A Local Compactification to Countable Markov Shifts

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Resumo

ROCHA, Henrique C. S. **Uma Compactificação Local de Shifts de Markov Enumeráveis**. 2022. 50 f. Dissertação (Mestrado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022.

Pouco mais de vinte anos atrás Ruy Exel e Marcelo Laca descobriram as álgebras de Exel-Laca, que naturalmente levam seus nomes, que são uma tentativa de estender as álgebras de Cuntz-Krieger para matrizes infinitas. Do ponto de vista da teoria de shifts de Markov, essas álgebras sao interessantes porque o espectro de uma sub-álgebra comutativa específica das álgebras de Exel-Laca aparece como uma local compactificação, ou compactificação a depender da matriz associada ao shift de Markov, de um shift de Markov que não é localmente compacto. Além disto, tais local compactificações deixam invariantes shifts que já são localmente compactos e sua construção independe de qualquer noção externa à matriz que define o shift de Markov. O objetivo deste trabalho é descrever em detalhes tais local compactificações para aqueles que não conhecem a teoria de álgebras C* e definir algumas noções básicas tais quais medidas conformes para estes espaços.

Palavras-chave: medidas conformes, dinâmica simbólica, sistemas dinâmicos.

Abstract

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Around twenty year ago Ruy Exel and Marcelo Laca first discovered the so-called Exel-Laca algebras which are an attempt to extend Cuntz-Krieger algebras to infinite matrices. From the point of view of the theory of countable Markov shifts, these algebras are interesting because they contain some commutative sub-algebras whose spectrum is a local compactification, or compactification depending on the matrix associated to the countable Markov shift, of a countable Markov which is not locally compact. Furthermore, such construction leaves invariant Markov shifts which are already compact and it does not depend on anything other than the matrix describing the Markov shift. The objective of this work is to describe in detail such local compactifications to those that are very knowledgeable on the theory of C*-alegbras and to define some basic notion such as conformal measures to such spaces.

Keywords: conformal measures, symbolic dynamics, dynamical systems.

Contents

List of Abbreviations

CMS	Countable Markov Shift
GCMS	Generalized Countable Markov
a.e.	almost everywhere
i.e.	id est, that is

List of Symbols

$\mathcal{S}$	Countable set of symbols/Alphabet
J	Infinite subset of symbols
F	Finite subset of symbols
i, j, r, s, t	Symbols/sLetters of $\mathcal{S}$
$A \in \{0,1\}^{\mathcal{S}\times\mathcal{S}}$	Matrix associated to a CMS
$A(i,j) \in \{0,1\}$	Element of A
$\mathbf{S}(s)$	Column of A associated to the symbol s
$\mathfrak{r}(s)$	Range of the symbol s, i.e., the set of symbols t such that $A(s,t) = 1$
$\mathfrak{s}(t)$	Sources of the symbol t, i.e., the set of symbols s such that $A(s,t) = 1$
$\#G$	Cardinality of $G \subset \mathcal{S}$
$\mathcal{P}^*(G)$	Set of non-empty subsets of $G \subset \mathcal{S}$
$\mathcal{P}^*_f(G)$	Set of non-empty finite subsets of $G \subset \mathcal{S}$
Σ_A	Space of sequences allowed by A
$x[m,n)$	Denotes the bloc $x_m x_{m+1} ... x_{n-1}$ of an infinite word $x \in \Sigma$
ω	Finite admissible word
$\omega[m,n)$	Denotes the bloc $\omega_m\omega_{m+1}...\omega_{n-1}$ of a finite word ω
X_A	Local compactification of Σ_A
F_A	Set of finite words of the local compactification of Σ_A
E_A	Set of empty words of the local compactification of Σ_A
$\mathcal{R}$	Admissible root
$[\omega]$	Cyllinder associated to the finite admissible word ω
σ	Shift application
ϕ	Real continuous function on a CMS or GCMS
$\mathrm{Var}_n\phi$	n-th variation of ϕ
L_ϕ	Ruelle's operator associated to ϕ
$P_g(\phi)$	Gurevich pressure of ϕ
μ, ν	σ-finite Borel measure on a CMS or GCMS
$\mu \sim \nu$	Denotes that $\nu(B) = 0 \iff \mu(B) = 0$
$\mathcal{B}_X, \mathcal{B}_\Sigma$	σ-algebra of Borel-measurable sets of X/Σ
B	Borel-measurable set
χ_B	Characteristic function of B
$\hat{T}$	Transfer operator

$\mathbb{R}[G]$	Real algebra generated by the set G
λ	Real number
$\|\|\|_\infty$	Supremum norm
$L^\infty(X)$	Set of bounded Borel-measurable real functions on X
$\mathbb{N}$	Natural numbers (starts at 1)
$\mathbb{N}_0$	Natural numbers with 0 included
$\mathbb{R}$	Real numbers
H	Hilbert space
$\mathcal{B}(H)$	Bounded operators on H
$\varphi_n \overset{*}{\rightharpoonup} \tilde{\varphi}$	"φ_n converges weakly to φ"

Chapter 1

Preliminaries

This work follows the footsteps of T. Raszeja [RAS20] in investigating the local compactification of a countable Markov shift that arises from studying some specific commutative sub-C^*-algebras of the Exel-Laca algebras first described by R. Exel and M. Laca [EL99] in 1999. In which case it is rather natural we start it with some preliminary definitions and results on the theory of CMS and of C^*-algebras. A very good reference to the first being the lecture notes by O. Sarig [SAR09] and the sufficient results to the second being present in any book of C^*-algebras such as [MUR90].

1.1 Countable Markov Shifts

Our aim in this section is to introduce the notion of a countable Markov shift. The first ingredients needed are a countable set $\mathcal{S}$ which we shall refer to as the alphabet and a matrix $A \in \{0,1\}^{\mathcal{S}\times\mathcal{S}}$. We endow the alphabet $\mathcal{S}$ with the discrete topology and we define the full shift space over $\mathcal{S}$ as the space

$$\Sigma_{full} \doteq \mathcal{S}^{\mathbb{N}_0}$$

endowed with its usual product topology. In a similar fashion, we define the shift space over $\mathcal{S}$ related to the matrix A as the subset of the full shift space given by

$$\Sigma_A \doteq \{x \in \Sigma_{full} : A(x_i, x_{i+1}) = 1 \text{ for all } i \in \mathbb{N}_0\}$$

and endow it with the subspace topology inherited from Σ_{full}. We note that the topology of Σ_{full} has a natural basis in the set of cyllinder subsets, i.e., subsets of the form

$$[\omega] \doteq \{x \in \Sigma_{full} : x_i = a_i \text{ for all } 0 \leq i \leq k-1\},$$

where $\omega = \omega_0\omega_1...\omega_{k-1} \in \mathcal{S}^k$ is a finite word with length k. The same is naturally also true for Σ_A. Finally, we define a continuous application $\sigma : \Sigma_{full} \to \Sigma_{full}$ by

$$\sigma(x)_i = x_{i+1}$$

and note that it defines a local homeomorphism, the same is once again true for the restriction of σ to Σ_A. Whenever no confusion over which space we are working with is possible, we shall denote the restriction of σ to Σ_A simply by σ.

It is evident that we need to demand that A satisfy some conditions so that the pair (Σ_A, σ) has interesting dynamical properties. We say that A is transitive if for any pair of symbols $(i,j) \in \mathcal{S} \times \mathcal{S}$, there exists a finite word ω such that

$$[i\omega j] \cap \Sigma_A \neq \emptyset.$$

We say that A is topologically mixing if for any pair of symbols $(i,j) \in \mathcal{S} \times \mathcal{S}$ there exists a number $N \in \mathbb{N}$ such that for any number $n > N$, there exists a finite word ω of length n such that

$$[i\omega j] \cap \Sigma_A \neq \emptyset.$$

If not stated otherwise, we shall assume henceforth that any matrix A is topologically mixing. We say that A is row-finite if

$$\#\{j \in \mathcal{S} : A(i,j) = 1\} < \infty$$

for all $i \in \mathcal{S}$ and note that Σ_A is compact if, and only if, $\mathcal{S}$ is finite, and that Σ_A is locally compact if, and only if, A is row-finite.

In general, our alphabet $\mathcal{S}$ shall be identified with the natural numbers and we shall write $\mathcal{S}$ or $\mathbb{N}$ interchangeably whenever no confusion is possible. Finally, we note that Σ_A is a metric space with a natural metric $d_\alpha : \Sigma_A \times \Sigma_A \to [0,\infty)$ given by

$$d_\alpha(x,y) \doteq \alpha^{\min\{i \in \mathbb{N}_0 : x_i \neq y_i\}},$$

where $0 < \alpha < 1$. It is straightforward to see that for this metric σ is a Hölder application and that Σ_A is complete.

1.1.1 Potentials

We shall refer to a continuous function on a CMS (countable markov shift) $\phi : \Sigma_A \to \mathbb{R}$ also by the term *potential*. In general, we will consider potentials such that

$$\mathrm{S}(\phi) \doteq \sum_{s \in \mathcal{S}} e^{\sup \phi([s])} < \infty. \tag{1.1}$$

For $n \in \mathbb{N}$, we define the n-th variation of ϕ by

$$\mathrm{Var}_n \phi \doteq \sup\{|\phi(x) - \phi(y)| : x_i = y_i \text{ for } 0 \leq i \leq n-1\}$$

and the n-th Birkhoff sum of ϕ by

$$\Sigma_A \ni x \mapsto \phi_n(x) \doteq \sum_{k=0}^{n-1} \phi \circ \sigma^k(x).$$

It is evident that ϕ is uniformly continuous if, and only if,

$$\lim_{n \to \infty} \mathrm{Var}_n \phi = 0.$$

In general, in the standard theory of CMS stricter conditions than uniform continuity are demanded. We say that a potential $\phi : \Sigma_A \to \mathbb{R}$ is **weakly Hölder continuous** if there exist $C > 0$ and $\theta \in (0,1)$ such that

$$\mathrm{Var}_n \phi \leq C\theta^n \tag{1.2}$$

for $n \geq 2$. We say that a potential $\phi : \Sigma_A \to \mathbb{R}$ has **summable variations** if

$$\sum_{n=2}^{\infty} \mathrm{Var}_n \phi < \infty. \tag{1.3}$$

Finally, we say that a potential $\phi : \Sigma_A \to \mathbb{R}$ satisfies **Walter's condition**, or is **Walter's**, if for all $k \in \mathbb{N}$

$$\sup_{n\in\mathbb{N}} \mathrm{Var}_{n+k}\phi_n < \infty \text{ and } \lim_{k\to\infty} \sup_{n\in\mathbb{N}} \mathrm{var}_{n+k}\phi_n = 0. \tag{1.4}$$

Lemma 1. *1. Every potential $\phi : \Sigma_A \to \mathbb{R}$ which is weakly Hölder continuous has summable variations.*

2. Every potential $\phi : \Sigma_A \to \mathbb{R}$ which has summable variations satisfies Walter's condition.

Proof. Suppose that $\phi : \Sigma_A \to \mathbb{R}$ is weakly Hölder continuous, then for some $C > 0$ and $\theta \in (0, 1)$, we have that

$$\sum_{n=2}^{\infty} \mathrm{var}_n \phi \leq \sum_{n=2}^{\infty} C\theta^n = \frac{C\theta^2}{1-\theta} < \infty$$

which proves the first item of the lemma.

