Regularity and comparison principles in complex analysis and locally integrable structures

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It is not incumbent upon you to complete the work,
but neither are you at liberty to desist from it.
(Avot 2:21)

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Resumo

Vinícius Novelli da Silva. **Regularidade e princípios de comparação em análise complexa e estruturas localmente integráveis**. Tese (Doutorado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2024.

Esta tese trata de três tópicos, relacionados a análise complexa em várias variáveis, geometria CR e sistemas de equações diferenciais parciais. Primeiramente, estudamos um princípio de comparação entre estruturas CR Levi-flat e sistemas de campos vetoriais reais. Em uma família de modelos dados por fibrados com fibra complexa, calculamos completamente a cohomologia do complexo diferencial associado. Introduzimos também alguns modelos compactos (parametrizados por um certo número de formas de tipo $(0, 1)$ $\bar{\partial}$-fechadas), e neste caso, estudamos questões de hipoelipticidade global e provamos, em vários casos, um isomorfismo de comparação entre os espaços de cohomologia. Na segunda parte, estudamos (no caso unidimensional) a álgebra de Fréchet $A^\infty(K)$ (introduzida recentemente por Cordaro, Della Sala e Lamel, para tratar de questões como a aplicação de Borel). Provamos algumas propriedades desta álgebra (como uma propriedade de localização, análoga ao caso uniforme $P(K)$) e estudamos a questão de determinar quando esta álgebra coincide com o espaço das séries formais com coeficientes em $C(K)$. Para tal, introduzimos como ferramenta fundamental uma generalização da transformada de Cauchy. Finalmente, estudamos uma questão a respeito da regularização de estruturas localmente integráveis. Inspirados em um resultado recente de Kossovskiy e Zaitsev, apresentamos uma família de estruturas (de tipo não-CR) e uma condição suficiente que garante quando tais estruturas são equivalentes a estruturas real-analíticas.

Palavras-chave: hipoelipticidade global. estruturas localmente integráveis. geometria CR.

Abstract

Vinícius Novelli da Silva. **Regularity and comparison principles in complex analysis and locally integrable structures**. Thesis (Doctorate). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2024.

This thesis deals with three topics, related to complex analysis in several variables, CR geometry and systems of partial differential equations. Firstly, we study a comparison principle between Levi-flat CR structures and systems of real vector fields. In a family of models given by fibre bundles with complex fiber, we compute completely the cohomology of the associated differential complex. We also introduce some compact models (parametrized by a certain number of forms of type $(0, 1)$ which are $\overline{\partial}$-closed) and, in this case, study questions of global hypoellipticity and prove, in several cases, a comparison isomorphism between the cohomology spaces. In the second part, we study (in the one-dimensional case) the Fréchet algebra $A^\infty(K)$ (introduced recently by Cordaro, Della Sala and Lamel, to treat questions about the Borel map). We prove some function-theoretic properties about this algebra (like a localization property, in analogy with the uniform case $P(K)$) and study the question of determining when this algebra coincide with the full algebra of formal power series with coefficients in $C(K)$. To do this, we introduce as the main tool a generalization of the Cauchy transform. Finally, we study a problem of regularizability of locally integrable structures. Inspired by a recent result of Kossovskiy and Zaitsev, we present a family of structures (of non-CR type) and a sufficent condition that guarantees when such structures are equivalent to real-analytic ones.

Keywords: global hypoellipticity. locally integrable structures. CR geometry.

Contents

Introduction

The theory of locally integrable structures (Treves, 1992, Berhanu, P. D. Cordaro, and Hounie, 2014) has developed as a very useful tool in the study of systems of complex first-order PDEs. One of the many exciting features this theory posesses is the ability to connect the general theory of integrable systems and a branch of Complex Analysis in Several Variables, called *Cauchy-Riemann Geometry* (CR Geometry for short), which is mainly concerned with (real) smooth submanifolds M of complex manifolds and the interplay between the real and complex structures reflected on M. It is precisely in this intersection that this thesis lies.

The thesis decomposes in three parts, related only in the sense that they study problems that live in (or around) this intersection. We will briefly describe the main contents of each chapter.

Chapter 1 deals with Levi-flat CR structures and comparison principles. An involutive structure $\mathcal{V} \subset \mathbb{C}T\Omega$ over a smooth manifold Ω is a Levi-flat CR structure if $\mathcal{V} \oplus \overline{\mathcal{V}}$ is also involutive (in particular, $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$). This yields two structures over Ω: $\mathcal{V}$ and $\mathcal{V} \oplus \overline{\mathcal{V}}$, which are both locally integrable by classical results. The objetive is to relate these two structures as much as possible, from two points of view: firstly, computing the cohomology of the associated differential complex and secondly, determining what regularization properties these structures have.

We study this problem in two models which are, in a sense, orthogonal. It is well-known that a Levi-flat CR structure induces over the base manifold a foliation by complex manifolds, called the *Levi foliation*. We can measure the complexity of the structure through the complexity of the associated foliation. The first class we study is one where this foliation is locally constant: more precisely, Ω is the total space of a fibre bundle $\Omega \to S$, where the fiber M is a fixed complex manifold and the structure group acts through biholomorphisms of M. In this situation, we compute completely the cohomology of the differential complexes of $\mathcal{V}$ and $\mathcal{V} \oplus \overline{\mathcal{V}}$ (Theorems 1.3.1 and 1.3.2), using as the main tool the Leray spectral sequence. Here, we are motivated by the intuition of having a complex structure depending smoothly on the parameters of S, so properties of the *completed tensor product* $C^\infty(S)\widehat{\otimes}X$ are of crucial importance (as this space describes the X-valued smooth functions on S). These ideas go back to classic work of Andreotti and Grauert, 1962. Our description of the cohomology allows us to verify, for instance, that $\overline{\partial}_b$ is globally solvable in top degree if and only if M has no compact connected component (see Corollary 1.3.2), a result which is expected to hold for every CR Levi-flat manifold.

The second model we study has a more complicated Levi foliation (it is a compact family of models). This case, whose study started in P. D. Cordaro, 1996, has as the corresponding real structure one which has been deeply studied in the literature (see, for instance, Bergamasco, P. D. Cordaro, and Malagutti, 1993, Greenfield and Wallach, 1972). Here, we have $\Omega = M \times \mathbb{T}^d$, where M is a compact Kähler manifold and one fixes d forms $\omega_1, \dots, \omega_d$ on M which are $\overline{\partial}$-closed. Then, we can define a natural CR Levi-flat structure on Ω with Levi foliation parametrized by the forms ω_j. For this family of models, we study the cohomology spaces. We cannot compute them explicitely (as in the fibre bundle case described before), but we can *compare* the cohomologies of $\mathcal{V}$ and $\mathcal{V} \oplus \overline{\mathcal{V}}$. In particular, we prove that these cohomologies satisfy a version of Hodge's theorem (Theorem 1.6.1). Moreover, we show that $H^1(\Omega; \mathcal{V})$ is Hausdorff if, and only if, $H^1(\Omega; \mathcal{V} \oplus \overline{\mathcal{V}})$ is Hausdorff, and in this case, they are isomorphic (Theorem 1.6.2). We also compare properties of global hypoellipticity for both structures, and introduce a notion of global hypoellipticity for forms of higher degree. Using recent results on characterization of global hypoellipticity for $\mathcal{V} \oplus \overline{\mathcal{V}}$ (proved in Araújo, Dattori da Silva, and Lessa Victor, 2022 and Araújo, Ferra, Jahnke, and Ragognette, 2023), we are able to obtain characterizations of global hypoellipticity for $\overline{\partial}_b$. Here, the main tool is the Laplacians associated to the natural metric on Ω and a Fourier decomposition of this (non-elliptic) operator into a countable sequence of elliptic operators.

Chapter 2 is about a function-theoretic approach to the Fréchet algebra $A^\infty(K)$: in P. D. Cordaro, Della Sala, and Lamel, 2019 and P. D. Cordaro, Della Sala, and Lamel, 2020, this Fréchet algebra was introduced (associated to a compact set $K \subset \mathbb{C}^m$), which measures, roughly, which holomorphic functions on $\operatorname{int} K$ admit *smooth* extensions to K, but without any assumption of regularity of the boundary of K. This turned out to be an important tool in CR geometry, and led to important advances (like determining which formal power series can be realized by a smooth CR function, or, in other words, properties concerning the *Borel map*). In this chapter, we carry out a study of this algebra in the 1-dimensional case, inspired by the classic function-theory developed for the simpler Banach algebra $P(K)$ (which is the closure, in the uniform topology, of restrictions of entire functions to K).

We introduce as the main tool a generalization of the classical Cauchy transform of a measure, but now associated to a *polynomial* (or finite sequence) of measures (see Definition 2.4.2). Studying formal properties of this transform yields many properties of the A^∞ algebra in the plane. In particular, we show that a smooth function f such that $\overline{\partial} f$ vanishes to infinite order over K induces naturally an element in $A^\infty(K)$, provided K is polynomially convex (see Theorem 2.4.2). We also establish a *localization property* for this algebra (see Theorem 2.5.2).

We also consider the problem of determining necessary and sufficient conditions on a compact set $K \subset \mathbb{C}$ for $A^\infty(K)$ to coincide with the full algebra of formal power series $C(K, \mathcal{F}_1)$. We propose conditions (see Theorem 2.6.1) that are necessary and we conjecture that these conditions are also sufficient. We use the generalized Cauchy transform to produce a family of examples of compact sets $K \subset \mathbb{C}$ for which the equality $A^\infty(K) = C(K; \mathcal{F}_1)$ holds (see Theorem 2.6.2).

Finally, in Chapter 3, we consider the problem of *regularizability* of a locally integrable structure. Namely, given a smooth locally integrable structure $(M, \mathcal{V})$ and a point $p \in M$, when does there exist an isomorphism of $\mathcal{V}$ near p onto a structure which is *real-analytic*? This problem was first considered for smooth strictly pseudoconvex hypersurfaces by Kossovskiy and Zaitsev, 2022, and they introduced a property (called *Condition E*) that is necessary and sufficient for regularizability to hold. We apply a technique of Marson, 1992 and obtain a family of locally integrable structures (see Definition 3.4.1) and a sufficient condition that implies they are also equivalent to real-analytic ones (see Theorem 3.5.1).

Chapter 1

A comparison principle between certain Levi-flat compact CR manifolds and systems of real vector fields

1.1 Introduction

Taking P. D. Cordaro, 1996 as a starting point, we study a correspondence between certain Levi-flat CR manifolds and the naturally associated real foliation. From the point of view of the theory of locally integrable structures (Treves, 1992, Berhanu, P. D. Cordaro, and Hounie, 2014), one has two different structures defined on the same manifold, and two associated complexes of differential operators: the tangential Cauchy-Riemann complex $\overline{\partial}_b$ and the tangential (along the leaves) de Rham complex $\mathbb{L}$. We study the cohomology spaces of these complexes in some models as well as we compare the property of global hypoellipticity in both cases.

In Section 2 we study these questions for Levi flat CR structures defined by locally trivial fiber bundles whose fibers are complex manifolds. In this situation we have at our disposal the Leray spectral sequence which, combined with a Künneth formula with parameters inspired by Andreotti and Grauert, 1962, leads to a fair description of the cohomologies for both structures. Among other things we prove that the global solvability for $\overline{\partial}_b$ in top forms is equivalent to the nonexistence of compact components of the fiber (cf. Corollary 1.3.2 below).

A much more involved model is the case of a product $M \times \mathbb{T}^d$ of a compact, complex manifold with the d-dimensional torus, where a natural CR Levi-flat structure can be defined after choosing suitable d-closed $(0, 1)$-forms on M. In this case, the associated real structure was studied by several authors (Bergamasco, P. D. Cordaro, and Malagutti, 1993, Bergamasco, P. D. Cordaro, and Petronilho, 1996, Araújo, Dattori da Silva, and Lessa Victor, 2022, Araújo, Ferra, Jahnke, and Ragognette, 2023). We introduce such models in Section 3 and describe some of their main properties. We refer the reader to both theorems 2.6.2 and 1.4.3, which will be pivotal in what follows.

In Section 4, we use a (non-elliptic) Laplacian comparison technique to prove equivalences of global hypoellipticity for $\mathbb{L}$ and $\overline{\partial}_b$. The introduction of the Laplacian allows us to define a notion of global hypoellipticity in higher degree forms for these complexes, which coincides with the usual notion on 0-forms. Our main result in this section is Corollary 1.5.1, which states that if M is balanced in degree q then the global hypoellipticity of $\overline{\partial}_b$ and that of $\mathbb{L}$ are equivalent in degree q.

Finally, in Section 5, we study the cohomology of these complexes, again with the Laplacian as the main tool. We prove a version of Hodge's theorem (in arbitrary degree) and, in degrees 0 and 1, we show an equivalence between global solvability (or validity of the Cousin property for solutions) between $\mathbb{L}$ and $\overline{\partial}_b$ (cf. Theorem 1.6.2 below).

1.2 Preliminaries

Let Ω be a smooth, paracompact manifold of dimension $N \geq 1$ which is endowed with a Levi-flat CR structure $\mathcal{V}$. This means that $\mathcal{V}$ is smooth involutive subbundle of $\mathbb{C}T\Omega$ satisfying the following properties:

1. $\mathcal{V} \cap \overline{\mathcal{V}} = 0$;
2. $\mathcal{V} \oplus \overline{\mathcal{V}}$ is an involutive bundle.

We shall denote by n the rank of $\mathcal{V}$ (the value of n is also called the CR dimension of $\mathcal{V}$) and by $d \doteq N - 2n$ the rank of the characteristic set of $\mathcal{V}$ (recall that the characteristic set of $\mathcal{V}$ is $\mathcal{V}^\perp \cap T^*\Omega$, which is real subundle of $T^*\Omega$ since $\mathcal{V}$ is CR bundle). We assume $d \geq 1$. Both $\mathcal{V}$ and $\mathcal{V} \oplus \overline{\mathcal{V}}$ define locally integrable structures on Ω. Indeed, for $\mathcal{V}$ this is result due to L. Nirenberg Nirenberg, 1958 whereas for $\mathcal{V} \oplus \overline{\mathcal{V}}$, it is consequence of Frobenius theorem.

Any point in Ω is the center of a coordinate system $(U; x_1, \dots, x_n, y_1, \dots, y_n, s_1, \dots s_d)$ such that $\mathcal{V}|_U$ is generated by the vectors fields $\partial/\partial\overline{z}_j$. Notice that then $(\mathcal{V} \oplus \overline{V})|_U$ is spanned by $\partial/\partial x_j, \partial/\partial y_k$. In particular, a smooth CR function on U is a smooth function $u(z,s)$ which is holomorphic with respect to z.

The essentially real structure $\mathcal{V} \oplus \overline{\mathcal{V}}$ defines a smooth foliation $\mathcal{F}$ in such a way, given any leaf $F \in \mathcal{F}$ (which has dimension $2n$), the restriction of $\mathcal{V}$ to F defines a complex structure on F. In other words each leaf of $\mathcal{F}$ is a complex manifold of dimension n.

Consider the bundles

$$G_j = \Lambda^j(\mathbb{C}T^*\Omega/\mathcal{V}^\perp), \quad j = 1, \dots, n;$$

$$H_k = \Lambda^k(\mathbb{C}T^*\Omega/(\mathcal{V} \oplus \overline{\mathcal{V}})^\perp), \quad k = 1, \dots, 2n.$$

It is well known that the de Rham complex in Ω induces differential complexes

$$\overline{\partial}_b : C^\infty(\Omega, G_j) \longrightarrow C^\infty(\Omega, G_{j+1}), \;\; \mathbb{L} : C^\infty(\Omega, H_k) \longrightarrow C^\infty(\Omega, H_{k+1}),$$

whose cohomologies will be denoted respectively by $H^j(\Omega; \overline{\partial}_b)$ and $H^k(\Omega; \mathbb{L})$.

Both complexes are locally exact. Hence if we further introduce the sheaf $\mathcal{A}_\Omega$ (resp. $\mathcal{B}_\Omega$) of germs of smooth solutions of the equation $\overline{\partial}_b u = 0$ (resp. $\mathbb{L}u = 0$) it follows from standard arguments in sheaf theory Godement, 1964 that

$$H^j(\Omega, \mathcal{A}_\Omega) = H^j(\Omega; \overline{\partial}_b), \quad H^k(\Omega, \mathcal{B}_\Omega) = H^k(\Omega; \mathbb{L}).$$

Of special interest is the vanishing of $H^1(\Omega, \mathcal{A}_\Omega)$, for this implies the validity of the first Cousin problem for solutions of $\mathcal{V}$.

1.3 Structures defined by locally trivial fibre bundles

In order to study the cohomology of the class of CR structures which we will describe next, it will be necessary to deal with Fréchet sheaves and their completed tensor products. Recall that a sheaf $\mathcal{F}$ over a topological space X is a *Fréchet sheaf* (respectively, *Fréchet-nuclear sheaf*) if the space of sections $\mathcal{F}(U)$ is a Fréchet (respectively, Fréchet-nuclear) space for every open set $U \subset X$ and the restriction maps $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ are continuous for all open sets $V \subset U \subset X$.

If $\mathcal{F}$ and $\mathcal{G}$ are Fréchet-nuclear sheaves over topological spaces X and Y, respectively, we denote by $\mathcal{F}\widehat{\otimes}\mathcal{G}$ the sheaf on $X \times Y$ associated to the presheaf $\mathcal{F}(U)\widehat{\otimes}\mathcal{G}(V)$, $U \subset X$ and $V \subset Y$ open (here, $\widehat{\otimes}$ denotes the completed tensor product in either the π or ε topologies).

The case which will be of most interest to us is the case where $\mathcal{F}$ is a fine sheaf (namely, the sheaf of smooth functions on a manifold). We record here some basic permanence properties of $\widehat{\otimes}$ when one of the spaces is $C^\infty(X)$, for X a smooth manifold (this space is Fréchet-nuclear for the usual topology, see Treves, 1967).

Proposition 1.3.1. *Let $\alpha : F \to F'$ be a continuous linear map between Fréchet-nuclear spaces and let X be a smooth manifold. Let $1\widehat{\otimes}\alpha : C^\infty(X)\widehat{\otimes}F \to C^\infty(X)\widehat{\otimes}F'$ be the induced map on the completed tensor product. Then,*

1. *The kernel of $1\widehat{\otimes}\alpha$ is given by $C^\infty(X)\widehat{\otimes} \ker \alpha$, which is naturally a (closed) subspace of $C^\infty(X)\widehat{\otimes}F$.*
2. *If α has closed range, then $1\widehat{\otimes}\alpha$ also has closed range. More precisely,*

$$\operatorname{ran}(1\widehat{\otimes}\alpha) = C^\infty(X)\widehat{\otimes} \operatorname{ran} \alpha.$$

Proof. The second item follows from the fact that the functor $E \mapsto C^\infty(X)\widehat{\otimes}E$ preserves short exact sequences of Fréchet-nuclear spaces (this remains true replacing $C^\infty(X)$ by any Fréchet-nuclear space). See, for instance, p. 435 in Demailly, n.d. or p. 205 in Andreotti and Grauert, 1962 for a direct proof in the case of $C^\infty(X)$.

This property does not (directly) imply item (1), because there we do not assume α has closed range. Here, we take advantage of the fact that $C^\infty(X)$ allows for the use of Fourier series (this argument is based on Andreotti and Grauert, 1962): indeed, if $X = \mathbb{T}^d$ is the d-dimensional torus, then the result is immediate from Fourier series decomposition:

$$\ker(1\widehat{\otimes}\alpha) = \left\{ \sum_{n\in\mathbb{Z}^d} f_n e^{in\theta} \in C^\infty(\mathbb{T}^d)\widehat{\otimes}F;\ \alpha(f_n) = 0 \text{ for all } n \right\} \simeq C^\infty(\mathbb{T}^d)\widehat{\otimes}\ker\alpha.$$

If X is a general (paracompact) smooth manifold, let $\{U_i\}$ be a locally finite covering of X by coordinate charts $h_i : U_i \to \mathbb{R}^d$ such that $h_i(U_i)$ is an open subset of the unit cube $I_d \subset \mathbb{R}^d$. Let $\{\rho_i\}$ be a smooth partition of unity subordinated to this covering. If $f \in C^\infty(X)\widehat{\otimes}F$ is such that $(1\widehat{\otimes}\alpha)(f) = 0$, then $f_i := \rho_i f$ is also in the kernel of $1\widehat{\otimes}\alpha$. It is clear that f_i can be identified with an element in $C^\infty(\mathbb{T}^d)\widehat{\otimes}F$, which yields $f = \sum f_i \in C^\infty(X)\widehat{\otimes}\ker\alpha$. The reverse inclusion is clear, since the algebraic tensor product $C^\infty(X)\otimes\ker\alpha$ is clearly contained in the (closed) kernel of $1\widehat{\otimes}\alpha$. □

We shall consider locally trivial fibre bundles $(\Omega, \Omega \xrightarrow{f} S, M)$, where Ω, the total space, is a smooth manifold of dimension N; S, the base space, is a smooth manifold of dimension d and M, the fibre space, is a complex manifold of complex dimension n. We take the structure group of this bundle to be $\mathrm{Aut}(M)$ (the group of biholomorphisms of M). It is easily seem that a natural Levi flat CR structure, of CR dimension equal to n, can be introduced on Ω. Indeed, if $U \subset S$ is an open set over which $(\Omega, \Omega \xrightarrow{f} S, M)$ trivializes, that is, there is a smooth diffeomorphism $h : U \times M \simeq f^{-1}(U)$ satisfying $(f \circ h)(x, z) = x$, $x \in U$ and $z \in M$, it follows that $h_\star(\{0\} \times T^{1,0}M)$ defines a Levi flat CR structure of CR dimension equal to n on $f^{-1}(U)$ and, when we vary U, they all match together to define a Levi flat CR structure of CR dimension equal to n on Ω.

We fix a (countable) cover $\mathcal{U} := (U_j)_{j\in\mathbb{Z}_+}$ of S such that the bundle is trivialized in each U_j. We denote it by $\phi_j : U_j \times M \to f^{-1}(U_j)$, with cocycle map $\phi_{ij} := \phi_i^{-1} \circ \phi_j : (U_i \cap U_j) \times M \to (U_i \cap U_j) \times M$ given by $\phi_{ij}(x, p) = (x, g_{ij}(x) \cdot p)$, where $g_{ij} : U_i \cap U_j \to \mathrm{Aut}(M)$ is smooth, for $i, j \in \mathbb{Z}_+$ such that $U_i \cap U_j \neq \emptyset$. The following lemma computes the cohomology of $\mathcal{A}_\Omega$ over such a set.

Lemma 1.3.1. *Let U_i be a trivializing open set. Then, if $H^q(M, \mathcal{O}_M)$ is Hausdorff for some $q \geq 1$, $H^q(f^{-1}(U_i), \mathcal{A}_\Omega|_{f^{-1}(U_i)})$ is also Hausdorff and there is a topological isomorphism*

$$T_i : H^q(f^{-1}(U_i), \mathcal{A}_\Omega|_{f^{-1}(U_i)}) \to C^\infty(U_i)\widehat{\otimes}H^q(M, \mathcal{O}_M).$$

Moreover, this isomorphism is compatible with restrictions to open subsets of U_i (that is, it induces a sheaf isomorphism).

