A generalization of the block decomposition for k-connected graphs

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Thesis presented to the
Institute of Mathematics and Statistics
of the University of São Paulo
in partial fulfillment
of the requirements
for the degree of
Master of Science

Program: Computer Science
Advisor: Yoshiharu Kohayakawa

During the development of this work, the author received financial support from CAPES

São Paulo
December, 2022

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This version of the thesis includes the corrections and modifications suggested by the Examining Committee during the defense of the original version of the work, which took place on December 15, 2022.

A copy of the original version is available at the Institute of Mathematics and Statistics of the University of São Paulo.

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Abstract

Jared León Malpartida. **A generalization of the block decomposition for k-connected graphs**. Thesis (Master's). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2022.

The decomposition of a connected graph by the set of its cut-vertices, sometimes called the "block decomposition" or "block tree" of a graph, is a well known and basic concept in graph theory. This decomposition, however, does not provide meaningful information when applied to a k-connected graph for $k \geqslant 2$. There has been a number of attempts to generalize the construction of the block decomposition of a graph for the case of k-connected graphs. Notably, Tutte constructed a tree that describes the mutual arrangement of 2-cutsets in a 2-connected graph. This decomposition has some similarities to the block decomposition of a connected graph. In other works, a block of a k-connected graph was defined as a maximal $(k+1)$-connected subgraph. Karpov described the decomposition of a k-connected graph by the set of k-cutsets that are not separated by any other k-cutset of the graph. Karpov also described some special properties of his decomposition for the case of a 2-connected graph. The decompositions defined by Karpov and Tutte for the case of a 2-connected graph share some similarities. In this work, we present a self-contained description of Karpov's decomposition. We also present some applications to the study of planarity, the chromatic number, critically 2-connected graphs, and the partition of certain 2-connected graphs into three connected subgraphs.

Keywords: block decomposition. block tree. k-connectivity.

Resumo

Jared León Malpartida. **Uma generalização da decomposição por blocos para grafos k-conexos**. Dissertação (Mestrado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022.

A decomposição de um grafo conexo pelo conjunto de seus vértices de corte, às vezes chamada de "decomposição por blocos" ou "árvore de blocos" de um grafo, é um conceito bem conhecido e básico na teoria dos grafos. Essa decomposição, no entanto, não fornece informações significativas quando é aplicada a um grafo k-conexo para $k \geqslant 2$. Tem havido uma série de tentativas de generalizar a construção da decomposição em blocos para grafos k-conexos. Notavelmente, Tutte construiu uma árvore que descreve o arranjo mútuo dos 2-cutsets em um grafo 2-conexo. Esta decomposição tem algumas semelhanças com a decomposição por blocos de um grafo conexo. Em outros trabalhos, um bloco de um grafo k-conexo é definido como um subgrafo $(k+1)$-conexo maximal. Karpov descreveu a decomposição de um grafo k-conexo pelo conjunto dos seus k-cutsets que não são separados por nenhum outro k-cutset do grafo. Karpov também descreveu algumas propriedades de sua decomposição para o caso de um grafo 2-conexo. As decomposições definidas por Karpov e Tutte para o caso de grafos 2-conexos compartilham algumas semelhanças. Neste trabalho, nós apresentamos uma descrição autocontida da decomposição de Karpov. Nós também apresentamos algumas aplicações para o estudo de planaridade, número cromático, grafos criticamente 2-conexos, e a partição de certos grafos 2-conexos em três subgrafos conexos.

Palavras-chave: decomposição por blocos. árvore de blocos. k-conexidade.

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Introduction

The block decomposition (also known as block tree) of a connected graph gives a partition of the set of vertices of the graph into (not necessarily disjoint) parts that induce maximal subgraphs that are 2-connected or copies of K_2 (also known as "non-separated subgraphs"), the blocks of the graph. These blocks are arranged in a tree-like structure. This decomposition is a well known and fundamental concept in graph theory. However, the usefulness of the block decomposition vanishes when applied to a 2-connected graph since the graph has no cut-vertices and hence, the only block of the graph is its set of vertices. There has been some attempts to generalize the classic block decomposition of a connected graph for k-connected graphs. One such attempt was made by Tutte, when he constructed a tree that describes the mutual arrangement of the 2-cutsets of a 2-connected graph (Tutte, 1966). Note that, although the set of 2-connected graphs is the set of k-connected graphs for $k \geqslant 2$, the graphs that are 3-connected do not contain 2-cutsets. Hence, Tutte's tree only provides meaningful information for graphs that are 2-connected but not 3-connected. Tutte's decomposition has some properties in common with the classic block decomposition of a connected graph. Two other paths have appeared in the search for a generalization. One has to do with defining blocks in terms of increased connectivity, and the other with defining blocks in terms of separability. For the former case, one approach that was taken was to define a block of a k-connected graph as a maximal $(k+1)$-connected subgraph (Matula, 1978; Harary and Kodama, 1964). This simple approach gives rise to properties that are rather different from the properties of the block decomposition of a connected graph, e.g., some vertices may not belong to some block, or the set of blocks may be empty in a non-empty graph. For the case of separability, one common idea is to consider a cutset S of a k-connected graph G and then define a block of the decomposition of G by the cutset S as the union of the set S and the vertex-set of a component of the graph $G - S$ (Hohberg, 1992; Karpov and Pastor, 2000). Where, in order to preserve the k-connectivity of the graphs induced by the blocks, one transforms the set S into a clique in all such graphs. In a more general case, one considers a sequence of k-cutsets $S_1, S_2, \dots, S_m$ and proceed in the following way. We first define the blocks of $G_1 = G$ by S_1 as the union of the set S_1 and the vertex-sets of the components of $G_1 - S_1$. If the next cutset S_2 is contained in one such blocks, let G_2 be the graph induced by this block. We repeat the process with G_2 and S_2, and continue this way with the remaining cutsets of the sequence. The sets that result of this process can be defined as the blocks of the decomposition of G by the sequence of k-cutsets $S_1, \dots, S_m$. Notice that when this sequence is the sequence of cut-vertices of G in any order, the decomposition corresponds to the classic block decomposition of G. A problem with this approach for $k \geqslant 2$ is that the order of the sequence matters, i.e., different orderings may produce difference decompositions if, for example, some cutsets of the sequence separate other cutsets in G. In this case, some of the cutsets are no longer contained in some block at the time of considering them. This approach is the base of a generalization proposed by Karpov, in which he described the decomposition of a k-connected graph by the set of cutsets that are not separated by any other cutset of the graph (Karpov, 2013). In his work, he also described some special properties for the case of a 2-connected graph, which share some similarities with Tutte's decomposition. In this work, we present a self-contained description of Karpov's decomposition using only concepts from elementary graph theory. We also present some applications of this decomposition to the study of planarity, the chromatic number, and critically 2-connected graphs.

Chapter 1

Preliminaries

In this chapter we define some concepts of separability of subsets of vertices. We also define the decomposition of a k-connected graph by an arbitrary set of its k-cutsets, which is a generalization of the block decomposition that we intend to study.

For a graph G, we denote by $v(G)$ the number of its vertices.

Let $G = (V, E)$ be a connected graph. A set $R \subset V$ is called a *cutset* if the graph $G - R$ is disconnected. If the cardinality of R is k, we also call R a k-cutset. If $k = 1$, we call the vertex contained in R a *cut-vertex*. In what follows when we deal with a singleton, say $X = v$, we may not distinguish the element v from the set X itself.

For an integer $k \geqslant 1$, the graph G is said to be *k-connected* if $|V| > k$ and G has no cutsets with fewer than k vertices. Note that there exists an integer $\ell \geqslant 1$ such that every connected graph except for the trivial graph, K_1, is k-connected for each $1 \leqslant k \leqslant \ell$.

A set $R \subset V$ *separates a set* $X \subset V$ *in* G if there are two vertices of $X \setminus R$ that belong to different components in the graph $G - R$. If the graph G is clear from the context, we simply say that R separates X. Observe that the graph $G[X]$ might not be connected (see Figure 1.1). Thus, if the graph $G[X \setminus R]$ is connected, then R does not separate X in G, but the converse is not necessarily true. Also, in order for R to separate X it is necessary that $|X| \geqslant 2$.

Let $R \subset V$, and let $X, Y \subset V$. Then set R *separates* X *and* Y *in* G if no component of the graph $G - R$ contains vertices of both X and Y. Note that when $X \subset R$ or $Y \subset R$, the set R separates X and Y in G. This definition is equivalent to the definition adopted by classic graph theory textbooks such as (Diestel, 2018).

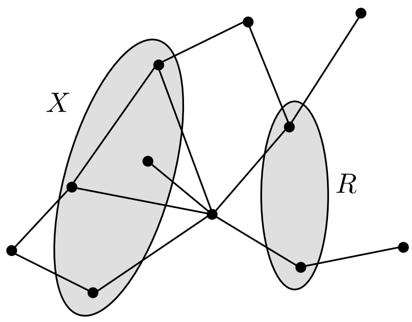

Figure 1.1: *The graph $G[X]$ is not connected, but the set R does not separate X.*

1.1 Decomposition of a k-connected graph by a set of cutsets

We now define the decomposition of a k-connected graph by an arbitrary set of k-cutsets of the graph.

Let $k \geqslant 1$ be fixed. Let G be a k-connected graph. Note that G has at least two vertices. Let $\mathrm{Cut}(G)$ and $\mathrm{Cut}_k(G)$ be the set of all cutsets and r-vertex cutsets of G respectively. Note that the set $\mathrm{Cut}_k(G)$ may possibly empty, e.g., when G is isomorphic to the graph K_{k+1}. Let $\mathfrak{S} \subset \mathrm{Cut}_k(G)$ be an arbitrary set of k-vertex cutsets of G. The decomposition of G by the set $\mathfrak{S}$ is defined as follows.

A set of vertices $A \subset V(G)$ is an $\mathfrak{S}$-*nonseparable set of* G if no cutset of $\mathfrak{S}$ separates A in G. Observe that the empty set is also a $\mathfrak{S}$-nonseparable set of G. Let $\mathrm{Ns}(G;\mathfrak{S})$ be the set of all non-empty $\mathfrak{S}$-nonseparable sets of G.

A set of vertices $A \subset V$ is an $\mathfrak{S}$-*block of* G if A is an inclusion-maximal element of $\mathrm{Ns}(G;\mathfrak{S})$. When the set $\mathfrak{S}$, is clear from the context, we call those elements *blocks of* G, or simply *blocks*. Clearly, no block is empty since every member of $\mathrm{Ns}(G;\mathfrak{S})$ is non-empty. The set of all $\mathfrak{S}$-blocks of G is denoted by $\mathrm{Block}(G;\mathfrak{S})$.

This is the more general version of the decomposition that we intend to study. We will now prove some properties of the blocks of the graph G.

Observe first the following.

Remark 1. (a) Given that $v(G) \geqslant 2$ and that a set that contains only one vertex is an $\mathfrak{S}$-nonseparable set of G, the set $\mathrm{Ns}(G;\mathfrak{S})$ is non-empty. Thus, the set $\mathrm{Block}(G;\mathfrak{S})$ is non-empty as well.

(b) For every $\mathfrak{S}$-nonseparable set of G, $A \in \mathrm{Ns}(G;\mathfrak{S})$ there is a block $A' \in \mathrm{Block}(G;\mathfrak{S})$ such that $A \subset A'$. This implies that every vertex in G is contained in a member of $\mathrm{Block}(G; \mathfrak{S})$.

Let A be a member of $\mathrm{Block}(G; \mathfrak{S})$. A vertex of A is said to be *internal* if no element of $\mathfrak{S}$ contains such vertex. Otherwise, the vertex is called a *boundary* vertex. Let $\mathrm{Int}(A)$ be the set of all internal vertices of A, which we call the *interior* of A. The *boundary* of A is the set $\mathrm{Bound}(A) = A \setminus \mathrm{Int}(A)$.

Proposition 2. *Let A, B be two different blocks of* $\mathrm{Block}(G; \mathfrak{S})$ *with nonempty intersection. Then there is at least one cutset $S \in \mathfrak{S}$ that separates $A \cup B$. For every such cutset, the set $A \cap B$ is contained in S.*

Proof. Given that A and B are different, the set $A \cup B$ contains both A and B properly, and so it cannot be an $\mathfrak{S}$-nonseparable set of G, so this implies the existence of a cutset $S \in \mathfrak{S}$ that separates $A \cup B$. Note that S also separates A from B since otherwise, S separates either A or B, which is not possible given that both sets are blocks of G (see Figure 1.2). Then there exist vertices $a \in A \setminus S$ and $b \in B \setminus S$ such that a and b belong to different components in the graph $G - S$. Suppose that there is a vertex $v \in (A \cap B) \setminus S$.

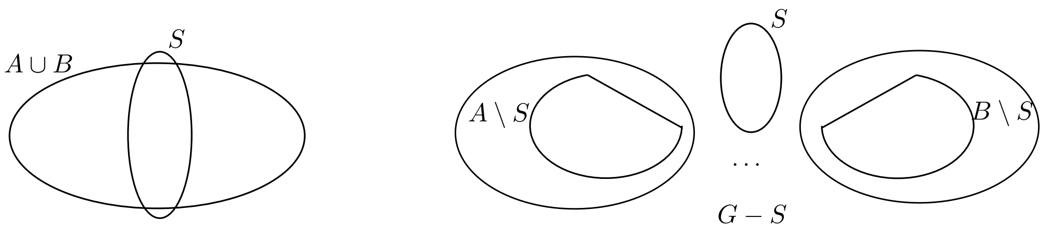

Figure 1.2: *The set S do not separate A and B individually.*

Then vertices a and v are in the same component in the graph $G - S$ since S does not separate set A. We also have that vertices v and b are in the same component in the graph $G - S$, which implies that a and b are in the same component in the graph $G - S$, a contradiction. We conclude that $(A \cap B) \setminus S = \emptyset$, and given that $A \cap B \neq \emptyset$, it follows that $A \cap B \subset S$. $\square$

Corollary 3. *A consequence of Proposition 2 is that for any pair of different blocks $A, B \in$* $\mathrm{Block}(G; \mathfrak{S})$*, the sets* $\mathrm{Int}(A)$ *and* $\mathrm{Int}(B)$ *are disjoint. Indeed, if a vertex belongs to both* $\mathrm{Int}(A)$ *and* $\mathrm{Int}(B)$*, then by Proposition 2 it belongs to some member of $\mathfrak{S}$, making it a boundary vertex of A and B.* $\square$

Corollary 3 implies that any vertex of G is either an internal vertex of a single block, or a boundary vertex of at least one block. When the set of k-vertex cutsets of G is clear from the context, we denote by $\mathrm{Int}(G)$ the set of vertices that belong to the interior of some block of G, and we denote by $\mathrm{Bound}(G)$ the set $V(G) \setminus \mathrm{Int}(G)$.

Proposition 4. *Let $S \in \mathfrak{S}$, and let $v \in S$. Then v is adjacent to a vertex in every component of the graph $G - S$.*

Proof. Suppose by contradiction that there is a component A of $G - S$ such that v is not adjacent to any vertex of A. We conclude that $k \geqslant 2$. Also, there is a path in G from v to a vertex of A given that G is connected. Then every such a path in G must contain vertices of $S \setminus \{v\}$. Therefore $S \setminus \{v\}$ separates v from A in G, contradicting the k-connectivity of G. □

Remark 5. Let $A \in \mathrm{Block}(G; \{S\})$ be an $\{S\}$-block of G. It is easy to see that A is the union of S and the vertices of a component of $G - S$. Indeed, such union is not separated by S in G, but adding one extra vertex of $V(G) \setminus A$ to A makes the set A separable by S in G (see Figure 1.3).

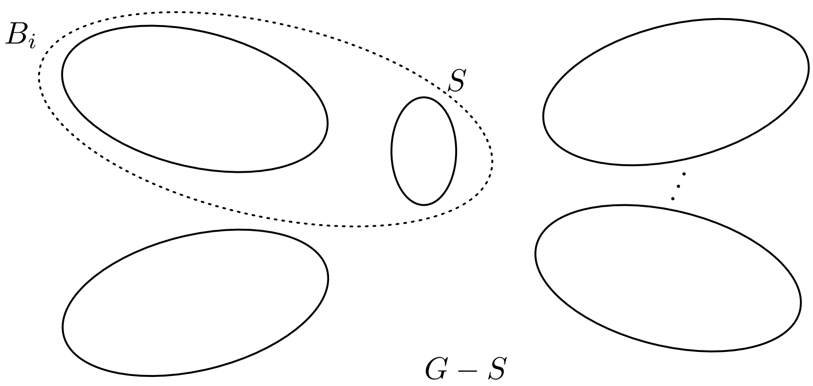

Figure 1.3: *The decomposition of G by $\{S\}$.*

The next result characterizes the boundary and interior of a block of G.

Proposition 6. *Let A be a block of* $\mathrm{Block}(G; \mathfrak{S})$*. Then the set* $\mathrm{Bound}(A)$ *consists of all vertices of A that are adjacent to a vertex in $V(G) \setminus A$.*

Proof. Let $v \in A$ and assume that v is adjacent to a vertex $u \in V(G) \setminus A$. The set $A \cup \{u\}$ is not a member of $\mathrm{Block}(G; \mathfrak{S})$ given the maximality of A. Then there is a cutset $S \in \mathfrak{S}$ such that S separates $A \cup \{u\}$. Given that S does not separate A, vertex u lies in a component different from the one containing $A \setminus S$ in the graph $G - S$ (see Figure 1.4). It follows that $v \notin A \setminus S$, otherwise u and $A \setminus S$ would be in the same component of $G - S$, because u and v are adjacent. Thus $v \in A \cap S \subset S$, i.e., v is a boundary vertex of A.

Let $v \in \mathrm{Bound}(A)$ be a boundary vertex of A, and let $T \in \mathfrak{S}$ be a cutset such that $v \in S$. Given that T does not separate A, all the vertices of $A \setminus T$ are in the same component in the graph $G - T$. By Proposition 4, v is adjacent to a vertex in every

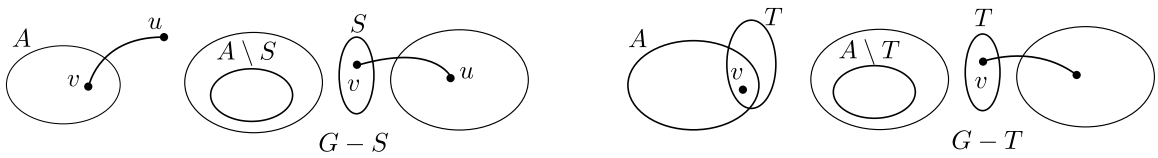

Figure 1.4: *Characterization of the boundary of a block.*

component in the graph $G - T$, at least one of which is different from the component that contains the vertices of $A \setminus T$. Then v is adjacent to a vertex of $V(G) \setminus A$. $\square$

Corollary 7. *Let $\mathfrak{S} \neq \emptyset$, and let $A \in \mathrm{Block}(G; \mathfrak{S})$ be a block of G. Then*

(a) $\mathrm{Bound}(A)$ *separates* $\mathrm{Int}(A)$ *and* $V(G) \setminus A$

(b) *If Int$(A) \neq \emptyset$, then* $|\mathrm{Bound}(A)| \geqslant k$ *and* $|A| \geqslant k+1$.

Proof. (a) Every path from a vertex of $\mathrm{Int}(A)$ to a vertex of $V(G) \setminus A$ contains an edge uv such that $u \in A$, $v \in V(G) \setminus A$. Then $u \in \mathrm{Bound}(A)$ and $\mathrm{Bound}(A)$ separates $\mathrm{Int}(A)$ from $V(G) \setminus A$.

