Quanto option pricing

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Precificação de opções quanto

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- Prof. Dr. Pedro Luiz Valls Pereira – FGV/São Paulo School of Economics;
- Prof. Dr. Ricardo Ratner Rochman – FGV/São Paulo School of Economics.

Abstract

This text aims to explore the quanto options pricing topic, as surprisingly little research has focused on it, despite its relevance. For this purpose, we propose a non-parametric pricing approach and compare it with the pricing of quanto options in a bi-dimensional Heston model framework, proposed by (Dimitroff, et al., 2009), and with the Black-Scholes framework, widely adopted by practitioners. The non-parametric approach aims to be as flexible as possible so that it adapts to a wider range of dependence relations among relevant variables when compared to parametrical models.

Keywords: Derivatives pricing, stochastic processes, options.

Resumo

Este texto tem por objetivo explorar o tópico de precificação de opções quanto, dado que este tópico tem sido objeto de pouca pesquisa, apesar de sua relevância. Para esse propósito, propomos uma abordagem não paramétrica de precificação e a comparamos com a precificação de opções quanto na abordagem de um modelo de Heston bidimensional, proposto por (Dimitroff, et al., 2009), e com a abordagem de Black-Scholes, comumente adotada na prática. A abordagem não paramétrica pretende ser tão flexível quanto possível, e ser adaptável a uma gama mais abrangente de relações de dependência entre as variáveis relevantes para a precificação, quando comparada com modelos paramétricos.

Palavras-Chave: Precificação de derivativos, processos estocásticos, opções.

Table of Contents

1 Introduction

The aim of this text is to present and compare three quanto option pricing models: the non-parametric model (proposed by the author), the common practitioners model (based on the Black-Scholes standard framework), and the Heston bi-dimensional model, from (Dimitroff, et al., 2009).

The approach based on the Black-Scholes standard framework is based on the standard "asset prices follow a geometric Brownian motion" and "volatility is constant" arguments. Stochastic volatility models, such as the Heston bi-dimensional model, are used to relax the "volatility is constant" assumption and better represent real-world volatility smile, in the quanto option pricing context. Nevertheless, as the approach based on the Black-Scholes standard framework provides a computationally low cost closed formula to price quanto options, while, for example, the Heston bi-dimensional approach demands a computationally higher cost simulation procedure, practitioners often resort to the former approach.

In the quanto option context, the dependence relation calibration among relevant variables, can considerably impact the pricing. The approach based on the Black-Scholes standard framework and the Heston bi-dimensional model use a simple constant correlation in order to address this issue. The non-parametric model, proposed by the author, intends to provide a flexible framework to define the dependence relation between the market variables used in the quanto option pricing. Besides, the non-parametric model, as it is the case in the Heston bi-dimensional model, is capable to adapt to a non-constant volatility smile. However, the non-parametric model is computationally expensive.

For the purpose of comparing the three quanto option pricing frameworks, we detail, through the text, the main stochastic calculus and quantitative finance results that support those models. In Section 2, we state some fundamental stochastic calculus

results - on the change of probability measure and Brownian motion topics for example - which will be used to derive a variety of equations throughout the text. Then, in Section 3, we present the main option pricing concepts, such as non-arbitrage and risk-neutral pricing. These concepts will as well be used in the derivation of the two quanto option pricing frameworks, mainly in the Black-Scholes one. In Section 4, we approach the change of numéraire, which is the main building block in the derivation of any quanto option pricing formula. In Section 5, we derive the Black-Scholes framework. In Section 6, we state how it is used by practitioners and provide the arguments sustaining its use. In Section 7, we present the Heston bi-dimensional framework, from (Dimitroff, et al., 2009). In Section 8 we propose a non-parametric approach for quanto option pricing. Finally, in Section 9, we compare all three models and analyze their pricing differences.

2 Stochastic calculus

In this text, we will model market variables such as stock prices and stock volatilities, with the help of stochastic processes. Stochastic calculus plays the role of a tool to solve pricing problems. (Meucci, 2011) provides a straightforward overview on the role of stochastic processes in derivatives pricing.

2.1 Brownian motion

Consider a variable, say, a stock price, for which the one-year change is given by a normal distribution with mean zero and variance equal to 1, namely $N(0,1)$. Assume that the changes are independent and identically distributed (i.i.d.). Hence, the two-year change of the stock price will be a sum of two i.i.d. normally distributed random variables, $N(0,1)$, which gives, as mean and variance are additive and

a sum of normally distributed random variables is also normally distributed, $N(0,2)$. Thus, we can generalize and conclude that the Δt-year change is given by $N(0, \Delta t)$. The process we have just described is known as Brownian motion. It is a particular type of Markov process with increments normally distributed with mean zero and variance equal to one per unit of time. Brownian motion is a building block for every quantitative finance model, as it is used to model stock return processes. In this text we will denote the Brownian motion process by $W(t)$.

Brownian motion is a continuous time process, so time steps (Δt) are infinitesimal. Additionally, each increment $\Delta W\ (t) \sim N(0, \Delta t)$. Hence, before each infinitesimal step Δt the result of a random variable, namely the increment, is determined - this result enables the processes to take one step ahead - then, the result of another random increment is determined, the process moves another infinitesimal step ahead, and so on. The sequence of results of the infinitely many increments from $t = 0$ to $t = t^* > 0$ determines the value of $W(t^*)$ and consequently, after this sequence of increments, $W(t^*)$ becomes known (non-random). As each increment is random, infinitely many sequences of increments are possible, that is, on each simulation one gets a different path realization.

We next present the definitions of concepts that are recurrently referenced in the text. In this text, $\mathbb{E}[\mathrm{X}]$ and $\mathbb{V}[\mathrm{X}]$ denote expectation and variance of a random variable X, respectively.

Definition 2.1. The probability space of a Brownian motion can be defined by a triple $(\Omega, \mathcal{F}, \mathbb{P})$. Ω is the sample space, that is, the set of infinitely many possible sequences of increments results. $\boldsymbol{\omega} = \omega_1\omega_2 \ldots$ denotes a generic element of Ω - ω_n is the result of the nth increment. As usual, $\mathcal{F}$ is a σ-algebra of Ω and $\mathbb{P}: \mathcal{F} \to [0,1]$ is a probability measure over $\mathcal{F}$. In this text, $\mathcal{F}(t)$ represents information available

upon time t, that is, all the increments of W from 0 to t are known and consequently $W(t)$ is not random.

Definition 2.2. Let X be a random variable defined on a nonempty sample space Ω. Let $\mathcal{F}$ be a σ-algebra of subsets of Ω. Let $\sigma(X)$ be the σ-algebra generated by X. If every set in $\sigma(X)$ is also in $\mathcal{F}$, we say that X is $\mathcal{F}$-measurable.

Definition 2.3. Let Ω be a nonempty set. Let T be a fixed positive number, and assume that for each $t \in [0, T]$ there is a σ-algebra $\mathcal{F}(t)$. Assume further that if $s \leq t$, then every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$. Then we call the collection of σ-algebras $\mathcal{F}(t)$, $0 \leq t \leq T$, a filtration.

Definition 2.4. Let Ω be a nonempty sample space equipped with a filtration $\mathcal{F}(t)$, $0 \leq t \leq T$. Let $X(t)$ be a collection of random variables indexed by $t \in [0, T]$. We say this collection of random variables is an adapted stochastic process if, for each t, the random variable $X(t)$ is $\mathcal{F}(t)$-measurable.

Definition 2.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, and let X be a random variable that is either nonnegative or integrable. The conditional expectation of X given $\mathcal{G}$, denoted $\mathbb{E}[X|\mathcal{G}]$, is any random variable that satisfies:

(i) (Measurability) $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable, and

(ii) (Partial averaging)

$$\int_A \mathbb{E}[X|\mathcal{G}]\,(\omega)\, d\mathbb{P}(\omega) = \int_A X\,(\omega)\, d\mathbb{P}(\omega), \tag{2.1.1}$$

for all $A \in \mathcal{G}$.

Proposition 2.1. (Unbiasedness of conditional expectation). $\mathbb{E}[X|\mathcal{G}]$ is an unbiased estimator of $\mathbb{E}[X]$.

Proof: Choosing $A = \Omega$ in (2.1.1), one finds:

$$\mathbb{E}\big[\mathbb{E}[X|\mathcal{G}]\big] = \mathbb{E}[X]. \tag{2.1.2}$$

Definition 2.6. A function $W(t)$ $(t \geq 0)$ is a Brownian motion if

(i) W is continous;

(ii)

$$W(0) = 0; \tag{2.1.3}$$

(iii) the increment of W in a small period of time Δt, namely $\Delta W \equiv W(t + \Delta t) - W(t)$ $(t \geq 0)$, is normally distributed and given by

$$\Delta W = \epsilon\sqrt{\Delta t}, \tag{2.1.4}$$

where $\epsilon \sim N(0,1)$;

(iv) and the increments ΔW are i.i.d.

Hence, from (2.1.4) it follows that

$$\mathbb{E}[\Delta W] = \sqrt{\Delta t}\,\mathbb{E}[\epsilon] = 0, \tag{2.1.5}$$

$$\mathbb{V}[\Delta W] = \Delta t\,\mathbb{V}[\epsilon] = \Delta t, \tag{2.1.6}$$

and from the fact that increments ΔW are i.i.d. (as per Definition 2.6), a Brownian motion is a Markov process.

The computation of the increment of W over a large interval T can be regarded as a sum of $N = \frac{T}{\Delta t}$ increments over small intervals Δt. Hence, according to 2.1.4:

$$W(t+T) - W(t) = \sum_{i=1}^{N} \epsilon_i \Delta t, \tag{2.1.7}$$

where $\epsilon_i \sim N(0,1)$, for all $i = \{1, \dots, N\}$.

Proposition 2.2. A Brownian motion is a Martingale.

Proof: Let $0 \leq s \leq t$, then

$$\begin{aligned}
\mathbb{E}[W(t)|\mathcal{F}(s)] &= \mathbb{E}[\big(W(t) - W(s)\big) + W(s)|\mathcal{F}(s)] && (2.1.8)\\
&= \mathbb{E}\big[\big(W(t) - W(s)\big)\big|\mathcal{F}(s)\big] + \mathbb{E}[W(s)|\mathcal{F}(s)] && (2.1.9)\\
&= \mathbb{E}[W(t) - W(s)] + W(s) && (2.1.10)\\
&= W(s). && (2.1.11)
\end{aligned}$$

(2.1.10) follows from the fact that once all the increments up to s are known, then $W(s)$ is not random.

(2.1.11) follows from (2.1.5) and (2.1.7), whence large increments have expected value equal to zero.

Definition 2.7. The covariation, $\langle f, g\rangle(T-t)$, of two adapted processes $f(t)$ and $g(t)$ is defined by (2.1.12). When $f = g$, $\langle f, f\rangle(T-t)$ is called quadratic variation.

$$\langle f, g\rangle(T-t) \equiv \lim_{\|\Pi \to 0\|} \sum_{j=0}^{n-1} \big(f(t_{j+1}) - f(t_j)\big)\big(g(t_{j+1}) - g(t_j)\big), \tag{2.1.12}$$

where $\Pi = \{t_0, t_1, \dots, t_n\}$ is a partition of $[t, T]$, $t \geq 0$.

Generalized Brownian motion

A random variable $X(t)$ following a generalized Brownian motion is defined by the stochastic differential equation:

$$dX(t) = \alpha(X, t)dt + \sqrt{V(X, t)}\, dW(t), \tag{2.1.13}$$

where α is known as drift rate and $\sqrt{V}$ is know as volatility. Both, α and V are allowed to be adapted processes.

Discretizing (2.1.13) and letting α and $\sqrt{V}$ be constants, one can gain some intuition on the meaning of this equation.

$$\Delta X = \alpha\, \Delta t + \sqrt{V}\, \epsilon\sqrt{\Delta t}. \tag{2.1.14}$$

The drift α represents the non-random part of the process, because if $\sqrt{V} = 0$, then $\mathbb{V}(\Delta X) = \mathbb{V}(\alpha\, \Delta t) = 0$. $\sqrt{V}$ adds randomness to the process, because if $\sqrt{V} \neq 0$ then $\mathbb{V}(\Delta X) = \mathbb{V}\big(\alpha\, \Delta t + \sqrt{V}\, \epsilon\sqrt{\Delta t}\big) = V\Delta t$.

The change in the X value, ΔX, is normally distributed:

$$\Delta X \sim N(\alpha\, \Delta t, V\Delta t). \tag{2.1.15}$$

Proposition 2.3. The covariation $\langle X_1, X_2\rangle(T-t)$ of two generalized Brownian motions $X_1(t)$ and $X_2(t)$ is given by

$$\begin{aligned}
\langle X_1, X_2\rangle(T-t) \quad &\equiv \quad \lim_{\|\Pi\to 0\|} \sum_{j=0}^{n-1} \big(X_1(t_{j+1}) - X_1(t_j)\big)\big(X_2(t_{j+1}) - X_2(t_j)\big) \\
&= \quad \rho \int_t^T \sqrt{V_1(X_1, t)V_2(X_2, t)}\ dt \\
&= \quad \int_t^T Cov[X_1, X_2]\ dt\,; \qquad (2.1.16)
\end{aligned}$$

where $\Pi = \{t_0, t_1, \dots, t_n\}$ is a partition of $[t, T]$ and ρ is the infinitesimal correlation between the increments of $X_1(t)$ and $X_2(t)$.

Proof:

As $\big(X_1(t_{j+1}) - X_1(t_j)\big)\big(X_2(t_{j+1}) - X_2(t_j)\big)$ is a random variable, we must prove that the expected value of

$\lim_{\|\Pi\to 0\|} \sum_{j=0}^{n-1}\big(X_1(t_{j+1}) - X_1(t_j)\big)\big(X_2(t_{j+1}) - X_2(t_j)\big)$ is

$\rho \int_t^T \sqrt{V_1(X_1,t)V_2(X_2,t)}\ dt$ and its variance is zero.

As $X_1(t)$ and $X_2(t)$ are generalized Brownian motions:

$$dX_i(t) = \alpha_i(X_i,t)dt + \sqrt{V_i(X,t)}\, dW_i(t), \quad \text{for } i = \{1,2\}. \tag{2.1.17}$$

Integrating both sides, yields:

$$X_i(t_2) - X_i(t_1) \quad = \quad \int_{t_1}^{t_2} \alpha_i(X_i,t)dt + \int_{t_1}^{t_2} \sqrt{V_i(X,t)}\, dW_i(t). \tag{2.1.18}$$

Hence,

$$\mathbb{E}\left[\lim_{\|\Pi\to 0\|} \sum_{j=0}^{n-1}\big(X_1(t_{j+1}) - X_1(t_j)\big)\big(X_2(t_{j+1}) - X_2(t_j)\big)\right] \tag{2.1.19}$$

$$= \lim_{\|\Pi\to 0\|} \sum_{j=0}^{n-1} \mathbb{E}\left[\left(\int_{t_j}^{t_{j+1}} \alpha_1(X_1,t)dt + \int_{t_j}^{t_{j+1}} \sqrt{V_1(X_1,t)}\, dW_1(t)\right)\left(\int_{t_j}^{t_{j+1}} \alpha_2(X_1,t)dt + \int_{t_j}^{t_{j+1}} \sqrt{V_2(X_2,t)}\, dW_2(t)\right)\right] \tag{2.1.20}$$

$$\begin{aligned} = \lim_{\|\Pi\to 0\|} \sum_{j=0}^{n-1} & \int_{t_j}^{t_{j+1}} \alpha_1(X_1,t)dt \int_{t_j}^{t_{j+1}} \alpha_2(X_1,t)dt \\ & + \int_{t_j}^{t_{j+1}} \alpha_1(X_1,t)dt\, \mathbb{E}\left[\int_{t_j}^{t_{j+1}} \sqrt{V_2(X_2,t)}\, dW_2(t)\right] \\ & + \int_{t_j}^{t_{j+1}} \alpha_2(X_1,t)dt\, \mathbb{E}\left[\int_{t_j}^{t_{j+1}} \sqrt{V_1(X_1,t)}\, dW_1(t)\right] \\ & + \mathbb{E}\left[\int_{t_j}^{t_{j+1}} \sqrt{V_1(X_1,t)}\, dW_1(t) \int_{t_j}^{t_{j+1}} \sqrt{V_2(X_2,t)}\, dW_2(t)\right]; \end{aligned} \tag{2.1.21}$$

with $\|\Pi \to 0\|$, $\int_{t_j}^{t_{j+1}} \alpha_1(X_1,t)dt \to 0$ and $\int_{t_j}^{t_{j+1}} \alpha_2(X_2,t)dt \to 0$. Thus, (2.1.19) equals

$$\sum_{j=0}^{n-1} \lim_{\|\Pi\to 0\|} \mathbb{E}\left[\int_{t_j}^{t_{j+1}} \sqrt{V_1(X_1,t)}\, dW_1(t) \int_{t_j}^{t_{j+1}} \sqrt{V_2(X_2,t)}\, dW_2(t)\right]; \tag{2.1.22}$$

with $\|\Pi \to 0\|$, $(t_{j+1} - t_j) \to 0$. Consequently, (2.1.19) equals

$$\sum_{j=0}^{n-1} \sqrt{V_1(X_1,t_j)}\sqrt{V_2(X_2,t_j)}\, \mathbb{E}\big[\big(W_1(t_{j+1}) - W_1(t_j)\big)\big(W_2(t_{j+1}) - W_2(t_j)\big)\big]. \tag{2.1.23}$$

It follows, from (2.1.4), that (2.1.19) equals

$$\sum_{j=0}^{n-1} \sqrt{V_1(X_1,t_j)}\sqrt{V_2(X_2,t_j)}\, \mathbb{E}\big[\epsilon_1\sqrt{t_{j+1} - t_j}\,\epsilon_2\sqrt{t_{j+1} - t_j}\big]. \tag{2.1.24}$$

As $\epsilon_1 \sim N(0,1)$, $\epsilon_2 \sim N(0,1)$ and allowing ϵ_1, ϵ_2 to be correlated with correlation equal to ρ , then $\mathbb{E}[\epsilon_1\epsilon_2] = Cov[\epsilon_1, \epsilon_2] = \rho\mathbb{V}[\epsilon_1]\mathbb{V}[\epsilon_2] = \rho$. Thus, (2.1.19) equals

$$\sum_{j=0}^{n-1} \sqrt{V_1(X_1,t_j)}\sqrt{V_2(X_2,t_j)}\, \rho(t_{j+1} - t_j)\,; \tag{2.1.25}$$

with $(t_{j+1} - t_j) \to 0$. It follows that (2.1.19) equals

$$\sum_{j=0}^{n-1} \rho \int_{t_j}^{t_{j+1}} \sqrt{V_1(X_1,t_j)}\sqrt{V_2(X_2,t_j)}\; dt = \rho \int_{t}^{T} \sqrt{V_1(X_1,t)V_2(X_2,t)}\; dt. \tag{2.1.26}$$

Hence,

$$\mathbb{V}\left[\lim_{\|\Pi\to 0\|} \sum_{j=0}^{n-1} \big(X_1(t_{j+1}) - X_1(t_j)\big)\big(X_2(t_{j+1}) - X_2(t_j)\big)\right] \tag{2.1.27}$$

$$= \lim_{\|\Pi\to 0\|} \sum_{j=0}^{n-1} \mathbb{V}\left[\left(\int_{t_j}^{t_{j+1}} \alpha_1(X_1,t)dt + \int_{t_j}^{t_{j+1}} \sqrt{V_1(X,t)}\, dW_1(t)\right)\left(\int_{t_j}^{t_{j+1}} \alpha_2(X_1,t)dt + \int_{t_j}^{t_{j+1}} \sqrt{V_2(X,t)}\, dW_2(t)\right)\right] \tag{2.1.28}$$

$$
\begin{aligned}
= \quad & \lim_{\|\Pi\to 0\|} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \alpha_1(X_1,t)dt \int_{t_j}^{t_{j+1}} \alpha_2(X_1,t)dt \\
& + \left(\int_{t_j}^{t_{j+1}} \alpha_1(X_1,t)dt\right)^2 \mathbb{V}\left[\int_{t_j}^{t_{j+1}} \sqrt{V_2(X,t)}\, dW_2(t)\right] \\
& + \left(\int_{t_j}^{t_{j+1}} \alpha_2(X_1,t)dt\right)^2 \mathbb{V}\left[\int_{t_j}^{t_{j+1}} \sqrt{V_1(X,t)}\, dW_1(t)\right] \\
& + \mathbb{V}\left[\int_{t_j}^{t_{j+1}} \sqrt{V_1(X,t)}\, dW_1(t) \int_{t_j}^{t_{j+1}} \sqrt{V_2(X,t)}\, dW_2(t)\right];
\end{aligned}
\tag{2.1.29}
$$

because $\|\Pi \to 0\|$, $\int_{t_j}^{t_{j+1}} \alpha_1(X_1,t)dt \to 0$ and $\int_{t_j}^{t_{j+1}} \alpha_2(X_2,t)dt \to 0$. Thus, (2.1.27) equals

$$
\sum_{j=0}^{n-1} \lim_{\|\Pi\to 0\|} \mathbb{V}\left[\int_{t_j}^{t_{j+1}} \sqrt{V_1(X_1,t)}\, dW_1(t) \int_{t_j}^{t_{j+1}} \sqrt{V_2(X_2,t)}\, dW_2(t)\right]. \tag{2.1.30}
$$

From (2.1.4), it follows that (2.1.30) equals

$$
\sum_{j=0}^{n-1} \lim_{\|\Pi\to 0\|} \mathbb{V}\left[\int_{t_j}^{t_{j+1}} \sqrt{V_1(X_1,t)}\, \epsilon_1\, dt \int_{t_j}^{t_{j+1}} \sqrt{V_2(X_2,t)}\epsilon_2\, dt\right] \tag{2.1.31}
$$

$$
= \sum_{j=0}^{n-1} \lim_{\|\Pi\to 0\|} \mathbb{V}\left[\epsilon_1\epsilon_2 \int_{t_j}^{t_{j+1}} \sqrt{V_1(X_1,t)}\; dt \int_{t_j}^{t_{j+1}} \sqrt{V_2(X_2,t)}\, dt\right] \tag{2.1.32}
$$

$$
= \sum_{j=0}^{n-1} \lim_{\|\Pi\to 0\|} \left(\int_{t_j}^{t_{j+1}} \sqrt{V_1(X_1,t)}\; dt \int_{t_j}^{t_{j+1}} \sqrt{V_2(X_2,t)}\, dt\right)^2 \mathbb{V}[\epsilon_1\epsilon_2] \tag{2.1.33}
$$

$$
\begin{aligned}
\leq \quad & \sum_{j=0}^{n-1} \lim_{\|\Pi\to 0\|} \left(\int_{t_m}^{t_M} \sqrt{V_1(X_1,t)}\; dt \int_{t_m}^{t_M} \sqrt{V_2(X_2,t)}\, dt\right) \\
& \left(\int_{t_j}^{t_{j+1}} \sqrt{V_1(X_1,t)}\; dt \int_{t_j}^{t_{j+1}} \sqrt{V_2(X_2,t)}\, dt\right) \mathbb{V}[\epsilon_1\epsilon_2];
\end{aligned}
\tag{2.1.34}
$$

where $\int_{t_m}^{t_M} \sqrt{V_1(X_1,t)}\ dt \int_{t_m}^{t_M} \sqrt{V_2(X_2,t)}\, dt =$

$\sup_j \int_{t_j}^{t_{j+1}} \sqrt{V_1(X_1,t)}\ dt \int_{t_j}^{t_{j+1}} \sqrt{V_2(X_2,t)}\, dt$. But, when $\|\Pi \to 0\|$,

$\int_{t_m}^{t_M} \sqrt{V_1(X_1,t)}\ dt \int_{t_m}^{t_M} \sqrt{V_2(X_2,t)}\, dt \to 0$. Hence, (2.1.27) vanishes.

■

2.2 Lévy's Theorem

Later in this text, and in a variety of quantitative finance models, it is necessary to turn to multi-dimension Brownian motion processes with correlation structures. Consequently, it is mandatory to verify if the defined stochastic processes are indeed Brownian motions, in order to make it possible to use the set of identities and properties defined in this section. In this context, Lévy's Theorem eases the identification of Brownian motions, as it requires few and easily verifiable conditions to guarantee that a process is a Brownian motion. Thus, we state Lévy's Theorem.

Theorem 2.1 (Lévy Theorem). Let $W(t)$, $t \geq 0$, be a martingale, relative to a filtration $\mathcal{F}(t)$, $t \geq 0$. If, (i) $W(0) = 0$, (ii) $W(t)$ is continuous, and its quadratic variation, $\langle W, W\rangle(T-t)$, satisfies (iii) $\langle W, W\rangle(T-t) = (T-t), \forall\, T > t \geq 0$. Then, $W(t)$ is a Brownian motion.

Proof: The proof can be found in (Karatzas, et al., 1991), page 157.

2.3 Change of measure

The proofs presented in this subsection follow (Shreve, 2004) closely.

Consider a finite sample space Ω on which we have two probability measures $\mathbb{P}$ and $\mathbb{Q}$. Assume that both $\mathbb{P}$ and $\mathbb{Q}$ give positive probability to every element of Ω and that $\omega \in \Omega$. Define the quotient:

$$Z(\boldsymbol{\omega}) = \frac{\mathbb{Q}(\boldsymbol{\omega})}{\mathbb{P}(\boldsymbol{\omega})}. \tag{2.3.1}$$

$Z(\boldsymbol{\omega})$ is called the *Radon-Nikodym* derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$. As $Z(\boldsymbol{\omega})$ depends on an outcome ω of a random experiment, $Z(\boldsymbol{\omega})$ is also a random variable.

Theorem 2.2 (*Radon-Nikodym* derivative). Assume that $\mathbb{P}(\boldsymbol{\omega}) > 0$ and $\mathbb{Q}(\boldsymbol{\omega}) > 0$ for every $\boldsymbol{\omega} \in \Omega$, and define the random variable Z by (2.3.1). Then, we have the following properties:

(i) $\mathbb{P}(Z > 0) = 1$;
(ii) $\mathbb{E}_{\mathbb{P}}[Z] = 1$;
(iii) for any random variable Y, $\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[ZY]$.

Proof:
(i) follows from the assumption $\mathbb{Q}(\boldsymbol{\omega}) > 0$ for every $\boldsymbol{\omega} \in \Omega$.
(ii) follows from

$$\mathbb{E}_{\mathbb{P}}[Z] = \sum_{\boldsymbol{\omega}\in\Omega} Z(\boldsymbol{\omega})\mathbb{P}(\boldsymbol{\omega}) = \sum_{\omega\in\Omega} \frac{\mathbb{Q}(\boldsymbol{\omega})}{\mathbb{P}(\boldsymbol{\omega})}\mathbb{P}(\boldsymbol{\omega}) = \sum_{\boldsymbol{\omega}\in\Omega} \mathbb{Q}(\boldsymbol{\omega}) = 1.$$

(iii) follows from

$$\mathbb{E}_{\mathbb{Q}}[Y] = \sum_{\boldsymbol{\omega}\in\Omega} Y(\boldsymbol{\omega})\mathbb{Q}(\boldsymbol{\omega}) = \sum_{\boldsymbol{\omega}\in\Omega} Y(\boldsymbol{\omega})\frac{\mathbb{Q}(\boldsymbol{\omega})}{\mathbb{P}(\boldsymbol{\omega})}\mathbb{P}(\boldsymbol{\omega}) = \sum_{\boldsymbol{\omega}\in\Omega} Y(\boldsymbol{\omega})Z(\boldsymbol{\omega})\mathbb{P}(\boldsymbol{\omega}) = \mathbb{E}_{\mathbb{P}}[ZY].$$

■

In the context of the Brownian motion continuous probability space, defined by the triple $(\Omega, \mathcal{F}, \mathbb{P})$, we have the following definition of Z, which is the continuous time analogous of (2.3.1):

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}}; \tag{2.3.2}$$

which results in a probability measure $\mathbb{Q}$, given by:

$$\int_A Z(\boldsymbol{\omega}) d\mathbb{P}(\boldsymbol{\omega}) = \int_A d\mathbb{Q}(\boldsymbol{\omega}). \tag{2.3.3}$$

Hence,

$$\mathbb{Q}(A) = \int_A Z(\boldsymbol{\omega}) d\mathbb{P}(\boldsymbol{\omega}); \quad \forall A \in \mathcal{F}. \tag{2.3.4}$$

In the context of continuous random variables, all the properties of Theorem 2.2 follow from similar arguments.

Definition 2.8 (*Radon-Nikodym* derivative process). In the context of (2.3.2), we define the process $Z(t)$ on the triple $(\Omega, \mathcal{F}, \mathbb{P})$. $Z(t)$ gives rise to a filtration $\mathcal{F}(t)$ defined on $0 \leq t \leq T$:

$$Z(t) \equiv \mathbb{E}_{\mathbb{P}}[Z|\mathcal{F}(t)]. \tag{2.3.5}$$

In particular,

$$Z = Z(T). \tag{2.3.6}$$

Lemma 2.1. If Y is an $\mathcal{F}(t)$ measurable random variable, then

$$\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[YZ(t)]. \tag{2.3.7}$$

Proof:

From statement (iii) of Theorem 2.2, we conclude that

$$\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[YZ]. \tag{2.3.8}$$

From the unbiasedness of conditional expectations (equation 2.1.2), we have

$$\mathbb{E}_{\mathbb{P}}[YZ] = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[YZ|\mathcal{F}(t)]] \tag{2.3.9}$$

$$= \mathbb{E}_{\mathbb{P}}[Y\mathbb{E}_{\mathbb{P}}[Z|\mathcal{F}(t)]]. \tag{2.3.10}$$

From (2.3.5), it follows that

$$\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[YZ(t)]. \tag{2.3.11}$$

■

Lemma 2.2. If Y is an $\mathcal{F}(t)$-measurable random variable and $0 \leq s < t \leq T$, then

$$\mathbb{E}_{\mathbb{Q}}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}_{\mathbb{P}}[YZ(t)|\mathcal{F}(s)]. \tag{2.3.12}$$

Proof: As the left-hand side of (2.3.12) is an expectation (see Definition 2.5), it must satisfy the partial averaging property, namely:

$$\int_A \mathbb{E}_{\mathbb{Q}}[Y|\mathcal{F}(s)]d\mathbb{Q} = \int_A Y\, d\mathbb{Q} \tag{2.3.13}$$

for all $A \in \mathcal{F}(s)$.