Proof. If $\phi_i : U_i \times M \to f^{-1}(U_i)$ is the trivialization, then the pullback sheaf $\phi_i^* \mathcal{A}_\Omega|_{f^{-1}(U_i)}$ is the sheaf $C^\infty_{U_i}\widehat{\otimes}\mathcal{O}_M$. In particular, the cohomology is isomorphic to the cohomology of the complex

$$C^\infty(U_i \times M; \Lambda^{q-1}) \xrightarrow{\overline{\partial}_{q-1}} C^\infty(U_i \times M; \Lambda^q) \xrightarrow{\overline{\partial}_q} C^\infty(U_i \times M; \Lambda^{q+1}).$$

of q-forms on M depending smoothly on parameters in U_i. This is equivalent to the complex

$$C^\infty(U_i)\widehat{\otimes}C^\infty(M;\Lambda^{q-1}) \xrightarrow{1\widehat{\otimes}\bar{\partial}_{q-1}} C^\infty(U_i)\widehat{\otimes}C^\infty(M;\Lambda^{q}) \xrightarrow{1\widehat{\otimes}\bar{\partial}_{q}} C^\infty(U_i)\widehat{\otimes}C^\infty(M;\Lambda^{q+1}). \quad (1.3.1)$$

We conclude from Proposition 1.3.1 that the cohomology of 1.3.1 is isomorphic to

$$\frac{C^\infty(U_i)\widehat{\otimes}\ker\bar{\partial}_q}{C^\infty(U_i)\widehat{\otimes}\operatorname{ran}\bar{\partial}_{q-1}} \simeq C^\infty(U_i)\widehat{\otimes}H^q(M,\mathcal{O}_M),$$

since the (completed) tensor product is an exact functor and $H^q(M,\mathcal{O}_M)$ is a Fréchet space. □

Assume that $H^q(M,\mathcal{O}_M)$ is Hausdorff for some $1 \le q \le n$. Then, we consider the sequence space

$$X := \prod_{j\in\mathbb{Z}_+} C^\infty\left(U_j, H^q(M,\mathcal{O}_M)\right) \simeq \prod_{j\in\mathbb{Z}_+} C^\infty(U_j)\widehat{\otimes}H^q(M,\mathcal{O}_M),$$

with its natural structure of Fréchet space. We define a natural subspace of X by

$$\mathcal{E}(\mathcal{U},M,\mathcal{O}_M) := \left\{ (v_i)_{i\in\mathbb{Z}_+} \in X;\ \left(T_i\circ T_j^{-1}\right) v_j\big|_{U_i\cap U_j} = v_i\big|_{U_i\cap U_j} \right\}. \quad (1.3.2)$$

It is clear that this is a closed subspace of X. With these notations in mind, we can state the main result.

Theorem 1.3.1. *Let $(\Omega,\Omega \xrightarrow{f} S, M)$ be as above. Let $\mathcal{U} = (U_j)_{j\in\mathbb{Z}_+}$ be a trivializing cover of S and assume that $H^q(M,\mathcal{O}_M)$ is Hausdorff for some $1\le q\le n$. Then, there is a topological isomorphism*

$$H^q(\Omega,\mathcal{A}_\Omega) \simeq \mathcal{E}(\mathcal{U},M,\mathcal{O}_M). \quad (1.3.3)$$

Proof. The key ingredient in the proof is the use of the Leray spectral sequence associated to $(\Omega,\Omega \xrightarrow{f} S, M)$. Let $R^q f_*\mathcal{A}$ denote the q-th cohomology sheaf of the direct image sheaf $f_*\mathcal{A}$, that is, the sheaf associated to the presheaf

$$V \mapsto H^q(f^{-1}(V),\mathcal{A}_\Omega), \quad V\subset S,\ \text{open}.$$

It is well known that there exists a spectral sequence (the Leray spectral sequence)

$$E_2^{p,q} = H^p(S,R^q f_*\mathcal{A}) \Longrightarrow H^{p+q}(\Omega,\mathcal{A}_\Omega).$$

Our first observation is that $R^q f_*\mathcal{A}$ and $C_S^\infty\widehat{\otimes}\mathbf{H}^q(M,\mathcal{O}_M)$ coincide over each U_i (here we denote by $\mathbf{H}^q(M,\mathcal{O}_M)$ the constant sheaf on S with stalks $H^q(M,\mathcal{O}_M)$): indeed, since sheafification is a functor (Kashiwara and Schapira, 1990, p. 85), this follows from Lemma 1.3.1.

In particular, we conclude that $R^q f_*\mathcal{A}_\Omega$ is locally fine, and therefore, locally soft. Then, it has no higher cohomology (see, for instance, Proposition 4.13 and Theorem 4.15, pages

204 and 205 in Demailly, n.d.). We obtain that

$$E_2^{p,q} = 0, \ \ p \geq 1,$$

and, therefore, the Leray spectral sequence degenerates. This implies a topological isomorphism

$$H^q(\Omega, \mathcal{A}_\Omega) \simeq H^0(S, R^q f_* \mathcal{A}_\Omega).$$

Observe that we have an embedding

$$\begin{aligned} \Phi : H^0(S, R^q f_* \mathcal{A}_\Omega) &\hookrightarrow \prod_{j \in \mathbb{Z}_+} R^q f_* \mathcal{A}_\Omega(U_i) \\ \sigma &\mapsto \left(\sigma|_{U_i}\right)_{i \in \mathbb{Z}_+} \end{aligned}$$

whose range is given by

$$\operatorname{Im} \Phi = \left\{ (u_i)_{i \in \mathbb{Z}_+} \in \prod_{j \in \mathbb{Z}_+} R^q f_* \mathcal{A}_\Omega(U_i); \ \ u_i|_{U_i \cap U_j} = u_j|_{U_i \cap U_j} \right\}.$$

Applying the isomorphisms from Lemma 1.3.1, we obtain an isomorphism of Fréchet spaces

$$\Psi : H^q(\Omega, \mathcal{A}_\Omega) \to \mathcal{E}(\mathcal{U}, M, \mathcal{O}_M).$$

□

By a similar argument, now recalling the fact that the spaces $H^q(M, \mathbb{C})$ are always Fréchet, thanks to de Rham theorem, we can obtain an analogous description of the space $H^q(\Omega, \mathcal{B}_\Omega)$ (compare with Theorem 14.18 in Bott and Tu, 1982). Note that in this case, the structure group can be the full group $\mathrm{Diff}(M)$ of diffeomorphisms of M, and the complex structure of the fibre is not required. Indeed, introducing the closed subspace

$$\mathcal{E}(\mathcal{U}, M, \mathbb{C}) \subset Y := \prod_{j \in \mathbb{Z}_+} C^\infty(U_j) \widehat{\otimes} H^q(M, \mathbb{C})$$

in exactly the same way as 1.3.2, but now considering the trivializations induced by the real structure on the bundle, we can state the following

Theorem 1.3.2. *Let $(\Omega, \Omega \xrightarrow{f} S, M)$ be as above. Let $\mathcal{U} = (U_j)_{j \in \mathbb{Z}_+}$ be a trivializing cover of S. Then, there is a topological isomorphism*

$$H^q(\Omega, \mathcal{B}_\Omega) \simeq \mathcal{E}(\mathcal{U}, M, \mathbb{C}) \tag{1.3.4}$$

for all $1 \leq q \leq 2n$.

We now list some consequences of these results.

Corollary 1.3.1. *Let $1 \leq q \leq n$. Then $H^q(M, \mathcal{O}_M) = 0$ if and only if $H^q(\Omega, \mathcal{A}_\Omega) = 0$. In particular, if M is Stein then $H^q(\Omega, \mathcal{A}_\Omega) = 0$ for $q = 1, \dots, n$ and, from Theorem 1.3.2, $H^k(\Omega, \mathcal{B}_\Omega) = 0$ for $k = n+1, \dots, 2n$.*

Proof. By Theorem 1.3.1 $H^q(M,\mathcal{O}_M)=0$ implies $H^q(\Omega,\mathcal{A}_\Omega)=0$. Conversely, assume that $H^q(\Omega,\mathcal{A}_\Omega)=0$ and let $\omega\in C^\infty_{(0,q)}(M)$ be $\overline{\partial}$-closed. Let $U\subset S$ be an open set such that there is a diffeomorphism $h:U\times M\simeq f^{-1}(U)$ satisfying $(f\circ h)(x,z)=x$, $x\in U$ and $z\in M$. Select $\psi\in C^\infty_c(U)$ which is equal to one at some point $x_0\in U$. Then $(h^{-1})^*(\psi\otimes\omega)$, extended as zero outside $f^{-1}(U)$, defines an element $\beta\in C^\infty(\Omega,G_q)$ which is $\overline{\partial}_b$-closed. Since $H^q(\Omega,\mathcal{A}_\Omega)=0$ there exists $\alpha\in C^\infty(\Omega,G_{q-1})$ such that $\overline{\partial}_b\alpha=\beta$. Then $\alpha_\bullet\doteq h^*(\alpha|_{f^{-1}U})$ solves $\overline{\partial}\alpha_\bullet=\psi\otimes\omega$ in $U\times M$. In particular $\gamma\doteq\alpha_\bullet(x_0,\cdot)\in C^\infty_{(0,q-1)}(M)$ solves $\overline{\partial}\gamma=\omega$ in M.

The remaining statements follow from well known results for Stein manifolds. □

Corollary 1.3.2. *The cohomology space $H^n(\Omega,\mathcal{A}_\Omega)$ is trivial if and only if M has no compact connected component.*

Proof. By Corollary 1.3.1, $H^n(\Omega,\mathcal{A}_\Omega)=0$ if and only if $H^n(M,\mathcal{O}_M)=0$, and this last space is trivial if and only if M has no compact component (cf. Malgrange, 1957).

□

Corollary 1.3.3. *Assume the bundle $\Omega\to S$ is trivial. Then, if $H^q(M,\mathcal{O}_M)$ is Hausdorff for some $1\le q\le n$, we have a topological isomorphism*

$$H^q(\Omega,\mathcal{A}_\Omega)\simeq C^\infty(S)\widehat{\otimes}H^q(M,\mathcal{O}_M).$$

Remark 1.3.1. Observe that this is a generalization of Proposition 7, page 208 in Andreotti and Grauert, 1962. Moreover, a similar result holds for $H^q(\Omega,\mathcal{B}_\Omega)$ (see the observations that precede Theorem 1.3.2).

1.4 A class of compact Levi-flat CR structures

In this part of the work, we concentrate on a different class of Levi-flat CR structures in which the complex structure of the (leaves of the) foliation is not fixed, but varies in a controlled way. We start by establishing some notation. Let M be a compact, connected complex manifold of (complex) dimension $n\ge 1$ and let $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional torus. Consider $\omega_1,\dots,\omega_d\in C^\infty(M;\Lambda^{0,1}T^*M)$ forms of type $(0,1)$ such that $\overline{\partial}\omega_k=0$ for all $k=1,\dots,d$. Let $(\theta_1,\dots,\theta_k)$ denote the usual angular coordinates on the torus. We define

$$T':=T^{1,0}M\oplus\operatorname{span}\{\alpha_1,\dots,\alpha_d\},$$

where $\alpha_k:=\mathrm{d}\theta_k+\omega_k$ for $k=1,\dots,d$ (we are identifying forms and bundles on M and on $\mathbb{T}^d$ with their pullbacks to $M\times\mathbb{T}^d$).

Proposition 1.4.1. *T' defines a locally integrable CR structure on $M\times\mathbb{T}^d$.*

Proof. T' is clearly a subbundle of $\mathbb{C}T^*(M\times\mathbb{T}^d)$, which is locally integrable from Dolbeault-Grothendieck's lemma. Moreover, we have $T'+\overline{T'}=\mathbb{C}T^*M$, since $\mathrm{d}\theta_k=\alpha_k-\omega_k$ is a section of $T'+\overline{T'}$ for every $k=1,\dots,d$, which shows T' is a CR structure. □

Denote by $\mathcal{V} \subset \mathbb{C}T(M\times\mathbb{T}^d)$ the subbundle orthogonal to T' (for the duality between one-forms and vector fields), which has rank n. Consider a system of holomorphic coordinates $(z_1, \dots, z_n)$ in an open set $\Omega \subset M$, and write $\omega_k = \sum_{j=1}^n \omega_{jk} \mathrm{d}\overline{z_j}$, for $k = 1, \dots, d$, where $\omega_{jk} \in C^\infty(\Omega)$. In such coordinates, a frame for the the bundle $\mathcal{V}$ over $\Omega \times \mathbb{T}^d$ is given by the set of vector fields

$$L_j = \frac{\partial}{\partial \overline{z_j}} - \sum_{k=1}^d \omega_{jk} \frac{\partial}{\partial \theta_k}, \quad j = 1, \dots, n.$$

We shall compute the characteristic set and the Levi form of $\mathcal{V}$. Fixing a point $p = (z, \theta) \in \Omega \times \mathbb{T}^d$, let $v \in T'_p$. Then, we can write

$$v = \sum_{j=1}^n \alpha_j \mathrm{d}z_j + \sum_{k=1}^d \xi_k \alpha_k = \sum_{j=1}^n \alpha_j \mathrm{d}z_j + \sum_{k=1}^d \xi_k \mathrm{d}\theta_k + \sum_{j=1}^n \sum_{k=1}^d \xi_k \omega_{jk} \mathrm{d}\overline{z_j}, \quad \alpha_j, \xi_k \in \mathbb{C}.$$

This covector is real if and only if $\xi_k \in \mathbb{R}$ for all $k = 1, \dots, d$ and

$$\alpha_j = \sum_{k=1}^d \xi_k \overline{\omega_{jk}}, \quad j = 1, \dots, n.$$

We conclude that the characteristic set $T^\circ = T' \cap T^*(M \times \mathbb{T}^d)$ has (real) dimension d, and is generated by the real forms $\mathrm{d}\theta_k + 2\mathsf{Re}\,\omega_k$, $k = 1, \dots, d$. Computing the Lie brackets $[L_j, \overline{L_k}]$, we obtain

$$[L_j, \overline{L_k}] = \sum_{l=1}^d \left(\frac{\partial \omega_{jl}}{\partial z_k} - \frac{\overline{\partial \omega_{kl}}}{\partial z_j} \right) \frac{\partial}{\partial \theta_l}, \quad j, k = 1, \dots, n.$$

Therefore, the matrix of the Levi form $\mathfrak{L}_{(p,v_k)}$ with respect to the basis $\{L_1, \dots, L_n\}$ at the characteristic vector $v_l = \mathrm{d}\theta_l + 2\mathsf{Re}\,\omega_l$ is given by

$$\mathfrak{L}_{(p,v_l)} = \frac{1}{2i} \left(\frac{\partial \omega_{jl}}{\partial z_k} - \frac{\overline{\partial \omega_{kl}}}{\partial z_j} \right)_{1 \le j,k \le n}.$$

We obtain the following result:

Proposition 1.4.2. *Let $\omega_1, \dots, \omega_d$ be smooth $(0,1)$-forms on M satisfying $\overline{\partial}\omega_k = 0$ for all $k = 1, \dots, d$. Then, the following are equivalent:*

1. *T' is Levi-flat.*

2. *$\mathrm{d}(\omega_k + \overline{\omega_k}) = 0$ for all $k = 1, \dots, d$.*

Proof. Just observe that, in a system of holomorphic coordinates $(z_1, \dots, z_N)$ where $\omega_l = \sum_{j=1}^N \omega_{jl} \mathrm{d}\overline{z_j}$, we have

$$\mathrm{d}(\omega_l + \overline{\omega_l}) = \sum_{j,k=1}^n \left(\frac{\partial \omega_{jl}}{\partial z_k} - \frac{\overline{\partial \omega_{kl}}}{\partial z_j} \right) \mathrm{d}\overline{z_k} \wedge \mathrm{d}z_j,$$

since $\overline{\partial}\omega_l = 0$. □

From now on, we assume the structure T' is Levi-flat ($\mathrm{d}(\omega_k + \overline{\omega}_k) = 0$ for all $k = 1, \ldots, d$). The tangential CR complex associated to such a structure the following: for $0 \le q \le n$,

$$\left(\overline{\partial}_b\right)_q : C^\infty\left(M \times \mathbb{T}^d; \Lambda^q \mathbb{C}T^*M\right) \to C^\infty\left(M \times \mathbb{T}^d; \Lambda^{q+1}\mathbb{C}T^*M\right) \tag{1.4.1}$$

given by

$$\overline{\partial}_b u = \overline{\partial} u - \sum_{k=1}^{d} \omega_k \wedge \frac{\partial u}{\partial \theta_k},$$

where the derivatives $\partial/\partial\theta_k$ are defined component-wise. One can also consider this complex acting on distributional sections, i.e., currents on $M \times \mathbb{T}^d$ (valued in $\mathbb{C}T^*M$). We would like to deduce regularity properties of solutions of $\overline{\partial}_b u = f$ from the corresponding properties of solutions of the real complex

$$\mathbb{L}_q : C^\infty\left(M \times \mathbb{T}^d; \Lambda^q \mathbb{C}T^*M\right) \to C^\infty\left(M \times \mathbb{T}^d; \Lambda^{q+1}\mathbb{C}T^*M\right)$$

given by

$$\mathbb{L}_q u = \mathrm{d}_M u - \sum_{k=1}^{d} (\omega_k + \overline{\omega_k}) \wedge \frac{\partial u}{\partial \theta_k}. \tag{1.4.2}$$

(this defines a differential complex since $\mathrm{d}(\omega_k + \overline{\omega_k}) = 0$ for all $k = 1, \ldots, d$). The main technique we will use to compare both complexes involves the Laplacians, and we discuss this next.

First, we discuss the complex acting on functions (i.e., $q = 0$). Fix an hermitian metric on M (with corresponding volume form denoted by $\mathrm{d}V$) and consider the usual flat metric on the torus $\mathbb{T}^d$. This metric induces hermitian products in the exterior algebra $\Lambda^\bullet(\mathbb{C}T^*M)$. We can then consider the Hilbert spaces $L^2(M; \mathrm{d}V)$ (respectively, $L^2(M \times \mathbb{T}^d; \mathrm{d}V\mathrm{d}\theta)$) and $L^2(M, \mathbb{C}T^*M; \mathrm{d}V)$ (respectively, $L^2(M \times \mathbb{T}^d; \mathbb{C}T^*M)$) given by the (equivalence classes of) measurable sections of the indicated bundles that satisfy

$$\int_M \langle f, f \rangle \mathrm{d}V < \infty, \text{ (respectively, } \int_{M \times \mathbb{T}^d} \langle f, f \rangle \mathrm{d}V \mathrm{d}\theta < \infty).$$

We shall use the notation $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ for the global hermitian product and $\langle \cdot, \cdot \rangle(x)$ for the product in $\Lambda^\bullet(\mathbb{C}T_p^*M)$. We can then consider the Hilbertian adjoints

$$\overline{\partial}_b^*, \mathbb{L}^* : C^\infty(M \times \mathbb{T}^d; \mathbb{C}T^*M) \to C^\infty(M \times \mathbb{T}^d).$$

We shall compute these adjoints explicitely:

Proposition 1.4.3. *Let $\omega_1, \ldots, \omega_d \in C^\infty(M \times \mathbb{T}^d; \Lambda^{0,1}M)$ be $(0,1)$-forms satisfying $\overline{\partial}\omega_k = 0$ for all $k = 1, \ldots, d$. Then,*

1. *The adjoint of the map $\omega_k \frac{\partial}{\partial \theta_k} : C^\infty(M \times \mathbb{T}^d) \to C^\infty(M \times \mathbb{T}^d; \mathbb{C}T^*M)$ is the map $\beta \mapsto -\left\langle \frac{\partial \beta}{\partial \theta_k}, \omega_k \right\rangle$.*
2. *$\langle \mathrm{d}_M u, \omega_k + \overline{\omega_k} \rangle(x) = \langle \partial u, \overline{\omega_k} \rangle(x) + \langle \overline{\partial} u, \omega \rangle(x)$ for all $u \in C^\infty(M \times \mathbb{T}^d)$, $x \in M \times \mathbb{T}^d$ and $k = 1, \ldots, d$ (the same formula holds for the global product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$).*

3. *$|\omega_k + \overline{\omega_k}|^2(p) = 2|\omega_k|^2(p)$ for all $p \in M$ and $k = 1, \dots, d$ (the same formula holds for the global product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$).*

Proof. For item 1), let $f \in C^\infty(M \times \mathbb{T}^d)$ and $\beta \in C^\infty(M \times \mathbb{T}^d; \mathbb{C}T^*M)$. Then,

$$\begin{aligned}
\left\langle\!\!\left\langle \omega_k \frac{\partial f}{\partial \theta_k}, \beta \right\rangle\!\!\right\rangle &= \int_{M \times \mathbb{T}^d} \left\langle \frac{\partial f}{\partial \theta_k}(z,\theta)\omega_k(z), \beta(z) \right\rangle \mathrm{d}V \mathrm{d}\theta \\
&= \int_{M \times \mathbb{T}^d} \frac{\partial f}{\partial \theta_k}(z,\theta) \langle \omega_k(z), \beta(z) \rangle \mathrm{d}V \mathrm{d}\theta \\
&= -\int_{M \times \mathbb{T}^d} f(z,\theta) \left\langle \omega_k(z), \frac{\partial \beta}{\partial \theta_k} \right\rangle \mathrm{d}V \mathrm{d}\theta \\
&= \left\langle\!\!\left\langle f, -\left\langle \frac{\partial \beta}{\partial \theta_k}, \omega_k \right\rangle \right\rangle\!\!\right\rangle.
\end{aligned}$$

For the items 2) and 3), just observe that forms with different bidegrees are orthogonal and that $\langle \overline{v}, \overline{w} \rangle(x) = \overline{\langle v, w \rangle(x)}$. $\square$

We shall now perform some computations in local coordinates. Let $\Omega \subset M$ be an open subset with holomorphic coordinates $(z_1, \dots, z_n)$. We write the hermitian metric in these coordinates as

$$h = \sum_{j,k=1}^{n} h_{jk} \mathrm{d}z_j \otimes \mathrm{d}\overline{z_k},$$

where (h_{jk}) is a hermitian matrix of functions in $C^\infty(\Omega)$ (which are given by $h_{jk} = \langle \partial/\partial z_j, \partial/\partial z_k \rangle$). We denote the inverse of this matrix by (h^{jk}) (it's a simple exercise to verify that $h^{jk} = \langle \mathrm{d}z_k, \mathrm{d}z_j \rangle$). Let $\phi, \psi \in C^\infty(\Omega; \Lambda^{0,1}M)$, which we write as

$$\phi = \sum_{j=1}^{N} \phi_j \mathrm{d}\overline{z_j}, \ \ \psi = \sum_{k=1}^{N} \psi_k \mathrm{d}\overline{z_k}, \ \ \phi_j, \psi_k \in C^\infty(\Omega).$$

Then, we have

$$\begin{aligned}
\langle \phi, \psi \rangle(z) &= \sum_{j,k=1}^{N} \phi_j(z) \overline{\psi_k}(z) \langle \mathrm{d}\overline{z_j}, \mathrm{d}\overline{z_k} \rangle \\
&= \sum_{j,k=1}^{n} \phi_j(z) \overline{\psi_k}(z) h^{jk}(z).
\end{aligned}$$

