Volumes, Areas and other Falconer-type problems

Belmiro Galo da Silva

Thesis presented
to the
Institute of Mathematics and Statistics
of the
University of São Paulo
to
obtain the title
of
PhD

Program: Applied Mathematics
Advisor: Prof. Dr. Edson Vargas
Co-advisor: Prof. Dr. Alex Iosevich

During the development of this work, the author received financial assistance from CAPES and CNPq

São Paulo, August 2021

Volumes, Areas and other Falconer-type problems

This is the original version of the thesis prepared by the candidate Belmiro Galo da Silva, such as submitted to the Judging Committee.

Volumes, Areas and other Falconer-type problems

This version of the thesis contains the suggested corrections and changes by the Committee during the defense of the original version of the work, held on 08/13/2021. A copy of the original version is available at Institute of Mathematics and Statistics, University of São Paulo.

Committee:

- Prof. Dr. Edson Vargas (Advisor)- IME-USP
- Prof. Dr. Sinai Robins - IME-USP
- Prof. Dr. Alex Iosevich (Co-advisor) - University of Rochester
- Prof. Dr. Sevak Mkrtchyan - University of Rochester
- Prof. Dr. Gonzalo Mateos - University of Rochester

Acknowledgment

I would like to express my gratitude to my supervisor Alex Iosevich, for all of his support and patient guidance. His help has not only made me a better mathematician, but also a better person. Thanks for believing in me. Life is good.

To all of the staff and permanent faculty, past and present, of the Department of Mathematics at University of Rochester, thank you. Special thanks to Joan Robinson, Cynthia Spencer and Professors Jonathan Pakianathan, Kalyani Madhu, Amanda Tucker, Sevak Mkrtchyan and Stephen Kleene for their support and guidance throughout my studies at Rochester. Also, special thanks to visiting Professors Charles Wolf and Alexander Carney.

Thanks you to Alex McDonald, my first collaborator. You helped me to expand my horizons and started understanding how research is done in academia.

I would like to thank all of the permanent faculty, past and present, of the Department of Mathematics at the University of São Paulo. Special thanks to my co-supervisor Edson Vargas and professors Gabriel Haeser, Edson de Faria, André de Carvalho for their support during part of my PhD at University of São Paulo.

Thanks to all graduate students of the Department of Mathematics and all the members of the Association of Latin American Students at the University of Rochester, past and present, for helping me feel at home over these years. Special thanks to Charles Brittenham, Jorge Olivares, Carlos Mena, Bai Lin and all ALAS volleyball friends.

Last, but not least, I would like to thank my family and friends for their love and for their full support. Special thanks to my friend Lucas Matos, my mom, Deborah Galo, my sister, Marcella Galo and my partner Sarah Hofheins.

This thesis is dedicated to the memory of my grandmother, Olga Galo, and my friend and first mentor, Arthur Matos. You are gone, but your belief in me has made this journey possible.

Abstract

Galo, B. **Volumes, Areas and other Falconer-type problems**. 2021. 40 p. Thesis - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2021.

In this thesis, we investigate the Falconer-type problems about point configurations and in different dimensions.

It is well-known the concept of the Hausdorff measure is a generalization of the Lebesgue measure and the Falconer distance problem aims to relate these two topics when it asks how large does the Hausdorff dimension of a compact set need to be to ensure the Lebesgue measure of the distance set.

In the first moment, we consider a k-point configurations in $\mathbb{R}^d$ and we prove that a compact set $E \subset \mathbb{R}^d$ determines a positive measure of such volume types if the Hausdorff dimension of E is greater than $d - \frac{d-1}{2k-d}$ generalizing some results in this field. This portion of the work represents joint work with Dr. Alex McDonald.

In the second moment, we study a Falconer-type problem on a 4-point configuration in the plane and we prove that a compact set $E \subset \mathbb{R}^2$ determines a positive measure of such Galo area types if the Hausdorff dimension of E is greater than $\frac{3}{2}$ extending some results from A. McDonald in [22].

Keywords: Falconer conjecture, volume type problems, Galo area type.

Contents

Introduction

One of the most classical questions in geometric measure theory is the Falconer's Distance Conjecture. This conjecture intrigues many mathematicians due to the fact that the technical statement of this conjecture is easy to understand and it relates basics concepts in the area such as the Lebesgue measure and the Hausdorff dimension in the sense of if we have a set which is sufficiently structured in some way and we apply a non-trivial map, the image is also structured. To state this conjecture we need to define a set in $\mathbb{R}^d$.

For a set $E \subset \mathbb{R}^d$, define its *distance set* to be

$$\Delta(E) = \{|x - y| : x, y \in E\}.$$

Definition 1. *For $E \subset \mathbb{R}^d$ non-empty, the diameter of E is defined as*

$$|E| = \sup \Delta(E).$$

We call the collection $\{A_i\}_{i \in J}$ a δ-cover of A, if $A \subseteq \bigcup_{i \in J} A_i$ and $0 < |A_i| \leqslant \delta$ for all $i \in J$.

Definition 2 (α-dimensional Hausdorff measure of E)**.** *Let $E \subseteq \mathbb{R}^d$, $\alpha > 0$and $\delta > 0$. We define the Hausdorff α-dimensional measure as*

$$\mathcal{H}_\alpha(E) := \lim_{\delta \to 0^+} \mathcal{H}_\alpha^\delta(E)$$

where $\mathcal{H}_\alpha^\delta(E) := \inf \left(\sum_{i=1}^{\infty} |A_i|^\alpha \right)$ and the infimum is taking over all countable δ-covers of E.

Thus, the Hausdorff dimension of $E \subset \mathbb{R}^d$ is

$$dim_{\mathcal{H}}(E) := \inf \{\alpha : \mathcal{H}_\alpha(E) = 0\} = \sup \{\alpha : \mathcal{H}_\alpha(E) = \infty\}.$$

In this work we will denote $\dim E = \dim_{\mathcal{H}}(E)$ the Hausdorff dimension of the set E. An interesting example is when we look at the Cantor middle third set and we notice its Hausdorff dimension is equal to $\frac{\log 2}{\log 3}$ but it has Lebesgue measure 0, see [17].

The Falconer's distance conjecture asks how large the Hausdorff dimension of a compact set E must be to ensure that $\Delta(E)$ has positive Lebesgue measure in the following way:

Conjecture 0.1 (Falconer's Conjecture)**.** *For a compact $E \subset \mathbb{R}^d$, $d \geqslant 2$. If*

$$dim_{\mathcal{H}}(E) > \frac{d}{2}$$

then $\Delta(E)$ *has positive Lebesgue measure.*

In [9], Falconer proved the following

Theorem 0.2. *Let* $E \subset \mathbb{R}^d$ *be compact. If*

$$\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$$

then $\Delta(E)$ *has positive Lebesgue measure.*

He also found a family of examples $\{E_s\}$ such that for any $s < \frac{d}{2}$, one has $\dim E_s > s$ and $\mathcal{L}_1(\Delta(E)) = 0$. This suggests what is now known as the Falconer distance problem, which asks for the smallest s such that $\dim E > s$ implies $\mathcal{L}_1(\Delta(E)) > 0$. Falconer's work implies this threshold is between $\frac{d}{2}$ and $\frac{d+1}{2}$, and it is conjectured that $\frac{d}{2}$ is in fact the correct threshold. The first major results were due to Wolff [23] and Erdogan [8], proving the threshold $\frac{d}{2} + \frac{1}{3}$ in the case $d = 2$ and $d \geqslant 3$, respectively. These were the best results until recently, when a number of improvements were made using the decoupling theorem of Bourgain and Demeter [2].

We can summarize the best results currently state that for compact $E \subset \mathbb{R}^d$, if

$$\dim E > \begin{cases} 5/4 & \text{for } d = 2 \quad [14] \\ 9/5 & \text{for } d = 3 \quad [4] \\ \frac{d}{2} + \frac{1}{4} & \text{for } d \geqslant 4 \text{ and } d \text{ even } [5] \\ \frac{d}{2} + \frac{1}{4} + \frac{1}{4(d-1)} & \text{for } d \geqslant 4 \text{ and } d \text{ odd } [6]. \end{cases}$$

then the distance set $\Delta(E)$ has positive Lebesgue measure.

A key generalization of the Falconer distance problem comes from considering geometric properties of point configurations. We first establish some notation. We will use superscripts to denote vectors and subscripts to denote components of vectors, so for a configuration $x \in (\mathbb{R}^d)^k$ we have $x = (x^1, \cdots, x^k)$ where each $x^j \in \mathbb{R}^d$ has components $x^j = (x^j_1, \cdots, x^j_d)$. The most direct generalization of the Falconer distance problem in this context is the problem of congruence classes of such configurations. For $k \leqslant d$, the congruence class of $x \in (\mathbb{R}^d)^{k+1}$ is determined by the $\binom{k+1}{2}$-tuple of distances $|x^i - x^j|$. Define $\Delta_k(E)$ to be the set of vectors $\{|x^i - x^j|\}_{1 \leqslant i < j < k+1}\}$ with $x^i \in E$ for all i. Note that the set $\Delta_1(E)$ coincides with $\Delta(E)$ defined above. Greenleaf, Iosevich, Liu, and Palsson [11] proved that $\Delta_k(E)$ has positive $\binom{k+1}{2}$ dimensional Lebesgue measure if $\dim E > d - \frac{d-1}{k+1}$. The proof strategy was built on the fact that two configurations are congruent if and only if there is an isometry mapping one to the other, which allowed me to study the problem in terms of the group action. The group action framework was instrumental in the proof of the discrete predecessor of the Falconer distance problem, known as the Erdos distinct distance problem, which asks for the minimum number of distances determined by a set of N points in $\mathbb{R}^d$. In that context the group action framework was introduced by Elekes and Sharir [7] and ultimately used by Guth and Katz to resolve the problem in the plane, obtaining the bound $N/\log N$ which is optimal up to powers of log [16].

The configuration congruence problem becomes more subtle when $k > d$. This is because the system of distance equations becomes overdetermined, and the space of congruence classes can no longer be identified with the space of distance vectors $\mathbb{R}^{\binom{k+1}{2}}$. Invoking the group action framework again, one would expect heuristically that the space of congruence classes should have dimension $d(k+1)-\binom{d+1}{2}$, since the space of configurations has dimension $d(k+1)$ and the space of isometries has dimension $\binom{d+1}{2}$. Chatziconstantinou, Iosevich, Mkrtchyan, and Pakianathan [3] proved that in fact this heuristic is correct, and obtained a non-trivial dimensional threshold. Their proof used the theory of combinatorial rigidity. Given a $(k+1)$-point configuration, they proved that the congruence class was determined (up to finitely many choices) if one fixes $d(k+1)-\binom{d+1}{2}$ strategically chosen distances. They then used the group action framework to prove that $\Delta_k(E)$ has positive $d(k+1)-\binom{d+1}{2}$ dimensional measure if $\dim E > d - \frac{1}{k+1}$.

The key to the results in [11] and [3] is the fact that the congruence relation can be described in terms of action of the isometry group on the space of configurations. It is therefore natural to study other point configuration problems where congruence is replaced by other geometric relations with a corresponding group action invariance.

Chapter 1

Main Results

The content of this chapter represents join work with Dr. Alex McDonald [15].

Let us start this chapter considering the volumes which are obtained by choosing any d points of a configuration. More precisely, we make the following definition.