(b) Part (a) implies that $\mathrm{Bound}(A) \in \mathrm{Cut}(G)$. Every member of $\mathrm{Cut}(G)$ has size at least k given that G is k-connected. The set $\mathrm{Int}(A)$ contains at least one vertex. $\square$

It is easy to see that every block has size at least 2 given that the graph has no isolated vertices, and the endpoints of every edge cannot be separated. Observe that, when the interior of a block is empty, the sizes of the blocks need not be bounded below as in part (b) of Corollary 7. For instance, the complete bipartite graph $G = K_{k,k}$ is k-connected, and the set $\mathrm{Cut}_k(G)$ contains two elements: the parts of the bipartition of $V(G)$. Each block of the decomposition of G by the set $\mathrm{Cut}_k(G)$ contains only two vertices: the endpoints of an edge of the graph. If any extra vertex of G is added to such a block, then two vertices of the new set are necessarily separated by one of the two cutsets of $\mathrm{Cut}_k(G)$.

Remark 8. If $\mathfrak{S} \neq \emptyset$, then $\mathrm{Bound}(A) \neq \emptyset$ for every $A \in \mathrm{Block}(G; \mathfrak{S})$.

The statements and proofs of Chapters 2 and 3 are reformulations of the original results by Karpov. We also present results that were not previously published.

Chapter 2

The decomposition tree

We now describe some properties of the above decomposition by a set of k-cutsets whose elements do not separate each other. First, we present the general form of the decomposition for k-connected graphs and then we show some special properties of this decomposition for 2-connected graphs and a specific set of pairwise independent 2-cutsets.

2.1 For a k-connected graph

Let an integer $k \geqslant 1$ be fixed. Let G be a k-connected graph. Two cutsets of $\mathrm{Cut}(G)$ are said to be *independent* if none of them separates the other in G. Otherwise, the cutsets are said to be *dependent* (see Figure 2.1). Clearly if all the cutsets are cut-vertices, then all of them are pairwise-independent.

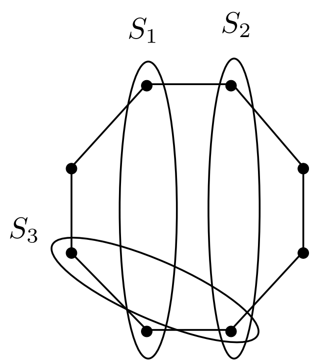

Figure 2.1: *The pairs S_1, S_2 S_2, S_3 are independent. The cutsets S_1 and S_3 are dependent.*

Proposition 9. *Let $S, T \in \mathrm{Cut}(G)$ be a pair of cutsets. Then either S and T are independent, or each of them separates the other in G.*

Proof. Assume that S does not separate T. We will prove that T does not separate S. Given that S is a cutset of G, the graph $G - S$ has at least two components. Clearly all the vertices in $T \setminus S$ are in the same component in the graph $G - S$. Let H be a component of $G - S$ that does not contain vertices of T (see Figure 2.2).

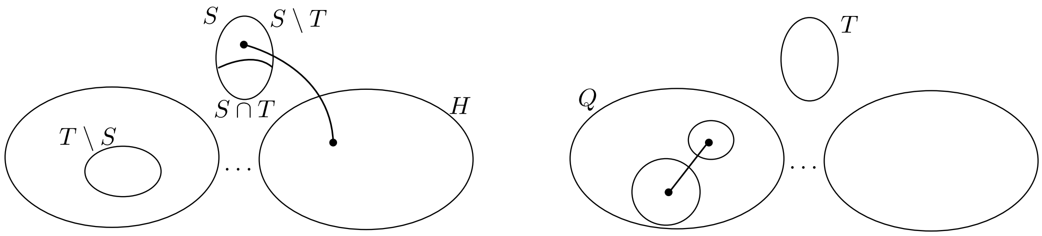

Figure 2.2: *Independent or mutually separable.*

Given that H is connected, all the vertices of $V(H)$ are in the same component Q in the graph $G - T$. By Proposition 4, every vertex of S is adjacent to at least one vertex of H in G. Thus, all vertices of $S \setminus T$ are in the component Q in the graph $G - T$. We conclude that T does not separate S in G, and the cutsets are independent. $\square$

Let $\mathfrak{S} \subset \mathrm{Cut}_k(G)$ be a set of pairwise independent cutsets. We now study some properties of the decomposition of G by the set $\mathfrak{S}$. In the end, we shall show that this decomposition produces a tree-like structure. More precisely, we will show that the properties of the sets $\mathrm{Block}(G; \mathfrak{S})$ and $\mathfrak{S}$ are identical to the properties of the sets $\mathcal{B}$ and $\mathcal{C}$ respectively in the following definition.

Definition 10. Let G be a connected graph. A pair $\mathcal{B}, \mathcal{C} \subset 2^{V(G)}$ is said to form a tree-like structure on G if the sets $\mathcal{B}$ and $\mathcal{C}$, and the $(\mathcal{B}, \mathcal{C})$-bipartite graph T, where $B \in \mathcal{B}$ is adjacent to $C \in \mathcal{C}$ in T if and only if $C \subset B$ have the following properties:

1. $$\bigcup_{B \in \mathcal{B}} B = V(G).$$

2. All the elements of $\mathcal{C}$ have the same size.

3. Let $A_1, A_2 \in \mathcal{B}$ be distinct sets such that $A_1 \cap A_2 \neq \emptyset$. Then $A_1 \cap A_2 \in \mathcal{C}$.

4. Let $A_1, A_2 \in \mathcal{B}$ be sets and suppose that there is an edge $a_1 a_2 \in E(G)$ with $a_1 \in A_1$, and $a_2 \in A_2$. Then $A_1 \cap A_2 \neq \emptyset$, and at least one of a_1,a_2 belongs to $A_1 \cap A_2$.

5. The graph T is a tree.

6. Every member of $\mathcal{C}$ has degree at least 2 in T (see Figure 2.3).

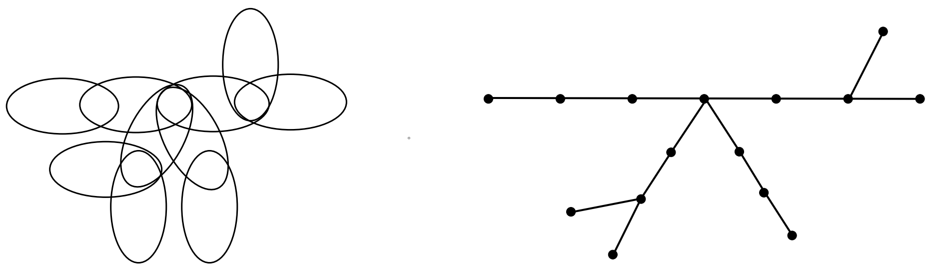

Figure 2.3: *A tree-like structure.*

The classic block decomposition of a connected graph and the set of its cut-vertices clearly forms a tree-like structure on G. However, the proof that the pair $\mathrm{Block}(G;\mathfrak{S})$ and $\mathfrak{S}$ forms a tree-like structure on G is technical. Before going into the proof of this fact, we first show some auxiliary properties of the set $\mathfrak{S}$ of pairwise independent cutsets that will be useful later.

Remark 11. Every cutset from $\mathfrak{S}$ is a member of $\mathrm{Ns}(G;\mathfrak{S})$ given that no other cutset from $\mathfrak{S}$ separates it. This implies that, by Remark 1 (b), every cutset of $\mathfrak{S}$ is contained in a member of $\mathrm{Block}(G;\mathfrak{S})$.

Let $G^{\mathfrak{S}}$ be the graph resulting by taking the graph G and for each cutset $S \in \mathfrak{S}$, adding all the edges that connect pairs of vertices of the set S. This graph will be useful to study some properties of the block decomposition of G by $\mathfrak{S}$ since, as we prove later, the $\mathfrak{S}$-blocks of G and $G^{\mathfrak{S}}$ are the same.

Proposition 12. *Two vertices of $V(G)$ are separated in G by a cutset $S \in \mathfrak{S}$ if and only if they are separated in $G^{\mathfrak{S}}$ by S.*

Proof. Clearly, if two vertices are separated in $G \subset G^{\mathfrak{S}}$, the same holds in G.

Now assume that two vertices, say x and y are separated in G by a cutset $S \in \mathfrak{S}$. For every cutset $T \in \mathfrak{S}$ different from S, we know that the vertices in $T \setminus S$ are contained in a single component of the graph $G - S$ given that S does not separate T. Hence, any edge in $E(G^{\mathfrak{S}}) \setminus E(G)$ cannot be incident with vertices in different components of the graph $G - S$. So the components of $G - S$ are $G^{\mathfrak{S}} - S$ are the same and hence, x and y are separated by S in $G^{\mathfrak{S}}$.

□

Corollary 13. *The following statements hold:*

(a) $\mathfrak{S} \subset \mathrm{Cut}_k(G^{\mathfrak{S}})$, *and the cutsets of $\mathfrak{S}$ are also independent in $G^{\mathfrak{S}}$.*

(b) $\mathrm{Ns}(G;\mathfrak{S}) = \mathrm{Ns}(G^{\mathfrak{S}};\mathfrak{S})$.

(c) $\mathrm{Block}(G;\mathfrak{S}) = \mathrm{Block}(G^{\mathfrak{S}};\mathfrak{S})$.

Proof. (a) Proposition 12 implies that, for every cutset $S \in \mathfrak{S}$, the vertex sets of the components of $G - S$ are the vertex sets of the components of $G^{\mathfrak{S}} - S$. Hence, every element of $\mathfrak{S}$ is also a k-cutset of $G^{\mathfrak{S}}$. Clearly no element of $\mathfrak{S}$ is separated in $G^{\mathfrak{S}}$ given that all of them are cliques.

(b) Let $A \in \mathrm{Ns}(G^{\mathfrak{S}};\mathfrak{S})$ be an $\mathfrak{S}$-nonseparable set of $G^{\mathfrak{S}}$, and suppose that $A \notin \mathrm{Ns}(G;\mathfrak{S})$. Then, there exists a cutset $S \in \mathfrak{S}$ that separates A in G. Let $a, b \in A \setminus S$ be vertices in different components in the graph $G - S$. Clearly the vertices a and b are separated in G by S. By Proposition 12, the vertices a and b are also separated by S in $G^{\mathfrak{S}}$. This contradicts that A is a $\mathfrak{S}$-nonseparable set of G, and hence $\mathrm{Ns}(G^{\mathfrak{S}};\mathfrak{S}) \subset \mathrm{Ns}(G;\mathfrak{S})$. It is easy to see that $\mathrm{Ns}(G;\mathfrak{S}) \subset \mathrm{Ns}(G^{\mathfrak{S}};\mathfrak{S})$ given that the graph $G^{\mathfrak{S}}$ contains all the edges of G.

(c) The set $\mathrm{Block}(G;\mathfrak{S})$ is the collection of all inclusion-maximal elements of $\mathrm{Ns}(G;\mathfrak{S}) = \mathrm{Ns}(G^{\mathfrak{S}};\mathfrak{S})$. □

Proposition 14. *Let A be a member of $\mathrm{Block}(G;\mathfrak{S})$ and assume that $\mathfrak{S} \neq \emptyset$. Then there exists some $S \in \mathfrak{S}$ such that $S \subset A$ with $S \neq A$.*

Proof. Given that $\mathrm{Block}(G;\mathfrak{S}) = \mathrm{Block}(G^{\mathfrak{S}};\mathfrak{S})$, we may prove the statement for $G^{\mathfrak{S}}$. Suppose by contradiction that there is a set $A \in \mathrm{Block}(G^{\mathfrak{S}};\mathfrak{S})$ that either does not contain a cutset of $\mathfrak{S}$, or the only cutset $F \in \mathfrak{S}$ it contains is $F = A$. Let $\mathfrak{S}'$ be defined as $\mathfrak{S}$ in the case that A does not contain a cutset and $\mathfrak{S} \setminus \{F\}$ otherwise. We now make two observations.

(a) No element of $\mathfrak{S}'$ is contained in A. This claim is obvious for the case where A does not contain any element of $\mathfrak{S}$. In the case where $A = F$, the set A has size k. Hence, it cannot contain a member of $\mathfrak{S}$ other than F.

(b) The set A intersects some element of $\mathfrak{S}'$. In the case where A does not contain a cutset of $\mathfrak{S}$, by Remark 8 it follows that $\mathrm{Bound}(A) \neq \emptyset$ given that $\mathfrak{S} \neq \emptyset$, and therefore set A intersects some element of $\mathfrak{S} = \mathfrak{S}'$. In the case where $A = F$, we have $\mathrm{Bound}(A) = A$. Let $a \in A$. By Proposition 6, a is adjacent to a vertex $b \in V(G) \setminus A$. Given the maximality of A, the set $A \cup \{b\}$ is not a block of $G^{\mathfrak{S}}$. Then, there exists a cutset $J \in \mathfrak{S}$ that separates $A \cup \{b\}$ in $G^{\mathfrak{S}}$. Note that $J \neq F$. Hence $J \in \mathfrak{S}'$. Clearly J does not separate A in $G^{\mathfrak{S}}$ given that A is a block of $G^{\mathfrak{S}}$. This implies that J separates $A \setminus J$ from b in $G^{\mathfrak{S}}$. The vertices of $A \setminus J$ belong to the same component of the graph $G^{\mathfrak{S}} - J$, and vertex b belongs to a different component of the graph $G^{\mathfrak{S}} - J$.

The fact that a and b are adjacent implies that $a \notin A \setminus J$ and therefore, that $a \in J$. It follows that A intersects an element of $\mathfrak{S}'$.

Let us choose a cutset $J \in \mathfrak{S}'$ such that $|A \cap J|$ is maximum, and let $Q = A \cap J$. By the observations (a) and (b) it follows that $1 \leqslant |Q| < k$. Let $\mathcal{C} \subset \mathfrak{S}'$ be the collection of cutsets of $\mathfrak{S}'$ such that their intersection with A is Q. Obviously $\mathcal{C}$ is nonempty since it has J as a member. Also, the set $A \setminus Q$ is nonempty. Indeed, in the case where A contains a cutset of $\mathfrak{S}$, we have $|A| = k$ and $|Q| < k$. In the case that A does not contain a cutset of $\mathfrak{S}$, if $A \setminus Q = \emptyset$, then $J \subset A$ since J cannot be separated by any other cutset, but this is clearly a contradiction.

Let (S, v, u, P) be a 4-tuple that satisfies the following conditions.

1. $S \in \mathcal{C}$.
2. $v \in S \setminus A$.
3. $u \in A \setminus Q$.
4. P is a vu-path in $G^{\mathfrak{S}}$ such that $V(P) \cap Q = \emptyset$.

It is easy to see that at least one such tuple exists. Indeed, for condition 1, the set $\mathcal{C}$ is nonempty. For condition 2, the set S is not contained in A for any $S \in \mathcal{C}$. For condition 3, the set $A \setminus Q$ is nonempty. For condition 4, note that $v, u \notin Q$ and since $|Q| < k$ clearly there is a vu-path in $G^{\mathfrak{S}} - Q$. Choose a tuple (S, v, u, P) such that P is as short as possible.

Given the maximality of the set A, the set $A \cup \{v\}$ is not a block of $G^{\mathfrak{S}}$. Then there exists a cutset $T \in \mathfrak{S}$ that separates $A \cup \{v\}$, i.e., the set $A \cup \{v\}$ contains vertices in different components in the graph $G^{\mathfrak{S}} - T$. Note that the vertices of $A \setminus T$ belong to the same component, and the vertex v belongs to a different component in such a graph. Clearly $S \neq T$ since $v \in S$. Observe however that $Q \subset T$. Indeed, if there is a vertex $v' \in Q \setminus T$, then v and v' are in the same component in the graph $G^{\mathfrak{S}} - T$ given that $v, v' \in S$ and S is a clique. But then vertex $v' \in Q \setminus T \subset A$ is in a different component than the vertices of $A \setminus T$ in such a graph, contradicting that A is a block of $G^{\mathfrak{S}}$. The fact that $Q \subset T$ implies that $T \in \mathcal{C}$ by the maximality of Q. Note that vertices v and u do not belong to T. Also, P contains some vertex t of T, otherwise v and u are in the same component in the graph $G^{\mathfrak{S}} - T$. Furthermore, $t \in T \setminus A$ since P does not use vertices of Q. Let P' be the tu-subpath of P (see Figure 2.4).

We claim that the tuple (T, t, u, P') is valid and contradicts the choice of the tuple (S, v, u, P). For the conditions, the tuple satisfies that $T \in \mathcal{C}$, $t \in T \setminus A$, $u \in A \setminus Q$, and P' is a tu-path in $G^{\mathfrak{S}}$ such that $V(P') \cap Q = \emptyset$ and P' is strictly shorter than P

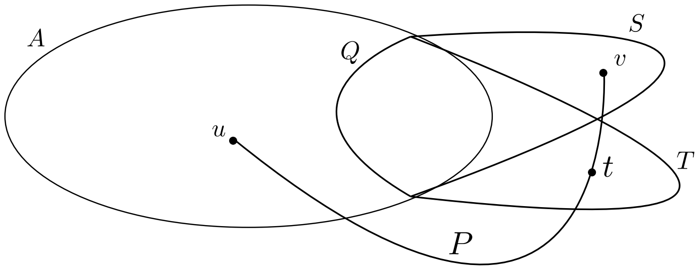

Figure 2.4: *Every block properly contains a cutset.*

since $v \neq t$, a contradiction. This shows that the supposition that there is a block of $\mathrm{Block}(G; \mathfrak{S})$ that does not contain a cutset of $\mathfrak{S}$ properly is false, and the proof is complete. $\square$

Remark 15. Every member of $\mathrm{Block}(G; \mathfrak{S})$ contains at least $k+1$ vertices. Indeed, if $\mathfrak{S} \neq \emptyset$, then by Proposition 14, every block of G contains some k-vertex cutset and some other vertex. If $\mathfrak{S} = \emptyset$, then the single block in $\mathrm{Block}(G; \mathfrak{S})$ is the set $V(G)$, which by k-connectivity has at least $k+1$ vertices.

Proposition 16. *Let $\mathfrak{T} \subset \mathfrak{S}$, let $A \in \mathrm{Block}(G; \mathfrak{T})$, and let $R \in \mathrm{Cut}(G^{\mathfrak{S}}[A])$. Then, $R \in \mathrm{Cut}(G)$.*

Proof. Suppose by contradiction that R is not a cutset of G. Given that R is a cutset of $G^{\mathfrak{S}}[A]$, let $x, y \in A$ be vertices such that R separates x from y in $G^{\mathfrak{S}}[A]$. The cutset R does not separate x from y in G, and hence, in $G^{\mathfrak{S}}$. Let P be a shortest xy-path in the graph $G^{\mathfrak{S}} - R$. Observe that the vertices of P do not all belong to A, otherwise P is an xy-path in $G^{\mathfrak{S}}[A] - R$ and R does not separate x from y in $G^{\mathfrak{S}}[A]$ (see Figure 2.5).

