Metastability in systems of interacting point processes with memory of variable length modeling social and neuronal networks

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Resumo

Kádmo de Souza Laxa. **Metaestabilidade em sistemas de processos pontuais com memória de alcance variável interagindo entre si modelando redes sociais e neuronais**. Tese (Doutorado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022.

Estudamos o comportamento metaestável de dois sistemas de processos pontuais com memória de alcance variável interagindo entre si. Um dos sistemas é um novo modelo para uma rede social altamente polarizada. Nesse sistema, os processos pontuais são marcados e indicam os instantes sucessivos em que um ator social expressa uma opinião “favorável” ou “contrária” sobre determinado assunto. Para este modelo, demonstramos que quando o coeficiente de polarização diverge, a rede social atinge o consenso instantaneamente e esse consenso tem um comportamento metaestável. Isso significa que a direção das pressões sociais sobre os atores muda globalmente após um tempo aleatório longo e imprevisível. O segundo sistema que consideramos modela uma rede de neurônios com disparos. Neste modelo, associados a cada neurônio existem dois processos pontuais, descrevendo seus instantes sucessivos de disparo e vazamento. Demonstramos que este sistema tem um comportamento metaestável quando o tamanho da população diverge. Isso significa que o instante em que o sistema fica preso pela lista de potenciais de membrana nulos adequadamente reescalado converge para um tempo aleatório exponencial de média 1.

Palavras-chave: Metaestabilidade. Processos pontuais com memória de alcance variável interagindo entre si. Redes sociais. Redes neuronais.

Abstract

Kádmo de Souza Laxa. **Metastability in systems of interacting point processes with memory of variable length modeling social and neuronal networks**. Thesis (Doctorate). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2022.

We study the metastable behavior of two systems of interacting point processes with memory of variable length. One of the systems is a new model for a highly polarized social network. In this system, the point processes are marked and indicate the successive times in which a social actor express a "favorable" or "contrary" opinion on a certain subject. For this model, we prove that when the polarization coefficient diverges, the social network reaches instantaneous consensus and this consensus has a metastable behavior. This means that the direction of the social pressures on the actors globally changes after a long and unpredictable random time. The second system we consider models a network of spiking neurons. In this model, associated to each neuron there are two point processes, describing its successive spiking and leakage times. We prove that this system has a metastable behaviour when the population size diverges. This means that the time at which the system gets trapped by the list of null membrane potentials suitably re-scaled converges to a mean one exponential random time.

Keywords: Metastability. Interacting point processes with memory of variable length. Social networks. Neuronal networks.

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Chapter 1

Introduction

In this PhD thesis we study the metastable behavior of two systems of interacting point processes with memory of variable length. One of the systems we consider is a new model for a highly polarized social network. The other system belongs to the class of stochastic models for networks of spiking neurons introduced in a discrete time framework in Galves and Löcherbach, 2013 and in a continuous time framework in De Masi *et al.*, 2014.

Systems of interacting point processes with memory of variable length are non-trivial extensions of both the class of interacting Markov processes introduced by Spitzer, 1970 and the class of stochastic chains with memory of variable length introduced by Rissanen, 1983. They appear as natural candidates to model networks of spiking neurons due to the fact that neurons reset the value of their membrane potentials every time they spike. They appear also as a good candidates to model social networks in a democratic society in which each social actor accepts resetting their personal beliefs under the influence of the reaction of the network every time they expresses an opinion.

The model for a highly polarized social network is a system of interacting marked point processes. Each point process indicates the successive times in which a social actor express a “favorable” or “contrary” opinion on a certain subject. The orientation and the rate at which an actor express an opinion is influenced by the social pressure exerted on them, modulated by a polarization coefficient. The social pressure on an actor is reset to 0, when they express an opinion, and simultaneously the social pressures on all the other actors change by one unit in the direction of the opinion that was just expressed.

For this model, we prove that when the polarization coefficient diverges, this social network reaches instantaneous consensus. Here by consensus we mean the set of lists in which all the social pressures push in the same direction. This consensus has a metastable behavior. This means that the direction of the social pressures on the actors globally changes after a long and unpredictable random time.

The second model we consider is a system of interacting point processes with memory of variable length modeling a finite but large network of spiking neurons with leakage. Associated to each neuron there are two point processes, describing its successive spiking and leakage times. For each neuron, the rate of the spiking point process is an exponential function of its membrane potential, with the restriction that the rate takes the value 0 when the membrane potential is 0, while the leakage rate is a constant. At each spiking time, the membrane potential of the neuron resets to 0, and simultaneously, the membrane potentials of the other neurons increase by one unit. At each leakage time, the membrane potential of the neuron is reset to 0, with no effect on the other neurons.

We prove that the system has a metastable behavior as the population size diverges. This means that the time at which the system gets trapped by the list of null membrane potentials suitably re-scaled converges to a mean one exponential random time.

A more detailed informal description of the results with references are presented in the beginning of Chapters 2 and 3.

This thesis is organized as follows. In Chapter 2 we study the model for a highly polarized social network. In Chapter 3 we study the model for a system of spiking neurons with leakage. Each chapter starts with a more detailed informal description of the models and the results obtained, with references.

Chapter 2

Fast Consensus and Metastability in a Highly Polarized Social Network

2.1 Introduction

Discrepancy between the results of electoral intentions carried out a few days before the actual voting and the electoral poll results during the first round of the 2018 presidential elections in Brazil was striking. See for instance Branco, 2018 and Franco, 2018.

At the time, it was conjectured that this discrepancy was the result of social-media campaigning days before the elections. See for instance Belli, 2018 and Mello, 2018; Mello, 2019. This conjecture rises a question: is social-media campaigning enough to change in a quite short period of time the voting intention of a significant portion of voters? To address this question we introduce a new stochastic model that mimics some important features of real world social networks.

The model we propose can be informally described as follows.

1. The model is a system with interacting marked point processes with memory of variable length.
2. Each point process indicates the successive times in which a social actor express either a "favorable" (+1) or "contrary" (−1) opinion on a certain subject.
3. The social pressure on an actor determines the orientation and the rate at which they express an opinion.
4. When an actor express their opinion, social pressure on them is reset to 0, and simultaneously, social pressures on all the other actors change by one unit in the direction of the opinion that was just expressed.
5. The orientation and the rate at which an actor express an opinion is influenced by the social pressure exerted on them, modulated by a polarization coefficient.

It is natural to conjecture that the formation of the conjectured "wave" that pushes the opinions in a direction is a consequence of the social pressure exerted on actors. In our model, this is represented by the fact that every time an actor express an opinion, the social pressure on them is reset to 0 and its new value depends on the group's reaction to the opinion that they have just expressed. This is the content of the fourth point of the informal description given above.

Starting with the classical voter model, introduced by Holley and Liggett, 1975, several articles addressed issues associated to opinion dynamics in a social network. See Wasserman and Faust, 1994; Castellano *et al.*, 2009 and Aldous, 2013 for a general review on this subject. However, to the best of our knowledge, a model with the features introduced here was not considered yet in the literature of social networks.

Actually, our model belongs to the same class of systems of interacting point process with memory of variable length that was introduced in discrete time by Galves and Löcherbach, 2013 and in continuous time by De Masi *et al.*, 2014 to model system of spiking neurons. This class of systems was since then studied in several articles, including Duarte, Ost, and Rodríguez, 2015; Brochini *et al.*, 2016; Duarte and Ost, 2016; Fournier and Löcherbach, 2016; Galves and Löcherbach, 2016; Yaginuma, 2016; Ferrari *et al.*, 2018; André, 2019; Baccelli and Taillefumier, 2019; Duarte, Galves, *et al.*, 2019; Galves, Löcherbach, Pouzat, and Presutti, 2019; André and Planche, 2021; Baccelli and Taillefumier, 2021; Nascimento, 2022; De Santis *et al.*, 2022; Baccelli, Davydov, *et al.*, 2022; Chariker and Lebowitz, 2022 and Löcherbach and Monmarché, 2022.

Let us now informally present our results. The existence of the process and the uniqueness of its invariant probability measure is the content of Theorem 2.1.

When the polarization coefficient diverges, the invariant probability measure concentrates on the set of consensus lists and the time the system needs to get there goes to zero. Here by a consensus list we mean any list in which all the social pressures push in the same direction. This is the content of Theorem 2.2.

In the social network, the consensus has a metastable behavior. This means that the direction of the social pressures on the actors globally change direction after a long and unpredictable random time. This is the content of Theorem 2.3.

The notion of metastability considered here is inspired by the so called *pathwise approach to metastability* introduced by Cassandro *et al.*, 1984. For more references and an introduction to the topic, we refer the reader to Olivieri and Vares, 2005; Hollander, 2009 and Fernández *et al.*, 2015.

This chapter is organized as follows. In Section 2.2 we introduce the model, the notation and state the main results. In Section 2.3 we introduce some extra notation and prove two auxiliary propositions. In Section 2.4, 2.5 and 2.6 we prove Theorems 2.1, 2.2 and 2.3, respectively.

2.2 Definitions, notation and main results

Let $\mathcal{A} = \{1, 2, ..., N\}$ be the set of social actors, with $N \geq 3$, and let $\mathcal{O} = \{-1, +1\}$ be the set of opinions that an actor can express, where $+1$ (respectively, -1) represents a *favorable* opinion (respectively, a *contrary* opinion).

Let $\beta \geq 0$ be the polarization coefficient of this network. The polarization coefficient of the network parametrizes the tendency of each actor $a \in \mathcal{A}$ to follow the social pressure that the actors belonging to $\mathcal{A} \setminus \{a\}$ exert on a.

To describe the time evolution of the social network we introduce a family of maps on the set of lists of social pressures. For any actor $a \in \mathcal{A}$, for any opinion $\mathfrak{o} \in \mathcal{O}$ and for any list $u = (u(a) \,:\, a \in \mathcal{A})$, where $u(a)$ is a integer number, we define the new list $\pi^{a,\mathfrak{o}}(u)$ as follows

$$\pi^{a,\mathfrak{o}}(u)(b) = \begin{cases} u(b) + \mathfrak{o}, \text{ if } b \neq a, \\ 0, \text{ if } b = a. \end{cases}$$

The time evolution of the social network can be described as follows.

- Assume that at time 0, the list of social pressures exerted on the actors is $u = (u(a) \,:\, a \in \mathcal{A})$.
- Independent exponential random times with parameters $\exp(\beta\mathfrak{o}u(a))$ are associated to each actor $a \in \mathcal{A}$ and each opinion $\mathfrak{o} \in \mathcal{O}$.
- Denote (A_1, O_1) the pair (actor, opinion) associated to the exponential random time that occurs first.
- At this random time, the list of social pressures changes from u to $\pi^{A_1,O_1}(u)$.
- At the new list of social pressures $\pi^{A_1,O_1}(u)$, independent exponential random times with parameters $\exp\left(\beta\mathfrak{o}\pi^{A_1,O_1}(u)(a)\right)$ are associated to each actor $a \in \mathcal{A}$ and opinion $\mathfrak{o} \in \mathcal{O}$.
- Denote (A_2, O_2) the pair (actor, opinion) associated to the exponential random time that occurs first, and so on.

Let $((A_n, O_n) \,:\, n \geq 1)$ be the sequence of pairs (actor, opinion), associated to the exponential random times realizing the successive minima and let $(T_n \,:\, n \geq 1)$ be the successive random times associated to them. Let also $(U_t^{\beta,u})_{t\in[0,+\infty)}$ be the time evolution defined as follows

$$U_t^{\beta,u} = u, \text{ if } 0 \leq t < T_1,$$

and for any $t \geq T_1$,

$$U_t^{\beta,u} = \pi^{A_m,O_m}(U_{T_{m-1}}^{\beta,u}), \text{ if } T_m \leq t < T_{m+1},$$

where $T_0 = 0$.

So defined, the system of social pressures $(U_t^{\beta,u})_{t\in[0,+\infty)}$ evolves as a Markov jump process taking values in the set

$$\mathcal{S} = \{u = (u(a) \,:\, a \in \mathcal{A}) \in \mathbb{Z}^N \,:\, \min\{|u(a)| \,:\, a \in \mathcal{A}\} = 0\}$$

and with infinitesimal generator defined as follows

$$\mathcal{G}f(u) = \sum_{\mathfrak{o}\in\mathcal{O}} \sum_{b\in\mathcal{A}} \exp\left(\beta \mathfrak{o} u(b)\right) \left[f(\pi^{b,\mathfrak{o}}(u)) - f(u)\right], \tag{2.1}$$

for any bounded function $f : \mathcal{S} \longrightarrow \mathbb{R}$.

The opinion dynamics of the social network is described by the system of interacting marked point process $((T_n, (A_n, O_n)) : n \geq 1)$ together with the initial list of social pressures $U_0^{\beta,u} = u$.

Observes that the rates of these marked point processes have a variable length dependency from the past. This comes from the fact that each time actor a express an opinion, the social pressure on them is reset to 0 and therefore, the actor *forgets the past.*

In a more formal way, for any pair $(a, \mathfrak{o})$, with $a \in \mathcal{A}$, $\mathfrak{o} \in \mathcal{O}$ and for any pair of real numbers $s < t$, let us define the counting measure

$$Z^{a,\mathfrak{o}}((s,t]) = \sum_{n=1}^{+\infty} \mathbf{1}\{s < T_n \leq t, (A_n, O_n) = (a, \mathfrak{o})\}.$$

For any time $t > 0$, let L_t^a be the last expression time of actor a before time t

$$L_t^a = \sup\{T_n \leq t : A_n = a\}.$$

We use the convention that $\sup \emptyset = 0$. Then, the social pressure on actor a at time $t > 0$ can be equivalently defined as

$$U_t^{\beta,u}(a) = \begin{cases} u(a) + \sum_{\mathfrak{o}\in\mathcal{O}} \sum_{b\in\mathcal{A}\setminus\{a\}} \mathfrak{o} Z^{b,\mathfrak{o}}((0,t]), & \text{if } L_t^a = 0, \\ \sum_{\mathfrak{o}\in\mathcal{O}} \sum_{b\in\mathcal{A}\setminus\{a\}} \mathfrak{o} Z^{b,\mathfrak{o}}((L_t^a, t]), & \text{if } L_t^a > 0. \end{cases}$$

A standard computation shows that the rate of the point process $N^{a,\mathfrak{o}}$ at time t, conditioned in the past history, is given by

$$\exp(\beta \mathfrak{o} U_t^{\beta,u}(a)),$$

and this value depends on the history of the system in the interval $(L_t^a, t]$ which has a variable length.

The process $(U_t^{\beta,u})_{t\in[0,+\infty)}$ is well defined for any $t \in [0, \sup\{T_m : m \geq 1\})$. The unique thing that must yet be clarified is whether this process is defined for any positive time t, i.e. if $\sup\{T_m : m \geq 1\} = +\infty$ or not. This is part of the content of the first theorem.

Theorem 2.1. *For any $\beta \geq 0$ and for any starting list $u \in \mathcal{S}$, the following holds.*

1. *The sequence $(T_m : m \geq 1)$ of jumping times of the process $(U_t^{\beta,u})_{t\in[0,+\infty)}$ satisfies*

   $$\mathbb{P}(\sup\{T_m : m \geq 1\} = +\infty) = 1,$$

   which assures the existence of the process for all time $t \in [0, +\infty)$.

2. *The process $(U_t^{\beta,u})_{t\in[0,+\infty)}$ has a unique invariant probability measure μ^β.*

We define the set of positive (respectively negative) consensus lists as

$$\mathcal{C}^+ = \{u \in \mathcal{S} \, : \, u \neq \vec{0}, u(a) \geq 0, \text{for all } a \in \mathcal{A}\}$$

and

$$\mathcal{C}^- = \{u \in \mathcal{S} \, : \, u \neq \vec{0}, u(a) \leq 0, \text{for all } a \in \mathcal{A}\},$$

where $\vec{0} \in \mathcal{S}$ is the null list. Consider also the set of positive and negative ladder lists

$$\mathcal{L}^+ = \{u \in \mathcal{S} \, : \, \{u(1), ..., u(N)\} = \{0, 1, ..., N-1\}\}$$

and

$$\mathcal{L}^- = \{u \in \mathcal{S} \, : \, \{u(1), ..., u(N)\} = \{0, -1, ..., -(N-1)\}\}.$$

The set of ladder lists is given by $\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^-$.

To state the next theorems, we define for any $u \in \mathcal{S}$ and for any $B \subset \mathcal{S}$, the reaching time $R^{\beta,u}(B)$ as follows

$$R^{\beta,u}(B) = \inf\{t > 0 \, : \, U_t^{\beta,u} \in B\}.$$

Theorem 2.2 states that the invariant measure gets concentrated in the set of the ladder lists as the polarization coefficient diverges. Moreover, for any non-null initial list, the time it takes for the process to reach the set of ladder lists goes to 0, as the polarization coefficient diverges.

Theorem 2.2.

1. *There exists a constant $C > 0$, such that for any $\beta \geq 0$ the invariant probability measure μ^β satisfies*

$$\mu^\beta(\mathcal{L}) \geq 1 - Ce^{-\beta}.$$

2. *For any fixed $\delta > 0$*

$$\sup_{u \in \mathcal{S} \setminus \{\vec{0}\}} \mathbb{P}\Big(R^{\beta,u}(\mathcal{L}) > e^{-\beta(1-\delta)}\Big) \longrightarrow 0, \textit{ as } \beta \longrightarrow +\infty.$$

Theorem 2.3 states that a highly polarized social network has a metastable behavior.

Theorem 2.3. *For any $v \in \mathcal{C}^+$,*

$$\frac{R^{\beta,v}(\mathcal{C}^-)}{\mathbb{E}[R^{\beta,v}(\mathcal{C}^-)]} \longrightarrow Exp(1) \textit{ in distribution, as } \beta \longrightarrow +\infty,$$

where $Exp(1)$ denotes the mean 1 exponential distribution.

2.3 Auxiliary notation and results

In this section we will prove some auxiliary results that will be used to prove Theorems 2.1, 2.2 and 2.3. To do this, we need to extend the notation introduced before.

Extra notation

- The Markov chain embedded in the process $(U_t^{\beta,u})_{t\in[0,+\infty)}$ will be denoted $(\tilde{U}_n^{\beta,u})_{n\geq 0}$. In other terms,
$$\tilde{U}_0^{\beta,u} = u \ \text{ and } \ \tilde{U}_n^{\beta,u} = U_{T_n}^{\beta,u}, \ \text{for any } n \geq 1.$$
- The invariant probability measure of the Markov chain $(\tilde{U}_n^{\beta,u})_{n\geq 0}$ will be denoted $\tilde{\mu}^\beta$.
- For any list $u \in \mathcal{S}$, the first return time of the embedded Markov chain $(\tilde{U}_n^{\beta,u})_{n\geq 0}$ to u will be denoted
$$\tilde{R}^{\beta,u}(u) = \inf\{n \geq 1 \, : \, \tilde{U}_n^{\beta,u} = u\}.$$
- For any $u \in \mathcal{S}$, the opposite list $-u \in \mathcal{S}$ is given by
$$(-u)(a) = -u(a), \text{ for all } a \in \mathcal{A}.$$
- Let $\sigma \, : \, \mathcal{A} \longrightarrow \mathcal{A}$ be a bijective map. For any $u \in \mathcal{S}$, the permuted list $\sigma(u) \in \mathcal{S}$ is given by
$$\sigma(u)(a) = u(\sigma(a)), \text{ for all } a \in \mathcal{A}.$$
- The following event will appear several times in what follows. For a fixed $u \in \mathcal{S}$ and for any $n \geq 1$,
$$M_n = \{A_n \in \text{argmax}\{|\tilde{U}_{n-1}^{\beta,u}(a)| \, : \, a \in \mathcal{A}\} \ \text{ and } \ O_n \tilde{U}_{n-1}^{\beta,u}(A_n) \geq 0\}.$$

Proposition 2.4.

1. *For any $m \geq 1$,*
$$\mathbb{P}\left(\bigcap_{j=1}^{m} M_j\right) \geq \zeta_\beta^m,$$
where
$$\zeta_\beta = \frac{e^\beta}{e^\beta + e^{-\beta} + 2(N-1)} \longrightarrow 1, \ \text{as } \beta \longrightarrow +\infty.$$
2. *For any $u \in \mathcal{S}$,*
$$\mathbb{P}\left(\tilde{U}_{3(N-1)}^{\beta,u} \in \mathcal{L} \,\middle|\, \bigcap_{j=1}^{3(N-1)} M_j\right) = 1.$$

Proof. To prove Part 1 of Proposition 2.4, we first observe that by the Markov property,
$$\mathbb{P}\left(\bigcap_{j=1}^{m} M_j\right) = \sum_{v\in\mathcal{S}} \mathbb{P}\left(\bigcap_{j=1}^{m-1} M_j, \tilde{U}_{m-1}^{\beta,u} = v\right) \mathbb{P}\left(M_m \middle| \tilde{U}_{m-1}^{\beta,u} = v\right). \tag{2.2}$$

Observe that the condition $O_m \tilde{U}_{m-1}^{\beta,u}(A_n) \geq 0$ in the definition of the event M_m is satisfied for any value of $O_m \in \{-1, +1\}$, whenever $\tilde{U}^{\beta,u}_{m-1}(A_m) = 0$. Therefore, the smallest value for

$$\mathbb{P}\left(M_m \mid \tilde{U}^{\beta,u}_{m-1} = v\right)$$

is obtained for any list in which all actors, except one, have null pressure, and the unique actor with non-null pressure has pressure +1 or −1. This implies that for any $v \in \mathcal{S}$,

$$\inf\left\{\mathbb{P}\left(M_m \mid \tilde{U}^{\beta,u}_{m-1} = v\right) \,:\, v \in \mathcal{S}\right\} = \mathbb{P}(M_m \mid \tilde{U}^{\beta,u}_{m-1} = (1, 0, ..., 0)) = \zeta_\beta.$$

Applying this lower bound m times in Equation (2.2), we conclude the proof of Part 1. □

The proof of Part 2 of Proposition 2.4 is based in two Lemmas. Before proving Lemmas 2.5 and 2.6, let us introduce some extra notation.