Then, the following equality must be verified. Plugging (2.3.12) into (2.3.13), we find

$$\int_A \frac{1}{Z(s)}\mathbb{E}_{\mathbb{P}}[YZ(t)|\mathcal{F}(s)]d\mathbb{Q} = \int_A Y\, d\mathbb{Q} \tag{2.3.14}$$

for all $A \in \mathcal{F}(s)$.

Indeed, (2.3.14) holds, since

$$\int_A \frac{1}{Z(s)}\mathbb{E}_{\mathbb{P}}[YZ(t)|\mathcal{F}(s)]d\mathbb{Q} = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{I}_{\omega\in A}\frac{1}{Z(s)}\mathbb{E}_{\mathbb{P}}[YZ(t)|\mathcal{F}(s)]\right], \tag{2.3.15}$$

where $\mathbb{I}$ is the indicator function.

As $\frac{1}{Z(s)}\mathbb{E}_{\mathbb{P}}[YZ(t)|\mathcal{F}(s)]$ is $\mathcal{F}(s)$-measurable, Lemma 2.1 implies that (2.3.15) equals

$$\mathbb{E}_{\mathbb{P}}\left[\mathbb{I}_{\boldsymbol{\omega}\in A}\mathbb{E}_{\mathbb{P}}[YZ(t)|\mathcal{F}(s)]\right] \tag{2.3.16}$$

$$= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}[\mathbb{I}_{\boldsymbol{\omega}\in A}YZ(t)|\mathcal{F}(s)]\right]. \tag{2.3.17}$$

From iterated-conditioning, it follows that (2.3.17) equals

$$\mathbb{E}_{\mathbb{P}}[\mathbb{I}_{\boldsymbol{\omega}\in A}YZ(t)], \tag{2.3.18}$$

and finally, from Lemma 2.1, it follows that (2.3.18) equals

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{I}_{\boldsymbol{\omega}\in A}Y] \tag{2.3.19}$$

$$= \int_A Y\, d\mathbb{Q}. \tag{2.3.20}$$

■

2.4 Gisarnov's Theorem

Girsanov's Theorem is needed on the risk neutral option pricing subject.

Theorem 2.3. Let $W_{\mathbb{P}}(t)$ be a Brownian motion on the triple $(\Omega, \mathcal{F}, \mathbb{P})$, let $\Theta(t)$ be an adapted process and $0 \leq t \leq T$. Define

$$Z(t) = \exp\left(-\int_0^t \Theta(u)dW_{\mathbb{P}}(u) - \frac{1}{2}\int_0^t \Theta^2(u)dW_{\mathbb{P}}(u)\right), \tag{2.4.1}$$

$$W_{\mathbb{Q}}(t) = W_{\mathbb{P}}(t) + \int_0^t \Theta(u)du. \tag{2.4.2}$$

Assume that

$$\mathbb{E}\left[\int_0^T \Theta^2(u)Z^2(u)du < \infty\right]. \tag{2.4.3}$$

Hence, Z is a *Radon-Nikodym* derivative $\frac{\partial\mathbb{Q}}{\partial\mathbb{P}}$, and under probability measure $\mathbb{Q}$, given by (2.3.4), the process $W_{\mathbb{Q}}(t)$ is a Brownian motion.

Proof: The proof can be found in (Shreve, 2004), page 212.

3 Option pricing core concepts

3.1 European call option definition

Definition 3.1. A European call option is a financial instrument that gives the holder the right, but not the obligation, to buy an underlying asset S at a predetermined price K, at maturity time T.

Hence, a European call option value, at maturity time T, is given by:

$$C(T) = \max\{S(T) - K, 0\}. \tag{3.1.1}$$

3.2 Underlying asset price stochastic process

Most models, Black-Scholes included, assume that returns $\left(\equiv \frac{S(t+1)-S(t)}{S(t)}\right)$ follow a Generalized Brownian motion process. Hence, from (2.1.13), and assuming drift and variance to be constants,

$$\frac{dS(t)}{S(t)} = \alpha\, dt + \sqrt{V}\, dW(t). \tag{3.2.1}$$

The process defined by (3.2.1) is used, in quantitative finance, to model stock returns, where α represents the expected return and $\sqrt{V}$ the standard deviation of the return per unit of time. $\sqrt{V}$ is known as volatility.

Hence, from the application of Ito's Lemma (Hull, 2015, page 313), to equation (3.2.1),

$$d\ln(S(t)) = \left(\alpha - \frac{V}{2}\right) dt + \sqrt{V}\, dW(t). \tag{3.2.2}$$

(3.2.2) is also a Generalized Brownian motion process. Thus, from (2.1.15) it follows that

$$\ln(S(T)) \sim N\left(\ln(S(0)) + \left(\alpha - \frac{V}{2}\right)T, VT\right) \tag{3.2.3}$$

$$\Rightarrow \quad S(T) \sim \text{Lognormal}\left(\ln(S(0)) + \left(\alpha - \frac{V}{2}\right)T, VT\right). \tag{3.2.4}$$

Hence, from (3.2.4), we conclude that Assumption 3.2.1 implies that the underlying asset price at time T is log-normally distributed.

Remark 3.1. We emphasize that the stochastic processes of Subsection (3.2) are under the real-world measure $\mathbb{P}$ - these processes should be observable from real-world stock market data.

3.3 No-arbitrage and risk-neutral option pricing

No-arbitrage pricing

In this section, we present a simple and strong argument to price a European call option, the no-arbitrage one.

Definition 3.2 (Arbitrage). An arbitrage portfolio is defined as follows:

(i) $\quad \forall \boldsymbol{\omega} \in \Omega;\ V_0(\boldsymbol{\omega}) = 0,$

(ii) $\quad \forall \boldsymbol{\omega} \in \Omega;\ V_T(\boldsymbol{\omega}) \geq 0,$

(iii) $\quad \exists \boldsymbol{\omega} \in \Omega;\ V_T(\boldsymbol{\omega}) > 0,$

where V_t is the value of the portfolio at time t.

Remark 3.2. An arbitrage portfolio has no risk, hence the return it earns must equal the risk-free interest rate.

Remark 3.3. From the argument that no arbitrage opportunity exists, two assets that surely have the same pay-off at time T, must have the same price.

A European call option can be priced by a simple argument: that no arbitrage opportunities exist. In order to use this argument, we set up a portfolio consisting of the option (whose price will be denoted by C) and the underlying asset, S, in such a way that there is no uncertainty about the value of the portfolio at maturity time T. Then from Remark 3.2, we know that such portfolio must yield the risk-free interest rate.

In order to illustrate the argument, and based on (Hull, 2015), we make the simplest possible assumption on market behavior: at time T the underlying asset can only take two values, $S_T = S_0 u$ or $S_T = S_0 d$ ($0 < d < 1$ and $u > 1$). In each case, the option value, $C_T = C_u$ and $C_T = C_d$, is given by equation (3.1.1). The setup is summarized in Figure 3.1.

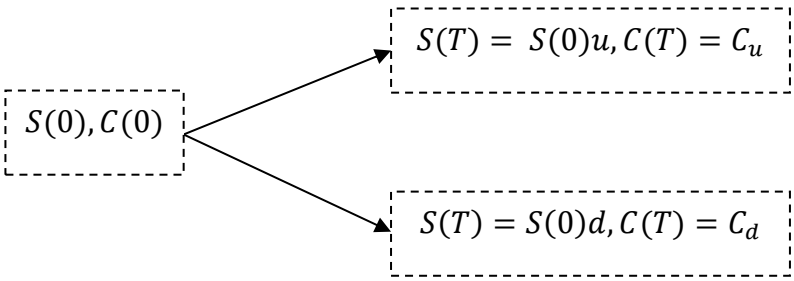

Figure 3.1

We build, at time zero, a portfolio of Δ assets (long position) and one option (short position)[1]. In order to make it riskless, its value at time T, $V(T)$, must not depend on market behavior, that is, if the asset price goes up or down the portfolio value must be the same. Equation (3.3.1) describes this condition.

$$V_T(u) = V_T(d). \tag{3.3.1}$$

It follows that

$$S(0)u\Delta - \mathrm{C_u} = S(0)d\Delta - \mathrm{C}_d, \tag{3.3.2}$$

$$\Delta = \frac{C_u - C_d}{S(0)u - S(0)d}. \tag{3.3.3}$$

From Remark 3.2, this portfolio must yield the risk-free interest rate, i.e.,

$$V_0 e^{rT} = V_T \tag{3.3.4}$$

$$(S(0)\Delta - \mathrm{C}(0))e^{rT} = S(0)u\Delta - \mathrm{C}_u \tag{3.3.5}$$

$$\mathrm{C}(0) = S(0)\Delta(1 - ue^{-rT}) + C_u e^{-rT}. \tag{3.3.6}$$

Hence, equation (3.3.6) gives the no-arbitrage price of the call option, which is based on the strong "no arbitrage opportunity exists" argument. Following is the main remark of this section: the counterintuitive irrelevance of the probability of the asset going up or down.

Remark 3.4. The no-arbitrage price of a call option, given by equation (3.3.6), was obtained with no assumption on the probability of the underlying asset going up or down. Hence, this probability is irrelevant for option pricing and can be chosen arbitrarily.

---

[1] A long (short) position is the buying (selling) of an asset with the expectation that it will rise in value (lose value).

Risk-neutral pricing

In this subsection, we show that risk-neutral pricing is merely a different approach of computing an option price based on the same non-arbitrage argument. We start with some intuition on the discrete case.

Substituting Δ from equation (3.3.3) in equation (3.3.6) gives

$$\mathrm{C}(0) \quad = \quad e^{-rT}(p_{\mathbb{Q}}C_u + (1 - p_{\mathbb{Q}})C_d), \tag{3.3.7}$$

where

$$p_{\mathbb{Q}} \quad \equiv \quad \frac{e^{rT} - d}{u - d}. \tag{3.3.8}$$

Remark 3.5. From Remark 3.4, the probability of an up movement in the underlying asset price is irrelevant. Hence, we are able to choose it equal to $p_{\mathbb{Q}}$, from equation (3.3.8). $p_{\mathbb{Q}}$ is known as the risk neutral probability, and gives rise to the risk-neutral pricing formula, equation (3.3.33) below (page 29). $\mathbb{Q}$ is known as the risk-neutral world - where the probability of an up movement is given by $p_{\mathbb{Q}}$.

Interpreting $p_{\mathbb{Q}}$ as a probability of up movement, (3.3.7) can be re-written as

$$\mathrm{C}(0) \quad = \quad e^{-rT}\mathbb{E}_{\mathbb{Q}}[C(T)]. \tag{3.3.9}$$

Proposition 3.1. Under the risk neutral measure, the discounted stock price process is a martingale.

Proof:

$$\mathbb{E}_{\mathbb{Q}}[e^{-rT}S(T)] \quad = \quad e^{-rT}(p_{\mathbb{Q}}S(0)u + (1 - p_{\mathbb{Q}})S(0)d) \tag{3.3.10}$$

$$= \quad S(0). \tag{3.3.11}$$

Back to the continuous case, integrating both sides of (3.2.2), we get

$$\int_0^t d\ln(S(t)) \quad = \quad \left(\alpha - \frac{V}{2}\right) \int_0^t dt + \int_0^t \sqrt{V} dW. \tag{3.3.12}$$

Hence $$\ln(S(t)) - \ln(S(0)) \quad = \quad \left(\alpha - \frac{V}{2}\right) t + \int_0^t \sqrt{V}\, dW, \tag{3.3.13}$$

i.e., $$S(t) \quad = \quad S(0)\exp\left(\left(\alpha - \frac{V}{2}\right) t + \int_0^t \sqrt{V} dW\right). \tag{3.3.14}$$

We now define a discounting process:

$$D(t) \quad = \quad e^{-rt}, \tag{3.3.15}$$

i.e., $$dD(t) \quad = \quad -rD(t)dt. \tag{3.3.16}$$

From (3.3.14) and (3.3.15), we have the discounted stock price process

$$D(t)S(t) \quad = \quad S(0)\, \exp\left(\left(\alpha - r - \frac{V}{2}\right) t + \sqrt{V} \int_0^t dW\right). \tag{3.3.17}$$

From (3.2.1), (3.3.16) and the application of the Itô product rule (see Corollary A.1 in Appendix A) to the right-hand side of (3.3.17), we get

$$d(D(t)S(t)) \quad = \quad (\alpha - r)D(t)S(t)dt + \sqrt{V}D(t)S(t)dW(t). \tag{3.3.18}$$

Thus $$d(D(t)S(t)) \quad = \quad \sqrt{V}D(t)S(t)\big(\Theta\, dt + dW(t)\big), \tag{3.3.19}$$

where

$$\Theta \quad \equiv \quad \frac{\alpha - r}{\sqrt{V}}. \tag{3.3.20}$$

Applying Girsanov's Theorem (2.3) to equation (3.3.19), with Θ in Theorem 2.3 defined by (3.3.20), we can change the Brownian motion measure from $\mathbb{P}$ to $\mathbb{Q}$:

$$d(D(t)S(t)) \quad = \quad \sqrt{V}D(t)S(t)dW_{\mathbb{Q}}(t); \tag{3.3.21}$$

where $W_{\mathbb{Q}}$ is a Brownian motion under the probability measure $\mathbb{Q}$, given by (2.3.4).

■

Definition 3.3. (Risk neutral measure) A probability measure $\mathbb{Q}$ is said to be risk-neutral if

(i) $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, i.e., if $\forall\, A \in \mathcal{F}, \mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0$; and

(ii) in the spirit of Proposition 3.1, under $\mathbb{Q}$, the discounted stock price $D(t)S(t)$ is a martingale.

Proposition 3.2. $\mathbb{Q}$ in (3.3.21) is the risk-neutral measure.

Proof: It follows from Girsanov's Theorem (2.3) that $\mathbb{Q}$ is equivalent to the original measure $\mathbb{P}$. Next, we show that the discounted stock price $D(t)S(t)$ is a martingale under $\mathbb{Q}$: it follows from (3.3.21) that

$$D(t)S(t) \quad = \quad S(0) + \int_0^t \sqrt{V}D(t)S(t)dW_{\mathbb{Q}}(t). \tag{3.3.22}$$

Here, $\int_0^t \sqrt{V}D(t)S(t)dW_{\mathbb{Q}}(t)$ is an Itô integral under $\mathbb{Q}$ and thus a martingale. Hence, $D(t)S(t)$ is a martingale under $\mathbb{Q}$.

■

Proposition 3.3. Under $\mathbb{Q}$, the underlying asset process (3.2.1) has drift equal to r.

Proof: It follows from (2.4.2) that

$$dW_{\mathbb{P}}(t) = dW_{\mathbb{Q}}(t) - \Theta(t)dt. \tag{3.3.23}$$

Inserting (3.3.23) into (3.2.1), and using Definition 3.3.20, we can conclude that

$$dS(t) \quad = \quad rS(t)dt + \sqrt{V}\,\mathrm{S(t)}dW_{\mathbb{Q}}(t). \tag{3.2.24}$$

■

We now consider an investor who, much in the spirit of non-arbitrage pricing, builds a portfolio, which we will denote by $P(t)$, consisting of $\Delta(t)$ shares of underlying asset, investing or borrowing at the risk-free constant interest rate r. $P(t)$ is the portfolio value and $\Delta(t)$ is the number of underlying assets at time t. The remaining cash value at time t is given by $P(t) - \Delta(t)S(t)$, this is the amount the investor will have to invest or borrow at the risk-free interest rate r.

Hence,

$$d(P(t)) \quad = \quad \Delta(t)dS(t) + r\big(P(t) - \Delta(t)S(t)\big)dt. \tag{3.3.25}$$

From Remark 3.4 we conclude that, for the option pricing purpose, one can arbitrarily choose among probability measures for $S(t)$. Here, we choose the measure $\mathbb{Q}$. Thus, using (3.2.24)

$$\begin{aligned} d(P(t)) \quad &= \quad \Delta(t)\left(rS(t)dt + \sqrt{V}S(t)dW_{\mathbb{Q}}(t)\right) + r\big(P(t) - \Delta(t)S(t)\big)dt \\ &= \quad rP(t)dt + \Delta(t)\sqrt{V}S(t)dW_{\mathbb{Q}}(t). \end{aligned} \tag{3.3.26}$$

From Itô's product rule, (3.3.16) and (3.3.26), it follows that

$$d(D(t)P(t)) \quad = \quad \Delta(t)\sqrt{V}D(t)S(t)dW_{\mathbb{Q}}(t). \tag{3.3.27}$$

Proposition 3.4. Under the risk neutral measure $\mathbb{Q}$ the discounted portfolio value process $D(t)P(t)$ is a martingale.

Proof: Integrating both sides of (3.3.27) from s to t $(0 \leq s < t \leq T)$ yields

$$D(t)P(t) \quad = \quad D(s)P(s) + \int_s^t \Delta(t)\sqrt{V}D(t)S(t)dW_{\mathbb{Q}}(t). \tag{3.3.28}$$

$W_{\mathbb{Q}}(t)$ is a Brownian motion under $\mathbb{Q}$. Hence, the integral in (3.3.28) is an Itô integral, with expectation value, under $\mathbb{Q}$, equal to zero. Therefore,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[D(t)P(t)|\mathcal{F}(s)] &= \mathbb{E}_{\mathbb{Q}}[D(s)P(s)|\mathcal{F}(s)] \\ &= D(s)P(s), \end{aligned} \tag{3.3.29}$$

which is the martingale property.

■

It follows from Proposition 3.4 that

$$D(t)P(t) \quad = \quad \mathbb{E}_{\mathbb{Q}}[D(T)P(T)|\mathcal{F}(t)]; \tag{3.3.30}$$

and from Proposition 3.6 that

$$D(t)P(t) \quad = \quad \mathbb{E}_{\mathbb{Q}}[D(T)C(T)|\mathcal{F}(t)]. \tag{3.3.31}$$

The amount $P(t)$ is the capital needed at time t in order to have, at time T, $P(T) = C(T)$. Hence, from Remark 3.3, it follows that $P(t)$ must be equal to the call option price at time t, $C(t)$. Hence, (3.3.31) yields:

$$D(t)C(t) \quad = \quad \mathbb{E}_{\mathbb{Q}}[D(T)C(T)|\mathcal{F}(t)]. \tag{3.3.32}$$

From (3.3.15), follows the well-known risk neutral pricing formula:

$$C(t) \quad = \quad e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[C(T)|\mathcal{F}(t)]. \tag{3.3.33}$$

From the derivation of equation (3.3.33) one can conclude that this formula can be used to compute the price at time t of any derivative that pays-off at maurity T - $C(T)$ representing the pay-off of such a derivative. This formula will be used to price a forward contract in Subsection 5.4.

The right-hand side of (3.3.33), in case a European call option (see Definition 3.1) is being priced, is equal to the well-known Black-Scholes pricing formula (Black, et al., 1973):

$$C(t) = \Phi(d_1)S(t) - \Phi(d_2)Ke^{-r(T-t)}, \tag{3.3.34}$$

where

$$d_1 = \frac{1}{\sqrt{V\,(T-t)}}\left[\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{V}{2}\right)(T-t)\right], \tag{3.3.35}$$

$$d_2 = d_1 - \sqrt{V(T-t)}\,. \tag{3.3.36}$$

From now on, we denote equation (3.3.34) by

$$C(t) \equiv BS\big(S(t), K, \sqrt{V}, T-t, r\,\big). \tag{3.3.37}$$

We use this notation to display the parameters explicitly and to make it clear that we are referring to the Black-Scholes equation.

Proposition 3.5. $D(t)C(t)$ is a martingale under $\mathbb{Q}$.

Proof: Assume $0 \le s < t \le T$.

From (3.3.32), it follows that

$$\mathbb{E}_{\mathbb{Q}}[D(t)C(t)|\mathcal{F}(s)] = \mathbb{E}_{\mathbb{Q}}\big[\mathbb{E}_{\mathbb{Q}}[D(T)C(T)|\mathcal{F}(t)]\big|\mathcal{F}(s)\big].$$

Iterated conditioning yields

$$\mathbb{E}_{\mathbb{Q}}[D(t)C(t)|\mathcal{F}(s)] \quad = \quad \mathbb{E}_{\mathbb{Q}}[D(T)C(T)|\mathcal{F}(s)],$$

and from equation (3.3.32) it follows that

$$\mathbb{E}_{\mathbb{Q}}[D(t)C(t)|\mathcal{F}(s)] \quad = \quad D(s)C(s), \tag{3.3.38}$$

which is the martingale property.

■

Proposition 3.6. There exists an initial capital $P(0)$ and a portfolio process $\Delta(t)$ $(0 \leq t \leq T)$ such that

$$P(T) \quad = \quad C(T) \tag{3.3.39}$$

almost surely.

Proof:

It follows from Proposition 3.5, that $D(t)C(t)$ is a martingale under $\mathbb{Q}$. Hence, applying Corollary A.3, we get that $D(t)C(t)$ has the following representation:

$$\begin{aligned} D(t)C(t) &= D(0)C(0) + \int_0^t \Gamma_{\mathbb{Q}}(u)dW_{\mathbb{Q}}(u) \\ &= C(0) + \int_0^t \Gamma_{\mathbb{Q}}(u)dW_{\mathbb{Q}}(u). \end{aligned} \tag{3.3.40}$$

Using (3.3.27), we find

$$\begin{aligned} D(t)P(t) &= D(0)P(0) + \int_0^t \Delta(u)\sqrt{V}D(u)S(u)dW_{\mathbb{Q}}(u) \\ &= P(0) + \int_0^t \Delta(u)\sqrt{V}D(u)S(u)dW_{\mathbb{Q}}(u). \end{aligned} \tag{3.3.41}$$

Equating the right-hand sides of (3.3.40) and (3.3.41), gives

$$P(t) = C(t) \ \forall\, 0 \le t \le T \Leftrightarrow P(0) = C(0), \tag{3.3.42}$$

and

$$\Delta(t) = \frac{\Gamma_{\mathbb{Q}}(u)}{\sqrt{V}D(t)S(t)} \ \forall\, 0 \le t \le T. \tag{3.3.43}$$

■

3.4 Volatility smile

In the derivation of the Black-Scholes formula, given in equation (3.3.34), two important assumptions were made, and we reinforce them here:

Assumption 3.1. The underlying asset price follows a Geometric Brownian motion given by (3.2.24). Under this assumption, it was argued that the call option price is given by the expectation in equation (3.3.33), which, in turn, resulted in the Black-Scholes pricing formula (3.3.34).

Assumption 3.2. The variance V in (3.2.24) is constant.

Remark 3.6. These assumptions and (3.2.4) suggest that the underlying asset price at time T is log-normally distributed:

$$S_T \sim \text{Lognormal}\left(\ln(S_0) + \left(r - \frac{V}{2}\right)T, VT\right). \tag{3.4.1}$$

Remark 3.7. As variance V is intrinsic from the underlying asset process, Assumption 3.2 implies that, if we consider variance V_i applied to the Black-Scholes formula (3.3.34) to obtain the price $C_i(K_i, T_i)$ of a European Call option over an unique underlying asset S, with strike K_i and time to maturity T_i, $i = \{1,2,\dots,n\}$, then even if $K_1 \neq \cdots \neq K_n$, $T_1 \neq \cdots \neq T_n$ and $C_1 \neq \cdots \neq C_n$, it follows that $V_1 = V_2 =$

$\cdots = V_n$. Hence, the variance that gives origin to the price of different options, in the Black-Scholes framework, is the same.

Remark 3.8. (Vagnani, 2009) shows that Assumption 3.1 holds in real-world. Nevertheless, Assumption 3.2 does not. This, in turn, implies that Remarks 3.6 and 3.7 are not verified in real-world. In real-world, an "implied volatility smile", as defined below, is observed.

Definition 3.9 (Implied volatility smile). The U-shaped curve in Figure 3.2 is called "Implied volatility smile". If real-world were as expected by the Black-Scholes framework, Figure 3.2 would be a horizontal line.
The volatilities $\sqrt{V_{imp}(K)}$ in Figure 3.2 are called implied volatilities. The implied volatility $\sqrt{V_{imp}(K)}$ is the volatility that, if applied to Black-Scholes equation (3.3.34), results in the market price of the option with strike K and time to maturity $T - t$.

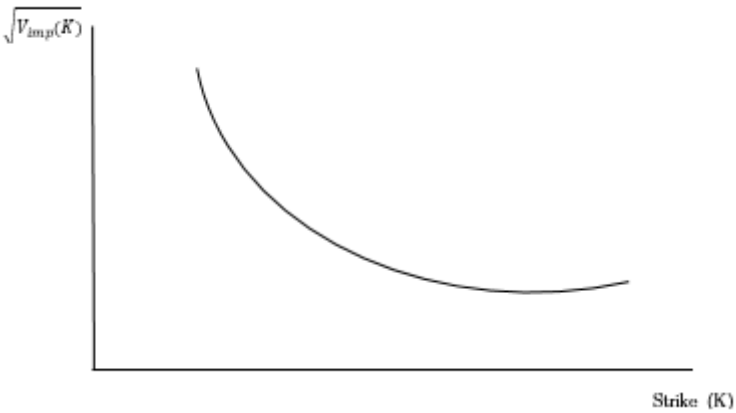

Figure 3.2: Implied volatility smile as a function of strike.

In contrast to Remark 3.6, market players believe that the risk-neutral distribution of S_T is not log-normal. The risk-neutral distribution of S_T, implied from market data, can be derived from the result in Proposition 3.7.

Proposition 3.7 (Implied risk-neutral distribution of $S(T)$).

This derivation is based on (Breeden, et al., 1978)

From (3.3.33), it follows that

$$C(t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[C(T)|\mathcal{F}(t)]; \tag{3.4.2}$$

using (3.1.1) yields

$$C(t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[\max\{S(T)-K,0\}|\mathcal{F}(t)]; \tag{3.4.3}$$

and defining $f(S_T)$ as the implied risk-neutral distribution of S_T it follows that

$$C(t) = e^{-r(T-t)}\int_K^{\infty}(S(T)-K)f(S_T)dS_T. \tag{3.4.4}$$

Leibniz formula yields

$$\begin{aligned} \frac{\partial C(t)}{\partial K} &= e^{-r(T-t)}\int_K^{\infty} -f(S(T))dS_T \\ \Rightarrow \quad \frac{\partial C^2(t)}{\partial K^2} &= e^{-r(T-t)}f(S(T)) \\ \Rightarrow \quad f(S(T)) &= e^{r(T-t)}\frac{\partial C^2(t)}{\partial K^2}. \end{aligned} \tag{3.4.5}$$

■

Hence, one can observe market option prices $C_i(t)$ as functions of its respective strikes K_i for n tradable options ($i = \{1, \dots, n\}$) maturing at time T and fit a curve to this function. Then, it is possible compute the second derivative of such function and derive the implied risk-neutral distribution of S_T from (3.4.5), which might not be log-normal, as assumed in the Black-Scholes framework.

In this context, stochastic volatility models are used to describe the volatility (and the implied risk-neutral distribution of S_T) observed in the real-world. This resolves

one of the drawbacks of the Black-Scholes framework, that is considering V to be constant. These models are the subject of Subsection 3.5 below.

3.5 Stochastic volatility models

Stochastic volatility models relax the Assumption 3.2, "V is constant", from the Black-Scholes framework. Instead, variance is allowed to be an adapted process $V(t)$, that is, at time t the available information (contained in $\mathcal{F}(t)$) is enough to determine the value of $V(t)$. In this context, (3.2.24) becomes

$$dS(t) = rS(t)dt + \sqrt{V(t)}\,\mathrm{S(t)}dW_1(t), \tag{3.5.1}$$

where the randomness of $V(t)$ is described by

$$dV(t) = \alpha(V(t),t)dt + \beta(V(t),t)dW_2(t). \tag{3.5.2}$$

Both $\alpha(V(t),t)$ and $\beta(V(t),t)$ depend on the specific stochastic volatility model being implemented. $\alpha(V(t),t)$ and $\beta(V(t),t)$ give rise to model parameters (denoted by φ) that shall be calibrated to market data. Such calibration, results in the following optimization problem:

$$\min_{\varphi} \sqrt{\sum_{K,T} \left(C_{mkt}(K,T) - C_m(K,T)\right)^2}, \tag{3.5.3}$$

where $C_{mkt}(K,T)$, $C_m(K,T)$ denote, respectively, the market and model European call options prices for strike K and time to maturity T.

(Mrázek, et al., 2016) provide a discussion about several techniques for calibration of stochastic volatility models.

Remark 3.10. The aim of the optimization problem (3.5.3) is to guarantee model prices to be consistent with the ones observed from market data. Consequently, the

model is capable of reproducing the market-observed risk-neutral distribution of S_T, addressed in Proposition 3.7.

From (3.5.1) and (3.5.2), we see that a bi-dimensional Brownian motion is involved, namely $(W_1(t), W_2(t))$. $W_1(t)$ and $W_2(t)$ are allowed to be correlated - their correlation will be denoted by ρ in this text.

Proposition 3.8. If W_1 and W_3 are two independent Brownian motions, then (i) $(\rho\, W_1(t) + \sqrt{1-\rho^2} W_3(t))$ is also a Brownian motion, and (ii) the correlation among dW_1 and $\rho\, dW_1(t) + \sqrt{1-\rho^2} dW_3(t)$ is ρ.

