Tight Quotients of Smale Diffeomorphisms on Surfaces

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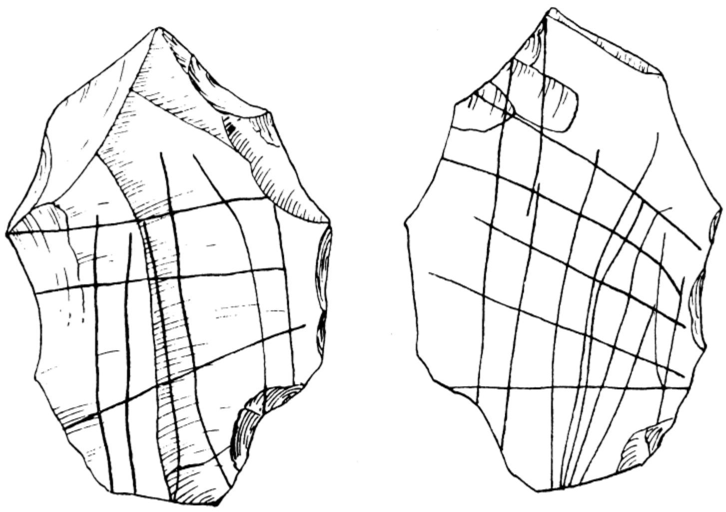

A shale flake with engraved trellis patterns on both sides, from Muden, KwaZulu-Natal, South Africa. An "individualistic product of mere experimenters" from Pleistocene [Mal56].

Agradecimentos

In mathematics you don't understand things. You just get used to them.
— John von Neumann

Agradeço aos meus pais, Silvia e Oswaldo, e às minhas irmãs, Julia e Mariana, por todo amor e suporte recebidos. Tenho sorte de tê-los por perto. Agradeço às minhas avós América e Dionetti pelos conselhos e cuidados. Agradeço aos meus sobrinhos e sobrinhas, tios e tias, primos e primas e à toda minha familía. Agradeço aos meus amigos e amigas. Sem todos vocês não teria sido possível concretizar este trabalho.

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Resumo

João Paulo Ferreira de Mello. **Quocientes Justos de Difeomorfismos de Smale em Superfícies**. Tese (Doutorado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2023.

Dado um difeomorfismo f sobre uma superfície fechada, dois pontos são ditos zero-entrópicos equivalentes se existe um contínuo contendo ambos os pontos onde o contínuo carrega zero entropia. Neste trabalho usamos este conceito para mostrar que a dinâmica quociente, pela relação de zero-entropia, de um difeomorfismo do tipo *shoe*, que é uma subclasse dos difeomorfismos de Smale em superfícies, é um homeomorfismo pseudo-Anosov generalizado sobre uma superfície fechada possivelmente possuindo pontos identificados.

Palavras-chave: difeomorfismos de smale. equivalência de zero-entropia. homeomorfismo pseuso-anosov generalizados.

Abstract

João Paulo Ferreira de Mello. **Tight Quotients of Smale Diffeomorphisms on Surfaces**. Thesis (Doctorate). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2023.

Given a diffeomorphism f over a closed surface, two points are said to be zero-entropy equivalence if there exist a continuum containing both points and the continuum carries zero entropy. In this work we use this concept to prove that the quotient dynamics, by the zero-entropy relation, of a *shoe* diffeomorphism, which is a subclass of Smale diffeomorphisms on surfaces, is a generalized pseudo-Anosov homeomorphism over a closed surface possibly having identified points.

Keywords: smale diffeomorphisms. zero-entropy equivalence. generalized pseudo-anosov homeomorphisms.

Contents

Introduction

In this work, the aim is to explore the connection between two fields in the study of dynamical systems, namely, surface dynamics and hyperbolic dynamics. While attempts to bridge these two areas have been made before, I drew inspiration from several works that helped guide my research. These include the works of A. Yu. Zhirov, specifically [ZP95], G. Ruas' thesis [RdM82], and A. S. de Carvalho's work [dC05]. In particular, two works that were instrumental in my research were the book by C. Bonatti and R. Langevin [BL98], with a focus on the final chapter by C. Bonatti and E. Jeandenans, and the works by A. S. de Carvalho and M. Paternain [dCP03].

We obtained in this thesis a semi-conjugacy, that preserve the topological entropy, between two dynamics, one defined in a surface and the other defined in a finite cactoid. This statement was inspired by the Theorem 8.3.1 of [BL98], due to Bonatti and Jeandenans. We will state exactly the statement of this theorem after we present the principal theorem of this work.

This work is divided into three chapters. The first chapter is devoted to give some definitions and state some theorems about topological entropy, metric entropy, symbolic dynamic and hyperbolic dynamic. The last two subsections are less common subjects and we encourage the reader to read the material. In the last section we define what is a generalized pseudo-Anosov homeomorphism on a finite cactoid. This is one of the dynamics involved in the main theorem.

The second chapter we discuss hyperbolic dynamics on surface and we define the Shoe diffeomorphisms. This is the second dynamic that we will use in the final result.

The third chapter, we discuss the way to obtain the semi-conjungacy. We will study the zero-entropy equivalence relation and we state and prove the final result: a shoe diffeomorphism on a closed surface is semi-conjugate to a generalized pseudo-Anosov on a finite cactoid. Moreover, the semi-conjugacy preserve the topological entropy.

Chapter 1

General Notions

1.1 Topology and Measure

In this chapter we present some well established notions and results in topology, measure theory, entropy, symbolic and hyperbolic dynamic theory. We do this for two reasons. Firstly, to fix notation. Secondly, to eventually remind the reader of some notion or result used in this text that may escape the reader's memory and that will be important to our argument.

A *probability compact metric space* is a triple (X, d, μ), where (X, d) is a compact metric space, and μ is a probability measure over the Borel sets. A *probability compact dynamic* is a quadruple (X, d, μ, f), where (X, d, μ) is a probability compact metric space, $f : (X, d) \to (X, d)$ is a homeomorphism and μ is f-invariant.

From now on, consider (X, d, μ, f) a probability compact dynamic. Let begin recalling the notions of topological entropy and metric entropy.

For every $n \in \mathbb{N}$, define new compact distances on X,

$$d_n(x, y) := \max\{d(f^i(x), f^i(y)) : \ 1 \le i < n\}, \text{ for } x, y \in X.$$

A set E is (n, ε)-*separated* if for any two distinct points $x, y \in E$, $d_n(x, y) \ge \varepsilon$. A set K, possibly non-invariant, is (n, ε)-*spanned* by a set F if for every $x \in K$ there exists $y \in F$ such that $d_n(x, y) \le \varepsilon$. Let $K \subseteq X$ be a compact subset and define the following quantities: $s(n, \varepsilon, K)$ is the maximal cardinality of an (n, ε)-separated subset of K; $r(n, \varepsilon, K)$ is the minimal cardinality of a set which (n, ε)-spans K; and $D(n, \varepsilon, K)$ is the minimum number of sets whose d_n-diameter is smaller than ε and whose union covers K. With these definitions, all the limits

$$\begin{aligned} h(f, K) &:= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln s(n, \varepsilon, K) \\ &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln r(n, \varepsilon, K) \\ &= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln D(n, \varepsilon, K), \end{aligned}$$

exist and are all equal, see [HK95].

The quantity $h(f, K)$ is called the *forward topological entropy carried by K under f*. The *backward topological entropy carried by K under f* is the quantity $h(f^{-1}, K)$. Finally, the *topological entropy carried by K under f*, and denoted by $h^f(K)$, is the quantity $h^f(K) := \max\{h(f, K), h(f^{-1}, K)\}$. If

K is f-invariant, then $h^f(K) = h(f, K) = h(f^{-1}, K)$. The *topological entropy of* f is defined as $h(f) := h^f(X) = \sup\{h^f(K) : K \subset X \text{ is compact}\}$.

Two points $x, y \in X$ are said to be *zero-entropy* related if there exist a continuum (compact and connected) C contained in X such that $x, y \in C$ and $h^f(C) = 0$. The zero-entropy relation is an equivalence relation. The reflexive and the symmetric properties are straightfoward. The transitivity follows from the equality $h(f, K \cup K') = \max\{h(f, K), h(f, K')\}$, where K and K' are compacts, and the fact that the union of connected sets with one point in common is also connected. If the compact K carry zero entropy, then $f(K)$ also carry zero entropy.

We denote by $\widetilde{X}$ the set of all zero-entropy equivalence classes, by $\widetilde{\pi} : X \to \widetilde{X}$ its canonical projection, and by $\widetilde{f} := \widetilde{\pi} \circ f \circ \widetilde{\pi}^{-1}$ the induced map $\widetilde{f} : \widetilde{X} \to \widetilde{X}$, called the *tight quotient* of f. It was proved in [dCP03] that if the map $f : S \to S$ is a $C^{1+\epsilon}$-diffeomorphism on a closed surface S, then the zero-entropy equivalence relation induces a monotone upper-semicontinuous decomposition of S. In particular, each zero-entropy equivalence class is a continuum. The notion of zero-entropy equivalence is one of the main ingredient of this work.

The set of all compact subsets of X is denoted by $\mathcal{K}(X)$. For all $C, C' \in \mathcal{K}(X)$, we define the *Hausdorff distance* between these sets as

$$d_H(C, C') := \max\left\{\max_{a \in C} \inf_{b \in C'} d(a, b), \max_{b \in C'} \inf_{a \in C} d(a, b)\right\}.$$

The map d_H is a compact distance on $\mathcal{K}(X)$, see [Sta67]. The compact metric space $(\mathcal{K}(M), d_H)$ is called the *hyperspace associated to* X. If we define $\hat{f}(K) := f(K)$, where $K \in \mathcal{K}(X)$, the map $\hat{f}$ is a homeomorphism and $(\mathcal{K}(X), \hat{f})$ is also a compact reversible dynamic system.

For the measure μ of X, a *measurable pre-partition* of X is a finite family $\mathcal{P} = \{P_1, \ldots, P_n\}$ of measurable sets such that $\mu(\cup_{i=1}^n P_i) = 1$ and $\mu(P_i \cap P_j) = 0$ if $i \neq j$. Two measurable pre-partitions $\mathcal{P}$ and $\mathcal{P}'$ are *equivalents* if for every $P \in \mathcal{P}$, there exists $P' \in \mathcal{P}'$ such that $\mu(P \setminus P') = 0$, and for every $P' \in \mathcal{P}'$, there exist $P \in \mathcal{P}$ such that $\mu(P' \setminus P) = 0$. The previous relation is an equivalence relation, and an equivalence classe of this equivalence relation is called a *measurable partition* of X. From a measurable partition $\mathcal{P}$ and $n \in \mathbb{N}$ we can construct new partitions,

$$\mathcal{P}_n := \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{P}),$$

where $\mathcal{P} \vee \mathcal{P}' := \{P \cap P' : P \in \mathcal{P}, P' \in \mathcal{P}'\}$.

The *entropy of a measurable partition* $\mathcal{P}$ *of* X *which respect to a measure* μ is given by

$$H_\mu(\mathcal{P}) := -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P).$$

The *metric entropy of* f *with respect to a measurable partition* $\mathcal{P}$ *of* X *and a measure* μ is

$$h_\mu(f, \mathcal{P}) := \lim_{n \to \infty} \frac{1}{n} H_\mu(\mathcal{P}_n).$$

We also define the *metric entropy of* f *with respect to a measure* μ by

$$h_\mu(f) := \sup\{h_\mu(f, \mathcal{P}) : \mathcal{P} \text{ is a measurable partition of } X\}.$$

The *Variational Principle*, see [HK95], establish the following relation between topological entropy and metric entropy:

$$h(f) = \sup\{h_\mu(f) \, : \; \mu \text{ is an } f\text{-invariant measure in } X\}.$$

If exist a measure μ such that $h(f) = h_\mu(f)$, then we say that μ is a *measure of maximal entropy*.

A *Omega-recurrent* point of (f, X) is an element of X such that belongs to its own *Omega*-limit set, i.e. there exist a strictly increasing sequence of natural number $(n_k)_{k\in\mathbb{N}}$ such that $f^{n_k}(x) \to x$ when $k \to \infty$. There is a similar notion, called *Alpha-recurrent* points, of elements in X such that belongs to its own *Alpha*-limit set. If a point is Omega-recurrent and Alpha-recurrent, then the point is called a *recurrent* point. The set of all recurrent points is a Borel f-invariant set and it is denoted by $\mathrm{Rec}(f)$.

1.1.1 Poincaré Recurrence theorem - probability compact dynamic version. *If (X, d, μ, f) is a probability compact dynamic, then $\mu(Rec(f)) = 1$.*

1.2 Symbolic dynamics

For each $k \in \mathbb{N}$, denote by $\Sigma_k = \{0, \ldots, k-1\}^{\mathbb{Z}}$ the set of all bi-infinite sequences $\phi = (\phi_i)_{i\in\mathbb{Z}} = \phi \, : \, \mathbb{Z} \to \{0, \ldots, k-1\}$. Define for the elements of Σ_k the distance

$$d_k(\phi, \phi') \, := \sum_{i=-\infty}^{\infty} \frac{\delta_i(\phi, \phi')}{k^{|i|}},$$

where $\delta_i(\phi, \phi') \, := 0$ if $\phi_i = \phi'_i$, and $\delta_i(\phi, \phi') \, := 1$ otherwise. It turns the metric space (Σ_k, d_k) into a Cantor space, i.e. a zero-dimensional, perfect and compact metric space. The topology generated by d_k is the product topology over Σ_k. The most natural dynamic on Σ_k is the *shift* map $\sigma \, : \, \Sigma_k \to \Sigma_k$, where $\sigma.(\phi_i)_{i\in\mathbb{Z}} = (\phi_{i+1})_{i\in\mathbb{Z}}$. The map σ is a homeomorphism.

Given $m, n \in \mathbb{Z}$, where $m \leq n$ and $i_m, i_{m+1}, \ldots, i_{n-1}, i_n \in \{1, \ldots, k\}$, we define the *cylinder* set

$$[i_m, \ldots, i_n] \, := \{(j_i)_{i\in\mathbb{Z}} \in \Sigma_k \, : \; j_l = i_l \text{ where } m \leq l \leq n\},$$

and we consider the *Sigma*-algebra in Σ_k generate by all cylinders. The topology generate by the set of all cylinder coincides with the product topology.

Consider a $k \times k$, $\{0, 1\}$-matrix A. Define the *subshift space of finite type* as the set

$$\Sigma_A \, := \{\phi \in \Sigma_k \, : \; A_{\phi_i, \phi_{i+1}} = 1\},$$

endowed with the metric d_k. The metric space (Σ_A, d_k) is also a Cantor set and it is σ-invariant.

A matrix A is said to be *reducible* if, by a permutation of the index set, it is possible to put it in triangular block form:

$$\begin{bmatrix} B & 0 \\ C & D \end{bmatrix}.$$

Otherwise, A is said to be *irreducible*. The matrix A is said to be *irreducible* and *aperiodic* if there exists a positive integer k such that A^k is positive, that is, all its entries are positive.

For aperiodic matrices, Perron [Per07] proved that the spectral radius λ_A is simple and greater than 1. Moreover, for this eigenvalue, there correspond positive left (row), and right (column) eigenvectors of A. The matrix A is aperiodic if and only if the dynamic (Σ_A, σ) is topologically mixing, see [DGS76]. From now on, let assume that the matrix A is aperiodic.

Consider $u = (u_1, \dots, u_k)$ and $v^T = (v_1, \dots, v_k)^T$ the left and right positive eigenvectors of A associated to λ_A, where $u.v = 1$. On the cylinders we define

$$\varrho([i_m, \dots, i_n]) := \lambda_A^{m-n}.\, u_{i_m}.\, A_{i_m i_{m+1}}. \dots .A_{i_{n-1} i_n}.\, v_{i_n},$$

that can be extended to a Borel measure over the *Sigma*-algebra generated by the cylinder sets. The measure ϱ is called the *Parry measure* of Σ_A. The Parry measure have some interesting properties: it is a Bernoulli measure, with $u.v$ as the probability vector; it is ergodic, it is nonatomic, it is the unique maximal entropy measure of σ, therefore, $h(\sigma) = h_\varrho(f) = \log \lambda_A$, and ϱ is positive in open sets, see [DGS76].

1.3 Hyperbolic Dynamic

We will present definitions and theorems in general uniformly hyperbolic dynamic theory. Since in the very early in this text we will restrict the discussion for dynamic on a closed surface, worth to say that the definitions and results presented here are not at all subject to this restriction. However, the two dimension dynamics have some unique properties that will be highlighted during the text.

Suppose S a closed (compact without boundary), connected, orientable, C^1-surface and $f : S \to S$ a C^1-diffeomorphism that preserves the orientation of S. Let take a Riemann metric $\langle ., . \rangle$ defined over the tangent space TS, and denote by $\|.\|$ and d the canonical norm and distance constructed from it.

An f-invariant, compact subset $\Lambda \subseteq S$ is called a *hyperbolic* set for f if there are constants $\lambda > 1$ and $c > 0$, and a decomposition of the tangent bundle of S restricted to Λ into a direct sum of two subbundles $T_\Lambda S = E^s_\Lambda \oplus E^u_\Lambda$, where, for all $x \in \Lambda$, $Df_x.E^s_x = E^s_{f(x)}$ and $Df_x.E^u_x = E^u_{f(x)}$, such that, for all $n \in \mathbb{N}$,

$$\|Df^n.v\| \le c.\lambda^{-n}.\|v\|, \text{ for all } v \in E^s_\Lambda,$$
$$\|Df^{-n}.v\| \le c.\lambda^{-n}.\|v\|, \text{ for all } v \in E^u_\Lambda.$$

The constant c reflect the particular choice of the Riemann metrics and it does not affect the hyperbolicity of Λ. In fact, by Mather's argument, see [Mat68], we can find a Riemann metric where $c = 1$. This is called an *adapted* Riemann metric of S. One interesting property of the adapted Riemann metric is that, for all $x \in \Lambda$, the angle between the tangent spaces E^s_x and E^u_x is equal to $\pi/2$, see [Bar12].

We say that a diffeomorphism f satisfies the *Axiom A* if its nonwandering set $\Omega(f)$ is hyperbolic, and the set of all periodic points of f is dense on $\Omega(f)$.

In dimension two (and one) it is only necessary suppose the hyperbolicity of $\Omega(f)$ since, by Newhouse and Palis [NP73], it implies in the density of the periodic points on $\Omega(f)$. In dimension greater than 2, we can find examples of dynamics where $\Omega(f)$ is hyperbolic but the set of all periodic points of f is not dense in $\Omega(f)$, see [Dan78] and [Kur79].

From now on, suppose that f satisfies the Axiom A and that $\langle . , . \rangle$ is adapted to $\Omega(f)$.

A hyperbolic set of f is *trivial* if it is finite set number of hyperbolic points. In the context we are working, every non-trivial hyperbolic set of f have an infinite number of orbits of periodic points contained on it.

Let Λ be a hyperbolic set for f. For all $\varepsilon > 0$ and $x \in \Lambda$, the *local stable* and *unstable manifold* are respectively defined as

$$\begin{aligned} W^s_\varepsilon(x) &:= \{y \in S \,:\, d(f^n(x), f^n(y)) < \varepsilon, \text{ for all } n \geq 0\}, \\ W^u_\varepsilon(x) &:= \{y \in S \,:\, d(f^{-n}(x), f^{-n}(y)) < \varepsilon, \text{ for all } n \geq 0\}. \end{aligned}$$

The *global stable* and *unstable manifolds* are respectively defined as

$$\begin{aligned} W^s(x) &:= \bigcup_{n\geq 0} f^{-n}\left(W^s_\varepsilon(f^n(x))\right), \\ W^u(x) &:= \bigcup_{n\geq 0} f^{n}\left(W^u_\varepsilon(f^{-n}(x))\right). \end{aligned}$$

Technically, we cannot called these sets manifolds yet. It is the next theorem that allow us to give such name.