$\mathcal{S}^+$ is the set of lists $u \in \mathcal{S}$ such that there exists $n(u) \in \{1, \dots, N-1\}$ and a sequence of $n(u)$ different actors $a_1(u), \dots, a_{n(u)}(u)$ satisfying

$$u(a_i(u)) = j, \text{ for } j = 1, ..., n(u) \quad \text{ and } \quad u(a) \geq -(n(u) - 1), \text{ for all } a \in \mathcal{A}.$$

In the same way, we define $\mathcal{S}^-$ as set of lists $u \in \mathcal{S}$ such that there exists $n(u) \in \{1, \dots, N-1\}$ and a sequence of $n(u)$ different actors $a_1(u), \dots, a_{n(u)}(u)$ satisfying

$$u(a_i(u)) = -j, \text{ for } j = 1, ..., n(u) \quad \text{ and } \quad u(a) \leq (n(u) - 1), \text{ for all } a \in \mathcal{A}.$$

Let

$$\tau = \inf\{n \geq 1 \,:\, A_n \in \{A_1, ..., A_{n-1}\} \cup \{a \in \mathcal{A} \,:\, u(a) = 0\}\}.$$

Note that the event $\{\tau = 1\} \cap M_1$ is only possible if $u = \vec{0}$. In this case, $\tilde{U}^{\beta,u}_1 \in \mathcal{S}^+ \cup \mathcal{S}^-$.

Lemma 2.5. *Assume that $\tau \geq 2$. For any initial list $u \in \mathcal{S}$, if the event M_τ occurs, then*

$$\tilde{U}^{\beta,u}_\tau \in \mathcal{S}^+ \cup \mathcal{S}^-.$$

Proof. At instant $\tau - 1$, we have that $\tilde{U}^{\beta,u}_{\tau-1}(A_{\tau-1}) = 0$ and for any $j = 1, \dots, \tau - 2$,

$$\tilde{U}^{\beta,u}_{\tau-1}(A_j) = O_{j+1} + \dots + O_{\tau-1},$$

which implies that

$$|\tilde{U}^{\beta,u}_{\tau-1}(A_j) - \tilde{U}^{\beta,u}_{\tau-1}(A_{j-1})| = |O_j| = 1.$$

Moreover, for any $a \in \mathcal{A}$ such that $u(a) = 0$, we have that

$$\tilde{U}^{\beta,u}_{\tau-1}(a) = O_1 + \dots + O_{\tau-1}.$$

Let,

$$m = \max\{|O_{\tau-1}|, |O_{\tau-2} + O_{\tau-1}|, \dots, |O_1 + \dots + O_{\tau-1}|\}.$$

It follows that there exists a set with $m+1$ actors

$$\{a_0, a_1, \dots, a_m\} \subset \{A_1, \dots, A_{\tau-1}\} \cup \{a \in \mathcal{A} \,:\, u(a) = 0\},$$

such that either

$$\{\tilde{U}^{\beta,u}_{\tau-1}(a_j) \,:\, j = 0, 1, \dots, m\} = \{0, 1, \dots, m\}$$

or

$$\{\tilde{U}^{\beta,u}_{\tau-1}(a_j) \,:\, j = 0, 1, \dots, m\} = \{0, -1, \dots, -m\}.$$

By assumption,

$$A_\tau \in \operatorname{argmax}\{|\tilde{U}^{\beta,u}_{\tau-1}(a)| \,:\, a \in \mathcal{A}\} \text{ and } O_n\tilde{U}^{\beta,u}_{\tau-1}(A_n) \geq 0.$$

This implies that

$$|\tilde{U}^{\beta,u}_{\tau-1}(A_\tau)| = m \geq |\tilde{U}^{\beta,u}_{\tau-1}(a)|, \text{ for all } a \in \mathcal{A},$$

and therefore, either

$$\{\tilde{U}^{\beta,u}_{\tau}(a_j) \,:\, j = 0, 1, \dots, m-1\} = \{1, \dots, m\} \text{ and } \tilde{U}^{\beta,u}_{\tau}(a) \geq -(m-1), \text{ for all } a \in \mathcal{A},$$

or

$$\{\tilde{U}^{\beta,u}_{\tau}(a_j) \,:\, j = 0, 1, \dots, m-1\} = \{0, -1, \dots, -m\} \text{ and } \tilde{U}^{\beta,u}_{\tau}(a) \leq m-1, \text{ for all } a \in \mathcal{A}.$$

This concludes the proof of Lemma 2.5.

□

Lemma 2.6. *For any $u \in \mathcal{S}^+$, if the event $\bigcap_{j=1}^{n(u)-1} M_j$ occurs, then $\tilde{U}^{\beta,u}_{n(u)-1} \in C^+$.*

Proof. If M_1 occurs, then

$$\tilde{U}^{\beta,u}_0(A_1) = \max\{|u(a)| \,:\, a \in \mathcal{A}\} \text{ and } O_1 = +1.$$

Moreover, by assumption, for any $j = 0, \dots, n(u)-1$ there exists an actor $a_j(u) \in \mathcal{A}$ such that $\tilde{U}^{\beta,u}_0(a_j(u)) = u(a_j(u)) = j$. Therefore,

$$\tilde{U}^{\beta,u}_1(a_j(u)) = u(a_j(u)) + 1 = j + 1.$$

As a consequence,

$$\{1, \dots, n(u)\} \subset \{\tilde{U}^{\beta,u}_1(a) \,:\, a \in \mathcal{A}\} \quad \text{and} \quad \tilde{U}^{\beta,u}_1(a) \geq -(n(u)-2), \text{ for all } a \in \mathcal{A}.$$

In general, for any $k = 1, \dots, n(u)-1$, if $\bigcap_{j=1}^{k} M_j$ occurs, then there exists a sequence of actors $a_0(\tilde{U}^{\beta,u}_{k-1})$, ..., $a_{n(u)}(\tilde{U}^{\beta,u}_{k-1})$ such that for any $j = 0, \dots, n(u)-1$, we have that

$$\tilde{U}^{\beta,u}_k(a_j(\tilde{U}^{\beta,u}_{k-1})) = j + 1,$$

and therefore,

$$\{1,\dots,n(u)\} \subset \{\tilde{U}_k^{\beta,u}(a) : a \in \mathcal{A}\} \quad \text{and} \quad \tilde{U}_k^{\beta,u}(a) \geq -(n(u)-(k+1)), \text{ for all } a \in \mathcal{A}.$$

This concluded the proof of Lemma 2.6. □

In the proof of Part 2 of Proposition 2.4 that follows, we will indicate the initial list $u \in \mathcal{S}$ when referring to the event M_n. In other words, for a fixed $u \in \mathcal{S}$ and for any $n \geq 1$, we will denote

$$M_n^u = \{A_n \in \operatorname{argmax}\{|\tilde{U}_{n-1}^{\beta,u}(a)| : a \in \mathcal{A}\} \text{ and } O_n\tilde{U}_{n-1}^{\beta,u}(A_n) \geq 0\}.$$

Proof. We will now prove Part 2 of Proposition 2.4.

Lemma 2.5 and the fact that $\tau \leq N$ for any initial list $u \in \mathcal{S}$, imply that if $\bigcap_{k=1}^N M_k^u$ occurs, then

$$\tilde{U}_N^{\beta,u} \in \mathcal{S}^+ \cup \mathcal{S}^-.$$

Lemma 2.6 and the fact that $n(v) \leq N-1$ for any $v \in \mathcal{S}^+ \cup \mathcal{S}^-$, imply that if $\bigcap_{k=1}^{N-2} M_k^v$ occurs, then

$$\tilde{U}_{N-2}^{\beta,v} \in \mathcal{C}^+ \cup \mathcal{C}^-.$$

We also have that for any $v' \in \mathcal{C}^+ \cup \mathcal{C}^-$, if $\bigcap_{k=1}^{N-1} M_k^{v'}$ occurs, then

$$\tilde{U}_{N-1}^{\beta,v'} \in \mathcal{L}.$$

Putting all this together, by the Markov property we conclude that for any $u \in \mathcal{S}$,

$$\mathbb{P}\left(\tilde{U}_{3(N-1)}^{\beta,u} \in \mathcal{L} \,\middle|\, \bigcap_{k=1}^{3(N-1)} M_k^u\right) =$$

$$\sum_{v\in\mathcal{S}^+\cup\mathcal{S}^-}\ \sum_{v'\in\mathcal{C}^+\cup\mathcal{C}^-} \mathbb{P}\left(\tilde{U}_N^{\beta,u} = v \,\middle|\, \bigcap_{k=1}^{N} M_k^u\right)\mathbb{P}\left(\tilde{U}_{N-2}^{\beta,v} = v' \,\middle|\, \bigcap_{k=1}^{N-2} M_k^v\right)\mathbb{P}\left(\tilde{U}_{N-1}^{\beta,v'} \in \mathcal{L} \,\middle|\, \bigcap_{k=1}^{N-1} M_k^{v'}\right).$$

We conclude the proof by noting that the sum above is equal 1. □

For any fixed $l \in \mathcal{L}^+$, let $c_{\beta,l}$ be the positive real number such that

$$\mathbb{P}(R^{\beta,l}(\mathcal{L}^-) > c_{\beta,l}) = e^{-1}. \tag{2.3}$$

Due to the symmetric properties of the process, it is clear that $c_{\beta,l} = c_{\beta,l'}$, for any pair of lists l and l' belonging to $\mathcal{L}^+$. Therefore, in what follows we will omit to indicate l in the notation of c_β. The next proposition gives an lower bound to c_β.

Proposition 2.7. *There exists $C_1 > 0$ such that for any $\beta \geq 0$,*

$$c_\beta \geq C_1 e^\beta.$$

Proof. For any fixed $l \in \mathcal{L}^+$, let

$$\tau_-^{(1)} = \inf\{T_n \,:\, O_n = -1, U^{\beta,l}_{T_{n-1}}(A_n) > 0\}.$$

Consider also

$$\tau_-^{(2)} = \inf\{T_n \,:\, E_n^- \cap (E_{n-1}^- \cup E_{n-2}^-)\},$$

where $E_n^- = \{O_n = -1, U^{\beta,l}_{T_{n-1}}(A_n) = 0\}$. By definition,

$$R^{\beta,l}(\mathcal{S} \setminus \mathcal{C}^+) \geq \min\{\tau_-^{(1)}, \tau_-^{(2)}\}.$$

For any $t > 0$,

$$\mathbb{P}(\tau_-^{(1)} > t) \geq \mathbb{P}(\mathrm{Exp}((N-1)e^{-\beta}) > t),$$

where $\mathrm{Exp}((N-1)e^{-\beta})$ is a random variable exponentially distributed with mean $e^{\beta}/(N-1)$.

Let $n_0^- = 0$ and for $j \geq 1$,

$$n_j^- = \inf\{n > n_{j-1}^- \,:\, O_n = -1, U^{\beta,l}_{T_{n-1}}(A_n) = 0\}.$$

We have that

$$\tau_-^{(2)} = \sum_{j=1}^{J}(T_{n_j^-} - T_{n_{j-1}^-}),$$

where $J = \inf\{j \geq 1 \,:\, n_j^- = n_{j-1}^- + 1 \text{ or } n_j^- = n_{j-1}^- + 2\}$.

Therefore, for any $t > 0$,

$$\mathbb{P}(\tau_-^{(2)} > t | \tau_-^{(1)} > t) \geq \mathbb{P}\left(\sum_{j=1}^{G} E_j > t\right),$$

where $(E_j)_{j\geq 1}$ is a sequence of i.i.d. random variables exponentially distributed with mean 1, and G is random variable independent from $(E_j)_{j\geq 1}$ with Geometric distribution assuming values in $\{1, 2, ...\}$ with parameter

$$\lambda_\beta = 2 \times \frac{(N-1)}{(N-1) + e^{\beta}}.$$

This implies that for any $t > 0$,

$$\mathbb{P}(\tau_-^{(2)} > t | \tau_-^{(1)} > t) \geq \mathbb{P}\left(\mathrm{Exp}(\lambda_\beta) > t\right),$$

where $\mathrm{Exp}(\lambda_\beta)$ is a random variable exponentially distributed with mean $1/\lambda_\beta$.

Therefore, for any $t > 0$,

$$\mathbb{P}(R^{\beta,l}(\mathcal{S} \setminus \mathcal{C}^+) > t) \geq \mathbb{P}[\mathrm{Exp}((N-1)e^{-\beta}) > t]\mathbb{P}[\mathrm{Exp}(\lambda_\beta) > t].$$

This implies that

$$e^{-1} = \mathbb{P}(R^{\beta,l}(\mathcal{L}^-) > c_\beta) \geq \mathbb{P}(R^{\beta,l}(\mathcal{S} \setminus \mathcal{C}^+) > c_\beta) \geq e^{-c_\beta\left((N-1)e^{-\beta}+\lambda_\beta\right)},$$

and therefore

$$c_\beta \geq \left((N-1)e^{-\beta} + \lambda_\beta\right)^{-1}.$$

With this we concluded the proof of Proposition 2.7.

□

2.4 Proof of Theorem 2.1

To prove Part 1 of Theorem 2.1 we first need to prove the following Lemma.

Lemma 2.8. *For any list $u \in \mathcal{S}$,*

$$\inf\left\{n \geq 1 \,:\, |U^{\beta,u}_{T_{n-1}}(A_n)| < N\right\} \leq N.$$

Proof. The initial list of social pressures u belongs to $\mathcal{S}$. Therefore, there exists $a_0 \in \mathcal{A}$ such that $u(a_0) = 0$. By definition, for any $m \geq 1$,

$$|U^{\beta,u}_{T_m}(a_0)| \leq \left|\sum_{j=1}^{m} O_j\right| \leq \sum_{j=1}^{m} |O_j| = m.$$

More generally, if $A_n = a$ then for any $m \geq 1$,

$$|U^{\beta,u}_{T_{n+m}}(a)| \leq \left|\sum_{j=1}^{m} O_{n+j}\right| \leq \sum_{j=1}^{m} |O_{n+j}| = m.$$

Therefore, if no actor express opinions twice in the first $N-1$ steps and moreover, $a_0 \neq A_j$ for $j = 1, ..., N-1$, i.e.

$$|\{a_0, A_1, ..., A_{N-1}\}| = N,$$

then necessarily the social pressure on actor expressing opinion at instant T_N is smaller than N in absolute value. This implies that

$$\inf\left\{n \geq 1 \,:\, |U^{\beta,u}_{T_{n-1}}(A_n)| < N\right\} \leq N.$$

Now, if

$$|\{a_0, A_1, ..., A_{N-1}\}| \leq N-1,$$

there exists $m \in \{1, ..., N-1\}$ such that

$$A_m \in \{a_0, A_1, ..., A_{m-1}\}.$$

This implies that

$$|U^{\beta,u}_{T_{m-1}}(A_m)| \leq m \leq N-1,$$

and therefore,

$$\inf\left\{ n \geq 1 \; : \; |U^{\beta,u}_{T_{n-1}}(A_n)| < N \right\} \leq m < N.$$

□

Define $T^<_0 = T^>_0 = 0$ and for any $k \geq 1$,

$$T^<_k = \inf\{ T_n > T^<_{k-1} \; : \; |U^{\beta,u}_{T_{n-1}}(A_n)| < N \},$$

$$T^>_k = \inf\{ T_n > T^>_{k-1} \; : \; |U^{\beta,u}_{T_{n-1}}(A_n)| \geq N \}.$$

Lemma 2.8 implies that $T^<_k$ is well defined for any $k \geq 1$. Now we can prove Part 1 of Theorem 2.1.

Proof. To prove Part 1 of Theorem 2.1, we will construct the process $(U^{\beta,u}_t)_{t\in[0,+\infty)}$ with jump times $\{T_n \; : \; n \geq 1\}$ as the superposition of two process with jump times $\{T^<_k \; : \; k \geq 1\}$ and $\{T^>_k \; : \; k \geq 1\}$ in the following way. For any $v \in \mathcal{S}$, the jump rates of these two processes are

$$q^<(v) = \sum_{a\in A} \mathbf{1}\{ v(a) < N \}(e^{\beta v(a)} + e^{-\beta v(a)})$$

and

$$q^>(v) = \sum_{a\in A} \mathbf{1}\{ v(a) \geq N \}(e^{\beta v(a)} + e^{-\beta v(a)}).$$

For any list $v \in \mathcal{S}$, we have that

$$q^<(v) \leq \lambda,$$

with

$$\lambda = N(e^{\beta(N-1)} + e^{-\beta(N-1)}).$$

For any $v \in \mathcal{S}$, define

$$\Phi^{<,-}_v(0) = \Phi^{<,+}_v(0) = 0 \text{ and } \Phi^{>,-}_v(0) = \Phi^{>,+}_v(0) = \lambda.$$

For any $a \in \mathcal{A}$, define

$$\Phi^{<,-}_v(a) = \Phi^{<,+}_v(a-1) + \mathbf{1}\{|v(a)| < N\} e^{-\beta v(a)},$$

$$\Phi^{<,+}_v(a) = \Phi^{<,-}_v(a) + \mathbf{1}\{|v(a)| < N\} e^{+\beta v(a)},$$

$$\Phi^{>,-}_v(a) = \Phi^{>,+}_v(a-1) + \mathbf{1}\{|v(a)| \geq N\} e^{-\beta v(a)},$$

$$\Phi^{>,+}_v(a) = \Phi^{>,-}_v(a) + \mathbf{1}\{|v(a)| \geq N\} e^{+\beta v(a)}.$$

Using this partitions indexed by $\mathcal{S}$ (see Figure 2.1), we now construct the process $(U^{\beta,u}_t)_{t\in[0,+\infty)}$ as follows. Consider an homogeneous rate 1 Poisson point process in the plane $[0,+\infty)^2$. Call $\mathcal{N}$ the counting measure of this process. Given the initial list $u \in \mathcal{S}$, define

$$T_1 = \inf\left\{ t > 0 \; : \; \mathcal{N}\Big((0,t] \times \{[0, q^<(u)) \cup [\lambda, \lambda + q^>(u))\} \Big) = 1 \right\}.$$

Denoting R_1 as the second coordinate of the mark (T_1, R_1) of $\mathcal{N}$, we have

$$A_1 = b,\, O_1 = -1, \quad \text{if} \quad R_1 \in [\Phi_u^{<,+}(b-1), \Phi_u^{<,-}(b)) \cup [\Phi_u^{>,+}(b-1), \Phi_u^{>,-}(b)),$$

$$A_1 = b,\, O_1 = +1, \quad \text{if} \quad R_1 \in [\Phi_u^{<,-}(b), \Phi_u^{<,+}(b)) \cup [\Phi_u^{>,-}(b), \Phi_u^{>,+}(b)).$$

At time T_1, we have $U_{T_1}^{\beta,u} = \pi^{A_1,O_1}(u)$.