We now prove that if $\phi : \Sigma_A \to \mathbb{R}$ has summable variations, then it is Walter's. Given $k, n \in \mathbb{N}$, let $x, y \in \Sigma_A$ satisfy $x[0, n + k) = y[0, n + k)$, then

$$|\phi_n(x) - \phi_n(y)| = \left| \sum_{i=0}^{n-1} \phi \circ \sigma^i(x) - \sum_{i=0}^{n-1} \phi \circ \sigma^i(y) \right| \leq \sum_{i=0}^{n-1} |\phi \circ \sigma^i(x) - \phi \circ \sigma^i(y)|.$$

We note that $\sigma^i(x)[0, n + k - i) = \sigma^i(y)[0, n + k - i)$ for all $0 \leq i \leq n - 1$, therefore

$$|\phi_n(x) - \phi_n(y)| \leq \sum_{i=0}^{n-1} \mathrm{var}_{n+k-i}\phi = \sum_{i=k+1}^{n+k} \mathrm{var}_i\phi,$$

that is

$$\mathrm{var}_{n+k}\phi_n \leq \sum_{i=k+1}^{n+k} \mathrm{var}_i\phi.$$

Finally,

$$\sup_{n\in\mathbb{N}} \mathrm{var}_{n+k}\phi_n \leq \sup_{n\in\mathbb{N}} \sum_{i=k+1}^{n+k} \mathrm{var}_i\phi = \sum_{i=k+1}^{\infty} \mathrm{var}_i\phi < \infty$$

and we conclude that ϕ is Walter's since

$$\lim_{k\to\infty} \sum_{i=k+1}^{\infty} \mathrm{var}_i\phi = 0$$

by hypothesis. □

1.1.2 Conformal Measures

In this subsection, we reproduce the results present on [SAR09] and omit some of the proofs, those interested may consult the original material. Our aim is to provide the definition of conformal measure and of Ruelle's operator. The following definitions shall be necessary to define the transfer operator which describes how densities under the application of a map with "good properties" related to such measure. We begin with the very general concept of non-singular measure/application.

Definition 1. *Let μ be a σ-finite Borel measure on Σ, we say that μ is σ-non-singular or that σ is non-singular on $(\Sigma, \mathcal{B}_\Sigma, \mu)$ if $\sigma_*\mu \sim \mu$, i.e., if B is Borel, then*

$$\sigma_*\mu(B) = \mu(\sigma^{-1}B) = 0 \iff \mu(B) = 0.$$

The following definitions appears somewhat mysterious at first but it should soon be clear why it is useful to describe such a measure.

Definition 2. *Suppose ν is σ-non-singular. We define $\nu \circ \sigma : \mathcal{B}_\Sigma \to [0, \infty]$ by*

$$\nu \circ \sigma(E) \doteq \sum_{a \in \mathcal{S}} \nu(\sigma(E \cap [a])). \tag{1.5}$$

Proposition 1. *Suppose ν is σ-non-singular, then*

1. *$\nu \circ \sigma$ defines a Borel measure on Σ;*
2. *for all non-negative functions $f : \Sigma \to \mathbb{R}$, we have that*
$$\int_\Sigma f \mathrm{d}(\nu \circ \sigma) = \sum_{a \in \mathcal{S}} \int_{\sigma[a]} f(ax) \mathrm{d}\nu(x);$$
3. *$\nu \ll \nu \circ \sigma$;*
4. *if for every state $a \in \mathcal{S}$ and every Borel set $E \subset [a]$, we have that $\nu(E) = 0 \iff \nu(\sigma E) = 0$, then $\nu \circ \sigma \sim \nu$.*

The following two definitions also justify themselves a posteriori.

Definition 3. *Suppose $\nu : \mathcal{B}_\Sigma \to [0, \infty]$ is σ-non-singular, we define the Jacobian of ν by*

$$g_\nu \doteq \frac{\mathrm{d}\nu}{\mathrm{d}(\nu \circ \sigma)}. \tag{1.6}$$

If $\nu \sim \nu \circ \sigma$, we define the log Jacobian of ν by $\log g_\nu = \log \frac{\mathrm{d}\nu}{\mathrm{d}(\nu \circ \sigma)}$.

Definition 4. *Suppose $\phi : \Sigma \to \mathbb{R}$ is Borel-measurable. We say that a (possibly infinite) Borel measure ν is ϕ-conformal if it is finite on cyllinders and if there is $\lambda > 0$ such that $g_\nu = \lambda^{-1} \exp \phi$ a.e.*

We are now able to define the transfer operator whose name shall be justified by the properties proved in the proposition below.

Definition 5. *The transfer operator, or Perron-Frobenius operator, of a non-singular map T on a σ-finite measure space $(\Omega, \mathcal{B}, \mu)$ is the operator $\hat{T} : L^1(\Omega, \mathcal{B}, \mu) \to L^1(\Omega, \mathcal{B}, \mu)$ given by*

$$\hat{T} f \doteq \frac{\mathrm{d}(\mu_f \circ T^{-1})}{\mathrm{d}\mu}, \tag{1.7}$$

where $\mathrm{d}\mu_f \doteq f \mathrm{d}\mu$.

Lemma 2. *The transfer operator is well-defined.*

Proof. To prove that the transfer operator is well-defined, we need to prove that $\mu_f \circ T^{-1} \ll \mu$ and that if $f \in L^1$, then $\hat{T}f$ is also L^1.

Let $E \in \mathcal{B}$ such that $\mu(E) = 0$, then, by the non-singularity of μ, we have that $\mu(T^{-1}E) = 0$. Therefore,

$$\mu_f \circ T^{-1}(E) \doteq \mu_f(T^{-1}E) = \int_{T^{-1}E} f \mathrm{d}\mu = 0$$

and we conclude.

Let $f \in L^1(\Omega, \mathcal{B}, \mu)$, then

$$\begin{aligned}\|\hat{T}f\|_1 &= \int \operatorname{sgn}(\hat{T}f) \cdot \hat{T}f \mathrm{d}\mu = \int \operatorname{sgn}(\hat{T}f) \mathrm{d}\mu_f \circ T^{-1} \\ &= \int \operatorname{sgn}(\hat{T}f) \circ T \mathrm{d}\mu_f = \int \operatorname{sgn}(\hat{T}f) \circ T f \mathrm{d}\mu \leq \|f\|_1,\end{aligned}$$

where sgn denotes the sign function. Thus, the lemma is proven. □

Proposition 2. *Suppose T is a non-singular map on a σ-finite measure space $(\Omega, \mathcal{B}, \mu)$, then*

1. *if $f \in L^1$, then $\hat{T}f$ is the unique L^1-function such that, for all $\varphi \in L^\infty$,*

$$\int \varphi \hat{T}f \mathrm{d}\mu = \int \varphi \circ T f \mathrm{d}\mu;$$

2. *$\hat{T}$ is positive, i.e., if $f \geq 0$ a.e., then $\hat{T}f \geq 0$ a.e.;*
3. *$\hat{T}$ is a bounded linear operator on L^1 with $\|\hat{T}\| = 1$;*
4. *$\int \hat{T}f \mathrm{d}\mu = \int f \mathrm{d}\mu$ for all $f \in L^1$;*
5. *if $f \geq 0$ and $\int f \mathrm{d}\mu = 1$, then $\hat{T}f = f \iff \mathrm{d}m \doteq f\mathrm{d}\mu$ is a T-invariant probability measure;*
6. *if μ is T-invariant, then $(\hat{T}f) \circ T = \mathbb{E}(f|T^{-1}\mathcal{B})$.*

Remark 1. *Property 1 is the one that justifies calling it the "transfer operator".*

We are now interested in describing the transfer operator in the case of a CMS. Suppose ν is a σ-non-singular σ-finite measure on $(\Sigma, \mathcal{B}_\Sigma)$, we recall that, for all $\varphi \in L^\infty(\nu)$,

$$\chi_{\sigma[a]}(x)\varphi(x) = \chi_{\sigma[a]}(x)\varphi \circ \sigma(ax).$$

Furthermore, recall that $\nu \ll \nu \circ \sigma$ and note that if $h \in L^1(\nu)$ then $g_\nu h = \frac{\mathrm{d}\nu}{\mathrm{d}\nu \circ \sigma} h \in L^1(\nu \circ \sigma)$.

Let $\varphi \in L^\infty(\nu)$ and $f \in L^1(\nu)$, then

$$\begin{aligned}\int \varphi \circ \sigma \cdot f \mathrm{d}\nu &= \int \varphi \circ \sigma \cdot f g_\nu \mathrm{d}\nu \circ \sigma = \sum_{a \in \mathcal{S}} \int_{\sigma[a]} (\varphi \circ \sigma \cdot f g_\nu)(ax) \mathrm{d}\nu(x) \\ &= \sum_{a \in \mathcal{S}} \int \chi_{\sigma[a]}(x)\varphi \circ \sigma(ax) f(ax) g_\nu(ax) \mathrm{d}\nu(x) \\ &= \int \varphi(x) \sum_{a \in \mathcal{S}} \chi_{\sigma[a]}(x) g_\nu(ax) f(ax) \mathrm{d}\nu(x) = \int \varphi(x) \left(\sum_{\sigma y = x} g_\nu(y) f(y) \right) \mathrm{d}\nu(x)\end{aligned}$$

and we conclude via item 1 of the previous proposition that, for a CMS,

$$\hat{T}f(x) = \sum_{\sigma y = x} g_\nu(y) f(y) \ \nu - \text{a.e.} \tag{1.8}$$

Inspired by this computation, we shall define Ruelle's operator.

Definition 6. *Let $\phi : \Sigma \to \mathbb{R}$, we define the Ruelle's operator L_ϕ associated to ϕ by*

$$L_\phi f(x) \doteq \sum_{\sigma y = x} e^{\phi(y)} f(y). \tag{1.9}$$

We have, thus, proved the following lemma.

Lemma 3. *Suppose that $\nu \sim \nu \circ \sigma$, then*

$$\hat{T}f = L_{\log g_\nu} f. \tag{1.10}$$

At this point we define the notion of conservative measure and prove a criterion relating the conservativity of a conformal measure to a series involving partition functions.

Definition 7. *Let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space and $T : \Omega \to \Omega$ be a measurable non-singular map. We say that a measurable set $W \in \mathcal{B}$ is wandering if the family $\{T^{-n}W\}_{n \in \mathbb{N}_0}$ is composed of pairwise disjoint subsets.*

Remark 2. *If $W \in \mathcal{B}$ is a wandering set, then*

$$\sum_{n \geq 0} \chi_W \circ T^n(x) \leq 1$$

for all $x \in \Omega$.

Definition 8. *We say that T is conservative or that μ is conservative if for every wandering subset $W \in \mathcal{B}$, we have that either $\mu(W) = 0$ or that $\mu(W^c) = 0$.*

Theorem 1. *T is conservative iff for all $E \in \mathcal{B}$ such that $\mu(E) > 0$, a.e. $x \in E$ is recurrent, i.e.,*

$$\#\{n \in \mathbb{N} : T^n x \in E\} = \infty.$$

Using this theorem by Halmos, it is possible to establish a criterion to check for the conservativy of a measure by studying its transfer operator.

Proposition 3. *Let T be a non-singular map on a σ-finite measure space $(\Omega, \mathcal{B}, \mu)$.*

1. *If there exists a non-negative function $f \in L^1$, such that*

$$\mu\left(\left\{x \in \Omega : \sum_{n=1}^{\infty} \hat{T}^n f(x) < \infty\right\}\right) = 0,$$

   then T is conservative.

2. *If there exists a strictly positive function $f \in L^1$, such that*

$$\mu\left(\left\{x \in \Omega : \sum_{n=1}^{\infty} \hat{T}^n f(x) < \infty\right\}\right) > 0,$$

   then T is not conservative.

Let us apply this general results to the context of topological Markov shifts to establish a connection between conservativity of a conformal measure and partition functions.