Proof:

Firstly, from (2.1.4):

$$\mathbb{V}[dW_1(t)] = dt\, \mathbb{V}[\epsilon_1] = dt \tag{3.5.4}$$

$$\mathbb{V}[dW_3(t)] = dt\, \mathbb{V}[\epsilon_3] = dt \tag{3.5.5}$$

Also, as $W_1(t)$ and $W_3(t)$ are independent:

$$Cov[dW_1(t), dW_3(t)] = 0. \tag{3.5.6}$$

We first show that

(i) $(\rho\, W_1(t) + \sqrt{1-\rho^2} W_3(t))$ is a Brownian motion.

Note that $(\rho\, W_1(t) + \sqrt{1-\rho^2} W_3(t))$ is zero at time zero:

$$(\rho\, W_1(0) + \sqrt{1-\rho^2} W_3(0)) \quad = \quad 0. \tag{3.5.7}$$

Moreover, $(\rho\, W_1(t) + \sqrt{1-\rho^2} W_3(t))$ is a continuous martingale, because it is a sum of two Brownian motions. Its quadratic variation is given by Proposition 2.3:

$$\langle \rho\, W_1(t) + \sqrt{1-\rho^2} W_3(t), \rho\, W_1(t) + \sqrt{1-\rho^2} W_3(t) \rangle\, (T-t)$$

$$= \int_t^T Cov\left[\rho\, W_1(t) + \sqrt{1-\rho^2} W_3(t), \rho\, W_1(t) + \sqrt{1-\rho^2} W_3(t)\right] dt$$

$$= \int_t^T \rho^2 \mathbb{V}[W_1(t)] + 2\rho\sqrt{1-\rho^2} Cov[W_1(t), W_3(t)] + (1-\rho^2)\mathbb{V}[W_3(t)]\, dt;$$

using (3.5.4), (3.5.5) and (3.5.6), it follows that

$$\langle \rho\, W_1(t) + \sqrt{1-\rho^2} W_3(t), \rho\, W_1(t) + \sqrt{1-\rho^2} W_3(t) \rangle\, (T-t)$$

$$= \int_t^T dt = (T-t). \qquad (3.5.8)$$

Hence, it follows from Lévy's Theorem (2.1) that $(\rho\, W_1(t) + \sqrt{1-\rho^2} W_3(t))$ is a Brownian motion.

We now establish

(ii) The correlation among dW_1 and $(\rho\, dW_1 + \sqrt{1-\rho^2} dW_3)$ is ρ.

We note that

$$Cov[dW_1(t), \rho\, dW_1(t) + \sqrt{1-\rho^2} dW_3(t)] = \rho \mathbb{V}[dW_1(t)] + \sqrt{1-\rho^2} Cov[dW_1(t), dW_3(t)].$$

Using (3.5.5) and (3.5.6), it follows that

$$Cov[dW_1(t), \rho\, dW_1(t) + \sqrt{1-\rho^2} dW_3(t)] = \rho\, dt. \qquad (3.5.9)$$

Additionally, $\mathbb{V}\left[\rho\, dW_1(t) + \sqrt{1-\rho^2} dW_3(t)\right] = \rho^2 dt \mathbb{V}[\epsilon_1] + (1-\rho^2) dt\, \mathbb{V}[\epsilon_3] = \rho^2 dt + (1-\rho^2) dt = dt.$

Hence,

$$Cor[dW_1(t), \rho\, dW_1(t) + \sqrt{1-\rho^2} dW_3(t)] = \frac{Cov[dW_1(t), \rho\, dW_1(t) + \sqrt{1-\rho^2} dW_3(t)]}{\sqrt{\mathbb{V}[dW_1(t)] \mathbb{V}\left[\sqrt{1-\rho^2} dW_3(t)\right]}}$$

$$= \frac{Cov[dW_1(t), \rho\, dW_1(t) + \sqrt{1-\rho^2} dW_3(t)]}{\sqrt{dtdt}}$$

$$= \rho. \qquad (3.5.10)$$

■

In order to induce the correlation among W_1 and W_2 in (3.5.1) and (3.5.2), we use Proposition 3.8 to rewrite (3.5.2):

$$dV(t) = \alpha(V(t),t)dt + \beta(V(t),t)\left(\rho dW_1(t) + \sqrt{1-\rho^2}dW_3(t)\right), \qquad (3.5.11)$$

with $W_1(t)$, $W_2(t)$ and $W_3(t)$ independent Brownian motions.

Heston model

The parsimonious Heston model (Heston, 1993) is a particular case of (3.5.1) and (3.5.11):

$$dS(t) = rS(t)dt + \sqrt{V(t)}\,\mathrm{S(t)}dW_1(t), \qquad (3.5.12)$$

$$dV(t) = \kappa\left(\bar{v} - V(t)\right)dt + \xi\sqrt{V(t)}\left(\rho dW_1(t) + \sqrt{1-\rho^2}dW_3(t)\right), \qquad (3.5.13)$$

where $\boldsymbol{\varphi} \equiv (\rho, \kappa, \bar{v}, V(0), \xi)$ is the vector of parameters.

The interpretation of each parameter is presented next.

$V(0)$ is called spot variance, that is the variance that takes place in the infinitesimal increment $\frac{S(0+\tau)-S(0)}{S(0)}$ $(\tau \to 0)$.

$\bar{v}$ is the long-term variance and κ is the rate of convergence. To see this, consider only the drift term of (3.5.13):

$$dV(t) = \kappa(\bar{v} - V(t))dt, \qquad (3.5.14)$$

which implies $$V(t) = V(0)\exp(-\kappa t) + \bar{v}. \qquad (3.5.15)$$

Hence, for $t \to \infty$, $V(t) \to \bar{v}$, and the greater κ the faster the convergence.

ρ is the correlation given by (3.5.10) and ξ is called volatility of the variance.

4 Change of numéraire

A numéraire is the unit of account in which an asset is denominated. A numéraire can be, for example, the currency of a country. In the context of quanto option pricing, the numéraire plays an important role as the underlying asset is quoted in a foreign (FOR) currency, but the payoff is given in domestic currency (DOM).

Definition 4.1. (Domestic money market account). The price of a share of the domestic money market account, given in domestic currency is

$$M(t) = e^{rt}, \tag{4.1}$$

where $M(t) \equiv \frac{\text{DOM}}{\text{ShareM}}$. $M(t)$ represents the amount of money (in domestic currency) that an investor would have if she had invested 1 DOM at the domestic risk-free interest rate r at time 0. ShareM stands for a share of the domestic money market account.

Equation (3.3.15) gives

$$M(t) = \frac{1}{D(t)}, \tag{4.2}$$

where $D(t) \equiv \frac{\text{ShareM}}{\text{DOM}}$.

Definition 4.2 (Foreign money market account). The price of a share of the foreign money market account, given in foreign currency is

$$M_f(t) = e^{r_f t} \tag{4.3}$$

$$\Rightarrow \quad dM_f(t) = r_f M_f(t) dt, \tag{4.4}$$

where $M_f(t) \equiv \frac{\text{FOR}}{\text{ShareMf}}$. $M_f(t)$ represents the amount of money (in foreign currency) that an investor would have if she had invested 1 FOR at the foreign risk-free interest rate r_f at time 0. ShareMf stands for a share of the foreign money market account. Hence

$$M_f(t) \equiv \frac{1}{D_f(t)}, \tag{4.5}$$

where $D_f(t) \equiv \frac{\text{ShareMf}}{\text{FOR}}$.

Remark 4.1. The discounted price of an asset, $D(t)S(t)$, is the asset price at time t denoted in shares of the domestic money market account $\left(\frac{\text{ShareM}}{\text{Asset}} \equiv \frac{\text{ShareM}}{\text{DOM}}\frac{\text{DOM}}{\text{Asset}}\right)$. From Proposition 3.2, under the domestic risk-neutral measure $\mathbb{Q}$, $D(t)S(t)$ is a martingale. Hence, we conclude that the discounted price of every asset, that is, the asset price expressed in units of domestic money market account, is a martingale under the domestic risk-neutral measure $\mathbb{Q}$.

We restate here the stochastic differential equation (3.2.1), emphasizing the probability measure.

$$dS(t) = \alpha\, S(t)dt + \sqrt{V_1}S(t)dW_{\mathbb{P}1}(t). \tag{4.6}$$

Hence, Itô's product rule yields

$$d(D(t)S(t)) = D(t)S(t)\left[(\alpha - r)dt + \sqrt{V_1}S(t)dW_{\mathbb{P}1}(t)\right]. \tag{4.7}$$

Assumption 4.1. We assume that the exchange rate, which we denote by $Q(t)$ ($\equiv \frac{\text{DOM}}{\text{FOR}}$), follows the stochastic differential equation

$$dQ(t) = \gamma\, Q(t)dt + \sqrt{V_2}Q(t)\left[\rho\, dW_{\mathbb{P}1}(t) + \sqrt{1-\rho^2}dW_{\mathbb{P}2}(t)\right]; \tag{4.8}$$

where $W_{\mathbb{P}1}(t)$ and $W_{\mathbb{P}2}(t)$ are independent Brownian motions.

From Proposition 3.8, $W_{\mathbb{P}3}(t) \equiv \rho\, W_{\mathbb{P}1}(t) + \sqrt{1-\rho^2} W_{\mathbb{P}2}(t)$ is also a Brownian motion and the correlation among $dW_{\mathbb{P}1}(t)$ and $dW_{\mathbb{P}3}$ is $\rho\, dt$. Thus, we allow asset price $S(t)$ and exchange rate $Q(t)$ to be correlated.

We express the foreign money market account in terms of the domestic money market account. Firstly, note that $D(t)M_f Q(t) \equiv \frac{\text{ShareM}}{\text{DOM}} \frac{\text{FOR}}{\text{ShareMf}} \frac{\text{DOM}}{\text{FOR}} \equiv \frac{\text{ShareM}}{\text{ShareMf}}$.

From equations (4.4), (4.8) and Itô's product rule, it follows that

$$d(M_f(t)Q(t)) = M_f(t)dQ(t) + Q(t)dM_f(t) + dM_f(t)dQ(t) \tag{4.9}$$

$$= M_f(t)Q(t)\left[(r_f+\gamma)dt + \sqrt{V_2}(\rho dW_{\mathbb{P}1}(t) + \sqrt{1-\rho^2}dW_{\mathbb{P}2})\right]. \tag{4.10}$$

Then, Itô's product rule and (3.3.16) yield

$$d(D(t)M_f(t)Q(t)) = D(t)M_f(t)Q(t)\left[(r_f - r + \gamma)dt + \sqrt{V_2}(\rho dW_{\mathbb{P}1}(t) + \sqrt{1-\rho^2}dW_{\mathbb{P}2})\right]. \tag{4.11}$$

A bi-dimensional Brownian motion $(W_{\mathbb{P}1}(t), W_{\mathbb{P}2}(t))$ drives the processes of $D(t)S(t)$ and $D(t)M_f(t)Q(t)$. As the numéraire of these processes is the domestic money market account, from Remark 4.1, under the domestic risk neutral measure $\mathbb{Q}$, these processes are martingales. Indeed, as $M_f(t)Q(t)$ is the price of an investment in the foreign money market account, converted to domestic currency, $M_f(t)Q(t)$ is also a tradable asset and Remark 4.1 can be applied to it.

Hence, through the multi-dimensional Girsanov's Theorem, it is possible to change the measure from $\mathbb{P}$ to $\mathbb{Q}$ and get the bi-dimensional Brownian motion $(W_{\mathbb{Q}1}(t), W_{\mathbb{Q}2}(t))$, under which both $D(t)S(t)$ and $D(t)M_f(t)Q(t)$ are martingales.

Thus, we define

$$dW_{\mathbb{Q}1}(t) \quad = \quad \Theta_1 dt + dW_{\mathbb{P}1}(t), \tag{4.12}$$

from which, it is possible to state the following condition:

$$\Theta_1 \quad = \quad \frac{\alpha - r}{\sqrt{V_1}}, \tag{4.13}$$

to obtain

$$dD(t)S(t) \quad = \quad \sqrt{V_1} D(t)S(t)dW_{\mathbb{Q}1}(t). \tag{4.14}$$

Additionally, we define

$$dW_{\mathbb{Q}2}(t) \quad = \quad \Theta_2 dt + dW_{\mathbb{P}2}(t), \tag{4.15}$$

from which, it is possible to state the condition

$$\sqrt{V_2}\rho\Theta_1 + \sqrt{V_2}\Theta_2\sqrt{1-\rho^2} \quad = \quad r_f - r + \gamma, \tag{4.16}$$

to obtain

$$d(D(t)M_f(t)Q(t)) \quad = \quad D(t)M_f(t)Q(t)\left[\sqrt{V_2}(\rho dW_{\mathbb{Q}1}(t) + \sqrt{1-\rho^2}dW_{\mathbb{Q}2})\right]. \tag{4.17}$$

The values of Θ_1 and Θ_2 are irrelevant. The important conclusion is that, from Girsanov's Theorem, the domestic risk-neutral measure $\mathbb{Q}$ exists. As the solution for Θ_1 and Θ_2 is unique, the measure $\mathbb{Q}$ is also unique.

It is then possible to derive the undiscounted process $S(t)$, by multiplying $D(t)S(t)$ by $M(t) \equiv \frac{1}{D(t)}$ and applying Itô's product rule:

$$dS(t) \quad = \quad rS(t)dt + \sqrt{V_1}S(t)dW_{\mathbb{Q}1}(t). \tag{4.18}$$

With the same argument,

$$d(M_f(t)Q(t)) \quad = \quad M_f(t)Q(t)\left[rdt + \sqrt{V_2}(\rho dW_{\mathbb{Q}1}(t) + \sqrt{1-\rho^2}dW_{\mathbb{Q}2})\right]. \qquad (4.19)$$

Multiplying (4.19) by $D_f(t) \equiv \frac{1}{M_f(t)}$ and applying Itô's product rule, gives

$$dQ(t) \quad = \quad Q(t)\left[(r - r_f)dt + \sqrt{V_2}\left(\rho dW_{\mathbb{Q}1}(t) + \sqrt{1-\rho^2}dW_{\mathbb{Q}2}\right)\right]. \qquad (4.20)$$

(4.18) and (4.20) are respectively the processes followed by the underlying asset $S(t)$ and the exchange rate $Q(t)$ under the domestic risk-neutral measure $\mathbb{Q}$.

5 Quanto option

5.1 Quanto call option definition

Definition 5.1 A quanto call option is a financial instrument that gives the holder the right, but not the obligation, to buy an underlying asset S_f, quoted in foreign currency (FOR), at a predetermined price K (in units of FOR currency), at maturity time T. The payoff amount, if positive, is converted to domestic currency (DOM) at a predetermined and fixed exchange rate q $\left(\equiv \frac{\text{DOM}}{\text{FOR}}\right)$. q is predetermined at the contract inception, i.e., at time $t < T$.

Hence, the quanto call option value, at maturity time T, is given by

$$C^q(T) = \max\{q(S_f(T) - K), 0\}. \qquad (5.1.1)$$

In this section, for the sake of clarity, we emphasize the currency of S by means of the notation S_{d}, for S expressed in domestic currency, and S_f, for S expressed in foreign currency.

5.2 Quanto call option pricing in the Black-Scholes framework

In this subsection, we derive what is the pricing formula of a quanto call option in the Black-Scholes framework, precisely, a formula that in the spirit of Assumptions (3.1) and (3.2), assumes that every asset (stock and exchange rate included) follows a Geometric Brownian motion with constant volatility. (Wystup, 2008), (Demeterfi, 1998) and (Baxter, et al., 1996) present similar derivation, but through different approaches.

From the risk-neutral pricing formula (3.3.33) and the quanto call payoff (5.1.1), it follows that:

$$C^q(t) \quad = \quad e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[\max\{q(S_{\mathrm{f}}(T)-K),0\}|\mathcal{F}(t)]. \tag{5.2.1}$$

Hence, we aim to derive the stochastic differential equation for $S_{\mathrm{f}}(T)$, but under the domestic risk-neutral measure $\mathbb{Q}$.

Under the domestic risk-neutral measure, we assume that

$$dS_{\mathrm{f}}(t) \quad = \quad \mu_{Sf}dt + \sqrt{V_1}S(t)dW_{\mathbb{Q}1}(t), \tag{5.2.2}$$

and from (4.20), we have

$$dQ(t) \quad = \quad Q(t)\big[(r-r_f)dt + \sqrt{V_2}dW_{\mathbb{Q}2}(t)\,\big], \tag{5.2.3}$$

where the following definition is made:

$$W_{\mathbb{Q}2}(t) \quad := \quad \rho_{S_fQ} W_{\mathbb{Q}1}(t) + \sqrt{1-\rho_{S_fQ}^2}\, W_{\mathbb{Q}3}(t), \tag{5.2.4}$$

where μ_{Sf} is the yet unknown drift of $S_{\mathrm{f}}(t)$ under the domestic risk-neutral measure, and ρ_{S_fQ} is the infinitesimal correlation among the increments of S_f and Q.

In order to derive μ_{Sf}, we express $S_{\mathrm{f}}(t)$ in domestic currency by multiplying by $Q(t)$, i.e.,

$$S_{\mathrm{d}}(t) \equiv Q(t)S_f(t). \tag{5.2.5}$$

Hence, from Itô's product rule it follows that

$$d(S_{\mathrm{d}}(t)) \equiv d(Q(t)S_f(t)) \tag{5.2.6}$$

$$\begin{aligned} &= Q(t)S_f(t)\left(\mu_{Sf} + r - r_f + \rho_{S_fQ}\sqrt{V_1}\sqrt{V_2}\right)dt + \sqrt{V_1}dW_{\mathbb{Q}1}(t) \\ &\qquad + \sqrt{V_2}dW_{\mathbb{Q}2}(t). \end{aligned} \tag{5.2.7}$$

In the spirit of Remark 4.1, $e^{-r(T-t)}S_{\mathrm{d}}(t)$ ($\frac{\text{ShareM}}{\text{Asset}} \equiv \frac{\text{ShareM}}{\text{DOM}}\frac{\text{DOM}}{\text{Asset}}$), under the domestic risk neutral measure, is a martingale. Hence, from Itô's product rule, the drift of $S_{\mathrm{d}}(t)$ is equal to r. Thus, from (5.2.7), it follows that

$$r = \mu_{Sf} + r - r_f + \rho_{S_fQ}\sqrt{V_1}\sqrt{V_2}, \tag{5.2.8}$$

$$\mu_{Sf} = r - \left(r - r_f + \rho_{S_fQ}\sqrt{V_1}\sqrt{V_2}\right); \tag{5.2.9}$$

whence (5.2.2) becomes

$$dS_{\mathrm{f}}(t) = r - \left(r - r_f + \rho_{S_fQ}\sqrt{V_1}\sqrt{V_2}\right)dt + \sqrt{V_1}S(t)dW_{\mathbb{Q}1}(t). \tag{5.2.10}$$

Therefore, we are able to compute expectation (5.2.1), as we now know the dynamics of $S_{\mathrm{f}}(t)$.

In contrast to (4.18), stochastic differential equation (5.2.10) can be seen as representing the dynamics of a dividend paying stock, with dividend yield $a \equiv \left(r - r_f + \rho_{S_f Q}\sqrt{V_1}\sqrt{V_2}\right)$. In this context, the solution to expectation (5.2.1) is well diffused in literature and can be found for example in (Hull, 2015), page 373. It is given by

$$C^q(t) = q\left[S_{\mathrm{f}}(t)e^{-\left(r-r_f+\rho_{S_f Q}\sqrt{V_1}\sqrt{V_2}\right)(T-t)}N\left(d_1\left((T-t),S_{\mathrm{f}}(t)\right)\right) - e^{r(T-t)}\,K\,N\left(d_2\left((T-t),S_{\mathrm{f}}(t)\right)\right)\right] \tag{5.2.11}$$

$$\equiv q\,BS\left(S_f(t)e^{-\left(r-r_f+\rho_{S_f Q}\sqrt{V_1}\sqrt{V_2}\right)(T-t)},K,\sqrt{V_1},T-t,r\right), \tag{5.2.12}$$

where d_1 and d_2 are given, respectively, by

$$d_1 = \frac{1}{\sqrt{V_1(T-t)}}\left[\ln\left(\frac{S_f(t)}{K}\right) + \left(r - \left(r - r_f + \rho_{S_f Q}\sqrt{V_1}\sqrt{V_2}\right) + \frac{V_1}{2}\right)(T-t)\right], \tag{5.2.13}$$

$$d_2 = d_1 - \sqrt{V_1(T-t)}. \tag{5.2.14}$$

and $BS(\cdot)$ is the notation defined in (3.3.37).

5.3 Quanto forward definition

In this subsection, we define the quanto forward, whose pricing formula will be used to support an argument given in Section 6.

Definition 5.2. A quanto forward is a financial instrument that gives the holder the obligation to buy an underlying asset S_f, quoted in foreign currency (FOR), at a

predetermined fixed price F^q (in units of FOR currency), at maturity time T. The payoff amount is converted to domestic currency (DOM) at a predetermined and fixed exchange rate $q\ \left(\equiv \frac{\text{DOM}}{\text{FOR}}\right)$. F^q and q are predetermined at the contract inception, precisely, at time $t < T$.

Hence, a quanto forward payoff, at maturity time T, is given by

$$F^q_{payoff} = q\big(S_f(T) - F^q\big). \tag{5.2.15}$$

Remark 5.1. A specificity of a forward contract is that the price to enter such contract is zero, the negotiation being only on the value of F^q.

5.4 Quanto forward pricing

From Remark 5.1, the pricing effort in forward contracts is in determining the value of F^q. F^q is determined at the contract inception, time $t < T$, thus we denote F^q by $F^q(t)$.

From the risk-neutral pricing formula (3.3.33), Remark 5.1, which states that the entry cost in a forward contract is zero, and from the payoff equation (5.2.15), we have

$$0 = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\big[F^q_{payoff}\big|\mathcal{F}(t)\big] \tag{5.2.16}$$

$$0 = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\big[q(S_f(T) - F^q(t))\big|\mathcal{F}(t)\big]$$

$$F^q(t) = \mathbb{E}_{\mathbb{Q}}\big[S_f(T)\big|\mathcal{F}(t)\big]. \tag{5.2.17}$$

From (5.2.10), it follows that

$$F^q(t) = S_f(t)e^{r-\left(r-r_f+\rho_{S_f Q}\sqrt{V_1}\sqrt{V_2}\right)} \tag{5.2.18}$$

$$= S_f(t)e^{r_f-\rho\sqrt{V_1}\sqrt{V_2}}. \tag{5.2.19}$$

$F^q(t)$ is the fair value of F^q to be fixed at time $t < T$.

6 Common quanto adjustments

Equation (5.2.12) represents a closed-form formula to price quanto call options. Nevertheless, it was derived under the Black-Scholes framework, that is, under the assumptions that the dynamics of $S_f(t)$ and $Q(t)$ are geometric Brownian motions with constant volatility. In the spirit of Remark 3.8, the assumption of constant volatility does not hold in real-world. Hence, the question here is which volatility should be used in order to be consistent with the market. For plain vanilla options, the answer for this question is to use implied volatility $\sqrt{V_{imp,S}(K)}$.

A common practitioners' approach, in the attempt of pricing quanto options consistently with market price of vanilla option, is to set in equation (5.2.12):

$$\rho_{S_fQ}\sqrt{V_1}\sqrt{V_2} \equiv \rho_{S_fQ}\sqrt{V_{1atm}}\sqrt{V_{2atm}} \tag{6.1}$$

and

$$\sqrt{V_1} \equiv \sqrt{V_{1strike}}, \tag{6.2}$$

where $\sqrt{V_{1atm}}$ (resp. $\sqrt{V_{2atm}}$) are the time T at-the-money implied volatilities of the stock (resp. exchange rate). Precisely, $\sqrt{V_{1atm}}$ (resp. $\sqrt{V_{2atm}}$) is obtained from the time T implied volatility smile of the stock $\sqrt{V_{imp,S}(K)}$ (resp. exchange rate $\sqrt{V_{imp,Q}(K)}$), by setting $\sqrt{V_{1atm}} = \sqrt{V_{imp,S}(K = S(t))}$ (resp. $\sqrt{V_{2atm}} = \sqrt{V_{imp,Q}(K = Q(t))}$). The implied volatility smile is defined in (3.9). Analogously, $\sqrt{V_{1strike}} = \sqrt{V_{imp,S}(K = \widetilde{K})}$, where $\widetilde{K}$ denotes the strike of the quanto call option being priced.

Hence, the value of $\sqrt{V_1}$ is set to $\sqrt{V_{1atm}}$ in (6.1), but in (6.2) it is set to $\sqrt{V_{1stike}}$. This is because if $\sqrt{V_1}$ was set to $\sqrt{V_{1stike}}$ in (6.1), then the quanto forward, as given by equation (5.2.19) would depend on an exogenous factor, the option strike, as the quanto forward also depends on $\rho_{S_fQ}\sqrt{V_1}\sqrt{V_2}$. This cannot be, as for each

option strike there would be a different $F^q(t)$ value, and then, as $F^q(t)$ is an expectation over S_T, in the spirit of Proposition 3.7, there would be multiple distributions for S_T, which is not the case.

Hence, a common quanto call option adjustment used by practitioners is, from the application of (6.1) and (6.2) in (5.2.12),

$$C_p^q(t) \quad = \quad q\, BS\left(S_f(t)e^{-\left(r-r_f+\rho_{S_f Q}\sqrt{V_{1atm}}\sqrt{V_{2atm}}\right)(T-t)}, K, \sqrt{V_{1strike}}, (T-t), r\right). \qquad (6.3)$$

7 Quanto option pricing in the Heston model

In this section we present the pricing of quanto options under a bi-dimensional Heston model, based on (Dimitroff, et al., 2011) and (Dimitroff, et al., 2009).

7.1 Risk neutral pricing formula derivation from a foreign investor perspective

For model simulation purposes that will later be clarified, we introduce the pricing of quanto options through foreign risk neutral measure (denoted by $\mathbb{Q}_f$) expectation. From (5.1.1), the quanto option payoff, in DOM currency is given by

$$C^q(T) = \max\{q\big(S_f(T) - K\big), 0\}. \qquad (7.1.1)$$

From a foreign investor perspective, the payoff, given in FOR currency, is

$$C_f^q(T) = Q^{-1}(T)\max\{q\big(S_f(T) - K\big), 0\}. \qquad (7.1.2)$$

From the risk neutral pricing formula (3.3.33), the option value, in FOR currency, at time $t < T$, is given by

$$C_f^q(t) \quad = \quad e^{-r_f(T-t)}\mathbb{E}_{\mathbb{Q}_f}\left[Q^{-1}(T)\max\{q(S_f(T)-K),0\}\middle|\mathcal{F}(t)\right]. \tag{7.1.3}$$

From a non-arbitrage argument, the time t quanto option value, in DOM currency, is given by

$$C^q(t) \quad = \quad Q(t)e^{-r_f(T-t)}\mathbb{E}_{\mathbb{Q}_f}\left[Q^{-1}(T)\max\{q(S_f(T)-K),0\}\middle|\mathcal{F}(t)\right]. \tag{7.1.4}$$

7.2 The bi-dimensional Heston process

From (7.1.4) it is now clear that the simulation of two stochastic processes, namely $S_f(t)$ and $Q^{-1}(t)$, under the foreign risk neutral measure $\mathbb{Q}_f$, is necessary. (Dimitroff, et al., 2009) propose a bi-dimensional Heston process for $(S_f(t), Q^{-1}(t))$. Both $S_f(t)$ and $Q^{-1}(t)$ follow a Heston model, as presented in Subsection 3.5, but $S_f(t)$ and $Q^{-1}(t)$ are allowed to be correlated – the correlation is denoted by ρ in this text. Precisely, the parameters of the Heston process followed by $S_f(t)$ are, from (3.5.12) and (3.5.13), $\boldsymbol{\varphi}_{S_f} = (\rho_1, \kappa_1, \overline{v_1}, V_1(0), \xi_1)$. Analogously, $\boldsymbol{\varphi}_{Q^{-1}} = (\rho_2, \kappa_2, \overline{v_2}, V_2(0), \xi_2)$. Thus

$$dS_f(t) \quad = \quad r_f S_f(t)dt + \sqrt{V_1(t)}\,\mathrm{S_f(t)}dW_{\mathbb{Q}_f 1}(t), \tag{7.2.1}$$

$$dV_1(t) \quad = \quad \kappa_1\left(\overline{v_1} - V_1(t)\right)dt + \xi_1\sqrt{V_1(t)}\left(\rho_1 dW_{\mathbb{Q}_f 1}(t) + \sqrt{1-\rho_1^2}dW_{\mathbb{Q}_f 2}(t)\right). \tag{7.2.2}$$

The process followed by $Q^{-1}(t)$ is given by

$$dQ^{-1}(t) \quad = \quad \left(r_f - r\right)Q^{-1}(t)dt + \sqrt{V_2(t)}\,Q^{-1}(t)dW_{\mathbb{Q}_f 5}(t), \tag{7.2.3}$$

$$dV_2(t) \quad = \quad \kappa_2\left(\overline{v_2} - V_2(t)\right)dt + \xi_2\sqrt{V_2(t)}\left(\rho_2 dW_{\mathbb{Q}_f 5}(t) + \sqrt{1-\rho_2^2}\,dW_{\mathbb{Q}_f 4}(t)\right). \tag{7.2.4}$$

In order to explicit the correlation ρ among the infinitesimal increments of $S_f(t)$ and $Q^{-1}(t)$, we define

$$W_{\mathbb{Q}_f 5}(t) \equiv \left(\rho W_{\mathbb{Q}_f 1}(t) + \sqrt{1-\rho^2} W_{\mathbb{Q}_f 3}(t)\right). \tag{7.2.5}$$

Whence,

$$dQ^{-1}(t) = (r_f - r)Q^{-1}(t)dt + \sqrt{V_2(t)}\, Q^{-1}(t)\left(\rho dW_{\mathbb{Q}_f 1}(t) + \sqrt{1-\rho^2} dW_{\mathbb{Q}_f 3}(t)\right), \tag{7.2.6}$$

$$\begin{aligned} dV_2(t) = {} & \kappa_2\left(\overline{v_2} - V_2(t)\right)dt \\ & + \xi_2\sqrt{V_2(t)}\left(\rho\rho_2 dW_{\mathbb{Q}_f 1}(t) + \rho_2\sqrt{1-\rho^2} dW_{\mathbb{Q}_f 3}(t)\right. \\ & \left. + \sqrt{1-\rho_2^2}\, dW_{\mathbb{Q}_f 4}(t)\right), \end{aligned} \tag{7.2.7}$$

where $W_{\mathbb{Q}_f 1}(t)$, $W_{\mathbb{Q}_f 2}(t)$, $W_{\mathbb{Q}_f 3}(t)$, $W_{\mathbb{Q}_f 4}(t)$ and $W_{\mathbb{Q}_f 5}(t)$ are independent Brownian motions. The correlations stock/variance (ρ_1), exchange-rate/variance (ρ_2) and stock/exchange-rate (ρ) follow from Proposition 3.8.