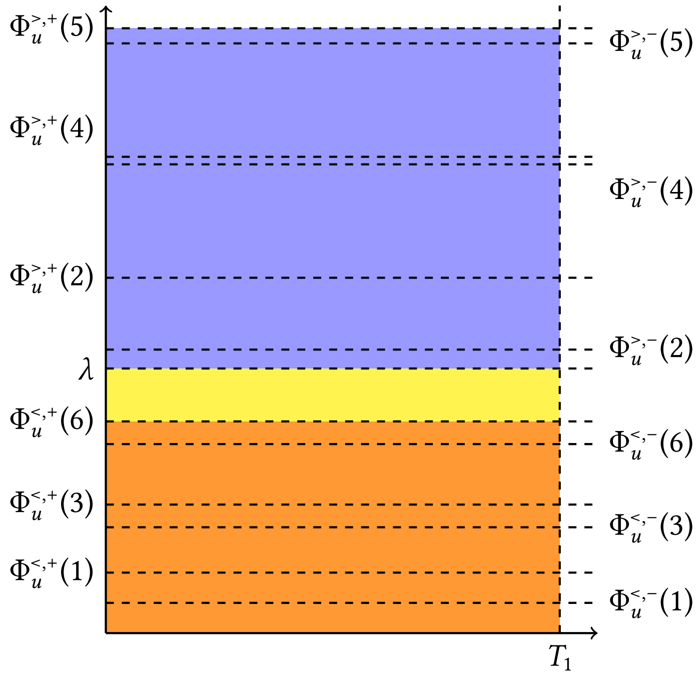

Figure 2.1: *The regions of plane* $[0, T_1] \times [0, \lambda + q^{>}(u)]$ *considered on the construction* T_1*. In this example,* $N = 6$ *and the list* $u \in \mathcal{S}$*, satisfies* $u(a) < N$ *for* $a = 1, 3, 6$ *and* $u(a) \geq N$ *for* $a = 2, 4, 5$*.*

More generally, for $n \geq 1$, we have

$$T_n = \inf\left\{ t > T_{n-1} \,:\, \mathcal{N}\Big((T_{n-1}, t] \times \Big\{ [0, q^{<}(U_{T_{n-1}}^{\beta,u})) \cup [\lambda, \lambda + q^{>}(U_{T_{n-1}}^{\beta,u})) \Big\} \Big) = 1 \right\}.$$

Denoting R_n as the second coordinate of the mark (T_n, R_n) of $\mathcal{N}$, we have

$$A_n = b,\, O_n = -1, \quad \text{if} \quad R_n \in \left[\Phi^{<,+}_{U_{T_{n-1}}^{\beta,u}}(b-1), \Phi^{<,-}_{U_{T_{n-1}}^{\beta,u}}(b)\right) \cup \left[\Phi^{>,+}_{U_{T_{n-1}}^{\beta,u}}(b-1), \Phi^{>,-}_{U_{T_{n-1}}^{\beta,u}}(b)\right),$$

$$A_n = b,\, O_n = +1, \quad \text{if} \quad R_n \in \left[\Phi^{<,-}_{U_{T_{n-1}}^{\beta,u}}(b), \Phi^{<,+}_{U_{T_{n-1}}^{\beta,u}}(b)\right) \cup \left[\Phi^{>,-}_{U_{T_{n-1}}^{\beta,u}}(b), \Phi^{>,+}_{U_{T_{n-1}}^{\beta,u}}(b)\right).$$

At time T_n, we have $U_{T_n}^{\beta,u} = \pi^{A_n,O_n}(U_{T_{n-1}}^{\beta,u})$ (see Figure 2.2).

For any $n \geq 1$, we have that

$$T_n^{<} = \inf\left\{ T_m > T_{n-1}^{<} \,:\, R_m \in \left[0, q^{<}(U_{T_{m-1}}^{\beta,u})\right) \right\},$$

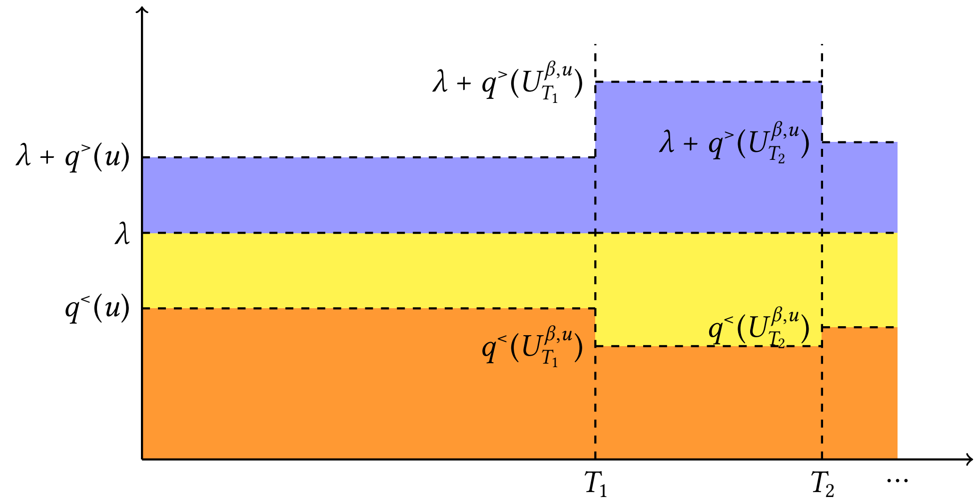

Figure 2.2: *The regions of the plane* $(T_n, T_{n+1}) \times [0, q^<(U^{\beta,u}_{T_n})]$ *and* $(T_n, T_{n+1}) \times [\lambda, \lambda + q^>(U^{\beta,u}_{T_n})]$*, for* $n = 0, 1, 2, \ldots$.

$$T^>_n = \inf\left\{ T_m > T^>_{n-1} \,:\, R_m \in \left[\lambda, \lambda + q^>(U^{\beta,u}_{T_{m-1}})\right) \right\}.$$

Define $T^\lambda_0 = 0$ and for any $n \geq 1$,

$$T^\lambda_n = \inf\left\{ t > T^\lambda_{n-1} \,:\, \mathcal{N}\left((T^\lambda_{n-1}, t] \times [0, \lambda)\right) = 1 \right\}.$$

By construction, $\{T^<_n \,:\, n \geq 1\} \subset \{T^\lambda_n \,:\, n \geq 1\}$. Also, $\{T^\lambda_n \,:\, n \geq 1\}$ are the marks of a homogeneous Poisson point process with rate λ. Since $\sup\{T^\lambda_n \,:\, n \geq 1\} = +\infty$, we have that $\sup\{T^<_n \,:\, n \geq 1\} = +\infty$. By Lemma 2.8, the event $\sup\{T^<_n \,:\, n \geq 1\} = +\infty$ implies that $\sup\{T^>_n \,:\, n \geq 1\} = +\infty$. $\{T_n \,:\, n \geq 1\}$ is the superposition of $\{T^<_n \,:\, n \geq 1\}$ and $\{T^>_n \,:\, n \geq 1\}$, and then, we conclude that $\sup\{T_n \,:\, n \geq 1\} = +\infty$. □

To conclude the proof of Theorem 2.1, we will prove that $(U^{\beta,u}_t)_{t\in[0,+\infty]}$ is an ergodic Markov process.

Proof. To prove Part 2 of Theorem 2.1, let $l \in \mathcal{L}^+$ satisfies $l(a) = a - 1$, for all $a \in \mathcal{A}$. For any $u \in \mathcal{S}$, we have that

$$l = \pi^{1,+1} \circ \pi^{2,+1} \circ \ldots \circ \pi^{N,+1}(u),$$

and then, if the event $\bigcap_{j=1}^N \{A_j = N - j + 1, O_j = +1\}$ occurs, then $\tilde{U}^{\beta,u}_N = l$.

For any $u' \in \mathcal{S}$,

$$\mathbb{P}\left(\tilde{U}^{\beta,u}_{n+2N} = l \,\middle|\, \tilde{U}^{\beta,u}_n = u' \right) \geq$$
$$\mathbb{P}\left(\bigcap_{a\in\mathcal{A}} \{|\tilde{U}^{\beta,u}_{n+N}(a)| < N\} \,\middle|\, \tilde{U}^{\beta,u}_n = u' \right) \mathbb{P}\left(\tilde{U}^{\beta,u}_{n+2N} = l \,\middle|\, \bigcap_{a\in\mathcal{A}} \{|\tilde{U}^{\beta,u}_{n+N}(a)| < N\} \right).$$

For any $u \in \mathcal{S}$, using Proposition 2.4 we have that

$$\mathbb{P}\left(\bigcap_{a\in\mathcal{A}}\{|\tilde{U}_N^{\beta,u}(a)| < N\}\right) \geq \mathbb{P}\left(\bigcap_{j=1}^{N} M_j\right) \geq \zeta_\beta^N.$$

Also, there exists $\epsilon^* > 0$ such that

$$\mathbb{P}\left(\tilde{U}_{n+2N}^{\beta,u} = l \,\middle|\, \bigcap_{a\in\mathcal{A}}\{|\tilde{U}_{n+N}^{\beta,u}(a)| < N\}\right) \geq$$

$$\min\left\{\mathbb{P}\left(\tilde{U}_{n+2N}^{\beta,u} = l \,\middle|\, \tilde{U}_{n+N}^{\beta,u} = v\right) \,:\, v \in \mathcal{S}, \bigcap_{a\in\mathcal{A}}\{|v(a)| < N\}\right\} = \epsilon^*.$$

Therefore, for any $u' \in \mathcal{S}$,

$$\mathbb{P}\left(\tilde{U}_{n+2N}^{\beta,u} = l \,\middle|\, \tilde{U}_n^{\beta} = u'\right) \geq \zeta_\beta^N \epsilon^*.$$

Recall that $\tilde{R}^{\beta,l}(l) = \inf\{n \geq 1 \,:\, \tilde{U}_n^{\beta,l} = l\}$. The last inequality implies that for any $t > 0$,

$$\mathbb{P}(\tilde{R}^{\beta,l}(l) > t) \leq \mathbb{P}(2N \times \mathrm{Geom}(\zeta_\beta^N \epsilon^*) > t),$$

where $\mathrm{Geom}(r)$ denotes a random variable with geometric distribution assuming values in $\{1, 2, \ldots\}$ and with parameter $r \in (0, 1)$. This implies that $\mathbb{E}(\tilde{R}^{\beta,l}(l)) < +\infty$ and then, $(\tilde{U}_n^{\beta,u})_{n\geq 0}$ is a positive-recurrent Markov chain.

The jump rate of the process $(U_t^{\beta,u})_{t\in[0,+\infty)}$ satisfies

$$\sum_{a\in\mathcal{A}}(e^{\beta v(a)} + e^{-\beta v(a)}) \geq 2N,$$

for any $v \in \mathcal{S}$. Putting all this together we conclude that $(U_t^{\beta,u})_{t\in[0,+\infty)}$ is ergodic.

□

2.5 Proof of Theorem 2.2

To prove Part 1 of Theorem 2.2 we will need the following Proposition.

Proposition 2.9. *For any $\beta > 0$ and $u \notin \mathcal{L}$ such that* $\max\{|u(a)| \,:\, a \in \mathcal{A}\} < N$ *we have that*

$$\tilde{\mu}^\beta(u) \leq C' e^{-\beta(N-1)},$$

where $C' > 0$.

Proof. First, we have that

$$\mathbb{E}(\tilde{R}^{\beta,u}(u)) \geq \sum_{m=3(N-1)}^{+\infty} \mathbb{P}(\tilde{R}^{\beta,u}(u) \geq m). \tag{2.4}$$

For any $u \notin \mathcal{L}$ such that $\max\{|u(a)| : a \in \mathcal{A}\} < N$, considering Proposition 2.4 we have that

$$\mathbb{P}\left(\{\tilde{U}_n^{\beta,u} \neq u, \text{ for } n = 1, 2, ..., 3(N-1)\} \cap \{\tilde{U}_{3(N-1)}^{\beta,u} \in \mathcal{L}\}\right) \geq \mathbb{P}\left(\bigcap_{j=1}^{3(N-1)} M_j\right).$$

Also, for any $l \in \mathcal{L}$ and for any $m \geq 1$, we have that

$$\mathbb{P}\left(\{\tilde{U}_n^{\beta,l} \neq u, \text{ for } n = 1, 2, ..., m\}\right) \geq \mathbb{P}\left(\bigcap_{j=1}^{m}\{O_j \tilde{U}_{j-1}^{\beta,l}(A_j) > 0\}\right).$$

Therefore, for any $m \geq 1$ we have that

$$\mathbb{P}(\tilde{R}^{\beta,u}(u) \geq 3(N-1) + m) \geq \mathbb{P}\left(\bigcap_{j=1}^{3(N-1)} M_j\right)\mathbb{P}\left(\bigcap_{j=1}^{m}\{O_j \tilde{U}_{j-1}^{\beta,l}(A_j) > 0\}\right). \tag{2.5}$$

For any $l \in \mathcal{L}$, we have that

$$\mathbb{P}\left(\bigcap_{j=1}^{m}\{O_j \tilde{U}_{j-1}^{\beta,l}(A_j) > 0\}\right) \geq \eta^m, \tag{2.6}$$

where

$$\eta = \frac{\sum_{j=1}^{N-1} e^{\beta j}}{\sum_{j=0}^{N-1}(e^{\beta j} + e^{-\beta j})}.$$

Therefore, by Proposition 2.4 and Equations (2.4), (2.5) and (2.6), it follows that

$$\mathbb{E}(\tilde{R}^{\beta,u}(u)) \geq \mathbb{P}\left(\bigcap_{j=1}^{3(N-1)} M_j\right)\left(1 + \sum_{m=1}^{\infty} \eta^m\right) \geq \frac{(2N)^{-3(N-1)}}{1-\eta}.$$

From the classical Kac's Lemma (see Kac, 1947), this implies that

$$\tilde{\mu}^\beta(u) \leq \frac{1-\eta}{(2N)^{-3(N-1)}}.$$

We have that

$$1 - \eta = \frac{\sum_{j=0}^{N-1} e^{-\beta j} + 1}{\sum_{j=0}^{N-1}(e^{\beta j} + e^{-\beta j})} \leq \frac{N+1}{e^{\beta(N-1)}} = (N+1)e^{-\beta(N-1)}.$$

Taking $C' = (N+1)(2N)^{3(N-1)}$, we conclude that

$$\tilde{\mu}^\beta(u) \leq C' e^{-\beta(N-1)}.$$

□

Corollary 2.10. *$\tilde{\mu}^\beta(\vec{0}) < C_1' e^{-N\beta}$ for all $\beta > 0$, where $C_1' > 0$.*

Proof. Let $v \in \mathcal{S}$ be the list in which $v(1) = 0$ and $v(a) = 1$, for $a \in \mathcal{A} \setminus \{1\}$. We have that $\{u \in \mathcal{S} : \mathbb{P}(\tilde{U}_1^{\beta,u} = \vec{0}) > 0\} = \{\sigma(v), -\sigma(v) \in \mathcal{S}$ such that $\sigma : \mathcal{A} \longrightarrow \mathcal{A}$ is a bijective map$\}$.

Therefore,

$$\tilde{\mu}^\beta(\vec{0}) = \sum_{u \in \mathcal{S}} \mathbb{P}(\tilde{U}_1^{\beta,u} = \vec{0})\tilde{\mu}^\beta(u) = 2N\mathbb{P}(\tilde{U}_1^{\beta,v} = \vec{0})\tilde{\mu}^\beta(v).$$

Since

$$\mathbb{P}(\tilde{U}_1^{\beta,v} = \vec{0}) = \frac{1}{2 + (N-1)(e^{+\beta} + e^{-\beta})} \le e^{-\beta},$$

from Proposition 2.9 we conclude that

$$\tilde{\mu}^\beta(\vec{0}) \le (N+1)(2N)^{3(N-1)} e^{-\beta(N-1)}(2N)e^{-\beta} = C_1' e^{-\beta N}.$$

□

Now we can prove the Part 1 of Theorem 2.2.

Proof. To prove Part 1 of Theorem 2.2, first note that for any $u \in \mathcal{S}$, the invariant measure μ^β satisfies

$$\mu^\beta(u) = \frac{\tilde{\mu}^\beta(u)}{q_\beta(u)} \left(\sum_{u' \in \mathcal{S}} \frac{\tilde{\mu}^\beta(u')}{q_\beta(u')} \right)^{-1},$$

where for any $v \in \mathcal{S}$,

$$q_\beta(v) = \sum_{a \in \mathcal{A}} (e^{\beta v(a)} + e^{-\beta v(a)})$$

is the jump rate of $(U_t^{\beta,u})_{t \in [0,+\infty)}$ at list v. Therefore,

$$\mu^\beta(\mathcal{L}) = \sum_{u \in \mathcal{L}} \frac{\tilde{\mu}^\beta(u)}{q_\beta(u)} \left(\sum_{u' \in \mathcal{S}} \frac{\tilde{\mu}^\beta(u')}{q_\beta(u')} \right)^{-1} = \frac{\tilde{\mu}^\beta(\mathcal{L})}{\sum_{j=0}^{N-1} (e^{\beta j} + e^{-\beta j})} \left(\sum_{u' \in \mathcal{S}} \frac{\tilde{\mu}^\beta(u')}{q_\beta(u')} \right)^{-1} =$$

$$\frac{1}{1 + \frac{\sum_{j=0}^{N-1}(e^{\beta j} + e^{-\beta j})}{\tilde{\mu}^\beta(\mathcal{L})} \sum_{u' \notin \mathcal{L}} \frac{\tilde{\mu}^\beta(u')}{q_\beta(u')}}.$$

By Proposition 2.4, we have that

$$\tilde{\mu}^\beta(\mathcal{L}) \ge (2N)^{-3(N-1)}.$$

We also have that

$$\sum_{j=0}^{N-1} (e^{\beta j} + e^{-\beta j}) \le 2N e^{\beta(N-1)},$$

and then,

$$\mu^{\beta}(\mathcal{L}) \geq \frac{1}{1 + (2N)^{3(N-1)+1} e^{\beta(N-1)} \sum_{u' \notin \mathcal{L}} \frac{\tilde{\mu}^{\beta}(u')}{q_{\beta}(u')}}$$

For any $u \in \mathcal{S}$ such that $\max\{|u(a)| : a \in \mathcal{A}\} \geq N$, we have that $q_{\beta}(u) \geq e^{\beta N}$, and then

$$\frac{\tilde{\mu}^{\beta}(u)}{q_{\beta}(u)} \leq \tilde{\mu}^{\beta}(u) e^{-\beta N}.$$

Observe that for any $u \neq \vec{0}$, $q_{\beta}(u) \geq e^{\beta}$. Therefore, by Proposition 2.9 and Corollary 2.10, it follows that for any $u \in \mathcal{L}^c$ such that $\max\{|u(a)| : a \in \mathcal{A}\} < N$, we have that

$$\frac{\tilde{\mu}^{\beta}(u)}{q_{\beta}(u)} \leq C_1' e^{-\beta N}.$$

Since $|\{u \in \mathcal{L}^c : \max\{|u(a)| : a \in \mathcal{A}\} < N\}| \leq N(2N-1)^{N-1}$, we have that

$$(2N)^{3(N-1)+1} e^{\beta(N-1)} \sum_{u' \notin \mathcal{L}} \frac{\tilde{\mu}^{\beta}(u')}{q_{\beta}(u')} \leq C e^{-\beta}.$$

where

$$C = (2N)^{3(N-1)+1}[1 + C_1' N(2N-1)^{N-1}].$$

We conclude that

$$\mu^{\beta}(\mathcal{L}) \geq \frac{1}{1 + Ce^{-\beta}} = 1 - \frac{Ce^{-\beta}}{1 + Ce^{-\beta}} \geq 1 - Ce^{-\beta}.$$

□

Now we can prove Part 2 of Theorem 2.2.

Proof. To prove Part 2 of Theorem 2.2, we first note that for any $u \in \mathcal{S} \setminus \{\vec{0}\}$, we have that

$$\mathbb{P}\Big(R^{\beta,u}(\mathcal{L}) > t\Big) \leq \mathbb{P}\left(R^{\beta,u}(\mathcal{L}) > t\,,\ \bigcap_{j=1}^{3(N-1)} M_j\right) + \mathbb{P}\left(\bigcup_{j=1}^{3(N-1)} M_j^c\right). \tag{2.7}$$

Part 2 of Proposition 2.4 implies that the right-hand side of Equation (2.7) is bounded above by

$$\mathbb{P}\left(T_{3(N-1)} > t\,,\ \bigcap_{j=1}^{3(N-1)} M_j\right) + \mathbb{P}\left(\bigcup_{j=1}^{3(N-1)} M_j^c\right).$$

By Part 1 of Proposition 2.4, we have that

$$\mathbb{P}\left(\bigcup_{j=1}^{3(N-1)} M_j^c\right) \leq 1 - \left(\frac{e^{\beta}}{e^{\beta} + e^{-\beta} + 2(N-1)}\right)^{3(N-1)}. \tag{2.8}$$

Note that the exit rate of any list different from the null list is always bigger than e^{β}.

Therefore,

$$\mathbb{P}\left(T_{3(N-1)} > t\ ,\ \bigcap_{j=1}^{3(N-1)} M_j\right) \leq \mathbb{P}\left(\sum_{n=1}^{3(N-1)} E_n > t\right), \tag{2.9}$$

where $(E_n)_{n\geq 1}$ is an i.i.d. sequence of exponentially distributed random variables with mean $1/e^{\beta}$. It follows that

$$\mathbb{P}\left(\sum_{n=1}^{3(N-1)} E_n > t\right) \leq \mathbb{P}\left(\bigcup_{n=1}^{3(N-1)} \left\{E_n > \frac{t}{3(N-1)}\right\}\right) \leq 3(N-1)e^{-e^{\beta}t/(3(N-1))}.$$

We conclude the proof by considering $t = e^{-\beta(1-\delta)}$ and noting that the bounds in (2.8) and (2.9) does not depend on u. □

The next corollary follows from Theorem 2.2.

Corollary 2.11. *For any fixed $\delta > 0$,*

$$\mathbb{P}\left(R^{\beta,\vec{0}}(\mathcal{L}) > \tau + e^{-\beta(1-\delta)}\right) \longrightarrow 0 \text{ as } \beta \longrightarrow \infty,$$

where τ exponentially distributed random time with mean $1/2N$.

Proof. Corollary 2.11 follows directly from Part 2 of Theorem 2.2 and the fact that given the initial list $\vec{0}$, the first jump time T_1 is an exponentially distributed random time with mean $1/2N$. □

2.6 Proof of Theorem 2.3

Recall that in Section 2.3, for any $l \in \mathcal{L}^+$ and for any $\beta \geq 0$ we considered c_β as the positive real number such that

$$\mathbb{P}(R^{\beta,l}(\mathcal{L}^-) > c_\beta) = e^{-1}.$$

To prove Theorem 2.3, we prove Proposition 2.12 which is interesting by itself.