Definition 9. *Let $s \in \mathcal{S}$ be a symbol, $n \in \mathbb{N}$ and $\phi : \Sigma \to \mathbb{R}$ be a potential. We define the n-th partition function of ϕ at s by*

$$Z_n(\phi, [s]) = \sum_{\substack{\sigma^n y = y \\ y_0 = s}} e^{\phi_n(y)}. \tag{1.11}$$

We note that there exists $N_s \in \mathbb{N}$ such that $Z_n(\phi, [s]) > 0$ for $n \geq N_s$ in case A is topologically mixing, which is the case we shall always be dealing with in this work. It is here that the extra properties demanded upon potentials will show their importance in establishing neat bounds for partition functions. Let us begin by proving that in case ϕ is Walter's, then for any $s, t \in \mathcal{S}$

$$\sum_{n=1}^{\infty} \lambda^{-n} Z_n(\phi, [s]) = \infty \iff \sum_{n=1}^{\infty} \lambda^{-n} Z_n(\phi, [t]) = \infty.$$

Since we suppose A transitive, there exist finite words ω and $\tilde{\omega}$ such that $s\omega t$ and $t\tilde{\omega}s$ are admissible. These words allow us to define an injection

$$\vartheta : \{y \in \Sigma : y_0 = s, T^n y = y\} \to \{y \in \Sigma : y_0 = t, T^{n+k} y = y\},$$

where $k = |\omega| + |\tilde{\omega}| + 2$ in following manner

$$y \mapsto \overline{t\tilde{\omega}y[0,n)s\omega},$$

where $y[0,n) = y_0 y_1 \ldots y_{n-1}$ and the overline means that it is repetition of this sequence of symbols.

The matter now becomes estimating, for $y \in \{y \in \Sigma : y_0 = s, T^n y = y\}$,

$$|\phi_{n+k}(\vartheta(y)) - \phi_n(y)|.$$

It turns out that if the potential is Walter's and $\sup|\phi|([s]) < \infty$ for all $s \in \mathcal{S}$, it is possible to extract a bound that does not depend on n. Indeed,

$$\begin{aligned}
|\phi_{n+k}(\vartheta(y)) - \phi_n(y)| &= |\phi_{n+k}(\overline{s\omega t\tilde{\omega}y[0,n)}) - \phi_n(y)| \\
&\leq |\phi_k(\overline{s\omega t\tilde{\omega}y[0,n)})| + |\phi_n(\overline{y[0,n)s\omega t\tilde{\omega}} - \phi_n(y))| \\
&\leq \sup|\phi_k|([s\omega t\tilde{\omega}]) + \sup_{n\in\mathbb{N}} \mathrm{Var}_{n+1}\phi_n = C(s,t) < \infty.
\end{aligned}$$

We conclude that,

$$Z_n(\phi, [s]) \leq e^{C(s,t)} Z_{n+k}(\phi, [t]).$$

Since the symbols s, t are arbitrary, we immediately get a similar inequality on the other side and we conclude that

$$\sum_{n=1}^{\infty} \lambda^{-n} Z_n(\phi, [s]) = \infty \iff \sum_{n=1}^{\infty} \lambda^{-n} Z_n(\phi, [t]) = \infty.$$

Having established this fact and this sort of strategy on extracting bounds of partition functions, we are in position to prove a criterion on the conservativity of a conformal measure.

Theorem 2. *Let ν be a non-singular measure on Σ which is finite on cyllinders and such that $\frac{\mathrm{d}\nu}{\mathrm{d}(\nu\circ\sigma)} = \lambda^{-1}e^{\phi}$. If ϕ is Walter's, then ν is conservative if, and only if,*

$$\sum_{n=1}^{\infty} \lambda^{-n} Z_n(\phi, [s]) = \infty$$

for all $s \in \mathcal{S}$.

Proof. The proof of this theorem is more or less a direct application of the criterion from the previous proposition applied to the function $f = \chi_{[s]}$, the characteristic function of the cyllinder $[s]$. Let us first note that, by hypothesis,

$$\hat{T}f(x) = L_{\log g_\nu} f(x) = \sum_{\sigma y = x} \lambda^{-1} e^{\phi(y)} f(y),$$

which implies by induction that

$$\hat{T}^n f(x) = \lambda^{-n} \sum_{\sigma^n y = x} e^{\phi_n(y)} f(y)$$

for any $n \in \mathbb{N}$ and non-negative measurable function. In particular,

$$\hat{T}^n \chi_{[s]}(x) = \lambda^{-n} \sum_{\sigma^n y = x} e^{\phi_n(y)} \chi_{[s]}(y) = \lambda^{-n} \sum_{\substack{\sigma^n y = x \\ y_0 = s}} e^{\phi_n(y)}. \tag{1.12}$$

We shall now establish a fine bound, much like we did before, relating $\hat{T}^n \chi_{[s]}(x)$ and $\lambda^{-n} Z_n(\phi, [s])$. Since A is transitive, there exists a word $\tilde{\omega}$ such that $x_0 \tilde{\omega} s$ is admissible. Hence, we can construct an injection $\vartheta : \sigma^{-n}x \cap [s] \to \{y \in \Sigma : T^{n+|\tilde{\omega}|+1} y = y, y_0 = s\}$ via

$$y = s\omega_1 ... \omega_{n-1} x \mapsto \overline{s\omega_1 ... \omega_{n-1} x_0 \tilde{\omega}}$$

and in the exact same fashion as before we get that

$$|\phi_{n+|\tilde{\omega}|+1}(\vartheta(y)) - \phi_n(y)| < C(s, x_0) < \infty$$

which implies that

$$\hat{T}^n \chi_{[s]}(x) \leq \left[\lambda^{|\tilde{\omega}|+1} e^{C(s,x_0)}\right] \lambda^{-n-|\tilde{\omega}|-1} Z_{n+|\tilde{\omega}|+1}(\phi, [s]).$$

On the other hand, there exists a $\omega = \omega_1 ... \omega_{k-1}$ such that $s\omega x_0$ is admissible and we get an injection $\vartheta' : \{y \in \Sigma : T^{n-k} y = y, y_0 = s\} \to \sigma^{-n} y \cap [s]$ via

$$y \mapsto y[0, n-k) s\omega x$$

which implies that

$$\left[\lambda^{-k} e^{C'(x_0,s)}\right] \lambda^{-(n-k)} Z_{n-k}(\phi, s) \leq \hat{T}^n \chi_{[s]}(x).$$

Therefore,

$$\sum_{n=1}^{\infty} T^n \chi_{[s]}(x) = \infty \iff \sum_{n=1}^{\infty} \lambda^{-n} Z_n(\phi, [s]) = \infty$$

and we have thus proved the "if" part of the statement.

Let us now prove the "only if" part. Suppose that $\sum_{n\geq 1} \lambda^{-n} Z_n(\phi, [s]) < \infty$, then by the computation before the theorem this is the case for all symbols $t \in \mathcal{S}$. Therefore,

$$\chi_{[t]}(x) \sum_{n=1}^{\infty} \hat{T}^n \chi_{[s]}(x) = C(s,t) < \infty.$$

Fix $\tilde{t} \in \mathcal{S}$ such that $\nu([\tilde{t}]) \neq 0$. We recall that by hypothesis the measure ν is finite on cyllinders. All that is left now is choosing a sequence $\{\varepsilon_s\}_{s\in\mathcal{S}}$ such that

$$\sum_{s\in\mathcal{S}} \varepsilon_s C(s,\tilde{t}) < \infty \text{ and } \sum_{s\in\mathcal{S}} \varepsilon_s \nu([s]) < \infty.$$

A straightforward computation shows that $f \doteq \sum_{s\in\mathcal{S}}$ is a strictly positive and integrable function such that

$$\sum_{s\in\mathcal{S}} \hat{T}^n f(x) < \infty$$

on $[\tilde{t}]$ and we conclude by the previous proposition. □

Let us finish this section by stating a theorem dealing with the existence of conformal measures on non-compact shifts for Walter's potential. Since the proof is very long, we omit it and refer to the Lecture Notes by Sarig.

Theorem 3. *Suppose $\phi : \Sigma \to \mathbb{R}$ satisfies Walter's property, then ϕ has a conservative conformal measure on Σ that is finite on cyllinders if, and only if, for some $s \in \mathcal{S}$*

1. $\log \lambda \doteq \limsup_{n\to\infty} \frac{1}{n} \log Z_n(\phi, [s]) < \infty$;
2. $\sum_{n=1}^{\infty} \lambda^{-n} Z_n(\phi, [s]) = \infty$.

Remark 3. *We call the quantity λ present in the theorem the Gurevich pressure of ϕ and denote it by $P_g(\phi)$. The sort of computation we did in the previous subsection proves that $P_g(\phi)$ does not depend on the symbol s if ϕ is Walters so this notation is justified.*

1.2 Very brief remarks on C*-Algebras

In this very brief section, we simply state the result that is important to define the local compactification of a non-local countable Markov shift that is the fact that the spectrum of a C*-algebra is locally compact (it is compact if the C*-algebra is unital). To those interested in the details of the matter, we recommend the third section of chapter 1 of [MUR90] and to those interested in a more abstract approach to the construction of the C*-algebras that we will deal with in the next chapter we recommend [EL99] and [RAS20].

Definition 10. *Let A be a complete normed complex algebra with an adjoint operation $\cdot^* : A \to A$, i.e., an application satisfying*

1. $(a^*)^* = a$ *for all* $a \in A$;
2. $(\lambda a)^* = \overline{\lambda} a^*$ *for all* $a \in A$ *and* $\lambda \in \mathbb{C}$;
3. $(a+b)^* = a^* + b^*$ *for all* $a, b \in A$;

4. $(ab)^* = b^*a^*$.

We say that A is a C-algebra if*

$$\|a\|^2 = \|a^*a\|$$

for all $a \in A$.

Let H be a Hilbert space, the typical example of a C*-algebra is a closed subset of the set of continuous operators on H which we denote by $\mathcal{B}(H)$ with the usual supremum norm. Furthermore, in case A is a C*-algebra without an unity there exists an unique way to unitize it while keeping it a C*-algebra. Finally, since we are not really interested in the theory of C*-algebras in this work, let us skip ahead to definition of character and that of the spectrum of a commutative C*-algebra.

Definition 11. *Let* A *be a commutative C*-algebra. We say that a non-zero continuous linear application* $\varphi : A \to \mathbb{C}$ *is a character if*

1. *it is compatible with the adjoint operation, i.e.,* $\varphi(a^*) = \overline{\varphi(a)}$ *for all* $a \in A$*;*
2. *it is an algebra homomorphism, i.e.,* $\varphi(ab) = \varphi(a)\varphi(b)$ *for all* $a, b \in A$.

We denote the set of characters of A *by* $\Omega(A)$ *and refer to it as the spectrum of* A.

In case A is unital, it turns out that every character has norm 1 and that endowed with the usual weak-* topology the set $\Omega(A)$ becomes a compact Hausdorff space. In the case where A is not unital, it is instead a locally compact Hausdorff space where any set of the form

$$K(a, \alpha) = \{\varphi \in \Omega(A) : |\varphi(a)| \geq \alpha\}$$

is compact, where $a \in A$ and $\alpha > 0$. Furthermore, the spectrum of $\tilde{A}$, the unitization of A, is the usual one-point compactification of the spectrum of $\Omega(A)$. Finally, a subset $U \in \Omega(A)$ is dense if, and only if, it separates the points of A.

Chapter 2

Generalized Countable Markov Shifts

We shall not, conceptually at least, dive in depth on the matter, those interested may consult work by T.Raszeja [RAS20] or R.Exel and M.Laca [EL99], but it is important to give a thorough operational description of the local compactifications of countable Markov shifts that arise from the study of the spectrum of specific commutative sub-C^*-algebras of the so-called Exel-Laca algebras.

2.1 An Interesting Commutative Operator Algebra

Given a topologically mixing countable Markov shift, let $H = l^2(\Sigma_A)$ denote the Hilbert space of square-summable sequences indexed by Σ_A. We begin by defining a family of continuous operators on H that somewhat captures the structure of Σ_A. Let $s \in \mathcal{S}$, we define $T_s \in \mathcal{B}(H)$ by

$$T_s(e_x) = A(s, x_0)e_{sx}, \tag{2.1}$$

where $\{e_x\}_{x\in\Sigma_A}$ denotes the canonical basis of H. A very straightforward computation yields us

$$T_s^*(e_x) = \chi_{[s]}(x)e_{\sigma(x)}. \tag{2.2}$$

Given a finite word $\omega \in \mathcal{S}^k$, we write

$$T_\omega \doteq T_{\omega_0}T_{\omega_1}...T_{\omega_{k-1}}$$

and note that $T_\omega \neq 0$ if, and only if, ω is admissible. In general, we have that

$$T_\omega(e_x) = \left[\prod_{l=0}^{k-2} A(\omega_i, \omega_{i+1})\right] A(\omega_{k-1}, x_0)e_{\omega x} \tag{2.3}$$

In a similar manner, we obtain that

$$T_\omega^*(e_x) = \chi_{[\omega]}(x)e_{\sigma^k(x)} \tag{2.4}$$

We may define two families of projections $\{P_s\}_{s\in\mathcal{S}}$ and $\{Q_s\}_{s\in\mathcal{S}}$ using the family of operators $\{T_s\}_{s\in\mathcal{S}}$ in the following manner:

$$P_s(e_x) \doteq T_sT_s^*(e_x) = \chi_{[s]}(x)e_x \tag{2.5}$$

and

$$Q_s(e_x) \doteq T_s^* T_s(e_x) = A(s, x_0) e_x \tag{2.6}$$

We note that $P_s P_{s'} = 0$ if $s \neq s'$ and that $Q_s Q_{s'} = Q_{s'} Q_s$ for all pairs $(s, s') \in \mathcal{S} \times \mathcal{S}$.