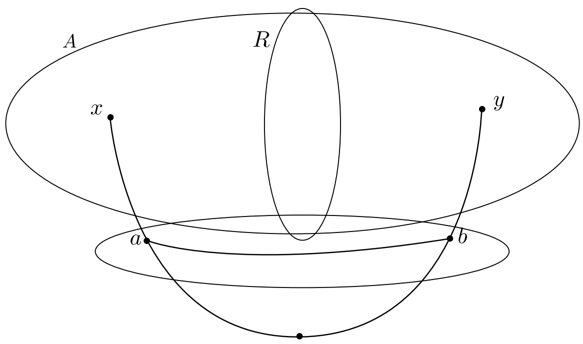

Figure 2.5: *The cutsets of the blocks are cutsets of the graph.*

Let z be a vertex of P that does not belong to set A. Given that A is a member of $\mathrm{Block}(G; \mathfrak{T})$, there is a cutset $T \in \mathfrak{T}$ such that T separates $A \cup \{z\}$. This implies that the vertices of P intersect the set T. Let a, b be the last vertices of P that belong to T going from z to x and from z to y in P respectively. Clearly, a and b are different, and adjacent in $G^{\mathfrak{S}}$ given that T is a clique in such a graph. Then, it is possible to replace the ab-section of P with the edge ab and create an xy-path shorter than P, a contradiction. This proves that R is a cutset of G. □

Corollary 17. *Let $\mathfrak{T} \subset \mathfrak{S}$, and let $A \in \mathrm{Block}(G; \mathfrak{T})$. Then the graph $G^{\mathfrak{S}}[A]$ is k-connected.*

Proof. By Remark 15, the graph $G^{\mathfrak{S}}[A]$ contains at least $k+1$ vertices. Thus, it is enough to show that $G^{\mathfrak{S}}[A]$ does not contain a cutset of size fewer than k. Suppose by contradiction that $R \in \mathrm{Cut}_{k-1}(G^{\mathfrak{S}}[A])$. Then, by Proposition 16, $R \in \mathrm{Cut}_{k-1}(G)$, which is not possible given that G is k-connected. Therefore $\mathrm{Cut}_{k-1}(G^{\mathfrak{S}}[A]) = \emptyset$. □

Let $\mathrm{T}(G; \mathfrak{S})$ be the bipartite graph in which one part is the set $\mathfrak{S}$, the other part is the set $\mathrm{Block}(G; \mathfrak{S})$, and two vertices $S \in \mathfrak{S}$ and $A \in \mathrm{Block}(G; \mathfrak{S})$ are adjacent if and only if $S \subset A$.

Proposition 18. *Let $S \in \mathfrak{S}$. Let $\mathrm{Block}(G; \{S\}) = \{A_1, \ldots, A_m\}$. For all, $i = 1, \ldots, m$, let $G_i = G^{\mathfrak{S}}[A_i]$, and let $\mathfrak{S}_i \subset \mathfrak{S}$ be the set of cutsets different from S that are contained in A_i. Then,*

(a) *The graph G_i is k-connected.*

(b) *For each $A_i \in \mathrm{Block}(G; \{S\})$, there is a unique $U_i \in \mathrm{Block}(G; \mathfrak{S})$ with $S \subset U_i \subset A_i$.*

(c) *The following holds:*

$$\mathrm{Block}(G; \mathfrak{S}) = \bigcup_{i=1}^{m} \mathrm{Block}(G_i; \mathfrak{S}_i), \tag{2.1}$$

and this union is disjoint.

Proof. Item (a) follows immediately from Corollary 17, i.e. by setting $\mathfrak{T} = \{S\}$.

We now prove items (b) and (c). Fix i in $\{1, \ldots, m\}$. By the independence between the members of $\mathfrak{S}$, each cutset from $\mathfrak{S} \setminus \{S\}$ belongs to exactly one of $\mathfrak{S}_1, \ldots, \mathfrak{S}_m$. The fact that S is a clique in G_i implies that it is also an element of $\mathrm{Ns}(G_i; \mathfrak{S}_i)$. By Remark 1 (b), the set S is contained in a member of $\mathrm{Block}(G_i; \mathfrak{S}_i)$. Furthermore, we shall show that S is contained in exactly one such an element of $\mathrm{Block}(G_i; \mathfrak{S}_i)$. Suppose by contradiction that there are different sets $X, Y \in \mathrm{Block}(G_i; \mathfrak{S}_i)$ that contain S. By Proposition 2, the set $X \cap Y$ is contained in some cutset of $\mathfrak{S}_i$. Then, set $S \subset X \cap Y$ is also contained in a cutset of $\mathfrak{S}_i$. Given that all cutsets of $\mathfrak{S}_i$ have size k, it follows that $X \cap Y = S$ and thus S belongs to $\mathfrak{S}_i$, which contradicts the definition of $\mathfrak{S}_i$. Let U_i be the member of $\mathrm{Block}(G_i; \mathfrak{S}_i)$ that contains cutset S.

For one side of Equation (2.1), we will show that $\mathrm{Block}(G_i; \mathfrak{S}_i) \subset \mathrm{Block}(G^{\mathfrak{S}}; \mathfrak{S}) = \mathrm{Block}(G; \mathfrak{S})$. Let $U \in \mathrm{Block}(G_i; \mathfrak{S}_i)$, and let $T \in \mathfrak{S}$ be given.

In order to show that $U \in \mathrm{Block}(G^{\mathfrak{S}}; \mathfrak{S})$, let us first prove that T does not separate U in $G^{\mathfrak{S}}$, i.e., $U \in \mathrm{Ns}(G^{\mathfrak{S}}; \{T\})$, and then we will prove that U is an inclusion-maximal element of $\mathrm{Ns}(G^{\mathfrak{S}}; \mathfrak{S})$, which will imply that U is an $\mathfrak{S}$-block of $G^{\mathfrak{S}}$. Consider first the case in which $T = S$. It follows that $U \subset A_i \in \mathrm{Block}(G^{\mathfrak{S}}; \{S\}) \subset \mathrm{Ns}(G^{\mathfrak{S}}; \{S\}) = \mathrm{Ns}(G^{\mathfrak{S}}; \{T\})$. Hence U is contained in a member of $\mathrm{Ns}(G^{\mathfrak{S}}; \{T\})$, which implies that U is also a member of $\mathrm{Ns}(G^{\mathfrak{S}}; \{T\})$. Consider now the case in which $T \neq S$. Suppose first that $T \subset A_i$. Given that $T \neq S$, we have $T \in \mathfrak{S}_i$. Then $U \in \mathrm{Block}(G_i; \mathfrak{S}_i) \subset \mathrm{Ns}(G_i; \mathfrak{S}_i) \subset \mathrm{Ns}(G_i; \{T\}) \subset \mathrm{Ns}(G^{\mathfrak{S}}; \{T\})$, where the last step holds since $G_i \subset G^{\mathfrak{S}}$. Therefore in this case we also have $U \in \mathrm{Ns}(G^{\mathfrak{S}}; \{T\})$. Suppose now that $T \not\subset A_i$. It follows that $|T \cap A_i| < k$ since T has k elements. By Corollary 17, the graph $G_i = G^{\mathfrak{S}}[A_i]$ is k-connected. Then, the set $T \cap A_i$ is not a cutset of G_i. This also implies that T does not separate the set $U \in A_i$ in $G^{\mathfrak{S}}$, i.e., $U \in \mathrm{Ns}(G^{\mathfrak{S}}; \{T\})$.

We now prove that U is an inclusion-maximal element of $\mathrm{Ns}(G^{\mathfrak{S}}; \mathfrak{S})$. Let $B \in \mathrm{Block}(G^{\mathfrak{S}}; \mathfrak{S})$ be a set and suppose by contradiction that $U \subset B$ with $U \neq B$. By Remark 15, set $U \subset A_i$ contains at least $k+1$ vertices, at least one of which belongs to $A_i \setminus S$ given that S has k elements. Observe that $B \subset A_i$. Indeed, the set B contains U, and therefore, contains at least one element of $A_i \setminus S$. Also, the set B is not separated by S in $G^{\mathfrak{S}}$ given that $B \in \mathrm{Block}(G^{\mathfrak{S}}; \mathfrak{S})$. By the maximality of U in $\mathrm{Ns}(G_i; \mathfrak{S}_i)$, there is a cutset $L \in \mathfrak{S}_i$ that separates set B in G_i, i.e., $B \notin \mathrm{Ns}(G_i; \{L\})$. This implies that there are two vertices $x, y \in B \setminus L$ such that every xy-path in G_i contains a vertex of L. The fact that L does not separate set B in $G^{\mathfrak{S}}$, i.e., $B \in \mathrm{Ns}(G^{\mathfrak{S}}; \mathfrak{S}) \subset \mathrm{Ns}(G^{\mathfrak{S}}; \{L\})$ implies that there is an xy-path in $G^{\mathfrak{S}}$ that contains vertices of $V(G^{\mathfrak{S}}) \setminus A_i$ and does not contain vertices of L. Note that such a path contains vertices of S. Let a and b be the first and last vertices of the path that belong to S going from x to y respectively. Given that S is a clique in $G^{\mathfrak{S}}$, the ab-section of the path can be replaced with the edge ab, obtaining an xy-path in G_i that does not contain vertices of L, a contradiction. We conclude that U is an inclusion-maximal element of $\mathrm{Ns}(G^{\mathfrak{S}}; \mathfrak{S})$. Thus, $U \in \mathrm{Block}(G^{\mathfrak{S}}; \mathfrak{S})$. This proves that $\mathrm{Block}(G_i; \mathfrak{S}_i) \subset \mathrm{Block}(G^{\mathfrak{S}}; \mathfrak{S})$ (see Figure 2.6).

We will now prove the other side of Equation (2.1), i.e., that $\mathrm{Block}(G^{\mathfrak{S}}; \mathfrak{S}) \subset$

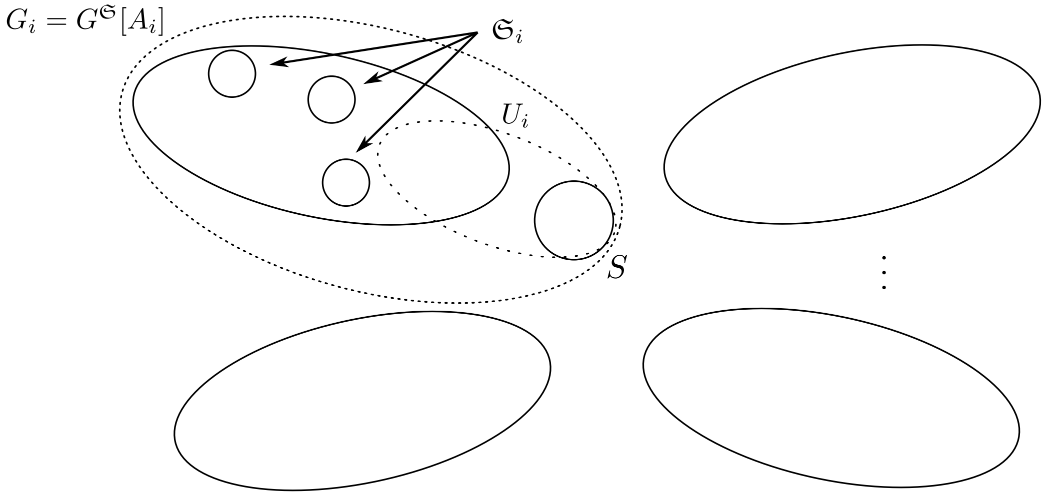

Figure 2.6: *The decomposition of the components of $G - S$.*

$\mathrm{Block}(G_i; \mathfrak{S}_i)$. For this, let $A \in \mathrm{Block}(G^{\mathfrak{S}}; \mathfrak{S})$. We shall show that $A \in \mathrm{Block}(G_i; \mathfrak{S}_i)$, for some i. For this, it is clear that the set S does not separate A in $G^{\mathfrak{S}}$. Indeed, $A \in \mathrm{Block}(G^{\mathfrak{S}}; \mathfrak{S}) \subset \mathrm{Ns}(G^{\mathfrak{S}}; \mathfrak{S}) \subset \mathrm{Ns}(G^{\mathfrak{S}}; \{S\})$. Consequently, $A \subset A_i$, for some i.

We will first prove that $A \in \mathrm{Ns}(G_i; \mathfrak{S}_i)$, and then we will show that it is an inclusion-maximal element of such a set, which will imply that $A \in \mathrm{Block}(G_i; \mathfrak{S}_i)$ as desired. Suppose by contradiction that $A \notin \mathrm{Ns}(G_i; \mathfrak{S}_i)$, i.e., there is a cutset $M \in \mathfrak{S}_i$ that separates A in G_i. Then, there are two vertices $x, y \in A \setminus M$ such that every xy-path in G_i contains a vertex of M. The fact that M does not separate set A in $G^{\mathfrak{S}}$, i.e., $A \in \mathrm{Ns}(G^{\mathfrak{S}}; \mathfrak{S}) \subset \mathrm{Ns}(G^{\mathfrak{S}}; \{M\})$ implies that there is an xy-path in $G^{\mathfrak{S}}$ that contains vertices of $V(G^{\mathfrak{S}}) \setminus A_i$ and does not contain vertices of M. Note that such a path contains vertices of S. Let a and b be the first and last vertices of the path that belong to S going from x to y respectively. Given that S is a clique in $G^{\mathfrak{S}}$, the ab-section of the path can be replaced with the edge ab, obtaining an xy-path in G_i that does not contain vertices of M, a contradiction. We conclude that $A \in \mathrm{Ns}(G_i; \mathfrak{S}_i)$.

We will now prove that A is an inclusion-maximal element of $\mathrm{Ns}(G_i; \mathfrak{S}_i)$. Suppose by contradiction that there is a set $A' \in \mathrm{Block}(G_i; \mathfrak{S}_i)$ such that $A \subset A'$ with $A \neq A'$. We already showed that $\mathrm{Block}(G_i; \mathfrak{S}_i) \subset \mathrm{Block}(G^{\mathfrak{S}}; \mathfrak{S})$. Then, $A, A' \in \mathrm{Block}(G^{\mathfrak{S}}; \mathfrak{S})$ but $A \subset A'$ with $A \neq A'$, a contradiction. Hence, the set A is an inclusion-maximal element of $\mathrm{Ns}(G_i; \mathfrak{S}_i)$, i.e., $A \in \mathrm{Block}(G_i; \mathfrak{S}_i)$.

To see that the union in the right-hand side of Equation (2.1) is disjoint, consider by contradiction two sets $A \in \mathrm{Block}(G_i; \mathfrak{S}_i)$ and $B \in \mathrm{Block}(G_j; \mathfrak{S}_j)$ with $i \neq j$ and $A = B$. Clearly, the set A cannot contain vertices of $A_j \setminus S$ given that S does not separate A in $G^{\mathfrak{S}}$. Hence, the only vertices of A that belong to the set A_j are the vertices in S. But the sets A and B have both at least $k+1$ vertices by Remark 15, a contradiction. This proves item (c).

Item (b) follows from the fact that a member of $\mathrm{Block}(G^{\mathfrak{S}};\mathfrak{S})$ that contains S is a member of $\mathrm{Block}(G_i;\mathfrak{S}_i)$ for some i, and the set $\mathrm{Block}(G_i;\mathfrak{S}_i)$ has exactly one set that contains S, namely, U_i. □

Remark 19. (a) The degree of $S \in \mathfrak{S}$ in the graph $\mathrm{T}(G;\mathfrak{S})$ is $|\mathrm{Block}(G;\{S\})| \geqslant 2$.

(b) Every degree-1 vertex of $\mathrm{T}(G;\mathfrak{S})$ is a member of $\mathrm{Block}(G;\mathfrak{S})$.

Proposition 20. *The following statements hold:*

(a) *The graph* $\mathrm{T}(G;\mathfrak{S})$ *is a tree.*

(b) *Two elements* $A, B \in \mathrm{Block}(G;\mathfrak{S})$ *are separated by a cutset* $S \in \mathfrak{S}$ *in* G *if and only if they are separated by* S *in* $\mathrm{T}(G;\mathfrak{S})$.

Proof. We prove both claims by induction on $|\mathfrak{S}|$. The case when $\mathfrak{S}$ is empty is trivial since in this case the graph $\mathrm{T}(G;\mathfrak{S})$ is a vertex being the only member of $\mathrm{Block}(G;\emptyset)$, which is $V(G)$. Given that $\mathrm{Block}(G;\mathfrak{S}) = \mathrm{Block}(G^{\mathfrak{S}};\mathfrak{S})$ and $\mathrm{T}(G;\mathfrak{S}) = \mathrm{T}(G^{\mathfrak{S}};\mathfrak{S})$, it is enough to prove the statements for $G^{\mathfrak{S}}$. Let $\mathrm{Block}(G^{\mathfrak{S}};\{S\}) = \{A_1,\dots,A_m\}$. The following is proven for $i=1,\dots,m$. By Proposition 18 (a), the graph $G_i = G^{\mathfrak{S}}[A_i]$ is k-connected. Let $\mathfrak{S}_i \subset \mathfrak{S}$ be the set of cutsets different from S that are contained in A_i. Let $U_i \in \mathrm{Block}(G_i;\mathfrak{S}_i)$ be the set that contains cutset S. We apply the induction hypothesis in the graph G_i with set of cutsets $\mathfrak{S}_i \subset \mathrm{Cut}_k(G_i)$. It follows that the graph $\mathrm{T}(G_i;\mathfrak{S}_i)$ is a tree. Each member of $\mathrm{Block}(G_i;\mathfrak{S}_i)$, that is not U_i, is adjacent to the same cutsets in $\mathrm{T}(G_i;\mathfrak{S}_i)$ and in $\mathrm{T}(G^{\mathfrak{S}};\mathfrak{S})$. In $\mathrm{T}(G^{\mathfrak{S}};\mathfrak{S})$, the edge joining U_i with S is added (see Figure 2.7). This completes the proof.

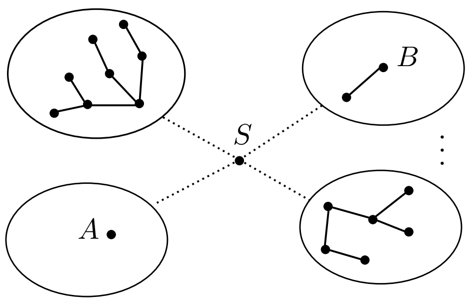

Figure 2.7: *The decomposition forms a tree-like structure.*

□

Note that this decomposition have the following property, which we already mentioned earlier.

Remark 21. The pair $\mathrm{Block}(G;\mathfrak{S})$ and $\mathfrak{S}$ form a tree-like structure on G.

2.2 For a 2-connected graph

Let $k \geqslant 1$ be fixed. Let G be a k-connected graph. A cutset $S \in \mathrm{Cut}_k(G)$ is said to be *isolated* if it is independent with all other cutsets from $\mathrm{Cut}_k(G)$. We denote by $\mathfrak{D}(G) \subset \mathrm{Cut}_k(G)$ the set of all isolated cutsets of G. The *block tree* $\mathrm{BT}(G)$ of G is the tree $\mathrm{T}(G;\mathfrak{D}(G))$. We also denote by $\mathrm{Block}(G)$ the set $\mathrm{Block}(G;\mathfrak{D}(G))$, and call its elements *blocks of the graph* G. A block of G is said to be a *leaf block* if it corresponds to a leaf of $\mathrm{BT}(G)$.

Observe that for the case of $k = 1$, the set $\mathfrak{D}(G)$ is the set of all cut-vertices of G. As we stated before, the presented decomposition corresponds to the classic block decomposition of a connected graph. We now study some properties of the decomposition for the case of $k = 2$. We omit some of the proofs in this section.