7.3 Parameters calibration

The parameters $\boldsymbol{\varphi} = (\rho_1, \kappa_1, \overline{v_1}, V_1(0), \xi_1)$ and $\boldsymbol{\varphi}_{Q^{-1}} = (\rho_2, \kappa_2, \overline{v_2}, V_2(0), \xi_2)$ can be calibrated to vanilla options (on both the stock and the exchange rate) market price. This issue is addressed in (Mikhailov, et al., 2003).

7.4 Stock/exchange-rate correlation calibration

In this subsection we address the stock/exchange-rate correlation (ρ) calibration. The main issue is that ρ is the correlation among the infinitesimal increments of $S_f(t)$ and $Q^{-1}(t)$, but what one can observe from simulation of the processes of

$S_f(t)$ and $Q^{-1}(t)$ is the correlation among the simulated log-returns of $S_f(t)$ and $Q^{-1}(t)$, which we denote by $\rho^{o,s}$(o stands for observed and s stands for simulated). Precisely, ρ enters in equation (7.2.7) and an unbiased estimator for $\rho^{o,s}$ is given by:

$$\hat{\rho}^{o,s}(\rho, \boldsymbol{\varphi}_{S_f}^{\mathbb{M}}, \boldsymbol{\varphi}_{Q^{-1}}^{\mathbb{M}}, \mathbb{M}) = \frac{\sum_{t_k}(X_1(t_k) - \overline{X_1})(X_2(t_k) - \overline{X_2})}{\sqrt{\sum_{t_k}(X_1(t_k) - \overline{X_1})^2 \sum_{t_k}(X_2(t_k) - \overline{X_2})^2}}, \tag{7.4.1}$$

where t_k is a partition of time. Without loss of generality, $t_k = \frac{k}{K}(T - t)$, where K is the number of periods in the interval $(T - t)$.

$$X_1(t_k) \equiv \ln\left(\frac{S_f(t_k)}{S_f(t_{k-1})}\right), \tag{7.4.2}$$

$$X_2(t_k) \equiv \ln\left(\frac{Q^{-1}(t_k)}{Q^{-1}(t_{k-1})}\right), \tag{7.4.3}$$

$$\overline{X_1} \equiv \frac{1}{K}\sum_{t_k} X_1(t_k), \tag{7.4.4}$$

$$\overline{X_2} \equiv \frac{1}{K}\sum_{t_k} X_2(t_k). \tag{7.4.5}$$

It is clear from (7.2.1), (7.2.2), (7.2.6) and (7.2.7) that $\rho^{o,s}$ depends on the parameters $\boldsymbol{\varphi}_{S_f}^{\mathbb{M}}, \boldsymbol{\varphi}_{Q^{-1}}^{\mathbb{M}}$ and on ρ. Additionally, $\rho^{o,s}$ depends indirectly on the probability measure $\mathbb{M}$ under which the trajectories are generated, as changing the probability measure changes the parameters. In equations (7.2.1), (7.2.2), (7.2.6) and (7.2.7), the trajectories are generated under the foreign risk-neutral measure $\mathbb{Q}_f$.

It is well known that $\hat{\rho}^{o,s}(\rho, \boldsymbol{\varphi}_{S_f}^{\mathbb{M}}, \boldsymbol{\varphi}_{Q^{-1}}^{\mathbb{M}}, \mathbb{M})$ is an unbiased estimator for $\rho^{o,s}(\rho, \boldsymbol{\varphi}_{S_f}^{\mathbb{M}}, \boldsymbol{\varphi}_{Q^{-1}}^{\mathbb{M}}, \mathbb{M})$, i.e.,

$$\mathbb{E}_{\mathbb{M}}\left[\hat{\rho}^{o,s}(\rho, \boldsymbol{\varphi}_{S_f}^{\mathbb{M}}, \boldsymbol{\varphi}_{Q^{-1}}^{\mathbb{M}}, \mathbb{M})\right] = \rho^{o,s}\left(\rho, \boldsymbol{\varphi}_{S_f}^{\mathbb{M}}, \boldsymbol{\varphi}_{Q^{-1}}^{\mathbb{M}}, \mathbb{M}\right). \tag{7.4.6}$$

The calibration objective is to solve the following optimization problem:

$$\min_{|\rho|\leq 1} \left\| \rho^{o,s}(\rho, \boldsymbol{\varphi}_{S_f}^{\mathbb{M}}, \boldsymbol{\varphi}_{Q^{-1}}^{\mathbb{M}}, \mathbb{M}) - \rho^{tgt} \right\|. \tag{7.4.7}$$

From (7.4.6), this can be rewritten as

$$\min_{|\rho|\leq 1} \left\| \mathbb{E}_{\mathbb{M}} \left[\hat{\rho}^{o,s}(\rho, \boldsymbol{\varphi}_{S_f}^{\mathbb{M}}, \boldsymbol{\varphi}_{Q^{-1}}^{\mathbb{M}}, \mathbb{M}) \right] - \rho^{tgt} \right\|; \tag{7.4.8}$$

where in ρ^{tgt}, tgt stands for target. ρ^{tgt} can be defined by an expert. We remark that commonly ρ^{tgt} is set to be equal to a historical correlation computed in the framework of equation (7.4.1), through a historical time interval $[t_h, t]$ (where $t_h < t$) in place of interval $[t, T]$.

As ρ^{tgt} is set by an expert or computed from historical real-world data, its value is set under the real-world measure $\mathbb{P}$. Therefore, calibration (7.4.8) shall take place with $\mathbb{M} = \mathbb{P}$. Precisely, the calibration procedure should solve the following optimization problem:

$$\min_{|\rho|\leq 1} \left\| \mathbb{E}_{\mathbb{P}} \left[\hat{\rho}^{o,s}(\rho, \boldsymbol{\varphi}_{S_f}^{*}, \boldsymbol{\varphi}_{Q^{-1}}^{*}, \mathbb{P}) \right] - \rho^{tgt} \right\|, \tag{7.4.9}$$

where $\boldsymbol{\varphi}_{S_f}^{*}$and $\boldsymbol{\varphi}_{Q^{-1}}^{*}$ are the parameters under measure $\mathbb{P}$.

In order to make calibration (7.4.9) viable, the parameters $\boldsymbol{\varphi}_{S_f}^{*}, \boldsymbol{\varphi}_{Q^{-1}}^{*}$ of stochastic differential equations (7.2.1), (7.2.2), (7.2.6) and (7.2.7) must be derived for the real-world measure $\mathbb{P}$. Under $\mathbb{P}$, the drift is modified by the market price of risk factors: $\lambda_{S_f} V_1(t)$ for $dS_f(t)$, $\lambda_{V_1} V_1(t)$ for $dV_1(t)$, $\lambda_{Q^{-1}} V_2(t)$ for $dQ^{-1}(t)$ and $\lambda_{V_2} V_2(t)$ for $dV_2(t)$. (Dimitroff, et al., 2011) utilize the (Heston, 1993) assumption on the form of the market price of risk factors, which is λV.

Define:

$$W_{\mathbb{P}1}(t) \;:=\; W_{\mathbb{Q}_f 1}(t) - \lambda_{S_f} \int_0^t \sqrt{V_1(u)}\, du, \tag{7.4.10}$$

$$W_{\mathbb{P}2}(t) \;:=\; W_{\mathbb{Q}_f 2}(t) - \frac{1}{\xi_1\sqrt{1-\rho_1^2}}\left(\lambda_{V_1} - \rho_1\xi_1\lambda_{S_f}\right)\int_0^t \sqrt{V_1(u)}du, \tag{7.4.11}$$

$$W_{\mathbb{P}3}(t) \;:=\; W_{\mathbb{Q}_f 5}(t) - \lambda_{\mathbb{Q}^{-1}}\int_0^t \sqrt{V_2(u)}du, \tag{7.4.12}$$

$$W_{\mathbb{P}4}(t) \;:=\; W_{\mathbb{Q}_f 4}(t) - \frac{1}{\xi_2\sqrt{1-\rho_2^2}}\left(\lambda_{V_2} - \rho_2\xi_2\lambda_{\mathbb{Q}^{-1}}\right)\int_0^t \sqrt{V_2(u)}du, \tag{7.4.13}$$

$$W_{\mathbb{P}5}(t) \;:=\; \left(\rho W_{\mathbb{P}1}(t) + \sqrt{1-\rho^2}W_{\mathbb{P}3}(t)\right). \tag{7.4.14}$$

From Girsanov's Theorem and Definition 7.4.14, Equations (7.2.1), (7.2.2), (7.2.6) and (7.2.7) can be rewritten as

$$dS_f(t) \;=\; \left[r + \lambda_{S_f}V_1(t)\right]S_f(t)dt + \sqrt{V_1(t)}\,\mathrm{S_f(t)}dW_{\mathbb{P}1}(t), \tag{7.4.15}$$

$$\begin{aligned} dV_1(t) \;=\; & \left[\kappa_1\left(\overline{v_1} - V_1(t)\right) + \lambda_{V_1}V_1(t)\right]dt \\ & + \xi_1\sqrt{V_1(t)}\left(\rho_1 dW_{\mathbb{P}1}(t) + \sqrt{1-\rho_1^2}dW_{\mathbb{P}2}(t)\right), \end{aligned} \tag{7.4.16}$$

$$\begin{aligned} dQ^{-1}(t) \;=\; & \left[\left(r_f - r\right) + \lambda_{\mathbb{Q}^{-1}}V_2(t)\right]Q^{-1}(t)dt \\ & + \sqrt{V_2(t)}\,Q^{-1}(t)\left(\rho dW_{\mathbb{P}1}(t) + \sqrt{1-\rho^2}dW_{\mathbb{P}3}(t)\right), \end{aligned} \tag{7.4.17}$$

$$\begin{aligned} dV_2(t) \;=\; & \left[\kappa_2\left(\overline{v_2} - V_2(t)\right) + \lambda_{V_2}V_2(t)\right]dt \\ & + \xi_2\sqrt{V_2(t)}\left(\rho\rho_2 dW_{\mathbb{Q}_f 1}(t) + \rho_2\sqrt{1-\rho^2}dW_{\mathbb{Q}_f 3}(t)\right. \\ & \left. + \sqrt{1-\rho_2^2}\,dW_{\mathbb{Q}_f 4}(t)\right). \end{aligned} \tag{7.4.18}$$

Proposition 7.1.

$$\lim_{K\to\infty}\hat{\rho}^{o,s}(\rho, \boldsymbol{\lambda}^*_{\boldsymbol{S_f}}, \boldsymbol{\lambda}^*_{\boldsymbol{Q^{-1}}}, \mathbb{M}) \;=\; \rho\frac{\int_t^T \sqrt{V_1(s)V_2(s)}\,ds}{\sqrt{\int_t^T V_1(s)\,ds}\sqrt{\int_t^T V_2(s)\,ds}}. \tag{7.4.19}$$

Proof:

We detail here the proof provided by (Dimitroff, et al., 2011).

We can rewrite

$$\hat{\rho}^{o,s}(\rho, \boldsymbol{\lambda}_{S_f}^{\mathbb{M}}, \boldsymbol{\lambda}_{Q^{-1}}^{\mathbb{M}}, \mathbb{M}) = \frac{\sum_{t_k} X_1(t_k)X_2(t_k) - K(\overline{X_1}\,\overline{X_2})}{\sqrt{\left(\sum_{t_k} X_1^2(t_k) - K\overline{X_1}^2\right)\left(\sum_{t_k} X_2^2(t_k) - K\overline{X_2}^2\right)}} \,. \tag{7.4.20}$$

Nonetheless, from Definitions 7.4.2 and 7.4.3, we have

$$\sum_{t_k} X_1(t_k)X_2(t_k) = \sum_{t_k} \left(\ln\left(S_f(t_k)\right) - \ln\left(S_f(t_{k-1})\right)\right)\left(\ln\left(Q^{-1}(t_k)\right) - \ln\left(Q^{-1}(t_{k-1})\right)\right), \tag{7.4.21}$$

and, for $K \to \infty$,

$$= \langle \ln\left(S_f(t)\right), \ln\left(Q^{-1}(t)\right)\rangle. \tag{7.4.22}$$

Because both $dS_f(t)$ and $dQ^{-1}(t)$ are in the format of (3.2.1), both $\ln\left(S_f(t)\right)$ and $\ln\left(Q^{-1}(t)\right)$ follow a generalized Brownian motion in the format of (3.2.2). The co-variation of two generalized Brownian motions is given in Proposition 2.3. Thus

$$\sum_{t_k} X_1(t_k)X_2(t_k) = \rho \int_t^T \sqrt{V_1(t)V_2(t)}\; dt. \tag{7.4.23}$$

Analogously, for $i = \{1,2\}$ and $K \to \infty$,

$$\sum_{t_k} X_i(t_k)X_i(t_k) = \int_t^T V_i(t)\, dt\ . \tag{7.4.24}$$

Inserting (7.4.23) and (7.4.24) into (7.4.20) and taking the limit $K \to \infty$, the statement follows.

■

The dynamics of (7.4.15), (7.4.16), (7.4.17) and (7.4.18) under the real-world measure $\mathbb{P}$ are only used to compute expectations on $\hat{\rho}^{o,s}$ in the calibration problem

(7.4.9). Nevertheless, (7.4.19) shows that, for $K \to \infty$, $\hat{\rho}^{o,s}$ does not depend on the drift of (7.4.15) and (7.4.17).

Thus, we can set

$$\lambda_{S_f} = \lambda_{\mathbb{Q}^{-1}} = 0\,, \tag{7.4.25}$$

as this will not affect the solution of (7.4.9).

Under $\mathbb{P}$, the Heston parameters $\boldsymbol{\varphi}^*_{S_f}$ and $\boldsymbol{\varphi}^*_{Q^{-1}}$ are derived, by setting the drifts in equations (7.4.15) and (7.4.17) equal to their Heston form, i.e., for $i = \{1,2\}$

$$\kappa_i\big(\overline{v_i} - V_i(t)\big) + \lambda_{V_i} V_i(t) = \kappa_i^*\big(\overline{v_i^*} - V_i(t)\big), \tag{7.4.26}$$

$$\kappa_i \overline{v_i} - \kappa_i V_i(t) + \lambda_{V_i} V_i(t) = \kappa_i^* \overline{v_i^*} - \kappa_i^* V_i(t), \tag{7.4.27}$$

$$\kappa_i \overline{v_i} + V_i(t)(-\kappa_i + \lambda_{V_i}) = \kappa_i^* \overline{v_i^*} - \kappa_i^* V_i(t). \tag{7.4.28}$$

Equality holds, if and only if, the two following conditions hold:

$$\kappa_i \overline{v_i} = \kappa_i^* \overline{v_i^*}, \tag{7.4.29}$$

and

$$\kappa_i - \lambda_{V_i} = \kappa_i^*. \tag{7.4.30}$$

Hence,

$$\kappa_i^* = \frac{\kappa_i \overline{v_i}}{\overline{v_i^*}}, \tag{7.4.31}$$

$$\rho_i^* = \rho_i, \tag{7.4.32}$$

$$\xi_i^* = \xi_i, \tag{7.4.33}$$

$$V_i^*(0) = V_i(0). \tag{7.4.34}$$

Thus, in order to obtain parameters $\boldsymbol{\varphi}^*_{S_f}$ and $\boldsymbol{\varphi}^*_{Q^{-1}}$, it suffices to estimate $\overline{v_i^*}$.

Proposition 7.2. (Dimitroff, et al., 2011)

$$\lim_{T\to\infty} \lim_{K\to\infty} \frac{1}{T} \sum_{k=1}^{K} X_i^2(t_k) = \overline{v_i^*}, \tag{7.4.35}$$

for $i = \{1,2\}$, where X_i are defined in (7.4.2) and (7.4.3).

The approach to estimate $\overline{v_i^*}$ is to apply (7.4.35) considering historical log-returns data for a large period and at high frequency. (Dimitroff, et al., 2011) use daily log-returns for a five-year long period.

From (7.4.25), (7.4.31), (7.4.32), (7.4.33), and (7.4.34), we are able to derive the parameters $\boldsymbol{\varphi}_{S_f}^*$ and $\boldsymbol{\varphi}_{Q^{-1}}^*$ of (7.4.15), (7.4.16), (7.4.17) and (7.4.18) under the real-world measure, thus we are able to solve calibration problem 7.4.9. Because $\mathbb{E}_{\mathbb{P}}\left[\hat{\rho}^{o,S}(\rho, \boldsymbol{\lambda}_{S_f}^*, \boldsymbol{\lambda}_{Q^{-1}}^*, \mathbb{P})\right]$ is an increasing function of ρ, a bisection algorithm is proposed by (Dimitroff, et al., 2011) to solve this problem.

7.5 Quanto option pricing

Once ρ and the foreign risk-neutral measure parameters, $\boldsymbol{\varphi}_{S_f} = (\rho_1, \kappa_1, \overline{v_1}, V_1(0), \xi_1)$ and $\boldsymbol{\varphi}_{Q^{-1}} = (\rho_2, \kappa_2, \overline{v_2}, V_2(0), \xi_2)$, are calibrated, we are able to perform the simulations of $S_f(t)$ and $Q^{-1}(t)$, based on their dynamics given by (7.2.1), (7.2.2), (7.2.6) and (7.2.7), and finally compute expectation in (7.1.4), in order to obtain the quanto call option price. We perform simulations under the Euler scheme.

8 Non-parametric approach

8.1 Introduction

Equation (7.1.4) sets a starting point for quanto option pricing. From this equation, a variety of methodologies can be used in order to compute the needed expectation. In this section, we propose a non-parametric methodology, intending to make the pricing of quanto options as adaptable as possible to both market conditions and the dependence relation between $S_f(T)$ and $Q^{-1}(T)$.

The expectation in equation (7.1.4) involves two random variables, namely $S_f(T)$ and $Q^{-1}(T)$, hence, one approach to solve it, is to estimate the bi-variate distribution of these random variables, under measure $\mathbb{Q}_f$. This distribution is denoted by $h(S_f(T),\ Q^{-1}(T))$ in this text. For this matter, Sklar's Theorem will be used.

Theorem 8.1 (Sklar's Theorem). Every multivariate cumulative distribution function (CDF),

$$H(x_1, \dots, x_d) = \mathbb{P}\{X_1 \leq x_1, \dots, X_d \leq x_d\}, \tag{8.1.1}$$

can be expressed in terms of its marginal $F_i(x_i) = \mathbb{P}\{X_i \leq x_i\}$, $i = \{1, \dots, d\}$, and a copula C, such that

$$H(x_1, \dots x_d) = C\big(F_1(x_1), \dots, F_d(x_d)\big). \tag{8.1.2}$$

Proof. The proof for this Theorem is provided in (Lo, 2018), Section 2.

■

For the theory of copulas, we refer to (Nelsen, 2006).

Proposition 8.1. Let x_1 and x_2 be random variables, and let F_1 and F_2 be their respective cumulative distribution functions. Let H be the joint cumulative distribution function, and let C be the copula in equation (8.1.2). Then $\mathbb{P}(X_1 \leq x_1 | X_2 = x_2) = \frac{\partial}{\partial F_2(x_2)} C(F_1(x_1), F_2(x_2))$.

Proof:

$$\mathbb{P}(X_1 \leq x_1 | X_2 = x_2) = \lim_{h \to 0} \mathbb{P}(X_1 \leq x_1 | x_2 \leq X_2 \leq x_2 + h) \tag{8.1.3}$$

$$= \lim_{h \to 0} \frac{H(x_1, x_2 + h) - H(x_1, x_2)}{F_2(x_2 + h) - F_2(x_2)} \tag{8.1.4}$$

$$= \lim_{h \to 0} \frac{C\big(F_1(x_1), F_2(x_2 + h)\big) - C\big(F_1(x_1), F_2(x_2)\big)}{F_2(x_2 + h) - F_2(x_2)} \tag{8.1.5}$$

$$= \lim_{h \to 0} \frac{C\big(F_1(x_1), F_2(x_2) + \Delta(h)\big) - C\big(F_1(x_1), F_2(x_2)\big)}{\Delta(h)} \tag{8.1.6}$$

$$= \frac{\partial}{\partial F_2(x_2)} C\big(F_1(x_1), F_2(x_2)\big). \tag{8.1.7}$$

■

Definition 8.1. Let x_1 and x_2 be random variables and let F_1 and F_2 be their respective cumulative distribution functions. Then, the copula density is given by $c(F_1(x_1), F_2(x_2)) = \frac{\partial^2}{\partial F_1(x_1) \partial F_2(x_2)} C(F_1(x_1), F_2(x_2))$. This can be immediately extended to a d-variables case

$$c(F_1(x_1), \dots, F_d(x_d)) = \frac{\partial^d}{\partial F_1(x_1) \dots \partial F_d(x_d)} C\big(F_1(x_1), \dots, F_d(x_d)\big). \tag{8.1.8}$$

Proposition 8.2.

$$h(x_1, \dots, x_d) = c(F_1(x_1), \dots, F_d(x_d)). f_1(x_1) \dots f_d(x_d), \tag{8.1.9}$$

where

$$c(F_1(x_1), \dots, F_d(x_d)) = \frac{\partial^d C\big(F_1(x_1), \dots, F_d(x_d)\big)}{\partial F_1(x_1) \dots \partial F_d(x_d)} \tag{8.1.10}$$

is the expression introduced in equation (8.1.8).

Proof: Taking derivatives on both sides of equation (8.1.2) in Sklar's Theorem (8.1) and considering that $\frac{\partial F_i(x_i)}{\partial x_i} = f_i(x_i)$, $i = \{1, \dots, d\}$, it follows that

$$\frac{\partial H(x_1, \dots x_d)}{\partial x_1 \dots \partial x_d} = \frac{\partial C\big(F_1(x_1), \dots, F_d(x_d)\big)}{\partial x_1 \dots \partial x_d}, \tag{8.1.11}$$

which gives equation (8.1.9).

■

Sklar's Theorem (8.1) and Equation (8.1.9) establish that the problem of estimating a bivariate distribution function h can be divided into two independent problems: (i) estimating the marginals distributions, and (ii) estimating a copula c, which plays the role of setting the dependence relation between the two random variables.

8.2 Obtaining the marginals

Proposition 8.3. Assume that for every $K \in (0, \infty)$ and given $T > 0$, there exists some $c^m_{Q^{-1}}(K,T)$ and $c^m_{S_f}(K,T)$. Then it is possible to obtain the market implied probability density functions, under the foreign risk neutral measure, for both the exchange rate $Q^{-1}(T)$ and the underlying asset $S_f(T)$. $c^m_u(K,T)$ denotes the consensual market price of a vanilla call option, quoted in foreign currency, over the underlying asset u, with strike K and maturity date equal to T.

Proof: The proof can be obtained immediately by setting, in Proposition 3.7, $\mathbb{Q}$ to be the foreign risk neutral measure and $S(T)$ to be $Q^{-1}(T)$ or $S_f(T)$.

■

If the conditions in Proposition 8.3 hold in practice, the first problem would be solved. Nevertheless, a market quote cannot be found for every K in a given date T. In practice, one finds market quotes $c^m_u(K_i,T)$ for a limited number n of strikes K_i, namely $\left(K_i, c^m_u(K_i,T)\right)$, $i = \{1, \dots, n\}$. Thus, in order to apply Proposition 3.7 and derive the market implied probability distribution function for both $Q^{-1}(T)$ and $S_f(T)$, it is necessary to interpolate ordered pairs $\left(K_i, c^m_u(K_i,T)\right)$, in order to get a continuum function of K, for a gven time T, namely $c^m_u(K,T)$. u is the underlying asset, $Q^{-1}(T)$ or $S_f(T)$.

In order to obtain an arbitrage-free probability density function, it is necessary to consider the non-arbitrage argument when implementing the required interpolation.

Remark 8.1. The arbitrage-free probability density function is constrained to be

(i) coherent with the observed market prices of vanilla options, i.e., when computing equation (3.3.33), taking $C(T)$ to be the pay-off of vanilla options liquid in the market, and using the arbitrage-free probability density function to compute the expectation in (3.3.33), one should obtain the same prices quoted in the market;

(ii) arbitrage-free.

Despite these two constraints, there is some flexibility in building such a probability density function.

In order to guarantee that the constraints mentioned in Remark 8.1 are satisfied, in this text, the probability density function of $Q^{-1}(T)$ and $S_f(T)$ will be obtained by using a stochastic process that is arbitrage-free by construction. Moreover, the parameters of such a stochastic process will be calibrated in such a manner that the constraint (i) in Remark 8.1 is also respected. For this purpose, the parsimonious single Heston model (Heston, 1993) will be calibrated for both, Q^{-1} and S_f, under the foreign risk-neutral measure. The model diffusion equations are (8.2.1) and (8.2.2) and its vector of parameters is (8.2.3). X is to be set to Q^{-1} or S_f for calibration purposes.

$$dX(t) = r_f X(t)dt + \sqrt{V(t)}\,\mathrm{X(t)}dW_{\mathbb{Q}_f}(t), \tag{8.2.1}$$

$$dV(t) = \kappa\big(\bar{v} - V(t)\big)dt + \xi\sqrt{V_1(t)}\left(\rho dW_{\mathbb{Q}_f}(t) + \sqrt{1-\rho^2}dW_{\mathbb{Q}_f}(t)\right), \tag{8.2.2}$$

$$\boldsymbol{\varphi} \equiv (\rho, \kappa, \bar{v}, V(0), \xi). \tag{8.2.3}$$

Once the dynamics in equations (8.2.1) and (8.2.2) are calibrated to vanilla options market quoted prices, simulations can be executed in order to derive the marginal probability density functions of $Q^{-1}(T)$ and $S_f(T)$, namely $f_{Q^{-1}}$ and f_{S_f}. In this text $F_{Q^{-1}}$ and F_{S_f} denote the cumulative distribution functions and $F_{Q^{-1}}^{-1}$ and $F_{S_f}^{-1}$ denote the inverse cumulative distribution functions of $Q^{-1}(T)$ and $S_f(T)$, respectively.

For another approach to estimate the risk-neutral probability density function of the future prices of an underlying asset, from the prices of options written on the asset, see (Monteiro, et al., 2008).

8.3 Dependence relation

In Section 8.2, it is stated that the marginals of $h(S_f(T),\ Q^{-1}(T))$ can be obtained from vanilla options market quoted prices. The marginals are model-dependent only for interpolation matters, but the model parameters are calibrated to vanilla options market quoted prices – no parameter is arbitrarily set.

From Proposition 8.2, the problem of estimating $h(S_f(T),\ Q^{-1}(T))$ will be completely solved once a copula c linking the random variables $S_f(T)$ and $Q^{-1}(T)$ is estimated. This copula stands for the dependence relation between the aforementioned random variables. In most of the current literature on quanto option pricing, the problem of establishing this dependence relation is solved by parametric approaches. For example, in the bi-variate Heston model presented in Section 7, the correlation ρ between $S_f(t)$ and $Q^{-1}(t)$ diffusions is the sole parameter needed to establish the dependence relation. Implicitly, that approach assumes that the copula c is a Gaussian copula with parameter ρ.

The objective of the proposed model is to solve this problem through a non-parametric approach. The main advantage of a non-parametric approach consists in its flexibility, as in this approach, data will calibrate the copula, instead of a set of parameters. Take as an example the model in (Dimitroff, et al., 2009): Only the correlation (a linear measure of dependence) between $S_f(t)$ and $Q^{-1}(t)$ diffusions is taken into account. Nevertheless, two different copulas (e.g., one with heavier tails than the other), which obviously give two different quanto option prices when applying equation (7.1.4), can have the same correlation and give the same quanto option price in the approach adopted by those authors.

Moreover, as the non-parametric approach is data driven and hence more flexible than parametric approaches, the parametric models that are present in literature can be compared to the non-parametric approach, in order to verify in which situations the pricing differences are relevant. This analysis can be used to decide when a simpler (parametric) model is still reliable and in which situations it can be used in practice. This analysis is done is Section 9. The main drawback of the non-parametric approach presented in this text is that it is computationally expensive, as much more simulations must be made when compared against a parametric approach.