Definition 3. *The* ***volume type*** *of* $x \in (\mathbb{R}^d)^k$ *is the vector*

$$\{\det(x^{j_1}, \cdots, x^{j_d})\}_{1 \leqslant j_1 < \cdots < j_d \leqslant k} \in \mathbb{R}^{\binom{k}{d}}.$$

For a set $E \subset \mathbb{R}^d$*, let*

$$\mathcal{V}_{k,d}(E) = \{\{\det(x^{j_1}, \cdots, x^{j_d})\}_{1 \leqslant j_1 < \cdots < j_d \leqslant k} : x^1, ..., x^k \in E\}$$

be the set of volume types determined by points in E*. Finally, let* $\mathcal{V}_{k,d} = \mathcal{V}_{k,d}(\mathbb{R}^d)$ *be the space of all volume types of* k*-point configurations in* $\mathbb{R}^d$*.*

Thus, the volume type of a k-point configuration $x \in (\mathbb{R}^d)^k$ encodes all volumes obtained by choosing any d points from x (see figure 1).

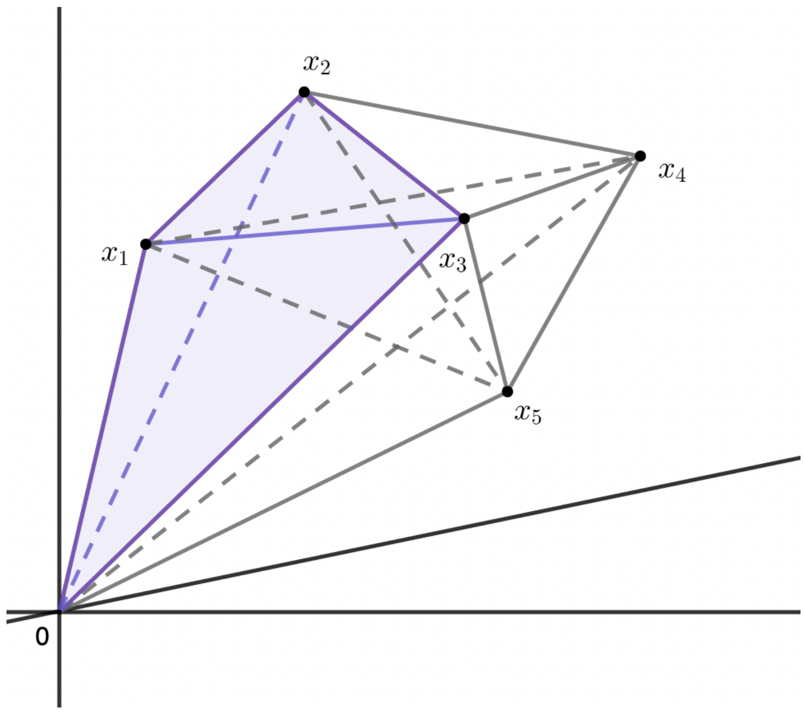

Figure 1.1: *5-point configuration* $x \in \left(\mathbb{R}^3\right)^5$.

When $k = d$, the space of volume types is simply $\mathcal{V}_{d,d} = \mathbb{R}$, which we may equip with the Lebesgue measure. In the case $k = d = 3$, Greenleaf, Iosevich, and Mourgoglou [12] proved that $\mathcal{V}_{3,3}(E)$ has positive measure if $\dim E > 13/5$. This threshold was later improved and generalized to higher dimension by Greenleaf, Iosevich, and Taylor [13] who considered the case $k = d$ for any $d \geqslant 3$ and proved $\mathcal{V}_{d,d}(E)$ has positive measure if $\dim E > d - 1 + \frac{1}{d}$. Notice in the case $d = 3$ this improves the 13/5 threshold to 7/3. When k is large, the problem is overdetermined and hence one needs to define an appropriate measure on the space of volume types. The second listed author [22] proved that $\mathcal{V}_{k+1,2}$ may be identified with a space of dimension $2k - 1$ (note that this is consistent with our previously described heuristic, since the space of $(k + 1)$-point configurations has dimension $2k + 2$ and the Lie group $\mathrm{SL}_2(\mathbb{R})$ has dimension 3) and that $\mathcal{V}_{k+1,2}$ has positive measure if $\dim E > 2 - \frac{1}{2k}$. The second author also obtained a non-trivial result in the two dimensional problem over finite fields and rings of the form $\mathbb{Z}/p^\ell\mathbb{Z}$ [21].

Our first goal is to generalize these results to the case where k, d are natural numbers satisfying $k \geqslant d \geqslant 2$ but are otherwise arbitrary. Our heuristic suggests that the dimension of $\mathcal{V}_{k,d}$ should be $d(k - d) + 1$. Our first theorem shows that this is indeed the case.

Theorem 1.1. *The set $\mathcal{V}_{k,d}$ is an embedded submanifold in $\mathbb{R}^{\binom{k}{d}}$ of dimension $d(k - d) + 1$.*

This will be proved in chapter 2. It follows that $\mathcal{V}_{k,d}$ is equipped with $(d(k-d)+1)$-dimensional Lebesgue measure, which we will denote by $\mathcal{L}_{d(k-d)+1}$. It also follows that if E is compact, $\mathcal{V}_{k,d}(E)$ is a compact subset of $\mathcal{V}_{k,d}$.

With this result, we are now ready to state our first main theorem.

Theorem 1.2. *Let $k \geqslant d \geqslant 2$ and let $E \subset \mathbb{R}^d$ be a compact set with Hausdorff dimension greater than $d - \frac{d-1}{2k-d}$. Then, $\mathcal{L}_{d(k-d)+1}(\mathcal{V}_{k,d}(E)) > 0$.*

We shall remark here that our decision to work with signed volume, rather than unsigned volume, is an arbitrary one. One can immediately deduce an unsigned version of Theorem 1.2 by decomposing the set $\mathcal{V}_{k,d}(E)$ into $2^{\binom{k}{d}}$ pieces according to the sign of each component and applying the pigeonhole principle. We also note that in the case $k = d$ our threshold is the same as the one in [13]. The general case is proved by reducing to the $k = d$ case, so a better exponent in that case would yield better general results.

Another classic object of study in the distance problem is chains of distances determined by a set. A configuration $x \in (\mathbb{R}^d)^k$ determines a $k-1$ chain of distances $|x^1 - x^2|, |x^2 - x^3|, ..., |x^{k-1} - x^k|$. Bennett, Iosevich, and Taylor [1] proved that if $\dim E > \frac{d+1}{2}$ then the set of distance chains determined by E has positive measure. This result was later generalized by Iosevich and Taylor [18] to apply to all trees.

Our other main theorem will pertain to chains of volumes. Since a volume is determined by d points rather than 2, we will consider chains in the sense of hypergraphs. Recall an r-regular hypergraph is a set of vertices and hyperedges, where each hyperedge connects r vertices (so, in particular, a 2-regular hypergraph is just a graph). A chain in a hypergraph is a seqeunce of vertices where each shares some hyperedge with the next.

Given a d-uniform hypergraph on vertices $\{1, ..., k\}$ and a configuration $x \in (\mathbb{R}^d)^k$, we may consider volumes determined by points $x^{j_1}, ..., x^{j_d}$ such that $(j_1, ..., j_d)$ forms a hyperedge. In this framework, Theorem 1.2 gives a result in the case where the hypergraph is complete. Our methods also allow us to obtain a result in the case of a chain. This is our next theorem.

Theorem 1.3. *Let $E \subset \mathbb{R}^d$ be compact, and let*

$$\mathcal{C}_{k,d} = \{\{\det(x^j, x^{j+1}, \cdots, x^{j+d-1})\}_{1 \leqslant j \leqslant k+1-d} : x^1, ..., x^k \in E\}.$$

If $\dim E > d - 1 + \frac{1}{d}$, *then* $\mathcal{L}_{(k+1-d)}(\mathcal{C}_{k,d}(E))$.

Here, we pause to make a couple remarks. First, note that if our set E is contained in a hyperplane through the origin it cannot determine any non-zero volume, so the optimal threshold cannot be smaller than $d-1$. Second, it is interesting to note that the threshold in Theorem 1.3 does not depend on k whereas the threshold in Theorem 1.2 tends to d as $k \to \infty$. Our final theorem shows that this cannot be avoided.

Theorem 1.4 (Sharpness)**.** *For any* $k \geqslant d \geqslant 2$ *and any*

$$s_{k,d} < d - \frac{d^2(d-1)}{d(k-1)+1},$$

there exists a compact set $E_{k,d} \subset \mathbb{R}^d$ *such that* $\dim E_{k,d} > s_{k,d}$ *and*

$$\mathcal{L}_{d(k-d)+1}(\mathcal{V}_{k,d}(E_k)) = 0.$$

1.1 Setting up the group action framework

We start by examining the relationship between volume types and the action of $\mathrm{SL}_d(\mathbb{R})$ on the space of configurations. Generically, the property of two configurations having the same volume type is equivalent to those configurations lying in the same orbit of this action. However, this equivalence breaks down for configurations which do not span $\mathbb{R}^d$. This leads to the following definition.

Definition 4. *A configuration* $x \in (\mathbb{R}^d)^k$ *is called* ***degenerate*** *if* $\{x^1, \cdots, x^d\}$ *is linearly dependent, and* ***non-degenerate*** *otherwise.*

We remark that we could broaden this notion of non-degeneracy to include configurations where any d points span $\mathbb{R}^d$, not just the first d points. However, in either case the set of degenerate configurations are negligible so we have chosen this definition to simplify our proofs and notation.

With our definition in place, we have the following lemma.

Lemma 1.5. *Let* $x, y \in (\mathbb{R}^d)^k$ *be non-degenerate. Then* x *and* y *have the same volume type if and only if there exists a unique* $g \in SL_d(\mathbb{R})$ *such that* $y = gx$ *(i.e., for each* j *we have* $y^j = gx^j$*).*

Proof. First, suppose x and y have the same volume types. Because x and y are non-degenerate,

$$D := \det(x^1, \cdots, x^d) = \det(y^1, \cdots, y^d) \neq 0.$$

Equivalently, the $d \times d$ matrix with columns $x^1 \cdots x^d$ is non-singular, same as $y^1 \cdots y^d$. We denote these matrices by $(x^1 \cdots x^d)$ and $(y^1 \cdots y^d)$, respectively. Let

$$g = (y^1 \cdots y^d)(x^1 \cdots x^d)^{-1}.$$

This equation means that $(gx^1 \cdots gx^d) = (y^1 \cdots y^d)$, so $gx^n = y^n$ for every $1 \leqslant n \leqslant d$. Let i be any index, and write

$$x^i = \sum_{n=1}^{d} a_n x^n, \quad y^i = \sum_{n=1}^{d} b_n y^n.$$

Observe that

$$\det(x^1, \cdots, x^{d-1}, x^i) = \det\left(x^1, \cdots, x^{d-1}, \sum_{n=1}^{d} a_n x^n\right) = \sum_{n=1}^{d} \det\left(x^1, \cdots, x^{d-1}, a_n x^n\right)$$

since the determinant behaves like a linear function on the rows of the matrix. Therefore,

$$\det(x^1, \cdots, x^{d-1}, x^i) = \det\left(x^1, \cdots, x^{d-1}, a_d x^d\right) = a_d D.$$

The same conclusion holds for

$$\det(y^1, \cdots, y^{d-1}, y^i) = \det\left(y^1, \cdots, y^{d-1}, \sum_{n=1}^{d} b_n y^n\right) = \sum_{n=1}^{d} \det\left(y^1, \cdots, y^{d-1}, b_n y^n\right) = b_d D.$$

By assumption x and y have the same volume type so we conclude $a_d = b_d$. An argument considering $\det(x^1, \cdots, x^{n-1}, x^{n+1}, \cdots, x^d, x^i)$ similarly shows that $a_n = b_n$ for every $1 \leqslant n \leqslant d$. Thus,

$$gx^i = g \sum_{n=1}^{d} a_n x^n = \sum_{n=1}^{d} a_n g x^n = \sum_{n=1}^{d} a_n y^n = y^i.$$

Note that $g \in \mathrm{SL}_d(\mathbb{R})$, since

$$\det g = \det((y^1, \cdots, y^d)(x^1, \cdots, x^d)^{-1}) = \det(y^1, \cdots, y^d) \det(x^1, \cdots, x^d)^{-1} = 1.$$

This proves existence. Uniqueness follows from the fact that the configuration contains a basis, so g is determined by its action on the configuration. The converse follows from the matrix equation

$$g(x^1, \cdots, x^d) = (y^1, \cdots, y^d)$$

and the fact that g has determinant 1.