Proposition 22. *Let G be a 2-connected graph, let $S \in \mathfrak{D}(G)$ be an isolated cutset of G, let $v \in S$, and let $d = d_{\mathrm{BT}(G)}(S)$. Then, $d_G(v) \geqslant \max\{d, 3\}$. If $d_G(v) = d$, then the vertices of S are not adjacent.*

Proof. By Proposition 18, $d = |\mathrm{Block}(G;\{S\})|$. By Proposition 4, v is adjacent to at least one vertex in every component of the graph $G - S$. Thus $d_G(v) \geqslant d$. Suppose now that $d_G(v) = d$. Then all vertices adjacent to v belong to the interiors of the members of $\mathrm{Block}(G;\{S\})$, and v is not adjacent to the other vertex of S.

Clearly $d_G(v) \geqslant 2$ by the 2-connectivity of G. Suppose that $d_G(v) = 2$, then $|\mathrm{Block}(G;\{S\})| = 2$, and the vertices of S are not adjacent. Therefore, the two vertices adjacent to v in G form a cutset of $\mathrm{Cut}_2(G)$ and such a cutset separates S, which implies that S is not isolated, a contradiction. It follows that $d_G(v) \geqslant 3$. □

Denote by G' the graph $G^{\mathfrak{D}(G)}$. As before, we will use this graph as a tool, but we also will prove some facts about G' that do not hold in general for G.

Remark 23. Proposition 12 implies in this case that $\mathrm{Cut}_2(G) = \mathrm{Cut}_2(G')$.

Proposition 24. *Let $S \in \mathrm{Cut}_2(G)$ be a cutset that is not isolated and let $S \subset A \in \mathrm{Block}(G)$. Then $S \in \mathrm{Cut}_2(G'[A])$ and S is not an isolated cutset of $G'[A]$ as well.*

Proof. Let $T \in \mathrm{Cut}_2(G)$ be a cutset of G that separates S in G. By Remark 23, $S, T \in \mathrm{Cut}_2(G')$. By Proposition 12, these cutsets are also dependent in G'. By Corollary 17, $G'[A]$ is 2-connected. Then the set $S \subset A$ cannot be separated in $G'[A]$

or in G' by deleting a set of vertices that contains fewer than two vertices from A. It follows that $T \subset A$. Because of this, S and T separate each other in $G'[A]$. Thus, $S, T \in \mathrm{Cut}_2(G'[A])$, and they are dependent in $G'[A]$. $\square$

Proposition 25. *Let $S = \{a, b\} \in \mathrm{Cut}_2(G)$ be a cutset that is not isolated. Then $|\mathrm{Block}(G; \{S\})| = 2$, and for each set $A \in \mathrm{Block}(G; \{S\})$, the graph $G[A]$ is not 2-connected and has a cut vertex that separates a from b.*

Proof. Let $T \in \mathrm{Cut}_2(G)$ be a cutset that separates S. Any ab-path in $G[A]$ contains a vertex of T. Then $\mathrm{Int}(A)$ contains vertices of T. Since the non-empty interiors of any two different blocks are disjoint by Corollary 3, and T intersects the interiors of each one of them, it follows that $|\mathrm{Block}(G; \{S\})| = 2$. The only vertex in $T \cap \mathrm{Int}(A)$ separates a from b in $G[A]$. $\square$

Proposition 26. *Let G be a 2-connected graph with no isolated 2-cutsets. Then either G is 3-connected or G is a cycle.*

Proof. Suppose that G is not 3-connected. We will prove that G is a cycle by proving that, for each cutset $S = \{a, b\} \in \mathrm{Cut}_2(G)$ and each set $A \in \mathrm{Block}(G; \{S\})$, the graph $G[A]$ is an ab-path. Observe that by Proposition 25, the set $\mathrm{Block}(G; \{S\})$ contains two elements. Hence proving this fact is sufficient.

The proof follows by induction on $|A|$. For the case that $|A| = 3$, the graph $G[A]$ is a path of length 2. For the induction step, let $H = G[A]$. By Proposition 25, the graph H is not 2-connected and has a cut vertex v that separates a from b in H. Let C_a and C_b be the set of vertices of the components of $H - v$ that contain a and b respectively. Given that the graph G is two connected, there are only two components in the graph $H - v$ since any other component that does not contain a or b would be a component of the graph $G - v$, which is not possible.

Let $C'_a = C_a \setminus \{a\}$. Suppose first that $C'_a \neq \emptyset$. Let $T_a = \{a, v\}$. The set T_a separates C'_a from the set $V(G) \setminus C'_a$. Hence $T_a \in \mathrm{Cut}_2(G)$. By the induction hypothesis, the graph $G[C'_a \cup T_a] = G[C_a \cup \{v\}]$ is an av-path. Suppose now that $C'_a = \emptyset$. Then $N_H(a) = \{v\}$ and the graph $G[C_a \cup \{v\}]$ is an av-path. This completes the proof. $\square$

Proposition 27. *Let $A \in \mathrm{Block}(G)$. Then, either $G'[A]$ is a cycle or a 3-connected graph.*

Proof. By Corollary 17, the graph $G'[A]$ is 2-connected. Suppose that $S \in \mathrm{Cut}_2(G'[A])$ exists. By Proposition 16, we have $S \in \mathrm{Cut}_2(G)$. Given that the set S separates the block A, it follows that the cutset S is not isolated. By Proposition 24, S is not an isolated cutset of $G'[A]$. Since the graph $G'[A]$ has no isolated cutsets, by Proposition 26, it is either 3-connected or a cycle. □

Proposition 28. *Let* $A \in \mathrm{Block}(G)$ *be a block such that* $G'[A]$ *is a cycle. Then all vertices of* $\mathrm{Int}(A)$ *have degree* 2 *in* G.

Proof. Let $v \in \mathrm{Int}(A)$. Given that $\mathrm{Bound}(A)$ is the set of vertices of A that have neighbors in $V(G) \setminus A$, all the edges of G that have v as one end have the other end in A. Therefore $d_G(v) = 2$. □

Proposition 29. *Let* $A \in \mathrm{Block}(G)$ *be a block such that* $G'[A]$ *is a cycle of length at least* 4. *Then any pair of non-adjacent vertices of* A *is a non-isolated cutset of* G. *All non-isolated cutsets of* G *are of this form.*

Proof. Let S be a pair of non-adjacent vertices in the cycle $G'[A]$. Clearly $S \in \mathrm{Cut}_2(G'[A])$. By Proposition 16 $S \in \mathrm{Cut}_2(G)$. Also, note that $S \notin \mathfrak{D}(G)$.

Let S be a non-isolated cutset of G. Given that no cutset of $\mathfrak{D}(G)$ separates S, it follows that there is a set A such that $S \subset A \in \mathrm{Block}(G)$. By Proposition 24, $S \in \mathrm{Cut}_2(G'[A])$. Then the graph $G'[A]$ is not 2-connected and hence, not 3-connected. Therefore by Proposition 27, the graph $G'[A]$ is a cycle of length at least 4. Also, the set S consists of two non-adjacent vertices of the cycle $G'[A]$. □

Chapter 3

Some applications

We now present some applications of the decomposition to the study of classic concepts in graph theory. For simplicity, we omit some proofs that are long or technical.

3.1 Planarity of 2-connected graphs

A classic result is that a connected graph is planar if and only if the graphs induced by its block (using the classic block decomposition) are planar. The necessary condition clearly holds for the studied decomposition on k-connected graphs as well. In 1937, Mac Lane gave a similar characterization of 2-connected planar graphs (Mac Lane, 1937). He studied a partition of a 2-connected graph into subgraphs which he called *atoms*. He used Kuratowski's theorem to show that a 2-connected graph is planar if and only if its atoms are also planar. It is possible to reformulate this theorem in terms of the block decomposition studied for 2-connected graphs.

In the following propositions, we denote by G' the graph $G^{\mathfrak{D}(G)}$, whenever G is a 2-connected graph.

Lemma 30. *Let G be a 2-connected graph. Then G contains a subdivision of $G'[A]$ for every block $A \in \mathrm{Block}(G)$.*

Proof. Observe that the edges $ab \in E(G'[A]) \setminus E(G)$ are such that $a, b \in A$ and $\{a, b\} \in \mathfrak{D}(G)$. For every edge $ab \in E(G'[A]) \setminus E(G)$ we perform the following operation in the graph $G'[A]$. Let $U_{a,b}$ be a component of $G - \{a, b\}$ such that $U_{a,b} \cap A = \emptyset$. By Proposition 4, the vertices a and b are adjacent to a vertex of $U_{a,b}$ in G. Then there exist an ab-path in G whose internal vertices belong to $U_{a,b}$ and do not belong to A.

We replace edge ab in $G'[A]$ by one such path $P_{a,b}$, adding its new vertices and edges. Doing so for every edge in $E(G'[A]) \setminus E(G)$, we obtain a new graph H, which is a subgraph of G. Thus, it remains to show that H is a subdivision of $G'[A]$.

Suppose by contradiction that there are two different cutsets $S, T \in \mathfrak{D}(G)$ such that the paths P_S and P_T contain a common internal vertex x. Consider the graph $G - S$. The internal vertices of P_S belong to a component U_S of $G - S$. Hence $x \in U_S$. There is a component Q different from U_S that contains the set $A \setminus S$. It follows that $T \setminus S \subset A \setminus S \subset V(Q)$. Given that $T \setminus S \neq \emptyset$ since $S \neq T$, and that the sets $T \setminus S$ and $\{x\}$ are in different components in the graph $G - S$, it follows that $V(P_T) \cap S = S$. Then there is an internal vertex of P_T that is contained in A. This contradicts that the internal vertices of P_T do not belong to A, and therefore, the graph H is a subdivision of $G'[A]$. □

Lemma 31. *Let H be a 3-connected graph, and let Q be a proper subdivision of H. Then Q is 2-connected but not 3-connected. Also, for any cutset $\{a, b\} \in \mathrm{Cut}_2(Q)$ of Q we have $a, b \in V(P)$, where P is a subdivided path of Q, and the vertices a and b are non-adjacent in P.*

Proof. Since Q is a proper subdivision of H, it contains a vertex of degree 2. Then Q is not 3-connected. Also, Q contains at least 5 vertices since H contains at least 4 vertices. Let $\{a, b\} \in \mathrm{Cut}_2(Q)$ be a cutset of Q. Suppose by contradiction that a and b do not belong to a subdivided path in Q. In the case that $a, b \in V(H)$, then a and b are not adjacent in H, otherwise both vertices belong to a (possibly properly) subdivided path of Q. Also, given that the graph $H - a - b$ is connected and that $ab \notin E(H)$, the graph $Q - a - b$ is also connected, and the set $\{a, b\}$ is not a 2-cutset of Q. Let us now consider the case in which $a \in V(H)$ and $b \notin V(H)$ belongs to a subdivided path of Q. Let F be the set of edges incident to a in H. Clearly b does not belong to a subdivision of an edge of F, otherwise both vertices belong to a subdivided path in Q. Also, given that the graph $H - a$ is connected, the graph $Q - a$ is also connected (see Figure 3.1 (a)), and vertex b is in a subdivision of some edge of $E(H) \setminus F$. Then the graph $Q - a - b$ is connected, a contradiction. In the case where $a, b \notin V(H)$, those vertices are not in the same subdivided path of Q, and it is also easy to see that the graph $Q - a - b$ is connected (see Figure 3.1 (b)), a contradiction. Therefore vertices a and b belong to a subdivided path in Q. Clearly vertices a and b cannot be adjacent in Q, otherwise the graph $Q - a - b$ is not connected. □

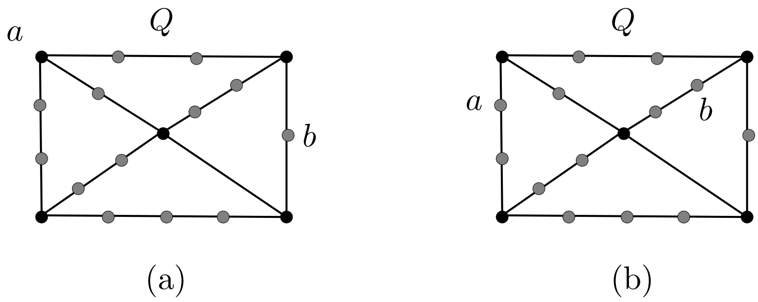

Figure 3.1: *In both cases, the graph $Q - a - b$ is connected.*

Lemma 32. *Let G be a 2-connected graph, and let H be a 3-connected graph. If G contains a subdivision of H, then there is some $A \in \text{Block}(G)$ such that $G'[A]$ contains a subdivision of H.*

Proof. The graph G' also contains a subdivision of H since $G \subset G'$. Let Q be a subgraph of G that is a subdivision of H with the minimum number of vertices. We will show that $Q \subset G'[A]$ for some $A \in \text{Block}(G; \mathfrak{D}(G')) = \text{Block}(G)$. Hence, it is enough to show that $V(Q) \in \text{Ns}(G'; \mathfrak{D}(G'))$. Let $\{a, b\} \in \mathfrak{D}(G')$ be a cutset of G'. We will prove that the cutset $\{a, b\}$ does not separate Q in G'. For this, we will only prove that $Q - a - b$ is connected. Suppose by contradiction that $\{a, b\} \in \text{Cut}_2(Q)$. Given that $\text{Cut}_2(H) = \emptyset$, it follows that Q is a proper subdivision of H, i.e., $Q \neq H$, which is 3-connected. By Lemma 31 we have $a, b \in V(P)$, where P is a subdivided path of Q. We also conclude that a and b are not adjacent in P. Note that the ab-section of P contains at least three vertices. Since a and b are adjacent in G', we can replace the ab-section of P with the edge ab and obtain a subgraph of G that is a subdivision of H with strictly less vertices than Q. This contradicts the minimality of Q and proves that $Q - a - b$ is connected. Then, $Q \subset G'[A]$ for some $A \in \text{Block}(G)$. $\square$

Theorem 33. *A 2-connected graph G is planar if and only if the graph $G'[A]$ is planar for every $A \in \text{Block}(G)$.*

Proof. Let us first show that if $G'[A]$ is not planar, for some $A \in \text{Block}(G)$, then G is not planar. By Kuratowski's theorem, $G'[A]$ contains a subdivision of K_5 or $K_{3,3}$. By Lemma 30, if $G'[A]$ contains a subdivision of K_5, then G also contains a subdivision of K_5 since G contains a subdivision of $G'[A]$. Similarly, if $G'[A]$ contains a subdivision of $K_{3,3}$, then G also contains a subdivision of $K_{3,3}$. This implies that G is not planar.

We now prove that if G is not planar, then there is some $A \in \text{Block}(G)$ such that $G'[A]$ is not planar. By Kuratowski's theorem, G contains a subdivision of K_5 or $K_{3,3}$. Given that both of these graphs are 3-connected, by Lemma 32 it follows that there is some $A \in \text{Block}(G)$ such that $G'[A]$ contains a subdivision of K_5 or $K_{3,3}$. Therefore, the graph $G'[A]$ is not planar. □

This result is equivalent with Mac Lane's theorem since the subgraphs that he defined as atoms of a 2-connected graph G are subdivisions of graphs $G'[A]$, where $A \in \text{Block}(G)$.

3.2 Colouring

In this section we present some upper bounds for the chromatic number and choice number of k-connected graphs in terms of these parameters of its blocks.

Lemma 34. *Let G be a graph, let $A \cup B = V(G)$ be sets of vertices of G. Suppose that the set $S = A \cap B \neq \emptyset$ is a clique, and suppose that every edge $ab \in E(G)$ with $a \in A$, and $b \in B$ has at least one end in S. Let $H_1 = G[A]$ and $H_2 = G[B]$. Then*

$$\chi(G) = \max\{\chi(H_1), \chi(H_2)\}. \tag{3.1}$$

Proof. Obviously $\chi(G) \geqslant \max\{\chi(H_1), \chi(H_2)\}$.

Given two proper colourings γ_1 and γ_2 of H_1 and H_2 respectively, we can observe that the partition they induce on the set S is the same, and every part contains exactly one element. Therefore, we can synchronize the colourings to obtain a colouring of G with $\max\{\chi(H_1), \chi(H_2)\}$ colours. □

Lemma 35. *Let G be a graph, let $A \cup B = V(G)$ be sets of vertices of G with $|A \cap B| = k \geqslant 1$. Suppose that every edge $ab \in E(G)$ with $a \in A$, and $b \in B$ has at least one end in $A \cap B$. Let $H_1 = G[A]$ and $H_2 = G[B]$. Then*

$$\chi(G) \leqslant \max\{k - 1 + \chi(H_1), \chi(H_2)\}. \tag{3.2}$$

Proof. Let $\gamma_1 : V(H_2) \to \{1, \dots, \chi(H_2)\}$ be a proper colouring of H_2. Consider the partition induced by γ_1 on the set $S = V(H_1) \cap V(H_2)$.

We will first consider the case in which there is a part containing a single vertex. Let v be such vertex, let $S' = S \setminus \{v\}$, and let C_1 be the set of colours of γ_1 that are used in S'. Clearly $\gamma_1(v) \notin C_1$ since the set $\{v\}$ is a part with a single element. Let $H_1' = H_1 - S'$. We will show that there is a proper colouring of H_1' with the colours of $C_2 = \{1, \dots, \chi(H_1) + k - 1\} \setminus C_1$. Given that $|S'| = k - 1$, it follows that $|C_1| \leqslant k - 1$.

Then, $|C_2| \geqslant \chi(H_1) + k - 1 - (k-1) = \chi(H_1)$. We also conclude that $\chi(H_1') \leqslant \chi(H_1)$ given that $H_1' \subset H_1$, which implies that $\chi(H_1') \leqslant |C_2|$.

Let $\gamma_2 : V(H_1') \to C_2$ be a proper colouring of H_1'. We now modify a copy of the colouring γ_2 to create a new proper colouring γ_3 of H_1' such that $\gamma_3(v) = \gamma_1(v)$. We can achieve this by synchronizing the colours of vertex v according to γ_1 and γ_2.

We can observe that there is no edge $ab \in E(G)$ such that $a \in V(H_1')$ and $b \in V(G) \setminus V(H_1')$ such that $\gamma_3(a) = \gamma_1(b)$. Indeed, suppose by contradiction that such an edge exists. Suppose first that $a = v \in V(H_1')$. By the definition of the colouring γ_3, we conclude that $\gamma_3(a) = \gamma_1(a)$. We also conclude that $\gamma_3(a) = \gamma_1(b)$, and hence $\gamma_1(a) = \gamma_1(b)$. This contradicts that γ_1 is a proper colouring of H_2. Suppose now that $a \neq v$. We can see that $a \notin S$ since $V(H_1') \cap S = \{v\}$. By hypothesis, the edge ab has at least one end in S, then $b \in S$ and furthermore, $b \neq v$ since $b \in V(G) \setminus V(H_1')$. Then $b \in S'$. By the definition of C_1, we have $\gamma_1(b) \in C_1$. Suppose first that $\gamma_3(a) = \gamma_2(a)$. In this case it follows that $\gamma_3(a) \in C_2$, and hence $\gamma_3(a) \notin C_1$ by the definition of $C_2 = \{1, \ldots, \chi(H_1) + k - 1\} \setminus C_1$. Therefore $\gamma_3(a) \neq \gamma_1(b)$, a contradiction. Suppose now that $\gamma_3(a) \neq \gamma_2(a)$, i.e., that a changed its colour from γ_2 to γ_3. It follows that either $\gamma_2(a) = \gamma_2(v)$ or $\gamma_2(a) = \gamma_1(v)$. In the first case we conclude that $\gamma_3(a) = \gamma_1(v)$. We also have that $\gamma_1(b) \neq \gamma_1(v)$ since vertex v is the only one with colour $\gamma_1(v)$ in S. Therefore, we conclude that $\gamma_1(b) \neq \gamma_3(a)$, a contradiction. In the second case, we have $\gamma_3(a) = \gamma_2(v) \in C_2$. Then, $\gamma_3(a) \notin C_1$ by the definition of $C_2 = \{1, \ldots, \chi(H_1) + k - 1\} \setminus C_1$. Given that $\gamma_1(b) \in C_1$, we conclude that $\gamma_3(a) \neq \gamma_1(b)$, a contradiction.