Let $(i, j) \in \mathcal{S} \times \mathcal{S}$, we are interested in computing $Q_i T_j$ for reasons that will become clear in a couple of paragraphs. Let $x \in \Sigma_A$, we have that

$$Q_i T_j(e_x) = A(i, j) A(j, x_0) e_{jx}$$

We conclude that

$$Q_i T_j = A(i, j) T_j \tag{2.7}$$

and by adjunction

$$T_j^* Q_i = A(i, j) T_j^*.$$

Let $\omega \in \mathcal{S}^k$ and $\tilde{\omega} \in \mathcal{S}^{\tilde{k}}$ be admissible words, we are also interested in computing $T_\omega^* T_{\tilde{\omega}}$. In order to so, we are going to divide the computation in three cases.

Case 1: $k > \tilde{k}$. Let $x \in \Sigma_A$, then

$$T_\omega^* T_{\tilde{\omega}}(e_x) = \begin{cases} e_{\sigma^k(\tilde{\omega} x)}, & \text{if } A(\tilde{\omega}_{\tilde{k}-1}, x_0) = 1 \text{ and } \tilde{\omega} x \in [\omega] \\ 0, & \text{otherwise.} \end{cases}$$

Since $k > \tilde{k}$, if $\tilde{\omega} x \in [\omega]$, then $\tilde{\omega}$ is a subword of ω, i.e., there exists an admissible word $\alpha \in \mathcal{S}^{k-\tilde{k}}$ such that $\omega = \tilde{\omega}\alpha$. In this case, $\tilde{\omega} x \in [\omega]$ iff $x \in [\alpha]$. Finally, we note that the fact that ω is admissible implies that if $x \in [\alpha]$, then

$$A(\tilde{\omega}_{\tilde{k}-1}, x_0) = A(\tilde{\omega}_{\tilde{k}-1}, \alpha_0) = 1.$$

On the other hand, $\sigma^k(\tilde{\omega} x) = \sigma^{k-\tilde{k}}(x)$, therefore

$$T_\omega^* T_{\tilde{\omega}}(e_x) = \begin{cases} e_{\sigma^{k-\tilde{k}}(x)}, & \text{if } \omega = \tilde{\omega}\alpha \text{ and } x \in [\alpha] \\ 0, & \text{otherwise.} \end{cases}$$

We conclude that

$$T_\omega^* T_{\tilde{\omega}} = \begin{cases} T_\alpha^*, & \text{if } \omega = \tilde{\omega}\alpha, \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

Case 2: $k = \tilde{k}$. We proceed as in the previous case. We get that $\tilde{\omega} x \in [\omega]$ iff $\tilde{\omega} = \omega$ and therefore

$$T_\omega^* T_{\tilde{\omega}}(e_x) = \begin{cases} e_x, & \text{if } \omega = \tilde{\omega} \text{ and } A(\omega_{k-1}, x_0) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We conclude that

$$T_\omega^* T_{\tilde{\omega}} = \begin{cases} Q_{\omega_{k-1}}, & \text{if } \omega = \tilde{\omega}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

Case 3: $k < \tilde{k}$. Let $x \in \Sigma_A$, then

$$T_\omega^* T_{\tilde{\omega}}(e_x) = \begin{cases} e_{\sigma^k(\tilde{\omega}x)}, & \text{if } A(\tilde{\omega}_{\tilde{k}-1}, x_0) = 1 \text{ and } \tilde{\omega}x \in [\omega] \\ 0, & \text{otherwise.} \end{cases}$$

Since $k < \tilde{k}$, $\tilde{\omega}x \in [\omega]$ iff ω is a subword of $\tilde{\omega}$, i.e., there exists an admissible word $\alpha \in \mathcal{S}^{\tilde{k}-k}$ such that $\tilde{\omega} = \omega\alpha$. In this case,

$$A(\tilde{\omega}_{\tilde{k}-1}, x_0) = A(\alpha_{\tilde{k}-k-1}, x_0).$$

On the other hand, $\sigma^k(\tilde{\omega}x) = \alpha x$, therefore

$$T_\omega^* T_{\tilde{\omega}}(e_x) = \begin{cases} e_{\alpha x}, & \text{if } \tilde{\omega} = \omega\alpha \text{ and } A(\alpha_{\tilde{k}-k-1}, x_0) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We conclude that

$$T_\omega^* T_{\tilde{\omega}} = \begin{cases} T_\alpha, & \text{if } \tilde{\omega} = \omega\alpha, \\ 0, & \text{otherwise.} \end{cases} \tag{2.10}$$

Let us write these results in the form of a lemma.

Lemma 4. *Let $\omega \in \mathcal{S}^k$ and $\tilde{\omega} \in \mathcal{S}^{\tilde{k}}$ be admissible words, then*

1. *if $k > \tilde{k}$, then* $T_\omega^* T_{\tilde{\omega}} = \begin{cases} T_\alpha^*, & \textit{if } \omega = \tilde{\omega}\alpha, \\ 0, & \textit{otherwise.} \end{cases}$

2. *if $k = \tilde{k}$, then* $T_\omega^* T_{\tilde{\omega}} = \begin{cases} Q_{\omega_{k-1}}, & \textit{if } \omega = \tilde{\omega}, \\ 0, & \textit{otherwise.} \end{cases}$

3. *if $k < \tilde{k}$, then* $T_\omega^* T_{\tilde{\omega}} = \begin{cases} T_\alpha, & \textit{if } \tilde{\omega} = \omega\alpha, \\ 0, & \textit{otherwise.} \end{cases}$

We shall now define a commutative and separable sub-algebra of $B(H)$, which we shall denote by D_A, and whose closure we shall denote by $\mathcal{D}_A$. We shall denote the empty word, i.e., the admissible (by vacuity) word of length zero by e. Let

$$\mathcal{A} \doteq \{(\alpha, F) : \alpha \text{ is a finite admissible word, } F \subset \mathcal{P}(\mathcal{S}) \text{ finite; } \alpha \neq e \text{ or } F \neq \emptyset\},$$

where $\mathcal{P}(\mathcal{S})$ denotes the power set of $\mathcal{S}$. It is evident that $\mathcal{A}$ is countable and for each $(\alpha, F) \in \mathcal{A}$, we shall define

$$e(\alpha, F) \doteq T_\alpha \left(\prod_{i \in F} Q_i \right) T_\alpha^*, \tag{2.11}$$

with the convention that $e(e,F) \doteq \prod_{i\in F} Q_i$ and $e(\alpha,\emptyset) \doteq T_\alpha T_\alpha^*$, and define

$$D_A = \operatorname{span}\{e(\alpha,F) : (\alpha,F) \in \mathcal{A}\} \tag{2.12}$$

and its closure on $\mathcal{B}(H)$ by $\mathcal{D}_A$. We note that for all $x \in \Sigma_A$, we have that

$$e(\alpha,F)e_x = \left[\prod_{i\in F} A(i, x_{|\alpha|})\right] \chi_{[\alpha]}(x) e_x$$

which implies that $e(\alpha,F)e(\beta,G) = e(\beta,G)e(\alpha,F)$ for all $(\alpha,F),(\beta,G) \in \mathcal{A}$. Therefore $\mathcal{D}_A$ defines a commutative algebra.

We note that $\mathcal{D}_A$ with the usual operator norm is then a C^*-algebra composed of diagonal operators which may, or may not, be unital. In fact, $\mathcal{D}_A$ has an unit if, and only if there is a finite number of symbols which cover the whole of $\mathcal{S}$, i.e., there is $F \subset \mathcal{S}$ finite such that for all $t \in \mathcal{S}$, we have that $A(s,t) = 1$ for some $s \in F$.

2.2 Stems and Roots

We are finally in position to give a description of a locally compact extension of countable Markov shifts which we shall call Generalized Countable Markov Shifts (GCMS). We recall the definition of character.

Definition 12. *A non-zero continuous linear functional $\varphi : \mathcal{D}_A \to \mathbb{C}$ is called a character if*

$$\varphi(ab) = \varphi(a)\varphi(b)$$

for all $a, b \in \mathcal{D}_A$.

Definition 13. *We define the GCMS associated to the matrix $A \in \{0,1\}^{\mathcal{S}\times\mathcal{S}}$ by*

$$X_A \doteq \Omega(\mathcal{D}_A) = \{\varphi : \mathcal{D}_A \to \mathbb{C} : \varphi \text{ is a character}\} \tag{2.13}$$

endowed with its usual weak-* topology, i.e., X_A is the spectrum of $\mathcal{D}_A$.

It follows from the theory of C^*-algebras that X_A is locally compact and that it is compact if, and only if, $\mathcal{D}_A$ is unital. Let us now show that there is a natural continuous injection of Σ_A onto X_A whose image is dense in X_A.

Proposition 4. *There is a continuous injection of the usual CMS Σ_A onto X_A whose image is dense.*

Proof. Given $x \in \Sigma_A$, we define $\varphi_x : \mathcal{D}_A \to \mathbb{C}$ by

$$\varphi_x(a) = \langle a(e_x), e_x\rangle_{l^2(\Sigma_A)}, \tag{2.14}$$

it is straightforward to see that $\varphi_x \in X_A$, i.e., that it defines a character on $\mathcal{D}_A$.

Let us now show that the mapping $x \mapsto \varphi_x$ is continuous. It is sufficient to show that if $(\alpha,F) \in \mathcal{A}$ and $\{x^n\}_{n\in\mathbb{N}} \subset \Sigma_A$ is a sequence converging to $x \in \Sigma_A$ that

$$\lim_{n\to\infty} \varphi_{x^n}(e(\alpha,F)) = \varphi_x(e(\alpha,F)) = \begin{cases} 1, & \text{if } x \in [\alpha] \text{ and } A(i, x_{|\alpha|}) = 1 \ \forall i \in F, \\ 0, & \text{otherwise.} \end{cases}$$

We note that $x^n \xrightarrow{n\to\infty} x$ iff for all $k \in \mathbb{N}$, there is $N(k) \in \mathbb{N}$ such that $n \geq N(k)$ implies that

$$x^n[0,k) \doteq \omega_0^n\omega_1^n...\omega_{k-1}^n = \omega_0\omega_1...\omega_{k-1} = x[0,k).$$

It is straightforward to see that if $n \geq N(|\alpha|+1)$, then $\varphi_{x^n}(e(\alpha,F)) = \varphi_x(e(\alpha,F))$ which proves the continuity of the application.

Let $x, \tilde{x} \in \Sigma_A$ satisfy $x \neq \tilde{x}$, there exists $k \in \mathbb{N}$ such that $x_k \neq \tilde{x}_k$. It follows that

$$\varphi_x(e(x[0,k+1),\emptyset)) = 1 \neq 0 = \varphi_{\tilde{x}}(e(x[0,k+1),\emptyset))$$

which proves the injectivity of the application.