Proposition 8.4. Let $Y = (Y_1, Y_2)$ denote a vector of two random variables. Let $\hat{f}(\boldsymbol{y})$, $\boldsymbol{y} = (y_1, y_2)$, denote the Kernel estimator of $f(\boldsymbol{y})$ (Y's PDF). Let K denote a Kernel function, and $\boldsymbol{z}$ be a $(N, 2)$ matrix of provided data, intending to represent the dependence relation between Y_1 and Y_2. Let $\boldsymbol{z}^{(n)}$, $n = \{1, \dots, N\}$, denote the nth line of $\boldsymbol{z}$. Hence,

$$\hat{f}(\boldsymbol{y}) = \frac{1}{Nh^2} \sum_{n=1}^{N} K\left(\frac{\boldsymbol{y} - \boldsymbol{z}^{(n)}}{h}\right) \tag{8.3.1}$$

$$\Rightarrow \quad \hat{F}(\boldsymbol{y}) \quad = \quad \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} \hat{f}(\boldsymbol{x})d\boldsymbol{x}, \tag{8.3.2}$$

where $\hat{F}(\boldsymbol{y})$ is the estimated bivariate cumulative distribution function of $\boldsymbol{y}$.

Proof: This Proposition follows from (Jean-David Fermanian, 2002), Section 3.1.

■

Hence, a copula C can be obtained from (8.3.3) below:

$$C(u_1, u_2) \quad \equiv \quad \hat{F}(\xi_1,\ \xi_2), \tag{8.3.3}$$

where $\xi_i = \inf_y\{y: \hat{F}_i(y) \geq u_i\}$, $u_1, u_2 \in [0,1]$. $\hat{F}_i(y)$ is the CDF of Y_i, $i = \{1,\ 2\}$.

The copula c can be computed applying (8.1.10).

Hence, setting $\boldsymbol{y}$ in Proposition 8.4 to be data on the behavior of S_f and Q^{-1} random variables concerning their dependence relation, gives copula C in (8.3.3), whence copula c can be computed applying (8.1.10). With this copula c and the already obtained marginals $f_{Q^{-1}}$ and f_{S_f}, $h(S_f(T),\ Q^{-1}(T))$ in (8.1.9) can be computed. Finally, the quanto option price from the non-parametric approach can be obtained by applying (7.1.4).

For other quanto option pricing approaches, see (Teng, et al., 2015) and (Kim, et al., 2015).

9 Numerical illustration

9.1 Relation between domestic and foreign correlation

Before considering numerical examples, we state the relation among $\rho_{S_f Q}$ and ρ in the following proposition.

Proposition 9.1. The relation between $\rho_{S_f Q}$, the correlation among the infinitesimal increments of S_f and Q, and ρ, the correlation among the infinitesimal incremements of S_f and Q^{-1}, is given by $\rho_{S_f Q} = -\rho$.

Proof:

Without loss of generality, only stochastic terms shall be considered.

From (5.2.3), it follows that

$$dQ(t) = Q(t)\sqrt{V_2}dW_{\mathbb{Q}2}(t). \tag{9.1.1}$$

From the comparison of (5.2.2) against (7.2.1), the difference among the processes of $S_f(t)$, and for any asset ($Q\ (t)$ included), under the domestic and risk-neutral measure, is only in the drift term. The format of the Brownian motion part remains unaltered. Thus,

$$dQ(t) = Q(t)\sqrt{V_2}dW_{\mathbb{Q}_f 2}(t). \tag{9.1.2}$$

We apply Itô's Lemma to $Q^{-1} = \frac{1}{Q}$. As $\frac{d(Q^{-1})}{dQ} = -\frac{1}{Q^2}$, $\frac{d^2(Q^{-1})}{dQ^2} = \frac{2}{Q^3}$, and $\frac{d(Q^{-1})}{dt} = 0$, we find

$$dQ^{-1}(t) = Q^{-1}(t)\left(-\sqrt{V_2}\right)dW_{\mathbb{Q}_f 2}(t). \tag{9.1.3}$$

From (5.2.4), we get

$$dQ^{-1}(t) = Q^{-1}(t)\sqrt{V_2}\left(-\rho_{S_f Q}dW_{\mathbb{Q}_f 1}(t) - \sqrt{1-\rho_{S_f Q}^2}\,dW_{\mathbb{Q}_f 3}(t)\right). \tag{9.1.4}$$

From (7.2.1), it follows that

$$dS_f(t) = r_f S_f(t)dt + \sqrt{V_1(t)}\,\mathrm{S(t)}dW_{\mathbb{Q}_f 1}(t). \tag{9.1.5}$$

Hence,

$$\rho = Cor\left[dW_{\mathbb{Q}_f1}(t), \left(-\rho_{S_fQ} dW_{\mathbb{Q}_f1}(t) - \sqrt{1-\rho_{S_fQ}^2}\, dW_{\mathbb{Q}_f3}(t)\right)\right] \tag{9.1.6}$$

$$= -Cor\left[dW_{\mathbb{Q}_f1}(t), \left(\rho_{S_fQ} dW_{\mathbb{Q}_f1}(t) + \sqrt{1-\rho_{S_fQ}^2}\, dW_{\mathbb{Q}_f3}(t)\right)\right], \tag{9.1.7}$$

and from the application of Proposition 3.8 we get

$$\rho = -\rho_{S_fQ}. \tag{9.1.8}$$

■

9.2 Numerical examples and analysis

Proposition 9.2. Let U_1, U_2, V_1, V_2 follow a uniform distribution on the $[0,\ 1]$ interval, i.e., $U_1, U_2, V_1, V_2 \sim Uniform\ [0,\ 1]$, then $\left(V_1, c_{V_1}^{-1}(V_2)\right) \sim C(u_1, u_2)$, where $C(u_1, u_2)$ is defined in equation (8.3.3).

Define:

$$c_{u_1}(u_2) \equiv \mathbb{P}\{U_2 \leq u_2 |\ U_1 = u_1\}. \tag{9.2.9}$$

From Proposition 8.1, it follows that

$$c_{u_1}(u_2) = \frac{\partial}{\partial u_1} C(u_1, u_2). \tag{9.2.10}$$

Then,

$$c_{u_1}(u_2) \sim V_2. \tag{9.2.11}$$

Applying the inverse on both sides gives

$$u_2 \sim c_{u_1}^{-1}(V_2), \tag{9.2.12}$$

and as $U_1 \sim Uniform\ [0,\ 1]$, finally

$$\left(V_1, c_{V_1}^{-1}(V_2)\right) \sim C(u_1, u_2). \tag{9.2.13}$$

■

Procedure 9.1. Sampling from copula $C(u_1, u_2)$ in equation (8.3.3)

I. Take V_1, $V_2 \sim Uniform\,[0,\ 1]$ and sample numerous variables v_1 and v_2 from V_1 and V_2. V_1 and V_2 are independent.

II. From (9.2.13), the obtained ordered pairs $\left(v_1, c_{v_1}^{-1}(v_2)\right)$ are a sample from $C(u_1, u_2)$.

In order to illustrate the pricing differences among the Black-Scholes, the bi-dimensional Heston and the non-parametric frameworks, we follow procedure (9.2) below.

Procedure 9.2. This numerical procedure is used to analyze pricing differences among the Black-Scholes framework, the bi-dimensional Heston framework and the proposed non-parametric approach.

I. Set Heston parameters $\boldsymbol{\varphi_{S_f}}$ and $\boldsymbol{\varphi_{Q^{-1}}}$;

II. Compute prices of foreign vanilla call options on both DOM currency and S_f, based on $\boldsymbol{\varphi_{S_f}}$ and $\boldsymbol{\varphi_{Q^{-1}}}$;

III. Derive implied volatility smiles for these options;

IV. Set ρ^{tgt}, $\overline{v_1^*}$ and $\overline{v_2^*}$;

V. Solve calibration problem (7.4.9);

VI. Compute quanto option prices under the Black-Scholes framework, using equation (6.3) and Proposition 9.1;

VII. Compute quanto option prices under the bi-dimensional Heston framework, as described in Subsection 7.5;

VIII. Compute quanto option prices under the non-parametric approach as follows:

i. Take as input a $(N, 2)$ matrix TS_Sf_Q1, representing data on the behavior of S_f and Q^{-1} random variables concerning their dependence relation;

ii. Use the kernel estimator procedure in Proposition 8.4, and compute a copula $C(u_1, u_2)$ using equation (8.3.3);

iii. Sample from copula $C(u_1, u_2)$ using procedure (9.1), obtaining numerous ordered pairs of quantiles $\left(v_1, c_{v_1}^{-1}(v_2)\right)$;

iv. These ordered pairs of quantiles are transformed into S_f and Q^{-1} values, by setting $\left(s_f\ (T),\ q^{-1}(T)\right) = \left(F_{S_f}^{-1}(v_1), F_{Q^{-1}}^{-1}\left(c_{v_1}^{-1}(v_2)\right)\right)$. Hence, $\left(s_f\ (T),\ q^{-1}(T)\right) \sim \left(S_f\ (T),\ Q^{-1}(T)\right)$.

v. Apply equation (7.1.4), such that $C^q(0) = Q(0)\ e^{-r_f(T)} q^{-1}(T)\ \max\{q(s_f(T) - K), 0\}$.

vi. Take the average of the numerous obtained values of $C^q(0)$ to be the price of the quanto option in the non-parametric framework.

IX. Plot the pricing differences.

We next follow the Procedure 9.2 for a set of different volatility smiles and inputted TS_Sf_Q1 matrix cases – pricing differences are analyzed for each case.

The Matlab script used for Procedure 9.2 implementation is presented in Appendix B.

Remark 9.1. The purpose of the numerical examples that follow is to illustrate the conceptual differences among the three models that are the object of the simulation

procedure (9.2). The mathematical concept of each of the models was presented in previous sections of this text.

The numerical examples are based on hypothetical, but parsimonious, market conditions. E.g.: $S_f(0) = 2500$ was the level of the S&P500 index at the beginning of 2018, $Q(0) = 3.1$ is an acceptable exchange rate between the United States dollar and an emerging market currency (e.g. Brazil at the aforementioned period), and $r = 0.1$ ($r_f = 0.01$) is an acceptable interest rate for an emerging country (developed country).

Case I

Case description:

TS_Sf_Q1: is set such that the generated copula is Gaussian.

Volatility smile: no volatility smile for both S_f and Q^{-1}.

$\rho^{tgt} = -0.7$, $\boldsymbol{\varphi}_{S_f} = \boldsymbol{\varphi}_{Q^{-1}} = (0, 0, 0, 0.2, 0)$, $T = 1$, $Q(0) = 3.1$, $S_f(0) = 2500$, $r = 0.1$, $r_f = 0.01$, $\overline{v_1^*} = \overline{v_2^*} = 0.1$, $q = 3$.

Case analysis:

In this case, models are submitted to Black-Scholes framework market conditions – no volatility smile for both S_f and Q^{-1}, volatility is constant for both assets, as can be checked from their vectors of parameters $\boldsymbol{\varphi}_{S_f}$ and $\boldsymbol{\varphi}_{Q^{-1}}$. These are the hypothesis of the Black-Scholes framework, and the bivariate Heston and non-parametric approaches are capable of adapting to it, the bivariate Heston model assuming intrinsically that the copula between S_f and Q^{-1} is Gaussian, and the non-parametric approach being set to assume Gaussian copula by properly setting TS_Sf_Q1 to it. Hence, no pricing differences should be observed, as it is attested by Figure 9.3. Minor differences are due to simulation imprecisions.

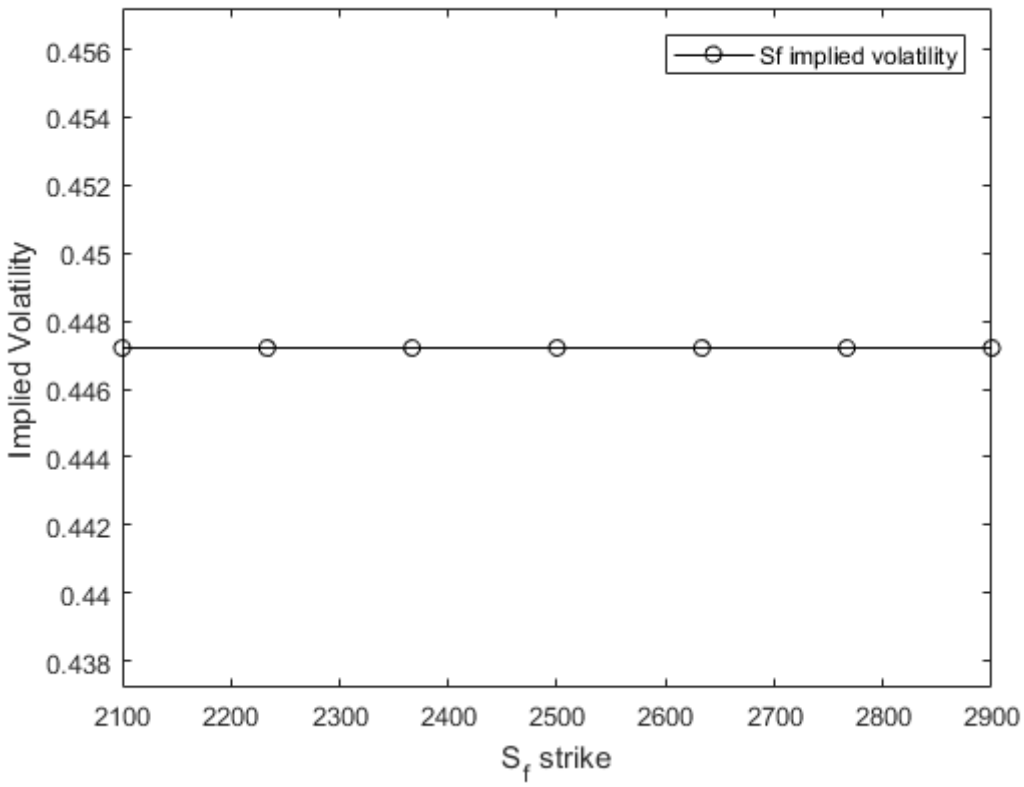

Figure 9.1: Implied volatilities for S_f, as a function of strike, under case I conditions.

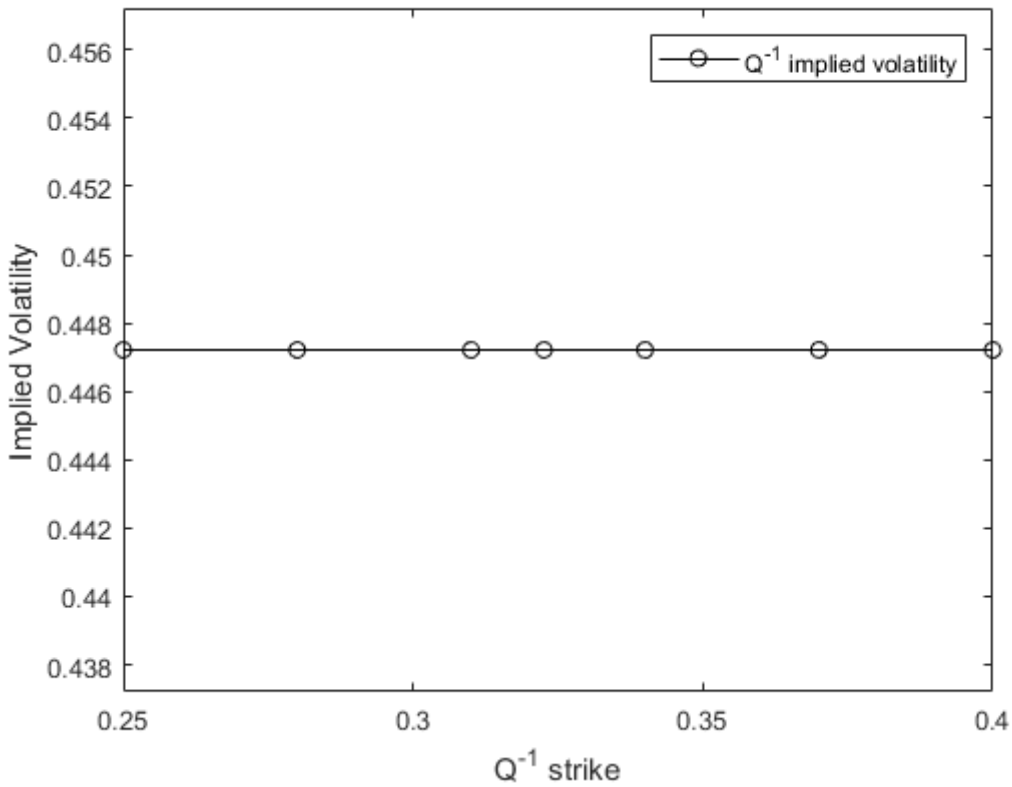

Figure 9.2: Implied volatilities for Q^{-1}, as a function of strike, under case I conditions.

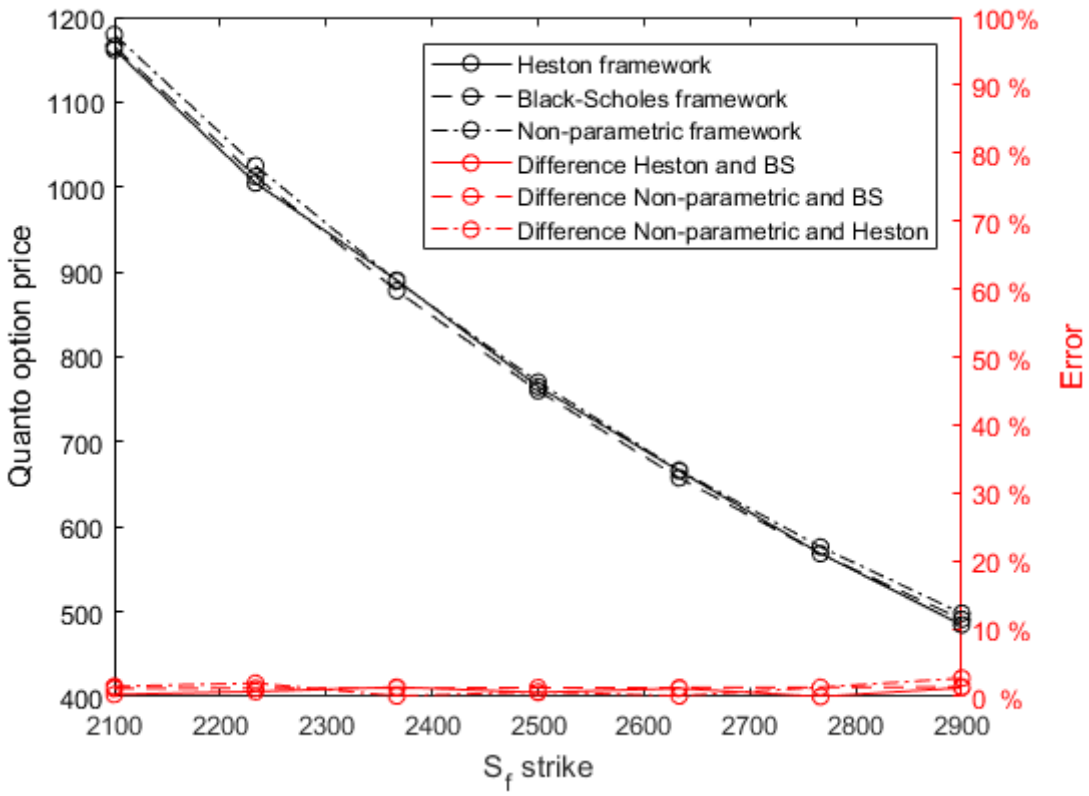

Figure 9.3: Pricing differences, under case I conditions.

Case II

Case description:

TS_Sf_Q1: is set such that the generated copula is Gaussian.

Volatility smile: no volatility smile for both S_f and Q^{-1}.

$\rho^{tgt} = -0.7$, $\boldsymbol{\varphi}_{S_f} = \boldsymbol{\varphi}_{Q^{-1}} = (0, 0, 0, 0.2, 0)$, $T = 3$, $Q(0) = 3.1$, $S_f(0) = 2500$, $r = 0.1$, $r_f = 0.01$, $\overline{v_1^*} = \overline{v_2^*} = 0.1$, $q = 3$.

Case analysis:

In this case, models are submitted to the same conditions of case I, except that now $T = 3$, instead of $T = 1$. This longer time to maturity intends to stress pricing differences, if any. As it is attested by Figure 9.6, and as theoretically expected, no

pricing differences are identified. Minor differences are due to simulation imprecisions.

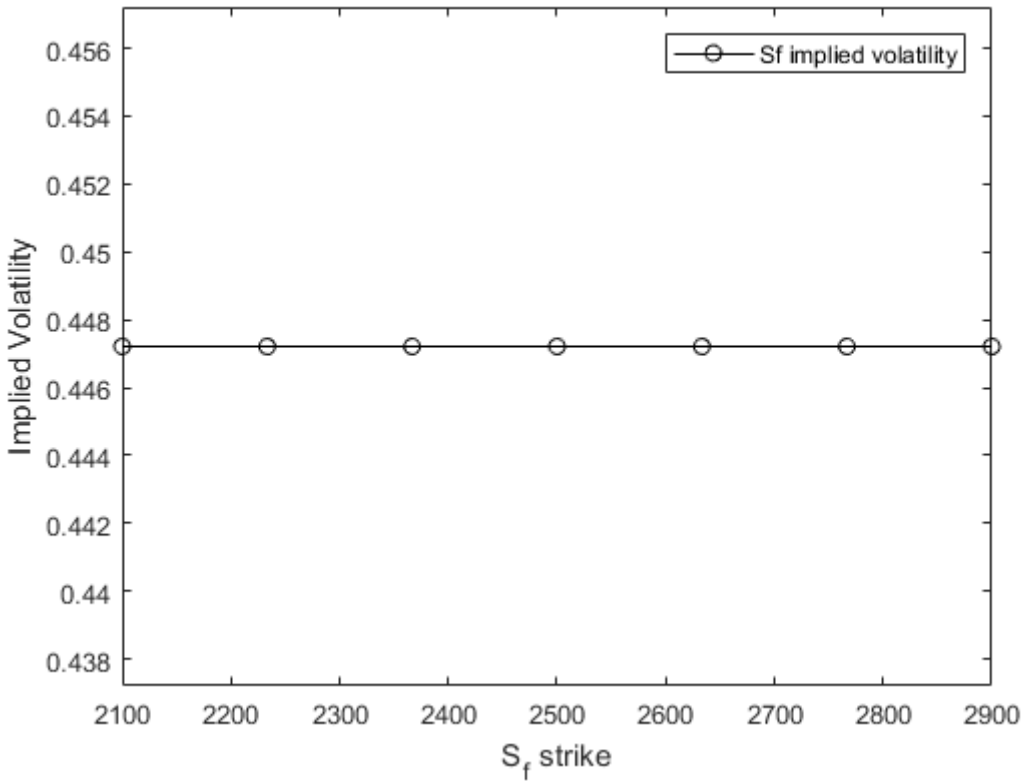

Figure 9.4: Implied volatilities for S_f, as a function of strike, under case II conditions.

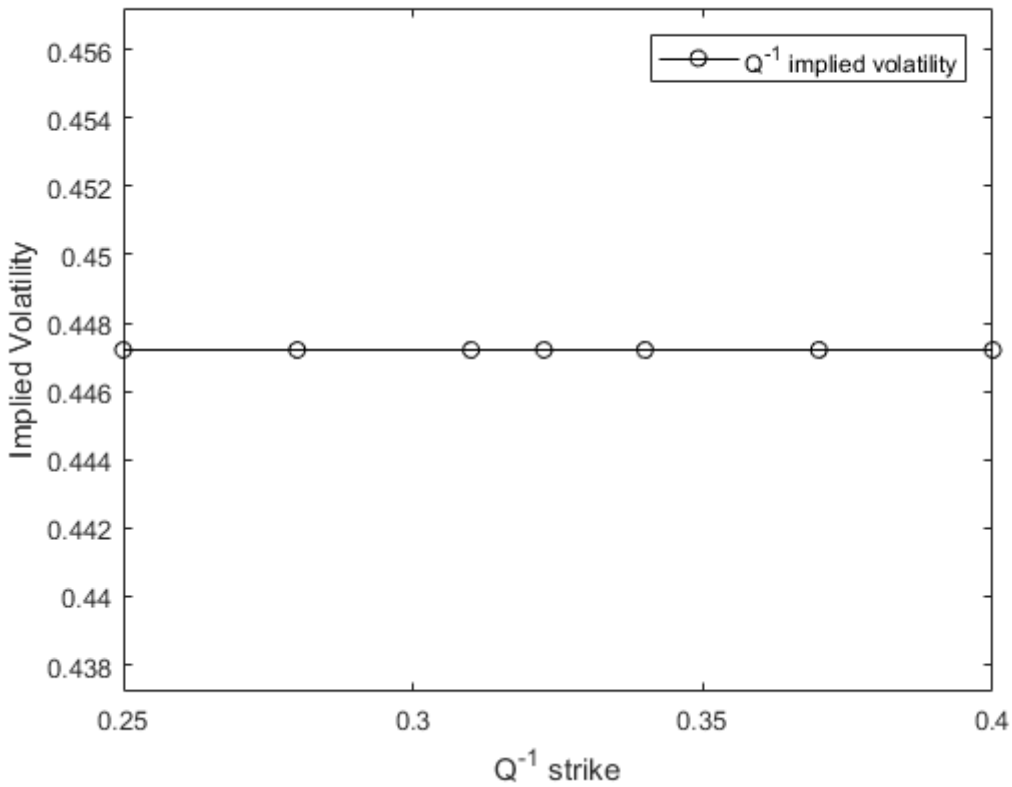

Figure 9.5: Implied volatilities for Q^{-1}, as a function of strike, under case II conditions.

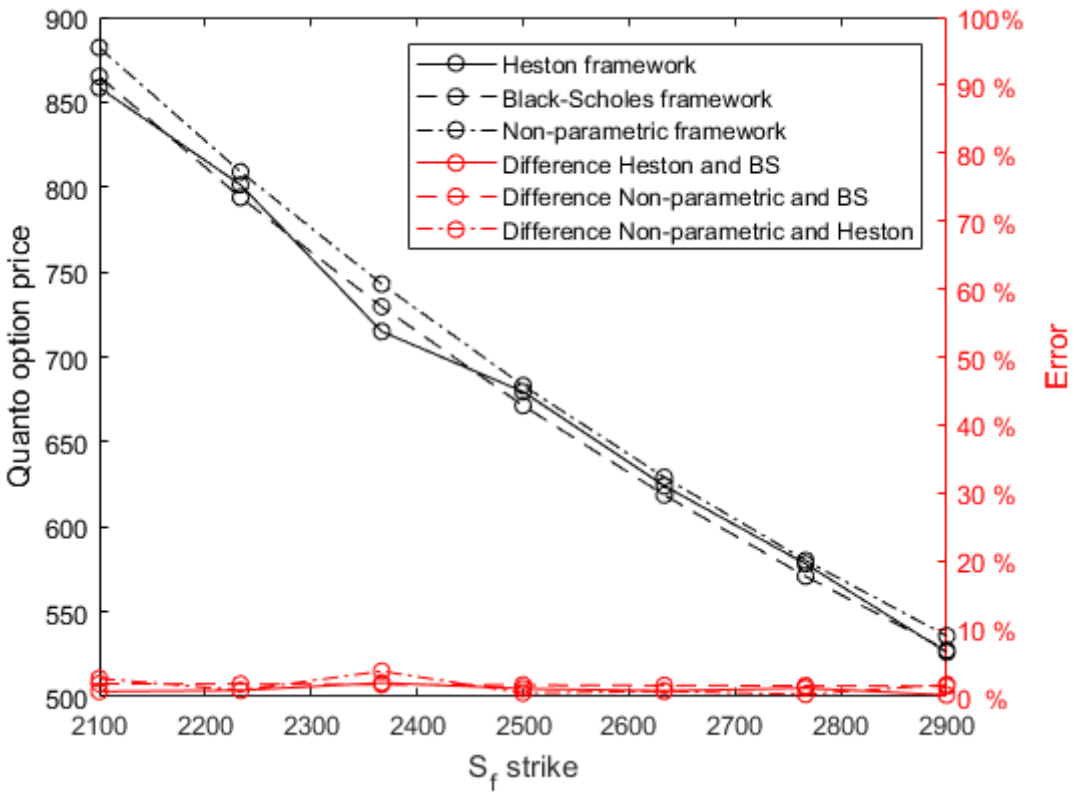

Figure 9.6: Pricing differences, under case II conditions.

Case III

Case description:

TS_Sf_Q1: is set such that the generated copula is Gaussian.

Volatility smile: co-inclining for S_f and Q^{-1}..

$\rho^{tgt} = -0.7$, $\boldsymbol{\varphi}_{S_f} = \boldsymbol{\varphi}_{Q^{-1}} = (-0.7, 1, 0.1, 0.2, 0.5)$, $T = 3$, $Q(0) = 3.1$, $S_f(0) = 2500$, $r = 0.1$, $r_f = 0.01$, $\overline{v_1^*} = \overline{v_2^*} = 0.1$, $q = 3$.

Case analysis:

In this case, models are submitted to co-inclining smile for S_f and Q^{-1}, as can be checked from theirs vectors of parameters $\boldsymbol{\varphi}_{S_f}$ and $\boldsymbol{\varphi}_{Q^{-1}}$, and Figures 9.7 and 9.8. These are the hypothesis of the bivariate Heston framework, and the bivariate

Heston and non-parametric approaches are capable of adapting to it - the bivariate Heston approach assuming intrinsically that the copula between S_f and Q^{-1} is Gaussian, and the non-parametric approach being set to assume Gaussian copula by properly setting TS_Sf_Q1 to it. Hence, no pricing differences should be observed between the bivariate Heston and the non-parametric frameworks. Nevertheless, the fact that a volatility smile is observed, and that the Black-Scholes framework assumes that the volatility is constant, pricing differences should be observed between the Black-Scholes and the other two pricing frameworks. These facts are attested by Figure 9.9. Minor differences between the bivariate Heston and the non-parametric framework are due to simulation imprecisions.