In the same way, if ϕ e ψ are of type $(1,0)$ (given by $\phi = \sum \phi_j \mathrm{d}z_j$ and $\psi = \sum \psi_j \mathrm{d}z_j$), we get

$$\langle \phi, \psi \rangle(z) = \sum_{j,k=1}^{n} \phi_j(z) \overline{\psi_k}(z) h^{kj}(z) = \sum_{j,k=1}^{n} \phi_j(z) \overline{\psi_k}(z) \overline{h^{jk}(z)}.$$

We write, for fixed $k = 1, \dots, d$,

$$\omega_k = \sum_{j=1}^{N} \omega_{jk} \mathrm{d}\overline{z_j}, \ \ \omega_{jk} \in C^\infty(\Omega).$$

We would like to compare $d^*(f(\omega_k + \overline{\omega_k}))$ with $\overline{\partial}^*(f\omega_k)$ (where $f \in C^\infty(M \times \mathbb{T}^d)$), under the hypothesis that $\omega_k + \overline{\omega_k}$ is closed. Let $u \in C_c^\infty(\Omega \times \mathbb{T}^d)$. Then,

$$\begin{aligned}\langle\!\langle u, \mathrm{d}^*\left(f(\omega_k + \overline{\omega_k})\right) - 2\overline{\partial}^*(f\omega_k)\rangle\!\rangle &= \langle\!\langle \mathrm{d}u, f\omega_k + f\overline{\omega_k}\rangle\!\rangle - 2\langle\!\langle \overline{\partial}u, f\omega_k\rangle\!\rangle \\ &= \langle\!\langle \partial u, f\overline{\omega_k}\rangle\!\rangle - \langle\!\langle \overline{\partial}u, f\omega_k\rangle\!\rangle.\end{aligned}$$

Writing $h(z) = \det(h_{jk}(z))$ gives us (see, for instance, page 146 in Kodaira and Akao, 1986)

$$\begin{aligned}&\langle\!\langle u, \mathrm{d}^*(f(\omega_k + \overline{\omega_k})) - 2\overline{\partial}^*(f\omega_k)\rangle\!\rangle = \\ &= 2^n \int_{U\times\mathbb{T}^d} \left\{ \sum_{j,l=1}^n \frac{\partial u}{\partial z_j}(z,\theta)\overline{f(z,\theta)}\omega_{lk}(z)\overline{h^{jl}(z)} - \frac{\partial u}{\partial \overline{z}_j}(z,\theta)\overline{f(z,\theta)\omega_{lk}(z)}h^{jl}(z)\right\} h(z)\mathrm{d}x\mathrm{d}\theta \\ &= 2^n \sum_{j,l=1}^n \int_{U\times\mathbb{T}^d} \left\{ \frac{\partial u}{\partial z_l}(z,\theta)\omega_{jk}(z) - \frac{\partial u}{\partial \overline{z}_j}(z,\theta)\overline{\omega_{lk}(z)}\right\} h^{jl}(z)h(z)\overline{f(z,\theta)}\mathrm{d}x\mathrm{d}\theta.\end{aligned}$$

Now we shall integrate by parts. When we integrate the terms of the form ω_{lk}, the term that will appear will be of the form $\sum_{j,l} \frac{\partial \omega_{jk}}{\partial z_l} - \overline{\frac{\partial \omega_{lk}}{\partial z_j}}$, which vanishes since $\mathrm{d}(\omega_k + \overline{\omega_k}) = 0$ (see the proof of Proposition 1.4.2). Therefore, we need only to integrate the terms of the form $h^{jl}h$ (which we call (I)) and $\overline{f}$ (which we call (II)). We obtain

$$\begin{aligned}\text{(I)} &= -2^n \sum_{j,l=1}^n \int_{U\times\mathbb{T}^d} \left\{ \frac{\partial(h^{jl}h)}{\partial z_l}\omega_{jk}(z) - \frac{\partial(h^{jl}h)}{\partial \overline{z}_j}\overline{\omega_{lk}(z)}\right\} u(z,\theta)\overline{f(z,\theta)}\mathrm{d}x\mathrm{d}\theta \\ &= 2^n \int_{U\times\mathbb{T}^d} u(z,\theta)\left\{ \overline{f(z,\theta)}\frac{1}{h(z)}\sum_{j,l=1}^n \left(\frac{\partial(h^{jl}h)}{\partial \overline{z}_j}\overline{\omega_{lk}(z)} - \frac{\partial(h^{jl}h)}{\partial z_l}\omega_{jk}(z)\right)\right\} h(z)\mathrm{d}x\mathrm{d}\theta \\ &= \int_{U\times\mathbb{T}^d} u(z,\theta)\cdot \overline{\left\{ f(z,\theta)\cdot\left(\frac{1}{h(z)}\sum_{j,l=1}^n \frac{\partial(\overline{h^{jl}h})}{\partial z_j}\omega_{lk}(z) - \frac{\partial(\overline{h^{jl}h})}{\partial \overline{z}_l}\overline{\omega_{jk}(z)}\right)\right\}} 2^n h(z)\mathrm{d}x\mathrm{d}\theta \\ &= \langle\!\langle u, f(R_k - \overline{R_k})\rangle\!\rangle,\end{aligned}$$

where $R_k \in C^\infty(\Omega)$ is given by

$$R_k(z) = \frac{1}{h(z)}\sum_{j,l=1}^n \frac{\partial(\overline{h^{jl}h})}{\partial z_j}\omega_{lk}(z).$$

For the term (II), we have

$$\begin{aligned}\text{(II)} &= -2^n \sum_{j,l=1}^n \int_{U\times\mathbb{T}^d} \left\{ \frac{\partial \overline{f}}{\partial z_l}(z,\theta)\omega_{jk}(z) - \frac{\partial \overline{f}}{\partial \overline{z}_j}(z,\theta)\overline{\omega_{lk}(z)}\right\} h^{jl}(z)h(z)u(z,\theta)\mathrm{d}x\mathrm{d}\theta \\ &= 2^n \int_{U\times\mathbb{T}^d} u(z,\theta)\overline{\left\{ \sum_{j,l=1}^n \frac{\partial f}{\partial z_j}(z,\theta)\omega_{lk}(z)\overline{h^{jl}(z)} - \sum_{j,l=1}^n \frac{\partial f}{\partial \overline{z}_l}(z,\theta)\overline{\omega_{jk}(z)h^{jl}(z)}\right\}} h(z)\mathrm{d}x\mathrm{d}\theta \\ &= \langle\!\langle u, \langle \partial f, \overline{\omega_k}\rangle - \langle \overline{\partial} f, \omega_k\rangle\rangle\!\rangle.\end{aligned}$$

Since $u \in C_c^\infty(\Omega \times \mathbb{T}^d)$ is arbitrary, we conclude that

$$\mathrm{d}^*\left(f(\omega_k + \overline{\omega_k})\right) - 2\overline{\partial}^*(f\omega_k) = \langle \partial f, \overline{\omega_k}\rangle - \langle \overline{\partial} f, \omega_k\rangle + f(R_k - \overline{R_k}) \tag{1.4.3}$$

in $\Omega \times \mathbb{T}^d$. With these results in hand, we compare the Laplacians $\overline{\partial}_b^*\overline{\partial}_b$ and $\mathbb{L}^*\mathbb{L}$ (when acting on functions supported on $\Omega \times \mathbb{T}^d$). From Proposition 1.4.3, we have

$$\overline{\partial}_b^* = \overline{\partial}^* + \sum_{k=1}^d \left\langle \frac{\partial}{\partial\theta_k}, \omega_k \right\rangle, \quad \mathbb{L}^* = \mathrm{d}^* + \sum_{k=1}^d \left\langle \frac{\partial}{\partial\theta_k}, \omega_k + \overline{\omega_k} \right\rangle.$$

We obtain then

$$\begin{aligned}
\overline{\partial}_b^*\overline{\partial}_b &= \left(\overline{\partial}^* + \sum_{k=1}^d \left\langle \frac{\partial\,\cdot}{\partial\theta_k}, \omega_k \right\rangle\right)\left(\overline{\partial} - \sum_{k=1}^d \omega_k \frac{\partial}{\partial\theta_k}\right) \\
&= \overline{\partial}^*\overline{\partial} - \sum_{k=1}^d \overline{\partial}^*\left(\omega_k \frac{\partial}{\partial\theta_k}\right) + \sum_{k=1}^d \left\langle \frac{\partial}{\partial\theta_k}\overline{\partial}, \omega_k \right\rangle - \sum_{k,k'=1}^d \langle \omega_{k'}, \omega_k\rangle \frac{\partial^2}{\partial\theta_k\partial\theta_{k'}}.
\end{aligned}$$

In the same way,

$$\begin{aligned}
\mathbb{L}^*\mathbb{L} &= \left(\mathrm{d}^* + \sum_{k=1}^d \left\langle \frac{\partial}{\partial\theta_k}, \omega_k + \overline{\omega_k} \right\rangle\right)\left(\mathrm{d} - \sum_{k=1}^d (\omega_k + \overline{\omega_k})\frac{\partial}{\partial\theta_k}\right) \\
&= \mathrm{d}^*\mathrm{d} - \sum_{k=1}^d \mathrm{d}^*\left((\omega_k + \overline{\omega_k})\frac{\partial}{\partial\theta_k}\right) + \sum_{k=1}^d \left\langle \frac{\partial}{\partial\theta_k}\mathrm{d}, \omega_k + \overline{\omega_k} \right\rangle - \sum_{k,k'=1}^d \langle \omega_{k'} + \overline{\omega_{k'}}, \omega_k + \overline{\omega_k}\rangle \frac{\partial^2}{\partial\theta_k\partial\theta_{k'}} \\
&= \mathrm{d}^*\mathrm{d} - \sum_{k=1}^d \mathrm{d}^*\left((\omega_k + \overline{\omega_k})\frac{\partial}{\partial\theta_k}\right) + \sum_{k=1}^d \left(\left\langle \frac{\partial}{\partial\theta_k}\partial, \overline{\omega_k} \right\rangle + \left\langle \frac{\partial}{\partial\theta_k}\overline{\partial}, \omega_k \right\rangle\right) - 2\sum_{k,k'=1}^d \langle \omega_{k'}, \omega_k\rangle \frac{\partial^2}{\partial\theta_k\partial\theta_{k'}}
\end{aligned}$$

Putting everything together, we obtain

$$\begin{aligned}
\mathbb{L}^*\mathbb{L} - 2\overline{\partial}^*\overline{\partial} &= \mathrm{d}^*\mathrm{d} - 2\overline{\partial}^*\overline{\partial} - \sum_{k=1}^d \left(\mathrm{d}^*\left((\omega_k + \overline{\omega_k})\frac{\partial}{\partial\theta_k}\right) - 2\overline{\partial}^*\left(\omega_k \frac{\partial}{\partial\theta_k}\right)\right) + \\
&\quad + \sum_{k=1}^d \left(\left\langle \frac{\partial}{\partial\theta_k}\partial, \overline{\omega_k} \right\rangle - \left\langle \frac{\partial}{\partial\theta_k}\overline{\partial}, \omega_k \right\rangle\right) \\
&= \mathrm{d}^*\mathrm{d} - 2\overline{\partial}^*\overline{\partial} - \sum_{k=1}^d \left(\left\langle \partial\frac{\partial}{\partial\theta_k}, \overline{\omega_k} \right\rangle - \left\langle \overline{\partial}\frac{\partial}{\partial\theta_k}, \omega_k \right\rangle + (R_k - \overline{R_k})\frac{\partial}{\partial\theta_k}\right) + \\
&\quad + \sum_{k=1}^d \left(\left\langle \frac{\partial}{\partial\theta_k}\partial, \overline{\omega_k} \right\rangle - \left\langle \frac{\partial}{\partial\theta_k}\overline{\partial}, \omega_k \right\rangle\right) \\
&= \mathrm{d}^*\mathrm{d} - 2\overline{\partial}^*\overline{\partial} - \sum_{k=1}^d (R_k - \overline{R_k})\frac{\partial}{\partial\theta_k},
\end{aligned}$$

in $\Omega \times \mathbb{T}^d$. Taking $f \equiv 1$ in (1.4.3) shows that $R_k - \overline{R_k} = \mathrm{d}^*(\omega_k + \overline{\omega_k}) - 2\overline{\partial}^*(\omega_k) = \partial^*\overline{\omega_k} - \overline{\partial}^*\omega_k$.

Therefore, we can invariantly write the identity

$$\mathbb{L}^*\mathbb{L} - 2\overline{\partial}_b^*\overline{\partial}_b = \square_{\mathrm{d}} - 2\square_{\overline{\partial}} + 2i\sum_{k=1}^{d}(\operatorname{Im}\overline{\partial}^*\omega_k)\frac{\partial}{\partial\theta_k}, \tag{1.4.4}$$

where $\square_{\mathrm{d}} = \mathrm{d}^*\mathrm{d}$ and $\square_{\overline{\partial}} = \overline{\partial}^*\overline{\partial}$ are the Laplace-Beltrami and Laplace-Dolbeault's operators, acting on functions (this follows because both these operators are local and are equal in small neighborhoods of every point of $M \times \mathbb{T}^d$).

If the manifold M admits a particular kind of metric, this expression can be simplified even further.

Definition 1.4.1. Let $0 \leq q \leq 2n$. A complex manifold M is *balanced in degree* q if it admits a hermitian metric h such that $\frac{1}{2}\square_{\mathrm{d}}^q := \frac{1}{2}\left(\mathrm{d}_q^*\mathrm{d}_q + \mathrm{d}_{q-1}\mathrm{d}_{q-1}^*\right) = (\overline{\partial}_q^*\overline{\partial}_q + \overline{\partial}_{q-1}\overline{\partial}_{q-1}^*) =: \square_{\overline{\partial}}^q$.

It is well-known that Kähler manifolds are balanced in every degree $0 \leq q \leq 2n$ (see Corollary 6.5, page 306 in Demailly, n.d.). Moreover, a result of Hsiung, 1966 shows that, conversely, if a complex manifold is balanced in degrees 0 and 1, then it is a Kähler manifold. However, for fixed values of q, the class of manifolds that are balanced in that particular degree might include non-Kähler manifolds (for example, in dimension $n \geq 3$, there are balanced manifolds in degree zero that are not Kähler, Michelsohn, 1982).

Theorem 1.4.1. *Assume the manifold M is balanced in degree* 0*. Then,*

$$\frac{1}{2}\mathbb{L}^*\mathbb{L} = \overline{\partial}_b^*\overline{\partial}_b,$$

when the adjoints are taken with respect to the balanced metric.

Proof. Let $k = 1, \dots, d$ be fixed. Since $\mathrm{d}(\omega_k + \overline{\omega_k}) = 0$, we can write in a sufficiently small neighborhood Ω of a point $z_0 \in M$ the equation $\omega_k + \overline{\omega_k} = \mathrm{d}f$, where $f \in C^\infty(\Omega; \mathbb{R})$, by Poincaré's lemma. In particular, we have $\overline{\partial}f = \omega_k$. Then, in Ω,

$$\overline{\partial}^*\omega_k = \overline{\partial}^*\overline{\partial}f = \frac{1}{2}\square_{\mathrm{d}}f,$$

which is a real-valued function (again, using $\square_{\mathrm{d}} = 2\square_{\overline{\partial}}$). The identity now follows from 1.4.4. □

Now we shall study what happens in larger degree (i.e., $q \geq 1$). We consider now the Hilbert spaces $L^2(M; \Lambda^{p,q}M; \mathrm{d}V)$; and $L^2(M \times \mathbb{T}^d; \Lambda^{p,q}; \mathrm{d}V\mathrm{d}\theta)$, for $0 \leq p, q \leq n$. We shall recall some elementary geometric operations.

Definition 1.4.2. Let N be a smooth manifold and $X \in \mathbb{C}T_pN$ be a tangent vector at $p \in N$. If $u \in \Lambda^q T_p^*N$ is a q-covector over on p ($q \geq 1$), then the contraction $X \lrcorner u$ is defined by

$$(X \lrcorner u)(v_1, \dots, v_{q-1}) := u(X, v_1, \dots, v_{q-1}), \;\; v_j \in \mathbb{C}T_pN.$$

Definition 1.4.3. Let (N, g) be an n-dimensional smooth Riemannian manifold. The musical isomorphism at $p \in N$ is defined as

$$\begin{aligned} \flat_p \, : \, \mathbb{C}T_pN &\to \mathbb{C}T_p^*N \\ v &\mapsto \left(\mathbb{C}T_pN \ni w \mapsto \langle w, v\rangle_p\right), \end{aligned}$$

with inverse denoted by $\sharp_p := \flat_p^{-1} : \mathbb{C}T_p^*N \to \mathbb{C}T_pN$.

Remark 1.4.1. Let $(x_1, \dots, x_n)$ be a system of coordinates in N. Then, writing $g_{ij} = \langle \partial/\partial x_i, \partial/\partial x_j\rangle$ for the coefficients of the Riemannian metric, we have the following: let $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$ be a tangent vector. Then, if $X^\flat = \sum_{i=1}^n \alpha_i \mathrm{d}x_i$,

$$\begin{aligned} \alpha_i = X^\flat(\partial/\partial x_i) &= \left\langle \frac{\partial}{\partial x_i}, X \right\rangle \\ &= \sum_{j=1}^n g_{ij}\overline{X_j}, \end{aligned}$$

i.e., $X^\flat = \sum_{i=1}^n \left(\sum_{j=1}^n g_{ij}\overline{X_j}\right) \mathrm{d}x_i$. In the same way, if $\omega = \sum_{i=1}^n \omega_i \mathrm{d}x_i$, we have $\omega^\sharp = \sum_{i=1}^n \left(\sum_{j=1}^n g^{ij}\overline{\omega_j}\right) \frac{\partial}{\partial x_i}$, where (g^{ij}) is the inverse matrix of (g_{ij}).

Now, if $\omega \in C^\infty(N; \mathbb{C}T^*N)$ is a one-form in N, we can define the interior product by ω using the musical isomorphism: if $\alpha \in C^\infty(N; \Lambda^q T^*N)$, $q \geq 1$, then

$$\omega \lrcorner \alpha := \omega^\sharp \lrcorner \alpha.$$

Lemma 1.4.1. *Let (N, g) be a Riemannian manifold and let $p \in N$.*

1. *Let $\omega_p \in CT_p^*N$, $\alpha_p \in \Lambda^q T_p^*N$ and $\eta_p \in \Lambda^{q-1}T_p^*N$, $q \geq 1$. Then,*

   $$\langle \omega_p \lrcorner \alpha_p, \eta_p\rangle = \langle \alpha_p, \omega_p \wedge \eta_p\rangle.$$

   In particular, if $\omega \in C^\infty(N; \mathbb{C}T^*N)$, $\alpha \in C^\infty(N; \Lambda^q T^*N)$ and $\eta \in C^\infty(N; \Lambda^{q-1}T^*N)$, we have $\langle\!\langle \omega \lrcorner \alpha, \eta\rangle\!\rangle = \langle\!\langle \alpha, \omega \wedge \eta\rangle\!\rangle$.

2. *The derivation property*

   $$\omega_p \lrcorner (u_p \wedge v_p) = (\omega_p \lrcorner u_p) \wedge v_p + (-1)^q u_p \wedge (\omega_p \lrcorner v_p)$$

   holds, for $\omega_p \in \mathbb{C}T_p^*N$, $u_p \in \Lambda^q T_p^*N$ and $v_p \in \Lambda^r T_p^*N$, $q + r \geq 1$.

3. *If $\omega_1, \omega_2 \in \mathbb{C}T_p^*N$, then*

   $$\omega_1 \lrcorner \omega_2 = \langle \omega_2, \omega_1\rangle.$$

Proof. Item 1) : Let $\{\xi_1, \dots, \xi_n\}$ be an orthonormal basis for $\mathbb{C}T_pN$. Then, $\{\xi_1^\flat, \dots, \xi_n^\flat\}$ is the dual basis, and is also an orthonormal basis for $\mathbb{C}T_p^*N$. The result is clear for $\omega_p = \xi_i^\flat$, $\alpha_p = \xi_I^\flat$ and $\eta_p = \xi_J^\flat$, $|I| = q$, $|J| = q - 1$, and this implies the general case by linearity. Item 2) is easily verified using this orthonormal basis. For the final item, writing $\omega_i = \sum_{j=1}^n \omega_{ij}\xi_j^\flat$,

$i = 1, 2$, we have

$$\omega_1 \lrcorner \omega_2 = \sum_{l,k=1}^{n} \overline{\omega_{1l}} \omega_{2k} \xi_l \lrcorner \xi_k^\flat = \sum_{l=1}^{n} \overline{\omega_{1l}} \omega_{2l} = \langle \omega_2, \omega_1 \rangle.$$

□

Remark 1.4.2. We remark that the interior product is local, i.e., if $U \subset M$ is an open set, $(\omega \lrcorner \alpha)\big|_U = (\omega\big|_U) \lrcorner (\alpha\big|_U)$.

We prove a basic identity for d^* and $\overline{\partial}^*$.

Lemma 1.4.2 (Leibniz formula)**.** *Let $U \subset M$ be an open set, $\alpha \in C^\infty(U; \Lambda^q T^* M)$ be a q-form (with $q \geq 1$) and $f \in C^\infty(U)$. Then,*

$$\mathrm{d}^* (f\alpha) = f (\mathrm{d}^* \alpha) - \left(\mathrm{d}\overline{f}\right) \lrcorner \alpha \text{ in } U \tag{1.4.5}$$

and

$$\overline{\partial}^* (f\alpha) = f \left(\overline{\partial}^* \alpha\right) - \left(\overline{\partial}\overline{f}\right) \lrcorner \alpha \text{ in } U. \tag{1.4.6}$$

Proof. We prove only (1.4.5) (the other one being analogous). Let $\beta \in C_c^\infty(U; \Lambda^{q-1} T^* M)$. Then,

$$\begin{aligned} \langle\!\langle \mathrm{d}^*(f\alpha), \beta \rangle\!\rangle &= \langle\!\langle \alpha, \overline{f} \mathrm{d}\beta \rangle\!\rangle \\ &= \langle\!\langle \alpha, \mathrm{d}(\overline{f}\beta) - (\mathrm{d}\overline{f}) \wedge \beta \rangle\!\rangle \\ &= \langle\!\langle f(\mathrm{d}^*\alpha) - (\mathrm{d}\overline{f}) \lrcorner \alpha, \beta \rangle\!\rangle. \end{aligned}$$

□

Now we can proceed to the main result.