Proposition 2.12. *For any $l \in \mathcal{L}^+$*

$$\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} \longrightarrow Exp(1), \text{ as } \beta \longrightarrow +\infty,$$

where $Exp(1)$ is a random variable exponentially distributed with mean 1.

To prove Proposition 2.12, we will first prove the following lemma.

Lemma 2.13. *For any $\beta \geq 0$, for any $l \in \mathcal{L}$ and for any $s > 0$,*

$$\mathbb{P}(U_s^{\beta,l} \in \mathcal{S} \setminus \mathcal{L}) \leq \frac{1-\mu^{\beta}(\mathcal{L})}{\mu^{\beta}(\mathcal{L})}.$$

Proof. For any $s > 0$,

$$\mu^\beta(\mathcal{L}) = \sum_{u\in\mathcal{L}} \mu^\beta(u)\mathbb{P}(U_s^{\beta,u} \in \mathcal{L}) + \sum_{u\in\mathcal{S}\setminus\mathcal{L}} \mu^\beta(u)\mathbb{P}(U_s^{\beta,u} \in \mathcal{L}).$$

By the symmetric properties of the process, it follows that for any $l, l' \in \mathcal{L}$,

$$\mathbb{P}(U_s^{\beta,l} \in \mathcal{L}) = \mathbb{P}(U_s^{\beta,l'} \in \mathcal{L}).$$

Moreover,

$$\sum_{u\in\mathcal{S}\setminus\mathcal{L}} \mu^\beta(u)\mathbb{P}(U_s^{\beta,u} \in \mathcal{L}) \leq 1 - \mu^\beta(\mathcal{L}).$$

This implies that

$$\mu^\beta(\mathcal{L}) \leq \mu^\beta(\mathcal{L})\mathbb{P}(U_s^{\beta,l} \in \mathcal{L}) + (1 - \mu^\beta(\mathcal{L})),$$

and therefore,

$$\mathbb{P}(U_s^{\beta,l} \in \mathcal{L}) \geq \frac{\mu^\beta(\mathcal{L}) - (1 - \mu^\beta(\mathcal{L}))}{\mu^\beta(\mathcal{L})}.$$

With this we concluded the proof of Lemma 2.13.

□

Proof. We will now prove Proposition 2.12. First of all, we will prove that for any $l \in \mathcal{L}^+$ and for any pair of positive real numbers $s, t \geq 0$, the following holds

$$\lim_{\beta\to+\infty} \left| \mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > s + t\right) - \mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > s\right)\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > t\right)\right| = 0. \tag{2.10}$$

To simplify the presentation of the proof, we will use the shorthand notation $R^{\beta,u}$ instead of $R^{\beta,u}(\mathcal{L}^-)$, for any $u \notin \mathcal{L}^-$.

Indeed, for a fixed $l \in \mathcal{L}^+$,

$$\left| \mathbb{P}\left(\frac{R^{\beta,l}}{c_\beta} > s + t\right) - \mathbb{P}\left(\frac{R^{\beta,l}}{c_\beta} > s\right)\mathbb{P}\left(\frac{R^{\beta,l}}{c_\beta} > t\right)\right| \leq$$

$$\sum_{u\in\mathcal{S}\setminus\mathcal{L}^-} \mathbb{P}\left(U_{c_\beta s}^{\beta,l} = u, \frac{R^{\beta,l}}{c_\beta} > s\right)\left|\mathbb{P}\left(\frac{R^{\beta,u}}{c_\beta} > t\right) - \mathbb{P}\left(\frac{R^{\beta,l}}{c_\beta} > t\right)\right|. \tag{2.11}$$

By the symmetric properties of the process, for any $u \in \mathcal{L}^+$,

$$\mathbb{P}\left(\frac{R^{\beta,u}}{c_\beta} > t\right) = \mathbb{P}\left(\frac{R^{\beta,l}}{c_\beta} > t\right).$$

Therefore, the left-hand Equation (2.11) is bounded above by

$$\sum_{u\in\mathcal{S}\setminus\mathcal{L}} \mathbb{P}\left(U_{c_\beta s}^{\beta,l} = u, \frac{R^{\beta,l}}{c_\beta} > s\right)\left|\mathbb{P}\left(\frac{R^{\beta,u}}{c_\beta} > t\right) - \mathbb{P}\left(\frac{R^{\beta,l}}{c_\beta} > t\right)\right| \leq$$

$$\mathbb{P}\left(U^{\beta,l}_{c_\beta s} \in \mathcal{S} \setminus \mathcal{L}, \frac{R^{\beta,l}}{c_\beta} > s\right) \leq \mathbb{P}\left(U^{\beta,l}_{c_\beta s} \in \mathcal{S} \setminus \mathcal{L}\right). \tag{2.12}$$

By Lemma 2.13, Equation (2.12) and Theorem 2.2 implies (2.10).

By definition,

$$\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > 1\right) = e^{-1}.$$

Iterating (2.10) with $t = s = 2^{-n}$, for $n = 1, 2, ...,$ we have that

$$\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > 2^{-n}\right) \longrightarrow e^{-2^{-n}}, \text{ as } \beta \longrightarrow +\infty.$$

More generally, we have that for any

$$t \in \left\{\sum_{n=1}^{m} b(n)2^{-n} \ : \ b(n) \in \{0, 1\}, n = 1, ..., m, m \geq 1\right\}$$

is valid that

$$\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > t\right) \longrightarrow e^{-t}, \text{ as } \beta \longrightarrow +\infty. \tag{2.13}$$

Any real number $r \in (0, 1)$ has a binary representation

$$r = \sum_{n=1}^{+\infty} b(n)2^{-n},$$

where for any $n \geq 1$, $b(n) \in \{0, 1\}$. Therefore, the monotonicity of

$$t \longrightarrow \mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > t\right)$$

implies that the convergence in (2.13) is valid for any $t \in (0, 1)$. Moreover, for any positive integer $n \geq 1$, Equation (2.10) implies that

$$\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > n\right) \longrightarrow e^{-n}, \text{ as } \beta \longrightarrow +\infty.$$

We conclude that (2.13) is valid for any $t > 0$. □

Remark 2.14. *For any $l \in \mathcal{L}^+$ and for any $\beta \geq 0$, the function $f_\beta : [0, +\infty) \longrightarrow [0, 1]$ given by*

$$f_\beta(t) = \mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > t\right)$$

is monotonic. Also, by Proposition 2.12, it converges pointwise as $\beta \longrightarrow +\infty$ to a continuous *function. Therefore, given* $\epsilon_\beta > 0$ *such that* $\lim_{\beta\to+\infty} \epsilon_\beta = 0$, *for any* $t > 0$ *we have that*

$$\lim_{\beta\to+\infty} \mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > t + \epsilon_\beta\right) = \lim_{\beta\to+\infty} \mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > t - \epsilon_\beta\right) = e^{-t}.$$

To prove Theorem 2.3, we will first prove the two following lemmas.

Lemma 2.15. *For any* $u \in \mathcal{S}$, *let*

$$\rho_u = \mathbb{P}\left(R^{\beta,u}(\mathcal{L}^+) < R^{\beta,u}(\mathcal{L}^-) \Big| \bigcap_{j=1}^{3(N-1)} M_j \right).$$

Then, for any $t > 0$,

$$\lim_{\beta\to+\infty} \sup_{u\in\mathcal{S}\setminus\mathcal{L}^-} \left| \mathbb{P}\left(\frac{R^{\beta,u}(\mathcal{L}^-)}{c_\beta} > t\right) - e^{-t}\rho_u \right| = 0.$$

Proof. Denoting $E_{\beta,u} = \{R^{\beta,u}(\mathcal{L}) < e^{\beta/2}, \bigcap_{j=1}^{3(N-1)} M_j\}$, we have that for any $u \notin \mathcal{L}$,

$$\mathbb{P}\left(\frac{R^{\beta,u}(\mathcal{L}^-)}{c_\beta} > t\right) = \mathbb{P}\left(\frac{R^{\beta,u}(\mathcal{L}^-)}{c_\beta} > t, E_{\beta,u}, R^{\beta,u}(\mathcal{L}^+) < R^{\beta,u}(\mathcal{L}^-)\right) + \tag{2.14}$$

$$\mathbb{P}\left(\frac{R^{\beta,u}(\mathcal{L}^-)}{c_\beta} > t, E_{\beta,u}, R^{\beta,u}(\mathcal{L}^-) < R^{\beta,u}(\mathcal{L}^+)\right) + \mathbb{P}\left(\frac{R^{\beta,u}(\mathcal{L}^-)}{c_\beta} > t, E^c_{\beta,u}\right).$$

By Proposition 2.7, there exists $\beta_t > 0$ such that for any $\beta > \beta_t$, $c_\beta t > e^{\beta/2}$. This implies that, for any $\beta > \beta_t$,

$$\mathbb{P}\left(\frac{R^{\beta,u}(\mathcal{L}^-)}{c_\beta} > t, E_{\beta,u}, R^{\beta,u}(\mathcal{L}^-) < R^{\beta,u}(\mathcal{L}^+)\right) = 0.$$

Considering $l \in \mathcal{L}^+$, for any $u \in \mathcal{S} \setminus \mathcal{L}^-$ and for all $\beta > \beta_t$, the left-hand side of Equation (2.14) is bounded bellow by

$$\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > t\right) \rho_u[1 - \mathbb{P}(E^c_{\beta,u})] \tag{2.15}$$

and bounded above by

$$\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > t - \frac{e^{\beta/2}}{c_\beta}\right) \rho_u + \mathbb{P}(E^c_{\beta,u}). \tag{2.16}$$

By Theorem 2.2 and Corollary 2.11,

$$\lim_{\beta\to+\infty} \sup_{u\in\mathcal{S}\setminus\mathcal{L}^-} \mathbb{P}\left(E^c_{\beta,u}\right) = 0.$$

By Proposition 2.7, it follows that $\lim_{\beta\to+\infty} e^{\beta/2}/c_\beta = 0$. Therefore, by Remark 2.14 we have that

$$\lim_{\beta\to+\infty} \mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > t\right) = \lim_{\beta\to+\infty} \mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > t - \frac{e^{\beta/2}}{c_\beta}\right) = e^{-t}.$$

We conclude the proof by noting that the limits in the last equation do not depend on u.

□

Lemma 2.16. *There exists $\alpha \in (0,1)$ and $\beta_\alpha > 0$ such that for any $\beta > \beta_\alpha$ and any $l \in \mathcal{L}^+$, the following upperbound holds*

$$\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > n\right) \le \alpha^n,$$

for any positive integer $n \geq 1$.

Proof. By Lemma 2.15, for any fixed $\alpha \in (e^{-1}, 1)$, there exists β_α such that for all $\beta > \beta_\alpha$ and for any $u \notin \mathcal{L}^-$,

$$\mathbb{P}\left(\frac{R^{\beta,u}(\mathcal{L}^-)}{c_\beta} > 1\right) \le \alpha < 1. \tag{2.17}$$

For any $l \in \mathcal{L}^+$ and for any $n \in \{2, 3, \dots\}$,

$$\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > n\right) = \sum_{u\in\mathcal{S}\setminus\mathcal{L}^-} \mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > n-1, U^{\beta,l}_{c_\beta(n-1)} = u\right) \mathbb{P}\left(\frac{R^{\beta,u}(\mathcal{L}^-)}{c_\beta} > 1\right).$$

Equation (2.17) implies that for any $\beta > \beta_\alpha$,

$$\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > n\right) \le \alpha \mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta} > n-1\right). \tag{2.18}$$

We finish the proof by iterating (2.18). □

Proof. We will now prove Theorem 2.3.

First of all, we will prove that for any $l \in \mathcal{L}^+$, the following holds

$$\frac{R^{\beta,l}(\mathcal{L}^-)}{\mathbb{E}[R^{\beta,l}(\mathcal{L}^-)]} \longrightarrow \text{Exp}(1) \text{ in distribution, as } \beta \longrightarrow +\infty. \tag{2.19}$$

Considering Proposition 2.12, we only need to show that

$$\lim_{\beta\to+\infty} \frac{\mathbb{E}[R^{\beta,l}(\mathcal{L}^-)]}{c_\beta} = 1.$$

Actually,

$$\lim_{\beta\to+\infty} \frac{\mathbb{E}[R^{\beta,l}(\mathcal{L}^-)]}{c_\beta} = \lim_{\beta\to+\infty} \int_0^{+\infty} \mathbb{P}(R^{\beta,l}(\mathcal{L}^-) > c_\beta s) ds.$$

Lemma 2.16 and the Dominated Convergence Theorem, allow us to put the limit inside the integral in the last term

$$\lim_{\beta\to+\infty}\int_0^{+\infty}\mathbb{P}(R^{\beta,l}(\mathcal{L}^-)>c_\beta s)ds=\int_0^{+\infty}\lim_{\beta\to+\infty}\mathbb{P}(R^{\beta,l}(\mathcal{L}^-)>c_\beta s)ds=\int_0^{+\infty}e^{-s}ds=1.$$

This and Proposition 2.12 imply (2.19).

For any $\beta>0$, for any $u\in C^+$ and for any $s>0$,

$$\mathbb{P}(R^{\beta,u}(C^-)\geq c_\beta s)=\mathbb{P}(R^{\beta,u}(C^-)\geq c_\beta s,E_{\beta,u})+\mathbb{P}(R^{\beta,u}(C^-)\geq c_\beta s,E_{\beta,u}^c),$$

where

$$E_{\beta,u}=\{R^{\beta,u}(\mathcal{L}^+)<\min\{1,R^{\beta,u}(C^-)\},R^{\beta,u}(\mathcal{L}^-)<R^{\beta,u}(C^-)+1\}.$$

For any $l\in\mathcal{L}^+$, we have

$$\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta}\geq s+\frac{1}{c_\beta},E_{\beta,u}\right)\leq\mathbb{P}(R^{\beta,u}(C^-)\geq c_\beta s,E_{\beta,u})\leq\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta}\geq s-\frac{1}{c_\beta},E_{\beta,u}\right).\tag{2.20}$$

By Theorem 2.2, for any $u\in C^+$,

$$\lim_{\beta\to+\infty}\mathbb{P}(E_{\beta,u})=1,$$

and then,

$$\lim_{\beta\to+\infty}\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta}\geq s+\frac{1}{c_\beta},E_{\beta,u}\right)=\lim_{\beta\to+\infty}\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta}\geq s+\frac{1}{c_\beta}\right),$$

$$\lim_{\beta\to+\infty}\mathbb{P}(R^{\beta,u}(C^-)\geq c_\beta s,E_{\beta,u})=\lim_{\beta\to+\infty}\mathbb{P}(R^{\beta,u}(C^-)\geq c_\beta s),$$

$$\lim_{\beta\to+\infty}\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta}\geq s-\frac{1}{c_\beta},E_{\beta,u}\right)=\lim_{\beta\to+\infty}\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta}\geq s-\frac{1}{c_\beta}\right).$$

By Proposition 2.7, $\lim_{\beta\to+\infty}c_\beta^{-1}=0$. Therefore, Remark 2.14 implies that

$$\lim_{\beta\to+\infty}\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta}\geq s+\frac{1}{c_\beta}\right)=\lim_{\beta\to+\infty}\mathbb{P}\left(\frac{R^{\beta,l}(\mathcal{L}^-)}{c_\beta}\geq s-\frac{1}{c_\beta}\right)=e^{-s}.$$

The conclusion follows from Equation (2.20) and by observing that the Dominated Convergence Theorem allow us to replace c_β by $\mathbb{E}[R^{\beta,u}(C^-)]$ as we did to prove that Equation (2.19) holds.

□

Chapter 3

Metastability in a Stochastic System of Spiking Neurons with Leakage

3.1 Introduction

We study a system of interacting point processes with memory of variable length modeling a finite but large network of spiking neurons with leakage. We prove that when the population size diverges this system has a metastable behavior.

The system we consider can be informally described as follows. Each neuron is associated to two point processes. The first point process indicates the successive spiking times of the neuron. The rate of this point process is an exponential function of the membrane potential of the neuron, with the restriction that the rate takes the value 0 when the membrane potential is 0. When a neuron spikes, its membrane potential resets to 0, and simultaneously, the membrane potentials of the other neurons increase by one unit.

The second point process associated to each neuron indicates its successive leakage times. This point process has a constant rate 1. At each leakage time of the neuron, its membrane potential is reset to 0, with no effect on the other neurons membrane potentials.

Let us now informally present our results. For any initial configuration of membrane potentials, the number of spiking and leakage times of the system is finite. Moreover, the process gets trapped after a finite time in the configuration in which the membrane potentials of all neurons are 0. This is the content of Theorem 3.1.

Let us suppose that the system starts with a configuration in which a sufficiently large set of neurons have strictly positive membrane potential. With such a starting point, as the number of neurons of the system diverges, the system instantaneously reaches a set of configurations in which all neurons but one have strictly positive membrane potentials and these membrane potentials are all different. The system are in this set with probability approaching to 1, for any instant before it gets trapped as the number of neurons of the system diverges. This is the content of Theorem 3.2.

The system has a metastable behavior, namely the time at which it gets trapped in the null membrane potentials configuration re-normalized by its mean value converges in distribution to a mean 1 exponential random time as the population size diverges. This is the content of Theorem 3.3. Theorem 3.3 assumes that the system starts with the same type of initial configuration considered in Theorem 3.2. This initial configuration condition prevents the system to be immediately attracted by the null configuration.

This system belongs to the class of models introduced by Ferrari *et al.*, 2018. In this article it was considered the case in which the spiking rate is 1 when the membrane potential is strictly positive and it is 0 otherwise, the leakage rate is constant and the set of neurons is the set of all integers, with each neuron interacting only with its two neighbors. In this framework it was proven that there exists a critical value for the leakage rate such that the system has either one or two extremal invariant measures when the leakage rate is either greater or smaller than the critical value, respectively. It was proven by André, 2019 that for a finite system with a sufficiently small leakage rate, the system displays a metastable behavior when the number of the neurons diverges (see also André and Planche, 2021 and André, 2022).

This system belongs to the class of systems of interacting point process with memory of variable length that was introduced in discrete time by Galves and Löcherbach, 2013 and in continuous time by De Masi *et al.*, 2014 to model systems of spiking neurons. The metastable behavior of systems of interacting point processes with memory of variable length was also analyzed by Yu and Taillefumier, 2022 and Löcherbach and Monmarché, 2022. Other aspects of systems of interacting point processes with memory of variable length in this class of models was considered in several articles, including Duarte, Ost, and Rodríguez, 2015; Brochini *et al.*, 2016; Duarte and Ost, 2016; Fournier and Löcherbach, 2016; Galves and Löcherbach, 2016; Yaginuma, 2016; Baccelli and Taillefumier, 2019; Duarte, Galves, *et al.*, 2019; Galves, Löcherbach, Pouzat, and Presutti, 2019; Baccelli and Taillefumier, 2021; Nascimento, 2022; Baccelli, Davydov, *et al.*, 2022; Chariker and Lebowitz, 2022 and De Santis *et al.*, 2022. For a self-contained and neurobiological motivated presentation of this class of variable length memory models for system of spiking neurons, both in discrete and continuous time, we refer the reader to Galves, Löcherbach, and Pouzat, 2021.

The notion of metastability considered here is inspired by the so called *pathwise approach to metastability* introduced by Cassandro *et al.*, 1984. For more references and an introduction to the topic, we refer the reader to Olivieri and Vares, 2005; Hollander, 2009 and Fernández *et al.*, 2015.

This chapter is organized as follows. In Section 3.2 we present the definitions, basic and extra notation and state the main results. In Section 3.3 we prove Theorem 3.1. In Section 3.4 we present a coupling construction and prove some auxiliary results. In Sections 3.5 and 3.6 we prove Theorems 3.2 and 3.3, respectively.

3.2 Definitions, notation and main results

Let $\mathcal{A}_N = \{1, 2, ..., N\}$ be the set of neurons, with $N \geq 2$ and denote

$$\mathcal{S}_N = \Big\{ u = (u(a) \,:\, a \in \mathcal{A}_N) \in \{0, 1, 2, \ldots\}^N \,:\, \min\{u(a) \,:\, a \in \mathcal{A}_N\} = 0 \Big\}$$

the set of lists of membrane potentials.

We want to describe the time evolution of the list of membrane potentials of a system of spiking neurons. To to this, for any neuron $a \in \mathcal{A}_N$, we define the maps $\pi^{a,*}$ and $\pi^{a,\dagger}$ on $\mathcal{S}_N$ as follows. For any $u \in \mathcal{S}_N$,

$$\pi^{a,*}(u)(b) = \begin{cases} u(b) + 1 & , \text{if } b \neq a, \\ 0 & , \text{if } b = a, \end{cases}$$

$$\pi^{a,\dagger}(u)(b) = \begin{cases} u(b) & , \text{if } b \neq a, \\ 0 & , \text{if } b = a. \end{cases}$$

The map $\pi^{a,*}$ represents the effect of a spike of neuron a in the system. When we apply the map $\pi^{a,*}$, the membrane potential of neuron a resets to 0 and the membrane potentials of all the other neurons increase by one unit.