1.3.1 Stable and Unstable Manifold Theorem [Shu86]. *Let Λ be a hyperbolic set for f. For all $\varepsilon > 0$ and $x \in \Lambda$, the local stable and unstable manifolds are C^1-embedded disks tangent at x to E^s_x and E^u_x respectively. Furthermore, the global stable/unstable manifolds are C^1-immersed submanifolds of S satisfying the following properties:*

- $f(W^{s/u}(x)) = W^{s/u}(f(x))$,
- $W^s(x) = \{y \in S :\; d(f^n(x), f^n(y)) \to 0$ *as* $n \to \infty\}$,
- $W^u(x) = \{y \in S :\; d(f^{-n}(x), f^{-n}(y)) \to 0$ *as* $n \to \infty\}$,
- *For all* $y \in W^s_\epsilon(x)$ *and* $z \in W^u_\epsilon(x)$ *we have:*

$$\begin{aligned} d(f(x), f(y)) &\leq \lambda^{-1}.d(x, y), \\ d(f^{-1}(x), f^{-1}(z)) &\leq \lambda^{-1}.d(x, z). \end{aligned}$$

Since we are assuming the dimension of S is two, for every $x \in \Lambda$, the dimension of $W^s(x)$ is equal to zero, one, or two. When the dimension of $W^s(x)$ is zero, then the dimension of $W^u(x)$ is two. In this case x is a periodic point and it is called a *source*. When the dimension of $W^s(x)$ is two, then the dimension of $W^u(x)$ is zero. In this case x is a periodic point and it is called a *sink*. A point x is called a *saddle* point if the dimension of its stable and unstable manifold are equal to one. A hyperbolic set is called a *saddle* hyperbolic set if it is formed exclusively by saddle points.

We say that f satisfies the *Strong Transversality Condition* if, for every $x \in \Lambda$, the sets $W^s(x)$ and $W^u(x)$ are transverse at x. A diffeomorphism is *Smale* if satisfies both, the Axiom A, and the Strong Transversality Condition. From now on, suppose that f is a Smale diffeomorphism.

A hyperbolic set Λ of f is *isolated*[1] if there is a neighborhood $\mathcal{U}$ of Λ such that $\Lambda = \cap_{n\in\mathbb{Z}} f^n(\mathcal{U})$. Every isolated hyperbolic set of f have a *local product structure*, i.e. there exist $\delta > 0$ and $\varepsilon > 0$ such that, for any two points $x, y \in \Lambda$, where $d(x, y) < \delta$, the intersection $W^s(x) \cap W^u(y)$ consists of one point. We denote by $\rho^{s/u}_x : B(x; \delta) \to W^{s/u}_\varepsilon(x)$ the canonical local projections induced by the local product structure. It is also possible to define a continuous map $[.,.] : \{(x, y) \in \Lambda \times \Lambda : d(x, y) \leq \delta\} \to \Lambda$, where $\{[x, y]\} = W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$. A *basic piece* Λ of f is a hyperbolic, isolated, and transitive subset of $\Omega(f)$.

[1] or *locally maximal*

1.3.2 Smale Spectral Decomposition Theorem [Sma67], [Bow04]. *If f is a Smale diffeomorphism, then there exist $n \in \mathbb{N}$ and a disjoint family $\{\Lambda_i : 1 \leq i \leq n\}$ of basic pieces of f such that $\Omega(f) = \cup_{i=1}^{n} \Lambda_i$. Later, Bowen proved that for each basic piece Λ_i exist $n_i \in \mathbb{N}$ and a finite family of hyperbolic sets $\{\Lambda_i^j : j \in \mathbb{Z}_{n_i}\}$ such that $\Lambda_i = \cup_{j=1}^{n_i} \Lambda_i^j$, $f^{n_i}(\Lambda_i^j) = \Lambda_i^{j+1}$, and $f^{n_i}|_{\Lambda_i^j}$ is topologically mixing.*

If Λ is a basic piece of f, we define $W^s(\Lambda)$ as the set of points on S that have its *Omega*-limit set contained on Λ. The set $W^u(\Lambda)$ is defined as the set of points on S that have its *Alpha*-limit set contained on Λ. We can use the *Shadowing Lemma* to prove that $W^s(\Lambda) = \cup_{x \in \Lambda} W^s(x)$ and $W^u(\Lambda) = \cup_{x \in \Lambda} W^u(x)$. A consequence of the previous theorem is $S = \cup_{i=1}^{n} W^s(\Lambda_i) = \cup_{i=1}^{n} W^u(\Lambda_i)$.

Two basic pieces Λ and Λ' of f are *heteroclinic related*, and denoted by $\Lambda \preccurlyeq \Lambda'$, if $W^s(\Lambda) \cap W^u(\Lambda') \neq \emptyset$, and it turns the set of all basic pieces of f into a finite partially ordered set, or just a *poset*. The Inclination Lemma, see [Pal69], allow us to prove that $\overline{W^u(\Lambda)} = \cup_{\Lambda' \preccurlyeq \Lambda} W^u(\Lambda')$ and $\overline{W^s(\Lambda)} = \cup_{\Lambda \preccurlyeq \Lambda'} W^s(\Lambda')$.

It is always possible to associate an oriented graph, called *Hasse* graph, to a poset $(Y, \leq)$. Given $a, b \in Y$, we say that *a covers b* if $b < a$ and there is no $c \in Y$ such that $b < c < a$. The Hasse graph is constructed by taking the elements of Y as the vertices and the edges are the ordered pair (a, b) where a covers b.

When the poset is the finite poset of all basic pieces of a Smale diffeomorphism f, the *Smale order* between two basic pieces is the cover relation of the heteroclinic relation of basic pieces. When two basic pieces are comparable with respect to the Smale order we say they are *Smale related.* The *Smale graph* is the Hasse graph associated to the finite poset of all basic pieces of f.

One important property of Smale diffeomorphism is concerned with its stability in the space of all dynamics. Let be more specific. Consider $\mathrm{Diff}^1(S)$ the space of all C^1-diffeomorphism of S endowed with the C^1-topology. We say that f is *structurally stable* if it has a neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is conjugate to f. By Mañé [Mañ82], a diffeomorphism $f \in \mathrm{Diff}^1(S)$ is Smale if and only if f is structurally stable. It was also proved by Mañé [Mañ87] a similar result but for dynamics occuring in dimension greater than 2.

Two Smale diffeomorphisms f and g are said to be Ω-*related* if $f|_{\Omega(f)}$ is conjugate to $g|_{\Omega(g)}$. We can consider another notion of stability using the relation above: f is Ω-*stable* if it has a neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is Ω-related to f. We say f satisfies the *no-cycle* condition if there is not cycles in the Smale graph associate to f.

It was proved by Mañé [Mañ82] and Palis [Pal87] that every Ω-stable diffeomorphism satisfies the Axiom A and the no-cycle condition. The converse was proved by Smale [Sma]. Every Smale diffeomorphism is Ω-stable and, thus, satisfies the no-cycle condition. Hence, for each basic piece Λ of f the set $W^s(\Lambda)$ is a lamination[2], and it will be called the *stable lamination* of Λ. The set $W^u(\Lambda)$ is also a lamination and it will be called the *unstable lamination.* For the hypothesis we are working, each leaf on either stable or unstable lamination is a C^1-embedding of the real line $\mathbb{R}$. Furthermore, $\Lambda = W^s(\Lambda) \cap W^u(\Lambda)$, and the stable and unstable laminations are transverse laminations.

Finally, the topological dimension of a basic piece Λ can be equal to zero, one, or two. In fact, it is not standard consider the dimension of a basic piece greater than zero. However, for our purpose, it will make sense to consider basic pieces with non-zero topological dimensions. If $\dim(\Lambda) = 2$, then $\Lambda = S = \mathbb{T}^2$, and f is an Anosov diffeomorphism, see [GMP16]. If $\dim(\Lambda) = 1$, then Λ is

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[2] A *lamination* is a topological space partioned into subsets (called "sheets" or "leaves") which look parallel in local charts.

equal either to $W^u(\Lambda)$ or $W^s(\Lambda)$. If $\Lambda = W^u(\Lambda)$, then Λ is called an *attractor*. If $\Lambda = W^s(\Lambda)$, then Λ is called a *repeller*. Both cases will be discussed further but, for the moment, consider that the dimension of all basic pieces of f is equal to zero.

By the properties of topological entropy, we can say that

$$h(f) = h(f|_{\Omega(f)}) = h^f(\Omega(f)) = \max_{i,j} h^f(\Lambda_i^j).$$

Furthermore, Bowen [Bow74] proved that the Borel measure define below is a measure of maximal entropy of f:

$$\beta = \lim_{n\to\infty} \sum_{x\in \mathrm{Fix}(f^n)} \frac{\delta_x}{\#\mathrm{Fix}(f^n)},$$

where δ_x is the probability measure supported by $\{x\}$, and $\mathrm{Fix}(f^n)$ is the set of all fixed points of f^n.

1.4 Roberts and Steenrod Theorems

The next theorems, due to Roberts and Steenrod [RS38], gives a characterization of the quotient space in the case of a upper semi-continuous equivalence relations. Before we discuss the theorems, let's do some definitions and remind the reader about some important facts.

A set of $n+1$ points $\{v_0, \dots, v_n\} \subset \mathbb{R}^{n+1}$ are said to be in *general position* if they are not contained in an $(n-1)$–dimensional hyperplane. An *n-simplex* is the smallest convex hull for the set $\{v_0, \dots, v_n\}$. The n-simplex is denoted by $[v_0, \dots, v_n]$ and each point v_i is called the *vertice* of the n-simplex. From this definition, we can see that each simplex is uniquely defined by its vertices. By choosing an enumeration of the vertices, we give the simplex an *orientation*. A *face* of an n-simplex ς[3] is an $(n-1)$–simplex contained in ς.

A *simplicial complex* $K = (V, \Sigma)$ is a collection of vertices V and simplices Σ contained in $\mathbb{R}^{n+1}$, for some $n \in \mathbb{N}$, satisfying the following properties:

- Every vertex $v \in V$ is the vertex of at least one and at most finitely many simplices in Σ.
- Every face of a simplex in Σ is itself an element of Σ.
- The intersection of two simplices is a common face of each, i.e. is itself a simplex in Σ.

A *finite complex* is a simplicial complex where Σ have a finite number of elements. The *geometric realization* of a simplicial complex $K = (V, \Sigma)$ is a subset $\mathcal{K} \subset \mathbb{R}^{n+1}$ obtained from the embedding of K in $\mathbb{R}^{n+1}$, together with the subspace topology. $\mathcal{K}$ is therefore a topological space.

A topological space M is *triangulable* if it is homeomorphic to the geometric realization of a simplicial complex K. We say that $\mathcal{K}$ is a *triangulation* of M. Every closed surface is triangulable, see [DM68].

Let M be a topological space with a triangulation $\mathcal{K}$. For each $0 \le i \le n$, let $C_i(M, \mathbb{Z}_2)$ be the free abelian group generated by the i-simplices s_α^i contained in K. Elements of $C_i(M, \mathbb{Z}_2)$, called *i-chains*, can be written as $\sum_\alpha m_\alpha s_\alpha^i$ with coefficients $m_\alpha \in \mathbb{Z}_2$.

We define the *boundary function* $\partial_i : C_i(M, \mathbb{Z}_2) \to C_{i-1}(M, \mathbb{Z}_2)$ that takes each oriented i-simplex ς with vertices $v_1, \dots, v_i$ to its boundary. More precisely, if the notation $\hat{v}_j$ denotes the

[3] This symbol is the lowercase sigma in word-final position.

elimination of that v_j from $[v_0, \dots, v_n]$, then we define

$$\partial_i(\varsigma) := \sum_j (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_i].$$

The boundary of an i-simplex is a $(i-1)$–chain, a sum of oriented $(i-1)$–dimensional simplices. Consequently, the boundary functions are homomorphisms from each free abelian group $C_i(M, \mathbb{Z}_2)$ to the following $C_{i-1}(M, \mathbb{Z}_2)$, and so we can create the chain complex below.

$$0 \to C_n(M, \mathbb{Z}_2) \xrightarrow{\partial_n} C_{n-1}(M, \mathbb{Z}_2) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(M, \mathbb{Z}_2) \xrightarrow{\partial_0} 0$$

It is possible to prove that the composition of two boundary maps is the zero homomorphism, $\partial_i \circ \partial_{i+1}(\xi) = 0$, for all $(i+1)$–chain $\xi \in C_{i+1}(M, \mathbb{Z}_2)$. This implies that the image of ∂_{i+1} is in the kernel of ∂_i. We will denote the image of $C_{i+1}(M, \mathbb{Z}_2)$ by ∂_{i+1} as $Z_i(M, \mathbb{Z}_2)$ and the kernel of ∂_i as $B_i(M, \mathbb{Z}_2)$.

The i^{th} *simplicial homology group* $H_i(M, \mathbb{Z}_2)$ of a triangulable manifold M is the abelian group obtained by the quotient group of the kernel of ∂_i by the image of ∂_{i+1},

$$H_i(M, \mathbb{Z}_2) = Z_i(M, \mathbb{Z}_2)/B_i(M, \mathbb{Z}_2).$$

Let K be a finite complex. The *mod 2 i–Betti number* $\mathcal{B}_i(K)$ is defined to be the rank[4] of the i^{th} simplicial homology group $H_i(M, \mathbb{Z}_2)$.

Let X be a compact metric space and $\mathcal{Q}$ a partition of X. The partition $\mathcal{Q}$ is called *monotone* if it is a partition into connected sets. It is called an *upper semi-continuous* if $x_n, y_n \in Q \in \mathcal{Q}$, and $x_n \to x$ and $y_n \to y$, when $n \to \infty$, imply that $x, y \in Q' \in \mathcal{Q}$. In this case, each element of $\mathcal{Q}$ is a continuum.

Consider S a closed surface. By Lemma 1 of [RS38], if $Q \subset S$ is a continuum and D is a connected component of $S \setminus Q$, then D contains a closed connected finite complex K where $\partial_2(K)$ is the formal sum of a finite number $\Gamma_1, \dots, \Gamma_s$ of pairwise disjoint simple closed curves (1-complex), and $D \setminus K = E_1 \cup \cdots \cup E_s$, where the E_i are mutually disjoint open cylinders and the boundary of E_i is the union of Γ_i and some subset of Q. We will say that E_i is a cylinders of $D \setminus K$ that *approach* D.

Let X a connected topological space. A point $p \in X$ for which $X \setminus \{p\}$ is not connected, is a *cut point* of X. An *endpoint* of X is a point which has arbitrarily small neighborhoods whose boundary is a single point. A cut point q *separate* two points $p, p' \in X$ if it is possible to write $X \setminus \{q\} = A \cup B$ where $p \in A$, $p' \in B'$ and $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. If $p \in X$ is neither a cut point nor an endpoint of X, the set of all points which cannot be separated from p by any other point is called a *(simple) link* of X.

Let X be a locally connected continuum. If each simple link of X is homeomorphic to $\mathbb{S}^2$ then X is called a *cactoid*. Let Y be a space that each link is homeomorphic to a surface and all but finite many links are homeomorphic to $\mathbb{S}^2$. In this case, the space Y is called a *generalized cactoid*. Suppose the space Y is a generalized cactoid and Z is obtained by identifying finitely many pairs of points of Y. Such Z is called a *finite generalized cactoid*. If Y is a closed surface, then we call Z a *finite cactoid*.

Consider S a closed surface and $\mathcal{Q}_S$ an upper semi-continous collection of continua filling S.

[4] The *rank* of a group is the smallest cardinality of a generating set of the group.

For a simplicial complex K, consider the quantity $\mathcal{B}_1(K)$ the mod 2 1–Betti number of the set K. Note that not only the compact surfaces are triangulable, the cactoid are also triangulable.

1.4.1 Theorem. *If, for each $Q \in \mathcal{Q}_S$, the set $S \setminus Q$ is connected and has just one cylinder approaching to Q, then $\mathcal{Q}_S$ is a closed surface, $\mathcal{B}(\mathcal{Q}_S) \leq \mathcal{B}(S)$, and if S is orientable so is $\mathcal{Q}_S$.*

1.4.2 Theorem. *With no restriction on $\mathcal{Q}_S$, there exists a finite number of spaces $C_0, C_1, \dots C_k$ where C_0 is a generalized cactoid and C_i is obtained by identifying just two points of C_{i-1} (i=1, …, k) such that $\mathcal{Q}_S$ is homeomorphic to C_k and $\mathcal{B}_1(\mathcal{Q}_S) \leq \mathcal{B}_1(S) - k$. Converselly , given $C_0, C_1, \dots C_k$ as above, there exits an S and a $\mathcal{Q}_S$ such that $\mathcal{Q}_S$ is homeomorphic to C_k and $\mathcal{B}_1(\mathcal{Q}_S) = \mathcal{B}_1(S) - k$.*

Note that in the previous theorem the space C_k is a finite generalized cactoid. Furthermore, if C_0 is a closed surface, then C_k is a finite cactoid.

1.5 Generalized Pseudo-Anosov Homeomorphisms

A *singular foliation* $\mathcal{F}$ on a closed surface S is a decomposition of S into a disjoint union of subsets of S, called the *leaves* of $\mathcal{F}$, and a countable set of points of S, called *singular points* of $\mathcal{F}$, such that the following conditions hold:

- For each non-singular point $p \in S$, there is a smooth chart from a neighborhood of p to $\mathbb{R}^2$ that takes leaves to horizontal (or vertical) line segments.
- Singular points can be divided into two groups: the *isolated* singular points, and the *accumulated* singular points. The first group of singular points is possibly countably infinite, and can be modeled on k-pronged singularities, with $k = 1$ or $k \geq 3$, as show in the figure below. The second group of singular points is finite and is accumulated by isolated singular points.

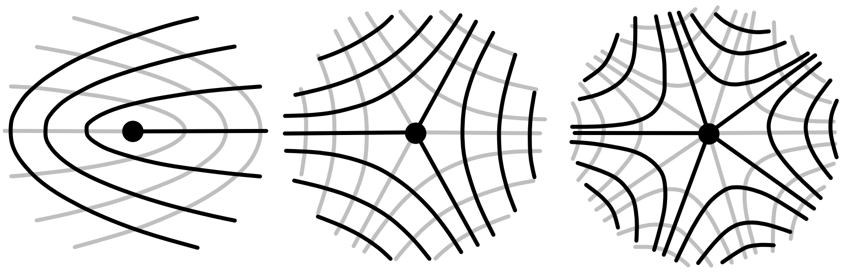

Figure 1.1: *A* 1*-pronged, a* 3*-pronged, and a* 5*-pronged singularities respectvelty*

Let $\mathcal{F}$ be a singular foliation on a surface S. A smooth arc α in S is *transverse* to $\mathcal{F}$ if it is transverse to each leaf of $\mathcal{F}$ at each point in its interior, and misses all the singular points of $\mathcal{F}$. Let $\alpha_1, \alpha_2 : [0,1] \to S$ be smooth arcs transverse to $\mathcal{F}$. A *leaf-preserving isotopy* from two arcs $\alpha_1 : [0,1] \to S$ to $\alpha_2 : [0,1] \to S$ is a map $H : [0,1] \times [0,1] \to S$ such that

- $H([0,1] \times \{0\}) = \alpha_1$ and $H([0,1] \times \{1\}) = \alpha_2$.
- $H([0,1] \times \{t\})$ is transverse to $\mathcal{F}$ for each $t \in [0,1]$.
- $H(\{0\} \times [0,1])$ and $H(\{1\} \times [0,1])$ are each contained in a single leaf of $\mathcal{F}$.

A *transverse measure* μ on $\mathcal{F}$ is a function that assigns a positive real number to each smooth arc transverse to $\mathcal{F}$, so that μ is invariant under leaf-preserving isotopy. A *measure singular foliation* on S is a pair $(\mathcal{F}, \mu)$ where $\mathcal{F}$ is a singular foliation of S equipped with a transvere measure μ.

Apropos from the above discussion, if $\mathcal{L}$ and $\mathcal{L}'$ are two transverse laminations on S, a *transverse measure* ν *on* $\mathcal{L}$ *adapted* to $\mathcal{L}'$ is a function that assigns a positive real number to each arc contained on $\mathcal{L}'$, so that ν is invariant under leaf-preserving isotopy restricted to all arcs contained in $\mathcal{L}'$.

Two foliations $\mathcal{F}$ and $\mathcal{F}'$ are called *transverse* if they share all singular points and at non-singular points they cross transversely.

A homeomorphism $\Phi : S \to S$ is called a *generalized pseudo-Anosov* if there exist a pair of Φ-invariant transverse measure singular foliations $(\mathcal{F}^s, \mu^s)$ and $(\mathcal{F}^u, \mu^u)$, the accumulated singularities are fixed points, and a real number $\lambda > 1$ such that:

$$\Phi * \mu^s = \lambda^{-1}\ \mu^s,$$
$$\Phi * \mu^u = \lambda\ \mu^u,$$

where $*$ is the pushfoward of measure.