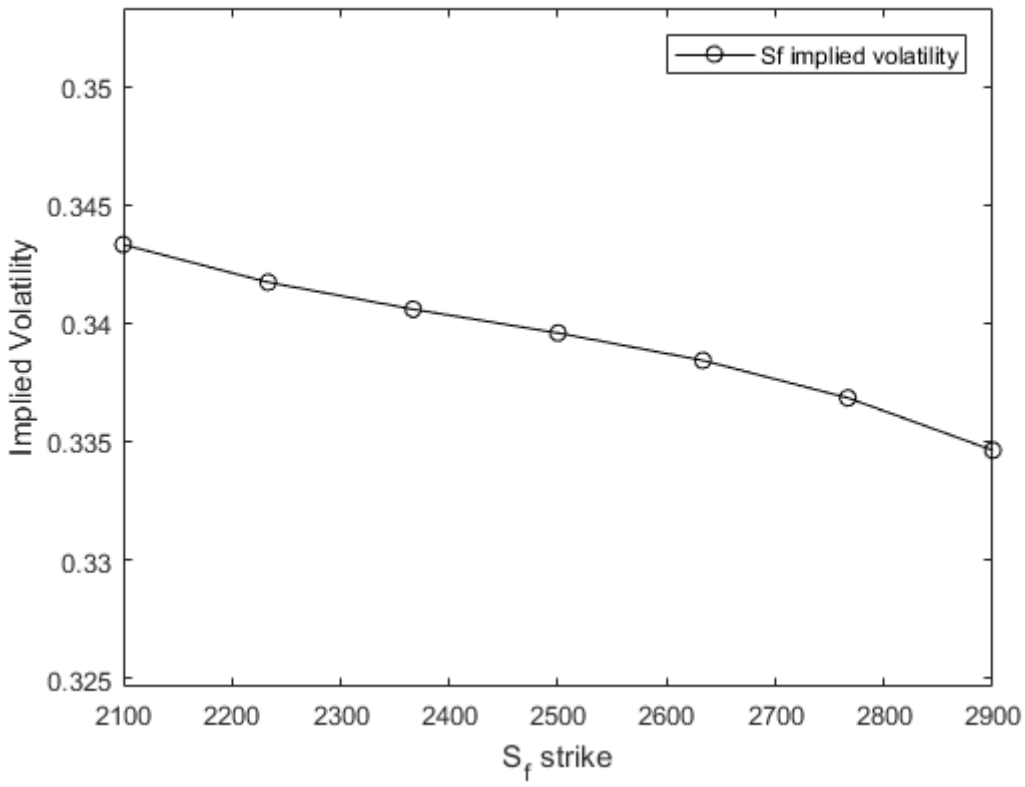

Figure 9.7: Implied volatilities for S_f, as a function of strike, under case III conditions.

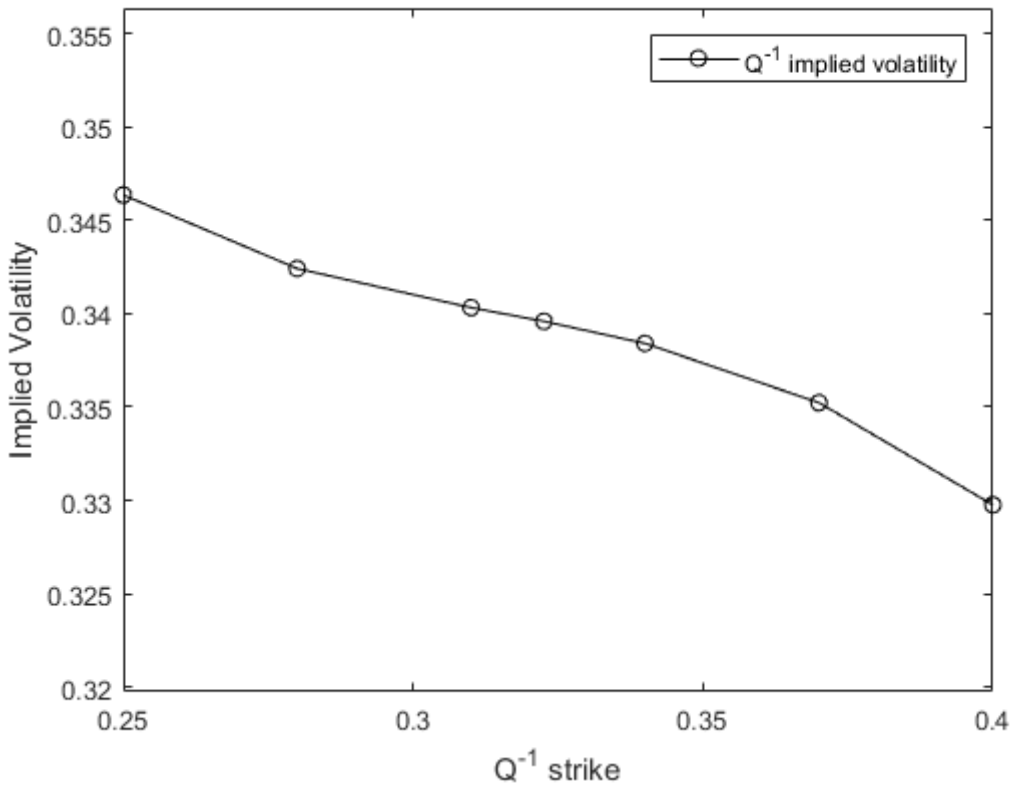

Figure 9.8: Implied volatilities for Q^{-1}, as a function of strike, under case III conditions.

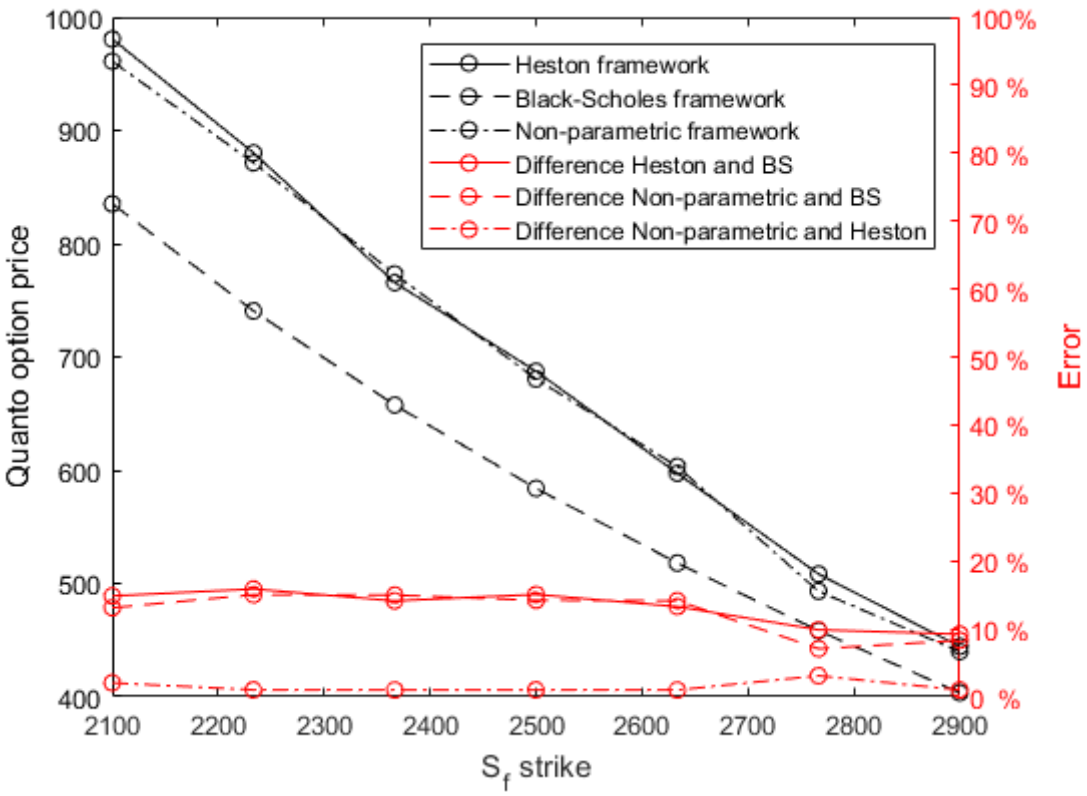

Figure 9.9: Pricing differences, under case III conditions.

Case IV

Case description:

TS_Sf_Q1: is set such that the generated copula is Gaussian.

Volatility smile: co-inclining smile for S_f and Q^{-1}..

$\rho^{tgt} = -0.7$, $\boldsymbol{\varphi}_{S_f} = \boldsymbol{\varphi}_{Q^{-1}} = (-0.7,1,0.1,0.2,0.5)$, $T = \frac{3}{12}$, $Q(0) = 3.1$, $S_f(0) = 2500$, $r = 0.1$, $r_f = 0.01$, $\overline{v_1^*} = \overline{v_2^*} = 0.1$, $q = 3$.

Case analysis:

In this case, models are submitted to the same conditions as in case III, expect that $T = \frac{3}{12}$, instead of $T = 3$. Hence, the analysis should be the same. Nevertheless, what one observes in practice is that there are minor pricing differences between

the Black-Scholes and the other two frameworks. Even if theoretically the Black-Scholes framework is not capable to adapting to the imposed conditions, as the time to maturity is short, the dependence relation between S_f and Q^{-1} does not play a considerably important role in quanto option pricing, and all the frameworks produce practically the same prices. These facts are attested by Figure 9.12. Minor differences between the bivariate Heston and the non-parametric framework are due to simulation imprecisions.

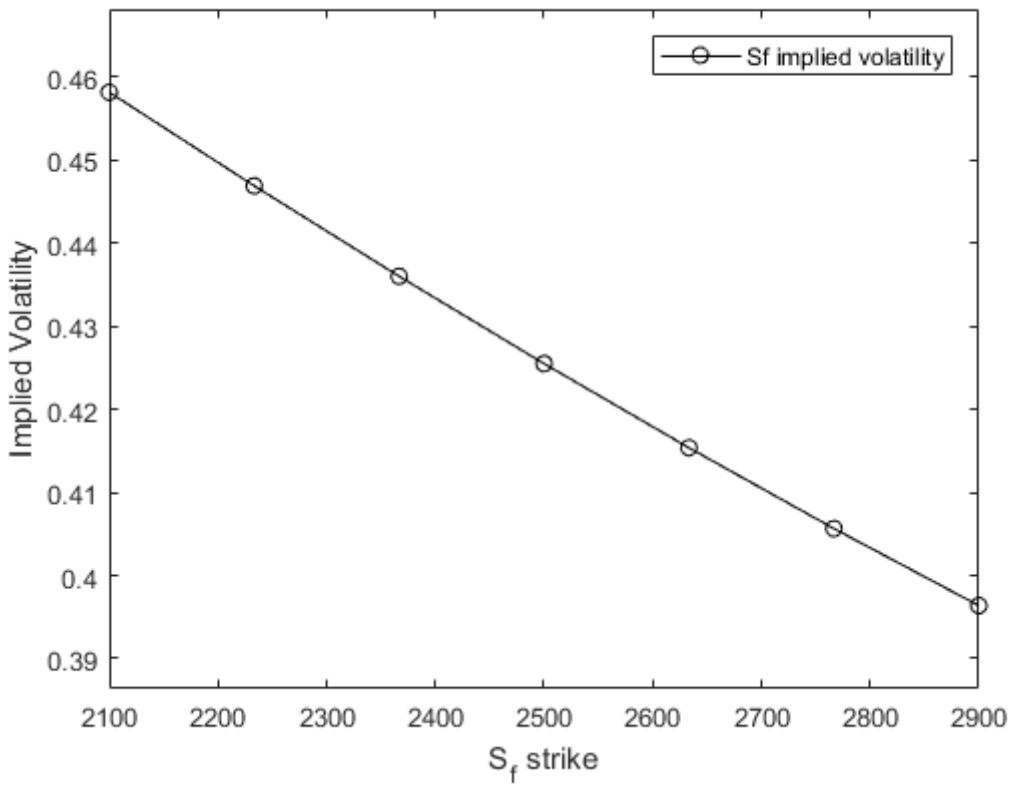

Figure 9.10: Implied volatilities for S_f, as a function of strike, under case IV conditions

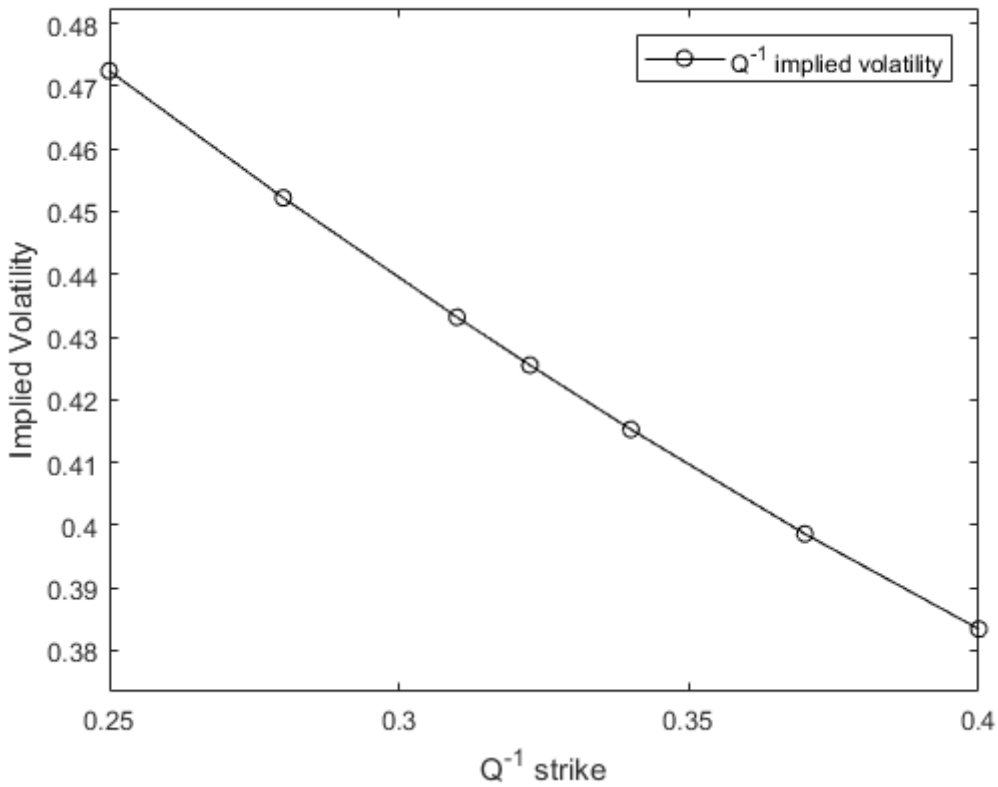

Figure 9.11: Implied volatilities for Q^{-1}, as a function of strike, under case IV conditions.

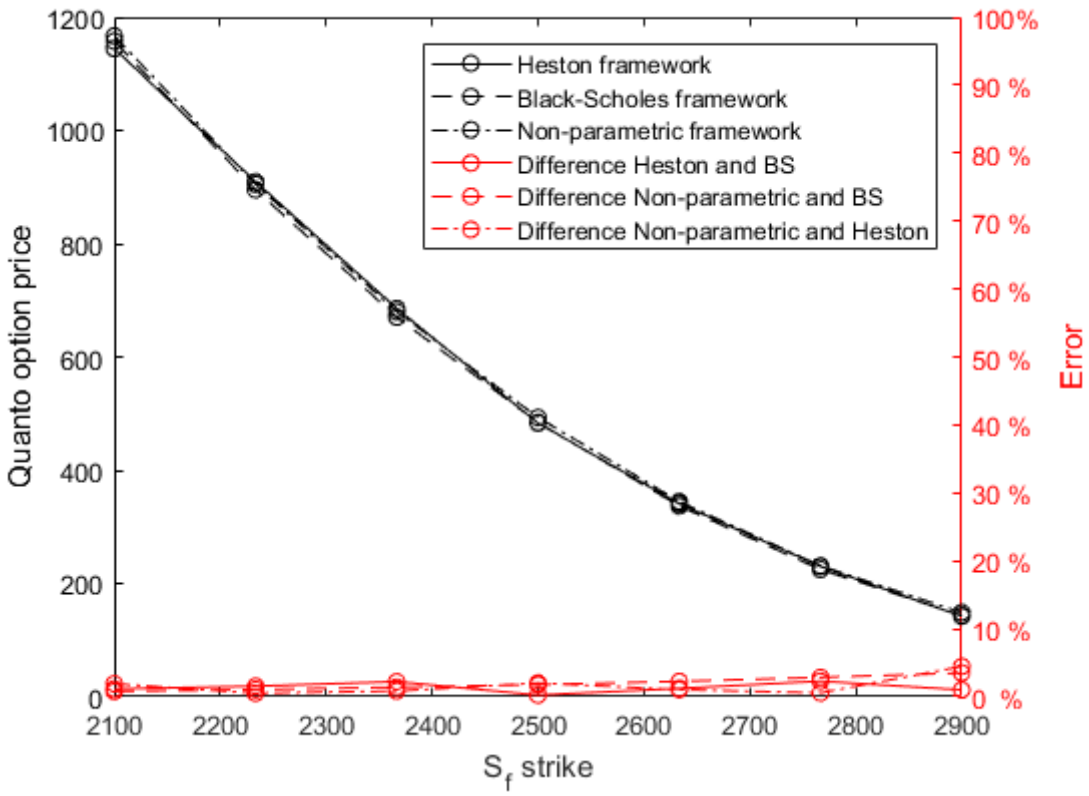

Figure 9.12: Pricing differences, under case IV conditions.

Case V

Remark 9.2 In cases V, VI and VII, we set TS_Sf_Q1 such that the generated copula is t-student with 3 degrees of freedom (see (McNeil, 2004) for the definition of the t-student copula). We do so, in order to illustrate the pricing differences between the non-parametric and the bi-variate Heston approaches, due to the fact that the tails of a t-copula are heavier than those from a Gaussian copula (which is intrinsic in the bi-variate Heston model). The lower the degrees of freedom, the heavier the tails of the t-copula. Hence, in order to maximize the differences between both models, we are interested in making the tails of the t-copula as heavy as possible.

However, we set the degrees of freedom to 3, instead of 1 or 2, because the variance of a student-t distribution with 1 or 2 degrees of freedom is infinite and, as depicted in Figures 9.13 and 9.14, even when sampling a very high number of pairs from a bivariate t-distribution, the obtained correlation can be much different from the correlation of the sampled distribution. Hence, in the 1 and 2 degrees of freedom cases, simulation imprecisions are a major issue, which can harm further model analysis.

As depicted in Figures 9.15 and 9.16, in the 3 and 4 degrees of freedom cases, the correlation between simulated vectors and the sampled distribution correlation are practically identical. Hence, we choose 3 degrees of freedom in cases V, VI and VII, for illustration purposes.

The Matlab script used to generate the aforementioned figures is displayed in Appendix C.

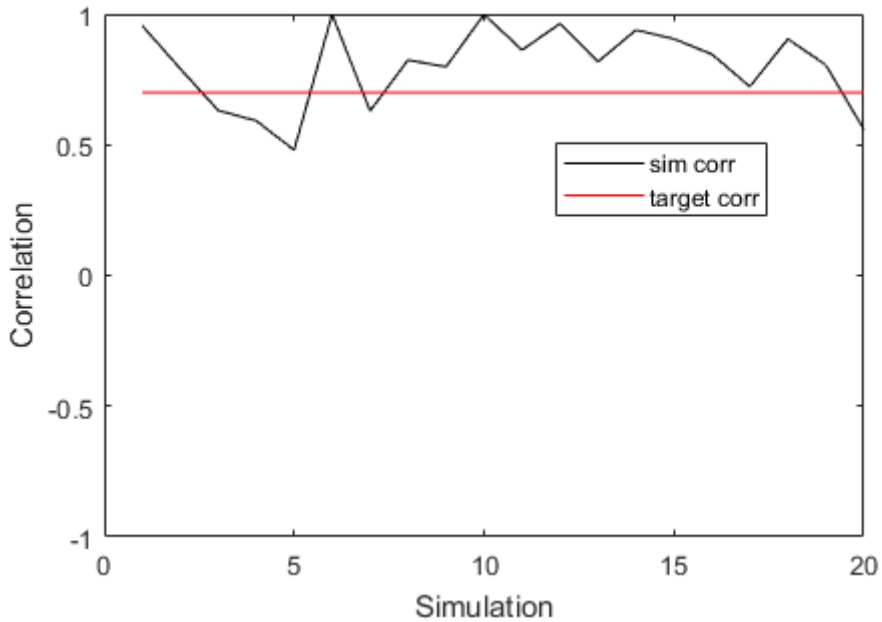

Figure 9.13: correlations obtained from a set of 20 simulations of a bivariate t-student distribution with 1 degree of freedom. The target correlation is set to 0.7 and the number of simulated pairs is set to 1000000.

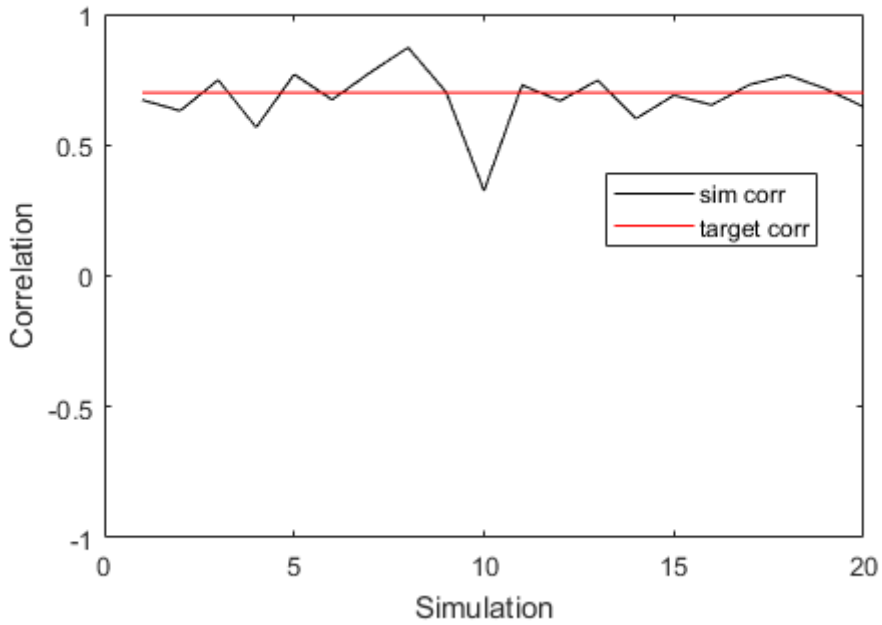

Figure 9.14: Correlations obtained from a set of 20 simulations of a bivariate t-student distribution with 2 degrees of freedom. The target correlation is set to 0.7 and the number of simulated pairs is set to 1000000.

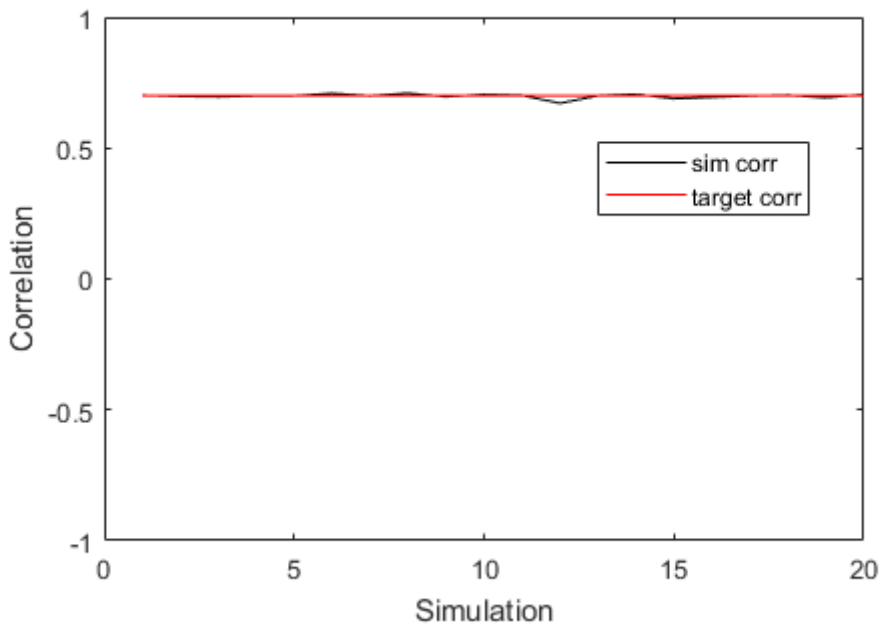

Figure 9.15: Correlations obtained from a set of 20 simulations of a bivariate t-student distribution with 3 degrees of freedom. The target correlation is set to 0.7 and the number of simulated pairs is set to 1000000.

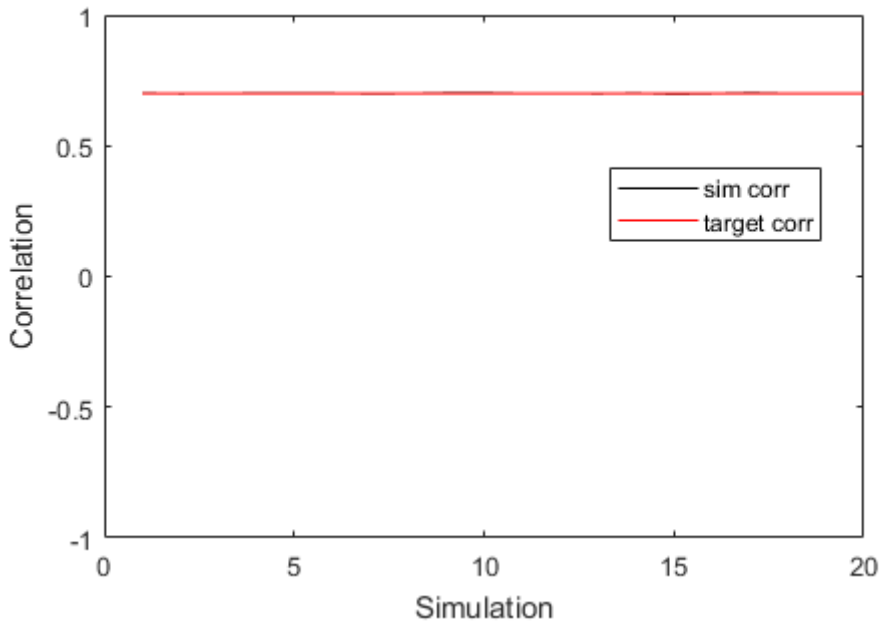

Figure 9.16: Correlations obtained from a set of 20 simulations of a bivariate t-student distribution with 4 degrees of freedom. The target correlation is set to 0.7 and the number of simulated pairs is set to 1000000.

Case description:

TS_Sf_Q1: is set such that the generated copula is a t-copula with 3 degrees of freedom.

Volatility smile: co-inclining smile for S_f and Q^{-1}.

$\rho^{tgt} = -0.7$, $\boldsymbol{\varphi}_{S_f} = \boldsymbol{\varphi}_{Q^{-1}} = (-0.7, 1, 0.1, 0.2, 0.5)$, $T = 3$, $Q(0) = 3.1$, $S_f(0) = 2500$, $r = 0.1$, $r_f = 0.01$, $\overline{v_1^*} = \overline{v_2^*} = 0.1$, $q = 3$.

Case analysis:

In this case, models are submitted to co-inclining smile for S_f and Q^{-1}, as can be checked from its vectors of parameters $\boldsymbol{\varphi}_{S_f}$ and $\boldsymbol{\varphi}_{Q^{-1}}$, and Figures 9.17 and 9.18. Nevertheless, TS_Sf_Q1 is set to generate a t-copula with 3 degrees of freedom. The

Black-Scholes framework is not capable of adapting to this case, because of the imposed volatility smile. On the other hand, the bi-variate Heston framework is not capable to adapt to this case either, because of the fact that it is only capable to adapt to the volatility smile, but not to the fact that the copula between S_f and Q^{-1} is a t-copula - with heavier tails than the Gaussian copula intrinsic in the bivariate Heston framework. Hence, pricing differences should be observed amongst all the three frameworks, as it is attested by Figure 9.19.

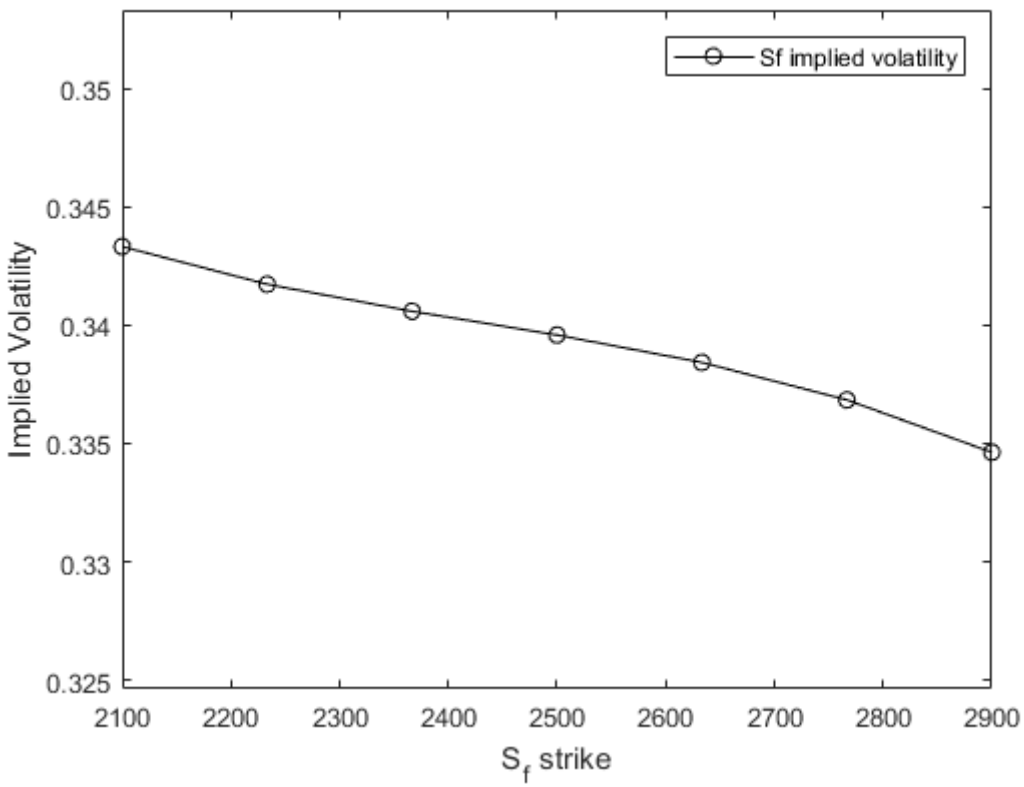

Figure 9.17: Implied volatilities for S_f, as a function of strike, under case V conditions.