□

We conclude this chapter by proving Theorem 1.1. Given manifolds M and N, a smooth map $\Phi : M \to N$ is an immersion if the derivative $D\Phi$ has full rank everywhere. A smooth embedding is an injective immersion which is also a topological embedding, i.e. a homeomorphism from M to $\Phi(M)$. A thorough treatment can be found in chapter 5 of [20]. In particular, we will use the following theorem.

Theorem 1.6 ([20], Theorem 5.31)**.** *The image of a smooth embedding is an embedded submanifold.*

Proof of Theorem 1.1. Let M be the subset of $(\mathbb{R}^d)^k$ consisting of configurations of the form

$$(e^1, ..., e^{d-1}, te^d, z^{d+1}, ..., z^k)$$

with $t \in \mathbb{R}\backslash\{0\}, z^i \in \mathbb{R}^d$, where e^i is the i-th standard basis vector in $\mathbb{R}^d$. We claim M has a unique representative of every non-degenerate volume type. To prove every volume type is represented, let $x \in (\mathbb{R}^d)^k$ be non-degenerate. Let $t = \det(x^1, ..., x^d)$ and let $g \in \mathrm{SL}_d(\mathbb{R})$ be such that $g(x^1, ..., x^d) = (e^1, ..., te^d)$. For $i > d$, let $z^i = gx^i$. This choice of t and z^i produces an element of M with the same volume type as x. To show this representation is unique, suppose $(e^1, ..., e^{d-1}, te^d, z^{d+1}, ..., z^k)$ and $(e^1, ..., e^{d-1}, t'e^d, w^{d+1}, ..., w^k)$ have the same volume type. Considering the volumes of the first d points, it is easy to see $t = t'$. If g is the element of $\mathrm{SL}_d(\mathbb{R})$ mapping the first configuration to the second, it follows that g fixes a basis and is therefore the identity.

M is a manifold of dimension $d(k-d)+1$, and we can take $t, z^{d+1}, ..., z^k$ as the coordinates of the point $(e^1, ..., e^{d-1}, te^d, z^{d+1}, ..., z^k)$. If $\Phi(t, z^{d+1}, ..., z^k)$ is the volume type of $(e^1, ..., e^{d-1}, te^d, z^{d+1}, ..., z^k)$, then we have a smooth injective map $\Phi : M \to \mathbb{R}^{\binom{k}{d}}$. We have

$$t = \det(e^1, ..., te^d), \qquad \text{and} \qquad tz_j^i = \det(e^1, ..., e^{j-1}, z^i, e^{j+1}, ..., te^d).$$

Let R_0 be the row of the matrix $D\Phi$ corresponding to the component $\det(e^1, ..., te^d)$, and for each $i, j > d$ let $R_{i,j}$ be the row corresponding to the component

$$\det(e^1, ..., e^{j-1}, z^i, e^{j+1}, ..., te^d).$$

Then R_0 has a 1 in the column corresponding to $\partial/\partial t$ and 0 elsewhere. The row $R_{i,j}$ has a t in the column corresponding to $\partial/\partial z_j^i$, a z_j^i in the column corresponding to $\partial/\partial t$, and 0 elsewhere. It is therefore clear that $D\Phi$ has full rank, so Φ is an immersion. It is also clear that Φ and Φ^{-1} are smooth, so Φ is an embedding. It follows from Theorem 1.6 that the image $\mathcal{V}_{k,d}$ is an embedded submanifold of $\mathbb{R}^{\binom{k}{d}}$. The dimension of $\mathcal{V}_{k,d}$ must be $\dim M = d(k-d)+1$.

Chapter 2

Bounds and Proofs

The content of this chapter represents join work with Dr. Alex McDonald [15].

2.1 Fourier integral operators and generalized Radon transforms

To prove our theorems, we will employ the usual strategy of defining pushforward measures supported on our sets $\mathcal{V}_{k,d}(E)$ and $\mathcal{C}_{k,d}(E)$, taking approximations to those measures, and obtaining a uniform L^2 bound on those approximations. This will reduce to using mapping properties of generalized Radon transforms, which we establish here. We will be following the framework introduced in [13].

Let X and Y be open subsets of $\mathbb{R}^{d\times(d-1)}$ and $\mathbb{R}^d$, respectively. A **symbol** of order m on $X \times Y \times \mathbb{R}$ is a smooth map $a : X \times Y \times \mathbb{R} \to \mathbb{R}$ satisfying the bound

$$\left|\frac{\partial^n}{\partial\theta^n} a(x,y,\theta)\right| \lesssim (1+|\theta|)^{m-n}$$

on compact subsets of $X \times Y$. Also, for smooth phase functions $\varphi : X \times Y \times \mathbb{R} \to \mathbb{R}$, define

$$C_\varphi = \left\{(x, \nabla_x\varphi(x,y,\theta), y, -\nabla_y\varphi(x,y,\theta) : \theta \neq 0, \frac{\partial}{\partial\theta}\varphi(x,y,\theta) = 0\right\}.$$

We view C_φ as a subset of $(T^*X\backslash\{0\}) \times (T^*Y\backslash\{0\})$. Given any subset $C \subset (T^*X\backslash\{0\}) \times (T^*Y\backslash\{0\})$ and any order $m \in \mathbb{R}$, define the class of Fourier integral operators of order m and with canonical relation C, denoted by $I^m(C)$, to be those with Schwartz kernels which are locally finite sums of kernels of the form

$$K(x,y) = \int e^{i\varphi(x,y,\theta)} a(x,y,\theta)\, d\theta$$

where C_φ is a relatively open subset of C and a is a symbol of order $m - \frac{1}{2} + \frac{d^2}{4}$. We will use the following result.

Theorem 2.1 ([13], Theorem 3.1)**.** *Let C be a canonical relation and let $A \in I^{r-\frac{d^2-2d}{4}}$ have compactly supported Schwartz kernel. Suppose the projections from $(T^*X\backslash\{0\}) \times (T^*Y\backslash\{0\})$ to each factor, restricted to C, have full rank (so the first is an immersion and the second is a submersion). Then A is a bounded operator $L^2(Y) \to L^2_{-r}(X)$.*

Let $\Phi, \eta : X \times Y \to \mathbb{R}$ be smooth and let η be compactly supported. A **generalized Radon transform** is an operator of the form

$$Af(x) = \int_{\Phi(x,y)=0} f(y)\eta(x,y)\, d\sigma_x(y),$$

where σ_x is the induced surface measure on the surface defined by $\Phi(x,y) = 0$. This can be written in terms of the delta distribution (and its Fourier transform) as an oscillatory integral; we have

$$\begin{aligned} Af(x) &= \int_{\Phi(x,y)=0} f(y)\eta(x,y)\, d\sigma_x(y) \\ &= \int \delta(\Phi(x,y)) f(y)\eta(x,y)\, dy \\ &= \int\int e^{2\pi i \Phi(x,y)\theta} f(y)\eta(x,y) 1(\theta)\, d\theta\, dy \end{aligned}$$

Therefore, A is a Fourier integral operator with phase function $2\pi\Phi(x,y)\theta$ and amplitude $\eta(x,y)\theta$. The symbol $\eta(x,y)\theta$ has order 0, so our generalized radon transforms are Fourier integral operators of order $\frac{2-d^2}{4}$. This means Theorem 2.1 applies with $r = -\frac{d-1}{2}$, assuming the condition on the canonical relation holds.

The generalized radon transforms we will be interested in are those given by the determinant function. Throughout this paper, $\mathcal{R}_t$ will denote the operator

$$\mathcal{R}_t f(x^1, \cdots x^{d-1}) = \int_{\det(x^1,\cdots,x^d)=t} f(x^d)\eta(x^1,\cdots,x^d)\, d\sigma_{t,x^1,\cdots,x^{d-1}}(x^d)$$

where $\sigma_{t,x^1,\cdots,x^{d-1}}$ is the surface measure. These operators are shown to satisfy the canonical relation hypothesis of Theorem 2.1 in [13], which implies the following Sobolev bound for $\mathcal{R}_t$.

Theorem 2.2. *The generalized Radon transform $\mathcal{R}_t$ defined above is a bounded operator $L^2(\mathbb{R}^d) \to L^2_{\frac{d-1}{2}}((\mathbb{R}^d)^{d-1})$.*

2.2 Frostman measures and Littlewood-Paley projections

The following theorem is frequently used to study the dimension of fractal sets; see, for example, [24].

Theorem 2.3 (Frostman's Lemma)**.** *Let $E \subset \mathbb{R}^d$ be compact. For any $s < \dim E$, there is a Borel probability measure μ supported on E satisfying*

$$\mu(B_r(x)) \lesssim r^s$$

for all $x \in \mathbb{R}^d$ and all $r > 0$.

A measure μ as in the theorem is called a Frostman probability measure of exponent s.

We will be interested in the Littlewood-Paley decomposition of Frostman measures. Let μ be a Frostman probability measure on $\mathbb{R}^d$ with exponent s and compact support. Then μ_j is the j-th Littlewood-Paley piece of μ, defined by $\widehat{\mu_j}(\xi) = \psi(2^{-j}\xi)\widehat{\mu}(\xi)$ where ψ is a Schwarz function supported in the range $\frac{1}{2} \leqslant |\xi| \leqslant 4$ and constantly equal to 1 in the range $1 \leqslant |\xi| \leqslant 2$. We will use the following bounds.

