C)$*, is given recursively by*

- $\{C\}$ *if* $C \in \mathsf{N_C}$*;*
- $\{C\} \cup \mathsf{Sub}(D) \cup \mathsf{Sub}(E)$ *if* $C \in \{D \sqcap E, D \sqcup E\}$*;*
- $\{C\} \cup \mathsf{Sub}(D)$ *if* $C \in \{\neg D, \exists r.D, \forall r.D\}$*.*

*The set of subconcepts of a set of concepts* $\Gamma$ *is defined by* $\mathsf{Sub}(\Gamma) = \bigcup_{C \in \Gamma} \mathsf{Sub}(C)$*.*

*The set of quantified concepts is the set of concepts nested in quantifiers. Formally,* $Qc(C) = \{D \mid \exists r.D \in \mathsf{Sub}(C)$ *or* $\forall r.D \in \mathsf{Sub}(C)\}$*. As before, the set of quantified concepts for a set of concepts* $\Gamma$ *is given by* $Qc(\Gamma) = \bigcup_{C \in \Gamma} Qc(C)$*. A set of concepts* $\Gamma$ *is said to be* closed under quantification *iff* $C \in \Gamma$ *for every* $C \in Qc(\Gamma)$*.*

**Example 2.3.** *Let* $\Gamma = \{\exists r.C \sqcap D, \forall s.(E \sqcup \neg F)\}$*. Then,* $\mathsf{Sub}(\Gamma) = \{\exists r.C \sqcap D, \exists r.C, D, C, \forall s.(E \sqcup \neg F), E \sqcup \neg F, E, \neg F, F\}$ *and* $Qc(\Gamma) = \{C \sqcap D, E \sqcup \neg F, \}$*.*

Some more expressive DLs have additional constructors, such as number restrictions. Others, such as the members of the $\mathcal{EL}$ family, have less. DLs can also have syntactical restrictions on *where* certain concepts can occur in knowledge representation. Those restrictions are usually proposed to tame the complexity of some DL.

## 2.2 Semantics

DLs are fragments of first-order logic (FOL) and therefore inherit its set-theoretical semantics. This semantics is supported by *interpretations*, pairs that define a universe of elements and mappings that connect those elements to concepts, roles, and individuals.

**Definition 2.4.** *An interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *is a pair composed of the non-empty set* $\Delta^{\mathcal{I}}$*, called the* domain *of the interpretation, and a mapping* $\cdot^{\mathcal{I}}$ *that maps basic concepts, roles and constants to the domain in the following way:*

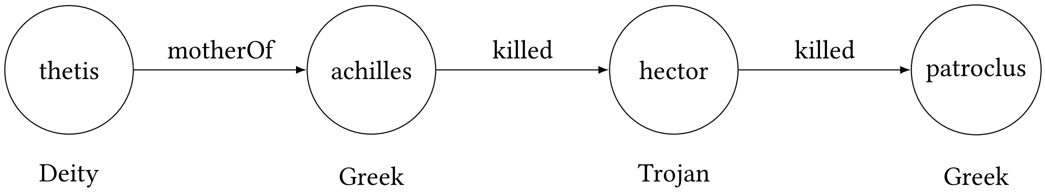


**Figure 2.1:** *A graphical interpretation of $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$. Nodes represent elements of the domain, and their labels cover concept membership. Edges represent the roles that label them.*

1. *$A \in \mathsf{N_C}$ to $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$;*
2. *$\top$ to $\Delta^{\mathcal{I}}$ and $\bot$ to $\emptyset$;*
3. *$r \in \mathsf{N_R}$ to $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$;*
4. *$a \in \mathsf{N_I}$ to $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$.*

*For complex concepts $C$, the* extension *is given compositionally by the following equivalences:*

| *Constructor* | *Syntax* | *Semantics* |
|---|---|---|
| *Negation* | $(\neg C)^{\mathcal{I}}$ | $\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$ |
| *Conjunction* | $(C \sqcap D)^{\mathcal{I}}$ | $C^{\mathcal{I}} \cap D^{\mathcal{I}}$ |
| *Disjunction* | $(C \sqcup D)^{\mathcal{I}}$ | $C^{\mathcal{I}} \cup D^{\mathcal{I}}$ |
| *Existential restriction* | $(\exists r.C)^{\mathcal{I}}$ | $\{d \in \Delta^{\mathcal{I}} \mid$ *there is some* $e \in \Delta^{\mathcal{I}}$ *such* $(r, e) \in r^{\mathcal{I}}$ *and* $e \in C^{\mathcal{I}}\}$ |
| *Value restriction* | $(\forall r.C)^{\mathcal{I}}$ | $\{d \in \Delta^{\mathcal{I}} \mid$ *for all* $e \in \Delta^{\mathcal{I}}$ *such that* $(d, e) \in r^{\mathcal{I}}, e \in C^{\mathcal{I}}\}$ |

The set of elements denoted by $C^{\mathcal{I}}$ is named the *extension* of $C$ in $\mathcal{I}$.

**Example 2.5.** *Let us consider a simple language that describes the Trojan War. Let* Deity, Mortal, Trojan *and* Greek *be concept names and* motherOf *and* killed *be roles. A possible interpretation for this vocabulary is $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where*

$$
\begin{aligned}
\Delta^{\mathcal{I}} &= \{thetis, achilles, hector, patroclus\};\\
\mathsf{Deity}^{\mathcal{I}} &= \{thetis\};\\
\mathsf{Mortal}^{\mathcal{I}} &= \{achilles, hector\};\\
\mathsf{Trojan}^{\mathcal{I}} &= \{hector\};\\
\mathsf{Greek}^{\mathcal{I}} &= \{achilles, patroclus\};\\
\mathsf{motherOf}^{\mathcal{I}} &= \{(thetis, achilles)\};\\
\mathsf{killed}^{\mathcal{I}} &= \{(achilles, hector), (hector, patroclus)\}.
\end{aligned}
$$

Interpretations can be seen as directed labeled graphs, where nodes represent the elements of the domain, and edges represent roles. A graphical representation of $\mathcal{I}$ is depicted in Figure 2.1.

Of course, this interpretation captures actual knowledge about the Trojan War only incidentally. Nothing in the vocabulary forces the domain to be comprised of this particular set of elements. Any set and mapping function would work in the same way. The only requirement is that the domain is not empty – therefore, any cardinality besides 0 is accepted, including infinite domains.

What interests us in modeling knowledge are the generalities that hold across all interpretations satisfying some condition and the patterns between concepts and roles that hold in them. We constrain the sets of interpretations to examine those properties by means of *knowledge bases*, which are collections of formulas called *axioms*. Axioms carry information on the intended interplay between the terms that make the vocabulary.

## 2.3 Representing Knowledge with DLs

There are two main ingredients in representing knowledge in DLs: (1) general concept inclusions (GCIs) and (2) concept or role assertions. Those ingredients make the two main components of a knowledge base: the terminological box (TBox), denoted by $\mathcal{T}$, and the assertional box (ABox), denoted by $\mathcal{A}$. Formally, a knowledge base (KB) is a pair $\mathcal{K} = (\mathcal{T}, \mathcal{A})$.[1]

The terminological component of a knowledge base encodes relationships between concepts. It should cover information such as *every parent has a child* and *(Greek) gods are immortal.* GCIs encode this kind of information through a *included in* symbol, $\sqsubseteq$, that relates two (simple or complex) concepts. Our examples could be written as $\mathsf{Parent} \sqsubseteq \exists \mathsf{has.Child}$ and $\mathsf{GreekGod} \sqsubseteq \mathsf{Immortal}$. Those formulas, called *axioms*, should be read as *every instance of the left-hand side concept is an instance of the right-hand side concept.* A TBox $\mathcal{T}$ is a set of such axioms.

If the terminological part of a KB can be seen as carrying information on the data structure, assertional knowledge, on the other hand, deals with the actual data, which is represented by individuals. The ABox contains *concept* and *role assertions*, which are attributions individual(s) to a concept or a role, respectively. *Hector killed Patroclus* and *Thetis is a deity and has a child* are examples of this kind of knowledge and would be represented in the DL formalism by $\mathsf{killed}(\mathsf{hector}, \mathsf{patroclus})$ and $(\mathsf{Deity} \sqcap \exists \mathsf{has.Child})(\mathsf{thetis})$, where $\mathsf{killed}$ and $\mathsf{has}$ are roles, $\mathsf{Deity}$ and $\mathsf{Child}$ are concepts and $\mathsf{hector}$, $\mathsf{patroclus}$, and $\mathsf{thetis}$ are individual names. An ABox $\mathcal{A}$ is a set of such assertions.

It will also be useful to port *subconcepts* and *quantified concepts* from Definition 2.2 to KBs. Adapting them is straightforward: $\mathsf{Sub}(\mathcal{T}) = \bigcup_{C \sqsubseteq D \in \mathcal{T}} (\mathsf{Sub}(C) \cup \mathsf{Sub}(D))$ and $\mathsf{Sub}(\mathcal{A}) = \bigcup_{C(a) \in \mathcal{A}} \mathsf{Sub}(C)$. Then, $\mathsf{Sub}(\mathcal{K}) = \mathsf{Sub}(\mathcal{T}) \cup \mathsf{Sub}(\mathcal{A})$. The same idea holds for

[1] Later in the thesis, we will deal with broader KBs, known as *defeasible knowledge bases* (DKBs). They increase the standard KBs by including a *defeasible terminological component.* Some more expressive DLs also have a third strict component in their KBs, role hierarchies, the role counterpart to the TBox. Those axioms are stored in the so-called RBoxes.

sets of quantified concepts:

$$
\begin{aligned}
Qc(C \sqsubsetapprox D) &:= Qc(C \sqsubseteq D) \\
&:= \{A \mid \exists r.A \in \mathsf{Sub}(C) \cup \mathsf{Sub}(D) \text{ or } \forall r.A \in \mathsf{Sub}(C) \cup \mathsf{Sub}(D)\} \\
Qc(\mathcal{T}) &:= \{Qc(C \sqsubseteq D) \mid C \sqsubseteq D \in \mathcal{T}\} \\
Qc(\mathcal{A}) &:= \{A \in \mathsf{sig}_{\mathsf{C}}(\mathcal{A}) \mid C(a) \in \mathcal{A} \text{ and } \exists r.A \text{ occurs in } C\}
\end{aligned}
$$

The axioms that make up a KB are connected to interpretations by the notion of satisfaction. If every element in the domain of some interpretation $\mathcal{I}$ conforms to some axiom, we say that this interpretation *satisfies* it. If an interpretation satisfies all axioms in some KB $\mathcal{K}$, we say that this interpretation is a *model* of $\mathcal{K}$. Formally:

**Definition 2.6** (Satisfaction & Model)**.** *Let $C \sqsubseteq D$ be a GCI and $C(a)$ be an assertion for some DL $\mathcal{L}$. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation. We say that $\mathcal{I}$* satisfies *$C \sqsubseteq D$* iff *$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, denoted by $\mathcal{I} \models C \sqsubseteq D$. We say that $\mathcal{I}$* satisfies *$C(a)$* iff *$a^{\mathcal{I}} \in C^{\mathcal{I}}$, denoted by $\mathcal{I} \models C(a)$.*

*Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a KB. If $\mathcal{I}$ satisfies all GCIs in $\mathcal{T}$ and all assertions in $\mathcal{A}$, we say that $\mathcal{I}$ is a model of $\mathcal{K}$, denoted by $\mathcal{I} \models \mathcal{K}$. We denote the* set of all models of *$\mathcal{K}$ by* $\mathsf{Mod}(\mathcal{K})$.

Axioms can be translated into the language of FOL through closed formulas, and interpretations for both formalisms are interchangeable given this translation. Formally:

| DL | FOL |
|---|---|
| $C \sqsubseteq D$ | $\forall x(C(x) \rightarrow D(x))$ |
| $C \sqsubseteq D \sqcap E$ | $\forall x(C(x) \rightarrow (D(x) \wedge E(x)))$ |
| $C \sqsubseteq D \sqcup E$ | $\forall x(C(x) \rightarrow (D(x) \vee E(x)))$ |
| $C \sqsubseteq \exists r.D$ | $\forall x(C(x) \rightarrow \exists y(r(x,y) \wedge D(y)))$ |
| $C \sqsubseteq \forall r.D$ | $\forall x(C(x) \rightarrow \forall y(r(x,y) \rightarrow D(y)))$ |

**Example 2.7.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a KB s.t.*

$$
\begin{aligned}
\mathcal{T} = \{&\exists \mathsf{motherOf}.\top \sqcup \exists \mathsf{fatherOf}.\top \sqsubseteq \mathsf{Parent}, \\
&\mathsf{Greek} \sqcup \mathsf{Trojan} \sqsubseteq \mathsf{Mortal}, \\
&\mathsf{Mortal} \sqcap \mathsf{Deity} \sqsubseteq \bot\};
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A} = \{&\mathsf{Deity(thetis)}, \mathsf{Greek(achilles)}, \mathsf{Greek(patroclus)}, \\
&\mathsf{Trojan(hector)}, \mathsf{motherOf(thetis, achilles)} \\
&\mathsf{killed(hector, patroclus)}, \mathsf{killed(achilles, hector)}\}.
\end{aligned}
$$

This KB constrains our interpretations to certain widely known facts about the Trojan War, namely, that Achilles, the son of the nymph Thetis, killed the trojan prince Hector to avenge the death of his companion Patroclus. Every interpretation that models $\mathcal{K}$ has to conform to this information, regardless of which elements make up its domain. Notice that interpretation $\mathcal{I}$ depicted in Example 2.5 is a model of $\mathcal{K}$.

Besides the stated facts, the KB also allows *infering* information that is not explicitly encoded. For example, one can conclude that there is (at least) one deity; that Thetis is a parent; that Achilles, Patroclus, and Hector are mortals; and that Thetis is not. We say that some formula – $C \sqsubseteq D$ or $C(a)$ – *follows* from a KB $\mathcal{K}$ iff every model $\mathcal{I}$ of $\mathcal{K}$ is also a model of the formula. This is denoted by $\mathcal{K} \models C \sqsubseteq D$ and $\mathcal{K} \models C(a)$, respectively.

## 2.4 Reasoning Tasks

Reasoning is deciding what follows from some given KB $\mathcal{K}$. FOL is undecidable, which means that there is no procedure guaranteed to output a correct answer to every query of this nature. However, reasoning in DLs is possible because they are *decidable* fragments of FOL, i.e. there are procedures guaranteed to terminate and give a correct answer to any query and KB in a DL $\mathcal{L}$, although the complexity of this task varies greatly depending on the DL being used. There are several different *reasoning tasks*

**Definition 2.8** (Consistency Checking)**.** *A KB $\mathcal{K}$ is said to be* consistent *if it is satisfiable, i.e.,* $\mathsf{Mod}(\mathcal{K}) \neq \emptyset$*. A concept $C$ is consistent w.r.t. $\mathcal{K}$ iff there is some model $\mathcal{I}$ of $\mathcal{K}$ s.t. $C^{\mathcal{I}} \neq \emptyset$.*

Notice that there may be inconsistent concepts in consistent KBs. The conflict only arises if the KB requires that the inconsistent concept is non-empty.

Another reasoning task is *subsumption checking*, which amounts to verifying whether one concept $C$ is *subsumed* by another, $D$.

**Definition 2.9** (Subsumption Checking)**.** *Let $\mathcal{K}$ be a KB and $C, D$ be concepts. We say that $C$ is* subsumed by $D$ *in $\mathcal{K}$, denoted by $\mathcal{K} \models C \sqsubseteq D$, iff, for every $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.*

Finally, we define *instance checking*, i.e. checking if some named individual $a$ necessarily belongs to some concept $C$.

**Definition 2.10** (Instance Checking)**.** *Let $\mathcal{K}$ be a KB, $C$ be a concept, and $a \in \mathsf{N_I}$ be a named individual. We say that $a$ is in $C$ in $\mathcal{K}$, denoted by $\mathcal{K} \models C(a)$, iff, for every $\mathcal{I} \in \mathsf{Mod}(\mathcal{K})$, $a \in C^{\mathcal{I}}$.*

Both subsumption and instance checking can be reduced to consistency checking. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be the KB, $C \sqsubseteq D$ be the subsumption, and $C(a)$ the instance to be checked. Then, $\mathcal{K} \models C \sqsubseteq D$ amounts to checking the consistency of $K' = (\mathcal{T} \cup \{C \sqcap \neg D\}, \mathcal{A})$, and $\mathcal{K} \models C(a)$ to checking the consistency of $K'' = (\mathcal{T}, \mathcal{A} \cup \{\neg C(a)\})$. This equivalence can be seen even in languages without full negation, such as those of the $\mathcal{EL}$ family equipped with the $\bot$ constant. Instead of the concept $C \sqcap \neg D$, we introduce a new auxiliary concept name Aux, and add $C \sqsubseteq \mathsf{Aux}$ and $\mathsf{Aux} \sqcap D \sqsubseteq \bot$ to $\mathcal{T}$. For instance checking, we may also add Aux, enrich $\mathcal{A}$ with $\mathsf{Aux}(a)$, and $\mathcal{T}$ with $\mathsf{Aux} \sqcap C \sqsubseteq \bot$.

## 2.5 The $\mathcal{EL}$ family

Taming complexity is an important part of DL research. Reasoning in general KBs[2] in the basic DL $\mathcal{ALC}$ is already ExpTime–complete. [Baa+17, Th. 5.11] However, there are more palatable options. $\mathcal{EL}$ is a lightweight DL that has a polynomicalcal reasoning algorithm. Plain $\mathcal{EL}$ cannot express contradictions, as it lacks full negation and the $\bot$ concept. Therefore, it is not a particularly interesting logic for defeasible reasoning. Nonetheless, introducing $\bot$ in the language opens space for inconsistency and defeasibility and does not increase the reasoning complexity [Bra04], [BBL05], [Pen19, p. 22].

This introduction characterizes the DLs $\mathcal{EL}_\bot$ and $\mathcal{ELI}_\bot$. We cover the main technical machinery and results for reasoning through canonical models. We present those results in our notation and with differences in formulation required by the purpose of the dissertation. Nonetheless, the literature already established most of what is presented here. For the $\mathcal{EL}_\bot$ calculi, the reader is referred to [BBL05] and [Baa+17]. The $\mathcal{EL}_\bot$ canonical model presentation that we consider is inspired by [LW10]. Reasoning for $\mathcal{ELI}_\bot$ is established in [BLB08] and also covered by [Baa+17]. Although inspired by the procedure presented in [Baa+17], the idea of prime set is original to our work. Pensel and Turhan [PT18b] proposed representative domains for $\mathcal{EL}_\bot$. The adaptation for $\mathcal{ELI}_\bot$ is also original to this dissertation.

### 2.5.1 $\mathcal{EL}_\bot$

The language of $\mathcal{EL}_\bot$ has the two constructors that make up $\mathcal{EL}$– conjunction and existential restrictions – and includes the constant $\bot$ to define concept disjointness.

**Definition 2.11** ($\mathcal{EL}_\bot$ Concept)**.** *Let* $\mathsf{N_C}$ *and* $\mathsf{N_R}$ *be two non-empty disjoint sets. Let* $A \in \mathsf{N_C}$ *and* $r \in \mathsf{N_R}$*. An* $\mathcal{EL}_\bot$ *concept* $C$ *is given by:*

$$C := \bot \mid A \mid C \sqcap D \mid \exists r.C$$

*Where* $C, D$ *range over* $\mathcal{EL}_\bot$ *concepts. An* $\mathcal{EL}_\bot$ *KB* $\mathcal{K}$ *is a KB s.t. every concept* $C$ *that occurs in it is an* $\mathcal{EL}_\bot$ *concept.*

The lack of disjunction in $\mathcal{EL}_\bot$ makes reasoning easier by eliminating the need for backtracking, which is not possible in DLs such as $\mathcal{ALC}$. Proving subsumptions can be done directly by deriving all subsumptions that follow from the KB, saturating it, a task known as *classification.* In $\mathcal{EL}_\bot$, this calculation can be done in polynomial time w.r.t. the size of $\mathcal{K}$. For a more detailed account of the consequence-based reasoning algorithm, the reader is referred to [BBL05] and [Baa+17].

One important feature of the DLs in the $\mathcal{EL}$ family is the *canonical model property*, i.e., any KB $\mathcal{K}$ has a special model that can be homomorphically embedded into every other model of $\mathcal{K}$. In $\mathcal{EL}_\bot$ and its extension $\mathcal{ELI}_\bot$, which will be introduced later in the

[2] If the TBox is acyclic, then the complexity goes down to PSpace [Baa+17, Th. 5.5]. A TBox $\mathcal{T}$ is said to be acyclic if (i) no concept name in $\mathcal{T}$ uses itself and (ii) concept names do not occur twice more than one time on the left-hand side of a GCI in $\mathcal{T}$ [Baa+17, pp. D. 2.9].

text, this model can be used to check subsumption and instances directly by looking at elements that represent the concept on the left-hand side of the inclusion or applied to the individual. Canonical models are defined w.r.t. *relevant contexts*, which delimit the domain to a given set of salient concepts. Relevant contexts are defined w.r.t. DKBs and queries.

**Definition 2.12** (Relevant context for $\mathcal{EL}_\bot$). *[Pen19, pp. D. 2.13.] Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB. The set $\mathbb{C} \subset \mathfrak{L}(\mathcal{EL}_\bot)$ is an $\mathcal{EL}_\bot$* relevant context *for $\mathcal{K}$ if it satisfies:*

- $Qc(\mathcal{K}) \subseteq \mathbb{C}$,
- *For every $C \in \mathbb{C}$, $\mathcal{K} \not\models C \sqsubseteq \bot$,*
- $Qc(\mathbb{C}) \subseteq \mathbb{C}$.

As a special case, we define the context of a KB $\mathcal{K}$, denoted by $\mathbb{C}(\mathcal{K})$, by the least set closed under quantification containing all satisfiable concepts $C, D$ s.t. $C \sqsubseteq D \in \mathcal{T}$ or $C(a) \in \mathcal{A}$. To include an arbitry query $C \sqsubseteq D$ in the context, one can add $C \sqsubseteq \top$ to $\mathcal{T}$.

**Example 2.13** (Relevant context for $\mathcal{EL}_\bot$). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB such that $\mathcal{T} = \{A \sqsubseteq \exists r.\exists s.B\}$ and $\mathcal{D} = \{B \mathrel{\substack{\sqsubset\\\sim}} C \sqcap D\}$. The set $\mathbb{C} = \{A, \exists s.B, B\}$ is a relevant context for $\mathcal{K}$. On the other hand, the set $\mathbb{C}' = \{A, \exists s.B\}$* is not *a relevant context for $\mathcal{K}$, as $Qc(\mathbb{C}') \not\subseteq \mathbb{C}'$.*

Adapting the definition given in [LW10], a direct definition of the canonical model for $\mathcal{EL}_\bot$ is given by:

**Definition 2.14** (Canonical Model for $\mathcal{EL}_\bot$). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a KB and $\mathbb{C}$ be a context over the language of $\mathcal{EL}_\bot$ consistent with $\mathcal{K}$. The canonical model of $\mathcal{K}$ over $\mathbb{C}$ is an interpretation $\mathcal{I}_{\mathcal{K},\mathbb{C}} = (\Delta^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}, \cdot^{\mathcal{I}_{\mathcal{K},\mathbb{C}}})$ s.t. $\Delta^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} = \mathbb{C} \cup \mathsf{sig}_\mathsf{I}(\mathcal{K})$ and $\cdot^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$ is defined by*

$$
\begin{aligned}
A^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} &= \{C \in \Delta^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} \mid \mathcal{K} \models C \sqsubseteq A\} \\
&\cup \{a \in \Delta^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} \mid \mathcal{K} \models A(a)\}; \\
r^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} &= \{(C, D) \in \Delta^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} \times \Delta^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} \mid \mathcal{K} \models C \sqsubseteq \exists r.D\} \\
&\cup \{(a, C) \in \Delta^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} \times \Delta^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} \mid \mathcal{K} \models (\exists r.C)(a)\}; \\
a^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} &= a.
\end{aligned}
$$

*As a special case, if $\mathbb{C} = \mathbb{C}(\mathcal{K})$, we denote $\mathcal{I}_{\mathcal{K},\mathbb{C}}$ by $\mathcal{I}_\mathcal{K}$.*

As mentioned before, it is possible to read subsumptions directly from canonical models by looking at concept representatives.

**Lemma 2.15.** *Let $\mathcal{K}$ be a KB and $\mathbb{C}$ be a context s.t. $C, D \in \mathbb{C}$, and $a \in \mathsf{sig}_\mathsf{I}(\mathcal{K})$. Let $\mathcal{I}_{\mathcal{K},\mathbb{C}}$ be the canonical model for $\mathcal{K}$ over $\mathbb{C}$. Then,*

1. $\mathcal{K} \models C \sqsubseteq D$ *iff* $C \in D^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$, *and*
2. $\mathcal{K} \models C(a)$ *iff* $a \in C^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$.

*Proof.* 1. The proof is on the structure of $D$. For the base case, let $D \in \mathsf{N_C}$. Then, by definition, $C \in D^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$ iff $\mathcal{K} \models C \sqsubseteq D$. There are two cases to examine in the inductive step because $D \in \{E \sqcap F, \exists r.E\}$, for two concepts $E, F$. If $D = E \sqcap F$,

$$\begin{aligned}
&C \in D^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} \Leftrightarrow \\
&C \in E^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} \text{ and } C \in F^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} \Leftrightarrow \\
&\mathcal{K} \models C \sqsubseteq E \text{ and } \mathcal{K} \models C \sqsubseteq F \Leftrightarrow \\
&\mathcal{K} \models C \sqsubseteq D
\end{aligned}$$

2. We show the directions separately. First, suppose that $C \in (\exists r.E)^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$. Then, there is some element $F$ s.t. $(C, F) \in r^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$ and $F \in E^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$. Notice that $F$ is a concept representative because, by construction, all role successors are concept representatives. Individuals only appear in role edges as predecessors. Then, by hypothesis, $\mathcal{K} \models C \sqsubseteq \exists r.F$ and $\mathcal{K} \models F \sqsubseteq E$. But this implies that $\mathcal{K} \models C \sqsubseteq \exists r.E$, proving one direction.

Now, suppose that $\mathcal{K} \models C \sqsubseteq \exists r.E$. Notice that $\exists r.E = D$ is in the context $\mathbb{C}$. Then, by the construction of the domain, $E \in \Delta^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$, as $E \in Qc(\exists r.E)$. Finally, by the definition of the extension of $r$, $(C, E) \in r^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$ and, because $\mathcal{K} \models E \sqsubseteq E$, $E \in E^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$. Therefore, $C \in D^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$.

(2) For the base case, once more, by definition, $a \in C^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$ iff $\mathcal{K} \models C(a)$.

For the inductive steps, the cases are the same as before. First, if $C = D \sqcap E$,

$$\begin{aligned}
&a \in C^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} \Leftrightarrow \\
&a \in D^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} \text{ and } a \in E^{\mathcal{I}_{\mathcal{K},\mathbb{C}}} \Leftrightarrow \\
&\mathcal{K} \models D(a) \text{ and } \mathcal{K} \models E(a) \\
&\mathcal{K} \models C(a)
\end{aligned}$$

The second case uses the same argument presented in (1). Suppose that $a \in (\exists r.D)^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$. Then, there is some $E \in \Delta^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$ s.t. $(a, E) \in r^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$ and $E \in D^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$. By hypothesis, $\mathcal{K} \models (\exists r.E)(a)$ and $\mathcal{K} \models D \sqsubseteq E$. From this, it follows that $\mathcal{K} \models (\exists r.D)$.

For the other direction, let $\mathcal{K} \models (\exists r.D)(a)$. Because $(\exists r.D) \in \mathbb{C}$, $D \in \mathbb{C}$ and $D \in \Delta^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$. Then, by definition, $(a, D) \in r^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$ and $D \in D^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$. Finally, $a \in (\exists r.D)^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}$, completing the proof. □

**Example 2.16.** *Let* $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ *be an* $\mathcal{EL}_\perp$ *KB s.t.* $\mathcal{T} = \{\mathsf{Trojan} \sqsubseteq \mathsf{Mortal} \sqcap \exists \mathsf{hasEnemy}.\mathsf{Greek}, \mathsf{Greek} \sqsubseteq \mathsf{Mortal}, \mathsf{Mortal} \sqcap \mathsf{Deity} \sqsubseteq \perp\}$ *and* $\mathcal{A} = \{\mathsf{Trojan}(\mathsf{hector})\}$*. Let* $\mathbb{C} = \{\mathsf{Trojan}, \mathsf{Greek}, \mathsf{Mortal}, \mathsf{Deity}\}$*. The canonical model* $\mathcal{I}_{\mathcal{K},\mathbb{C}}$ *of* $\mathcal{K}$ *over* $\mathbb{C}$ *can be visualized by the graph diagram depicted in Figure 2.2.*

From Example 2.16, it is possible to see that the expected relationships between concepts hold – e.g., $\mathsf{Trojan} \sqsubseteq \exists \mathsf{hasEnemy}.\mathsf{Greek}$ –, but it is also possible to draw new conclusions, such as $\mathsf{Trojan} \sqsubseteq \exists \mathsf{hasEnemy}.\mathsf{Mortal}$ and $\mathsf{Mortal}(\mathsf{hector})$.

### 2.5.2 $\mathcal{ELI}_\perp$

Most of the letters in the nomenclature of DLs have special meanings, and this is the case for the $\mathcal{I}$ in $\mathcal{ELI}_\perp$, which stands for *inverted roles.* $\mathcal{ELI}_\perp$ adds a construct to refer to

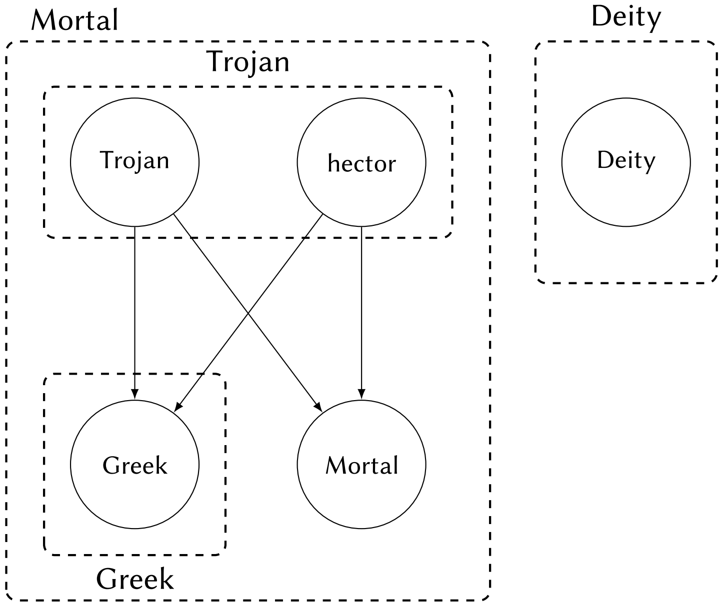


**Figure 2.2:** *A graphical interpretation of* $\mathcal{I}_{\mathcal{K},\mathbb{C}} = (\Delta^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}, \cdot^{\mathcal{I}_{\mathcal{K},\mathbb{C}}})$. *Nodes represent the elements of the domain, and the labeled colored rectangles show the extension of the named concepts. Finally, edges represent the extension of the role* hasEnemy.

the inverse of a role $r$, denoted by $r^-$, to the language of $\mathcal{EL}_\bot$. Roles are interpreted by pairs of elements from the domain and, for every role, we can define its inverse by taking the inverse of the pairs. Formally, $(a, b) \in r^\mathcal{I}$ iff $(b, a) \in r^\mathcal{I}$.

At first glance, this addition is useful to represent meaningful inverses, e.g., hasChild$^-$ is hasParent, and so on. However, inverse roles are not limited to this simple use and their addition results in a great increase in the expressivity and complexity of $\mathcal{EL}_\bot$. Employing the equivalence $\exists r^-.C \sqsubseteq D \equiv C \sqsubseteq \forall r.D$, the language of $\mathcal{ELI}_\bot$ is able to represent *value restrictions* on the right-hand side of GCIs. From now on, whenever a GCI of the form $C \sqsubseteq \forall r.D$ appears in the context of $\mathcal{ELI}_\bot$, it should be seen as an abbreviation for $\exists r^-.C \sqsubseteq D$.

Throughout this dissertation, we will work with syntactically restricted TBoxes called *normalized TBoxes*. This syntactical restriction does not result in a loss of expressivity since there are methods to normalize any TBox that output a conservative extension of the original one, i.e., it preserves subsumption for the shared concepts. It is possible to normalize any $\mathcal{ELI}_\bot$ TBox by a linear number of applications of normalization rules, and the size of the new TBox is linear on the size of the original non-normalized one. [Baa+17, Lemma 4.2]

**Definition 2.17** ($\mathcal{ELI}_\bot$ TBox normal form)**.** *Let* $A, A', B \in \mathsf{N_C} \cup \{\bot, \top\}$ *and* $r \in \mathsf{N_R}^-$. *An* $\mathcal{ELI}_\bot$ *TBox* $\mathcal{T}$ *is in* TBox normal form *(written* $\mathsf{T\text{-}NF}(\mathcal{T})$*), if all of its axioms have one of the following forms:*

$$A \sqsubseteq B \qquad A \sqcap A' \sqsubseteq B \qquad A \sqsubseteq \exists r.B \qquad A \sqsubseteq \forall r^-.B$$

A *simple* ABox $\mathcal{A}$ is an ABox which contains only assertions concerning named

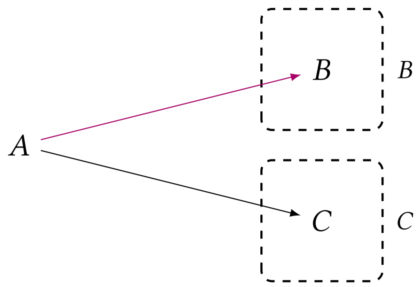


**Figure 2.3:** *A graphical interpretation of the $\mathcal{EL}_\bot$ canonical model $\mathcal{I}_{\mathcal{K},\mathbb{C}} = (\Delta^{\mathcal{I}_{\mathcal{K},\mathbb{C}}}, \cdot^{\mathcal{I}_{\mathcal{K},\mathbb{C}}})$ defined over the KB $\mathcal{K}$ from Example 2.18. The nodes $A, B$, and $C$ represent the elements of the domain and the labelled dashed rectangles show the extension of named concepts. Finally, edges represent the extension of the role $r$. The violation arises from the $(A, C)$ edge and is marked in red.*

concepts, i.e., for every $C(a) \in \mathcal{A}$, $C \in \mathsf{N_C}$. It is also possible to simplify any given ABox by introducing new concept names $C_{aux}$ for each $C(a) \in \mathcal{A}$, substituting the assertion $C(a)$ for $C_{aux}(a)$, and adding $C_{aux} \sqsubseteq C$ to the TBox. From now on, we consider all the ABoxes to be simple and the TBoxes to be normalized unless stated otherwise.

The introduction of inverse roles undermines the procedure for building the $\mathcal{EL}_\bot$ canonical model from Definition 2.14. The intuition for this failure is that the $\mathcal{EL}_\bot$ canonical model has *too many* role edges, and $\mathcal{ELI}_\bot$ introduces restrictions on edges. This excess of edges originates in the requirement that, for every $\mathcal{K} \models C \sqsubseteq \exists r.D$, there is a corresponding pair $(C, D) \in r^{\mathcal{I}_\mathcal{K}}$. A simple example shows why this is the case.

**Example 2.18.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ where $\mathcal{T} = \{A \sqsubseteq \exists r.B, A \sqsubseteq \forall r.C\}$ and $\mathcal{A} = \emptyset$.*

The interpretation stemming from Definition 2.14 would not be a model, as the edge $r(A, B)$ violates $A \sqsubseteq \forall r.C$. It is not possible to fix this violation by adding the element $B$ to the extension of the concept $C$ because it is not true that $B \sqsubseteq C$.

The key to overcoming this obstacle is noticing the inner workings of value restrictions, namely, that they make existing roles "collect" all the concepts nested inside the corresponding restrictions. For $\mathcal{K}$ from Example 2.18, the $B$ $r$-successor for $A$ is not only a $B$, but also a $C$ – hence, it can be represented by $B \sqcap C$. This intuition can be put to work by departing from concept representatives to *sets of named concepts* representing their conjunction.[3] Let $M \subseteq \mathsf{N_C}$ be a set of named concepts. The *corresponding concept* of $M$ is

$$\lceil M \rceil := \begin{cases} \bigsqcap_{A \in M} A & \text{, if } M \neq \emptyset \\ \top & \text{, otherwise} \end{cases}$$

Each domain element is the representative of its corresponding concept. Single concepts $C$ can still be represented by the singleton $\{C\}$. By convention, $\lceil \emptyset \rceil = \top$, which allows representing existential requirements such as $C \sqsubseteq \exists r.\top$.

Given the transition from concept representatives to sets of named concepts, it is necessary to decide once again which elements will compose the domain of the canonical model. A *representative domain* for some KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ contains all the combinations of

[3] More inclusive sets of concepts can be used and may even be more economical. However, the construction of typicality models for $\mathcal{ELI}_\bot$ will demand this restriction to names.

named concepts appearing within quantification, singletons of the named concepts in the Tbox, and the individuals from the Abox.

**Definition 2.19** (Representative Domain for $\mathcal{ELI}_\bot$). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ELI}_\bot$ KB. The* representative domain *for $\mathcal{K}$ is given by*

$$\Delta^{\mathcal{K}} = \{\{A\} \mid A \in \mathsf{sig}_\mathsf{C}(\mathcal{K})\} \cup \mathcal{P}(Qc(\mathcal{K})) \cup \mathsf{sig}_\mathsf{I}(\mathcal{A})$$

Finding the right edges for the canonical model amounts to finding witness that are maximal according to the subset relation. Going back to the Example 2.3, the $r$ edge departing from $A$ (now $\{A\}$) should land neither in $\{B\}$, nor in $\{C\}$, but in $\{B, C\}$. In other words, the $B$ edge should collect all value restrictions imposed by the KB. Formally, they are defined as follows:

**Definition 2.20** (Prime successor). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB, $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ an interpretation, $C \in \mathfrak{L}(\mathcal{ELI}_\bot)$ be a concept, and $r \in \{s, s^-\}$ with $s \in \mathsf{sig}_\mathsf{R}(\mathcal{K})$. Then, $N \in \Delta^\mathcal{K}$ is a* prime $r$-successor *for $C$ in $\mathcal{I}$ iff:*

1. *$\mathcal{K} \models C \sqsubseteq \exists r.\lceil N \rceil$, and*
2. *There is no $N' \in \Delta^\mathcal{K}$ s.t.*
   (a) *$N \subset N'$, and*
   (b) *$\mathcal{K} \models C \sqsubseteq \exists r.\lceil N' \rceil$*

By employing the representative domain as its domain and restricting role edges through primeness, it is possible to give a suitable direct definition of a canonical model for $\mathcal{ELI}_\bot$.

**Definition 2.21** (Canonical Model for $\mathcal{ELI}_\bot$). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ELI}_\bot$ KB. Then, $\mathcal{I}_\mathcal{K} = (\Delta^{\mathcal{I}_\mathcal{K}}, \cdot^{\mathcal{I}_\mathcal{K}})$ is the canonical model for $\mathcal{K}$, where $\Delta^{\mathcal{I}_\mathcal{K}} = \Delta^\mathcal{K}$ is the representative domain for $\mathcal{K}$ and*

$$\begin{aligned}
A^{\mathcal{I}_\mathcal{K}} &= \{M \subseteq \mathsf{N_C} \mid \mathcal{K} \models \lceil M \rceil \sqsubseteq A\} \cup \{a \in \mathsf{sig}_\mathsf{I}(\mathcal{K}) \mid \mathcal{K} \models A(a)\} \\
r^{\mathcal{I}_\mathcal{K}} &= \{(M, N) \in \mathcal{P}(\mathsf{N_C}) \times \mathcal{P}(\mathsf{N_C}) \mid \mathcal{K} \models \lceil M \rceil \sqsubseteq \exists r.\lceil N \rceil \text{ and } N \text{ is prime for } M \text{ and } r\} \\
&\quad \cup \{(N, M) \in \mathcal{P}(\mathsf{N_C}) \times \mathcal{P}(\mathsf{N_C}) \mid \mathcal{K} \models \lceil M \rceil \sqsubseteq \exists r^-.\lceil N \rceil \text{ and } N \text{ is prime for } M \text{ and } r^-\} \\
&\quad \cup \{(a, M) \in \mathsf{sig}_\mathsf{I}(\mathcal{K}) \times \mathcal{P}(\mathsf{N_C}) \mid \mathcal{K} \models (\exists r.\lceil M \rceil)(a) \text{ and } M \text{ is prime for } a \text{ and } r\} \\
&\quad \cup \{(M, a) \in \mathcal{P}(\mathsf{N_C}) \times \mathsf{sig}_\mathsf{I}(\mathcal{K}) \mid \mathcal{K} \models (\exists r^-.\lceil M \rceil)(a) \text{ and } M \text{ is prime for } a \text{ and } r^-\} \\
a^{\mathcal{I}_\mathcal{K}} &= a
\end{aligned}$$

The reader should note that the four-part definition of the extension of roles is necessary to cover all four edge-generating scenarios – prime successors of straight and inverted roles, and prime successors of an individual with respect to straight and inverted roles.

**Example 2.22** (Canonical Model for $\mathcal{ELI}_\bot$). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ELI}_\bot$ KB s.t. $\mathcal{T} = \{\mathsf{Trojan} \sqcap \mathsf{Soldier} \sqsubseteq \exists \mathsf{killed}.\mathsf{Greek}, \mathsf{Soldier} \sqsubseteq \forall \mathsf{killed}.\mathsf{Soldier}\}$ and $\mathcal{A} = \{\mathsf{Trojan}(\mathsf{hector}), \mathsf{Soldier}(\mathsf{hector})\}$.*

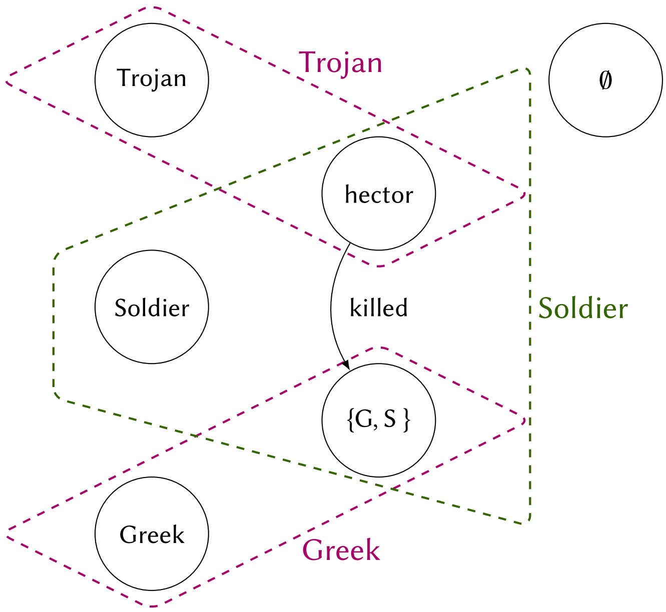


**Figure 2.4:** *A graphical representation of $\mathcal{I}_\mathcal{K} = (\Delta^{\mathcal{I}_\mathcal{K}}, \cdot^{\mathcal{I}_\mathcal{K}})$. Nodes represent the elements of the domain, and the labeled colored rectangles show the extension of named concepts. For the sake of simplicity, the brackets in singletons are omitted; hence,* Soldier *stands for* {Soldier}*. The set with more elements contains only the initials for graphical purposes. As before, edges represent the roles that label them.*

$\mathcal{I}_\mathcal{K} = (\Delta^{\mathcal{I}_\mathcal{K}}, \cdot^{\mathcal{I}_\mathcal{K}})$ *is the canonical model according to Definition 2.21. Then, the* representative domain *is*

$$\Delta^{\mathcal{I}_\mathcal{K}} = \{\emptyset, \{\mathsf{Trojan}\}, \{\mathsf{Greek}\}, \{\mathsf{Soldier}\}, \{\mathsf{Greek}, \mathsf{Soldier}\}, \mathsf{hector}\}$$

*A graphical representation of the model is depicted in Diagram 2.4.*

The problem pointed out in Example 2.18 vanished by the requirement of primeness for roles. The representative of hector has only one killed successor, and it is neither the Greek nor the Soldier representative, but the Greek $\sqcap$ Soldier. The construction of the representative domain does not include Trojan $\sqcap$ Soldier, despite it being in the axioms. The required combinations are limited to those whose elements can appear together as a prime successor to avoid an unnecessary expansion of the domain. Hence, the required combinations are sets of elements that appear nested in quantifiers.

Unlike the construction from $\mathcal{EL}_\bot$, it is not possible to answer any query by looking directly into the model. If this were the case, $\mathcal{I}_\mathcal{K}$ would answer affirmatively to Greek $\sqsubseteq \forall$killed.Greek, simply because, in this model, greeks do not kill anyone, making the subsumption vacuously true. This does not pose a serious threat to the canonical model's purpose, as the property still holds between conjunctions of named concepts, and it is possible to represent any query by named concepts with the introduction of auxiliary concepts and some supplementary axioms. Suppose $C \sqsubseteq D$ is the query. Then, Let $C_{aux}, D_{aux} \in \mathsf{N_C}$ s.t. $C_{aux}, D_{aux} \notin \mathsf{sig_C}(\mathcal{K})$. Let $\mathcal{T}^* = \mathcal{T} \cup \{C_{aux} \sqsubseteq C, D \sqsubseteq D_{aux}\}$. It is easy to see that $\mathcal{T} \models C \sqsubseteq D$ iff $\mathcal{T}^* \models C_{aux} \sqsubseteq D_{aux}$, and that the second query contains only named concepts. This also applies to the full KB $\mathcal{K}$.[4]

Now, this construction will be shown to have the necessary properties. The argument is divided into two main parts. First, it is necessary to prove that it is indeed a model of the KB. Then, after this is established, the property of representing subsumptions by concept representatives and instance relationships by individual representatives is shown.

**Lemma 2.23.** *$\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an $\mathcal{ELI}_\bot$ KB. Then, the interpretation $\mathcal{I}_\mathcal{K} = (\Delta^{\mathcal{I}_\mathcal{K}}, \cdot^{\mathcal{I}_\mathcal{K}})$ as defined in Definition 2.21 is a model of $\mathcal{K}$.*

*Proof.* As stated before, $\mathcal{K}$ is a normalized KB. Therefore, its axioms have one of the following forms:

$$A \sqsubseteq B \mid A_1 \sqcap A_2 \sqsubseteq B \mid A \sqsubseteq \exists r.B \mid A \sqsubseteq \forall r.B$$

To show this property, we examine each of the forms for generic elements of the domain $\Delta^{\mathcal{I}_\mathcal{K}}$. We consider the two classes of elements – sets of named concepts and individuals – separately.

(1) For an element $M$ s.t. $M \subseteq \mathrm{sig}_\mathsf{C}(\mathcal{K})$.

($A \sqsubseteq B$) If $M \in A^{\mathcal{I}_\mathcal{K}}$, $\mathcal{K} \models \lceil M \rceil \sqsubseteq A$, and, therefore, $\mathcal{K} \models \lceil M \rceil \sqsubseteq B$. By construction, $M \in B^{\mathcal{I}_\mathcal{K}}$.

($A_1 \sqcap A_2 \sqsubseteq B$) The argument is the same as before, with the additional step from $M \in (A_1 \sqcap A_2)^{\mathcal{I}_\mathcal{K}}$ to $M \in A_1^{\mathcal{I}_\mathcal{K}}$ and $M \in A_2^{\mathcal{I}_\mathcal{K}}$.

($A \sqsubseteq \exists r.B$) Notice that $\{B\} \in Qc(\mathcal{K})$ and, therefore, $\{B\} \in \Delta^{\mathcal{I}_\mathcal{K}}$. Due to the inclusion of sets of quantified concepts in the domain, this guarantees that there is some $N \subseteq \mathrm{sig}_\mathsf{C}(\mathcal{K})$ s.t. $B \in N$, $\mathcal{K} \models \lceil M \rceil \sqsubseteq \exists r.\lceil N \rceil$, and $N$ is prime for $M$. By construction of the domain, $(M, N) \in r^{\mathcal{I}_\mathcal{K}}$ and, because $B \in N$, $\mathcal{K} \models \lceil N \rceil \sqsubseteq B$, and $N \in B^{\mathcal{I}_\mathcal{K}}$. Taken together, these facts imply $M \in (\exists r.B)^{\mathcal{I}_\mathcal{K}}$.

($A \sqsubseteq \forall r.B$) Suppose that $(M, N) \in r^{\mathcal{I}_\mathcal{K}}$, for the elements $M$ and $N$ s.t. $M \in A^{\mathcal{I}_\mathcal{K}}$. There are two possible explanations for the origin of this edge. Either $\mathcal{K} \models \lceil M \rceil \sqsubseteq \exists r.\lceil N \rceil$, and $N$ is prime, or $\mathcal{K} \models \lceil N \rceil \sqsubseteq \exists r^-.\lceil M \rceil$, and $M$ is prime.

On the first case, because $N$ is prime for $M$ and $\mathcal{K}$, and because $B \in Qc(\mathcal{K})$, $B \in N$. Otherwise, the prime successor would be $N \cup \{B\}$. Therefore $\mathcal{K} \models \lceil M \rceil \sqsubseteq B$ and, by construction, $M \in B^{\mathcal{I}_\mathcal{K}}$.

In the second case, we show that $\mathcal{K} \models \lceil N \rceil \sqsubseteq B$. Suppose, by contradiction, that this is not the case. Therefore, there is an interpretation $\mathcal{I}$ s.t. $\mathcal{I} \models \mathcal{K}$ and $n \in \Delta^\mathcal{I}$ with $n \in \lceil N \rceil^\mathcal{I}$ and $n \notin B^\mathcal{I}$. Because $\mathcal{K} \models \lceil N \rceil \sqsubseteq \exists r^-.\lceil M \rceil$, there is some $m \in \Delta^\mathcal{I}$ s.t. $m \in \lceil M \rceil^\mathcal{I}$ and $(m, n) \in r^\mathcal{I}$. However, this would result in the violation of the axiom $A \sqsubseteq \forall r.B$, and $\mathcal{I}$ would not be a model of $\mathcal{K}$.

Finally, because $\mathcal{K} \models \lceil N \rceil \sqsubseteq B$, $N \in B^{\mathcal{I}_\mathcal{K}}$.

□

[4] Given that $C$ and $D$ can be any concept whatsoever, the resulting TBox $\mathcal{T}^*$ may have to be renormalized.

**Lemma 2.24.** *Let* $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ *be an* $\mathcal{ELI}_\bot$ *KB,* $\mathcal{I}_\mathcal{K} = (\Delta^{\mathcal{I}_\mathcal{K}}, \cdot^{\mathcal{I}_\mathcal{K}})$ *be its canonical model,* $A, B \in \mathsf{sig}_\mathsf{C}(\mathcal{K})$ *and* $a \in \mathsf{sig}_\mathsf{I}(\mathcal{K})$.

1. $\mathcal{K} \models A \sqsubseteq B$ *iff* $\{A\} \in B^{\mathcal{I}_\mathcal{K}}$;
2. $\mathcal{K} \models A(a)$ *iff* $a \in A^{\mathcal{I}_\mathcal{K}}$.

*Proof.* Both properties follow directly from Definition 2.21. □

# Chapter 3

# Nonmonotonic Reasoning

Classical logics are monotonic. Formally, this means that if some set of formulas $\Gamma$ entails a formula $\Phi$, every proper superset of $\Gamma$ also entails $\Phi$. Informally, monotonicity means increasing the presupposed information augments the entailed conclusions. Although this property is cogent for various domains, such as mathematics, human reasoning often lacks it.

Suppose a doctor is treating a patient whose symptoms are similar to the textbook definition of an extremely contagious disease. The doctor can assume that the patient is indeed suffering from the disease, therefore isolating them from others even though some tests are necessary to confirm the diagnosis. In this scenario, the doctor operates under the assumption that the patient suffers from the contagious disease but can retract this new information if the tests come back negative.

Another scenario where nonmonotonicity comes into play is legal reasoning. It is often assumed that everyone is innocent unless proven otherwise. This principle is nonmonotonic, as proving the guilt of someone would entail retracting the previous conclusion that they were innocent.

Those two examples highlight scenarios where nonmonotonicity is useful: dealing with incomplete information and with the transformation of the information. In the first case, artificial intelligence systems may benefit from drawing conclusions before having all the available information, as gathering it can be costly, and there may be urgency in taking some action. In the second case, some conclusions may become invalid in light of some change in the world or the system's knowledge.

There are several reasoning patterns in which nonmonotonicity plays an important role, such as inductive reasoning, abductive reasoning, and typicality-based reasoning. Inductive reasoning is characterized by extrapolating tendencies and rules from available data and is widely used within scientific reasoning. In abductive reasoning, the agent tries to come up with the best explanation for a series of facts, even though the facts themselves may be insufficient for a doubt-free conclusion. This method was employed by the famous detective Sherlock Homes, even though Holmes himself incorrectly called his way of drawing inferences *deduction*. Finally, typicality-based reasoning is very prevalent in concept formation and categorization. In a nutshell, this reasoning pattern consists in judging potential instances of a concept by comparing it to a *prototype*, a fictional object that embodies the most common and salient features of the members, even if some of them are not *necessarily* present in every member. As a popular adage says: if it has the nose of a pig, the ears of a pig, and the tail of a pig, it must be a pig. On the other direction, we may assume that a member of a given class has the prototypical qualities attributed to it. Hence, if we know that some animal is a bird, we usually conclude that it flies, even though some birds, such as ostriches and penguins, do not.

It is debatable whether robust AI systems must imitate human reasoning patterns. However, some form of nonmonotonic reasoning seems unavoidable as the area progresses. Tasks that deal with incomplete information and transforming scenarios are ubiquitous in the problems tackled by AI systems. Self-driving cars have to take unavoidable decisions even when they do not possess all the information required for a decision within a safe margin. Systems for medical diagnosis have to operate under transforming information, as new exams may rule out some conditions and point to others.

In the realm of knowledge representation and reasoning, nonmonotonicity is of utmost importance. Several areas of knowledge are not representable under a monotonic paradigm. In medical ontologies, for example, it may be necessary to represent things that are often true, but that admit exceptions. Humans have five fingers in each hand unless they suffer from polydactyly, for example. Areas that deal with hierarchies of rules may also need nonmonotonic representation capacity in order to formalize their presuppositions. In legal reasoning, a more general law may overrule the more specific ones. A law may regulate violence, but another more specific one can create an exception if the violence is a form of self-defense.

In the rest of this section, two nonmonotonic reasoning paradigms will be presented: the KLM framework and circumscription. Those paradigms were chosen because they greatly impacted typicality-based reasoning in DLs, even though they were proposed to different formalisms. The eventual inadequacies of translating them to DLs is a central topic in the thesis. For a more in-depth exploration of nonmonotonic logics, including several other nonmonotonic calculi, the reader is referred to [AW97] and [Bre91].

## 3.1 The KLM Framework for NMR systems

The KLM framework is the name given to an influential hierarchy of nonmonotonic reasoning systems proposed by Krauss, Lehman, and Magidor (1990), followed up in other papers by the same authors, such as [LM92], and [Leh95]. The authors' motivation was to outline nonmonotonic systems by sets of positive properties instead of focusing on what they lacked. Their study is based on proof-theoretical characterizations of nonmonotonic systems with Gentzen-style sequents. Furthermore, Krauss, Lehman, and Magidor lay out semantical characterizations for each of the five systems in their hierarchy, providing representation theorems for each.

The authors hoped that their hierarchy would provide a benchmark to evaluate a wide array of NMR systems. It should serve to compare autoepistemic logic and circumscription, techniques already popular at the time of the publication. This goal was reasonably successful, and the KLM properties are widely used in NMR research up to this day. However, the framework also has shortcomings that can be limiting, as will be discussed.

The language of the KLM framework is that of propositional calculus. Therefore, there are no rules involving quantifiers. To account for defeasibility, the authors introduce a symbol for *conditional assertions*, $\mid\sim$. Conditional assertions express a binary relationship between formulas. Given two formulas $\phi, \psi$, the conditional assertion $\phi \mid\sim \psi$ is understood as "'*if* $\phi$*, normally* $\psi$", or as "$\psi$ *is a plausible consequence of* $\phi$" [KLM90, p. 7]. Sets of conditional assertions are *consequence relations* [KLM90, p. 7].

The introduction of conditional assertions allows the partitioning of knowledge into two realms. One realm has a stable, always true set of rules, while the other represents regularities that usually hold but are not necessarily true, i.e., the defeasible information.

The original paper discusses five different logical systems, presented from the weaker to the strongest. Those systems are: **C**, **CL**, **CM**, **P** and **M**, the last one corresponding to plain monotonic reasoning.

**Definition 3.1** (Cumulative Reasoning – System **C**)**.** *[KLM90, pp. D. 1]*

*A consequence relation* $\mid\sim$ *is cumulative* iff *it complies with the axiom schema* reflexivity *and the rules below:*

*(Reflexivity)* $$\phi \mid\sim \phi$$

*(Left Logical Equivalence)* $$\frac{\models \phi \equiv \psi \qquad \phi \mid\sim \chi}{\psi \mid\sim \chi}$$

*(Right Weakening)* $$\frac{\models \phi \rightarrow \psi \qquad \chi \mid\sim \phi}{\chi \mid\sim \psi}$$

*(Cut)* $$\frac{\phi \wedge \psi \mid\sim \chi \qquad \phi \mid\sim \psi}{\phi \mid\sim \chi}$$

*(Cautious Monotonicity)* $$\frac{\phi \mid\sim \psi \qquad \phi \mid\sim \chi}{\phi \wedge \psi \mid\sim \chi}$$

The intuition for the rules are the following:

- *reflexivity* expresses the fact that every formula normally implies itself. It is a basic principle that can only be broken in a scenario involving theory change [KLM90].
- *Left Logical Equivalence* expresses the fact that logically equivalent formulas should conditionally imply the same thing. Conditional assertions are impervious to syntactical differences on the left-hand side.
- *Right Weakening* expresses the fact that something that conditional assertions carry all the logical implications of their consequent, i.e., if something normally follows from a situation, everything that *logically* follows also follows. If pets are usually dogs, it also follows that pets are usually mammals.