Let us now show that $\{\varphi_x : x \in \Sigma_A\}$ is dense in X_A. As discussed in the introduction, it is sufficient to show that $\{\varphi_x : x \in \Sigma_A\}$ separates the points of $\mathcal{D}_A$, or similarly that if $\varphi_x(a) = 0$ for all $x \in \Sigma_A$, then $a = 0$. We note that if $\varphi_x(a) = 0$, then $ae_x = 0$ since all operators in $\mathcal{D}_A$ are diagonal. Suppose now that $a \in \mathcal{D}_A$ is such that $\varphi_x(a) = 0$ for all $x \in \Sigma_A$, then, by the previous observation, we have that $ae_x = 0$ for all $x \in \Sigma_A$, which is equivalent to $a = 0$ and we conclude the proof of the proposition. □

Let us begin by characterizing what are the sequences $\{x^n\}_{n\in\mathbb{N}} \subset \Sigma_A$ for which we get no new points, i.e., the sequences $\{\varphi_{x^n}\}_{n\in\mathbb{N}}$ for which the only accumulation points are of the form φ_x for some $x \in \Sigma_A$.

Proposition 5. *Let $\{x^n\}_{n\in\mathbb{N}} \subset \Sigma_A$ be a sequence such that*

$$\#\{x_k^n : n \in \mathbb{N}\} < \infty$$

for all $k \in \mathbb{N}_0$. Then, every accumulation point of $\{\varphi_{x^n}\}_{n\in\mathbb{N}}$ is of the form φ_x for some $x \in \Sigma_A$.

Remark 4. *We already know that a sequence of this form always has a converging subsequence in an usual CMS.*

Proof. Let $I_l \doteq \{x_l^n : n \in \mathbb{N}\}$ for $l \geq 0$, then

$$\sum_{\alpha_0\in I_0}\sum_{\alpha_1\in I_1}...\sum_{\alpha_l\in I_l}\chi_{[\alpha_0\alpha_1...\alpha_l]}(x^n) = 1$$

for all $l \geq 0$ and $n \in \mathbb{N}$.

Now let $\tilde{\varphi}$ be an accumulation point of $\{\varphi_n\}_{n\in\mathbb{N}}$, then there exists a subsequence $\{\varphi_{n_k}\}_{k\in\mathbb{N}}$ such that $\varphi_{n_k} \stackrel{*}{\rightharpoonup} \tilde{\varphi}$. In particular, for all $\alpha_0\alpha_1...\alpha_l \in I_0 \times I_1 \times ... \times I_l$ and $l \geq 0$, we have that

$$\lim_{k\to\infty}\varphi_{x^{n_k}}(e(\alpha_0\alpha_1...\alpha_l,\emptyset))$$

exists. The computation above shows that for all $l \geq 0$, we have that

$$\tilde{\varphi}\left(e\left(\alpha_0^l\alpha_1^l...\alpha_l^l,\emptyset\right)\right) = \lim_{k\to\infty}\varphi_{x^{n_k}}\left(e\left(\alpha_0^l\alpha_1^l...\alpha_l^l,\emptyset\right)\right) = 1$$

for some $\alpha_0^l\alpha_1^l...\alpha_l^l \in I_0 \times I_1 \times ... \times I_l$.

Let us consider the sequence $\{x_0^{n_k}\}_{k\in\mathbb{N}}$, since $\tilde{\varphi}(e(\alpha_0^0,\emptyset)) = 1$, there exists $K_0 \in \mathbb{N}$ such that $x_0^{n_k} = \alpha_0^0$ for $k \geq K_0$. Proceeding inductively in this manner we get that if $l < l'$, then

$$\alpha_0^{l'}\alpha_1^{l'}...\alpha_l^{l'} = \alpha_0^l\alpha_1^l...\alpha_l^l$$

and that $\lim_{k\to\infty} x_l^{n_k} = \alpha_l^l$. By construction we have that $A\left(\alpha_l^l, \alpha_{l+1}^{l+1}\right) = 1$ for all $l \geq 0$ and therefore

$$\lim_{k\to\infty} x^{n_k} = \alpha_0^0\alpha_1^1...\alpha_l^l... \in \Sigma_A.$$

By continuity of $x \mapsto \varphi_x$, we conclude that $\{\varphi_{x^{n_k}}\}_{k\in\mathbb{N}}$ converges weakly to φ_x, where $x = \alpha_0^0\alpha_1^1...\alpha_l^l...$. □

Corollary 1. *If Σ_A is locally compact, then $X_A = \Sigma_A$.*

Proof. If Σ_A is locally compact, then every sequence in Σ_A satisfies the hypothesis of the previous proposition and we conclude by the density of $\{\varphi_x : x \in \Sigma_A\}$ in X_A. □

We have shown that if $\#\{x_k^n : n \in \mathbb{N}\} < \infty$ for all $k \in \mathbb{N}_0$, then every accumulation point of $\{\varphi_{x^n}\}_{n\in\mathbb{N}}$ is of the form φ_x for some $x \in \Sigma_A$. We are now, therefore, interested in studying sequences $\{x^n\}_{n\in\mathbb{N}}$ for which there is $k \in \mathbb{N}_0$ such that $\#\{x_k^n : n \in \mathbb{N}\} = \infty$. Note that if

$$k' \doteq \inf\{k \in \mathbb{N}_0 : \#\{x_k^n : n \in \mathbb{N}\} = \infty\} > 0,$$

by arguing in the same manner as in this section, we are dealing essentially with the case $x^n = \omega y^n$ where $\omega \in \mathcal{S}^{k'}$ is an admissible word, which is the case that we deal next. Furthermore, we may suppose by taking subsequences that no symbol $s \in \mathcal{S}$ appears an infinite amount of times on the sequence $\{x_{k'}^n\}_{n\in\mathbb{N}}$. Finally, we leave for a future subsubsection the case $k' = 0$ with no symbol $s \in \mathcal{S}$ appearing an infinite amount of times on the sequence $\{x_0^n\}_{n\in\mathbb{N}}$.

2.2.1 Finite Words

Let us now describe most of the extra points that appear in case Σ_A is not locally compact, which for reason that should become clear below we will name finite words. We will denote the set of finite words of X_A by F_A.

Definition 14. *Let $s \in \mathcal{S}$, we define the range of s by*

$$\mathfrak{r}(s) \doteq \{t \in \mathcal{S} : A(s,t) = 1\} \tag{2.15}$$

and the source of s by

$$\mathfrak{s}(s) \doteq \{r \in \mathcal{S} : A(r,s) = 1\}. \tag{2.16}$$

In a similar manner, given $F \subset \mathcal{S}$, we write

$$\mathfrak{r}(F) \doteq \{t \in \mathcal{S} : A(s,t) = 1, \forall s \in F\} = \bigcap_{s\in F} \mathfrak{r}(s)$$

and, given $J \subset \mathcal{S}$, we write

$$\mathfrak{s}(J) \doteq \{r \in \mathcal{S} : A(r,s) = 1, \forall s \in J\} = \bigcap_{s\in J} \mathfrak{s}(s).$$

Definition 15. *We say that $s \in \mathcal{S}$ is an infinite emitter if*

$$\#\mathfrak{r}(s) = \infty.$$

We call a finite admissible word $\omega \in \mathcal{S}^n$ a finite stem if ω_{n-1} is an infinite emitter.

Let us consider a sequence of elements of Σ_A of the form $x^n = \omega j_n y^n$, where ω is a finite stem and such that no symbol $s \in \mathcal{S}$ appears an infinite number of times in $\{j_n\}_{n\in\mathbb{N}}$. Since $\varphi_{x^n}(e(\omega, \emptyset)) = 1$ for all $n \in \mathbb{N}$, it follows that $\{\varphi_{x^n}\}_{n\in\mathbb{N}}$ is relatively compact in X_A. Hence, there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ and $\tilde{\varphi} \in X_A$ such that $\varphi_{x^{n_k}} \xrightarrow{*} \tilde{\varphi}$ which is a nontrivial character.

We begin by computing $\tilde{\varphi}(e(\alpha, F))$ in the case $\alpha \neq \omega$. We have that

$$\varphi_{x^n}(e(\alpha, F)) = \left[\prod_{i\in F} A\left(i, x^n_{|\alpha|}\right)\right] \chi_{[\alpha]}(x^n). \tag{2.17}$$

Let us first analyze the $\chi_{[\alpha]}(x^n)$ part. Since no symbol $s \in \mathcal{S}$ appears an infinite amount of times in $\{j_n\}_{n\in\mathbb{N}}$, we have that $n \mapsto \chi_{[\alpha]}(x^n)$ is eventually constant and equal to zero if $|\alpha| > |\omega|$. On the other hand, if $|\alpha| < |\omega|$, then the sequence $n \mapsto \chi_{[\alpha]}(x^n)$ is constant and the same is true for the sequence $n \mapsto x^n_{|\alpha|}$. Finally, if $|\alpha| = |\omega|$ and $\alpha \neq \omega$, then $\chi_{[\alpha]}(x^n) = 0$ for all $n \in \mathbb{N}$. So far we have obtained that

$$\tilde{\varphi}(e(\alpha, F)) = \begin{cases} 1, & \text{if } \alpha\gamma = \omega \text{ s.t. } \gamma \neq e \text{ and } A(i, \omega_{|\alpha|}) = 1\ \forall i \in F, \\ 0, & \text{otherwise.} \end{cases}$$

Nothing interesting has come up so far. Let us now treat the case $\alpha = \omega$, here we have that

$$\varphi_{x^n}(e(\alpha, F)) = \left[\prod_{i\in F} A\left(i, x^n_{|\alpha|}\right)\right] \chi_{[\alpha]}(x^n) = \prod_{i\in F} A(i, j_n),$$

we now need to restrict ourselves to the converging subsequence $\{\varphi_{n_k}\}_{k\in\mathbb{N}}$. Due to convergence we get that either of the following is true:

1. there exists $K(F) \in \mathbb{N}$ such that $j_{n_k} \in \mathfrak{r}(F)$ for all $k \geq K(F)$, in which case $\tilde{\varphi}(e(\omega, F)) = 1$;
2. or there exists $K(F) \in \mathbb{N}$ such that $j_{n_k} \notin \mathfrak{r}(F)$ for all $k \geq K(F)$, in which case $\tilde{\varphi}(e(\omega, F)) = 0$.

This motivates us to define what we shall refer to as a *root* in the following way

$$\mathcal{R} \doteq \{F \subset \mathcal{S} : F \text{ is finite and } \exists K(F) \in \mathbb{N} : j_{n_k} \in \mathfrak{r}(F) \text{ for } k \geq K(F)\}. \tag{2.18}$$

It is evident that $\mathcal{R} \neq \emptyset$ since $\{\omega_{|\omega|-1}\} \in \mathcal{R}$. We conclude that:

$$\tilde{\varphi}(e(\alpha, F)) = \begin{cases} 1, & \text{if } \alpha = \omega \text{ and } F \in \mathcal{R}, \\ 1, & \text{if } \alpha\gamma = \omega \text{ s.t. } \gamma \neq e \text{ and } A(i, \omega_{|\alpha|}) = 1\ \forall i \in F, \\ 0, & \text{otherwise.} \end{cases} \tag{2.19}$$

It is also evident that $\tilde{\varphi} \neq \varphi_x$ for any $x \in \Sigma_A$ since

$$\tilde{\varphi}(e(x[0, |\omega| + 1), \emptyset)) = 0 \neq 1 = \varphi_x(e(x[0, |\omega| + 1), \emptyset)),$$

so it describes a "new" point that somewhat behaves as the finite word ω.

Remark 5. *There are no finite words ending in finite emitters.*

2.2.2 Roots

The calculations in the previous subsubsection also prove that if $s \in \mathcal{S}$ is an infinite emitter and $J \subset \mathfrak{r}(s)$ is such that $\#J = \infty$, then there exists an infinite $\tilde{J} \subset J$ such that either of the following is true for any finite $F \subset \mathcal{S}$:

1. there exists $G_F \subset \tilde{J}$ finite such that $\tilde{J} \setminus G_F \subset \mathfrak{r}(F)$,

2. there exists $G_F \subset \tilde{J}$ finite such that $(\tilde{J} \setminus G_F) \cap \mathfrak{r}(F) = \emptyset$.