Lemma 2.4. *Let μ be a compactly supported Frostman probability measure with exponent s, and let $(f\mu)_j$ be the j-th Littlewood Paley piece of the measure $f\mu$ for a function f. Then*

$$\|(f\mu)_j\|_{L^\infty} \lesssim 2^{j(d-s)}\|f\|_{L^\infty(\mu)}$$

and

$$\|(f\mu)_j\|_{L^2}^2 \lesssim 2^{j(d-s)}\|f\|_{L^2(\mu)}^2$$

Proof. Firstly, let us prove the L^∞ bound. Since $\|(f\mu)_j\|_{L^\infty} \leqslant \|f\|_{L^\infty(\mu)}\|\mu_j\|_{L^\infty}$ it suffices to prove the bound in the case $f = 1$. Observe that

$$(f\mu)_j(x) = 2^{2j}\check{\psi}(2^j\cdot) * f\mu(x)$$

Since ψ is a Schwarz function, we have $\psi(x) \lesssim (1+|x|)^{-2}$. Therefore,

$$|\mu_j(x)| \lesssim 2^{2j}\int (1+2^j|x-y|)^{-2}d\mu(y)$$

Splitting this integral into two parts: $2^j|x-y|<1$ and $2^j|x-y|>1$. We have

$$\begin{aligned}
&2^{2j}\int_{2^j|x-y|<1}(1+2^j|x-y|)^{-2}d\mu(y)\\
&\lesssim 2^{2j}\mu(\{y : 2^j|x-y|<1\})\\
&\lesssim 2^{j(d-s)}
\end{aligned}$$

and

$$\begin{aligned}
&2^{2j}\int_{2^j|x-y|>1}(1+2^j|x-y|)^{-2}d\mu(y)\\
&= 2^{2j}\sum_{i=0}^{\infty}\int_{2^i\leqslant 2^j|x-y|\leqslant 2^{i+1}}(1+2^j|x-y|)^{-2}d\mu(y)\\
&\lesssim 2^{2j}\sum_{i=0}^{\infty}2^{-2i}\mu(\{y : 2^i\leqslant 2^j|x-y|\leqslant 2^{i+1}\})\\
&\lesssim 2^{j(d-s)}\sum_{i=0}^{\infty}2^{i(s-2)}\\
&\lesssim 2^{j(d-s)}
\end{aligned}$$

Thus, we get the first result as claimed. To prove the L^2 bound, we first observe that

$$\begin{aligned}
\|(f\mu)_j\|_{L^2}^2 &= \|\widehat{(f\mu)_j}\|_{L^2}^2\\
&= \int |\widehat{f\mu}(\xi)|^2\psi_j^2(\xi)d\xi\\
&= 2^{jd}\int\int \widehat{\psi^2}(2^j(x-y))f(x)f(y)d\mu(x)d\mu(y)
\end{aligned}$$

where we have used Fourier inversion in the last line. Break the integral into two parts corresponding to $|x-y|<2^{-j}$ and $|x-y|>2^{-j}$, where C is a large constant. Since ψ is a Schwartz function, it suffices to bound the first part. Let $K_j = 2^{2j}\chi_{\{|x-y|<2^{-j}\}}$ and let $T_jf(x) = \int K_j(x,y)f(y)d\mu(y)$. Our goal is to prove $\langle T_jf, f\rangle_{L^2(\mu)} \lesssim 2^{j(d-s)}\|f\|_{L^2(\mu)}$. By Cauchy-Schwarz, it suffices to show the norm of T_j as an operator $L^2(\mu)\to L^2(\mu)$ is bounded by $2^{j(d-s)}$. This follows from Schur's test, as

$$\int K(x,y)d\mu(x) = \int K(x,y)d\mu(y) \lesssim 2^{j(d-s)}.$$

□

The generalized Radon transform applied to μ_j also has Fourier transform concentrated at scale 2^j. This together with Theorem 2.2 allows us to prove the following bounds. Here and throughout, given $f_1, ..., f_n : X \to \mathbb{R}$, the function $f_1 \otimes \cdots \otimes f_n$ is the function $X^n \to \mathbb{R}$ given by

$$f_1 \otimes \cdots \otimes f_n(x^1, ..., x^n) = f_1(x^1) \cdots f_n(x^n)$$

Lemma 2.5. *Let φ be a smooth function which is supported on $[-1, 1]$ and equal to 1 on $[-1/2, 1/2]$, and let $\varphi^\varepsilon(t) = \varepsilon^{-1}\varphi(\varepsilon^{-1}t)$. Let $\eta : (\mathbb{R}^d)^d \to \mathbb{R}$ be a smooth cutoff function supported in the region $|x^i - e^i| < c$ where e^i is the i-th standard basis vector and c is a small positive constant. Finally, let $\mathcal{R}_t^\varepsilon$ be the approximate generalized Radon transform defined by*

$$\mathcal{R}_t^\varepsilon f(x^1, ..., x^{d-1}) = \int f(x^d)\eta(x^1, ..., x^d)\varphi^\varepsilon(\det(x^1, ..., x^d) - t)\, dx^d.$$

If c is sufficiently small, we have the following.

(i)
$$\|\mathcal{R}_t^\varepsilon(f\mu)_j\|_{L^2}^2 \lesssim 2^{j(1-s)}\|f\|_{L^2(\mu)}^2.$$

(ii) If $j, j_1, ..., j_{d-1}$ are any indices such that $|j - j_i| > 5$ for any i, then for every number N and functions $f, f_1, ..., f_{d-1}$ we have

$$\left\langle \mathcal{R}_t^\varepsilon(f\mu)_j, (f_1\mu)_{j_1} \otimes \cdots \otimes (f_{d-1}\mu)_{j_{d-1}} \right\rangle \lesssim_N 2^{-N\cdot\max(j, j_1, ..., j_{d-1})},$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $L^2(\mathbb{R}^{d-1})$.

Proof. We first prove that the Fourier transform of $\mathcal{R}_t^\varepsilon(f\mu)_j$ decays rapidly outside the region $|x^j| \approx 2^j$. After we prove this, both statements follow from Plancherel and Theorem 2.2. By Fourier inversion, we have

$$\mathcal{R}_t^\varepsilon\mu_j(x^1, ..., x^{d-1}) = \int\int\int e^{2\pi i\xi^d\cdot x^d}e^{2\pi i\tau(\det(x)-t)}\widehat{(f\mu)_j}(x^d)\widehat{\varphi}(\varepsilon\tau)\eta(x)\, dx^d\, d\xi^d\, d\tau,$$

and therefore

$$\widehat{\mathcal{R}_t^\varepsilon\mu_j}(\xi^1, ..., \xi^{d-1}) = \int\int\int e^{2\pi i(\tilde{\xi}\cdot x+\tau(\det(x)-t))}\widehat{(f\mu)_j}(\xi^d)\widehat{\varphi}(\varepsilon\tau)\eta(x)\, dx\, d\xi^d\, d\tau$$

where $\tilde{\xi} = (\xi^1, ..., \xi^{d-1}, -\xi^d)$. This integral can be written

$$\int\int \widehat{(f\mu)_j}(\xi^d)\widehat{\varphi}(\varepsilon\tau)I(\tau, \xi)\, d\tau\, d\xi^d,$$

where

$$I(\tau, \xi) = \int e^{2\pi i(\tilde{\xi}\cdot x+\tau(\det(x)-t))}\eta(x)\, dx.$$

This is an oscillatory integral with phase function

$$\Phi_{\tau,\xi}(x) = \tilde{\xi} \cdot x + \tau(\det(x) - t).$$

We observe

$$\nabla\Phi_{\tau,\xi}(x) = \tilde{\xi} + \tau \cdot \nabla\det(x).$$

For x in the support of η, we have $\frac{1}{2} < |\nabla_{x^i}\det(x) - e^i| < 2$ if the constant c in the statement of the theorem is sufficiently small. Therefore, if $\Phi_{\tau,\xi}$ has critical points then we must have $\frac{1}{2}|\xi^i| \leqslant \tau \leqslant 2|\xi^i|$ for all i. If $2^{j-2} < |\xi^d| < 2^{j+2}$ and $2^{j_i-2} < |\xi^i| < 2^{j_i+2}$ with $|j - j_i| > 5$, then $\Phi_{\tau,\xi}$ has no critical points and by non-stationary phase (for example [24], proposition 6.1) we have

$$I(\tau,\xi) \lesssim_N 2^{-N\cdot\max(j,j_1,\dots,j_{d-1})}.$$

It follows from this and Lemma 3.2 that

$$\left\langle \mathcal{R}_t^{\varepsilon}(f\mu)_j, (f_1\mu)_{j_1} \otimes \cdots \otimes (f_{d-1}\mu)_{j_{d-1}} \right\rangle = \left\langle \widehat{\mathcal{R}_t^{\varepsilon}(f\mu)}_j, \widehat{(f_1\mu)}_{j_1} \otimes \cdots \otimes \widehat{(f_{d-1}\mu)}_{j_{d-1}} \right\rangle$$

$$\begin{aligned} &= \int\int \widehat{(f_1\mu)}_{j_1}(\xi^1)\cdots\widehat{(f_{d-1}\mu)}_{j_{d-1}}(\xi^{d-1})\widehat{(f\mu)}_j(\xi^d)\widehat{\varphi}(\varepsilon\tau)I(\tau,\xi)\,d\tau\,d\xi \\ &\lesssim_N 2^{-N\cdot\max(j,j_1,\dots,j_{d-1})}. \end{aligned}$$

It also follows that

$$\begin{aligned} \|\mathcal{R}_t^{\varepsilon}(f\mu)_j\|_{L^2}^2 &= \|\widehat{\mathcal{R}_t^{\varepsilon}(f\mu)}_j\|_{L^2}^2 \\ &\lesssim 2^{-j(d-1)}\int_{|\xi|\approx 2^j} |\xi|^{d-1}\widehat{\mathcal{R}_t^{\varepsilon}(f\mu)}_j(\xi)d\xi \\ &= 2^{-j(d-1)}\|\mathcal{R}_t^{\varepsilon}(f\mu)_j\|_{L^2_{\frac{d-1}{2}}}^2 \\ &\lesssim 2^{j(1-s)}\|f\|_{L^2(\mu)}^2 \end{aligned}$$

□

2.3 Proofs

2.3.1 Proof of Theorem 1.2

Many Falconer type problems can be attacked by defining an appropriate pushforward measure and proving it is in L^2. The following lemma establishes this framework.

Lemma 2.6. *Let $\mathcal{M}$ be an n-dimensional submanifold of $\mathbb{R}^m$ equipped with n-dimensional Lebesgue measure $\mathcal{L}_n$ and consider a map $\Phi : (\mathbb{R}^d)^k \to \mathcal{M}$. For $E \subset \mathbb{R}^d$, let*

$$\Delta_\Phi(E) = \{\Phi(x) : x \in E^k\}.$$

If μ is a probability measure supported on a compact set E and

$$\varepsilon^{-n}\int\cdots\int_{|\Phi(x)-\Phi(y)|\lesssim\varepsilon} d\mu^k(x)\,d\mu^k(y) \lesssim 1,$$

then $\mathcal{L}_n(\Delta_\Phi(E)) > 0$.

Proof. Define a probability measure ν on $\mathcal{M}$ by the relation

$$\int f(t)\,d\nu(t) = \int f(\Phi(x))\,d\mu^k(x).$$

It suffices to prove ν is absolutely continuous with respect to $\mathcal{L}_n$. Let φ be a symmetric Schwartz function on $\mathbb{R}^m$ supported on the ball of radius 2 and equal to 1 on the unit ball. Let $\varphi^\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ and let $\nu^\varepsilon = \varphi^\varepsilon * \nu$. Then

$$\int_A \nu^\varepsilon(t)dt \leqslant \mathcal{L}_n(A)^{1/2}\|\nu^\varepsilon\|_{L^2},$$

where dt denotes integration with respect to n-dimensional Lebesgue measure. This reduces matters to proving an upper bound on $\|\nu^\varepsilon\|_{L^2}$ which is uniform in ε. We have

$$\begin{aligned}\nu^\varepsilon(t) &= \int \varphi^\varepsilon(t'-t)d\nu(t') \\ &= \int \varphi^\varepsilon(\Phi(x)-t)d\mu^k(x) \\ &\approx \varepsilon^{-n}\int_{|\Phi(x)-t|\leqslant\varepsilon/2} d\mu^k(x).\end{aligned}$$

Thus,

$$\begin{aligned}||\nu^\varepsilon||_{L^2}^2 &\approx \varepsilon^{-2n}\int\left(\int\cdots\int_{|\Phi(x)-t|\leqslant\varepsilon/2} d\mu^k(x)d\mu^k(y)\right)dt \\ &= \varepsilon^{-2n}\int\cdots\int_{|\Phi(x)-\Phi(y)|\leqslant\varepsilon}\left(\int_{|\phi(x)-t|\leqslant\varepsilon/2} dt\right)d\mu^k(x)d\mu^k(y) \\ &\approx \varepsilon^{-n}\int\cdots\int_{|\phi(x)-\phi(y)|\leqslant\varepsilon} d\mu^k(x)d\mu^k(y) \\ &\lesssim 1\end{aligned}$$

□

To apply this approach to our current problem, we first reduce to the case where our set $E \subset \mathbb{R}^d$ has some additional structure.