Given that vertex v is the only vertex of G that belongs to the domain of both γ_1 and γ_3, and that they assign v the same colour, i.e., $\gamma_1(v) = \gamma_3(v)$; we can join colourings γ_1 and γ_3 to make a proper colouring of G with the colours of the set $C_3 = \{1, \ldots, \chi(H_2)\} \cup C_2$. Observe that $|C_3| \leqslant \max(C_3) = \max(\chi(H_2), \max(C_2))$. By the definition of $C_2 = \{1, \ldots, \chi(H_1) + k - 1\} \setminus C_1$, it follows that $\max(C_2) \leqslant \chi(H_1) + k - 1$. Then $\chi(G) \leqslant |C_3| \leqslant \max(k - 1 + \chi(H_1), \chi(H_2))$.

We now consider the case in which γ_1 induces a partition on the set S such that every part contains at least two members. Let C_4 be the set of colours of γ_1 that are used in S. It follows that $|C_4| \leqslant k/2 \leqslant k-1$ since $k \geqslant 2$. Let $H_1'' = H_1 - S$. Observe that there is a proper colouring of H_1'' with the colours of $C_5 = \{1, \ldots, \chi(H_1) + k - 1\} \setminus C_4$ since $|C_5| \geqslant \chi(H_1) + k - 1 - (k-1) = \chi(H_1)$ and the fact that $\chi(H_1'') \leqslant \chi(H_1)$ given that $H_1'' \subset H_1$, which implies that $\chi(H_1'') \leqslant |C_5|$. Let $\gamma_4 : V(H_1') \to C_5$ be a proper colouring of H_1'.

Observe that there is no edge $ab \in E(G)$ such that $a \in V(H_1'')$ and $b \in V(G) \setminus V(H_1'') = V(H_2)$ such that $\gamma_4(a) = \gamma_1(b)$. Suppose by contradiction that such an edge exists. Given that at least one end of ab lies in S and that $a \notin S$, we conclude that $b \in S$. Then $\gamma_1(b) \in C_4$. But since $\gamma_4(a) \notin C_4$ by the definition of $C_5 = \{1, \ldots, \chi(H_1) + k - 1\} \setminus C_4$, we conclude that $\gamma_4(a) \neq \gamma_1(b)$, a contradiction.

Given that the domains of γ_1 and γ_4 do not share any vertex in common, we can join colourings γ_1 and γ_4 to produce a proper colouring of G with the colours of the set $C_6 = \{1, \ldots, \chi(H_2)\} \cup C_5$. Observe that $|C_6| \leqslant \max(C_6) = \max(\chi(H_2), \max(C_5))$. By the definition of $C_5 = \{1, \ldots, \chi(H_1) + k - 1\} \setminus C_4$, it follows that $\max(C_5) \leqslant \chi(H_1)+k-1$. Then $\chi(G) \leqslant \max\{k-1+\chi(H_1), \chi(H_2)\}$. This completes the proof. □

We present a second proof by Cláudio Leonardo Lucchesi.

Let J be a graph and let u and v be distinct non-adjacent vertices of J. Denote by $J + uv$ the graph obtained from J by the addition of a new edge joining u and v. Denote by $J/\{u, v\}$ the graph obtained from J by contracting the set $\{u, v\}$ to a single vertex.

The following two propositions will be useful for the proof of Lemma 35.

Proposition 36. *Let u and v be two non-adjacent vertices of a graph J. The following properties hold:*

$$\chi(J + uv) \in \{\chi(J), \chi(J) + 1\} \text{ and } \chi(J/\{u, v\}) \in \{\chi(J), \chi(J) + 1\}.$$

Moreover, at least one of $\chi(J + uv)$ and $\chi(J/\{u, v\})$ is equal to $\chi(J)$.

Proof. This proof is easy and is left as an exercise for the reader. □

Proposition 37. *Let I the set of pairs of non-adjacent vertices of a graph G and let G^* be the graph obtained from G by the addition of the edges uv, for each pair $\{u, v\} \in I$. Suppose that for every pair $\{u, v\} \in I$, $\chi(G/\{u, v\}) = \chi(G) + 1$. Then $\chi(G^*) = \chi(G)$.*

Proof. By induction on $|I|$. We prove the following two sub-statements.

(a) For every pair $\{u, v\} \in I$, $\chi(G + uv) = \chi(G)$.

Proof. By hypothesis, $\chi(G/\{u, v\}) = \chi(G)+1$. By Proposition 36, it follows that $\chi(G+uv) = \chi(G)$. □

If I is empty then the assertion holds trivially. Thus, suppose that I is non-empty. Let $\{u,v\} \in I$. Let $G' = G + uv$ and let $I' = I \setminus \{\{u,v\}\}$. Then $\chi(G') = \chi(G)$. If I is a singleton then the assertion holds. We may thus assume that I' is nonempty.

(b) For each pair $\{w,x\} \in I'$, $\chi(G'/\{w,x\}) = \chi(G') + 1$.

Proof. Suppose that $\chi(G'/\{w,x\}) \neq \chi(G') + 1$. From Proposition 36 we infer that $\chi(G'/\{w,x\}) = \chi(G')$. Then G' has a proper $\chi(G')$-colouring, Γ, in which w and x have the same colour. But $\chi(G') = \chi(G)$ and Γ is a proper colouring of G. This conclusion is a contradiction to the hypothesis that $\chi(G/\{w,x\}) = \chi(G) + 1$. $\square$

By induction, with G' playing the role of G and I' playing the role of I, we conclude that $\chi(G^*) = \chi(G') = \chi(G)$. $\square$

Proof of Lemma 35. Let $S = A \cap B$ and let I denote the set of pairs of non-adjacent vertices of S. The proof is by induction on $|I|$, the number of pairs of non-adjacent vertices of $A \cap B$.

Case 1 $I = \emptyset$. Then S is a clique, and the assertion follows immediately from Lemma 34.

Case 2 The set I contains a pair $\{u,v\}$ such that $\chi(H_1 + uv) = \chi(H_1)$ and $\chi(H_2 + uv) = \chi(H_2)$. By Proposition 36 and by induction,

$$\begin{aligned}\chi(G) \leqslant \chi(G+uv) &\leqslant \max\{k-1+\chi(H_1+uv), \chi(H_2+uv)\}\\ &= \max\{k-1+\chi(H_1), \chi(H_2)\}.\end{aligned}$$

Case 3 The set I contains a pair $\{u,v\}$ such that $\chi(H_2/\{u,v\}) = \chi(H_2)$. By Proposition 36 and by induction,

$$\begin{aligned}\chi(G) \leqslant \chi(G/\{u,v\}) &\leqslant \max\{k-2+\chi(H_1/\{u,v\}, \chi(H_2/\{u,v\})\}\\ &\leqslant \max\{k-1+\chi(H_1), \chi(H_2)\}.\end{aligned}$$

Case 4 The set I contains a pair $\{u,v\}$ such that $\chi(H_2/\{u,v\}) = \chi(H_2) + 1 \leqslant k - 1 + \chi(H_1)$. By Proposition 36 and by induction,

$$\begin{aligned}\chi(G) \leqslant \chi(G/\{u,v\}) &\leqslant \max\{k-2+\chi(H_1/\{u,v\}, \chi(H_2/\{u,v\})\}\\ &\leqslant \max\{k-1+\chi(H_1), \chi(H_2)+1\}\\ &= k-1+\chi(H_1).\end{aligned}$$

Case 5 The previous cases are not applicable. By Proposition 36, we have the following equalities, for each pair $\{u, v\} \in I$:

$$\chi(H_1/\{u,v\}) = \chi(H_1), \;\; \chi(H_1 + uv) = \chi(H_1) + 1, \tag{3.3}$$

$$\chi(H_2 + uv) = \chi(H_2), \;\; \chi(H_2/\{u,v\}) = \chi(H_2) + 1, \tag{3.4}$$

$$\text{We also have } \chi(H_2) \geqslant k - 1 + \chi(H_1). \tag{3.5}$$

By Proposition 37, H_2 has a proper $\chi(H_2)$-colouring, Γ_2, in which the k vertices of S have k distinct colours. Let $v \in S$, let $S' = S - v$. From Equation (3.5) we deduce that $\chi(H_1) \leqslant \chi(H_2) - (k-1)$. Thus, there exists a proper $\chi(H_1)$-colouring of $H_1 = S'$, Γ_1', which does not use any of the $k-1$ colours of the vertices of S'. Moreover, we may fix the colour of v in Γ_1' to coincide with its colour in Γ_2. Thus, Γ_1' may be extended to a proper $\chi(H_2)$-colouring Γ_1 of H_1 such that each vertex of S has the same colour in both Γ_1 and in Γ_2. Thus, G has a proper $\chi(H_2)$-colouring. The assertion holds. □

For a graph G, we denote by $\chi_L(G)$ its list chromatic number.

Lemma 38. *Let G be a graph, let $A \cup B = V(G)$ be sets of vertices of G with $|A \cap B| = k \geqslant 1$. Suppose that every edge $ab \in E(G)$ of G with $a \in A$, and $b \in B$ has at least one end in $A \cap B$. Let $H_1 = G[A]$ and $H_2 = G[B]$. Then*

$$\chi_L(G) \leqslant \max\{k + \chi_L(H_1), \chi_L(H_2)\}. \tag{3.6}$$

Proof. Let $m = \max\{k + \chi_L(H_1), \chi_L(H_2)\}$. Let $L = (L(v))_{v \in V(G)}$ be a set of lists assigned to each vertex of G, where $|L(v)| = m$ for each $v \in V(G)$. We will prove that G is L-choosable. Given that $|L(v)| = m \geqslant \chi_L(H_2)$ for every $v \in V(H_2)$, there is a proper colouring γ_1 of H_2. such that $\gamma_1(v) \in L(v)$. Let C be the set of colours that are used by γ_1 in $S = V(H_1) \cap V(H_2)$. Clearly $|C| \leqslant |S| = k$. Let $H_1' = H_1 - S$. For every vertex $u \in V(H_1')$, let $L'(u) = L(u) \setminus C$. For each such vertex we have $|L'(u)| \geqslant |L(u)| - |C| \geqslant k + \chi_L(H_1) - k = \chi_L(H_1)$. Therefore there is a proper colouring γ_2 of the vertices of H_1' such that $\gamma_2(u) \in L'(u)$ for every $u \in V(H_1')$ since $\chi_L(H_1') \leqslant \chi_L(H_1)$.

We can observe that there is no edge $ab \in E(G)$ such that $a \in V(H_1')$ and $b \in V(H_2)$ with $\gamma_2(a) = \gamma_1(b)$. Indeed, if such an edge exists, it follows that $b \in S$ by hypothesis. And then $\gamma_1(b) \in C$, hence $\gamma_1(b) \notin L'(a)$, which implies that $\gamma_2(a) \neq \gamma_1(b)$, a contradiction.

Given that no vertex belongs to the domain of both γ_1 and γ_2, we can join colourings γ_1 and γ_2 to produce a proper colouring of G such that every vertex is assigned to a colour of its list in L. This implies that G is L-choosable. □

Lemma 39. *Let G be a graph, and let $\mathcal{B}, \mathcal{C} \subset 2^{V(G)}$ be families of subsets of $V(G)$. Let T be the $(\mathcal{B}, \mathcal{C})$-bipartite graph where $B \in \mathcal{B}$ is adjacent to $C \in \mathcal{C}$ in T if $C \subset B$. Let $k \geqslant 1$ be an integer, and let the following conditions be met:*

1. $$\bigcup_{B \in \mathcal{B}} B = V(G).$$

2. *For every $C \in \mathcal{C}$, $|C| = k$.*

3. *Let $A, B \in \mathcal{B}$ be different sets such that $A \cap B \neq \emptyset$. Then $A \cap B \in \mathcal{C}$.*

4. *Let $A, B \in \mathcal{B}$ such that there is an edge $ab \in E(G)$ with $a \in A$, and $b \in B$. Then $A \cap B \neq \emptyset$, and at least one of a,b belongs to $A \cap B$.*

5. *The graph T is a tree.*

6. *Every member of $\mathcal{C}$ has degree at least 2 in T.*

Then,

$$\chi(G) \leqslant k - 1 + \max_{B \in \mathcal{B}} \chi(G[B]). \tag{3.7}$$

Proof. We prove the statement by induction on $|\mathcal{B}|$. Clearly $\mathcal{B}$ is nonempty by condition 1. For the case $|\mathcal{B}| = 1$, let $\{B\} = \mathcal{B}$. It follows that $B = V(G)$, then $\chi(G) = \chi(G[B]) \leqslant k - 1 + \chi(G[B])$. Let G, $\mathcal{B}$, $\mathcal{C}$, and T be given. The tree T has at least two vertices since $v(T) \geqslant |\mathcal{B}| \geqslant 2$. Let $A \in \mathcal{B}$ be a leaf of T, and let $S \in \mathcal{C}$ be only the vertex adjacent to A in T, i.e., $S \subset A$. Let $H = V(G) \setminus (A \setminus S)$. Clearly $A \cap H = S$.

We will prove that for every edge $ab \in E(G)$ such that $a \in A$, and $b \in H$, at at least one of a, b belongs to S. Indeed, by condition 1, vertex b belongs to some set $B \in \mathcal{B}$. Then, by condition 4, $A \cap B \neq \emptyset$. If $A = B$, then clearly $b \in S$ since $b \in H$, and the statement holds. If $A \neq B$, by condition 3 we conclude that $A \cap B \in \mathcal{C}$, and given that S is the only element of $\mathcal{C}$ contained in A, it follows that $A \cap B = S$. Therefore, at least one of a, b belongs to $A \cap B = S$.

Let $G' = G[H]$. By Lemma 35, we conclude that

$$\chi(G) \leqslant \max\left\{k - 1 + \chi(G[A]), \chi(G')\right\}. \tag{3.8}$$

Let $\mathcal{B}' = \mathcal{B} \setminus \{A\}$. Let $\mathcal{C}'$ be defined as $\mathcal{C} \setminus \{S\}$ in the case where $d_T(S) = 2$; otherwise, i.e., when $d_T(S) \geqslant 3$, let $\mathcal{C}' = \mathcal{C}$. Let T' be defined as $T - A - S$ in the case where $d_T(S) = 2$; otherwise let $T' = T - A$. We will prove that the conditions of the statement hold in order to apply the induction hypothesis in G', $\mathcal{B}'$, $\mathcal{C}'$, and T'.

Condition 1. Let $v \in V(G')$ be a vertex of G'. By condition 1, v is contained in some member of $\mathcal{B}$. If such member is different from A, then it belongs to $\mathcal{B}'$ and the condition is maintained. If such member is equal to A, we can observe that $v \in S$ given that $v \in V(G') = H$ and that $A \cap H = S$. By condition 6, the set S is contained in at least 2 members of $\mathcal{B}$, at least one of which is different from A, and the condition is maintained.

Condition 2. Every member of $\mathcal{C}$ has size k. Every member of $\mathcal{C} \setminus \{S\}$ has size k as well. The set $\mathcal{C}'$ is defined as one of these two sets.

Condition 3. Let $C, D \in \mathcal{B}'$ be different sets such that $C \cap D \neq \emptyset$. By condition 3, we conclude that $C \cap D \in \mathcal{C}$. If $\mathcal{C}' = \mathcal{C}$, then the condition is maintained. Otherwise it follows that $\mathcal{C}' = \mathcal{C} \setminus \{S\}$. Suppose by contradiction that $C \cap D = S$. We have $A \neq C, D$. Then, S is contained in at least 3 members of $\mathcal{B}$, i.e., $d_T(S) \geqslant 3$. But then $\mathcal{C}' = \mathcal{C}$, a contradiction. Therefore $C \cap D \neq S$, and we still have that $C \cap D \in \mathcal{C}'$.

Condition 4. Let $C, D \in \mathcal{B}'$ such that there is an edge $ab \in E(G') \subset E(G)$ with $a \in C$ and $b \in B$. By condition 4, $C, D \in \mathcal{B}$, and we still have that $C \cap D \neq \emptyset$, and that at least one of a, b belongs to $A \cap B$.

Condition 5. The graph $T - A$ is a tree since A is a leaf. The graph $T - A - S$ is also a tree provided that $d_T(S) = 2$. The graph T' is defined as one of these two graphs.

Condition 6. In the case that $T' = T - A$, it follows that $d_T(S) \geqslant 3$. Then $d_{T'}(S) \geqslant 2$. The remaining members of $\mathcal{C}'$ have the same degree in T' since S is the only vertex adjacent to A in T. In the case that $T' = T - A - S$, we conclude that S is not adjacent to any other cutset of $\mathcal{C}'$ since T is a $(\mathcal{B}, \mathcal{C})$-bipartite graph. Then, the degree of any member of $\mathcal{C}'$ in T' is the same in T, and the condition is maintained.

By the induction hypothesis, we have

$$\chi(G') \leqslant k - 1 + \max_{B \in \mathcal{B}'} \chi(G'[B]). \tag{3.9}$$

We combine 3.8 with 3.9 to get:

$$\begin{aligned}\chi(G) &\leqslant \max\left\{k-1+\chi(G[A]), k-1+\max_{B\in\mathcal{B}'}\chi(G'[B])\right\}\\ &= k-1+\max\left\{\chi(G[A]), \max_{B\in\mathcal{B}'}\chi(G'[B])\right\}\\ &= k-1+\max_{B\in\mathcal{B}}\chi(G[B]).\end{aligned}$$

This completes the proof. □

Theorem 40. *Let G be a k-connected graph, and let $\mathfrak{S}\subset \mathrm{Cut}_k(G)$ be a family of pairwise independent k-cutsets of G. Then,*

$$\chi(G) \leqslant \chi(G') = \max_{A\in\mathrm{Block}(G;\mathfrak{S})}\chi(G'[A]). \tag{3.10}$$

Proof. This proof is easy and uses induction on the tree like structure of the decomposition of G. It also uses Lemma 34. □

Theorem 41. *Let G be a k-connected graph, and let $\mathfrak{S}\subset \mathrm{Cut}_k(G)$ be a family of pairwise independent k-cutsets of G. Then,*

$$\chi(G) \leqslant k-1+\max_{A\in\mathrm{Block}(G;\mathfrak{S})}\chi(G[A]). \tag{3.11}$$

Proof. Clearly the blocks of G form a tree like structure as defined in Definition 10, i.e., by assigning $\mathcal{B}=\mathrm{Block}(G;\mathfrak{S})$, $\mathcal{C}=\mathfrak{S}$, and $T=\mathrm{T}(G;\mathfrak{S})$. Thus, the hypothesis of Lemma 39 holds for this case. The conclusion is immediate. □

Theorem 42. *Let G be a k-connected graph, and let $\mathfrak{S}\subset \mathrm{Cut}_k(G)$ be a family of pairwise independent k-cutsets of G. Then,*

$$\chi_L(G) \leqslant k+\max_{A\in\mathrm{Block}(G;\mathfrak{S})}\chi_L(G[A]). \tag{3.12}$$

Proof. We can prove a similar statement as in Lemma 39, where we have $\chi_L(G)\leqslant k+\max_{B\in\mathcal{B}}\chi_L(G[B])$. We can then proceed as in the proof of Theorem 41. □

3.3 Critically 2-connected graphs

A graph G is *critically k-connected* if G is k-connected with $v(G) \geqslant k+2$ and for any vertex $v \in V(G)$ of G, the graph $G - v$ is not k-connected.