Consider $\Phi : S \to S$ is a generalized pseudo-Anosov and $\pi : S \to \Pi$ a monotone upper semi-continuous projection, where C_0 (in the Theorem 1.4.2) is a closed surface, and the points that are identified to obtain the sequence of spaces $C_1, \dots, C_k$ are accumulated singularities in C_0. The map $\pi \circ \Phi \circ \pi^{-1}$ is a homeomorphism and will also be called a *generalized pseudo-Anosov*, but now defined on a finite cactoid.

Chapter 2

Shoe Diffeomorphism on Surfaces

We will obtain some consequences of all hypothesis that we did so far. With some more restrictions to the dynamic (S, f) we obtain a special type of diffeomorphism that we called *shoe* diffeomorphism. Some propositions have already been proved by Bonatti, Langevin and Jeandenans in [BL98]. In these cases we will reference the pages where the proof can be found.

Just to reinforce, we are assuming that $f : S \to S$ is a C^1-Smale diffeomorphism on a closed orientable surface S, that preserve the orientation of S, and f have just zero-dimensional basic pieces.

2.1 Shoe diffeomorphisms

We say that f is a *shoe* diffeomorphism if:

- f have only one non-trivial basic piece Λ,
- the dynamic $(\Lambda, f|_\Lambda)$ is topologically mixing,
- all saddle basic pieces are Smale related only with sources and sinks.

From now on, consider f a shoe diffeomorphim, and Λ the only non-trivial basic piece of f.

By the assumptions we made so far, for all $x \in \Lambda$, the set $W^s(x)$ is a non-compact one-dimensional manifold and, thus, it is homeomorphic to $\mathbb{R}$. An *s-interval* is any connected subset of $W^s(x)$. We will use the same nomenclature and notation of intervals on $\mathbb{R}$ for s-intervals on $W^s(x)$. For instance, any compact s-interval of $W^s(x)$ have extreme points p and q, and it is denoted by $[p,q]^s_x$. An *isolated* s/u-interval I is an s/u-interval with extreme points in Λ and there exist an open s/u-interval J such that $I \subset J$ and $J \cap \Lambda \subset I$.

2.1.1 Lemma. *If f is a shoe diffeomorphism, then $W^s(\Lambda) \cup W^u(\Lambda)$ is path-connected.*

Proof. Consider $x, y \in \Lambda$ and neighborhoods $A, B \subset S$ of x and y, respectively, that admits a product structure. Consider $\varepsilon > 0$ small enough such that $W^{s/u}_\varepsilon(x) \subset A$ and $W^{s/u}_\varepsilon(y) \subset B$. Since $f|_\Lambda$ is topologically mixing, we can find an $N \in \mathbb{N}$ such that $f^n(A) \cap B \neq \emptyset$ and $f^n(A) \cap A \neq \emptyset$, for all $n \geq N$. Moreover, we can find $\overline{N} \geq N$ where the space $f^{\overline{N}}(W^u_\varepsilon(x))$ cross $W^s_\varepsilon(x)$ and $W^s_\varepsilon(y)$ at one point, let's say, at the points $\overline{x}$ and $\overline{y}$, respectively. The curve $[x, \overline{x}]^s_x \cup [\overline{x}, \overline{y}]^u_{f^{\overline{N}}(x)} \cup [y, \overline{y}]^s_y$ is continuous and connects the point x and y. Thus, $W^s(\Lambda) \cup W^u(\Lambda)$ is path-connected. □

In the context we are working, a region $R \subset S$ is a *rectangle* if there exist compact intervals $I, J \subset \mathbb{R}$ and a homeomorphism $\varphi : I \times J \to R$ such that:

- $\varphi(\partial I \times J) \subset W^u(\Lambda)$ and $\varphi(I \times \partial J) \subset W^s(\Lambda)$.
- For all $t \in J$, $\varphi(I \times \{t\})$ is either disjoint of $W^s(\Lambda)$ or $\varphi(I \times \{t\}) \subset W^s(\Lambda)$.
- For all $t \in \varphi(\{t\} \times J)$ is either disjoint of $W^u(\Lambda)$ or $\varphi(\{t\} \times J) \subset W^u(\Lambda)$.

For a rectangle R, we denote by $\partial^s R = \varphi(I \times \partial J)$ the stable edges of R, and $\partial^u R = \varphi(\partial I \times J)$ the unstable edges of R. A *sub-rectangle of* R is a rectangle contained on R with stable edges, or unstable edges, contained in $\partial^s R$, or $\partial^u R$. If $x \in R \cap \Lambda$, the R*-stable/unstable manifold* of x is the unique connected subset of $W^{s/u}(\Lambda) \cap R$ that contains x. Denote by $\rho^{s/u}_{x,R} : R \cap \Lambda \to W^{s/u}_R(x)$ the canonical projections of points of $R \cap \Lambda$ over $W^{s/u}_R(x)$. A *sub-rectangle of* R is a rectangle contained on R with stable edges, or unstable edges, contained in $\partial^s R$, or $\partial^u R$.

For any point $p \in W^s(\Lambda)$, there exist $x \in \Lambda$ such that $p \in W^s(x)$. We also denote by $W^s(p)$ the set $W^s(x)$. Every point $p \in W^s(\Lambda)$ divide its leaf into two connected components and each one is called a *separatrix* of p. When one of the separatrices does not intersect Λ, it is called a *free* separatrix of p. We can do analogous definitions to points on $W^u(\Lambda)$.

Since $W^s(\Lambda)$ is a lamination, for any $p \in W^s(\Lambda)$, it is possible to find a neighborhood $\mathcal{N}_p$, with a product structure, and a homeomorphism $\varphi_p : Q = (-1,1)^2 \to \mathcal{N}_p$, such that $\varphi_p(0,0) = p$ and $\varphi_p^{-1}(\mathcal{N}_p \cap W^s(\Lambda)) = (-1,1) \times F^u_p$, where F^u_p is a Cantor set. Analougous facts holds for points in the lamination $W^u(\Lambda)$. If C is a Cantor set that is a subspace of a topological space X, then the *endpoints* of C with respect to X, is all the points in the boundary of a connected component of the complementary set $X \backslash C$. It will be important later the fact that the set of all endpoints of a Cantor set is a dense subset of this Cantor set.

A point $p \in W^s(\Lambda)$ is an *s-boundary* point if 0 is an endpoint of F^u_p. We can similarly define the *u-boundary* points on $W^u(\Lambda)$. A *corner* is both an s-boundary and an u-boundary point.

The next theorem is due to [NP73] and the proof can be found in pag. 42–45 of [BL98].

2.1.2 Proposition. *Let f be a shoe diffeomorphism. Then,*

1. *If $x \in W^{s/u}(\Lambda)$ is an s/u-boundary point, then all iterates of x by f is also an s/u-boundary point. Moreover, all points belonging to $W^{s/u}(x)$ are s/u-boundary points.*
2. *A point $x \in \Lambda$ is an s/u-boundary periodic point if, and only if, one, and only one, of the separatrices of $W^{u/s}(x)$ is free.*
3. *A point $p \in W^{s/u}(\Lambda)$ is an s/u-boundary point if it is contained in one of the s/u-separatrices of an s/u-boundary periodic point.*
4. *There is a finite, and not null, s/u-boundary periodic points on Λ.*

An *s/u-boundary separatrices* is a non-free s/u-separatrix of an s/u-boundary periodic point. Let respectively denote by $\mathrm{Per}^s(\Lambda)$ and $\mathrm{Per}^u(\Lambda)$ the set of all s-boundary and u-boundary periodic points on Λ, and $\mathrm{BPer}(\Lambda) = \mathrm{Per}^s(\Lambda) \cup \mathrm{Per}^u(\Lambda)$ the set of all boundary periodic points on Λ. Note that the corner periodic points are the points on $\mathrm{Per}^s(\Lambda) \cap \mathrm{Per}^u(\Lambda)$. For each $p \in \mathrm{Per}^{s/u}(\Lambda)$, denote by $\widehat{W}^{u/s}_p$ the unique unstable/stable free separatrices of p. Each free separatrix is homeomorphic to an open interval, with one of its extreme point the boundary periodic point, and the other extremity a sink or source, depending on the nature of the free separatrix. Let's denote by $q^{s/u}_p$ the sink/source that is the other extreme point of $\widehat{W}^{u/s}_p$.

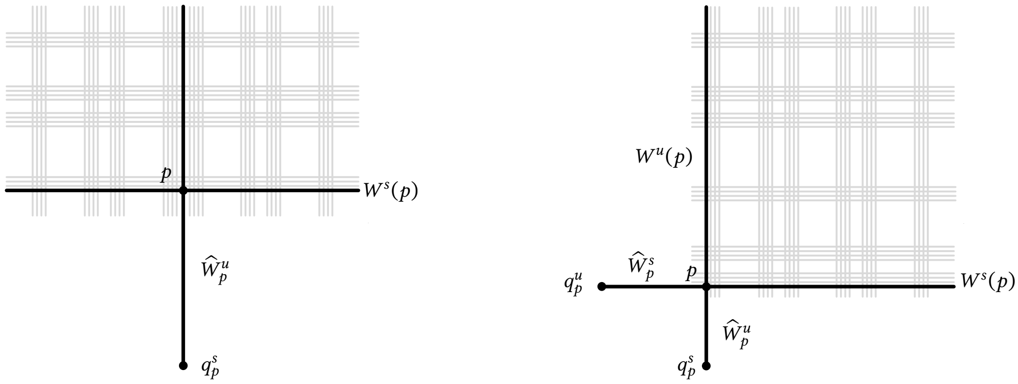

Figure 2.1: *A boundary and corner periodic points.*

We will make one more assumption about shoe diffeomorphism. We will assume that all points on $\mathrm{BPer}(\Lambda)$ and all trivial basic pieces of the shoe diffeomorphism f are fixed points. We do this just to simplify the writing and avoiding cumbersome notation. We can always get a new shoe diffeomorphism satisfying these hypothesis from a previous shoe f just considering a new map f^{lcm}, where lcm is the least common multiple of the periods of the points on $\mathrm{BPer}(\Lambda)$ and periods of the trivial basic pieces of f.

2.1.3 Corollary. *Any s/u-boundary periodic point but not corner have two non-free s/u-boundary separatrices. A corner periodic point also have two non-free boundary separatrices but they are of different nature. Moreover, any boundary periodic point does not have three free separatrices.*

Proof. The proof is a direct application of (2) of Proposition 2.1.2. □

An *u-arc*[1] γ of Λ is a compact u-interval $[x_1, x_2]^u$ where the interior is disjoint of Λ, i.e. $(x_1, x_2)^u \cap \Lambda = \emptyset$, and $x_1, x_2 \in \Lambda$. We define *s-arcs* of Λ similarly as u-arcs. Note that if γ is an arc, then $f(\gamma)$ is also an arc, and its extreme points are over the same boundary separatrices. Two u-arcs γ and γ' are said to be *equivalent* if either $\gamma = \gamma'$, or there exist a rectangle R, where $\partial^u(R) = \{\gamma, \gamma'\}$. An analogous arc equivalence holds for s-arcs. The map f acts on the classes of equivalence of arcs. Thus, we are able to work with the orbits of each class of equivalence of arcs by f.

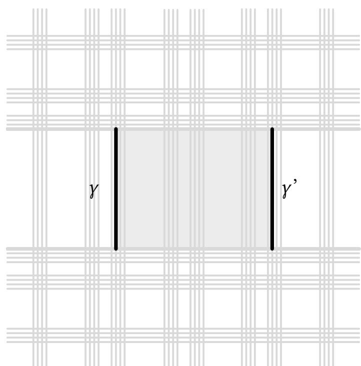

Figure 2.2: *Two equivalent arcs.*

2.1.4 Proposition. *The set of orbits (by f) of equivalence classes of arcs is finite.*

The proof of this proposition can be found in pag. 48–49 of [BL98].

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[1] In any case, when we use the word arc, it means a space homeomorphic to a closed interval.

A *domain* of an equivalence class of arcs is the union of all rectangles that turns the arcs equivalent. An arc is called an *extreme* arc of the domain if it is contained on the boundary of the domain. Note that not all equivalence class of arcs have an extreme arc. When an equivalence class of arcs have extreme arcs, then we say that this equivalence class is a *limited* equivalent class of arcs.

For any u-arc γ of Λ, with extremities x_1 and x_2 contained respectively in the s-boundary separatrix $W^s_{p_1}$ and $W^s_{p_2}$, we can define the closed curve σ_γ[2] in the following way: if $W^s_{p_1} \neq W^s_{p_2}$, then $\partial^u \sigma_\gamma = \{\gamma, f(\gamma)\}$ and $\partial^s \sigma_\gamma = \{[x_1, f(x_1)]^s_{p_1}, [x_2, f(x_2)]^s_{p_2}\}$; if $W^s_{p_1} = W^s_{p_2}$, then $\partial^u \sigma_\gamma = \{\gamma\}$ and $\partial^s \sigma_\gamma = \{[x_1, x_2]^s_{p_1}\}$. Similar definitions can be made for s-arcs.

2.1.5 Proposition. *For any arc γ, the closed simple curve σ_γ is the boundary of a topological disk $T(\sigma_\gamma)$ disjoint of Λ, and the family $\{f^n(T(\sigma_\gamma)) : n \in \mathbb{Z}\}$ form a disjoint family of topological disks.*

The prove for this proposition can be found in pag. 55–59 of [BL98].

2.2 Iterates of Arcs

Let $\gamma = [x_1, x_2]^u$ be an u-arc and $W^s_{p_1}$ and $W^s_{p_2}$ the two s-boundary separatrices containing, respectively, the extremities x_1 and x_2 of γ. Let also be $\widehat{W}^u_{p_1}$ and $\widehat{W}^u_{p_2}$ the free u-separatrices associated to p_1 and p_2, and $q^s_{p_1}$ and $q^s_{p_2}$ the sinks associated to each free separatrix.

2.2.1 Lemma. *The interior of γ, the set $\mathring{\gamma} := (x_1, x_2)^u$, is contained in an unique basin of attraction $q = q^s_{p_1} = q^s_{p_2}$ of a sink of f.*

Proof. Suppose the lemma is false, i.e. there exist two points of $\mathring{\gamma}$ such that each one is contained in two distinct basin of attraction. However, every basin of attraction is an open set and have stable manifolds of saddle periodic points as its boundary. Thus, if the hypothesis we made is true, should exist a point on $\mathring{\gamma}$ that is contained in a stable manifold of a saddle periodic point, what is a contradiction with the fact that γ is an arc. If γ is contained in the basin of atraction of a sink q, then $q = q^s_{p_1} = q^s_{p_2}$. □

Note that $W^s_{p_1}$ could be distinct of $W^s_{p_2}$, but there is the possibility that $W^s_{p_1} = W^s_{p_2}$. In this case γ bend over $W^s_{p_1}$. In both cases $\widehat{W}^u_{p_1} \cup \widehat{W}^u_{p_2} \cup \{p_1, p_2\} \cup \{q^s_{p_1}, q^s_{p_2}\}$ is a compact set of S, as well as γ.

2.2.2 Proposition. *The iterates of γ converge, in the Hausdorff metric, to $\widehat{W}^u_{p_1} \cup \widehat{W}^u_{p_2} \cup \{p_1, p_2\} \cup \{q^s_{p_1}, q^s_{p_2}\}$, i.e. $d_H(f^n(\gamma), \widehat{W}^u_{p_1} \cup \widehat{W}^u_{p_2} \cup \{p_1, p_2\} \cup \{q\}) \to 0$, when $n \to \infty$.*

Proof. Consider $\mathcal{N}$ an neighborhood of p_1 that admits a product structure. Since $f^n(x_1) \to p_1$, we can find an $N \in \mathbb{N}$, such that $f^n(x_1) \in \mathcal{N}$, for all $n > N$. The proof of this proposition follows from the following observations:

- Suppose δ_1 is a compact sub-arc of γ with x_1 as extreme point and $f^n(\delta_1) \subset \mathcal{N}$, where $n > N$, and δ_1' is a compact sub-arc of $\widehat{W}^u_{p_1} \cup \{p_1\}$ with p_1 as extreme point. Then, $d_H(f^n(\delta_1), f^n(\delta_1')) \to 0$, when $n \to \infty$.
- For a δ_1' satisfying the same properties, then $d_H(f^n(\delta_1'), \widehat{W}^u_{p_1} \cup \{p_1\} \cup \{q\}) \to 0$, when $n \to \infty$.

[2] Not be confused with the shift map σ.

From these two observations we can conclude that $d_H(f^n(\delta_1), \widehat{W}^u_{p_1} \cup \{p_1\} \cup \{q\}) \to 0$, when $n \to \infty$. An analogous result can be obtained for δ_2 a compact sub-arc of γ with x_2 as extreme point and $f^n(\delta_2) \subset \mathcal{N}$, where $n > N$.

Consider now $\gamma' = \gamma \backslash (\delta_1 \cup \delta_2)$. The open arc γ' is completely contained in the basin of atraction of q, so $d_H(f^n(\overline{\gamma'}), q) \to 0$, when $n \to \infty$. Thus, $d_H(f^n(\gamma), \widehat{W}^u_{p_1} \cup \widehat{W}^u_{p_2} \cup \{p_1, p_2\} \cup \{q\}) \to 0$, when $n \to \infty$.

□

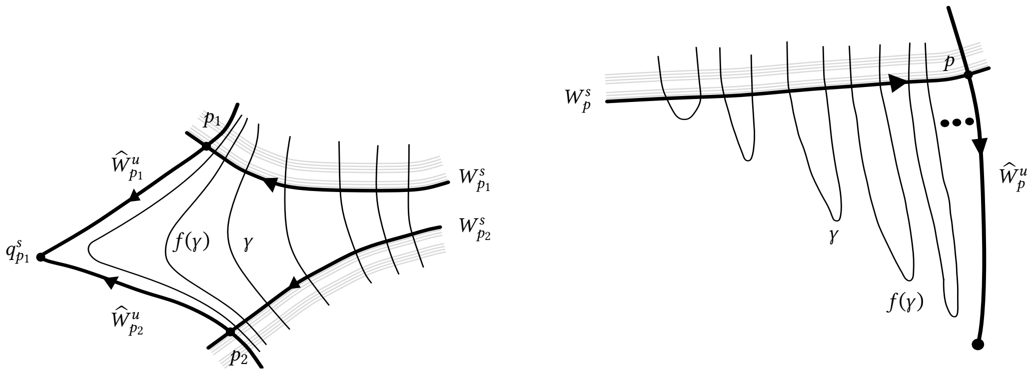

Figure 2.3: *The two cases of Proposition 2.2.2.*

As well as in the other results, we can obtain an analogous result for the iterates of an s-arc.

2.3 Margulis Measure

We are now able to present an important, and well explored, property of hyperbolic dynamic, namely, the existence of a Markov partition. Just like the others results presented early, the existence of Markov partition is not at all restricted to surface dynamic.

A *Markov partition* for Λ is a finite family of rectangles $\mathcal{M} = \{R_i : 1 \leq i \leq m\}$ such that:

- For all $(i, j) \in \{1, \ldots, m\}^2$, every connected component of the intersection $R_i \cap f(R_j)$ is a sub-rectangle of R_i (resp. $f(R_j)$) with stable edges (resp. unstable edges) included in $\partial^s R_i$ (resp. $\partial^u f(R_j)$), and crossing each of the connected component of $\partial^s R_i$ (resp. $\partial^u f(R_j)$).
- For all sequence $(i_n)_{n\in\mathbb{Z}} \in \{1, \ldots, m\}^{\mathbb{Z}}$, each connected component of the intersection $\cap_{n\in\mathbb{N}} f^n(R_{i_n})$ contains at most one point of Λ.

Since we are assuming that the topological dimension of Λ is equal to zero, then all the rectangles in a Markov partition form a disjoint family. A result by Bowen [Bow70] proved that there exist a Markov Partition for Λ. The existence of a Markov Partition for Λ allow us to find a conjugacy between the dynamic $(\Lambda, f|_\Lambda)$ and a subshift of finite type, in the following way: consider the $m \times m$ matrix $M = (m_{ij})$, where m_{ij} is the number of sub-rectangles of R_i with image by f contained in R_j. Note that if some entry of M is greater than 1, than we can consider a new Markov Partition of Λ made of sub-rectangles of the original Markov Partition such that all the entries of M are 0 or 1. Thus, we can suppose, without loss of generality, that M is a $\{0, 1\}$-matrix. It is possible to prove that $(\Lambda, f|_\Lambda)$ is conjugate to (Σ_M, σ) by a homeomorfism $h : \Lambda \to \Sigma_M$ and $\varrho = h * \beta$.