Theorem 1.4.2. *Assume that M is balanced in degree $q \in \{0, \dots, 2N\}$. Then,*

$$\frac{1}{2} \square_{\mathbb{L}}^q = \square_{\overline{\partial}_b}^q. \tag{1.4.7}$$

Proof. The previous section covers the case $q = 0$, so we assume $q \geq 1$. Fix a point $p \in M$ and let $U_p \subset M$ be an open neighborhood of p such that there exist real-valued functions $f_1, \dots, f_d \in C^\infty(U_p; \mathbb{R})$ such that $\mathrm{d} f_k = \omega_k + \overline{\omega_k}$ for all $k = 1, \dots, d$ (this neighborhood exists by Poincaré's lemma, since $\mathrm{d}(\omega_k + \overline{\omega_k}) = 0$ for all $k = 1, \dots, d$). We shall prove that

$$\frac{1}{2} \square_{\mathbb{L}}^q u = \square_{\overline{\partial}_b}^q u$$

for all $u \in C_c^\infty(U_p \times \mathbb{T}^d; \Lambda^q T^* M)$. Note that, by applying a partition of unity, this implies that $\frac{1}{2}\Box^q_{\mathbb{L}} = \Box^q_{\overline{\partial}_b}$. Let $u \in C^\infty_c(U_p \times \mathbb{T}^d; \Lambda^q \mathbb{C}T^*M)$. Then,

$$\begin{aligned}
\Box^q_{\mathbb{L}} u &= \mathbb{L}^*_q \mathbb{L}_q u + \mathbb{L}_{q-1}\mathbb{L}^*_{q-1} u \\
&= \mathbb{L}^*_q \left(\mathrm{d}u - \sum_{k=1}^d (\omega_k + \overline{\omega_k}) \wedge \frac{\partial u}{\partial \theta_k} \right) + \mathbb{L}_{q-1} \left(\mathrm{d}^* u + \sum_{k=1}^d (\omega_k + \overline{\omega_k}) \lrcorner \frac{\partial u}{\partial \theta_k} \right) \\
&= \mathrm{d}^*\mathrm{d}u - \sum_{k=1}^d \mathrm{d}^* \left((\omega_k + \overline{\omega_k}) \wedge \frac{\partial u}{\partial \theta_k} \right) + \sum_{k=1}^d (\omega_k + \overline{\omega_k}) \lrcorner \frac{\partial (\mathrm{d}u)}{\partial \theta_k} \\
&\quad - \sum_{k,k'=1}^d (\omega_k + \overline{\omega_k}) \lrcorner \left((\omega_{k'} + \overline{\omega_{k'}}) \wedge \frac{\partial^2 u}{\partial \theta_k \partial \theta_{k'}} \right) + \mathrm{d}\mathrm{d}^* u + \sum_{k=1}^d \mathrm{d} \left((\omega_k + \overline{\omega_k}) \lrcorner \frac{\partial u}{\partial \theta_k} \right) \\
&\quad - \sum_{k=1}^d (\omega_k + \overline{\omega_k}) \wedge \frac{\partial (\mathrm{d}^* u)}{\partial \theta_k} - \sum_{k,k'=1}^d (\omega_k + \overline{\omega_k}) \wedge \left((\omega_{k'} + \overline{\omega_{k'}}) \lrcorner \frac{\partial^2 u}{\partial \theta_k \partial \theta_{k'}} \right).
\end{aligned}$$

From Lemma 1.4.1,

$$\begin{aligned}
&\sum_{k,k'=1}^d (\omega_k + \overline{\omega_k}) \lrcorner \left((\omega_{k'} + \overline{\omega_{k'}}) \wedge \frac{\partial^2 u}{\partial \theta_k \partial \theta_{k'}} \right) = \\
&= \sum_{k,k'=1}^d ((\omega_k + \overline{\omega_k}) \lrcorner (\omega_{k'} + \overline{\omega_{k'}})) \wedge \frac{\partial^2 u}{\partial \theta_k \partial \theta_{k'}} - (\omega_{k'} + \overline{\omega_{k'}}) \wedge \left((\omega_k + \overline{\omega_k}) \lrcorner \frac{\partial^2 u}{\partial \theta_k \partial \theta_{k'}} \right),
\end{aligned}$$

which implies that (again, using 1.4.1 and the orthogonality relations)

$$\begin{aligned}
&\sum_{k,k'=1}^d (\omega_k + \overline{\omega_k}) \lrcorner \left((\omega_{k'} + \overline{\omega_{k'}}) \wedge \frac{\partial^2 u}{\partial \theta_k \partial \theta_{k'}} \right) + \sum_{k,k'=1}^d (\omega_k + \overline{\omega_k}) \wedge \left((\omega_{k'} + \overline{\omega_{k'}}) \lrcorner \frac{\partial^2 u}{\partial \theta_k \partial \theta_{k'}} \right) = \\
&= 2 \sum_{k,k'=1}^d (\omega_k \lrcorner \omega_{k'}) \wedge \frac{\partial^2 u}{\partial \theta_k \partial \theta_{k'}}.
\end{aligned}$$

Now we exploit the locality of the operators. From (1.4.2), we have

$$\begin{aligned}
\sum_{k=1}^d \Box^q_{\mathrm{d}} \left(f_k \frac{\partial u}{\partial \theta_k} \right) &= \sum_{k=1}^d \left(\mathrm{d} \left(f_k \frac{\partial \mathrm{d}^* u}{\partial \theta_k} - (\omega_k + \overline{\omega_k}) \lrcorner \frac{\partial u}{\partial \theta_k} \right) + \mathrm{d}^* \left((\omega_k + \overline{\omega_k}) \wedge \frac{\partial u}{\partial \theta_k} + f_k \frac{\partial \mathrm{d}u}{\partial \theta_k} \right) \right) \\
&= \sum_{k=1}^d \left((\omega_k + \overline{\omega_k}) \wedge \frac{\partial \mathrm{d}^* u}{\partial \theta_k} + f_k \frac{\partial \mathrm{d}\mathrm{d}^* u}{\partial \theta_k} \right) - \sum_{k=1}^d \mathrm{d} \left((\omega_k + \overline{\omega_k}) \lrcorner \frac{\partial u}{\partial \theta_k} \right) + \\
&\quad + \sum_{k=1}^d \mathrm{d}^* \left((\omega_k + \overline{\omega_k}) \wedge \frac{\partial u}{\partial \theta_k} \right) + \sum_{k=1}^d f_k \frac{\partial \mathrm{d}^* \mathrm{d}u}{\partial \theta_k} - \sum_{k=1}^d (\omega_k + \overline{\omega_k}) \lrcorner \frac{\partial \mathrm{d}u}{\partial \theta_k}.
\end{aligned}$$

We obtain, then,

$$\sum_{k=1}^{d}\left(\Box_{\mathrm{d}}^{q}\left(f_k\frac{\partial u}{\partial\theta_k}\right)-f_k\frac{\partial(\Box_{d}^{q}u)}{\partial\theta_k}\right)=\sum_{k=1}^{d}(\omega_k+\overline{\omega_k})\wedge\frac{\partial \mathrm{d}^*u}{\partial\theta_k}-\sum_{k=1}^{d}\mathrm{d}\left((\omega_k+\overline{\omega_k})\lrcorner\frac{\partial u}{\partial\theta_k}\right)$$
$$+\sum_{k=1}^{d}\mathrm{d}^*\left((\omega_k+\overline{\omega_k})\wedge\frac{\partial u}{\partial\theta_k}\right)-\sum_{k=1}^{d}(\omega_k+\overline{\omega_k})\lrcorner\frac{\partial \mathrm{d}u}{\partial\theta_k}.$$

Going back to the expression of $\Box_{\mathbb{L}}^{q}u$, we obtain the fundamental identity

$$\Box_{\mathbb{L}}^{q}u=\Box_{\mathrm{d}}^{q}u+\sum_{k=1}^{d}\left(f_k\frac{\partial(\Box_{\mathrm{d}}^{q}u)}{\partial\theta_k}-\Box_{\mathrm{d}}^{q}\left(f_k\frac{\partial u}{\partial\theta_k}\right)\right)-2\sum_{k,k'=1}^{d}(\omega_k\lrcorner\omega_{k'})\wedge\frac{\partial^2u}{\partial\theta_k\partial\theta_{k'}}.$$

In the same way, for $\Box_{\overline{\partial}_b}^{q}=(\overline{\partial}_b)_q^*(\overline{\partial}_b)_q+(\overline{\partial}_b)_{q-1}(\overline{\partial}_b)_{q-1}^*$ we have

$$\begin{aligned}
\Box_{\overline{\partial}_b}^{q}u&=(\overline{\partial}_b)_q^*(\overline{\partial}_b)_qu+(\overline{\partial}_b)_{q-1}(\overline{\partial}_b)_{q-1}^*u\\
&=(\overline{\partial}_b)_q^*\left(\overline{\partial}u-\sum_{k=1}^{d}\omega_k\wedge\frac{\partial u}{\partial\theta_k}\right)+(\overline{\partial}_b)_{q-1}\left(\overline{\partial}^*u+\sum_{k=1}^{d}\omega_k\lrcorner\frac{\partial u}{\partial\theta_k}\right)\\
&=\overline{\partial}^*\overline{\partial}u-\sum_{k=1}^{d}\overline{\partial}^*\left(\omega_k\wedge\frac{\partial u}{\partial\theta_k}\right)+\sum_{k=1}^{d}\omega_k\lrcorner\frac{\partial(\overline{\partial}u)}{\partial\theta_k}-\sum_{k,k'=1}^{d}\omega_k\lrcorner\left(\omega_{k'}\wedge\frac{\partial^2u}{\partial\theta_k\partial\theta_{k'}}\right)+\\
&+\overline{\partial}\overline{\partial}^*u+\sum_{k=1}^{d}\overline{\partial}\left(\omega_k\lrcorner\frac{\partial u}{\partial\theta_k}\right)-\sum_{k=1}^{d}\omega_k\wedge\frac{\partial(\overline{\partial}^*u)}{\partial\theta_k}-\sum_{k,k'=1}^{d}\omega_k\wedge\left(\omega_{k'}\lrcorner\frac{\partial^2u}{\partial\theta_k\partial\theta_{k'}}\right).
\end{aligned}$$

Making exactly the same maneuvers we did for $\Box_{\mathbb{L}}^{q}$ yields the identity (recall that $\overline{\partial}f_k=\omega_k$ for all $k=1,\dots,d$)

$$\Box_{\overline{\partial}_b}^{q}u=\Box_{\overline{\partial}}^{q}u+\sum_{k=1}^{d}\left(f_k\frac{\partial(\Box_{\overline{\partial}}^{q}u)}{\partial\theta_k}-\Box_{\overline{\partial}}^{q}\left(f_k\frac{\partial u}{\partial\theta_k}\right)\right)-\sum_{k,k'=1}^{d}(\omega_k\lrcorner\omega_{k'})\wedge\frac{\partial^2u}{\partial\theta_k\partial\theta_{k'}}.$$

The hypothesis that $\frac{1}{2}\Box_{\mathrm{d}}^{q}=\Box_{\overline{\partial}}^{q}$ yields the result. ☐

We will, in this section, determine criteria that allow us to decide whether or not a differential form f on $M\times\mathbb{T}^d$ (valued in $\mathbb{C}T^*M$) is smooth by analyzing the decay (in L^∞ or L^2 norms) of the Fourier coefficients $\widehat{f}(j)$, $j\in\mathbb{Z}^d$. To do so, we need to consider the operators $\mathbb{L}$ and $\overline{\partial}_b$ acting on currents: we use the notation $\mathcal{D}'(M\times\mathbb{T}^d;\Lambda^p\mathbb{C}T^*M)$ for the space of currents on $M\times\mathbb{T}^d$ of degree p, valued in $\mathbb{C}T^*M$. Any such current can be written locally a differential p-form on M with distributional coefficients defined on $M\times\mathbb{T}^d$.

We define the following spaces, for $1\leq q\leq 2n$:

$$X_P^q=\left\{\alpha\in\mathcal{D}'(M\times\mathbb{T}^d;\Lambda^q\mathbb{C}T^*M);\ P_q\alpha\text{ and }P_{q-1}^*\alpha\text{ are smooth sections}\right\}.$$

where P is either $\mathbb{L}$ or $\overline{\partial}_b$. If $\alpha \in X_P^q$, then we can write a Fourier decomposition

$$\alpha = \sum_{j \in \mathbb{Z}^d} \widehat{\alpha}(j) e^{i\langle j,\theta\rangle}, \ \theta \in \mathbb{T}^d,$$

where $\widehat{\alpha}(j) \in \mathcal{D}'(M; \Lambda^q \mathbb{C}T^*M)$ is a q-current in M. A simple verification shows that

$$P\alpha = \sum_{j \in \mathbb{Z}^d} P_j(\widehat{\alpha}(j)) e^{i\langle j,\theta\rangle},$$

where P_j is $D_j := \mathrm{d} - i\sum_{k=1}^d j_k(\omega_k + \overline{\omega_k}) \wedge \cdot$ if $P = \mathbb{L}$ and $\delta_j := \overline{\partial} - i\sum_{k=1}^d j_k\omega_k \wedge \cdot$ if $P = \overline{\partial}_b$ (we omit the degree q in the notation for simplicity). Since P_j form elliptic complexes for every fixed $j \in \mathbb{Z}^d$, we conclude that $\widehat{\alpha}(j) \in C^\infty(M; \Lambda^q\mathbb{C}T^*M)$ for every $j \in \mathbb{Z}^d$.

We now turn to main result of this section.

Theorem 1.4.3. *Let P denote either $\mathbb{L}$ or $\overline{\partial}_b$. Let $\alpha \in X_P^q$, for $q \geq 1$. Then, the following are equivalent:*

1. *$\alpha \in C^\infty(M \times \mathbb{T}^d; \Lambda^q\mathbb{C}T^*M)$.*
2. *For every $A \in \mathbb{Z}_+$,*
$$\sup_{j \in \mathbb{Z}^d} (1 + |j|)^A \left\| \widehat{\alpha}(j) \right\|_{L^\infty(M)} < \infty.$$
3. *For every $A \in \mathbb{Z}_+$,*
$$\sup_{j \in \mathbb{Z}^d} (1 + |j|)^A \left\| \widehat{\alpha}(j) \right\|_{L^2(M)} < \infty.$$

Proof. We first adress the case $P = \mathbb{L}$. It is clear that 1) $\implies$ 2) $\implies$ 3). To show 3) $\implies$ 1), it is enough to show that for every $s, A \in \mathbb{Z}_+$,

$$\sup_{j \in \mathbb{Z}^d} (1 + |j|)^A \|\widehat{\alpha}(j)\|_{W^s(M)} < \infty, \tag{1.4.8}$$

where $W^s(M)$ is the usual L^2 Sobolev space of order s. We shall prove (1.4.8) by induction on $s \in \mathbb{Z}_+$. If $s = 0$, this is just our hypothesis 3). Assume now that this estimate holds for $s \in \mathbb{Z}_+$. Let $f := \mathbb{L}\alpha \in C^\infty(M \times \mathbb{T}^d; \Lambda^{q+1}\mathbb{C}T^*M)$ and $f^* := \mathbb{L}^*\alpha \in C^\infty(M \times \mathbb{T}^d; \Lambda^{q-1}\mathbb{C}T^*M)$. We can write $f = \sum_{j \in \mathbb{Z}_d} f_j e^{i\langle j,\theta\rangle}$ and $f^* = \sum_{j \in \mathbb{Z}^d} f_j^* e^{i\langle j,\theta\rangle}$, where $f_j = \mathbb{L}_j\widehat{\alpha}(j)$ and $f_j^* = \mathbb{L}_j^*\widehat{\alpha}(j)$ for all $j \in \mathbb{Z}^d$. Then, $\widehat{\alpha}(j)$ satisfies the two following equations:

$$\begin{cases} \mathrm{d}\widehat{\alpha}(j) = i\sum_{k=1}^d j_k(\omega_k + \overline{\omega_k}) \wedge \widehat{\alpha}(j) + f_j \\ \mathrm{d}^*\widehat{\alpha}(j) = i\sum_{k=1}^d j_k(\omega_k + \overline{\omega_k}) \lrcorner \widehat{\alpha}(j) + f_j^*. \end{cases}$$

Since the de Rham complex is elliptic, there is[1] a constant $C > 0$ such that

$$\|\beta\|_{W^{s+1}(M)} \leq C\left(\|\mathrm{d}\beta\|_{W^s(M)} + \|\mathrm{d}^*\beta\|_{W^s(M)} + \|\beta\|_{L^2(M)}\right), \ \beta \in C^\infty(M; \Lambda^q\mathbb{C}T^*M). \tag{1.4.9}$$

[1] Indeed, since the Laplacian $\Delta = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$ is an elliptic operator of order 2 (acting on q-forms), we have the fundamental estimate $\|\beta\|_{s+1} \leq C(\|\mathrm{d}^*\mathrm{d}\beta + \mathrm{d}\mathrm{d}^*\beta\|_{s-1} + \|\beta\|_0)$, which immediately implies the result using the continuity of d and d^* from W^s to W^{s-1}.

Applying it with $\beta = \widehat{\alpha}(j)$ yields

$$\|\widehat{\alpha}(j)\|_{W^{s+1}(M)} \leq C_1 \left((1+|j|)\|\widehat{\alpha}(j)\|_{W^s(M)} + \|f_j\|_{W^s(M)} + \|f_j^*\|_{W^s(M)} + \|\widehat{\alpha}(j)\|_{L^2(M)}\right),$$

where $C_1 > 0$ is independent of j. Since $\mathbb{L}\alpha$ and $\mathbb{L}^*\alpha$ are both smooth, this implies 1.4.8 for $s+1$. The case where $P = \overline{\partial}$ is done in a similar way (observing that the Dolbeault complex is also elliptic on M). □

1.5 Global hypoellipticity

We shall now apply the techniques developed in the previous section to relate regularity properties of $\mathbb{L}$ and $\overline{\partial}_b$. The main property that will concern us is the following:

Definition 1.5.1. Let (N, g) be a Riemannian manifold and E_1, E_2, E_3 be smooth vector bundles over N, endowed with euclidean (or hermitian) metrics. Let C be a complex of linear partial differential operators

$$C : \mathcal{D}'(N; E_1) \xrightarrow{Q} \mathcal{D}'(N; E_2) \xrightarrow{P} \mathcal{D}'(N; E_3)$$

with smooth coeficients. We say C is *globally hypoelliptic* if every distributional section $u \in \mathcal{D}'(N; E_2)$ such that $Pu \in C^\infty(N; E_3)$ and $Q^*u \in C^\infty(N; E_1)$ belongs to $C^\infty(M; E_2)$ (here, Q^* denotes the formal adjoint of Q with respect to the metric structures).

In our setting, we use the following terminology:

Definition 1.5.2. Let $\overline{\partial}_b$ (respectively, $\mathbb{L}$) be the differential complex defined by 1.4.1 (respectively, 1.4.2), and fix a metric on $M \times \mathbb{T}^d$ of the form $g_M \oplus g_{\mathbb{T}^d}$, where g_M is a hermitian (respectively, Riemannian) metric on M and $g_{\mathbb{T}^d}$ is the flat metric. We say this complex is *globally hypoelliptic* in degree q ($1 \leq q \leq 2n$) if the complex

$$\mathcal{D}'\left(M \times \mathbb{T}^d; \Lambda^{q-1}\mathbb{C}T^*M\right) \xrightarrow{(\overline{\partial}_b)_{q-1}} \mathcal{D}'\left(M \times \mathbb{T}^d; \Lambda^{q}\mathbb{C}T^*M\right) \xrightarrow{(\overline{\partial}_b)_{q}} \mathcal{D}'\left(M \times \mathbb{T}^d; \Lambda^{q+1}\mathbb{C}T^*M\right)$$

is globally hypoelliptic (respectively, if the complex given by $\{\mathbb{L}_{q-1}, \mathbb{L}_q\}$ is globally hypoelliptic). We say $\overline{\partial}_b$ (respectively, $\mathbb{L}$) is globally hypoelliptic in degree 0 if it is globally hypoelliptic in the usual sense (see, for example, Bergamasco, P. D. Cordaro, and Malagutti, 1993).

It is not clear to what extent this definition depends on the Riemannian metric on the manifold $M \times \mathbb{T}^d$. We can show, however, the following characterization:

Theorem 1.5.1. *Let $u \in \mathcal{D}'(M \times \mathbb{T}^d; \Lambda^q\mathbb{C}T^*M)$. Let P stand for either $\mathbb{L}$ or $\overline{\partial}_b$. Then, the following are equivalent:*

1. *$P_q u \in C^\infty(M \times \mathbb{T}^d; \Lambda^{q+1}\mathbb{C}T^*M)$ and $P^*_{q-1}u \in C^\infty(M \times \mathbb{T}^d; \Lambda^{q-1}\mathbb{C}T^*M)$.*
2. *$\square_P^q u \in C^\infty(M \times \mathbb{T}^d; \Lambda^q\mathbb{C}T^*M)$.*

In particular, the complex $\mathbb{L}$ (respectively, $\overline{\partial}_b$) is globally hypoelliptic in degree q if and only if *the corresponding Laplacian*

$$\square^q_{\mathbb{L}} := \mathbb{L}^*_q \mathbb{L}_q + \mathbb{L}_{q-1}\mathbb{L}^*_{q-1} \,:\, \mathcal{D}'\left(M \times \mathbb{T}^d; \Lambda^q \mathbb{C}T^*M\right) \to \mathcal{D}'\left(M \times \mathbb{T}^d; \Lambda^q \mathbb{C}T^*M\right)$$

is globally hypoelliptic in the usual sense (respectively, the Laplacian $\square^q_{\overline{\partial}_b}$ *is globally hypoelliptic). If* $q = 0$*, the Laplacians are defined by* $\square^0_{\mathbb{L}} := \mathbb{L}^*_0\mathbb{L}_0$ *(respectively,* $\square^0_{\overline{\partial}_b} := (\overline{\partial}_b)^*_0(\overline{\partial}_b)_0$*).*

Proof. Fix a degree $0 \le q \le 2n$ and consider the operator $\overline{\partial}_b$ (the proof for $\mathbb{L}$ is the same). It is clear that if $\square^q_{\overline{\partial}_b}$ is globally hypoelliptic, then $\overline{\partial}_b$ is globally hypoelliptic in degree q. To show the converse, assume $\overline{\partial}_b$ is globally hypoelliptic in degree q and let $u \in \mathcal{D}'(M \times \mathbb{T}^d; \Lambda^q \mathbb{C}T^*M)$ be a distributional sectional such that $\square^q_{\overline{\partial}_b} u \in C^\infty(M \times \mathbb{T}^d; \Lambda^q \mathbb{C}T^*M)$. Write the Fourier decomposition

$$u = \sum_{j \in \mathbb{Z}^d} \widehat{u}(j) e^{i\langle j, \theta\rangle}, \;\; \widehat{u}(j) \in C^\infty(M; \Lambda^q \mathbb{C}T^*M).$$

(observe that the Fourier coefficients of u are smooth since $(\square^q_{\overline{\partial}_b})_j \widehat{u}(j) \in C^\infty(M; \Lambda^q \mathbb{C}T^*M)$ for every $j \in \mathbb{Z}^d$, and these are elliptic complexes for every fixed $j \in \mathbb{Z}^d$). We shall first prove that $(\overline{\partial}_b)^*_q(\overline{\partial}_b)_q u \in C^\infty(M \times \mathbb{T}^d; \Lambda^q \mathbb{C}T^*M)$. Since

$$(\overline{\partial}_b)^*_{q-1}(\overline{\partial}_b)^*_q(\overline{\partial}_b)_q u = 0 \text{ and } (\overline{\partial}_b)_q(\overline{\partial}_b)^*_q(\overline{\partial}_b)_q = (\overline{\partial}_b)_q \square^q_{\overline{\partial}_b} u \in C^\infty(M \times \mathbb{T}^d; \Lambda^q \mathbb{C}T^*M),$$

we have $(\overline{\partial}_b)^*_q(\overline{\partial}_b)_q u \in X^q_{\overline{\partial}_b}$. By Theorem 1.4.3, we have to estimate the L^2 norm of the Fourier coefficients of $(\overline{\partial}_b)^*_q(\overline{\partial}_b)_q u$, which are given by $\delta^*_j \delta_j \widehat{u}(j)$ (we omit the degree q from the notation for simplicity). Fix $j \in \mathbb{Z}^d$. Then,

$$\left\|\delta^*_j \delta_j \widehat{u}(j)\right\|^2_{L^2(M)} = \left\langle\!\left\langle \delta_j \widehat{u}(j), \delta_j\left(\square_{\delta_j} \widehat{u}(j)\right)\right\rangle\!\right\rangle.$$

Since $M \times \mathbb{T}^d$ is a compact manifold, there is a number $s \in \mathbb{R}$ such that $\overline{\partial}^q_b u \in H^s(M \times \mathbb{T}^d; \Lambda^q \mathbb{C}T^*M)$. In particular, we have

$$\sum_{j \in \mathbb{Z}^d} (1 + |j|)^s \|\delta_j \widehat{u}(j)\|_{W^s(M)} < \infty.$$

We conclude then, from the generalized Cauchy-Schwarz inequality (see, for example, Proposition A.1.1 in Folland and Kohn, 1972)

$$\left\|\delta^*_j \delta_j \widehat{u}(j)\right\|^2_{L^2(M)} \le \|\delta_j \widehat{u}(j)\|_{W^s(M)} \|\delta_j(\square_{\delta_j} \widehat{u}(j))\|_{W^{-s}(M)}.$$

Therefore, given $A \in \mathbb{Z}_+$,

$$(1 + |j|)^{2A} \left\|\delta^*_j \delta_j \widehat{u}(j)\right\|^2_{L^2(M)} \le \left\{(1 + |j|)^s \|\delta_j \widehat{u}(j)\|_{W^s(M)}\right\}\left\{(1 + |j|)^{2A-s} \|\delta_j(\square_{\delta_j} \widehat{u}(j))\|_{W^{-s}(M)}\right\}$$

Since $\overline{\partial}_b(\square^q_{\overline{\partial}_b} u)$ is smooth, the supremum of the expression above is finite over $j \in \mathbb{Z}^d$. We conclude by theorem 1.4.3 that $(\overline{\partial}_b)^*_q(\overline{\partial}_b)_q u$ is smooth. In the same way one proves that $(\overline{\partial}_b)_{q-1}(\overline{\partial}_b)^*_{q-1} u$ is also smooth. We obtain then that $(\overline{\partial}_b)_q u \in X^{q+1}_{\overline{\partial}_b}$ (also $(\overline{\partial}_b)^*_{q-1} u \in X^{q-1}_{\overline{\partial}_b}$).