Definition 16. *Let $s \in \mathcal{S}$ be an infinite emitter and $J \subset \mathcal{S}$ be an infinite set of symbols such that $\#J \setminus \mathfrak{r}(s) < \infty$ and such that for any finite $F \subset \mathcal{S}$, there exists $G_F \subset J$ finite such that either $J \setminus G_F \subset \mathfrak{r}(F)$ or $(J \setminus G_F) \cap \mathfrak{r}(F) = \emptyset$. We say that J is compatible with a root and we define the root associated to J by*

$$\mathcal{R}_J \doteq \{F \subset \mathcal{S} : F \textit{ is finite and } \exists G_F \subset J \textit{ finite s.t. } J \setminus G_F \subset \mathfrak{r}(F)\}. \tag{2.20}$$

Remark 6. *Note that if $F \in \mathcal{R}$, where $\mathcal{R}$ is a root, then all subsets $F' \subset F$ are also elements of $\mathcal{R}$. In particular, if $s \in F$, then $\{s\} \in \mathcal{R}$.*

Without further inspection of the specific matrix A, it is not at all evident how many such roots exist. They are in fact in a one-to-one correspondence with the acumulation points of the sources of symbols in $\mathcal{S}$. Let us make this statement clear.

Given a symbol $s \in \mathcal{S}$, we define the vector of its sources $\mathbf{S}(s) \in \{0,1\}^{\mathcal{S}}$ by

$$\mathbf{S}(s)(i) = \begin{cases} 1, & \text{if } i \in \mathfrak{s}(s), \\ 0, & \text{otherwise}, \end{cases} \tag{2.21}$$

where we endow $\{0,1\}^{\mathcal{S}}$ with its usual topology. What are then the acumulation points of the set $\mathbf{S}(\mathcal{S})$ in $\{0,1\}^{\mathcal{S}}$? First, note that there is a non-zero accumulation point of $\mathbf{S}(S)$ iff there is an infinite emitting symbol, i.e., iff Σ_A is not locally compact. Indeed, if there are no infinite emitters, then

$$\#\{\mathbf{S}(s)(i) = 1 : s \in \mathcal{S}\} = \#\mathfrak{r}(i) < \infty$$

for all $i \in \mathcal{S}$, i.e., 0 is the only accumulation point of $\{\mathbf{S}(s)(i) : s \in \mathcal{S}\} \subset \{0,1\}$ for all $i \in \mathcal{S}$ which implies that 0 is the only accumulation point of $\mathbf{S}(\mathcal{S})$. On the other hand, suppose $\tilde{s} \in \mathcal{S}$ is an infinite emitter, then

$$\infty = \#\mathbf{S}(\mathfrak{r}(\tilde{s})) = \#\{\mathbf{S}(s) : s \in \mathfrak{r}(\tilde{s})\}$$

and $S(s)(\tilde{s}) = 1$ for all $s \in \mathbf{S}(\mathfrak{r}(\tilde{s}))$. Since $\{0,1\}^{\mathcal{S}}$ is compact, there is an accumulation point $\mathbf{S}(\infty)$ of $\mathbf{S}(\mathfrak{r}(\tilde{s}))$ and it is evident that $\mathbf{S}(\infty) \neq 0$ since $\mathbf{S}(\infty)(\tilde{s}) = 1$.

Remark 7. *Note that $\mathbf{S}(\infty)(s) = 1$ for some accumulation point $\mathbf{S}(\infty)$ of $\mathbf{S}(\mathcal{S})$ if, and only if, $s \in \mathcal{S}$ is an infinite emitter.*

Now, let $\mathbf{S}(\infty)$ be a non-zero accumulation point of $\mathbf{S}(\mathcal{S})$, then there exists an injective sequence $\{j_k\}_{k \in \mathbb{N}} \subset \mathcal{S}$ such that

$$\mathbf{S}(\infty) = \lim_{k \to \infty} \mathbf{S}(j_k).$$

Define $G \doteq \{s \in \mathcal{S} : \mathbf{S}(s) = 1\}$ and $J \doteq \{j_k : k \in \mathbb{N}\}$. Let $F \subset G$ be a finite subset of $\mathcal{S}$, then

$$\begin{aligned}\#\{j \in J : j \notin \mathfrak{r}(F)\} &= \#\{k \in \mathbb{N} : \mathbf{S}(j_k)(s) = 0 \text{ for some } s \in F\}\\ &\leq \sum_{s \in F} \#\{k \in \mathbb{N} : \mathbf{S}(j_k)(s) = 0\} < \infty.\end{aligned}$$

On the other hand, suppose $F \subset \mathcal{S}$ be a finite subset such that $\mathbf{S}(s) = 0$ for some $s \in F$, then

$$\#\{j \in J : j \in \mathfrak{r}(F)\} \leq \#\{j \in J : \mathbf{S}(j)(s) = 1\} < \infty.$$

We have thus proved that J is compatible with a root and that the root it is compatible with is given by

$$\mathcal{R}_J = \{F \subset \mathcal{S} : F \text{ is finite and } F \subset G\} = \{F \subset \mathcal{S} : F \text{ is finite and } \mathbf{S}(\infty)(s) = 1\ \forall s \in F\}. \tag{2.22}$$

Let us prove the converse, suppose that $J \subset \mathcal{S}$ is an infinite set of symbols such that $\#J \setminus \mathfrak{r}(s) < \infty$, for some infinite emitter $s \in \mathcal{S}$, and such that for any finite $F \subset \mathcal{S}$, there exists $G_F \subset J$ finite such that either $J \setminus G_F \subset \mathfrak{r}(F)$ or $(J \setminus G_F) \cap \mathfrak{r}(F) = \emptyset$. Enumerate $J = \{j_n\}_{n \in \mathbb{N}}$, let us show that $\lim_{n\to\infty} \mathbf{S}(j_n) = \mathbf{S}(\infty_J)$ exists and that

$$\mathbf{S}(\infty_J)(s) = \begin{cases} 1, & \text{if } \#J \setminus \mathfrak{r}(s) < \infty,\\ 0, & \text{if } \#J \cap \mathfrak{r}(s) < \infty.\end{cases} \tag{2.23}$$

Since $\{0,1\}^{\mathcal{S}}$ is compact, it is sufficient to show that $\mathbf{S}(\infty_J)$ is the only accumulation of $\mathbf{S}(J)$. Note that this is a consequence of the following two observations

$$\#\{n \in \mathbb{N} : \mathbf{S}(j_n)(s) = 1\} < \infty$$

if $\{s\} \notin \mathcal{R}_J$ and

$$\#\{n \in \mathbb{N} : \mathbf{S}(j_n)(s) = 0\} < \infty$$

if $\{s\} \in \mathcal{R}_J$. Hence, 0 is the only accumulation point of $\{\mathbf{S}(j_n)(s) : n \in \mathbb{N}\}$ if $\{s\} \notin \mathcal{R}_J$ and similarly 1 is the only accumulation point of $\{\mathbf{S}(j_n)(s) : n \in \mathbb{N}\}$ if $\{s\} \in \mathcal{R}_J$. We have thus proved the following theorem.

Theorem 4. *A collection of finite sets of symbols $\mathcal{R}$ is a root if, and only if, there is a non-zero accumulation point $\mathbf{S}(\infty)$ of $\mathbf{S}(\mathcal{S})$ such that*

$$\mathcal{R} = \{F \subset \mathcal{S} : F \textit{ is finite and } \mathbf{S}(\infty)(s) = 1\ \forall s \in F\}. \tag{2.24}$$

Corollary 2. *Let $\mathcal{R}$ be a root. Then, J is compatible with $\mathcal{R}$ if, and only if,*

$$\lim_{j \in J} \mathfrak{s}(j) = \bigcup_{F \in \mathcal{R}} F \doteq R. \tag{2.25}$$

The set of roots also does something similar to partitioning the set of symbols $\mathcal{S}$ in the following sense: let $\mathcal{R}$ and $\mathcal{R}'$ be distinct roots associated respectively to the infinite subsets of symbols J and J', which we naturally suppose are compatible with roots. Since $\mathcal{R} \neq \mathcal{R}'$, we may suppose that there exists $s \in \mathcal{S}$ such that $\{s\} \in \mathcal{R} \setminus \mathcal{R}'$, in which case we have by definition that

$$\#J \setminus \mathfrak{r}(s) < \infty \text{ and } \#J' \cap \mathfrak{r}(s) < \infty.$$

Therefore, there exists two finite, maybe empty, subsets of symbols G and G' such that

$$J \subset \mathfrak{r}(s) \cup G \text{ and } (J' \setminus G') \cap \mathfrak{r}(s) = \emptyset.$$

Hence,

$$J \cap J' \subset (\mathfrak{r}(s) \cup G) \cap J' = (\mathfrak{r}(s) \cap J') \cup (G \cap J') \subset G' \cup G.$$

Therefore,

$$\#J \cap J' < \infty.$$

So J and J' are essentially disjoint sets.

Proposition 6. *Suppose there are two distinct roots $\mathcal{R}$ and $\mathcal{R}'$ associated respectively to the infinite subsets of symbols J and J'. Then,*

$$\#J \cap J' < \infty. \tag{2.26}$$

Corollary 3. *Let $\mathfrak{R}$ be a countable family of roots, then there exists $\mathcal{J} = \{J_{\mathcal{R}} : \mathcal{R} \in \mathfrak{R}\}$ such that $J_{\mathcal{R}}$ is compatible with $\mathcal{R}$ for all $\mathcal{R} \in \mathfrak{R}$ and such that $J_{\mathcal{R}} \cap J_{\mathcal{R}'} \neq \emptyset$ if, and only if, $\mathcal{R} = \mathcal{R}'$.*

Proof. We begin by enumerating $\mathfrak{R}$ and for each $\mathcal{R}_n$ we take J'_n compatible with $\mathcal{R}_n$. For $n = 1$, we put

$$J_{\mathcal{R}_1} = J'_1$$

and, for $n > 1$, we put

$$J_{\mathcal{R}_n} = J'_n \setminus \bigcup_{k=1}^{n-1} J_{\mathcal{R}_k}.$$

This is sufficient since

$$\#J'_n \cap \left(\bigcup_{k=1}^{n-1} J_{\mathcal{R}_k} \right) \leq \sum_{k=1}^{n-1} \#J'_n \cap J_{\mathcal{R}_k} < \infty$$

due to the previous proposition. $\square$

Let us finish this subsection analyzing the matter of convergence of roots.

Proposition 7. *Suppose that $\{\mathcal{R}_n\}_{n\in\mathbb{N}}$ is a sequence of roots such that $\lim_{n\to\infty} \mathcal{R}_n$ exists. Then,*

$$\mathcal{R} \doteq \lim_{n\to\infty} \mathcal{R}_n$$

is a root if it is not equal to the empty set. Furthermore, for each $n \in \mathbb{N}$, let $\mathbf{S}(\infty_n)$ be the accumulation point of $\mathbf{S}(\mathcal{S})$ associated to the root $\mathcal{R}_n$. Then,

$$\lim_{n\to\infty} \mathbf{S}(\infty_n) = \mathbf{S}(\infty)$$

exists, is not equal to 0 *and $\mathcal{R}$ is the root associated to $\mathcal{S}(\infty)$. The converse is also true.*

Proof. Let $\{\mathcal{R}_n\}_{n\in\mathbb{N}}$ be a sequence of roots and suppose that its limit exists. We note that, by the definition of limit of a sequence of sets, for $s \in \mathcal{S}$, we have either that $\{s\} \in \mathcal{R}_n$ for all $n \geq N_s$ or that $\{s\} \notin \mathcal{R}_n$ for all $n \geq N_s$, for some $N_F \in \mathbb{N}$. Then, we have that

$$\lim_{n\to\infty} \mathbf{S}(\infty_n)(s) = \begin{cases} 1, & \text{if } \{s\} \in \mathcal{R}, \\ 0, & \text{if } \{s\} \notin \mathcal{R}, \end{cases}$$

exists for all $s \in \mathcal{S}$ since $\{s\} \in \mathcal{R}_n$ iff $\mathbf{S}(\infty_n) = 1$ and $\{s\} \notin \mathcal{R}_n$ iff $\mathbf{S}(\infty_n) = 0$. Due to the compactness of $\{0,1\}^{\mathcal{S}}$, we have that $\lim_{n\to\infty} \mathbf{S}(\infty_n) = \mathbf{S}(\infty)$ exists and is an accumulation point of $\mathbf{S}(\mathcal{S})$. It is evident that $\mathcal{R}$ is associated to $\mathbf{S}(\infty)$ and it is, therefore, a root. The proof of the converse is identical. $\square$

2.2.3 Compatibility between roots and finite stems

We have seen previously how a sequence of infinite words converges towards a finite word, we also saw that this finite *stem* is not sufficient to describe this limit point and we have therefore introduced the notion of *root*. On the other hand, we have not explored in depth the relation between finite stems and roots, this is precisely the topic of this subsubsection.