2.3.1 Proposition. *Consider the laminations $W^s(\Lambda)$ and $W^u(\Lambda)$ on S. Then, there is a transverse measure ν^u on $W^u(\Lambda)$ adapted to $W^s(\Lambda)$, called the Margulis stable measure, satisfying the following* *conditions:*

- *The measure of a segment on $W^s(\Lambda)$ is strictly positive if, and only if, its interior cross $W^u(\Lambda)$.*
- *There exist a constant $\lambda > 1$ such that, for all segments α contained on $W^s(\Lambda)$ and transverse to $W^u(\Lambda)$, holds $\nu^u(f^{-1}(\alpha)) = \lambda.\nu^u(\alpha)$.*

Proof. Consider $\{R_1, \dots, R_k\}$ a Markov Partition for Λ and M the $\{0,1\}$-matrix as described above. Consider also $u = (u_1, \dots, u_k)$ and $v^T = (v_1, \dots, v_k)^T$ the left and right positive eigenvectors of M associated to $\lambda_M > 1$, where $u.v = 1$.

For $n \in \{0, 1, 2, \dots\}$ and $i \in \{1, \dots, k\}$, consider now all the intervals I_i^n of $W^s(\Lambda)$, where $I_i^n \subset f^n(R_i)$ and both extreme points of I_i^n are elements of $\partial^u(f^n(R_i))$. For these intervals we define, $\nu^u(I_i^n) := \lambda_M^{-n}.v_i$. Intervals I that do not intersect Λ we define $\nu^u(I) := 0$. Note that for both type of intervals I the relation $\nu^u(f^{-1}(I)) = \lambda_M.\nu^u(I)$ holds.

Consider now an isolated s-interval $I \subset W^s(\Lambda)$. In this case, it is possible to split the interval I into a union of finite s-arcs, and k_I finite intervals $I_{i_j}^{n_j}$, where $j \in \{1, \dots, k_I\}$, and $I_{i_j}^{n_j} \subset f^{n_j}(R_{i_j})$ and both extreme points of $I_{i_j}^{n_j}$ are elements of $\partial^u(f^{n_j}(R_{i_j}))$. By what was define until now we have $\nu^u(I) := \sum_{j=1}^{k_I} \nu^u(I_{i_j}^{n_j})$. The equality $\nu^u(f^{-1}(I)) = \lambda_M.\nu^u(I)$ is true by the previous equality and by the fact that the inverse of s-arcs continue to be an s-arcs.

Consider now $I \subset W^s(\Lambda)$ any compact s-interval. This interval can be written as a union of an isolated s-interval $\hat{I}$ and at most two s-intervals that do not intersect Λ. In this case $\nu^u(I) := \nu^u(\hat{I})$, and obviously satisfies $\nu^u(f^{-1}(I)) = \lambda_M.\nu^u(I)$.

Finally, note that we define ν^u for all compact s-intervals contained in $W^s(\Lambda)$. For each leaf of $W^s(\Lambda)$ we can proceed like the construction of the Lebesgue measure on the real line $\mathbb{R}$, and extend the measure ν^u to the *Sigma*-algebra on this leaf generated by the compact intervals in this leaf. □

Following an analogous argument, we can prove the existence of a transverse measure ν^s of $W^s(\Lambda)$ adapted to $W^u(\Lambda)$, where $\nu^s(f(\alpha)) = \lambda.\nu^s(\alpha)$. Both the measures ν^s and ν^u are unique, up a multiplication by a constant. The product measure $\nu^s \times \nu^u$ is defined over the *Sigma*-algebra generate by all sub-rectangles contained in any rectangle of a Markov Partition $\{R_1, \dots, R_k\}$ of Λ. Moreover, the measure $\nu^s \times \nu^u$ is supported over Λ. Note that $\nu^s \times \nu^u(\cup_{i=1}^k R_i) = 1$. In this case, for an element R of the *Sigma*-algebra generate by all sub-rectangles, we have $\nu^s \times \nu^u(R) = \beta(R \cap \Lambda)$.

2.4 Tied and Sewn relations

An *arc chain* is a curve α formed by alternating boundary s-arcs and boundary u-arcs. Each arc that formed an arc chain is called a *side* of the arc chain. We say that a side is *adjacent* to other side if they share a point. Each side can have at most two adjacent sides. A *finite* arc chain is an arc chain with a finite number of sides.

A *closed arc chain* is a simple closed curve formed by alternating boundary s-arcs and boundary u-arcs. Every closed arc chain is a finite arc chain. Moreover, every closed arc chain have an even number of sides.

2.4.1 Proposition. *The following statement are true: A closed arc chain ψ is the boundary of an open topological disk $P(\psi)$ that do not intersect Λ. Moreover, the iterates of these disks $\{f^n(P(\psi)) : n \in \mathbb{Z}\}$ form a pairwise disjoint family. The set of orbits (by f) of any non-trivial polygon is finite.*

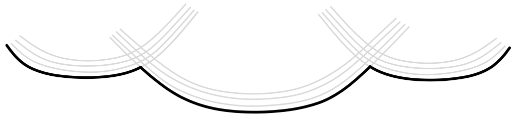

Figure 2.4: *Part of an arc chain.*

The proof of this statment can be found in pag. 68-69 of [BL98].

If a finite arc chain is not a closed arc chain, then we called it an *open arc chain*. We say that two arcs are *finite linked* if they are contained in an open finite arc chain.

A *n-polygon* is the union of a closed arc chain ψ, with $2n$ sides, and its associated topological disk $P(\psi)$. A *trivial* polygon is a 2-polygon (or a rectangle) and *non-trivial* otherwise.

2.4.2 Lemma. *The set of orbits (by f) of any non-trivial polygon is finite.*

The proof of this statment can be found in pag. 69 of [BL98].

We say that two distinct s-boundary separatrices W_1^s and W_2^s are *tied* if there exist an u-arc γ with one extreme point over W_1^s and the other extreme point over W_2^s. We use the same word (tied) to refer a similar notion that we can define for u-boundary separatrices.

The next propositions 2.4.3, 2.4.5 and 2.4.6 were also proved in [BL98], but with different terminology. We provide the proofs of these propositons for the convenience of the reader.

2.4.3 Propostion. *If the boundary separatrices W_1 and W_2 are tied, then there is no distinct boundary separatrix W_3 tied to W_1 or W_2. In other words, W_1 is only tied to W_2 and vice versa.*

Proof. Consider $W_{p_1}^s$ and W_2^s two distinct s-boundary separatrices tied by an u-arc γ, and $x \in W_{p_1}^s$ an extreme point of γ. Consider now another u-arc γ' with extreme point $y \in [x, p_1]_{p_1}^s$. Since $f^n(x) \to p_1$, when $n \to \infty$, there exist $N \in \mathbb{N}$ such that $y \in [f^N(x), f^{N+1}(x)]_{p_1}^s$. By the Proposition 2.1.5, the set $\sigma_{f^N(\gamma)}$ is a closed curve and is the boundary of a topological disk $T(\sigma_{f^N(\gamma)})$. Hence, the interior of γ' should be contained in $T(\sigma_{f^N(\gamma)})$ and, thus, there are only two possibility for the other extreme point of γ': or belongs to $W_{p_1}^s$ or belongs to W_2^s.

Suppose now that there exist a third s-boundary separatrix W_3^s, distinct to $W_{p_1}^s$ and distinct to W_2^s, but tied to $W_{p_1}^s$ by an u-arc γ'', and $z \in W_{p_1}^s$ the extreme point of γ'' contained in $W_{p_1}^s$. Since $f^n(z) \to p_1$, when $n \to \infty$, there exist $N \in \mathbb{N}$ such that $f^N(z) \in [x, p_1]_{p_1}^s$. Then, we can do a similar argument of the argument above, and prove that $f^N(\gamma'')$ should have the other extremity on $W_{p_1}^s$ or W_2^s, which leads us to a contradiction. We can do an analogous argument to prove the proposition for tied u-boundary separatrices. □

The next Corollary of the previous Proposition will be important later.

2.4.4 Corollary. *Every non-trivial polygon have at most two s-sides that tied two distinct u-boundary points.*

Proof. If the u-sides a finite arc chain ψ are all contained in just one u-boundary separatrix, then the corollary is true. Furthermore, if ψ have just two sides, then the statement is also true. Suppose now that ψ have $2n$ sides, where $n \geq 3$.

By the Proposition 2.4.3, the u-sides of ψ can be contained in at most two u-boundary separatrix. If this happen to ψ, then at least two s-sides tied the u-boundary separatrices. If there is a third

s-side that tied the two u-boundary separatrices, then one of these s-sides are in between the others two, a contradiction. □

It is possible to state and prove an analogous Corollary about u-sides of a polygon.

Consider now an s-boundary separatrix W^s where all arcs with one extreme point over W^s have, in fact, both extreme points over W^s. In this case, the u-arcs can be compare in the following way: if γ and γ' are two u-arcs having their extreme points over W^s, we say that γ is *subordinated* to γ' if $T(\sigma_\gamma) \subset T(\sigma_{\gamma'})$. Note that this relation make the set of all u-arcs having their extreme points over W^s a poset. Denote by $\alpha(W^s)$ the set of all maximal elements of this poset. The set $\alpha(W^s)$ is f-invariant and have infinite elements.

2.4.5 Proposition. *Consider W^s and $\alpha(W^s)$ as above. There exist a unique u-boundary separatrix W^u such that $\alpha(W^s) \subset W^u$. On the other hand, all s-arc having one extreme points over W^u have in fact both extreme points over W^u, and $\alpha(W^u) \subset W^s$.*

Proof. Denote by $\mathcal{A}(W^s)$ the set of all u-boundary separatrices W^u such that exist an u-arc $\alpha \subset W^u$ where $\alpha \in \alpha(W^s)$. We need to proof that $\mathcal{A}(W^s)$ have just one element. Firstly, the set $\mathcal{A}(W^s)$ is not null. More than that, since there exist just a finite number of boundary separatrices, the set $\mathcal{A}(W^s)$ is a finite set.

Let suppose that there exist more than one element in $\mathcal{A}(W^s)$. Hence, by the finiteness of $\mathcal{A}(W^s)$, there should exist an s-arc $\gamma \subset W^s$ that tied two distinct u-boundary separatrices $W_1^u, W_2^u \in \mathcal{A}(W^s)$. By Proposition 2.4.3, the set $\mathcal{A}(W^s)$ have just two elements, namely, W_1^u and W_2^u. Note that $f^n(\gamma)$, where $n \in \mathbb{Z}$, also tied W_1^u and W_2^u.

Consider an s-arc $\gamma \in \alpha(W^s)$ that tied $W_1^u, W_2^u \in \mathcal{A}(W^s)$. Take another s-arc $\gamma' \in \alpha(W^s)$ that tied $W_1^u, W_2^u \in \mathcal{A}(W^s)$, such that γ and γ' are finite linked by the finite arc chain α' such that they have just one adjacent side in α', and there is no other s-arc in α' that ties W_1^u and W_2^u. Thus, there is a finite number of s-arcs in α' with both extreme points over the same u-boundary separatrix, let's say W_1^u. If x and x' are extreme points of γ and γ', respectively, and contained in W_2^u. Note that all the s-arcs contained in $\alpha(W^s)\backslash\alpha'$ must have all its extreme points in $[x, x']^u$. However, this implies in the existence of a u-boundary periodic point, a contradiction. Thus, there is just only one element, denoted by W^u, in $\alpha(W^s)$.

Any s-arc that have one extreme point in W^u have both extreme point in W^u. Now we wat to prove that $\alpha(W^u) \subset W^s$. Let's suppose that γ is an extreme arc for its equivalent class of arcs, and γ is contained in an s-boundary separatrix W_1^s. If $W_1^s \neq W^s$, then we can find an u-arc contained in W^u with extreme point on W^s and on W_1^s, a contradiction. □

Two boundary separatrices of oposite nature W^s and W^u are *sewn* if it is related as in the Proposition described above.

2.4.6 Proposition. *Consider that W^s and W^u are sewn boundary separatrices, and p^s and p^u its associated boundary periodic point. Then, $\alpha(W^s, W^u) := \alpha(W^s) \cup \alpha(W^u)$ is an infinite arc chain, and $\alpha(W^s, W^u) \cup \widehat{W}^u_{p^s} \cup \widehat{W}^s_{p^u} \cup \{p^s, p^u\} \cup \{q^s_{p^s}, q^u_{p^u}\}$ is a continuum. Furthermore, every infinite arc chain is associated to a sewn boundary separatrices.*

Proof. The way $\alpha(W^s)$ and $\alpha(W^u)$ was constructed in Proposition 2.4.5, it is easy to see that $\alpha(W^s, W^u)$ is an infinite arc chain. By Proposition 2.2.2, we know that, for any arc $\alpha \subset \alpha(W^s, W^u)$, we have $d_H(f^n(\alpha), \widehat{W}^u_{p^s} \cup \{p^s\} \cup \{q^s_{p^s}\}) \to 0$ and $d_H(f^{-n}(\alpha), \widehat{W}^s_{p^u} \cup \{p^u\} \cup \{q^u_{p^u}\}) \to 0$, when $n \to \infty$. This shows that $\alpha(W^s, W^u) \cup \widehat{W}^u_{p^s} \cup \widehat{W}^s_{p^u} \cup \{p^s, p^u\} \cup \{q^s_{p^s}, q^u_{p^u}\}$ is a compact set.

Suppose now that $\alpha(W^s, W^u) \cup \widehat{W}^u_{p^s} \cup \widehat{W}^s_{p^u} \cup \{p^s, p^u\} \cup \{q^s_{p^s}, q^u_{p^u}\}$ can be written as a union of two disjoint non-empty open sets A and B. Here, open sets of $\alpha(W^s, W^u) \cup \widehat{W}^u_{p^s} \cup \widehat{W}^s_{p^u} \cup \{p^s, p^u\} \cup \{q^s_{p^s}, q^u_{p^u}\}$ are induced by the open sets of S. Note that $\alpha(W^s, W^u)$ is a one-dimensional non-compact topological space and, thus, is homeomorphic to $\mathbb{R}$. Thus, $\alpha(W^s, W^u)$ is connected and if the set A contain just one element of $\alpha(W^s, W^u)$, it should be contained entirely on A.

Suppose without loss of generality that $\alpha(W^s, W^u) \subset A$. It remains for the set B contain one of the sets $\widehat{W}^u_{p^s}$, $\widehat{W}^s_{p^u}$, $\{p^s\}$, $\{p^u\}$, $\{q^s_{p^s}\}$ or $\{q^u_{p^u}\}$. However, any point contained in any of these sets is accumulated by points of $\alpha(W^s, W^u)$. Thus, if B' is an open set of S that contains any point of any set listed above, then it contains infinite elements of $\alpha(W^s, W^u)$. Hence, if B is an open set of $\alpha(W^s, W^u) \cup \widehat{W}^u_{p^s} \cup \widehat{W}^s_{p^u} \cup \{p^s, p^u\} \cup \{q^s_{p^s}, q^u_{p^u}\}$ and not contain $\alpha(W^s, W^u)$, then $B = \emptyset$. It proves that $\alpha(W^s, W^u) \cup \widehat{W}^u_{p^s} \cup \widehat{W}^s_{p^u} \cup \{p^s, p^u\} \cup \{q^s_{p^s}, q^u_{p^u}\}$ is connected and, thus, a continuum.

Finally, if we proceed as in the proof of Proposition 2.4.5, we can show that any infinite arc chain α is contained in the union of two boundary separatrices of oposite nature, say $\alpha \subset W^s_\alpha \cup W^u_\alpha$. The existence of an arc tiying W^s to a distinct s-boundary separatrix, or an s-arc tying W^u to a distinct u-boundary separatrix, make impossible the existence of an infinite arc chain contained in the union of these two boundary separatrices. □

We can finally state an important result that we will use during the rest of the text.

2.4.7 Theorem. *A boundary separatrix is either tied to a single distinct boundary separatrix of same nature, or it is sewn to a single boundary separatrix of opposite nature.*

Proof. For a boundary separatrix, there are two options: either there exist an arc with one extreme point at the boundary separatrix and the other extreme point in another distinct boundary separatrix of the same nature, and in this case the boundary separatrices are tied; or all the arcs with extreme point on the boundary separatrix have both extremities on the boundary separatrix, and in this case the boundary separatrix is sewn to other boundary separatrix of oposite nature. The uniquiness of the tieness and the sewness are part of the Propostions 2.4.3 and 2.4.5. □

2.5 Zippers and Humps

Zippers

Consider now two tied s-boundary separatrices $W^s_{p_1}$ and $W^s_{p_2}$, where $p_1, p_2 \in \mathrm{Per}^s(\Lambda)$. An *$s$-zipper* is the region of S, with interior disjoint of Λ, and delimited by $W^s_{p_1}$ and $W^s_{p_2}$, $\widehat{W}^u_{p_1}$ and $\widehat{W}^u_{p_1}$, p_1 and p_2, and $q := q^s_{p_1} = q^s_{p_2}$. The s-zipper is denoted by $\mathcal{Z}^s(W^s_{p_1}, W^s_{p_2})$. An *undone s-zipper* is an s-zipper such that every u-arc realizing the tying between $W^s_{p_1}$ and $W^s_{p_2}$ are arc equivalence, and a *stuck s-zipper*, otherwise. We can make similar definitions of u-zippers.

Note that the domain of an u/s-arc of an undone s/u-zipper does note have any extreme arc, thus the domain of any arc realizing the tying coincide with the whole s/u-zipper. On the other hand, a stuck s/u-zipper only happen when some u/s-arc have its extreme points over the same s/u-boundary separatrix. Thus, each equivalent class of u/s-arcs tying the s/u-boundary separatrices have two extreme arcs.

Consider now $\mathcal{Z}^s(W^s_{p_1}, W^s_{p_2})$ a stuck s-zipper and γ an extreme u-arc realizing the tying between $W^s_{p_1}$ and $W^s_{p_1}$. Denote by $x_1 \in W^s_{p_1}$ and $x_2 \in W^s_{p_2}$ the extremities of γ. By the Propositon 2.1.5, the region $T(\sigma_\gamma)$ is a topological disk. Moreover, the region $T(\sigma_\gamma)$ is a fundamental domain of $\mathcal{Z}^s(W^s_{p_1}, W^s_{p_2})$, thus, $\mathcal{Z}^s(W^s_{p_1}, W^s_{p_2}) = \cup_{n \in \mathbb{Z}} f^n(T(\sigma_\gamma))$.

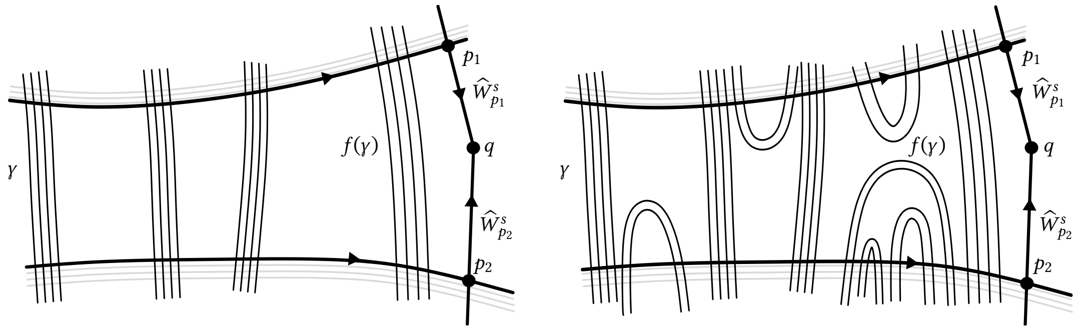

Figure 2.5: *Undone zipper and a stuck zipper.*

2.5.1 Lemma. *There is a finite number k_γ of equivalent classes of u-arcs tying $W^s_{p_1}$ and $W^s_{p_2}$ contained in $\sigma_\gamma \cup T(\sigma_\gamma)$. Moreover, k_γ is independent of γ.*

Proof. The existence of an infinite number of equivalent classes of u-arcs tying $W^s_{p_1}$ and $W^s_{p_2}$ contained in $\sigma_\gamma \cup T(\sigma_\gamma)$ implies the existence of an s-boundary periodic points on $[x_1, f(x_1)]^s_{p_1}$ and another s-boundary periodic points on $[x_2, f(x_2)]^s_{p_2}$, a contradiction.