Now, writing

$$\|\delta_j \widehat{u}(j)\|^2_{L^2(M)} = \langle\!\langle \widehat{u}(j), \delta_j^* \delta_j \widehat{u}(j) \rangle\!\rangle$$

and using smoothness of $(\overline{\partial}_b)_q^*(\overline{\partial}_b)_q u$, we obtain again by Theorem 1.4.3 that $(\overline{\partial}_b)_q u$ is smooth. In the same way, $(\overline{\partial}_b)_{q-1}^* u$ is smooth. Since $\overline{\partial}_b$ is globally hypoelliptic in degree q, the section u is smooth. The result is proved.

□

We can now state the main result of this section:

Corollary 1.5.1. *Let $0 \leq q \leq 2n$ and $u \in \mathcal{D}'(M \times \mathbb{T}^d; \Lambda^q \mathbb{C}T^*M)$. Assume that M is a balanced manifold in degree q (see definition 1.4.1). Then, the following are equivalent:*

1. *$(\overline{\partial}_b)_q u \in C^\infty(M \times \mathbb{T}^d; \Lambda^{q+1}\mathbb{C}T^*M)$ and $(\overline{\partial}_b)_{q-1}^* u \in C^\infty(M \times \mathbb{T}^d; \Lambda^{q-1}\mathbb{C}T^*M)$.*

2. *$\mathbb{L}_q u \in C^\infty(M \times \mathbb{T}^d; \Lambda^{q+1}\mathbb{C}T^*M)$ and $\mathbb{L}_{q-1}^* u \in C^\infty(M \times \mathbb{T}^d; \Lambda^{q-1}\mathbb{C}T^*M)$.*

In particular, $\overline{\partial}_b$ is globally hypoelliptic in degree q if and only if $\mathbb{L}$ is globally hypoelliptic in degree q.

In many interesting cases, a characterization for global hypoellipticity (in degree 0) is known for $\mathbb{L}$. For instance, using Theorem 7.3 in Araújo, Dattori da Silva, and Lessa Victor, 2022 we obtain the following

Corollary 1.5.2. *Assume that M is balanced in degree 0. Then, $\overline{\partial}_b$ is globally hypoelliptic in degree 0 if and only if the system $\boldsymbol{\omega} + \overline{\boldsymbol{\omega}} = (\omega_1 + \overline{\omega_1}, \dots, \omega_d + \overline{\omega_d})$ is neither rational nor Liouville.*

Remark 1.5.1. We refer to the work Araújo, Dattori da Silva, and Lessa Victor, 2022 for the definition of rational and Liouville systems. We also remark that this result (concerning global hypoellipticity of $\mathbb{L}$ in degree 0) for $d = 1$ was obtained by Bergamasco, P. D. Cordaro, and Malagutti, 1993.

1.6 Global solvability

We shall now compare the cohomology of the complexes (1.4.2) and (1.4.1). We use the following notation:

$$\mathcal{H}^q(M \times \mathbb{T}^d; P) = \left\{ u \in C^\infty(M \times \mathbb{T}^d; \Lambda^q \mathbb{C}T^*M);\ \Box_P u = 0 \right\},$$

where $P \in \{\mathbb{L}, \overline{\partial}_b\}$. We have the natural (injective) map

$$i_q^P : \mathcal{H}^q(M \times \mathbb{T}^d; P) \to H^q(M \times \mathbb{T}^d; P),$$

sending a P-harmonic smooth form to its cohomology class (again, for $P \in \{\mathbb{L}, \overline{\partial}_b\}$). We have the following result:

Theorem 1.6.1. *Let $P \in \{\mathbb{L}, \overline{\partial}_b\}$ and $0 \leq q \leq 2n$. Then, the following are equivalent:*

1. *$H^q(M \times \mathbb{T}^d; P)$ is a Fréchet space.*

2. *i^P_q is a topological isomorphism.*

Proof. The implication 2) $\implies$ 1) is immediate. For the converse direction, it remains to see i^P_q is surjective. Let $[u] \in H^q(M \times \mathbb{T}^d; P)$ be a cohomology class, and choose a representative $u \in C^\infty(M \times \mathbb{T}^d; \Lambda^q \mathbb{C}T^*M)$ such that $Pu = 0$. We write the Fourier series decomposition

$$u = \sum_{j \in \mathbb{Z}^d} u_j e^{ij\theta},$$

where $u_j \in C^\infty(M; \Lambda^q \mathbb{C}T^*M)$ is such that $P_j u_j = 0$ for all $j \in \mathbb{Z}^d$ (P_j is either D_j or δ_j, as in section 4). Recalling these complexes are elliptic, there is a unique P_j-harmonic representative $v_j \in C^\infty(M; \Lambda^q \mathbb{C}T^*M)$ in the P_j-cohomology class of u_j, i.e., $u_j - v_j = P_j w_j$ for some $(q-1)$-form $w_j \in C^\infty(M; \Lambda^{q-1}\mathbb{C}T^*M)$ and $\square_{P_j} v_j = 0$. We also know that $\|v_j\|_{L^2(M)} \leq \|u_j\|_{L^2(M)}$ for all $j \in \mathbb{Z}^d$. Therefore, by Theorem 1.4.3, we conclude that

$$v = \sum_{j \in \mathbb{Z}^d} v_j e^{ij\theta}$$

defines an element in $\mathcal{H}^q(M \times \mathbb{T}^d; P)$. It remains to see that $i^P_q(v) = [u]$. Indeed, for every $N \geq 1$, we have

$$\sum_{|j| \leq N} (u_j - v_j) e^{ij\theta} = P\left(\sum_{|j|\leq N} w_j e^{ij\theta} \right).$$

This sequence of elements in the (closed) range of P converges to $u - v$, which must be also in the range of P by (1). The result is proved. □

Remark 1.6.1. These results do not depend on any metric properties of M, only on the ellipticity of the de Rham and Dolbeault's complexes.

We move to the main result of this section.

Theorem 1.6.2. *Let M be balanced in degree 0. Then, the following are equivalent:*

1. *$H^1(M \times \mathbb{T}^d; \mathbb{L})$ is a Fréchet space.*
2. *$H^1(M \times \mathbb{T}^d; \overline{\partial}_b)$ is a Fréchet space.*

Under these conditions, we have $H^1(M \times \mathbb{T}^d; \mathbb{L}) \simeq H^1(M \times \mathbb{T}^d; \overline{\partial}_b)$. We also have $H^0(M \times \mathbb{T}^d; \mathbb{L}) = H^0(M \times \mathbb{T}^d; \overline{\partial}_b)$.

The main technical ingredient is the following:

Proposition 1.6.1. *Let M be balanced in degree 0. Then, for every $k \in \mathbb{Z}_+$, there is a constant $C_k > 0$ such that*

$$\left\|\overline{\partial}_b u\right\|_{W^k} \leq \|\mathbb{L}u\|_{W^k} \leq C_k \left\|\overline{\partial}_b u\right\|_{W^k}, \quad u \in C^\infty(M \times \mathbb{T}^d). \tag{1.6.1}$$

Proof. The inequalities 1.6.1 follows from 1.4.1 for $k = 0$, so we assume $k \geq 1$. The first inequality in 1.6.1 is immediate, since $\overline{\partial}_b u$ is the projection of $\mathbb{L}u$ onto the $(0,1)$-forms. To prove the second estimate, we argue by contradiction: assume the estimate is false. Then, we can find $k \geq 1$ and a sequence $u_n \in C^\infty(M \times \mathbb{T}^d)$ such that $\|\mathbb{L}u_n\|_{W^k} = 1$ for all n and $\overline{\partial}_b u_n \to 0$ in W^k.

By Rellich's lemma, we can (after passing to a subsequence) assume that $\mathbb{L}u_n \to v$ in W^{k-1}, where $v \in W^{k-1}(M \times \mathbb{T}^d; \mathbb{C}T^*M)$. Applying $\mathbb{L}^*$ yields (using 1.4.1)

$$2\overline{\partial}_b^*\overline{\partial}_b u_n = \mathbb{L}^*\mathbb{L}u_n \to \mathbb{L}^* v \text{ in } \mathcal{D}'.$$

We also have $\overline{\partial}_b^*\overline{\partial}_b u_n \to 0$, so by uniqueness of the distributional limit, we conclude that $\mathbb{L}^* v = 0$, i.e., $v \in \ker \mathbb{L}^* \cap W^{k-1}(M \times \mathbb{T}^d; \mathbb{C}T^*M)$. Since $k \geq 1$, we have $v \in L^2$ and since the range of $\mathbb{L}$ is orthogonal to the kernel of $\mathbb{L}^*$, we conclude that $v = 0$, which contradicts the fact that $\|\mathbb{L}u_n\|_{W^k} = 1$ for all n. □

A consequence of this proposition is the following

Corollary 1.6.1. *Assume M is balanced in degree 0. Then, the following are equivalent:*

1. *$\mathbb{L} : C^\infty(M \times \mathbb{T}^d) \to C^\infty(M \times \mathbb{T}^d; \mathbb{C}T^*M)$ has closed range.*
2. *$\overline{\partial}_b : C^\infty(M \times \mathbb{T}^d) \to C^\infty(M \times \mathbb{T}^d; \mathbb{C}T^*M)$ has closed range.*

Proof. We apply the following characterization from Köthe, 1979, page 18: a continuous linear map $T : E \to F$ between Fréchet spaces has closed range if, and only if, the following holds:

$$\begin{aligned} &\text{For every sequence } (u_n) \text{ in } E \text{ such that } Tu_n \to 0, \\ &\text{there is a sequence } v_n \in E \text{ such that } Tu_n = Tv_n \text{ and } v_n \to 0. \end{aligned} \tag{1.6.2}$$

Assume $\mathbb{L}$ has closed range and let $u_n \in C^\infty(M \times \mathbb{T}^d)$ be such that $\overline{\partial}_b u_n \to 0$. From 1.6.1, we have $\mathbb{L}u_n \to 0$ in the C^∞ topology. From 1.6.2, there is a sequence $v_n \in C^\infty(M \times \mathbb{T}^d)$ such that $\mathbb{L}u_n = \mathbb{L}v_n$ and $v_n \to 0$. However, the kernel of $\mathbb{L}$ and $\overline{\partial}_b$ are equal in $C^\infty(M \times \mathbb{T}^d)$ (again from 1.6.1), so 1.6.2 implies that $\overline{\partial}_b$ has closed range. The argument in the other direction is identical. □

Now, recalling that $H^1(M \times \mathbb{T}^d; \mathbb{L})$ is Fréchet if and only if $\mathbb{L}$ has closed range (same for $\overline{\partial}_b$), the statement 1.6.2 follows immediately.

In recent work Araújo, Ferra, Jahnke, and Ragognette, 2023, a characterization for closedness of the range of $\mathbb{L}$ was obtained in terms of a diophantine condition on the forms. Using this result, we can state the following

Corollary 1.6.2. *Assume M is balanced in degree 0. Then, $\overline{\partial}_b : C^\infty(M \times \mathbb{T}^d) \to C^\infty(M \times \mathbb{T}^d; \mathbb{C}T^*M)$ has closed range if and only if the collection $\boldsymbol{\omega} + \overline{\boldsymbol{\omega}} = (\omega_1 + \overline{\omega_1}, \dots, \omega_d + \overline{\omega_d})$ is weakly non-simultaneously approximable.*

Remark 1.6.2. We refer to the paper Araújo, Ferra, Jahnke, and Ragognette, 2023 for the definition of a weakly non-simultaneously approximable system.

We say $\mathbb{L}$ (respectively, $\overline{\partial}_b$) is *globally solvable* if $H^1(M \times \mathbb{T}^d; \mathbb{L}) = 0$ (respectively, $H^1(M \times \mathbb{T}^d; \overline{\partial}_b) = 0$). In view of the previous results, we obtain the following

Corollary 1.6.3. *Let M be balanced in degree 0. Then, the following are equivalent:*

1. *$\mathbb{L}$ is globally solvable.*
2. *$\overline{\partial}_b$ is globally solvable.*

Chapter 2

$A^\infty(K)$: a function-theoretic approach

2.1 Introduction

In P. D. Cordaro, Della Sala, and Lamel, 2019 and P. D. Cordaro, Della Sala, and Lamel, 2020, the authors introduced the following object: given a compact set $K \subset \mathbb{C}^m$, consider the $\mathbb{C}$-algebra homomorphism

$$\begin{aligned} \gamma\, :\, \mathcal{O}(\mathbb{C}^m) &\to C(K)[[X_1, \dots, X_m]] \\ f &\mapsto \sum_{\alpha \in \mathbb{Z}_+^m} \frac{(\partial^\alpha f)\big|_K}{\alpha!} X^\alpha, \end{aligned} \tag{2.1.1}$$

where $\partial^\alpha = \partial_{z_1}^{\alpha_1} \dots \partial_{z_m}^{\alpha_m}$. The closure of the range of this homomorphism (in the usual Fréchet algebra topology of the space of formal power series $C(K)[[X_1, \dots, X_m]]$) is denoted by $A^\infty(K)$. This construction is inspired by the classical Banach algebra $P(K) \subset C(K)$ (which is the closure, in the uniform topology, of the range of the restriction map $\mathcal{O}(\mathbb{C}^m) \to C(K)$). In interior points of K, the A^∞ algebra does not carry any extra information, but on the boundary of K, it captures more subtle behaviour concerning the derivatives of holomorphic functions.

In this chapter, we shall study this algebra from a more function-theoretic point of view, focusing on $m = 1$. We establish a few properties of this algebra, using as the main tool a *generalized Cauchy transform*. This sheds light on smooth versions of Mergelyan's theorem. We also establish a *localization property* for the A^∞ algebra in the plane.

Finally, we consider the problem of determing for which compact sets does the equality

$$A^\infty(K) = C(K; \mathcal{F}_1)$$

hold. A similar question is well-known for the algebra $P(K)$: it coincides with $C(K)$ if and only if K is polynomially convex and has empty interior (as it follows from the standard Mergelyan theorem). We conjecture that $A^\infty(K) = C(K; \mathcal{F}_1)$ if and only if K is polynomially convex and contains no rectifiable arcs. This generalizes a conjecture of Bishop, 1958. We establish it in a special case, where K has zero measure projections onto both coordinate axes.

2.2 Topological preliminaries

Let $K \subset \mathbb{C}^m$ be a compact set. We denote by $C(K;\mathcal{F}_m)$ the Fréchet space of formal power series in m indeterminates, with coeficients in $C(K)$ (the space of continuous complex-valued functions on K). We will record a useful characterization for the strong dual of $C(K;\mathcal{F}_m)$ (whose basis for the filter of neighborhood of the origin is formed by polars of bounded subsets of $C(K;\mathcal{F}_m)$).

Let $M(K) = C(K)^*$ be the Banach space of Radon measures on K (i.e., complex regular Borel measures on K, with the total variation norm). We denote by

$$M(K)[X_1,\dots,X_m]_{\leq d} = \prod_{|\alpha|\leq d} M(K) = \left\{ \sum_{|\alpha|\leq d} \mu_\alpha X^\alpha;\ \mu_j \in M(K) \right\}$$

the (Banach) space of polynomials of measures on K with degree at most d. We shall put on $M(K)[X_1,\dots,X_m]$ (the space of polynomials with coefficients in $M(K)$) the inductive limit topology

$$M(K)[X_1,\dots,X_m] = \varinjlim_{d\to\infty} M(K)[X_1,\dots,X_m]_{\leq d}.$$

Consider the bilinear map

$$\begin{aligned} T\ :\ C(K;\mathcal{F}_m) \times M(K)[X_1,\dots,X_m] &\to \mathbb{C} \\ \left(\sum_{\alpha\in\mathbb{Z}_+^m} f_\alpha X^\alpha, \sum_{|\alpha|\leq d} \mu_\alpha X^\alpha \right) &\mapsto \sum_{|\alpha|\leq d} \int_K f_\alpha \mathrm{d}\mu_\alpha. \end{aligned} \tag{2.2.1}$$

Then, we have the following theorem:

Theorem 2.2.1. *The map 2.2.1 induces an isomorphism between the strong dual of $C(K;\mathcal{F}_m)$ and the LF space $M(K)[X_1,\dots,X_m]$.*

Remark 2.2.1. This proof is a simple adaptation of the isomorphism between the space of polynomials and the dual of the space of formal power series, Theorem 22.1 in Treves, 1967.

Proof. Note that, if $p = \sum_{|\alpha|\leq d} \mu_\alpha X^\alpha \in M(K)[X_1,\dots,X_m]$ and $\mathbf{f} = \sum_{\alpha\in\mathbb{Z}_+^m} f_\alpha X^\alpha \in C(K;\mathcal{F}_m)$, then

$$|T(\mathbf{f},p)| \leq \left(\sum_{|\alpha|\leq d} \|\mu_\alpha\|_{M(K)} \right) \left(\sum_{|\alpha|\leq d} \|f_\alpha\|_{C(K)} \right).$$

Therefore, the linear map

$$\begin{aligned} \widetilde{T} : M(K)[X_1, \dots, X_m] &\to C(K; \mathcal{F}_m)^* \\ p &\mapsto (\mathbf{f} \mapsto T(\mathbf{f}, p)) \end{aligned}$$

is well-defined and continuous. We shall show that $\widetilde{T}$ is an isomorphism. Let us start with the injectivity: assume that, for a polynomial $p \in M(K)[X_1, \dots, X_m]$, the functional $\mathbf{f} \mapsto \widetilde{T}(\mathbf{f}, p)$ is identically zero. Then, in particular, given $\alpha \in \mathbb{Z}_+^m$ with $|\alpha| \leq \deg p$, we have $\int_K f \mathrm{d}\mu_\alpha = 0$for all $f \in C(K)$ (considering the element of $C(K; \mathcal{F}_m)$ that has f in the α-th position and zero elsewhere). From this it follows that $\mu_\alpha = 0$ and, since α is arbitrary, $p = 0$.

Now we check the surjectivity. Let $L \in C(K; \mathcal{F}_m)^*$ be arbitrary. Considering the map $i_\alpha : C(K) \to C(K; \mathcal{F}_m)$ given by $i_\alpha(f) = \sum_{\beta \in \mathbb{Z}_+^m} (\delta_{\alpha\beta} f) X^\beta$, performing the composition $L \circ i_\alpha$, we obtain an element of $C(K)^*$, which gives rise to a unique measure $\mu_\alpha \in M(K)$ such that

$$L(i_\alpha(f)) = \int_K f \mathrm{d}\mu_\alpha, \;\; f \in C(K).$$

The continuity of L implies that there is an integer $k \in \mathbb{Z}_+$ and a constant $C > 0$ such that

$$|L(\mathbf{f})| \leq C \sum_{|\alpha| \leq k} \|f_\alpha\|_{C(K)}, \quad \mathbf{f} = \sum_{\alpha \in \mathbb{Z}_+^m} f_\alpha X^\alpha \in C(K; \mathcal{F}_m).$$

This implies that, if $|\alpha| > k$, $L(i_\alpha) \equiv 0$, which shows $\mu_\alpha = 0$ if $|\alpha| > k$. We define then $p := \sum_{|\alpha| \leq k} \mu_\alpha X^\alpha \in M(K)[X]$. Clearly, $T(\mathbf{f}, p) = L(\mathbf{f})$, and $\widetilde{T}$ is surjective.

It remains to show the continuity of the inverse of $\widetilde{T}$. Let $V \subset M(K)[X_1, \dots, X_m]$ be a convex neighborhood of zero. We will show $\widetilde{T}(V)$ is a neighborhood of zero in $C(K; \mathcal{F}_m)^*$ (i.e., it contains the polar of a bounded subset of $C(K; \mathcal{F}_m)$).

For every $d \in \mathbb{Z}_+$, the intersection $V \cap M(K)[X_1, \dots, X_m]_{\leq d}$ contains a set of the form

$$W_d = \left\{ \sum_{|\alpha| \leq d} \mu_\alpha X^\alpha ; \; \sum_{|\alpha| \leq d} \|\mu_\alpha\|_{M(K)} \leq \rho_d \right\},$$

for some $\rho_d > 0$. We can take the sequence $(\rho_d)_{d \in \mathbb{Z}_+}$ strictly decreasing and converging to zero. Let $W = \bigcup_{d \in \mathbb{Z}_+} W_d$ and $\widetilde{W} = \Gamma(W)$ be the convex hull of W. By definition of the LF topology of $M(K)[X_1, \dots, X_m]$, $\widetilde{W}$ is a neighborhood of zero in $M(K)[X_1, \dots, X_m]$ (since it is a convex set such that its intersection with each $M(K)[X_1, \dots, X_m]_{\leq d}$ is a neighborhood of zero in $M(K)[X_1, \dots, X_m]_{\leq d}$). It is clear that $\widetilde{W} \subset V$. Consider now the set

$$\mathcal{B} = \left\{ \mathbf{f} = \sum_{\alpha \in \mathbb{Z}_+^m} f_\alpha X^\alpha ; \; \sup_{|\alpha| \leq d} \|f_\alpha\|_{C(K)} \leq \frac{1}{\rho_d'}, \; d \in \mathbb{Z}_+ \right\} \subset C(K; \mathcal{F}_m),$$

where $\rho_d' = \rho_d / \left(d \cdot 2^{d+1}\right)$. It is clear that $\mathcal{B}$ is bounded in $C(K; \mathcal{F}_m)$. Let $L \in \mathcal{B}^0$, which we already know (since $\widetilde{T}$ is bijective) is of the form $\widetilde{T}(v)$ for a unique $v \in M(K)[X_1, \dots, X_m]$. We have to show $v \in V$. Write $v = \sum_{|\alpha| \leq k} \mu_\alpha X^\alpha$ (we use the convention $\mu_\alpha = 0$ if $|\alpha| > k$).