Given a finite stem ω ending in $\tilde{s} \in \mathcal{S}$, which is an infinite emitter by definition, we are interested in determining what are the roots $\mathcal{R}$ (or equivalently, the accumulation points of sources) for which $\varphi_{\omega,\mathcal{R}} \in X_A$, where

$$\varphi_{\omega,\mathcal{R}}(e(\alpha, F)) = \begin{cases} 1, & \text{if } \alpha = \omega \text{ and } F \in \mathcal{R} \text{ or } F = \emptyset, \\ 1, & \text{if } \alpha\gamma = \omega \text{ s.t. } \gamma \neq e \text{ and } A(i, \omega_{|\alpha|}) = 1 \ \forall i \in F, \\ 0, & \text{otherwise.} \end{cases}$$

for all $(\alpha, F) \in \mathcal{A}$.

Theorem 5. *Let ω be an finite stem ending in $\tilde{s} \in \mathcal{S}$ and $\mathcal{R}$ be a root, then $\varphi_{\omega,\mathcal{R}} \in X_A$ if, and only if, $\mathcal{R} = \mathcal{R}(\mathbf{S}(\infty))$ for some accumulation point $\mathbf{S}(\infty)$ of $\mathbf{S}(\mathcal{S})$ such that $\mathbf{S}(\infty)(\tilde{s}) = 1$, where*

$$\mathcal{R}(\mathbf{S}(\infty)) = \{F \subset \mathcal{S} : F \textit{ is finite and } \mathbf{S}(\infty)(s) = 1 \ \forall s \in F\}. \tag{2.27}$$

Proof. Suppose $\varphi_{\omega,\mathcal{R}} \in X_A$, then there exists a sequence $\{x^n\}_{n\in\mathbb{N}} \subset \Sigma_A$ such that φ_{x^n} converges weakly to $\varphi_{\omega,\mathcal{R}}$ such that $n \mapsto x^n_{|\omega|}$ is injective. It is easy to see that there exists $N \in \mathbb{N}$ such that

$$x^n[0, |\omega|) = \omega$$

for $n \geq N$ since

$$\lim_{n\to\infty} \varphi_{x^n}(e(\omega, \emptyset)) = \varphi_{\omega,\mathcal{R}}(e(\omega, \emptyset)) = 1.$$

In particular, $x^n_{|\omega|-1} = \tilde{s}$ for $n \geq N$.

We are going to show that $\{\mathbf{S}(x^n_{|\omega|})\}_{n\in\mathbb{N}}$ converges to $\mathbf{S}(\infty)$, that $\mathbf{S}(\tilde{s}) = 1$ and, finally, that $\mathcal{R} = \mathcal{R}(\mathbf{S}(\infty))$. We recall that we can partition $\mathcal{S}$ in the following manner, either $\{s\} \in \mathcal{R}$ or $\{s\} \notin \mathcal{R}$, in which case, $F \cap \{s\} = \emptyset$ for all $F \in \mathcal{R}$. Furthermore, we note that this implies that $F \in \mathcal{R}$ iff $F = \{s_1, ..., s_m\}$ where $\{s_i\} \in \mathcal{R}$ for all $1 \leq i \leq m$.

Let $s \in \mathcal{S}$ be a symbol such that $\{s\} \notin \mathcal{R}$. Then, by hypothesis, we have that

$$0 = \varphi_{\omega,\mathcal{R}}(e(\omega, \{s\})) = \lim_{n\to\infty} \varphi_{x^n}(e(\omega, \{s\})) = \lim_{n\to\infty} A\left(s, x^n_{|\omega|}\right).$$

Therefore, 0 is the only accumulation point of $\{\mathbf{S}(x^n_{|\omega|})(s)\}_{n\in\mathbb{N}}$, i.e., $\#\{n \in \mathbb{N} : \mathbf{S}(x^n_{|\omega|})(s) = 1\} < \infty$. On the other hand, if $s \in \mathcal{S}$ is a symbol such that $\{s\} \in \mathcal{R}$, then, once again by hypothesis, we have that

$$1 = \varphi_{\omega,\mathcal{R}}(e(\omega, \{s\})) = \lim_{n\to\infty} \varphi_{x^n}(e(\omega, \{s\})) = \lim_{n\to\infty} A\left(s, x^n_{|\omega|}\right).$$

Therefore, $\#\{n \in \mathbb{N} : \mathbf{S}(x^n_{|\omega|})(s) = 0\} < \infty$. Taking into consideration the observations of the previous paragraph, we conclude that

$$\lim_{n\to\infty} \mathbf{S}\left(x^n_{|\omega|}\right) = \mathbf{S}(\infty)$$

exists and that $\mathcal{R} = \mathcal{R}(\mathbf{S}(\infty))$. Finally, since $x^n_{|\omega|-1} = \tilde{s}$ for $n \geq N$, we have that $A(\tilde{s}, x^n_{|\omega|}) = 1$ for $n \geq N$, therefore $\mathbf{S}(\infty)(\tilde{s}) = 1$.

Let us prove the converse. Let $\mathbf{S}(\infty)$ be an accumulation point of $\mathbf{S}(\mathcal{S})$ such that $\mathbf{S}(\infty)(\tilde{s}) = 1$, then there exists $n \mapsto j_n \in \mathcal{S}$ injective such that

$$\lim_{n\to\infty} \mathbf{S}(j_n) = \mathbf{S}(\infty)$$

and $\mathbf{S}(j_n)(\tilde{s}) = 1$ for all $n \in \mathbb{N}$. Define $\{x_n\}_{n\in\mathbb{N}} \subset \Sigma_A$ by

$$x^n = \omega j_n y^n.$$

Following the usual computations that we have done in this section, we obtain that

$$\varphi_{x^n} \xrightharpoonup{*} \varphi_{\omega,\mathcal{R}(\mathbf{S}(\infty))}$$

and the theorem is proved. □

2.2.4 Empty words

So far we have shown that in case Σ_A is not locally compact, there are elements of X_A that are not of the form φ_x for $x \in \Sigma_A$ and that we may describe such elements by what we called stems and roots. Nonetheless, there is still one type of sequence $\{\varphi_{x^n}\}_{n\in\mathbb{N}}$ that we have not yet discussed which gives rise to what we shall refer to as *empty words*, the case

$$k' \doteq \inf\{k \in \mathbb{N}_0 : \#\{x^n_k : n \in \mathbb{N}\} = \infty\} = 0$$

with no symbol $s \in \mathcal{S}$ appearing an infinite amount of times in $\{x^n_0\}_{n\in\mathbb{N}}$.

First, we note that, in this case,

$$\lim_{n\to\infty} \varphi_{x^n}(e(\alpha, F)) = 0$$

if $\alpha \neq e$. It is, therefore, sufficient to study the family of sequences $\{a_n^F\}_{n\in\mathbb{N}}$

$$a_n^F \doteq \prod_{i\in F} A(i, x_0^n) = \varphi_{x^n}(e(e,F)),$$

where $\emptyset \neq F \subset \mathcal{S}$ is a finite subset of symbols, something very similar to what we have done in studying roots.

If for some $\tilde{F} \subset \mathcal{S}$ finite, we have that 1 is an accumulation point of $\{a_n^{\tilde{F}}\}_{n\in\mathbb{N}}$. Then, by a similar procedure as in the previous subsubsection, we can find $J = \{x_0^{n_k}\}_{k\in\mathbb{N}} \subset \{x_0^n : n \in \mathbb{N}\}$ infinite such that for all $F \subset \mathcal{S}$ finite, we have that either $\#J \cap \mathfrak{r}(F) < \infty$ or $\#J \setminus \mathfrak{r}(F) < \infty$. In which case, we have that $\tilde{F} \in \mathcal{R}_J$ and

$$\lim_{k\to\infty} \varphi_{x^{n_k}} = \varphi_{e,\mathcal{R}_J},$$

where

$$\varphi_{e,\mathcal{R}_J}(e(\alpha, F)) = \begin{cases} 1, & \text{if } \alpha = e \text{ and } F \in \mathcal{R}_J, \\ 0, & \text{otherwise.} \end{cases} \tag{2.28}$$

On the other hand, let $\mathcal{R}$ be a root, then there exists $\mathbf{S}(\infty)$ accumulation point of $\mathbf{S}(\mathcal{S})$ such that

$$\mathcal{R} = \{F \subset \mathcal{S} : F \text{ is finite and } \mathbf{S}(\infty)(s) = 1 \ \forall s \in F\}.$$

Choose an injective sequence $\{j_n\}_{n\in\mathbb{N}}$ of symbols such that $\mathbf{S}(\infty) = \lim_{n\to\infty} \mathbf{S}(j_n)$ and a sequence $\{x^n\}_{n\in\mathbb{N}} \subset \Sigma_A$ such that $x_0^n = j_n$. It is straightforward to prove that

$$\varphi_{x^n} \overset{*}{\rightharpoonup} \varphi_{e,\mathcal{R}}.$$

The following proposition has been proved.

Proposition 8. *The set $E_A \subset X_A$ of empty words is in a one-to-one correspondence with the set of roots.*

Sometimes it can happen that the only accumulation point of $\{a_n^F\}_{n\in\mathbb{N}}$ is 0 for all $F \subset \mathcal{S}$ finite. Note that this happens if, and only if, 0 is an accumulation point of $\mathbf{S}(\mathcal{S})$. This is precisely the case where X_A is not compact and we have that $\{\varphi_{x^n}\}_{n\in\mathbb{N}}$ has no accumulation point in X_A.

2.2.5 Sequences of finite and empty words

We have seen how infinite words converge to finite words, let us now explore the opposite phenomenon, i.e., how finite words converge to infinite words. Let $x \in \Sigma_A$, it is important to determine whether there is a sequence

$$F_A \ni \varphi_{\omega^n,\mathcal{R}_n} \overset{*}{\rightharpoonup} \varphi_x.$$

It is necessary that

$$\lim_{n\to\infty} \varphi_{\omega^n,\mathcal{R}_n}\left(e\left(x[0,k),\emptyset\right)\right) = \varphi_x\left(e\left(x[0,k),\emptyset\right)\right) = 1$$

for all $k \in \mathbb{N}$. This is true if, and only if, for all $k \in \mathbb{N}$, there is $N(k) \in \mathbb{N}$ such that if $n \geq N(k)$, then

$$|\omega^n| \geq k \text{ and } \omega^n[0,k) = x[0,k).$$

We shall now show that this is sufficient. Note that this is always possible since A is transitive.