- *Cut* backs the *cumulative* aspect that names system **C**. It is a principle valid in monotonic reasoning that does not imply monotonicity. Roughly, it says that if a bigger set of hypotheses allows conditionally inferring something, but one hypothesis also allows conditionally inferring the others, this hypothesis should be sufficient to derive the conclusion. In KLM's words: "Its meaning, it should be stressed, is that a plausible conclusion is as secure as the assumptions that ground it. Therefore it may be added (this is the origin of the term cumulative) into the assumptions. There is no loss of confidence along the chain of derivation." [KLM90, p. 12]
- *Cautious Monotonicity* says that facts that are conditionally inferred from some hypothesis should not invalidate other facts also conditionally inferred from the same hypothesis. If birds normally fly and have feathers, flying birds normally have feathers.

**Definition 3.2** (Cumulative Reasoning with Loop – System **CL**). *A consequence relation* $\mid\!\sim$ *is cumulative with loop* iff *it is cumulative and complies with the following rule:*

$$\textit{(Loop)} \qquad \frac{\phi_0 \mid\!\sim \phi_1, \phi_1 \mid\!\sim \phi_2, \ldots, \phi_{k-1} \mid\!\sim \phi_k, \phi_k \mid\!\sim \phi_0}{\phi_0 \mid\!\sim \phi_k}$$

The introduction of *loop* has a semantical motivation. Models for cumulative logic have a preference relation defined over *worlds*, and introducing this rule is shown to be equivalent to imposing transitivity over this order. Regardless, the rule states that if there is a loop of conditional assertions, any of them can be derived from any other. [KLM90, p. 22]

**Definition 3.3** (Preferential Reasoning – System **P**). *A consequence relation* $\mid\!\sim$ *is preferential* iff *it is cumulative and complies with the following rule:*

$$\textit{(Or)} \qquad \frac{\phi \mid\!\sim \chi \qquad \psi \mid\!\sim \chi}{\phi \vee \psi \mid\!\sim \chi}$$

The rule *or* states that if two hypotheses separately conditionally derive some fact, knowing that at least one of them is true is enough to derive the fact. Rain normally makes the ground wet, and snow usually makes the ground wet. If the forecast says that it will either rain or snow, we should prepare our waterproof boots.

The system **P** is a powerful one and allows deriving *loop* as a theorem. Before appearing in the KLM paper, it was studied by several authors, such as [Ada65] and [PG88].

**Definition 3.4** (Cumulative Monotonic Reasoning – System **CM**). *A consequence relation* $\mid\!\sim$ *is cumulative monotonic* iff *it is cumulative and satisfies the following rule:*

$$\textit{(Monotonicity)} \qquad \frac{\models \phi \to \psi \qquad \psi \mid\!\sim \chi}{\phi \mid\!\sim \chi}$$

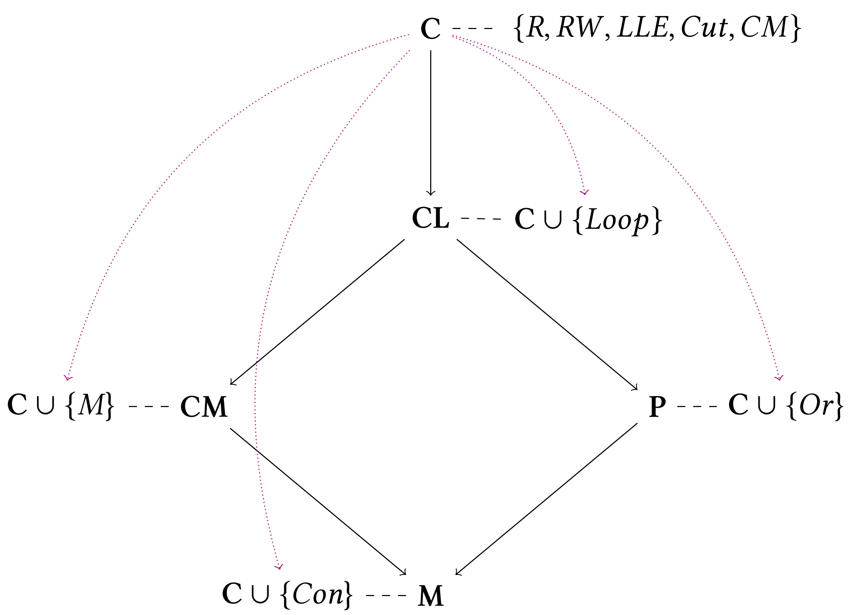


**Figure 3.1:** *Diagram representing the NMR systems presented in [KLM90]. Dashed black lines attribute systems to their axioms. Dotted purple lines highlight how* C *is the minimal system, and straight black arrows depict the relationship between their inferential strengths.*

Cumulative monotonic reasoning is a strong reasoning framework that allows deriving loop, therefore strictly extending **CL**. However, it is incomparable to **P**. This rule seems undesirable for nonmonotonic systems, although **CMR** is not fully monotonic in itself. The rule can be instantiated to generate inferences with undesirable conclusions, such as *penguins are birds* and *birds normally fly*. Therefore, penguins normally fly.

**Definition 3.5** (Monotonic Reasoning – System **M**). *A consequence relation* $\mid\!\sim$ *is monotonic* iff *it is cumulative and complies with the following rule:*

*(Contraposition)* $$\frac{\phi \mid\!\sim \psi}{\neg\psi \mid\!\sim \neg\phi}$$

Monotonic reasoning needs no introduction and is proved to be strictly stronger than both **P** and **CM** [KLM90, pp. L. 34 & 35]. The hierarchy of those systems is depicted in Figure 3.1.

## 3.2 Extensions of the KLM Framework: Rational and Lexicographic Closures

Much debate followed the original KLM paper. Two extensions of the framework are of special importance to DLs of typicality: *rational reasoning* and the *lexicographic closure*.

The discussion around rational reasoning started in [KLM90] and was further developed in (1992). Preferential reasoning is considered the core of a good NMR system. However, the authors noticed that it does not comply with three properties deemed desirable for NMR systems. The properties are:

$$\text{(Negation Rationality)} \qquad \frac{\phi \wedge \chi \not\mid\!\sim \psi \qquad \phi \wedge \neg\chi \not\mid\!\sim \psi}{\phi \not\mid\!\sim \psi}$$

$$\text{(Disjunctive Rationality)} \qquad \frac{\phi \not\mid\!\sim \chi \qquad \psi \not\mid\!\sim \chi}{\phi \vee \psi \not\mid\!\sim \chi}$$

$$\text{(Rational Monotonicity)} \qquad \frac{\phi \wedge \psi \not\mid\!\sim \chi \qquad \phi \not\mid\!\sim \neg\psi}{\phi \not\mid\!\sim \chi}$$

Their negative aspect is what unifies the three rationality postulates and distinguishes them from the KLM properties. They cannot be expressed as Horn rules, i.e., they are not derivations of some formula from other formulas. On the contrary, they "(...) deduce the absence of an assertion from the absence of other assertions." [LM92, p. 16] The intuitions for the rules are the following:

- *Negation rationality* guarantees that defeasible inferences should not be drawn exclusively from the lack of information. If $\psi$ usually follows from $\phi$, it should usually follow from at least one of the more specific, disjoint scenarios $\phi \wedge \chi$ and $\phi \wedge \neg\chi$.
- On a similar fashion, *disjunctive rationality* states that disjunctive defeasible inferences should be supported by at least one of the disjuncts. If animals that are bats or birds usually fly, we expect that at least one of *bats usually fly* and *birds usually fly* to be valid.
- *Rational monotonicity* is a bit more intricate. Lehman and Magidor justify it by highlighting how it minimizes information retraction: "(...) it says that an agent should not have to retract any previous defeasible conclusion when learning about a new fact the negation of which was not previously derivable." .

Put together with preferential reasoning, they are arranged in order of increasing strength. Rational monotonicity, the stronger one, implies disjunctive rationality, which, in turn, implies negation rationality. The authors propose *rational reasoning*: a framework that satisfies all properties from preferential reasoning plus rational monotonicity. This framework is coupled with a proper semantics with a representation theorem.

Some reasonable principles are desirable for defeasible reasoning but also resist codification as sequent-based rules such as the ones discussed. To address some of these concerns, Lehmann 1995 discusses another improvement over rational reasoning: the *lexiographic closure*. The term closure alludes to the approach that considers sets of conditional assertions closed under certain properties. The informal principles of defeasible reasoning evoked in the paper are [Leh95, p. 63]:

1. *Presumption of typicality* states that further specifying something should hold less specific conclusions unless there is reason to conclude otherwise. Formally, given

$\phi \mid\sim \psi$, we do not know whether we should conclude that $\phi \wedge \chi \mid\sim \psi$ or $\phi \mid\sim \neg\chi$, and presumption of typicality favors the former.

2. *Presumption of independence* is a further refinement of presumption of typicality. It states that we should expect conclusions to be independent, even when they are derived from the same premises. Presumption of independence is a central concept for typicality DLs, as it regulates concept inheritance. If *birds normally fly* and *birds normally have feathers*, we suppose that flying and being feathered are independent. Hence, if we face a penguin, a bird that does not fly, we still should be able to conclude that it has feathers.

3. *Priority to typicality* says that between presumptions of typicality and independence, typicality prevails. It is a principle that deals with further refinement in concepts that have some kind of clash. In the well-known penguin example, we know that *birds normally fly* and that *penguins (& birds) normally do not fly*. Hence, when we face a particular penguin species, the blue penguin, we should conclude that it does not fly.

4. *Respect for specificity* expresses the overriding of general information by a more specific one. It says that, when facing conflicting inferences, one of them with an antecedent that is a refinement of the other, we should opt for the more specific one. *Penguins (& birds) normally do not fly* conflicts with *Birds normally fly*, and we should discard the latter, i.e., penguins should not fly.

Lexicographic closure is a syntactic-based reasoning procedure that refines rational closure complying with those principles.[1] Because it is syntactic based, different but logically-equivalent formulas can yield different outputs. Whether this is a feature or a bug is a topic of considerable discussion. An argument for treating logically-equivalent information differently, depending on how it is presented, is the following: suppose that $\phi$ and $\psi$ are statements about an airplane's altitude and speed. $\phi \wedge \psi$ is a report given by a single sensor, while $\phi$ and $\psi$ are two separate reports originating in two independent sensors. Although their information is the same, it should be treated separately. The reader can evaluate the strength of this line of reasoning. It should be noticed, however, that human imagination is wide, and scenarios that invalidate principles of reasoning are not a scarce commodity.

[1] The procedure for generating the lexicographic closure will be discussed in more detail in its version for DLs in Section 4.3.1.

# Part II

# Description Logics of Typicality

# Chapter 4

# A Survey of Description Logics of Typicality

Description logics that deal with typicality have a long story. The DL and NMR research communities developed a wide array of techniques to incorporate aspects of the prototype theory of concepts at several levels. Those levels include the expansion of the language with special operators and constructions, the introduction of new kinds of concept and role inclusion, and algorithmic techniques to draw non-monotonic inference.

In this chapter, we present an overview of the area's state-of-the-art. The text is the result of a survey that systematically examined more than 230 papers from more than 231 authors. The main presentation is divided in three topics: representation of typicality, which covers the tools developed to represent typciality within DLs; semantics, which exposes the new semantics proposed to the extended DLs; and reasoning, which presents some proof-theory methods for the extended DLs. After the main exposition, the chapter has a section covering issues and open problems. Finally, a meta-analysis present some interesting data on the papers examined. The meta-data of all the papers examined was transformed into a graph database, which reveals research and collaboration patterns of the community.

Sections on *typicality models* semantics and reasoning, present in the original version of the survey, were ommited, as they are explored in finer detail in Part III, Chapters 5, 6, 7, and 9.

## 4.1 Representing Typicality in DLs

Typicality has many forms. It may appear as reasoning that assumes normality; in the partitioning of a set's members into more and less typical elements; in the degrees of representativeness that an individual has for a concept; in the determination of a hypothetical maximally representative element, i.e., a prototype. Therefore, there are numerous ways of representing typicality within DLs. In this section, we cover the most relevant approaches, discuss their way of presenting typicality, and debate some of the advantages and shortcomings of each method.

### 4.1.1 Defeasible Inclusions

Representing the relationship between concepts in the form of GCIs is one of the core capacities of DLs. The GCI $C \sqsubseteq D$ expresses that $C$s are $D$s, which can be readily translated to the language of first-order logic as $\forall x(C(x) \rightarrow D(x))$. It is possible to portray typicality as the knowledge of what is expected to be the case but is not necessarily so, such as birds *typically* fly. Therefore, it is natural to represent it as a special kind of GCI that is not necessarily true of every element belonging to the antecedent, $C$. This special form of GCI is called a *Defeasible Concept Inclusion* (*DCI*).

This approach is one of the oldest and most prolific ways of incorporating typicality into DLs and dates back to the late 2000s [BHM09] and early 2010s [CS10]. In the latter, DCIs are represented with a symbol for defeasible sequents, $\mid\!\sim$, alluding to the KLM [KLM90] framework for defeasible reasoning. Nonetheless, the most common option in the literature is the defeasible inclusion denoted by $C \mathrel{\underset{\sim}{\sqsubset}} D$, which conveys that $C$s are typically $D$s. DCIs are generally stored in an isolated component of the KB called *defeasible box* (*DBox*), usually denoted by $\mathcal{D}$.

One feature of typicality-informed reasoning is its dependence on contexts. Because categorization stems from experience, what counts as typical may be a function of what is normally seen by a given agent. DCIs, however, cannot capture this, as $C \mathrel{\underset{\sim}{\sqsubset}} D$ means "$C$s are usually $D$s" in every context. There are diverse strategies for dealing with this. Some try to circumvent this limitation by attaching indexes to the DCIs. Those indexes can be role names, as proposed by Britz & Varzinczak (2017), ((2017)), (2018). One of the advantages of such an approach is that there is a finite and fixed set of contexts ready for use. Additionally, some rationale backs it up in a broader sense: roles determine relationships between individuals, and those relationships determine *contexts* of interaction.

Some more expressive DLs have role hierarchies – i.e., inclusion between roles – alongside the standard GCIs. They can represent things such as isFather $\sqsubseteq$ isParent, i.e., that fatherhood implies parenthood. Because instances of relationships can be more or less typical, it is natural to extend the DCIs framework to role inclusions. Britz & Varzinczak (2016), (2017) and (2017) present DLs that can represent inclusions such as isParent $\mathrel{\underset{\sim}{\sqsubset}}$ isProgenitor, which express that the parents of someone are usually, although not necessarily, their progenitors.

The idea of defeasibility is not limited to inclusions. Britz & Varzinczak (2016), (2017), (2017) propose adding *defeasible constructors* to the language, which provides alternative ways of conveying defeasible knowledge. Some examples are the *defeasible value restriction*, $\forall\!\!\!\sim r.C$, which defines concepts whose usual role successors for the role $r$ are $C$s; and the *defeasible* (*qualified*) *number restrictions*, $\gtrsim nr.C$, $\lesssim nr.C$ and $\simeq nr.C$, which defines concepts that typically have at least, at most and exactly $n$ role successors, respectively, for the role the $r$, that are $C$s.

### 4.1.2 Typicality Operators

Another way to represent typicality within DLs is to enrich the language with a "typicality operator". This technique was introduced in [Gio+07] and appeared in many other works, such as [Gio+08], [Gio+09] and [Gio+13]. Different research groups – see, for example, [BMV11], and [Var18] – also adopted the idea, although the choice of symbols varies throughout the literature. The typicality operator, denoted by $\mathbf{T}(\cdot)$, is a modal-like operator that, when applied to a concept $C$, designates the most typical, normal, or common members of $C$.[1] A sparrow is in $\mathbf{T}(\mathsf{Bird})$, but a penguin is not.

Unlike DCIs, the typicality operator introduces new concepts to the language. Concepts are characterized by GCIs. Suppose we have a concept referring to typical birds, denoted by $\mathbf{T}(\mathsf{Bird})$. To characterize typical birds, we need GCIs such as $\mathbf{T}(\mathsf{Bird}) \sqsubseteq \mathsf{Flying}$, a construction that resembles DCIs in many ways. Their intended meaning is slightly different. While $C \sqsubsim D$ means that $C$s are typically $D$s, $\mathbf{T}(C) \sqsubseteq D$ means that typical $C$s are $D$s. However, for most practical purposes, it is possible to translate $\mathbf{T}(C) \sqsubseteq D$ to $C \sqsubsim D$, and vice-versa. This translation is common in the literature. It is used to adapt many procedures defined for typicality operators to defeasible inclusions or the other way around. Nonetheless, there are some meaningful differences. Most DLs containing the typicality operator are syntactically limited to allow it only on the left-hand side of GCIs, as in the example given in the last paragraph. However, there are some more expressive logics in which it can also appear on the right-hand side. Giordano & Dupré (2018) give an example: "friends of Mary are typical students", which can be translated to the language of DL as $\exists \mathsf{friendOf}.\{\mathsf{mary}\} \sqsubseteq \mathbf{T}(\mathsf{Student})$.[2], a GCI that cannot be translated by the equivalence $\mathbf{T}C \sqsubseteq D \equiv C \sqsubsim D$, as the typicality operator occurs on the right-hand side instead of the left-hand side.

Typicality operators also enable characterizing atypical instances of a given concept. The DL presented in [BNM16] has a dual constructor, denoted by $\cdot^{\mathcal{E}}$, that identifies atypical instances of a concept. Introducing such constructors may not be strictly necessary, as it is possible to define atypicality through other constructors, such as in $C \sqcap \neg\mathbf{T}(C)$. This translation rests on the assumption that all information on atypicality is negative, i.e. atypical instances are those that do not have typical properties. A challenge to this assumption would be that, in some cases, atypical individuals are defined by positive traits. Swans are typically white, but atypical swans are not only non-white but also black. In this modeling scenario, having an operator to characterize atypicality could be useful.

Abnormal concepts also appear in circumscribed DLs, such as the ones described in [BLW06] and [BLW09]. Circumscription is a semantical approach to non-monotonicity that constrains the models of some KB by minimizing the extension of some predicates specified by a *circumscription pattern*. There is no real atypicality operator in circumscribed DLs, which have the same constructors as the monotonic DLs that generate them. However, one straightforward way of implementing a circumscription pattern is to add atypical versions of some concepts – e.g., $Ab_{\mathsf{Bird}}$, for Bird – to construct defeasible inclusions.

[1] Analogous approaches can be seen in [BS17], where the operator is called *normality* operator, and denoted by $N$; in [Var18], where the operator is denoted by $\bullet$; and in [BNM16], where $\delta C$ denotes "the concept $C$ by default" [BNM16, p. 249].

[2] Notice that this DL also has *nominals*, e.g. $\{\mathsf{mary}\}$.

This is done in the same way as the translation from the typicality operator to defeasible inclusions. To express that typical birds fly, the following axiom is added to the KB: $\mathsf{Bird} \sqcap \neg Ab_{\mathsf{Bird}} \sqsubseteq \mathsf{Flying}$. Then, the circumscription pattern is defined to minimize all the atypical concepts. We cover circumscribed DLs in more detail in Section 4.2.5.

Probabilities are another way to enrich the typicality operators. Pozzato (2015), (2016), (2017), (2018) develops a framework that associates probabilities to concept inclusions with typicality operators on their left-hand side, indexing GCIs in a similar way to the contextual indices introduced in 4.1.1. Given a number $p \in (0, 1)$, $\mathbf{T}(C) \sqsubseteq_p D$ represents that the typical $C$s are usually $D$s with a probability of $p$. This language adds finer detail to the otherwise qualitative description of *typically*.

Following defeasible role hierarchies, there are also typicality operators for roles. The typicality operator $\bullet$ in [Var18] can be applied to concepts and roles. In the second case, it selects the most typical pairs of a role. Similarly to concepts, role pairs can be more or less typical, referring to other pairwise relationships. For example, parents are also usually biological parents, which can be expressed by $\bullet\mathsf{parent\_of} \sqsubseteq \mathsf{biological\_parent\_of}$. However, this is only sometimes the case, as other arrangements exist, such as adoption. Typical roles work in the same way as typical concepts. It is assumed that there are more and less typical representatives of a role and that the most typical present certain properties that are not necessary for membership.

### 4.1.3 Reiter-like Defaults

One of the most well-known approaches to non-monotonic reasoning is the adoption of the so-called default rules, proposed by Raymond Reiter in (1980), transposed to DLs by Baader and Hollunder (1995). Default rules have three components and are usually denoted by $(\alpha : \beta_1 \ldots \beta_2 \setminus \gamma)$, where $\alpha$ is called the *prerequisite*; $\beta_i$ are called *justifications*; and $\gamma$ is called the *conclusion*. The conclusion should follow from the prerequisite and the consistency of the justifications. Defaults are usually interpreted in epistemic terms, i.e., one should believe $\gamma$ if they believe $\alpha$ and every $\beta_i$ is consistent with their knowledge.

Baader and Hollunder (1995) introduce *terminological default theories*, which are combinations of the DL $\mathcal{ALCF}$ with open defaults theories from Reiter. A terminological default theory is a pair $(\mathcal{A}, \mathcal{D})$, where $\mathcal{A}$ is a regular ABox and $\mathcal{D}$, is "a finite set of rules whose prerequisites, justifications and consequents are concept terms." [BH95, p. 155]

Frota *et al.* (2014) takes a slightly different approach, where a *description default theory* is a pair $(W, \mathcal{D})$, where $W$ is the union of an $\mathcal{ALC}$ TBox and ABox. In the same vein, Kolovski, Parsia, and Katz (2006) define *terminological default theories* by $(W, \mathcal{D})$. In this case, $W$ stands for a $\mathcal{SHOIN}$ KB instead of an $\mathcal{ALC}$ KB.

Some approaches combine DLs with non-monotonic, closed-world rules. In this field, Motik & Rosati (2010) developed the *MKNF*$^+$framework – *minimal knowledge and negation as failure*. This framework, which is based on *answer set programming* (ASP), deals with *hybrid KBs*, a combination of standard KBs with prolog rules. Rules have the form $H_1 \vee \cdots \vee H_k \leftarrow B_1, \ldots, B_m, \mathit{not}\, B_{m+1}, \ldots, \mathit{not}\, B_n$, where $B_i$ are *body literals* and $H_i$, *head literals* [MR10, p. 8]. There are also two classes of negation: $\neg$, the classical negation, and *not*, the non-monotonic negation as failure. This framework allows expressing default rules by setting $\gamma \leftarrow \alpha, \bigwedge_{i=1}^{n} not\neg\beta_i$. The authors present the following example:

$$\mathsf{HeartOnLeft}(x) \leftarrow \mathsf{Vertebrate}(x), not\neg\mathsf{HeartOnLeft}(x)$$

Given an individual, Peter, which is a vertebrate, Vertebrate(Peter), we conclude that it has its heart on the left side of the body (HeartOnLeft(Peter)). However, if the hybrid KB also has (or allows to derive) ¬HeartOnLeft(Peter), the rule is not applied. This scenario is a good example of reasoning based on typicality, as vertebrates usually have their hearts on the left side. However, there also is a rare condition in humans, named *situs inversus*, whose bearers have the heart on the other side.

### 4.1.4 Weighted Concept Combination

Most of the techniques presented so far enable drawing conclusions from individuals deemed typical representatives of a given class. For instance, we may conclude that some bird flies because we have reasons to believe it is a typical bird. Weighted concept combination inverts this direction by allowing inferences that employ (typical) characteristics to group individuals similarly to classification algorithms.

Porello *et al.* (2019) introduce a family of constructors denoted by $\mathbb{W}$, which they call *tooth operators*. Each operator has an associated vector of weights, $w \in \mathbb{R}^n$, and a threshold value, $t \in \mathbb{R}$. Given a list of $\mathcal{ALC}$ concepts, $C_1, \ldots, C_n$, the combination $\mathbb{W}^t_w(C_1, \ldots, C_n)$ designates a new concept. The intuition is that each individual has a value, calculated as the sum of the weights of the concepts $C_i$ it instantiates. The operator selects those individuals whose value surpasses the threshold $t$. This definition covers some well-known theories of conceptualization from cognitive sciences, such as the prototype view, the exemplar view, the knowledge view, Gärdenfors' conceptual spaces, and Barsalou's theory of frames [Por+19, p. 2]. As an example, the authors give a model of *elephant*:

$$E = \mathbb{W}^t((\mathsf{Large}, w_1), (\mathsf{Heavy}, w_2), (\mathsf{hasTrunk}, w_3), (\mathsf{Grey}, w_4))$$

Tooth operators are very general and can model several kinds of operators as special cases, such as Boolean operators – for example, $C \sqcap D = \mathbb{W}^2_{(1,1)}(C, D)$ for concept conjunction and $C \sqcup D = \mathbb{W}^1_{(1,1)}(C, D)$ for concept disjunction – and majority voting, i.e., a concept defined by those individuals that satisfy the majority of the concepts in a list – $C = \mathbb{W}^{\frac{n}{2}}_{(1,\ldots,1)}(C_1, \ldots, C_n)$. The authors introduce a formalization of the prototype of a given concept $C$. The prototype is derived from sets of attributes, which are disjoint properties. Color is an attribute whose properties are concepts representing particular colors, such as red and green. Each attribute has an associated weighted, $d_i$, which represents its importance for the categorization of $C$-members – its "diagnositicy" – and each property has a salience, $s^j_i$, i.e. the degree to which that particular property is representative of $C$ – its "typicality" –. The prototype of $C$, $\pi_C$, is then defined by:

$$\pi_C := \{(Q^1_1, s^1_1 \cdot d^1), \ldots, (Q^1_r, s^1_r \cdot d^1), \ldots, (Q^n_q, s^n_1 \cdot d^1), \ldots, (Q^n_1, s^n_1 \cdot d^n), \ldots, (Q^n_m, s^n_m \cdot d^n)\}$$

By defining a threshold, $t$, one can use the $\mathbb{W}$ operator to define the focal region as

$$C := \mathbb{W}\{(Q_1^1, s_1^1 \cdot d^1), \ldots, (Q_r^1, s_r^1 \cdot d^1), \ldots, (Q_q^n, s_1^n \cdot d^1), \ldots, (Q_1^n, s_1^n \cdot d^n), \ldots, (Q_m^n, s_m^n \cdot d^n)\}$$

Finally, maximal typical individuals, such as those selected by typicality operators, $\mathbf{T}(C)$, can be modeled by selecting the maximal elements according to the construction above. This is done by defining the threshold as the maximal value, i.e. $\mathbb{W}^{\max}$.

Another approach that subscribes to weighting characteristics to determine concept membership is [BE16]. The authors take inspiration from the conceptual spaces proposed by Gärdenfors (2000), and define prototypes as focal points of some space. Typicality is determined by measuring the distance between a particular individual and this focal point. This operation is performed by the automata-based reasoning for DL, except, in this case, the automata have weights.

Weighted automata receive a tree-shaped interpretation with the individual at the root and return a non-negative integer representing the distance of the individual to the focal point. This procedure paves the way for the definition of the so-called *threshold concepts*, denoted by $P_{\sim n}(\mathcal{A})$, where $\sim \in \{<, \leq, >, \geq\}$, $\mathcal{A}$ is the weighted automaton, and $n$ is the threshold. Threshold concepts encompass all the elements within $\sim n$ distance of $C$, according to $\mathcal{A}$ [BE16, p. 2]. As an example, the authors model the concept of a cup, "(...) a small container with handles, which can hold liquids and is made of plastic or porcelain" [BE16, p. 8]. In the proposed formalism, this description is denoted by: $Container \sqcap Small \sqcap \exists hasPart.Handle \sqcap \forall holds.Liquid \sqcap \forall material.(Glass \sqcap Porcelain)$, which is a crisp definition and is translated to the weighted version, with modal-like $\Box$ and $\Diamond$ communicating necessary and likely properties to

$$(Container \vee 3) \wedge (Small \vee 1) \wedge (\Diamond(hasPart \wedge Handle) \vee 1) \wedge \Box(\neg holds \vee (Liquid \vee 2))$$
$$\wedge \Box(\neg material \vee ((Glass \vee 1) \vee (Porcelain \vee 1)))$$

The intuition is that an object will either have a property – e.g. being a container – or pay the corresponding price in the form of an increase in its overall distance to the focal point. The greater the number is, the farthest away from the focal point the individual will be. Threshold concepts can be articulated not only in terms of a maximal distance, which amounts to its typicality but also in terms of a minimum distance, which amounts to exceptionally. A more nuanced take on typicality arises by grading distance with integers, as several levels of proximity to the focal point are possible.

This approach has been further developed – e.g. [BG17] – under the name of *threshold concepts*.

## 4.2 Semantics

Despite the vast literature on DLs of typicality, there has yet to be a consensus on its semantical characterization. In this section, we explore several variations of semantics based on preference relations and those based on canonical models.

### 4.2.1 Preferential Semantics

Constructions such as $\mathbf{T}(C)$ and defeasible inclusions such as $C \sqsubsetsim D$ pick the most typical elements of $C$, something that can be accomplished through *selection functions*. Those functions first appear in the context of DLs of typicality in [Gio+07]. The paper proposes extending regular interpretations with a function $f : \mathcal{P}(\Delta) \to \mathcal{P}(\Delta)$ that, given a set representing the domain of a concept, $C$, outputs a subset of this set that contains its most typical elements. Giordano *et al.* (2007) postulate five properties that such a function should obey. Those properties are modeled after the KLM conditional logic [Gio+07, p. 261]:

$$(f_{\mathbf{T}}-1)\ f_{\mathbf{T}}(S) \subseteq S$$
$$(f_{\mathbf{T}}-2)\ \text{if } S \neq \emptyset\text{, then also } f_{\mathbf{T}}(S) \neq \emptyset$$
$$(f_{\mathbf{T}}-3)\ \text{if } f_{\mathbf{T}}(S) \subseteq R\text{, then } f_{\mathbf{T}}(S) = f_{\mathbf{T}}(S \cap R)$$
$$(f_{\mathbf{T}}-4)\ f_{\mathbf{T}}(\bigcup S_i) \subseteq \bigcup f_{\mathbf{T}}(S_i)$$
$$(f_{\mathbf{T}}-5)\ \bigcap f_{\mathbf{T}}(S_i) \subseteq f_{\mathbf{T}}(\bigcup S_i)$$

One way of realizing such a function is by defining a strict partial order over the domain. This order must satisfy the *smoothness condition*, which ensures that there is a minimal element for every set of elements. Formally, "(…) for all $S \subseteq \Delta$, for all $x \in S$, either $x \in Min_<(S)$ or $\exists y \in Min_<(S)$ such that $y < x$" [Gio+07, p. 262]. An interpretation is a triple $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}}, <)$, where $\Delta$ and $\cdot^{\mathcal{I}}$ are the domain and interpretation function, and $<\subseteq \Delta \times \Delta$, is an order over the domain as specified. [3]

The formalism employed an auxiliary tool to bridge the gap between the typicality operator and the order $<$: the modal operator $\square$, modeled after Gödel-Lob's logic of provability. The semantics of the operator is given by $(\square C)^{\mathcal{I}} = \{x \in \Delta \mid \text{ for every } y \in \Delta, \text{if } y < x \text{ then } y \in C^{\mathcal{I}}\}$, i.e., it selects the elements of the domain for which every smaller element with respect to $<$ is in $C$. The semantics for the typicality operator is given by $\mathbf{T}(C) = \square\neg C \sqcap C$, i.e., the elements of $C$ for which all preferred elements are not in $C$. This formalism grounds the DL $\mathcal{ALC} + \mathbf{T}_{\text{min}}$, a non-monotonic extension of $\mathcal{ALC}$ with the typicality operator.

### 4.2.2 Multipreferentiality

Suppose the following scenario: Alice and Bruce are both high schoolers and athletes. Alice is a good student, and her sport is chess; Bruce, on the other hand, is a terrible student and plays basketball. Arguably, Alice is a typical student but an atypical athlete, as chess is an uncommon and exceptional sport in many ways. On the other hand, Bruce is an atypical student but a typical athlete. A single preference order would fall short of representing this toy-model scenario: either Alice < Bruce (or vice-versa), in which case Alice (or Bruce) would be both a typical athlete and a typical student, and Bruce (or Alice)

[3] Although we only mentioned the interpretation of $\mathbf{T}(C)$, these techniques can be easily transposed to defeasible DLs. A straightforward way is to say that a defeasible inclusion $C \sqsubsetsim D$ is verified by an interpretation $\mathcal{I}$ if for every minimal $x \in C^{\mathcal{I}}$ – selected by an appropriate $f$ or an order $<$ – $x \in D^{\mathcal{I}}$.

would be none, or they would be incomparable and would be both typical athletes and students.

A possible solution is to consider interpretations with several preference orders to account for different contexts. Although this problem was diagnosed early in the research, solutions only appeared recently. Gliozzi (2016) and Giordano & Gliozzi (2018) explore interpretations based on several local preference relationships that can transmit information to a more general, global one. Formally, the *enriched rational interpretation* is a structure: $\mathcal{M} = (\Delta, <_{A_1}, \ldots, <_{A_n}, <, \cdot)$, where $A_i$, $1 \leq i \leq n$, are concepts of the language [GG18, p. 5]. These concepts are on the right-hand side of inclusions of the form $\mathbf{T}(C) \sqsubseteq A_i$. GCIs of this kind are valid when all the minimal C-elements in the domain, according to the global order, $<$, and to the local one, $<_{A_i}$, are also $A_i$-elements. The global order is partially dependent on the concept-indexed orders. The *principle of specificity* conditions the global order to the partial ones in the following way: if two elements $x, y$ are such that $x <_{A_i} y$ and $y <_{A_j} x$, for two different concepts $A_i, A_j$, the global order favors the one that falsifies less specific properties.

Similarly, Britz and Varzinczak (2017) discuss the idea of a context-based defeasible concept inclusion mediated by several orderings, each representing a single context. One of the main differences between this approach and Giordano and Gliozzi's is that roles index the orderings instead of concepts. Furthermore, there is no global preference relationship, and each order, $<_r$, where $r$ is a role, corresponds to a single defeasible inclusion, $\sqsubseteq_r$. Formally, an ordered interpretation is defined by $\mathcal{O} := (\Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}})$, where $\Delta^{\mathcal{O}}$ is the domain; $\cdot^{\mathcal{O}}$ is the interpretation mapping and $\ll^{\mathcal{O}} := (\ll_1^{\mathcal{O}}, \ldots, \ll_{|R|}^{\mathcal{O}})$ is a set of orders on roles satisfying the smoothness condition. Therefore, $\ll_i^{\mathcal{O}} \subseteq r_i^{\mathcal{O}} \times r_i^{\mathcal{O}}$, for $i = 1, \ldots, \mid R \mid$.

The path from orderings over pairs to orderings over individuals is indirect. It is given by an intermediate ordering defined by

$$\prec_r^{\mathcal{O}} := \{(x, y) \mid \text{ there is some } (x, z) \in r^{\mathcal{O}} \text{ such that for all } (y, v) \in r^{\mathcal{O}} [((x, z), (y, v)) \in \ll_r^{\mathcal{O}}]\}$$

In other words, $x \prec_r^{\mathcal{O}} y$ is the case if a given pair with $x$ as its first element is minimal for every pair with $y$ as its first element. With $x \prec_r^{\mathcal{O}} y$ defined, it is possible to fetch minimal elements as usual. Notice, however, that minimal elements will always be minimal to some role-defined context.

### 4.2.3 Role Preferentiality

Given some domain $\Delta$, it is possible to introduce a preferential order on $\Delta \times \Delta$ to account for logics that can express defeasible role hierarchies. Varcinczak (2018) defines *bi-ordered interpretations* by: $\mathcal{B} := (\Delta^{\mathcal{B}}, \cdot^{\mathcal{B}}, <^{\mathcal{B}}, \ll^{\mathcal{B}})$, where $\Delta^{\mathcal{B}}$ is the domain; $\cdot^{\mathcal{B}}$ is an interpretation mapping; $<^{\mathcal{B}} \subseteq \Delta^{\mathcal{B}} \times \Delta^{\mathcal{B}}$ is a typicality ordering over the domain and $\ll^{\mathcal{B}} \subseteq (\Delta^{\mathcal{B}} \times \Delta^{\mathcal{B}}) \times (\Delta^{\mathcal{B}} \times \Delta^{\mathcal{B}})$ is a typicality ordering on domain's pairs, i.e., role extensions.

Bi-ordered semantics is the counterpart of single-ordered semantics that handles role preferentiality. In single-ordered semantics, the minimal elements of any given concept must comply with the defeasible or typical axioms associated with the concept. Therefore, not all birds fly, but the minimal elements in the extension of bird fly. The same relation holds for roles. Typical axioms on roles, such as *parents are biological progenitors*, must be respected by the minimal pairs in the extension of Parent, according to $\ll$.

Although bi-ordered semantics are technically multipreferential, as they have more than one preference relation, they suffer from the same problems as traditional, single-ordered preferential semantics. The two orders cover different aspects of the model and encompass all concepts in the case of $<$ and all role edges in the case of $\ll$.

### 4.2.4 Non-monotonic Consequence and Semantics

Standard preferential semantics alone do not guarantee non-monotonic reasoning, something noticed early in the research on typicality DLs. Giordano et al. (2007), who introduced the DL $\mathcal{ALC}$+T, pointed out that it did not support several intuitive conclusions. Suppose that some KB states that typical birds fly and that Tweety is a bird. It is desirable to (non-monotonically) conclude that Tweety flies; however, considering all preferential models, this is not inferred because we do not know whether Tweety is a typical bird.

From the semantical point of view, a popular strategy is to restrict the models to a fraction of the valid interpretations. There are several ways of achieving this. Giordano (2008) presents a preferential semantics to characterize $\mathcal{ALC} + \mathbf{T}_{\text{min}}$. The semantics minimizes atypical instances by considering only models with a minimal amount of atypical individuals, which is done on the language level by keeping track of individuals belonging to $\neg\Box\neg C$ concepts – i.e., atypical individuals for the concept $C$. Let $\mathcal{L}_T$ be the set of concepts occurring in the KB and $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I}, <)$. Then, the set of atypical instances of concepts in $\mathcal{I}$ is defined by:

$$\mathcal{I}^{\Box^-}_{\mathcal{L}_T} := \{(x, \neg\Box\neg C) \mid x \in (\neg\Box\neg C)^\mathcal{I}, \text{with } x \in \Delta, C \in \mathcal{L}_T\}$$

A preference relation over the models, denoted by $\mathcal{L}_T$, compares their $\mathcal{M}^{\Box^-}_{\mathcal{L}_T}$ sets.

**Definition 4.1.** *[Gio+08, def. 3] Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I}, <_\mathcal{I})$ and $\mathcal{J} = (\Delta^\mathcal{J}, \cdot^\mathcal{J}, <_\mathcal{J})$ be two preferential models of a KB $\mathcal{K}$. Let $\mathcal{I}^{\Box^-}_{\mathcal{L}_T}, \mathcal{J}^{\Box^-}_{\mathcal{L}_T}$ be the sets of atypical instances of $\mathcal{I}$ and $\mathcal{J}$ with respect to the concepts in $\mathcal{K}$, respectively. Then, $\mathcal{I} <_{\mathcal{L}_T} \mathcal{J}$* iff *$\Delta^\mathcal{I} = \Delta^\mathcal{J}$ and (2) $\mathcal{I}^{\Box^-}_{\mathcal{L}_T} \subset \mathcal{J}^{\Box^-}_{\mathcal{L}_T}$.*

The downside to this strategy is that it depends on the language instead of relying solely on the structure of the models. A strategy to overcome this problem is to employ the concept of *ranks*, an option taken made by Giordano *et al.* (2013) and Bonatti *et al.* (2015). Those ranks are grounded in the specificity of the domain's elements, and there is more than one way of defining them.

**Definition 4.2** (Interpretation-based rank)**.** *[Gio+13, p. 9] Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I}, <^\mathcal{I})$ be a preferential interpretation. The rank of an element $x \in \Delta^\mathcal{I}$ is denoted by $\mathsf{Rank}_\mathcal{I}(x)$ and is defined as the longest chain $x_0 <^\mathcal{I} \cdots <^\mathcal{I} x$, where $x_0$ is minimal with respect to $<^\mathcal{I}$.*

*Let $C$ be a concept in the language of some DL. Then, the rank of $C$ in $\mathcal{I}$ is defined by* $\mathsf{Rank}_\mathcal{I}(C) = min\{\mathsf{Rank}_\mathcal{I}(x) : x \in C^\mathcal{I}\}$

It is possible to define a preference relation over models by comparing the ranks of the domain elements. Formally, for models $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, <^{\mathcal{I}})$ and $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}}, <^{\mathcal{J}})$, $\mathcal{I} \prec \mathcal{J}$ *iff*:

1. $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$.
2. For all $x \in \Delta^{\mathcal{I}}$, $\mathsf{Rank}_{\mathcal{I}}(x) \leq \mathsf{Rank}_{\mathcal{J}}(x)$.
3. There exists $y \in \Delta^{\mathcal{I}}$ such that $\mathsf{Rank}_{\mathcal{I}}(y) < \mathsf{Rank}_{\mathcal{J}}(y)$.

### 4.2.5 Circumscription

McCarthy (1980) introduced circumscription to model reasoning that jumps to conclusions from non-expressed information, a task very similar to prototype-based inference. Circumscription works by restricting the class of models of some knowledge base to those that minimize a set of predicates specified by a *circumscription pattern*. Brewka (1987) introduced a semantic based on circumscription for frame languages, and Bonatti *et al.* [BLW06], [BLW09], [BFS10], [BFS11a], [BFS11b], [Bon+15a], [BS17] explored questions regarding complexity and decidability for a wide array of circumscribed DLs.

A circumscription pattern CP for a knowledge base $\mathcal{K}$ over a language $L$ is a tuple defined by $\mathsf{CP} = (\prec, M, F, V)$, where $M$, $F$, and $V$ are mutually disjoint subsets of $\mathsf{N_C} \cup \mathsf{N_R}$, characterizing minimized, fixed and varying predicates and $\prec$ is a partial order over $M$ representing a preference for minimization. We denote the circumscription of a KB $\mathcal{K}$ by the circumscription pattern CP by $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. Typicality-based reasoning can be done by stipulating atipicality concepts and defining a circumscription pattern that minimizes them. The axioms on typical birds can be formalized by:

$$\mathsf{Bird} \sqsubseteq \mathsf{Flying} \sqcup Ab_{\mathsf{Bird}} \tag{4.1}$$
$$\mathsf{Penguin} \sqsubseteq \neg\mathsf{Flying} \tag{4.2}$$

A circumscription pattern modeling typicality-based reasoning should include $Ab_{\mathsf{Bird}}$ in $M$, but the rest of the specification will yield different results. If every other predicate were kept fixed, the circumscribed KB would entail $\mathsf{Penguin} \equiv Ab_{\mathsf{Bird}}$. The only reason for the existence of atypical birds is the absence of flying, and the only bird that requires this property, according to the KB, is the penguin. If, on the other hand, the concept Penguin were allowed to vary, then the KB would entail $\mathsf{Penguin} \sqsubseteq \bot$, as every model that had a penguin would also have an atypical bird and therefore could be further improved by the removal of this element from the extension of both concepts.

Conflicts between the minimization of different predicates may arise. If the penguin axiom (6) were substituted by a defeasible version, such as $\mathsf{Penguin} \sqsubseteq \neg\mathsf{Flying} \sqcup Ab_{\mathsf{Penguin}}$, and the atypicality concept were included in $M$ as expected, there would be more than one way of minimizing models with penguins in the domain. The first option is prioritizing the minimization of $Ab_{\mathsf{Penguin}}$. In this case, the element would be a typical penguin but an atypical bird. As expected, it would not fly. On the other hand, prioritizing the minimization of $Ab_{\mathsf{Bird}}$ would result in a circumscribed model with an atypical penguin that is a typical bird, i.e. a flying penguin. The order $\prec$ determines the options that the pattern enforces.

Circumscription is more flexible than some of the other non-monotonic semantics covered in this survey. Therefore, it relies heavily on specifications provided by the knowledge engineers. If the latter alternative seems to lead to undesirable conclusions (i.e., flying penguins), it is due to the principle of specificity. According to this principle, the circumscription pattern should drop more general axioms in favor of more specific ones, which can be achieved by defining $\prec$ according to the subsumption hierarchy, similarly to the rational chain construction described in Section 4.3.1.

Defeasible reasoning tasks such as subsumptions and instance checkings are defined with the introduction of some auxiliary concepts. Suppose that the following axiom was added to the KB: $\mathsf{Flying} \sqsubseteq \exists \mathsf{has.Wings}$. If all the normal predicates were kept fixed, $\mathsf{Circ_{CP}}(\mathcal{K}) \not\models \mathsf{Bird} \sqsubseteq \exists \mathsf{has.Wings}$ would be the case, as models containing penguins would be counterexamples to this inclusion. A defeasible subsumption query should answer whether birds *typically* have wings. Questions such as this can be answered with the introduction of the special operator $CWA_{\mathcal{K}}(A) = A \sqcap \bigsqcap\{\neg B \mid B \in \mathsf{N_C} \text{ and } A \not\sqsubseteq_{\mathcal{K}} B\}$. It selects the $A$ elements that belong only to $A$ supersets, which can be seen as $A$'s without further specification [BFS10]. A model-theoretic version of this idea based on an interpretation $\mathcal{I}$ is given by the set $\lfloor A \rfloor^{\mathcal{I}}$, which is comprised of every $d \in \Delta^{\mathcal{I}}$ such that $d \in A^{\mathcal{I}}$ and $d \in B^{\mathcal{I}}$ iff $\mathcal{K} \models A \sqsubseteq B$, for $A, B \in \mathsf{N_C}$. The defeasible subsumption query is defined by $\mathsf{Circ_{CP}}(\mathcal{K}) \models_{CW} A \sqsubseteq D$ iff for all models $\mathcal{I}$ of $\mathsf{Circ_{CP}}(\mathcal{K})$, $\lfloor A \rfloor^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ [BFS10, def. 3].

A similar procedure applies to instance checking. Instead of checking named individuals, one checks the concept to which they belong in an extended KB. $AtCls_{\mathcal{K}}(a) = \bigsqcap\{A \mid \mathcal{K} \models A(a)\}$ captures the concepts to which an individual $a$ belongs. An extended version of the KB is defined by adding $CWA_{\mathcal{K}}(AtCls_{\mathcal{K}}(a))$ statements for all individuals in the *KB*. The extended KB is denoted by $CWA(\mathcal{K})$. Then, defeasibly checking $C(a)$ is finally defined w.r.t. this extended KB by $\mathsf{Circ_{fix}}(CWA(\mathcal{K})) \models C(a)$, or, model theoretically, by $\{A \in \mathsf{N_C} \mid a^{\mathcal{I}}\} = \{A \in \mathsf{N_C} \mid \mathcal{K} \models A(a)\}$ implying $a^{\mathcal{I}} \in C^{\mathcal{I}}$, for all models $\mathcal{I} \in \mathsf{Circ_{fix}}(\mathcal{K})$ [BFS10, Def. 6 & 7].

These reasoning tasks are defined w.r.t. $\mathcal{EL}_{\perp}$ and can lose important properties in more expressive circumscribed DLs. It can be the case that subconcepts $A_1, \ldots, A_n$ completely cover some concept $C$. In this case, there are no $C$ individuals that do not belong to more specific subconcepts. This is not a problem for $\mathcal{EL}_{\perp}$ as this coverage of $C$ by $A_1, \ldots, A_n$ cannot be expressed in it [BFS10, p. 70].

Circumscription is not immune to inheritance blocking. Consider the penguin knowledge base with the additional axiom $\mathsf{Bird} \sqsubseteq Ab_{\mathsf{Bird}} \sqcup \mathsf{Feathered}$. Such a KB would not entail $\mathsf{Penguin} \sqsubseteq \mathsf{Feathered}$. Consider the model $\mathcal{I}$ with $\mathsf{Penguin}^{\mathcal{I}} = Ab_{\mathsf{Bird}}{}^{\mathcal{I}} \neq \emptyset$ and $\mathsf{Feathered}^{\mathcal{I}} = \mathsf{Flying}^{\mathcal{I}}$, i.e., a model in which penguins do not have feathers. This model cannot be improved, as all penguins are in the atypical birds' extension, and therefore adding them to the extension of Feathered would not create a preferred model. In principle, it is possible to overcome inheritance blocking in such a scenario by having more than one atypicality predicate. In this case, one atypicality would be independent of the other, and if they are indeed logically independent, one could be minimized despite of the other. This technique is similar to the introduction of several preference relations in the preferential interpretations presented in Section 4.2.2.

Circumscribed DLs are also one of the few non-monotonic semantics to extend information to anonymous individuals introduced by existential quantification [BFS11b]. However, the information transfer via roles is limited, and defeasible conclusions are not necessarily pushed through quantifiers. Hence, $A \sqsubseteq B$ does not imply $\exists r.A \sqsubseteq \exists r.B$. Let us consider the penguin KB with the following additional axiom: $\mathsf{Cat} \sqsubseteq \exists \mathsf{eats}.\mathsf{Bird}$, and a circumscription pattern minimizing $Ab_{\mathsf{Bird}}$. There are two different outcomes depending on whether Penguin is a fixed of a varying predicate. If it is fixed, a model $\mathcal{I}$ with $\Delta^{\mathcal{I}} = \{c, p\}$, $c \in \mathsf{Cat}^{\mathcal{I}}$, $(c, p)$, and $p \in \mathsf{Penguin}^{\mathcal{I}}$ cannot be further improved. The element $p$ is an atypical bird – $p \in Ab_{\mathsf{Bird}}$ – but it cannot be removed from the extension of this concept, as it is a penguin, penguins are atypical birds, and the penguin predicate is fixed. If, on the other hand, Penguin is a varying predicate, one would get that $\mathsf{Penguin} \equiv \bot$, as every model $\mathcal{J}$ with elements in $Ab^{\mathcal{J}}_{\mathsf{Bird}}$ would be improved by removing them from the atypical concept extension. This would also imply removing elements from Penguin as penguins are atypical birds by definition.

Decidability and complexity are amongst the more pressing challenges for circumscription in DLs. Most of the results are based on logics with the finite model property [BFS11b]; although some results were provided for DLs lacking this property in [Bon+15a]. Some restrictions must be imposed over the circumscription patterns to secure decidability. Roles names should neither be minimized nor fixed. They should vary freely, i.e., $\mathsf{N_R} \subseteq V$ [BLW09]. Special cases of circumscription patterns are defined by taking this restriction into account. The pattern $\mathsf{Circ_{var}}$ is defined by letting roles and non-minimized concepts vary freely, and $\mathsf{Circ_{fix}}$ or $\mathsf{Circ_{N_C}}$ denotes the pattern where all non-minimized concept names are fixed (but the roles still vary freely).

Even lightweight circumscribed DLs, such as $\mathcal{EL}_\bot$, can be intractable. Unrestricted circumscribed $\mathcal{EL}_\bot$ is ExpTime-hard [BFS11b]. However, there are interesting fragments with lower complexities. Notably, $\mathsf{Circ_{fix}}$ applied to the $LL$ fragment of $\mathcal{EL}_\bot$ (which is defined by disallowing unqualified existentials on the left-hand side of inclusions) is polynomial [BFS11b]. Most of the results rely on the finite model property. However, some DLs lacking this feature have been investigated. In particular, Bonatti *et al.* (2015) shows that $\mathcal{ALCFI}$ and $DL-Lite^F_{bool}$ with fixed roles are decidable. The complexity of reasoning within these logics is unknown up to this date [Bon+15a].

## 4.3 Reasoning Methods

This section examines some techniques developed to perform reasoning tasks that include typicality. Most techniques considered are adaptations of pre-existing methods developed for other logics, such as classical propositional logic. In some cases, they translate defeasible into standard reasoning, which allows employing efficient DL reasoning techniques. In other cases, they propose new methods like the dual tableaux systems developed for $\mathcal{ALC} + \mathbf{T}_{\min}$. We investigate two main reasoning tasks – checking defeasible subsumption and instance checking – and focus on two families of techniques: materialization-based reasoning and tableaux.

### 4.3.1 Materialization-Based Reasoning

Materialization-based reasoning rests on the assumption that it is possible to check defeasible subsumption by performing certain classical queries. The overall idea is to enrich the concept on the left-hand side of the subsumption with supplementary information. This supplementary information is the transformation of terminological axioms into concepts by an element satisfies a GCI $C \sqsubseteq D$ *iff* $\neg C \sqcup D$. The same analogy holds for DCIs such as $C \mathrel{\underset{\sim}{\sqsubset}} D$. Notice that any element that satisfies with the GCI or DCI is a member of the concept, and vice-versa.

**Definition 4.3** (Disjunctive materialization). *Let $C \sqsubseteq D$ be a GCI, $C \mathrel{\underset{\sim}{\sqsubset}} D$ be a DCI, and $S$ be a set of GCIs or DCIs. The materialization of these elements is defined by:*

- $\overline{C \sqsubseteq D} := \neg C \sqcup D$
- $\overline{C \mathrel{\underset{\sim}{\sqsubset}} D} := \neg C \sqcup D$
- $\overline{S} = \bigsqcap_{C \mathrel{\underset{\sim}{\sqsubset}} D \in S} \overline{C \mathrel{\underset{\sim}{\sqsubset}} D} \sqcap \bigsqcap_{C \sqsubseteq D \in S} \overline{C \sqsubseteq D}$

In order to check if $C \mathrel{\underset{\sim}{\sqsubset}} D$ follows from some DKB, one checks whether the materialization of a selected set of DCIs in conjunction with $C$ is classically subsumed by $D$. Some materialization-based reasoning methods select more than one subset of $\mathcal{D}$ to materialize with a given concept $C$ and define skeptical or credulous reasoning over this set. The literature contains several methods for selecting the DCIs to materialize alongside concepts. We call those methods *strengths*, and give a general characterization for them all based on a selection function.

**Definition 4.4** (Consistent-selection function). *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB, $C \in \mathfrak{L}(\mathcal{L})$ be a concept in the language of a DL $\mathcal{L}$, and* s *be a strength. A consistent selection function for* s *and $\mathcal{K}$ is a function $\mathsf{sel}_{\mathcal{K},\mathsf{s}} : \mathfrak{L}(\mathcal{L}) \cup \mathsf{sig}_{\mathsf{I}}(\mathcal{K}) \to \mathcal{P}(\mathcal{D})$ such that, if $\mathcal{U} \in \mathsf{sel}_{\mathcal{K},\mathsf{s}}(C)$, then $\mathcal{K} \not\models C \sqcap \overline{\mathcal{U}} \sqsubseteq \bot$.*

*If $\mathcal{L} \in \{\mathcal{EL}_\bot, \mathcal{ELI}_\bot\}$, the condition is altered to $\overline{\mathcal{K}} \not\models C \sqcap \overline{\mathcal{U}} \sqsubseteq \bot$ to account for non-disjunctive materialization. Finally, when the consistent-selection function selects a singleton, we employ a slight abuse of notation and define $\mathsf{sel}_{\mathcal{K},\mathsf{s}}(C) = \mathcal{U}$ instead of $\{\mathcal{U}\}$.*

**Definition 4.5** (Materialization-based subsumption). *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB, $C, D \in \mathfrak{L}(\mathcal{L})$ be concepts of some DL $\mathcal{L}$,* s *be a strength, and $\mathsf{sel}_{\mathcal{K},\mathsf{s}}()$ be a materialization-based reasoning method. Then, $\mathcal{K} \models_{\mathsf{mat},\mathsf{s}} C \mathrel{\underset{\sim}{\sqsubset}} D$* iff *$\mathcal{K} \models C \sqcap \overline{\mathcal{U}} \sqsubseteq D$ for every $\mathcal{U} \in \mathsf{sel}_{\mathcal{K},\mathsf{s}}(C)$.*

Several methods of materialization-based reasoning were proposed to capture different properties of typicality reasoning and to address particular shortcomings of the already-existing varieties in the literature. Some of them conform to specific *closures*, i.e., conform to some inferential properties, such as the *rational* or the *lexicographic closure*. These algorithms are sometimes referred to by the term *closure*.

#### Materialization-Based Rational Closure

Casini & Straccia (2010) proposed materialization-based rational reasoning to define a rational reasoning procedure in the sense of the hierarchy proposed by Kraus, Lehmann, and Magidor (1990). Informally known as the KLM hierarchy, it characterizes several defeasible reasoning procedures in terms of their inferential properties and correspondent semantics bridged by representation theorems. The hierarchy is based on *defeasible sequents*, such as $\alpha \mathrel{|\!\sim} \beta$, for propositional formulas $\alpha$ and $\beta$. The following principles characterize *preferential logic*, the fundamental system of the hierarchy.

$$\alpha \mathrel{|\!\sim} \alpha \; (Ref) \qquad \frac{\models \alpha \leftrightarrow \beta, \quad \alpha \mathrel{|\!\sim} \gamma}{\beta \mathrel{|\!\sim} \gamma} \; (LLE) \qquad \frac{\alpha \mathrel{|\!\sim} \beta, \quad \alpha \mathrel{|\!\sim} \gamma}{\alpha \mathrel{|\!\sim} \beta \wedge \gamma} \; (And)$$

$$\frac{\alpha \mathrel{|\!\sim} \gamma, \quad \beta \mathrel{|\!\sim} \gamma}{\alpha \vee \beta \mathrel{|\!\sim} \gamma} \; (Or) \qquad \frac{\models \alpha \rightarrow \beta, \quad \gamma \mathrel{|\!\sim} \alpha}{\gamma \mathrel{|\!\sim} \beta} \; (RW) \qquad \frac{\alpha \mathrel{|\!\sim} \beta, \quad \alpha \mathrel{|\!\sim} \gamma}{\alpha \wedge \beta \mathrel{|\!\sim} \gamma} \; (CM)$$

Rational reasoning emerges from combining the preferential closure with *rational monotonicity*, a negative property. As emphasized by Kraus, Lehmann, and Magidor, the preferential properties cover positive aspects of defeasible reasoning. They speak about what should follow from a DKB. Rational monotonicity, on the other hand, says that something should *not* follow from the, DKB given the absence of some information.

$$\frac{\alpha \wedge \beta \not\mathrel{|\!\sim} \gamma, \alpha \not\mathrel{|\!\sim} \neg\beta}{\alpha \not\mathrel{|\!\sim} \gamma} \; (RM)$$

Rational monotonicity serves to limit the update brought by new information. The intuition is that the only information that should bring updates is the ones whose negation was previously expected. [KLM90, p. 33]

A technique of translating those principles to DLs is swapping the propositions for concepts and the defeasible sequents by DCIs. This approach was popularized by Casini & Straccia (2010), although they employed a symbol for defeasible sequents instead of the standard defeasible inclusion and a slightly different set of rules.

$$C \sqsubsetapprox C \; (Ref) \qquad \frac{\models C = D, \quad C \sqsubsetapprox E}{D \sqsubsetapprox E} \; (LLE) \qquad \frac{C \sqsubsetapprox D, \quad C \sqsubsetapprox E}{C \sqsubsetapprox D \sqcap E} \; (And)$$

$$\frac{C \sqsubsetapprox E, \quad D \sqsubsetapprox E}{C \sqcup D \sqsubsetapprox E} \; (Or) \qquad \frac{\models C \sqsubseteq D, \quad E \sqsubsetapprox C}{E \sqsubsetapprox D} \; (RW) \qquad \frac{C \sqsubsetapprox D, \quad C \sqsubsetapprox E}{C \sqcap D \sqsubsetapprox E} \; (CM)$$

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$$\frac{C \sqcap D \not\sqsubsetapprox E, \quad C \not\sqsubsetapprox \neg D}{C \not\sqsubsetapprox E} \; (RM)$$

Bonatti & Sauro (2017) discuss this translation by pointing out that the original KLM postulates were meta-level properties, while the common translations to DLs *internalize* the properties in the object-level. They suggest a non-internalized version that translates the defeasible sequents $\sqsubsetapprox$ to the consequence relationship defined by some DKB, which considers whole formulas instead of the concepts on the two sides of GCIs and DCIs, to be considered alongside the internalized version.