Consider now γ' another extreme u-arc realizing the tying between $W^s_{p_1}$ and $W^s_{p_1}$. For each equivalence class of u-arcs that tied $W^s_{p_1}$ and $W^s_{p_1}$ and is contained in $\sigma_{\gamma'} \cup T(\sigma_{\gamma'})$, we can find an integer such that the iterate of this equivalent class of arcs by this integer lies on $\sigma_\gamma \cup T(\sigma_\gamma)$. This proves that $k_{\gamma'} \leq k_\gamma$. Following an analogous argument, we can show that $k_\gamma \leq k_{\gamma'}$, and, thus, $k_\gamma = k_{\gamma'}$. □

By the previous Lemma, there are $k \in \mathbb{N}$ equivalence classes of u-arcs contained on $\sigma_\gamma \cup T(\sigma_\gamma)$ and tying $W^s_{p_1}$ and $W^s_{p_2}$. Let's denote by $\{\omega_i \ : \ 1 \leq i \leq k\}$ such family of equivalence classes of u-arcs. If $W^s_{p_1}$ is oriented towards its associated periodic points, we are assuming that the enumeration of the family $\{\omega_i \ : \ 1 \leq i \leq k\}$ is coherent with the orientation of $W^s_{p_1}$, i.e. ω_1 is further away than ω_2 from p_1, and ω_2 is further away than ω_3 from p_1, ando so on until the last region ω_k, that is the closest equivalent class to p_1. Note that if we choose $W^s_{p_2}$ instead of $W^s_{p_1}$ does not change the enumeration, since in the case we are working, the orientation of $W^s_{p_1}$ and $W^s_{p_2}$ are coherent, see Corollary 2.4.3, item (3), page 57 in [BL98].

The spaces obtained removing from $\sigma_\gamma \cup T(\sigma_\gamma)$ the interior of the domain of each ω_i is a union of k regions delimited by four intervals of diferent nature and having u-arcs in its interior with both extreme points over the same separatrix. Each region described above is denoted by D^i_γ, for $i \in \{1, \cdots, k\}$, and ϕ^i_1 and ϕ^i_2 are the sides of D^i_γ respectively contained on $W^s_{p_1}$ and $W^s_{p_2}$. Again, we are considering that the enumeration of the regions D^i_γ are also coherent with the orientation of $W^s_{p_1}$ and $W^s_{p_2}$ towards its associated periodic points.

Let M^i_j be the set of all equivalent classes of arcs having an u-arc with both extremities over ϕ^i_j ($j = 1$ or 2).

2.5.2 Lemma. *For each $i \in \{1, \cdots, k\}$, at least one M^i_1 or M^i_2 is non-empty. Furthermore, if M^i_j is non-empty, then is finite.*

Proof. Each unstable side of D^i_γ are not arc equivalent by hypothesis. Therefore, there must be an u-arc with both extremities over the same s-boundary separatrix. Hence, each M^i_1 is non-empty. If M^i_1 have infinite elements, then there must have an s-boundary periodic point on ϕ^i_1, a contradiction. □

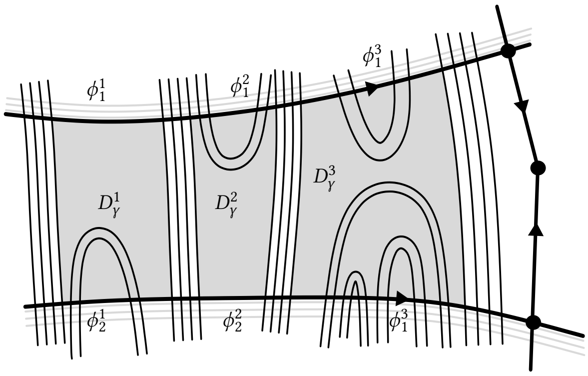

Figure 2.6: *Regions D_γ^i for the stuck zipper in the Figure 2.6.*

The space obtained removing from D_γ^i the interior of the domains of each element in M_j^i, for both $j = 1$ and $j = 2$, is a union of non-trivial polygons. There is exactly one polygon $\mathcal{P}^i$ that have its stable sides included both in ϕ_1^i and in ϕ_2^i. The rest of the polygons have its stable sides exclusively include either in ϕ_1^i or in ϕ_2^i. Denote by $P_\gamma^i = \{\mathcal{P}_l^i \ : \ 0 \leq l \leq |M_1^i| + |M_2^i|\}\}$, where $\mathcal{P}_0^i = \mathcal{P}^i$, the set of all polygons contained on D_γ^i. The set $P_\gamma = \cup_{i=1}^k P_\gamma^i$ is the set of all polygons contained in $\sigma_\gamma \cup T(\sigma_\gamma)$. The set $P(W_1^s, W_2^s) = \cup_{n\in\mathbb{Z}} P_{f^n(\gamma)}$ are all the polygons contained in $\mathcal{Z}^s(W_{p_1}^s, W_{p_2}^s)$.

Of course we can do similar definitions and obtain similar propositions about tied u-boundary separatrices and the u-zippers associated to it.

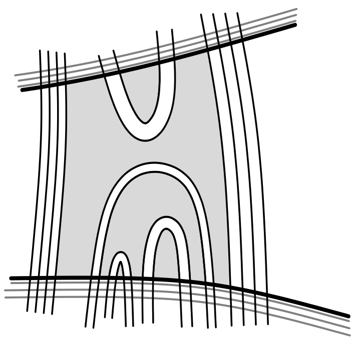

Figure 2.7: *The five polygons in region D_γ^3 for the stuck zipper in the Figure 2.6.*

Humps

Consider now an s-boundary separatrix W^s sewn to an u-boundary separatrix W^u. If we oriented W^s towards its associated s-boundary periodic point p^s and orient W^u away its associated u-boundary periodic point p^u, the curve $\alpha(W^s, W^u)$ can also be oriented: the curve $\alpha(W^s, W^u)$ is oriented from the u-boundary periodic point toward the s-boundary periodic point. Taking advantage of the orientation of $\alpha(W^s, W^u)$ we define the following order on $\alpha(W^s, W^u)$: given $\gamma, \gamma' \subset \alpha(W^s, W^u)$, we write $\gamma \prec \gamma'$ if γ is behind γ', with respect to the orientation of $\alpha(W^s, W^u)$. Note that γ and γ' can be of different nature.

For each $\gamma \in \alpha(W^s, W^u)$, each $T(\sigma_\gamma)$ is called a *hump*. The spaces $\alpha(W^s, W^u)$ and $\alpha^h(W^s, W^u) := \cup_{\gamma\subset\alpha(W^s,W^u)}(T(\sigma_\gamma))$ are f-invariants. We can also define the region $\alpha_\gamma^h(W^s, W^u) \subset \alpha^h(W^s, W^u)$ of all regions $\sigma_{T(\sigma_{\gamma'})}$ such that $\gamma \preceq \gamma' \prec f(\gamma)$. Thus, $\alpha^h(W^s, W^u) = \cup_{n\in\mathbb{Z}} f^n(\alpha_\gamma^h(W^s, W^u))$.

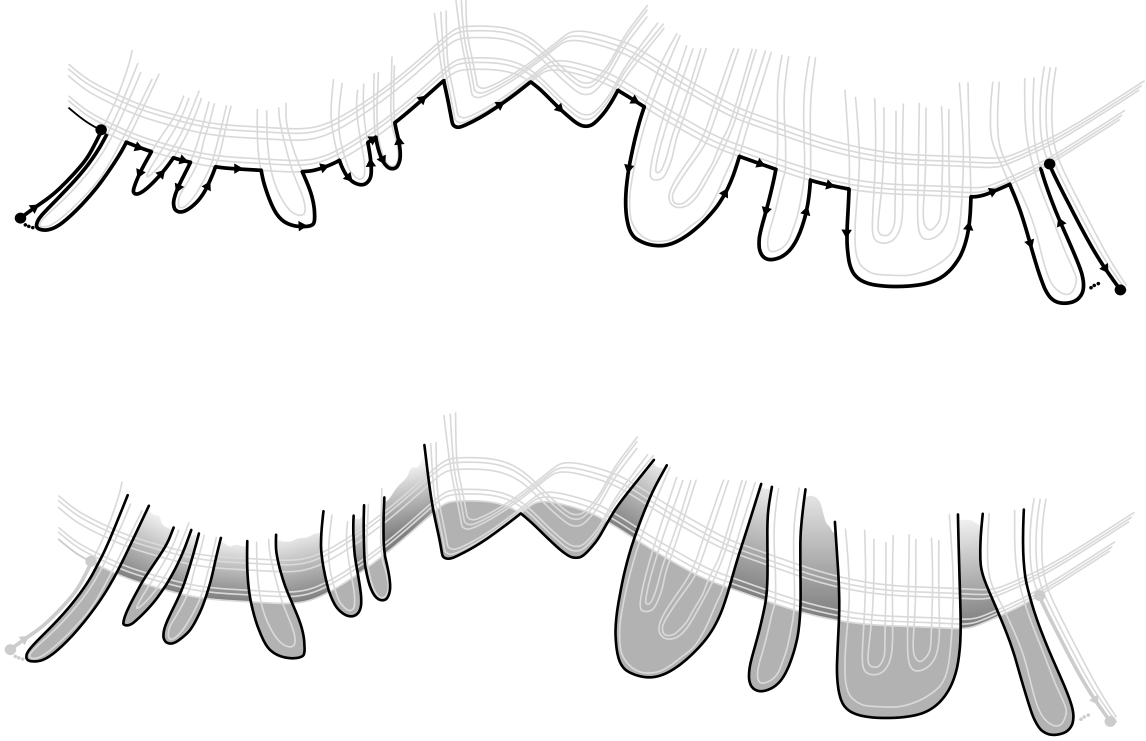

Figure 2.8: *The curve $\alpha(W^s, W^u)$ and the humps $\alpha^h(W^s, W^u)$.*

2.5.3 Lemma. *There is a finite number $\kappa_\gamma \in \mathbb{N}$ such that $\alpha^h_\gamma(W^s, W^u)$ have κ_γ humps. Moreover, κ_γ is independent of γ.*

The proof of this Lemma is similar to the proof of Lemma 2.5.2. Let's write $\alpha^h_\gamma(W^s, W^u) = \cup_{i=1}^{\kappa} T(\sigma_{\gamma_i})$, where $\gamma_i \preceq \gamma_j$ when $i \leq j$.

2.5.4 Lemma. *For each hump $T(\sigma_{\gamma_i}) \in \alpha^h_\gamma(W^s, W^u)$, where $1 \leq i \leq \kappa$, there exist a finite number $\tau_i \in \mathbb{N}$ of distincts equivalence classes of arcs contained in $T(\sigma_{\gamma_i})$.*

Proof. As there is an arc $\gamma \subset T(\sigma_{\gamma_i})$ that is arc equivalent to γ_i, then there exist at least one equivalent class of arcs contained in $T(\sigma_{\gamma_i})$. As in previous cases, the existence of infinite equivalent classes of arcs contained in $T(\sigma_{\gamma_i})$ implies the presence of a boundary periodic point on W^s, or W^u, a contradiction. Thus, there exist a finite number $\tau_i \in \mathbb{N}$ of distincts equivalent classes of arcs contained in $T(\sigma_{\gamma_i})$. □

Let's denote by $\{\omega^j_{\gamma_i} : 1 \leq j \leq \tau_i\}$ the family of equivalence classes of arcs contained in $T(\sigma_{\gamma_i})$. For each $T(\sigma_{\gamma_i}) \in \alpha^h_\gamma(W^s, W^u)$, the space obtained removing from $T(\sigma_{\gamma_i})$ the interior of the domain of each $\omega^i_{\gamma_i}$ is a union of τ_i non-trivial polygons. We will denote by $\mathcal{L}_{\gamma_i}$ the unique polygon that have one of its side in the same equilalence class of arc of γ_i. Denote by $L_{\gamma_i} = \{\mathcal{L}^j_{\gamma_i} : 1 \leq j \leq \tau_i - 1\}$, where $\mathcal{L}^1_{\gamma_i} = \mathcal{L}_{\gamma_i}$, the set of all polygons contained in $T(\sigma_{\gamma_i})$. The set $L_\gamma = \cup_{i=1}^{\kappa} L_{\gamma_i}$ is the set of all polygons contained in $\alpha^h_\gamma(W^s, W^u)$. The set $L(W^s, W^u) = \cup_{n \in \mathbb{Z}} L_{f^n(\gamma)}$ is the set of all the polygons contained on $\alpha^h(W^s, W^u)$.

Finally, we can prove the following proposition.

2.5.5 Proposition. *If a non-trivial polygon is contained in a stuck s-zipper, then it is contained in a stuck u-zipper. Moreover, every non-trivial polygon is either contained in a stuck zipper or in a hump.*

Proof. For any non-trivial polygon, by the Corollary 2.4.4, there exist at least two u-sides that ties two distinct s-boundary separatrices. If a non-trivial polygon is contained in an s-zipper, then there are two u-sides that tied two distinct s-boundary separatrices. On the other hand, by the Corollary 2.4.4, there are two s-sides of the polygon that tied two u-boundary separatrices. Hence, the polygon is contained in the u-zipper associated to these two u-boundary separatrices.

If a polygon is not contained in a zipper, then, by Corollary 2.4.4, its s-sides are contained in only one s-boundary separatrix W^s, and its u-sides are contained in only one u-boundary separatrix W^u. If the s-boundary separatrix W^s is tied to another s-boundary separatrix, then it implies that the polygon is contained in an s-zipper, a contradiction. Thus, every u-side of the polygon is contained in $T(\sigma_\gamma)$, where the u-arc $\gamma \subset \alpha(W^s)$ satisfies $\gamma' \prec \gamma$, when γ' is an u-side of the polygon. □

2.6 Contact Between Boundary Periodic Point

By Theorem 2.4.7, a boundary separatrix is either tied to another boundary separatrix of the same nature, or sewn to a boundary separatrix of oposite nature. We say that two boundary periodic points are in *contact* if the boundary separatrix of one of these boundary periodic point is tied or sewn to the boundary separatrix of the other boundary periodic point. Note that a boundary periodic point can be in contact to itself.

From the contact relation between boundary periodic points we can construct a graph $\mathcal{G}$ where each vertex is a boundary periodic point and each edge is the pair formed by the boundary separatrices that turn the boundary periodic points in contact.

2.6.1 Proposition. *There is a finite number $g_\Lambda \in \mathbb{N}$ of graphs $\mathcal{G}$, and each graph $\mathcal{G}$ is a cycle graph*[3].

Proof. By the Corollary 2.1.3, each boundary periodic point have two boundary separatrices. Thus, each boundary periodic point is in contact to two other boundary periodic points, or it is in contact with itself. In both cases, the degree of all vertices is two. W will denote by $\mathcal{G}$ each connected component of the graph formed by all the boundary periodic point. Since there are just a finite number of boundary periodic point, then the number of connected component $\mathcal{G}$ is finite, let's say g_Λ, and each connected component of $\mathcal{G}$ have an infinite number of vertices. □

Consider $\mathcal{G}$ a cycle graph as described above and $\{p_i^{\mathcal{G}};\ 1 \le i \le v_{\mathcal{G}}\}$ all the boundary periodic points associated to each vertex of $\mathcal{G}$, where $v_{\mathcal{G}}$ is the number of vertex of $\mathcal{G}$. Let suppose that the enumeration of the $p_i^{\mathcal{G}}$'s are also cyclic, i.e., $p_1^{\mathcal{G}}$ is in contact to $p_2^{\mathcal{G}}$, $p_2^{\mathcal{G}}$ is in contact to $p_3^{\mathcal{G}}, \ldots, p_{v_{\mathcal{G}}}^{\mathcal{G}}$ is in contact to $p_1^{\mathcal{G}}$. If the contact between the periodic points is due to a tied, then we associate to this contact the space formed by the union of the ree separatrices associated to the zipper, the boundary periodic points associated to the tied boundary separatrices and the source/sink also associated to the zipper.

formed by the free separatrices associated to the zipper, the boundary periodic points, and the source/sink also associated to the zipper. Otherwise, if the contact between the periodic points is due to a sewn, then we associate to this contact the infinite arc chain α and the free separatrices. The union of all spaces associated to each contact of vertices of $\mathcal{G}$ as we described is denote by $C(\mathcal{G})$. See Figure 2.9 and 2.11.

2.6.2 Lemma. *The space $C(\mathcal{G})$ is a continuum.*

[3] A *cycle graph* is any connected graph with vertices connected to exactly two other vertices.

Proof. Consider W^s and W^u are sewn boundary separatrices and p^s and p^u are the boundary periodic points associated to each boundary separatrices. By Proposition 2.4.6, the space $\alpha(W^s, W^u) \cup \widehat{W}^u_{p^s} \cup \widehat{W}^s_{p^u} \cup \{p^s, p^u\} \cup \{q^s_{p^s}, q^u_{p^u}\}$ is a continuum. Moreover, each free separatrix is obviously a continuum. Hence, the space $C(\mathcal{G})$ is a union of continua. Since each continuum in this union share a point (in fact a boundary periodic point) with another continuum, the union is a connected space.

The compactness of $C(\mathcal{G})$ follows from the fact that $C(\mathcal{G})$ is a finite union of compact, which implies that $C(\mathcal{G})$ is also compact. □

Define now the space $S(\Lambda) := \overline{W^s(\Lambda) \cup W^u(\Lambda)} \cup \{P(\psi) : \psi \text{ is a closed arc chain}\}$. The space $S(\Lambda)$ is an f-invariant continuum in S.

2.6.3 Proposition. *There exist an f-invariant compact surface with boundary without a finite number of points, denoted by $\Delta(\Lambda)$, and containing $S(\Lambda)$, such that:*

a) The points missing in the boundary of $\Delta(\Lambda)$ are sources and sinks. Furthermore, the union of $\Delta(\Lambda)$ and the sources and sinks that are missing in the boundary of $\Delta(\Lambda)$ is equal to $\overline{\Delta(\Lambda)}$.

b) There exist a finite number of connected component of $\overline{\Delta(\Lambda)} \backslash S(\Lambda)$. Each connected component is homeomorphic to $\mathbb{R} \times [0, +\infty)$ and the dynamic restrict to each connected component is conjugate to a translation.

c) Each connected component of $\overline{\Delta(\Lambda)} \backslash S(\Lambda)$ is delimited by the boundary of $\overline{\Delta(\Lambda)}$, and by either a union of an infinite arc chain and the accumulated free separatrices, or by the union of two separatrices of a corner.

The proof of this propositon can be found in [BL98], pag. 72–80.

By the assumptions we made for the diffeomorphism f, for all connected component $\mathcal{D}$ of $S \backslash \overline{\Delta(\Lambda)}$, the space $\overline{\mathcal{D}}$ is a compact surface with boundary and the dynamic of f restricted to $\overline{\mathcal{D}}$ is a Morse-Smale dynamic.

2.6.4 Corollary. *Each connected component of $S \backslash \overline{\Delta(\Lambda)}$ is contained in only one connected component of $S \backslash S(\Lambda)$. Moreover, each connected component of $S \backslash S(\Lambda)$ contains only one connected component of $S \backslash \overline{\Delta(\Lambda)}$.*

Let's denote by n_Λ the number of boundary components of $\overline{\Delta(\Lambda)}$, and by m_Λ the number of connected component of $S \backslash \overline{\Delta(\Lambda)}$. If $\{\mathcal{S}_j : 1 \leq j \leq m_\Lambda\}$ is the set of all connected component of $S \backslash S(\Lambda)$, then $\sum_{j=1}^{m_\Lambda} t_{\mathcal{S}_j} \leq g_\Lambda \leq m_\Lambda \leq n_\Lambda$.