Critically k-connected graphs were studied in (Chartrand *et al.*, 1972) and (Hamidoune, 1980). In (Hamidoune, 1980), it was proven that any critically k-connected graph has at least two vertices of degree less than $(3k-1)/2$. This implies that any critically 2-connected graph has at least two vertices of degree 2. Using the block decomposition for 2-connected graphs, we will prove that any critically 2-connected graph has at least four vertices of degree 2.

Proposition 43. *Let G be a 2-connected graph. Then the vertices not contained in the cutsets of $\mathrm{Cut}_2(G)$ are the internal vertices of the blocks $A \in \mathrm{Block}(G)$ where $G'[A]$ is 3-connected or a triangle.*

Proposition 44. *Let G be a 2-connected graph. Then G is critically 2-connected if and only if $\mathrm{Int}(A) = \emptyset$ for all of its blocks $A \in \mathrm{Block}(G)$ such that $G'[A]$ is 3-connected or a triangle.*

Proof. Let G be a critically 2-connected graph. Let $v \in V(G)$ be a vertex of G. We will prove that v is either a boundary vertex, or belongs to a block that induces a cycle on G' with at least four vertices. It follows that $G' = G - v$ is connected, but not 2-connected and has at least three vertices. Let $u \in V(G')$ be a vertex of G' such that $G' - u = G - v - u$ is disconnected. Clearly $\{v, u\} \in \mathrm{Cut}_2(G)$. Assume first that $\{v, u\}$ is an isolated cutset. In this case, by Remark 11 there is a block that contains the cutset, and the vertices v and u are boundary vertices of such block. Assume now that $\{v, u\}$ is not an isolated cutset. In this case, by Proposition 29, there is a block B such that $\{v, u\} \subset B$ and the graph $G'[B]$ is a cycle on at least four vertices. In both cases, vertex v is either a boundary vertex, or belongs to a block that induces a cycle on G' with at least four vertices. Then v cannot be an internal vertex of a block that induces a 3-connected graph or a triangle in G'. Therefore the interior of any such blocks are empty.

Let G be a 2-connected graph such that $\mathrm{Int}(A) = \emptyset$ for all of its blocks $A \in \mathrm{Block}(G)$ such that $G'[A]$ is 3-connected or a triangle. Let $v \in V(G)$ be a vertex of G. We will prove that v is contained in a 2-cutset of G. It follows that v is either a boundary vertex of some block, or belongs to some block that induces a cycle on G' with at least four vertices. In the case that v is a boundary vertex, it belongs to some isolated cutset $\{v, u\} \in \mathfrak{D}(G)$. In the case that v belongs to some block that induce a cycle on G' with at least four vertices, by Proposition 29, the vertex v together with a non-adjacent vertex in the cycle $G'[A]$ forms a cutset of G that is not isolated. In both cases, vertex v is contained in some cutset of $\mathrm{Cut}_2(G)$, which implies that the graph $G - v$ is not 2-connected, and contains at least three vertices. Therefore G is critically 2-connected. $\square$

Proposition 45. *Let G be a critically 2-connected graph and let A be a leaf of $\mathrm{BT}(G)$ adjacent to cutset S in $\mathrm{BT}(G)$. Then A is a cycle with at least four vertices and all vertices of A, except for the two vertices of S have degree 2 in G.*

Proof. Clearly $\mathrm{Int}(A) \neq \emptyset$, therefore by Proposition 44, $G'[A]$ cannot be a 3-connected graph or a triangle. By Proposition 27, $G'[A]$ is a cycle with at least four vertices. By Proposition 28, all the vertices of $\mathrm{Int}(A)$, i.e., all vertices of the cycle except for the vertices of S, have degree 2 in G. $\square$

Corollary 46. *Any critically 2-connected graph has at least four vertices of degree 2.*

Proof. Let G be a critically 2-connected graph. If G has at least one isolated cutset, i.e., $\mathfrak{D}(G) \neq \emptyset$, then the tree $\mathrm{BT}(G)$ has at least two leaves. By Proposition 45, each of the leaves of $\mathrm{BT}(G)$ contain at least two vertices of degree 2 in G.

In the case that $\mathfrak{D}(G) = \emptyset$, then by Proposition 26, G is 3-connected or a cycle. Clearly G cannot be 3-connected and critically 2-connected. Then G is a cycle on at least four vertices. $\square$

Chapter 4

Decomposition of certain 2-connected graphs into three connected subgraphs

Between 1976 and 1977, Győri and Lovász independently proved the following theorem (Győri, 1978; Lovász, 1977).

Theorem 47. *Let $k \geqslant 2$ be an integer, let G be a k-connected graph on n vertices, let $v_1, \ldots, v_k$ be distinct vertices of G, and let $n_1, \ldots, n_k$ be positive integers with $n_1 + \cdots + n_k = n$. Then G has disjoint connected subgraphs $G_1, \ldots, G_k$ such that, for $i = 1, 2, \ldots, k$, the graph G_i has n_i vertices and $v_i \in V(G_i)$.*

A natural question to ask is whether $(k-1)$-connected graphs also share this property (for $k \geqslant 3$). That is, if such a graph admits k disjoint connected subgraphs of any sizes that cover the set of vertices of the graph. We now show a counter-example for this property.

Fix an integer $t \geqslant 2$. Let G be a $(k-1)$-connected graph of order n that contains a cutset S of size $k-1$ such that the graph $G - S$ has at least $k+1$ components, each containing exactly t vertices. We have $n \geqslant (k+1)t + (k-1) > k(t+1)$. Then, there exist integers $n_1, \ldots, n_k$ such that $n_1 + \cdots + n_k = n$ and such that $n_i \geqslant t+1$. We claim that the graph G cannot be partitioned into disjoint connected subgraphs of orders $n_1, \ldots, n_k$. Suppose that such subgraphs $G_1, \ldots, G_k$ exist. Let $W_1, W_2, \ldots$ be the components of the graph $G - S$. Then one of the subgraphs, G_i is contained in a component W_j given that S only contains $k-1$ vertices and there are k such subgraphs, and hence G_i does not contain vertices of S. But then we have $v(G_i) \leqslant v(W_j) = t$, which contradicts that $v(G_i) = n_i \geqslant t + 1$.

This counter-example suggest that, one possible obstruction for partitioning a $(k-1)$-connected graph into k disjoint connected subgraphs is the existence of $(k-1)$-cutsets that separate the graph into more than k pieces. Karpov proposed the following conjecture in (Karpov, 2017).

Conjecture 48. *Let $k \geqslant 3$ be an integer, and let G be a $(k-1)$-connected graph on n vertices such that any $(k-1)$-cutset of G splits the graph into at most k components. Let $n_1, \ldots, n_k$ be positive integers with $n_1 + \cdots + n_k = n$. Then G has disjoint connected subgraphs $G_1, \ldots, G_k$ such that for $i = 1, 2, \ldots, k$, the graph G_i has n_i vertices.*

It is easy to see that the conjecture is not true for $k = 2$. For instance, consider the graph that consists on taking the triangle K_3 and then adding three new vertices of degree one, each adjacent to a different vertex of of the triangle. This graph is an example of a connected graph where any of its cut vertices (the vertices of the triangle) separate the graph into two pieces: an isolated vertex and a path on four vertices, but it cannot be partitioned into two disjoint connected subgraphs of order three each (one of the subgraphs must contain two vertices of the triangle).

Karpov also proved the conjecture for the case $k = 3$ using the decomposition for a 2-connected graph described in this work. In this section, we present his proof for this case.

We first prove some auxiliary tools.

Proposition 49. *Let G be a connected graph, and suppose that the graph $\mathrm{BT}(G)$ has exactly two leaves B_1 and B_2 (using the block decomposition for $k = 1$, i.e., the classic decomposition). Let $v_1 \in \mathrm{Int}(B_1)$ and $v_2 \in \mathrm{Int}(B_2)$, and let n_1 and n_2 be positive integers such that $n_1 + n_2 = n$. Then G has disjoint connected subgraphs G_1 and G_2 such that $v_1 \in V(G_1)$, $v_2 \in V(G_2)$ and with $v(G_1) = n_1$ and $v(G_2) = n_2$.*

Proof. Note that the graph $G + v_1v_2$ is 2-connected. Then by Theorem 47 the desired subgraph exists. □

Proposition 50. *Let $A \in \mathrm{Block}(G)$ be a block of a 2-connected graph G such that the graph $G^{\mathfrak{D}(G)}[A]$ is 3-connected, and let $v \in \mathrm{Int}(A)$. Then the graph $G - v$ is 2-connected.*

Proof. Given that v is an internal vertex of a block of G, it does not belong to an isolated cutset of G. By Proposition 28 and Proposition 29, the vertex v is also not part of a non-isolated cutset of G. Therefore, the vertex v does not belong to any 2-cutset of G, and the graph $G - v$ is 2-connected. □

Proposition 51. *Let G be a 2-connected graph, and let $S = \{a, b\} \in \mathfrak{D}(G)$ be an isolated cutset of G. Let A be the union of some blocks of* $\mathrm{Block}(G; S)$. *Then the graph $G[A] + ab$ is 2-connected.*

Proof. Let $A = B_1 \cup \cdots \cup B_t$ be the union of blocks of $\mathrm{Block}(G; S)$. By Corollary 17, the graphs $G[B_i] + ab = G^{\{S\}}[B_i]$ are 2-connected. Then removing one vertex from the graph $G[A] + ab$ clearly leaves the graph connected. □

The following result is a more general version of Theorem 47 for the case $k = 2$.

Proposition 52. *Let G be a 2-connected graph of order n. Let $V_1 \subset V(G)$ be a set of vertices such that the graphs $G[V_1]$ and $G - V_1$ are connected, and let $v_2 \in V(G) \setminus V_1$. Let n_1, n_2 be positive integers such that $n_1 + n_2 = n$ and $|V_1| \leqslant n_1$. Then there exist disjoint connected subgraphs G_1 and G_2 of G with $v(G_1) = n_1$ and $V(G_2) = n_2$, and such that $V_1 \subset V(G_1)$ and $v_2 \in V(G_2)$.*

Proof. Let V_1 be fixed and proceed by induction on n_1. For the base case $n_1 = |V_1|$, the graphs $G[V_1]$ and $G - V_1$ satisfy the desired conditions. Suppose that $n_1 \geqslant |V_1| + 1$ and that the statement is true for smaller values of n_1. Then there exist connected subgraphs G_1 and G_2 of G of orders $n_1 - 1$ and $n_2 + 1$ that contain the set V_1 and the vertex v_2 respectively. Observe that if there is a vertex $v \in V(G_2)$ with $v \neq v_2$, with v adjacent to a vertex of G_1, and with the graph $G_2 - v$ connected, then the graphs $G_1 + v$ and $G_2 - v$ satisfy the desired conditions.

Consider the block decomposition of the graph G_2 (for the case $k = 1$, which coincides with the classic block decomposition). In the case that the block tree of G_2 is trivial, then clearly any vertex of G_2 that is different from v_2 does not disconnect G_2. Also, given that G is 2-connected, there must be at least two vertices in G_2 that are adjacent to a vertex in G_1, at least one of which is not vertex v_2. In the case that the block tree of G_2 is not trivial, then it has at least two leaf blocks. If v_2 is not a cut-vertex of G_2, then there is a leaf block B that does not contain vertex v_2. If v_2 is a cut-vertex of G_2, then let B be any leaf block. Note that $v_2 \notin \mathrm{Int}(B)$. Let u be a vertex of G_2 that separates B from the rest of the graph G_2. Then the set $\mathrm{Int}(B)$ cannot be separated from the rest of the graph G by u. Then there is a vertex in the set $\mathrm{Int}(B)$ that is adjacent to a vertex in G_1. This vertex clearly does not disconnect the graph G_2 and is different from v_2. □

Let G be a 2-connected graph with a weight function $w : V(G) \to \mathbb{Z}_{\geqslant 1}$, and let n_2 and n_3 be positive integers such that the sum of the weights of all vertices of G

is n_2+n_3. For a vertex $z_1 \in V(G)$ such that $w(z_1) \leqslant \max\{n_2, n_3\}$, a vertex $x \in V(G)$, and an integer $k \leqslant w(x)-1$; a (z_1, x, k)-*operation* consists on decreasing the weight of x by k, and increasing the weight of z_1 by k. Note that this operation creates a new function w'.

Proposition 53. *Let G, w, n_2 and n_3 be as above. Then, for any vertex $z_1 \in V(G)$ such that $w(z_1) \leqslant \max\{n_2, n_3\}$, there exist a vertex $x \in V(G)$ and an integer k such that after a (z_1, x, k)-operation, the graph G has disjoint connected subgraphs G_2 and G_3 such that, for $i = 1, 2$, the sum of the weights of the vertices of G_i is n_i (using the new weight function).*

Proof. Suppose that $w(z_1) \leqslant n_2$. For a set $U \subset V(G)$, let $w(U)$ be the sum of the weights of all vertices in U. Let $V_2 = \{v\}$, with $v \in V(G) \setminus \{z_1\}$, and $V_3 = V(G) \setminus \{v\}$. The graphs $G[V_2]$ and $G[V_3]$ are connected. In the case that $w(v) = w(V_2) \geqslant n_2$, then we perform a $(z_1, v, w(v) - n_2)$-operation, after which the statement clearly holds. Suppose now that $w(V_2) < n_2$. We will describe an algorithm to find the operation that will be performed. In each step of the algorithm, we start with the sets V_2 and V_3 and want to find a vertex $u \in V_3$ such that u is adjacent to a vertex in V_2 and the graph $G[V_3] - u$ is connected. Then the new sets are $V_2' = V_2 \cup \{u\}$ and $V_3' = V_3 \setminus \{u\}$, which are connected. In the case that $w(V_2) + w(u) \geqslant n_2$, then we perform a $(z_1, u, w(u) - n_2 + w(V_2))$-operation, so the final weight of vertex u is $n_2 - w(V_2)$, and the graphs $G[V_2']$ and $G[V_3']$ are the desired ones. In the case that $w(V_2) + w(u) < n_2$, we perform another iteration of the algorithm. At each step, the size of the set V_2 increases. Therefore, the algorithm stops after at most $v(G)$ iterations.

At each iteration, we find vertex u in the following way. If the graph $G[V_3]$ is 2-connected or has two vertices, then clearly such a vertex exist. If the graph $G[V_3]$ has at least three vertices and is connected, but not 2-connected, then we perform the block decomposition of $G[V_3]$ (for the case $k = 1$). Then the graph $G[V_3]$ has at least two leaf blocks, one of which, L does not contain vertex z_1 in its interior. Let d be a cut-vertex of $G[V_3]$ that separates L from the rest of the graph $G[V_3]$. The vertex d cannot separate L from the rest of the graph. Then there is a vertex u in the interior of L that is adjacent to a vertex of V_2. Vertex u is the vertex used for the current iteration (see Figure 4.1). $\square$

The following result offers a sufficient condition for partitioning a 2-connected graph into three disjoint connected subgraphs.

Proposition 54. *Let G be a 2-connected graph of order n, and let n_1, n_2, n_3 be positive*

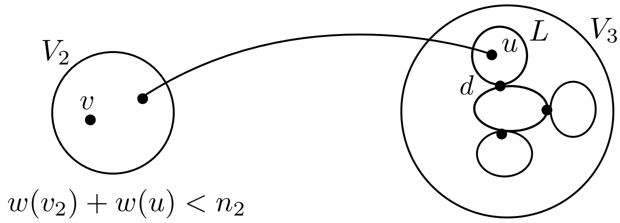

Figure 4.1: *An iteration that increases V_2.*

integers with $n_1+n_2+n_3 = n$. Suppose that there is an isolated 2-cutset $S \in \mathfrak{D}(G)$ and a block $A \in \mathrm{Block}(G; \{S\})$ such that $n_1 \leqslant |\mathrm{Int}(A)| \leqslant n_1 + \min\{n_2, n_3\} - 1$. Then G has three disjoint connected subgraphs of orders n_1, n_2, and n_3 respectively.

Proof. Suppose without loss of generality that $n_2 \leqslant n_3$. Let $\{a, b\} = S$, let $G' = G[A]+ab$. By Proposition 51, the graph G' is 2-connected. Also, the graphs $G'[S]$ and $G' - S$ are connected, and G' has at least $n_1 + 2$ vertices. Then by Proposition 52, G' has disjoint connected subgraphs G_1 and G_2' of such that $v(G_1) = n_1$, $2 \leqslant v(G_2') \leqslant n_3 + 1$, and $S \subset V(G_2')$ (the graph G_1 will be one of the resulting graphs offered by the lemma).

In the case that $ab \notin E(G)$, we conclude that the graph $G[V(G_2')]$ has exactly two components H_a and H_b containing vertices a and b respectively. In the case that $ab \in E(G)$, then the graph $G[V(G_2')]$ is connected and has disjoint connected subgraphs H_a and H_b containing vertices a and b. In both cases, given the order of the graph G_2' and that the graphs H_a and H_b contain at least one vertex, we conclude that $v(H_a) \leqslant n_2$ and $v(H_b) \leqslant n_3$. respectively (see Figure 4.2).

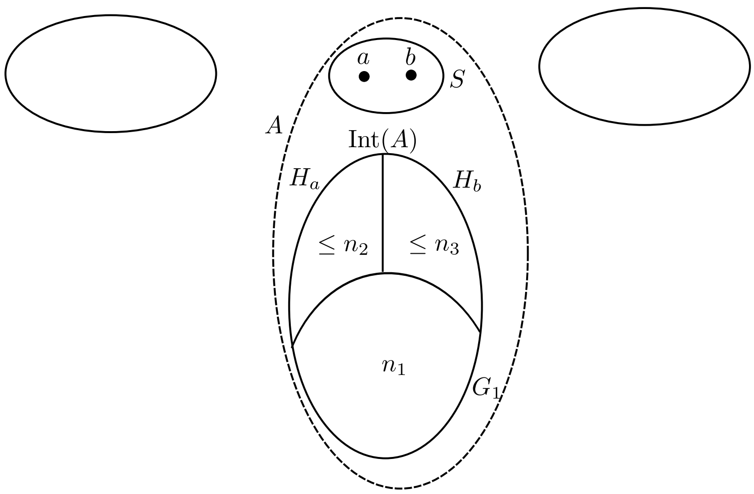

Figure 4.2: *Decomposition of a block.*

Let $m_2 = n_2+1-v(H_a)$ and $m_3 = n_3+1-v(H_b)$. Note that both $m_2, m_3 \geqslant 1$, and that that the graph $G^* = G-\mathrm{Int}(A)+ab$ has m_2+m_3 vertices. Also, by Proposition 51, the graph G^* is 2-connected. By Proposition 52, the graph G^* has disjoint connected subgraphs F_2 and F_3 such that $v(F_2) = m_2$, $v(F_3) = m_3$ and that $a \in V(F_2)$ and $b \in V(F_3)$. Observe that $V(H_a) \cap V(F_2) = \{a\}$ and $V(H_b) \cap V(F_3) = \{b\}$. Let $G_2 = H_a \cup F_2$ and let $G_3 = H_b \cup F_3$. Clearly both graphs are connected, and $v(G_2) = n_2$, and $v(G_3) = n_3$. Then the graphs G_1, G_2 and G_3 are disjoint connected subgraphs of G with the desired orders. □

4.1 Proof of Conjecture 48 for the case $k = 3$

We now use the block decomposition theory to prove Conjecture 48.