**Exceptionality** The reasoning procedure presented in [CS10] rests on the concept of *exceptionality*, which paves the way for a stratification of the DBox by the generality of its antecedents, respecting the *principle of specificity*. A DCI $C \precsim D$ is *exceptional* to some (strict) KB $\mathcal{K}$ *iff* $\mathcal{K} \models C \sqsubseteq \bot$, i.e., if its antecedent is unsatisfiable in the KB. This concept grounds the definition of a chain of more and more specific DCIs, which gives rise to a stratification of the DBox by specificity.

**Definition 4.6** (Exceptionality Chain)**.** *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB. Then, its* exceptionality chain $\mathcal{E}_0, \ldots, \mathcal{E}_n$ *is defined inductively by:*

- $\mathcal{E}_0 = \mathcal{D}$
- $\mathcal{E}_{i+1} = \{C \precsim D \mid (\mathcal{A}, \mathcal{T}) \models C \sqcap \overline{\mathcal{E}_i} \sqsubseteq \bot\}$

*We say that $\mathcal{K}$ is well-separated when $\mathcal{E}_n = \emptyset$. When that is not the case, $\mathcal{E}_n$ is the fixpoint and $\mathcal{E}_n = \mathcal{E}_{n+1}$.*

A well-separated DKB is simply one that does not have any inconsistent antecedent. It is possible to generate an equivalent well-separated DKB $\mathcal{K}'$ from any DKB $\mathcal{K}$ by setting $\mathcal{T}' = \mathcal{T} \cup \{C \sqsubseteq \bot \mid C \precsim D \in \mathcal{E}_n\}$. [Bri+13] Because DKBs are finite, $n \in \mathbb{N}$.

Exceptionality chains ground a proof-theoretical definition of ranks, contrasted to Definition 4.2, which is based on semantics.

**Definition 4.7** (Exceptionality-chain based rank)**.** *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB and $\mathcal{E}_0, \ldots, \mathcal{E}_n$ be its exceptionality chain. Let $C \precsim D$ be a DCI. The* exceptionality-chain based rank *is given by:*

- $\mathsf{Rank}_{\mathcal{K}}(C)$ = *the smallest $i$ such that* $(\mathcal{A}, \mathcal{T}) \not\models C \sqcap \overline{\mathcal{E}_i} \sqsubseteq \bot$*,*
- $\mathsf{Rank}_{\mathcal{K}}(C \precsim D) = \mathsf{Rank}_{\mathcal{K}}(C)$*.*

Materialization-based rational reasoning is defined by materializing $\mathcal{E}_i$ in conjunction with the $i$-ranked concept on the left-hand side of the checked DCI. Formally:

**Definition 4.8** (Rational Consistent-selection Function and Closure)**.** *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB and $C, D \in \mathfrak{L}(\mathcal{L})$ be two concepts. Then, $\mathsf{sel}_{\mathcal{K},\mathbf{rat}}(C) = \mathcal{E}_i$ for $\mathsf{Rank}_{\mathcal{K}}(C) = i$.*

*We say that $\mathcal{K}$ entails a DCI $C \precsim D$ by rational materialization-based reasoning, denoted by $\mathcal{K} \models_{\mathsf{mat,rat}} C \precsim D$, iff $\mathcal{K} \models C \sqcap \overline{\mathsf{sel}_{\mathcal{K},\mathbf{rat}}(C)} \sqsubseteq D$.*

Casini & Straccia (2010) show that this procedure defines an internalized rational consequence relation. The algorithm had a problem later rectified in [Bri+21].

Rational closure has good computational and inferential properties. However, it does not lend support to some reasonable conclusions. One of the most pressing problems is the inheritance blocking problem. Consider the DKB illustrated in Figure 4.1.

Preserving as many defeasible inclusions as possible for atypical elements is desirable. In the penguins-birds scenario, we would like to conclude that penguins have nice feathers because they are birds, and having nice feathers is a typical property of birds that does not conflict with any property of penguins. They cannot inherit all characteristics from typical birds because flying is inconsistent with what we know of them; that is to say, they do not fly. However, having nice feathers has nothing to do with that. It is *presumed to*

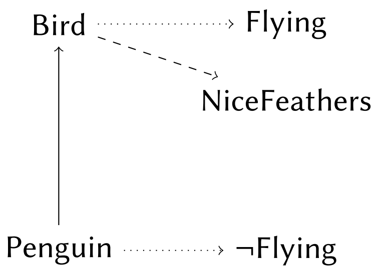


**Figure 4.1:** *Straight arrows represent strict concept subsumptions. Dotted and dashed arrows represent defeasible subsumptions. The two dotted arrows create a conflict for penguins, and the dashed one could be inherited by it.*

*be independent* from flying. Nonetheless, because the algorithmic procedure of rational closure ranks inclusions by the generality of their antecedent, it is impossible to infer Penguin $\mathrel{\sqsubset\!\!\!\sim}$ NiceFeathers from the DKB.

This problem was identified early by Lehmann, and there are several solutions to it in the literature, such as the *lexicographic closure* [Leh95] and [CS12], the *relevant closures* (basic and minimal) [Cas+14b], the *skeptical closure* [GG20], and the *multipreferential closure* [Gli16], [GG19c], [GG19b].

### Lexicographic Closure

Lexicographic Closure was originally proposed in [LM92] and was adapted to DLs in [CS12]. It models *presumptive reasoning*, a reasoning pattern according to which objects are assumed to be typical unless there is evidence to the contrary. The reasoning procedure selects the subsets of $\mathcal{D}$ to be materialized by a lexicographic order based on the rank of the DCIs in each subset of $\mathcal{D}$.

Formally, let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB and $\mathcal{D}^k = \{C \mathrel{\sqsubset\!\!\!\sim} D \in \mathcal{D} \mid \mathsf{Rank}_{\mathcal{K}}(C \mathrel{\sqsubset\!\!\!\sim} D) = k\}$ be the set of $k$-ranked defeasible inclusions in $\mathcal{D}$. For every $\mathcal{U} \subseteq \mathcal{D}$, let $\langle n_0, \ldots, n_k \rangle_{\mathcal{U}}$ be a string of numbers where $n_0 = |\mathcal{U} \cap \mathcal{D}^k|$ and $n_i = |\mathcal{U} \cap \mathcal{D}^{k-i}|$, i.e., the number of $i$-ranked conditionals in $\mathcal{U}$. Precedence is given to the more specific concepts.[4] Hence, a *seriousness* ordering over subsets of $\mathcal{D}$ is defined by:

$$\mathcal{U} \prec_{\mathsf{lex}} \mathcal{U}' \text{ if and only if } \langle n_0, \ldots, n_k \rangle_{\mathcal{U}'} < \langle n_0, \ldots, n_k \rangle_{\mathcal{U}}$$

where $<$ is a lexicographical order, i.e., $\langle n_0, \ldots, n_k \rangle \geq \langle m_0, \ldots, m_k \rangle$ *if and only if* (i) $n_i \geq m_i$ for every $0 \leq i \leq k$ or (ii) if $n_i < m_i$ for an $i$, $0 \leq i \leq k$, then there exists a $j$, $0 \leq j < i \leq k$, such that $n_j > m_j$.

**Definition 4.9** (Lexicographic Consistent-selection Function and Closure)**.** *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB. Then, $\mathcal{U} \in \mathsf{sel}_{\mathcal{K},\mathbf{lex}}(C)$ iff $\mathcal{U} \subseteq \mathcal{D}$ is a maximal subset of $\mathcal{D}$ according to $\prec_{\mathsf{lex}}$ whose materialization is consistent with $C$.*

*We say that $\mathcal{K}$ entails a DCI $C \mathrel{\sqsubset\!\!\!\sim} D$ by lexicographic materialization-based reasoning,*

[4] *Specificity* is a widely accepted reasoning principle, *e.g.*, " (...) we would like to infer that individuals have the properties which are typical of the most specific concept to which they belong." [Gio+07, p. 268]

*denoted by* $\mathcal{K} \models_{\mathsf{mat,lex}} C \sqsubsetsim D$*, iff* $\mathcal{K} \models C \sqcap \overline{\mathcal{U}} \sqsubseteq D$*, for every* $\mathcal{U} \in \mathsf{sel}_{\mathcal{K},\mathbf{lex}}(C)$.

Unlike rational closure, which is based on a total order defined by the exceptionality chain, the lexicographic ordering may select more than one subset of $\mathcal{D}$ to materialize with a given concept. The reasoning defined by this procedure is skeptical: it only considers what follows from all the selected sets.

This reasoning procedure overcomes inheritance blocking as portrayed in Figure 4.1 by favoring the subset $\mathcal{U} = \{\mathsf{Penguin} \sqsubsetsim \neg\mathsf{Flying}, \mathsf{Bird} \sqsubsetsim \mathsf{NiceFeathers}\}$, represented by $\langle 1, 1\rangle_{\mathcal{U}}$, over $\mathcal{U}' = \{\mathsf{Penguin} \sqsubsetsim \neg\mathsf{Flying}\}$, represented by $\langle 1, 0\rangle_{\mathcal{U}'}$. The entailment relation strengthens the rational closure [Cas+13, Prop. 3], as it preserves all its entailments. However, its complexity is also higher, as it has to check several subsets of $\mathcal{D}$ to complete the seriousness ordering through classical reasoning. The authors do not present a proof but conjecture that its complexity is ExpTime for defeasible $\mathcal{ALC}$ DKBs, the same complexity class for reasoning over strict $\mathcal{ALC}$ KBs.

### MP-Closure

MP-Closure is the reasoning procedure for the multipreferential semantics presented in section 4.2.2. Unlike rational closure, it does not correspond exactly to the satisfaction forwarded by the semantics, as it is only sound but not complete for multipreferential semantics. Its intuition is similar to lexicographic closure, sharing the same overall procedure. It begins with the maximal subset determined by the rational closure and adding more DCIs, giving preference to those related to more specific concepts. The main difference between the two is that the comparison is not based on the *number* of DCIs (i.e., the cardinality of the sets containing DCIs of a given rank), as in the lexicographic closure, but rather on strict subset inclusion.

Formally, given sets of DCIs $\mathcal{U}$ and $\mathcal{U}'$, stratified by rank by $\mathcal{U}_i = \mathcal{U} \cap D_i$, $\mathcal{U}' \prec_{\mathsf{MP}} \mathcal{U}$ if and only if there is $h$ s.t. $\mathcal{U}_h \subset \mathcal{U}'_h$ and (ii) for every $j > h$, $\mathcal{U}_{j'} = \mathcal{U}_j$ [GG18, p. 12]. Then, the inference relation is defined in the same way as the lexicographic case, i.e., by taking the maximal subset of $\mathcal{D}$ according to this order

**Definition 4.10** (MP Consistent-selection Function and MP-Closure)**.** *Let* $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ *be a DKB. Then,* $\mathcal{U} \in \mathsf{sel}_{\mathcal{K},\mathbf{MP}}(C)$ *iff* $\mathcal{U} \subseteq \mathcal{D}$ *is a maximal subset of* $\mathcal{D}$ *according to* $\prec_{\mathsf{MP}}$*-maximal whose materialization is consistent with* $C$.

*We say that* $\mathcal{K}$ *entails a DCI* $C \sqsubsetsim D$ *by MP materialization-based reasoning, denoted by* $\mathcal{K} \models_{\mathsf{mat,MP}} C \sqsubsetsim D$*, iff* $\mathcal{K} \models C \sqcap \overline{\mathcal{U}} \sqsubseteq D$*, for every* $\mathcal{U} \in \mathsf{sel}_{\mathcal{K},\mathbf{MP}}(C)$.

It is easy to see that this procedure solves the inheritance blocking problem exemplified in diagram 4.1 similarly to the lexicographic closure. What remains to be explained is where the difference between the two resides. Figure 4.2 depicts a DKB adapted from [GG19a] that highlights the locus of disagreement.

Lexicographic closure answers affirmatively to the query $A \sqsubsetsim H$, because $\mathcal{U} = \{A \sqsubsetsim \neg B, C \sqsubsetsim G, C \sqsubsetsim F\}$ has more elements than $\mathcal{U}' = \{A \sqsubsetsim \neg B, C \sqsubsetsim D\}$. However, the sets $\mathcal{U}$ and $\mathcal{U}'$ are not comparable w.r.t. subset inclusion. Hence, MP-Closure answers the query negatively, as $\mathcal{U}'$ does not contain $C \sqsubsetsim G$, a link in the path leading from $A$ to $H$.

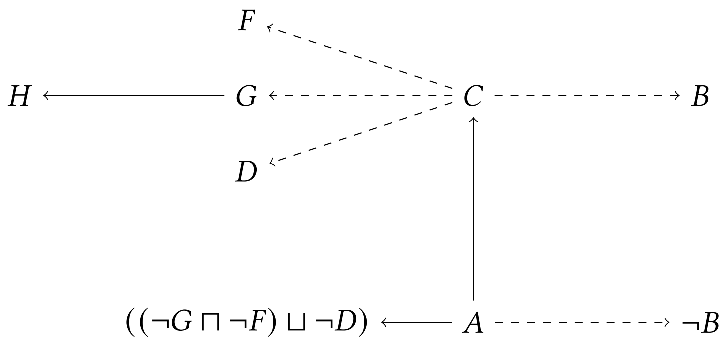


**Figure 4.2:** *Straight lines represent GCIs, and dashed lines represent DCIs.*

### Relevant Closure

Relevant reasoning is a strong form of defeasible reasoning based on *justifications*. It first appeared in [Cas+14b] and subsequently showed up in other papers such as [PT18a] and [GG19c]. *Justifications* are "minimal set of sentences responsible for a conflict" [Cas+14b, p. 2]. The idea is to partition a DBox $\mathcal{D}$ by separating the subsumptions that contribute to an inconsistency from those that do not and then applying the rational closure procedure only to the relevant part. By doing this, the procedure preserves the rest from being discarded.

Casini *et al.* (2014) propose two ways of defining this relevant part. The first leads to *basic relevant closure*; the second, to *minimal relevant closure*, which is stronger than the former. Given the set $R \subseteq \mathcal{D}$, which is the *relevant* part of $\mathcal{D}$ with respect to some $C$, the DBox is partitioned into two by $Rel^k(R) = (R, R^-)$, where $R^- = \mathcal{D} \setminus R$. The relevant closure preserves the strict part, $\mathcal{T}$, and the irrelevant (defeasible) one, $R^-$. Entailment is defined by running the rational closure algorithm on the remaining part to find some $R' \subseteq R$ consistent with the query's antecedent. Alternatively, it is possible to define it directly by means of a consistent-selection function. We consider such function for the two classes of relevant closures presented in (2014).

Given a concept $C$ and a DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$, a subset $\mathcal{J} \subseteq \mathcal{D}$ is a $C$–justification with respect to $\mathcal{K}$ *iff*:

1. $(\mathcal{A}, \mathcal{T}) \models C \sqcap \overline{\mathcal{J}} \sqsubseteq \bot$ and,
2. $(\mathcal{A}, \mathcal{T}) \not\models C \sqcap \overline{\mathcal{J}'} \sqsubseteq \bot$ for every $\mathcal{J}' \subset \mathcal{J}$.

*Basic relevant closure* defines the relevant part by taking all the axioms from every justification. Everything that can generate a conflict may be eliminated, regardless of specificity. Let $\mathsf{Rel}(C) = \{\mathcal{J} \mid \mathcal{J}$ is a $C$-justification$\}$ be the set of all the $C$-justifications for some DKB $\mathcal{K}$. Then, the relevant part of $\mathcal{D}$ for the basic relevant closure is defined as $\mathsf{R}(C) = \bigcup \mathsf{Rel}(C)$.

*Minimal relevant closure* favors specificity by taking the *rank* of the defeasible inclusions into account. Let $\mathcal{J}_{\min} = \{D \precsim E \mid \mathsf{Rank}_{\mathcal{K}}(D) \leq \mathsf{Rank}_{\mathcal{K}}(F)$ for every $F \precsim G \in \mathcal{J}\}$ be the minimally ranked DCIs of each justification $\mathcal{J}$. Then, the relevant part for minimal relevant closure is defined by $\mathsf{Rel}_{\min}(C) = \bigcup_{\mathcal{J} \in \mathsf{Rel}(C)} \mathcal{J}_{\min}$, which is the union of the minimal elements of each $\mathcal{J}$.

**Definition 4.11** (Relevant Consistent-selection Function and Closure)**.** *Let* $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ *be a DKB and* $C, D$ *be two concepts. Let* $R \subseteq \mathcal{D}$ *and* $R^{-} \subseteq \mathcal{D}$ *be the relevant and the irrelevant parts of the DBox. Then,* $\mathsf{sel}_{\mathcal{K},\mathbf{relBa}}(C) = \mathcal{D} \setminus \mathsf{Rel}(C)$ *and* $\mathsf{sel}_{\mathcal{K},\mathbf{relMin}}(C) = \mathcal{D} \setminus \mathsf{Rel}_{\min}(C)$*. The {basic, minimal} relevant closure are defined by:*

$$\mathcal{K} \models_{\mathsf{mat},\mathbf{relBa}} C \mathrel{\underset{\sim}{\sqsubset}} D \text{ if and only if } \mathcal{K} \models C \sqcap \overline{\mathsf{sel}_{\mathcal{K},\mathbf{relBa}}(C)} \sqsubseteq D$$
$$\mathcal{K} \models_{\mathsf{mat},\mathbf{relMin}} C \mathrel{\underset{\sim}{\sqsubset}} D \text{ if and only if } \mathcal{K} \models C \sqcap \overline{\mathsf{sel}_{\mathcal{K},\mathbf{relMin}}(C)} \sqsubseteq D$$

To better compare the versions, let us consider yet another extension of the penguin example in Figure 4.3.

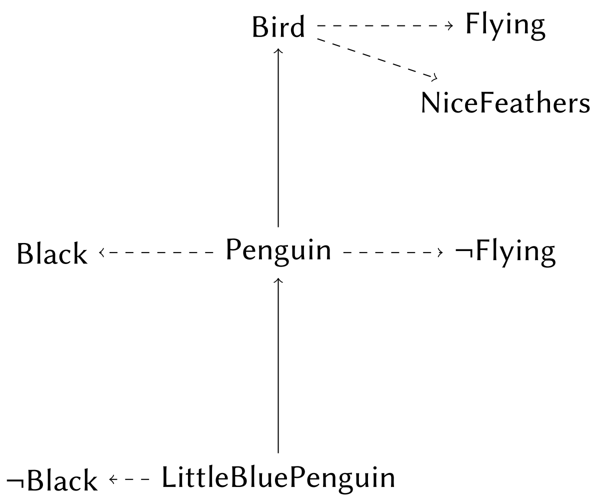


**Figure 4.3:** *Straight lines represent GCIs, and dashed lines represent DCIs.*

As already shown, rational closure suffers from inheritance blocking. Hence, when considering a query whose antecedent is Penguin, it discards all the defeasible inclusions stemming from its inconsistency-generating superclass, Bird. Basic relevant closure considers all the justifications: $\mathcal{J}_1 = \{\mathsf{Bird} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Flying}, \mathsf{Penguin} \mathrel{\underset{\sim}{\sqsubset}} \neg\mathsf{Flying}\}$ and $\mathcal{J}_2 = \{\mathsf{Penguin} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Black}, \mathsf{LittleBluePenguin} \mathrel{\underset{\sim}{\sqsubset}} \neg\mathsf{Black}\}$. Therefore, as an improvement of rational closure, it overcomes inheritance blocking to yield $\mathsf{Penguin} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{NiceFeathers}$, as $\mathsf{Bird} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{NiceFeathers}$ is not in any justification.

This procedure can be refined even further if one takes out only a fraction of the DCIs in each justification. Little Blue Penguin (*Eudyptula minor*) is a species of penguins whose members are not black, contrary to the most typical penguins. As the species name suggests, they are blue. However, they inherit other typical properties of penguins, *viz.* inability to fly. Because $\mathsf{Penguin} \mathrel{\underset{\sim}{\sqsubset}} \neg\mathsf{Flying}$ is deemed relevant to the query for basic relevant closure, it may be discarded when performing reasoning. Queries with LittleBluePenguin on the left-hand side discard all relevant DCIs whose rank is smaller than one and therefore discard $\mathsf{Penguin} \mathrel{\underset{\sim}{\sqsubset}} \neg\mathsf{Flying}$. Minimal relevant closure solves this by considering only minimal-ranked inclusions in justifications. Hence, in the example, $\mathsf{Rel}_{\min}(\mathsf{LittleBluePenguin}) = \{\mathsf{Bird} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Flying}, \mathsf{Penguin} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Black}\}$, and $\mathsf{Penguin} \mathrel{\underset{\sim}{\sqsubset}} \neg\mathsf{Flying}$ will not be discarded, as desired.

Although lexicographic and the two relevant closures are similarly motivated, they yield different results. Giordano and Gliozzi (2019) compare some closure algorithms adapted to DL. They present a slightly different version of the example depicted in Figure 4.4.

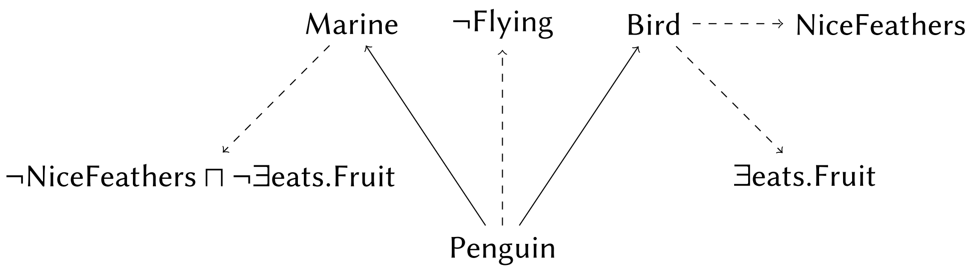


**Figure 4.4:** *Straight lines represent GCIs and DCIs are represented by dashed lines.*

In this KB, both Bird and Marine have rank 0, as they have the same specificity degree. Hence, relevant closures cannot distinguish between the elements of $J_1 = \{\mathsf{Bird} \mathrel{\underset{\sim}{\sqsubset}} \exists\mathsf{eats.Fruit}, \mathsf{Marine} \mathrel{\underset{\sim}{\sqsubset}} \neg\mathsf{NiceFeathers} \sqcap \neg\exists\mathsf{eats.Fruit}\}$ and $J_2 = \{\mathsf{Bird} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{NiceFeathers}, \mathsf{Marine} \mathrel{\underset{\sim}{\sqsubset}} \neg\mathsf{NiceFeathers} \sqcap \neg\exists\mathsf{eats.Fruit}\}$. Therefore, neither $\mathsf{Penguin} \sqsubseteq \exists\mathsf{eats.Fruit}$, nor $\mathsf{Penguin} \sqsubseteq \neg\exists\mathsf{eats.Fruit}$ follows from them. This does not happen in lexicographic closure. Let $\mathcal{U}_1 = \{\mathsf{Bird} \mathrel{\underset{\sim}{\sqsubset}} \exists\mathsf{eats.Fruit}, \mathsf{Bird} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{NiceFeathers}, \mathsf{Penguin} \mathrel{\underset{\sim}{\sqsubset}} \neg\mathsf{Fly}\}$ and $\mathcal{U}_2 = \{\mathsf{Marine} \mathrel{\underset{\sim}{\sqsubset}} \neg\mathsf{NiceFeathers} \sqcap \neg\exists\mathsf{eats.Fruit}, \mathsf{Penguin} \mathrel{\underset{\sim}{\sqsubset}} \neg\mathsf{Flying}\}$. Then, $\langle 2, 1\rangle_{\mathcal{U}_1} > \langle 1, 1\rangle_{\mathcal{U}_2}$, which entails $\mathcal{U}_1 \prec \mathcal{U}_2$. Hence, by the lexicographic closure, $\mathsf{Penguin} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{eats.Fruit}$ undesirably follows from the KB.

## Skeptical Closure

Giordano & Gliozzi (2020) introduced *skeptical closure* as a refinement of rational closure that is weaker than lexicographic closure but also computationally lighter. The algorithm makes a polynomial number of calls to the underlying classical reasoner. In some closures that extend rational closure, this number is exponential [GG20].

This reasoning procedure is grounded in two different notions of compatibility between concepts and DCIs: *individual* and *global compatibility*. Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB, $C$ be a concept such that $\mathsf{Rank}_{\mathcal{K}}(C) = k$, and $\mathcal{E}_0, \ldots, \mathcal{E}_n$ be the rational chain over $\mathcal{K}$. Then $\mathcal{U}^C = \{E \mathrel{\underset{\sim}{\sqsubset}} F \in \mathcal{D} \mid (\mathcal{A}, \mathcal{T}) \not\models C \sqcap \overline{\mathcal{E}_k \cup \{E \mathrel{\underset{\sim}{\sqsubset}} F\}} \sqsubseteq \bot\}$ is the set of defeasible subsumptions that are *individually compatible* with $C$ and the materialization of $\mathcal{E}_k$, which is selected by the rational closure algorithm.[5]

However, this definition does not account for the interaction between the DCIs in the set, which may render the final result inconsistent with $C$. To account for this, the authors introduce the notion of *global compatibility*. Two sets of DCIs $S, S'$ are *globally compatible* with respect to $C$ *if and only if* their union is consistent with $C$: $(\mathcal{A}, \mathcal{T}) \not\models C \sqcap \overline{\mathcal{E}_k \cup S \cup S'} \sqsubseteq \bot$. Let $\mathcal{U}_i^C = \{E \mathrel{\underset{\sim}{\sqsubset}} F \in \mathcal{U}^C \mid \mathsf{Rank}_{\mathcal{K}}(E \mathrel{\underset{\sim}{\sqsubset}} F) = i\}$ be the restriction of $\mathcal{U}^C$ to $i$-ranked DCIs. The set of DCIs to be materialized in conjunction with $C$ is

[5] The original formulation had $\mathbf{T}(E) \sqsubseteq F$ instead of $E \mathrel{\underset{\sim}{\sqsubset}} F$. The adaptation of the definitions seen here serves the purpose of facilitating the comparison of different reasoning methods.

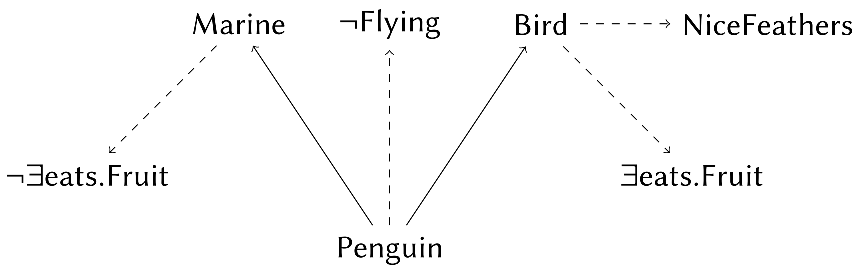


**Figure 4.5:** *Straight lines represent GCIs, and dashed lines represent DCIs.*

[GG20]:

$$S^{sk,C} = \mathcal{E}_k \cup S^C_{k-1} \cup S^C_{k-2} \cup \cdots \cup S^C_h$$

where $h$ is the least $j$ such that $0 \leq j < k$ and $S^C_j$ is globally compatible with $C$ with respect to $S^{sk,C} = \mathcal{E}_k \cup S^C_{k-1} \cup S^C_{k-2} \cup \cdots \cup S^C_{j+1}$.

Skeptical closure diverges from lexicographic closure because it runs out of DCIs to select if the formulas are individually but not globally compatible with $C$. Consider the KB depicted in Figure 4.5.

Penguin has rank 1 and both Marine and Bird have rank 0. When considering the skeptical closure with regards to Penguin, we have only that $S^{sk,\text{Penguin}} = \mathcal{E}_1$, which is the same set selected by the rational closure. This happens because $S^{\text{Penguin}}_0 = \{\text{Bird} \sqsubset\!\!\!\sim \exists\text{eats.Fruit}, \text{Bird} \sqsubset\!\!\!\sim \text{NiceFeathers}, \text{Marine} \sqsubset\!\!\!\sim \neg\exists\text{eats.Fruit}\}$ is composed of DCIs that are individually compatible with Penguin but generate inconsistencies when put together.

Under the skeptical closure, the DKB does not imply Penguin $\sqsubset\!\!\!\sim$ NiceFeathers, a desirable conclusion entailed by stronger closures. In the lexicographic closure, the sets $D_1 = \{\text{Penguin} \sqsubset\!\!\!\sim \neg\text{Flying}, \text{Bird} \sqsubset\!\!\!\sim \text{NiceFeathers}, \text{Bird} \sqsubset\!\!\!\sim \exists\text{eats.Fruit}\}$ and $D_2 = \{\text{Penguin} \sqsubset\!\!\!\sim \neg\text{Flying}, \text{Bird} \sqsubset\!\!\!\sim \text{NiceFeathers}, \text{Marine} \sqsubset\!\!\!\sim \neg\exists\text{eats.Fruit}\}$ correspond to $\langle 1, 2\rangle_{D_i}$, while $S^{sk,\text{Penguin}}$ corresponds to $\langle 1, 0\rangle_{S^{sk,\text{Penguin}}}$. Henceforth, they dominate the latter according to the lexicographical order, and both $D_{1,2} \prec S^{sk,\text{Penguin}}$, yielding Penguin $\sqsubset\!\!\!\sim$ NiceFeathers.

### Materialization-based Instance Checking

*Instance checking* is a reasoning task that checks if a given named individual from the ABox belongs to some concept. The defeasible version of this task seeks to answer if the individual is typically a concept member. An agent that knows that Tweety is a bird should draw the defeasible conclusion that it flies, even though the KB does not entail this conclusion strictly.

Lifting defeasible reasoning from the terminological to the assertional knowledge is not trivial. The root of this difficulty is similar to quantification neglect. Individuals incorporating defeasible information can affect each other through role connections. Therefore, the order in which the defeasible information is applied to the individuals is important, and ABoxes can have multiple incomparable closure extensions. The following example is adapted from Casini & Straccia (2010):

**Example 4.12.** *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB with $\mathcal{T} = \emptyset$, $\mathcal{A} = \{(a, b) : R\}$ and $\mathcal{D} = \{\top \sqsubseteq\!\!\!\sim A \sqcap \forall r.\neg A\}$.*

The DCI in $\mathcal{D}$ can be applied to any of the two individuals alone but not to both. It is easy to see that if it is applied to $a$, it cannot be applied to $b$, and vice-versa.

A technique to overcome this obstacle consists in defining a preference order over the individuals in the ABox and conditioning the entailment to this order. Let $s = (a_0, \ldots, a_m)$ be such an order. We define the rank of an individual $a$ as $\mathsf{Rank}_{\mathcal{K}}(a) = \mathsf{Rank}_{\mathcal{K}}(C)$, for the maximally ranked $C$ such that $\mathcal{K} \models C(a)$. Then, a rational expansion of the ABox is defined by iteratively applying the DCIs to the individuals in $\mathcal{A}$ according to the order $s$. Notice that, at each step, a new $\mathcal{A}'$ is defined, and the ranks of the individuals may have to be recalculated.

**Definition 4.13** (Ordered Defeasible Rational Instance Checking)**.** *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB and $s = (a_0, \ldots, a_m)$ be a total order over the individuals in $\mathcal{A}$. Let $a$ be some individual in the ABox. Let $\mathcal{K} = \mathcal{K}_0, \ldots, \mathcal{K}_{m+1}$ be a series of ABoxs expansions respecting the order $s$. Then: $\mathcal{K} \models^{s}_{\mathbf{rat}} C(a)$ iff $\mathcal{K}_{m+1} \models C(a)$.*

Casini & Straccia (2010) show that the relation in Definition 4.13 satisfies the KLM rationality postulates.

Eliminating the need for orders is made possible through the use of skeptical reasoning techniques. Casini *et al.* (2013) define an entailment that is not conditioned to a single order $s$ but to the intersection of all possible orders.

**Definition 4.14** (Skeptical Defeasible Rational Instance Checking)**.** *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB and $a$ be some individual in the ABox. Then: $\mathcal{K} \models^{Ske}_{\mathbf{rat}} C(a)$ iff $\mathcal{K} \models^{s}_{\mathbf{rat}} C(a)$ for every order $s$ over the individuals in $\mathcal{A}$.*

This technique has two shortcomings. On the one hand, the consequence relation lacks rational monotonicity, which is present in the entailment relation conditioned to a single order $s$. On the other hand, it can increase the complexity of reasoning in practice, while keeping the problem at the same complexity class. In the worst case, the $\mathcal{ALC}$-based algorithm has to perform $n!$ ExpTime-complete decision procedures [Cas+13]. A partial solution to this increase in complexity is limiting the order to individuals connected by roles, as they are the only ones that can transfer defeasible information back and forth. Casini *et al.* (2013) introduce *clusters* of individuals to this end. Clusters are sets of individuals that inhabit the same connected parts of the graph whose nodes are individuals and edges are role edges.

The reasoning procedures mentioned above are adaptations of the rational closure reasoning procedure. However, there are also similar techniques lifted to stronger closures. A version concerned with lexicographic closure is examined in [CS12].

### 4.3.2 Tableaux

Tableau-based reasoning algorithms are a popular reasoning method for DLs. They work by reducing subsumption to concept unsatisfiability and are very useful for DLs such as $\mathcal{ALC}$ and its extensions [Baa+03]. Formally, tableaux are trees whose nodes are constraint systems that model the expansion of the ABox by applying terminological knowledge to its individuals. A node can branch when a disjunctive axiom is applied to some individual, such as the axiom $C \sqsubseteq D \sqcup E$ for some $a : C$, and it closes when an open contradiction – $a : C$ and $a : \neg C$, or $a : \bot$ – is found in the expanded ABox. It is possible to check concept subsumption $C \sqsubseteq D$ by introducing $a : \neg C \sqcap D$ in the ABox for a fresh individual $a$. Tableaux for DLs of typicality are built over the foundations of the systems for standard DLs and have special modifications to keep track of DCIs or typicality operators.

### Biphasic Tableaux for DLs with Typicality Operators

Giordano *et al.* (2008) developed a tableaux system for the logics $\mathcal{ALC} + \mathbf{T}$ and $\mathcal{ALC}+\mathbf{T}_{\text{min}}$. Their proposal is a biphasic system composed of two independent procedures, $\mathcal{TAB}_{PH1}^{\mathcal{ALC}+\mathbf{T}}$and $\mathcal{TAB}_{PH2}^{\mathcal{ALC}+\mathbf{T}}$. The first procedure verifies if the checked instance has a model that satisfies the KB. If such a model exists, the second system tries to minimize it in terms of atypicality ascription formulas, $x : \neg\Box\neg C$, a definition in line with the semantics for the $\mathbf{T}$ operator discussed in Section 4.2.1. Because the non-monotonic logic $\mathcal{ALC}+\mathbf{T}_{\text{min}}$ has a semantics based on a preferred set of models, the tableau procedure checks whether (i) there are any models to the KB and, if there are, (ii) whether those models belong to the set of preferred models.

The nodes in $\mathcal{TAB}_{PH1}^{\mathcal{ALC}+\mathbf{T}}$are constraint systems with two components: $S$ and $U$. The first stores information on the individuals in the ABox. Namely, to which concepts and roles they belong. The second garners terminological knowledge. It contains the TBox axioms labeled with the set of individuals to which they were applied. Let $a \in \text{sig}_{\mathsf{I}}(\mathcal{A})$ be an individual from the KB and $L \subseteq \text{sig}_{\mathsf{I}}(\mathcal{A})$ be a set of individuals from the KB such that $a \notin L$. The *unfold rule* exemplifies their interplay:

$$\frac{(S \mid U \cup \{C \sqsubseteq D^{L}\})}{(S \cup \{a : \neg C \sqcup D\} \mid U \cup \{C \sqsubseteq D^{L\cup\{a\}}\})}\ (unfold)$$

A supplementary order $\prec$ takes care of the chronological order in which the procedure introduces new constants, which is necessary to ensure termination. The tableau procedure deals with typicality through the interplay between the typicality operator and the modality $\Box$. Besides the rules for standard $\mathcal{ALC}$, there are four rules covering typicality, which we discuss at a high level:

1. The rule (*cut*) branches the constraint system for every element $a$ that is neither in $\Box\neg C$, nor in $\neg\Box\neg C$. Each one of the two branches covers one possibility. The intuition is that the model should decide, for every element, whether it is minimal or not for every concept.

2. The rule ($\Box^{-}$) adds typical elements to a concept $C$ if there is no typical member of $C$, but there is an atypical one, $x$. It creates $n$ branches, where $n$ is the number of the elements occurring in $S$ including $x$. Let those elements be $x, v_1, \ldots, v_{n-1}$. In every new branch, $v_i$ will be the typical member of $C$. Finally, there is an additional branch with a new element $y$ as the typical member of $C$. Notice that $x$ remains as an atypical member of $C$ in all branches, as this was already a fact before the

application of the rule.

3. The rule $(T^-)$ governs the negative interplay between the typicality operator and the modality $\Box$. If an element $x$ belongs to $\neg\mathbf{T}(C)$, the rule creates two new branches: one in which $x : \neg C$, and another in which $x : \neg\Box\neg C$. Either $x$ is not a member of $C$ or a non-minimal member of $C$.

4. Finally, the rule $(T^+)$ governs the positive interplay between the typicality operator and the modality. If an element $x$ belongs to $\mathbf{T}(C)$, it also belongs to $C$ and $\Box\neg C$, i.e., it is a minimal member of $C$.

Given an $\mathcal{ALC}$ KB $K = (\mathcal{T}, \mathcal{A})$, the constraint system is initialized by: $S = \{a : C \mid a : C \in \mathcal{A}\} \cup \{(a, b) : r \mid (a, b) : r \in \mathcal{A}\}$ and $\mathcal{U} = \{C \sqsubseteq D^{\emptyset} \mid C \sqsubseteq D \in \mathcal{T}\}$. When there are no more rules to apply to some open branch, it gives rise to a model of $\mathcal{K}$. If every branch closes due to some contradiction, $\mathcal{K}$ is unsatisfiable.

Giordano *et al.* 2007 proved the system to be sound, complete, and to terminate. However, the model it outputs is not guaranteed to minimize the number of atypical instances $x : \neg\Box\neg C$. Minimizing this parameter is the purpose of $\mathcal{TAB}_{PH2}^{\mathcal{ALC}+\mathbf{T}}$.

A preferred model is minimal for models sharing its domain. The tableau procedure $\mathcal{TAB}_{PH2}^{\mathcal{ALC}+\mathbf{T}}$operates on constraint systems taken from open branches from $\mathcal{TAB}_{PH1}^{\mathcal{ALC}+\mathbf{T}}$. The rules that introduced new elements – $(\Box^+)$, which accounts for typical elements, and $(\exists^+)$, which account for elements introduced by existential restrictions – are modified to operate only on existent elements. The new system keeps track of the atypical elements by counting $x : \neg\Box\neg C$ formulas, which is kept in a third component of the constraint system, $K$. This component is initialized with all the atypicality formulas in the open branch in $\mathcal{TAB}_{PH1}^{\mathcal{ALC}+\mathbf{T}}$. As those formulas arise in the new branch in $\mathcal{TAB}_{PH2}^{\mathcal{ALC}+\mathbf{T}}$, they are removed from $K$. Two new clash rules cover the comparison between atypical individuals in the original open branch and the new constraint system in $\mathcal{TAB}_{PH2}^{\mathcal{ALC}+\mathbf{T}}$.

$$(S \mid U \mid \emptyset) \quad (\text{Clash})_{\emptyset}$$
$$(S \cup \{x : \neg\Box\neg C\} \mid U \mid K) \text{ and } x : \neg\Box\neg C \notin K \quad (\text{Clash})_{\Box^-}$$

The first rule covers comparable models. When all the atypicality formulas from the original open branch are used, the new model is at least as atypical, and therefore cannot be an improvement over it. The second rule deals with incomparable models. When an atypicality formula not in the original branch is added, it closes the new $\mathcal{TAB}_{PH2}^{\mathcal{ALC}+\mathbf{T}}$branch. The complete dual query algorithm is sound and complete regarding $\mathcal{ALC} + \mathbf{T}_{\min}$ entailment relation, and its complexity is in CO-NEXP$^{\text{NP}}$[Gio+08]. We summarize the complete procedure as follows:

1. Let $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ be a KB in the language of $\mathcal{ALC} + \mathbf{T}$ and $Q = a : C$ be a query,

2. Define $\mathcal{K}' = (\mathcal{A} \cup \neg Q, \mathcal{T})$,

3. Run $\mathcal{TAB}_{PH1}^{\mathcal{ALC}+\mathbf{T}}$over $\mathcal{K}'$,

4. Run $\mathcal{TAB}_{PH2}^{\mathcal{ALC}+\mathbf{T}}$in every open branch for which there are no more rules to be applied,

5. If there is an open branch in $\mathcal{TAB}_{PH2}^{\mathcal{ALC}+\mathbf{T}}$ with no applicable rules, $Q$ does *not* follow from $\mathcal{K}$, as there is a preferred counterexample to it. Otherwise, if every branch closes, $Q$ follows from $\mathcal{K}$.

Although this method is tailored to $\mathcal{ALC} + \mathbf{T}_{\min}$, there are analogous systems to other DLs such as $\mathcal{EL}_\bot\mathbf{T}_{\min}$ [Gio+11a] and DL-Lite$_c\mathbf{T}_{\min}$ [Gio+11b]. The overall strategy is the same, but the complexity is $\prod_2^p$ for DL-Lite$_c\mathbf{T}_{\min}$ and the syntactical restricted Left Local $\mathcal{EL}_\bot\mathbf{T}_{\min}$ KBs[6].

There are differences in the dual tableaux systems for those logics when compared to $\mathcal{ALC} + \mathbf{T}_{\min}$. In part, the new rules reflect divergences in the strict part of the logics, as $\mathcal{ALC}$ is more expressive than both $\mathcal{EL}_\bot$ and DL-Lite$_c$. However, the distinction also impacts the rules dealing with typicality. The rule that introduces new individuals to witness existential quantification is altered because those logics have the *small model property*. Therefore, for each $\exists r.C$, there needs to be exactly one witness, denoted by $x_C$. The rule ($\Box^-$), which introduces new individuals to account for typicality, is altered to deal with more than one concept at a single step. This stronger version of the rule can introduce a single new individual $y$ for $n$ concepts, $C_i, \ldots, C_n$. Finally, DL-Lite$_c\mathbf{T}_{\min}$ has two additional rules for existential formulas to deal with inverse roles. However, those formulas are limited to the form $x : \exists r^-.\top$.

### Tableaux for DLs with concept and role typicality

There are tableaux for stronger non-monotonic DLs that are not direct adaptations of the system in [Gio+08]. These logics include $d\mathcal{SROIQ}$ [BV17b], $\mathcal{ALCH}^\bullet$ [Var18], and $d\mathcal{ALC}$ [BV19]. Both $d\mathcal{SROIQ}$ and $\mathcal{ALCH}^\bullet$ are considerably more expressive than $\mathcal{ALC} + \mathbf{T}_{\min}$. These logics can represent role typicality, and their semantics are based on bi-preferential models. Another difference is neither resorts to the intermediary modality $\Box$ to characterize typicality. Instead, they deal with the minimization of atypical individuals more directly. In $\mathcal{ALCH}^\bullet$, this is done by two auxiliary relations, $<$ and $\ll$, that correspond to individuals and roles, respectively. Those relations keep track of the typicality of the individuals and edges introduced by tableaux expansion rules.

Just as in $\mathcal{TAB}_{PH\{1,2\}}^{DL+\mathbf{T}}$, the rules in this system expand a set storing assertional knowledge. The rules for the classical operators are in line with standard DL tableaux, and it also implements blocking to avoid unending loops. In addition to the assertional knowledge, nodes also store two relations on individuals and pairs of individuals to account for typicality. Those relations are denoted by $<$ and $\ll$, respectively.

Four rules address typicality. Two regulate the typicality of named individuals and the remaining ones address the typicality of role edges.

1. The rule ($\overset{+}{c}$) ensures that any typical element $a : \bullet C$ is the minimal member of $C$ according to $<$. Therefore, if there are elements $b$ such that $b < a$, they are added to $\neg C$.

[6] The subsumptions of the LL KB are of the form $C_E^{LL} \sqsubseteq C_R$, where $C := A \mid \top \mid \bot \mid C \sqcap C$, $C_R := C \mid C_R \sqcap C_R \mid \exists R.C$ and $C_L^{LL} := C \mid C_L^{LL} \sqcap C_L^{LL} \mid \exists R.\top \mid \mathbf{T}(C)$ [Gio+11a, Def.1, Def. 6].

2. The rule $(\bar{c})$ takes care of atypical individuals $a : \neg \bullet C$ for concepts without typical witness $b : C, b < a$. If the constraint system does not guarantee that they are not in $C$ (i.e., $a : \neg C \notin \mathcal{A}$), then the system creates two new branches. In the first one, the element is put out of $C$ by $a : \neg C$. In the second, a new element $c$ is added as the typical witness: $a : C, c : C$ with $c < a$.

The rules $(\overset{+}{r})$ and $(\bar{r})$ are analogous for roles and the order $\ll$. Clashes occur when there are open contradictions either in concept or role ascriptions. Just like before, a clash closes its branch. If there are no more rules to be applied to an open branch, then the KB is satisfiable, and the branch gives a possible model. The procedure to define a preferential interpretation is more direct than the one for $\mathcal{TAB}^{DL+\mathbf{T}}_{PH\{1,2\}}$ because a bi-ordering is already given by $<$ and $\ll$.

This tableaux system is a sound decision procedure for the satisfiability of $\mathcal{ALCH}^{\bullet}$ KBs [Var18]. There are no complexity results, although Varzinczak 2018 conjectures that it is ExpTime.

Finally, Britz & Varzinczak (2017) present a tableaux system for the notably more expressive DDL $d\mathcal{SROIQ}$. Besides role hierarchies and typicality, it has several additional constructors, both classical and defeasible (discussed briefly in Section 4.1.1). It has an RBox that stores both classical and defeasible role inclusion axioms. Another feature is that it can express defeasible and classical role axioms (e.g., $Sym(r)$ states that $r$ is symmetric, and $dSym(r)$ states that it is *usually* symmetric). Concept-wise, it also supports defeasible versions of value, existential, at-least, at-most, and self restrictions. It has a preferential (ordered) semantics equipped with a single order on individuals, $\prec^{O}$, and a collection of orders on pairs of individuals, $\ll^{O} := (\ll^{O}_{1}, \ldots, \ll^{O}_{|\mathsf{N_R}|})$, where $\mathsf{N_R}$ is the set of role names.

The proposed tableaux system abandons constraint systems in favor of a more complex structure to deal with the increased expressivity. The rules operate over a labeled completion graph, where nodes represent individuals and labels store information on concept and role membership, and also concept and role normality. Let $\mathsf{N_{R^-}} = \{r^- \mid r \in \mathsf{N_R}\}$ be the set of inverse roles and $\mathsf{N_{\mathbf{R}}} = \mathsf{N_R} \cup \mathsf{N_{R^-}}$. Formally, the completion graph is defined by $\mathcal{G} := (V, E, M, \mathcal{L}, \mathcal{N} \neq)$, where:

- $V$ is a set of nodes that represent the individuals of the domain;
- $E \subseteq V \times V$ is a set of edges, which will represent roles via labeling;
- $M \subseteq E \times E$ is a relation on edges;
- $\mathcal{L}$ is the labeling function that assigns labels to nodes (i.e., the concepts to which they belong) and edges (i.e., roles that include them);
- $\mathcal{N} \subseteq E \times \mathsf{N_{\mathbf{R}}}$, for which $(e, r) \in \mathcal{N}$ only if $r \in \mathcal{L}(e)$, is a relation between edge pairs and roles that signalizes edge-normality (i.e., a given edge is a normal instance of a given role);
- $\neq \subseteq V \times V$ is a symmetric relation on nodes.

To check satisfiability of a concept $C$, let $o_1, \ldots, o_k$ denote the nominals occurring in $C$ and set the completion graph $\mathcal{G} = (\{v_0, \ldots, v_k\}, \emptyset, \emptyset, \mathcal{L}, \emptyset, \emptyset)$, where $\mathcal{L}(v_0) := \{C\}$ and

$\mathcal{L}(v_i) = \{o_i\}$, for $1 \leq i \leq k$. Tableaux-like rules iteratively expand the graph following an order that ensures termination. The expansion rules are equipped with blocking to ensure termination and merging, which fuse nodes. A more comprehensive set of clash rules closes branches that contain some contradiction. Besides usual contradictions dealing with inconsistent concept membership, the conditions presented here also cover both strict and defeasible role axioms, such as $\mathsf{Dis}(r, s)$, which implies the disjointness of roles $r$ and $s$. If a clash appears, the expansion halts, and the concept is unsatisfiable. Otherwise, it is satisfiable, and a model can be constructed from the graph. The resulting procedure is sound and complete regarding $d\mathcal{SROIQ}$, but no complexity analysis exists. [BV17b]

## 4.4 Open problems

Research on DLs of typicality has been done consistently for more than a decade now. Nevertheless, there are still important problems and unsatisfactory aspects in existing solutions. We classify those problems into two broad classes. On the one hand, some affect nonmonotonic reasoning in general. Those issues are carried over to DLs of typicality because most of the methods are adaptations from techniques proposed to other frameworks. It is unlikely that they will be solved in the context of DLs, although research in this area can push the community to new horizons. One promising research program is investigating the combination of defeasible reasoning and quantification, and DLs offer good case studies due to their combination of first-order properties with decidability.

On the other hand, there are problems specific to DLs. As mentioned, many techniques that deal with typicality within DLs are adaptations of solutions proposed in other contexts. Because most of them originated within propositional logic, they fall short of dealing with the quantificational aspect of DLs. Investigating those aspects is an important avenue of research with several interesting problems.

We present a list of some of the most serious open problems concerning DLs of typicality.

- **Inheritance blocking**. Inheritance blocking is probably the most widely reckoned problem, and it is not limited to DLs that deal with typicality but is also present in many defeasible reasoning systems. It is a flaw inherent to rational reasoning. It can be described as the loss of defeasible information irrelevant to the inconsistencies that arise from the KB, as depicted in Figure 4.1. Inheritance blocking is arguably the most discussed problem within the community. Several solutions overcome it, such as the lexicographic and relevant materialization-based reasoning methods discussed in Section 4.3.1. The major downside of most of the solutions in the literature is that they come with a great increase in complexity, something undesirable in the context of DLs. Some approaches try to circumvent this increase in complexity while avoiding inheritance blocking, such as skeptical closure covered in Section 4.3.1. The extent to which they achieve this is better discussed in the referred section.
- **Context and single preference orderings**. Multiple solutions for dealing with typicality and defeasible reasoning within DLs are based on preference orders over domain elements. Extracting typicality information from a single order limits what can be expressed by the underlying logic. In the real world, contextual information

plays an active role in reasoning related to typicality. Something typical in one context may not be in another one. Strategies to deal with this topic include semantics equipped with multiple preference orderings, such as those presented in [Gli16], [GG18], and [BV17a]. Multiple preferences are promising, but there still are open questions w.r.t. them. There is no closure corresponding to the framework presented by Gliozzi and Giordano (2018) – the MP-closure is just sound but is not complete w.r.t. their multipreferential semantics. It is also not clear which kind of reasoning problem they should tackle. Solving problems like the scenario described in Section 4.2.2 may require substantially weakening the consequence relation. The price of this trade-off may be too high for a calculus aiming for tractability.

- **Quantification neglect**. One widely known shortcoming of defeasible DLs is that several frameworks cannot push defeasible information through quantifiers. Hence, a KB may entail the defeasible inclusion $\mathsf{Bird} \mathrel{\sqsubseteq\!\!\!\sim} \mathsf{Flies}$, but not $\exists \mathsf{eats}.\mathsf{Bird} \mathrel{\sqsubseteq\!\!\!\sim} \exists \mathsf{eats}.\mathsf{Flies}$. The semantics based on typicality models that will be covered in Part III deals with this problem for the defeasible version of the lightweight DL $\mathcal{EL}_\bot$. The technique depends on the existence of canonical models and, therefore, cannot be extended to DLs such as $\mathcal{ALC}$. Even having the canonical model property may not be enough, as the semantics rely on stronger properties, such as preserving the extension of concepts over the subset relation defined over models. Recently, Câmara & Turhan (2022) and (2023) proposed the first steps for a generalization for $\mathcal{ELI}_\bot$. How to deal with this problem in more expressive DLs remains an open problem in need of further investigation.

- **Reasoning principles that include quantification**. The KLM postulates are a widely used set of principles to evaluate defeasible reasoning, including defeasible DLs and those that represent typicality. Internalized and external versions of these postulates were proposed to DLs several times, notably in [CS10]. However, as previously stated, the nature of the principles is propositional, and, therefore, they do not capture the behavior of defeasible information when combined with quantifiers. Even though there are some suggestions for reasoning defeasibly in the presence of quantifiers, there has yet to be a set of postulates to evaluate the results. Establishing such principles is an important step in future research.

- **Reasoning with data**. Currently, most known approaches focus on the terminological part of the KBs. Reasoning with data is left aside, which is not surprising, as the problem is notoriously hard, mostly for the same reasons giving rise to quantification neglect. The interaction of individuals (and elements) incorporating different levels of defeasible information can create incompatible scenarios that need to be taken into account by the underlying framework. One strategy to overcome this challenge is conditioning expansions of the ABox to a preference order over individuals, as done in [CS10]. This method places a burden on the knowledge engineers. On the other hand, defining a skeptical consequence relation over all these orders can greatly increase the complexity of the calculus. Besides instance checking, more complex tasks such as *query answering* have yet to be approached from the perspective of typicality DLs. Investigating those tasks could widen the possible applications of DDLs.

- **Tools and datasets**. As pointed out by [BS17], there is a lack of genuine DKBs, mostly due to the absence of robust nonmonotonic inference technologies for DLs. This status quo results in two problems that are the faces of a single coin. On the one hand, there are no useful benchmarks against which to test possible implementations of the existing methods. Hence, researchers must develop artificial ones, as in [Bon+15b] and [Gui20]. On the other hand, the lack of implementations is a problem in itself. It is important for logics with a wide array of applications, such as DLs, to have robust and working implementations. Another aspected pointed out by [Bon+15b] is that most existing systems can only deal with moderate-sized DKBs.

## 4.5 Research landscape

This survey reported in this chapter analyzed roughly 230 papers from 231 authors. We used *Neo4J*[7] to create and plot a graph database of authorship and collaboration for analyzing the research community behavior. In this section, we present a brief overview of the analysis. We start with a plot of the full collaboration graph, depicted in Figure 4.6. The graph has two types of nodes. Dark nodes represent authors, light nodes represent papers, and edges represent authorship.

It is possible to partition the publications into two broad classes. On the one hand, there are groups or researchers that published one (or a few) paper(s) appearing in the survey. On the other hand, there are very active clusters of researchers with a considerable number of relevant publications.

A possible way to further partition the papers in the first group is to separate them into two more classes: applied and theoretical. The first class is comprised of research that employs defeasible DLs to solve a particular practical problem instead of developing formalisms to deal with the shortcomings of DLs of typicality. Examples of this are [Med+16], which uses defeasible DLs to create a conversational interface in the healthcare context, and [GN18], which uses a defeasible DL to solve problems within access control. This is an interesting avenue of research, especially considering that solid implementations are still rare in the research landscape. Most logics covered in this survey were not tested in real-world applications, and many do not have publicly available implementations.

The theoretical papers can be further separated into two groups. The first group contains papers presenting the transposition of techniques from other research areas to DLs of typicality. On the other hand, the second group has papers examining DLs of typicality in their own terms. An example of a study in the former category is [GCS10], in which the authors apply formal argumentation analysis techniques – which is the main research area of their group – to inconsistent ontology handling. With a few exceptions, research groups and individual researchers that published just a few papers on the subject had less impact than those that made several contributions, which is not surprising.