The map $\pi^{a,\dagger}$ represents the leakage effect on the membrane potential of neuron a. When we apply the map $\pi^{a,\dagger}$, the membrane potential of neuron a resets to 0 and the membrane potentials of all the other neurons remain the same.

The time evolution of the system of neurons can be described as follows. Denote the initial list of membrane potentials $U_0^{N,u} = u \in \mathcal{S}_N$. The list of membrane potentials $(U_t^{N,u})_{t\in[0,+\infty)}$ evolves as a Markov jump process taking values in the set $\mathcal{S}_N$ and with infinitesimal generator $\mathcal{G}$ defined as follows

$$\mathcal{G}f(u) = \sum_{b\in\mathcal{A}_N} e^{u(b)}\mathbf{1}\{u(b) > 0\} \left[f(\pi^{b,*}(u)) - f(u) \right] + \sum_{b\in\mathcal{A}_N} \left[f(\pi^{b,\dagger}(u)) - f(u) \right],$$

for any bounded function $f \,:\, \mathcal{S}_N \longrightarrow \mathbb{R}$.

Observe that the null list $\vec{0}_N \in \mathcal{S}_N$, defined as

$$\vec{0}_N(a) = 0, \text{ for any } a \in \mathcal{A}_N$$

is a trap for the process. The goal is to study the time the process takes to get trapped and its behavior before get trapped.

To state our main results, we need to introduce some notation. Let

$$\tau^{N,u} = \inf\{t > 0 \,:\, U_t^{N,u} = \vec{0}_N\}$$

and define $\mathcal{N}^{N,u}$ as the number of spikes and leakages of membrane potential of the process, namely

$$\mathcal{N}^{N,u} = \left| \left\{ s \in (0, \tau^{N,u}] \,:\, U_s^{N,u} \neq \lim_{t \to s^-} U_t^{N,u} \right\} \right|.$$

We consider also the set

$$S_N^{(0)} = \{ u \in \mathcal{S}_N \,:\, |\{ a \in \mathcal{A}_N \,:\, u(a) > 0 \}| \geq \lfloor N^{1/2} \rfloor \}$$

and the set

$$\mathcal{W}_N = \left\{ u \in \mathcal{S}_N \,:\, \left\{ 1, \dots, N - \lfloor N^{1/2} \rfloor \right\} \subset \{ u(a) \,:\, a \in \mathcal{A}_N \};\ \bigcap_{a \in \mathcal{A}_N} \bigcap_{b \neq a} \{ u(a) \neq u(b) \} \right\}.$$

We can now state our main results.

Theorem 3.1. *For any $N \geq 2$ and for any initial list $u \in \mathcal{S}_N$, it follows that*

$$\mathbb{P}(\mathcal{N}^{N,u} < +\infty) = 1$$

and

$$\mathbb{P}(\tau^{N,u} < +\infty) = 1.$$

Theorem 3.2. *For any $t > 0$, it follows that*

$$\inf_{u \in S_N^{(0)}} \mathbb{P} \left(U_t^{N,u} \in \mathcal{W}_N \mid \tau^{N,u} > t \right) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

Theorem 3.3. *For any sequence $(u_N \in S_N^{(0)} \,:\, N \geq 2)$,*

$$\frac{\tau^{N,u_N}}{\mathbb{E}[\tau^{N,u_N}]} \longrightarrow Exp(1) \text{ in distribution, as } N \longrightarrow +\infty,$$

where $Exp(1)$ denotes a mean 1 exponential distributed random variable.

To prove our results it is convenient to extend the notation introduced before.

Extra notation

- Let $T_0 = 0$ and for $n = 1, \dots, \mathcal{N}^{N,u}$, denote T_n the successive jumping times of the process $(U_t^{N,u})_{t \in [0,+\infty)}$, namely

$$T_n = \inf \left\{ t > T_{n-1} \,:\, U_t^{N,u} \neq U_{T_{n-1}}^{N,u} \right\}.$$

- For $n = 1, \dots, \mathcal{N}^{N,u}$, we define $A_n \in \mathcal{A}_N$ and $O_n \in \{*, \dagger\}$ as the pair such that

$$U_{T_n}^{N,u} = \pi^{A_n, O_n} \left(U_{T_{n-1}}^{N,u} \right).$$

- The leakage times are defined as $T_0^\dagger = 0$ and for $n \geq 1$,

$$T_n^\dagger = \inf\{T_m > T_{n-1}^\dagger \,:\, O_m = \dagger\}.$$

- The spiking times are defined as $T_0^* = 0$ and for $n \geq 1$,

$$T_n^* = \inf\{T_m > T_{n-1}^* \,:\, O_m = *\}.$$

- For any time interval $I \subset [0, +\infty)$, the counting measures indicating the number of leakage times and spiking times that occurred during the time interval I are defined as

$$Z^\dagger(I) = \sum_{m=1}^{+\infty} \mathbf{1}\{T_m^\dagger \in I\} \quad \text{and} \quad Z^*(I) = \sum_{m=1}^{+\infty} \mathbf{1}\{T_m^* \in I\}.$$

- For any $u \in \mathcal{S}_N$, we define $a_1^u, \dots, a_N^u \in \mathcal{A}_N$ in the following way

$$a_1^u \in \operatorname{argmin}\{u(a) \,:\, a \in \mathcal{A}_N\},$$

$$a_2^u \in \operatorname{argmin}\{u(a) \,:\, a \in \mathcal{A}_N \setminus \{a_1^u\}\},$$

$$\dots$$

$$a_N^u \in \operatorname{argmin}\{u(a) \,:\, a \in \mathcal{A}_N \setminus \{a_1^u, a_2^u, \dots, a_{N-1}^u\}\}.$$

  To avoid ambiguity, we use the following convention: if $u(a_j^u) = u(a_{j+1}^u)$, then $a_j^u < a_{j+1}^u$.

- The set of ladder lists is defined as

$$\mathcal{L}_N = \Big\{u \in \mathcal{S}_N \,:\, \{u(a) \,:\, a \in \mathcal{A}_N\} = \{0, 1, \dots, N-1\}\Big\}.$$

- Let $\sigma : \mathcal{A}_N \longrightarrow \mathcal{A}_N$ be a bijective map. For any $u \in \mathcal{S}_N$, the permuted list $\sigma(u) \in \mathcal{S}_N$ is defined as

$$\sigma(u)(a) = u(\sigma(a)), \text{ for all } a \in \mathcal{A}_N.$$

- For any $\lambda > 0$, $\xi^{\{\lambda\}}$ and $(\xi_j^{\{\lambda\}} \,:\, j = 1, 2, \dots)$ will always be, respectively, a random variable exponentially distributed with mean λ^{-1} and a sequence of independent random variables exponentially distributed with mean λ^{-1}.

3.3 Proof of Theorem 3.1

In this section we will prove Theorem 3.1. First we need to prove the following lemma.

Lemma 3.4. *For any $u \in \mathcal{S}_N \setminus \{\vec{0}_N\}$, it follows that*

$$\mathbb{P}(U_{T_{N-1}}^{N,u} \in \mathcal{L}_N) \geq \left(\frac{1}{2(N-1)}\right)^{N-1}.$$

Proof. For any initial list $u \in \mathcal{S}_N \setminus \{\vec{0}_N\}$, the occurrence of the event $\{A_1 = a^u_N, O_1 =*\}$ implies that $U^{N,u}_{T_1} \in \mathcal{L}_N$ in the case $N = 2$, and implies that

$$a_1^{U^{N,u}_{T_1}} = 0,\ a_2^{U^{N,u}_{T_1}} = 1,\ a_j^{U^{N,u}_{T_1}} \geq 1, \text{ for } j = 3, \dots, N,$$

in the case $N \geq 3$. As a consequence, the occurrence of the event

$$\left\{A_1 = a^u_N, O_1 =*, A_2 = a_N^{U^{N,u}_{T_1}}, O_2 =*\right\}$$

implies that implies that $U^{N,u}_{T_2} \in \mathcal{L}_N$ in the case $N = 3$, and it implies that

$$a_1^{U^{N,u}_{T_2}} = 0,\ a_2^{U^{N,u}_{T_2}} = 1,\ a_3^{U^{N,u}_{T_2}} = 2,\ a_j^{U^{N,u}_{T_2}} \geq 2, \text{ for } j = 4, \dots, N.$$

in the case $N \geq 4$. Iterating this, we conclude that the occurrence of the event

$$\bigcap_{j=1}^{N-1} \{A_j = a_N^{U^{N,u}_{T_{j-1}}}, O_j =*\}$$

implies that $U^{N,u}_{T_{N-1}} \in \mathcal{L}_N$. Therefore,

$$\mathbb{P}(U^{N,u}_{T_{N-1}} \in \mathcal{L}_N) \geq \mathbb{P}\left(\bigcap_{j=1}^{N-1} \{A_j = a_N^{U^{N,u}_{T_{j-1}}}, O_j =*\}\right). \tag{3.1}$$

The smallest value for

$$\mathbb{P}(A_1 = a^u_N, O_1 =*)$$

is obtained for any initial list u in which all neurons, except one, have membrane potential equal 1. This implies that

$$\inf\left\{\mathbb{P}(A_1 = a^v_N, O_1 =* \mid U^{N,v}_0 = v) : v \in \mathcal{S}_N \setminus \{\vec{0}_N\}\right\} \geq \frac{1}{2(N-1)}.$$

We conclude the proof by using Markov property and applying this lower bound $N-1$ times in Equation (3.1). □

Proof. We will now prove Theorem 3.1.

For any $N \geq 2$ and for any $u, u' \in \mathcal{S}_N \setminus \{\vec{0}_N\}$, we have that

$$\mathbb{P}\left(\mathcal{N}^{N,u} \leq n + 2(N-1) \,\middle|\, U^{N,u}_{T_n} = u'\right) \geq$$
$$\mathbb{P}\left(U^{N,u}_{T_{n+N-1}} \in \mathcal{L}_N \,\middle|\, U^{N,u}_{T_n} = u'\right)\mathbb{P}\left(\mathcal{N}^{N,u} \leq n + 2(N-1) \,\middle|\, U^{N,u}_{T_{n+N-1}} \in \mathcal{L}_N\right).$$

Using together the Markov property and Lemma 3.2, we get

$$\mathbb{P}\left(U^{N,u}_{T_{n+N-1}} \in \mathcal{L}_N \,\middle|\, U^{N,u}_{T_n} = u'\right) \geq [2(N-1)]^{-(N-1)}.$$

The invariance by permutation of the process implies that, for any $l, l' \in \mathcal{L}_N$, we have

$$\mathbb{P}\left(\mathcal{N}^{N,u} \leq n + 2(N-1) \,\middle|\, U^{N,u}_{T_{n+N-1}} = l\right) = \mathbb{P}\left(\mathcal{N}^{N,u} \leq n + 2(N-1) \,\middle|\, U^{N,u}_{T_{n+N-1}} = l'\right).$$

Calling

$$\epsilon' = \mathbb{P}\left(\mathcal{N}^{N,u} \leq n + 2(N-1) \,\middle|\, U^{N,u}_{T_{n+N-1}} = l\right),$$

we conclude that

$$\mathbb{P}\left(\mathcal{N}^{N,u} \leq n + 2(N-1) \,\middle|\, U^{N,u}_{T_{n+N-1}} \in \mathcal{L}_N\right) = \epsilon' > 0.$$

Therefore, for any $u' \in \mathcal{S}_N$,

$$\mathbb{P}\left(\mathcal{N}^{N,u} \leq n + 2(N-1) \,\middle|\, U^{N,u}_{T_n} = u'\right) \geq [2(N-1)]^{-(N-1)}\epsilon'.$$

The last inequality implies that for any $n \geq 1$,

$$\mathbb{P}(\mathcal{N}^{N,u} \geq n) \leq \mathbb{P}\left(2(N-1) \times \mathrm{Geom}\left([2(N-1)]^{-(N-1)}\epsilon'\right) \geq n\right),$$

where $\mathrm{Geom}(r)$ denotes a random variable with geometric distribution assuming values in $\{1, 2, \ldots\}$ and with mean $1/r$. This implies that $\mathbb{P}(\mathcal{N}^{N,u} < +\infty) = 1$, concluding the first part of the proof.

The jump rate of the process $(U^{N,u}_t)_{t\in[0,+\infty)}$ satisfies

$$\sum_{a\in\mathcal{A}_N} \mathbf{1}\{u'(a) > 0\}(e^{u'(a)} + 1) \geq e + 1,$$

for any $u' \in \mathcal{S}_N \setminus \{\vec{0}_N\}$. Putting all this together we conclude that $\mathbb{P}(\tau^{N,u} < +\infty) = 1$.

□

3.4 A coupling construction

In this section we will prove the following proposition.

Proposition 3.5. *The following holds*

$$\lim_{N\to+\infty} \sup_{t\geq 0} \sup_{w,w'\in\mathcal{W}_N} \left|\mathbb{P}(\tau^{N,w} > t) - \mathbb{P}(\tau^{N,w'} > t)\right| = 0.$$

To prove Proposition 3.5, we need to introduce a coupling construction of the processes $(U^{N,u'}_t)_{t\in[0,+\infty)}$ and $(U^{N,v'}_t)_{t\in[0,+\infty)}$ starting from two different lists $u', v' \in \mathcal{S}_N$.

We want to describe the time evolution of $(U^{N,u'}_t, U^{N,v'}_t)_{t\in[0,+\infty)}$. To to this, for any index $j \in \{1, \ldots, N\}$, we define the maps $\pi^{j,\min}$, $\pi^{j,\max}$ and $\pi^{j,\dagger}$ on $\mathcal{S}^2_N$ as follows. For any $(u, v) \in \mathcal{S}^2_N$,

$$\pi^{j,\min}(u, v) = (\pi^{a^u_j,*}(u), \pi^{a^v_j,*}(v)),$$

$$\pi^{j,\max}(u,v) = \begin{cases} (\pi^{a_j^u,*}(u), v) & \text{, if } u(a_j^u) > v(a_j^v), \\ (u, \pi^{a_j^v,*}(v)) & \text{, if } v(a_j^v) > u(a_j^u), \end{cases}$$

$$\pi^{j,\dagger}(u,v) = (\pi^{a_j^u,\dagger}(u), \pi^{a_j^v,\dagger}(v)).$$

The map $\pi^{j,\min}(u,v)$ represents the simultaneous effect of a spike of neuron a_j^u in the system $(U_t^{N,u'})_{t\geq 0}$ and a spike of neuron a_j^v in the system $(U_t^{N,v'})_{t\in[0,+\infty)}$.

The map $\pi^{j,\max}(u,v)$ represents the effect of either a spike of neuron a_j^u in the system $(U_t^{N,u'})_{t\geq 0}$ in the case in which $u(a_j^u) > v(a_j^v)$, or a spike of neuron a_j^v in the system $(U_t^{N,v'})_{t\in[0,+\infty)}$ in the case in which $v(a_j^v) > u(a_j^u)$.

The map $\pi^{j,\dagger}(u,v)$ represents the simultaneous leakage effect on the membrane potential of neuron a_j^u in the system $(U_t^{N,u'})_{t\geq 0}$ and on the membrane potential of neuron a_j^v in the system $(U_t^{N,v'})_{t\in[0,+\infty)}$.

The pair of lists of membrane potentials $(U_t^{N,u'}, U_t^{N,v'})_{t\in[0,+\infty)}$ evolves as a Markov jump process taking values in the set $\mathcal{S}_N^2$ and with infinitesimal generator $\mathcal{G}_C$ defined as follows

$$\mathcal{G}_C f(u,v) = \sum_{j=1}^{N} e^{\min\{u(a_j^u), v(a_j^v)\}} \mathbf{1}\{\min\{u(a_j^u), v(a_j^v)\} > 0\} \left[f(\pi^{j,\min}(u,v)) - f(u,v)\right] +$$

$$\sum_{j=1}^{N} e^{|u(a_j^u)-v(a_j^v)|} \mathbf{1}\{u(a_j^u) \neq v(a_j^v)\} \left[f(\pi^{j,\max}(u,v)) - f(u,v)\right] + \sum_{j=1}^{N} \left[f(\pi^{j,\dagger}(u,v)) - f(u,v)\right],$$

for any bounded function $f : \mathcal{S}_N^2 \longrightarrow \mathbb{R}$.

For the coupling construction we introduce some extra notation.

Extra notation - coupling construction

- Define
$$\tau^N(u,v) = \inf\{s > 0 : (U_s^{N,u}, U_s^{N,v}) = (\vec{0}_N, \vec{0}_N)\}.$$

- Define $\mathcal{N}^N(u,v)$ as the number of spikes and leakages of membrane potential of the coupling process, namely
$$\mathcal{N}^N(u,v) = \left|\left\{ s \in (0, \tau^N(u,v)] : (U_s^{N,u}, U_s^{N,v}) \neq \left(\lim_{t\to s^-} U_t^{N,u}, \lim_{t\to s^-} U_t^{N,v}\right)\right\}\right|.$$

- Let $T_0(u,v) = 0$ and for $n = 1, \ldots, \mathcal{N}^N(u,v)$ denote $T_n(u,v)$ the successive jumping times of the process $(U_t^{N,u}, U_t^{N,v})_{t\in[0,+\infty)}$, namely
$$T_n(u,v) = \inf\left\{ t > T_{n-1}(u,v) : (U_t^{N,u}, U_t^{N,v}) \neq \left(U_{T_{n-1}(u,v)}^{N,u}, U_{T_{n-1}(u,v)}^{N,v}\right)\right\}.$$

- For each $n = 1, \ldots, \mathcal{N}^N(u,v)$, we define $J_n(u,v) \in \{1, \ldots, N\}$ and $K_n(u,v) \in$

$\{\min, \max, \dagger\}$ as the pair such that

$$(U^{N,u}_{T_n(u,v)}, U^{N,v}_{T_n(u,v)}) = \pi^{J_n(u,v),K_n(u,v)} \left(U^{N,u}_{T_{n-1}(u,v)}, U^{N,v}_{T_{n-1}(u,v)} \right).$$

- For any $j \geq 1$, we define the event

$$E_j(u,v) = \bigcap_{n=2(j-1)\lceil N^{1/2}\rceil+1}^{2j\lceil N^{1/2}\rceil} \{J_n(u,v) = N, K_n(u,v) \neq \dagger\}.$$

- The number of jumping times of the process $(U^{N,u}_t, U^{N,v}_t)_{t\in[0,+\infty)}$ until the first leakage time is defined as

$$\mathcal{N}^N_{\dagger}(u,v) = \inf\{n : K_n(u,v) = \dagger\}.$$

- The number of jumping times of the process $(U^{N,u}_t, U^{N,v}_t)_{t\in[0,+\infty)}$ until the coupling time is defined as

$$\mathcal{N}^N_C(u,v) = \inf\left\{ n : \text{exists } \sigma : \mathcal{A}_N \longrightarrow \mathcal{A}_N \text{ bijective, such that } U^{N,u}_{T_n(u,v)} = \sigma\left(U^{N,v}_{T_n(u,v)}\right) \right\}.$$

Remark 3.6. *There exists a bijective map $\sigma : \mathcal{A}_N \longrightarrow \mathcal{A}_N$ such that*

$$U^{N,u}_s = \sigma\left(U^{N,v}_s\right), \textit{ for all } s \geq T_{\mathcal{N}^N_C(u,v)}(u,v).$$

Moreover, if there exists $t \geq 0$ such that $U^{N,u}_t \in \mathcal{L}_N$ and $U^{N,v}_t \in \mathcal{L}_N$, then $t \geq T_{\mathcal{N}^N_C(u,v)}(u,v)$.

The proof of Proposition 3.5 is based on the three following lemmas about the coupling construction.

Lemma 3.7. *For any lists $w, w' \in \mathcal{W}_N$, if the event $E_1(w,w')$ occurs, then*

$$\mathcal{N}^N_C(w,w') \leq 2\lceil N^{1/2}\rceil.$$

Proof. The occurrence of the event

$$E_1(w,w') = \bigcap_{n=1}^{2\lceil N^{1/2}\rceil} \left\{J_n(w,w') = N, K_n(w,w') \neq \dagger\right\}$$

implies that in the first $2\lceil N^{1/2}\rceil$ steps of the coupling construction there are neurons spiking and at each step, the neuron that spikes is the neuron with greatest membrane potential.

For the first step, denoting $u_1 = U^{N,w}_{T_1(w,w')}$ and $u'_1 = U^{N,w'}_{T_1(w,w')}$, we have two possible cases:

- If $J_1(w,w') = N$ and $K_1(w,w') = \min$, then

$$\left\{1, \ldots, N - \lfloor N^{1/2}\rfloor + 1\right\} \subset \{u_1(a) : a \in \mathcal{A}_N\}; \bigcap_{a\in\mathcal{A}_N} \bigcap_{b\neq a} \{u_1(a) \neq u_1(b)\}$$

and

$$\left\{1, \dots, N - \lfloor N^{1/2} \rfloor + 1\right\} \subset \{u_1'(a) : a \in \mathcal{A}_N\}; \bigcap_{a \in \mathcal{A}_N} \bigcap_{b \neq a} \{u_1'(a) \neq u_1'(b)\}.$$

- If $J_1(w, w') = N$ and $K_1(w, w') = \max$, then either $u_1' = w'$ and

$$\left\{1, \dots, N - \lfloor N^{1/2} \rfloor + 1\right\} \subset \{u_1(a) : a \in \mathcal{A}_N\}; \bigcap_{a \in \mathcal{A}_N} \bigcap_{b \neq a} \{u_1(a) \neq u_1(b)\}$$

in the case $u_1(a_N^{u_1}) > u_1'(a_N^{u_1'})$, or $u_1 = w$ and

$$\left\{1, \dots, N - \lfloor N^{1/2} \rfloor + 1\right\} \subset \{u_1'(a) : a \in \mathcal{A}_N\}; \bigcap_{a \in \mathcal{A}_N} \bigcap_{b \neq a} \{u_1'(a) \neq u_1'(b)\}$$

in the case $u_1'(a_N^{u_1'}) > u_1(a_N^{u_1})$.