Lemma 2.7. *Let $k \geqslant d$ and let $E \subset \mathbb{R}^d$ be a compact set with Hausdorff dimension $\dim E > d-1$. Then there exist subsets $E_1, ..., E_k \subset E$ and a constant c with $\dim E_j = \dim E$ and the property that for any choice of d points $x^1, ..., x^d$ in different cells E_j, we have $\det(x^1, \cdots, x^d) > c$.*

Proof. Let μ be a Frostman probability measure on E with exponent $s > d-1$ and let N be a large integer to be determined later. The idea of the proof is that the 2^{-N}-neighborhood of a compact piece of a hyperplane has negligible μ-measure, so we can construct our sets E_j recursively by throwing away bad parts of E.

Given a point $x \in \mathbb{R}^d$, let $B(x)$ denote the ball of radius 2^{-N} centered at x. Let $\mathcal{S}_0$ be a finite set such that $\{B(x) : x \in \mathcal{S}_0$ covers E, and let $\mathcal{S} \subset \mathcal{S}_0$ be the subset obtained by discarding any x such that $\mu(B(x)) = 0$. Without loss of generality we may assume that none of our balls contains the origin.

Let $x^1, x^2 \in \mathcal{S}$ be arbitrary points such that the balls $B(x^1)$ and $B(x^2)$ have distance $> 2^{-N}$. For $2 \leqslant j \leqslant d-1$, suppose $x^1, ..., x^j$ have been defined and are linearly independent. Let X denote the 2^{-N+10}-neighborhood of $\mathrm{span}(x^1, ..., x^j)$ intersected with the ball of radius $\sup E$. Then $\mu(X) \lesssim 2^{-N(s-j)}$. Since $s > j$, for large N this is small, so we can choose $x^{j+1} \in E\backslash X$. It follows that $B(x^{j+1})$ does not intersect the 2^{-N} neighborhood of $\mathrm{span}(x^1, ..., x^j)$. For $d \leqslant j < k$, suppose $x^1, ..., x^j$ have been defined and have the property that for any $j_1, ..., j_d \leqslant j$, $B(x^j)$ does not intersect the 2^{-N}-neighborhood of span$(x^{j_1}, ..., x^{j_{d-1}})$. Again, if N is sufficiently large then the union of all $\binom{j}{d-1}$ approximate hylperplanes determined by any $d-1$ of the points $x^1, ..., x^j$ has small μ measure, so we can choose x^{j+1} to avoid all of them as well. It is clear that the collection $E_j := B(x^j)$ has the desired properties.

□

To prove Theorem 1.2, by Lemmas 2.6 and 2.7 it suffices to bound

$$\varepsilon^{-d(k-d)-1} \int\int_{|\Phi(x)-\Phi(y)|\lesssim\varepsilon} d\mu^k(x)\, d\mu^k(y) \tag{1}$$

independent of ε. We follow the approach used in [11] and [22] to reduce matters to the $k = d$ case. We first decompose the $d\mu^k(y)$ factor into Littlewood-Paley pieces, reducing (1) to

$$\approx \varepsilon^{-d(k-d)-1} \sum_{j_1,\dots,j_k} \int\int_{|\Phi(x)-\Phi(y)|\lesssim\varepsilon} \mu_{j_1}(y^1)\cdots\ \mu_{j_k}(y^k)\, dy^1 \cdots dy^k\, d\mu^k(x). \tag{2}$$

Here $\{\mu_j\}$ are the Littlewood Paley pieces of μ, as defined in chapter 3. Now that we have an integral in dy, we want to use the group action framework discussed in chapter 2 to turn this into an integral over $\mathrm{SL}_d(\mathbb{R})$. The idea is that for fixed x, integrating over the region $|\Phi(x) - \Phi(y)| < \varepsilon$ is equivalent to integrating over $y \sim gx$ as g varies. If ε is sufficiently small then $\det(y^1, \cdots, y^d) \neq 0$ for y in this region. Every such y has the same area type as a configuration of the form

$$(x_1^1, \cdots, x_{d-1}^d, t_{d^2}, \cdots, t_{kd}).$$

Moreover, there is an open set $U_d \subset \mathbb{R}^{d^2-1}$ such that for every $(g_1^1, \cdots, g_{d-1}^d) \in U_d$ there exists a unique $g \in \mathrm{SL}_d(\mathbb{R})$ whose matrix has those entries, and the lower right entry is a rational function of the others. This gives a rational change of variables

$$y = g(x_1^1, \cdots, x_{d-1}^d, t_{d^2}, \cdots, t_{kd}),$$

where g is viewed in terms of its coordinates. Since x lives in a fixed compact subset of configuration space, the Jacobian determinant is ≈ 1 and (2) is

$$\approx \varepsilon^{-d(k-d)-1} \int\int\int_{B_\varepsilon} \left(\sum_{j_1,\dots,j_k} \mu_{j_1} \otimes \cdots \otimes \mu_{j_k} \right) (g(x_1^1, \cdots, x_{d-1}^d, t_{d^2}, \cdots, t_{kd}))\, dg\, dt\, d\mu^k(x), \tag{3}$$

where the two inner integral signs represent integration over the first $d^2 - 1$ coordinates of g and the $d(k-d)+1$ coordinates $\{t_i\}$, respectively. The t_i coordinates are integrated over the ball B_ε raidus ε centered at the last $d(k-d)+1$ coordinates of x. Taking the limit as $\varepsilon \to 0$, this is

$$\sum_{j_1,\dots,j_k} \int\int \cdots \int \mu_{j_1}(gx^1) \cdots \mu_{j_k}(gx^k)\, d\mu(x^1) \cdots\ d\mu(x^k)\, dg. \tag{4}$$

Here we make a couple simple reductions. First, μ_j is a Schwarz function satisfying the L^∞ bound $\|\mu_j\|_{L^\infty} \lesssim 2^{j(d-s)}$ (see for example [22], Lemma 3) which we use to reduce from general $k \geqslant d$ to the $k = d$ case. Moreover, the sum over $j_1, ..., j_k$ can be reduced to the sum over indices satisfying $j_1 \geqslant \cdots \geqslant j_k \geqslant 0$, as negative indices clearly contribute $O(1)$ to the sum and other permutations of indices only change the sum by a multiplicative constant. Applying the L^∞ bound and running the sum in the indices $j_{d+1}, ..., j_k$, it follows that (4) is

$$\lesssim \sum_{j_1 \geqslant \cdots \geqslant j_d} 2^{j_d(d-s)(k-d)} \int\int \cdots \int \mu_{j_1}(gx^1) \cdots \mu_{j_d}(gx^d)\, d\mu(x^1) \cdots d\mu(x^d)\, dg. \tag{5}$$

This reduces matters to the $k = d$ case. Using the same change of variables in the other direction, this is

$$\begin{aligned}
&\approx \varepsilon^{-1} \sum_{j_1 \geqslant \cdots \geqslant j_d} 2^{j_d(d-s)(k-d)} \int \cdots \int_{|\det(x^1,\ldots,x^d)-\det(y^1,\ldots,y^d)|<\varepsilon} \mu_{j_1}(y^1) \cdots \mu_{j_d}(y^d)\, dy\, d\mu^k(x) \\
&\approx \varepsilon^{-2} \sum_{j_1 \geqslant \cdots \geqslant j_d} 2^{j_d(d-s)(k-d)} \int\int \cdots \int_{\substack{|\det(x^1,\cdots,x^d)-t|<\varepsilon \\ |\det(y^1,\cdots,y^d)-t|<\varepsilon}} \mu_{j_1}(y^1) \cdots \mu_{j_d}(y^d)\, dy\, d\mu^k(x)\, dt. \\
&\approx \sum_{j_1 \geqslant \cdots \geqslant j_d} 2^{j_d(d-s)(k-d)} \int \left(\varepsilon^{-1} \int_{|\det(x^1,\cdots,x^d)-t|<\varepsilon} d\mu^k(x) \right) \langle \mathcal{R}_t^\varepsilon \mu_{j_1}, \mu_{j_2} \otimes \cdots \otimes \mu_{j_d} \rangle\, dt
\end{aligned} \tag{6}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2((\mathbb{R}^d)^{d-1})$ and $\mathcal{R}_t^\varepsilon$ is the approximation to the generalized Radon transform discussed in chapter 2. Let

$$\nu_{k,d}^\varepsilon(t) = \int_{|\det(x^{i_1},\cdots,x^{i_d})-t|<\varepsilon} d\mu^k(x).$$

The quantity in (1) we are trying to bound is $\|\nu_{k,d}^\varepsilon\|_{L^2}^2$, and we have proved.

$$\|\nu_{k,d}^\varepsilon\|_{L^2}^2 \lesssim \sum_{j_1 \geqslant \cdots \geqslant j_d} 2^{j_d(d-s)(k-d)} \int \nu_{d,d}^\varepsilon(t) \langle \mathcal{R}_t^\varepsilon \mu_{j_1}, \mu_{j_2} \otimes \cdots \otimes \mu_{j_d} \rangle\, dt,$$

Let

$$S = \sum_{j_1 \geqslant \cdots \geqslant j_d \geqslant 0} 2^{j_d(d-s)(k-d)} \sup_t \langle \mathcal{R}_t^\varepsilon \mu_{j_1}, \mu_{j_2} \otimes \cdots \otimes \mu_{j_d} \rangle.$$

If S is finite, we have

$$\|\nu_{k,d}^\varepsilon\|_{L^2}^2 \lesssim \|\nu_{d,d}^\varepsilon\|_{L^2}.$$

Plugging in $k = d$ on the left, we have a uniform bound on $\|\nu_{d,d}^\varepsilon\|_{L^2}$ which in turn implies a uniform bound on $\|\nu_{k,d}^\varepsilon\|_{L^2}$ for all $k \geqslant d$. So, it suffices to prove S is finite under the hypotheses of Theorem 1.2. By Lemma 2.5 it is clear that the part of the sum corresponding to indices with $j_d < j_1 - 5$ converges. It also follows from Lemma 2.5 and Cauchy-Schwarz that

$$\sup_t \langle \mathcal{R}_t^\varepsilon \mu_j, \mu_j \otimes \cdots \otimes \mu_j \rangle \lesssim 2^{\frac{j}{2}(1-s+(d-s)(d-1))}.$$

Therefore,

$$S \lesssim \sum_{j \geqslant 0} 2^{j\left((d-s)(k-d) + \frac{1-s+(d-s)(d-1)}{2}\right)}.$$

The sum will converge if $s > d - \frac{d-1}{2k-d}$, as claimed.

2.3.2 Proof of Theorem 1.3

To prove Theorem 1.3, it is enough to establish the following bound. The theorem then follows from Lemma 2.6.

Lemma 2.8. *Let φ^ε be an approximation to the identity on $\mathbb{R}$, and let*

$$J^{\varepsilon}_{t,k} = \int \left(\prod_{j=1}^{k+1-d} \varphi^{\varepsilon}(\det(x^j, ..., x^{j+d-1}) - t_j) \right) d\mu^k(x).$$

For every $k \geqslant d$ there is a constant C_k (which does not depend on t or ε) such that $J^{\varepsilon}_{t,k} \leqslant C_k$.