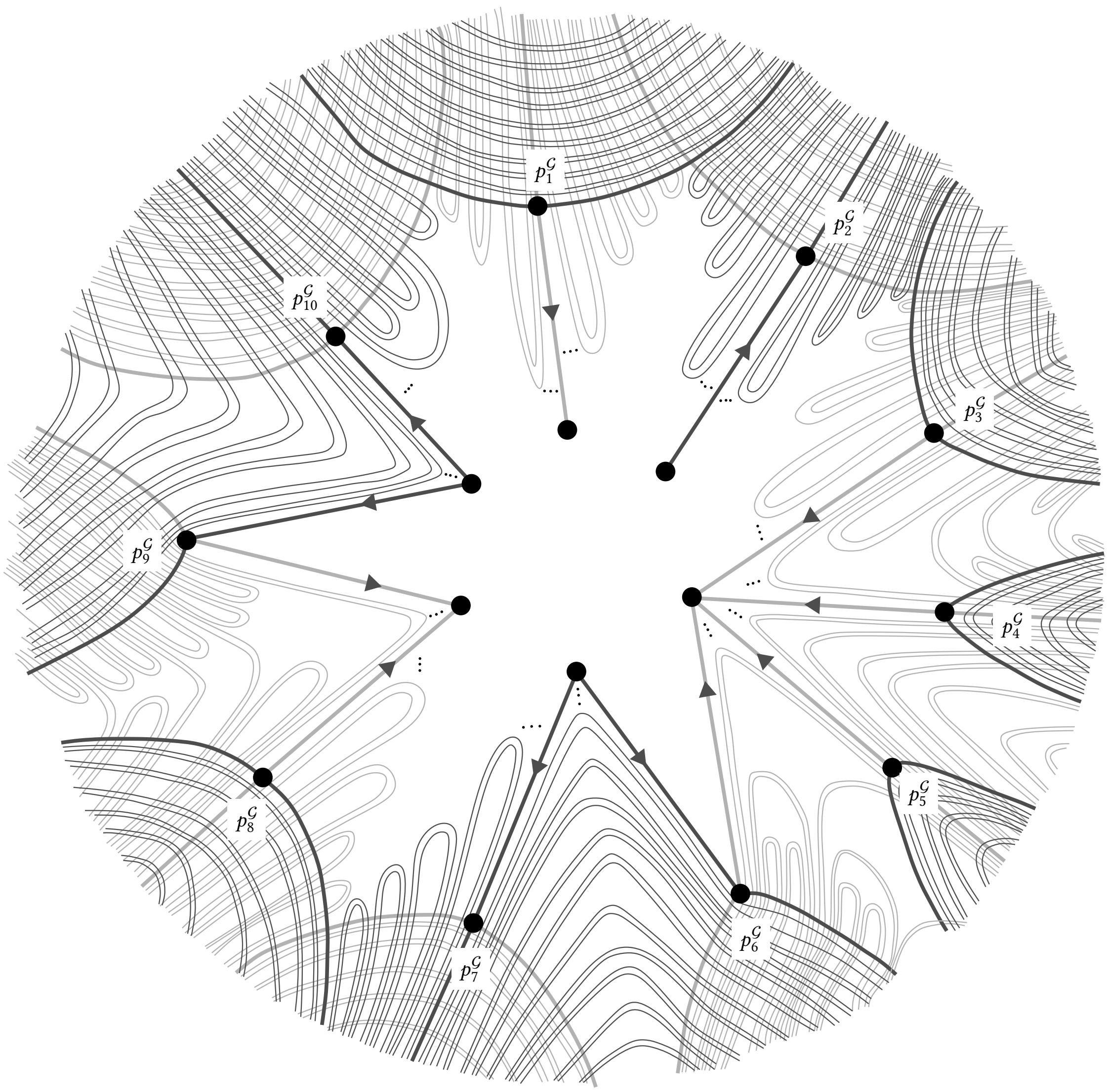

Figure 2.9: *The points $p_{i-1}^{\mathcal{G}}$, $p_i^{\mathcal{G}}$ and $p_{i+1}^{\mathcal{G}}$ are in contact, for $i \in \{1, \dots, 10\}$. For example, the point $p_4^{\mathcal{G}}$ is tied to $p_3^{\mathcal{G}}$, which in turn is sew to $p_2^{\mathcal{G}}$.*

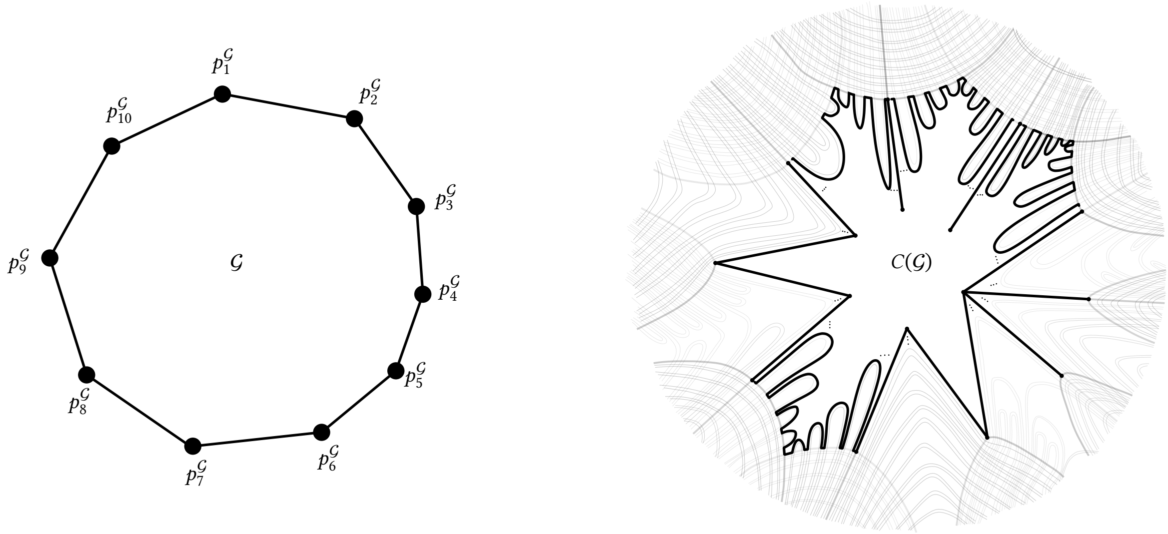

Figure 2.10: *The graph $\mathcal{G}$ and the space $C(\mathcal{G})$ associated to Figure 2.9.*

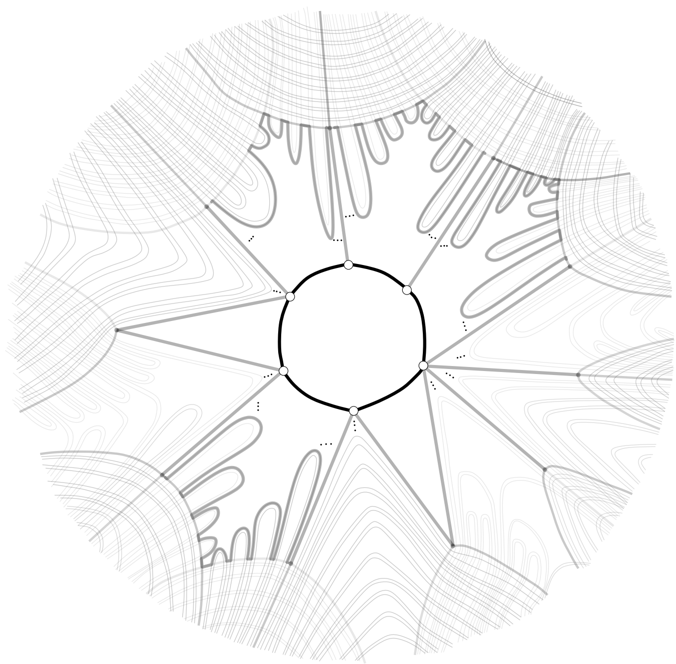

Figure 2.11: *One boundary of the compact surface $\Delta(\Lambda)$ and the missing points on the boundary.*

Chapter 3

Zero-Entropy Equivalence

As we anticipated in the first chapter, the zero-entropy relation will play an important role in this work. Just to remember, two points $x, y \in X$ are said to be *zero-entropy* related if there exist a continumm (compact and connected) C contained on X such that $x, y \in C$ and $h^f(C) = 0$. As we wrote, the zero-entropy relation is an equivalence relation. We denote by $\widetilde{X}$ the set of all zero-entropy equivalent classes, by $\widetilde{\pi} : X \to \widetilde{X}$ its canonical projection, and by $\widetilde{f} = \widetilde{\pi} \circ f$ the induced map $\widetilde{f} : \widetilde{X} \to \widetilde{X}$. The map $\widetilde{f}$ is called the *tight quotient* of the map f.

3.1 Monotone upper semi-continuous equivalence relation

Let X be a compact metric space and $\mathcal{Q}$ a partition of X. The partition $\mathcal{Q}$ is called *monotone* if it is a partition into connected sets. It is called an *upper semi-continuous* if $x_n, y_n \in Q_n \in \mathcal{Q}$, and $x_n \to x$ and $y_n \to y$, when $n \to \infty$, imply that $x, y \in Q' \in \mathcal{Q}$.

We will need the following Lemma, that the reader can find the proof in [dCP03].

3.1.1 Lemma. *Let (X, d) be a compact metric space, $f : X \to X$ be a homeomorphism and $C \subset X$ be a compact set satisfying $h(f, C) > 0$. Then there exists an f-invariant ergodic Borel probability measure μ_C on X, such that $h_{\mu_C}(f) > 0$ and such that μ_C-almost every point belongs to a non-degenerate set in $\omega(C)$*[1].

By the hypothesis we made so far, the closure of each leaf of $W^{s/u}(\Lambda)$ is a continuum that carries entropy. Thus, there exist continua in S that carries entropy.

A point $x \in \Lambda$ is called *inaccessible* if it is neither an s-boundary point, nor an u-boundary point. An *accessible* point is all points in Λ that are not inaccessible. An *inaccessible arc* is any arc where its extreme points are formed by two strictly boundary periodic points.

3.1.2 Lemma. *If $C \subset S$ is a compact set where $h(f, C) > 0$, then, if $A \subset \Lambda$ is the set of all accessible point, $\mu_C(A) = 0$.*

Proof. Since $h_{\mu_C}(f) > 0$ and the measure μ_C is ergodic, then μ_C is non-atomic. Consider $\{R_1, \dots, R_k\}$ a Markov Partition for Λ. If $I_i^{s/u}$ is one connected component of $\partial^{s/u}(R_i)$, then the set of endpoints contained in $I_i^{s/u}$, that will be denoted by $E_i^{s/u}$, is countable.

---

[1] Here the set $\omega(C)$ is the *Omega*-limit set of C associated to the dynamic $(\hat{f}, \mathcal{K}(X))$, where $\mathcal{K}(X)$ is the hyperspace associated to X.

Since $A \subset (\cup_{i=1}^{k}(E_i^s \times I_i^u) \cap W^s(\Lambda)) \cup (\cup_{i=1}^{k}(I_i^s \times E_i^u) \cap W^u(\Lambda))$, and this union is also countable, the measure by μ_C of all accessible points is zero. □

3.1.3 Lemma. *Let K, K' be two continua of S such that $d_H(f^n(K), f^n(K')) \to 0$, when $n \to \infty$. Then, $h(f, K) = h(f, K')$.*

Proof. Consider $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $d_H(f^n(K), f^n(K')) < \varepsilon/2$ for all $n \geq N$. Note that if we cover $f^n(K)$ by open sets of diameter ε, then it is also covers the compact $f^n(K')$. Hence, $D(n, \varepsilon, K') \leq D(n, \varepsilon, K)$, for all $n \geq N$. We can proceed in similar way to prove that $D(n, r, K) \leq D(n, r, K')$. Thus, $h(f, K) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln D(n, \varepsilon, K) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln D(n, \varepsilon, K') = h(f, K')$. □

3.1.4 Lemma. *Consider a compact s/u-interval $I \subset W^{s/u}(\Lambda)$. If I is contained in a s/u-arc, then $h^f(I) = 0$.*

Proof. First of all, note that the closure of any free separatrix carry zero entropy.

Consider I a compact u-interval contained in an u-arc. The interior of I do not cross $W^s(\Lambda)$. By Proposition 2.2.2, this u-arc converges, in Hausdorff distance, to the closure of a free separatrix. By Lemma 3.1.3, the forward topological entropy carried by this u-arc under f is zero. Furthermore, the length of the inverse iterates of this u-arc converge to zero. Hence, the backward topological entropy carried by this u-arc under f is also zero. Thus, $h^f(I) = 0$. □

The proof of the next theorem was inspired by the proof of a similar proposition presented in [dCP03]. In the case of this paper, the authors proved, among other things, that for a $C^{1+\epsilon}$ diffeomorphism on a surface, the zero-entroy equivalence is a upper semi-continuous equivalence class. The next theorem was necessary since we are assuming that the diffeomorphism f is C^1 diffeomorphism.

3.1.5 Theorem. *If $f : S \to S$ is a shoe diffeomorphism, then the zero-entropy equivalence relation in S is a monotone upper-semicontinuous equivalence relation.*

Proof. The zero-entropy equivalence is monotone by the construction of the equivalence relation. It remains to show that the zero-entropy equivalence relation is upper semi-continuous.

Consider two sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subset S$ such that x_n is zero-entropy equivalence to y_n, and $x_n \to x \in S$ and $y_n \to y \in S$, when $n \to \infty$. Consider C_n a continuum containing x_n and y_n and carrying zero entropy. By compactness of the space $(\mathcal{K}(M), d_H)$, there exist a continuum C that contains x and y, and a sequence $(C_{m_i})_{i \in \mathbb{N}}$ such that $d_H(C_{m_i}, C) \to 0$, when $i \to \infty$.

Suppose that $h(f, C) > 0$ and consider μ_C the measure given by Lemma 3.1.1. By Poincaré Recurrence theorem 1.1.1, by Lemma 3.1.1, and by Lemma 3.1.2., the set of all recurrent, inaccessible and that belongs to a non-degenerate set in $\omega(C)$, have full measure.

Consider x a recurrent, inaccessible and $x \in E \in \omega(C)$, where E is non-degenerate. Since $E \in \omega(C)$, then there exist a strictly increasing sequence of natural number $(n_k)_{k \in \mathbb{N}}$ such that $d_H(f^{n_k}(C), E) \to 0$, when $k \to \infty$. By the continuity of $\hat{f}$, for all $\epsilon > 0$ there exist $\delta > 0$ such that for all compact D, $d_H(D, C) < \delta$ implies $d_H(\hat{f}^n(D), \hat{f}^n(C)) < \epsilon/2$. Consider a natural number N greater enough such that $d_H(f^{n_k}(C), E) < \epsilon/2$, for all $n_k > N$. Thus, for all compact D, $d_H(D, C) < \delta$ implies $d_H(\hat{f}^n(D), E) < \epsilon$.

Consider a point $y \in E$, such that $\text{diam}(E)/2 < d(x, y) < \text{diam}(E)$. Then, there exist a point $w \in f^{n_k}(D)$ such that $d(y, w) < \epsilon$. Thus, $d(w, x) \geq d(x, y) - d(w, y) > \text{diam}(E)/2 - \epsilon$.

Consider a rectangular neighborhood $\mathcal{N}_x$ of x such that $\operatorname{diam}(E) > 2\operatorname{diam}(\mathcal{N}_x)$, and $\partial^{s/u}\mathcal{N}_x$ are contained in boundary sepratrices.

Consider points $z, w \in f^{n_k}(D)$ such that $z \in \mathcal{N}_x$ and $d(x, w) > \operatorname{diam}(E)/2 - \epsilon$. Suppose also that ϵ is small enough such that $\epsilon < \operatorname{diam}(E)/2 - \operatorname{diam}(\mathcal{N}_x)$. Thus, $d(x, w) > \operatorname{diam}(\mathcal{N}_x)$. The point z is contained in $\mathcal{N}_x$ and w is not contained in $\mathcal{N}_x$. Consider $\mathcal{N}_x'$ a rectangular neighborhood of x that also contains w.

Consider U a continuum contained in $\mathcal{N}_x'$. Define $U_1 = U \cap \operatorname{Int}(\mathcal{N}_x)$ and $U_2 = U \cap \operatorname{Int}(\mathcal{N}_x' \backslash \mathcal{N}_x)$. Define also $U_1' = \overline{U}_1 \cap \delta\mathcal{N}_x$ and $U_2' = \overline{U}_2 \cap \delta\mathcal{N}_x$. We say the continuum U *cross* $\mathcal{N}_x$ if there exit $a_1 \in C_1'$ and $a_2 \in C_2'$, and a continuum $T \subset C \cap \partial\mathcal{N}_x$ with $a_1, a_2 \in T$. The continuum T can be degenerate.

Suppose that $U \subset \mathcal{N}_x'$ is a continuum that cross $\mathcal{N}_x$. Thus, we can take points $a_1 \in C_1'$ and $a_2 \in C_2'$, and a continuum $T \subset C \cap \partial\mathcal{N}_x$ with $a_1, a_2 \in T$. Consider two sequences $(u_i)_{n\in\mathbb{N}} \subset U_1$ and $(u_i')_{n\in\mathbb{N}} \subset U_2$, such that $u_i \to a_1$ and $u_i' \to a_2$, when $n \to \infty$.

Suppose that the continuum T contain some compact s/u-interval I that have its interior disjoint of Λ. Consider an endpoint $e \in \operatorname{Int}(I)$. Since the point e is a s/u-boundary point, thus there exist a s/u-boundary periodic point q such that $e \in W^{s/u}(q)$. Hence, $d_H(f^n(I), \overline{W^{u/s}(q)}) \to 0$, when $n \to \infty$. Thus, by Lemma 3.1.3, $h(f, I) = h(f, \overline{W^{u/s}(q)}) > 0$.

Suppose now that the continuum T does not contain any compact s/u-interval that have its interior disjoint of Λ. Thus, T is contained in an s/u-arc $\gamma = [p_1, p_2]_q^{s/u} \subset W^{s/u}(q)$, where q is an s/u-boundary periodic point. Suppose that q_1 and q_2 are u/s-boundary periodic point such that $p_1 \in W^{u/s}(q_1)$ and $p_2 \in W^{u/s}(q_2)$. Define Q the sub-rectangle contained in $\mathcal{N}_x'$ where $\partial^u Q = \{W^u_{\mathcal{N}_x'}(p_1), W^u_{\mathcal{N}_x'}(p_2)\}$ and $\partial^s Q \subset \partial^s \mathcal{N}_x'$. The sets $Q_1 = Q \cap C_1$ and $Q_2 = Q \cap C_2$ are compact sets. Since $d_H(f^n(W^u_{\mathcal{N}_x'}(p_i)), \overline{W^{u/s}(q)}) \to 0$, when $n \to \infty$, and $i = 1, 2$, then $d_H(f^n(C_1 \cup C_2), \overline{W^{u/s}(q)}) \to 0$, when $n \to \infty$. Hence, the compact $C_1 \cup C_2$ is a compact set contained in U that carry positive entropy. Thus, U carry positive entropy. We showed that if a continuum cross $\mathcal{N}_x$, then it carry positive entropy.

Suppose we consider U a continuum that is contained in $\mathcal{N}_x'$ and contains the points z and w. By what was proved before, this continuum cross $\mathcal{N}_x$, thus it carries positive entropy. Hence, z and w are not zero-entropy equivalent. With that, we can conclude that $0 < h(f, f^{n_k}(D)) = h(f, D)$.

Consider now $N' \in \mathbb{N}$ such that $d_H(C_{n_i}, C) < \delta$. Thus, by the previous conclusion, $0 < h(f, C_{n_i})$, which is a contradiction. Thus, $h(f, C) = 0$, and x and y are zero-entropy equivalent, and the zero-entropy equivalence relation is upper-semicontinuous. □

The space $\widetilde{S}$ can be made into a compact metric space if we consider the distance $\widetilde{d}(\widetilde{x}, \widetilde{y}) := d_H(\widetilde{\pi}^{-1}(\widetilde{x}), \widetilde{\pi}^{-1}(\widetilde{y}))$, for all $\widetilde{x}, \widetilde{y} \in \widetilde{S}$.

3.1.6 Corollary. *Every zero-entropy equivalence class have at least one point of* Λ.

Proof. Since the zero-entropy equivalence class is upper semicontinuous, then each zero-entropy equivalence class is closed. Note that if a zero-entropy equivalence class does not contain a point in Λ, then it also does not contain any point in $\overline{W^s(\Lambda) \cup W^u(\Lambda)}$. Thus, if a zero-entropy equivalence class does not contain a point in Λ, then it is a connected component of $S \backslash \overline{W^s(\Lambda) \cup W^u(\Lambda)}$. However, each connected component of $S \backslash \overline{W^s(\Lambda) \cup W^u(\Lambda)}$ is an open set. Thus, a connected component of $S \backslash \overline{W^s(\Lambda) \cup W^u(\Lambda)}$ is either the empty set, or S. Both leads to contradictions. □

3.2 Equivalence Classes for the Zero-Entropy Equivalence

Consider B a zero-entropy equivalence class. Let's denote by $B_{\text{Per}} = B \cap \text{Per}(\Lambda)$ and $B_{\text{NPer}} = (B \cap \Lambda)\backslash\text{Per}(\Lambda)$.