Iterating this, we conclude that if the event $E_1(w, w')$ occurs, then

$$U^{N,w}_{T_{2\lceil N^{1/2} \rceil}(w,w')} \in \mathcal{L}_N \quad \text{and} \quad U^{N,w'}_{T_{2\lceil N^{1/2} \rceil}(w,w')} \in \mathcal{L}_N. \tag{3.2}$$

By Remark 3.6, (3.2) implies that $\mathcal{N}_C^N(w, w') \leq 2\lceil N^{1/2} \rceil$. □

Lemma 3.8. *For any $n \geq 1$ and for any $w, w' \in \mathcal{W}_N$,*

$$\mathbb{P}\left(\mathcal{N}_C^N(w, w') \leq 2n\lceil N^{1/2} \rceil < \mathcal{N}_\dagger^N(w, w')\right) \geq$$

$$\mathbb{P}\left(Geom\left(\zeta^{2\lceil N^{1/2} \rceil}\right) \leq n\right) \left(\frac{e^{\lfloor N^{1/2} \rfloor}}{e^{\lfloor N^{1/2} \rfloor} + 2(N-1)}\right)^{2n\lceil N^{1/2} \rceil},$$

where $\zeta = 1 - e^{-1}$ and $Geom\left(\zeta^{2\lceil N^{1/2} \rceil}\right)$ is a random variable with geometric distribution assuming values in $\{1, 2, \dots\}$ and with mean $1/\zeta^{2\lceil N^{1/2} \rceil}$.

Proof. For any $n \geq 1$ and for any $w, w' \in \mathcal{W}_N$, the occurrence of the event

$$\bigcap_{m=1}^{2n\lceil N^{1/2} \rceil} \left\{J_m(w, w') \in \{N - \lfloor N^{1/2} \rfloor + 1, \dots, N\}, K_m(w, w') \neq \dagger\right\}$$

implies that $U_m^{N,w} \in \mathcal{W}_N$ and $U_m^{N,w'} \in \mathcal{W}_N$, for all $m = 1, \dots, 2n\lceil N^{1/2} \rceil$. This implies that for any $w, w' \in \mathcal{W}_N$,

$$\mathbb{P}\left(\mathcal{N}_C^N(w, w') \leq 2n\lceil N^{1/2} \rceil < \mathcal{N}_\dagger^N(w, w')\right) \geq$$

$$\mathbb{P}\left(\bigcup_{m=1}^{n} E_m(w, w'), \bigcap_{m=1}^{2n\lceil N^{1/2} \rceil} \left\{J_m(w, w') \in \{N - \lfloor N^{1/2} \rfloor + 1, \dots, N\}, K_m(w, w') \neq \dagger\right\}\right).$$

For any lists $u, v \in \mathcal{W}_N$ we have that

$$\mathbb{P}(J_1(u,v) = N, K_1(u,v) \neq \dagger) = \frac{e^{\max\{u(a_N^u), v(a_N^v)\}}}{\sum_{j=2}^{N} e^{\max\{u(a_j^u), v(a_j^v)\}}} \mathbb{P}(K_1(u,v) \neq \dagger). \tag{3.3}$$

The left term of the right-hand side in Equation (3.3) is bounded below by

$$\frac{e^{(N-1)}}{\sum_{j=1}^{N-1} e^j} \geq \zeta.$$

Therefore,

$$\mathbb{P}\left(E_1(w,w'), \bigcap_{m=1}^{2\lceil N^{1/2}\rceil} \left\{ J_m(w,w') \in \{N - \lfloor N^{1/2} \rfloor + 1, \dots, N\}, K_m(w,w') \neq \dagger \right\} \right) \geq$$

$$\zeta^{2\lceil N^{1/2}\rceil} \mathbb{P}\left(\bigcap_{m=1}^{2\lceil N^{1/2}\rceil} \left\{ J_m(w,w') \in \{N - \lfloor N^{1/2} \rfloor + 1, \dots, N\}, K_m(w,w') \neq \dagger \right\} \right),$$

and more generally, for any $n = 1, 2, \dots$,

$$\mathbb{P}\left(\bigcup_{m=1}^{n} E_m(w,w'), \bigcap_{m=1}^{2n\lceil N^{1/2}\rceil} \left\{ J_m(w,w') \in \{N - \lfloor N^{1/2} \rfloor + 1, \dots, N\}, K_m(w,w') \neq \dagger \right\} \right) \geq$$

$$\left(1 - \left(1 - \zeta^{2\lceil N^{1/2}\rceil}\right)^n\right) \mathbb{P}\left(\bigcap_{m=1}^{2n\lceil N^{1/2}\rceil} \left\{ J_m(w,w') \in \{N - \lfloor N^{1/2} \rfloor + 1, \dots, N\}, K_m(w,w') \neq \dagger \right\} \right).$$

To conclude the proof, note that for any lists $u, v \in \mathcal{W}_N$, we have $\max\{u(a_N^u), v(a_N^v)\} \geq N - 1$ and $u\left(a_{N-\lfloor N^{1/2}\rfloor}^u\right) = v\left(a_{N-\lfloor N^{1/2}\rfloor}^v\right) = N - \lfloor N^{1/2} \rfloor - 1$. This implies that

$$\mathbb{P}\left(\bigcap_{m=1}^{2n\lceil N^{1/2}\rceil} \left\{ J_m(w,w') \in \{N - \lfloor N^{1/2} \rfloor + 1, \dots, N\}, K_m(w,w') \neq \dagger \right\} \right) \geq$$

$$\left(\frac{e^{\lfloor N^{1/2}\rfloor}}{e^{\lfloor N^{1/2}\rfloor} + 2(N-1)} \right)^{2n\lceil N^{1/2}\rceil}.$$

$\square$

Lemma 3.9. *The following holds*

$$\inf_{w,w' \in \mathcal{W}_N} \mathbb{P}\left(\mathcal{N}_C^N(w,w') < e^{N^{1/2}} N^{-2} < \mathcal{N}_\dagger^N(w,w') \right) \longrightarrow 1, \ \textit{as } N \longrightarrow +\infty.$$

Proof. For any $w, w' \in \mathcal{W}_N$, taking $n = \lfloor e^{N^{1/2}} N^{-2} \rfloor / (2\lceil N^{1/2} \rceil)$ in Lemma 3.8, we have that

$$\mathbb{P}\left(\mathcal{N}_C^N(w, w') < \lfloor e^{N^{1/2}} N^{-2} \rfloor < \mathcal{N}_\dagger^N(w, w')\right) \geq \tag{3.4}$$

$$\left(1 - \left(1 - \zeta^{2\lceil N^{1/2} \rceil}\right)^{\lfloor e^{N^{1/2}} N^{-2} \rfloor / (2\lceil N^{1/2} \rceil)}\right) \left(\frac{e^{\lfloor N^{1/2} \rfloor}}{e^{\lfloor N^{1/2} \rfloor} + 2(N-1)}\right)^{\lfloor e^{N^{1/2}} N^{-2} \rfloor} \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

To finish the proof, just note that the bound of Equation 3.4 does not depend on the initial lists $w, w' \in \mathcal{W}_N$ and take $n = \lceil e^{N^{1/2}} N^{-2} \rceil / (2\lceil N^{1/2} \rceil)$ in Lemma 3.8. □

Corollary 3.10. *The following holds*

$$\sup_{w, w' \in \mathcal{W}_N} \mathbb{P}(T_{\mathcal{N}_C^N(w,w')}(w, w') > e^{-(N - N^{1/2})}) \longrightarrow 0, \ \textit{as } N \longrightarrow +\infty.$$

Proof. For any $w, w' \in \mathcal{W}_N$ and for any $t > 0$,

$$\mathbb{P}(T_{\mathcal{N}_C^N(w,w')}(w, w') > t) \leq$$

$$\mathbb{P}(T_{\mathcal{N}_C^N(w,w')}(w, w') > t, \mathcal{N}_C^N(w, w') < e^{N^{1/2}} N^{-2} < \mathcal{N}_\dagger^N(w, w')) +$$
$$\mathbb{P}(\{\mathcal{N}_C^N(w, w') < e^{N^{1/2}} N^{-2} < \mathcal{N}_\dagger^N(w, w')\}^c).$$

Lemma 3.9 implies that

$$\sup_{w, w' \in \mathcal{W}_N} \mathbb{P}(\{\mathcal{N}_C^N(w, w') < e^{N^{1/2}} N^{-2} < \mathcal{N}_\dagger^N(w, w')\}^c) \longrightarrow 0, \text{ as } N \longrightarrow +\infty.$$

Moreover,

$$\mathbb{P}(T_{\mathcal{N}_C^N(w,w')}(w, w') > t, \mathcal{N}_C^N(w, w') < e^{N^{1/2}} N^{-2} < \mathcal{N}_\dagger^N(w, w')) \leq$$

$$\mathbb{P}(T_{\lfloor e^{N^{1/2}} N^{-2} \rfloor}(w, w') > t, \mathcal{N}_C^N(w, w') < e^{N^{1/2}} N^{-2} < \mathcal{N}_\dagger^N(w, w')). \tag{3.5}$$

For any initial lists w, w' and for any $s > 0$, if the event $\mathcal{N}_\dagger^N(w, w') > e^{N^{1/2}} N^{-2}$ occurs, then

$$\mathbb{P}(T_j(w, w') - T_{j-1}(w, w') > s) \leq \mathbb{P}(\xi^{e^{(N-1)}} > s), \text{ for any } j = 1, \ldots, \lfloor e^{N^{1/2}} N^{-2} \rfloor.$$

Therefore, taking $t = e^{-(N - N^{1/2})}$ the right-hand side of Equation (3.5) is bounded above by

$$\mathbb{P}\left(\sum_{j=1}^{\lfloor e^{N^{1/2}} N^{-2} \rfloor} \xi_j^{\{e^{(N-1)}\}} > e^{-(N - N^{1/2})}\right) \longrightarrow 0, \text{ as } N \longrightarrow +\infty. \tag{3.6}$$

We conclude the proof by putting Equations (3.5) and (3.6) together and noting that the bound on (3.6) does not depend on the lists $w, w' \in \mathcal{W}_N$.

□

Remark 3.11. *By Remark 3.6 and Equation (3.2), we can replace $\mathcal{N}_C^N(w,w')$ by*

$$\inf\{s>0\,:\,\{U_s^{N,w}\in\mathcal{L}_N\}\cap\{U_s^{N,w'}\in\mathcal{L}_N\}\}$$

in Lemmas 3.7, 3.8 and 3.9. This implies that

$$\sup_{w\in\mathcal{W}_N}\mathbb{P}(\inf\{s>0\,:\,U_s^{N,w}\in\mathcal{L}_N\}>e^{-(N-N^{1/2})})\longrightarrow 0,\ \text{as } N\longrightarrow+\infty.$$

Proof. We have now all the ingredients to prove Proposition 3.5.

For any $t>0$ and for any $w,w'\in\mathcal{W}_N$,

$$\mathbb{P}(\tau^{N,w}>t)\leq\mathbb{P}(\tau^{N,w}>t,\mathcal{N}_C^N(w,w')<\mathcal{N}_\dagger^N(w,w'))+\mathbb{P}(\mathcal{N}_C^N(w,w')>\mathcal{N}_\dagger^N(w,w'))$$

Now, note that

$$\mathbb{P}(\tau^{N,w}>t,\mathcal{N}_C^N(w,w')<\mathcal{N}_\dagger^N(w,w'))=\mathbb{P}(\tau^{N,w'}>t,\mathcal{N}_C^N(w,w')<\mathcal{N}_\dagger^N(w,w')).$$

This implies that

$$\mathbb{P}(\tau^{N,w}>t)-\mathbb{P}(\tau^{N,w'}>t)\leq\mathbb{P}(\mathcal{N}_C^N(w,w')>\mathcal{N}_\dagger^N(w,w')).$$

Analogously,

$$\mathbb{P}(\tau^{N,w'}>t)-\mathbb{P}(\tau^{N,w}>t)\leq\mathbb{P}(\mathcal{N}_C^N(w,w')>\mathcal{N}_\dagger^N(w,w')),$$

and therefore,

$$|\mathbb{P}(\tau^{N,w}>t)-\mathbb{P}(\tau^{N,w'}>t)|\leq\mathbb{P}(\mathcal{N}_C^N(w,w')>\mathcal{N}_\dagger^N(w,w')).$$

By Lemma 3.9, we conclude that

$$\sup_{t\geq 0}\sup_{w,w'\in\mathcal{W}_N}|\mathbb{P}(\tau^{N,w}>t)-\mathbb{P}(\tau^{N,w'}>t)|\leq\sup_{w,w'\in\mathcal{W}_N}\mathbb{P}(\mathcal{N}_C^N(w,w')<\mathcal{N}_\dagger^N(w,w'))\longrightarrow 0,$$

as $N\longrightarrow+\infty$, and with this we concluded the proof. □

3.5 Proof of Theorem 3.2

To prove Theorem 3.2 we need to introduce the auxiliary process $(\tilde{U}_t^{N,u})_{t\in[0,+\infty)}$ that evolves as a Markov jump process taking values in the set $\tilde{\mathcal{S}}_N=\mathcal{S}_N\setminus\{\vec{0}_N\}$ with initial list $u\in\tilde{\mathcal{S}}_N$ and with infinitesimal generator $\tilde{\mathcal{G}}$ defined as follows

$$\tilde{\mathcal{G}}f(u)=\sum_{b\in\mathcal{A}_N}e^{u(b)}\mathbf{1}\{u(b)>0\}\left[f(\pi^{b,*}(u))-f(u)\right]+\sum_{b\in\mathcal{A}_N}\mathbf{1}\{\pi^{b,\dagger}(u)\neq\vec{0}_N\}\left[f(\pi^{b,\dagger}(u))-f(u)\right],$$

for any bounded function $f\,:\,\tilde{\mathcal{S}}_N\longrightarrow\mathbb{R}$.

Remark 3.12. *In general, the processes $(\tilde{U}_t^{N,u})_{t\in[0,+\infty)}$ and $(U_t^{N,u})_{t\in[0,+\infty)}$ have the same jump* *rates. The only exception is that* $(\tilde{U}_t^{N,u})_{t\in[0,+\infty)}$ *can not jump from a list in which only one neuron has non-null membrane potential to the null list.*

This implies that the processes $(U_t^{N,u})_{t\in[0,+\infty)}$ *and* $(\tilde{U}_t^{N,u})_{t\in[0,+\infty)}$ *can be coupled in such way that*

$$\tilde{U}_s^{N,u} = U_s^{N,u}, \textit{for all } s \in \left[0, \tau^{N,u}\right).$$

For the auxiliary process $(\tilde{U}_t^{N,u})_{t\in[0,+\infty)}$, let us introduce some extra notation.

Extra notation - auxiliary process

- Denote $\tilde{T}_0 = 0$ and for $n = 1, 2, ...$, denote $\tilde{T}_n$ the successive jumping times of the process $(\tilde{U}_t^{N,u})_{t\in[0,+\infty)}$, namely
$$\tilde{T}_n = \inf\left\{ t > \tilde{T}_{n-1} \ : \ \tilde{U}_t^{N,u} \neq \tilde{U}_{\tilde{T}_{n-1}}^{N,u} \right\}.$$
- For $n = 1, 2, ...$, we define $\tilde{A}_n \in \mathcal{A}_N$ and $\tilde{O}_n \in \{*, \dagger\}$ as the pair such that
$$\tilde{U}_{\tilde{T}_n}^{N,u} = \pi^{\tilde{A}_n,\tilde{O}_n}\left(\tilde{U}_{\tilde{T}_{n-1}}^{N,u}\right).$$
- The leakage times are defined as $\tilde{T}_0^\dagger = 0$ and for $n \geq 1$,
$$\tilde{T}_n^\dagger = \inf\{\tilde{T}_m > \tilde{T}_{n-1}^\dagger \ : \ \tilde{O}_m = \dagger\}.$$
- The spiking times are defined as $\tilde{T}_0^* = 0$ and for $n \geq 1$,
$$\tilde{T}_n^* = \inf\{\tilde{T}_m > \tilde{T}_{n-1}^* \ : \ \tilde{O}_m = *\}.$$
- For any time interval $I \subset [0, +\infty)$, the counting measures indicating the number of leakage times and spiking times that occurred during the time interval I are defined as
$$\tilde{Z}^\dagger(I) = \sum_{j=1}^{+\infty} \mathbf{1}\{\tilde{T}_j^\dagger \in I\} \quad \text{and} \quad \tilde{Z}^*(I) = \sum_{j=1}^{+\infty} \mathbf{1}\{\tilde{T}_j^* \in I\}.$$
- In the next proposition, we prove that $(\tilde{U}_t^{N,u})_{t\in[0,+\infty)}$ has an unique invariant probability measure. We use the symbol μ^N to denote this probability measure.

Proposition 3.13. *The process* $(\tilde{U}_t^{N,u})_{t\in[0,+\infty)}$ *is ergodic.*

Proof. Let $l \in \mathcal{L}_N$ satisfies $l(a) = a - 1$, for all $a \in \mathcal{A}_N$. For any $u \in \tilde{\mathcal{S}}_N$, we have that

$$l = \pi^{1,*} \circ \pi^{2,*} \circ ... \circ \pi^{N,*}(u),$$

and then, if the event $\bigcap_{j=1}^N \{\tilde{A}_j = N - j + 1, \tilde{O}_j = *\}$ occurs, then $\tilde{U}_N^{N,u} = l$.

Let

$$\tilde{\mathcal{N}}^{N,u} = \inf\{n \geq 1 \ : \ \tilde{U}_{\tilde{T}_n}^{N,u} = l\}.$$

As in Theorem 3.1, for any $u' \in \tilde{\mathcal{S}}_N$, we have that

$$\mathbb{P}\Big(\tilde{\mathcal{N}}^{N,u} \leq n + 2N - 1 \,\Big|\, \tilde{U}^{N,u}_{\tilde{T}_n} = u'\Big) \geq$$
$$\mathbb{P}\left(\tilde{U}^{N,u}_{\tilde{T}_{n+N-1}} \in \mathcal{L}_N \Big| \, \tilde{U}^{N,u}_{\tilde{T}_n} = u'\right) \mathbb{P}\left(\tilde{\mathcal{N}}^{N,u} \leq n + 2N - 1 \,\Big|\, \tilde{U}^{N,u}_{\tilde{T}_{n+N-1}} \in \mathcal{L}_N\right).$$

Moreover, the right-hand side of the equation above is bounded above by $[2(N-1)]^{-(N-1)}\tilde{\epsilon}$, where

$$\tilde{\epsilon} = \min\left\{\mathbb{P}\left(\bigcap_{j=1}^{N}\{\tilde{A}_{n+j} = N - j + 1, \tilde{O}_{n+j} = *\} \,\Big|\, \tilde{U}^{N,u}_{T_n} = v\right) \,:\, v \in \mathcal{L}_N\right\}.$$

We conclude that for any $n \geq 1$,

$$\mathbb{P}(\tilde{\mathcal{N}}^{N,l} \geq n) \leq \mathbb{P}\Big((2N-1) \times \mathrm{Geom}\Big([2(N-1)]^{-(N-1)}\tilde{\epsilon}\Big) \geq n\Big),$$

where $\mathrm{Geom}(r)$ denotes a random variable with geometric distribution assuming values in $\{1, 2, \ldots\}$ and with mean $1/r$. This implies that $\mathbb{E}(\tilde{\mathcal{N}}^{N,l}) < +\infty$ and then, $(\tilde{U}^{N,u}_{\tilde{T}_n})_{n\geq 0}$ is a positive-recurrent Markov chain. The jump rate of the process $(\tilde{U}^{N,u}_t)_{t\in[0,+\infty)}$ satisfies

$$\sum_{a\in\mathcal{A}_N} \mathbf{1}\{u'(a) > 0\}\left(e^{u'(a)} + \mathbf{1}\{\pi^{a,\dagger}(u') \neq \vec{0}_N\}\right) \geq e,$$

for any $u' \in \tilde{\mathcal{S}}_N$. Putting all this together we conclude that $(\tilde{U}^{N,u}_t)_{t\in[0,+\infty)}$ is ergodic. □

The proof of Theorem 3.2 is based on two lemmas.