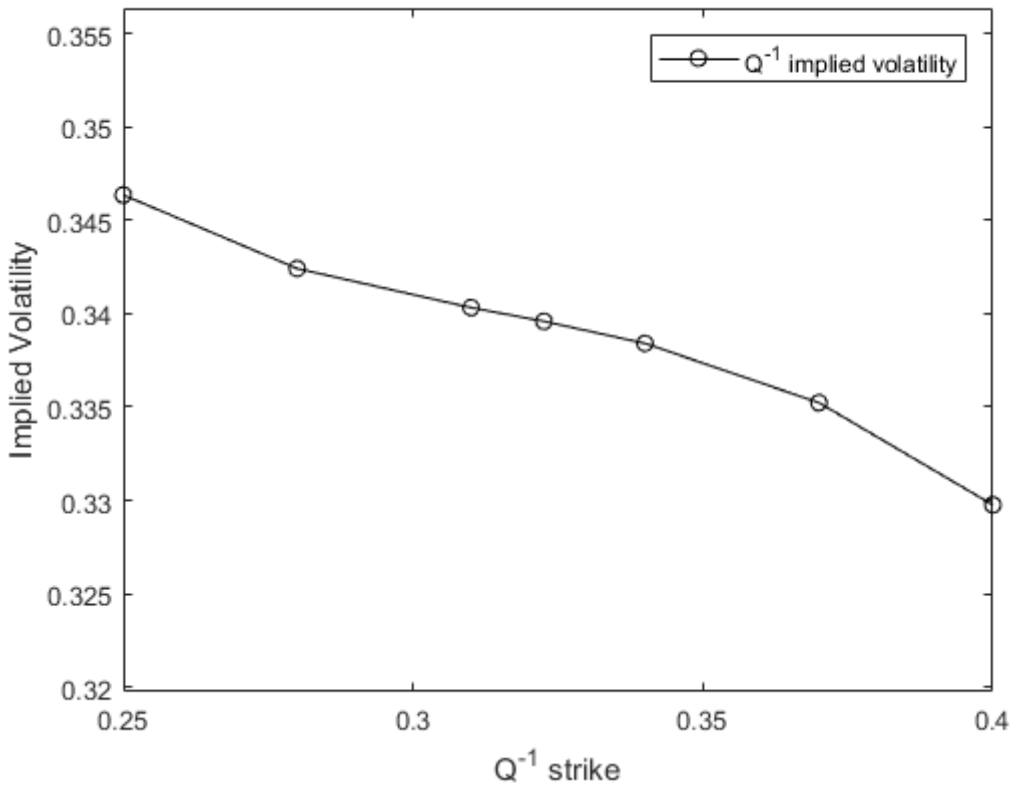

Figure 9.18: Implied volatilities for Q^{-1}, as a function of strike, under case V conditions.

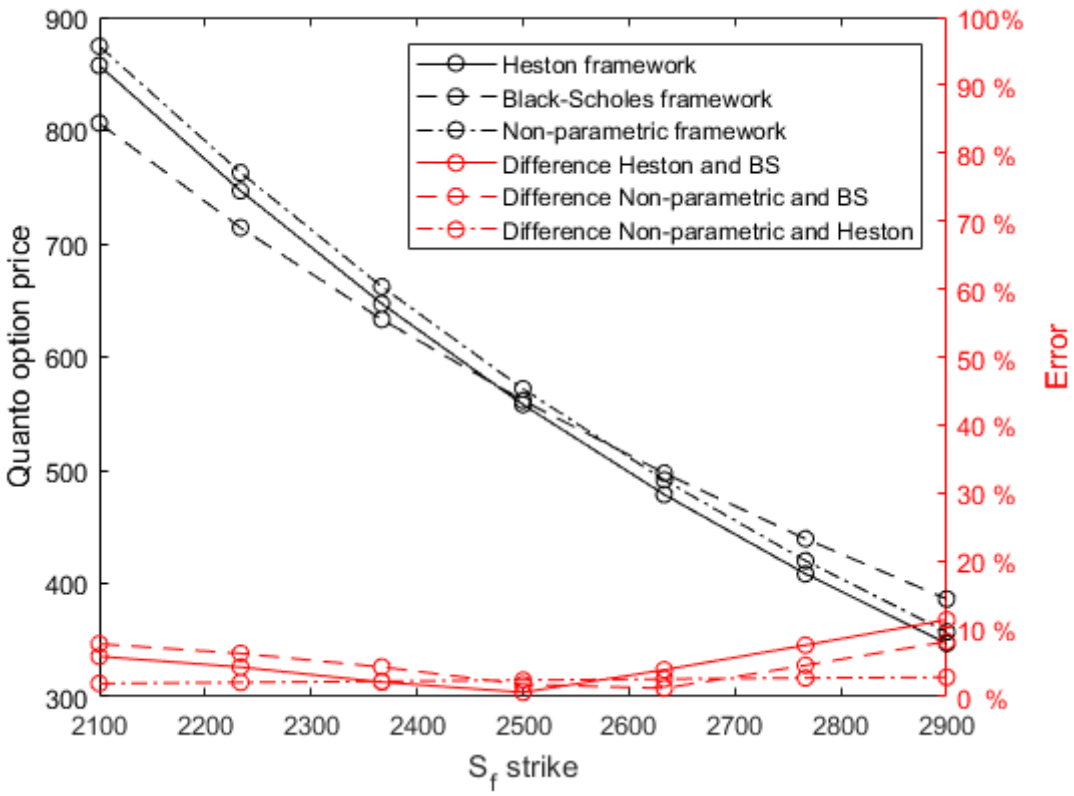

Figure 9.19: Pricing differences, under case V conditions.

Case VI

Case description:

TS_Sf_Q1: is set such that the generated copula is a t-copula with 3 degrees of freedom.

Volatility smile: co-inclining smile for S_f and Q^{-1}.

$\rho^{tgt} = -0.7$, $\boldsymbol{\varphi}_{S_f} = \boldsymbol{\varphi}_{Q^{-1}} = (-0.7, 1, 0.1, 0.2, 0.5)$, $T = 6$, $Q(0) = 3.1$, $S_f(0) = 2500$, $r = 0.1$, $r_f = 0.01$, $\overline{v_1^*} = \overline{v_2^*} = 0.1$, $q = 3$.

Case analysis:

In this case, the conditions are the same as in case V, except that $T = 6$, instead of $T = 3$. This allows to stress the pricing differences among the frameworks due to

their different capability of adapting to the imposed case conditions. Hence, pricing differences should be observed among all the three frameworks, as it is attested by Figure 9.22.

Remark 9.3. In this same case, we also used the following values: $S_f(0) = 3000$, $Q(0) = 3.5$, $\overline{v_1^*} = \overline{v_2^*} = 0.15$, in order to verify if the pricing differences persist. With the aforementioned new values, the pricing differences persisted and the conclusion remains the same.

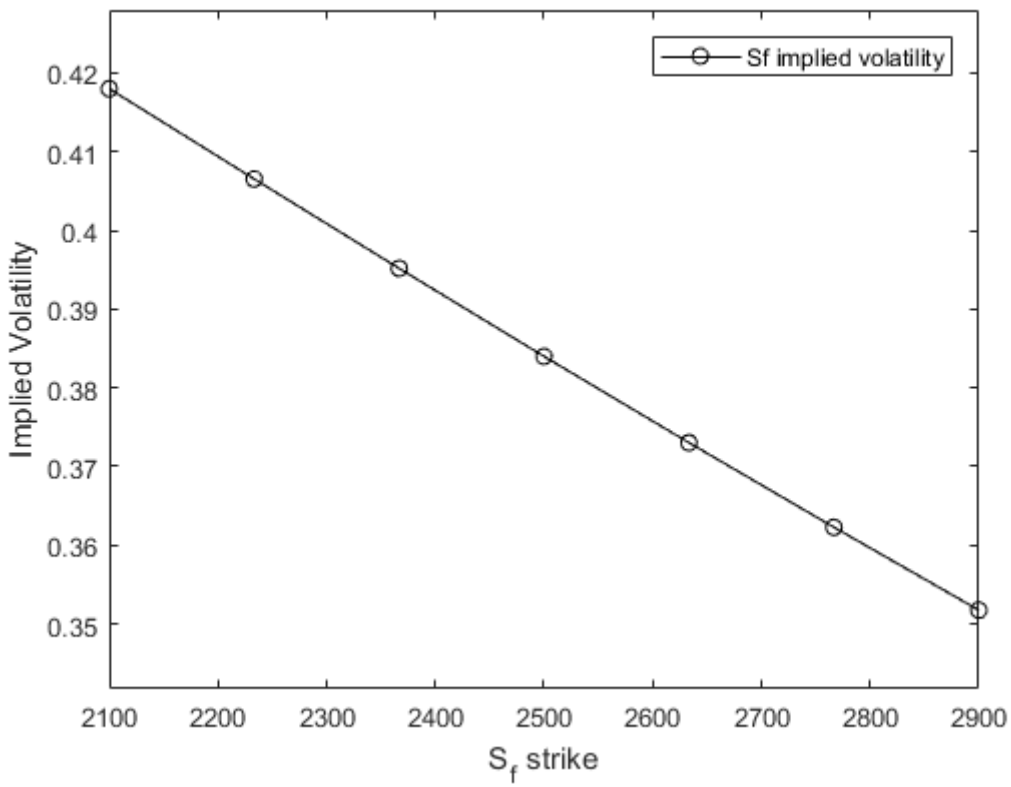

Figure 9.20: Implied volatilities for S_f, as a function of strike, under case VI conditions.

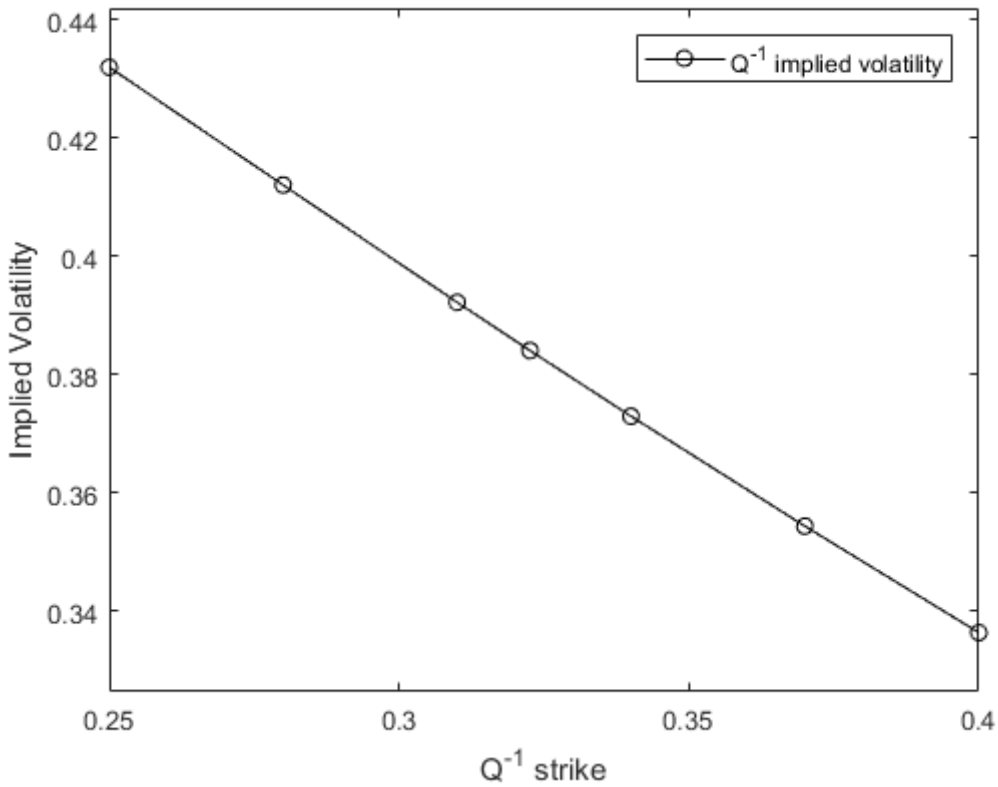

Figure 9.21: Implied volatilities for Q^{-1}, as a function of strike, under case VI conditions.

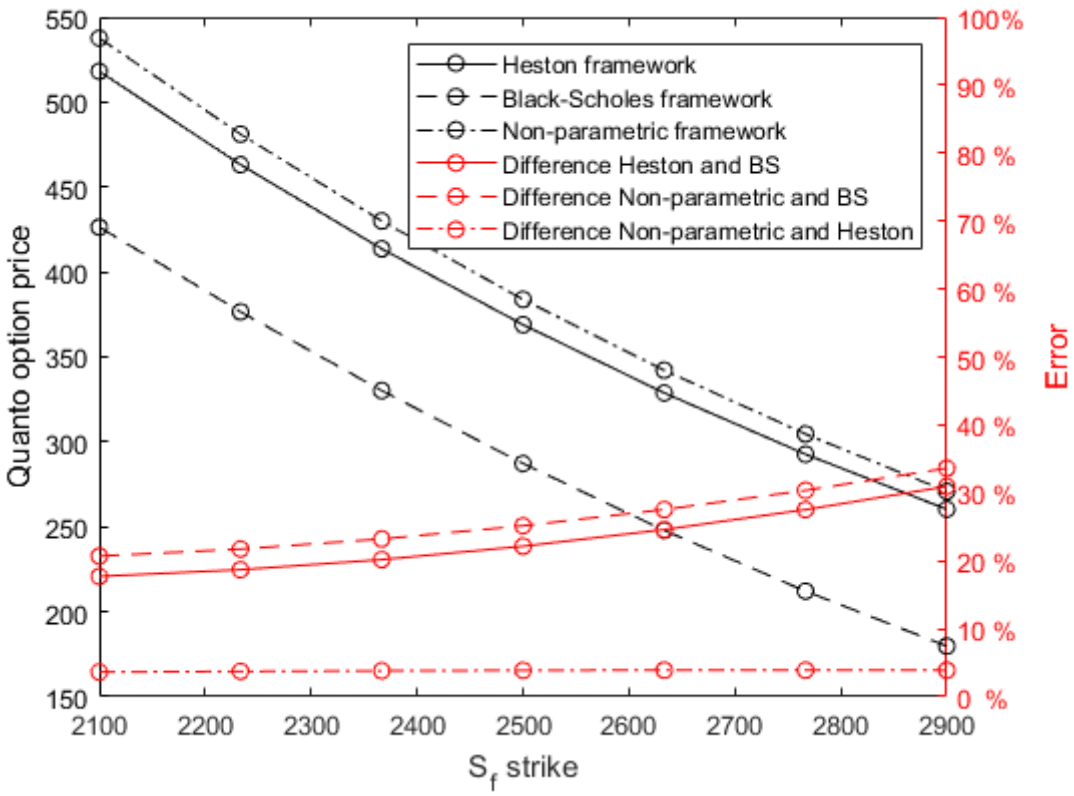

Figure 9.22: Pricing differences, under case VI conditions.

Case VII

Case description:

TS_Sf_Q1: is set such that the generated copula is a t-copula with 3 degrees of freedom.

Volatility smile: co-inclining smile for S_f and Q^{-1}.

$\rho^{tgt} = -0.7$, $\boldsymbol{\varphi}_{S_f} = \boldsymbol{\varphi}_{Q^{-1}} = (-0.7, 1, 0.1, 0.2, 0.5)$, $T = \frac{3}{12}$, $Q(0) = 3.1$, $S_f(0) = 2500$, $r = 0.1$, $r_f = 0.01$, $\overline{v_1^*} = \overline{v_2^*} = 0.1$, $q = 3$.

Case analysis:

In this case, the conditions are the same as in case V, except that $T = \frac{3}{12}$, instead of $T = 3$. This allows to verify the pricing differences among the frameworks due to their different capability of adapting to the imposed case conditions, for a short

time to maturity. Pricing differences should theoretically be observed among all the three frameworks. Nevertheless, as it is attested by Figure 9.25, practically no difference is observed. This is due to the fact that in the short-term, neither the dependence relation between S_f and Q^{-1} nor the volatility smile play a major role in the pricing of quanto options.

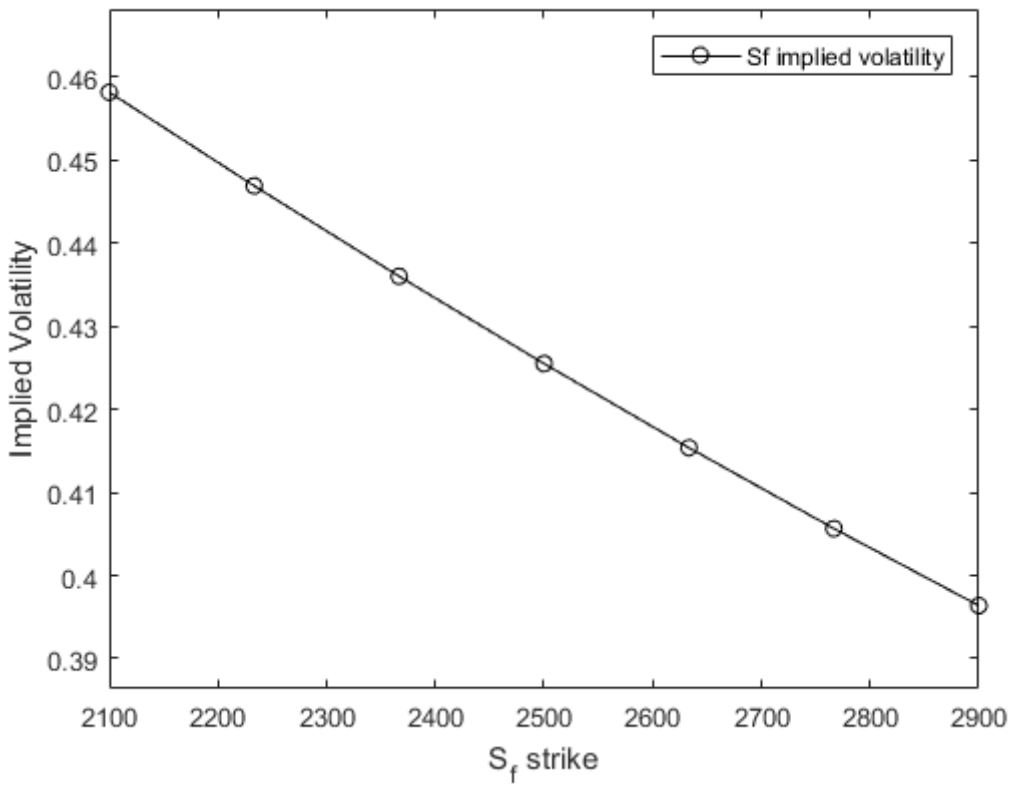

Figure 9.23: Implied volatilities for S_f, as a function of strike, under case VII conditions.

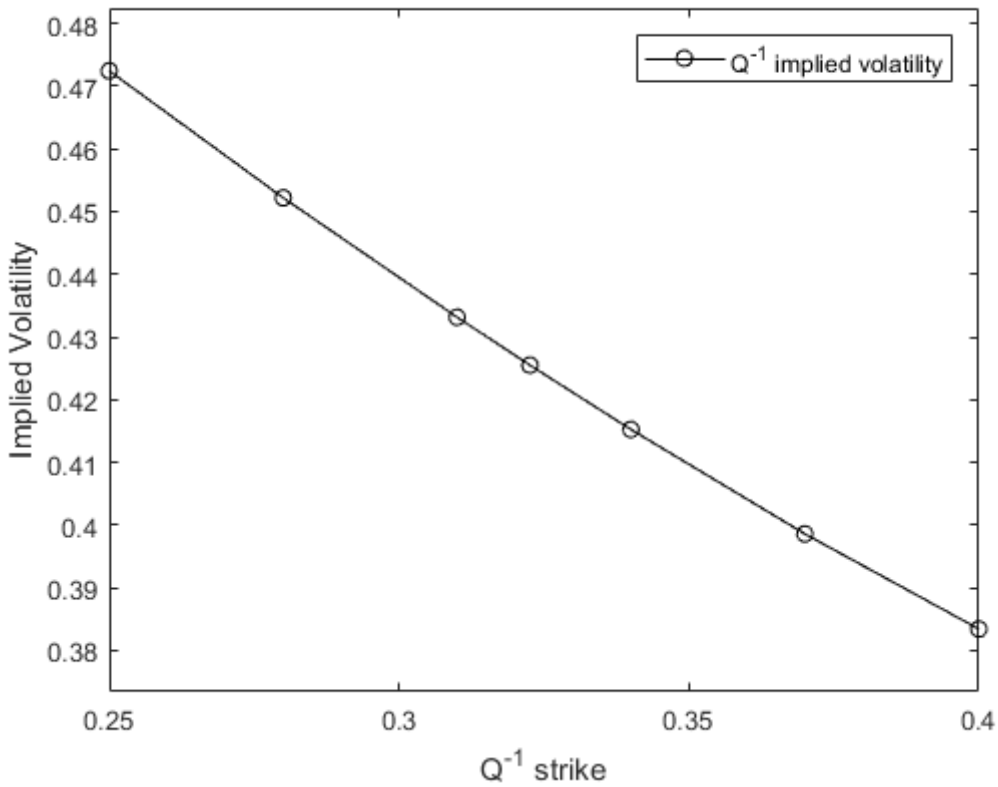

Figure 9.24: Implied volatilities for Q^{-1}, as a function of strike, under case VII conditions.

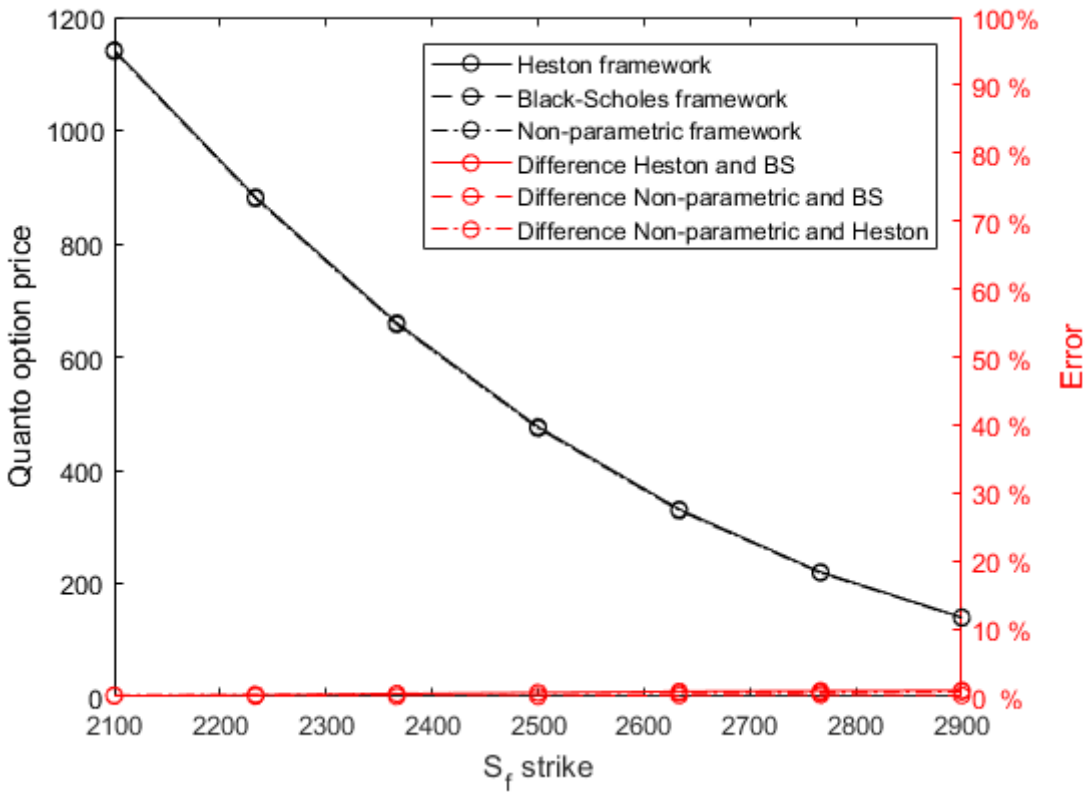

Figure 9.25: Pricing differences, under case VII conditions.

10 Conclusion

We have proposed a non-parametric framework for quanto option pricing and have given numerical examples in order to illustrate the pricing differences among (6.3), the Heston bi-dimensional framework, and the non-parametric framework (proposed by the author). Analyzing the results, we conclude that: (i) except for short dated contracts, quanto option requires explicit modeling for accurate pricing, (ii) the flexibility provided by the non-parametric approach, concerning the dependence relation modeling, results in non-negligible pricing differences when contrasted to less flexible parametric models. On the proposed non-parametric model, we conclude that: (i) it provides a flexible framework to define the dependence relation

between the market variables used in quanto option pricing, (ii) it can adapt to the observed volatility smiles from the relevant market variables, as the marginals of the probability density function h shall be calibrated based on plain vanilla options prices, and (iii) the main drawback of the proposed model is that it is computationally more expensive than the others models it was compared against.

Appendix A

Corollary A.1 (Itô product rule). Let $X(t)$ and $Y(t)$ be Itô processes. Then

$$d(X(t)Y(t)) \quad = \quad X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t). \tag{A.1}$$

Proof: The proof can be found in (Shreve, 2004), page 168.

We state the following theorem, presented in (Shreve, 2004, page 221), without proof.

TheoremA.2 (Martingale representation). Let $W_{\mathbb{P}}(t)$, $0 \leq t \leq T$, be a Brownian motion on a triple $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{F}(t)$ be the filtration generated by this Brownian motion.

If $M_{\mathbb{P}}(t)$ is a martingale with respect to the filtration $\mathcal{F}(t)$, that is if

(i) $M_{\mathbb{P}}(t)$ is $\mathcal{F}(t)$-measureble and;

(ii) for $0 \leq s \leq t \leq T$

$$\mathbb{E}_{\mathbb{P}}[M_{\mathbb{P}}(t)|\mathcal{F}(s)] \quad = \quad M_{\mathbb{P}}(s), \tag{A.2}$$

then, there is an adapted process $\Gamma_{\mathbb{P}}(u)$, $0 \leq u \leq T$, such that

$$M_{\mathbb{P}}(t) \quad = \quad M_{\mathbb{P}}(0) + \int_0^t \Gamma_{\mathbb{P}}(u)dW_{\mathbb{P}}(u). \tag{A.3}$$

Corollary A.3 (from Theorem A.2)

Let $M_{\mathbb{Q}}(t)$, $0 \leq t \leq T$, be a martingale under $\mathbb{Q}$. Then, there is an adapted process $\Gamma_{\mathbb{Q}}(u)$, $0 \leq u \leq T$, such that

$$M_{\mathbb{Q}}(t) \quad = \quad M_{\mathbb{Q}}(0) + \int_0^t \Gamma_{\mathbb{Q}}(u)dW_{\mathbb{Q}}(u). \tag{A.4}$$

Proof:

Firstly, we recall Girsanov's Theorem (2.3)

Let $W_{\mathbb{P}}(t)$ be a Brownian motion on the triple $(\Omega, \mathcal{F}, \mathbb{P})$, $\Theta(t)$ be an adapted process and $0 \leq t \leq T$. Define

$$Z(t) \coloneqq \exp\left(-\int_0^t \Theta(u)dW_{\mathbb{P}}(u) - \frac{1}{2}\int_0^t \Theta^2(u)dW_{\mathbb{P}}(u)\right), \tag{A.5}$$

$$W_{\mathbb{Q}}(t) \coloneqq W_{\mathbb{P}}(t) + \int_0^t \Theta(u)du, \tag{A.6}$$

and assume that

$$\mathbb{E}\left[\int_0^T \Theta^2(u)Z^2(u)du < \infty\right]. \tag{A.7}$$

Then $Z = \frac{\partial \mathbb{Q}}{\partial \mathbb{P}}$ is a Radon-Nikodym derivative, and, under the probability measure $\mathbb{Q}$ given by (2.3.4), the process $W_{\mathbb{Q}}(t)$ is a Brownian motion.

The filtration $\mathcal{F}$ is generated by $W_{\mathbb{P}}$, not $W_{\mathbb{Q}}$.

We now present the proof of the corollary.