The class of papers authored by groups of researchers very active in the area is spread along some clusters of publications. There are at least four major clusters with some uniformity regarding their authors and the content of their papers. The clusters are

[7] https://neo4j.com/

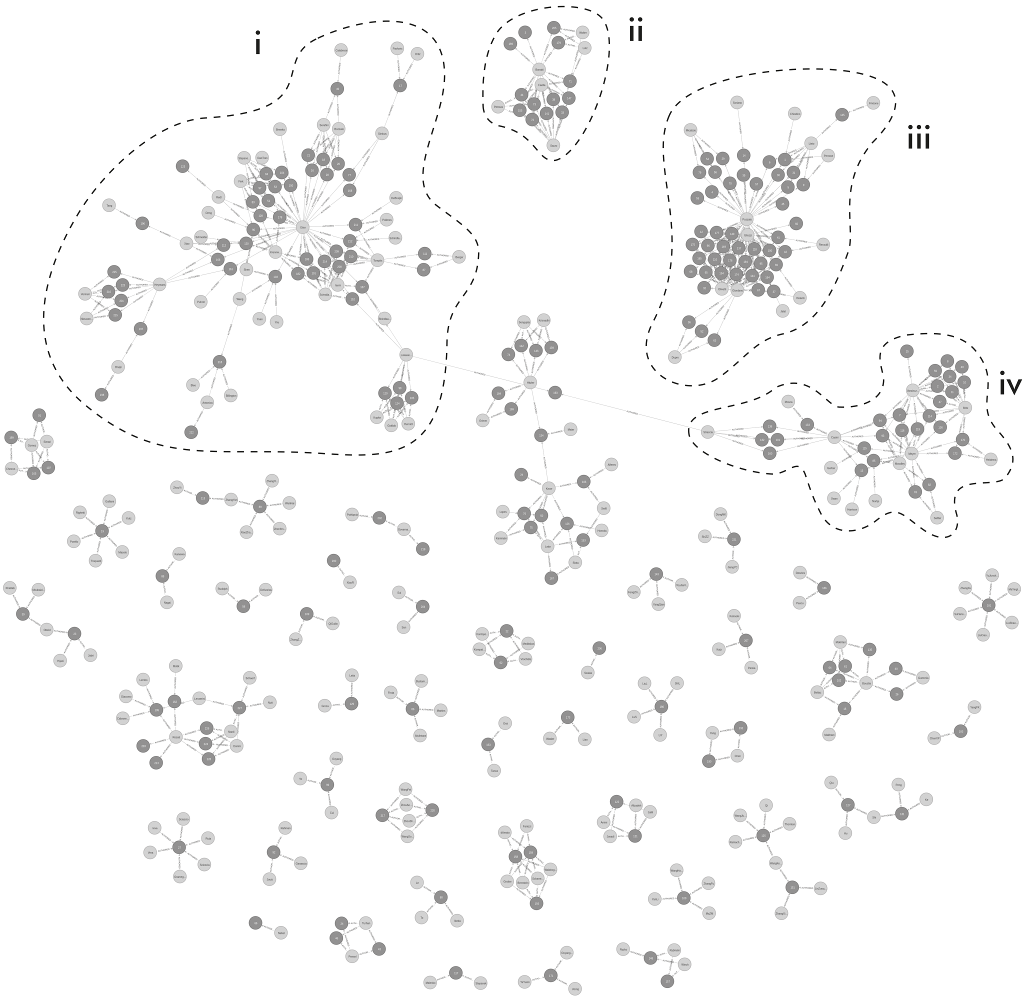


**Figure 4.6:** *The plot of the full collaboration graph depicts a small number of big clusters of prolific research groups and a wide array of lone papers published by small number of authors. The clusters are outlined by dashed lines.*

partially determined by geographic proximity, as researchers that work close to each other have more opportunities to collaborate. However, geography is not the only factor that comes into play. There is some cohesion in how the papers of each cluster deal with the problems, something that can be seen as a *school of thought*. We give a brief overview of the contributions of each group as follows:

(i) **Typicality Operator**: this cluster is composed of researchers from Italian universities. Most of its papers deal with DLs that are extended with typicality operators, a technique first proposed in [Gio+07].

(ii) **Circumscription and Overriding**: this cluster is centered around the University of Napoli, in Italy, although there are researches from other places, such as the United Kingdom. The papers in it deal mostly with circumscription, introduced for DLs in [BLW06]. Another follow up is the logics of overriding and $DL^N$, DLs with a normality (i.e. typicality) operator whose semantics is related to circumscription.

Although the number of the publications in this cluster is not so high as the other three examined in this list, it contains several publications of high impact.

(iii) **Defeasible DLs**: this cluster is less geographically cohesive. It includes researchers from several countries, including Germany, France, and South Africa. Even though the strategies vary a little more than those of the first cluster, the predominant approach is that of defeasible DLs. Another characteristic is the presence of several variations of algorithmic materialization-based reasoning. Those DLs that have some defeasible components in the KB, such as defeasible concept inclusions, defeasible sequents or even defeasible constructors. Some papers such as [Var18] also deal with typicality operators.

(iv) **DL and Rules**: this is the most diversified cluster emphasized in Figure 4.6. The distribution of its authors is also more sparse than the others, containing several authors that did not collaborate with each other. Most of its papers deal with a combination of standard DLs and a logical programming inspired rule layer. This integration is considered by some experts a vital enterprise for the semantic web framework [Eit+06]. They appeared in this dissertation because these rules have a nonmonotonic nature, partly because of their non-classical negation as failure. In some cases, these formalisms can represent well-known nonmonotonic machinery such as *defaults*, as in [KHM12]. However, because they are not primarily concerned with typicality and differ substantially from the other approaches, we do not cover them in detail in this survey.

A first analysis could favor the view that the clusters work in isolation without being aware of the work being done elsewhere. However, this is not the case. This impression is possible because we plotted only authorship and direct collaboration. Nonetheless, a quick overview of the references section of each paper surveyed reveals that information flows between different research groups. A more comprehensive analysis should take this information into account as well, which could help establish the most influential papers and the flow of ideas. Such an analysis surpasses the scope of the current work but remains an interesting future work.

# Part III

# Typicality Models for Defeasible Description Logics

# Chapter 5

# A Brief Introduction to Typicality Models

Semantics based on typicality models is a nonmonotonic semantics for DLs firstly proposed by Pensel & Turhan in (2017), (2017), (2018), (2018) and Pensel (2019) to alleviate quantification neglect for several nonmonotonic semantics defined for the lightweight DDL $\mathcal{EL}_\bot$. In particular, it extends the capacities of materialization-based reasoning procedures.

As the name suggests, the semantics is based on a special class of interpretations – *typicality interpretations*. Those interpretation mirror the canonical models for the strict version of its grounding DL. There are several types of semantics based on typicality models (TM), which are defined by constraining the set of the models considered. The different flavors are classified in two axis by the following parameters:

- **strength** of reasoning, denoted by **s**, points to the materialization-based entailment backing the TM construction. In this dissertation, $\mathbf{s} \in \{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\}$, i.e., it includes classes of models encompassing rational, relevant, and lexicographic closures.
- **coverage** of reasoning, denoted by **c**, corresponds to the depth to which defeasible information is applied. In this dissertation, $\mathbf{c} \in \{\mathbf{prop}, \mathbf{nest}\}$. The two model classes represent *propositional* reasoning, in which defeasible information is not transmitted through quantifiers, and *nested* reasoning, in which defeasible information spreads through arbitrarily long quantification chains.

The elements of typicality interpretations represent concepts and individuals similarly to the canonical model of the base DL being considered. This dissertation considers two DDLs: $\mathcal{EL}_\bot$ and $\mathcal{ELI}_\bot$. As exposed in Section 2.5, the domains in those canonical models are different. In $\mathcal{EL}_\bot$, the elements are $C \in \mathfrak{L}(\mathcal{EL}_\bot)$. On the other hand, in $\mathcal{ELI}_\bot$, the elements are sets of named concepts and they represent the conjunction of those concepts. The elements in a *typicality domain* have a second dimension representing sets of DCIs. We call the part of an element representing concepts its *concept set*, and the part representing DCIs its *typicality set*. The form of an arbitrary domain element is

$$x_{\mathcal{U}}$$

where $\mathcal{X}$ is either a concept (in TMs for $\mathcal{EL}_\bot$) or a set of concepts (in TMs for $\mathcal{ELI}_\bot$), and $\mathcal{U}$ is a set of DCIs. In canonical models for $\mathcal{EL}_\bot$ and $\mathcal{ELI}_\bot$, membership for concept representatives aligns with subsumption for the same concept. The idea is the same for typicality interpretations, but the elements represent a concept in conjunction with (the materialization of) some defeasible information.

Typicality interpretations are built from typicality domains coupled with a special definition of satisfaction for defeasible formulas. A typicality interpretation $\mathcal{I}$ satisfies a defeasible inclusion $C \sqsubseteq\!\!\!\sim D$ if the most typical instances of $C$ in the domain are in $D^{\mathcal{I}}$. The degree of typicality is measured by the subset relation over the defeasible sets of the elements.

The starting point of semantics based on typicality models is the *minimal typicality model*, a canonical model construction for materialization-based reasoning semantics. The minimal typicality model defines propositional coverage of strength **s**, which is proved to be equivalent to **s**-materialization-based reasoning. One property of the minimal typicality model is that individuals witnessing existential quantification are always atypical, i.e., they are always of the form $\mathcal{X}_\emptyset$, which is a concrete manifestation of quantification neglect.

Semantics based on typicality models addresses this shortcoming by propagating defeasible information by means of two-stepped *upgrades*. The first step, called *update*, amounts to adding new edges landing into more typical instances of the same concept. Suppose an element had a successor $\mathcal{X}_\emptyset$. In this case, a possible update would be $\mathcal{X}_\mathcal{U}$, with $\mathcal{U} \neq \emptyset$. We do not delve into detail now, as this procedure is considerably different for $\mathcal{EL}_\bot$ and $\mathcal{ELI}_\bot$.

The second component of an upgrade procedure is fixing the interpretation in a meaningful way to recover the model property. Unsurprisingly, altering the relational structure of a model may create axiom violations, which need to be addressed. Solving those violations is done through different techniques in each logic. In $\mathcal{EL}_\bot$, the algorithm is called a *model completion* and, in $\mathcal{ELI}_\bot$, *model recovery*. The former only adds elements to concepts and edges to the interpretation, while the latter can also delete edges.

Finally, iterating the upgrade procedure eventually halts when the model runs short of viable update candidates. The result is a *saturated typicality model*, and the set of such models characterizes nested coverage. The upgrade procedure may have divergent upgrades. Therefore, a single minimal typicality model may give rise to a set containing several saturated typicality models. Nested reasoning is done skeptically over this set. The consequences of the semantics are the formulas valid through all models in the set.

The plan for the remainder of Part III is as follows:

- Chapter 6 develops semantics based on typicality models for the DDL $\mathcal{EL}_\bot$ and parameters $\{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\} \times \{\mathbf{prop}, \mathbf{nest}\}$;
- Chapter 7 develops semantics based on typicality models for the DDL $\mathcal{ELI}_\bot$ and parameters $\{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\} \times \{\mathbf{prop}, \mathbf{nest}\}$;
- Chapter 8 presents considerations on the inferential power of each of the presented semantics, including a detailed comparison between them all.

## 5.1 Materialization in the $\mathcal{EL}$ family

This short introduction made several mentions to materialization-based reasoning. Typicality models are measured against materialization-based semantics, and are formulated to solve a shortcoming common to all of then – quantification neglect. However, materialization as defined in Section 4.3 is not compatible with the logics in the $\mathcal{EL}$ family. As $\mathcal{EL}_\bot$ and $\mathcal{ELI}_\bot$ admit neither disjunction nor negation, the material implication is realized by the use of new GCIs in the TBox. Materialization by TBox extension from [PT18b] is oblivious to the DL in use and is used here for $\mathcal{ELI}_\bot$.

**Definition 5.1** (Materialization)**.** *Let* $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ *be a DKB and* $\mathsf{N_C}^{aux} \subseteq \mathsf{N_C}$ *be a set of auxiliary concept names such that* $\mathsf{sig}(\mathcal{K}) \cap \mathsf{N_C}^{aux} = \emptyset$*. Then, let:*

- $\overline{E \mathrel{\underset{\sim}{\sqsubset}} F} := A_{E \mathrel{\underset{\sim}{\sqsubset}} F}$*, with* $A_{E \mathrel{\underset{\sim}{\sqsubset}} F} \in \mathsf{N_C}^{aux}$*.*
- *DBox materialization:* $\overline{\mathcal{D}} := \bigsqcap_{(E \mathrel{\underset{\sim}{\sqsubset}} F) \in \mathcal{D}} \overline{E \mathrel{\underset{\sim}{\sqsubset}} F}$*, and*
- *DKB materialization:* $\overline{\mathcal{K}} := \big(\mathcal{T} \cup \{(\overline{E \mathrel{\underset{\sim}{\sqsubset}} F} \sqcap E) \sqsubseteq F \mid E \mathrel{\underset{\sim}{\sqsubset}} F \in \mathcal{D}\},\ \emptyset\big)$

The materialization of DBoxes is also used for sets of DCIs in general. In this case, the same naming function is used, ensuring that the names $\overline{E \mathrel{\underset{\sim}{\sqsubset}} F}$ in the conjunction $\overline{\mathcal{U}}$ are the same in $\overline{\mathcal{D}}$, where $\mathcal{U}$ and $\mathcal{D}$ coincide, and therefore trigger the intended axioms. Note that DKB materialization rewrites the DBox into the new TBox, where each DCI is represented by a GCI that, intuitively, refers to the typical members of the concept on the left-hand side. The DBox in $\overline{\mathcal{K}}$ is always empty, thus referring to classical reasoning.

# Chapter 6

# Typicality Models for $\mathcal{EL}_\bot$

THE main results for semantics based on typicality models for $\mathcal{EL}_\bot$ were developed by Pensel & Turhan in (2017), (2017), (2018), (2018) and Pensel (2019). This chapter considers a summary of those results, although the notation and the ways of defining certain concepts may differ in the presentation. The chapter also introduces some new advancements to typicality models for $\mathcal{EL}_\bot$:

1. the development of a strength for the lexicographic closure, denoted by **lex**,
2. a comparison between different all entailments parametrized $\{\mathbf{prop}, \mathbf{nest}\} \times \{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\}$.

Lexicographic closure was previously not considered by typicality models, despite being an influential and time-tested materialization-based semantic. Regarding the second item, the comparisons that appeared in the literature were limited to fixed strengths, i.e., for a given **s**, a comparison between $\models_{\mathsf{prop,s}}$ and $\models_{\mathsf{nest,s}}$. Here, we extend them to all combinations.

## 6.1 Preliminaries

The backbone of semantics based on typicality models is the domain. As outlined in Chapter 5, *typicality domains* for $\mathcal{EL}_\bot$ are composed of two-dimensional elements $C_{\mathcal{U}}$, where $C \in \mathfrak{L}(\mathcal{EL}_\bot)$ and $\mathcal{U}$ is a set of DCIs. The concept set is taken from the context over $\mathcal{K}$ denoted by $\mathbb{C}(\mathcal{K})$. The typicality set is defined according to the materialization-based semantics **s**. Each semantics has its restrictions on what sets of DCIs should be in the domain. For now, the only general restriction is $\mathcal{U} \subseteq \mathcal{D}$ for every $\mathcal{U}$ in the typicality set.

**Definition 6.1** (Typicality Domains for $\mathcal{EL}_\bot$). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ and $\mathbb{C}(\mathcal{K})$ be a context over $\mathcal{K}$. A* typicality domain *for $\mathcal{K}$, denoted by $\Delta^{T(\mathcal{K})}$, is characterized by $\Delta^{T(\mathcal{K})} \subseteq \mathbb{C}(\mathcal{K}) \times \mathcal{D}$ s.t. for every $C \in \mathbb{C}(\mathcal{K})$, $C_\emptyset \in \Delta^{T(\mathcal{K})}$.*

*A typicality domain is* consistent *with respect to $\mathcal{K}$ iff for every $C_{\mathcal{U}} \in \Delta^{T(\mathcal{K})}$, $\overline{\mathcal{K}} \not\models C \sqcap \overline{\mathcal{U}} \sqsubseteq \bot$.*

From now on, we consider only consistent typicality domains. Typicality interpretations are simply interpretations built on top of typicality domains.

**Definition 6.2** (Typicality Interpretations for $\mathcal{EL}_\perp$). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB. The interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ is a* typicality interpretation *for $\mathcal{K}$* iff *$\Delta^\mathcal{I}$ is a typicality domain for $\mathcal{K}$.*

Semantics based on typicality models are defined to capture strict and defeasible subsumption. In order to do so, it couples the standard definition of GCI satisfaction with additional criteria for the satisfaction of DCIs. To check the satisfaction of some DCI, we restrict the elements being considered to the most typical instances of the concepts.

**Definition 6.3** (Satisfaction). *Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be a typicality interpretation and $C, D \in \mathfrak{L}(\mathcal{EL}_\perp)$ be two concepts. We say that*

- $\mathcal{I} \models C \sqsubseteq D$ iff $C^\mathcal{I} \subseteq D^\mathcal{I}$, *and*
- $\mathcal{I} \models C \mathrel{\underset{\sim}{\sqsubset}} D$ iff $C_\mathcal{U} \in D^\mathcal{I}$ *for every* $\mathcal{U} \subseteq \mathcal{D}$ *such that* $\nexists \mathcal{V} \supset \mathcal{U}$ *with* $C_\mathcal{V} \in \Delta^\mathcal{I}$.

Typicality interpretations also have a particular definition of *satisfying* a DKB. Besides the usual notion of satisfaction for strict axioms, they require that every element satisfies the DCIs in its typicality set. The intuition is that an element $C_\mathcal{U}$ represents the concept $C \sqcap \overline{\mathcal{U}}$. By complementing $C$ with the materialization of the DCIs in $\mathcal{U}$, the interpretation guarantees that it satisfies every $E \mathrel{\underset{\sim}{\sqsubset}} F \in \mathcal{U}$.

**Definition 6.4** (Model for $\mathcal{K}$). *Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be a typicality interpretation and $\mathcal{K} = (\mathcal{T}, \mathcal{D})$. $\mathcal{I}$ is a model of $\mathcal{K}$, denoted by $\mathcal{I} \models \mathcal{K}$, iff*

- *$\mathcal{I} \models C \sqsubseteq D$ for every $C \sqsubseteq D$, and*
- *If $C_\mathcal{U} \in E^\mathcal{I}$, then $C_\mathcal{U} \in F^\mathcal{I}$ for every $E \mathrel{\underset{\sim}{\sqsubset}} F \in \mathcal{U}$.*

We want elements to represent the conjunction of the concept and the set of DCIs from their name. An element $C_\mathcal{U}$ should represent the concept $C$ combined with the defeasible knowledge in $\mathcal{U}$. To this end, we further restrict our attention to a special class of typicality models: *standard typicality models*. The standard property simply requires every $C$ representative to belong to $C$, and that every existential restriction within the context to be represented in the model by one edge with an atypical successor.

**Definition 6.5** (Standard property). *[Pen19] An typicality interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ defined over a context $\mathbb{C}(\mathcal{K})$ is* standard *iff:*

1. *$C_\mathcal{U} \in C^\mathcal{I}$, for every $C_\mathcal{U} \in \Delta^\mathcal{I}$,*
2. *If $C_\mathcal{U} \in (\exists r.D)^\mathcal{I}$ and $D \in \mathbb{C}(\mathcal{K})$, then $(C_\mathcal{U}, D_\emptyset) \in r^\mathcal{I}$.*

From now on, every time we speak of typicality models for a DKB $\mathcal{K}$, we assume that they are standard, unless specified otherwise. Example 6.6 depitcs a standard typicality model for $\mathcal{K}$.

**Example 6.6** (Typicality Interpretation). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB with $\mathcal{T} = \{\mathsf{Deity} \sqsubseteq \mathsf{Being}, \mathsf{Human} \sqsubseteq \mathsf{Being}\}$ and $\mathcal{D} = \{\mathsf{Being} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Mortal}, \mathsf{Being} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Corporeal}, \mathsf{Deity} \sqcap \mathsf{Mortal} \mathrel{\underset{\sim}{\sqsubset}} \perp\}$. Figure 6.1 represents a fragment of $\mathcal{I} = (\mathcal{T}^\mathcal{I}, \cdot^\mathcal{I})$, a standard model of $\mathcal{K}$.*

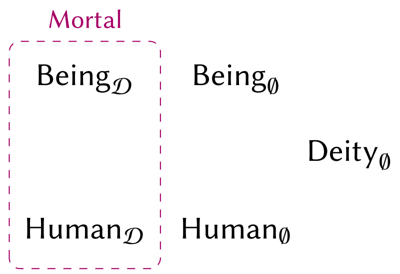


**Figure 6.1:** *Diagram representing a fragment of $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$. Elements are represented in the $C_\mathcal{U}$ format, and the dashed rectangle represents the extension of* Mortal$^\mathcal{I}$.

*Notice that:*

- $\mathcal{I} \models$ Human $\mathrel{\sqsubseteq\!\!\!\sim}$ Mortal, *as the most typical instance of* Human *in $\mathcal{I}$ belongs to* Mortal$^\mathcal{I}$,
- *Every $C_\mathcal{U}$ satisfies the DCIs in its typicality set. For example,* Being$_\mathcal{D} \in$ Being$^\mathcal{I}$, *and therefore* Being$_\mathcal{D} \in$ Mortal$^\mathcal{I}$. *On the other hand,* Being$_\emptyset$ *is not in* Mortal$^\mathcal{I}$.
- *There is no instance of* Deity *combined with $\mathcal{D}$, as this would yield an inconsistent element belonging to* Mortal *and, therefore, to $\bot$.*

There is no requirement that elements satisfy only the concept inclusions required by the DKB. An interpretation $\mathcal{J}$ that is exactly as the interpretation $\mathcal{I}$ from Example 6.6 except by Being$_\emptyset \in$ Mortal$^\mathcal{J}$ would still be a model. Although Being$_\emptyset$ is not *required* to be a mortal, it could be one without generating a contradiction. Models whose membership is interchangeable to subsumption for their concept representative are said to be canonical. This canonicity must be defined for some entailment relation. In the following section, we define the *minimal typicality model*, which is proven to be canonical for three different strengths of materialization-based reasoning: rational, relevant, and lexicographic.

## 6.2 Minimal Typicality Model for $\mathcal{EL}_\bot$

The minimal typicality model is based on the canonical model for $\mathcal{EL}_\bot$ portrayed in Definition 2.14. The minimal typicality model is essentially a multi-layered copy of the canonical model, where the layers are subsets of $\mathcal{D}$. Each element $C$ has several representatives instead of only one, and each of those representatives is enriched with a set of DCIs.

Combining concepts with sets of DCIs is the main idea of the materialization-based semantics discussed in 4.3.1. This intuition makes it possible to represent different semantics by carefully tailoring the domain to contain the correct elements according to some materialization-based semantics $\mathsf{s}$. Before delving into the particular strengths, we consider a general definition of the minimal typicality model that covers all of them. Then, we consider what it means for the minimal typicality model to be parametrized by a strength $\mathsf{s}$. Each strength defines a different minimal typicality model agreeing with the general definition and corresponding to some materialization-based semantics. We examine each construction and show their canonicity for the equivalent strength.

**Definition 6.7** (Minimal typicality model for $\mathcal{EL}_\bot$)**.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB, $\mathbb{C}(\mathcal{K})$ be the context of $\mathcal{K}$ and $\Delta^{T(\mathcal{K})}$ be a typicality domain defined over $\mathbb{C}(\mathcal{K})$. The minimal typicality model $\mathcal{I}^{\mathcal{K}}_{\text{min}} = (\Delta^{T(\mathcal{K})}, \cdot^{\mathcal{I}^{\mathcal{K}}_{\text{min}}})$ is defined directly by setting:*

$$A^{\mathcal{I}^{\mathcal{K}}_{\text{min}}} := \{C_{\mathcal{U}} \in \Delta^{T(\mathcal{K})} \mid \overline{\mathcal{K}} \models C \sqcap \overline{\mathcal{U}} \sqsubseteq A\}$$
$$r^{\mathcal{I}^{\mathcal{K}}_{\text{min}}} := \{(C_{\mathcal{U}}, D_{\emptyset}) \mid \overline{\mathcal{K}} \models C \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.D\}$$

It is important to show that this construction is a model of $\mathcal{K}$.

**Lemma 6.8.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and $\mathcal{I}^{\mathcal{K}}_{\text{min}}$ be the minimal typicality model for a typicality domain $\Delta^{T(\mathcal{K})}$. Then, $\mathcal{I}^{\mathcal{K}}_{\text{min}} \models \mathcal{K}$.*

*Proof.* To show $\mathcal{I}^{\mathcal{K}}_{\text{min}} \models \mathcal{K}$ we need to show that:

1. $C_{\mathcal{U}} \in E^{\mathcal{I}^{\mathcal{K}}_{\text{min}}} \Rightarrow C_{\mathcal{U}} \in F^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$, for every $E \sqsubseteq F \in \mathcal{T}$, and
2. $C_{\mathcal{U}} \in E^{\mathcal{I}^{\mathcal{K}}_{\text{min}}} \Rightarrow C_{\mathcal{U}} \in F^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$, for every $E \precsim F \in \mathcal{U}$.

(1) The proof is on the structure of $F$. For the base, $F \in \text{sig}_{\mathsf{C}}(\mathcal{K})$. Then, the result follows by the construction of the minimal typicality model. For the inductive step, suppose the result holds for concepts $F$ with $\text{Size}(F) = i$.[1] We show that it also holds for any $G \in \mathfrak{L}(\mathcal{EL}_\bot)$ such that $\text{Size}(G) = i + 1$. Notice that there are only two ways to increase the size of a formula in $\mathcal{EL}_\bot$: $G \in \{H_1 \sqcap H_2, \exists r.H\}$. In the first case, $\mathcal{K} \models E \sqsubseteq G = H_1 \sqcap H_2$ implies that $\mathcal{K} \models E \sqsubseteq H_1$ and $\mathcal{K} \models E \sqsubseteq H_2$. By the induction hypothesis, $C_{\mathcal{U}} \in H_1^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$ and $C_{\mathcal{U}} \in H_2^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$, which together imply $C_{\mathcal{U}} \in (H_1 \sqcap H_2)^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$. For the second case, notice that $H \in \mathbb{C}(\mathcal{K})$ because $H \in Qc(\mathcal{K})$. Then, $H_{\emptyset} \in \Delta^{T(\mathcal{K})}$, $(C_{\mathcal{U}}, H_{\emptyset}) \in r^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$ and $H_{\emptyset} \in H^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$. Therefore, $C_{\mathcal{U}} \in (\exists r.H)^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$.

(2) Let $C_{\mathcal{U}} \in \Delta^{T(\mathcal{K})}$ be an arbitrary element in the typicality domain, $E \precsim F \in \mathcal{U}$, and $C_{\mathcal{U}} \in E^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$. The materialized DKB $\overline{\mathcal{K}}$ contains $E \sqcap \overline{E \precsim F} \sqsubseteq F$. By construction, $C_{\mathcal{U}} \in (\overline{E \precsim F})^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$, because $\overline{\mathcal{K}} \models C \sqcap \overline{\mathcal{U}} \sqsubseteq \overline{E \precsim F}$. Then, $\overline{\mathcal{K}} \models C \sqcap \overline{\mathcal{U}} \sqsubseteq F$. We can use the same argument made in (1) because the extended DKB $\overline{\mathcal{K}}$ determines membership in the minimal typicality model. By induction over $\text{Size}(F)$, we conclude that $C_{\mathcal{U}} \in F^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$. □

The general definition of the minimal typicality model covered by Definition 6.7 makes no requirements over the domain, except that it is a consistent typicality domain. An additional requirement for each **s** secures canonicity for the correspondent minimal typicality. After presenting this requirement, we present the role of the strength to the semantics based on typicality models. This parameter serves to define the *shape* of the typicality domain, namely, the elements that it contains.

Materialization-based reasoning of strength **s** defines entailment for DCIs by enriching the concept on the left-hand side with the materialization of a subset of $\mathcal{D}$ selected by the consistent-selection function $\text{sel}_{\mathcal{K},\mathbf{s}}(C)$. On the other hand, typicality models satisfy

[1] Intuitively, $\text{Size}(C)$ denotes the size of a concept, where $\text{Size}(C) = 1$ if $C$ is a named concept or a constant, $\text{Size}(C \sqcap D) = \text{Size}(C) + \text{Size}(D) + 1$, and $\text{Size}(\exists r.C) = \text{Size}(\forall r.C) = \text{Size}(C) + 1$.

DCIs by limiting subsumption to a subset of the elements in the extension of the left-hand side concept: the most typical instances of the concept representative. To bridge those two reasoning methods, we must ensure that the most typical instances of each concept representative in the domain are exactly those in $\mathsf{sel}_{\mathcal{K},\mathbf{s}}(C)$.

**Theorem 6.9** (Canonicity of $\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}$)**.** *Let* $\mathbf{s} \in \{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\}$ *be a strength,* $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ *a DKB. Let* $\mathcal{I}^{\mathcal{K}}_{\mathsf{min}} = (\Delta^{T(\mathcal{K})}, \cdot^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}})$ *be a minimal typicality model of* $\mathcal{K}$ *such that for every* $C \in \mathbb{C}(\mathcal{K})$*,* $C_{\mathcal{U}}$ *is maximally typical in* $\Delta^{T(\mathcal{K})}$ *iff* $\mathcal{U} \in \mathsf{sel}_{\mathcal{K},\mathbf{s}}(C)$*. Then, for every* $M \in \Delta^{\mathcal{K}}$*:*

1. $\mathcal{K} \models_{\mathsf{mat,s}} C \sqsubseteq D$ *with* iff $C_\emptyset \in D^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}$*, and*

2. $\mathcal{K} \models_{\mathsf{mat,s}} C \mathrel{\underset{\sim}{\sqsubset}} D$ iff $C_{\mathcal{U}} \in D^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}$ *for every maximally typical instance* $C_{\mathcal{U}}$ *of* $M$*.*

*Proof.* (1) ($\Rightarrow$) by construction, $C_\emptyset \in C^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}$. Then, Lemma 6.8 implies that $C_\emptyset \in D^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}$. ($\Leftarrow$) We show by induction on the structure of $D$. *Base*: $D \in \mathsf{N_C}$. Then, $C_\emptyset \in D^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}$ iff $\overline{\mathcal{K}} \models C \sqsubseteq D$ iff $\mathcal{K} \models_{\mathsf{mat,s}} C \sqsubseteq D$. *Inductive step*: let $\mathsf{Size}(D) = i + 1$. Then, $D \in \{E_1 \sqcap E_2, \exists r.E\}$. In the first case, $C_\emptyset \in (E_1 \sqcap E_2)^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}$ iff $C_\emptyset \in E_1^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}$ and $C_\emptyset \in E_2^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}$ iff $\overline{\mathcal{K}} \models C \sqsubseteq E_1$ and $\overline{\mathcal{K}} \models C \sqsubseteq E_2$ iff $\overline{\mathcal{K}} \models C \sqsubseteq D$ iff $\mathcal{K} \models_{\mathsf{mat,s}} C \sqsubseteq D$. If $D = \exists r.E$, then $C_\emptyset \in (\exists r.E)^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}$ implies there is some element $F \in \Delta^{\mathcal{K}}$ such that $(C_\emptyset, F_\emptyset) \in r^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}$ and $F \in E^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}$. Therefore, $\overline{\mathcal{K}} \models C \sqsubseteq \exists r.F$ and $\overline{\mathcal{K}} \models F \sqsubseteq E$. These facts imply $\overline{\mathcal{K}} \models C \sqsubseteq \exists r.E$ and $\mathcal{K} \models_{\mathsf{mat,s}} C \sqsubseteq \exists r.E$.

(2) $\mathcal{K} \models_{\mathsf{mat,s}} C \mathrel{\underset{\sim}{\sqsubset}} D$ iff $\overline{\mathcal{K}} \models C \sqcap \overline{\mathcal{U}}$, for every $\mathcal{U} \in \mathsf{sel}_{\mathcal{K},\mathbf{s}}(C)$. Note that, by hypothesis, the maximally typical instances of $C$, for every $C \in \mathbb{C}(\mathcal{K})$, are $C_{\mathcal{U}}$, for $\mathcal{U} \in \mathsf{sel}_{\mathcal{K},\mathbf{s}}(C)$. Then, $C_{\mathcal{U}} \in D^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}$ and $\mathcal{I}^{\mathcal{K}}_{\mathsf{min}} \models C \mathrel{\underset{\sim}{\sqsubset}} D$. □

We employ the minimal typicality model for a given strength $\mathbf{s}$ to define typicality models' based concept subsumption of *propositional coverage.*

**Definition 6.10** (Propositional coverage for defeasible $\mathcal{EL}_\bot$)**.** *Let* $\mathbf{s} \in \{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\}$ *be a strength,* $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ *a DKB. Let* $\mathcal{I}^{\mathcal{K}}_{\mathsf{min}} = (\Delta^{T(\mathcal{K})}, \cdot^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}})$ *be a minimal typicality model of* $\mathcal{K}$ *such that for every* $C \in \mathbb{C}(\mathcal{K})$*,* $C_{\mathcal{U}}$ *is maximally typical in* $\Delta^{T(\mathcal{K})}$ *iff* $\mathcal{U} \in \mathsf{sel}_{\mathcal{K},\mathbf{s}}(C)$*.*

$$\mathcal{K} \models_{\mathsf{prop,s}} C \mathrel{\underset{\sim}{\sqsubset}} D \text{ iff } \mathcal{I}^{\mathcal{K}}_{\mathsf{min}} \models C \mathrel{\underset{\sim}{\sqsubset}} D$$
$$\mathcal{K} \models_{\mathsf{prop,s}} C \sqsubseteq D \text{ iff } \mathcal{I}^{\mathcal{K}}_{\mathsf{min}} \models C \sqsubseteq D$$

Notice that, by the equivalence proved in Theorem 6.9, propositional reasoning of strength $\mathbf{s}$ is equivalent to materialization-based reasoning defined over $\mathsf{sel}_{\mathcal{K},\mathbf{s}}$.

### 6.2.1 Domain Shapes

The minimal condition expressed by Theorem 6.9 is sufficient to model materialization-based closure with typicality models. However, the purpose of those models is lifting reasoning to transmit defeasible information through quantifiers. In order to do this, we update the elements witnessing existential restrictions to more typical instances. An edge $(C_{\mathcal{U}}, D_\emptyset) \in r^{\mathcal{I}}$, which represents $C \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.D$, may be upgraded to $(C_{\mathcal{U}}, D_{\mathcal{V}}) \in r^{\mathcal{I}}$, for some $\mathcal{V} \neq \emptyset$.

However, including only the atypical and the maximally typical instances of every concept according to $\mathsf{sel}_{\mathcal{K},\mathbf{s}}$ is not sufficient to faithfully capture the partitioning of the domain achieved by materialization-based reasoning. An individual may be incapable of having a maximally typical successor $D_{\mathcal{V}}$, for $\mathcal{V} \in \mathsf{sel}_{\mathcal{K},\mathbf{s}}(D)$, but may be consistent with a successor of intermediate typicality $D_{\mathcal{V}'}$, for $\mathcal{V}' \subset \mathcal{V}$ and $\mathcal{V}' \neq \emptyset$. Choosing those elements of intermediate typicality impacts the upgrade procedure that leads to reasoning of nested coverage. In this subsection, we explore the construction of the domains for $\mathbf{s} \in \{\mathbf{rat}, \mathbf{lex}, \mathbf{rel}\}$, which is shaped by the choice of those elements of intermediate typicality.

## Rational Domain

The rational domain is grounded on the intuition that the typicality sets are taken from the exceptionality chain, as defined in 4.6. The partition of $\mathcal{D}$ used by rational reasoning is a list of ever-decreasing subsets of $\mathcal{D}$. The domain is then populated by having the elements $C_{\mathcal{E}_i}$ such that $\mathsf{Rank}_C(\mathcal{K}) \leq i$.

**Definition 6.11** (Rational typicality domain)**.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and $\mathcal{E}_0, \ldots, \mathcal{E}_n$ be its exceptionality chain. Let $\mathbb{C}(\mathcal{K})$ be a context over $\mathcal{K}$. The* rational typicality domain *of $\mathcal{K}$ is:*

$$\Delta^{T(\mathcal{K})}_{\mathbf{rat}} := \{C_{\mathcal{E}_i} \in \mathbb{C}(\mathcal{K}) \times \{\mathcal{E}_0, \ldots, \mathcal{E}_n\} \mid \overline{\mathcal{K}} \not\models C \sqcap \overline{\mathcal{E}_i} \sqsubseteq \bot \textit{ for every } C \in \mathbb{C}(\mathcal{K})\}$$

**Example 6.12.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be the DKB portrayed in Example 6.6. The exceptionality chain for this DKB is: $\mathcal{E}_0 = \mathcal{D}$, $\mathcal{E}_1 = \{\mathsf{Deity} \sqcap \mathsf{Mortal} \sqsubseteq \bot\}$, $\mathcal{E}_2 = \emptyset$. The rational typicality domain for this DKB can be visualized as:*

| $\Delta^{T(\mathcal{K})}_{\mathbf{rat}}$ | $\mathcal{E}_0$ | $\mathcal{E}_1$ | $\emptyset$ |
|---|---|---|---|
| Deity | | $\mathsf{D}_{\mathcal{E}_1}$ | $\mathsf{D}_{\emptyset}$ |
| Human | $\mathsf{H}_{\mathcal{E}_0}$ | $\mathsf{H}_{\mathcal{E}_1}$ | $\mathsf{H}_{\emptyset}$ |
| Being | $\mathsf{B}_{\mathcal{E}_0}$ | $\mathsf{B}_{\mathcal{E}_1}$ | $\mathsf{B}_{\emptyset}$ |
| Mortal | $\mathsf{M}_{\mathcal{E}_0}$ | $\mathsf{M}_{\mathcal{E}_1}$ | $\mathsf{M}_{\emptyset}$ |

**Figure 6.2:** *Rational typicality domain for the DKB in Example 6.12. Concept representatives are represented by the concept's first letter. Colored elements are the most typical instances of their representatives. The element $\mathsf{D}_{\mathcal{E}_0}$ is absent, as it is unsatisfiable.*

Combining minimal typicality domains with the rational domains produces minimal rational typicality domains.

**Definition 6.13** ($\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rat}}$, $\models_{\mathsf{prop},\mathbf{rat}}$)**.** *A* rational minimal typicality model *of a DKB $\mathcal{K}$ is a minimal typicality model of a DKB $\mathcal{K}$ over the rational typicality domain $\Delta^{T(\mathcal{K})}_{\mathbf{rat}}$, defined as $\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rat}} := (\Delta^{T(\mathcal{K})}_{\mathbf{rat}}, \cdot^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rat}}})$.*

Semantics for rational strength, propositional coverage *by typicality models, denoted by $\models_{\mathsf{prop},\mathbf{rat}}$, is the combination of semantics of propositional coverage with* $\mathbf{s} = \mathbf{rat}$.

**Example 6.14.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be the DKB portrayed in Example 6.6 with an additional strict axiom:* Deity $\sqsubseteq \exists$observes.Human. *The exceptionality chain for this DKB is: $\mathcal{E}_0 = \mathcal{D}$, $\mathcal{E}_1 = \{$Deity $\sqcap$ Mortal $\sqsubseteq \bot\}$, $\mathcal{E}_2 = \emptyset$. The rational typicality domain for this DKB can be visualized as:*

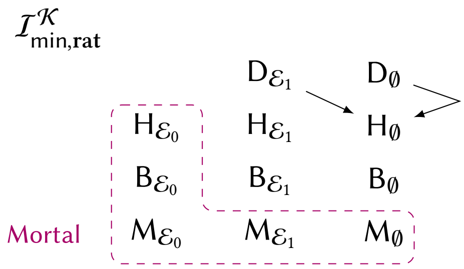


**Figure 6.3:** *Minimal typicality domain with the rational model. The colored and dashed path represents the extension of* Mortal*. The arrows represent edges for the role* observes.

This definition ensures that the rational typicality model is canonical for materialization-based rational reasoning.

**Corollary 6.15** (Canonicity of $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}}$, equivalence of prop and mat)**.** *Let $\mathcal{K}$ be a DKB and $C \in \mathbb{C}(\mathcal{K})$. Then:*

$$\mathcal{K} \models_{\mathsf{mat,rat}} C \mathrel{\underset{\sim}{\sqsubset}} D \textit{ iff } \mathcal{K} \models_{\mathsf{prop,rat}} C \mathrel{\underset{\sim}{\sqsubset}} D$$

*Proof.* The result follows directly from Theorem 6.9 (canonicity of $\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}$) and Definition 6.11 (rational domain for $\mathcal{EL}_\bot$). For every $C \in \mathbb{C}(\mathcal{K})$, $C_{\mathcal{E}_i}$ is the most typical instance in the domain, where $\mathsf{Rank}_C(\mathcal{K}) = i$. Therefore, $C \in D^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}}}$ iff $\mathcal{K} \models_{\mathsf{prop,rat}} C \mathrel{\underset{\sim}{\sqsubset}} D$ iff $\mathcal{K} \models_{\mathsf{mat,rat}} C \mathrel{\underset{\sim}{\sqsubset}} D$. □

## Relevant and Lexicographic Domains

Relevant and lexicographic closures stratify the DCIs into finer sets to avoid problems such as inheritance blocking. The two consistent-selection functions achieve more fine-grained selections while keeping the precedence of ranking. Therefore, the two materialization-based closures extend rational reasoning by including every DCI from $\mathsf{sel}_{\mathcal{K},\mathbf{rat}}$ and possibly some more of lesser ranks.

In order to capture the finer detail of $\mathsf{sel}_{\mathcal{K},\mathbf{rel}}$ and $\mathsf{sel}_{\mathcal{K},\mathbf{lex}}$, we move from a matrix-shaped domain to the full lattice over $\mathcal{D}$ given by the subset relation. To populate the domain, we take every maximally typical $C_{\mathcal{U}}$ as selected by the consistent-selection function and include $C$-representatives paired with all subsets of $\mathcal{U}$.

**Definition 6.16** (Relevant typicality domain)**.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB. The* relevant typicality domain *of $\mathcal{K}$ is defined as:*

$$\Delta^{T(\mathcal{K})}_{\mathbf{rel}} := \{C_{\mathcal{U}} \in \mathbb{C}(\mathcal{K}) \times \mathcal{P}(\mathcal{D}) \mid \mathcal{U} \subseteq \mathsf{sel}_{\mathcal{K},\mathbf{rel}}(C) \textit{ for every } C \in \mathbb{C}(\mathcal{K})\}$$

**Definition 6.17** (Lexicographic typicality domain)**.** *The* lexicographic typicality domain *of $\mathcal{K}$ is defined as:*

$$\Delta^{T(\mathcal{K})}_{\mathbf{lex}} := \{C_{\mathcal{U}} \in \mathbb{C}(\mathcal{K}) \times \mathcal{P}(\mathcal{D}) \mid \mathsf{sel}_{\mathcal{K},\mathbf{lex}}(C) = \{\mathcal{U}_1, \ldots, \mathcal{U}_n\} \text{ and } \mathcal{U} \subseteq \mathcal{U}_i \text{ for some } 1 \leq i \leq n \text{ for every } C \in \mathbb{C}(\mathcal{K})\}$$

**Example 6.18.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be the same DKB defined in Example 6.12. We consider a fragment of the relevant and lexicographic domains. The domains coincide for this DKB. The fragment is depicted in Figure 6.4.*

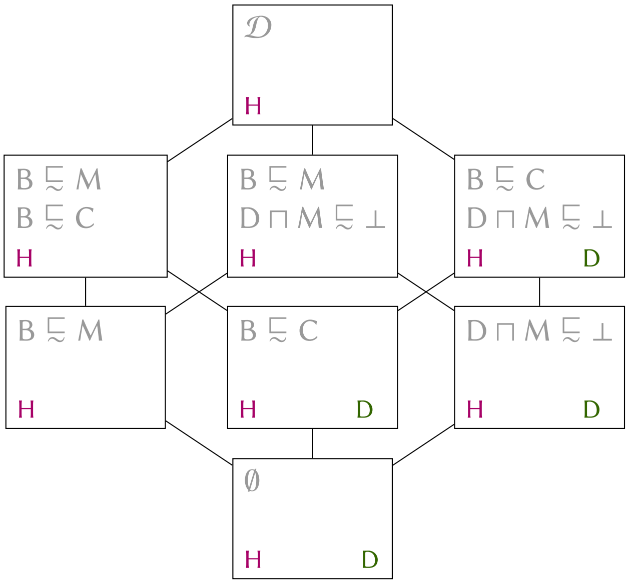


**Figure 6.4:** *Graphical representation of the relevant and lexicographic domains, which coincide in this case. The lattice over $\mathcal{D}$ represents all possible typicality sets. Letters are the initial of the concepts* Being, Corporeal, Human, Mortal. *Grey DCIs characterize each typicality set, and the colored letters* H *and* D *are instances of the concepts* Human *and* Deity*, respectively.*

Notice that the most typical instance of *human* remains $\mathsf{Human}_{\mathcal{D}}$, but the most typical instance of *deity* becomes $\mathsf{Deity}_{\mathcal{U}}$, for $\mathcal{U} = \{\mathsf{Being} \mathrel{\tilde{\sqsubseteq}} \mathsf{Corporeal}, \mathsf{Deity} \sqcap \mathsf{Mortal} \mathrel{\tilde{\sqsubseteq}} \bot\}$, which is stronger than the most typical instance in the rational domain.

Once more, pairing **rel**, **lex** domains with the definition of minimal typicality model generates minimal typicality models of **rel**, **lex** strength.

**Definition 6.19** ($\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rel}}$, $\models_{\mathsf{prop},\mathbf{rel}}$)**.** *Let $\mathcal{K}$ be a DKB and $\Delta^{T(\mathcal{K})}_{\mathbf{rel}}$ be the relevant domain defined over the context $\mathbb{C}(\mathcal{K})$. The* relevant minimal typicality model *of $\mathcal{K}$ over $\Delta^{T(\mathcal{K})}_{\mathbf{rel}}$ is denoted by $\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rel}} := (\Delta^{T(\mathcal{K})}_{\mathbf{rel}}, \cdot^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rel}}})$ and defined as the minimal typicality model with the domain $\Delta^{T(\mathcal{K})}_{\mathbf{rel}}$. The semantics of propositional coverage with $\mathbf{s} = \mathbf{rel}$ is defined by the relevant minimal typicality model and is denoted by $\models_{\mathsf{prop},\mathbf{rel}}$.*

**Definition 6.20** ($\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{lex}}$, $\models_{\mathsf{prop},\mathbf{lex}}$)**.** *Let $\mathcal{K}$ be a DKB and $\Delta^{T(\mathcal{K})}_{\mathbf{lex}}$ be the relevant domain defined over the context $\mathbb{C}(\mathcal{K})$. The* lexicographic minimal typicality model *of $\mathcal{K}$ over $\Delta^{T(\mathcal{K})}_{\mathbf{lex}}$*

*is denoted by* $\mathcal{I}^{\mathcal{K}}_{\text{min},\textbf{lex}} := (\Delta^{T(\mathcal{K})}_{\textbf{lex}}, \cdot^{\mathcal{I}^{\mathcal{K}}_{\text{min,lex}}})$ *and defined as the minimal typicality model with the domain* $\Delta^{T(\mathcal{K})}_{\textbf{lex}}$*. The semantics of propositional coverage with* $\mathbf{s} = \textbf{lex}$ *is defined by the relevant minimal typicality model and is denoted by* $\models_{\text{prop},\textbf{lex}}$.

Finally, we show that the minimal relevant and lexicographic typicality models are canonical for relevant and lexicographic materialization-based reasoning.

**Corollary 6.21** (Canonicity of $\mathcal{I}^{\mathcal{K}}_{\text{min},\textbf{rel}}$ and $\mathcal{I}^{\mathcal{K}}_{\text{min},\textbf{lex}}$, equivalence of prop and mat)**.** *Let* $\mathcal{K}$ *be a DKB and* $C \in \mathbb{C}(\mathcal{K})$*. Then:*

$$\mathcal{K} \models_{\text{mat},\textbf{rel}} C \mathrel{\underset{\sim}{\sqsubset}} D \textit{ iff } \mathcal{K} \models_{\text{prop},\textbf{rel}} C \mathrel{\underset{\sim}{\sqsubset}} D$$
$$\mathcal{K} \models_{\text{mat},\textbf{lex}} C \mathrel{\underset{\sim}{\sqsubset}} D \textit{ iff } \mathcal{K} \models_{\text{prop},\textbf{lex}} C \mathrel{\underset{\sim}{\sqsubset}} D$$

*Proof.* The result follows directly from Theorem 6.9 (canonicity of $\mathcal{I}^{\mathcal{K}}_{\text{min}}$) and Definitions 6.16 and 6.17 (relevant and lexicographic domains for $\mathcal{EL}_\bot$). For every $C \in \mathbb{C}(\mathcal{K})$, the most typical instances in the relevant and lexicographic domains are $C_{\mathcal{U}}$ and $C_{\mathcal{U}'}$, where $\mathcal{U} = \text{sel}_{\mathcal{K},\textbf{rel}}(C)$ and $\mathcal{U}' \in \text{sel}_{\mathcal{K},\textbf{lex}}(C)$. Therefore, $C_{\mathcal{U}} \in D^{\mathcal{I}^{\mathcal{K}}_{\text{min,rel}}}$ iff $\mathcal{K} \models_{\text{prop},\textbf{rel}} C \mathrel{\underset{\sim}{\sqsubset}} D$ iff $\mathcal{K} \models_{\text{mat},\textbf{rel}} C \mathrel{\underset{\sim}{\sqsubset}} D$ and $C_{\mathcal{U}'} \in D^{\mathcal{I}^{\mathcal{K}}_{\text{min,lex}}}$ iff $\mathcal{K} \models_{\text{prop},\textbf{lex}} C \mathrel{\underset{\sim}{\sqsubset}} D$ iff $\mathcal{K} \models_{\text{mat},\textbf{lex}} C \mathrel{\underset{\sim}{\sqsubset}} D$. □

## 6.3 Upgrading Typicality Interpretations

Canonical models for $\mathcal{EL}_\bot$ represent existential restrictions by role edges. In any given edge $(C, D) \in r^{\mathcal{I}}$, $C$ is called the *predecessor* of the edge, and $D$ is the *successor* of the edge. Due to the lack of inverse roles in $\mathcal{EL}_\bot$, edges always represent an existential requirement of the predecessor, i.e., $\mathcal{K} \models C\sqcap \sqsubseteq \exists r.D$, for a given KB $\mathcal{K}$.

In the minimal typicality model, the successors of the edges are always atypical. Hence, the corresponding edge for $\overline{\mathcal{K}} \models C \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.D$ is $(C_{\mathcal{U}}, D_{\emptyset}) \in r^{\mathcal{I}}$. This feature preserves materialization-based reasoning's incapacity to push defeasible information through quantifiers. We update existing edges to more typical instances of the same concept to overcome quantification neglect, as including more typical successors pushes defeasible information through quantifiers by making the model satisfy new existential restrictions.

Adding new edges can result in an interpretation that is not a model of the DKB. Defining a repair procedure that fixes those violations while maintaining the equivalence between concept membership and subsumption is necessary. In $\mathcal{EL}_\bot$, this can be done by imbue the interpretation with new edges and increased its concept membership. There is no need to subtract elements from the extension of concepts or edges, which makes the procedure to reconquer the model property straightforward. In virtue of this additive character, the procedure is called *model completion.*

This section introduces (i) a technique for selecting edge updates and updating a typicality model and (ii) the model completion algorithm that restores the model property after an update.

### 6.3.1 Updating Typicality Interpretations

For a given DKB $\mathcal{K}$, an existential requirement of the form $\overline{\mathcal{K}} \models C \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.D$ is represented by the edge $(C_\mathcal{U}, D_\emptyset) \in r^\mathcal{I}$. Pushing defeasible information through this requirement means that $C_\mathcal{U}$ should have a non-atypical $D$-successor, i.e., some $D_\mathcal{V}$, for $\mathcal{V} \neq \emptyset$. Generally speaking, a possible update for an edge $(C_\mathcal{U}, D_\mathcal{V}) \in r^\mathcal{I}$ is an element $D_{\mathcal{V}'}$ with $\mathcal{V} \subset \mathcal{V}'$. To select update candidates, we pick an existing $r$-edge $(M_\mathcal{U}, D_\mathcal{V})$ and look to some $D_{\mathcal{V}'}$ in the domain such that $\mathcal{V} \subset \mathcal{V}'$. Finally, the new edge is added to the updated interpretation.

We make the extra requirement that the predecessor – the element on the left-hand side of the existential restriction – is also non-atypical. Atypical elements do not have their successors updated. Those elements remain as the faithful representatives of the plain concept that they name, without incoporate any defeasible information. As shown by [Pen19, p. 103], admiting the upgrade of atypical predecessors can lead to strange conclusions. Also, another technical motivation to keep thos elements intact is that the atypical elements act like ideal representatives of strict subsumption within typicality models. Therefore, $\mathcal{K} \models C \sqsubseteq D$ iff $C_\emptyset \in D^{\mathcal{I}^\mathcal{K}_{\mathsf{min}}}$ to the minimal typicality model and its descendants.

**Definition 6.22** (Update Candidates, Updated Interpretation)**.** *Let $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ be a typicality interpretation. The set of r-update candidates for $\mathcal{I}$ is:*

$$\mathsf{UpCan}_r(\mathcal{I}) := \{(C_\mathcal{U}, D_\mathcal{V}) \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid (C_\mathcal{U}, D_{\mathcal{V}'}) \in r^\mathcal{I}, \mathcal{V}' \subset \mathcal{V} \textit{ and } \mathcal{U} \neq \emptyset\}$$

*The interpretation $\mathcal{J} = (\Delta^\mathcal{I}, \cdot^\mathcal{J})$ is an $r$-*updated interpretation *to $\mathcal{I}$ iff*

$$A^\mathcal{J} = A^\mathcal{I}$$
$$r^\mathcal{J} = r^\mathcal{I} \cup \{(C_\mathcal{U}, D_\mathcal{V})\} \textit{ for one } (C_\mathcal{U}, D_\mathcal{V}) \in \mathsf{UpCan}_r(\mathcal{I})$$
$$s^\mathcal{J} = s^\mathcal{I} \textit{ for all } s \in \mathsf{sig}_\mathsf{R}(\mathcal{K}), s \neq r$$

*An $r$-update of $\mathcal{I}$ for the candidate $(C_\mathcal{U}, D_\mathcal{V}) \in \mathsf{UpCan}_r(\mathcal{I})$ is denoted by* $\mathsf{UD}^r(\mathcal{I}, (C_\mathcal{U}, D_\mathcal{V}))$.

The following example illustrates a typicality update.

**Example 6.23.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be the DKB and $\mathcal{I}^\mathcal{K}_{\mathsf{min,rat}}$ be the typicality interpretation from Example 6.14. The interpretation $\mathcal{J}$, depicted in Figure 6.5, is an* observer*-update over $\mathcal{I}^\mathcal{K}_{\mathsf{min,rat}}$. The new edge connects $\mathsf{D}_{\mathcal{E}_1}$ to $\mathsf{H}_{\mathcal{E}_0}$. Because $\mathsf{D}_{\mathcal{E}_1}$ is the most typical instance of* Deity*, the new interpretation satisfies $\mathcal{J} \models \mathsf{Deity} \mathrel{\underset{\sim}{\sqsubset}} \exists \mathsf{observes}.\mathsf{Mortal}$.*

### 6.3.2 Recovering the Model Property

New edges can break the model property of an interpretation by giving rise to a violation of an axiom $\exists r.E \sqsubseteq F$. A new edge can add an element $C_\mathcal{U}$ to the extension of $\exists r.E$, which may be outside the extension of $F$. By satisfying this requirement – i.e., adding $C_\mathcal{U}$ to $F$ – the new interpretation can break arbitrary violations $D \sqsubseteq E$, for any complex $E \in \mathfrak{L}(\mathcal{EL}_\bot)$.

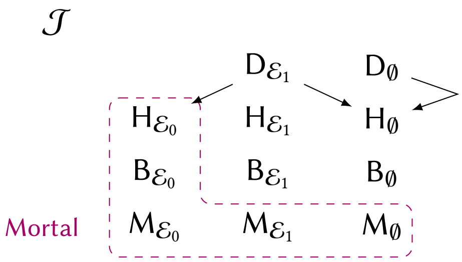


**Figure 6.5:** *The typicality interpretation $\mathcal{J}$ is an* observe-*updated interpretation to $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}}$. The new edge adds a* Mortal *successor for* $\mathsf{D}_{\mathcal{E}_1}$. *Therefore, $\mathcal{J}$ satisfies a DCI which was not satisfied by $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}}$.*

*Minimal model completion* is a repair procedure that takes as an input an updated model and returns an interpretation satisfying the DKB. It enacts the minimal amount of change required for the input interpretation to satisfy the axioms. The algorithm is very similar to the reasoning procedure for classic $\mathcal{EL}_\bot$ outlined in [Baa+17]. The most critical difference between the two methods is that axioms are applied directly to the elements in the minimal model completion, instead of an extended version of the KB that only induces a (canonical) model, as it is for monotonic $\mathcal{EL}_\bot$. The current version of the algorithm we present here is an adaptation of the algorithm presented in [Pen19, p. 119].

**Definition 6.24** (Minimal model completion)**.** *[Pen19, p. 91] Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB, and $\mathcal{J}$ be an update over a typicality model $\mathcal{I}$ defined over the context $\mathbb{C}(\mathcal{K})$. Let $\bowtie \in \{\sqsubseteq, \precsim\}$ be a generic inclusion operator. And consider that $\mathcal{EL}_\bot$ concepts can be represented by the general form $F = A_1 \sqcap \cdots \sqcap A_n \sqcap \exists r_1.G_1 \sqcap \cdots \sqcap \exists r_m.G_m$, with $A_i \in \mathsf{N_C} \cup \{\bot, \top\}$ for $1 \leq i \leq n$ and $G_i \in \mathfrak{L}(\mathcal{EL}_\bot)$, for $1 \leq j \leq m$.*

*Repeat until $\mathcal{J} \models \mathcal{K}$ or the algorithm returns clash.*

1. *For every $C_{\mathcal{U}} \in \Delta^{\mathcal{I}}$ and every $E \bowtie F \in \mathcal{T} \cup \mathcal{U}$:*
2. *If $C_{\mathcal{U}} \in E^{\mathcal{J}} \setminus F^{\mathcal{J}}$, for $F = A_1 \sqcap \cdots \sqcap A_n \sqcap \exists r_1.G_1 \sqcap \cdots \sqcap \exists r_m.G_m$:*
   (a) *Add $C_{\mathcal{U}}$ to $A_i^{\mathcal{J}}$, for $1 \leq i \leq n$.*
   (b) *Add $(C_{\mathcal{U}}, E_{j\emptyset})$ to $r_j^{\mathcal{J}}$, for $1 \leq j \leq m$.*
   (c) *If $\bot \in \{A_1, \ldots, A_n, G_1, \ldots, G_m\}$, return clash*
   (d) *For every $H \in \mathbb{C}(\mathcal{K})$:*
      - *If $C_{\mathcal{U}} \in (\exists r.H)^{\mathcal{I}}$ and $(C_{\mathcal{U}}, H_\emptyset) \notin r^{\mathcal{I}}$, add $(C_{\mathcal{U}}, H_\emptyset)$ to $r^{\mathcal{I}}$.*

*We denote the minimal model completion of $\mathcal{J}$ by* $\mathsf{mmc}(\mathcal{J}, \mathcal{K})$.

Step $(d)$ is called *standarization* and builds the additional edges to guarantee that the algorithm's output has the standard property. By construction, $E_{j\emptyset} \in \Delta^{\mathcal{I}}$, for every $E_j$ occuring in an existential restriction from the DKB. Notice that $F \in \mathbb{C}(\mathcal{K})$, and $E_j \in Qc(\mathbb{C}(\mathcal{K}))$. Finally every typicality domain $\Delta$ over a context $\mathbb{C}(\mathcal{K})$ satisfies $\Delta \subseteq \mathbb{C}(\mathcal{K}) \times \{\emptyset\}$ by definition. This ensures that the addition of role edges is always fulfilled if $E$ is a satisfiable concept.

The minimal model completion always terminates, outputting either a model extending the input interpretation or *clash*. The algorithm saturates a finite domain interpretation guided by terminological knowledge which is also finite. Therefore, there is a hard limit for the procedure. The algorithm would necessarily halt when $A^{\mathcal{I}} = \Delta$ for every $A \in \mathsf{sig}_{\mathsf{C}}(\mathcal{K})$ and $r^{\mathcal{I}} = \Delta \times \Delta$ for every $r \in \mathsf{sig}_{\mathsf{R}}(\mathcal{K})$.