Given $f \in C(K)$, $f \neq 0$, consider the element

$$\mathbf{f}_\alpha = i_\alpha \left(\frac{f}{\rho'_{|\alpha|}\|f\|_{C(K)}} \right) \in C(K; \mathcal{F}_m).$$

Clearly $\mathbf{f}_\alpha \in \mathcal{B}$, therefore

$$\left|\widetilde{T}(v)\left(\mathbf{f}_\alpha\right)\right| = \frac{1}{\rho'_{|\alpha|}\|f\|_{C(K)}} \left| \int_K f \,\mathrm{d}\mu_\alpha \right| \leq 1.$$

Since $f \in C(K) \setminus \{0\}$ is abitrary, we conclude that $\|\mu_\alpha\|_{M(K)} \leq \rho'_{|\alpha|}$ for all $|\alpha| \leq d$. In particular, given $h \in \mathbb{Z}_+$, consider the homogeneous part of degree h of v given by

$$v_h := \sum_{|\alpha|=h} \mu_\alpha X^\alpha.$$

We have

$$\sum_{|\alpha|=h} \|\mu_\alpha\|_{M(K)} \leq h\rho'_h = \frac{\rho_h}{2^{h+1}}.$$

Therefore, $2^{h+1}v_h \in W_h$. However, since $\sum_{h=0}^k 2^{-(h+1)} < 1$, we conclude that $v = \sum_{h=0}^d v_h = \sum_{h=0} 2^{-(h+1)}(2^{h+1}v_h)$ belongs to the convex hull $\widetilde{W}$ of W, which is contained in V. Therefore, $\mathcal{B}^0 \subset \widetilde{T}(V)$, and the proof is complete. $\square$

2.3 The algebras $A^\infty(K)$ and $R^\infty(K)$ in the plane

From now on, we assume that $m = 1$. We recall the definition of the Fréchet algebras we will work with. Let $\gamma : \mathcal{O}(\mathbb{C}) \to C(K; \mathcal{F}_1)$ given by

$$\gamma(f) = \sum_{j=0}^\infty \frac{f^{(j)}|_K}{j!} X^j,$$

where $f^{(j)} = \partial^j f / \partial z^j$. We define $A^\infty(K) = \overline{\gamma(\mathcal{O}(\mathbb{C}))}$. Similarly, the algebra $R^\infty(K)$ is the closure (in $C(K; \mathcal{F}_1)$) of the range of γ, now defined on rational functions on $\mathbb{C}$ without poles in K.

2.4 The (generalized) Cauchy transform

In this section, we shall record the basic properties of the Cauchy transform and its properties. Then, we present a generalization of this concept for polynomials of measures and an application to a classical question of approximation of $\overline{\partial}$-flat functions in compact sets.

Definition 2.4.1. Let $K \subset \mathbb{C}$ be a compact set and $\mu \in M(K)$. The *Cauchy transform* of μ is the function

$$\widehat{\mu}(z) = \int_K \frac{\mathrm{d}\mu(\zeta)}{\zeta - z}, \ z \in \mathbb{C} \setminus K.$$

Theorem 2.4.1. *The Cauchy transform $\widehat{\mu}$ satisfies*

1. $\widehat{\mu} \in L^p_{loc}(\mathbb{C})$ *for all* $1 \leq p < 2$.
2. $\widehat{\mu}$ *is holomorphic in the complement of* K *(in fact, in the complement of the support of* μ*).*
3. $\widehat{\mu}(\infty) = 0$.
4. $\widehat{\mu}'(\infty) = -\mu(K)$.

Remark 2.4.1. The function $\widehat{\mu}$ exists in the complement of K in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Proof. The first item follows since $|\zeta|^{-\alpha}$ is locally integrable in $\mathbb{C}$ for $1 \leq \alpha < 2$. The second item follows from differentiation under the integral sign, and the other itens are immediate calculations. □

A very important property of the Cauchy transform is the following:

Proposition 2.4.1. *Let* $\mu \in M(K)$*. Then, if* $\widehat{\mu} = 0$ *a.e. in* $\mathbb{C}$*, then* $\mu = 0$*.*

Proof. To show μ is the zero measure, it is enough to show that, for every function $g \in C^1_c(\mathbb{C})$, $\int g \mathrm{d}\mu = 0$. Therefore, given $g \in C^1_c(\mathbb{C})$, it follows from Cauchy's integral formula that

$$g(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial g(\zeta)}{\partial \bar{\zeta}} \frac{\mathrm{dm}(\zeta)}{\zeta - z}, \; z \in \mathbb{C},$$

where m is the Lebesgue measure in $\mathbb{C}$. Then, we obtain

$$\int g(z) \mathrm{d}\mu(z) = -\int_{\mathbb{C}} \frac{\partial g(\zeta)}{\partial \bar{\zeta}} \widehat{\mu}(\zeta) \mathrm{dm}(\zeta).$$

Therefore, if $\widehat{\mu}$ is zero a.e., then $\mu = 0$. □

Corollary 2.4.1. *Let* $K \subset \mathbb{C}$ *be a compact set with zero Lebesgue measure. Let* $\mu \in M(K)$*. Then,* $\mu = 0$ *if and only if* $\widehat{\mu} = 0$ *in* $\mathbb{C} \setminus K$*.*

Now we shall generalize this to polynomials of measures. In section 2.2, we showed that $p \in C(K; \mathcal{F}_1)^*$ can be written as $p = \sum_{j=0}^d \mu_j X^j$, where $\mu_j \in M(K)$. Let $f \in \mathcal{O}(\mathbb{C})$, which we identify with an element in $A^\infty(K)$ via

$$\mathbf{f} = \sum_{j=0}^{\infty} \frac{f^{(j)}|_K}{j!} X^j.$$

Let $V \subset \mathbb{C}$ be a regular open set with $K \subset V$. Then, from Cauchy's integral formula for derivatives, we have

$$\begin{aligned} p(\mathbf{f}) &= \sum_{j=0}^{d} \int_K \frac{f^{(j)}(\zeta)}{j!} \mathrm{d}\mu_j(\zeta) \\ &= \sum_{j=0}^{d} \int_K \left(\frac{1}{2\pi i} \int_{\partial V} \frac{f(z)}{(z-\zeta)^{j+1}} \mathrm{d}z \right) \mathrm{d}\mu_j(\zeta) \\ &= \frac{1}{2\pi i} \int_{\partial V} f(z) \left(\sum_{j=0}^{d} \int_K \frac{\mathrm{d}\mu_j(\zeta)}{(z-\zeta)^{j+1}} \right) \mathrm{d}z. \end{aligned} \tag{2.4.1}$$

Then, in analogy with the Cauchy transform, we make the following definition.

Definition 2.4.2. Let $p = \sum_{j=0}^{d} \mu_j X^j \in M(K)[X]$. We define the *generalized Cauchy transform* of the polynomial p as the holomorphic function

$$\widetilde{p}(z) = \sum_{j=0}^{d} \int_K \frac{\mathrm{d}\mu_j(\zeta)}{(z-\zeta)^{j+1}}, \quad z \in \mathbb{C} \setminus K.$$

If $p = \mu$ consists of a single measure, then $\widetilde{p} = -\widehat{\mu}$, where $\widehat{\mu}$ is the Cauchy transform of μ. Observe that, if the degree of p is greater than or equal to 1, the Cauchy transform$\widetilde{p}$ is no longer locally integrable in $\mathbb{C}$. However, $\widetilde{p}$ always extend holomorphically to $z = \infty$, with $\widetilde{p}(\infty) = 0$.

Let us use the notation $E(z) := -1/z \in L^1_{loc}(\mathbb{C}) \subset \mathcal{D}'(\mathbb{C})$ (then, $-\pi^{-1}E$ is the fundamental solution of the Cauchy-Riemann operator in the plane).

Lemma 2.4.1. *Let $\mu \in M(K)$. Then, given $j \in \mathbb{Z}_+$, the distribution*

$$h_j(\mu) := \frac{(-1)^{j+1}}{j!} \mu * E^{(j)} \in \mathcal{D}'(\mathbb{C})$$

coincides, in $\mathbb{C} \setminus K$, with the holomorphic function

$$\mathbb{C} \setminus K \ni z \mapsto \int_K \frac{\mathrm{d}\mu(\zeta)}{(z-\zeta)^{j+1}} = \widetilde{\mu X^j}(z). \tag{2.4.2}$$

Proof. Let $\psi \in C_c^\infty(\mathbb{C})$ be arbitrary. Then,

$$\begin{aligned} \langle h_j(\mu), \psi \rangle &= \frac{(-1)^{j+1}}{j!} \left(\mu * E^{(j)} * \check{\psi} \right)(0) \\ &= \frac{(-1)^{j+1}}{j!} \left(\mu * E * \check{\psi}^{(j)} \right)(0) \\ &= \frac{(-1)^{j+1}}{j!} (\mu * h)(0), \end{aligned}$$

where $h \in C^\infty(\mathbb{C})$ is given by

$$\begin{aligned} h(z) &= \int_{\mathbb{C}} E(z-w)\check{\psi}^{(j)}(w)\mathrm{dm}(w) \\ &= (-1)^j \int_{\mathbb{C}} \frac{\psi^{(j)}(-w)}{w-z}\mathrm{dm}(w) \\ &= (-1)^j \int_{\mathbb{C}} \frac{\psi^{(j)}(w)}{-z-w}\mathrm{dm}(w). \end{aligned}$$

Therefore,

$$\langle h_j(\mu), \psi\rangle = -\frac{1}{j!}\int_K \int_{\mathbb{C}} \frac{\psi^{(j)}(w)}{z-w}\mathrm{dm}(w)\mathrm{d}\mu(z)$$

Suppose that $\operatorname{supp}\psi \subset \mathbb{C} \setminus K$. Then, for each $z \in K$, we have

$$\int_{\mathbb{C}} \frac{\psi^{(j)}(w)}{z-w}\mathrm{dm}(w) = (-1)^j j! \int_{\mathbb{C}} \frac{\psi(w)}{(z-w)^{j+1}}\mathrm{dm}(w)$$

(partial integration). Therefore,

$$\langle h_j(\mu), \psi\rangle = -\frac{1}{j!}\int_K \int_{\mathbb{C}} \frac{\psi^{(j)}(w)}{z-w}\mathrm{dm}(w)\mathrm{d}\mu(z) = (-1)^{j+1}\int_K \int_{\mathbb{C}} \frac{\psi(w)}{(z-w)^{j+1}}\mathrm{dm}(w)\mathrm{d}\mu(z) = \int_{\mathbb{C}} \psi(w)\widetilde{\mu X^j}(w)\mathrm{dm}(w),$$

i.e., in $\mathbb{C} \setminus K$, $h_j(\mu)$ coincides with the holomorphic function 2.4.2. □

Using this lemma, given a polynomial $p = \sum_{j=0}^d \mu_j X^j \in M(K)[X]$, we can represent its generalized Cauchy transform by

$$\widetilde{p} = \sum_{j=0}^{d} \frac{(-1)^{j+1}}{j!}\mu_j * E^{(j)}. \tag{2.4.3}$$

The interest in the function $\widetilde{p}(z)$ comes from the following lemma.

Lemma 2.4.2. *Let $K \subset \mathbb{C}$ be compact and polynomially convex. Then, $p \in M(K)[X]$ is such that $p\big|_{A^\infty(K)} = 0$ if and only if, $\widetilde{p}(z) = 0$ for all $z \in \mathbb{C} \setminus K$.*

Proof. The identity 2.4.1 shows that, if $\widetilde{p} = 0$ in $\mathbb{C}\setminus K$, then $p\big|_{A^\infty(K)} = 0$. Conversely, suppose that $p\big|_{A^\infty(K)} = 0$ and let $z \in \mathbb{C} \setminus K$. Then, the function $f(\zeta) := (z-\zeta)^{-1}$ is homolorphic in a neighborhood Ω of K, and since K is polynomially convex, there is an open polynomial polyhedron Π contained Ω and containing K (page 283 in Stolzenberg, 1963). Since $\overline{\Pi}$ is polynomially convex,there is a sequence of entire functions $(f_\nu)_{\nu\in\mathbb{Z}_+}$ such that $f_\nu \to f$ uniformly on $\overline{\Pi}$. From Cauchy's estimates, we have $f_\nu^{(j)}/j! \to (z-\zeta)^{-(j+1)}$ uniformly for $\zeta \in K$, for all $j \in \mathbb{Z}_+$. We conclude then that

$$\mathbf{f} = \sum_{j=0}^{\infty} \frac{1}{(z-\zeta)^{j+1}}\bigg|_{\zeta\in K} X^j$$

belongs to $A^\infty(K)$ for all $z \in \mathbb{C} \setminus K$, and $\widetilde{p}(z) = p(\mathbf{f}) = 0$. The lemma is proved. □

Remark 2.4.2. Observe that the implication $\widetilde{p} = 0$ in $\mathbb{C} \setminus K \implies p\big|_{A^\infty(K)} = 0$ does not use the hypothesis that K is polynomially convex.

To conclude this section, we shall show an application of this result that yields a simple proof of a generalization of the smooth version of Mergelyan's theorem (ver P. D. Cordaro, Della Sala, and Lamel, 2020):

Theorem 2.4.2. *Let $K \subset \mathbb{C}$ be polynomially convex. Let $F \in C_c^\infty(\mathbb{C})$ be such that*

$$\partial^\alpha \left(\frac{\partial F}{\partial \overline{z}} \right)(z) = 0$$

for all $z \in K$ and $\alpha \in \mathbb{Z}_+^2$. Then,

$$\boldsymbol{F} = \sum_{j=0}^{\infty} \frac{F^{(j)}\big|_K}{j!} X^j \in C(K; \mathcal{F}_1)$$

belongs to $A^\infty(K)$.

Proof. We shall prove that, if $p \in M(K)[X]$ is orthogonal to $A^\infty(K)$, then $p(\mathbf{F}) = 0$. Let $p = \sum_{j=0}^d \mu_j X^j \in A^\infty(K)^\perp$. Then,

$$\begin{aligned}
\left\langle \widetilde{p}, \frac{\partial F}{\partial \overline{z}} \right\rangle &= \sum_{j=0}^{d} \frac{(-1)^{j+1}}{j!} \left\langle \mu_j * E^{(j)}, \frac{\partial F}{\partial \overline{z}} \right\rangle \\
&= \sum_{j=0}^{d} \frac{(-1)^{j+1}}{j!} \mu_j * E^{(j)} * \widecheck{\frac{\partial F}{\partial \overline{z}}}(0) \\
&= \sum_{j=0}^{d} \frac{(-1)^{j}}{j!} \mu_j * E^{(j)} * \frac{\partial \check{F}}{\partial \overline{z}}(0) \\
&= \sum_{j=0}^{d} \frac{(-1)^{j}}{j!} \mu_j * \left(\frac{\partial \check{F}}{\partial \overline{z}} * E \right)^{(j)}(0) \\
&= -\pi \sum_{j=0}^{d} \frac{(-1)^{j}}{j!} \mu_j * \left(\check{F} \right)^{(j)}(0) \\
&= -\pi \sum_{j=0}^{d} \frac{1}{j!} \mu_j * \widecheck{F^{(j)}}(0) \\
&= -\pi \sum_{j=0}^{d} \langle \mu_j, f_j \rangle \\
&= -\pi p(\mathbf{F}).
\end{aligned}$$

Since supp $\widetilde{p} \subset K$ and $\overline{\partial} F$ vanishes to infinite order on K, we conclude that $p(\mathbf{F}) = 0$. This implies the result by Hahn-Banach's theorem: indeed, assume that $\mathbf{F} \notin A^\infty(K)$. Consider the quotient map $\pi : C(K;\mathcal{F}_1) \to C(K;\mathcal{F}_1)/A^\infty(K)$. Then, $\pi(\mathbf{F}) \neq 0$. Let $L \subset C(K;\mathcal{F}_1)/A^\infty(K)$ be the one-dimensional subspace generated by $\pi(\mathbf{F})$. We have a map $T : L \to \mathbb{C}$ given by $T(\lambda\phi(\mathbf{F})) = \lambda$. Since $C(K;\mathcal{F}_1)/A^\infty(K)$ is Hausdorff, T is continuous, and by the Hahn-Banach theorem, it admits a continuous extension $\widetilde{f} : C(K;\mathcal{F}_1)/A^\infty(K) \to \mathbb{C}$. Defining $p := \widetilde{f} \circ \pi$, we obtain an element of $M(K)[X]$ which is orthogonal to $A^\infty(K)$ and such that $p(\mathbf{F}) = 1$.

□

2.5 Localization property

The algebra $R(K)$ has the following fundamental property (of *localization*): if $f \in C(K)$ is a continuous function in the compact set $K \subset \mathbb{C}$ such that, for every $x \in K$, there is a compact neighborhood $U_x \subset \mathbb{C}$ such that $f\big|_{U_x \cap K} \in R(U_x \cap K)$, then $f \in R(K)$. We would like to verifiy a similar property for the algebra $R^\infty(K)$ (or $A^\infty(K)$, if K is polynomially convex). To do this, we apply Sakai's strategy Sakai, 1972, using the $\overline{\partial}$ problem. First, we need a result about estimates for derivatives for the Cauchy-Riemann operator in the plane.

Theorem 2.5.1. *Let $\Omega \subset \mathbb{C}$ be a regular open set and $f \in C^\infty(\overline{\Omega})$. Then, the function*

$$(T_\Omega f)(\zeta) := -\frac{1}{\pi}\int_\Omega \frac{f(z)}{z-\zeta}\mathrm{d}m(z),\ \ \zeta \in \Omega \tag{2.5.1}$$

belongs to $C^\infty(\overline{\Omega})$, with $\partial u/\partial \overline{z} = f$ in Ω and

$$\|T_\Omega f\|_{C^k(\overline{\Omega})} \leq C_k \|f\|_{C^k(\overline{\Omega})}, \tag{2.5.2}$$

where $C_k > 0$ is a constant depending only on $k \in \mathbb{Z}_+$ and which can be taken uniform for domains in a compact neighborhood of $\overline{\Omega}$.

To prove this result, we need a lemma:

Lemma 2.5.1. *Assume that $f \in C^\infty(\overline{\Omega})$. Then, for every integer $m \in \mathbb{Z}_+$, there is a function $\Phi_m \in C^\infty(\overline{\Omega})$ that vanishes in the boundary of $\Omega$and such that $\partial\Phi_m/\partial\overline{z}$ and f coincide in $\partial\Omega$ up to order m.*

Remark 2.5.1. A function $F \in C^\infty(\overline{\Omega})$ vanishes $\partial\Omega$ to order m if $(\partial^\alpha F)\big|_{\partial\Omega} = 0$ for all $\alpha \in \mathbb{Z}_+^2$ with $|\alpha| \leq m$. Two functions in $C^\infty(\overline{\Omega})$ coincide up to order m in $\partial\Omega$ if their difference vanishes to order m in $\partial\Omega$. Since the boundary is C^∞, we can find a defining function $\rho \in C_c^\infty(\mathbb{C})$ with the following properties: $\Omega = \{z \in \mathbb{C};\ \rho(z) < 0\}$, $\partial\Omega = \{z \in \mathbb{C};\ \rho(z) = 0\}$ and $\mathrm{d}\rho \neq 0$ in $\partial\Omega$. A simple exercise is the following: $F \in C^\infty(\overline{\Omega})$ vanishes to order m in $\partial\Omega$ if and only if there is a function $\theta \in C^\infty(\overline{\Omega})$ such that $F = \theta\rho^{m+1}$ in $\overline{\Omega}$.

Proof of lemma 2.5.1: Let ρ be a defining function for $\partial\Omega$. We construct the functions Φ_m by induction in m. Let $\chi \in C_c^\infty(\mathbb{C})$ be equal 1 in a neighborhood of $\partial\Omega$ and that vanishes in a neighborhood of $\{z \in \mathbb{C};\ \partial\rho/\partial\overline{z}(z) = 0\}$ (observe that, since ρ is real, this set coincides with the set $\mathrm{d}\rho = 0$). To build Φ_0, the idea is to take $\Phi_0 = \theta_0\rho$ and choose θ_0 appropriately.