Proof. We first prove a bound in the case $k = d$. Since $J^{\varepsilon}_{t,d} \approx \sum_j \|\mathcal{R}^{\varepsilon}_t \mu_j\|_{L^1(\mu^{d-1})}$, it is enough to prove $\|\mathcal{R}^{\varepsilon}_t \mu_j\|_{L^2(\mu^{d-1})} \lesssim 2^{-cj}$ for some positive c. To accomplish this, fix t and let $T^{\varepsilon}_j f = \mathcal{R}^{\varepsilon}_t (f\mu)_j$. We want to bound the norm of T^{ε}_j as an operator $L^2(\mu) \to L^2(\mu^{d-1})$. To do this, let $g \in L^2(\mu^{d-1})$ be given by $g(x) = g_1(x^1) \cdots g_{d-1}(x^{d-1})$ with $g_i \in L^2(\mu)$. Using Littlewood-Paley decomposition, Lemma 2.5, and Cauchy-Schwarz we have

$$\begin{aligned}\left\langle T^{\varepsilon}_j f, g \right\rangle_{L^2(\mu^{d-1})} &\lesssim \langle \mathcal{R}^{\varepsilon}_t (f\mu)_j, (g_1\mu)_j \otimes \cdots \otimes (g_{d-1}\mu)_j \rangle \\ &\lesssim 2^{\frac{j}{2}(1-s+(d-s)(d-1))} \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu^{d-1})}.\end{aligned}$$

It follows that the operator norm, and hence $\|\mathcal{R}^{\varepsilon}_t \mu_j\|_{L^1(\mu^{d-1})}$, is bounded by $2^{\frac{j}{2}(1-s+(d-s)(d-1))}$, and this series converges when $s > d - 1 + \frac{1}{d}$. This gives the desired bound in the case $k = d$.

For $k > d$, let

$$\chi^{\varepsilon}_{t,k}(x) = \prod_{j=1}^{k+1-d} \varphi^{\varepsilon}(\det(x^j, ..., x^{j+d-1}) - t_j).$$

We have

$$\begin{aligned}J^{\varepsilon}_{t,k} &= \int \chi^{\varepsilon}_{t,k}(x) d\mu^k(x) \\ &= \int \chi^{\varepsilon}_{\tilde{t},k-1}(\tilde{x}) \varphi^{\varepsilon}(\det(x^{k+1-d}, ..., x^k) - t_{k+1-d}) d\mu^{k-1}(\tilde{x}) d\mu(x^k) \\ &\approx \sum_j \int \chi^{\varepsilon}_{\tilde{t},k-1}(\tilde{x}) \mathcal{R}^{\varepsilon}_{t_{k+1-d}} \mu_j(x^{k+1-d}, ..., x^{k-1}) d\mu^{k-1}(\tilde{x}) \\ &\lesssim (J^{\varepsilon}_{\tilde{t},k-1})^{1/2} \sum_j \|\mathcal{R}_{t_{k+1-d}} \mu_j\|_{L^2(\mu^{k-1})} \\ &\lesssim (J^{\varepsilon}_{\tilde{t},k-1})^{1/2}\end{aligned}$$

□

Let $\Phi(x) = (\det(x^1, ..., x^d), ..., \det(x^{k+1-d}, ..., x^k))$. We have

$$\varepsilon^{-(k+1-d)} \int \int_{|\Phi(x)-\Phi(y)|<\varepsilon} d\mu^k(x) d\mu^k(y) \lesssim \int J^{\varepsilon}_{\Phi(x),k} d\mu^k(x) \lesssim 1.$$

Theorem 1.3 then follows from Lemma 2.6.

2.3.3 Proof of Sharpness Theorem

We conclude this paper by proving Theorem 1.4. Let $\Lambda_{q,s}$ be the $q^{-\frac{d}{s}}$-neighborhood of $\frac{1}{q}\left(\mathbb{Z}^d \bigcap \left([\frac{q}{2}, q] \times [0, q]^{d-1}\right)\right)$, the right half of the lattice in the $(d-1)$-dimensional unit cube with spacing $\frac{1}{q}$ (see figure 2). By Theorem 8.15 in [10] we can choose a sequence q_n that increases sufficiently rapidly such that

$$\dim\left(\bigcap_n \Lambda_{q_n,s}\right) = s$$

Thus, for large q we may regard $\Lambda_{q,s}$ as an approximation to a set of Hausdorff dimension s. Let us modify this situation to fit our problem. By Lemma 1.8 in [10] we have

Lemma 2.9 ([10], Lemma 1.8)**.** *Let ψ be Lipschitz and surjective, and let $\mathcal{H}^s$ be the s-dimensional Hausdorff measure. Then $\mathcal{H}^s(F) \lesssim \mathcal{H}^s(E)$.*

As consequence of this lemma we have $\dim F \leqslant \dim E$. If ψ is bijective and Lipschitz in both directions, then $\dim F = \dim E$. Let $E_{q,s}$ (figure 2.2) be the image of $\Lambda_{q,s}$ under the spherical map

$$\psi(x_1, x_2, \cdots, x_d) =$$
$$= x_1\left(\cos\left(\frac{\pi x_2}{2}\right), \sin\left(\frac{\pi x_2}{2}\right)\cos\left(\frac{\pi x_3}{2}\right), \cdots, \prod_{i=2}^{d-1}\sin\left(\frac{\pi x_i}{2}\right)\cos\left(\frac{\pi x_d}{2}\right), \prod_{i=2}^{d}\sin\left(\frac{\pi x_i}{2}\right)\right)$$

.

It is not hard to check this map is injective on $[\frac{1}{2}, 1] \times [0,1]^{d-1}$ and therefore bijective as a map $\Lambda_{q,s} \to E_{q,s}$.

Let us fix a sequence q_n such that $\dim\left(\bigcap_n E_{q_n,s}\right) = s$ and call $E_s = \bigcap_n E_{q_n,s}$. It remains to prove $\mathcal{L}_{d(k-d)+1}\left(\mathcal{V}_{k,d}\left(E_s\right)\right) = 0$.

We begin by counting the number of volume types determined by the image of $\frac{1}{q}\left(\mathbb{Z}^d \bigcap \left([\frac{q}{2}, q] \times [0,q]^{d-1}\right)\right)$ under ψ (i.e., the spherical lattice points themselves and not the thickened set). It is clear that every volume type of this set is obtained by considering configurations with x^1 restrained to the first axis, and $x^2, ..., x^k$ unrestrained. Thus there are $\approx q$ choices for x^1 and $\approx q^d$ choices for $x^2, ..., x^k$. It follows that

$$\mathcal{L}_{d(k-d)+1}\left(\mathcal{V}_{k,d}(E_{q,s})\right) \lesssim \left(q^{-\frac{d}{s}}\right)^{d(k-d)+1} q^{d(k-1)+1}$$

This tends to 0 as $q \to \infty$ provided $s < d - \frac{d^2(d-1)}{d(k-1)+1}$. □

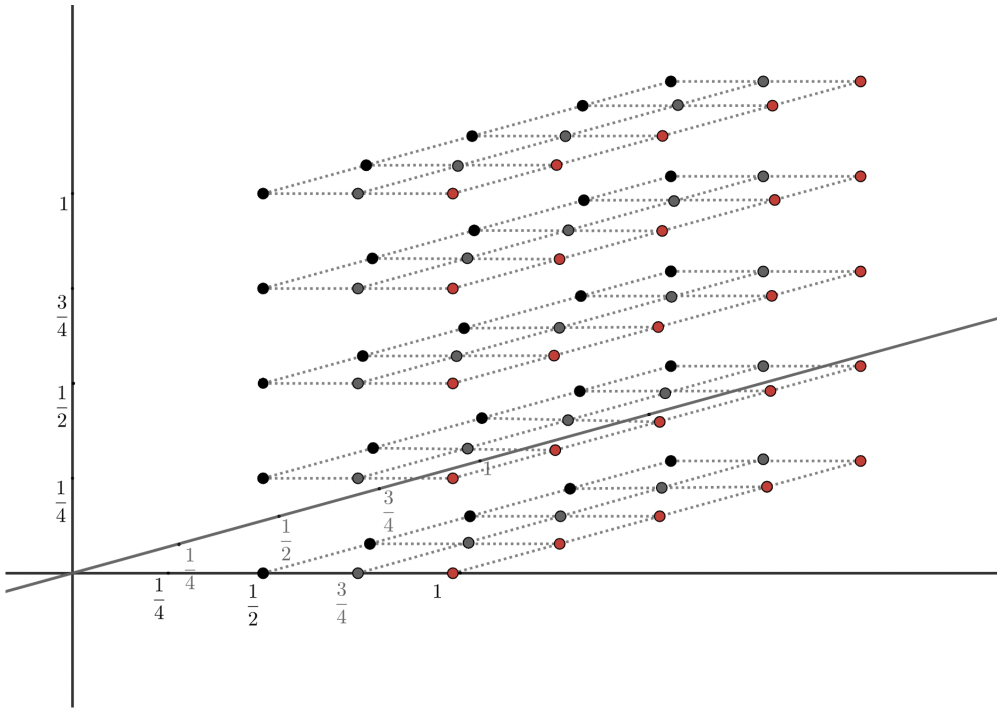

Figure 2.1: $\Lambda_{4,s}$ *for* $d = 3$.

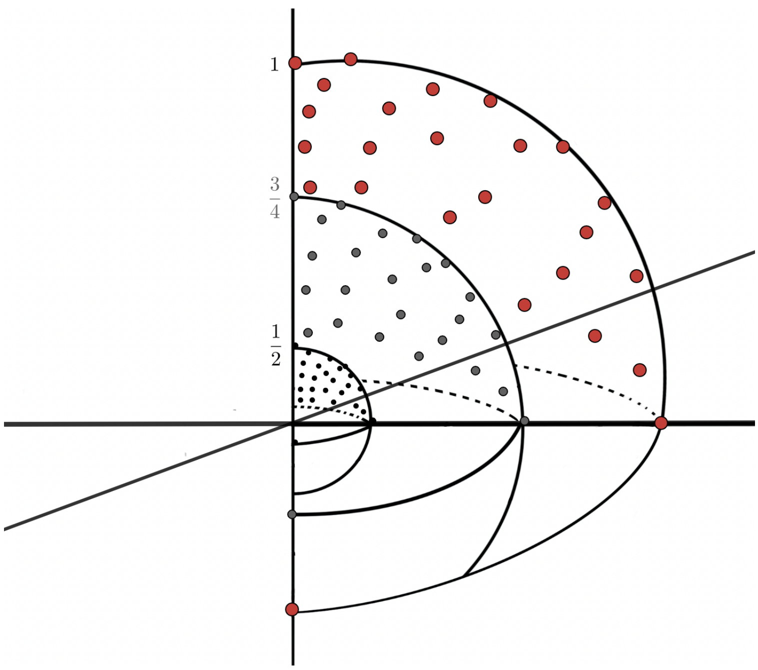

Figure 2.2: $E_{4,s}$ *for* $d = 3$.

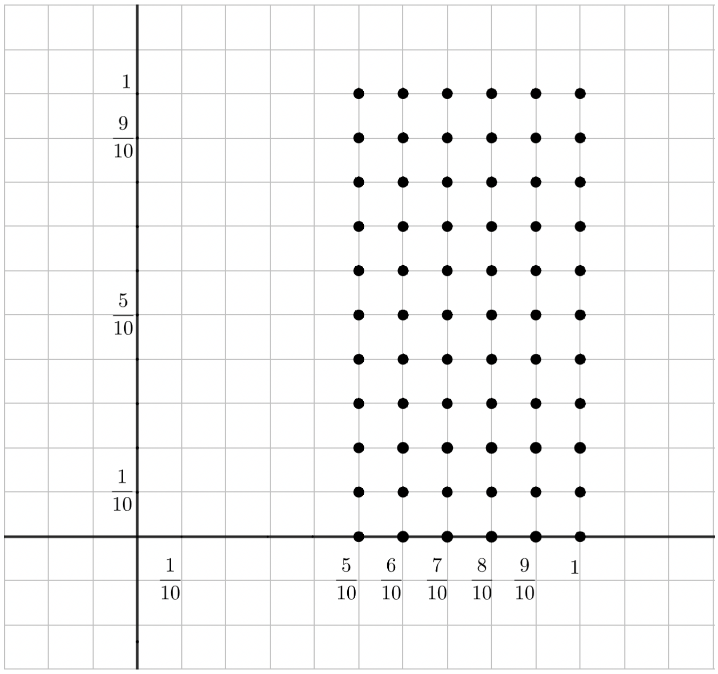

Figure 2.3: $\Lambda_{10,s}$ *for* $d = 2$.