3.2.1 Lemma. *If $B_{Per} = \emptyset$, then B_{NPer} is finite. Moreover, the number of elements of B_{NPer} is either one, or an even number. In other words, B is a polygon.*

Proof. Consider that B_{NPer} is infinite. Note that B_{NPer} is not f-invariant. Consider p an accumulation point of B. Denote by $(x_i^n)_{n,i\in\mathbb{N}}$ sequences contained in $f^n(B)$ and converging to $f^n(p)$. Since each $f^n(B)$ is a closed set, then $f^n(p) \in f^n(B)$. The sequence $(x_n^n)_{n\in\mathbb{N}}$ converge to a periodic point q. Thus, the sequence $(f^{-n}(x_n^n))_{n\in\mathbb{N}} \subset B$ converge to a periodic point q' contained in B. But we assume that $B_{\text{Per}} = \emptyset$.

Since B_{NPer} is non empty, then B_{NPer} is finite. Suppose first that B_{NPer} contain an inaccessible point. Then, following the proof of Theorem 3.1.5, B is a singleton and contain just this point. Suppose now that B_{NPer} contain just boundary points.

If B_{NPer} contain a strictly boundary point, then, there must exist a distinct strictly boundary point in B_{NPer}, and B_{NPer} contain just these two point. This is because the unique continuum that carry zero entropy is the arc that have at each extremities one strictly boundary point contained in B_{NPer}. In other words, its is an inaccessible arc.

If B_{NPer} contain just corners points, then we can associate to B_{NPer} a closed arc chain. This is because, each corner is zero-entropy related with two extreme points of its two access. Clearly these two extreme points are also contained in B_{NPer}. Since the access of a corner have oposite nature, the union of the points in B_{NPer} and the arcs that have the corners of B_{NPer} as extremities, forms a closed arc chain. By Proposition 2.4.1, the number of arcs is even, thus the number of points in B_{NPer} is also even. □

If $B_{\text{Per}} \neq \emptyset$, then we can divide $B_{\text{Per}} = B_{\text{BPer}} \cup (B_{\text{Per}} \backslash B_{\text{BPer}})$, where B_{BPer} is the set of all boundary periodic points contained in B_{Per}.

3.2.2 Lemma. *a) If $B_{Per}\backslash B_{BPer} \neq \emptyset$, then B is a singleton, and the unique element of B is an inaccessible periodic point.*

b) If $B_{Per} = B_{BPer}$, then B_{Per} is a union of the vertices of the circular graphs induced by the contact between boundary periodic points. In other words, $B = C(\mathcal{G})$, where $\mathcal{G}$ is a circular graph.

Proof. a) The point contained is B_{Per} is inaccessible. Hence, it is not zero-entropy equivalent with any other point. Thus, B_{Per} have just this inaccessible periodic point.

b) Consider $x \in B_{\text{BPer}}$ and consider that x is in contact to a boundary periodic point y.

If this contact is due to a tiedness between boundary separatrices of x and y, then the union of the boundary periodic points x and y, its free separatrices, and the sources and sinks associated to the free separatrices is a continuum containing x and y.

If this contact is due to a sewness between the boundary separatrices W^s and W^u, the the union of the boundary periodic points x and y, its free separatrices, and the union of $\alpha(W^s, W^u)$ is a continuum contained x and y.

Both continua described in the previous paragraphs carry zero entropy. Thus, if x is a point of B_{Per} and is in contact to a boundary periodic point y, then $y \in B_{\text{Per}}$. Thus, B_{Per} is a union of the vertices of the circular graphs induced by the contact between boundary periodic points.

□

If for a zero-entropy equivalence class B holds $B_{\mathrm{Per}} = B_{\mathrm{BPer}}$, consider $B_{\mathrm{source/sinks}}$ the set of all associated sources/sinks to each free separatrices of points in B_{BPer}. Note that in this case the set $B_{\mathrm{source/sinks}}$ is non-empty.

We can describe all the equivalence classes for the zero-entropy equivalence relation.

3.2.3 Theorem. *Consider $f : S \to S$ a shoe diffeomorphism on a closed surface S. The only possible equivalence classes for the zero-entropy equivalence are the following:*

1. *An inaccessible point or an inaccessible arc.*
2. *A polygon $\psi \cup P(\psi)$, where ψ is a closed arc chain.*
3. *The continua $C(\mathcal{G})$, where $\mathcal{G}$ is a circular graph and $C(\mathcal{G})_{source/sinks}$ have just one element.*
4. *Each $\overline{\mathcal{S}} = \mathcal{S} \cup (\cup_{i=1}^{t_S} C(\mathcal{G}_i))$, where $\mathcal{S}$ is a connected component of $S \backslash S(\Lambda)$, and each $\mathcal{G}_i$ is a circular graph.*

Proof. Denote by B a zero-entropy equivalence class.

By Lemma 3.2.1 and item (a) of Lemma 3.2.2, if B have one point of Λ, then B is an inaccessible point, that can be either periodic or non-periodic. Moreover, if B have only two non-periodic strictly boundary points, then B is an inaccessible arc. If B have an even number of non-periodic corners, then B is a polygon.

Suppose now that $B_{\mathrm{Per}} = B_{\mathrm{BPer}}$. If $B_{\mathrm{sources/sinks}}$ is a singleton, then all the contacts between B_{BPer} is due to a tied relation between the boundary separatrices of the points of B_{BPer}. Moreover, all the points in B_{BPer} are strictly boundary periodic point that are vertices of the same circular graph $\mathcal{G}$. The zero-entropy class B is the continuum $C(\mathcal{G})$.

Finally, suppose that $B_{\mathrm{Per}} = B_{\mathrm{BPer}}$ and $B_{\mathrm{sources/sinks}}$ have more than one point. By item (b) of Lemma 3.2.2, consider $\{\mathcal{G}_i : 1 \leq i \leq t_B\}$, where t_B is the number of circular graphs in which the elements of B_{BPer} are vertices. Note that each continuum $C(\mathcal{G}_i)$ carry zero entropy, and $\cup_{i=1}^{t_B} C(\mathcal{G}_i) \subset B$.

Consider $\overline{\Delta(\Lambda)}$ the compact surface with boundary given by the Proposition 2.6.3. The set $B_{\mathrm{sources/sinks}}$ is contained in connected components of $\partial\overline{\Delta(\Lambda)}$. Denote by B_Δ the set of all connected components of $\partial\overline{\Delta(\Lambda)}$ that contain the points of $B_{\mathrm{sources/sinks}}$. There exist a connected component $\mathcal{S}'$ of $S\backslash\overline{\Delta(\Lambda)}$ where the connected components of $\partial\overline{\mathcal{S}'}$ are exactly the same elements of B_Δ. Otherwise, the non-existence of such connected component would imply the existence of $a, b \in B_{\mathrm{sources/sinks}}$ such that there is no continuum containing a and b, an absurd, since $B_{\mathrm{sources/sinks}} \subset B$.

As we observe early, the dynamic of f restricted to $\overline{\mathcal{S}'}$ is a Morse-Smale dynamic. Thus, $\overline{\mathcal{S}'}$ carry zero entropy, and $\overline{\mathcal{S}'} \subset B$. Consider $\mathcal{S}$ the connected component of $S\backslash S(\Lambda)$ that contains $\mathcal{S}'$. By item (b) of Proposition 2.6.3, each connected component of $\mathcal{S}\backslash\mathcal{S}'$ is an f-invariant set that is homeomorphic to $\mathbb{R}^2$ and the dynamic restricted to this connected component is conjugate to a translation. Thus, the compact set $\overline{\mathcal{S}}$ carry zero entropy and $\overline{\mathcal{S}} \subset B$. By a connectedness argument, we can show that $B = \mathcal{S} \cup (\cup_{i=1}^{t_B} C(\mathcal{G}_i))$. □

3.3 Neighborhoods of the Equivalence Classes

An *alternate curve* is a curve formed by alternating s-intervals and u-intervals. Arc chains are special type of alternate curves. A *regular* neighborhood of a zero-entropy equivalence class

B is a closed set $\mathcal{O} \subset S$, such that $B \subset \mathrm{Int}(\mathcal{O})$ and the boundary of B is formed by a finite number of finite alternate closed curves. A *minimal* neighborhood is a regular neighborhood that minimize the number of sides of the alternate curve that formed each connected component of the boundary.

3.3.1 Proposition. *Every zero-entropy equivalence class have a minimal regular neighborhood.*

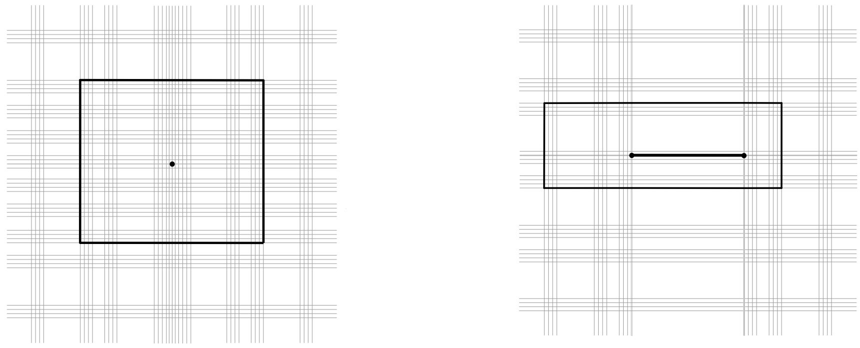

Figure 3.1: *Minimal regular neighborhoods of an inaccessible point and an inaccessible arc.*

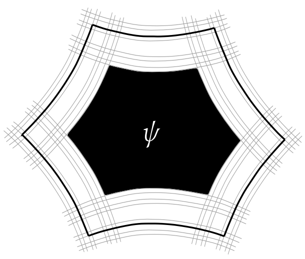

Figure 3.2: *Minimal regular neighborhood of a polygon ψ.*

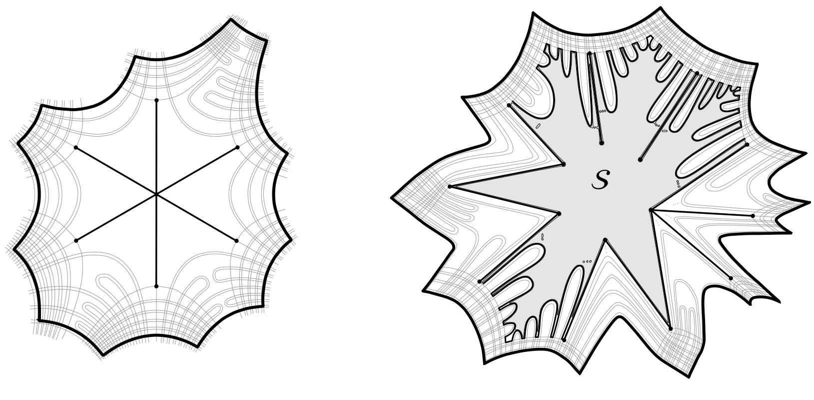

Figure 3.3: *Minimal regular neighborhoods.*

3.4 Final Theorem

3.4.1 Theorem. *Consider $f : S \to S$ a shoe diffeomorphism on a closed surface S. The space $\widetilde{S}$ is a finite cactoid and $\widetilde{f}$ is a generalized pseudo-Anosov homeomorphism. Moreover, $\widetilde{\pi} : (S, f) \to (\widetilde{S}, \widetilde{f})$ is a semi-conjugacy, and $h(f) = h(\widetilde{f})$.*

Proof. Let's first describe the topology of $\widetilde{S}$. Consider the case when $S \setminus S(\Lambda)$ does not have any connected components. In this case, for all zero-entropy equivalence class B, there is only one cylinder of $S \setminus B$ approaching B. By Theorem 1.4.1, the space $\widetilde{S}$ is an orientable closed surface and $\mathcal{B}_1(S) = \mathcal{B}_1(\widetilde{S})$. Note that $\mathcal{B}_1(S) < \mathcal{B}_1(\widetilde{S})$ is only possible if $S \setminus S(\Lambda)$ have connected components.

Consider the case when $S \setminus S(\Lambda)$ have connected components. Denote by $\mathcal{S}_j$, for $1 \leq j \leq t$, each connected component of $S \setminus S(\Lambda)$, where t is a positive integer that represent the number of connected components of $S \setminus S(\Lambda)$. As we see in Theorem 3.2.3, each $\overline{\mathcal{S}_j}$ is a zero-entropy equivalence class, and $\overline{\mathcal{S}_j} \setminus \mathcal{S}_j = \cup_{i=1}^{t_{\mathcal{S}_j}} C(\mathcal{G}_i)$, where $\mathcal{G}_i$ is a circular graph and $t_{\mathcal{S}_j}$ is the number cylinders of $S \setminus \overline{\mathcal{S}_j}$ that approach to $\overline{\mathcal{S}_j}$. Note that $\mathcal{B}_1(\overline{\Delta(\Lambda)}) = 2g_\Lambda + b_\Lambda - 1$, where g_Λ is the genus of $\overline{\Delta(\Lambda)}$ and b_Λ is the number of connected component of the boundary of $\overline{\Delta(\Lambda)}$. The number $b_\Lambda = \sum_{j=1}^{t} t_{\mathcal{S}_j}$.

Denote by $\mathcal{S}(\Lambda)$ the union of the space $\overline{\Delta(\Lambda)}$ and topological disks, where the boundary of each topological disk is some connected component of $\overline{\Delta(\Lambda)}$. We can "glue" this topological disk respecting the dynamic. Thus, suppose that the dynamic on each topological disk is a Morse-Smale dynamic in the closure of the disk and the sources and sinks associated to each connected component of $\overline{\Delta(\Lambda)}$ are sources and sinks in the boundary of the disk. The map $\overline{f}$ is defined as f in all points in $\overline{\Delta(\Lambda)}$ and some Morse-Smale dynamic in the glued topological disks. Note that $\mathcal{S}(\Lambda)$ is a closed surface and $(\mathcal{S}(\Lambda), \overline{f})$ is a shoe dynamic.

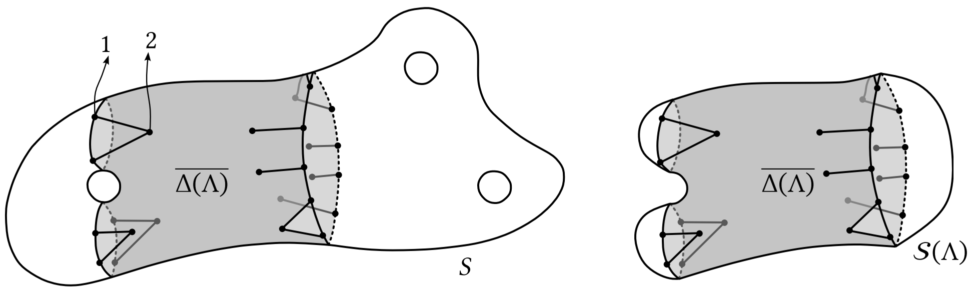

Figure 3.4: *The number* 1 *is a source/sink associated to a free separatrix. The number* 2 *is a boundary periodic point associated to* Λ.

With these changes, for each zero-entropy equivalence class B, there exist just one cylinder of $\mathcal{S}(\Lambda) \setminus B$ approaching to B. By the Theorem 1.4.1, $\widetilde{\pi}(\mathcal{S}(\Lambda))$ is a closed surface and $\mathcal{B}_1(\widetilde{\pi}(\mathcal{S}(\Lambda))) \leq \mathcal{B}_1(S)$.

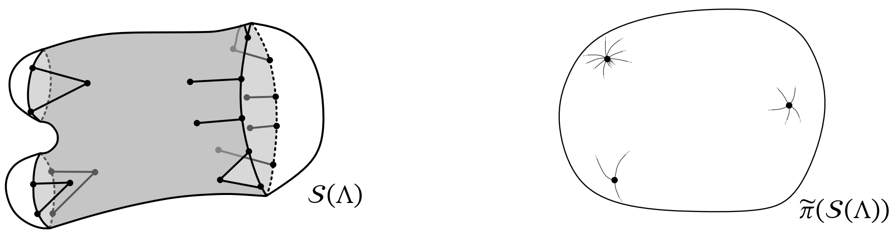

Figure 3.5: *In the right image, we represent the accumulated singular points. The arcs leaving theses points is an attempt to represent the stable and unstable manifold of these points. For an accurate representation it would be necessary to know what are the zippers and humps of each contact between the periodic points.*

Consider $C_0 = \widetilde{\pi}(\mathcal{S}(\Lambda))$. We construct a finite cactoid via the sequence of spaces $C_1, \dots, C_{t_{\mathcal{S}_1}}$ (like in the Theorem 1.4.2), obtaining the finite cactoid $C_{t_{\mathcal{S}_1}}$, where all $C(\mathcal{G}_i)$, for $1 \leq i \leq t_{\mathcal{S}_1}$, are identified to a point. From the finite cactoid $C_{t_{\mathcal{S}_1}}$ we can obtain, via the sequence of spaces $C_{t_{\mathcal{S}_1}+1}, \dots, C_{t_{\mathcal{S}_1}+t_{\mathcal{S}_2}}$, obtaining the finite cactoid $C_{t_{\mathcal{S}_1}+t_{\mathcal{S}_2}}$, where all $C(\mathcal{G}_i)$, for $1 \leq i \leq t_{\mathcal{S}_2}$, are identified to a point. If we do the same argument until the last $t_{\mathcal{S}_t}$ we will obtain a finite cactoid $C_{t_{\mathcal{S}_1}+\dots+t_{\mathcal{S}_t}} = \widetilde{S}$.

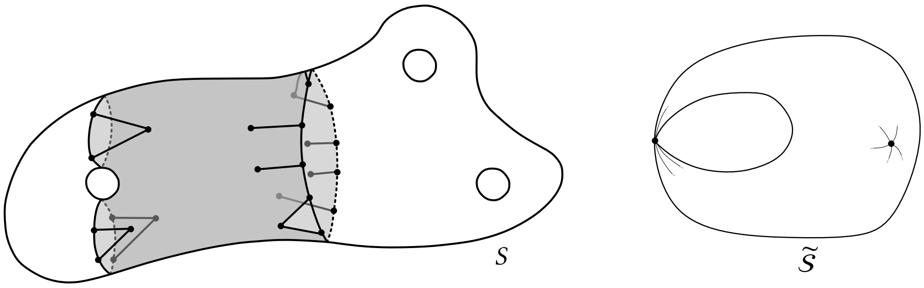

Figure 3.6: *As in the previous image, the right image is just an attempt to represent the space after the collapse by the zero-entropy equivalence relation.*

For simplicity, we will denote by W the space $\overline{W^s(\Lambda) \cup W^u(\Lambda)}$. Consider now a rectangle $N \subset S$ and l_N^u and l_N^s are the length and width, measure by ν^s and ν^u, of N. The image $\widetilde{\pi}(N \cap W)$ is diffeomorphic to $[0, l_N^s] \times [0, l_N^u]$, by a diffeomorphism $\varphi : \widetilde{\pi}(N \cap W) \to [0, l_N^s] \times [0, l_N^u]$. Let's give an explanation how to obtain the previous assertion. Firstly, note that, for constants $a, b \in \mathbb{R}$, the set $[0, a] \times [0, b] \subset \mathbb{R}^2$ can be parametrized by the set $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 = a, x_3 + x_4 = b \text{ and } 0 \leq x_i \leq a, \text{ for } 1 \leq i \leq 4\}$. For each $p \in N \cap W$, denote by $l_1^{s/u}(p)$ and the $l_2^{s/u}(p)$ the measures of each s/u-separatrix of $W_N^{s/u}(p)$. Thus, define $\vartheta : N \cap W \to [0, l_N^s] \times [0, l_N^u]$ as $\vartheta(p) := (l_1^s(p), l_2^s(p), l_1^u(p), l_2^s(p))$. The map ϑ is a differentiable projection[2] and sends $W^s(\Lambda) \cap N$ to horizontal lines, and sends $W^u(\Lambda) \cap N$ in vertical lines. See Figure 3.7

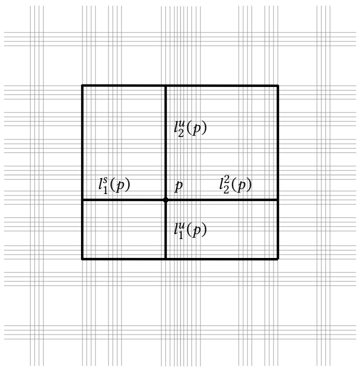

Figure 3.7: *Each $l_1^{s/u}(p)$ and the $l_2^{s/u}(p)$ of a point $p \in N \cap W$.*

Two points $p, p' \in N \cap W$ are zero-entropy equivalence if, and only if, $\vartheta(p) = \vartheta(p')$. Since the only type of zero-entropy equivalence class contained in N are inaccessible points, inaccessible arcs, and 2-gons (or just rectangles with interior disjoint of $W^s(\Lambda) \cup W^u(\Lambda)$), then each zero-entropy equivalence class contained in N is one-to-one, via a map φ such that $\varphi \circ \widetilde{\pi} = \vartheta$, to the space $[0, l_N^s] \times [0, l_N^u]$. Since the map $\widetilde{\pi}|_N$ and ϑ are differentiable at all points, then ψ is also differentiable. Furthermore, $\vartheta * (\nu^s \times \nu^u)$ is the Lebesgue measure restricted to $[0, l_N^s] \times [0, l_N^u]$.