Lemma 3.14. *The invariant probability measure μ^N of the auxiliary process satisfies*

$$\mu^N(\mathcal{W}_N) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

Proof. To prove Lemma 3.14, we will first show that there exists sets $S_N^{(1)}, S_N^{(2)}$ and $S_N^{(3)}$ such that

$$S_N^{(1)} \supset S_N^{(2)} \supset S_N^{(3)} \supset \mathcal{W}_N$$

and for any $j \in \{1, 2, 3\}$,

$$\mu^N(S_N^{(j)}) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

Let

$$S_N^{(1)} = \{u \in \tilde{\mathcal{S}}_N \,:\, |\{a \in \mathcal{A}_N \,:\, u(a) = 0\}| \leq N^{\frac{1}{2}}\}$$

and consider the following events

$$E^{(1)}_{N,1} = \{\tilde{Z}^*([0, N^{\frac{1}{2}}]) \geq 1\},$$

$$E^{(1)}_{N,2} = \{\tilde{Z}^{\dagger}([0, N^{\frac{1}{2}}]) \leq N^2\},$$

$$E^{(1)}_{N,3} = \bigcap_{j=1}^{N^2/\lfloor \frac{N^{1/2}}{2} \rfloor} \left\{ \tilde{Z}^* \left(\left[\tilde{T}^{\dagger}_{(j-1)\lfloor \frac{N^{1/2}}{2} \rfloor + 1}, \tilde{T}^{\dagger}_{j\lfloor \frac{N^{1/2}}{2} \rfloor} \right] \right) \geq 1 \right\}.$$

For any $u \in \tilde{S}_N$, the rate in which the process has a leakage is bounded above by N. Moreover, the rate in which the process has a spike is bounded bellow by e. This implies that

$$\mathbb{P}(E^{(1)}_{N,1}) \geq \mathbb{P}\left(\xi^{\{e\}} \leq N^{\frac{1}{2}} \right) \longrightarrow 1, \text{ as } N \longrightarrow +\infty, \tag{3.7}$$

$$\mathbb{P}(E^{(1)}_{N,2}) \geq \mathbb{P}\left(\sum_{j=1}^{N^2} \xi^{\{N\}}_j \geq N^{\frac{1}{2}} \right) \longrightarrow 1, \text{ as } N \longrightarrow +\infty. \tag{3.8}$$

For any list $u \in \tilde{S}_N$ and for any instant $n \geq 1$, we have that

$$\mathbb{P}\left(\tilde{O}_n =* \left| \tilde{U}^{N,u'}_{\tilde{T}_{n-1}} = u \right. \right) \geq \frac{1}{2}.$$

This implies that for any initial list $u \in \tilde{S}_N$ and for any $j = 1, \dots, N^2/\lfloor \frac{N^{1/2}}{2} \rfloor$,

$$\mathbb{P}\left(\tilde{Z}^* \left(\left[\tilde{T}^{\dagger}_{(j-1)\lfloor \frac{N^{1/2}}{2} \rfloor + 1}, \tilde{T}^{\dagger}_{j\lfloor \frac{N^{1/2}}{2} \rfloor} \right] \right) \geq 1 \right) \geq \mathbb{P}\left(\text{Geom}\left(\frac{1}{2}\right) \leq \left\lfloor \frac{N^{1/2}}{2} \right\rfloor - 1 \right),$$

where $\text{Geom}\left(\frac{1}{2}\right)$ is a random variable with geometric distribution assuming values in $\{1, 2, \dots\}$ and with mean 2. Therefore,

$$\mathbb{P}(E^{(1)}_{N,3}) \geq \left(1 - 2^{-\lfloor \frac{N^{1/2}}{2} \rfloor + 1} \right)^{N^2/\lfloor \frac{N^{1/2}}{2} \rfloor} \longrightarrow 1, \text{ as } N \longrightarrow +\infty. \tag{3.9}$$

If the event $E^{(1)}_{N,1} \cap E^{(1)}_{N,2} \cap E^{(1)}_{N,3}$ occurs, then until time $N^{1/2}$ the process has at least one spiking time and at most $\lfloor N^{1/2} \rfloor - 1$ successive leakage times (with no spiking times in between). This implies that $\tilde{U}^{N,u}_{N^{1/2}} \in S^{(1)}_N$. Since the inequalities of Equations (3.7) , (3.8) and (3.9) holds for any $u \in \tilde{S}_N$ and they do not depend on u, it follows that

$$\sup_{u \in \tilde{S}_N} \mathbb{P}(\tilde{U}^{N,u}_{N^{1/2}} \notin S^{(1)}_N) \longrightarrow 0, \text{ as } N \longrightarrow +\infty,$$

and as a consequence,

$$\mu^N(S^{(1)}_N) = \sum_{u \in \tilde{S}_N} \mu^N(u) \mathbb{P}(\tilde{U}^{N,u}_{N^{1/2}} \in S^{(1)}_N) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

Let

$$S^{(2)}_N = \left\{ u \in \tilde{S}_N \ : \ \text{there exists } a_1, \dots, a_{\lceil N^{1/2} \rceil} \in \mathcal{A}_N, \text{ such that } 1 \leq u(a_1) < \dots < u\left(a_{\lceil N^{1/2} \rceil}\right) \right\}$$

and consider the following events

$$E^{(2)}_{N,1} = \bigcap_{j=1}^{N^2/\lfloor \frac{N^{1/2}}{2} \rfloor} \left\{ \tilde{Z}^* \left(\left[\tilde{T}^{\dagger}_{(j-1)\lfloor \frac{N^{1/2}}{2} \rfloor + 1}, \tilde{T}^{\dagger}_{j\lfloor \frac{N^{1/2}}{2} \rfloor} \right] \right) \geq 1 \right\},$$

$$E^{(2)}_{N,2} = \{\tilde{Z}^{\dagger}([0, N^{-\frac{1}{4}}]) \leq N^2\},$$

$$E^{(2)}_{N,3} = \{\tilde{Z}^{*}([0, N^{-\frac{1}{4}}]) \geq \lceil N^{1/2} \rceil\}.$$

As in Equation (3.8) and (3.9),

$$\mathbb{P}(E^{(2)}_{N,1}) \geq \left(1 - 2^{-\lfloor \frac{N^{1/2}}{2} \rfloor + 1}\right)^{N^2/\lfloor \frac{N^{1/2}}{2} \rfloor} \longrightarrow 1, \text{ as } N \longrightarrow +\infty, \tag{3.10}$$

$$\mathbb{P}(E^{(2)}_{N,2}) \geq \mathbb{P}\left(\sum_{j=1}^{N^2} \xi_j^{\{N\}} \geq N^{-\frac{1}{4}} \right) \longrightarrow 1, \text{ as } N \longrightarrow +\infty. \tag{3.11}$$

For any initial list $u \in S^{(1)}_N$, the occurrence of $E^{(2)}_{N,1} \cap E^{(2)}_{N,2}$ implies that until time $N^{-1/4}$ the rate in which the process has a spike is bounded bellow by $e(N - 2\lceil N^{1/2} \rceil)$. This implies that

$$\mathbb{P}(E^{(2)}_{N,3}) \geq \mathbb{P}\left(\sum_{j=1}^{\lceil N^{1/2} \rceil} \xi_j^{\{e(N - 2\lceil N^{1/2} \rceil)\}} \leq N^{-\frac{1}{4}} \right) \mathbb{P}\left(E^{(2)}_{N,1} \cap E^{(2)}_{N,2} \right) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

For any initial list $u \in S^{(1)}_N$, if the event $E^{(2)}_{N,1} \cap E^{(2)}_{N,2} \cap E^{(2)}_{N,3}$ occurs, then until time $N^{-1/4}$ the process has at least $\lceil N^{1/2} \rceil$ spiking times and at most $\lfloor N^{1/2} \rfloor - 1$ successive leakage times (with no spiking times in between). This implies that $\tilde{U}^{N,u}_{N^{-1/4}} \in S^{(2)}_N$. Therefore,

$$\sup_{u \in S^{(1)}_N} \mathbb{P}(\tilde{U}^{N,u}_{N^{-1/4}} \notin S^{(2)}_N) \longrightarrow 0, \text{ as } N \longrightarrow +\infty,$$

and as a consequence,

$$\mu^N(S^{(2)}_N) = \sum_{u \in \tilde{S}_N} \mu^N(u) \mathbb{P}(\tilde{U}^{N,u}_{N^{-1/4}} \in S^{(2)}_N) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

Let

$$S^{(3)}_N = \{u \in \tilde{S}_N \ : \ u(a^u_j) \geq j - 1, \text{ for all } j = 1, \ldots, N\}$$

and consider the following events

$$E^{(3)}_{N,1} = \{\tilde{Z}^{\dagger}([0, N^{-2}]) = 0\},$$

$$E^{(3)}_{N,2} = \{\tilde{Z}^{*}([0, N^{-2}]) \geq N\}.$$

For any $u \in S^{(2)}_N$,

$$\mathbb{P}(E^{(3)}_{N,1}) \geq \mathbb{P}\left(\xi^{\{N\}} \geq N^{-2}\right) \longrightarrow 1, \text{ as } N \longrightarrow +\infty,$$

For any initial list $u \in S_N^{(2)}$, the occurrence of the event $E_{N,1}^{(3)}$ implies that until time N^{-2} the rate in which the process has a spike is bounded bellow by $e^{\lfloor N^{1/2} \rfloor}$. This implies that

$$\mathbb{P}(E_{N,2}^{(3)}) \geq \mathbb{P}\left(\sum_{j=1}^{N} \xi_j^{\{e^{\lfloor N^{1/2} \rfloor}\}} \leq N^{-2} \right) \mathbb{P}\left(E_{N,1}^{(3)} \right) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

For any initial list $u \in S_N^{(2)}$, if the event $E_{N,1}^{(3)} \cap E_{N,2}^{(3)}$ occurs, then until time N^{-2} the process has at least N spiking times and does not have any leakage. This implies that $\tilde{U}_{N^{-2}}^{N,u} \in S_N^{(3)}$. Therefore,

$$\sup_{u \in S_N^{(2)}} \mathbb{P}(\tilde{U}_{N^{-2}}^{N,u} \notin S_N^{(3)}) \longrightarrow 0, \text{ as } N \longrightarrow +\infty,$$

and as a consequence,

$$\mu^N(S_N^{(3)}) = \sum_{u \in \tilde{S}_N} \mu^N(u) \mathbb{P}(\tilde{U}_{N^{-2}}^{N,u} \in S_N^{(3)}) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

Recall that

$$\mathcal{W}_N = \left\{ u \in \mathcal{S}_N \,:\, \left\{1, \dots, N - \lfloor N^{1/2} \rfloor \right\} \subset \{u(a) \,:\, a \in \mathcal{A}_N\}; \bigcap_{a \in \mathcal{A}_N} \bigcap_{b \neq a} \{u(a) \neq u(b)\} \right\}$$

and consider the following events

$$E_{N,1}^{(4)} = \{\tilde{Z}^{\dagger}([0, e^{-(N-N^{1/4})}]) = 0\},$$

$$E_{N,2}^{(4)} = \left\{ \sum_{j=1}^{+\infty} \mathbf{1}\{\tilde{T}_j \leq e^{-(N-N^{1/4})}, \tilde{U}_{\tilde{T}_{j-1}}^{N,u}(\tilde{A}_j) \leq N - \lfloor N^{1/2} \rfloor, O_j = *\} = 0 \right\},$$

$$E_{N,3}^{(4)} = \{\tilde{Z}^{*}([0, e^{-(N-N^{1/4})}]) \geq N + \lceil N^{1/2} \rceil\},$$

$$E_{N,4}^{(4)} = \bigcap_{j=1}^{N + \lceil N^{1/2} \rceil} \left\{ \tilde{U}_{\tilde{T}_{j-1}}^{N,u}(\tilde{A}_j) \geq \max\left\{ \tilde{U}_{\tilde{T}_{j-1}}^{N,u}(a) \,:\, a \in \mathcal{A}_N \right\} - \lfloor N^{1/2} \rfloor \right\}.$$

For any $u \in \tilde{S}_N$,

$$\mathbb{P}(E_{N,1}^{(4)}) \geq \mathbb{P}\left(\xi^{\{N\}} \geq e^{-(N-N^{1/4})} \right) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

The rate in which the process has a spike of a neuron that in the moment of the spike have membrane potential smaller or equal $N - \lfloor N^{1/2} \rfloor$ is bounded above by $N e^{N - \lfloor N^{1/2} \rfloor}$. Therefore,

$$\mathbb{P}(E_{N,2}^{(4)}) \geq \mathbb{P}\left(\xi^{\{N e^{N - \lfloor N^{1/2} \rfloor}\}} > e^{-(N-N^{1/4})} \right) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

For any $u \in S_N^{(3)}$, the occurrence of the event $E_{N,1}^{(4)}$ implies that the rate in which the process has a spike is bounded bellow by $e^{(N-1)}$. Therefore,

$$\mathbb{P}(E^{(4)}_{N,3}) \geq \mathbb{P}\left(\sum_{j=1}^{N+\lceil N^{1/2}\rceil} \xi_j^{\{e^{(N-1)}\}} \leq e^{-(N-N^{1/4})}\right)\mathbb{P}\left(E^{(4)}_{N,1}\right) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

Moreover, the probability

$$\mathbb{P}\left(\tilde{U}^{N,u}_{\tilde{T}_0}(\tilde{A}_1) \geq \max\{u(a) : a \in \mathcal{A}_N\} - \lfloor N^{1/2}\rfloor\right) \tag{3.12}$$

is minimized when the difference between the membrane potential of the neuron with greatest potential and the membrane potential of the other neurons is $\lfloor N^{1/2}\rfloor + 1$. This implies that (3.12) is bounded bellow by

$$\frac{e^{\lfloor N^{1/2}\rfloor}}{e^{\lfloor N^{1/2}\rfloor} + 2(N-1)},$$

and therefore,

$$\mathbb{P}(E^{(4)}_{N,4}) \geq \left(\frac{e^{\lfloor N^{1/2}\rfloor}}{e^{\lfloor N^{1/2}\rfloor} + 2(N-1)}\right)^{N+\lceil N^{1/2}\rceil}\mathbb{P}\left(E^{(4)}_{N,1} \cap E^{(4)}_{N,3}\right) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

For any initial list $u \in S^{(3)}_N$, if the event $E^{(4)}_{N,1} \cap \ldots \cap E^{(4)}_{N,4}$ occurs, then until time $e^{-(N-N^{1/4})}$ the process has at least $N + \lceil N^{1/2}\rceil$ spiking times, does not have any leakage of membrane potential and does not have any spike of a neuron with membrane potential smaller or equal $N - \lfloor N^{1/2}\rfloor$. This implies that

$$\left\{1, \ldots, N - \lfloor N^{1/2}\rfloor\right\} \subset \left\{\tilde{U}^{N,u}_{e^{-(N-N^{1/4})}}(a) : a \in \mathcal{A}_N\right\}.$$

Moreover, the occurrence of the events $E^{(4)}_{N,1} \cap E^{(4)}_{N,3} \cap E^{(4)}_{N,4}$ implies that all neurons spikes at least once in the first $N + \lceil N^{1/2}\rceil$ steps of the process. This implies that $\tilde{U}^{N,u}_{e^{-(N-N^{1/4})}}(a) \neq \tilde{U}^{N,u}_{e^{-(N-N^{1/4})}}(a')$, for all $a \neq a'$. We conclude that $\tilde{U}^{N,u}_{e^{-(N-N^{1/4})}} \in \mathcal{W}_N$.

Therefore,

$$\sup_{u \in S^{(3)}_N} \mathbb{P}\left(\tilde{U}^{N,u}_{e^{-(N-N^{1/4})}} \notin \mathcal{W}_N\right) \longrightarrow 0, \text{ as } N \longrightarrow +\infty,$$

and as a consequence,

$$\mu^N(\mathcal{W}_N) = \sum_{u \in \tilde{S}_N} \mu^N(u)\mathbb{P}\left(\tilde{U}^{N,u}_{e^{-(N-N^{1/4})}} \in \mathcal{W}_N\right) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

□

From Lemma 3.14 it follows Corollaries 3.15 and 3.16, that are used to prove Theorems 3.2 and 3.3.

Corollary 3.15. *The following holds*

$$\inf_{u \in \tilde{S}_N} \mathbb{P}\left(\inf\left\{ \tilde{T}_n \,:\, \tilde{U}^{N,u}_{\tilde{T}_n} \in \mathcal{L}_N \right\} \leq t(N) + e^{-(N-N^{1/2})}\right) \longrightarrow 1, \; \textit{as } N \longrightarrow +\infty,$$

where $t(N) = N^{1/2} + N^{-1/4} + N^{-2} + e^{-(N-N^{1/4})}$.

Proof. First, note that

$$\inf_{u \in \tilde{S}_N} \mathbb{P}\left(\inf\left\{ \tilde{T}_n \,:\, \tilde{U}^{N,u}_{\tilde{T}_n} \in \mathcal{W}_N \right\} \leq t(N)\right) \geq$$

$$\inf_{u \in \tilde{S}_N} \mathbb{P}\left(\tilde{U}^{N,u}_{N^{1/2}} \in S^{(1)}_N, \; \tilde{U}^{N,u}_{N^{1/2}+N^{-1/4}} \in S^{(2)}_N, \; \tilde{U}^{N,u}_{N^{1/2}+N^{-1/4}+N^{-2}} \in S^{(3)}_N, \; \tilde{U}^{N,u}_{t(N)} \in \mathcal{W}_N \right).$$

Remark 3.11 implies that for any $w \in \mathcal{W}_N$,

$$\inf_{w \in \mathcal{W}_N} \mathbb{P}\left(\inf\left\{ \tilde{T}_n \,:\, \tilde{U}^{N,w}_{\tilde{T}_n} \in \mathcal{L}_N \right\} \leq e^{-(N-N^{1/2})}\right) \longrightarrow 1, \; \text{as } N \longrightarrow +\infty.$$

We conclude the proof by putting all this together with Lemma 3.14 and Markov property. □

Corollary 3.16. *The following holds*

$$\inf_{u \in S^{(0)}_N} \mathbb{P}\left(\inf\left\{ T_n \,:\, U^{N,u}_{T_n} \in \mathcal{L}_N \right\} \leq t'(N)\right) \longrightarrow 1, \; \textit{as } N \longrightarrow +\infty,$$

where $t'(N) = N^{-1/4} + N^{-2} + e^{-(N-N^{1/4})} + e^{-(N-N^{1/2})}$.

Proof. Note that starting from any list $u \in S^{(0)}_N$, as in the proof of Lemma 3.14 we have that

$$\mathbb{P}\left(E^{(2)}_{N,1} \cap E^{(2)}_{N,2} \cap E^{(2)}_{N,3}\right) \longrightarrow 1, \; \text{as } N \longrightarrow +\infty.$$

Then, as in Corollary 3.15 we have that

$$\inf_{u \in S^{(0)}_N} \mathbb{P}\left(\inf\left\{ \tilde{T}_n \,:\, \tilde{U}^{N,u}_{\tilde{T}_n} \in \mathcal{L}_N \right\} \leq t'(N)\right) \longrightarrow 1, \; \text{as } N \longrightarrow +\infty. \tag{3.13}$$

By the definition of the events $E^{(2)}_{N,1}, E^{(3)}_{N,1}$ and $E^{(4)}_{N,1}$, we have that

$$\inf_{u \in S^{(0)}_N} \mathbb{P}\left(\inf\left\{ \tilde{T}_n \,:\, \left| a \in \mathcal{A}_N \,:\, \tilde{U}^{N,u}_{\tilde{T}_n}(a) > 0 \right| = 1 \right\} > t'(N)\right) \longrightarrow 1, \; \text{as } N \longrightarrow +\infty.$$

By Remark 3.12 and the coupling construction, it follows that

$$\inf_{u \in S^{(0)}_N} \mathbb{P}\left(U^{N,u}_t = \tilde{U}^{N,u}_t, \; \text{for all } t \in [0, t'(N)]\right) \longrightarrow 1, \; \text{as } N \longrightarrow +\infty.$$

Therefore, we can replace $\tilde{U}$ by U and $\tilde{T}_n$ by T_n on Equation (3.13) and with this we concluded the proof.

□

Lemma 3.17. *For any $N \geq 2$, for any list $l \in \mathcal{L}_N$ and for any $s > 0$,*

$$\mathbb{P}(\tilde{U}_s^{N,l} \in \mathcal{W}_N^c) \leq \frac{\mu^N(\mathcal{W}_N^c)}{\mu^N(\mathcal{W}_N)} + \delta(N,s),$$

where $\lim_{N\to+\infty}\delta(N,s) = 0$, *for any* $s > 0$.