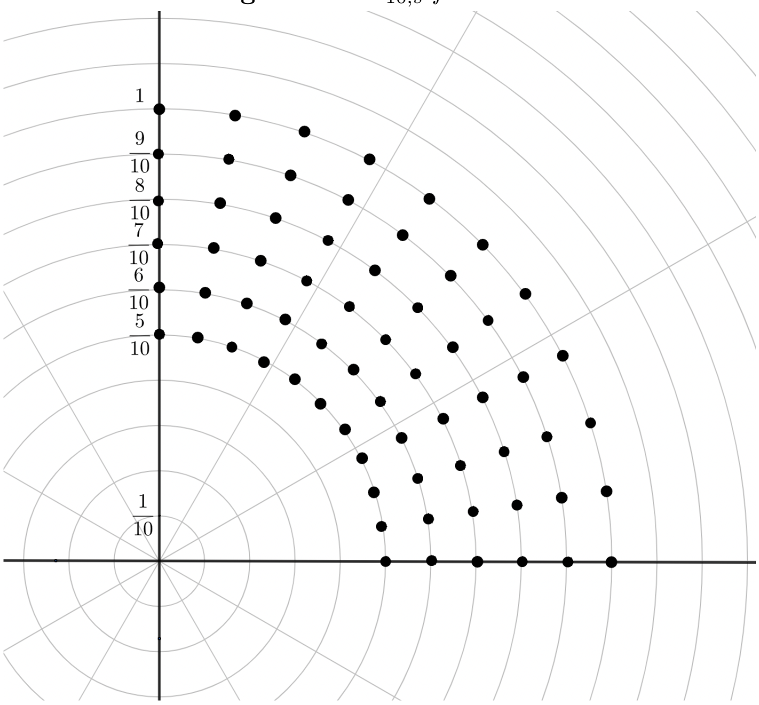

Figure 2.4: $E_{10,s}$ *for* $d = 2$.

Chapter 3

Configurations in the plane

In this chapter, we consider 4-point configuration in the plane, $d = 2$, under a similar notion of equivalence as A. McDonald in [22]. The substantial difference here is that our 4-point configuration in the plane are triangles with an extra leg. It means we will look for the triangle spanned by the first three points in our configuration, x,y, and z, and then we add one more point, w and look at the area of the triangle spanned by x and w. More precisely, we have the following definition

Definition 5. *The **Galo area type** of $(x, y, z, w) \in (\mathbb{R}^2)^4$ is the vector*

$$\left\{xy^{\perp}, yz^{\perp}, zx^{\perp}, xw^{\perp}\right\} \in \mathbb{R}^4.$$

For a set $E \subset \mathbb{R}^2$, let

$$\mathcal{G}(E) = \{(xy^{\perp}, yz^{\perp}, zx^{\perp}, xw^{\perp}) | x, y, z, w \in E\}$$

be the set of Galo area types determined by 4-points in E.
Finally, let $\mathcal{G} := \mathcal{G}(\mathbb{R}^2)$ be the space of all Galo area types in $\mathbb{R}^2$.

It follows that a configuration has vertex area type 0 if and only if all the points of the configuration lie on a common line through the origin.
We will use here the same concept of **degeneracy** defined in [22]:

Definition 6. *A 4-point configuration $v = (x, y, z, w) \in (\mathbb{R}^2)^4$ is called **degenerate** if $\{x, y\}$ are linearly dependent and **non-degenerate** otherwise.*

Thus, if you look at only for **non-degenerate** configurations we can also conclude v and u have the same Galo area type if and only if there is a unique $g \in SL_2(\mathbb{R})$ such that $gv = u$. Thus, we choose a measure on the space of Galo area types to be the 4-dimensional Lebesgue measure, $\mathcal{L}_4$. Thus, we can state our main theorem:

Theorem 3.1. *Let $E \subset \mathbb{R}^2$ be a compact set with $\dim E > \frac{3}{2}$. Then,*

$$\mathcal{L}_4(\mathcal{G}(E)) > 0.$$

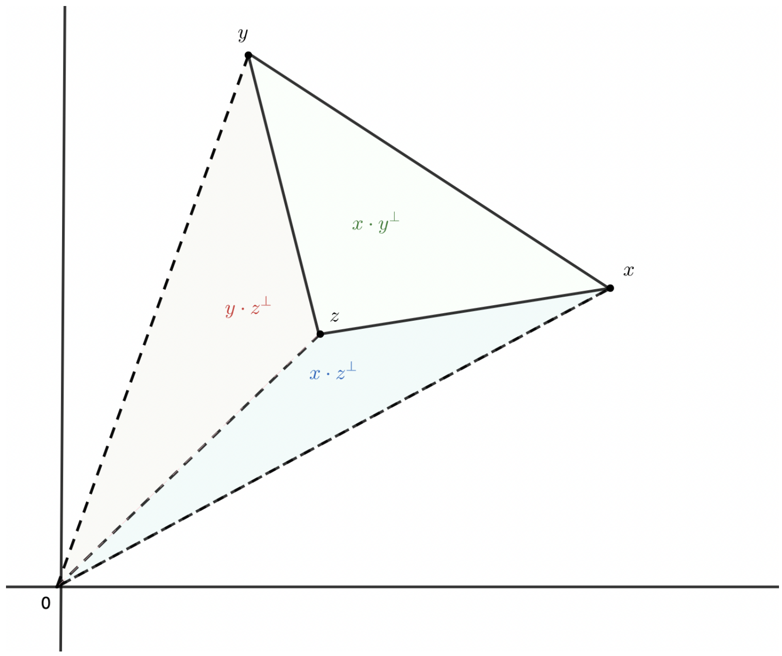

Figure 3.1: 3 *point configuration used in [22]*

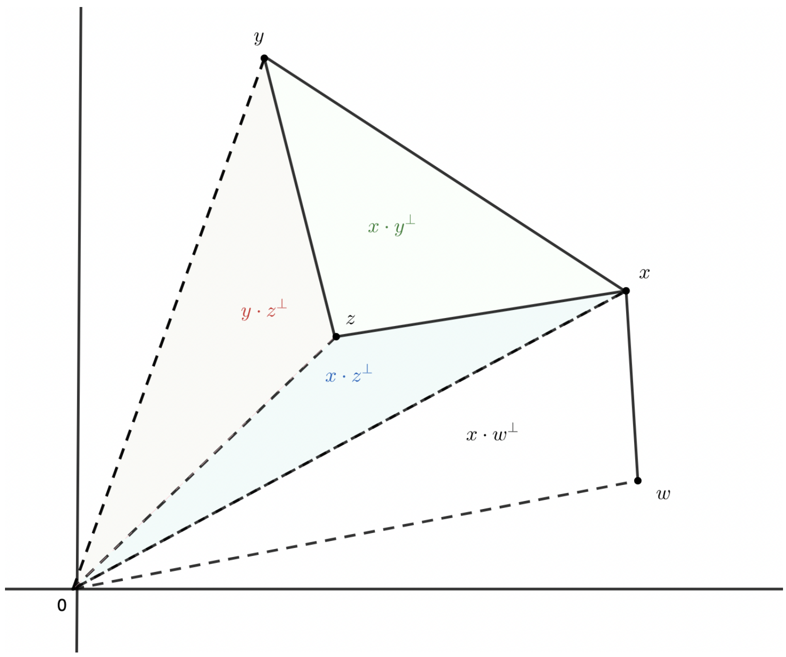

Figure 3.2: 4 *point configuration.*

3.1 Proof

In order to estimate our measure in $\mathcal{G}$

$$\frac{1}{e^4}\int_{|xy^\perp-a|<\varepsilon}\int_{|yz^\perp-b|<\varepsilon}\int_{|zx^\perp-c|<\varepsilon}\int_{|xw^\perp-t|<\varepsilon} d\mu(x)\mu(y)\mu(z)\mu(w)$$

Let us define

$$F_t(x) = \frac{1}{\varepsilon}\int_{|xw^\perp-t|<\varepsilon} d\mu(w)$$

It will be equivalent to estimate

$$\frac{1}{e^3}\int_{|xy^\perp-a|<\varepsilon}\int_{|yz^\perp-b|<\varepsilon}\int_{|zx^\perp-c|<\varepsilon}\mu(y)\mu(z)F_t(x)d\mu(x).$$

Firstly, let us compute the L^2 norm squared in a, b, c (not at t at this point) since we know two configurations are equivalent if they have the same area type and if $(x, y, z) \in (\mathbb{R}^2)^3$ and $(x', y', z') \in (\mathbb{R}^2)^3$ are non-degenerate configurations. Then (x, y, z) and (x', y', z') have the same area type if and only if there is a unique $g \in SL_2(\mathbb{R})$ such that, we have

$$(x', y', z') =^\varepsilon (gx, gy, gz)$$

Let $E \subset \mathbb{R}^2$ be compact with Hausdorff dimension greater than $\frac{3}{2}$ and and let μ be a Frostman probability measure with exponent $s > \frac{3}{2}$. We repeat the process made in [22] to reduce matters to bounding the integral

$$\int\int\int \mu^\varepsilon(gx)\mu^\varepsilon(gy)\mu^\varepsilon(gz)F_t(gx)F_t(x)d\mu(x)d\mu(y)d\mu(z)$$

Thus, rewriting the quantity above and integrate it in dg we have

$$\int\left(\left(\int \mu^\varepsilon(gy)d\mu(y)\right)^2\int \mu^\varepsilon(gx)F_t(gx)F_t(x)d\mu(x)\right)dg$$

The fundamental idea behind is that if we can estimate this quantity

$$\int \mu^\varepsilon(gx)F_t(gx)F_t(x)d\mu(x)$$

then the quantity the we have left is the squared integral

$$\left(\int \mu^\varepsilon(gy)d\mu(y)\right)^2$$

which corresponds to a single area problem and all bounds and details was solved by A. McDonald in [22].

Using the Littlewood-Paley decomposition of Frostman measures. Let μ be a Frostman probability measure on $\mathbb{R}^2$ with exponent s and compact support. Then μ_j is the j-th Littlewood-Paley piece of μ, defined by $\widehat{\mu_j}(\xi) = \psi(2^{-j}\xi)\widehat{\mu}(\xi)$ where ψ is a Schwarz function supported in the range $\frac{1}{2} \leqslant |\xi| \leqslant 4$ and constantly equal to 1 in the range $1 \leqslant |\xi| \leqslant 2$. We will use the following bounds.