**Lemma 6.25** (Termination of the minimal model completion)**.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a typicality interpretation whose domain is defined over $\mathbb{C}(\mathcal{K})$. The minimal model completion algorithm for $\mathcal{K}$ and $\mathcal{I}$ terminates.*

*Proof.* Let $\mid \Delta \mid = k$ and $\mid \mathcal{T} \cup \mathcal{D} \mid = l$, for $k, l \in \mathbb{N}$. Each $F$ such that $E \bowtie F \in \mathcal{T} \cup \mathcal{D}$ is of the form $F = A_1 \sqcap \cdots \sqcap A_n \sqcap \exists r_1.G_1 \sqcap \cdots \sqcap \exists r_m.G_m$, for $n, m \in \mathbb{N}$. Let an *addition operation* be the operations (a), (b), and (d) performed by the algorithm.

Every loop of the minimal model completion selects one element $e$ and one $E \bowtie F \in \mathcal{T} \cup \mathcal{D}$. It applies at most $n + m + \mid \mathbb{C}(\mathcal{K}) \mid$ addition operations to $e$[2]. After that, it is impossible for $e$ to violate $E \bowtie F$, as $e \in F^{\mathcal{I}}$ and the algorithm does not decrease extensions. Therefore, the hard bound on the number of passes over the main loop is $kl$, and each pass executes at most $n + m + \mid \mathbb{C}(\mathcal{K}) \mid$ operations. After every pass, the number of possible executions decreases by one. The algorithm cannot run out of executions without running into a clash and resulting in an interpretation $\mathcal{I}$ that does not satisfy $\mathcal{K}$. If this were the case, there would still be a violation pointing to some element $e \in E^{\mathcal{I}} \setminus F^{\mathcal{I}}$, and there would be at least one more pass available to the algorithm. □

To meaningfully compare models of a DKB, we introduce set-theoretical inclusion for interpretations. Given two interpretations $\mathcal{I}, \mathcal{J}$ over the same domain $\Delta$, we say that $\mathcal{I} \subseteq \mathcal{J}$ iff $A^{\mathcal{I}} \subseteq A^{\mathcal{J}}$ for every $A \in \mathsf{N_C}$ and $r^{\mathcal{I}} \subseteq r^{\mathcal{J}}$ for every $r \in \mathsf{N_R}$. The output of minimal model completion is correct. It returns the least model of $\mathcal{K}$ extending the input interpretation $\mathcal{I}$ when there is at least one extension of $\mathcal{I}$ modeling $\mathcal{K}$, and returns *clash* otherwise.

**Lemma 6.26.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an update over some standard typicality model of $\mathcal{K}$.*

1. *For every standard interpretation $\mathcal{J} \supseteq \mathcal{I}$ sharing the domain $\Delta^{\mathcal{I}}$ such that $\mathcal{J} \models \mathcal{K}$, $\mathcal{J} \subseteq \mathsf{mmc}(\mathcal{I}, \mathcal{K})$.*
2. *If $\mathsf{mmc}(\mathcal{I}, \mathcal{K}) = clash$, then $\nexists \mathcal{J} \supseteq \mathcal{I}$ such that $\mathcal{J} \models \mathcal{K}$.*

*Proof.* Let $Add_1, \ldots, Add_k$ be the addition operations transforming $\mathcal{I}$ into $\mathsf{mmc}(\mathcal{I}, \mathcal{K})$. We denote $\mathcal{I}_i$ as $\mathcal{I}_{i-1}$ after $Add_i$ is applied, and $\mathcal{I}_0 = \mathcal{I}$. Any extension interpretation $\mathcal{J}$ of $\mathcal{I}$ can arise from $\mathcal{I}$ by a similar series of additions, which we denote by $Add^*_1, \ldots, Add^*_l$. For (1), we show by induction on $i$ that $Add_i \in \{Add^*_1, \ldots, Add^*_l\}$, for $1 \leq i \leq k$. For the base case, consider that $Add_1$ is triggered by a violation $C_{\mathcal{U}} \in (D \setminus E)^{\mathcal{I}}$, with $D \bowtie E \in \mathcal{T} \cup \mathcal{U}$ and $D = A_1 \sqcap \ldots A_n \sqcap \exists r_1.F_1 \sqcap \cdots \sqcap \exists r_m.F_m$. The addition operation is either adding $C_{\mathcal{U}}$ to $A^{\mathcal{I}}_j$, $1 \leq j \leq n$ or adding $(C_{\mathcal{U}}, F_{j'\emptyset})$ to $r^{\mathcal{I}}_{j'}$, for $1 \leq j' \leq m$. The interpretation $\mathcal{J}$ extends $\mathcal{I}$ and is a model of $\mathcal{K}$. Therefore, $C_{\mathcal{U}} \in A_j$, for $1 \leq j \leq n$. Since $\mathcal{J}$ is standard, $(C_{\mathcal{U}}, F_{j'\emptyset}) \in r^{\mathcal{J}}_{j'}$, for $1 \leq j' \leq m$.

---

[2] The $\mathbb{C}(\mathcal{K})$ being the possible standardization steps.

For the induction step, consider that $Add_j \in \{Add_1^*, \dots, Add_l^*\}$ for $1 \leq j \leq i < k$. We show that $Add_{i+1} \in \{Add_1^*, \dots, Add_l^*\}$. Let $\mathcal{I}_i$ be the intermediate model generated from $\mathcal{I}$ by applying $Add_1, \dots, Add_i$. Then, $Add_{i+1}$ is triggered by a violation of the form $C_{\mathcal{U}} \in (D \setminus E)^{\mathcal{I}_i}$, with $D \bowtie E \in \mathcal{T} \cup \mathcal{U}$ and $E = A_1 \sqcap \dots A_n \sqcap \exists r_1.F_1 \sqcap \dots \sqcap \exists r_m.F_m$. Notice that, by induction hypothesis, $C_{\mathcal{U}} \in (D \setminus E)^{\mathcal{J}}$. Then, the same argument from the base case holds here. Either $Add_{i+i}$ adds $C_{\mathcal{U}}$ to $A_j^{\mathcal{I}_i}$, $1 \leq j \leq n$, or it adds $(C_{\mathcal{U}}, F_\emptyset)$ to $r^{\mathcal{I}_i}$. Those additions have to hold for $\mathcal{J}$ because it is a standard model of $\mathcal{K}$.

(2) We show by contradiction that, in this case, there is no standard model $\mathcal{J}$ of $\mathcal{K}$. Suppose, by absurd, that there is such a $\mathcal{J}$. We denote the additions to transform $\mathcal{I}$ into $\mathcal{J}$ by $Add_1^* \dots Add_l^*$. Consider the path $Add_1, \dots, Add_k$ that leads $\mathcal{I}$ into *clash* in the minimal model completion algorithm. As before, we can conclude, by induction, that $Add_i \in \{Add_1^*, \dots, Add_l^*\}$, for $1 \leq i \leq k$. Notice that *clash* is triggered by some $C_{\mathcal{U}} \in D^{\mathcal{I}_k}$ such that $D \bowtie E \in \mathcal{T} \cup \mathcal{U}$, with $E = A_1 \sqcap \dots \sqcap A_n \sqcap \exists r_1.F_1 \sqcap \exists r_m.F_m$, $\bot \in \{A_1, \dots, A_n, F_1, \dots, F_m\}$. But $C_{\mathcal{U}} \in D^{\mathcal{J}}$; therefore, $\mathcal{J}$ is not a model of $\mathcal{K}$. □

### 6.3.3 Upgrade Steps

The goal of the upgrade procedure is saturating a typicality model with defeasible information. Each pass of a typicality update and subsequent minimal model completion is another step towards this goal. To characterize a procedure that goes from the minimal typicality model to a saturated typicality model, we must define an iterable operator that bind updates and completions. However, the order in which the available updates are applied at any given point impacts the final result. Some updates are incompatible between themselves, which is not surprising, as they bring new information into the model. Consider the following example.

**Example 6.27.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB where $\mathcal{T} = \{A \sqsubseteq \exists r.B, A \sqsubseteq \exists r.C, \exists r.(E \sqcap D) \sqsubseteq \bot\}$ and $\mathcal{D} = \{B \precsim D, C \precsim E\}$. Let $\mathcal{I}$ be the standard model of $\mathcal{K}$ depicted below. There are*

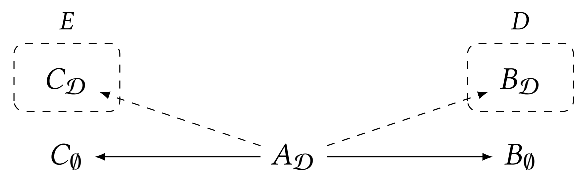


**Figure 6.6:** *Diagram representing a typicality interpretation with two update candidates for the role r: $(A_{\mathcal{D}}, C_{\mathcal{D}})$ and $(A_{\mathcal{D}}, B_{\mathcal{D}})$. Dashed rounded rectangles represent the extension of the concepts they delimit and straight arrows are edges in $r^{\mathcal{I}}$.*

*two incompatible update paths to take. If $\mathcal{I}$ is updated with the edge $(A_{\mathcal{D}}, C_{\mathcal{U}}) \in r^{\mathcal{I}}$, the it loses the other update candidate, $(A_{\mathcal{D}}, D_{\mathcal{D}}) \in r^{\mathcal{I}}$, and vice-versa.*

We need a typicality upgrade operator that takes sets of typicality models as input to deal with multiple incompatible upgrade paths. To that end, we define an operator that receives a single standard typicality model and outputs either

- a set of models when it is possible to upgrade the input, or
- the input itself when there is no possible upgrade.

The existence of viable upgrades is called *typicality extensibility*. We say that $\mathcal{I}$ is *typicality extensible* for $\mathcal{K}$ iff there is some $(d, e) \in \mathsf{UpCan}_r(\mathcal{I})$ for some $r \in \mathsf{sig}_\mathsf{R}(\mathcal{K})$ such that $\mathsf{mmc}(\mathsf{UD}^r(\mathcal{I}, (d, e)), \mathcal{K}) \neq \mathit{clash}$.

A more general version of the operator is defined for sets by applying the single-concept operator to each set member and outputting the union of each of the outputs. We define a saturation pipeline that starts with the singleton of the minimal typicality model, $\{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}\}$, and ends in a set of *saturated typicality models*.

**Definition 6.28** (Typicality Upgrade Operator)**.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB, $\mathcal{I}$ be a standard typciality model of $\mathcal{K}$, and $S = \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$ be a set of standard typicality models of $\mathcal{K}$ sharing the domain $\Delta^\mathcal{I}$ over the context $\mathbb{C}(\mathcal{K})$. A* typicality upgrade *for $\mathcal{I}$ and $\mathcal{K}$ and a role is*

$$\mathsf{TU}(\mathcal{I}, \mathcal{K}) := \begin{cases} \mathcal{I} \text{ if } \mathcal{I} \text{ is not } \textit{typicality extensible} \\ \{\mathsf{mmc}(\mathsf{UD}^r(\mathcal{I}, (d, e)), \mathcal{K}) \mid (d, e) \in \mathsf{UpCan}_r(\mathcal{I}) \text{ and } r \in \mathsf{sig}_\mathsf{R}(\mathcal{K})\} \text{ otherwise} \end{cases}$$

*A* typicality upgrade *over the set S of typicality models is given by*

$$\mathsf{TU}(S, \mathcal{K}) := \bigcup_{\mathcal{I}_i \in S} \mathsf{TU}(\mathcal{I}_i, \mathcal{K})$$

**Example 6.29.** *Let $\mathcal{K}$ be a DKB and $\mathcal{I}$ be an arbitrary standard typicality model for $\mathcal{K}$. Figure 6.7 depicts the pipeline that characterizes a full typicality upgrade step over $\mathcal{I}$, i.e. a single application of $\mathsf{TU}(\mathcal{I}, \mathcal{K})$.*

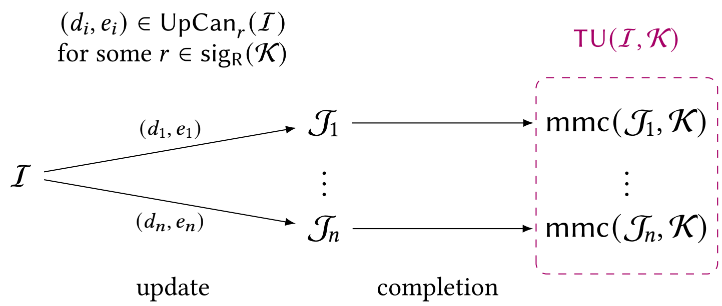


**Figure 6.7:** *Diagram representing the pipeline of a full typicality upgrade step with a typicality model $\mathcal{I}$ for a DKB $\mathcal{K}$ as inputs.*

In order to saturate a standard typicality model $\mathcal{I}$ with defeasible information from the DKB $\mathcal{K}$, the upgrade operator is applied iteratively to one input until it reaches a fixpoint denoted by $\mathsf{TU}_{\mathsf{max}}(\mathcal{I}, \mathcal{K})$. The fixpoint is a set of standard typicality models lacking update candidates that can be successfully incorporated into the models. There may be remaining update candidates, but after their addition, it is impossible to regain the model property through model completion. Those models extend the minimal typicality model, and we call them *saturated typicality models*. We define *nested reasoning* over this class of preferred models.

To formally define this framework, we start with a definition of the fixpoint set and a proof that applying the typicality operator iteratively, beginning with the minimal typicality model for a strength $\mathbf{s} \in \{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\}$ eventually comes to a halt. After this, we characterize nested reasoning and discuss its properties.

**Lemma 6.30** (Termination of the upgrade procedure). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}$ be the minimal typicality model for a strength* $\mathbf{s} \in \{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\}$. *Applying the upgrade operator iteratively to $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}$ eventually reaches a fixpoint $\mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}, \mathcal{K}) = \{\mathcal{J}_1, \ldots, \mathcal{J}_n\}$ such that $\mathsf{TU}(\mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}, \mathcal{K}), \mathcal{K}) = \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}, \mathcal{K})$.*

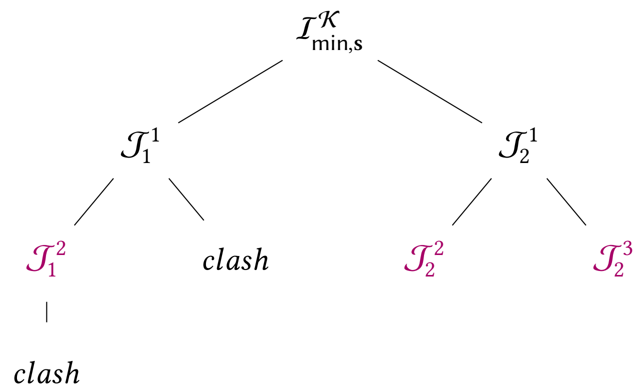


**Figure 6.8:** *Tree representing the upgrade procedure starting from $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}$. The colored leafs are the elements of $\mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}, \mathcal{K})$.*

*Proof.* The argument rests on two fundamental considerations:

1. All the models share a single domain, $\Delta^{T(\mathcal{K})}$.
2. An upgrade step only increases the extensions of its input.

The iteration of the upgrade procedure can be visualized as a tree. Each node is a standard typicality model $\mathcal{J}$ giving rise to $n = \sum_{r \in \mathsf{sig}_\mathsf{R}(\mathcal{K})} \mid \mathsf{UpCan}_r(\mathcal{J}) \mid$ branches. Each level of the tree is defined by the addition of one new edge to the model. Therefore, the maximum depth is $\mid \mathsf{sig}_\mathsf{R}(\mathcal{K}) \mid \times \Delta^{T(\mathcal{K})} \times \Delta^{T(\mathcal{K})}$, as the procedure does not remove edges. When a node (i) has no update candidates or (ii) has only candidates whose addition leads to a *clash* in the model completion (i.e., is not typicality extensible), it remains in the set.

This tree is finite by definition, as both the branches per node and the depth have hard limits. Each upgrade step generating a new branch is based on procedures that terminate (update and model completion). Therefore, the procedure is guaranteed to terminate. □

The termination of the upgrade procedure ensures that the set of saturated typicality models is computable and well-defined. Formally, for a given minimal typicality model $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}$ for the DKB $\mathcal{K}$ and strength $\mathbf{s}$, we define the set of saturated typicality models as $\mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}, \mathcal{K})$. This class of preferred models characterizes the reasoning of nested coverage.

### 6.3.4 Nested Reasoning

*Nested reasoning* is essentially a reasoning coverage grounded on a set of preferred models. In practice, defeasible information is pushed through quantifiers by upgrading the typicality of the edges in a typicality model. The edges stand on a one-on-one relation with defeasible subsumption; therefore, upgrading the edges may augment the set DCIs satisfied by the model.

Reasoning based on typicality models of propositional coverage is rooted in the minimal typicality model, a canonical model construction for materialization-based reasoning. The result of saturating this model with defeasible information is a strengthening of the initial consequences of the DKB under strength **s**. Note, however, that the final output is a set of models. The reasoning of nested coverage is defined skeptically over this set, as we want to consider only the information common to all possible upgrades. Alternatively, the necessity for skeptical reasoning could be overcome if an order over edges (i.e., upgrade candidates) were adopted. This would be equivalent to considering a single path from the root to one leaf in the upgrade tree.

**Definition 6.31** (Reasoning of nested coverage)**.** *Let* $\mathbf{s} \in \{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\}$ *be a strength,* $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ *a DKB. Let* $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}} = (\Delta^{T(\mathcal{K})}, \cdot^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}})$ *be a minimal typicality model of* $\mathcal{K}$ *such that for every* $C \in \mathbb{C}(\mathcal{K})$*,* $C_{\mathcal{U}}$ *is maximally typical in* $\Delta^{T(\mathcal{K})}$ *iff* $\mathcal{U} \in \mathsf{sel}_{\mathcal{K},\mathbf{s}}(C)$*.*

$$\mathcal{K} \models_{\mathsf{nest,s}} C \mathrel{\underset{\sim}{\sqsubset}} D \text{ iff } \mathcal{J} \models C \mathrel{\underset{\sim}{\sqsubset}} D \text{ for every } \mathcal{J} \in \mathsf{TU}_{\max}(\mathcal{I}, \mathcal{K})$$
$$\mathcal{K} \models_{\mathsf{nest,s}} C \sqsubseteq D \text{ iff } \mathcal{J} \models C \sqsubseteq D \text{ for every } \mathcal{J} \in \mathsf{TU}_{\max}(\mathcal{I}, \mathcal{K})$$

**Example 6.32.** *Consider the DKB and the minimal rational model presented in 6.14. As all the possible updates are compatible between themselves, there is just one saturated model in* $\mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}}, \mathcal{K})$*. Figure 6.9 depicts the fragment of* $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}}$ *that changes with the upgrade. The element* $\mathsf{Deity}_{\mathcal{E}_1}$ *has new outgoing edges. The update to more typical humans causes it to have a mortal successor. The standard property then extends this edge to* $\mathsf{Mortal}_\emptyset$*, and this new edge is further updated.*

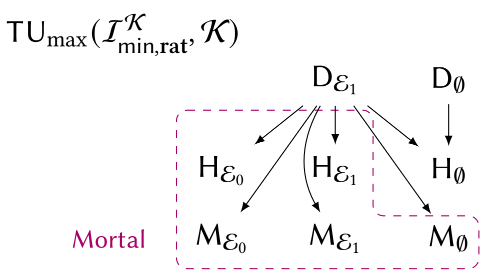


**Figure 6.9:** *Fragment of the saturated typicality models of rational strength. There is only one fully saturated model for this particular DKB and its* $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}}$*. As before, arrows represent edges of the role* observe.

Notice that this example effectively solves quantification neglect in this particular scenario. The DKB entails $\mathsf{Human} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Mortal}$ and $\mathsf{Deity} \sqsubseteq \exists \mathsf{observes}.\mathsf{Human}$. However, the minimal typicality model does not satisfy $\mathsf{Deity} \mathrel{\underset{\sim}{\sqsubset}} \exists \mathsf{observes}.\mathsf{Mortal}$, despite there being no obstacle to reaching such a conclusion. On the other hand, all the typicality saturated models satisfy Deity $\precsim$ ∃observes.Mortal, as Deity$\mathcal{E}_{1\in}$(∃observes.Mortal)$^{\mathcal{J}}$, for every $\mathcal{J} \in \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rat}}, \mathcal{K})$.

This result extends to the same DKB and $\mathbf{s} \in \{\mathbf{rel}, \mathbf{lex}\}$. The new domains do not introduce any obstacle to the upgrade described for Example 6.32. Therefore the saturated models satisfy the same formula, and nested reasoning for these strengths satisfies the same DCI. However, because those domains have more elements of intermediate typicality and there are more edges in the final product. This impacts the number of upgrades needed to reach the fixpoint that defines nested reasoning.

As a final characterization of nested reasoning, we show that it effectively extends the entailment of DCIs of propositional coverage. Despite this extension of entailment, nested reasoning preserves strict entailment, as expected. Any GCI derived from propositional and nested reasoning can be derived by classical DL reasoning over the TBox alone.

**Theorem 6.33.** *Let $\mathcal{K}$ be a DKB and* $\mathbf{s} \in \{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\}$.

1. $\mathcal{K} \models_{\mathsf{prop,s}} C \sqsubseteq D \Leftrightarrow \mathcal{K} \models_{\mathsf{nest,s}} C \sqsubseteq D$
2. $\mathcal{K} \models_{\mathsf{prop,s}} C \precsim D \Rightarrow \mathcal{K} \models_{\mathsf{nest,s}} C \precsim D$
3. $\mathcal{K} \models_{\mathsf{prop,s}} C \precsim D \not\Leftarrow \mathcal{K} \models_{\mathsf{nest,s}} C \precsim D$

*Proof.* (1) By Theorem 6.9 (canonicity of $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}$) and the definition of propositional reasoning, $\mathcal{K} \models_{\mathsf{prop,s}} C \precsim D$ iff $C_\emptyset \in D^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}}$. Notice that atypical instances are not updated by definition and only updated elements change their membership during an upgrade. Therefore, $C_\emptyset \in D^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}}$ iff $C_\emptyset \in D^{\mathcal{J}}$, for $J \in \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}, \mathcal{K})$. Moreover, $C_\emptyset$ is the least member of $C$ considering membership. Every other member of $C$ belongs to at least the same concepts as $C_\emptyset$, and possibly more. Suppose there were some element $d$ such as $d \in (C \setminus D)^{\mathcal{J}}$. However, $\mathcal{K} \models C \sqsubseteq D$, and therefore $\mathcal{J}$ would not be a model of $\mathcal{K}$. Hence, $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$, for every $J \in \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}, \mathcal{K})$, and $\mathcal{K} \models_{\mathsf{nest,s}} C \sqsubseteq D$.

(2) Entailment of defeasible subsumptions of the form $C \precsim D$ is decided by concept membership of the most typical instances of $C$. During the upgrade procedure, no element is removed from the extension of any concept. Then, $\mathcal{K} \models_{\mathsf{prop,s}} C \precsim D$ iff $C_{\mathcal{U}} \in D^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}}$, for every maximally typical $C_{\mathcal{U}} \in \Delta_{\mathsf{s}}^{T(\mathcal{K})}$ iff $C_{\mathcal{U}} \in D^{\mathcal{J}}$, for every $\mathcal{J} \in \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min,s}}, \mathcal{K})$ and every maximally typical $C_{\mathcal{U}} \in \Delta_{\mathsf{s}}^{T(\mathcal{K})}$ iff $\mathcal{K} \models_{\mathsf{nest,s}} C \precsim D$.

(3) Consider Example 6.32 and further commentary on relevant and lexicographic strengths for the same DKB. □

### 6.3.5 Comparing Semantics for $\mathcal{EL}_\bot$

This section presents a wide panorama comparing the semantics based on typicality models for all the combinations of $\{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\} \times \{\mathbf{prop}, \mathbf{nest}\}$ for the DL $\mathcal{EL}_\bot$. Prior research considered only the contrast between propositional and nested reasoning of the same (rational or relevant) strength [PT18b], [PT18a], [Pen19]. The relationship presented here covers the newly introduced lexicographic strength, as shown by Theorem 6.33. The result establishes that the reasoning of nested coverage effectively extends the reasoning of propositional coverage for the three strengths considered. Moreover, as materialization-based reasoning is equivalent to the reasoning of propositional coverage, nested coverage also extends it.

To compare different strengths whithin propositional coverage, we can travel an indirect route. Because every reasoning of propositional coverage and strength **s** is equivalent to **s**-materialization-based reasoning, we can draw upon the comparisons already present in the materialization-based literature. In particular, [Cas+14a] states that $\models_{\mathsf{mat},\mathbf{lex}} > \models_{\mathsf{mat},\mathbf{rel}} > \models_{\mathsf{mat},\mathbf{rat}}$. Therefore, $\models_{\mathsf{prop},\mathbf{lex}} > \models_{\mathsf{prop},\mathbf{rel}} > \models_{\mathsf{prop},\mathbf{rat}}$.

A reasonable expectation is that the reasoning hierarchy mentioned above is preserved under nested coverage. After all, it holds in propositional coverage, and the reasoning of nested coverage is proven to extend it. It could be argued that, more than probable, this is even desirable, as the original purpose of the stronger closures was to extend rational reasoning [Leh95]. However, this is surprisingly not the case.

As it turns out, $\models_{\mathsf{nest},\mathbf{rat}}$, $\models_{\mathsf{nest},\mathbf{rel}}$, and $\models_{\mathsf{nest},\mathbf{lex}}$ are incomparable between themselves. The hierarchy established in propositional reasoning holds for the entailment of some DCIs. However, having larger and finer domains can also negatively impact the entailment. For example, suppose that an upgrade $(d_1, d_2)$ is present in all typicality saturated models of rational strength. Relevant strength introduces more elements and enables another update, $(e_1, e_2)$. As seen before, updates can block other previously available updates. In this case, suppose that $(e_1, e_2)$ blocks $(d_1, d_2)$. Then, there are two branches in the relevant upgrade procedure; one with $(d_1, d_2)$, and the other with $(e_1, e_2)$. A conclusion derived in the rational paradigm can be dissolved in the larger $\mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rel}}, \mathcal{K})$. We illustrate this point with concrete examples.

**Example 6.34.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be the DKB specified as follows: $\mathcal{T} = \{A \sqsubseteq \exists r.B, A \sqsubseteq \exists r.C, B \sqsubseteq D, \exists r.F \sqcap \exists r.G \sqsubseteq \bot\}$ and $\mathcal{D} = \{B \sqcap E \lesssim \bot, C \lesssim G, D \lesssim E, D \lesssim F\}$. The exceptionality chain for $\mathcal{K}$ is $\mathcal{E}_0 = \mathcal{D}$, $\mathcal{E}_1 = \{B \sqcap E \lesssim \bot\}$, $\mathcal{E}_2 = \emptyset$.*

*Figure 6.10 illustrates how increasing the domain can dissolve some entailments obtained skeptically. The models in $\mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rat}}, \mathcal{K})$ satisfy $A \lesssim \exists r.G$ because the most typical instances of $A$, $A_{\mathcal{D}}$, belong to $(\exists r.G)^{\mathcal{J}}$. The larger, more fine-grained, relevant domain enables a new upgrade path by including $B_{\mathcal{U}}$.[3] The saturated models resulting from this path lack the edge $(A_{\mathcal{D}}, C_{\mathcal{D}}) \in r^{\mathcal{J}}$, and therefore do not satisfy $A \lesssim \exists r.G$. Even if the path is still available for the relevant domain, it is not common to all upgrade branches, and the DCI is not entailed anymore. Notice that the relevant saturated models are also stronger than the rational ones in certain aspects. For example, they entail $B \lesssim F$ through $B_{\mathcal{U}} \in F^{\mathcal{J}}$. This is a "propositional" defeasible inclusion, as it occurs in the depth 0 of quantification. However, similar effects can be obtained to quantified concepts that serve as witness of existential restrictions.*

The same effect is verified when comparing nested relevant and lexicographic strengths. Materialization-based lexicographic reasoning is stronger than relevant reasoning. Therefore, lexicographic reasoning of propositional coverage may entail some DCIs not entailed

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[3] The effect discussed here for this particular example extends to the lexicographic strength as well.

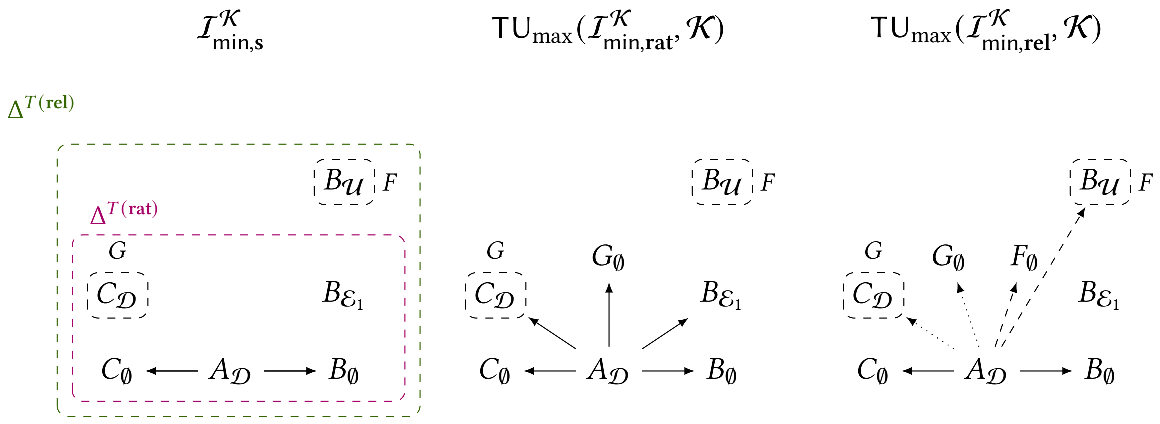


**Figure 6.10:** *Series of three diagrams representing (i) the minimal typicality models for* **rat** *and* **rel***; (ii) the common part of rational saturated typicality models; (iii) two distinct and incomparable parts of the relevant saturated typicality models. Arrows are edges of the role $r$. Dashed black squares represent concept extensions. The colored dashed squares in the first diagram represent the rational and relevant domains. In the last diagram, dotted and dashed arrows represent two incompatible upgrade paths.*

by relevant reasoning of the same coverage. Those entailments are preserved in nested semantics and exemplify situations where lexicographic is stronger than relevant nested reasoning.

However, increasing the domain that ensures the domination of relevant by lexicographic reasoning within propositional coverage may also dissolve some conclusions attainable by relevant reasoning. We may reproduce the same effect by carefully curating an example where an element is only present in the lexicographic domain and upgrading an edge to this element blocks an upgrade taken by all saturated relevant typicality models.

**Example 6.35.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be the DKB specified as follows: $\mathcal{T} = \{A \sqsubseteq \exists r.B, A \sqsubseteq \exists r.C, B \sqcap F_1 \sqcap F_3 \sqsubseteq \bot, B \sqcap F_2 \sqcap F_3 \sqsubseteq \bot, \sqsubseteq \bot, B \sqsubseteq D, \exists r.F_1 \sqcap \exists r.G \sqsubseteq \bot\}$ and $\mathcal{D} = \{B \sqcap E \mathrel{\underset{\sim}{\sqsubset}} \bot, C \mathrel{\underset{\sim}{\sqsubset}} G, D \mathrel{\underset{\sim}{\sqsubset}} E, D \mathrel{\underset{\sim}{\sqsubset}} F_1, D \mathrel{\underset{\sim}{\sqsubset}} F_2, D \mathrel{\underset{\sim}{\sqsubset}} F_3\}$. The exceptionality chain for $\mathcal{K}$ is $\mathcal{E}_0 = \mathcal{D}$, $\mathcal{E}_1 = \{B \sqcap E \mathrel{\underset{\sim}{\sqsubset}} \bot\}$, $\mathcal{E}_2 = \emptyset$.*

*The visualization is the same as in Picture 6.10. The only difference is that, instead of $B_{\mathcal{U}} \in F$, we have $B_{\mathcal{U}} \in F_{1,2}$ in the lexicographic typicality models, and $B_{\mathcal{E}_i}$ is the most typical instance of $B$ in the relevant ones.*

Let $\mathbf{s} \in \{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\}$. The results presented in this section are summarized by:

- Materialization-based **s** reasoning is equivalent to typicality models-based **s**-propositional reasoning;
- For both materialization-based reasoning and typicality models-based propositional reasoning, **lex** > **rel** > **rat**;
- Nested reasoning extends propositional reasoning for any given **s**;
- Nested reasonings of different strengths are incomparable between themselves.

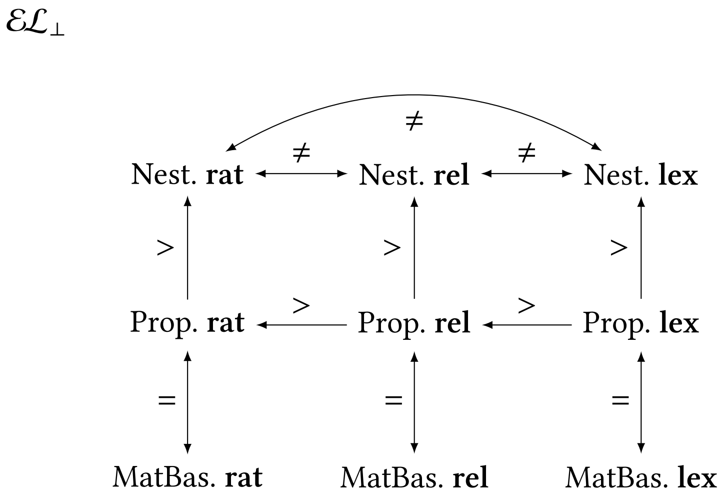


**Figure 6.11:** *Diagram with the strength comparison between materialization-based and typicality-models-based defeasible subsumption checking of all strengths and coverages. The > and = relations are transitive.*

Figure 6.11 depicts these results graphically. A deeper discussion of the issues concerning the reasoning of nested coverage is found in Chapter 8.

## 6.4 Epilogue: Rational defeasible instance checking

Until now, we considered only reasoning tasks related to subsumption, whether strict or defeasible. This last section briefly outlines a technique for incorporating assertional knowledge into the DKB and performing defeasible instance checking. The motivation for this section is to pave the way for instance checking for $\mathcal{ELI}_\bot$, discussed in Chapter 7.

Pensel and Turhan (2018) propose a method to bring defeasible instance checking to semantics based on typicality models. This method departs from a DKB with an ABox $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$. The context $\mathbb{C}(\mathcal{K})$ is increased by $\mathbb{C}(\mathcal{K}) := \mathbb{C}(\mathcal{K}) \cup \mathsf{sig_I}(\mathcal{A})$. The procedure has two main stages. The first one defines an augmented minimal typicality model with individuals. To accomplish this, we must characterize an augmented typicality domain incorporating individual representatives. The extended minimal typicality model allows defining defeasible instance checking of propositional coverage. In the second step, we define nested reasoning by simply saturating the augmented minimal typicality model as before. The introduction of individuals does not compromise or alter the upgrade procedure.

Before delving into the definitions, we must adapt the $\mathcal{EL}$ family materialization from Definition 5.1 to DKBs with ABoxes. For a terminological DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$, $\overline{\mathcal{K}}$ is defined as $\overline{\mathcal{K}} := \big(\mathcal{T} \cup \{(\overline{E \mathrel{\sqsubset\!\!\!\sim} F} \sqcap E) \sqsubseteq F \mid E \mathrel{\sqsubset\!\!\!\sim} F \in \mathcal{D}\},\ \emptyset\big)$. The defeasible terminological knowledge is transferred to the augmented TBox. For a full DKB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$, the idea remains fundamentally the same, as the materialization procedure does not impact the

ABox. Therefore, when we refer to the materialization of such a DKB, we are referring to $\overline{\mathcal{K}} = (\mathcal{A}, \mathcal{T} \cup \{(\overline{E \mathrel{\tilde{\sqsubseteq}} F} \sqcap E) \sqsubseteq F \mid E \mathrel{\tilde{\sqsubseteq}} F \in \mathcal{D}\}, \emptyset)$.

We start by defining the new domain. The structure of individual representatives is the same as the one for concept representatives. An element $a_{\mathcal{U}}$ – $a \in \mathsf{sig_I}(\mathcal{K})$ and $\mathcal{U} \subseteq \mathcal{D}$ – represents the individual $a$ satisfying the DCIs in $\mathcal{U}$. The main difference between individual and concept representatives is that only one representative per individual exists in the domain. Therefore, each individual representative is already maximally typical. To choose the set $\mathcal{U}$ to be materialized with $a$ we follow the iterative expansion of the ABox proposed in [CS10]. This construction relies in a preference order $o$ over individuals, and the entailment relation that is defined is associated with this order.

For simplifying the construction, we consider only *simple* ABoxes. A simple ABox $\mathcal{A}$ is an ABox s.t. for every axiom $C(a) \in \mathcal{A}$, $C \in \mathsf{N_C}$. It is possible to transform any $\mathcal{EL}_\bot$ (or $\mathcal{ELI}_\bot$) by introducing auxiliary concepts. So, given a complex $D$ such that $D(a)$ is in the ABox, we (i) remove $D(a)$ from $\mathcal{A}$; (ii) add $Aux_D(a)$ to the new ABox; (iii) and $Aux_D \sqsubseteq D$ to an augmented TBox.

**Definition 6.36** (Rational consistent-selection function for individuals). *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB and $\mathcal{A}$ be a simple ABox. Let $o = (a_1, \ldots, a_n)$ be a total order over $\mathsf{sig_I}(\mathcal{A})$. Let $\mathcal{E}_1, \ldots, \mathcal{E}_m$ be the exceptionality chain over $\mathcal{K}$. We say that $\mathcal{E}_i$ is consistent with a w.r.t. $\mathcal{A}$ iff $(\mathcal{A} \cup \{\overline{E \mathrel{\tilde{\sqsubseteq}} F}(a) \mid E \mathrel{\tilde{\sqsubseteq}} F \in \mathcal{E}_i\}, \overline{(\mathcal{T}, \mathcal{D})}) \not\models \bot(a)$. In other words, if a is satisfiable under the materialization of the terminological part of the DKB, the axioms in $\mathcal{A}$, and the axioms in $\mathcal{E}_i$. Then, the chain $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_n = \mathcal{A}^*$ is a set of ABoxes defined by:*

$$\mathcal{A}_0 = \mathcal{A}$$
$$\mathcal{A}_j = \mathcal{A}_{j-1} \cup \{\overline{E \mathrel{\tilde{\sqsubseteq}} F}(a) \mid E \mathrel{\tilde{\sqsubseteq}} F \in \mathcal{E}_i \text{ and } \mathcal{E}_i \text{ is consistent with } a\}$$

$\mathsf{sel}_{\mathcal{K},\mathbf{rat},o}(a_j) = \mathcal{E}_i$*, where i is the smallest index such that $\mathcal{E}_i$ is consistent with $a_j$ in $\mathcal{A}_j$. We abbreviate $\mathsf{sel}_{\mathcal{K},\mathbf{rat},o}(a_j)$ by $\mathcal{E}_{a_j}^{\mathcal{K}_o}$*

Notice that $\mathcal{A}_i$ remains simple for any $i > 0$, because $\overline{E \mathrel{\tilde{\sqsubseteq}} F} \in \mathsf{N_C}$. Building the enriched minimal typicality model is done by the following steps.

1. Enrich $\mathcal{A}$ according to an order over individuals $o$ as specified in Definition 6.36.
2. The result, $\mathcal{A}^*$, induces an interpretation $\mathcal{I}_{\mathcal{A}^*,\mathcal{T}}$. This interpretation will embody the information on individuals in $\mathcal{A}^*$. Although it is not necessarily a model, it will be when united with the terminalogical canonical model.
3. Finally, define a minimal typicality model by taking the union of the minimal typicality model and $\mathcal{I}_{\mathcal{A}^*,\mathcal{T}}$.

We start by detailing the inner workings of step 2. The construction of $\mathcal{I}_{\mathcal{A}^*,\mathcal{T}}$ is as follows:

**Definition 6.37** (ABox Interpretation). *[PT18b] Let $\mathcal{K} = (\mathcal{A}^*, \mathcal{T}, \mathcal{D})$ be a DKB expanding $(\mathcal{A}, \mathcal{T}, \mathcal{D})$ as described in 6.36 according to the order over individuals o. Then $\mathcal{I}_{\mathcal{A}^*,\mathcal{K}} =$*

$(\Delta^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}, \cdot^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}})$ *with:*

$$\begin{aligned}
\Delta^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}} &= \{a_{j_{\mathcal{E}^{\mathcal{K}_o}_{a_j}}} \mid a_j \in \mathsf{sig_I}(\mathcal{A})\} \cup \{E_\emptyset \mid E \in Qc(\mathcal{K})\} \\
a_j^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}} &= a_{j_{\mathcal{E}^{\mathcal{K}_o}_{a_j}}} \\
A^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}} &= \{a_{j_{\mathcal{E}^{\mathcal{K}_o}_{a_j}}} \in \Delta^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}} \mid \overline{\mathcal{K}} \models A(a_j)\} \\
r^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}} &= \{r(a_{j_{\mathcal{E}^{\mathcal{K}_o}_{a_j}}}, a_{k_{\mathcal{E}^{\mathcal{K}_o}_{a_k}}}) \in \mathcal{A}^*\} \cup \{(a, E_\emptyset) \mid \overline{\mathcal{K}} \models (\exists r.E)(a)\}
\end{aligned}$$

Membership of concept representatives (e.g. $E_\emptyset$) is not covered in $\mathcal{I}_{\mathcal{A}^*,\mathcal{T}}$, as it is entirely covered by the minimal typicality model $\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}$. The only way in which the addition of individuals can break the resulting model is by introducing them as predecessors in role edges. Existential restrictions on the left-hand side of terminological axioms can represent value restrictions for the inverse role by the equivalence $\exists r.C \sqsubseteq D \equiv C \sqsubseteq \forall r^-.D$. Axioms with this form cannot be violated in this construction, as $(\exists r.C)(a) \in \mathcal{A}$ and $\exists r.C \sqsubseteq D \in \mathcal{T}$ together imply $(\mathcal{T}, \mathcal{A}) \models D(a)$, and, therefore, $a_{j_{\mathcal{E}^{\mathcal{K}_o}_{a_j}}} \in D^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$.

We make two important remarks on $\mathcal{I}_{\mathcal{A}^*,\mathcal{T}}$. In the first place, it is *not* (necessarily) a model of $\mathcal{K}$. Furthermore, it is *quasi-disjoint* with the minimal typicality model $\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}$. The first one is explained by the absence of consideration for the concept membership of concept representatives. The second guarantees we can take the union of the two interpretations without undesirable consequences. We can take unions of two interpretations $\mathcal{I} \cup \mathcal{J}$ by setting the domain to $\Delta^{\mathcal{I}} \cup \Delta^{\mathcal{J}}$ and the concept extensions to $C^{\mathcal{I} \cup \mathcal{J}} = C^{\mathcal{I}} \cup C^{\mathcal{J}}$.

Informally, quasi-disjointness means that even though $\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}$ shares some domain elements with the minimal typicality model (i.e. the atypical concept representatives in $Qc(\mathcal{K})$), there is no real information on these shared elements in $\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}$. They do not belong to the extension of any concept, nor have successors. Their presence serves solely as existential witnesses for the individual representatives, and all the concept membership information will come from the minimal typicality model. The property of quasi-disjointness ensures that it is possible to unite both interpretations. Formally,

**Definition 6.38** (Quasi-disjointness)**.** *[PT18b, p. 33] Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, $\mathcal{J} = (\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}})$ be two interpretations. We say that $\mathcal{I}$ is quasi-disjoint from $\mathcal{J}$ iff*

1. *$\forall A \in \mathsf{N_C}$, $A^{\mathcal{I}} \cap \Delta^{\mathcal{J}} = \emptyset$; and*
2. *$\forall r^{\mathcal{J}} \in \mathsf{N_R} \cap (\Delta^{\mathcal{I}} \times (\Delta^{\mathcal{I}} \cup \Delta^{\mathcal{J}})) = \emptyset$.*

For $\mathcal{I}$ quasi-disjoint from $\mathcal{J}$, for any $\mathcal{EL}_\bot$ concept $C$, $C^{\mathcal{I} \cup \mathcal{J}} \cap \Delta^{\mathcal{J}} = C^{\mathcal{J}}$ [PT18b, p. 33]. This property does not hold for $\mathcal{ELI}_\bot$ concepts. Fortunately, it is possible to show comparable, weaker properties that will serve our purposes. Pensel and Turhan (2018) employ it to prove the two following properties of $\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}$:

1. $\mathcal{K} \models_{\mathsf{mat,rat},o} C(a)$ iff $\mathcal{I}^{\mathcal{K}}_{\mathsf{min}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}} \models C(a)$,
2. $\mathcal{I}^{\mathcal{K}}_{\mathsf{min}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}} \models (\mathcal{A}, \mathcal{T})$.

The first one states that the model resulting from the union of the minimal typicality model and the ABox interpretation is canonical for instance checking for an order $o$. The second states that it is indeed a model.

As a last remark, the introduction of individuals does affect the upgrade procedure. Let $a_{\mathcal{U}}, b_{\mathcal{V}}$ bet two individual representatives in the domain of $\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}$. There are three kinds of role edges to any given individual $a_{\mathcal{U}}$:

1. $r(a_{\mathcal{U}}, b_{\mathcal{V}})$,
2. $r(b_{\mathcal{V}}, a_{\mathcal{U}})$,
3. $r(a_{\mathcal{U}}, C_{\mathcal{U}})$.

Edges restricted to individuals, such as (1) and (2), are not upgradeable, as every individual has exactly one representative in any given typicality interpretation. Edges with concept representatives as successors can be upgraded, and the upgrade has the potential to spread change throughout the model. It is true that, due to the lack of inverse roles and universal quantification in $\mathcal{EL}_{\perp}$, information only travels backward through roles, as there may be axioms such as $\exists r.F \sqsubseteq E$, which is equivalent to $F \sqsubseteq \forall r^{-}.E$. However, model completions affecting the predecessors (e.g. including it in the extension of $E$) can block upgrades that would be feasible otherwise. Consider the following example:

**Example 6.39.** *Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a typicality interpretion partially depicted in Figure 6.12, and a TBox containing the axioms $C \sqsubseteq \forall r^{-}.E$, $\exists r.E \sqcap \exists r.F \sqsubseteq \perp$, and $D \sqsubseteq \forall r^{-}.F$.*

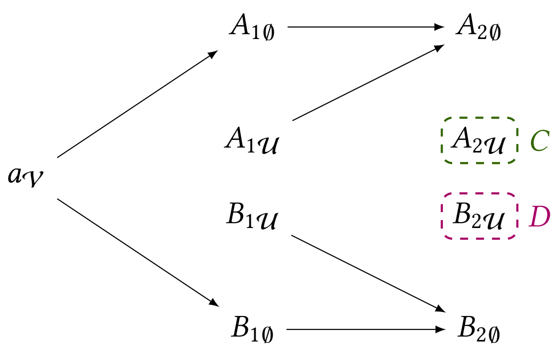


**Figure 6.12:** *A diagram representing the fragment of an upgradable model with individual representatives. The arrows represent edges of the role r.*

This fragment of the interpretation has the following possible $r$-updates: $(a_{\mathcal{V}}, A_{1\mathcal{U}})$, $(a_{\mathcal{V}}, B_{1\mathcal{U}})$, $(A_{1\mathcal{U}}, A_{2\mathcal{U}})$, and $(B_{2\mathcal{U}}, B_{2\mathcal{U}})$. However, it is not possible to enact all of them. In fact, considering only this fragment of the interpretation depicted in the diagram, it is possible to choose exactly three updates.

The upgrades of $A_{1\mathcal{U}}$ and $B_{1\mathcal{U}}$ successors increase their concept membership (after the minimal model completion). In light of those changes, it is impossible to accommodate them simultaneously as successors to $a$. If, on the other hand, the $a$ upgrades are done before, then the upgrades to $A_{1\mathcal{U}}$ and $B_{1\mathcal{U}}$ are blocked, which is how the introduction of individuals affect the upgrade procedure even for concepts.

# Chapter 7

# Typicality Models for $\mathcal{ELI}_\bot$

THE minimal requirement for a successfully defining typicality models for a given DDL is the canonical model property, as typicality models are themselves multi-layered canonical models. This makes Horn-DLs as ideal candidates for the task. Those class of DLs can be very expressive while maintaining the canonical model property, as they brush away unlimited negation and disjunction. They are successfully employed in more complex reasoning tasks, such as query answering [ORS11], which would also be an interesting improvement for semantics based on typicality models.

However, this minimal requirement of having the canonical model property is far from being sufficient to characterize semantics based on typicality models. The model-theoretic presentation of typicality models in [Pen19] relies on several properties from $\mathcal{EL}_\bot$ that are absent from more expressive DLs, such as the preservation under intersection for models. For this reason, the presentation here focuses on the algorithmic view of earlier publications, such as [PT17a], that defined actual upgrade procedures instead of preferences between models. The increase in expressivity brought by the inclusion of inverse roles, which can represent limited forms of universal restrictions, undermines the whole procedure. We list some of the most crucial issues:

- The standard property is not attainable in canonical models for $\mathcal{ELI}_\bot$.
- A same edge $(C, D)$ can represent two different existential restrictions: $C \sqsubseteq \exists r.D$ or $D \sqsubseteq \exists r^-.C$.
- Violations involving universal restrictions may require the removal of edges, harming one of the main tenets of the minimal model completion algorithm.

In adapting typicality models to this hostile territory, we will keep the general idea of canonical models composed of pairwise elements representing concepts and a set of DCIs and the idea of updating the roles to increase typicality and push defeasible information through quantifiers. However, the technicalities of how to achieve this are entirely novel, and the machinery required to lift the semantics to $\mathcal{ELI}_\bot$ is considerably heavier. As the idiom goes, *the devil is in the details.*

This chapter presents what is intended to be a first step into the realm of semantics based on typicality models for more expressive DLs. It presents techniques that hopefully will be useful in adapting formulating typicality models for Horn-DLs, as it addresses several of the difficulties introduced by those logics.

## 7.1 Foundations of $\mathcal{ELI}_\bot$ Typicality Models

Typicality models are extending canonical models for monotone DLs. The goal is to develop typicality models that are also canonical models, i.e. allow to check entailments by satisfaction in this model. In particular, both strict and defeasible subsumption are indicated by concept membership of particular domain elements. Typicality models for defeasible $\mathcal{ELI}_\bot$ are built from canonical models for standard $\mathcal{ELI}_\bot$, and the latter depend on normalized KBs. Therefore, we need to extend the normalization to encompass the knowledge component exclusive to defeasible $\mathcal{ELI}_\bot$, i.e. DBoxes.

### 7.1.1 Normal form for $\mathcal{ELI}_\bot$ DKBs

The computation of canonical models requires a DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ to be in normal form, which is achieved in two steps. The first normalizes the DBox $\mathcal{D}$ and augments the TBox. The second is the TBox normalization from Definition 2.17 applied to the augmented TBox.

**Definition 7.1** (DKB normal form)**.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB. For every $C \mathrel{\underset{\sim}{\sqsubset}} D \in \mathcal{D}$, let $A_C, A_D \in \mathsf{N_C} \setminus \mathsf{sig_C}(\mathcal{K})$ be concept names. Then:*

$$\begin{aligned} \mathsf{NF}(\mathcal{D}) &:= \{A_C \mathrel{\underset{\sim}{\sqsubset}} A_D \mid C \mathrel{\underset{\sim}{\sqsubset}} D \in \mathcal{D}\} \\ \mathcal{T}_{\mathsf{aux}} &:= \{C \sqsubseteq A_C \mid A_C \mathrel{\underset{\sim}{\sqsubset}} A_D \in \mathsf{NF}(\mathcal{D})\} \cup \\ &\quad\ \{A_D \sqsubseteq D \mid A_C \mathrel{\underset{\sim}{\sqsubset}} A_D \in \mathsf{NF}(\mathcal{D})\} \\ \mathsf{NF}(\mathcal{T}) &:= \mathsf{T\text{-}NF}(\mathcal{T} \cup \mathcal{T}_{\mathsf{aux}}) \end{aligned}$$

*and* $\mathsf{NF}(\mathcal{K}) := (\mathsf{NF}(\mathcal{T}), \mathsf{NF}(\mathcal{D}))$ *is the DKB in* normal form.

The normalization introduces names for complex concepts, i.e. the DBox only contains DCIs for names and the TBox associates these names to the complex concepts.

Materialization-based defeasible reasoning uses the *rank* of concepts and inclusions. We show that the proposed normalization keeps the same rank for the “proxy” DCIs. More precisely, we show that, for every $k$-ranked $C$ in some DCI $C \mathrel{\underset{\sim}{\sqsubset}} D \in \mathcal{D}$, the corresponding auxiliar concept $A_C$ is also $k$-ranked. This result is intuitive and expected, as DKB normalization just introduces fresh names for complex concepts in DCIs.

**Lemma 7.2.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and $\mathcal{D} = \mathcal{E}_0$, $\mathcal{E}_0 \supset \mathcal{E}_1 \supset \cdots \supset \mathcal{E}_n$ its exceptionality chain. For $\mathsf{NF}(\mathcal{K})$ let $\mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_m$ be its exceptionality chain. Then, $C \mathrel{\underset{\sim}{\sqsubset}} D \in \mathcal{E}_i$ if and only if $A_C \mathrel{\underset{\sim}{\sqsubset}} A_D \in \mathcal{F}_i$ for all $0 \leq i \leq n$ and $n = m$.*

*Proof.* The proof is by induction on the index $i$ of the exceptionality chain of $\mathcal{K}$.

*Base:* $i = 0$. Since $\mathcal{E}_0 = \mathcal{D}$ and $\mathsf{NF}(\mathcal{E}_0) = \mathsf{NF}(\mathcal{D})$, by construction, $C \mathrel{\underset{\sim}{\sqsubset}} D \in \mathcal{D}$ if and only if $A_C \mathrel{\underset{\sim}{\sqsubset}} A_D \in \mathsf{NF}(\mathcal{D})$ holds.

*Inductive step.* We prove the contra-positive for each implication.

($\Longrightarrow$) Let $C \mathrel{\tilde\sqsubseteq} D \in \mathcal{D}$ and $C \mathrel{\tilde\sqsubseteq} D \notin \mathcal{E}_{i+1}$. This means that $\overline{\mathcal{K}} \not\models C \sqcap \overline{\mathcal{E}_i} \sqsubseteq \bot$, and, thus, there exists a model $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ of $\overline{\mathcal{K}}$ such that $\exists e \in \Delta^\mathcal{I}$ and $e \in (C \sqcap \overline{\mathcal{E}_i})^\mathcal{I}$. From $\mathcal{I}$ we construct an interpretation $\mathcal{J}$ such that $\mathcal{J} \models \overline{\mathsf{NF}(\mathcal{K})}$ and $\mathcal{J} \not\models A_C \sqcap \overline{\mathcal{F}_i} \sqsubseteq \bot$. We define $\mathcal{J}$ by extending $\mathcal{I}$ to the concepts introduced by normalization. Let $\mathcal{J} = (\Delta^\mathcal{I}, \cdot^\mathcal{J})$ with:

$$\begin{aligned} A^\mathcal{J} &:= A^\mathcal{I} \text{ for every } A \in \mathrm{sig}_\mathsf{C}(\overline{\mathcal{K}}) \\ A_C{}^\mathcal{J} &:= C^\mathcal{I} \text{ for every } A_C \in \mathrm{sig}_\mathsf{C}(\overline{\mathsf{NF}(\mathcal{K})}) \setminus \mathrm{sig}_\mathsf{C}(\overline{\mathcal{K}}) \\ (\overline{A_C \mathrel{\tilde\sqsubseteq} A_D})^\mathcal{J} &:= (\overline{C \mathrel{\tilde\sqsubseteq} D})^\mathcal{I} \text{ for every } \overline{A_C \mathrel{\tilde\sqsubseteq} A_D} \in \mathrm{sig}_\mathsf{C}(\overline{\mathsf{NF}(\mathcal{K})}) \setminus \mathrm{sig}_\mathsf{C}(\overline{\mathcal{K}}) \end{aligned}$$

We need to show that (1) $\mathcal{J} \models \overline{\mathsf{NF}(\mathcal{K})}$ and (2) $\mathcal{J} \not\models A_C \sqcap \overline{\mathcal{F}_i} \sqsubseteq \bot$. To show (1): the names in $\mathrm{sig}_\mathsf{C}(\overline{\mathcal{K}})$ have the same extensions in $\mathcal{I}$ and $\mathcal{J}$ and satisfy the axioms in $\overline{\mathcal{K}}$. Names from $\mathrm{sig}_\mathsf{C}(\overline{\mathsf{NF}(\mathcal{K})}) \setminus \mathrm{sig}_\mathsf{C}(\overline{\mathcal{K}})$ occur in three types of axioms:

1. $C \sqsubseteq A_C$,
2. $A_D \sqsubseteq D$, and
3. $A_C \sqcap \overline{A_C \mathrel{\tilde\sqsubseteq} A_D} \sqsubseteq A_D$.

Axioms of the first two forms are satisfied in $\mathcal{J}$, as $C^\mathcal{I} = C^\mathcal{J} \subseteq A_C{}^\mathcal{J}$, and $A_D{}^\mathcal{J} \subseteq D^\mathcal{J} = D^\mathcal{I}$. For the third from, consider some $e \in (A_C \sqcap \overline{A_C \mathrel{\tilde\sqsubseteq} A_D})^\mathcal{J}$. This implies that $e \in A_C{}^\mathcal{J}$ and $e \in (\overline{A_C \mathrel{\tilde\sqsubseteq} A_D})^\mathcal{J}$. Then, $e \in C^\mathcal{I}$ and $e \in (\overline{C \mathrel{\tilde\sqsubseteq} D})^\mathcal{I}$. Since $\mathcal{I} \models \overline{\mathcal{K}}$, $e \in D^\mathcal{I}$ holds, and thus $e \in D^\mathcal{J}$ implying $e \in A_D{}^\mathcal{J}$.

To show (2) we consider $e \in (C \sqcap \overline{\mathcal{E}_i})^\mathcal{I}$. By construction, $e \in A_C{}^\mathcal{J}$ and $e \in (\overline{A_E \mathrel{\tilde\sqsubseteq} A_F})^\mathcal{J}$ for every $E \mathrel{\tilde\sqsubseteq} F \in \mathcal{E}_i$. Therefore, by induction hypothesis, $e \in \overline{\mathcal{F}_i}^\mathcal{J}$ and $\mathcal{J} \not\models A_C \sqcap \overline{\mathcal{F}_i} \sqsubseteq \bot$. Hence, $A_C \mathrel{\tilde\sqsubseteq} A_D \notin \mathcal{F}_{i+1}$.