By the existence of a finite Markov partition $\{N_1, \dots, N_m\}$ for Λ, we can prove that $\widetilde{\pi}(\mathcal{S}(\Lambda))$ can be foliated by the image of $(\cup_i N_i) \cap W^{s/u}(\Lambda)$ by the map ϑ. These two foliation of $\widetilde{S}$ are measurable transverse foliations, where the measures are given by $\vartheta * \nu^{s/u}$.

Let's describe the possible types of singularities of $\widetilde{\widetilde{f}}$, and finally prove that $\widetilde{\widetilde{f}}$ is a generalized pseudo-Anosov homeomorphism.

Every minimal neighborhood of a non-trivial n-polygon ψ is mapped to a minimal neighborhood

---

[2] Surjective and continuous.

of a n-gon singularity. Thus, every isolated singularity can be modeled by a n-gon as in the Figure 1.1.

It remains to show what happens to a minimal neighborhood N of $\mathcal{S} \cup C(\mathcal{G}_{\mathcal{S}})$, where $\mathcal{S}$ is a connected component of $\mathcal{S}(\Lambda)\backslash S(\Lambda)$. Consider γ an s/u-arc contained in N. Then the positive/negative iterates of γ by $\overline{f}$ converges, in the Hausdorff distance, to $C(\mathcal{G}_{\mathcal{S}})$. Hence, every polygon contained in N must converge, in the Hausdorff distance, to $C(\mathcal{G}_{\mathcal{S}})$ in the future, or in the past. Thus, there is only one point $p \in \widetilde{\pi}(N)$ such that all the other singularities contained in $\widetilde{\pi}(N)$ converge to p for positive, or negative iterate of $\overline{f}$. In other words, the point $\widetilde{\pi}(\mathcal{S} \cup C(\mathcal{G}_{\mathcal{S}}))$ is an accumulated singularity of $\overline{f}$. Since we have just a finite number of connected components of $\mathcal{S}(\Lambda)\backslash S(\Lambda)$, then there is just a finite number of accumulated singularities. Note that will be the accumulated points that will be identificated to obtain the finite cactoid $\widetilde{S}$.

With all that, we proved that $\overline{f} : \widetilde{\pi}(\mathcal{S}(\Lambda)) \to \widetilde{\pi}(\mathcal{S}(\Lambda))$ is a generalized pseudo-Anosov of a closed surface $\widetilde{\pi}(\mathcal{S}(\Lambda))$. Proceeding as in Theorem 1.4.2, we will identify the accumulated points to obtain a generalized pseudo-Anosov homeomorphism in the finite cactoid $\widetilde{S}$.

Consider that $\widetilde{S}$ is (n, ϵ)-spanned by a finite set $F \subset \widetilde{S}$, containing $r(n, \epsilon, \widetilde{S})$ elements. Hence, the family of compact sets $\widetilde{\pi}^{-1}(B_{\widetilde{d}}(\widetilde{x}, \epsilon))$, where $\widetilde{x} \in \widetilde{S}$ and $B_{\widetilde{d}}(\widetilde{x}, \epsilon)) = \{\widetilde{y} \in \widetilde{S} : \widetilde{d}(\widetilde{x}, \widetilde{y}) \leq \epsilon\}$, covers the set S. Thus, $D(n, \epsilon, S) \leq r(n, \epsilon, \widetilde{S})$, what implies that

$$h(f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln D(n, \varepsilon, S) \leq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \ln r(n, \varepsilon, \widetilde{S}) = h(\widetilde{f}).$$

However, since $\widetilde{\pi}$ is a semi-conjugacy, we have that $h(\widetilde{f}) \leq h(f)$. This, proves that $h(f) = h(\widetilde{f})$. □

As we wrote in the introduction, the previous theorem was inspired by the Theorem 8.3.1 of [BL98]. It state that if the diffeomorphism f does not have any 2-gon, or in its terminology, the diffeomorphism f does not admit *impasses*, then the generalized pseudo-Anosov is in fact a pseudo-Anosov homeomorphism on a compact surface.

Note that by the Theorem 3.4.1, if there is no 2-gon, then there is no stuck zipper and there is not possible any sewn between boundary separatrices. Hence, for all circular graph $\mathcal{G}$, the continuum $C(\mathcal{G})$ is formed just by boundary periodic point, free separatrices and sources and sinks associated to each undone zipper. Finally, the point of $\widetilde{S}$ that $C(\mathcal{G})$ is sent will be a n-gon, where n depends on how many stable and unstable free separatrices is contained in $C(\mathcal{G})$. Furthermore, the dynamic $(\widetilde{f}, \widetilde{S})$ is a pseudo-Anosov homeomorphism, as in the result obtained by Bonatti and Jeandenans in [BL98].

Let's remember that we are supposing that Λ have topological dimension 0. Le's suppose now that the basic piece Λ have dimension 1, i.e. is a hyperbolic attractor/repeller. Hence, $\Lambda = W^u(\Lambda)$ if Λ is an attractor, and $\Lambda = W^s(\Lambda)$ if Λ is a repeller. Moreover, $W^s(\Lambda)$ is a foliation, if Λ is an attractor, and $W^u(\Lambda)$ is a foliation, if Λ is a repeller. Note that the existence of impasses implies that both stable and unstable laminations are not foliations. This because, the an impass occurs in the "gaps" between the leaves of the laminations. If the laminations does not have any "gap", i.e. is a foliation, then it is not possible the existence of impasses. By what was discuss in the previous paragraph, the dynamic $(\widetilde{f}, \widetilde{S})$ is also a pseudo-Anosov homeomorphism on a closed surface.

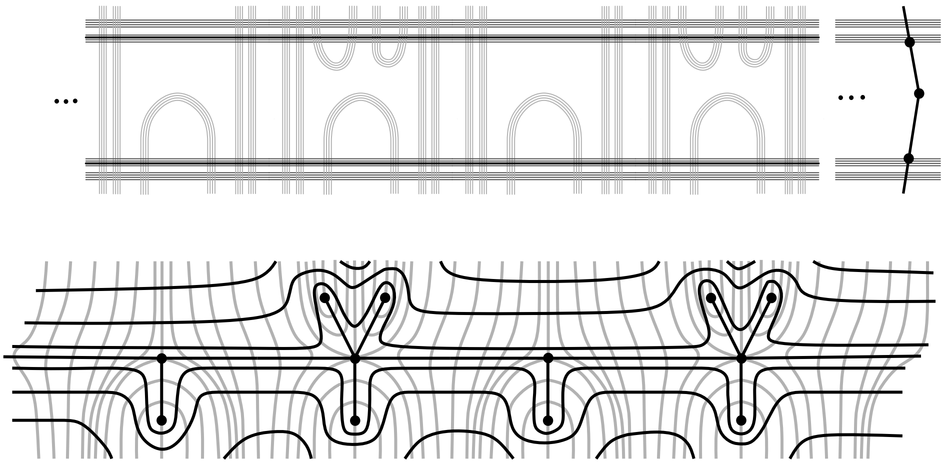

Figure 3.8: *The collapsing of a stuck zipper.*

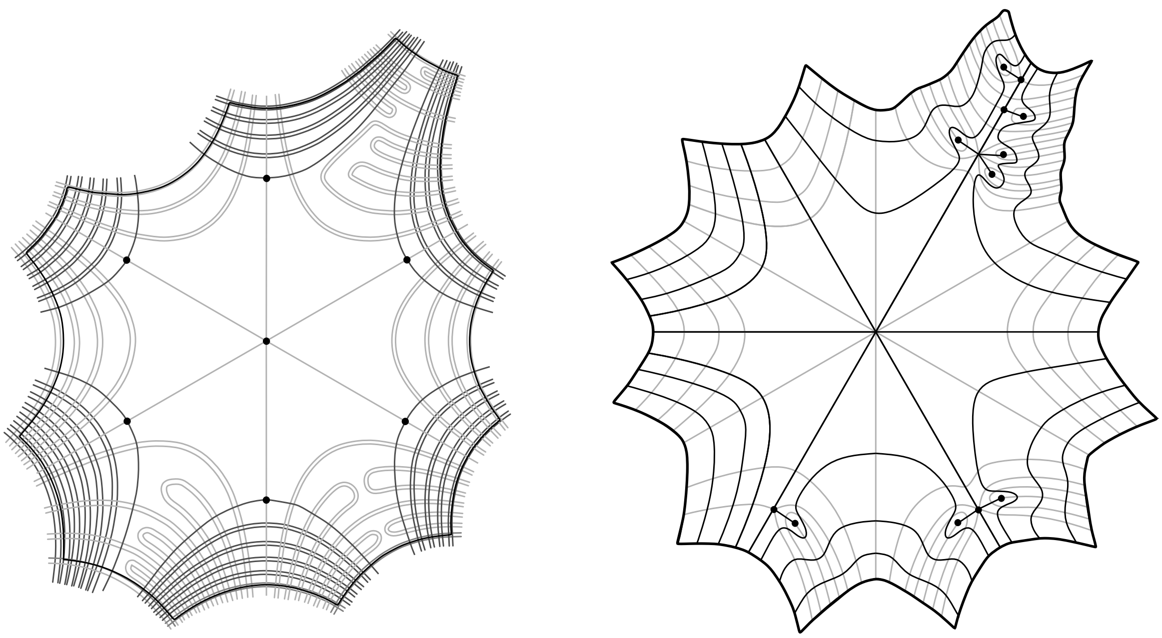

Figure 3.9: *The collapsing of a zero-entropy equivalence class of type (3).*

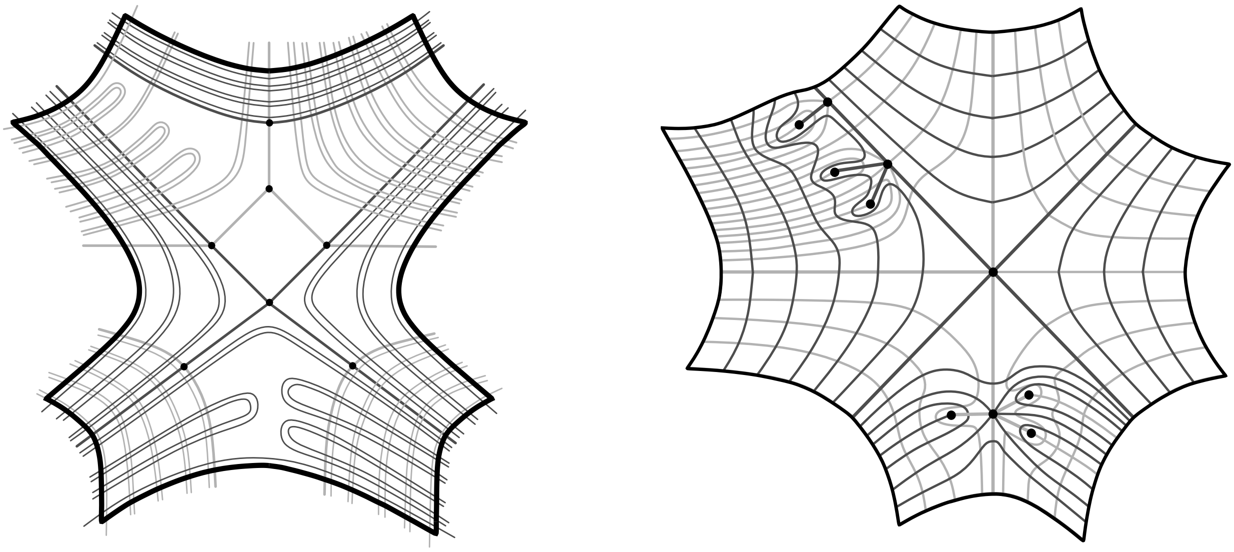

Figure 3.10: *The collapsing of a zero-entropy equivalence class of type (4). Note that all the contacts are due to a tiedness.*

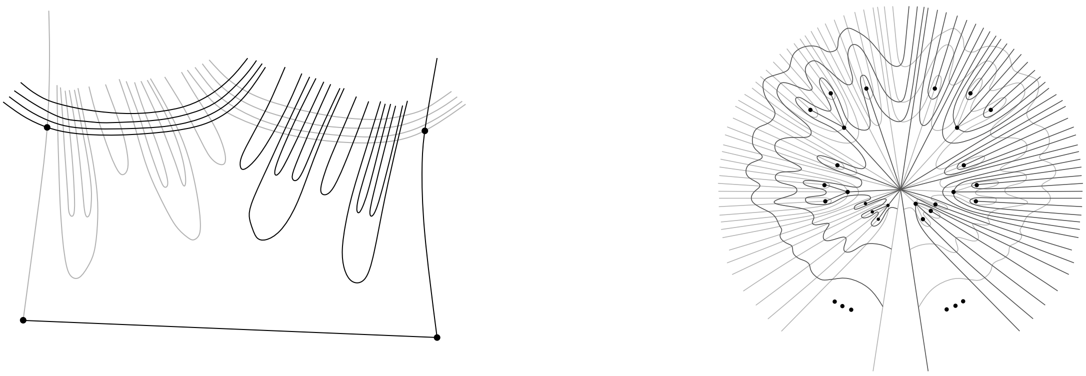

Figure 3.11: *The collapsing of humps.*

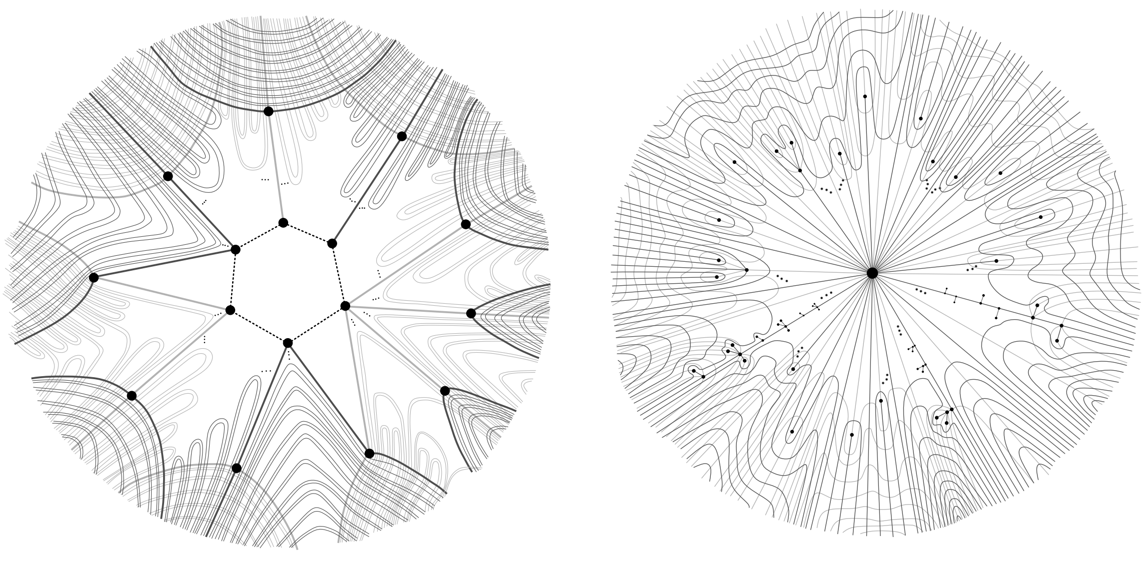

Figure 3.12: *The collapsing of a zero-entropy equivalence class of type (4).*

Bibliography

[Bar12] L. Barreira. *Ergodic Theory, Hyperbolic Dynamics and Dimension Theory.* Springer-Verlag Berlin Heidelberg, 2012.

[BL98] C. Bonatti and R. Langevin. *Difféomorphismes de Smale des surfaces.* Number 250 in Astérisque. Société mathématique de France, 1998.

[Bow70] R. Bowen. Markov partitions for axiom a diffeomorphisms. *American Journal of Mathematics*, 92(3):725–747, 1970.

[Bow74] R. Bowen. Some systems with unique equilibrium states. *Mathematical Systems Theory*, 8(3):193–202, September 1974.

[Bow04] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Springer, 1ª edition, 2004.

[Dan78] A. Dankner. On smale's axiom a dynamical systems. *The Annals of Mathematics*, 107(3):517, May 1978.

[dC05] A. S. de Carvalho. Extensions, quotients and generalized pseudo-anosov maps. *Graphs and patterns in mathematics and theoretical physics*, 73:315–338, 2005.

[dCP03] A. S. de Carvalho and M. Paternain. Monotone quotients of surface diffeomorphisms. *Mathematical Research Letters*, 10(5):603–619, 2003.

[DGS76] M. Denker, C. Grillenberger, and K. Sigmund. *Ergodic Theory on Compact Spaces.* Springer Berlin Heidelberg, 1976.

[DM68] P. H. Doyle and D. A. Moran. A short proof that compact 2-manifolds can be triangulated. *Inventiones Mathematicae*, 5(2):160–162, June 1968.

[GMP16] V. Grines, T. Medvedev, and O. Pochinka. *Dynamical Systems on 2- and 3-Manifolds.* Springer International Publishing, 2016.

[HK95] B. Hasselblatt and A. Katok. *Introduction to the Modern Theory of Dynamical Systems.* Cambridge University Press, April 1995.

[Kur79] M. Kurata. Markov partitions of hyperbolic sets. *Journal of the Mathematical Society of Japan*, 31(1):39–51, 1979.

[Mal56] B. D Malan. An engraved flake from Muden, Natal. *The South African Archaeological Bulletin*, 11(43):63–65, 1956.

[Mañ82] R. Mañé. An ergodic closing lemma. *Annals of Mathematics*, 116, 11 1982.

[Mañ87] R. Mañé. A proof of the c1 stability conjecture. *Publications mathématiques de l'IHÉS*, 66(1):161–210, December 1987.

[Mat68] J. Mather. Characterization of anosov diffeomorphisms. *Indag. Math*, 30:479–483, 1968.

[NP73] S. Newhouse and J. Palis. Hyperbolic nonwandering sets on two-dimensional manifolds. In *Dynamical Systems*, pages 293–301. Academic Press, 1973.

[Pal69] J. Palis. On morse-smale dynamical systems. *Topology*, 8(4):385–404, September 1969.

[Pal87] J. Palis. On the C^1 Ω-stability conjecture. *Publications Mathématiques de l'IHÉS*, 66:211–215, 1987.

[Per07] O. Perron. Zur theorie der matrices. *Mathematische Annalen*, 64(2):248–263, June 1907.

[RdM82] A. A. G. Ruas and W. C. de Mello. *Atratores hiperbólicos de codimensão um e classes de isotopia em superfícies.* PhD thesis, 1982.

[RS38] J. H. Roberts and N. E. Steenrod. Monotone transformations of two-dimensional manifolds. *The Annals of Mathematics*, 39(4):851, October 1938.

[Shu86] M. Shub. *Global Stability of Dynamical Systems.* Springer, 1986.

[Sma] S. Smale. The Ω-Stability Theorem. In *The Collected Papers of Stephen Smale*, pages 759–767. World Scientific Publishing Co. Pte. Ltd. and Singapore University Press.

[Sma67] S. Smale. Differentiable dynamical systems. *Bull. A.M.S*, 73:747–817, 1967.

[Sta67] K. Stacy. *On the Hausdorff hyperspace of a compact metric space.* PhD thesis, 1967.

[ZP95] A. Yu. Zhirov and R. V. Plykin. On the relationship between one-dimensional hyperbolic attractors of surface diffeomorphisms and generalized pseudo-anosov diffeomorphisms. *Mathematical Notes*, 58(1):779–781, July 1995.