Let G be a 2-connected graph on n vertices such that any 2-cutset of G splits the graph into at most 3 components. Let n_1, n_2, n_3 be positive integers with $n_1+n_2+n_3 = n$, and assume without loss of generality that $n_1 \geqslant n_2 \geqslant n_3$.

In the case that $\mathfrak{D}(G) = \emptyset$, by Proposition 26 the graph G is either a cycle or 3-connected. If G is a cycle, it has disjoint connected subgraphs G_1, G_2 and G_3 such that for $i = 1, 2, 3$, the graph G_i has n_i vertices. If G is a 3-connected graph, then by Theorem 47, the graph also admits the same decomposition [1].

For the rest of the proof, we assume that $\mathfrak{D}(G) \neq \emptyset$, and hence, that the block tree $\mathrm{BT}(G)$ of G is non-trivial.

Let ℓ be a leaf of the graph $\mathrm{BT}(G)$. In order to define parent and children relationships between vertices of $\mathrm{BT}(G)$, consider an orientation of all the edges of $\mathrm{BT}(G)$ away from vertex ℓ such that every other vertex is accessible from a directed path starting in vertex ℓ. For an edge uv with orientation (u, v), we call vertex u the *parent* of vertex v, and v is a *child* of vertex u. It is easy to see that every vertex v of $\mathrm{BT}(G)$ has at most one parent. Let $M(\ell) = V(G)$. For a vertex $A \in V(\mathrm{BT}(G))$ such that $A \in \mathrm{Block}(G)$, let $M(A) = B$, with $B \in \mathrm{Block}(G; \{p(A)\})$ and $A \subset B$ (observe that $p(A) \in \mathfrak{D}(G)$). For a vertex $S \in V(\mathrm{BT}(G))$ such that $S \in \mathfrak{D}(G)$, let $M(S)$ be the union of the members of $\mathrm{Block}(G; \{S\})$ with the exception of the one containing ℓ (see Figure 4.3). For a vertex $v \in V(\mathrm{BT}(G))$, let $m(v) = |M(v)|$.

We will choose a special vertex of $\mathrm{BT}(G)$. Consider an ℓA-path of $\mathrm{BT}(G)$ maximal with respect to the property that $m(A) \geqslant n_2 + 2$ and $m(B) \leqslant n_2 + 1$ for every child B

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[1] If G is a 3-connected graph and $F \subset E(G)$ is a minimal set of edges such that $G - F$ is not 3 connected, then every 2-cutset of the graph $G - F$ splits it into two pieces. Thus, we can avoid applying Theorem 47 and continue with the proof considering the graph $G - F$ given that it satisfies the conditions of the conjecture.

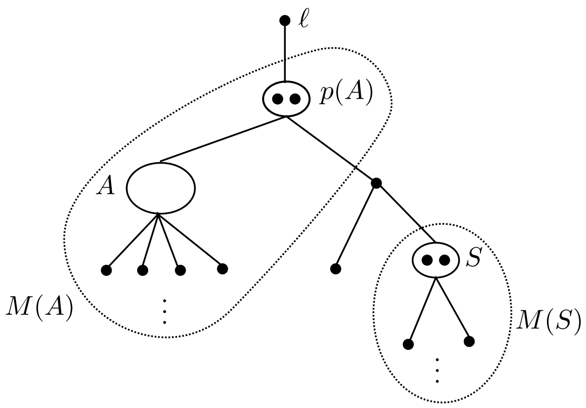

Figure 4.3: *Branching of the graph* BT(G).

of A. This path clearly exists since $m(\ell) = n \geqslant n_2 + 2$, so we can take the trivial path consisting of vertex ℓ.

Note that the vertex A can be either a block or an isolated 2-cutset of G. We will now split the proof into two cases depending on these possibilities.

Suppose that the set $A = S$ is an isolated cutset, i.e., $S \in \mathfrak{D}(G)$. Let $B_\ell \in \text{Block}(G; \{S\})$ be the block that contains ℓ. Let $B_g \in \text{Block}(G; \{S\})$ be a block with maximum cardinality different from B_ℓ. By Proposition 18 (b), there is a block $B_d \in \text{Block}(G)$ such that B_d is adjacent to S in $\text{BT}(G)$ and $B_d \subset B_g$. Note that B_d is a child of S (see Figure 4.4). Therefore, $M(B_d) = B_g$ and $m(B_d) = |\text{Int}(B_g)| + 2$, and given the special property of vertex S, it follows $m(B_d) \leqslant n_2 + 1$ and hence, $|\text{Int}(B_g)| \leqslant n_2 - 1$.

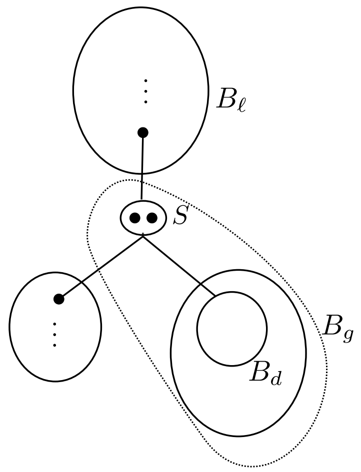

Figure 4.4: *When* $A = S$ *is a 2-cutset.*

We claim that $|\text{Int}(B_g)| \geqslant n_3$ or $|\text{Int}(B_\ell)| \geqslant n_1$. Suppose by contradiction that $|\text{Int}(B_g)| \leqslant n_3 - 1$ and that $|\text{Int}(B_\ell)| \leqslant n_1 - 1$. It follows that $|\text{Block}(G; \{S\})| \leqslant 3$ by the hypothesis of the conjecture. Also, by the choice of B_g each block of $\mathrm{Block}(G;\{S\})$ different from B_ℓ has at most $|\mathrm{Int}(B_g)| \leqslant n_3 - 1$ vertices on its interior. Then $n-2 = |V(G) \setminus S| \leqslant n_1 - 1 + 2(n_3 - 1) \leqslant n_1 + n_2 + n_3 - 3 = n - 3$, a contradiction.

In the case that $|\mathrm{Int}(B_g)| \geqslant n_3$, it follows that $n_3 \leqslant |\mathrm{Int}(B_g)| \leqslant n_2 - 1 < n_3 + n_2 - 1 \leqslant n_3 + n_1 - 1$, and thus we can apply Proposition 54 with $S \in \mathfrak{D}(G)$ and $B_g \in \mathrm{Block}(G;\{S\})$ to get the desired decomposition of G.

In the case that $|\mathrm{Int}(B_\ell)| \geqslant n_1$, given that $m(S) \geqslant n_2 + 2$ and that $|\mathrm{Int}(B_\ell)| + m(T) = n = n_1 + n_2 + n_3$, we conclude that $|\mathrm{Int}(B_\ell)| \leqslant n_1 + n_3 - 2 < n_1 + n_2 - 1$, and we also can apply Proposition 54 with $S \in \mathfrak{D}(G)$ and $B_\ell \in \mathrm{Block}(G;\{S\})$ to get the desired decomposition of G. This completes the proof for the case when $A = S$ is an isolated cutset.

Suppose now that A is a block of G, i.e., $A \in \mathrm{Block}(G)$. Let $T_1, T_2, \ldots, T_h$ be the vertices adjacent to A in the graph $\mathrm{BT}(G)$ such that $T_1 = p(A)$. For $i = 1, \ldots, h$, let B_i be the union of all the blocks in $\mathrm{Block}(G;\{T_i\})$ that do not contain A. Let $\mathrm{Int}(B_i) = B_i \setminus T_i$ (given that B_i is not a block, this was not previously defined), and let $m'_i = |\mathrm{Int}(B_i)|$. For $i = 2, \ldots, h$, we conclude that

$$m'_i = m(T_i) - 2 \leqslant n_2 + 1 - 2 = n_2 - 1. \tag{4.1}$$

We also have $\mathrm{Int}(B_1) \cup M(A) = V(G)$, where the sets $\mathrm{Int}(B_1)$ and $M(A)$ are disjoint. Then $m'_1 + m(A) = |\mathrm{Int}(B_1)| + |M(A)| = n = n_1 + n_2 + n_3$. Given that $m(A) \geqslant n_2 + 2$, it follows that

$$m'_1 = n_1 + n_2 + n_3 - m(A) \leqslant n_1 + n_3 - 2. \tag{4.2}$$

Let j be an integer such that m'_j is maximum among all $m'_1, \ldots, m'_h$. Let $A_1, \ldots, A_h$ be the sets $B_1, \ldots, B_h$ with $A_1 = B_j$, $A_j = B_1$, and $A_i = B_i$ for $i = 2, \ldots, h$, with $i \neq j$. For $i = 1, \ldots, h$, let $m_i = |\mathrm{Int}(A_i)|$.

Equation (4.1) and Equation (4.2) hold for $m_1, \ldots, m_h$, indeed, in the case that $j = 1$, the claim is trivial. If $j > 1$, for $i = 2, \ldots, h$ we have

$$m_i \leqslant m_1 = m'_j \leqslant n_2 - 1. \tag{4.3}$$

Also, it follows that

$$m_1 = m'_j \leqslant n_2 - 1 \leqslant n_1 - 1 \leqslant n_1 + n_3 - 1 - 1 = n_1 + n_3 - 2. \tag{4.4}$$

Let $T_1 = \{y_1, z_1\}, T_2 = \{y_2, z_2\}, \ldots, T_h = \{y_h, z_h\}$. Let $F = G[A] + y_1z_1 + y_2z_2 + \cdots + y_hz_h$ (see Figure 4.5). Let $w(y_iz_i) = m_i = |\mathrm{Int}(A_i)|$, and $w(e) = 0$ for any other edge e of G (the edge y_1z_1 has maximum weight m_1).

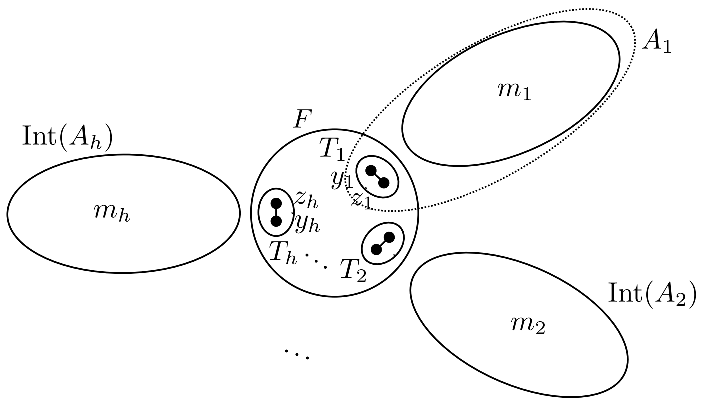

Figure 4.5: *When A is a block.*

By Proposition 27, the graph F is a cycle, or it is 3-connected. We consider both cases separately.

We first consider the case where F is a cycle. Let $F = a_0a_1 \ldots a_\ell$, and assume that the vertices $y_1, z_1, y_2, z_2, \ldots, y_h, z_h$ are present in cyclic order, with $a_0 = y_1$ and $a_1 = z_1$. We now construct a new cycle C by taking a copy of the cycle F and, for each edge a_ia_{i+1} adding $w(a_ia_{i+1})$ new vertices of degree 2 (this new cycle represent the whole graph G that we want to split in three pieces).

Note that if we "perform three cuts" to the cycle C such that it is decomposed into three paths with n_1, n_2 and n_3 vertices respectively and with no two cuts on the same a_ia_{i+1}-section of the cycle, then we can obtain the desire decomposition of G. Indeed, suppose that we perform a cut in the a_ia_{i+1}-segment of C. If there are no internal vertices in the segment, i.e., if $w(a_ia_{i+1}) = 0$, then we put vertex a_i in one subgraph and vertex a_{i+1} in another subgraph. If the segment contains internal vertices, then $w(a_ia_{i+1}) > 0$ and $\{a_i, a_{i+1}\}$ is one of the sets T_i. Suppose that the cut occurs after the jth vertex starting from vertex a_i. Then, by Proposition 52, the 2-connected graph $G[A_i] + a_ia_{i+1}$ has disjoint connected subgraphs G_1 and G_2 such that $v(G_1) = j$ and $a_i \in V(G_1)$, $a_{i+1} \in V(G_2)$. Performing this decomposition for each of the three cuts creates three disjoint subgraphs of G with sizes n_1, n_2 and n_3 respectively (see Figure 4.6).

We now prove that there is a choice of three cuts on the cycle C that decomposes

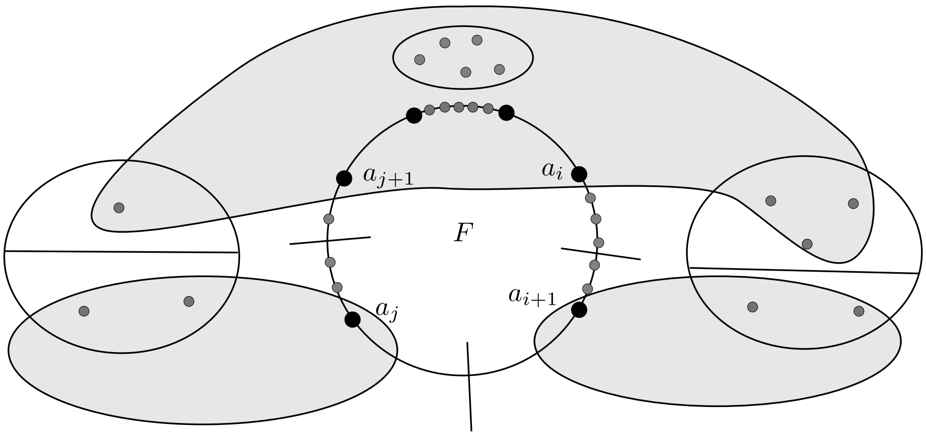

Figure 4.6: *Three cuts on F.*

it into three paths with n_1, n_2 and n_3 vertices respectively and with no two cuts on the same $a_i a_{i+1}$-section of the cycle.

In the case that $w(a_0 a_1) \geqslant n_1 - 1$, the first cut is made in the edge incident to vertex a_0 outside the $a_o a_1$-section. The second cut is made just after the n_1th vertex starting from the first cut (the first vertex is a_0) and going in cyclic order. Given that there are $w(a_0 a_1)+2 \geqslant n_1+1$ vertices in the $a_0 a_1$-section, the second cut occurs in this section and the first cut occurs outside of it. The third cut is made just after the n_3th vertex starting from the second cut and going in cyclic order. By Equation (4.4), the third cut is not on the $a_0 a_1$-section. Also, by Equation (4.3), this cut is not on the $a_\ell a_0$-section (see Figure 4.7 left).

In the case that $w(a_0 a_1) < n_1 - 1$, the first cut is made in the edge incident to vertex a_0 inside the $a_0 a_1$-section. The second cut is made just after the n_1th vertex starting from the first cut and going in cyclic order. Given that there are $w(a_0 a_1)+2 \leqslant n_1$ vertices on the $a_0 a_1$-section and that the vertex a_0 lies before the first cut, the second cut is in some $a_i a_{i+1}$-section, where $i > 0$. The third cut is made just after the n_2th vertex starting from the second cut and going in cyclic order. Given that there are $w(a_i a_{i+1}) < n_2 - 1$ internal vertices in the $a_i a_{i+1}$-section, the third cut is outside of it. Also, note that the third cut is outside the $a_0 a_1$-section (see Figure 4.7 right). This completes the proof for the case where the graph F is a cycle.

For the rest of the proof, we assume that the graph $F = G[A]+y_1z_1+y_2z_2+\cdots+y_hz_h$ is 3-connected. For a set $U \subset V(G)$ of vertices of G, let $U' = U \cap V(F) = U \cap A$. An edge $y_i z_i$ is said to be *U-problematic* if it is incident to exactly one vertex in U'. We now define the weights of the vertices of F. For a U-problematic edge $y_i z_i$ such that $y_i \in U' \subset U$, let $w(y_i) = 1 + |\mathrm{Int}(A_i) \cap U|$, and $w(z_i) = 1 + m_i - |\mathrm{Int}(A_j) \cap U|$. All other vertices of F have weight 1 (see Figure 4.8). Let $w_{U'}$ be the sum of the

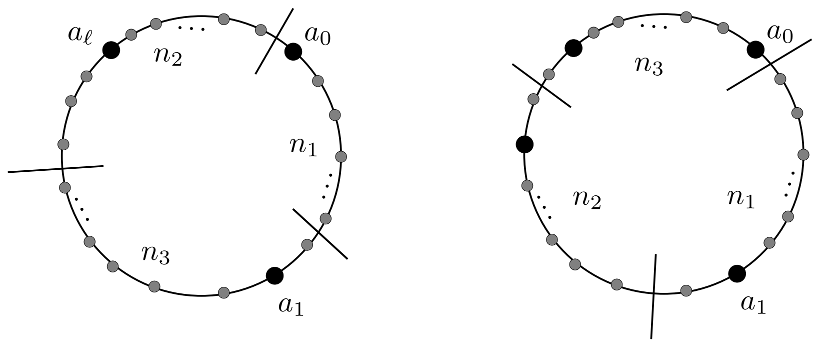

Figure 4.7: *Choice of the cuts on F.*

weights of all the vertices inside U' plus the sum of the weights of all edges of the form $y_i z_i$ with $y_i, z_i \in U'$.

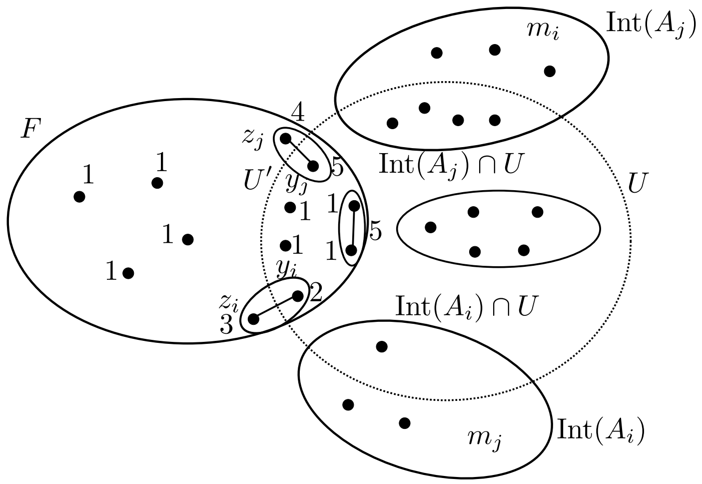

Figure 4.8: *Example of the weights of F.*

We call a set $U \subset V(G)$ *promising* if all the following conditions hold:

1. $|U| \leqslant n_1$.

2. edge $y_1 z_1$ is U-problematic, with $y_1 \in U$ and $z_1 \notin U$.

3. $w(y_1) \geqslant w(v)$ for all $v \in A \setminus \{z_1\}$.

4. The graphs $G[U]$, $F[U']$, $G - U$, and $F - U'$ are connected.

5. The graph $F - U'$ does not have cut-vertices (in particular, if it has more than 2 vertices, then it is 2-connected).

It is clear that, for any promising set U, if any edge $y_i z_i$ is such that $y_i, z_i \in U$, then $\mathrm{Int}(A_i) \subset U$, otherwise the graph $G - U$ separates $\mathrm{Int}(A_i)$ from the rest of the graph, and hence it is disconnected, contradicting condition 4. Observe also that $w_{U'} = |U|$ given that the sets $\mathrm{Int}(A_1), \mathrm{Int}(A_2), \ldots, \mathrm{Int}(A_h)$ are disjoint.