($\Longleftarrow$) Let $A_C \mathrel{\tilde\sqsubseteq} A_D \in \mathsf{NF}(\mathcal{D})$ and $A_C \mathrel{\tilde\sqsubseteq} A_D \notin \mathcal{F}_{i+1}$, which together imply that $\overline{\mathsf{NF}(\mathcal{K})} \not\models A_C \sqcap \overline{\mathcal{F}_i} \sqsubseteq \bot$. Then, there is some model $\mathcal{I} \models \overline{\mathsf{NF}(\mathcal{K})}$ and some element $e \in \Delta^\mathcal{I}$ such that $e \in (A_C \sqcap \overline{\mathcal{F}_i})^\mathcal{I}$. We construct an interpretation $\mathcal{J}$ from $\mathcal{I}$ as follows: define $(\overline{G \mathrel{\tilde\sqsubseteq} H})^\mathcal{J} := (\overline{A_G \mathrel{\tilde\sqsubseteq} A_H})^\mathcal{I}$ for all $A_G \mathrel{\tilde\sqsubseteq} A_H \in \mathsf{NF}(\mathcal{D})$ and set $C^\mathcal{J} := C^\mathcal{I} \cup \{e\}$. For the remaining concepts $A$ shared by both signatures, set $A^\mathcal{J} := A^\mathcal{I}$. We show that:

1. $\mathcal{J} \models \overline{\mathcal{K}}$, and
2. $\mathcal{J} \not\models C \sqcap \overline{\mathcal{E}_i} \sqsubseteq \bot$.

The only GCIs in $\overline{\mathcal{K}} \setminus \overline{\mathsf{NF}(\mathcal{K})}$ are of the form $E \sqcap \overline{E \mathrel{\tilde\sqsubseteq} F} \sqsubseteq F$. Suppose $e' \in (E \sqcap \overline{E \mathrel{\tilde\sqsubseteq} F})^\mathcal{J}$. Then, $e' \in E^\mathcal{I}$ holds, implying $e' \in A_E{}^\mathcal{I}$ and $e' \in (\overline{A_E \mathrel{\tilde\sqsubseteq} A_F})^\mathcal{I}$. Because $\mathcal{I} \models \overline{\mathsf{NF}(\mathcal{K})}$, $e' \in A_F{}^\mathcal{I}$ which implies $e' \in F^\mathcal{I}$, and $e' \in F^\mathcal{J}$. For (2), by induction hypothesis, we have $A_E \mathrel{\tilde\sqsubseteq} A_F \in \mathcal{F}_i$ iff $E \mathrel{\tilde\sqsubseteq} F \in \mathcal{E}_i$. Then, by construction of $\mathcal{J}$, $e \in \overline{\mathcal{F}_i}^\mathcal{I} \Rightarrow e \in \overline{\mathcal{E}_i}^\mathcal{J}$ and, finally, $e \in (C \sqcap \overline{\mathcal{E}_i})^\mathcal{J}$. □

**Corollary 7.3.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and $C$ be a concept occurring in the left-hand side of some DCI in $\mathcal{D}$. Then, $r_\mathcal{K}(C) = r_{\mathsf{NF}(\mathcal{K})}(A_C)$, where $C \sqsubseteq A_C \in \mathcal{T}_\mathsf{aux}$.*

*Proof.* (Sketch) Let $\mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \mathcal{F}_n$ be the exceptionality chain of $\mathsf{NF}(\mathcal{K})$. Furthermore, let $r_{\mathcal{K}}(C) = i$. To show that $r_{\mathsf{NF}(\mathcal{K})}(A_C) = i$, we show that:

1. $\overline{\mathsf{NF}(\mathcal{K})} \not\models A_C \sqcap \overline{\mathcal{F}_i} \sqsubseteq \bot$.
2. $\overline{\mathsf{NF}(\mathcal{K})} \models A_C \sqcap \overline{\mathcal{F}_{i-1}} \sqsubseteq \bot$, if $i > 0$.

For (1), note that $C \sqcap \overline{\mathcal{E}_i}$ is satisfiable. For a model $\mathcal{I} \models \overline{\mathcal{K}}$ s.t. $(C \sqcap \overline{\mathcal{E}_i})^{\mathcal{I}} \neq \emptyset$, we build a model $\mathcal{J} \models \overline{\mathsf{NF}(\mathcal{K})}$ using the same construction as in the proof of Lemma 7.2. Because $A_C{}^{\mathcal{J}} = C^{\mathcal{I}}$ and $(\overline{A_C \mathrel{\underset{\sim}{\sqsubset}} A_D})^{\mathcal{J}} = (\overline{C \mathrel{\underset{\sim}{\sqsubset}} D})^{\mathcal{I}}$, for every $A_C \mathrel{\underset{\sim}{\sqsubset}} A_D \in \mathcal{F}_i$, we have $(A_C \sqcap \overline{\mathcal{F}_i})^{\mathcal{J}} \neq \emptyset$.

We show (2) by contradiction. Let $i > 0$ and suppose that $A_C$ was consistent with $\overline{\mathcal{F}_{i-1}}$ w.r.t. $\overline{\mathsf{NF}(\mathcal{K})}$. Then, there would be a model $\mathcal{I} \models \overline{\mathsf{NF}(\mathcal{K})}$ s.t. $(A_C \sqcap \overline{\mathcal{F}_{i-1}})^{\mathcal{I}} \neq \emptyset$. By the same technique from the last proof, we could build an interpretation $\mathcal{J}$ s.t. $\mathcal{J} \models \overline{\mathcal{K}}$ and $(C \sqcap \overline{\mathcal{E}_i})^{\mathcal{J}} \neq \emptyset$. In this case, $i$ would not be the lesser index for which $\mathcal{E}_i$ is consistent with $C$ and $r_{\mathcal{K}}(C) \neq i$. □

These results show that neither the exceptionality construction nor the exceptionality chain are affected by the normalization of the DKB.

### 7.1.2 Typicality models for $\mathcal{ELI}_\bot$

*We assume* from now on that we want to test for $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ whether $\mathcal{K} \models C \mathrel{\underset{\sim}{\sqsubset}} D$ holds and that $\mathcal{T}$ contains $A \sqsubseteq C, B \sqsubseteq D$ for $A, B \in \mathsf{N_C}$ and that $\mathcal{K}$ is in DKB normal form.

The basis for the domain of typicality interpretations is the representative domain of a DKB, which collects named concepts and all concepts occurring in the scope of quantifiers. The *set of quantified concepts* (2.3) of a DBox captures concepts nested within quantifiers similarly to the TBox, but operating over DCIs instead of GCIs. Formally,

$$Qc(C \mathrel{\underset{\sim}{\sqsubset}} D) := Qc(C \sqsubseteq D)$$
$$Qc(\mathcal{D}) := \bigcup_{C \mathrel{\underset{\sim}{\sqsubset}} D \in \mathcal{D}} Qc(C \mathrel{\underset{\sim}{\sqsubset}} D)$$

The basis of typicality domains for $\mathcal{ELI}_\bot$ is the representative domain (Definition 2.19). Notice that our DKBs (i) have a DBox and (ii) do not have ABoxes. The definition remains the same, except for the Abox, that goes away. Formally, $\Delta^{\mathcal{K}} = \{\{A\} \mid A \in \mathsf{sig_C}(\mathcal{T} \cup \mathcal{D})\} \cup \mathcal{P}(Qc(\mathcal{T} \cup \mathcal{D}))$

Typicality interpretations have two-dimensional domains as their elements $M_{\mathcal{U}}$ are pairs, where $M \in \Delta^{\mathcal{K}}$ is the *concept set* and $\mathcal{U}$ is a *typicality set*, i.e. a subset of $\mathcal{D}$. Intuitively, such an element represents the instances of concept $C$ that conform with the DCIs in $\mathcal{U}$. For example, the element $\{\mathsf{Bird}\}_\emptyset$ represents atypical birds, while $\{\mathsf{Bird}\}_{\{\mathsf{Bird} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Flies}\}}$ represents more typical birds satisfying $\mathsf{Bird} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Flies}$.

**Definition 7.4** (Typicality domain, typicality interpretation)**.** *Let* $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ *be a DKB and* $\Delta^{\mathcal{K}}$ *be its representative domain. A* typicality domain $\Delta^{T(\mathcal{K})}$ of $\mathcal{K}$ *is defined as* $\Delta^{T(\mathcal{K})} \subseteq$

$\Delta^{\mathcal{K}} \times \mathcal{P}(\mathcal{D})$ *s.t.* $\forall M \in \Delta^{\mathcal{K}}$, $M_\emptyset \in \Delta^{T(\mathcal{K})}$. *An interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *is a* typicality interpretation *if and only if* $\Delta^{\mathcal{I}}$ *is a typicality domain.*

The *maximally typical instances of* are defined in the same way as in $\mathcal{EL}_\bot$. The maximally typical instances of $\lceil M \rceil$ in a typicality interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ are those $M_{\mathcal{U}} \in \Delta^{\mathcal{I}}$ s.t. there is no $\{M\}_{\mathcal{V}} \in \Delta^{\mathcal{I}}$ with $\mathcal{U} \subset \mathcal{V}$. Intuitively, satisfaction of $\lceil M \rceil \sqsubsetapprox A$ holds in $\mathcal{I}$ if the most typical instances of $\lceil M \rceil$ satisfy belong to $A$. While maximally typical instances of a concept are unique in rational and relevant typicality domains, this need not be for lexicographic domains, which are compatible with several maximally typical elements for a single concept.

**Definition 7.5** (Satisfaction, Typicality Model)**.** *Let* $C, D$ *be concepts,* $A, B \in \mathsf{N_C}$ *and* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *a typicality interpretation. Then*

- $\mathcal{I} \models C \sqsubseteq D$ *iff* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$,
- $\mathcal{I} \models M \sqsubsetapprox A$ *iff* $M_{\mathcal{U}} \in A^{\mathcal{I}}$ *for every maximally typical instance of* $M$ *in* $\Delta^{\mathcal{I}}$.

$\mathcal{I}$ *is a* model of a (normalized) DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ *if and only if*

- $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ *for all* $(C \sqsubseteq D) \in \mathcal{T}$ *and*
- $M_{\mathcal{U}} \in A^{\mathcal{I}}$ *implies* $M_{\mathcal{U}} \in B^{\mathcal{I}}$ *for all* $(A \sqsubsetapprox B) \in \mathcal{U}$.

While the strength is accommodated by the shape of the domain, the coverage is captured by *minimal typicality models* in case of propositional and by *saturated typicality models* in case of nested semantics—as for defeasible $\mathcal{EL}_\bot$.

## 7.2 Minimal Typicality Models

We extend the minimal typicality models for $\mathcal{EL}_\bot$ to inverse roles to define the semantics of propositional coverage and strength $\mathbf{s} \in \{rational, relevant, lexicographic\}$. The goal is devising canonical models for defeasible reasoning under propositional coverage and the chosen strength.

In this section, we start with a general definition of the minimal typicality model for an undefined strength $\mathbf{s}$. We show that such general structures are indeed models of $\mathcal{K}$ and that they are canonical in the sense that (defeasible) subsumption relations can be read-off from the concept memberships of the elements. Lastly, we define kinds of typicality domains that then realize the strength of reasoning.

The starting point are the canonical models used to decide subsumption in monotone $\mathcal{ELI}_\bot$. These models have subsets from $\mathsf{sig_C}(\mathcal{T})$ as their domain elements. The domain of typicality models for defeasible $\mathcal{ELI}_\bot$ is two-dimensional as its elements are pairs of concept and typicality sets. In canonical models for monotone $\mathcal{ELI}$, the relational structure is such that an element is connected to those other elements that are maximal (subset relation) $r$-successors. The idea is that the element $\{A_1, \dots, A_n\}$ represents exactly the conjunction of the concepts $A_1, \dots, A_n$. In addition, for typicality interpretations, each element satisfies the DCIs from its typicality set.

During the upgrade procedure, elements are added to new concepts. As a result, their prime successors can change. To keep track of these changes and ensure primeness, we introduce the concept of $\mathsf{N_C}$-*type*.

**Definition 7.6** ($\mathsf{N_C}$-type). *Let $\mathcal{K}$ be a DKB and $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ an interpretation. For $d \in \Delta^\mathcal{I}$, the* $\mathsf{N_C}$-type *of $d$ w.r.t. $\mathcal{K}$ and $\mathcal{I}$ is defined as* $\mathsf{N_C}\text{-type}_\mathcal{K}(d, \mathcal{I}) := \{A \in \mathsf{sig_C}(\mathcal{K}) \mid d \in A^\mathcal{I}\}$.

We use the $\mathsf{N_C}$-type of an element to redefine primeness for typicality interpretations, adapting Definition 2.20. With this concept, we identify the prime $r$-successors of a given element according to the concepts to which it belongs in the interpretation. Primeness is defined for concepts. In typicality interpretations, primeness is defined w.r.t. the $\mathsf{N_C}$-type of the element.

**Definition 7.7** (Prime successor). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB, $\mathcal{I} = (\Delta^{T(\mathcal{K})}, \cdot^\mathcal{I})$ a typicality interpretation, and $r \in \{s, s^-\}$ with $s \in \mathsf{sig_R}(\mathcal{K})$. Then, $N_\mathcal{V} \in \Delta^{T(\mathcal{K})}$ with $N \in \Delta^\mathcal{K}$ is a* prime $r$-successor *of $M_\mathcal{U}$ in $\mathcal{I}$* iff

1. $\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_\mathcal{K}(M_\mathcal{U}, \mathcal{I}) \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.\lceil N \rceil$, *and*
2. *There is no $N' \in \Delta^\mathcal{K}$ s.t.*
   (a) $N \subset N'$, *and*
   (b) $\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_\mathcal{K}(M_\mathcal{U}, \mathcal{I}) \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.\lceil N' \rceil$

As in the minimal typicality domain for $\mathcal{EL}_\bot$, all the role-successors in minimal typicality models for $\mathcal{ELI}_\bot$ are atypical elements, i.e., elements with $\emptyset$ as their typicality set.

**Definition 7.8** (Minimal Typicality Model). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and $\Delta^{T(\mathcal{K})}$ a typicality domain of $\mathcal{K}$. The* minimal typicality model $\mathcal{I}^\mathcal{K}_{\mathsf{min}} = (\Delta^{T(\mathcal{K})}, \cdot^{\mathcal{I}^\mathcal{K}_{\mathsf{min}}})$ *is defined as:*

$$\begin{aligned} A^{\mathcal{I}^\mathcal{K}_{\mathsf{min}}} :=& \{M_\mathcal{U} \mid \overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq A\} \\ r^{\mathcal{I}^\mathcal{K}_{\mathsf{min}}} :=& \{(M_\mathcal{U}, N_\emptyset) \mid N \text{ is a prime } r\text{-successor for } M_\mathcal{U} \text{ in } \mathcal{I}^\mathcal{K}_{\mathsf{min}}\} \cup \\ & \{(N_\emptyset, M_\mathcal{U}) \mid N \text{ is a prime } r^-\text{-successor for } M_\mathcal{U} \text{ in } \mathcal{I}^\mathcal{K}_{\mathsf{min}}\} \end{aligned}$$

It still needs to be shown that the structures just defined are indeed models of the DKB $\mathcal{K}$. To do so, we (1) show that $\mathcal{I}^\mathcal{K}_{\mathsf{min}}$ satisfies GCIs in $\overline{\mathcal{K}}$ and, (2) show that every element $M_\mathcal{U}$ satisfies all DCIs in its typicality set $\mathcal{U}$.

**Lemma 7.9.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and $\mathcal{I}^\mathcal{K}_{\mathsf{min}} = (\Delta^{T(\mathcal{K})}, \cdot^{\mathcal{I}^\mathcal{K}_{\mathsf{min}}})$ its minimal typicality model. Then, $C^{\mathcal{I}^\mathcal{K}_{\mathsf{min}}} \subseteq D^{\mathcal{I}^\mathcal{K}_{\mathsf{min}}}$ for every $C \sqsubseteq D \in \overline{\mathcal{K}}$.*

*Proof.* Since $\mathcal{K}$ is in normal form, there are four kinds of GCIs in $\mathcal{K}$ to consider.

*Case 1: $A \sqsubseteq B$.* Let $M_\mathcal{U} \in A^{\mathcal{I}^\mathcal{K}_{\mathsf{min}}}$. Then, $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq A$, which implies $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq B$, thus $M_\mathcal{U} \in B^{\mathcal{I}^\mathcal{K}_{\mathsf{min}}}$.

*Case 2:* $A_1 \sqcap A_2 \sqsubseteq B$. $M_{\mathcal{U}} \in (A_1 \sqcap A_2)^{\mathcal{I}^{\mathcal{K}}_{\min}}$ implies $M_{\mathcal{U}} \in A_1^{\mathcal{I}^{\mathcal{K}}_{\min}}$ and $M_{\mathcal{U}} \in A_2^{\mathcal{I}^{\mathcal{K}}_{\min}}$, which implies $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq B$, thus $M_{\mathcal{U}} \in B^{\mathcal{I}^{\mathcal{K}}_{\min}}$.

*Case 3:* $A \sqsubseteq \exists r.B$. There are two cases two consider: $r \in \mathsf{N_R}$ and $r \in \mathsf{N_R}^- \setminus \mathsf{N_R}$.

1. $r \in \mathsf{N_R}$. Let $M_{\mathcal{U}} \in A^{\mathcal{I}^{\mathcal{K}}_{\min}}$. Then, $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq A$, and thus $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.B$. Note that $B \in Qc(\mathcal{K})$ implies $\{B\} \in \Delta^{\mathcal{K}}$ and $\{B\}_{\emptyset} \in \Delta^{T(\mathcal{K})}$. This guarantees the existence of a $N \in \Delta^{\mathcal{K}}$ s.t. $B \in N$ and $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.\lceil N \rceil$ and $N$ is a prime $r$-successor for $\lceil M \rceil \sqcap \overline{\mathcal{U}}$ in $\mathcal{I}^{\mathcal{K}}_{\min}$. By construction, $(M_{\mathcal{U}}, N_{\emptyset}) \in r^{\mathcal{I}^{\mathcal{K}}_{\min}}$ and $N \in B^{\mathcal{I}^{\mathcal{K}}_{\min}}$, which imply $M_{\mathcal{U}} \in (\exists r.B)^{\mathcal{I}^{\mathcal{K}}_{\min}}$.
2. If $r \in \mathsf{N_R}^- \setminus \mathsf{N_R}$, the proof is the same as in the last case with $r$ and $r^-$ exchanged.

*Case 4:* $A \sqsubseteq \forall r.B$. Once again, $r \in \mathsf{N_R}$ or $r \in \mathsf{N_R}^- \setminus \mathsf{N_R}$.

1. $r \in \mathsf{N_R}$. Let $M_{\mathcal{U}} \in A^{\mathcal{I}^{\mathcal{K}}_{\min}}$. Suppose that there is some $N_{\mathcal{U}'}$ s.t. $(M_{\mathcal{U}}, N_{\mathcal{U}'}) \in r^{\mathcal{I}^{\mathcal{K}}_{\min}}$. We show that $N_{\mathcal{U}'} \in B^{\mathcal{I}^{\mathcal{K}}_{\min}}$. There are two possible origins for this edge. First, suppose that $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.\lceil N \rceil$, and $N$ is a prime $r$-successor of $M_{\mathcal{U}}$. By definition of $\mathcal{I}^{\mathcal{K}}_{\min}$, we have $\mathcal{U}' = \emptyset$ and $B \in N$. Now, $B \in N$ and $\overline{\mathcal{K}} \models \lceil N \rceil \sqsubseteq B$ imply $N_{\emptyset} \in B^{\mathcal{I}^{\mathcal{K}}_{\min}}$. Second, if $\overline{\mathcal{K}} \models \lceil N \rceil \sqcap \overline{\mathcal{U}'} \sqsubseteq \exists r^-.\lceil M \rceil$ and $M$ is a prime $r^-$-successor for $N_{\mathcal{U}'}$. To show $\overline{\mathcal{K}} \models \lceil N \rceil \sqcap \overline{\mathcal{U}'} \sqsubseteq B$ by contradiction, assume there exists a model $\mathcal{J}$ of $\overline{\mathcal{K}}$ with some $d \in \Delta^{\mathcal{J}}$ s.t. $d \in (\lceil N \rceil \sqcap \overline{\mathcal{U}'})^{\mathcal{J}}$ and $d \notin B^{\mathcal{J}}$. Due to $\overline{\mathcal{K}} \models \lceil N \rceil \sqcap \overline{\mathcal{U}'} \sqsubseteq \exists r^-.\lceil M \rceil$, there must be some $e \in \lceil M \rceil^{\mathcal{J}}$ s.t. $(e, d) \in r^{\mathcal{J}}$. Now $\overline{\mathcal{K}} \models \lceil M \rceil \sqsubseteq A$ implies $m \in A^{\mathcal{J}}$, which violates the GCI $A \sqsubseteq \forall r.B$ from $\overline{\mathcal{K}}$, contradicting the assumption that $\mathcal{J}$ is a model.
2. $r \in \mathsf{N_R}^- \setminus \mathsf{N_R}$. The proof is analogous. This follows from the construction of the domain. Roles are built from two sets – one for required successors and the other for required predecessors. For the axiom with a named role, we examine members of the first set by considering that they were prime $r$-successors and therefore maximal. For the edges originating in the second set, we showed by contradiction that they must be subsumed by $B$. We employ the same arguments inverting the sets for the axiom $A \sqsubseteq \exists r^-.B$.

□

Now, we show that the elements satisfy the DCIs in their typicality sets.

**Lemma 7.10.** *Let* $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ *be a DKB and* $\mathcal{I}^{\mathcal{K}}_{\min} = (\Delta^{T(\mathcal{K})}, \cdot^{\mathcal{I}^{\mathcal{K}}_{\min}})$ *its minimal typicality model. Then, for every* $M_{\mathcal{U}} \in \Delta^{T(\mathcal{K})}$ *and for every* $A \mathrel{\underset{\sim}{\sqsubset}} B \in \mathcal{U}$*, it follows that* $M_{\mathcal{U}} \in A^{\mathcal{I}^{\mathcal{K}}_{\min}} \Rightarrow M_{\mathcal{U}} \in B^{\mathcal{I}^{\mathcal{K}}_{\min}}$.

*Proof.* Suppose that $M_{\mathcal{U}} \in A^{\mathcal{I}^{\mathcal{K}}_{\min}}$ for some $A \mathrel{\underset{\sim}{\sqsubset}} B \in \mathcal{U}$, then $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq A$ by definition of $\mathcal{I}^{\mathcal{K}}_{\min}$. Since $\overline{A \mathrel{\underset{\sim}{\sqsubset}} B}$ is a conjunct in $\overline{\mathcal{U}}$, we have that $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \overline{A \mathrel{\underset{\sim}{\sqsubset}} B}$. By definition of $\overline{\mathcal{K}}$, we have $\overline{\mathcal{K}} \models \overline{A \mathrel{\underset{\sim}{\sqsubset}} B} \sqcap A \sqsubseteq B$, which implies that $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq B$, and thus $M_{\mathcal{U}} \in B^{\mathcal{I}^{\mathcal{K}}_{\min}}$. □

Together, the last two lemmas imply that minimal typicality models are indeed models, i.e. that $\mathcal{I}_{\mathsf{min}}^{\mathcal{K}} \models \mathcal{K}$ holds.

Next, we show that minimal typicality models are canonical if they fulfill a condition on their typicality domain. In typicality models, DCIs are satisfied if all most typical instances of a concept satisfy it. Therefore, to make a minimal typicality model a canonical model for **s**, the maximally typical instances of the concepts in the representative domain must be exactly the ones selected by the consistent selection function for **s**. More formally, the typicality domain must (i) contain all the elements $M_{\mathcal{U}}$, where $\mathcal{U}$ is the set of DCIs to be materialized with $M$ in **s**, and (ii) do *not* contain any maximally typical instance of $M$, $M_{\mathcal{U}'}$, s.t. $\mathcal{U}' \notin \mathsf{sel}_{\mathcal{K},\mathbf{s}}(\lceil M \rceil)$.

**Theorem 7.11** (Canonicity of $\mathcal{I}_{\mathsf{min}}^{\mathcal{K}}$)**.** *Let* $\mathbf{s} \in \{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\}$ *be a strength,* $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ *a DKB, and* $A \in \mathsf{sig}_\mathsf{C}(\mathcal{K})$*. Let* $\mathcal{I}_{\mathsf{min}}^{\mathcal{K}} = (\Delta^{T(\mathcal{K})}, \cdot^{\mathcal{I}_{\mathsf{min}}^{\mathcal{K}}})$ *be a minimal typicality model of* $\mathcal{K}$ *s.t. for every* $M \in \Delta^{\mathcal{K}}$*,* $M_{\mathcal{U}}$ *is maximally typical in* $\Delta^{T(\mathcal{K})}$ *iff* $\mathcal{U}$ *is selected by* $\mathsf{sel}_{\mathcal{K},\mathbf{s}}(\lceil M \rceil)$ *to be materialized with* $\lceil M \rceil$*. Then, for every* $M \in \Delta^{\mathcal{K}}$*:*

1. $\mathcal{K} \models_{\mathsf{mat,s}} \lceil M \rceil \sqsubseteq A$ *with* iff $M_\emptyset \in A^{\mathcal{I}_{\mathsf{min}}^{\mathcal{K}}}$*, and*
2. $\mathcal{K} \models_{\mathsf{mat,s}} \lceil M \rceil \mathrel{\widetilde{\sqsubseteq}} A$ iff $M_{\mathcal{U}} \in A^{\mathcal{I}_{\mathsf{min}}^{\mathcal{K}}}$ *for every maximally typical instance* $M_{\mathcal{U}}$ *of* $M$*.*

*Proof. Claim 1.* By construction, $\mathcal{K} \models_{\mathsf{mat,s}} \lceil M \rceil \sqsubseteq A$ iff $M_\emptyset \in A^{\mathcal{I}_{\mathsf{min}}^{\mathcal{K}}}$.

*Claim 2.* $\mathcal{K} \models_{\mathsf{mat,s}} \lceil M \rceil \mathrel{\widetilde{\sqsubseteq}} A$ holds iff $M_{\mathcal{U}} \in A^{\mathcal{I}_{\mathsf{min}}^{\mathcal{K}}}$ for every maximally typical instance of $A$, where $A \in \Delta^{T(\mathcal{K})}$. Thus, by the requirement on the maximal instances in the theorem, $\mathcal{U}$ is selected by $\mathsf{sel}_{\mathcal{K},\mathbf{s}}(\lceil M \rceil)$ to be materialized with $M$ and thus we have $M_{\mathcal{U}} \in A^{\mathcal{I}_{\mathsf{min}}^{\mathcal{K}}}$ for every $\mathcal{U}$ selected by **s**. By the definition of satisfaction (Def. 7.5), this yields $\mathcal{I}_{\mathsf{min}}^{\mathcal{K}} \models M \mathrel{\widetilde{\sqsubseteq}} A$. □

We define *propositional coverage* by minimal typicality models equipped with particular domains. We speak about those domains by referring to their *shape*, which we explore in the next subsection. The shapes covering **rat**, **rel**, and **lex**, are roughly equivalent to their $\mathcal{EL}_\bot$ counterparts. The typicality sets are combined with concept representatives in the same way. The domains differ in how they choose those concept representatives. In $\mathcal{EL}_\bot$, they are defined by the context over $\mathcal{K}$, while, in $\mathcal{ELI}_\bot$, they come from the relevant domain.

**Definition 7.12** (Semantics of propositional coverage)**.** *Let* $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ *be a DKB,* **s** *be a strength, and* $\Delta_{\mathbf{s}}^{T(\mathcal{K})}$ *be the domain for* **s**.

1. $\mathcal{K} \models_{\mathsf{prop,s}} \lceil M \rceil \sqsubseteq A$ iff $\mathcal{I}_{\mathsf{min}}^{\mathcal{K}} \models \lceil M \rceil \sqsubseteq A$
2. $\mathcal{K} \models_{\mathsf{prop,s}} \lceil M \rceil \mathrel{\widetilde{\sqsubseteq}} A$ iff $\mathcal{I}_{\mathsf{min}}^{\mathcal{K}} \models \lceil M \rceil \mathrel{\widetilde{\sqsubseteq}} A$

### 7.2.1 Domain Shapes Determine Strength of Reasoning

The minimal requirements for a typicality model domain for a DKB $\mathcal{K}$ and strength **s** are to contain (i) atypical instances for every concept set in the relevant domain, (ii) all combinations of $M_{\mathcal{U}}$ selected by $\mathsf{sel}_{\mathcal{K},\mathbf{s}}(\lceil M \rceil)$ and those instances being exactly the maximally typical instances of $M$ in the domain. Intuitively, for each concept set, the presence of the atypical and the most typical elements is sufficient. These requirements already ensure that a minimal typicality model is a canonical model for materialization-based reasoning of strength **s**, as shown in Theorem 7.11. Sometimes, it is advantageous to admit elements of "intermediate" typicality also to achieve reasoning of a particular strength. We examine materialization-based reasoning, in regard of the domain shapes that achieve reasoning of rational, relevant, and lexicographic strengths.

### Rational Domain

Intuitively, the rational domain uses as the second dimension the exceptionality chain (cf. 4.6) of the DKB. For every domain element $M$ there is a maximal set $\mathcal{E}_i$ (of minimal $i$) consistent with $M$. Since $\mathcal{E}_j \subseteq \mathcal{E}_i$, for any $0 \leq i < j$, $M$ is also consistent with $\mathcal{E}_j$. Therefore, a rational domain can contain all pairs $M_{\mathcal{E}_j}$ s.t. $0 \leq i \leq j \leq n$. In this case, $j = n$ defines the atypical instance $M_\emptyset$.

**Definition 7.13** (Rational typicality domain)**.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB and $\mathcal{E}_0, \ldots, \mathcal{E}_n$ be its exceptionality chain. The* rational typicality domain *of $\mathcal{K}$ is:*

$$\Delta^{T(\mathcal{K})}_{\mathbf{rat}} := \{M_{\mathcal{E}_i} \in \Delta^{\mathcal{K}} \times \{\mathcal{E}_0, \ldots, \mathcal{E}_n\} \mid \overline{\mathcal{K}} \not\models \lceil M \rceil \sqcap \overline{\mathcal{E}_i} \sqsubseteq \bot\}$$

With the rational domain, we can define rational minimal typicality models.

**Definition 7.14** ($\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rat}}$, $\models_{\mathsf{prop},\mathbf{rat}}$)**.** *A* rational minimal typicality model *of a DKB $\mathcal{K}$ is a minimal typicality model of a DKB $\mathcal{K}$ over the rational typicality domain $\Delta^{T(\mathcal{K})}_{\mathbf{rat}}$, defined as $\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rat}} := (\Delta^{T(\mathcal{K})}_{\mathbf{rat}}, \cdot^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}})$.*

*Semantics based on typicality models with rational strength and propositional coverage is denoted by $\models_{\mathsf{prop},\mathbf{rat}}$ and defined as the semantics of propositional coverage with* **s** = **rat**.

**Example 7.15.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB with*

$$\begin{aligned}\mathcal{T} =&\{\mathsf{Deity} \sqsubseteq \mathsf{Being}, \mathsf{Human} \sqsubseteq \mathsf{Being}, \mathsf{Human} \sqsubseteq \exists \mathsf{worships.Deity},\\ &\mathsf{Human} \sqsubseteq \forall \mathsf{worships.Powerful}, \mathsf{Immortal} \sqcap \mathsf{Mortal} \sqsubseteq \bot\} \text{ and}\\ \mathcal{D} =&\{\mathsf{Being} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Mortal}, \mathsf{Being} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Corporeal}, \mathsf{Deity} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Immortal}\}.\end{aligned}$$

*The exceptionality chain for $\mathcal{K}$ is: $\mathcal{E}_0 = \mathcal{D}$, $\mathcal{E}_1 = \{\mathsf{Deity} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Immortal}\}$ and $\mathcal{E}_2 = \emptyset$. Figure 7.1 depicts the domain $\Delta^{T(\mathcal{K})}_{\mathbf{rat}}$ for $\mathcal{K}$ with the same matrix structure of the rational domain for $\mathcal{EL}_\bot$. In this case, the matrix is the Cartesian product between $\Delta^{\mathcal{K}}$ and the exceptionality chain. A difference regarding the $\mathcal{EL}_\bot$ version is the existence of* {Deity, Powerful} *instances representing combinations of concepts occurring within quantifiers. In this example, this combination is needed to witness the* worship *successor for instances of* Human.

The rational domain still fulfills the requirement for canonicity.

**Lemma 7.16** (Canonicity of $\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rat}}$)**.** *Let $\mathcal{K}$ be a DKB, $M \in \Delta^{\mathcal{K}}$ and $A \in \mathsf{sig}_\mathsf{C}(\mathcal{K})$.*

$$\mathcal{K} \models_{\mathsf{mat},\mathbf{rat}} \lceil M \rceil \mathrel{\underset{\sim}{\sqsubset}} A \text{ iff } \mathcal{K} \models_{\mathsf{prop},\mathbf{rat}} \lceil M \rceil \mathrel{\underset{\sim}{\sqsubset}} A$$

*Proof.* The result follows directly from Theorem 7.11 and the construction of $\Delta^{T(\mathcal{K})}_{\mathbf{rat}}$. For

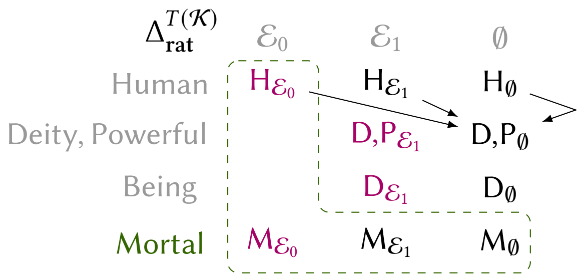


**Figure 7.1:** *Minimal rational typicality domain for $\mathcal{ELI}_\bot$. The green dashed area represents the extension of the concept* Mortal *in the model. The major differences w.r.t. the $\mathcal{EL}_\bot$ counterpart are the concept sets and the primeness requirment for role edges.*

every $M \in \Delta^{\mathcal{K}}$, $M_{\mathcal{E}_i}$ is a maximally typical instance of $M$ for the largest $\mathcal{E}_i$ in the exceptionality chain consistent with $\lceil M \rceil$. For the minimal typicality model, $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{E}_i} \sqsubseteq A$ *iff* $M_{\mathcal{E}_i} \in A^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}}}$. □

### Relevant and Lexicographic Domains

The relevant and lexicographic domains are also built from the same intuition of their counterparts for $\mathcal{EL}_\bot$, and the difference rests on the concept set. Instead of populating the full lattice over $\mathcal{D}$ with conceps in a context $\mathbb{C}(\mathcal{K})$, the relevant and lexicographic domains for $\mathcal{ELI}_\bot$ populate the same lattice with the sets in the representative domain $\Delta^{\mathcal{K}}$. Specifically, for every $M \in \Delta^{\mathcal{K}}$, the typicality domain contains the maximal pair(s) selected by the strength $\mathbf{s}$, $M_{\mathcal{U}}$, and also every $M_{\mathcal{U}'}$ with $\mathcal{U}' \subset \mathcal{U}$. The relevant and the lexicographic strengths differ in the selection of $\mathcal{U}$.

**Definition 7.17** (Relevant typicality domain, lexicographic typicality domain)**.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB. The* relevant typicality domain *of $\mathcal{K}$ is defined as:*

$$\Delta^{T(\mathcal{K})}_{\mathbf{rel}} := \{M_{\mathcal{U}} \in \Delta^{\mathcal{K}} \times \mathcal{P}(\mathcal{D}) \mid \mathcal{U} \subseteq \mathsf{sel}_{\mathcal{K},\mathbf{rel}}(\lceil M \rceil)\}$$

**Definition 7.18.** *The* lexicographic typicality domain *of $\mathcal{K}$ is defined as:*

$$\Delta^{T(\mathcal{K})}_{\mathbf{lex}} := \{M_{\mathcal{U}} \in \Delta^{\mathcal{K}} \times \mathcal{P}(\mathcal{D}) \mid \mathsf{sel}_{\mathcal{K},\mathbf{lex}}(\lceil M \rceil) = \{\mathcal{U}_1, \ldots, \mathcal{U}_n\} \text{ and } \mathcal{U} \subseteq \mathcal{U}_i \text{ for some } 1 \leq i \leq n\}$$

**Definition 7.19** ($\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rel}}$, $\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{lex}}$, $\models_{\mathsf{prop},\mathbf{rel}}$, $\models_{\mathsf{prop},\mathbf{lex}}$)**.** *The minimal typicality model of a DKB $\mathcal{K}$ over a*

- *domain $\Delta^{T(\mathcal{K})}_{\mathbf{rel}}$, is a* relevant minimal typicality model *of $\mathcal{K}$, defined as $\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rel}} := (\Delta^{T(\mathcal{K})}_{\mathbf{rel}}, \cdot^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}})$.*
- *domain $\Delta^{T(\mathcal{K})}_{\mathbf{lex}}$, is a* lexicographic minimal typicality model *of $\mathcal{K}$, defined as $\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{lex}} := (\Delta^{T(\mathcal{K})}_{\mathbf{lex}}, \cdot^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}})$.*

*Semantics based on typicality models with propositional coverage and strength*

- $\mathbf{s} = \mathbf{rel}$ *is denoted by* $\models_{\mathsf{prop},\mathbf{rel}}$.
- $\mathbf{s} = \mathbf{lex}$ *is denoted by* $\models_{\mathsf{prop},\mathbf{lex}}$.

Again, concept membership of domain elements of intermediate typicality does not impact the model's canonicity.

**Lemma 7.20** (Canonicity of $\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rel}}$ and of $\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{lex}}$)**.** *Let $\mathcal{K}$ be a DKB, $M \in \Delta^{\mathcal{K}}$ and $A \in \mathsf{sig}_{\mathsf{C}}(\mathcal{K})$. Then*

1. $\mathcal{K} \models_{\mathsf{mat},\mathbf{rel}} \lceil M \rceil \mathrel{\tilde{\sqsubseteq}} A$ *iff* $\mathcal{K} \models_{\mathsf{prop},\mathbf{rel}} \lceil M \rceil \mathrel{\tilde{\sqsubseteq}} A$.
2. $\mathcal{K} \models_{\mathsf{mat},\mathbf{lex}} \lceil M \rceil \mathrel{\tilde{\sqsubseteq}} A$ *iff* $\mathcal{K} \models_{\mathsf{prop},\mathbf{lex}} \lceil M \rceil \mathrel{\tilde{\sqsubseteq}} A$.

*Proof.* Both claims follow directly from Theorem 7.11 and the definitions of $\Delta^{T(\mathcal{K})}_{\mathbf{rel}}$ and $\Delta^{T(\mathcal{K})}_{\mathbf{lex}}$. The element $M_{\mathsf{sel}_{\mathcal{K},\mathbf{rel}}(\lceil M \rceil)}$ and the elements $M_{\mathcal{U}_i}$, $1 \leq i \leq n$, for $\mathsf{sel}_{\mathcal{K},\mathbf{lex}}(\lceil M \rceil) = \{\mathcal{U}_1, \ldots, \mathcal{U}_n\}$ are the most typical instances of $M$ in their respective domains. Hence, $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathsf{sel}_{\mathcal{K},\mathbf{rel}}(\lceil M \rceil)} \sqsubseteq A$ iff $M_{\mathsf{sel}_{\mathcal{K},\mathbf{rel}}(\lceil M \rceil)} \in A^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rel}}}$ and $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{U}_i} \sqsubseteq A$ for every $1 \leq i \leq n$ iff $M_{\mathcal{U}_i} \in \mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{lex}}$. □

In this section we have devised minimal typicality models for defeasible $\mathcal{ELI}_\bot$. The computation of the minimal typicality model for the chosen strength gives a reasoning method for defeasible subsumption for propositional coverage for that strength. Furthermore, should the DBox of a DKB be empty, the typicality domains would contain only the atypical elements $M_\emptyset$ and thus minimal typicality models would essentially coincide with the canonical models for monotonic $\mathcal{ELI}_\bot$.

## 7.3 Saturated Typicality Models

In minimal typicality models the role successors required by an existential restriction are all atypical, i.e. have empty typicality sets and need not to satisfy any DCI. A stronger form of typicality model is needed if defeasible information is to be applied to these elements. In this section, we develop saturated typicality models for $\mathcal{ELI}_\bot$ extending those for $\mathcal{EL}_\bot$ from [PT18b; Pen19]. Intuitively, saturated typicality models "saturate" all required role successors with defeasible information and thereby maximize typicality of these elements. We first describe how saturated typicality models can be computed and then describe the entailment defined over them, reasoning of *nested coverage*. We show that nested reasoning successfully tackles quantification neglect for each strength $\mathbf{s} \in \{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\}$.

### 7.3.1 Computation of Saturated Typicality Models

The general approach to compute a saturated typicality model of a given DKB $\mathcal{K}$ (and strength $\mathbf{s}$) is to:

1. Compute a minimal typicality model $\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}$.
2. Perform *model upgrades* exhaustively. Each upgrade has the following two parts:

(a) *model update*: for an edge in the current model, introduce a new edge to an element of the same concept set and a larger typicality set, i.e. increase the number of DCIs that the role successor must satisfy.

(b) *model recovery*: introduce and remove the minimal amount of information necessary to ensure that the resulting interpretation is a (canonical) model.

Clearly, an upgrade can lead to a set of different models as the update step can have different results. Thus, the construction of saturated typicality models works with sets of models. Some updated interpretations may be unrecoverable through the model recovery procedure. Such interpretations are then discarded and only effectively recovered interpretations – i.e. models satisfying some properties – are upgraded further.

The computation method for saturated typicality models for $\mathcal{EL}_\bot$ from Chapter 6 relies on two properties of $\mathcal{EL}_\bot$:

1. $\mathcal{EL}_\bot$ axioms can require successors only for named roles, but not for inverse roles.
2. Recovery of the model property can be achieved by increasing the extensions of concepts and roles.

These properties do not hold for $\mathcal{ELI}_\bot$. In $\mathcal{ELI}_\bot$, axioms can additionally require successors of inverse roles. So, an edge between elements in $\mathcal{ELI}_\bot$ models can be "initiated" by the successor or the predecessor of a named role or even from both elements independently. Thus an edge $(M_\mathcal{U}, N_\mathcal{V}) \in r^\mathcal{I}$ cannot indicate whether $\overline{\mathcal{K}}$ entails $\lceil M \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.\lceil N \rceil$ or $\lceil N \rceil \sqcap \overline{\mathcal{V}} \sqsubseteq \exists r^-.\lceil M \rceil$ and it is unclear during model upgrades whether to update $M_\mathcal{U}$ or $N_\mathcal{V}$. While model updates for $\mathcal{EL}_\bot$ always target the successor of a named role, they can also affect the predecessor in $\mathcal{ELI}_\bot$. To address this problem, we introduce a formalism to record, for each edge in a typicality interpretation, which of its end-point(s) initiated it.

We notice a similar effect of the introduction of value restrictions in the model recovery step. When an $r$-edge $(M_\mathcal{U}, N_\mathcal{V})$ violates a value restriction, say $A \sqsubseteq \forall r.B$, there are two ways to handle this. Either $N_\mathcal{V}$ can be added to the extension of $B$ or the edge can be moved to $N \cup \{B\}_\mathcal{V}$. The correct choice to recover the model depends on which existential requirement the edge represents in the interpretation.

Model recovery in $\mathcal{ELI}_\bot$ may have to remove edges from an interpretation. Consider an element $d$ added to $A^\mathcal{I}$ during model update. Axioms such as $A \sqsubseteq \forall r.B$ can force *moving* all existing $r$-edges starting at $d$ from $M_\mathcal{U}$ to $M \cup \{B\}_\mathcal{U}$. So, merely completing extensions as in $\mathcal{EL}_\bot$ is not enough.[1] We propose a new method for recovering the model property that accounts for the removal of edges and also accommodates the requirement of primeness for role successors. This method is called *model recovery*.

### Updating Typicality Interpretations

The update step identifies an edge in the given typicality interpretation that comports a more typical successor and adds a new edge s.t. one of the endpoints is shared with the original edge and the other one is a more typical representative of the element in

[1] Model recovery for $\mathcal{EL}_\bot$ is even called model *completion*.

the original edge's other endpoint. For example, for an edge $(d, e) \in r^{\mathcal{I}}$, the update may add $(d', e) \in r^{\mathcal{I}'}$, where $d'$ is a more typical representative of the concept represented by $d$.

To do so, an update has to identify for each edge which of the two elements *initiates* the edge to have the other as its prime successor. To do so, an update has to identify which of the edge's two elements *initiates* it. The other element should be a required prime successor of the initiator. For example, suppose an edge $(d, e)$. If $e$ *initiates* this edge, then element $d$ is to be updated. We introduce a labeling of edges in an interpretation to keep track of which elements initiate each edge.

**Definition 7.21** (Initiator labeling)**.** *Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a typicality interpretation and $\mathcal{K}$ be a DKB. For $r \in \mathsf{N_R}$ an edge $(M_{\mathcal{U}}, N_{\mathcal{V}}) \in r^{\mathcal{I}}$ is*

- p-initiated *if $\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_{\mathcal{K}}(M_{\mathcal{U}}, \mathcal{I})\rceil \sqsubseteq \exists r.\lceil N \rceil$, $N$ is a prime $r$-successor w.r.t. $\overline{\mathcal{K}}$, and $\lceil \mathsf{N_C}\text{-type}_{\mathcal{K}}(M_{\mathcal{U}}, \mathcal{I})\rceil$.*
- s-initiated *if $\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_{\mathcal{K}}(N_{\mathcal{V}}, \mathcal{I})\rceil \sqsubseteq \exists r^{-}.\lceil M \rceil$ and $M$ is a prime $r^{-}$-successor w.r.t. $\overline{\mathcal{K}}$, and $\lceil \mathsf{N_C}\text{-type}_{\mathcal{K}}(N_{\mathcal{V}}, \mathcal{I})\rceil$.*

*An* initiator labeling *for $\mathcal{I}$ w.r.t. $\mathcal{K}$ is a function $\mathsf{Init}_{\mathcal{I}} : \Delta^{\mathcal{I}} \times \mathsf{sig_R}(\mathcal{K}) \times \Delta^{\mathcal{I}} \to \mathcal{P}(\{\mathsf{s}, \mathsf{p}\})$ respecting the following conditions:*

1. *If $\mathsf{p} \in \mathsf{Init}_{\mathcal{I}}(d, r, e)$, then $(d, e) \in r^{\mathcal{I}}$ is $\mathsf{p}$-initiated,*
2. *If $\mathsf{s} \in \mathsf{Init}_{\mathcal{I}}(d, r, e)$, then $(d, e) \in r^{\mathcal{I}}$ is $\mathsf{s}$-initiated,*
3. *$\mathsf{Init}_{\mathcal{I}}(d, r, e) = \emptyset$ iff $(d, e) \notin r^{\mathcal{I}}$.*

*We call the pair $(\mathcal{I}, \mathsf{Init}_{\mathcal{I}})$ a labeled interpretation.*

Note that generally there can be several initiator labeling functions for the same pair of a DKB and an interpretation. For $(d, e) \in r^{\mathcal{I}}$ and $r \in \mathsf{N_R}$, we abbreviate $\mathsf{p} \in \mathsf{Init}_{\mathcal{I}}(d, r, e)$ by $(d, e) \in r^{\mathcal{I}}_{\mathsf{p}}$ and $\mathsf{s} \in \mathsf{Init}_{\mathcal{I}}(d, r, e)$ by $(d, e) \in r^{\mathcal{I}}_{\mathsf{s}}$.

**Example 7.22.** *Let $\mathcal{K} = (\mathcal{T}, \emptyset)$ be a DKB with $\mathcal{T} = \{A \sqsubseteq \exists r.B, B \sqsubseteq \exists r^{-}.A\}$. Consider the typicality interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ s.t. $\{A\}, \{B\}$ are in the representative domain of $\mathcal{K}$. Figure 7.2 depicts a fragment of $\mathcal{I}$ with $\mathsf{Init}_{\mathcal{I}}$ restricted to $r$. Edges may be labeled with $\mathsf{p}$, $\mathsf{s}$, or both. The dashed edge is a possible update that can be either labeled with $\mathsf{p}$, if the element $\{A\}_{\mathcal{U}}$ was updated, or with $\mathsf{s}$, if the update was on the element $\{B\}_{\mathcal{U}}$.*

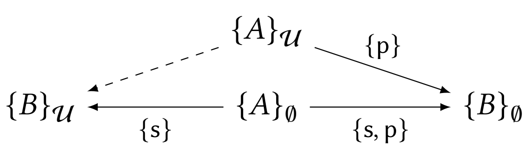


**Figure 7.2:** *Fragment of $\mathcal{I}$, with $r$-edges labeled by $\mathsf{Init}_{\mathcal{I}}$.*

Generally the initiator labeling needs not be unique for an interpretation. However, it is for minimal typicality models, which are the initial input to model upgrades. The labeling $\mathsf{Init}_{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}$ can be determined from the DKB and the structure of $\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}$, since all required role successors are represented by atypical elements.

**Definition 7.23** (Init-min$_{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$). *Let* $\mathcal{I}^{\mathcal{K}}_{\text{min}} = (\Delta^{T(\mathcal{K})}, \Delta^{\mathcal{I}^{\mathcal{K}}_{\text{min}}})$ *be a minimal typicality model for some DKB* $\mathcal{K}$. *The* initiator labeling for $\mathcal{I}^{\mathcal{K}}_{\text{min}}$ *is the function* $\text{Init-min}_{\mathcal{I}^{\mathcal{K}}_{\text{min}}} : \Delta^{T(\mathcal{K})} \times \text{sig}_{\text{R}}(\mathcal{K}) \times \Delta^{T(\mathcal{K})} \to \mathcal{P}(\{\text{s}, \text{p}\})$ *that maps each* $(M_{\mathcal{U}}, r, N_{\mathcal{V}})$ *to the set containing:*

- p, *if* $(M_{\mathcal{U}}, N_{\mathcal{V}}) \in r^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$ *is* p*-initiated and* $\mathcal{V} = \emptyset$ *and*
- s, *if* $(M_{\mathcal{U}}, N_{\mathcal{V}}) \in r^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$ *is* s*-initiated and* $\mathcal{U} = \emptyset$.

We show that this construction satisfies Definition 7.21 and that it captures the *correct* labels according to the minimal typicality model intuition that concept representatives witnessing existential restrictions must be atypical.

**Lemma 7.24.** *Let* $\mathcal{I}^{\mathcal{K}}_{\text{min}}$ *be a minimal typicality model for* $\mathcal{K}$ *and* $\text{Init-min}_{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$ *its initiator labeling. Then* $\text{Init-min}_{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$ *is an initiator labeling function.*

*Proof.* By construction,

1. Every edge in $\mathcal{I}^{\mathcal{K}}_{\text{min}}$is is p-initiated or s-initiated.
2. For every edge $(M_{\mathcal{U}}, N_{\mathcal{V}}) \in r_{\text{p}}^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$, $\mathcal{V} = \emptyset$.
3. For every edge $(M_{\mathcal{U}}, N_{\mathcal{V}}) \in r_{\text{s}}^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$, $\mathcal{U} = \emptyset$.

Conditions 1-3 here guarantee that the labels p and s correspond to p-initiated and s-initiated edges, respectively. This covers Conditions 1 and 2 of Definition 7.21 (initiator labeling functions). The definition of $\text{Init-min}_{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$ maps p to p-initiated edges, and similarly for s. For Condition 3 of the initiator labeling, note that, for every $r$ and every pair $(M_{\mathcal{U}}, N_{\mathcal{V}}) \in \Delta^{T(\mathcal{K})} \times \Delta^{T(\mathcal{K})}$, $\text{Init}_{\mathcal{I}^{\mathcal{K}}_{\text{min}}}(M_{\mathcal{U}}, r, N_{\mathcal{V}})$ is non-empty only when the edge is p-initiated and $\mathcal{V} = \emptyset$ or the edge is s-initiated and $\mathcal{U} = \emptyset$. However, in every edge of the minimal typicality model, either $\mathcal{U}$ or $\mathcal{V}$ are $\emptyset$. Then, all edges (and only them) have a label s or p (or both). □

One of the fundamental properties of the minimal typicality model is that existential requirements are witnessed by atypical instances of the successor. If $\overline{\mathcal{K}} \models \lceil M \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \lceil N \rceil$, for a prime successor $N$, the corresponding edge will be $(M_{\mathcal{U}}, N_{\emptyset}) \in r^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$. The initiator labeling function for the minimal typicality model captures this intuition correctly by labeling the edges with their intended initiators. If $\mathcal{U} \neq \emptyset$, the edge $(M_{\mathcal{U}}, N_{\emptyset}) \in r^{\mathcal{I}^{\mathcal{K}}_{\text{min}}}$ is labeled with $\{\text{p}\}$ even in the case where $\overline{\mathcal{K}} \models \lceil N \rceil \sqsubseteq \exists r^-.\lceil M \rceil$, for a prime $r^-$ successor $M$.

An *update candidate* for a typicality model $\mathcal{I}$ is pair consisting of a domain element $d$ and an edge including $d$. This edge can be added to $\mathcal{I}$ to increase the typicality of a successor of $d$ already present in $\mathcal{I}$. We define a function that returns a set of update candidates for any labeled interpretation $(\mathcal{I}, \text{Init}_{\mathcal{I}})$.

**Definition 7.25** (Update candidates). *Let* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *be a typicality interpretation and* $\text{Init}_{\mathcal{I}}$ *an initiator labeling. Let* $d, M_{\mathcal{U}}, M_{\mathcal{V}} \in \Delta^{\mathcal{I}}$ *and* $r \in \text{N}_{\text{R}}$. *The* set of $r$-update candidates *for* $(\mathcal{I}, \mathsf{Init}_{\mathcal{I}})$ *is:*

$$\begin{aligned}\mathsf{UpCan}_r(\mathcal{I}, \mathsf{Init}_{\mathcal{I}}) := &\{((d, M_{\mathcal{U}}), d) \mid (d, M_{\mathcal{V}}) \in r^{\mathcal{I}}_{\mathsf{p}} \text{ and } \mathcal{V} \subset \mathcal{U}\} \cup \\ &\{((M_{\mathcal{U}}, d), d) \mid (M_{\mathcal{V}}, d) \in r^{\mathcal{I}}_{\mathsf{s}} \text{ and } \mathcal{V} \subset \mathcal{U}\}.\end{aligned}$$

*Element $d$ is called* update root *and the other element from the edge is the* update target.

A model update augments a given typicality interpretation by performing an update, i.e., it adds the edge connecting update root and target, and updates the initiator labeling accordingly. Observe that such a change preserves the concept set of the successor. Therefore, the update target is a prime successor iff the successor that it supersedes is a prime successor.

**Definition 7.26** (Typicality Model Update)**.** *Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a typicality model,* $\mathsf{Init}_{\mathcal{I}}$ *an initiator labeling, $r \in \mathsf{N_R}$, and $((d_1, d_2), d_i) \in \mathsf{UpCan}_r(\mathcal{I}, \mathsf{Init}_{\mathcal{I}})$ an update candidate for some root $d_i \in \{d_1, d_2\}$. A* typicality model $r$-update *is*

$$\mathsf{UD}_r(\mathcal{I}, \mathsf{Init}_{\mathcal{I}}, (d_1, d_2), d_i) := (\mathcal{J}, \mathsf{Init}_{\mathcal{J}})$$

*where $\mathcal{J} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{J}})$ with*

$$\begin{aligned}A^{\mathcal{J}} &:= A^{\mathcal{I}} \\ r^{\mathcal{J}} &:= r^{\mathcal{I}} \cup \{(d_1, d_2)\} \\ s^{\mathcal{J}} &:= s^{\mathcal{I}}, \text{ for all } s \in \mathsf{N_R}, s \neq r\end{aligned}$$

$\mathsf{init} := \mathsf{p}$ *if $d_i = d_1$, and* $\mathsf{init} := \mathsf{s}$ *if $d_i = d_2$; and* $\mathsf{Init}_{\mathcal{J}}$ *is the following mapping:*

$$\begin{aligned}\mathsf{Init}_{\mathcal{J}}(d_1, r, d_2) &:= \mathsf{Init}_{\mathcal{I}}(d_1, r, d_2) \cup \{\mathsf{init}\} \\ \mathsf{Init}_{\mathcal{J}}(e_1, r, e_2) &:= \mathsf{Init}_{\mathcal{I}}(e_1, r, e_2) \text{ if } e_1 \neq d_1 \text{ or } e_2 \neq d_2 \\ \mathsf{Init}_{\mathcal{J}} &:= \mathsf{Init}_{\mathcal{I}}, \text{ for all } s \neq r\end{aligned}$$

It is easy to see that $\mathsf{Init}_{\mathcal{J}}$ is a initiator labeling of $\mathcal{J}$. Note that $\mathsf{Init}_{\mathcal{J}}$ differs from the initiator labeling $\mathsf{Init}_{\mathcal{I}}$ by one additional edge label. Let this labeled edge be $(M_{\mathcal{U}}, N_{\mathcal{V}}) \in r^{\mathcal{J}}$. Suppose, without loss of generality, that the root of the update is $M_{\mathcal{U}}$. Because the pair is part of an update candidate, there must be some $N_{\mathcal{V}'}$ s.t. $\mathcal{V}' \subset \mathcal{V}$ and $(M_{\mathcal{U}}, N_{\mathcal{V}'}) \in r^{\mathcal{I}}_{\mathsf{p}}$, which means that the edge is p-initiated. Since the $\mathsf{N_C}$-type of the elements do not change between $\mathcal{I}$ and $\mathcal{J}$, this also holds for $\mathcal{J}$. More formally, $\mathsf{N_C}\text{-type}_{\mathcal{K}}(M_{\mathcal{U}}, \mathcal{I}) = \mathsf{N_C}\text{-type}_{\mathcal{K}}(M_{\mathcal{U}}, \mathcal{J})$ and $\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_{\mathcal{K}}(M_{\mathcal{U}}, \mathcal{I}) \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.\lceil N \rceil$, for a prime $N$, imply $\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_{\mathcal{K}}(M_{\mathcal{U}}, \mathcal{J}) \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.\lceil N \rceil$, with $N$ prime.