Since

$$\frac{\partial\,(\theta_0\rho)}{\partial\bar{z}} = \frac{\partial\theta_0}{\partial\bar{z}}\rho + \theta_0\frac{\partial\rho}{\partial\bar{z}},$$

we let $\theta_0 := \chi f/\partial_{\bar{z}}\rho \in C^\infty(\overline{\Omega})$. It is clear then that $\partial_{\overline{z_0}}\Phi_0 = f$ in $\partial\Omega$. Observe that

$$\|\Phi_0\|_{C^k(\overline{\Omega})} \leq \|\chi\rho/(\partial_{\bar{z}}\rho)\|_{C^k(\overline{\Omega})}\,\|f\|_{C^k(\overline{\Omega})},\ k \in \mathbb{Z}_+.$$

Suppose now that Φ_l have been constructed for $l = 0, \dots, m-1$, which satisfy the conclusion of the lemma. Then,

$$\frac{\partial\Phi_{m-1}}{\partial\bar{z}} - f = \Psi_{m-1}\rho^m,\ \Psi_{m-1} \in C^\infty(\overline{\Omega}).$$

We complete the inductive step setting $\Phi_m = \Phi_{m-1} - \theta_m\rho^{m+1}$ and choosing θ_m appropriately. Notice that

$$\begin{aligned}\frac{\partial\Phi_m}{\partial\bar{z}} - f &= \frac{\partial\Phi_{m-1}}{\partial\bar{z}} - f - \frac{\partial(\theta_m\rho^{m+1})}{\partial\bar{z}}\\ &= \Psi_{m-1}\rho^m - \frac{\partial\theta_m}{\partial\bar{z}}\rho^{m+1} - \theta_m(m+1)\rho^m\frac{\partial\rho}{\partial\bar{z}}.\end{aligned}$$

We define $\theta_m = \chi\Psi_{m-1}/\left((m+1)\partial_{\bar{z}}\rho\right) \in C^\infty(\overline{\Omega})$. □

Theorem 2.5.2 (Localization). *Let $K \subset \mathbb{C}$ be a polynomially convex compact set and $\boldsymbol{f} \in C(K;\mathcal{F}_1)$. Suppose that, for every $x \in K$, there is a compact neighborhood $U_x \subset \mathbb{C}$ of x such that $\boldsymbol{f} \in A^\infty(U_x \cap K)$. Then $\boldsymbol{f} \in A^\infty(K)$.*

Proof. Write $\mathbf{f} = \sum_{l=0}^\infty f_l X^l$ and let $d \in \mathbb{Z}_+$. We will assume that f_l have been continuosly extended to a neighborhood of K. We shall prove that $\mathbf{f} \in A^d(K)$ (note that this implies the result). Let $U_1, \dots, U_N$ be a covering of K with the properties of the statement, in a way that $\{\operatorname{int} U_j\}_{j=1}^N$ is also a covering of K. Let $\{\chi_j\}_{j=1}^N$ be a partition of unity subordinate to this open cover and let $U \subset \mathbb{C}$be an open neighborhood of K where $\sum \chi_j = 1$. Let $m \in \mathbb{Z}_+$ be arbitrary and $\varepsilon > 0$. By hypothesis, we can find polynomials $p_j \in \mathcal{O}(\mathbb{C})$ and a neighborhood $U_j \supset V_j \supset K \cap U_j$ such that

$$\sum_{l=0}^m \left\|\frac{p_j^{(l)}}{l!} - f_l\right\|_{C(V_j)} \leq \varepsilon,\ j = 1, \dots, N.$$

Write $V = \bigcup V_j$. Define the function $g := \sum_{j=1}^N \chi_j p_j \in C_c^\infty(\Omega)$, where $\Omega = \bigcup U_j$. Observe that

$$\begin{aligned}\sum_{l=0}^m \left\|\frac{g^{(l)}}{l!} - f_l\right\|_{C(V)} &= \sum_{l=0}^m \sup_V \left|\sum_{j=1}^N \frac{\left(\chi_j p_j\right)^{(l)}}{l!} - f_l\right|\\ &\leq \sum_{l=0}^m \sup_V \left|\sum_{j=1}^N \chi_j\frac{p_j^{(l)}}{l!} - f_l\right| + \sum_{l=0}^m \sup_V \left|\sum_{k=0}^{l-1} \frac{1}{(l-k)!}\sum_{j=1}^N \chi_j^{(l-k)}\left(\frac{p_j^{(k)}}{k!} - f_k\right)\right|\\ &\leq C(m)\varepsilon\end{aligned}$$

Moreover, a similar calculation implies that

$$\|\bar{\partial} g\|_{C^m(V)} \leq C'(m)\varepsilon.$$

Choose, finally, $\chi \in C_c^\infty(V)$ which is 1 in a neighborhood of K. We have $\|\chi\bar{\partial} g\|_{C^m(\overline{V})} \leq C''(m)\varepsilon$, and therefore, $\|T(\chi\bar{\partial} g)\|_{C^k(\overline{U})} \leq A\varepsilon$, where T is the Cauchy-Green operator (and A can be taken independent of the open set by lemma 2.5.1). The holomorphic function $g - T(\chi\bar{\partial} g)$ is, therefore, holomorphic in a neighnborhood of K and approximates f (up to order m) in a neighborhood of K. The result now follows by Runge's theorem. □

2.6 Simultaneous approximation of jets by holomorphic functions

Let $K \subset \mathbb{C}$ be a compact set. A well-known theorem of Lavrentiev implies that

$$P(K) = C(K)$$

if and only if K has empty interior and connected complement (this also follows from Mergelyan's theorem). In classical uniform algebra theory, there are a number of criteria (involving, say, analytic capacity) that are also equivalent to the property $P(K) = C(K)$. In our study of $A^\infty(K)$ in a similar to spirit to $P(K)$, the following question is then natural:

Question 2.6.1. *Let $K \subset \mathbb{C}$ be a compact set. What are necessary and sufficient conditions on K that guarantee that*

$$A^\infty(K) = C(K; \mathcal{F}_1)?$$

This is, evidently, much stronger than $P(K) = C(K)$, given that we are supposed to approximate an arbitrary jet of continuous functions on K by the jet of an entire function. There are some clear necessary conditions for this to hold.

2.6.1 Necessary conditions

Let $\gamma : [0,1] \to \mathbb{C}$ be a Lipschitz function and assume that there is a point $t \in [0,1]$ such that γ is not locally constant at t. Then, there is a continuous function $f \in C(\mathbb{C})$ such that $f \circ \gamma$ is not Lipschitz in any neighborhood of t.

Indeed, the condition of not being locally constant implies that, for every $n \geq 0$, there is a point $t_n \in (t - (n+1)^{-1}, t + (n+1)^{-1}) \cap [0,1]$ such that $\gamma(t_n) \neq \gamma(t)$. Passing to a subsequence, we can even assume that $\gamma(t_n) \not\gamma(t_{n'})$ for all $n \neq n'$. Consider the following compact set:

$$K = \{\gamma(t_n);\, n \in \mathbb{Z}_+\} \cup \{\gamma(t)\}.$$

Now, consider the following continuous function on K:

$$f(x) = \begin{cases} 0, & x = \gamma(t), \\ |t_n - t|^{1/2}, & x = \gamma(t_n),\ n \in \mathbb{Z}_+. \end{cases}$$

By Tietze's extension theorem, this function admits a continuous extension $f : \mathbb{C} \to \mathbb{R}$. We claim that $f \circ \gamma$ is not Lipschitz in any neighborhood of t. Indeed, if that were true, it would follow in particular that

$$\frac{f(\gamma(t_n)) - f(\gamma(t))}{t_n - t}$$

would be bounded, which is not the case.

Proposition 2.6.1. *Let $K \subset \mathbb{C}$ be a compact set. Let $\gamma : [0,1] \to \mathbb{C}$ be a Lipschitz continuous function with $\gamma([0,1]) \subset K$. Then, if*

$$\boldsymbol{f} = \sum_{j \in \mathbb{Z}_+} f_j X^j \in A^\infty(K),$$

then $f_0 \circ \gamma$ is Lipschitz in $[0,1]$.

Proof. This fact is a consequence of Lemma 6.11 of Bierstone and Milman, 2004, but the proof is very simple. Indeed, let $(p_l)_{l \in \mathbb{Z}_+}$ be a sequence of holomorphic polynomials with $\frac{1}{j!}\partial^j p_l \to f_j$ uniformly on K for every j. Let L be the LIpschitz constant of γ. Given points $x, y \in [0,1]$ with $x \leq y$, we write for all $M \geq 1$, $M \in \mathbb{Z}_+$,

$$|p_l \circ \gamma(x) - p_l \circ \gamma(y)| \leq \sum_{k=0}^{M-1} \left| p_l\left(\gamma(x + \frac{j}{M}(y-x))\right) - p_l\left(\gamma(x + \frac{(j+1)}{M}(y-x))\right)\right|.$$

From the mean-value inequality, given $z, w \in \mathbb{C}$, we have

$$|p_l(z) - p_l(w)| \leq C \sup_{x \in [z,w]} \left|\frac{\partial p_l}{\partial z}(x)\right| |z - w|,$$

where $C > 0$ is a universal constant and $[z, w]$ is the line segment connecting z and w. Replacing this in the previous estimate and letting $M \to \infty$ implies that

$$|f_0(\gamma(y)) - f_0(\gamma(x))| \leq \left(CL \sup_K |f_1|\right) |x - y|.$$

$\square$

We obtain the following corollary:

Corollary 2.6.1. *Let $K \subset \mathbb{C}$ be compact and assume that theere is a Lipschitz curve $\gamma : [0,1] \to K$ which is not locally constant at $t_0 \in [0,1]$. Then, $A^\infty(K) \neq C(K; \mathcal{F}_1)$.*

Moreover, from P. D. Cordaro, Della Sala, and Lamel, 2020, it is known that the spectrum of the Fréchet algebra $A^\infty(K)$ is the polynomial hull $\widehat{K}$. From this, we obtain the following

Theorem 2.6.1. *Let $K \subset \mathbb{C}$. Then, in order for $A^\infty(K)$ to be equal to $C(K; \mathcal{F}_1)$, the following conditions are necessary:*

1. *K is polynomially convex, i.e., $\widehat{K} = K$ (in this case, this means that the complement $\mathbb{C} \setminus K$ is connected).*
2. *K contains no non-constant Lipschitz curves: if $\gamma : [0,1] \to K$ is a Lipschitz curve, then γ is constant.*

We conjecture that these conditions are also sufficient. This was conjectured by Bishop, 1958 in the case of Jordan arcs (i.e., K is homeomorphic to the interval $[0,1]$). It follows from work of Smirnov, 1993 (see also Smirnov and Khavin, 1998) that the conjecture is true in the case of $A^1(K)$: this means perfoming the construction above but only considering convergence of the entire function and its first derivative.

2.6.2 Sufficient conditions

We shall present, in this section, a proof of the following theorem:

Theorem 2.6.2. *Let $K \subset \mathbb{C}$ be compact. Assume that both projections of K onto the coordinate axes have zero Lebesgue measure. Then, $A^\infty(K) = C(K; \mathcal{F}_1)$.*

This result is contained in Bishop, 1959 (since such compact sets are totally disconnected), but the proof is of a different nature. We need some preparation to obtain sufficient conditions for a locally integrable holomorphic function in the complement of K to admit holomorphic extension. These conditions are contained in Garnett, 1972.

Lemma 2.6.1. *Let γ be a rectifiable Jordan curve that bounds an open set $D \subset \mathbb{C}$. If μ is a compactly supported measure on $\mathbb{C}$ such that*

$$\int_\gamma U_{|\mu|}(s)\mathrm{d}s < \infty,$$

where $U_{|\mu|}(z) = \int_{\mathbb{C}} |\zeta - z|^{-1}\mathrm{d}|\mu|(\zeta)$ is the Newtonian potential of μ, then

$$-\frac{1}{2\pi i}\int_\gamma \widehat{\mu}(s)\mathrm{d}s = \mu(D) = \mu(\overline{D}).$$

Proof. This is a consequence of Fubini's theorem:

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \widehat{\mu}(s)\mathrm{d}s &= \frac{1}{2\pi i}\int_\gamma\int_{\mathbb{C}} \frac{\mathrm{d}\mu(z)}{z-s}\mathrm{d}s \\
&= \int_{\mathbb{C}}\left(\frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}s}{z-s}\right)\mathrm{d}\mu(z) \\
&= -\mu(D),
\end{aligned}$$

where the function $|z-s|^{-1}$ is integrable with respect to the product measure $\mathrm{d}|\mu| \times \mathrm{d}s$ precisely because of the integrability hypothesis of the Newtonian potential in γ. Moreover, $\mu(D) = \mu(\overline{D})$ because $\int_\gamma U_{|\mu|} < \infty$ implies that $|\mu|(\Gamma) = 0$. □

This lemma is useful because, since $U_{|\mu|}$ is locally integrable in $\mathbb{C}$, it will be locally integrable in almost every rectangle. To explore this fact, we use the following

Definition 2.6.1. Let $\mathcal{R}$ be a collection of open rectangles in $\mathbb{C}$ with sides parallel to the coordinate axes. We say $\mathcal{R}$ is an *admissible network* if the set of pairs (z, w), where z and w are the lower left and upper right vertices, respectively, of a rectangle $R \in \mathcal{R}$ has full measure in

$$\{(z, w) \in \mathbb{C} \times \mathbb{C};\ \operatorname{Re} w > \operatorname{Re} z,\ \ \operatorname{Im} w > \operatorname{Im} z\}.$$

This admissibility condition implies that the network $\mathcal{R}$ is highly fine (i.e., it contains many rectangles). Note that a countable intersection of admissible networks is an admissible network. Another fact we will use is the following: if $f \in L^1_{loc}(\mathbb{C})$, then there is an admissible network $\mathcal{R}$ such that

$$\int_{\partial R} |f(s)| \mathrm{d}s < \infty$$

for all $R \in \mathcal{R}$. Indeed, the local integrability condition implies, by Fubini's theorem, that f is locally integrable over almost all vertical and horizontal lines. The collection of rectangles formed by such lines yields $\mathcal{R}$.

Lemma 2.6.2 (Royden). *Let $D \subset \mathbb{C}$ be open and $F \in L^1_{loc}(D)$. Suppose that*

$$\int_{\partial R} |F(s)| \mathrm{d}s < \infty, \quad \int_{\partial R} F(s) \mathrm{d}s = 0$$

for all the rectangles R over an admissible network $\mathcal{R}$ (where the rectangles have closure contained in D). Then, F coincides with a holomorphic function almost everywhere on D. In particular, if $D = \mathbb{C}$ and F vanishes at infinity, $F = 0$ almost everywhere.

Proof. Let $\chi \in C_c^\infty(\mathbb{C})$ be such that $0 \leq \chi \leq 1$, $\int \chi \mathrm{dm} = 1$ and $\operatorname{supp} \chi \subset B(0, 1)$. Consider the approximations of the identity $\chi_\rho(z) = \rho^{-2}\chi(z/\rho)$. Let $D_\rho = \{z \in D;\ d(z, D^c) > \rho\}$. Then, $F * \chi_\rho \in C^\infty(D_\rho)$. Fix $R \in \mathcal{R}$. Since $\overline{R} \subset D$, for $\rho > 0$ sufficiently small, we have

$$\begin{aligned}
\int_{\partial R} \left(F * \chi_\rho\right)(s) \mathrm{d}s &= \int_{\partial R} \int_C F(s - \zeta) \chi_\rho(\zeta) \mathrm{dm}(\zeta) \mathrm{d}s \\
&= \int_{\mathbb{C}} \chi_\rho(\zeta) \left(\int_{\partial R} F(s - \zeta) \mathrm{d}s \right) \mathrm{dm}(\zeta) \\
&= \int_{\mathbb{C}} \chi_\rho(\zeta) \left(\int_{\partial(R - \zeta)} F(s) \mathrm{d}s \right) \mathrm{dm}(\zeta),
\end{aligned}$$

where the last equality holds because almost every translate $R - \zeta$ of $R \in \mathcal{R}$ belongs to $\mathcal{R}$. Then, we conclude thaat $F * \chi_\rho$ is holomorphic in D_ρ by Morera's theorem. Letting $\rho \to 0$, the functions $F * \chi_\rho$ converge uniformly over the compact sets of D to a holomorphic function on D. Since $F * \chi_\rho \to F$ when $\rho \to 0$, the theorem is proved. □

From this theorem, we have the following result.

Theorem 2.6.3. *Let $D \subset \mathbb{C}$ be open and $E \subset D$ closed. Assume that f is locally integrable on D and holomorphic on $D \setminus E$. Suppose that $\int_\gamma f(s) \mathrm{d}s = 0$ for all closed rectifiable curves γ with trace in $D \setminus E$. If the projections of E onto both coordinate axes have zero measure, then f extends holomorphically to D.*

Proof. The condition on E implies that the set of rectangles on D whose boundary is disjoint with E is admissible.

□

We can now proceed with the proof of theorem 2.6.2.

Proof of 2.6.2. Let $p = \sum_{j=0}^{d} \in M(K)[X]$. Suppose that $p|_{A^\infty(K)} = 0$ or, by lemma 2.4.2, that $\widehat{p}(z) = 0$ for $z \in \mathbb{C} \setminus K$. We want to show that $p \equiv 0$. Observe that, if $d = 0$, the result follows from 2.4.1 (and from the fact that K has null measure).

Assume then that $d \geq 1$. The condition $\widetilde{p}(z) = 0$ for $z \in \mathbb{C} \setminus K$ implies that

$$\widehat{\mu_0}(z) = \sum_{j=1}^{d} \int_K \frac{\mathrm{d}\mu_j(\zeta)}{(z-\zeta)^{j+1}}, \ z \in \mathbb{C} \setminus K. \tag{2.6.1}$$

The main point is that the right-hand side of 2.6.1 has a primitive on $\mathbb{C} \setminus K$. Therefore, if γ is a closed rectifiable curve with trace in$\mathbb{C} \setminus K$, it follows that

$$\int_\gamma \widehat{\mu_0}(s)\mathrm{d}s = 0.$$

Therefore, by theorem 2.6.3, the function $\widehat{\mu_0}$ admits holomorphic extension to K. But since $\widehat{\mu_0}(\infty) = 0$, we conclude that $\widehat{\mu_0} \equiv 0$, and by corollary 2.4.1, $\mu_0 = 0$. It follows then that

$$\widetilde{p}(z) = \sum_{j=1}^{d} \int_K \frac{\mathrm{d}\mu_j(\zeta)}{(z-\zeta)^{j+1}}, \ z \in \mathbb{C} \setminus K.$$

Note now that $\widetilde{p}$ has a primitive in $\mathbb{C} \setminus K$ (which is connected), therefore, there is a constant $c \in \mathbb{C}$ such that

$$-\sum_{j=0}^{d-1} \frac{1}{j+1} \int_K \frac{\mathrm{d}\mu_{j+1}(\zeta)}{(z-\zeta)^{j+1}} = c, \ z \in \mathbb{C} \setminus K.$$

Evaluating at $z = \infty$, we conclude that $c = 0$, from which the Cauchy transform $\widehat{\mu_1}$ thas zero integral along every curve with trace in $\mathbb{C} \setminus K$, and from before, $\mu_1 = 0$. Proceeding inductively, we obtain $\mu_0 = \dots = \mu_d = 0$, and $p = 0$. The proof is complete. □

Chapter 3

On the regularity of locally integrable structures

3.1 Introduction

In this chapter, we consider the following question.

Question 3.1.1. *Let $(M, \mathcal{V}, p)$ be a germ of a smooth locally integrable structure near a point $p \in M$. Find necessary and sufficient conditions on $\mathcal{V}$ to be equivalent to a germ of a real-analytic locally integrable structure.*

In this situation, we say that $\mathcal{V}$ is *analytically regularizable* near $p \in M$. This problem is of interest in cases where techniques exist that only treat the real-analytic case.

To our knowledge, this problem was first considered in Kossovskiy and Zaitsev, 2022 for smooth strictly pseudoconvex hypersurfaces in $\mathbb{C}^n$. They introduce a property (called *condition (E)*), which we recall in section 3.5. The main result is then the following.

Theorem 3.1.1 (Kossovskiy, Zaitsev)**.** *Let $M \subset \mathbb{C}^n$ be a smooth, strictly pseudoconvex hypersurface, and let $p \in M$. Then M is CR equivalent near p to a real-analytic hypersurface if and only if M satisfies condition (E).*

Our goal for this chapter is to extend this result to different classes of locally integrable structures (of non-CR type). In order to do this, we use as the main tool a technique by Marson, 1992 that associates to a locally integrable structure a CR one. Then, we have to perform a construction that takes advantage of the large automorphism algebra of the associated CR structure in order to produce a regularizing map that is simple enough to descend to an equivalence of the original structure. This yields the main result, Theorem 3.5.1.

3.2 The local picture of a locally integrable structure

Let $(M, \mathcal{V})$ be a locally integrable structure. Let $p \in M$ and $n = \dim \mathcal{V}_p$, $N = \dim M$, $m = N - n$ and $d = \dim_{\mathbb{R}} T_p^\circ = \dim_{\mathbb{R}} \left(\mathcal{V}_p^\perp \cap T_p^* M\right)$. We will discuss the problem of finding appropriate coordinates for $\mathcal{V}$ in a neighborhood of p. This is known as the *fine local embedding*.

This depends on some elementary results of complex linear algebra, which we prove next.

Lemma 3.2.1. *Let $V \subset \mathbb{C}^N$ be a complex vector subspace. Let $V^\circ = V \cap \mathbb{R}^N$. Then, $V^\circ \otimes_{\mathbb{R}} \mathbb{C} \simeq V^\circ + iV^\circ = V \cap \overline{V}$.*

Proof. Let $v \in V \cap \overline{V}$, then there is $w \in V$ such that $\bar{v} = w$. In particular, $\mathsf{Re}\, v = \frac{v+\bar{v}}{2} = \frac{v+w}{2} \in V$. Therefore, we have a well-defined real linear transformation

$$\mathsf{Re}\ : V \cap \overline{V} \to V^\circ,$$

which is clearly surjective (since $V^\circ \subset V \cap \overline{V}$ and $\mathsf{Re}\big|_{V^\circ}$ is just the identity). The kernel of this map is the set of $v \in V \cap \overline{V}$ such that $\mathsf{Re}\, v = 0$, which implies $v = iw$ for $w \in \mathbb{R}^N$, and since $w = -iv \in V$, we obtain $\ker \mathsf{Re}\big|_{V \cap \overline{V}} = iV^\circ$. Since every short exact sequence of finite dimensional vector spaces splits, we obtain

$$V \cap \overline{V} \simeq V^0 + iV^0.$$

Since $V^\circ + iV^\circ$ is naturally a subspace of $V \cap \overline{V}$, the isomorphism above is an equality, which proves our contention. $\square$

The second one is the following:

Lemma 3.2.2. *Let $V \subset \mathbb{C}^N$ be a complex vector subspace and $V^\circ = V \cap \mathbb{R}^N$. Let $m = \dim_{\mathbb{C}} V$, $d = \dim_{\mathbb{R}} V^\circ$ and $\nu = m - d$. Let $V_1 \subset \mathbb{C}^N$ be a complex vector subspace such that $V = (V \cap \overline{V}) \oplus V_1$ and pick*

$$\{\zeta_1, \dots, \zeta_\nu\} \text{ a complex basis for } V_1, \quad \{\xi_{\nu+1}, \dots, \xi_m\} \text{ a real basis for } V^\circ.$$

Then, $\{\zeta_1, \dots, \zeta_\nu, \xi_{\nu+1}, \dots, \xi_m\}$ is a basis of V such that

$$\{\zeta_1, \dots, \zeta_\nu, \overline{\zeta_1}, \dots, \overline{\zeta_\nu}, \xi_{\nu+1}, \dots, \xi_m\}$$

are linearly independent over $\mathbb{C}$ (in particular, $2\nu + d = m + \nu \leq N$).

Proof. First, observe that $V_1 \cap \overline{V_1} = \{0\}$, since $V_1 \cap \overline{V_1} \subset V \cap \overline{V}$, $V_1 \cap \overline{V_1} \subset V_1$, and $(V \cap \overline{V}) \cap V_1 = \{0\}$. In particular, $V_1 \cap \mathbb{R}^N =: V_1^\circ = \{0\}$. Therefore, $\{\zeta_1, \dots, \zeta_\nu, \overline{\zeta_1}, \dots, \overline{\zeta_\nu}\}$ are linearly independent over $\mathbb{C}$. Moreover, since $V \cap \overline{V} = V^\circ + iV^\circ$ by lemma 3.2.1, the set $\{\xi_{\nu+1}, \dots, \xi_m\}$ is a $\mathbb{C}$-basis for $V \cap \overline{V}$. Thus, $\{\zeta_1, \dots, \zeta_\nu, \xi_{\nu+1}, \dots, \xi_m\}$ is a $\mathbb{C}$-basis for V.

Finally, we claim that $V \cap \overline{V_1} = ((V \cap \overline{V}) \oplus V_1) \cap \overline{V_1} = \{0\}$. Indeed, let $v \in V$ be such that $\bar{v} = w \in V_1$. Then, if we write $\mathsf{Re}\, v = \frac{v+\bar{v}}{2}$, we conclude that $\mathsf{Re}\, v \in V^\circ$. In particular, $\mathsf{Im}\, v \in V^\circ$, and then $v \in V^\circ + iV^\circ = V \cap \overline{V}$. Since $(V \cap \overline{V}) \cap V_1 = (V \cap \overline{V}) \cap \overline{V_1} = 0$, we conclude that $v = 0$. This shows that the set

$$\{\zeta_1, \dots, \zeta_\nu, \overline{\zeta_1}, \dots, \overline{\zeta_\nu}, \xi_{\nu+1}, \dots, \xi_m\}$$

consists of linearly independent vectors, and the proof is complete.