Let us first show that a promising set exist. In the case that $m_1 \leqslant n_1 - 1$, let $U = \{y_1\} \cup \mathrm{Int}(A_1)$. In the case where $m_1 > n_1 - 1$, by Proposition 52 the graph $G[A_1] + y_1 z_1$ has disjoint connected subgraphs with vertex sets W_1 and W_2 with $y_1 \in W_1$ and $z_1 \in W_2$ and such that $|W_1| = n_1$; we let $U = W_1$. Clearly the first two conditions are satisfied. Also, since F is 3-connected, it follows that $F - U = F - y_1$ is 2-connected, therefore conditions 4 and 5 are satisfied. Suppose that $y_i z_i$ is a U-problematic edge, where $i \geqslant 2$. Then, $w(y_i)$ and $w(z_i)$ are both at most $m_i + 1$. By the choice of the set U, either $w(y_1) = m_1 + 1 \geqslant m_i + 1$, or $w(y_1) = n_1 \geqslant n_2 \geqslant m_i + 1$. Then the weight of any vertex in the set $A \setminus \{z_1\}$ is at most $w(y_1)$, so condition 3 is also satisfied, and the set U is promising.

Observe that, by Equation (4.4), it follows that $w(z_1) \leqslant \max\{0, 1 + m_1 - n_1\} \leqslant n_3$. Also, the graph $F - U'$ has at least two vertices, otherwise it only contains vertex z_1, which is not possible.

For a promising set U, let G' be the graph obtained by taking a copy of the graph $G - U$ and then, for each U-problematic edge $y_i z_i$, deleting all the vertices of each set $\mathrm{Int}(A_i)$.

Proposition 55. *Let U be a promising set such that $|A \setminus U| \geqslant 3$, then the graphs $F - U'$ and G' are 2-connected.*

Proof. The graph $F - U'$ has at least 3 vertices, and by condition 5, it does not have cut-vertices, therefore it is 2-connected. Let Z be the set of edges of the form $y_i z_i$ such that $y_i, z_i \in U'$. Note that $G' = (F - U' - Z) \cup \bigcup_{y_i z_i \in Z} G[\mathrm{Int}(A_i)]$. Given that $F - U'$ and the graphs $G[\mathrm{Int}(A_i)]$ are both 2-connected, the graph G' is also 2-connected. □

Remark 56. By the above argument, if the graph $F - U'$ is 2-connected, then the graph G' is also 2-connected.

A promising set U is *perfect* if $|U| = n_1$.

Proposition 57. *If there exists a perfect set $U \subset V(G)$, then G has disjoint connected subgraphs G_1, G_2 and G_3 with n_1, n_2 and n_3 vertices respectively.*

Proof. Observe first that the sum of the weights of all vertices of the graph G' is $n_2 + n_3$. For any U-problematic edge $y_i z_i$, with $y_i \in U'$ and $z_i \notin U$, we conclude that $w(z_j) = 1 + |\mathrm{Int}(A_j) \setminus U|$. Then the sum of the weights of the vertices in $V(G')$ is the number of vertices in $V(G) \setminus U$, which is $n_2 + n_3$.

We will apply Proposition 53 in the 2-connected graph G' using z_1 as a fixed vertex (for understanding, Figure 4.9 will be helpful). Observe that by Equation (4.4) we conclude that $w(z_1) \leqslant \max\{0, m_1 - n_1 + 1\} \leqslant n_3 \leqslant n_2$. Also, we showed that the sum of the weights of the vertices of the graph G' is $n_2 + n_3$. Roughly speaking, we will first perform the operation described in Proposition 53, and then obtain disjoint subsets of vertices $V_2', V_3' \subset V(G')$ such that the graphs $G'[V_2']$ and $G'[V_3']$ are connected and the sum of the weights of their vertices is n_2 and n_3 respectively. The set U will be modified when performing the operation while still inducing a connected subgraph and without changing the number of vertices.

Proposition 53 gives us a vertex $u \in A \setminus U'$, and an integer d. Suppose that we will perform a (z_1, u, d)-operation by decreasing the weight of u and increasing the weight of z_i by d each. It follows that $w(u) \geqslant d + 1$, then $u \in \{y_i, z_i\}$, where $y_i z_i$ is a U-problematic edge, with $i \geqslant 2$. Let $u = z_i$ and $y_i \in U'$, and let q_i be such that $w(z_i) = w(u) = q_i + 1$. The 2-connected graph $G[A_i] + y_i z_i$ has disjoint connected subgraphs with vertex sets W_i and U_i such that $|W_i| = q_i + 1$, $z_i \in W_i$ and $y_i \in U_i$, and such that $W_i \subset V(G) \setminus U$ and $U_i \subset U$. Also, by Proposition 52 we can divide the graph $G[A_i] + y_i z_i$ into disjoint connected subgraphs with vertex sets W_i' and U_i' such that $|W_i'| = 1 + q_i - d$, $z_i \in W_i'$ and $y_i \in U_i$ and then replace the subset U_i by U_i' in U, and the subset W_i by W_i' in $V(G) \setminus U$.

For increasing the weight of z_1 by d, the 2-connected graph $G[A_1] + y_1 z_1$ has disjoint connected subgraphs with vertex sets W_1 and U_1 such that $z_1 \in W_1$ and $y_1 \in U_1$, and such that $W_1 \subset V(G) \setminus U$ and $U_1 \subset U$. By condition 3 of the promising set U, we have $|U_1| = w(y_1) \geqslant w(z_i) \geqslant d + 1$. By Proposition 52 we can divide the graph $G[A_1] + y_1 z_1$ into disjoint connected subgraphs with vertex sets W_1' and U_1' such that $|U_1'| = |U_1| - d$, and such that $z_1 \in W_1'$ and $y_1 \in U_1'$ and then replace the subset U_1 by U_1' in U, and the subset W_1 by W_1' in $V(G) \setminus U$. Clearly the new set U induces a connected graph and still has n_1 vertices (see Figure 4.9).

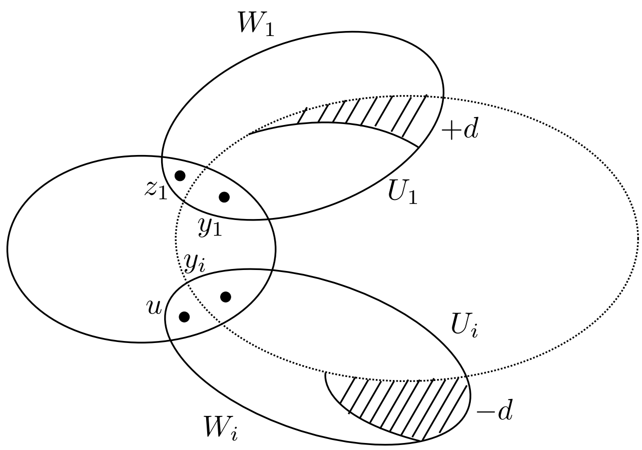

Figure 4.9: *Performing a (z_1, u, d)-operation in the outside world.*

We now construct the three desired disjoint connected subgraphs of G. Let $V_1 = U$. the graph $G[V_1]$ is connected with $|V_1| = n_1$ vertices. By Proposition 53, the graph G' has disjoint connected subgraphs with vertex sets V_2' and V_3' such that $w(V_2') = \sum_{x \in V_2'} w(x) = n_2$ and $w(V_3') = \sum_{x \in V_3'} w(x) = n_3$.

We will construct set V_2 (the same process can construct set V_3). Let $v \in V_2'$ be a vertex such that $w(v) \geqslant 2$, and suppose that $v \in \{y_j, z_j\}$, where $y_j z_j$ is a U-problematic edge. The set $W_v = A_j \setminus U$ is connected, and $z_j \in W_j$ with $|W_j| = w(z_j)$. Let W be the union of the sets W_v for each vertex $v \in V_2'$ such that $w(v) \geqslant 2$. Let $V_2 = V_2' \cup W$. Observe that we replace each vertex z_j by $w(z_j)$ new vertices that induce a connected subgraph. Note that the set $V_3 = (V(G) \setminus V_1) \setminus V_2$ can be constructed in the same way. Thus, the graphs $G[V_2]$ and $G[V_3]$ are connected with n_2 and n_3 vertices respectively. $\square$

Proposition 58. *Let $U \subset V(G)$ be a promising set of maximum size. Then, either U is perfect, or for all U-problematic edges $y_i z_i$,* $\mathrm{Int}(A_i) \subset U$.

Proof. Assume that U is not perfect, and suppose by contradiction that there is a U-problematic edge $y_i z_i$ such that $\mathrm{Int}(A_i) \not\subset U$. Let $y_i \in U$ and $z \notin U$. The 2-connected graph is divided into disjoint connected subgraphs with vertex sets U_j and W_j such that $y_j \in U_j$ and $z_j \in W_j$, and $U_j \subset U$, $W_j \cap U = \emptyset$, and $|W_j| \geqslant 2$. By Proposition 52, the 2-connected graph $G[A]+y_i z_i$ has disjoint connected subgraphs with vertex sets W_j' and U_j' with $|U_j'| = |U_j| + 1$ and $y_j \in U_j'$, $z_j \in W_j'$. The set U can be replaced with the set $(U \setminus U_j) \cup U_j'$, which is still a promising set and with one more vertex than the set U. This contradicts the maximality of the set U. $\square$

Let U be a promising set of maximum size. By Proposition 58, the set U is perfect or for all U-problematic edges $y_i z_i$, $\mathrm{Int}(A_i) \subset U$. If the set U is perfect, then the proof is complete by Proposition 57. Suppose that $|U| < n_1$ and that for all U-problematic edges $y_i z_i$, $\mathrm{Int}(A_i) \subset U$.

Proposition 59. *The graphs $F - U'$ and $G - U$ are 2-connected.*

Proof. Given that the graph $F - U'$ has at least two vertices, if it is not 2-connected, then by Proposition 55 and Equation (4.4) it is the graph K_2. If the edge on the graph $F - U'$ is not of the form $y_i z_i$, then $G - U = F - U'$, hence the graph $G - U$ has two vertices, and we have $|U| \geqslant n_1$, which is a contradiction. If the edge of the graph $F - U'$ is of the form $y_i z_i$, then $i \geqslant 2$ and $m_i = |\mathrm{Int}(A_i)| \leqslant n_2 - 1$ by Equation (4.3). Therefore $V(G - U) = V(A_j)$ and the graph $G - U$ has at most $n_2 + 1 \leqslant n_2 + n_3$ vertices. In this case we also conclude that $|U| \geqslant n_1$, a contradiction. Then the graph $F - U'$ is 2-connected.

Observe that the graph $G - U$ is the graph G' since for all U-problematic edges $y_i z_i$, $\mathrm{Int}(A_i) \subset U$. By Remark 56, the graph $G - U$ is 2-connected. □

Let $H = F - U'$ be a 2-connected graph. In the case that the graph H is not 3-connected and also is not a cycle, then it has isolated 2-cutsets, and we perform the block decomposition (for the case $k = 2$) on H. In this case, the graph $\mathrm{BT}(H)$ has at least two leaf blocks, and we let $B \in V(\mathrm{BT}(H))$ be a leaf block such that $z_1 \notin \mathrm{Int}(B)$.

Proposition 60. *The graph H is a cycle, or the graph $H^{\mathfrak{D}(H)}[B]$ is a cycle.*

Proof. Suppose by contradiction that the graph H is 3-connected or that the graph $H^{\mathfrak{D}(H)}[B]$ is 3-connected.

Observe that, given that the graph F is 3-connected, and that $U \neq \emptyset$, no set $M \subset V(G)$ can be separated from U' in the graph F by less than three vertices, and hence it can also not be separated from U in G by less than three vertices. We will use this observation in the following two cases.

Consider the case in which H is 3-connected. Then by the observation above, the graph H has a vertex $v \neq z_1$ that is adjacent to a vertex in U, and the graph $H - v = F - (U' \cup \{v\})$ is 2-connected.

Consider now the case in which the graph H is not 3-connected, but the graph $H^{\mathfrak{D}(H)}[B]$ is 3-connected. Then by the observation above, there exists a vertex $v \in \mathrm{Int}(B)$ that is adjacent to a vertex in U and such that $v \neq z_1$. By Proposition 50, the graph $H - v = F - (U' \cup \{v\})$ is 2-connected.

In both cases, the graphs $F[U' \cup \{v\}]$ and $G[U \cup \{v\}]$ are connected. By Proposition 59, the graph $G - U$ is 2-connected, and so the graph $G - U - v$ is connected. Note that the set $U \cup \{v\}$ is promising. This contradicts the maximality of U. □

Suppose that the graph $H^{\mathfrak{D}(H)}[B] = C$ is a cycle. Let $C = a_0 a_1 \ldots a_\ell a_{\ell+1}$. Let $\mathrm{Bound}(B) = a_0, a_{\ell+1}$. In the case that $z_1 \in \mathrm{Bound}(B)$, let $z_1 = a_0$. If H is a cycle, then let $H = C = a_0 a_1 \ldots a_\ell$, and let $a_0 = z_1$.

Observe that each of the vertices $a_1, \ldots, a_\ell$ are adjacent to a vertex in U' given that the graph F is 3-connected and that these vertices have degree 2 in the graph $H = F - U'$. Fix $0 \leqslant i < j \leqslant \ell + 1$, and let $S_{i,j}$ be union of the vertices $a_{i+1}, \ldots, a_{j-1}$

(note the exclusion of vertices a_i and a_j) and all the vertices in $\mathrm{Int}(A_l)$, where edge $y_l z_l$ is one of $a_i a_{i+1}, a_{i+1} a_{i+2}, \ldots, a_{j-1} a_j$ (note the inclusion of vertices a_i and a_j). Let $s_{ij} = |S_{i,j}|$.

It follows that $s_{0,\ell} > n_1 - |U|$. Indeed, suppose that $s_{0,\ell} \leqslant n_1 - |U|$. Then the set $U \cup S_{0,\ell}$ is promising with size at most n_1, which contradicts the maximality of U.

Let us construct a new cycle C' as before, by adding to each edge $a_i a_{i+1}$, $w(a_i a_{i+1})$ new vertices of degree 2. Let b be the vertex in the $a_0 a_1$ section of C that is adjacent to vertex a_0. We perform a cut on the edge $a_0 b$ (in the cycle C') and obtain a path P.

We first consider the case in which $s_{0,1} < n - |U|$. In this case we consider the subpath P_1 of P that contains vertex b and has exactly $n_1 - |U|$ vertices. For this, we perform a cut in P in the edge outside P_1 that is adjacent to the endpoint of P_1 opposite to b. Suppose that such cut was performed on the $a_i a_{i+1}$-section of P. Note that $i \geqslant 1$. We start constructing set V_1 by first putting all the vertices of the set $U \cap S_{0,i}$, which induces a connected graph given that a_1 is adjacent to a vertex in U. We will now include in V_1 the remaining $t = n_1 - |U| - s_{0,i} \geqslant 0$ vertices of the $a_i a_{i+1}$-section of P_1. in the following way. In the case where the edge $a_i a_{i+1}$ is not of the form $y_j z_j$ is trivial since there is nothing more to add, therefore assume that $a_i a_{i+1} = y_j z_j$, with $j \geqslant 2$. By Proposition 52, the 2-connected graph $G[A] + a_i a_{i+1}$ has disjoint connected subgraphs with vertex sets W and W' such that $a_i \in W$, $a_{i+1} \in W'$, and $|W| = t + 1$. We now add all the vertices of W to V_1, i.e., let $V_1 = U \cup S_{0,i} \cup W$. Note that $|V_1| = n_1$ and that $G[V_1]$ is connected. We conclude that $|W'| \leqslant m_j + 1 \leqslant n_2$. Let $V_2 = W'$. The graph $G - U - S_{0,i+1}$ is connected. Also, given that $G - U$ is 2-connected, the vertex a_{i+1} is an internal vertex of a leaf block of $G - U - S_{0,i+1}$. By Proposition 49, the 2-connected graph $G - U - S_{0,i+1}$ has disjoint connected subgraphs Z and V_3 such that $|V_3| = n_3$ and $a_{i+1} \in V_3$. The graphs $G[V_1]$, $G[V_2]$, and $G[V_3]$ are connected and have the desired sizes.

We now consider the case in which $s_{0,1} \geqslant n_1 - |U|$. We have $s_{0,1} > 0$, and the edge $a_0 a_1$ is of the form $y_j z_j$, for some $j \geqslant 2$. We also have that $s_{0,1} = m_j$. Let $t = n_1 - |U|$. Observe that $3 + s_{0,1} + s_{1,2} \leqslant v(G - U) = n_2 + n_3 + t$ since the graph $G - U$ contains the vertices a_0, a_1, and a_2 and all the vertices of $\mathrm{Int}(A_j)$ and $\mathrm{Int}(A_l)$ in the case that the edge $a_1 a_2$ is of the form $y_l z_l$; and also because $v(G) = n_1 + n_2 + n_3$ and $|U| = n_1 - t$.

Let t_0 and t_2 be non-negative integers such that $t_0 + t_2 + 1 = t$, $t_0 \leqslant s_{0,1} + 1$, $q_3 =$

$s_{0,1} - t_0 + 1 \leqslant n_3$, $t_2 \leqslant s_{1,2} + 1$, $q_2 = s_{1,2} - t_2 + 1 \leqslant n_2$. These integers exist given the last observation and Equation (4.3). By Proposition 52, the 2-connected graph $G[A_j] + a_0 a_1$ has disjoint connected subgraphs with vertex sets T_0 and W_3 such that $a_1 \in T_0$, $a_0 \in W_3$ and $|T_0| = t_0 + 1$ and $|W_3| = q_3 \leqslant n_3$. We initially construct set V_3 by putting the vertices in W_3. In the case that the edge $a_1 a_2$ is not of the form $y_l z_l$, then $t_2 = 0$ and $q_2 = 1$, and we set $T_2 = \{a_1\}$ and $W_2 = \{a_2\}$. In the case that the edge $a_1 a_2$ is of the form $y_l z_l$, by Proposition 52 the 2-connected graph $G[A_l] + a_1 a_2$ has disjoint connected subgraphs T_2 and W_2 such that $a_1 \in T_2$, $a_2 \in W_2$ with $|T_2| = t_2 + 1$, and $|W_2| = q_2 \leqslant n_2$. We start constructing set V_2 by putting first the vertices of W_2. Let $T = T_0 \cup T_2$. Note that the set $V_1 = U \cup T$ has n_1 vertices since T has t vertices. Also, the set V_1 is connected given that $a_1 \in T$ is adjacent to a vertex in U.

Finally, let $G^* = G - U - S_{0,2}$. The graph G^* is 2-connected, and has $n_2 + n_3 + 2 - q_2 - q_3$ vertices. By Proposition 52, the graph G^* has disjoint connected subgraphs with vertex sets W_2' and W_3' such that $a_2 \in W_2'$, $a_0 \in W_3'$ with $|W_2'| = n_2 + 1 - q_2$ and $|W_3'| = n_3 + 1 - q_3$. We finish the construction of sets V_2 and V_3 by adding the vertices of W_2' and W_3' respectively, i.e., $V_2 = W_2 \cup W_2'$ and $V_3 = W_3 \cup W_3'$. The graphs $G[V_1]$, $G[V_2]$, and $G[V_3]$ are connected and have sizes n_1, n_2, and n_3 respectively.

We have the following theorem.

Theorem 61 (Karpov, 2017). *Let G be a 2-connected graph on n vertices such that any 2-cutset of G splits the graph into at most 3 components. Let n_1, n_2, n_3 be positive integers with $n_1 + n_2 + n_3 = n$. Then G has disjoint connected subgraphs G_1, G_2, G_3 such that for $i = 1, 2, 3$, the graph G_i has n_i vertices.*

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