The semantics of saturated typicality models considers exhaustively all upgrades of the minimal typicality model for some strength **s**. Since some choices of update candidates can block updates of other candidates, it is necessary to consider all of candidates in parallel. Let $(\mathcal{I}, \mathsf{Init}_{\mathcal{I}})$ be a labeled typicality interpretation and $\mathcal{K}$ a DKB. The set of all typicality interpretation updates of $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ w.r.t. $\mathcal{K}$ is :

$$\mathsf{Upd}_\mathcal{K}(\mathcal{I}, \mathsf{Init}_\mathcal{I}) := \bigcup_{r \in \mathsf{sig}_\mathsf{R}(\mathcal{K})} \big\{ (\mathcal{J}, \mathsf{Init}_\mathcal{J}) \in \mathsf{UD}_r(\mathcal{I}, \mathsf{Init}_\mathcal{I}, (d_1, d_2), d_i) \mid ((d_1, d_2), d_i) \in \mathsf{UpCan}_r(\mathcal{I}, \mathsf{Init}_\mathcal{I}) \big\}.$$

### 7.3.2 Model Recovery

After an update, an interpretation may fail to be a model of $\mathcal{K}$. A new edge $(M_\mathcal{U}, N_\mathcal{V}) \in r^\mathcal{I}$, with $N_\mathcal{V} \in A^\mathcal{I}$, can violate axioms of the form $A \sqsubseteq \forall r^-.B$, if $M_\mathcal{U} \notin B^\mathcal{I}$. Axioms that cover the transmission of information in the other direction of the new edge – e.g., from $M_\mathcal{U}$ to $N_\mathcal{V}$ – will not threaten the model property at first. By construction, successors are already prime regarding their concept set. For a root $M_\mathcal{U}$ and an updated interpretation $\mathcal{I}$, if $\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_\mathcal{K}(M_\mathcal{U}, \mathcal{I}) \rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \forall r.B$, then any $N_\mathcal{V}$ neighboring the root would be such that $B \in N$ and, henceforth, $N_\mathcal{V} \in B^\mathcal{I}$.

To solve such violations, one needs to add the root to the extension of the required concept, increasing its $\mathsf{N_C}$-type. In the example above, this means adding $M_\mathcal{U}$ to $B$. However, this simple action opens Pandora's box and triggers all kinds of violations. The reader should remember that the left-hand side of terminological axioms in normalized TBoxes is always either a named concept or a conjunction of two named concepts. Adding some element to the extension of some named concept can trigger any terminological axiom. Now, we will investigate how to address these violations to recover the model property.

Violations concerning named concept inclusions – i.e., $A \sqsubseteq B$ and $A_1 \sqcap A_2 \sqsubseteq B$ – are simple to solve. Adding the element belonging to the concept(s) on the left-hand side to the extension of the right-hand side concept fixes them. Existential requirements are also relatively straightforward to correct. Solving a violation of $A \sqsubseteq \exists r.B$ is done by creating a new edge to the prime superset of $B$ required by the $\mathsf{N_C}$-type of the triggered element. For a prime $N \ni B$, the edge lands on $N_\emptyset$.

The strategies examined so far saturate the interpretation, i.e., they add elements to new extensions and create new edges between elements. Some violations dealing with value restrictions can be solved by altering any of the two elements connected by a given edge. Therefore, one has to *choose* which element to change to address the problem. Let $\mathcal{I}$ be a typicality interpretation with $(M_\mathcal{U}, N_\mathcal{V}) \in r^\mathcal{I}$ s.t. $M_\mathcal{U} \in A$, $N_\mathcal{V} \notin B^\mathcal{I}$ and $A \sqsubseteq \forall r.B \in \mathcal{T}$. One way of solving this violation would be to add $N_\mathcal{V}$ to the extension of $B$. Another way to address it is to substitute the edge $(M_\mathcal{U}, N_\mathcal{V})$ for $(M_\mathcal{U}, N \cup \{B\}_\mathcal{V})$, which does not only correct the violation, because $N \cup \{B\}_\mathcal{U} \in B$, but it also ensures that this role successor is again a prime successor, as $N \cup \{B\}$. As it turns out, both fixes are necessary for different situations.

The difference lies in which element *initiates* the edge, which is coded in the initiator labeling. The initiator lays out which existential restriction creates the edge, and a successful solution to a violation of value restriction should take this into account. Elements that initiate an edge cannot be cut off from it because they represent concepts whose required existential restrictions are represented by the edge. Therefore, if the initiator causes the violation, it needs to be changed to solve it. On the other hand, if the other element is the cause of the violation, the edge may be moved to a more inclusive successor without major issues. The edge only incidentally landed on the element, and it does not represent an existential restriction to it. Moving the edge to a more inclusive successor ensures primeness of required successors.

Coming back to the example previously stated, if $M_{\mathcal{U}}$ owns the edge and requires that every $r$-successor is a $B$, then this information should be reflected in the successor. The solution is moving the edge to $(M_{\mathcal{U}}, N \cup \{B\}_{\mathcal{V}})$, preserving the primeness of the successors. On the other hand, if the edge is initiated by $N_{\mathcal{V}}$, the violation should be solved by adding it to the extension of $B$. In this case, there is a back-and-forth dynamic between the two elements: $N_{\mathcal{V}}$ requires a predecessor, which requires that every successor is of some kind, as illustrated in Figure 7.3.

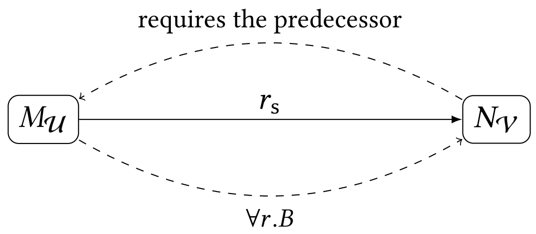


**Figure 7.3:** *Illustration of the back-and-forth dynamics between two elements connected by an edge. $N_{\mathcal{V}}$ requires the predecessor $M_{\mathcal{U}}$, which requires its successors to be $B$.*

In the model recovery procedure, the $\mathsf{N_C}$-types of an element increases in only two scenarios. If the $\mathsf{N_C}$-type was previously upgraded with some concept $A$, then it can trigger axioms $A \sqsubseteq B$ and $A_1 \sqcap A_2 \sqsubseteq B$ and be further increased. The other situation that governs an increase in the $\mathsf{N_C}$-types is an increase in the $\mathsf{N_C}$-types of a neighbor. However, this change in the neighborhood can only affect an element when the element itself initiates the edge. Therefore, change is kept in a connected part of the edge initiator's graph represented by the initiator labeling.

To formally characterize this notion, we introduce the concepts of *role causation path* and *dependency set*. The first captures a chain of elements connected by edges that they initiate, and the second is defined for elements $e$ and captures the set of all elements connected to $e$ by such chains. Intuitively, those concepts can show that the model recovery procedure is repairing only problems either in the root of the upgrade or in elements that depend on the root. The change that a model recovery brings is limited to the dependency set of the root of the update.

**Definition 7.27** (Edge Owner Path, Dependency Set). *Let $(\mathcal{I}, \mathsf{Init}_{\mathcal{I}})$ be a labeled typicality interpretation for some $\mathcal{K}$. Let $d_0, \dots, d_n \in \Delta^{\mathcal{I}}$ and the roles $r_0, \dots, r_n \in \mathsf{sig}_{\mathsf{R}}(\mathcal{K})$. The sequence $d_0, \dots, d_n$ is an* initiator path *iff $(d_i, d_{i+1}) \in r_{i\,\mathrm{p}}^{\mathcal{I}}$ or $(d_{i+1}, d_i) \in r_{i\,\mathrm{s}}^{\mathcal{I}}$, for every $0 \leq i \leq n-1$.*

*The* dependency set *of an element $d \in \Delta^{\mathcal{I}}$ is:*

$$\mathsf{DS}(d) := \{e \in \Delta^{\mathcal{I}} \mid e_0 \dots, e_n \textit{ is an initiator path}, e = e_0, e_n = d\}$$

The dependency set collects all the elements that *depend* on its input. This dependency is defined as a series of labeled edges. It can be understood, inductively, by acknowledging that an element $d$ depends on itself. Then, all the elements that require $d$ either as a successor or predecessor depend on it. As a next step, all the elements that depend on those that depend on $d$, and so forth.

A model recovery should:

1. address all violations of GCIs from $\overline{\mathcal{K}}$ caused directly or indirectly by the last update and change the interpretation only minimally;
2. preserve the concept set of prime successors;
3. only change the $\mathsf{N_C}$-types of elements in the dependency set of the upgrade's root.

A fix of a violation can give rise to other violations. Furthermore, an update or a fix can result in an interpretation that cannot be recovered—in that case it contains a clash.

**Definition 7.28** (Clash)**.** *Let $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ be a labeled interpretation, where $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ and $M_\mathcal{U}, N_\mathcal{V} \in \Delta^\mathcal{I}$. $\mathcal{I}$ contains a*

- direct clash, *if $\{C \sqsubseteq \bot, C \sqsubseteq \exists r.\bot\} \cap \overline{\mathcal{K}} \neq \emptyset$ for some concept $C$, $M_\mathcal{U} \in C^\mathcal{I}$.*
- successor clash, *if $A \sqsubseteq \forall r.\bot \in \overline{\mathcal{K}}$, $M_\mathcal{U} \in A^\mathcal{I}$, and $A \in (\exists r.\top)^\mathcal{I}$.*
- successor domain clash, *if $A \sqsubseteq \forall r.B \in \overline{\mathcal{K}}$, $(M_\mathcal{U}, N_\mathcal{V}) \in r_\mathsf{p}^\mathcal{I}$, $B \notin N$, and $N \cup \{B\}_\mathcal{V} \notin \Delta^\mathcal{I}$.*
- predecessor domain clash, *if $A \sqsubseteq \forall r^-.B \in \overline{\mathcal{K}}$, $(N_\mathcal{V}, M_\mathcal{U}) \in r_\mathsf{s}^\mathcal{I}$, $B \notin N$, and $N \cup \{B\}_\mathcal{V} \notin \Delta^\mathcal{I}$.*

We define *model recovery* as the result of a series of individual *fixes*, which are controlled changes caused by the violation of an axiom. There is one fix rule for each normalized GCI besides the ones containing value restrictions. Those can be amended by two different procedures depending on edge ownership.

**Definition 7.29** (Fix rules)**.** *Let $(\mathcal{I}', \mathsf{Init}'_\mathcal{I})$ be a typicality model of some DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ and its initiator labeling and let $(\mathcal{I}, \mathsf{Init}_\mathcal{I}) \in \mathsf{Upd}_\mathcal{K}(\mathcal{I}', \mathsf{Init}_{\mathcal{I}'})$ with $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, $M_\mathcal{U}, N_\mathcal{V} \in \Delta^\mathcal{I}$, and let $l \in \{\mathsf{p}, \mathsf{s}\}$.*

*To* remove *$(M_\mathcal{U}, N_{\mathcal{U}'})$ from $r_l^\mathcal{I}$ results in*

1. *$\mathsf{Init}_\mathcal{I}(M_\mathcal{U}, r, N_\mathcal{V}) := \mathsf{Init}_\mathcal{I}(M_\mathcal{U}, r, N_\mathcal{V}) \setminus \{l\}$ and*
2. *if $\mathsf{Init}_\mathcal{I}(M_\mathcal{U}, r, N_\mathcal{V}) = \emptyset$, then $r^\mathcal{I} := r^\mathcal{I} \setminus \{(M_\mathcal{U}, N_\mathcal{V})\}$.*

*The* fix rules *for $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ and $\mathcal{K}$ are the following:*

(R$\sqsubseteq$) ***If*** *$A \sqsubseteq B \in \overline{\mathcal{K}}$, $M_\mathcal{U} \in A^\mathcal{I}$, and $M_\mathcal{U} \notin B^\mathcal{I}$,* ***then*** *add $M_\mathcal{U}$ to $B^\mathcal{J}$.*

(R$\sqcap$) ***If*** *$A_1 \sqcap A_2 \sqsubseteq B \in \overline{\mathcal{K}}$, $M_\mathcal{U} \in (A_1 \sqcap A_2)^\mathcal{I}$, and $M_\mathcal{U} \notin B^\mathcal{I}$,* ***then*** *add $M_\mathcal{U}$ to $B^\mathcal{J}$.*

(R$\exists$) ***If*** *$A \sqsubseteq \exists r.B \in \overline{\mathcal{K}}$, $M_\mathcal{U} \in A^\mathcal{I}$, and $\nexists N.B \in N$, and $(M_\mathcal{U}, N_\mathcal{V}) \in r_\mathsf{p}^\mathcal{I}$, for any $\mathcal{V} \subseteq \mathcal{D}$,* ***then*** *add $(M_\mathcal{U}, \{B\}_\emptyset)$ to $r_\mathsf{p}^\mathcal{J}$.*

(R$\exists^-$) ***If*** $A \sqsubseteq \exists r^-.B \in \overline{\mathcal{K}}$, $M_\mathcal{U} \in A^\mathcal{I}$, *and* $\nexists N.B \in N$ *and* $(N_\mathcal{V}, M_\mathcal{U}) \in r_\mathrm{s}^\mathcal{I}$, *for any* $\mathcal{V} \subseteq \mathcal{D}$, ***then*** *add* $(\{B\}_\emptyset, M_\mathcal{U})$ *to* $r_\mathrm{s}^\mathcal{J}$.

(R$\forall$R) ***If*** $A \sqsubseteq \forall r.B \in \overline{\mathcal{K}}$, $M_\mathcal{U} \in A^\mathcal{I}$ *and* $\exists N, \mathcal{V}.(M_\mathcal{U}, N_\mathcal{V}) \in r_\mathrm{p}^\mathcal{I}$, $B \notin N$, *and* $N \cup \{B\}_\mathcal{V} \in \Delta^\mathcal{I}$, ***then*** *replace in* $r_\mathrm{p}^\mathcal{J}$ *the edge* $(M_\mathcal{U}, N_\mathcal{V})$ *by* $(M_\mathcal{U}, N \cup \{B\}_\mathcal{V})$.

(R$\forall$R$^-$) ***If*** $A \sqsubseteq \forall r^-.B \in \overline{\mathcal{K}}$, $M_\mathcal{U} \in A^\mathcal{I}$ *and* $\exists N, \mathcal{V}.(N_\mathcal{V}, M_\mathcal{U}) \in r_\mathrm{s}^\mathcal{I}$, $B \notin N$, *and* $N \cup \{B\}_\mathcal{V} \in \Delta^\mathcal{I}$, ***then*** *replace in* $r_\mathrm{s}^\mathcal{J}$ *the edge* $(N_\mathcal{V}, M_\mathcal{U})$ *by* $(N \cup \{B\}_\mathcal{V}, M_\mathcal{U})$.

(R$\forall$C) ***If*** $A \sqsubseteq \forall r.B \in \overline{\mathcal{K}}$, $M_\mathcal{U} \in A^\mathcal{I}$, $N_\mathcal{V} \notin B^\mathcal{I}$, *and* $(M_\mathcal{U}, N_\mathcal{V}) \in r_\mathrm{s}^\mathcal{I}$, ***then*** *add* $N_\mathcal{V}$ *to* $B^\mathcal{J}$.

(R$\forall$C$^-$) ***If*** $A \sqsubseteq \forall r^-.B \in \overline{\mathcal{K}}$, $M_\mathcal{U} \in A^\mathcal{I}$, $N_\mathcal{V} \notin B^\mathcal{I}$, *and* $(N_\mathcal{V}, M_\mathcal{U}) \in r_\mathrm{p}^\mathcal{I}$, ***then*** *add* $N_\mathcal{V}$ *to* $B^\mathcal{J}$.

We illustrate the different effects of the fix rules for violated value restrictions.

**Example 7.30** (Application of fix rules)**.** *Let* $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ *be an updated interpretation for a DKB* $\mathcal{K}$ *and* $M_\mathcal{U}$ *be the root of the update. Let* $A \sqsubseteq \forall r^-.B, B \sqsubseteq \forall r.C \in \overline{\mathcal{K}}$. *Figure 7.4 shows two subsequent applications of the fix rules: first* (R$\forall$C$^-$) *results in* $\mathcal{I}_1$ *and the application of* (R$\forall$R) *results in* $\mathcal{I}_2$.

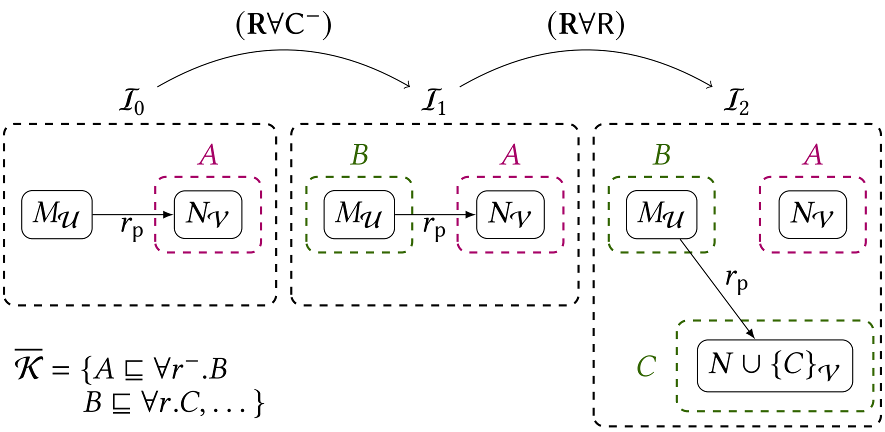


**Figure 7.4:** *Diagrams representing a series of three interpretations.* $\mathcal{I}_{i+1}$ *is a fix of* $\mathcal{I}_i$.

Each fix rule addresses a single violation, but may also create new ones. Thus these rules are applied exhaustively to an updated interpretation and the final result is a *model recovery*.

**Definition 7.31** (Model Recovery)**.** *Let* $(\mathcal{I}', \mathsf{Init}'_\mathcal{I})$ *be a typicality model of some DKB* $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ *and its initiator labeling and let* $(\mathcal{I}, \mathsf{Init}_\mathcal{I}) \in \mathsf{Upd}_\mathcal{K}(\mathcal{I}', \mathsf{Init}_{\mathcal{I}'})$ *with* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$. *The pair* $(\mathcal{J}, \mathsf{Init}_\mathcal{J})$ *of an interpretation and its initiator labeling is a* model recovery *of* $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ *iff it is the result of applying the fix rules to* $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ *exhaustively and does not contain a clash. The set of all model recoveries for* $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ *w.r.t.* $\mathcal{K}$ *is denoted by:*

$$\mathsf{ModRec}_\mathcal{K}(\mathcal{I}, \mathsf{Init}_\mathcal{I}) := \{(\mathcal{J}, \mathsf{Init}_\mathcal{J}) \mid \mathcal{J} \text{ is a model recovery of } (\mathcal{I}, \mathsf{Init}_\mathcal{I}) \text{ w.r.t.} \mathcal{K}\}$$

A model recovery is indeed a model.

**Lemma 7.32.** *Let* $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ *be a pair of an updated typicality model and its initiator labeling. Then,* $\mathcal{J} \models \mathcal{K}$ *for all* $(\mathcal{J}, \mathsf{Init}_\mathcal{J}) \in \mathsf{ModRec}_\mathcal{K}(\mathcal{I}, \mathsf{Init}_\mathcal{I})$.

*Proof.* Since $\mathcal{K} \subseteq \overline{\mathcal{K}}$, any axiom from $\mathcal{K}$ that may be violated is contained in $\overline{\mathcal{K}}$. We consider the four GCI types in a normalized DKB:

$$A \sqsubseteq B, \quad A_1 \sqcap A_2 \sqsubseteq B, \quad A \sqsubseteq \exists r.B, \quad A \sqsubseteq \forall r.B\ .$$

For the first three, any $M_\mathcal{U} \in A^\mathcal{J}$ (or $(A_1 \sqcap A_2)^\mathcal{J}$) is included in $B^\mathcal{J}$ or $(\exists r.B)^\mathcal{J}$, because the rules (**R**$\sqsubseteq$), (**R**$\sqcap$), (**R**$\exists$), and (**R**$\exists^-$) have been applied exhaustively. If $B = \bot$ for any of these GCIs, then $\mathcal{J}$ is not a model recovery because $\mathcal{J}$ would contain a (direct) clash.

Now, we consider axioms of the form $A \sqsubseteq \forall r.B$. For every $(M_\mathcal{U}, N_\mathcal{V}) \in r^\mathcal{J}$, the labeling $\mathsf{Init}_\mathcal{J}(M_\mathcal{U}, r, N_\mathcal{V})$ is non-empty by definition. If $(M_\mathcal{U}, N_\mathcal{V}) \in r_\mathsf{p}^\mathcal{J}$ and $M_\mathcal{U} \in A^\mathcal{J}$, the violation can trigger (**R**$\forall$R) in case $N \cup \{B\}_\mathcal{V} \in \Delta^\mathcal{J}$. If $N \cup \{B\}_\mathcal{V} \notin \Delta^\mathcal{J}$, then $\mathcal{J}$ contains a successor domain clash and $\mathcal{J}$ is not a model recovery. In the case where $(M_\mathcal{U}, N_\mathcal{V}) \in r_\mathsf{s}^\mathcal{J}$ and $M_\mathcal{U} \in A^\mathcal{J}$, then $N_\mathcal{V}$ is added to $B^\mathcal{J}$ by rule (**R**$\forall$C).

Note that, in the dual case where $r \in \mathsf{N_R}^-$, the argument is the same, except the rules applied are (**R**$\forall$R$^-$) and (**R**$\forall$C$^-$) and the possible clash is a predecessor instead of a successor domain clash. □

Besides being a model, a model recovery preserves the property of canonical models, i.e. subsumption can be read-off from concept membership and role edges represent prime existential restrictions for at least one of its elements.

There may be several ways of fixing an interpretation because moving edges can inadvertently solve more than one violation at a time. This effect occurs due to the non-monotonic nature of defeasible information. In a monotonic context, all the axioms that hold for the representative of some $M$ also have to hold for representatives of $M'$ s.t. $M \subset M'$. However, this property does not hold for defeasible axioms. In some cases, defeasible axioms may hold only for *less* specific elements, e.g., birds fly, while penguins, which are also birds, do not. Therefore, moving an edge from $M$ to $M'$ can alleviate violations triggered by concepts to which $M$ belongs, but $M'$ does not. The order of the fixes can impact the final product, as different orders may have to address different problems, leading to distinct outcomes. We consider the following example:

**Example 7.33** (Multiple model recoveries)**.** *Let* $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ *be an updated typicality interpretation and its initiator labeling, and* $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ *be a DKB. Let* $M_\mathcal{U}$ *be the root of the update. The normalized DKB* $\overline{\mathcal{K}}$ *contains the following axioms:*

$$A_1 \sqsubseteq \forall r^-.A_2, \quad A_2 \sqsubseteq \forall r.C, \quad B_1 \sqsubseteq \forall r^-.B_2, \quad B_2 \sqsubseteq \forall r.D.$$

*Consider the fragment of* $\mathcal{I}$ *depicted by Figure 7.5. The initial configuration presents two axiom violations:* ***Ax1*** *and* ***Ax4****. They are solved by adding* $M_\mathcal{U}$ *to* $A_2$ *and* $B_2$*, respectively. The first fix creates a new violation of* ***Ax2****, and the second one creates a violation of* ***Ax3****. However, it is important to notice that solving* ***Ax2*** *before addressing* ***Ax4*** *will eliminate the violation of* ***Ax4****, as the solution is moving the* $r$ *edges from* $\{A\}_{\mathcal{U}_1}$ *and* $\{B\}_{\mathcal{U}_2}$ *to* $\{A, C\}_{\mathcal{U}_1}$

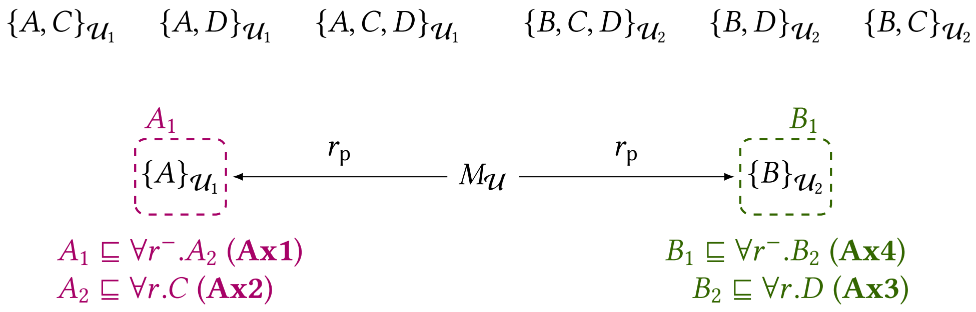


**Figure 7.5:** *Initial configuration of an interpretation to be recovered. At this stage, it violates two axioms:* ***Ax1*** *and* ***Ax2****. Solving any of them will yield new violations, and the order in which those violations are solved results in a different outcome.*

*and $\{B, C\}_{\mathcal{U}_1}$, and those elements do not belong to $B_1$. Similarly, solving* ***Ax3*** *before* ***Ax1*** *will eliminate the violation of the later axiom.*

*Hence, according to the definition, this configuration has three distinct model recoveries, depending on the order of the fixes. The three orders are given by solving the violations in the following orders:*

1. ***Ax1****,* ***Ax2****.*
2. ***Ax4****,* ***Ax3****.*
3. ***Ax1****,* ***Ax4*** *and $\{$**Ax2**, **Ax3**$\}$ in any order.*

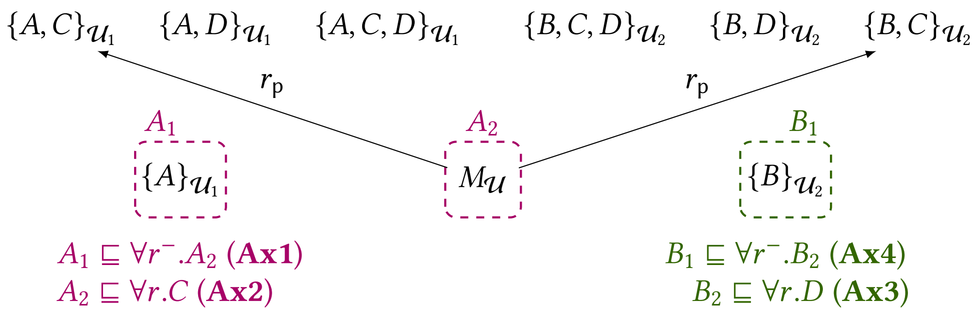


**Figure 7.6:** *A possible outcome of the model recovery. In this case, the order of the fixes is* ***Ax1****,* ***Ax2****.*

Now, we show two important features of the model recovery: (1) it preserves a fundamental property of canonical models for $\mathcal{ELI}_\bot$ and (2) that it only increases the $\mathsf{N_C}$-types of the elements and does so only to elements in the dependency set of the root. The rest of the elements remain intact.

The first property is important because it guarantees that the structure of the model aligns concept and role membership with subsumption. In the minimal typicality model, concept membership for named concepts is equivalent to subsumption w.r.t. the concepts and the DKB, and each edge is required and prime w.r.t. to one of its elements. Besides, each prime existential restriction required by the DKB has a corresponding edge in the model. Property 2. ensures that the procedure does not throw away membership (and therefore subsumption) in order to fix the model, and also that changes affect only the updated element and elements connected to it.

**Definition 7.34** (Quasi-canonicty). *Let $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ be a typicality interpretation and its initiator labeling. Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB. The pair $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ is* quasi-canonical *w.r.t. $\mathcal{K}$ iff the following conditions hold:*

1. *If $\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_\mathcal{K}(M_\mathcal{U}, \mathcal{I})\rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.\lceil N\rceil$ for a prime $r$-successor $N$, then $\exists N_\mathcal{V} \in \Delta^\mathcal{I}$ s.t.*
   - *$(M_\mathcal{U}, N_\mathcal{V}) \in r_\mathsf{p}^\mathcal{I}$, if $r \in \mathsf{N_R}$ and*
   - *$(N_\mathcal{V}, M_\mathcal{U}) \in r^{-\mathcal{I}}_\mathsf{s}$, if $r \in \mathsf{N_R}^-$.*
2. *If $(M_\mathcal{U}, N_\mathcal{V}) \in r_\mathsf{l}^\mathcal{I}$, for some non-empty $\mathsf{l} \subseteq \{\mathsf{p}, \mathsf{s}\}$, then*
   - *$\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_\mathcal{K}(M_\mathcal{U}, \mathcal{I})\rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.\lceil N\rceil$ and $N$ is a prime $r$-successor of $M_\mathcal{U}$, if $\mathsf{p} \in \mathsf{l}$ and*
   - *$\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_\mathcal{K}(N_\mathcal{V}, \mathcal{I})\rceil \sqcap \overline{\mathcal{V}} \sqsubseteq \exists r^-.\lceil M\rceil$ and $M$ is a prime $r^-$-successor of $N_\mathcal{V}$, if $\mathsf{s} \in \mathsf{l}$.*

Now, we show that quasi-canonicity holds in model recoveries.

**Lemma 7.35.** *Let $(\mathcal{J}, \mathsf{Init}_\mathcal{J})$ be a model recovery of $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ w.r.t. the DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ and let $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ be quasi-canonical. Then, $(\mathcal{J}, \mathsf{Init}_\mathcal{J})$ is quasi-canonical.*

*Proof.* We start with property (1). Without loss of generality, we consider only the case in which $r \in \mathsf{N_R}$. The proof is symmetrical for inverse roles. Let $\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_\mathcal{K}(M_\mathcal{U}, \mathcal{J})\rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.\lceil N\rceil$, for a prime $r$-successor $N$, for some $M_\mathcal{U} \in \Delta^\mathcal{I}$. There are two cases to examine: (i) if there is some edge $(M_\mathcal{U}, N'_\mathcal{V}) \in r_\mathsf{p}^\mathcal{I}$ s.t. $N' \subset N$, or (ii) if there is no edge like this.

In the first case, because $\mathcal{J}$ is a model recovery of $\mathcal{I}$, $(M_\mathcal{U}, N_\mathcal{V}) \in r_\mathsf{p}^\mathcal{J}$. Otherwise, the fix rule (**R**∀R) would be applicable to it. In the second case, $(M_\mathcal{U}, N_\emptyset) \in r_\mathsf{p}^\mathcal{J}$, because the fix rule (**R**∃) would be applicable to $\mathcal{J}$ for some $B' \in N$ s.t. $\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_\mathcal{K}(M_\mathcal{U}, \mathcal{I})\rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.B'$. Then, every $C \in N$, $C \neq B'$, would lead to one application of (**R**∀R), resulting in the edge $(M_\mathcal{U}, N_\emptyset) \in r_\mathsf{p}^\mathcal{J}$. If $r \in \mathsf{N_R}^-$, then the argument is the same, but the rules are (**R**∀R$^-$) and (**R**∃$^-$).

For property (2), suppose $(M_\mathcal{U}, N_\mathcal{V}) \in r_\mathsf{p}^\mathcal{J}$. We consider two scenarios. The first is when the same edge is also in $\mathcal{I}$, and the second is when it is new.

If $(M_\mathcal{U}, N_\mathcal{V}) \in r_\mathsf{p}^\mathcal{I}$, then $\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_\mathcal{K}(M_\mathcal{U}, \mathcal{I})\rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.\lceil N\rceil$ and $N$ is a prime $r$-successor. If $N$ were not a prime $r$-successor for $\lceil \mathsf{N_C}\text{-type}_\mathcal{K}(M_\mathcal{U}, \mathcal{J})\rceil \sqcap \overline{\mathcal{U}}$ – i.e., if there was some $N' \supset N$ s.t. $\overline{\mathcal{K}} \models \lceil \mathsf{N_C}\text{-type}_\mathcal{K}(M_\mathcal{U}, \mathcal{I})\rceil \sqcap \overline{\mathcal{U}} \sqsubseteq \exists r.\lceil N'\rceil$ – then the rule (**R**∀R) would be applicable to $\mathcal{J}$, which would not be a model recovery.

On the other hand, if $(M_\mathcal{U}, N_\mathcal{V}) \notin r_\mathsf{p}^\mathcal{I}$, the fix rules added the edge. Let $R_0, \dots, R_n$ be the rules applied to $\mathcal{I}$ generate $\mathcal{J}$. Consider the rule $R_i$ s.t. $\mathcal{I}$ before its application is $(M_\mathcal{U}, N_\mathcal{V}) \notin r_\mathsf{p}^\mathcal{I}$, but after its application is $(M_\mathcal{U}, N_\mathcal{V}) \in r_\mathsf{p}^\mathcal{I}$. Two rules can add edges:

(**R**∃) and (**R**∀R). If (**R**∃) was applied with the axiom $A \sqsubseteq \exists r.B$, and $\{B\} \subset N$, then there would be a further application of (**R**∀R) rules moving the exge $(M_{\mathcal{U}}, \{B\}_{\emptyset}) \in r_{\mathrm{p}}^{\mathcal{I}}$ to $(M_{\mathcal{U}}, \{B \cup \{B'\}\}_{\emptyset}) \in r_{\mathrm{p}}^{\mathcal{I}}$, for every axiom $A \sqsubseteq \forall r.B'$. When there is no more (**R**∀R) left to apply, the resulting edge $(M_{\mathcal{U}}, N_{\emptyset}) \in r_{\mathrm{p}}^{\mathcal{J}}$ will be a prime $r$-successor. Note that, in this case, $\mathcal{V} = \emptyset$.

If, on the other hand, the edge is introduced by (**R**∀R), we note that (i) the edge $(M_{\mathcal{U}}, N \setminus \{B\}_{\mathcal{V}}) \in r_{\mathrm{p}}^{\mathcal{I}}$ that triggered the rule is required, but $N \setminus \{B\}$ is not a prime $r$-successor, as there is an axiom $A \sqsubseteq \forall r.B$ triggering the rule. Then, either $N$ is a prime successor, or there is another axiom $A' \sqsubseteq \forall r.B'$ that triggers the rule again. This cannot be the case, as no rule is applicable to $\mathcal{J}$, and therefore $N$ is a prime $r$-successor. □

We have shown that the model recovery preserves the canonical structure of the edges, namely, that they represent all required prime successors by edges. Now, we show that the changes realized to recover the model property after a typicality update affect primarily the root of the update and, secondarily, the elements that are causally connected to the root. No other element is affected by the recovery.

**Lemma 7.36.** *Let $(\mathcal{J}, \mathsf{Init}_{\mathcal{J}})$ be a model recovery of $(\mathcal{I}, \mathsf{Init}_{\mathcal{I}})$ w.r.t. the DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$. Let $M_{\mathcal{U}}$ be the root of the recovery and $\mathsf{DS}(M_{\mathcal{U}})$ be its dependency set. Then:*

1. $\mathsf{N_C}\text{-type}_{\mathcal{K}}(N_{\mathcal{V}}, \mathcal{I}) \subseteq \mathsf{N_C}\text{-type}_{\mathcal{K}}(N_{\mathcal{V}}, \mathcal{J})$ *for every* $N_{\mathcal{V}} \in \mathsf{DS}(M_{\mathcal{U}})$*;*
2. $\mathsf{N_C}\text{-type}_{\mathcal{K}}(N_{\mathcal{V}}, \mathcal{I}) = \mathsf{N_C}\text{-type}_{\mathcal{K}}(N_{\mathcal{V}}, \mathcal{J})$ *for every* $N_{\mathcal{V}} \notin \mathsf{DS}(M_{\mathcal{U}})$*.*

*Proof.* We prove that the properties hold by examining the series of fixes leading to the recovered model $\mathcal{J}$. Let $R_0, \ldots, R_n$ be the set of rules leading from $\mathcal{I}$ to $\mathcal{J}$. We denote the series of intermediate labeled interpretations by $(\mathcal{I}, \mathsf{Init}_{\mathcal{I}}) = (\mathcal{I}_0, \mathsf{Init}_{\mathcal{I}_0}), \ldots, (\mathcal{I}_n, \mathsf{Init}_{\mathcal{I}_n}) = (\mathcal{J}, \mathsf{Init}_{\mathcal{J}})$. Every $(\mathcal{I}_i, \mathsf{Init}_{\mathcal{I}_i})$ results from the application of the fix rule $R_{i-1}$ to $(\mathcal{I}_{i-1}, \mathsf{Init}_{\mathcal{I}_{i-1}})$, for $1 \leq i \leq n$.

First, notice that no fix rule removes elements from the extension of a named concept. Therefore, $\mathsf{N_C}$-types can only increase. We prove by induction on $i$ that the eventual increase satisfies properties (1) and (2).

For the base case, consider that the only fix rules applicable to $(\mathcal{I}_0, \mathsf{Init}_{\mathcal{I}_0})$ are (**R**∀C) and (**R**∀C$^-$), which increase the $\mathsf{N_C}$-type of the root, i.e., add it to the extension of some concept. That is because the only difference between $\mathcal{I}$ and the model whose update gave rise to it is the new edge initiated by the root. All the other fixes are triggered by membership to a named concept $A$ (or, similarly, to two named concepts $A_1$, $A_2$). No violation of this kind can exists in $\mathcal{I}$ because concept extention is unaltered in the update and the original interpretation that gave rise to $(\mathcal{I}, \mathsf{Init}_{\mathcal{I}})$ through the update is a model.

For the inductive step, consider that properties (1) and (2) hold for $\mathcal{I}_i$. Hence, the only $\mathsf{N_C}$-type increases w.r.t. previous interpretations are in $\mathsf{DS}(M_{\mathcal{U}})$. The first two rules that increase the $\mathsf{N_C}$-type of a given element – (**R**⊑) and (**R**⊓) – can only be applied to elements in $\mathsf{DS}(M_{\mathcal{U}})$, as they are themselves triggered by the increase in the $\mathsf{N_C}$-type of their target element. The remaining rules that increase the $\mathsf{N_C}$-type of an element are (**R**∀C) and (**R**∀C$^-$). Notice, however, that those rules are triggered by the increase of the $\mathsf{N_C}$-type of the element on the other endpoint of an edge. For some element $N_{1\mathcal{U}_1}$ to be affected by such rules, it should be connected by an edge to some $N_{2\mathcal{U}_2}$ that had its $\mathsf{N_C}$-type increased. This connection should be either $(N_{1\mathcal{U}_1}, N_{2\mathcal{U}_2}) \in r_\mathrm{p}^{\mathcal{I}_i}$ or $(N_{2\mathcal{U}_2}, N_{1\mathcal{U}_1}) \in r_\mathrm{s}^{\mathcal{I}_i}$. By hypothesis, $N_{2\mathcal{U}_2}$ belongs to $\mathsf{DS}(M_\mathcal{U})$, as it $\mathsf{N_C}$-type increased. Then, the roles that could trigger the rule imply that $N_{1\mathcal{U}_1} \in \mathsf{DS}(M_\mathcal{U})$ as well. □

A full upgrade step has two parts: interpretation update and model recovery. The initial input is a pair $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$. The procedure computes all the possible update candidates for this pair and generates a set containing all possible updates. Model recovery is applied to each one of these interpretations. Notice that, for a single labeled interpretation, there may be

- more than one model recovery; or
- no model recovery, if every possible sequence of fix rule's application to a clash.

In the latter case, the procedure outputs the input model, i.e., the update is ignored. The final product is a set $(\mathcal{J}_1, \mathsf{Init}_{\mathcal{J}_1}), \ldots, (\mathcal{J}_n, \mathsf{Init}_{\mathcal{J}_n})$ of labeled models extending $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$. With a slight abuse of notation, we define this procedure over sets of interpretations instead of single interpretations, enabling iteration.

**Definition 7.37** (Full upgrade step). *Let $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ be a labeled typicality model of the DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$. A* full upgrade step *over $(\mathcal{I}, \mathsf{Init}_\mathcal{I})$ and $\mathcal{K}$ is defined by:*

$$\mathsf{TU}(\mathcal{I}, \mathsf{Init}_\mathcal{I}, \mathcal{K}) = \bigcup_{(\mathcal{J}, \mathsf{Init}_\mathcal{J}) \in \mathsf{Upd}_\mathcal{K}(\mathcal{I}, \mathsf{Init}_\mathcal{I})} \mathsf{ModRec}_\mathcal{K}(\mathcal{J}, \mathsf{Init}_\mathcal{J})$$

*With a slight abuse of notation, we define the full upgrade step to a set $S = \{(\mathcal{I}_0, \mathsf{Init}_{\mathcal{I}_0}), \ldots, (\mathcal{I}_n, \mathsf{Init}_{\mathcal{I}_n})\}$ of interpretations and their initiator labelings by*

$$\mathsf{TU}(S, \mathcal{K}) = \bigcup_{i=0}^{n} \mathsf{TU}(\mathcal{I}_i, \mathsf{Init}_{\mathcal{I}_i}, \mathcal{K})$$

**Example 7.38.** *Let $\mathcal{K}$ be a DKB and $\mathcal{I}$ be an arbitrary typicality model for $\mathcal{K}$. Figure 6.7 depicts the pipeline that characterizes a full typicality upgrade step over $\mathcal{I}$, i.e. a single application of $\mathsf{TU}(\mathcal{I}, \mathcal{K})$.*

*The pipeline is similar to $\mathcal{EL}_\bot$'s pipeline. The main differences are that (i) it operates over labeled interpretations, (ii) the individual steps (update and recovery) are different, although defined in the same spirit, and (iii) the recovery procedure also branches.*

We show that iterating the upgrade steps terminates in a fixpoint when the initial input is a minimal typicality model.

**Theorem 7.39** (Termination of the upgrade steps). *Let $(\mathcal{I}^\mathcal{K}_{\mathsf{min}}, \mathsf{Init}_{\mathcal{I}^\mathcal{K}_{\mathsf{min}}})$ be the minimal typicality model over some typicality domain and its initiator labeling for a DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$. Iterating the $\mathsf{TU}$ operator reaches a fixpoint denoted by $\mathsf{TU}_{\max}(\mathcal{I}, \mathsf{Init}_\mathcal{I}, \mathcal{K})$.*

*Proof.* First, we notice that when adding an edge does not break the model property, the resulting interpretation is the only model recovery of itself. Notice that the minimal

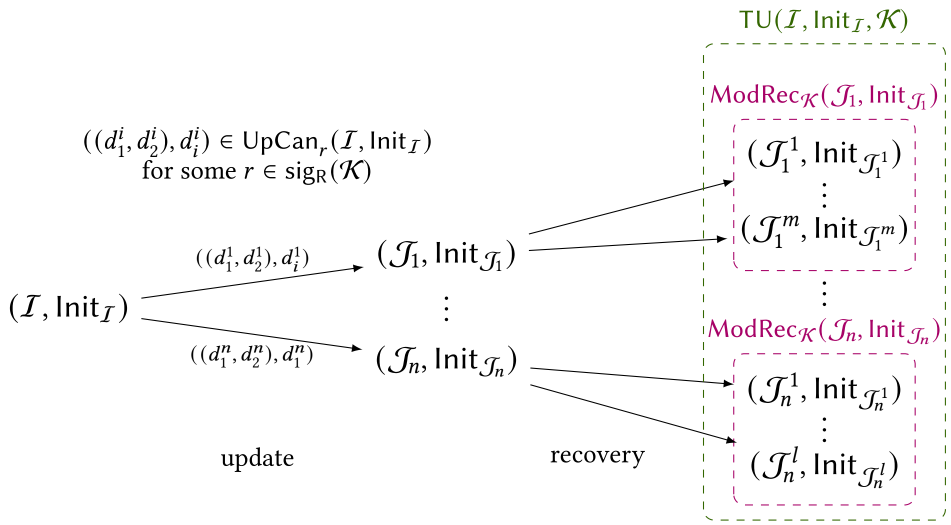


**Figure 7.7:** *Diagram representing the pipeline of a full typicality upgrade step with a labeled typicality model* $(\mathcal{I}, \mathsf{Init}_{\mathcal{I}})$ *for a DKB* $\mathcal{K}$ *as inputs.*

typicality interpretation satisfies all the model recovery requirements, and every model recovery does the same. Hence, by adding an edge, the only rules that can be triggered are (**R**∀C) and (**R**∀C$^-$), which are only triggered when the interpretation is *not* a model. Therefore, the upgrade step returns the same set if no updates exist.

Now, we should establish that adding edges comes to a halt. This is not obvious *prima facie* because the procedure removes edges besides adding them. Therefore, it is necessary to show that no endless loops add and remove the same edges indefinitely.

Partial orders can be defined by the subset relation over the two sets (concept and typicality) that make up the elements, and both orders have a single supremum. The concept set goes from singletons to the collection of all names in the signature. The typicality set starts from the empty set and moves up to $\mathcal{D}$. The granularity of the middle elements depends on the strength of the domain.

It is possible to visualize a role edge starting from an element $M_{\mathcal{U}}$ as a connection between the element itself and a set $N$ of named concepts supplemented with a series of sets of defeasible inclusions $\emptyset, \mathcal{U}_1, \ldots, \mathcal{U}_n$. Notice that the update only adds a new $\mathcal{U}_i$ to this list, which is finite. For example, if $(M_{\mathcal{U}}, N_{\emptyset}) \in r_{\mathsf{p}}^{\mathcal{I}}$, a possible update is $(M_{\mathcal{U}}, N_{\mathcal{U}}) \in r_{\mathsf{p}}^{\mathcal{I}}$, for some $\mathcal{U} \subseteq \mathcal{D}$, $\mathcal{U} \neq \emptyset$. The original edge to the atypical successor remains there.

The swapping of edges that can occur during the model recovery moves this edge upwards on the partial order over the concept set, as it is motivated by primeness. Hence, it blocks all lesser elements from being added in the future and carries all the already present components from the typicality set. Continuing the example, if the $N'$ is the new prime successor, with $N \subset N'$, then the edges $(M_{\mathcal{U}}, N_{\emptyset})$ and $(M_{\mathcal{U}}, N_{\mathcal{U}})$ are removed from $r_{\mathsf{p}}^{\mathcal{I}}$, and new edges $(M_{\mathcal{U}}, N'_{\emptyset})$ and $(M_{\mathcal{U}}, N'_{\mathcal{U}})$ are added.

Both stages of the upgrade – the update and the model recovery – have hard limits. For the update, saturation occurs when every $\mathcal{U} \subseteq \mathcal{D}$ is added to a certain role. For the model recovery, it occurs when the prime successor is $\mathsf{sig}_{\mathsf{C}}(\mathcal{K})$. At each step, the updated role moves over the partial order towards these hard limits, guaranteeing termination. The diagram in Figure 7.8 illustrates the behavior of the upgrade. □

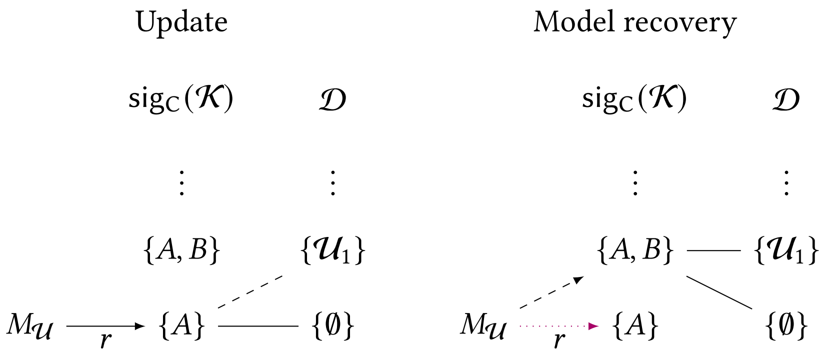


**Figure 7.8:** *Diagram representing the two steps of an upgrade with $M_\mathcal{U}$ as the root. The elements were broken down into two columns. The dashed lines represent changes during each step, and the colored dotted edge symbolizes an edge being removed. During update stage, the procedure saturates the second component of the successor. In this case, it adds the edge to $\{A\}_{\mathcal{U}_1}$. On the model recovery, it moves the edges upwards in the first column. In this example, it moves the edges pointing to elements with $\{A\}$ as its first component to $\{A, B\}$.*

## 7.4 Nested Reasoning

The minimal typicality model highlights that materialization-based reasoning does not generally push information through roles. Hence, all required successors are atypical. The upgrade procedure we presented intends to remedy this by upgrading edges when possible, and possibility is bounded by consistency and adequacy to the quasi-canonical behavior of concept and role memberships.

The intuition backing *nested reasoning* is that a typicality upgrade for a role successor should be done except when it cannot be made compatible with the rest of the typicality model. Formally, this reasoning coverage is defined by the reasoning emerging from a preferred set of models – those in $\mathsf{TU}_{\max}(\mathcal{I}^\mathcal{K}_{\min}, \mathsf{Init}_{\mathcal{I}^\mathcal{K}_{\min}}, \mathcal{K})$, for the minimal typicality model $\mathcal{I}^\mathcal{K}_{\min}$ and some strength $\mathsf{s}$ over the DKB $\mathcal{K}$. Limiting the considered models to those ensure that every role successor has been made as typical as possible.

**Definition 7.40** (Nested Reasoning)**.** *Let $\mathcal{K}$ be a DKB; $A \in \mathsf{sig}_\mathsf{C}(\mathcal{K})$ and $M \in \Delta^\mathcal{K}$ be a set of named concepts in the representative domain. Let $\mathsf{s}$ be a strength and $(\mathcal{I}^\mathcal{K}_{\min}, \mathsf{Init}_{\mathcal{I}^\mathcal{K}_{\min}})$ the labeled minimal typicality model for $\mathsf{s}$. We define* $\mathsf{s}$ nested entailment *by:*

$$\mathcal{K} \models_{\mathsf{nest,s}} \lceil M \rceil \sqsubseteq A \textit{ iff } \quad \lceil M \rceil^\mathcal{I} \subseteq A^\mathcal{I}, \textit{for every } \mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^\mathcal{K}_{\min}, \mathsf{Init}_{\mathcal{I}^\mathcal{K}_{\min}}, \mathcal{K})$$

$$\mathcal{K} \models_{\mathsf{nest,s}} \lceil M \rceil \mathrel{\substack{\sqsubset\\\sim}} A \textit{ iff } \quad M_\mathcal{U} \in A^\mathcal{I} \textit{ for every maximally typical instance of } M \textit{ in the domain, for every } \mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^\mathcal{K}_{\min}, \mathsf{Init}_{\mathcal{I}^\mathcal{K}_{\min}}, \mathcal{K})$$

Nested reasoning presents some interesting properties. It extends propositional defeasible reasoning, preserving all its entailments. For the strict part, both the propositional and the nested coverages are equivalent to the standard entailment, making them supra-classical.

**Lemma 7.41.** *Let* $\mathcal{K}$ *be a DKB and* $\mathsf{s}$ *be a strength,* $A \in \mathsf{sig}_\mathsf{C}(\mathcal{K})$ *and* $M \in \Delta^{\mathcal{K}}$ *be a set of named concepts in the representative domain. Then:*

1. $\mathcal{K} \models_{\mathsf{prop,s}} \lceil M \rceil \precsim A \Rightarrow \mathcal{K} \models_{\mathsf{nest,s}} \lceil M \rceil \precsim A$;

2. $\mathcal{K} \models \lceil M \rceil \sqsubseteq A \Leftrightarrow \mathcal{K} \models_{\mathsf{prop,s}} \lceil M \rceil \sqsubseteq A \Leftrightarrow \mathcal{K} \models_{\mathsf{nest,s}} \lceil M \rceil \sqsubseteq A$.

*Proof.* *(1).* The maximally typical instances of any $M \in \Delta^{\mathcal{K}}$ are the same in the minimal typicality model and any extension emerging from the upgrade procedure, as they share the same domain. To establish the claim, it suffices to notice that (i) defeasible subsumption is established by membership w.r.t. named concepts of the maximally typical representatives, and (ii) the upgrade procedure only increases the $\mathsf{N_C}$-type of the elements. Hence, if $M_{\mathcal{U}} \in A^{\mathcal{I}^{\mathcal{K}}_{\min}}$, $M_{\mathcal{U}} \in A^{\mathcal{I}}$, for any $\mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}, \mathsf{Init}_{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}, \mathcal{K})$.

*(2).* We show the property by proving *monotone reasoning* $\Rightarrow$ *propositional typicality-models based reasoning* $\Rightarrow$ *nested typicality-models based reasoning* $\Rightarrow$ *monotone reasoning*. We consider a generic strength $\mathsf{s}$.

Suppose that $\mathcal{K} \models \lceil M \rceil \sqsubseteq A$. For every element $N_{\mathcal{V}} \in \Delta^{\mathcal{I}^{\mathsf{s}}_{\min}}$ s.t. $N_{\mathcal{V}} \in \lceil M \rceil^{\mathcal{I}^{\mathsf{s}}_{\min}}$, $\overline{\mathcal{K}} \models \lceil N \rceil \sqcap \overline{\mathcal{V}} \sqsubseteq \lceil M \rceil$, and therefore $\overline{\mathcal{K}} \models \lceil N \rceil \sqcap \overline{\mathcal{V}} \sqsubseteq A$. Hence, $N_{\mathcal{V}} \in A^{\mathcal{I}^{\mathsf{s}}_{\min}}$, establishing the first claim.

For the second, notice that the $\mathsf{N_C}$-types of the elements only increase during the upgrade procedure. Therefore, a counterexample to the strict subsumption could only arise from an element added to the extension of $\lceil M \rceil$ during the upgrade procedure. Let $N_{\mathcal{V}}$ be such an element. However, every $\mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}, \mathsf{Init}_{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}, \mathcal{K})$ is a model of $\overline{\mathcal{K}}$, and $\overline{\mathcal{K}} \models \lceil M \rceil \sqsubseteq A$. Hence, $N_{\mathcal{V}} \in A^{\mathcal{I}}$, for every $\mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}, \mathsf{Init}_{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}, \mathcal{K})$.

For the last part, notice that the atypical representative of $M$, $M_\emptyset$, is defined for the minimal typicality model as only belonging to the concepts $A$ s.t. $\overline{\mathcal{K}} \models \lceil M \rceil \sqsubseteq A$, and it is not upgraded during the upgrade procedure. Hence, for every $\mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}, \mathsf{Init}_{\mathcal{I}^{\mathcal{K}}_{\mathsf{min}}}, \mathcal{K})$, $\mathsf{N_C}\text{-type}_{\mathcal{K}}(M_\emptyset, \mathcal{I}) = \mathsf{N_C}\text{-type}_{\mathcal{K}}(M_\emptyset, \mathcal{I}^{\mathsf{s}}_{\mathsf{min}})$. Therefore, $M_\emptyset$ is the minimal member of the concept $\lceil M \rceil$, and $M_\emptyset \in A^{\mathcal{I}}$ *iff* $\mathcal{K} \models \lceil M \rceil \sqsubseteq A$, completing the proof.

Notice also that typicality models allow checking strict subsumptions $\lceil M \rceil \sqsubseteq A$ by looking at concept membership of $M$ atypical instances, i.e. $M_\emptyset$. □

To put nested reasoning in perspective, we delve into the domain shapes explored in this dissertation. We show that each of the three strengths – rational, relevant, and lexicographic – extend their propositional counterpart, dealing with quantification neglect. Then, we compare nested reasoning for each of the strengths in the same spirit as the analogous comparison for $\mathcal{EL}_\bot$ in Section 6.3.5.

### 7.4.1 Nested Rational Reasoning

**Definition 7.42** (Nested Rational Reasoning)**.** *Let $\mathcal{K}$ be a DKB; $A \in \mathsf{sig}_\mathsf{C}(\mathcal{K})$ and $M \in \Delta^\mathcal{K}$ be a set of named concepts in the representative domain. Let $(\mathcal{I}^\mathcal{K}_{\mathsf{min,rat}}, \mathsf{Init}_{\mathcal{I}^\mathcal{K}_{\mathsf{min,rat}}})$ be the labeled minimal typicality domain for* **rat**. *We define* rational nested entailment *by:*

$$\mathcal{K} \models_{\mathsf{nest,rat}} \lceil M \rceil \sqsubseteq A \textit{ iff } \quad \lceil M \rceil^\mathcal{I} \subseteq A^\mathcal{I}, \textit{for every } \mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^\mathcal{K}_{\mathsf{min,rat}}, \mathsf{Init}_{\mathcal{I}^\mathcal{K}_{\mathsf{min,rat}}}, \mathcal{K})$$

$$\mathcal{K} \models_{\mathsf{nest,rat}} \lceil M \rceil \mathrel{\underset{\sim}{\sqsubset}} A \textit{ iff } \quad M_\mathcal{E} \in A^\mathcal{I} \textit{ for every maximally typical instance of } M \textit{ in the domain}, \forall \mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^\mathcal{K}_{\mathsf{min,rat}}, \mathsf{Init}_{\mathcal{I}^\mathcal{K}_{\mathsf{min,rat}}}, \mathcal{K})$$

**Example 7.43.** *Consider the DKB described in Example 7.15. Suppose we want to test whether $\mathcal{K} \models_{\mathsf{nest,rat}} \mathsf{Human} \mathrel{\underset{\sim}{\sqsubset}} \exists \mathsf{worships.Immortal}$. First, we introduce an auxiliary concept Aux and the GCI $\exists \mathsf{worships.Immortal} \sqsubseteq Aux$ to the TBox. Notice that this GCI is equivalent to $\mathsf{Immortal} \sqsubseteq \forall \mathsf{worships}^-.Aux$. We test whether $\mathcal{K} \models_{\mathsf{nest,rat}} \mathsf{Human} \mathrel{\underset{\sim}{\sqsubset}} Aux$. The most typical instance of Human is $\{\mathsf{Human}\}_{\mathcal{E}_0}$, for $\mathcal{E}_0 = \mathcal{D}$.*

*In the minimal typicality model, $(\{\mathsf{Human}\}_{\mathcal{E}_0}, \{\mathsf{Deity}\}_\emptyset) \in \mathsf{worships}^{\mathcal{I}^\mathcal{K}_{\mathsf{min,rat}}}$ and $\{\mathsf{Human}\}_{\mathcal{E}_0} \notin Aux^{\mathcal{I}^\mathcal{K}_{\mathsf{min,rat}}}$. However, this edge can be upgraded to end in $\{\mathsf{Deity}\}_{\mathcal{E}_1}$, and this upgrade does not interfere with any other possible upgrades, i.e. it is present in all upgrade paths. The model recovery adds $\{\mathsf{Human}\}_{\mathcal{E}_0}$ to Aux, and therefore, for every $\mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^\mathcal{K}_{\mathsf{min}}, \mathsf{Init}_{\mathcal{I}^\mathcal{K}_{\mathsf{min}}}, \mathcal{K})$, $\{\mathsf{Human}\}_{\mathcal{E}_0} \in Aux^\mathcal{I}$, as illustrated in Figure 7.9.*

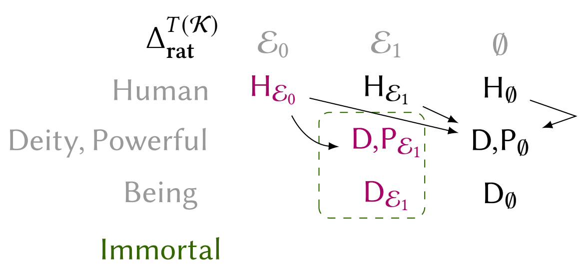


**Figure 7.9:** *Diagrams representing a fragment shared by all the models in $\mathsf{TU}_{\max}(\mathcal{I}^\mathcal{K}_{\mathsf{min}}, \mathsf{Init}_{\mathcal{I}^\mathcal{K}_{\mathsf{min}}}, \mathcal{K})$. The new edge $(\mathsf{Human}_{\mathcal{E}_0}, \{\mathsf{Deity}, \mathsf{Powerful}\}_{\mathcal{E}_1}) \in \mathsf{worships}^\mathcal{I}$ results in $\mathsf{Human}_{\mathcal{E}_0} \in Aux^\mathcal{I}$.*

Now, we can establish the relationship between rational nested and propositional reasonings. By consequence of Lemma 7.16, this relationship also characterizes the relationship between nested rational and materialization-based rational reasonings.