□

We now apply lemma 3.2.2 with $V = T'_p$ (and $V^\circ = T^\circ_p$). This yields a basis $\{\zeta_1, \dots, \zeta_\nu, \xi_{\nu+1}, \dots, \xi_m\}$ of T'_p with the properties stated in 3.2.2. Let $G_1, \dots, G_m$ be a complete set of first integrals for $\mathcal{V}$ in a neighborhood of p. Then, there is a matrix $(c_{jk})_{1 \le j,k \le m} \in \mathrm{GL}_m(\mathbb{C})$ such that

$$\zeta_j = \sum_{k=1}^{m} c_{jk} \mathrm{d}G_k(p), \;\; 1 \le j \le \nu$$

and

$$\xi_j = \sum_{k=1}^{m} c_{jk} \mathrm{d}G_k(p), \;\; \nu + 1 \le j \le m.$$

We now perform an affine (in particular, biholomorphic) change in the first integrals, by setting

$$Z_j = \sum_{k=1}^{m} c_{jk}(G_k - G_k(p)), \;\; j = 1, \dots, \nu$$

and

$$W_l = \sum_{k=1}^{m} c_{(\nu+l)k}(G_k - G_k(p)), \;\; j = \nu + 1, \dots, d.$$

It's clear that $\{\mathrm{d}Z_1, \dots, \mathrm{d}Z_\nu, \mathrm{d}W_1, \dots, \mathrm{d}W_d\}$ span T' in a neighborhood of p. Now let

$$x_j = \mathsf{Re}\, Z_j, \;\; y_j = \mathsf{Im}\, Z_j, \;\; s_l = \mathsf{Re}\, W_l, \;\; 1 \le j \le \nu, \;\; 1 \le l \le d.$$

Then, since $\mathrm{d}Z_j(0) = \zeta_j$ and $\mathrm{d}W_l = \xi_{\nu+l}$ for $1 \le j \le \nu$ and $1 \le l \le d$, it follows from lemma 3.2.2 that we can find real-valued functions $t_1, \dots, t_{n'}$, where $n' = N - 2\nu - d$ such that the set

$$\{x_1, \dots, x_\nu, y_1, \dots, y_\nu, s_1, \dots, s_d, t_1, \dots, t_{n'}\}$$

gives a coordinate system for M in a neighborhood of p (which vanishes at p). In these coordinates, there are smooth real-valued functions $\phi_1, \dots, \phi_d$ vanishing at 0, with derivatives vanishing at 0 such that

$$\begin{cases} Z_j(x, y) = x_j + iy_j = z_j, \;\; j = 1, \dots, \nu, \\ W_k(x, y, s, t) = s_k + i\phi_k(x, y, s, t), \;\; k = 1, \dots, d. \end{cases}$$

Moreover, $\{\mathrm{d}s_1\big|_p, \dots, \mathrm{d}s_d\big|_p\}$ is a basis for T°_p. Note that, from the construction of these coordinates, we have

$$\frac{\partial W_k}{\partial s_{k'}}(0) = \delta_{kk'}, \;\; 1 \leq k, k' \leq d.$$

We can find then, in a neighborhood of 0 in $\mathbb{R}^{2\nu+d+n'}$, smooth complex vector fields

$$M_k = \sum_{k'=1}^{d} a_{kk'}(x, y, s, t)\frac{\partial}{\partial s_{k'}}, \;\; k = 1, \dots, d,$$

(which are real at 0) such that

$$M_k W_{k'} = \delta_{kk'}.$$

If we write $z_j = x_j + iy_j$ for $1 \leq j \leq \nu$, then we can introduce the vector fields

$$\begin{cases} L_j = \dfrac{\partial}{\partial \overline{z_j}} - i \displaystyle\sum_{k=1}^{d} \dfrac{\partial \phi_k}{\partial \overline{z_j}}(z, s, t)M_k, \;\; j = 1, \dots, \nu, \\ \\ \widetilde{L}_l = \dfrac{\partial}{\partial t_l} - i \displaystyle\sum_{k=1}^{d} \dfrac{\partial \phi_k}{\partial t_l}(z, s, t)M_k, \;\; l = 1, \dots, n'. \end{cases} \tag{3.2.1}$$

It is clear that

$$L_j Z_{j'} = L_j W_k = \widetilde{L}_l Z_{j'} = \widetilde{L}_l W_k = 0$$

for all $1 \leq j, j' \leq \nu$, $1 \leq k \leq d$ and $1 \leq l \leq n'$. They are also linearly independent, and since $\nu + n' = n = \dim \mathcal{V}_p$, we conclude that

$$L_1, \dots, L_\nu, \widetilde{L}_1, \dots, \widetilde{L}_{n'} \text{ span } \mathcal{V} \text{ in a neighborhood of the origin.}$$

The one-forms

$$\mathrm{d}z_1, \dots, \mathrm{d}z_\nu, \mathrm{d}\overline{z_1}, \dots, \mathrm{d}\overline{z_\nu}, \mathrm{d}W_1, \dots, \mathrm{d}W_d, \mathrm{d}t_1, \dots, \mathrm{d}t_{n'}$$

span $\mathbb{C}T^*M$ in a neighborhood near the origin, and the dual basis of this basis is given by

$$L_1^\flat, \dots, L_\nu^\flat, L_1, \dots, L_\nu, M_1, \dots, M_d, \widetilde{L}_1, \dots, \widetilde{L}_{n'},$$

where

$$L_j^\flat = \frac{\partial}{\partial z_j} - i \sum_{k=1}^{d} \frac{\partial \phi_k}{\partial z_j}(z, s, t)M_k, \;\; j = 1, \dots, \nu.$$

These vector fields generate $\mathbb{C}TM$ near p and are pairwise commuting. We denote this neighborhood of the origin (or p) from now on by Ω.

We shall now discuss the Levi form. We start with the following lemma:

Lemma 3.2.3. *Let $\omega_0 \in T_p^\circ$. Then, the number*

$$\frac{1}{2i}\langle \omega_0, [L_1, \overline{L_2}]\big|_p \rangle$$

depends only on the values $L_1(p)$ and $L_2(p)$, for L_1, L_2 sections of $\mathcal{V}$ near p.

Proof. It is enough to prove that, if $L_1(p) = 0$ and L_2 is any section of $\mathcal{V}$ near p, then $(2i)^{-1}\langle \omega_0, [L_1, \overline{L_2}]\big|_p\rangle = 0$. This follows from the fact that ω_0 is a real covector. In a neighborhood U of p there is a smooth basis (of linearly independent vector fields)$\{\widetilde{L_1}, \dots, \widetilde{L_n}\}$ of $\mathcal{V}$. If $L_1(p) = 0$, then

$$L = \sum_{j=1}^{n} c_j \widetilde{L_j} \text{ where } c_j \in C^\infty(U) \text{ and } c_j(p) = 0 \text{ for all } j = 1, \dots, n.$$

In this case, if L_2 is any other section, we have an identity

$$\begin{aligned}
[L_1, \overline{L_2}](p) = -[\overline{L_2}, L_1](p) &= -\sum_{j=1}^{n} [\overline{L_2}, c_j \widetilde{L_j}](p) \\
&= -\sum_{j=1}^{n} (\overline{L_2} c_j)(p)\widetilde{L_j}(p) + c_j(p)[\overline{L_2}, \widetilde{L_j}](p).
\end{aligned}$$

We conclude that, when L_1 vanishes at p, the bracket $[L_1, \overline{L_2}](p)$ belongs to $\mathcal{V}_p$. The lemma is proved. $\square$

We can now define the hermitian form (at the characteristic point (p, ω))

$$\begin{aligned}
\mathcal{B}_{(p,\omega)} \,:\, \mathcal{V}_p \times \mathcal{V}_p &\to \mathbb{C} \\
(v_1, v_2) &\mapsto \frac{1}{2i}\langle \omega, [V_1, \overline{V_2}]_p\rangle,
\end{aligned}$$

where V_1, V_2 are sections of $\mathcal{V}$ satisfying $V_1(p) = v_1$ and $V_2(p) = v_2$. We let $\mathcal{L}_{(p,\omega)}(v)$ the associated quadratic form. This is called the *Levi form* of $\mathcal{V}$ at the characteristic point (p, ω).

A simple computation (see, for instance, Treves, 1992) shows that in the local coordinates. the Levi form at $\omega = \sum_{k=1}^{d} \sigma_k (\mathrm{d}s_k)_0$ is given by

$$\mathcal{L}(\zeta, \tau) = \sum_{i,j=1}^{\nu} \frac{\partial^2 \Phi}{\partial \overline{z_i} \partial z_j}(0)\zeta_i \overline{\zeta_j} + 2\mathrm{Re}\left(\sum_{i=1}^{\nu}\sum_{k=1}^{n-\nu} \frac{\partial^2 \Phi}{\partial \overline{z_l} \partial t_k}(0)\zeta_i \overline{\tau_k}\right) + \sum_{k,l=1}^{n-\nu} \frac{\partial^2 \Phi}{\partial t_k \partial t_l}(0)\tau_k \overline{\tau_l}, \tag{3.2.2}$$

where $(\zeta, \tau) \in \mathbb{C}^\nu \times \mathbb{C}^{n-\nu}$ and $\Phi = \sum_{k=1}^{d} \sigma_k \phi_k$.

3.3 The associated CR structure

In this section, we describe a procedure introduced by Marson, 1992 that yields, in the coordinate system just introduced, an associated (generic) CR manifold to $(M, \mathcal{V})$. We maintain the notation of the previous section. Let $X \subset \mathbb{R}^{n'}$ be a neighborhood of 0 and set $\Omega^\bullet = X \times \Omega \subset \mathbb{R}^{n'+N}$. Let $M^\bullet = Z^\bullet(\Omega^\bullet)$ be the submanifold parametrized by the mapping

$$Z^\bullet \,:\, \Omega^\bullet \to \mathbb{C}^{m+n'}$$

given by

$$\begin{cases} Z_j^\bullet(x,y,s,t) = x_j^\bullet + it_j, \;\; j = 1,\dots,n', \\ \\ Z_{n'+j}^\bullet(x,y,s,t) = Z_j(x,y) = x_j + iy_j, \;\; j = 1,\dots,\nu, \\ \\ Z_{n'+\nu+k}^\bullet(x,y,s,t) = W_k(x,y,s,t) = s_k + i\phi_k(x,y,s,t), \;\; k = 1,\dots,d. \end{cases}$$

This map is easily seen to have full rank, and induces a CR structure $\mathcal{V}^\bullet$ on $\Omega^\bullet$, with dual structure bundle $T'^\bullet$ being generated by $\left\{ \mathrm{d}Z_1^\bullet, \dots, \mathrm{d}Z_{m+n'}^\bullet \right\}$. We write $z_l^\bullet = x_l^\bullet + it_l$ and $z_j = x_j + iy_j$ for all $1 \leq l \leq n'$ and $1 \leq j \leq \nu$. Then, we can consider the vector fields

$$\begin{cases} L_l^\bullet = \dfrac{\partial}{\partial \overline{z_l^\bullet}} - i\displaystyle\sum_{k=1}^{d} \dfrac{\partial \phi_k}{\partial \overline{z_l^\bullet}} M_k, \;\; l = 1,\dots,n', \\ \\ L_j = \dfrac{\partial}{\partial \overline{z_j}} - i\displaystyle\sum_{k=1}^{d} \dfrac{\partial \phi_k}{\partial \overline{z_j}} M_k, \;\; j = 1,\dots,\nu. \end{cases} \tag{3.3.1}$$

It's clear that $L_k^\bullet Z_l^\bullet = L_j Z_l^\bullet = 0$ for all $1 \leq k \leq n'$ and $1 \leq l \leq \nu$ (we maintain, for simplicity, the name L_j for the second set of vector fields, but formally they are the pullback of the vector fields L_j in Ω by the projection $X \times \Omega \to \Omega$). Therefore,

$$L_1^\bullet, \dots, L_{n'}^\bullet, L_1, \dots, L_\nu \text{ generate } \mathcal{V}^\bullet \text{ over } \Omega^\bullet.$$

We can describe the image of $Z^\bullet$ explicitely as

$$M^\bullet = \left\{ (z^\bullet, z, w) \in \mathbb{C}^{n'} \times \mathbb{C}^\nu \times \mathbb{C}^d ;\; \mathsf{Im}\, w = \phi(z, \overline{z}, \mathsf{Im}\, z^\bullet, \mathsf{Re}\, w) \right\}$$

This structure is called an *associated CR structure* to $(\mathcal{V}, p)$. We shall denote the Levi form of $\mathcal{V}^\bullet$ by $\mathcal{L}(\zeta^\bullet, \zeta)$, where $\zeta^\bullet \in \mathbb{C}^{n'}$ and $\zeta \in \mathbb{C}^\nu$. Using 3.2.2, we obtain

$$\mathcal{L}^\bullet(\zeta^\bullet,\zeta) = \sum_{i,j=1}^{n'} \frac{\partial^2 \Phi}{\partial \overline{z_i^\bullet} \partial z_j^\bullet}(0)\zeta_i^\bullet \overline{\zeta_j^\bullet} + \sum_{i=1}^{n'}\sum_{j=1}^{\nu} \frac{\partial^2 \Phi}{\partial \overline{z_i^\bullet} \partial z_j}(0)\zeta_i^\bullet \overline{\zeta_j} + \sum_{i=1}^{\nu}\sum_{j=1}^{n'} \frac{\partial^2 \Phi}{\partial \overline{z_i} \partial z_j^\bullet}(0)\zeta_i \overline{\zeta_j^\bullet} + \sum_{i,j=1}^{\nu} \frac{\partial^2 \Phi}{\partial \overline{z_i} \partial z_j}(0)\zeta_i \overline{\zeta_j}.$$

Since Φ is independent of $x^\bullet$, we get

$$\mathcal{L}^\bullet(\zeta^\bullet,\zeta) = \frac{1}{4}\sum_{i,j=1}^{n'} \frac{\partial^2 \Phi}{\partial t_i \partial t_j}(0)\zeta_i^\bullet \overline{\zeta_j^\bullet} + \frac{i}{2}\sum_{i=1}^{n'}\sum_{j=1}^{\nu} \frac{\partial^2 \Phi}{\partial t_i \partial z_j}(0)\zeta_i^\bullet \overline{\zeta_j} - \frac{i}{2}\sum_{i=1}^{\nu}\sum_{j=1}^{n'} \frac{\partial^2 \Phi}{\partial \overline{z_i} \partial t_j}\zeta_i \overline{\zeta_j^\bullet} + \sum_{i,j=1}^{\nu} \frac{\partial^2 \Phi}{\partial \overline{z_i} \partial z_j}(0)\zeta_i \overline{\zeta_j}.$$

If $\mathcal{L}(\zeta, \tau)$ denotes the Levi form of $\mathcal{V}$ at 0 (in the characteristic vector $\Phi = \sum_{k=1}^d \sigma_k \phi_k$), we obtain

$$\mathcal{L}(\zeta,\tau) = \mathcal{L}^\bullet(-2i\tau, \zeta), \;\; \zeta \in \mathbb{C}^\nu, \;\; \tau \in \mathbb{C}^{n'}.$$

In particular, they are equivalent quadratic forms.

3.4 Classes of non-degenerate locally integrable structures

We would like to introduce a class of (germs of) locally integrable structures such that the associated (germ of) CR submanifold is a strictly pseudoconvex hypersurface.

Definition 3.4.1. Let $(\mathcal{V}, p)$ be a germ of locally integrable structure. We say this germ is *of strict pseudoconvex type* if the characteristic set has dimension 1, $\dim_{\mathbb{R}} T_p^{\circ} = 1$, and the Levi form of the structure is positive definite at p.

Remark 3.4.1. This structures are of *hypersurface type*. The systematic study of such structures started in P. Cordaro and Treves, 1991.

We remark that, in the case the structure is elliptic at p (meaning that $T_p^{\circ} = 0$), then it is elliptic in a neighborhood of p and we can find coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, t_1, \dots, t_{n'})$ centered at p such that $Z_j(x, y, t) = x_j + y_j$ are first integrals for $\mathcal{V}$ (see Theorem I.10.1 in Berhanu, P. D. Cordaro, and Hounie, 2014). In particular, it is always locally equivalent to a real-analytic structure.

The discussion in the previous section implies the following

Proposition 3.4.1. *A germ of locally integrable structure $(\mathcal{V}, p)$ is of strict pseudoconvex type if and only if an associated CR structure $M^{\bullet}$ is a germ of a C^{∞}-smooth strictly pseudoconvex hypersurface.*

The problem we are concerned with is deciding when does a C^{∞}-germ $(\mathcal{V}, p)$ of a locally integrable structure (of pseudoconvex type) is equivalent to a real-analytic germ $(\widetilde{\mathcal{V}}, q)$. We shall provide a sufficient condition for this property to hold, which is based on condition E for a strictly pseudoconvex germ of hypersurface $M^{\bullet}$ associated to $\mathcal{V}$.

3.5 On Condition E

Let $\pi : J^{1,n} \to \mathbb{C}^{n+1}$ be the bundle of 1-jets of complex hypersurfaces of $\mathbb{C}^{n+1}$, which is a projective holomorphic bundle over $\mathbb{C}^{n+1}$ with fiber dimension n and let $M \subset \mathbb{C}^{n+1}$ be a smooth strictly pseudoconvex real hypersurface. The complex tangent bundle $T^{\mathbb{C}}M$ induces an embedding

$$\begin{aligned} \phi : M &\to J^{1,n} \\ p &\mapsto [p, T_p^{\mathbb{C}}M]. \end{aligned}$$

Then, the image

$$\phi(M) = M_J \subset J^{1,n}$$

is a smooth $(2n+1)$-dimensional real submanifold in the $(2n+1)$-dimensional complex manifold $J^{1,n}$. When M is Levi-nondegenerate, M_J is totally real (Webster). We define

$$\widehat{M} = \pi^{-1}(M) \subset J^{1,n},$$

which is a pseudoconvex real hypersurface. The manifold M_J is a smooth submanifold of $\widehat{M}$. Moreover, $\widehat{M}$ is locally CR equivalent to $M \times \mathbb{C}^n$ (i.e., $\widehat{M}$ is holomorphically degenerate). Let U^+ be the pseudoconvex side of M and let

$$\widehat{U}^+ = \pi^{-1}(U^+)$$

be that of $\widehat{M}$ (which is locally biholomorphic to $U^+ \times \mathbb{C}$).

Fix a point $p \in M$. Since M is smooth, at each point $q \in M$ near p we have a formal complexification as a formal complex formal hypersurface in $\mathbb{C}^{n+1} \times \overline{\mathbb{C}^{n+1}}$, obtained by complexifying the formal Taylor series of its defining function at q. We can then define the formal Segre variety S_q of M at q.

We can then consider the 2-jets $j^2_q S_q$ of these formal Segre varieties, which produces an embedding of M into the bundle $J^{2,n}$ of 2-jets of complex hypersurfaces in $\mathbb{C}^{n+1}$. This space canonically fibers over $J^{1,n}$, i.e., we have a fiber bundle $\pi^2_1 : J^{2,n} \to J^{1,n}$. This embedding produces a section

$$s : M_J \to J^{2,n}.$$

Definition 3.5.1. We say that M satisfies *condition E* at p if, for some choice of neighborhood U of p, the section s extends as a smooth section of π^2_1 over the pseudoconvex side $\widehat{U^+} \cup \widehat{M}$, which is holomorphic in $\widehat{U^+}$.

We are now ready to state the main result of this section:

Theorem 3.5.1. *Let $(\mathcal{V}, p)$ be a germ of a locally integrable structure of strict pseudoconvex type. Assume an associated CR estruture $M^\bullet$ satisfies condition E. Then, $\mathcal{V}$ is equivalent (at p) to a germ of real-analytic integrable structure.*

3.6 Proof of the main result

We work in the coordinates described in section 2. We have, in a neighborhood of the origin, a smooth real-valued function ϕ vanishing at 0, with derivatives vanishing at 0 such that

$$\begin{cases} Z_j(x,y) = x_j + iy_j = z_j, \quad j = 1, \dots, \nu, \\ W(x,y,s,t) = s + i\phi(x,y,s,t) \end{cases}$$

and such that $\mathrm{d}Z_j$ and $\mathrm{d}W$ span, in a neighborhood of 0, the bundle T'. We have a smooth, strictly pseudoconvex hypersurface

$$M^\bullet = \left\{ (z^\bullet, z, w) \in \mathbb{C}^{n'} \times \mathbb{C}^\nu \times \mathbb{C};\ \mathsf{Im}\, w = \phi(z, \overline{z}, \mathsf{Im}\, z^\bullet, \mathsf{Re}\, w) \right\} \tag{3.6.1}$$

which we assume satisfies condition E. From Zaitsev-Kossovskiy's theorem, there is a CR diffeomorphism $f : M^\bullet \to N$ (near the origin), where $(N^\bullet, q)$ is a real-analytic hypersurface in $\mathbb{C}^{n'+\nu+1}$. Consider the (abelian) subalgebra $\mathfrak{a} \subset \mathrm{aut}(M^\bullet, p)$ generated by the real vector fields $\left\{ \frac{\partial}{\partial z^\bullet_l} + \overline{\frac{\partial}{\partial z^\bullet_l}} \right\}$, $l = 1, \dots, n'$.

This implies that the real-analytic hypersurface $N^\bullet$ has an abelian subalgebra $\widetilde{\mathfrak{a}} \subset$

$\operatorname{hol}(N^\bullet, q)$ of real dimension n'. Since $N^\bullet$ is real-analytic, there are germs $V_1, \dots, V_{n'}$ of holomorphic vector fields in $\mathbb{C}^{n'+\nu+1}$ such that

$$\operatorname{span}\{\operatorname{Re} V_1, \dots, \operatorname{Re} V_{n'}\} = \widetilde{a}$$

and $\operatorname{Re} V_1, \dots, \operatorname{Re} V_{n'}$ are tangent to $N^\bullet$. Applying the holomorphic version of Frobenius' theorem, we can find holomorphic coordinates $(Z^\bullet, Z, W) \in \mathbb{C}^{n'+\nu+1}$ near q and a real-analytic function ψ (vanishing, along with its derivatives at the origin) such that $N^\bullet$ is given by

$$\left\{(Z^\bullet, Z, W) \in \mathbb{C}^{n'+\nu+1}; \operatorname{Im} W = \psi(Z, \overline{Z}, \operatorname{Im} Z^\bullet, \operatorname{Re} W)\right\}$$

near q. Moreover, since f maps $\mathfrak{a}$ to $\widetilde{\mathfrak{a}}$, we conclude that f is of the form

$$f(z^\bullet, z, w) = (z^\bullet + f_1(z, w), f_2(z, w)).$$

We now consider the real-analytic locally integrable structure $(N, \mathcal{W})$ given by the first integrals

$$\begin{cases} \widetilde{Z}_j(x, y, s, t) = x_j + iy_j, \ \ j = 1, \dots, \nu, \\ \widetilde{W}(x, y, s, t) = s + i\psi(x, y, s, t). \end{cases}$$

Then, the induced map $F : M \to N$, given by

$$F(x, y, s, t) = (x, y, \operatorname{Re} f_2(Z(x, y), W(x, y, s, t)), t + \operatorname{Im} f_1(Z(x, y), W(x, y, s, t))),$$

defines an isomorphism between the structures $\mathcal{V}$ and $\mathcal{W}$, for the pullback of the first integrals $\widetilde{Z}_j$ and $\widetilde{W}_j$ are solutions of $\mathcal{V}$. We conclude that M is equivalent (near p) to a real-analytic structure.

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