Lemma 3.2. *Let μ be a compactly supported Frostman probability measure with exponent s, and let $(F_t\mu)_j$ be the j-th Littlewood Paley piece of the measure $F_t\mu$ for a function F_t. Then*

$$\|(F_t\mu)_j\|_{L^\infty} \lesssim 2^{j(2-s)}\|F_t\|_{L^\infty(\mu)}$$

and

$$\|(F_t\mu)_j\|_{L^2}^2 \lesssim 2^{j(2-s)}\|F_t\|_{L^2(\mu)}^2$$

Proof. Firstly, let us prove the L^∞ bound. Since $\|(F_t\mu)_j\|_{L^\infty} \leqslant \|F_t\|_{L^\infty(\mu)}\|\mu_j\|_{L^\infty}$ it suffices to prove the bound in the case $F = 1$. Observe that

$$(F_t\mu)_j(x) = 2^{2j}\check{\psi}(2^j\cdot) * F\mu(x)$$

Since ψ is a Schwarz function, we have $\psi(x) \lesssim (1 + |x|)^{-2}$. Therefore,

$$|\mu_j(x)| \lesssim 2^{2j} \int (1+2^j|x-y|)^{-2} d\mu(y)$$

Splitting this integral into two parts: $2^j|x-y|<1$ and $2^j|x-y|>1$. We have

$$\begin{aligned}
&2^{2j} \int_{2^j|x-y|<1} (1+2^j|x-y|)^{-2} d\mu(y) \\
&\lesssim 2^{2j}\mu(\{y : 2^j|x-y|<1\}) \\
&\lesssim 2^{j(2-s)}
\end{aligned}$$

and

$$\begin{aligned}
&2^{2j} \int_{2^j|x-y|>1} (1+2^j|x-y|)^{-2} d\mu(y) \\
&= 2^{2j} \sum_{i=0}^{\infty} \int_{2^i \leqslant 2^j|x-y| \leqslant 2^{i+1}} (1+2^j|x-y|)^{-2} d\mu(y) \\
&\lesssim 2^{2j} \sum_{i=0}^{\infty} 2^{-2i} \mu(\{y : 2^i \leqslant 2^j|x-y| \leqslant 2^{i+1}\}) \\
&\lesssim 2^{j(2-s)} \sum_{i=0}^{\infty} 2^{i(s-2)} \\
&\lesssim 2^{j(2-s)}
\end{aligned}$$

Thus, we get the first result as claimed. To prove the L^2 bound, we first observe that

$$\begin{aligned}
\|(F_t\mu)_j\|_{L^2}^2 &= \|\widehat{(F_t\mu)_j}\|_{L^2}^2 \\
&= \int_{|\xi|\sim 2^j} |\widehat{F_t\mu}(\xi)|^2 \psi_j^2(\xi) d\xi \\
&= 2^{2j} \int\int \widehat{\psi^2}(2^j(x-y)) F_t(x) F_t(y) d\mu(x) d\mu(y) \\
&\leqslant C||F_t||_{L^2}^2 \cdot 2^{j(2-s)}
\end{aligned}$$

where we have used Fourier inversion in the third line. Break the integral into two parts corresponding to $|x-y|<2^{-j}$ and $|x-y|>2^{-j}$, where C is a large constant. Since ψ is a Schwartz function, it suffices to bound the first part. Let $K_j = 2^{2j}\chi_{\{|x-y|<2^{-j}\}}$ and let $T_jF_t(x) = \int K_j(x,y)F_t(y)d\mu(y)$.

The last line came from Lemma 2.5 [1], using $\langle T_jF_t, F_t\rangle_{L^2(\mu)} \lesssim 2^{j(2-s)}\|F_t\|_{L^2(\mu)}$ and by Cauchy-Schwarz, it suffices to show the norm of T_j as an operator $L^2(\mu) \to L^2(\mu)$ is bounded by $2^{j(2-s)}$. This follows from Schur's test, as

$$\int K(x,y)d\mu(x) = \int K(x,y)d\mu(y) \lesssim 2^{j(2-s)}.$$

□

To show

$$\int ||F_t||^2_{L^2(\mu)} dt \leqslant C$$

We are going to use te fact that

$$\begin{aligned} ||F_t||^2_{L^2(\mu)} &= \int \left(\frac{1}{\varepsilon} \int_{|x\cdot w^{\perp} - t| < \varepsilon} d\mu(w) \right)^2 d\mu(x) \\ &= \frac{1}{\varepsilon^2} \int\int\int_{\substack{|x\cdot u^{\perp} - t| < \varepsilon \\ |x\cdot v^{\perp} - t| < \varepsilon}} d\mu(x) d\mu(y) d\mu(v) \end{aligned}$$

with that we have 2-chain of areas, and for areas it is well known this quantity is bounded for $s > \frac{3}{2}$.

If we call $\phi(x, y) = x \cdot y^{\perp} = x_1 y_2 - x_2 y_1$ we can show it satisfies the non-vanishing Monge-Ampere determinant assumption:

$$\det \begin{pmatrix} 0 & \nabla_x \phi \\ -(\nabla_y \phi)^T & \frac{\partial^2 \phi}{dx dy} \end{pmatrix} = \det \begin{pmatrix} 0 & y_2 & -y_1 \\ x_2 & 0 & -1 \\ x_1 & 1 & 0 \end{pmatrix} = x_1 y_2 - x_2 y_1 = x \cdot y^{\perp}$$

It implies for $t \neq 0$ it does not vanish on the set $\{(x, y) | \phi(x, y) = t\}$.

Thus, we can estimate $\int ||F_t||^2_{L^2(\mu)} dt$ using the results from A. Iosevich, K. Taylor and Uriarte-Tuero in [19]. Since they work with the assumption

$$s_\lambda > d + 1 - s_\mu$$

and the measure λ in their paper works as μ for our case and $d = 2$ and it gives us:

$$s_\mu > 2 + 1 - s_\mu$$

$$s_\mu > \frac{3}{2}$$

This result is a bound which is adequate when $s > \frac{3}{2}$, hence completes the proof of the main theorem. □

3.2 Future steps

We could repeat this process and expend this result to any rigid structure adding chains or trees to each vertex.

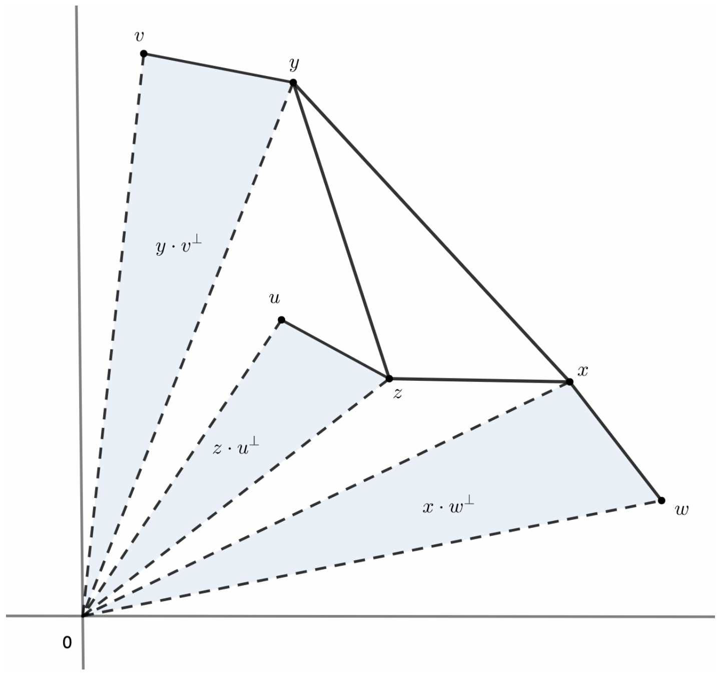

Figure 3.3: *6-point configuration* $h \in (\mathbb{R}^2)^6$

This is the next step of my research in this area. In addition to consider a k-point configurations in $\mathbb{R}^d$ and define a Galo volume type configurations.

Bibliography

[1] M. Bennett, A. Iosevich, K. Taylor, *Finite chains inside thin subsets of* $\mathbb{R}^d$, Anal. PDE 9 (2016), no. 3, 597–614. 2, 22

[2] J. Bourgain, C. Demeter, *The proof of the* ℓ^2 *decoupling conjecture*, Ann. of Math. (2) 182 (2015), no. 1, 351-389. viii

[3] N. Chatzikonstantinou, A. Iosevich, S. Mkrtchyan, J. Pakianathan, *Rigidity, graphs, and Hausdorff dimension*, https://arxiv.org/abs/1708.05919 (2017) ix

[4] X. Du, L. Guth, Y. Ou, H. Wang, B. Wilson, and R. Zhang, *Weighted restriction estimates and application to Falconer distance set problem*, (arXiv:1802.10186) (2018). viii

[5] X. Du, A. Iosevich, Y. Ou, H. Wang, R. Zhang, *An improved result for Falconer's distance set problem in even dimensions*, arXiv preprint https://arxiv.org/abs/2006.06833 viii

[6] X. Du, R. Zhang, *Sharp* L^2 *estimates of the Schrodinger maximal function in higher dimensions*, Ann. of Math. (2) 189 (2019), no. 3, 837–861. viii

[7] G. Elekes, M. Sharir, *Incidences in 3 dimensions and distinct distances in the plane*, Combin. Probab. Comput. 20 (2011), no. 4, 571–608. viii

[8] B. Erdogan, *A bilinear Fourier extension theorem and applications to the distance set problem*, Int. Math. Res. Not. (2006) viii

[9] K.J. Falconer, *On the Hausdorff dimensions of distance sets*, Mathematika 32 (1985), no. 2, 206–212 (1986). viii

[10] K.J. Falconer, *The geometry of Fractal sets*, Cambridge Tracts in Mathematics, 85. Cambridge University Press, Cambridge, 1986. 16

[11] A. Greenleaf, A. Iosevich, B. Liu, E. Palsson, *A group-theoretic viewpoint on Erods-Falconer problems and the Mattila integral*, Rev. Mat. Iberoam. 31 (2015), no. 3, 799–810. viii, ix, 13

[12] A. Greenleaf, A. Iosevich, M. Mourgoglou, Forum Math. 27 (2015), no. 1, 635–646. 2

[13] A. Greenleaf, A. Iosevich, K. Taylor, *On* k*-point configuration sets with nonempty interior*, preprint at https://arxiv.org/abs/2005.10796, 2020 2, 7, 8

[14] L. Guth, A. Iosevich, Y. Ou, H. Wang, *On Falconer's distance set problem in the plane*, Invent. Math. 219 (2020), no. 3, 779–830. viii

[15] B. Galo, A. McDonald *Volumes spanned by* k *point configurations in* $\mathbb{R}^d$, preprint at arXiv:2102.02323. 1, 7

[16] L. Guth, N.H. Katz, *On the Erdos distinct distances problem in the plane*, Ann. of Math. (2) 181 (2015), no. 1, 155–190. viii

[17] F. Hausdorff, *Dimension und ausseres Mass*, Math. Ann. **79** (1919), 157-179. vii

[18] A. Iosevich, K. Taylor *Finite treens inside thin subsets of* $\mathbb{R}^d$, Springer Proc. Math. Stat., 291, Springer, Cham, 2019 2

[19] A. Iosevich, K. Taylor and I. Uriarte-Tuero *Pinned geometric configurations in the euclidean space and Riemannian Manifolds*, 2016, arxiv. 1610.00349. 23

[20] J.M. Lee, *Introduction to smooth manifolds*, Graduate Texts in Mathematics 218, Springer New York, 2003. 4

[21] A. McDonald, *Areas of triangles and* SL_2 *actions in finite rings*, BULLETIN of the L.N. Gumilyov Eurasian National University Mathematics Series, Computer science, Mechanics, No.2 (127) / 2019 2

[22] A. McDonald, *Areas spanned by point configurations in the plane*, to appear in Proceedings of the AMS, preprint at https://arxiv.org/abs/2008.13720 iii, 2, 13, 19, 20, 21

[23] T. Wolff, *Decay of circular means of Fourier transforms of measures*, Int. Math. Res. Not. 10 (1999) 547-567. viii

[24] T. Wolff, *Lectures on Harmonic Analysis*, American Mathematical Society, University Lecture Series Vol 29, 2003. 8, 11