**Theorem 7.44.** *Let $\mathcal{K}$ be a DKB, $M \in \Delta^\mathcal{K}$ be a set of named concepts in the representative domain, and $A \in \mathsf{sig}_\mathsf{C}(\mathcal{K})$ be a name. Then:*

1. $\mathcal{K} \models_{\mathsf{prop,rat}} \lceil M \rceil \mathrel{\underset{\sim}{\sqsubset}} A \Rightarrow \mathcal{K} \models_{\mathsf{nest,rat}} \lceil M \rceil \mathrel{\underset{\sim}{\sqsubset}} A$,
2. $\mathcal{K} \models_{\mathsf{prop,rat}} \lceil M \rceil \mathrel{\underset{\sim}{\sqsubset}} A \not\Leftarrow \mathcal{K} \models_{\mathsf{nest,rat}} \lceil M \rceil \mathrel{\underset{\sim}{\sqsubset}} A$

*Proof.* Claim 1 is a special case of the Lemma 7.41, 1. Example 7.43 ensures claim 2. □

Nested rational reasoning solves the problem of quantification neglect by upgrading role successors whenever it is possible. However, it does so in a domain shaped by the exceptionality chain, which means that the granularity of the defeasible information is too coarse to capture some subtleties. It can extend defeasible information to role successors, but this defeasible information is still affected by problems such as inheritance blocking. Although nested rational reasoning entails *humans typically worship immortals*, as shown by Example 7.43, it does not entail that *humans typically worship corporeal beings*, even though deities are beings and beings are typically corporeal. We introduce nested reasoning of relevant and lexicographic strengths to address this shortcoming.

## 7.4.2 Nested Relevant and Lexicographic Reasoning

**Definition 7.45** (Nested Relevant Reasoning). *Let $\mathcal{K}$ be a DKB; $A \in \mathsf{sig}_C(\mathcal{K})$ and $M \in \Delta^{\mathcal{K}}$ be a set of named concepts in the representative domain. Let $(\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rel}}, \mathsf{Init}_{\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathsf{rel}}})$ be the minimal labeled typicality model for* **rel** *strength. We define* relevant nested entailment *by:*

$$\mathcal{K} \models_{\mathsf{nest},\mathbf{rel}} \lceil M \rceil \sqsubseteq A \textit{ iff } \quad \lceil M \rceil^{\mathcal{I}} \subseteq A^{\mathcal{I}}, \textit{for every } \mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rel}}, \mathsf{Init}_{\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathsf{rel}}}, \mathcal{K})$$

$$\mathcal{K} \models_{\mathsf{nest},\mathbf{rel}} \lceil M \rceil \precsim A \textit{ iff } \quad M_{\mathcal{U}} \in A^{\mathcal{I}} \textit{ for every maximally typical instance of } M \textit{ in the domain}, \forall \mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{rel}}, \mathsf{Init}_{\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathsf{rel}}}, \mathcal{K})$$

**Definition 7.46** (Nested Relevant Reasoning). *Let $\mathcal{K}$ be a DKB; $A \in \mathsf{sig}_C(\mathcal{K})$ and $M \in \Delta^{\mathcal{K}}$ be a set of named concepts in the representative domain. Let $(\mathcal{I}^{\mathsf{lex}}_{\mathsf{min},\mathbf{lex}}, \mathsf{Init}_{\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathsf{lex}}})$ be the minimal labeled typicality model for* **lex** *strength. We define* lexicographic nested entailment *by:*

$$\mathcal{K} \models_{\mathsf{nest},\mathbf{lex}} \lceil M \rceil \sqsubseteq A \textit{ iff } \quad \lceil M \rceil^{\mathcal{I}} \subseteq A^{\mathcal{I}}, \textit{for every } \mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{lex}}, \mathsf{Init}_{\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathsf{lex}}}, \mathcal{K})$$

$$\mathcal{K} \models_{\mathsf{nest},\mathbf{lex}} \lceil M \rceil \precsim A \textit{ iff } \quad M_{\mathcal{U}} \in A^{\mathcal{I}} \textit{ for every maximally typical instance of } M \textit{ in the domain}, \forall \mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathbf{lex}}, \mathsf{Init}_{\mathcal{I}^{\mathcal{K}}_{\mathsf{min},\mathsf{lex}}}, \mathcal{K})$$

Nested relevant and lexicographic reasonings tackle inheritance blocking and quantification neglect simultaneously. Inheritance blocking is solved by increasing the granularity of the typicality set, while the upgrade procedure propagates typical information through role edges. Example 7.47 illustrates this feature.

**Example 7.47.** *Consider the DKB described in Example 7.15. Let* $\mathsf{s} \in \{\mathbf{rel}, \mathbf{lex}\}$.

*We want to test whether* $\mathcal{K} \models_{\mathsf{nest,s}} \mathsf{Human} \precsim \exists \mathsf{worships}.\mathsf{Corporeal}$. *First, we introduce an auxiliary concept $Aux$ and the GCI* $\mathsf{Corporeal} \sqsubseteq \forall \mathsf{worships}^{-}.Aux$ *(which is equivalent to* $\exists \mathsf{worships}.\mathsf{Corporeal} \sqsubseteq Aux$*) to the TBox. Then, we test whether* $\mathcal{K} \models_{\mathsf{nest,s}} \mathsf{Human} \precsim Aux$. *The most typical instance of* Human *in the relevant domain is* $\{\mathsf{Human}\}_{\mathcal{D}}$.

*The minimal typicality relevant model has the* worships *edge* $(\{\mathsf{Human}\}_{\mathcal{D}}, \{\mathsf{Deity}, \mathsf{Powerful}\}_{\emptyset})$, *which connects the most typical instance of* Human *to the atypical powerful deity. The most typical successor attainable by the upgrade procedure connects* $(\{\mathsf{Human}\}_{\mathcal{D}}, \{\mathsf{Deity}, \mathsf{Powerful}\}_{\mathcal{U}})$, *where* $\mathcal{U} = \{\mathsf{Being} \precsim \mathsf{Corporeal}, \mathsf{Deity} \precsim \mathsf{Immortal}\}$.

*This is not incompatible with any other upgrade, and therefore* $\{\mathsf{Human}\}_{\mathcal{D}} \in Aux^{\mathcal{I}}$ *for every* $\mathcal{I} \in \mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\min,s}, \mathsf{Init}_{\mathcal{I}^{\mathcal{K}}_{\min,s}}, \mathcal{K})$. *Hence,* $\mathcal{K} \models_{\mathsf{nest,s}} \mathsf{Human} \mathrel{\underset{\sim}{\sqsubset}} Aux$, *which is equivalent to* $\mathcal{K} \models_{\mathsf{nest,s}} \mathsf{Human} \mathrel{\underset{\sim}{\sqsubset}} \exists\mathsf{worships}.\mathsf{Corporeal}$. *This configuration is represented in Figure 7.10.*

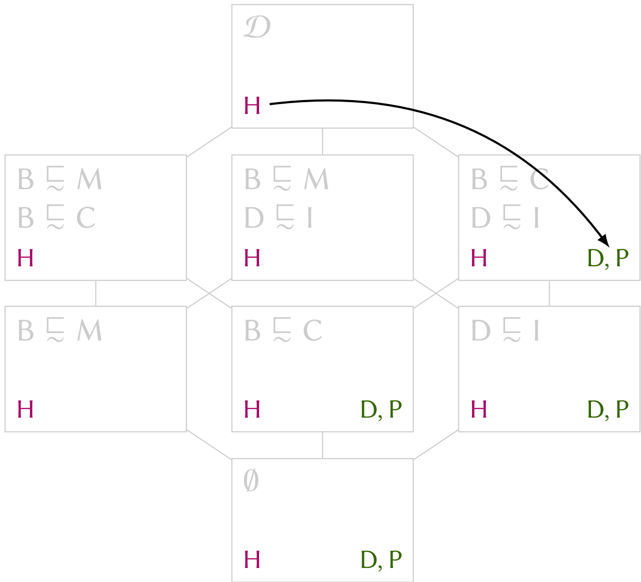


**Figure 7.10:** *Common fragment of all interpretations in* $\mathsf{TU}_{\max}(\mathcal{I}^{\mathcal{K}}_{\min}, \mathsf{Init}_{\mathcal{I}^{\mathcal{K}}_{\min}}, \mathcal{K})$. *The thick arrow connecting* $\mathsf{Human}_{\mathcal{D}}$ *to* $\{\mathsf{Deity}, \mathsf{Powerful}\}_{\mathcal{U}}$ *to* $\{\mathsf{Deity}, \mathsf{Powerful}\}_{\mathcal{U}}$, *where* $\mathcal{U} = \{\mathsf{Being} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Corporeal}, \mathsf{Deity} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{Immortal}\}$ *is an edge of the role* worships *that is common to all saturated typicality models.*

As with nested rational reasoning, nested relevant and lexicographic reasonings extends their propositional counterparts and, therefore, they also extend their materialization-based counterparts. The argument is the same as before.

**Theorem 7.48.** *Let* $\mathcal{K}$ *be a DKB,* $M \in \Delta^{\mathcal{K}}$ *be a set of named concepts in the representative domain,* $A \in \mathsf{sig}_{\mathsf{C}}(\mathcal{K})$ *be a name, and* $\mathbf{s} \in \{\mathbf{rel}, \mathbf{lex}\}$. *Then:*

1. $\mathcal{K} \models_{\mathsf{prop,s}} \lceil M \rceil \mathrel{\underset{\sim}{\sqsubset}} A \Rightarrow \mathcal{K} \models_{\mathsf{nest,s}} \lceil M \rceil \mathrel{\underset{\sim}{\sqsubset}} A$,
2. $\mathcal{K} \models_{\mathsf{prop,s}} \lceil M \rceil \mathrel{\underset{\sim}{\sqsubset}} A \not\Leftarrow \mathcal{K} \models_{\mathsf{nest,s}} \lceil M \rceil \mathrel{\underset{\sim}{\sqsubset}} A$

*Proof.* Claim 1 is a special case of the Lemma 7.41, 1. Example 7.47 ensures claim 2. □

## 7.5 Comparing Semantics for $\mathcal{ELI}_\bot$

The hierarchy of defeasible semantics is established for materialization-based reasoning. Both relevant and lexicographic strengths extend rational reasoning [Cas+14a], and lexicographic is stronger than the relevant closure [Cas+14a]. These results transfer to typicality-models semantics of propositional coverage. However, as in $\mathcal{EL}_\bot$, the same order is not preserved under nested coverage. We already showed how nested coverage of any strength extends its propositional counterpart. Now, we compare the different strengths, laying out a complete hierarchy of typicality-models based reasoning of nested coverage.

**Rational compared to relevant and lexicographic strengths** First, we show that the rational strength is not comparable to relevant and lexicographic strengths in the nested coverage.

**Theorem 7.49.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB, $M \in \Delta^{\mathcal{K}}$ be a set of named concepts in the representative domain, and $A \in \mathrm{sig}_{\mathrm{C}}(\mathcal{K})$ be a concept name.*

1. $\mathcal{K} \models_{\mathsf{nest},\mathbf{rat}} \lceil M \rceil \mathrel{\lessapprox} A \not\Rightarrow \mathcal{K} \models_{\mathsf{nest},\mathbf{rel}} \lceil M \rceil \mathrel{\lessapprox} A$;
2. $\mathcal{K} \models_{\mathsf{nest},\mathbf{rat}} \lceil M \rceil \mathrel{\lessapprox} A \not\Leftarrow \mathcal{K} \models_{\mathsf{nest},\mathbf{rel}} \lceil M \rceil \mathrel{\lessapprox} A$;
3. $\mathcal{K} \models_{\mathsf{nest},\mathbf{rat}} \lceil M \rceil \mathrel{\lessapprox} A \not\Rightarrow \mathcal{K} \models_{\mathsf{nest},\mathbf{lex}} \lceil M \rceil \mathrel{\lessapprox} A$;
4. $\mathcal{K} \models_{\mathsf{nest},\mathbf{rat}} \lceil M \rceil \mathrel{\lessapprox} A \not\Leftarrow \mathcal{K} \models_{\mathsf{nest},\mathbf{lex}} \lceil M \rceil \mathrel{\lessapprox} A$.

*Proof.* We prove the claims by specifying a DKB in which rational nested reasoning diverges from both relevant and lexicographic nested reasonings.

Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be the DKB defined as[2]:

$$\begin{aligned}\mathcal{T} =& \{A \sqsubseteq \exists r.B, B \sqsubseteq \exists r.C, B \sqsubseteq D, G_0 \sqsubseteq \forall r^-G_1,\\ & G_1 \sqsubseteq \forall r^-.G_2, G_2 \sqcap \exists r.F \sqsubseteq \perp, D \sqcap \exists r.G_0 \sqsubseteq \perp\}\\ \mathcal{D} =& \{B \sqcap E \mathrel{\lessapprox} \perp, D \mathrel{\lessapprox} E, D \mathrel{\lessapprox} F, C \mathrel{\lessapprox} G_0\}\end{aligned}$$

The exceptionality chain for this DKB is given by $\mathcal{E}_0 = \mathcal{D}, \mathcal{E}_1 = \{B \sqcap E \mathrel{\lessapprox} \perp\}$. The set $\mathcal{U}$ such that $\mathrm{sel}_{\mathcal{K},\mathbf{rel}}(B) = \mathcal{U}$ and $\mathrm{sel}_{\mathcal{K},\mathbf{lex}}(B) = \{\mathcal{U}\}$ is $\mathcal{U} = \{B \sqcap E \mathrel{\lessapprox} \perp, C \mathrel{\lessapprox} G_0, D \mathrel{\lessapprox} F\}$.

The most typical instance of $A$ in all domains is $\{A\}_{\mathcal{D}}$. It the rational domain, it can upgrade its $B$ $r$-successor to $\{B\}_{\mathcal{E}_1}$, which can upgrade its $C$ successor to $\{D\}_{\mathcal{D}}$. Hence, in *all* upgrade paths, $A$ ends up with in $\exists r.\exists r.G_0$, which is equivalent to $G_2$.

However, in both the lexicographic and the relevant domains, there is an alternative path incompatible with the first one. Upgrading $A$'s $B$ $r$-successor to $\{B\}_{\mathcal{U}}$ creates an impasse: unlike its less typical counterpart, $\{B\}_{\mathcal{U}}$ cannot upgrade its $C$ $r$-successor. However, the rational path is not achievable once $A$ gets this successor, as $\{C\}_{\mathcal{D}}$ would push $G_2$ back to $A$, which is not consistent with $D$ successors. Figure 7.11 depicts this effect, highlighting both upgrade paths.

Notice that, in this example:

- $\mathcal{K} \models_{\mathsf{nest},\mathbf{rat}} A \mathrel{\lessapprox} G_2$, $\mathcal{K} \not\models_{\mathsf{nest},\mathbf{rel}} A \mathrel{\lessapprox} G_2$, and $\mathcal{K} \not\models_{\mathsf{nest},\mathbf{lex}} A \mathrel{\lessapprox} G_2$,
- $\mathcal{K} \not\models_{\mathsf{nest},\mathbf{rat}} B \mathrel{\lessapprox} F$, $\mathcal{K} \models_{\mathsf{nest},\mathbf{rel}} B \mathrel{\lessapprox} F$, and $\mathcal{K} \models_{\mathsf{nest},\mathbf{lex}} B \mathrel{\lessapprox} F$.

□

In this case, the stronger part of the relevant and rational strengths comes from the propositional framework. However, this is not the only way in which lexicographic and relevant strengths can be stronger under nested coverage. These strengths can extend

[2] We use a non-normalized DKB to make the example more intuitive. The effect would be the same in a normalized DKB.

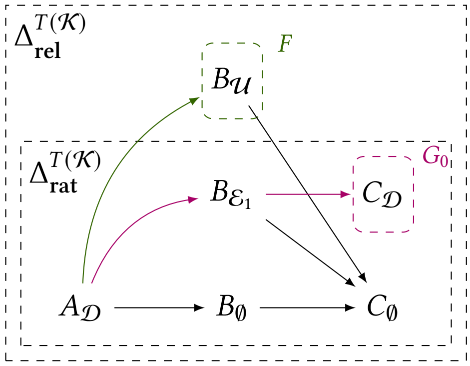


**Figure 7.11:** *Diagram representing an overlapping fragment of the upgrade possibilities in the rational domain and in the lexicographic/relevant domains. green arrows represent a complete upgrade path only available to the relevant and lexicographic domains. The purple path is available to all the domain shapes.*

rational reasoning by tackling inheritance blocking nested within quantifiers. Therefore, their entailment relations are truly incomparable in nested coverage.

**Comparison between relevant and lexicographic** The comparison between relevant and lexicographic strengths is the only one left. It was established by [Cas+14a] that lexicographic trumps relevant in the materialization-based setting, and this result transfers to typicality-models based reasoning of propositional strength via Lemma 7.20. This hierarchy is not imported to the nested coverage, and both strengths define incomparable entailment relations.

The effect is very similar to what generated incomparability w.r.t. the rational strength: lexicographic is stronger in the propositional setting, which means it has a larger domain. A large domain may give rise to new entailments by allowing upgrades to more typical required successors. On the other hand, this increase in domain size can also undermine some conclusions by virtue of the skeptical nature of nested reasoning. An upgrade that was the only path in some strength, therefore leading to some conclusion, can be dissolved in a larger domain that introduces other incompatible paths. The models corresponding to the entailment are still there, but they are not the only ones in the set of preferred models.

**Theorem 7.50.** *Let $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ be a DKB, $M \in \Delta^{\mathcal{K}}$ be a set of named concepts in the representative domain, and $A \in \mathsf{sig}_\mathsf{C}(\mathcal{K})$ be a concept name.*

1. $\mathcal{K} \models_{\mathsf{nest},\mathbf{rel}} \lceil M \rceil \sqsubseteq\!\!\!\sim A \not\Rightarrow \mathcal{K} \models_{\mathsf{nest},\mathbf{lex}} \lceil M \rceil \sqsubseteq\!\!\!\sim A$;
2. $\mathcal{K} \models_{\mathsf{nest},\mathbf{rel}} \lceil M \rceil \sqsubseteq\!\!\!\sim A \not\Leftarrow \mathcal{K} \models_{\mathsf{nest},\mathbf{lex}} \lceil M \rceil \sqsubseteq\!\!\!\sim A$.

*Proof.* The proof relies on the DKB that shows both items of the theorem. Let $\mathcal{K}' = (\mathcal{T}', \mathcal{D}')$ be defined from the DKB $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ from Example 7.49 as follows:

$$\begin{aligned}\mathcal{T}' &= \mathcal{T} \setminus \{F \sqcap \exists r.G_0 \sqsubseteq \bot\} \\ &\quad \cup \{B \sqcap F_1 \sqcap F_3 \sqsubseteq \bot, B \sqcap F_2 \sqcap F_3 \sqsubseteq \bot, F_1 \sqcap \exists r.G_0 \sqsubseteq \bot\} \\ \mathcal{D}_2 &= (\mathcal{D} \setminus \{D \mathrel{\sqsubseteq\!\!\!\!\sim} F\}) \\ &\quad \cup \{D \mathrel{\sqsubseteq\!\!\!\!\sim} F_1, D \mathrel{\sqsubseteq\!\!\!\!\sim} F_2, D \mathrel{\sqsubseteq\!\!\!\!\sim} F_3\}\end{aligned}$$

Intuitively, $F$ is broken down into $F_1$, $F_2$, and $F_3$. Taken together, they are not compatible with $B$. However, $F_1$ and $F_2$ can be applied $B$, as does $F_3$ alone. This partitioning highlights the difference between lexicographic closure, which "counts" the number of DCIs, and relevant, which wipes same-ranked DCIs leading to inconsistency.

The exceptionality chain for this DKB is given by $\mathcal{E}_0 = \mathcal{D}, \mathcal{E}_1 = \{B \sqcap E \mathrel{\sqsubseteq\!\!\!\!\sim} \bot\}$. The set $\mathcal{U}$ s.t. $\mathsf{sel}_{\mathcal{K},\mathbf{lex}}(B) = \{\mathcal{U}\}$ is $\mathcal{U} = \mathcal{D} \setminus \{D \mathrel{\sqsubseteq\!\!\!\!\sim} E, D \mathrel{\sqsubseteq\!\!\!\!\sim} F_3\}$. All upgrade paths starting from the minimal typicality relevant model upgrade $(\{A\}_\mathcal{D}, \{C\}_\emptyset) \in r^{\mathcal{I}^\mathcal{K}_{\mathsf{min,rel}}}$ to $(\{A\}_\mathcal{D}, \{C\}_\mathcal{D}) \in r^\mathcal{I}$, and therefore, have $\{A\}_\mathcal{D} \in G_2^\mathcal{I}$. Hence, $\mathcal{K} \models_{\mathsf{nest},\mathbf{rel}} A \mathrel{\sqsubseteq\!\!\!\!\sim} G_2$. However, the minimal lexicographic model has a possible upgrade from $(\{A\}_\mathcal{D}, \{B\}_\emptyset) \in r^{\mathcal{I}^\mathcal{K}_{\mathsf{min,lex}}}$ to $(\{A\}_\mathcal{D}, \{B\}_\mathcal{U}) \in r^\mathcal{I}$ that blocks the aforementioned upgrade path, resulting in $\mathcal{K} \not\models_{\mathsf{nest},\mathbf{lex}} A \mathrel{\sqsubseteq\!\!\!\!\sim} G_2$.

Notice, however, that the more typical $B$ instance in the domain brings some additional conclusions to the preferred models from lexicographic strength, as $\{B\}_\mathcal{U} \in F_2$. Therefore, $\mathcal{K} \models_{\mathsf{nest},\mathbf{lex}} B \mathrel{\sqsubseteq\!\!\!\!\sim} F_2$, while $\mathcal{K} \not\models_{\mathsf{nest},\mathbf{rel}} B \mathrel{\sqsubseteq\!\!\!\!\sim} F_1$.

□

To summarize the comparison, the materialization-based hierarchy between rational, relevant, and propositional strengths holds on typicality-models semantics of propositional coverage. Furthermore, the nested coverage of each strength extends its propositional (and, therefore, its materialization-based) counterpart. However, the hierarchy does not hold between different strengths in typicality-models based reasoning of nested coverage. The different strengths examined in this dissertation are not comparable in this coverage. These results are analogous to the ones for $\mathcal{EL}_\bot$ and are summarized in Figure 7.12.

## 7.6 Epilogue: Lifting rational defeasible instance checking based on typicality models to $\mathcal{ELI}_\bot$

The epilogue of Chapter 6 presented a procedure to extend instance checking for semantics based on rational models of rational strength for $\mathcal{EL}_\bot$. This section presents a strategy to port this technique to the typicality models framework for $\mathcal{ELI}_\bot$.

The introduction of inverse roles changes the representation of individuals in two ways. First, it introduces a fourth type of edge: $(C_\mathcal{U}, a_\mathcal{V})$, where $a \in \mathsf{sig_I}(\mathcal{K})$ is an individual. In $\mathcal{EL}_\bot$, individuals only appeared on the left-hand side of roles. In $\mathcal{ELI}_\bot$, they can also be featured on the right-hand side due to the occurrence of inverse roles.

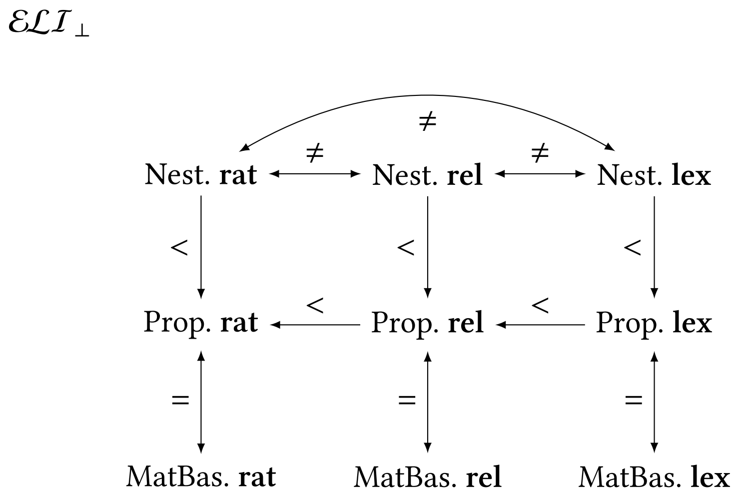


**Figure 7.12:** *Diagram with the strength comparison between materialization-based and typicality-models-based defeasible subsumption checking of all strengths and coverages. The < and = relations are transitive.*

The second novelty is that value restrictions can cover both directions of any edge. In $\mathcal{EL}_\bot$, they were limited to the predecessors, as they were expressed through the equivalence $\exists r.A \sqsubseteq D \equiv A \sqsubseteq \forall r^-.D$. Therefore, $\mathcal{ELI}_\bot$ is compatible with axioms of the form $C \sqsubseteq \forall r.D$, which also affects the elements representing individuals in typicality domains. This increase in expressivity is reflected in the construction of the domain and in the upgrade procedure.

### 7.6.1 Building the minimal typicality model with individuals

The construction of the enriched ABox according to some order $o$ remains unchanged. The original algorithm was devised to $\mathcal{ALC}$, which is more expressive than $\mathcal{ELI}_\bot$. However, the interpretation induced by the ABox proposed for $\mathcal{EL}_\bot$ in 7.51 has to undergo some changes to account for the changes in the domain's structure. Formally, the new definition is given by:

**Definition 7.51** (ABox Interpretation for $\mathcal{ELI}_\bot$)**.** *Let $\mathcal{K} = (\mathcal{A}^*, \mathcal{T}, \mathcal{D})$ be a DKB expanding $(\mathcal{A}, \mathcal{T}, \mathcal{D})$ as described in 6.36 according to the order over individuals $o$. With $\mathcal{I}_{\mathcal{A}^*,\mathcal{K}} = (\Delta^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}, \cdot^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}})$. Then:*

$$
\begin{aligned}
\Delta^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}} &= \{a_{j_{\mathcal{E}^{\mathcal{K}_o}_{a_j}}} \mid a_j \in \mathsf{sig}_\mathsf{I}(\mathcal{A})\} \cup \{M_\emptyset \mid M \in \mathcal{P}(\mathsf{sig}(\mathcal{T}))\} \\
a_j^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}} &= a_{j_{\mathcal{E}^{\mathcal{K}_o}_{a_j}}} \\
A^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}} &= \{a_{j_{\mathcal{E}^{\mathcal{K}_o}_{a_j}}} \in \Delta^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}} \mid \overline{\mathcal{K}} \models A(a_j)\} \\
r^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}} &= \{r(a_{j_{\mathcal{E}^{\mathcal{K}_o}_{a_j}}}, a_{k_{\mathcal{E}^{\mathcal{K}_o}_{a_k}}}) \in \mathcal{A}^*\} \\
&\quad \cup \{(a_{j_{\mathcal{E}^{\mathcal{K}_o}_{a_j}}}, M_\emptyset) \mid \overline{\mathcal{K}} \models (\exists r.\lceil M \rceil)(a_j) \textit{ for a prime } M \textit{ } r\textit{-successor for } a_j \textit{ in } \overline{\mathcal{K}}\} \\
&\quad \cup \{(M_\emptyset, a_{j_{\mathcal{E}^{\mathcal{K}_o}_{a_j}}}) \mid \overline{\mathcal{K}} \models (\exists r^-.\lceil M \rceil)(a_j) \textit{ for a prime } M \textit{ } r^-\textit{-successor for } a_j \textit{ in } \overline{\mathcal{K}}\}
\end{aligned}
$$

We show two fundamental properties of this new version of ABox interpretation. First, we show that the concept membership of the individuals in the interpretation coincides perfectly with what is entailed by rational materialization-based instance checking. Second, we show that the union of this interpretation with the minimal typicality model is a model of $\overline{\mathcal{K}}$.

**Lemma 7.52.** *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB, $o$ be a total order over $\mathsf{sig_I}(\mathcal{A})$, and $\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}$ be an ABox interpretation as defined in Definition 7.51. Let $A \in \mathsf{sig_C}(\mathcal{K})$. Then:*

$$\mathcal{K} \models_{\mathsf{mat,rat},o} A(a) \text{ iff } \mathcal{I}_{\mathcal{A}^*,\mathcal{K}} \cup \mathcal{I}^{\mathcal{K}}_{\mathsf{min}} \models A(a)$$

*Proof.* This follows directly from the definition. Materialization-based instance checking is defined by selecting a $\mathcal{E}_i$ by iteratively expanding the ABox according to the order $o$. For a given $a_j$, this set is $\mathcal{E}^{\mathcal{K}_o}_{a_j}$. Then, $\mathcal{K} \models_{\mathsf{mat,rat},o} A(a_j)$ iff $(\mathcal{A}^*, \overline{(\mathcal{T}, \mathcal{D})}) \models A(a_j)$ iff $a_{j_{\mathcal{E}^{\mathcal{K}_o}_{a_j}}} \in A^{\mathcal{I}_{\mathcal{A}^*,\mathcal{K}} \cup \mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}}}$ iff $\mathcal{I}_{\mathcal{A}^*,\mathcal{K}} \cup \mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}} \models A(a)$. ◻

Now, we show that $\mathcal{I}_{\mathcal{A}^*,\mathcal{K}} \cup \mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}}$ is indeed a model of the DKB.

**Theorem 7.53.** *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be a DKB and $o$ an order over the individuals in $\mathcal{A}$. Let $\mathcal{I}_{\mathcal{A}^*,\mathcal{T}}$ be an ABox interpretation as defined in Definition 7.51. It follows that*

$$\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}} \models \mathcal{K}$$

*Proof.* Lemma 7.9 shows that $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}} \models C \sqsubseteq D$ for every $C \sqsubseteq D \in \overline{(\mathcal{T}, \mathcal{D})}$. The interpretation $\mathcal{I}_{\mathcal{A}^*,\overline{\mathcal{T}}}$ is quasi-disjoint to $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}}$. Hence, all information on concept representatives comes from $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}}$, and $\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}}$is a model of $\mathcal{K}$. What remains to be shown is that the individuals introduced by $\mathcal{I}_{\mathcal{A}^*,\mathcal{T}^{\mathcal{D}}}$ do not break the model property. In principle, those individuals could break the model property in two ways.

The first is by directly violating an axiom, e.g. $a_{\mathcal{U}} \in (A^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}} \setminus B^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}})$ for some $A \sqsubseteq B \in \mathcal{T}$. The second is by violating a value restriction by being connected to some concept representative, e.g., $(a_{\mathcal{U}}, M_\emptyset) \in r^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$, $A \sqsubseteq \forall r^-.B \in \mathcal{T}$, $M_\emptyset \in A^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$, and $a_{\mathcal{U}} \notin B^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$.

None of these conditions hold. To show it, we proceed by cases. First, we check whether the individuals respect all the axioms in $\mathcal{K}$. Then, we check whether the role edges connecting individuals and concept representatives violate something required by $\mathcal{K}$. We recall that the terminological knowledge is normalized, and the ABox is simple. Therefore, there are four forms of axioms. We proceed by cases. Let $a_{\mathcal{U}} \in A^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$ (or, for the conjunctive axiom, $a_{\mathcal{U}} \in A_1^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$ and $a_{\mathcal{U}} \in A_2^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$). Notice that $\overline{\mathcal{K}}$ is an extension of $\mathcal{K}$. Therefore, $\mathcal{K} \models C \sqsubseteq D$ implies $\overline{\mathcal{K}} \models C \sqsubseteq D$.

Case 1: $A \sqsubseteq B$ and $A_1 \sqcap A_2 \sqsubseteq B$. The hypothesis $a_{\mathcal{U}} \in A^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$ implies $\mathcal{K} \models A(a)$. But $\mathcal{K} \models A \sqsubseteq B$, therefore, $\mathcal{K} \models B(a)$, and $a_{\mathcal{U}} \in B^{\mathcal{I}^{\mathcal{K}}_{\mathsf{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$. The argument for $A_1 \sqcap A_2 \sqsubseteq B$ is identical, with one extra step going from $A_1 \sqcap A_2$ to $A_1$ and $A_2$.

Case 2: $A \sqsubseteq \exists r.B$. Remember that, by construction, $\overline{\mathcal{K}} \models A(a)$. This fact implies $\overline{\mathcal{K}} \models (\exists r.B)(a)$. Notice also that $B \in Qc(\mathcal{K})$, therefore $\{B\}_\emptyset \in \Delta^{T(\mathcal{K})}$ and there is some $M \supseteq \{B\}$ that is the prime $r$-successor for $a$ in $\overline{\mathcal{K}}$. Then, $(a_\mathcal{U}, M_\emptyset) \in r^{\mathcal{I}^{\mathcal{K}}_{\text{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$ and $M_\emptyset \in B^{\mathcal{I}^{\mathcal{K}}_{\text{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$. Notice that the argument stays the same if $r$ is an inverted role.

Case 3: $A \sqsubseteq \forall r.B$. From the hypothesis, we have $\overline{\mathcal{K}} \models (\forall r.B)(a)$. Suppose that there is some element $e$ in the domain s.t. $(a, e) \in r^{\mathcal{I}^{\mathcal{K}}_{\text{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$. The element $e$ may be a concept or an individual representative. If it is a concept representative $M_\emptyset$, we know that $\overline{\mathcal{K}} \models (\exists r.\lceil M \rceil)(a)$, for a prime $M$. The primeness of $M$ implies $B \in M$, otherwise $M$ would not be prime, as $\overline{\mathcal{K}} \models (\exists r.\lceil M \cup \{B\} \rceil)(a)$. Therefore, $M_\emptyset \in B^{\mathcal{I}^{\mathcal{K}}_{\text{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{T}^\mathcal{D}}}$. On the other hand, if $e$ is an individual representative – $b \in \text{sig}_\text{I}(\mathcal{K})$ – then $(a, b) : r \in \mathcal{A}^*$. This, in conjunction with $\overline{\mathcal{K}} \models A(a)$, imply $\overline{\mathcal{K}} \models (B)(b)$. Therefore, $b \in B^{\mathcal{I}^{\mathcal{K}}_{\text{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$.

Case 4: $A(a) \in \mathcal{A}^*$. This follows directly from the construction of $\mathcal{I}_{\mathcal{A}^*,\mathcal{K}}$.

Now, we proceed to investigate the edges between individuals and concept representatives. Suppose[3] $(a_\mathcal{U}, M_\emptyset) \in r^{\mathcal{I}^{\mathcal{K}}_{\text{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$, $M_\emptyset \in A^{\mathcal{I}^{\mathcal{K}}_{\text{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$, and $A \sqsubseteq \forall r^-.B \in \mathcal{T}$. Consider the following two facts.

**(A)** $(a, M_\emptyset) \in r^{\mathcal{I}^{\mathcal{K}}_{\text{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$ implies $(\mathcal{A}^*, \overline{\mathcal{T}}) \models (\exists r.M)(a)$, with a maximal $M$.

**(B)** $M_\emptyset \in A^{\mathcal{I}^{\mathcal{K}}_{\text{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$ implies $(\mathcal{A}^*, \overline{\mathcal{T}}) \models \lceil M \rceil \sqsubseteq A$.

Taken together, **(A)** and **(B)** yield $(\mathcal{A}^*, \overline{\mathcal{T}}) \models B(a)$, which, in turn, implies $a \in B^{\mathcal{I}^{\mathcal{K}}_{\text{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$.

Once again, if the situation is inverted and the edge is $(M_\emptyset, a_\mathcal{U}) \in r^{\mathcal{I}^{\mathcal{K}}_{\text{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}}$, the argument is the same. □

### 7.6.2 A commentary on the upgrade procedure

We have shown that the expanded minimal typicality model $\mathcal{I}^{\mathcal{K}}_{\text{min,rat}} \cup \mathcal{I}_{\mathcal{A}^*,\mathcal{K}}$ is canonical regarding instance checking for materialization-based rational reasoning. Now, we consider the effect of this increase in the input model in the upgrade procedure. We do not rebuild the whole upgrade but instead merely comment on why the extended model does not spoil it.

In general, individual representatives behave as special cases of concept representatives. This means that their introduction increases the number of upgrade paths, as the order of upgrades now includes edges containing individuals. Other than that, the fundamental idea behind the algorithm remains the same. Let us consider some remarks on the idiosyncrasy of individual representatives in the upgrade procedure:

1. Upgrades of edges initiated by individuals can block upgrades of concept representatives.
2. Individuals initiate every edge to which they belong. From this fact, it follows that:

[3] Notice that, by construction, all edges between individuals and concept representatives land on atypical concept representatives.

(a) Let $a$ be an individual and $M_\mathcal{U}$ be a concept representative. If an edge $(a_\mathcal{V}, M_\mathcal{U}) \in r^\mathcal{I}$ or $(M_\mathcal{U}, a_\mathcal{V}) \in r^\mathcal{I}$ breaks the model property due to the violation of some value restriction from $M_\mathcal{U}$, the solution is always moving the endpoint $M_\mathcal{U}$ to a suitable $M'_\mathcal{U}$, $M \subset M'$. If, on the other hand, the violation is in $a$, then the solution is always adding $a_\mathcal{V}$ to the extension of the missing concept.

(b) Let $a, b$ be individuals. If an edge $(a_\mathcal{V}, b_\mathcal{U})$ breaks the model property (due to the violation of some value restriction), the solution is adding $a_\mathcal{V}$ or $b_\mathcal{U}$ to the extension of the missing concept.

The first remark is not exclusive to $\mathcal{ELI}_\bot$. Upgrades may add an individual representative to some concept, which may block other upgrades of concepts connected to this particular individual representative. This flow of information depends on axioms with existential restrictions on the left-hand side, i.e. value restrictions over the inverted role. The following interpretation exemplifies this property. This remark only highlights that the introduction of individuals does indeed amplify the upgrade branches leading to saturated typicality models.

Now, we consider the second remark. Edge labels stem from existential restrictions. Edges are labeled by $\mathsf{Init}_\mathcal{I}$ regarding the elements whose concept (or individual) requires the other concept as a prime existential witness. Because $\mathcal{ELI}_\bot$ lacks nominals, requirements involving individuals are limited to (i) $(\exists r.A)(a) \in \mathcal{A}^*$ and (ii) $r(a, b) \in \mathcal{A}^*$. The first one, in conjunction with the rest of the DKB, implies the existence of some maximal $M \supseteq \{A\}$ such that $\overline{\mathcal{K}} \models (\exists r.\lceil M \rceil)(a)$. Therefore, the corresponding edge is labeled with p, belonging to the representative of $a$. For the second kind of requirement, the edge is labeled with $\{\mathrm{s}, \mathrm{p}\}$, as it belongs to both individual representatives. Every violation arising from an edge connecting two individuals can only be fixed by adding the violating representatives to the extension of the missing concepts.

There are three kinds of violations involving individuals. They are characterized by the elements in the edge that give rise to the violation. They are *individual–individual*, *individual–concept*, and *concept-individual* (the first element is the one that breaks some axiom or consequence of the KB).

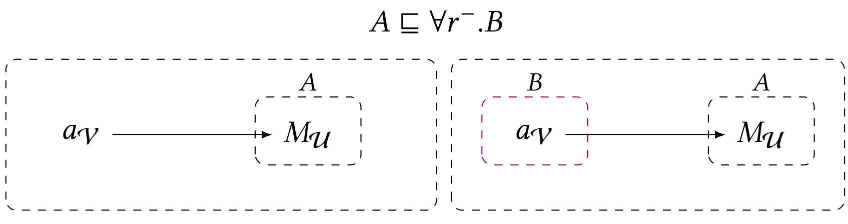


**Figure 7.13:** *A concept representative may be added to some concept A, imposing restrictions on its neighbors. When the neighbor is an individual representative, it is added to the extension of the missing concept because it will always initiate the edge.*

To conclude the discussion, we show that this expansion of typicality models deals with quantification neglect in the rational case. Consider the following example.

**Example 7.54.** *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{D})$ be the DKB defined in 7.15 with the additional ABox*

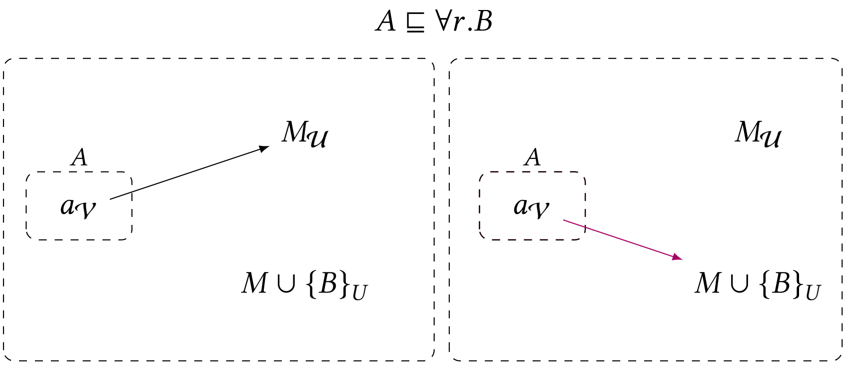


**Figure 7.14:** *An individual representative may be added to some concept C, which may impose restrictions on its neighbors. When those neighbors are concept representatives, the individual owns the edge. Hence, the repair consists in moving the edge to a more inclusive concept representative.*

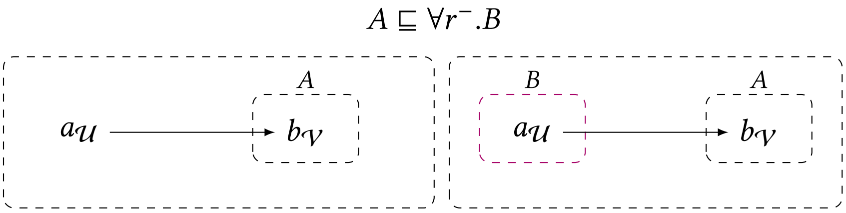


**Figure 7.15:** *Edges between two individuals are labeled with* $\{<, \mathsf{s}\}$. *Moving them is impossible, as individuals have only one representative per domain. Hence, the solution is to add the individual to the extension of the missing concept.*

$\mathcal{A} = \{\mathsf{Human}(\mathsf{hector})\}$. *Figure 7.16 represent the minimal typicality model fragments and the common part of all saturated models.*

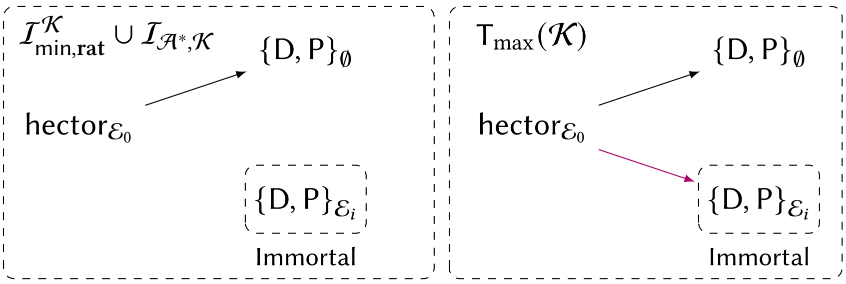


**Figure 7.16:** *The left figure depicts a fragment of the augmented minimal typicality model, while the left one depicts a common part to all saturated models. Arrows represent the role* worship.

*The individual* hector *is a human and therefore worships powerful deities. However, deities are only typically immortal, a piece of information not transmitted to his worship successors. The upgrade procedure connects Hector with the more typical representative of a powerful deity, and therefore we may conclude that Hector worships immortals. As a matter of fact, some of them sided with him in the Trojan War, although this was not enough to spare him from his tragic destiny.*

# Chapter 8

# Discussion

Chapters 5, 6 and 7 presented semantics based on typicality models for the DDLs $\mathcal{EL}_\bot$, and $\mathcal{ELI}_\bot$, focusing on the problem of defeasible subsumption. We defined semantics of strength **s** and propositional coverage, which are proved to be equivalent to **s** materialization-based reasoning, and a machinery to saturate the models with defeasible information and define **s** reasoning of nested coverage, which solves the problem of quantification neglect. Even though the upgrade procedures that lead to the saturated models are different, they are based more or less on the same intuitions.

In this chapter, we discuss some issues common to the two frameworks. More specifically, we shed light on the comparison between all the parametrized strengths for $\mathcal{EL}_\bot$ and $\mathcal{ELI}_\bot$ depicted in 6.11 and 7.12, respectively. Different strengths are incomparable within nested coverage, which is an unexpected property, since they are hierarchically stacked in propositional coverage. Discussing these properties is a preliminary foray into the difficulties of reasoning defeasibly in first-order logic.

Until now, there was no explicit comparison between semantics of different strengths on nested coverage for $\mathcal{EL}_\bot$. Pensel & Turhan (2018) and Pensel (2019) compared different coverages under the same strength. The incomparability results presented in this dissertation for $\mathcal{ELI}_\bot$ nested semantics also extends to the less expressive $\mathcal{EL}_\bot$. In this case, the minimum expressivity for creating incompatible upgrade branches requires only value restrictions for inverse roles, which are already expressible in the language of $\mathcal{EL}_\bot$ by the equivalence $\exists r.A \sqsubseteq B \equiv A \sqsubseteq \forall r^-.B$. The DKB used to exemplify the incomparability effect in Example 7.43 is written in the language of $\mathcal{EL}_\bot$.

There is a straightforward explanation for this effect. Given the skeptical nature of nested reasoning, introducing more elements may drown conclusions that were unanimous in smaller domains. This may seem strange, but it is grounded on a shortcoming of materialization-based reasoning for DLs. When applying defeasible information, materialization-based reasoning treats the concepts as isolated atoms. A penguin can be more or less typical, have or not have feathers, but this will not affect any other concept of the DKB. However, in nested coverage, defeasible information flows between different concepts through their representatives. Hence, if a penguin is deemed to have feathers, all other beings related to penguins are now related to animals that have feathers.

This is significant, as reasoning with defeasible information is nonmonotonic in nature. Hence, information has a negative aspect. Being related to a penguin *with feathers* can hinder other defeasible conclusions otherwise reachable to a particular concept. Stronger materialization-based paradigms create more detailed depictions of the world, ultimately resulting in the withdrawal of conclusions drawn from less inclusive representative domains.

Considering this explanation, it is possible to argue that more inclusive strengths, such as relevant and lexicographic, are improvements over rational nested reasoning, even if they are not strictly stronger. Compared to rational reasoning, the consequences they gain solve shortcomings stemming from a too-coarse partition of the defeasible part of a DKB. On the other hand, the information derived only by rational strength results from its "lack of imagination". The limited nature of rational domains makes rational nested reasoning derive some consequences by being unable to depict suitable counterexamples. The idea that rational reasoning may be too strong to represent defeasible reasoning in DLs is not new. Giordano *et al.* (2010) defended it at the propositional level. However, reasoning power is not the only important feature when judging knowledge representation systems, and rational reasoning is still attractive due to its nice computational properties and well-behavedness.

An opposing perspective is that nested coverage should keep the hierarchy between rational, relevant, and lexicographic reasoning intact. Keeping the hierarchy would align with the original presentations of the materialization-based presentations of these methods, which introduced them as strict improvements over rational reasoning. In this case, nested semantics has to be adapted. Swapping the skeptical reasoning procedure for a *credulous* one is one possibility. Strengths such as relevant and lexicographic generate larger typicality domains. Hence, all upgrade paths available to rational reasoning are also in them, and a brave reasoning procedure would inherit them. However, a pure credulous reasoning procedure would also be prone to deriving inconsistent conclusions, as it would take the information from incompatible upgrade paths.

A middle ground is conditioning the entailment relation to an upgrade order. Pensel (2019) suggested something similar to this. If the initial section of the order is a full upgrade path for the rational domain, the full order for stronger domains would effectively extend rational strength even for the nested coverage. This approach is analogous to the method for defeasible instance checking originally developed in [CS10]. The parallels between the two procedures are unsurprising, as both problems – reasoning with individuals and pushing defeasible information through quantifiers – stem from what is fundamentally the same complication: defeasible information flows through role edges, and no element is an island.

A slightly different approach is to initially input the upgrade procedure for relevant or lexicographic strengths with the output of the upgrade procedure of the rational domain, where both domains coincide. This procedure still characterizes a skeptical semantics, but it limits the final models to those that augment rational reasoning by starting the upgrade procedure from models that already have all the information derived from rational strength. Although this procedure works, it has an ad-hoc flavor, as there is no theoretical justification basing it except the motivation to keep the entailments generated by rational reasoning.

Overall, this scenario highlights the challenges for reasoning defeasibly in a first-order setting. Nonmonotonic reasoning is a notoriously difficult problem, and researchers have been searching for solutions for decades. In the context of DDLs, the leading frameworks have a propositional nature and are grounded in the KLM (propositional) hierarchy. Propositional methods to defeasible reasoning are often limited in scope, as they isolate elements from each other and do not consider the complex interaction between elements mediated by n-ary relations. First-order approaches attempt to address this limitation but must account for the interaction between elements of differing typicalities, which is a difficult problem conceptually and computationally. Isolating elements from each other can be good from a computational point of view, but it is difficult to find a strong epistemological foundation for this option. After all, if the language allows for expressing relations between individuals, those related elements are expected to vary in their level of typicality. Reasoning gets more accurate as we expand the degree of typicality individuals can attain. The conclusions lost in this process reflect a weakness of the more coarse domains.

# Chapter 9

# Conclusions and Future Work

In this dissertation, we investigated the representation and reasoning based on typicality within description logics. More specifically, we dealt with the problem of incorporating first-order properties into existing defeasible semantics for DLs, taking advantage of their full expressivity. The solution presented here is a semantics based on a special class of models – typicality models – which are two-dimensional canonical models tailored for the logics $\mathcal{EL}_\bot$ and $\mathcal{ELI}_\bot$. Those models represent satisfaction of DCIs through preferred concept representatives as measured by their second dimension, *typicality sets*. Semantics based on typicality models is shown to tackle quantification neglect for several different materialization-based semantics. The logic $\mathcal{ELI}_\bot$ was chosen as an intermediary and hopefully a useful middle step to broaden the typicality models framework to the more expressive class of Horn-DLs. The introduction of inverted roles results in a powerful increase in expressibility and greatly increases complexity. We summarize the main results and contributions.

- Chapter 4 presents an extensive overview of the most relevant literature dealing with typicality and description logics. To the best of our knowledge, the academic literature lacked such a survey. Our intention was to collect the most pressing results, put them in perspective, measure their advantages and shortcomings, and identify critical challenges.
- Chapter 6 covers typicality models for the $\mathcal{EL}_\bot$. This framework was originally presented in [PT17a], [PT18a], [Pen19] and is repackaged in this chapter with some new results. In particular, we
  - present the new strength **lex**, which extends the semantics based on typicality models to lexicographic materialization-based reasoning, a well-established and powerful framework for defeasible reasoning. This introduction gives rise to two new semantics, $\models_{\mathsf{prop},\mathbf{lex}}$ and $\models_{\mathsf{nest},\mathbf{lex}}$, the first which is shown to be equivalent to lexicographic materialization-based reasoning (Theorem 6.9) and the second which is shown to extend it, tackling quantification neglect (Theorem 6.33).
  - develop a full comparison of all existing semantics, showing the unintuitive result of breaking the hierarchy in the context of nested semantics.

- Chapter 7 develops a new framework for typicality models in the logic $\mathcal{ELI}_\bot$. Those new typicality models defined six different semantics for defeasible $\mathcal{ELI}_\bot$, parametrized along the two axes $\{\mathbf{prop}, \mathbf{nest}\} \times \{\mathbf{rat}, \mathbf{rel}, \mathbf{lex}\}$. This framework is roughly inspired by the original formulation for $\mathcal{EL}_\bot$, but has considerable differences. We highlight the main technical constructs:
  - new definition of typicality domains and satisfaction, which transforms the concept set of the original typicality models.Typicality domains for $\mathcal{EL}_\bot$ are defined through contexts, which pick arbitrary concepts to represent. Typicality domains for $\mathcal{ELI}_\bot$ drop those arbitrary concepts in favor of sets of named concepts representing their conjunction. This choice aims to handle challenges brought by the introduction of value restrictions. The new typicality domain leads to a new definition for a notion of satisfaction based on named concepts only, which can be extended to arbitrary concepts by the introduction of auxiliary names;
  - a new definition of minimal typicality domains based on the insights from canonical models for $\mathcal{ELI}_\bot$. This definition abandons the *standard* property and restricts edges to maximal successors, which we call *prime*;
  - the introduction of *initiator labelings*, a formal machinery to keep track of what a given edge represents. Put together with an interpretation, this machinery gives rise to *labeled interpretations*, which are the main ingredient of the upgrade procedure for $\mathcal{ELI}_\bot$;
  - a new definition of typicality update that takes initiator labelings into account in order to decide which element to update, and that updates the labels as well;
  - *model recovery*, which is a tableaux-inspired procedure to recover the model property when it is lost after an update. Model recovery is considerably more complex than its $\mathcal{EL}_\bot$ counterpart, *model completion*, because it is not limited to adding edges to the interpretation and elements to the concept extension. It has to remove some edges to maintain the primeness of successors and operates with the initiator labeling to accomplish this.
  - a sketch of the technical apparatus needed to incorporate assertional reasoning into defeasible $\mathcal{ELI}_\bot$ through ABox interpretations.
- Finally, we presented a preliminary discussion of the nature and challenges of defeasible reasoning in a first-order setting.

## 9.1 Future Work

The dissertation is but an exploratory foray into a large an yet to be explored the forest. There are several topics to be addressed, some of which we initially hoped to include here. We present some of the most important open problems and future work suggestions, ranked subjectively by difficulty.

### Low-hanging fruits

- Define strengths for other materialization-based reasoning semantics, such as MP-Closure;

  Some of the materialization-based semantics presented in Section 4.3.1 are not yet covered by typicality models. Studying and developing domains for those semantics is an interesting way of emphasizing the generality of the typicality models framework.

- Check nested semantics against the KLM principles;

  We discussed the shortcomings of analyzing DDLs with the KLM hierarchy. However, a first-order set of principles is still unavailable, and the principles are widely used. Therefore, evaluating semantics based on typicality models against those principles and positioning the entailment relations in the hierarchy is an interesting endeavor.

### Medium difficulty

- Instance checking for other strengths;

  In Chapter 7, we presented a sketch of how to incorporate assertional knowledge and instance checking to typicality models for $\mathcal{ELI}_\bot$ for rational strength. We did not consider stronger semantics, such as **rel** and **lex**. We conjecture that there is no insurmountable obstacle to this. In fact, Pensel (2019) sketches a procedure for relevant strength for $\mathcal{EL}_\bot$.

- Study the complexity of subsumption checking for defeasible $\mathcal{ELI}_\bot$ under nested **s**-semantics;

  Pensel (2019) presents some preliminary results for nested reasoning in $\mathcal{EL}_\bot$. Specifically, skeptical, rational nested reasoning is shown to be co-NP-complete, and relevant nested reasoning is shown to be in co-NExp [Pen19, p. 153]. Ut to this date, there are no established results for nested reasoning for $\mathcal{ELI}_\bot$, although we have some reasons to expect a considerable increase, namely: the increase of the domain, potentially exponential, and the complexity of the recovery procedure, which allows for multiple recoveries, contrasting to the single minimal model completion developed for $\mathcal{EL}_\bot$.

- Examine the possibility of extending the results from $\mathcal{ELI}_\bot$ to Horn-$\mathcal{ALC}$;

  One of the initial goals of our exploration was to lift the typicality models framework to the class of Horn-DLs, which posed more difficulties than we expected. There is a reason to suspect that the step to Horn-$\mathcal{ALC}$ is not insurmountable. Sabellek (2019) presents normalized Horn-$\mathcal{ALC}$ Tboxes whose axioms have the same forms as normalized $\mathcal{ELI}_\bot$ axioms. We conjecture that this points to the existence of a viable solution.

- Develop efficient implementations for the current existing algorithms.

There are no currently usable implementations for the algorithms described here. Implementing them would put the usefulness of our results to test.

### Hard difficulty

- Extend typicality models to expressive Horn-DLs;

  Besides Horn-$\mathcal{ALC}$, there is no self-evident path to lift the $\mathcal{ELI}_{\bot}$ machinery to more expressive Horn-DLs, such as Horn-$\mathcal{SHOIQ}$ and Horn-$\mathcal{SROIQ}$. Investigating this is a promising avenue of research.

- Adapt typicality models machinery to more complex reasoning tasks, such as query answering;

  One of the motivations to lift typicality models for the Horn fragments of DLs is to perform more complex reasoning tasks with defeasible information. One of those tasks is query answering, as developed in [ORS11]. There are some technical challenges to this task. Traditionally, query answering is done over possibly infinite tree-shaped models. Our upgrade procedure, however, is iterative and depends on finiteness to ensure termination. There is no obvious solution to this dilemma.

### Aspirational

- Develop a new set of KLM-like postulates to benchmark first-order defeasible logics;

  Throughout this dissertation, we made several allusions to the limitations of applying the KLM postulates to DLs. In short, the postulates are propositional and fall short of modeling first-order defeasible information. To this date, however, there is no widely set of postulates for defeasible reasoning encompassing quantifiers. To the best of our judgment, developing such principles is a crucial step to the area's progress.

- Develop techniques for defeasible reasoning with quantifications that apply to non-Horn-DLs;

  We pointed to the possibility of extending typicality models to a wide array of Horn-DLs. This choice is based on the fact that those logics have the canonical model property, a necessary but insufficient condition to develop typicality models. However, this property is lost due to disjunctive-like combinations of constructors featured in non-Horn DLs. It is impossible to represent the requirement $A \sqsubseteq B \sqcup C$ by a single membership. Therefore, typicality models are insufficient for developing defeasible reasoning for a wider class of DLs. However, we believe that this method's insights can be propagated to different approaches. One of the main intuitions is that preference relations over models should take edges into account as well. Preferential semantics for DLs usually limit their preference relation to elements in isolation. Incorporating edges – i.e. pairs of elements – into this formalism can be a first step toward a more general solution.

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