From (A.5) we find

$$dZ(t) \quad = \quad -Z(t)\Theta(t)dW(t). \tag{A.8}$$

From Itô's Lemma, with $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$, we get

$$d\left(\frac{1}{Z(t)}\right) \quad = \quad f'\big(Z(t)\big)dZ(t) + \frac{1}{2}f''\big(Z(t)\big)dZ(t)dZ(t) \tag{A.9}$$

$$= \frac{\Theta(t)}{Z(t)} dW(t) + \frac{\Theta^2(t)}{Z(t)} dt. \tag{A.10}$$

Lemma 2.2 implies that, for $0 \leq s < t$,

$$M_{\mathbb{Q}}(s) = \mathbb{E}_{\mathbb{Q}}[M_{\mathbb{Q}}(t)|\mathcal{F}(s)] \tag{A.11}$$

$$= \frac{1}{Z(s)} \mathbb{E}_{\mathbb{P}}\big[Z(t)M_{\mathbb{Q}}(t)\big|\mathcal{F}(s)\big]. \tag{A.12}$$

Hence,

$$M_{\mathbb{Q}}(s)Z(s) = \mathbb{E}_{\mathbb{P}}\big[Z(t)M_{\mathbb{Q}}(t)\big|\mathcal{F}(s)\big]; \tag{A.13}$$

which shows that $M_{\mathbb{P}}(t) = M_{\mathbb{Q}}(t)Z(t)$ is a martingale under $\mathbb{P}$:

$$dM_{\mathbb{Q}}(t) = d\left(M_{\mathbb{P}}(t)\frac{1}{Z(t)}\right). \tag{A.14}$$

From (A.3) and the Itô product rule, we get that

$$dM_{\mathbb{Q}}(t) = \frac{\Gamma_{\mathbb{P}}(t)}{Z(t)} dW_{\mathbb{P}}(t) + \frac{M_{\mathbb{P}}\Theta(t)}{Z(t)} dW_{\mathbb{P}}(t) + \frac{M_{\mathbb{P}}\Theta^2(t)}{Z(t)} dt + \frac{\Gamma_{\mathbb{P}}(t)\Theta(t)}{Z(t)} dt \tag{A.15}$$

$$= \frac{\Gamma_{\mathbb{P}}(t) + M_{\mathbb{P}}\Theta(t)}{Z(t)} (dW_{\mathbb{P}}(t) + \Theta(t)dt). \tag{A.16}$$

Setting

$$\Gamma_{\mathbb{Q}}(t) := \frac{\Gamma_{\mathbb{P}}(t) + M_{\mathbb{P}}\Theta(t)}{Z(t)}, \tag{A.17}$$

(A.16) becomes

$$dM_{\mathbb{Q}}(t) = \Gamma_{\mathbb{Q}}(t)\, dW_{\mathbb{Q}}(t), \tag{A.18}$$

which proves the corollary.

■

Appendix B

main.m

```
clear all;
clc;
% Skew case and maturity setting
skew_case=2; % 0 for no skew, 1 for no exchange rate skew, 2 for co-
inclining skew
mmaturity= 3/12;

% Prices of foreign vanilla call options (quoted in FOR currency)
%on both DOM currency and Sf (denoted C_Q1 and C_Sf respectively). Im-
plied
%volatilities computation.

% Initial values
Q_0=3.1;
Q1_0=1/Q_0; % Q-1 current value
Sf_0=2500; % Sf current value
r=0.1; % domestic risk-free interest rate
rf=0.01; % foreign risk-free interest rate


% Heston Parameters for Sf
mkappa=1;
mtheta=0.1;
mrho=-0.7;
msigma=0.5;
mv0=0.2;

% Heston Parameters for Q-1
mkappa_2=1;
mtheta_2=0.1;
mrho_2=-0.7;
msigma_2=0.5;
mv0_2= 0.2;

n = 5;
strk_C_Sf=zeros(n+2,1);
C_Sf=zeros(n+2,1);
iv_C_Sf=zeros(n+2,1);
strk_C_Q1=zeros(n+2,1);
C_Q1=zeros(n+2,1);
iv_C_Q1=zeros(n+2,1);

if skew_case==0
    i=1;
    for strk= 2100: ((2900-2100)/(n+1)):2900
         strk_C_Sf (i)= strk;
         C_Sf (i) = blsprice(Sf_0, strk, rf,mmaturity,sqrt(mv0),0) ;
         iv_C_Sf (i)= blsimpv(Sf_0,strk_C_Sf (i),rf,mmaturity,C_Sf
(i));
         i=i+1;
    end

    i=1;
    for strk= [1/4: ((1/2.5-1/4)/n):1/2.5, Q1_0]
```

```
        strk_C_Q1 (i)= strk;
        C_Q1 (i) =  blsprice (Q1_0, strk, rf,mmatu-
rity,sqrt(mv0_2),0);
        iv_C_Q1 (i)= blsimpv(Q1_0,strk_C_Q1 (i),rf,mmaturity,C_Q1
(i));
        i=i+1;
    end
elseif skew_case==1

    i=1;
    for strk= 2100: ((2900-2100)/(n+1)):2900
        strk_C_Sf (i)= strk;
        C_Sf (i) =
HestonPrice('Call',mkappa,mtheta,0,mrho,msigma,mmatu-
rity,strk,Sf_0,rf,mv0) ;
        %PutCall, kappa, theta, lambda, rho, sigma, tau, K, S, r, v0
        iv_C_Sf (i)= blsimpv(Sf_0,strk_C_Sf (i),rf,mmaturity,C_Sf
(i));
        i=i+1;
    end

    i=1;
    for strk= [1/4: ((1/2.5-1/4)/n):1/2.5, Q1_0]
        strk_C_Q1 (i)= strk;
        C_Q1 (i) =  blsprice (Q1_0, strk, rf,mmatu-
rity,sqrt(mv0_2),0);
        iv_C_Q1 (i)= blsimpv(Q1_0,strk_C_Q1 (i),rf,mmaturity,C_Q1
(i));
        i=i+1;
    end

 elseif skew_case==2
   i=1;
    for strk= 2100: ((2900-2100)/(n+1)):2900
        strk_C_Sf (i)= strk;
        C_Sf (i) =
HestonPrice('Call',mkappa,mtheta,0,mrho,msigma,mmatu-
rity,strk,Sf_0,rf,mv0) ;
        iv_C_Sf (i)= blsimpv(Sf_0,strk_C_Sf (i),rf,mmaturity,C_Sf
(i));
        i=i+1;
    end

    i=1;
    for strk= [1/4: ((1/2.5-1/4)/n):1/2.5, Q1_0]
        strk_C_Q1 (i)= strk;
        C_Q1 (i) =
HestonPrice('Call',mkappa_2,mtheta_2,0,mrho_2,msigma_2,mmatu-
rity,strk,Q1_0,rf,mv0_2) ;
        iv_C_Q1 (i)= blsimpv(Q1_0,strk_C_Q1 (i),rf,mmaturity,C_Q1
(i));
        i=i+1;
    end
end

% Single-asset Heston  model parameters

if skew_case==0
    %v0, Kappa, Theta, Vol of Variance (sigma), Correlation (rho)
    phi1=[mv0,0,0,0,0];
```

```
    phi2=[mv0_2,0,0,0,0];
elseif skew_case==1
    phi1=[mv0,mkappa,mtheta,msigma,mrho];
    phi2=[mv0_2,0,0,0,0];
elseif skew_case==2
    phi1=[mv0,mkappa,mtheta,msigma,mrho];
    phi2=[mv0_2,mkappa_2,mtheta_2,msigma_2,mrho_2];
end

%  Real-world parameters of the single-asset Heston Models

V_bar_star_1= 0.1;
V_bar_star_2= 0.1;
p_aa_go=-0.7; % p target: target observed aseet-asset correlation set-
ting

phi1_star=
[phi1(1),phi1(2)*phi1(3)/V_bar_star_1,V_bar_star_1,phi1(4),phi1(5)];
phi2_star=
[phi2(1),phi2(2)*phi2(3)/V_bar_star_2,V_bar_star_2,phi2(4),phi2(5)];


% Multi-asset Heston  models calibration - infinitesimal asset-asset
correlation
% (p_inf)calibration

TTM =5;
N=TTM*1000; %Number of time steps per path
M=10; %Number of paths

% Spot values vectors
S1= zeros (1,N+1);
S2= zeros (1,N+1);
S1(1)=Sf_0;
S2(1)=Q1_0;

% Current variances definition
V1= zeros (1,N+1);
V2= zeros (1,N+1);
V1(1)=phi1_star(1);
V2(1)=phi2_star(1);

err=0.001; % error accepted for the asset-asset correlation calibra-
tion

delta_t=TTM/N;


% p_aa_inf: Asset-asset infinitesimal correlation
% p_aa_sim: Asset-asset observed correlation obtained from model simu-
lation

vec_p_aa_inf=[-1,1,0]; % Vector with extreme values of p_aa_inf. This
vector will be bisectioned.
mat_p_aa_sim=zeros(M,3);  % mat_p_aa_sim initialization. This matrix's
columns are the asset-asset correlations
%observed from M model simulations when p_aa_inf= vec_p_aa_inf(i)
(i=1,2,3)
vec_p_aa_sim=[0,0,0]; % vec_p_aa_sim initialization.
%This vectors's elements are the average of each mat_p_aa_sim column,
```

```
%for the sake of correlation expected value computation.

% Bisection algorithm
while abs(p_aa_go-vec_p_aa_sim(3))>err
    vec_p_aa_inf(3)=0.5*(vec_p_aa_inf(1)+vec_p_aa_inf(2));% p_aa_inf
mean value
    for j= 1:3
        p_aa_inf=vec_p_aa_inf(j);% p_aa_inf to be used in path simula-
tions
        % Path simulations:
        for a=1:M
            % Random standard normal variables generation
            Z1=normrnd(0,1,[1 N]);
            Z1_tilde=normrnd(0,1,[1 N]);
            Z2=normrnd(0,1,[1 N]);
            Z2_tilde=normrnd(0,1,[1 N]);
            % Paths construction (Euler Scheme)
            for i = 1:N
                S1(i+1)=S1(i)*exp((rf-
0.5*V1(i))*delta_t+sqrt(V1(i)*delta_t)*Z1(i));
                S2(i+1)=S2(i)*exp((rf-r-
0.5*V2(i))*delta_t+sqrt(V2(i)*delta_t)*...
                        (p_aa_inf*Z1(i)+sqrt(1-p_aa_inf^2)*Z2(i)));
                V1(i+1)=max(V1(i)+phi1_star(2)*(phi1_star(3)-
V1(i))*delta_t+ ...

phi1_star(4)*sqrt(V1(i)*delta_t)*(phi1_star(5)*Z1(i)+ ...
                        sqrt(1-(phi1_star(5))^2)*Z1_tilde(i)),0);
                V2(i+1)=max(V2(i)+phi2_star(2)*(phi2_star(3)-
V2(i))*delta_t+...

phi2_star(4)*sqrt(V2(i)*delta_t)*(phi2_star(5)*...
                        p_aa_inf*Z1(i)+phi2_star(5)*sqrt(1-
p_aa_inf^2)*Z2(i)+...
                        sqrt(1-(phi2_star(5))^2)*Z2_tilde(i)),0);
            end
            % Simulated paths observed log-returns correlation
            log_ret1=log(S1(2:length(S1))./S1(1:length(S1)-1));
            log_ret2=log(S2(2:length(S2))./S2(1:length(S2)-1));
            mat_p_aa_sim(a,j)=corr(log_ret1',log_ret2');
        end
        vec_p_aa_sim(j)=mean(mat_p_aa_sim(:,j));
    end

    % vec_p_aa_inf bisection:
    p_aa_inf_mean=vec_p_aa_inf(3);
    if vec_p_aa_sim(3)>p_aa_go
        vec_p_aa_inf=[vec_p_aa_inf(1),vec_p_aa_inf(3)];
    else
        vec_p_aa_inf=[vec_p_aa_inf(3),vec_p_aa_inf(2)];
    end
end
% Optimal p_aa_inf: such that |p_aa_sim(p_aa_inf)-p_aa_go|< error
p_aa_inf=p_aa_inf_mean;

% Quanto option pricing under the bi-dimensional Heston model.
numSimul=10000;
TTM =mmaturity;
N=TTM*5000*2; %Number of time steps per path
delta_t=TTM/N;
```

```
M=1*numSimul; %Number of paths
q=3; % Predetermined quanto exchange rate
sim_Quanto_Price=zeros (1,M);
Heston_Quanto_Price= zeros (n+2,1);
Pract_Quanto_Price =  zeros (n+2,1);


S1 = zeros(N+1,1);
S2 = zeros(N+1,1);
V1 = zeros(N+1,1);
V2 = zeros(N+1,1);
S1(1)=Sf_0;
S2(1)=Q1_0;
V1(1)=phi1(1);
V2(1)=phi2(1);
p_SQ= -p_aa_inf; % infinitesimal correlation among S and Q (DOM/FOR)

i_atm_Q1= find (strk_C_Q1 == Q1_0 );
i_atm_Sf= find (strk_C_Sf == Sf_0 );
vol_atm_Sf = iv_C_Sf(i_atm_Sf);
vol_atm_Q1 = iv_C_Q1 (i_atm_Q1);

for j= 1:length (strk_C_Sf)

    strk=strk_C_Sf (j);
    for a=1:M
            % Random standard normal variables generation
            Z1=normrnd(0,1,[1 N]);
            Z1_tilde=normrnd(0,1,[1 N]);
            Z2=normrnd(0,1,[1 N]);
            Z2_tilde=normrnd(0,1,[1 N]);
            % Paths construction (Euler Scheme)
            for i = 1:N
                S1(i+1)=S1(i)*exp((rf-
0.5*V1(i))*delta_t+sqrt(V1(i)*delta_t)*Z1(i));
                S2(i+1)=S2(i)*exp((rf-r-
0.5*V2(i))*delta_t+sqrt(V2(i)*delta_t)* ...
                        (p_aa_inf*Z1(i)+sqrt(1-(p_aa_inf)^2)*Z2(i)));
                V1(i+1)=max(V1(i)+phi1(2)*(phi1(3)-
V1(i))*delta_t+phi1(4)*sqrt(V1(i)*delta_t)* ...
                        (phi1(5)*Z1(i)+sqrt(1-
(phi1(5))^2)*Z1_tilde(i)),0);
                V2(i+1)=max(V2(i)+phi2(2)*(phi2(3)-
V2(i))*delta_t+phi2(4)*sqrt(V2(i)*delta_t)* ...
                        (phi2(5)*p_aa_inf*Z1(i)+phi2(5)*sqrt(1-
p_aa_inf^2)*Z2(i)+sqrt(1-(phi2(5))^2)*Z2_tilde(i)),0);
            end
            % Simulated quanto option price
             sim_Quanto_Price(a)=1/Q1_0 * exp(-
rf*TTM)*S2(length(S2))*max(q*(S1(length(S1))-strk),0);

    end
    Heston_Quanto_Price (j)= mean(sim_Quanto_Price);

    % Practitioners' quanto adjustment
    vol_stk_Sf = iv_C_Sf(j);
    Pract_Quanto_Price (j) = q* (blsprice(Sf_0*exp(-(r-
rf+p_SQ*vol_atm_Sf*vol_atm_Q1)*TTM), strk,r, TTM,vol_stk_Sf));

end
```

```
cq0_Cop = cq0_Copula(q,Sf_0, r, rf, p_SQ, TTM,strk_C_Sf, 1*100000,
TTM*10000, 100000, 1000000, phi1, phi2, Q_0, 1000, 1000, 10000, 10000,
'tCopula', 3, TS_Sf_Q1);



% Plot pricing differences

figure

dif = abs((Pract_Quanto_Price-Heston_Quanto_Price) ./
Heston_Quanto_Price)*100;
dif1 = abs((Pract_Quanto_Price-cq0_Cop) ./ cq0_Cop)*100;
dif2 = abs((Heston_Quanto_Price-cq0_Cop) ./ cq0_Cop)*100;
yyaxis left
plot (strk_C_Sf,Heston_Quanto_Price,'k-
o',strk_C_Sf,Pract_Quanto_Price,'k--o',strk_C_Sf,cq0_Cop,'k-.o')
ax=gca;
ax.YColor= 'k';
ax.YLabel.String='Quanto option price';
yyaxis right
plot (strk_C_Sf,dif,'r-o', strk_C_Sf,dif1,'r--o', strk_C_Sf,dif2,'r-
.o')
ax=gca;
ax.YColor= 'r';
ax.YLim = [0, 100];
ax.YTickLabel=[char(ax.YTickLabel), char(ones(length(ax.YTick-
Label),1)*'%')];
ax.YLabel.String='Error';
legend('Heston framework','Black-Scholes framework','Non-parametric
framework', 'Difference Heston and BS', 'Difference Non-parametric and
BS', 'Difference Non-parametric and Heston')

ax.XLabel.String='S_{f} strike';


% Plot volatility smiles
figure
plot(strk_C_Sf,iv_C_Sf,'k-o');
ax=gca;
ax.YLim = [min(iv_C_Sf)-0.01, max(iv_C_Sf)+0.01];
ax.YLabel.String='Implied Volatility';
legend('Sf implied volatility');
ax.XLabel.String='S_{f} strike';

figure
strk_C_Q1 = [strk_C_Q1(1:3);strk_C_Q1(7);strk_C_Q1(4:6)];
iv_C_Q1 = [iv_C_Q1(1:3);iv_C_Q1(7);iv_C_Q1(4:6)];
plot(strk_C_Q1,iv_C_Q1,'k-o');
ax=gca;
ax.XLim =[min(strk_C_Q1), max(strk_C_Q1)];
ax.YLim = [min(iv_C_Q1)-0.01, max(iv_C_Q1)+0.01];
ax.YLabel.String='Implied Volatility';
legend('Q^{-1} implied volatility');
ax.XLabel.String='Q^{-1} strike';
```

cq0_Copula.m

```
function [cq0_Cop] = cq0_Copula(q,Sf_0, r, rf, p_SQ, T,strk, M, N,
nSim_Cop_Gauss, nSimQtl,phi_Sf, phi_Q1, Q_0, nPts_Cop_Sf, nPts_Cop_Q1,
nPts_Marg_Sf, nPts_Marg_Q1, copula, df, TS_Sf_Q1)


% Settings
rho= -p_SQ;
Q1_0=1/Q_0 ;


if isempty(TS_Sf_Q1)
    clear TS_Sf_Q1
end


% Initial settings
[spec_Sf, spec_Q1, gs_Sf, gs_Q1, grid_Sf, grid_Q1, pts, TS_Sf_Q1, len-
GridSf,lenGridQ1] =  initialSettingV2('copula', 'Gaussian', 'rho',rho,
'nSim_Cop_Gauss', nSim_Cop_Gauss,
'nPts_Cop_Q1',nPts_Cop_Q1,'nPts_Cop_Sf',nPts_Cop_Sf);

if strcmp(copula, 'Gaussian')

    [qSfSim, qQ1Sim] = copulaQuantilesSimGaussian (rho, nSimQtl);

elseif strcmp(copula, 'tCopula')

    [qSfSim, qQ1Sim] = copulaQuantilesSim_t (rho, nSimQtl, df);

else

    % Descriptive statistics
    [f, pdf, qSf, qQ1, pts] =  descriptiveStatistics(TS_Sf_Q1, pts,
gs_Sf, gs_Q1);

    % Copula plotting
    surfFromVec (qSf, qQ1, f, lenGridSf,lenGridQ1)

    % Quantiles simulation
    [qSfSim, qQ1Sim] = copulaQuantilesSim (qSf, qQ1, pdf, nSimQtl);


end


HestonSf = singleHestonSim(M, N, rf, Sf_0, T, phi_Sf);
HestonQ1 = singleHestonSim(M, N, rf-r, Q1_0, T, phi_Q1);


[cdf_Sf, f_Sf, ~, pts_Sf]= randomVarSummV2(HestonSf, nPts_Marg_Sf);
[cdf_Q1, f_Q1, ~, pts_Q1]= randomVarSummV2(HestonQ1, nPts_Marg_Q1);


Sf_Sim_Cop = qtlToVal(cdf_Sf, pts_Sf, qSfSim);
Q1_Sim_Cop = qtlToVal(cdf_Q1, pts_Q1, qQ1Sim);
```

```
cq0_Cop = quantoPriceSim(Q_0, rf, q, strk, T, Q1_Sim_Cop, Sf_Sim_Cop);



end
```

copulaQuantilesSim.m

```
function [q1Sim_out, q2Sim_out] = copulaQuantilesSim (q1, q2, pdf,
nSim)

len = length(pdf);

pairNum = transpose(1:1:len);

simPairs = datasample(pairNum, nSim,'Replace', true,'Weights',pdf);

q1Sim_out = q1(simPairs);
q2Sim_out = q2(simPairs);


end
```

copulaQuantilesSim_t.m

```
function [q1Sim_out, q2Sim_out] = copulaQuantilesSim_t (rho, nSim, df)

Z = mvtrnd([1, rho; rho, 1], df, nSim);

U =tcdf(Z,df);

q1Sim_out=U(:,1);
q2Sim_out=U(:,2);

end
```

copulaQuantilesSimGaussian.m

```
function [q1Sim_out, q2Sim_out] = copulaQuantilesSimGaussian (rho,
nSim)

Z = mvnrnd([0 0], [1 rho; rho 1], nSim);
U = normcdf(Z,0,1);

q1Sim_out=U(:,1);
q2Sim_out=U(:,2);

end
```

descriptiveStatistics.m

```
function [f_out, pdf_out, q1_out, q2_out, pts_out] = descriptiveSta-
tistics(pair_Values, pts, gs1, gs2)


[f, pts] = ksdensity(pair_Values,  pts);
f_out = f;
pts_out = pts;



recSurf = gs1*gs2;
pdf = f .* recSurf;
pdf = pdf .* (1/sum(pdf));
pdf_out = pdf;


rv1 = pts(:,1);
rv1 = unique (rv1, 'sorted');
rv2 = pts(:,2);
rv2 = unique (rv2, 'sorted');


lenValues=length(rv1);
cdf_rv1= zeros (lenValues,1);

for i = 1:1:lenValues
    curVal = rv1 (i);
    idxLeq = pts (:,1) <= curVal;
    cdf_rv1 (i) = sum (pdf(idxLeq));
end

lenValues=length(rv2);
cdf_rv2= zeros (lenValues,1);

for i = 1:1:lenValues
    curVal = rv2 (i);
    idxLeq = pts (:,2) <= curVal;
    cdf_rv2 (i) = sum (pdf(idxLeq));

end


q1 = pts (:,1);
q2 = pts (:,2);

lenValues=length(rv1);
for i = 1:1:lenValues
    curVal = rv1 (i);
    curQuant = cdf_rv1 (i);
    idxEq = q1 == curVal;
    q1 (idxEq)= curQuant;

end

lenValues=length(rv2);
```

```
for i = 1:1:lenValues
    curVal = rv2 (i);
    curQuant = cdf_rv2 (i);
    idxEq = q2 == curVal;
    q2 (idxEq)= curQuant;

end

q1_out = q1;
q2_out= q2;

end
```

initialSettingV2.m

```
function [spec_Sf_out, spec_Q1_out, gs_Sf_out, gs_Q1_out, grid_Sf_out,
grid_Q1_out, pts_out, TS_Sf_Q1_out, lenGridSf_out,lenGridQ1_out] = in-
itialSettingV2(varargin)


% Input parser
default_nSim_Cop_Gauss = 100000;
default_spec_Sf = normrnd(0,1,[default_nSim_Cop_Gauss,1]);
default_spec_Q1 = normrnd(0,1,[default_nSim_Cop_Gauss,1]);
default_nPts_Cop_Sf = 1000;
default_nPts_Cop_Q1 = 1000;
default_rho = 0;
defalut_copula = 'Gaussian';


p = inputParser;

addParameter(p,'spec_Sf',default_spec_Sf);
addParameter(p,'spec_Q1',default_spec_Q1);
addParameter(p,'nPts_Cop_Sf',default_nPts_Cop_Sf);
addParameter(p,'nPts_Cop_Q1',default_nPts_Cop_Q1);
addParameter(p,'rho',default_rho);
addParameter(p,'copula',defalut_copula);
addParameter(p,'nSim_Cop_Gauss',default_nSim_Cop_Gauss);

parse(p,varargin{:});

% Local variables

spec_Sf = p.Results.spec_Sf;
spec_Q1 = p.Results.spec_Q1;
nPts_Cop_Sf = p.Results.nPts_Cop_Sf;
nPts_Cop_Q1 = p.Results.nPts_Cop_Q1;
rho = p.Results.rho;
copula = p.Results.copula;
nSim_Cop_Gauss = p.Results.nSim_Cop_Gauss;

% Specialist vectors setting
if strcmp(copula, 'Gaussian') && rho ~= 0
    mu = [0 0];
```

```
    sigma = [1 rho; rho 1];
    R = mvnrnd(mu,sigma,nSim_Cop_Gauss);
    spec_Sf = R(:,1);
    spec_Q1 = R(:,2);

end

spec_Sf_out = spec_Sf;
spec_Q1_out = spec_Q1;

TS_Sf_Q1_out = [spec_Sf_out, spec_Q1_out] ;

% Grids setting
UB_Sf= 1.1*max(spec_Sf);
LB_Sf=0.9*min(spec_Sf);
gs_Sf = (UB_Sf-LB_Sf)/nPts_Cop_Sf;

gs_Sf_out = gs_Sf;
grid_Sf_out = LB_Sf:gs_Sf:UB_Sf;
lenGridSf_out = length(grid_Sf_out);


UB_Q1= 1.1*max(spec_Q1);
LB_Q1=0.9*min(spec_Q1);
gs_Q1 = (UB_Q1-LB_Q1)/nPts_Cop_Q1;

gs_Q1_out = gs_Q1;
grid_Q1_out = LB_Q1:gs_Q1:UB_Q1;
lenGridQ1_out = length(grid_Q1_out);


% Points for density estimation setting
[x1,x2] = meshgrid(grid_Sf_out, grid_Q1_out);
x1 = x1(:);
x2 = x2(:);
pts_out = [x1 x2];

end
```

qtlToVal.m

```
function [val_x_Vec] = qtlToVal(cdf, pts, qtlVec)

lenQtl = length(qtlVec);
val_x_Vec = zeros (lenQtl,1);

for i = 1:1:lenQtl
    cur = qtlVec(i);
    [~, idx] = min(abs(cdf - cur));
    val_x_Vec (i)= pts(idx);
end
```

quantoPriceGBM.m

```
function [cq0GBM] = quantoPriceGBM(Sf_0, r_Sf,
sigma_Sf,Q1_0,r_Q1,sigma_Q1,K, T, q, rho, rf, nSim)


mu_par_logN_Sf = log(Sf_0)+r_Sf*T-0.5*sigma_Sf^2*T;
sigma_par_logN_Sf = sigma_Sf*sqrt(T);

mu_par_logN_Q1= log(Q1_0)+r_Q1*T-0.5*sigma_Q1^2*T;
sigma_par_logN_Q1 = sigma_Q1*sqrt(T);
corrMat= [1,rho;rho,1];

y=MvLogNRand
([mu_par_logN_Sf;mu_par_logN_Q1],[sigma_par_logN_Sf;sigma_par_logN_Q1]
,nSim,corrMat);

SfT= y(:,1);
Q1T= y(:,2);

Q0=1/Q1_0;

cq0GBM=quantoPriceSim(Q0, rf, q, K, T, Q1T, SfT);

end
```

quantoPriceSim.m

```
function [cq0] = quantoPriceSim(Q0, rf, q, K, T, Q1T, SfT)

lenK = length(K);
cq0 = zeros (lenK,1);

for i=1:1:lenK
    curK=K(i);
    cq0 (i) = mean((Q0 * exp(-rf*T)* Q1T) .* (max(q*(SfT-curK),0)));
end
```

randomVarSummV2.m

```
function [cdf, f, pdf, pts] = randomVarSummV2(x,nPts_Marg)
LB= min(x);
UB=max(x);
gs= (UB-LB)/nPts_Marg;
pts= LB:gs:UB;
f = ksdensity(x,pts);
pdf = f .* gs;
pdf = pdf .* (1/sum(pdf)); % PDF sums up to 1


lenValues=length(pts);
cdf= zeros (lenValues,1);

for i = 1:1:lenValues
    curVal = pts (i);
    idxLeq = pts <= curVal;
```

```
    cdf (i) = sum (pdf(idxLeq));

end
```

singleHestonSim.m

```
% "Single Heston Simulation" function creates a vector of simulated
values
% of a random variable that follows a single Heston model.

% Inputs:
% M: number of paths
% N: number of steps in a path
% rfir: risk-free interest rate
% S0: current value of variable S
% T: simulate up to time T
% phi: Heston model parameters vector: v0 (initial variance), kappa,
theta,
%      sigma (volatility of variance), rho (asset-variance correla-
tion).


% Outputs:
% ST: vector of simulation results of S

function [ST] = singleHestonSim(M, N, rfir, S0, T, phi)

% S and V vectors definition
S= zeros (1,N+1);
V= zeros (1,N+1);
S(1)=S0;


% Heston model parameters setting
V(1)=phi(1); % current value of variance
kappa = phi(2);
theta = phi(3);
sigma = phi(4); % volatility of variance
rho = phi(5); % asset-variance correlation

% Path simulations:
ST= zeros(M,1);
delta_t=T/N;

for j=1:M
    % Random standard normal variables generation
    Z=normrnd(0,1,[1 N]);
    Z_tilde=normrnd(0,1,[1 N]);

    % Paths construction (Euler Scheme)
    for i = 1:N
        S(i+1)=S(i)*exp((rfir-
0.5*V(i))*delta_t+sqrt(V(i)*delta_t)*Z(i));

        V(i+1)=max(V(i)+ kappa*(theta-V(i))*delta_t+ ...
            sigma*sqrt(V(i)*delta_t)*(rho*Z(i)+ ...
            sqrt(1-(rho)^2)*Z_tilde(i)),0);
```

```
    end
    % Simulation final value
    ST (j) = S(N);

end
```

Appendix C

graphsCorrT.m

```
for df=1:1:4

rho=0.7;
nSim=1000000;
lng = 20;
corVec= zeros(lng,1);
rhoVec= rho*ones(lng,1);

for i=1:1:lng
    Z = mvtrnd([1, rho; rho, 1], df, nSim);
    corVec (i)= corr(Z(:,1),Z(:,2));

end
figure
plot(1:lng, corVec,'k-',1:lng, rhoVec,'r-')
ax=gca;
ax.YLabel.String='Correlation';
ax.XLabel.String='Simulation';
legend('sim corr','target corr')
ax.YLim = [-1, 1];

end
```

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