Proof. For any $s > 0$,

$$\mu^N(\mathcal{W}_N) = \sum_{u\in\mathcal{W}_N} \mu^N(u)\mathbb{P}(\tilde{U}_s^{N,u} \in \mathcal{W}_N) + \sum_{u\in\tilde{\mathcal{S}}_N\setminus\mathcal{W}_N} \mu^N(u)\mathbb{P}(\tilde{U}_s^{N,u} \in \mathcal{W}_N).$$

By Remark 3.12, for any $l \in \mathcal{L}_N$ and $w \in \mathcal{W}_N$ we have

$$\mathbb{P}(\tilde{U}_s^{N,w} \in \mathcal{W}_N) \leq \mathbb{P}(\tilde{U}_s^{N,l} \in \mathcal{W}_N) + \mathbb{P}\left(\{T_{\mathcal{N}_C^N(l,w)}(l,w) > s\} \cup \{\mathcal{N}_C^N(l,w) > \mathcal{N}_\dagger^N(l,w)\}\right).$$

Considering

$$\delta(N,s) = \sup_{l\in\mathcal{L}_N}\sup_{w\in\mathcal{W}_N} \mathbb{P}\left(\{T_{\mathcal{N}_C^N(l,w)}(l,w) > s\} \cup \{\mathcal{N}_C^N(l,w) > \mathcal{N}_\dagger^N(l,w)\}\right),$$

by Lemma 3.9 and Corollary 3.10 it follows that $\lim\limits_{N\to+\infty}\delta(N,s) = 0$, for any $s > 0$. Moreover,

$$\sum_{u\in\tilde{\mathcal{S}}_N\setminus\mathcal{W}_N} \mu^N(u)\mathbb{P}(\tilde{U}_s^{N,u} \in \mathcal{L}_N) \leq 1 - \mu^N(\mathcal{W}_N).$$

This implies that

$$\mu^N(\mathcal{W}_N) \leq \mu^N(\mathcal{W}_N)(\mathbb{P}(\tilde{U}_s^{N,l} \in \mathcal{W}_N) + \delta(N,s)) + (1 - \mu^N(\mathcal{W}_N)),$$

and therefore,

$$\mathbb{P}(U_s^{N,l} \in \mathcal{W}_N) \geq \frac{\mu^N(\mathcal{W}_N) - (1 - \mu^N(\mathcal{W}_N))}{\mu^N(\mathcal{W}_N)} - \delta(N,s).$$

With this we concluded the proof of Lemma 3.17.

□

Proof. Now we will prove Theorem 3.2.

By Remark 3.12 and the invariance by permutation of the process it follows that for any $u \in S_N^{(0)}$, for any $l \in \mathcal{L}_N$ and for any $t > 0$,

$$\mathbb{P}\left(U_t^{N,u} \in \mathcal{S}_N \setminus \mathcal{W}_N \,\middle|\, \tau^{N,u} > t\right) \leq \mathbb{P}(\inf\{t > 0 : U_t^{N,u} \in \mathcal{L}_N\} > t/2) + \sup_{s\in[t/2,t]} \mathbb{P}\left(\tilde{U}_s^{N,l} \in \tilde{\mathcal{S}}_N \setminus \mathcal{W}_N\right).$$

By Corollary 3.16,

$$\sup_{u \in S_N^{(0)}} \mathbb{P}(\inf\{t > 0 \,:\, U_t^{N,u} \in \mathcal{L}_N\} > t/2) \longrightarrow 0, \text{ as } N \longrightarrow +\infty.$$

By Lemma 3.17,

$$\sup_{s \in [t/2,t]} \mathbb{P}\left(\tilde{U}_s^{N,l} \in \tilde{S}_N \setminus \mathcal{W}_N\right) \leq \frac{\mu^N(\mathcal{W}_N^c)}{\mu^N(\mathcal{W}_N)} + \delta(N, t/2).$$

By Lemmas 3.14 and 3.17 it follows that

$$\lim_{N \longrightarrow +\infty} \delta(N, t/2) = \lim_{N \longrightarrow +\infty} \mu^N(\mathcal{W}_N^c) = 0$$

and with this we concluded the proof.

□

Remark 3.18. *For any $N \geq 2$, $\mathbb{P}(\inf\{t > 0 \,:\, U_t^{N,u} \in \mathcal{L}_N\} > t/2)$ and $\delta(N, t/2)$ decreases with t. This implies that for any $(t_N \,:\, N \geq 2)$ such that $\lim_{N \longrightarrow +\infty} t_N = +\infty$, we have*

$$\inf_{u \in S_N^{(0)}} \mathbb{P}\left(U_{t_N}^{N,u} \in \mathcal{W}_N \mid \tau^{N,u} > t_N\right) \longrightarrow 1, \text{ as } N \longrightarrow +\infty.$$

3.6 Proof of Theorem 3.3

For any fixed $l \in \mathcal{L}_N$, let $c_{N,l}$ be the positive real number such that

$$\mathbb{P}(\tau^{N,l} > c_{N,l}) = e^{-1}. \tag{3.14}$$

Due to the invariance by permutation of the process, it is clear that $c_{N,l} = c_{N,l'}$, for any pair of lists l and l' belonging to $\mathcal{L}_N$. Therefore, in what follows we will omit to indicate l in the notation of c_N.

To prove Theorem 3.3, we first prove the following proposition that gives a bound for c_N.

Proposition 3.19. *For any $N \geq 3$,*

$$c_N \geq \frac{N - 1 + e^{(N-2)}}{(N-1)^3}.$$

Proof. For a initial list $l \in \mathcal{L}_N$, let

$$\tau_-^N = \inf\left\{ T_n \,:\, O_n = \dagger, \bigcup_{j=1}^{N-1} \{O_{n-j} = \dagger\} \right\}.$$

We have

$$\tau_-^N = \sum_{j=1}^{G}(T_j^\dagger - T_{j-1}^\dagger),$$

where $G = \inf\{j \,:\, Z^*([T_{j-1}^\dagger, T_j^\dagger]) \leq N-2\}$.

The rate in which the process has a leakage is bounded above by $N-1$. Therefore, for any $j \geq 1$ and for any $s > 0$,

$$\mathbb{P}(T_j^\dagger - T_{j-1}^\dagger > s) \geq \mathbb{P}(\xi^{\{N-1\}} > s).$$

Recall that

$$S_N^{(3)} = \{u \in \mathcal{S}_N \,:\, u(a_j^u) \geq j-1, \text{ for any } j = 1, \dots, N\}.$$

For any initial list $w \in \mathcal{W}_N$, we have that

$$U_t^{N,w} \in S_N^{(3)}, \text{ for any } t < T_1^\dagger.$$

Moreover, for any initial list $u \in \mathcal{S}_N \setminus \{\vec{0}_N\}$, if $O_1 = \dots = O_{N-1} = *$, then $U_{T_{N-1}}^{N,u} \in S_N^{(3)}$. Together with Markov property, this implies that for any $m \geq 1$ and for any $j \geq 1$,

$$\mathbb{P}(Z^*([T_{j-1}^\dagger, T_j^\dagger]) \leq N-2 \mid T_{j-1}^\dagger = T_m, G \geq j) = \mathbb{P}\left(\bigcup_{j=1}^{N-1}\{O_{j+m} = \dagger\} \,\middle|\, O_m = \dagger, U_{T_{m-1}}^{N,u} \in S_N^{(3)}\right).$$

The probability on the right-hand side of equation above is bounded above by

$$\lambda_N = (N-1) \times \frac{N-1}{N-1+e^{(N-2)}}.$$

Therefore, for any $s > 0$,

$$\mathbb{P}(\tau_-^N > s) \geq \mathbb{P}\left(\sum_{j=1}^{\mathrm{Geom}(\lambda_N)} \xi_j^{\{N-1\}} > t\right),$$

where $\mathrm{Geom}(\lambda_N)$ is a random variable independent of $(\xi_j^{\{N-1\}})_{j\geq 1}$ with Geometric distribution assuming values in $\{1, 2, \dots\}$ with mean $1/\lambda_N$. This implies that

$$\mathbb{P}(\tau_-^N > s) \geq \mathbb{P}\left(\xi^{\{\lambda_N(N-1)\}} > s\right).$$

Therefore,

$$e^{-1} = \mathbb{P}(\tau^{N,l} > c_N) \geq \mathbb{P}(\tau_-^N > c_N) \geq e^{-c_N \lambda_N (N-1)},$$

and then,

$$c_N \geq \frac{1}{\lambda_N(N-1)}.$$

□

To prove Theorem 3.3, we prove Proposition 3.20 which is interesting by itself.

Proposition 3.20. *For any sequence* $(l_N \in \mathcal{L}_N \, : \, N \geq 2)$,

$$\frac{\tau^{N,l_N}}{c_N} \longrightarrow Exp(1), \ \text{as } N \longrightarrow +\infty,$$

where $Exp(1)$ *is a random variable exponentially distributed with mean* 1.

Proof. First of all, we will prove that for any sequence $(l_N \in \mathcal{L}_N \, : \, N \geq 2)$ and for any pair of positive real numbers $s, t \geq 0$, the following holds

$$\lim_{N \longrightarrow +\infty} \left| \mathbb{P}\left(\frac{\tau^{N,l_N}}{c_N} > s + t\right) - \mathbb{P}\left(\frac{\tau^{N,l_N}}{c_N} > s\right) \mathbb{P}\left(\frac{\tau^{N,l_N}}{c_N} > t\right) \right| = 0. \tag{3.15}$$

Indeed, for any $N \geq 2$ and for any $l \in \mathcal{L}_N$,

$$\left| \mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > s + t\right) - \mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > s\right) \mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > t\right) \right| \leq$$

$$\sum_{u \in \mathcal{S}_N \setminus \{\vec{0}_N\}} \mathbb{P}\left(U^{N,l}_{c_N s} = u, \frac{\tau^{N,l}}{c_N} > s\right) \left| \mathbb{P}\left(\frac{\tau^{N,u}}{c_N} > t\right) - \mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > t\right) \right|. \tag{3.16}$$

The right-hand side of Equation (3.16) is equal

$$\sum_{u \in \mathcal{W}_N} \mathbb{P}\left(U^{N,l}_{c_N s} = u, \frac{\tau^{N,l}}{c_N} > s\right) \left| \mathbb{P}\left(\frac{\tau^{N,u}}{c_N} > t\right) - \mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > t\right) \right| +$$

$$\sum_{u \in \mathcal{S}_N \setminus \{\mathcal{W}_N \, \cup \, \vec{0}_N\}} \mathbb{P}\left(U^{N,l}_{c_N s} = u, \frac{\tau^{N,l}}{c_N} > s\right) \left| \mathbb{P}\left(\frac{\tau^{N,u}}{c_N} > t\right) - \mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > t\right) \right| \leq$$

$$\sup_{w \in \mathcal{W}_N} \left| \mathbb{P}(\tau^{N,l} > c_N t) - \mathbb{P}(\tau^{N,w} > c_N t) \right| + \mathbb{P}\left(U^{N,l}_{c_N s} \in \mathcal{S}_N \setminus \mathcal{W}_N, \tau^{N,l} > c_N s\right). \tag{3.17}$$

By Theorem 3.2, Remark 3.18 and Propositions 3.5 and 3.19, Equation (3.17) and the invariance by permutation of the process implies (3.15).

By definition, for any $N \geq 2$ and for any $l \in \mathcal{L}_N$,

$$\mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > 1\right) = e^{-1}.$$

Iterating (3.15) with $t = s = 2^{-n}$, for $n = 1, 2, \ldots$, we have that for any sequence $(l_N \in \mathcal{L}_N \, : \, N \geq 2)$,

$$\mathbb{P}\left(\frac{\tau^{N,l_N}}{c_N} > 2^{-n}\right) \longrightarrow e^{-2^{-n}}, \ \text{as } N \longrightarrow +\infty.$$

More generally, we have that for any

$$t \in \left\{ \sum_{n=1}^{m} b(n)2^{-n} \ : \ b(n) \in \{0,1\}, n = 1,...,m, m \geq 1 \right\}$$

is valid that

$$\mathbb{P}\left(\frac{\tau^{N,l_N}}{c_N} > t\right) \longrightarrow e^{-t}, \text{ as } N \longrightarrow +\infty. \tag{3.18}$$

Any real number $r \in (0,1)$ has a binary representation

$$r = \sum_{n=1}^{+\infty} b(n)2^{-n},$$

where for any $n \geq 1$, $b(n) \in \{0,1\}$. Therefore, the monotonicity of

$$t \longrightarrow \mathbb{P}\left(\frac{\tau^{N,l_N}}{c_N} > t\right)$$

implies that the convergence in (3.18) is valid for any $t \in (0,1)$. Moreover, for any positive integer $n \geq 1$, Equation (3.15) implies that

$$\mathbb{P}\left(\frac{\tau^{N,l_N}}{c_N} > n\right) \longrightarrow e^{-n}, \text{ as } N \longrightarrow +\infty.$$

We conclude that (3.18) is valid for any $t > 0$. □

Remark 3.21. *For any $N \geq 2$ and for any $l_N \in \mathcal{L}_N$, the function $f_N : [0,+\infty) \longrightarrow [0,1]$ given by*

$$f_N(t) = \mathbb{P}\left(\frac{\tau^{N,l_N}}{c_N} > t\right)$$

is monotonic. Also, by Proposition 3.20, it converges pointwise as $N \longrightarrow +\infty$ to a continuous function. Therefore, for any $(\epsilon_N : N \geq 2)$ such that $\lim_{N\longrightarrow+\infty} \epsilon_N = 0$, for any $t > 0$ and for any sequence $(l_N \in \mathcal{L}_N : N \geq 2)$, we have

$$\lim_{N\longrightarrow+\infty} \mathbb{P}\left(\frac{\tau^{N,l_N}}{c_N} > t + \epsilon_N\right) = \lim_{N\longrightarrow+\infty} \mathbb{P}\left(\frac{\tau^{N,l_N}}{c_N} > t - \epsilon_N\right) = e^{-t}.$$

To prove Theorem 3.3, we need the two following lemmas.

Lemma 3.22. *For any $t > 0$,*

$$\lim_{N\longrightarrow+\infty} \sup_{u\in\mathcal{S}_N\setminus\{\vec{0}_N\}} \mathbb{P}\left(\frac{\tau^{N,u}}{c_N} > t\right) \leq e^{-t}.$$

Proof. For any $u \in \mathcal{S}_N \setminus \mathcal{L}_N$ and for any $N \geq 2$, considering the event

$$E_{N,u} = \left\{ \min\{\tau^{N,u}, \inf\{T_n : U^{N,u}_{T_n} \in \mathcal{L}_N\}\} \leq N^{1/2} + N^{-1/4} + N^{-2} + e^{-(N-N^{1/4})} + e^{-(N-N^{1/2})} \right\},$$

we have that

$$\mathbb{P}\left(\frac{\tau^{N,u}}{c_N} > t\right) = \mathbb{P}\left(\frac{\tau^{N,u}}{c_N} > t, E_{N,u}, \inf\{T_n : U^{N,u}_{T_n} \in \mathcal{L}_N\} < \tau^{N,u}\right) + \tag{3.19}$$

$$\mathbb{P}\left(\frac{\tau^{N,u}}{c_N} > t, E_{N,u}, \tau^{N,u} < \inf\{T_n : U^{N,u}_{T_n} \in \mathcal{L}_N\}\right) + \mathbb{P}\left(\frac{\tau^{N,u}}{c_N} > t, E^c_{N,u}\right).$$

By Proposition 3.19, there exists $N_t > 0$ such that for any $N > N_t$, we have that $c_N t > N^{1/2} + N^{-1/4} + N^{-2} + e^{-(N-N^{1/4})} + e^{-(N-N^{1/2})}$. This implies that, for any $N > N_t$ and for any $u \in \mathcal{S}_N \setminus \mathcal{L}_N$,

$$\mathbb{P}\left(\frac{\tau^{N,u}}{c_N} > t, E_{N,u}, \tau^{N,u} < \inf\{T_n : U^{N,u}_{T_n} \in \mathcal{W}_N\}\right) = 0.$$

Considering $l \in \mathcal{L}_N$, for any $u \in \mathcal{S}_N \setminus \{\mathcal{L}_N \cup \vec{0}_N\}$ and for any $N > N_t$, the left-hand side of Equation (3.19) is bounded above by

$$\mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > t - \frac{N^{1/2} + N^{-1/4} + N^{-2} + e^{-(N-N^{1/4})} + e^{-(N-N^{1/2})}}{c_N}\right) + \mathbb{P}(E^c_{N,u}). \tag{3.20}$$

By Remark 3.12 and Corollary 3.15, it follows that

$$\lim_{N\to+\infty} \sup_{u\in\mathcal{S}_N\setminus\{\vec{0}_N\}} \mathbb{P}\left(E^c_{N,u}\right) = 0.$$

By Proposition 3.19, it follows that

$$\lim_{N\to+\infty} \frac{N^{1/2} + N^{-1/4} + N^{-2} + e^{-(N-N^{1/4})} + e^{-(N-N^{1/2})}}{c_N} = 0.$$

Therefore, by Remark 3.21 we have that

$$\lim_{N\to+\infty} \sup_{l\in\mathcal{L}_N} \mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > t - \frac{N^{1/2} + N^{-1/4} + N^{-2} + e^{-(N-N^{1/4})} + e^{-(N-N^{1/2})}}{c_N}\right) = e^{-t}.$$

We conclude the proof by noting that the limits in the last equation do not depend on u.

□

Lemma 3.23. *There exists $\alpha \in (0,1)$ and $N_\alpha > 0$ such that for any $N > N_\alpha$ and any $l \in \mathcal{L}_N$, the following upperbound holds*

$$\mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > n\right) \leq \alpha^n,$$

for any positive integer $n \geq 1$.

Proof. By Lemma 3.22, for any fixed $\alpha \in (e^{-1}, 1)$, there exists N_α such that for all $N > N_\alpha$,

$$\sup_{u \in \mathcal{S}_N \setminus \{\vec{0}_N\}} \mathbb{P}\left(\frac{\tau^{N,u}}{c_N} > 1\right) \leq \alpha < 1. \tag{3.21}$$

For any $l \in \mathcal{L}_N$ and for any $n \in \{2, 3, \dots\}$,

$$\mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > n\right) = \sum_{u \in \mathcal{S}_N \setminus \{\vec{0}_N\}} \mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > n-1, U^{N,l}_{c_N(n-1)} = u\right) \mathbb{P}\left(\frac{\tau^{N,u}}{c_N} > 1\right).$$

Equation (3.21) implies that for any $N > N_\alpha$,

$$\mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > n\right) \leq \alpha \mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > n-1\right). \tag{3.22}$$

We finish the proof by iterating (3.22). □

Proof. We will now prove Theorem 3.3.

First of all, we will prove that for any sequence $(l_N \in \mathcal{L}_N : N \geq 2)$, the following holds

$$\frac{\tau^{N,l_N}}{\mathbb{E}[\tau^{N,l_N}]} \longrightarrow \mathrm{Exp}(1) \text{ in distribution, as } N \longrightarrow +\infty. \tag{3.23}$$

Considering Proposition 3.20, we only need to show that

$$\lim_{N \to +\infty} \frac{\mathbb{E}[\tau^{N,l_N}]}{c_N} = 1.$$

Actually,

$$\lim_{N \to +\infty} \frac{\mathbb{E}[\tau^{N,l_N}]}{c_N} = \lim_{N \to +\infty} \int_0^{+\infty} \mathbb{P}(\tau^{N,l_N} > c_N s) ds.$$

Lemma 3.23 and the Dominated Convergence Theorem, allow us to put the limit inside the integral in the last term

$$\lim_{N \to +\infty} \int_0^{+\infty} \mathbb{P}(\tau^{N,l_N} > c_N s) ds = \int_0^{+\infty} \lim_{N \to +\infty} \mathbb{P}(\tau^{N,l_N} > c_N s) ds = \int_0^{+\infty} e^{-s} ds = 1.$$

This and Proposition 3.20 imply (3.23).

For any $N \geq 2$, for any $u \in S_N^{(0)}$ and for any $s > 0$,

$$\mathbb{P}(\tau^{N,u} > c_N s) = \mathbb{P}(\tau^{N,u} > c_N s, E_{N,u}) + \mathbb{P}(\tau^{N,u} > c_N s, E^c_{N,u}),$$

where

$$E_{N,u} = \{\inf\{t : U^{N,u}_t \in \mathcal{L}_N\} \leq 1\}.$$

For any $l \in \mathcal{L}_N$, by Markov property and the invariance by permutation of the process we have

$$\mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > s\right)\mathbb{P}(E_{N,u}) \leq \mathbb{P}(\tau^{N,u} > c_N s, E_{N,u}) \leq \mathbb{P}\left(\frac{\tau^{N,l}}{c_N} > s - \frac{1}{c_N}\right)\mathbb{P}(E_{N,u}). \tag{3.24}$$

By Corollary 3.16,

$$\lim_{N\longrightarrow+\infty} \inf_{u\in S_N^{(0)}} \mathbb{P}(E_{N,u}) = 1,$$

and then, for any sequence $(u_N \in S_N^{(0)} : N \geq 2)$,

$$\lim_{N\longrightarrow+\infty} \mathbb{P}(\tau^{N,u_N} > c_N s, E_{N,u_N}) = \lim_{N\longrightarrow+\infty} \mathbb{P}(\tau^{N,u_N} > c_N s).$$

Proposition 3.19 and Remark 3.21 implies that for any sequence $(l_N \in \mathcal{L}_N : N \geq 2)$,

$$\lim_{N\longrightarrow+\infty} \mathbb{P}\left(\frac{\tau^{N,l_N}}{c_N} > s\right) = \lim_{N\longrightarrow+\infty} \mathbb{P}\left(\frac{\tau^{N,l_N}}{c_N} > s - \frac{1}{c_N}\right) = e^{-s}.$$

The conclusion follows from Equation (3.24) and by observing that the Dominated Convergence Theorem allow us to replace c_N by $\mathbb{E}[\tau^{N,u_N}]$ as we did to prove that Equation (3.23) holds. $\square$

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