Proposition 9. *Let $\{\varphi_{\omega^n,\mathcal{R}_n}\}_{n\in\mathbb{R}} \subset F_A$ be a sequence such that for all $k \in \mathbb{N}$, there is $N(k) \in \mathbb{N}$ such that if $n \geq N(k)$, then*

$$|\omega^n| \geq k \textit{ and } \omega^n[0,k) = x[0,k).$$

Then,

$$\varphi_{\omega^n,\mathcal{R}_n} \xrightarrow{*} \varphi_x.$$

Remark 8. *This proposition shows that the root element of finite words disappears when passing to infinite words as expected.*

Proof. Let $(\alpha, F) \in \mathcal{A}$ and take $n \geq N(|\alpha|)$, then

$$\varphi_x(e(\alpha,F)) = \prod_{i\in F} A\left(i, x_{|\alpha|}\right)\chi_{[\alpha]}(x) = \prod_{i\in F} A\left(i, \omega^n_{|\alpha|}\right)\chi_{[\alpha]}(\omega^n) = \varphi_{\omega^n,\mathcal{R}_n}(e(\alpha,F))$$

and we conclude. □

Corollary 4. *The set of finite words F_A is dense in X_A.*

Let us now study how roots interact with each other when it comes to the convergence of sequences, i.e., how they relate to the topology of X_A. Let $\varphi_{\omega,\mathcal{R}} \in F_A$, we need to determine what are the properties that a sequence $\{\varphi_{\omega^n,\mathcal{R}_n}\}_{n\in\mathbb{R}} \subset F_A$ must satisfy so that $\varphi_{\omega^n,\mathcal{R}_n} \xrightarrow{*} \varphi_{\omega,\mathcal{R}}$.

Suppose that $\varphi_{\omega^n,\mathcal{R}_n} \xrightarrow{*} \varphi_{\omega,\mathcal{R}}$. At first, note that

$$1 = \varphi_{\omega,\mathcal{R}}\left(e(\omega,\emptyset)\right) = \lim_{n\to\infty} \varphi_{\omega^n,\mathcal{R}_n}(e(\omega,\emptyset)).$$

Therefore, there exists $N \in \mathbb{N}$ such that

$$\omega^n[0,|\omega|) = \omega$$

for $n \geq N$. On the other hand, for all $s \in \mathcal{S}$

$$0 = \varphi_{\omega s,\mathcal{R}}\left(e(\omega,\emptyset)\right) = \lim_{n\to\infty} \varphi_{\omega^n,\mathcal{R}_n}(e(\omega s,\emptyset)).$$

Two scenarios are possible.

If there is an infinite amount on indexes for which $|\omega^n| > |\omega|$, then no symbol $s \in \mathcal{S}$ appears an infinite amount of times on the sequence $\{\omega^n_{|\omega|}\}_{k\in\mathbb{N}}$, where $\{n_k : k \in \mathbb{N}\}$ is the set of indexes for which $|\omega^n| > |\omega|$. On the other hand, we have that $\{\varphi_{\omega^{n_k},\mathcal{R}_{n_k}}\}$ also converges weakly to $\varphi_{\omega,\mathcal{R}}$. Hence, by the usual arguments exposed in this section, the set of symbols

$$J = \left\{\omega^{n_k}_{|\omega|} : k \in \mathbb{N}\right\}$$

is compatible with the root $\mathcal{R}$. The converse is also true, if $\{\varphi_{\omega^n,\mathcal{R}_n}\}_{n\in\mathbb{N}}$ is such that

1. $|\omega^n| > |\omega|$ for all $n \in \mathbb{N}$,
2. $\omega^n[0, |\omega|) = \omega$ for $n \geq N$ for some $N \in \mathbb{N}$,
3. and $J \doteq \left\{\omega^n_{|\omega|} : n \in \mathbb{N}\right\}$ compatible with $\mathcal{R}$.

Then, $\varphi_{\omega^n, \mathcal{R}_n} \xrightarrow{*} \varphi_{\omega, \mathcal{R}}$. The proof is more or less identical to how a sequence of elements of Σ_A converge to a finite word.

Suppose now that $\#\{n \in \mathbb{N} : |\omega^n| > |\omega|\} < \infty$. Then, we have proved that $\omega^n = \omega$ for $n \geq N$ for some $N \in \mathbb{N}$. Take $F \in \mathcal{R}$ finite and nonempty, then

$$1 = \varphi_{\omega,\mathcal{R}}(e(\omega, F)) = \lim_{n\to\infty} \varphi_{\omega,\mathcal{R}_n}(e(\omega, F)),$$

i.e., $F \in \mathcal{R}_n$ for all $n \geq N_F$ for some $N_F \in \mathbb{N}$ or equivalently $F \in \liminf_{n\to\infty} \mathcal{R}_n$. Similarly, if $F \notin \mathcal{R}$ finite and nonempty, then

$$0 = \varphi_{\omega,\mathcal{R}}(e(\omega, F)) = \lim_{n\to\infty} \varphi_{\omega,\mathcal{R}_n}(e(\omega, F)),$$

i.e., $F \notin \mathcal{R}$ for any $n \geq N_F$ for some $N_F \in \mathbb{N}$, or equivalently $F \notin \limsup_{n\to\infty} \mathcal{R}_n$. Note that this implies that $\lim_{n\to\infty} \mathcal{R}_n$ exists and that

$$\mathcal{R} = \lim_{n\to\infty} \mathcal{R}_n.$$

The converse is also true, and once again the proof is more or less identical to what we have done before in detail: if $\{\varphi_{\omega^n,\mathcal{R}_n}\}_{n\in\mathbb{N}}$ is such that

1. $\omega^n = \omega$ for $n \geq N$ for some $n \in \mathbb{N}$,
2. and $\lim_{n\in\mathbb{N}} \mathcal{R}_n = \mathcal{R}$.

Then, $\varphi_{\omega^n, \mathcal{R}_n} \xrightarrow{*} \varphi_{\omega, \mathcal{R}}$. We conclude that a sequence of finite words converging to another finite word would be of either one of those types described above or a mixture of both.

Sequences of finite words converging to empty words follow the same routine as sequences of infinite words converging to empty words, where only the first symbol matters. More precisely, if $\{\varphi_{\omega^n,\mathcal{R}_n}\}$ converges weakly to $\varphi_{e,\mathcal{R}}$, then no symbol appears an infinite amount of times on the sequence $\{\omega^n_0\}_{n\in\mathbb{N}}$ and $J = \{\omega^n_0 : n \in \mathbb{N}\}$ is compatible with the root $\mathcal{R}$. The converse is also true.

Let us now take a quick look at sequences composed of empty words. It should be evident by this point that the set of empty words E_A is closed in X_A since $\varphi_{e,\mathcal{R}}(e(s, \emptyset)) = 0$ for any $s \in \mathcal{R}$ and $\varphi_{e,\mathcal{R}} \in E_A$. Similarly to the case of finite words converging to finite words, a sequence $\{\varphi_{e,\mathcal{R}_n}\}_{n\in\mathbb{N}}$ converges weakly to $\varphi_{e,\mathcal{R}}$ if, and only if, $\mathcal{R} = \lim_{n\to\infty} \mathcal{R}_n$.

Corollary 5. *The set of roots is separable in the sense that there exists a countable family of roots $\mathfrak{R}$ such that for all roots $\mathcal{R}$ we have that*

$$\mathcal{R} = \lim_{k\to\infty} \mathcal{R}_n \tag{2.29}$$

for some $\{\mathcal{R}_n\}_{n\in\mathbb{N}} \subset \mathfrak{R}$.

Proof. Since there is a countable subset dense on X_A (for example the set of all infinite periodic words), there is a countable dense subset of E_A. Let $\{\varphi_{e,\mathcal{R}} : \mathcal{R} \in \mathfrak{R}\}$ be such a set, take $\varphi_{e,\mathcal{R}_\infty} \in E_A$, there exists $\{\mathcal{R}_n\}_{n\in\mathbb{N}} \subset \mathfrak{R}$ such that

$$\varphi_{e,\mathcal{R}_n} \xrightharpoonup{*} \varphi_{e,\mathcal{R}_\infty}$$

and therefore

$$\mathcal{R}_\infty = \lim_{k\to\infty} \mathcal{R}_n$$

and we conclude. □

2.3 Continuous Functions

While the notation $\varphi_{\omega,\mathcal{R}}$ was convenient when proving results regarding the convergence of sequences of elements of X_A, it is in general more convenient to write $\omega_\mathcal{R}$ instead of $\varphi_{\omega,\mathcal{R}}$.

Definition 17. *Let $\omega \in \mathcal{S}^k$, for $k \in \mathbb{N}$, be an admissible word, we define the generalized cyllinder $[\omega]$ associated to ω by*

$$[\omega] = \{x \in \Sigma_A : x[0,k) = \omega\} \sqcup \{\alpha_\mathcal{R} \in F_A : |\alpha| \geq k \text{ and } \alpha[0,k) = \omega\}. \tag{2.30}$$

Due to the results proved in the previous subsection, we have that generalized cyllinders are clopen and sequentially compact and, therefore, also compact in X_A since X_A is metrizable as it is the spectre of a commutative separable C^*-algebra. We should also note that

$$\Sigma_A \sqcup F_A = \bigsqcup_{s\in\mathcal{S}} [s], \tag{2.31}$$

which proves the following proposition.

Proposition 10. *Let Y be a topological space and $F \doteq \{f_s : s \in \mathcal{S}\}$ be a family of functions such that $f_s : [s] \to Y$ is continuous for all $s \in \mathcal{S}$. Then, $f : \Sigma_A \sqcup F_A \to Y$ defined by*

$$f(y) \doteq f_{y_0}(y) \tag{2.32}$$

is continuous.

2.3.1 The Generalized Shift Application

Having fully characterized the topology (in the language of convergence of sequences) of X_A, we are now interested in extending the usual shift application $\sigma : \Sigma_A \to \Sigma$ to the generalized setting. In general, it will not be possible to extend it continuously to the whole X_A, but instead we will always be able to define a continuous shift application

$$\sigma : \Sigma_A \sqcup F_A \to X_A \tag{2.33}$$

with the property that $\sigma(F_A) = F_A \sqcup E_A$ and the usual $\sigma(\Sigma_A) = \Sigma_A$.

So what is the natural candidate for $\sigma(\omega_\mathcal{R})$? It is $\sigma(\omega)_\mathcal{R}$, where $\sigma(\omega) = \omega_1 ... \omega_{|\omega|-1}$. Proving that this definition extends the usual shift map σ continuously is more or less identical to the work already done in this section so we omit the details.

Remark 9. *In case $\omega = \tilde{s}$ for some $\tilde{s} \in \mathcal{S}$, we define $\sigma(\omega) = e$, the empty stem.*

Definition 18. *We define the generalized shift application $\sigma : \Sigma_A \sqcup F_A \to X_A$ by*

$$\sigma(x) = \begin{cases} \sigma(x), & \text{if } x \in \Sigma_A, \\ \sigma(\omega)_{\mathcal{R}}, & \text{if } x = \omega_{\mathcal{R}} \in F_A. \end{cases} \tag{2.34}$$

Remark 10. *Due to the previous proposition, the generalized shift application is continuous.*

An interesting identity relating the set of finite words F_A and the preimages of the empty words arises from the generalized shift application. More specifically,

$$F_A = \bigsqcup_{\mathcal{R} \text{ root}} \bigsqcup_{k \geq 1} \sigma^{-k}\left(\{e_{\mathcal{R}}\}\right). \tag{2.35}$$

Having this identity in mind, for a root $\mathcal{R}$, we define its *$\mathcal{R}$-family* $Y_{\mathcal{R}}$ by

$$Y_{\mathcal{R}} \doteq \bigsqcup_{k \geq 0} \sigma^{-k}\left(\{e_{\mathcal{R}}\}\right). \tag{2.36}$$

Note that $\sigma^{-1}\left(Y_{\mathcal{R}}\right) = Y_{\mathcal{R}}$ and $\sigma\left(Y_{\mathcal{R}} \cap U\right) = Y_{\mathcal{R}}$. Furthermore,

$$\sigma^{-1}\left(\{e_{\mathcal{R}}\}\right) = \left\{s_{\mathcal{R}} : s \in \mathcal{S} \text{ s.t. } \{s\} \in \mathcal{R}\right\}. \tag{2.37}$$

It becomes rather easy to describe the set of finite words in this manner, it is sufficient to know all the roots, which can be done simply by studying the matrix A.

Remark 11. *Every $\mathcal{R}$-family is countable. Hence, F_A is uncountable if, and only if, there is uncountable number of distinct roots.*

2.3.2 The Algebra of Real Continuous Functions on Generalized Cyllinders

Our objective here is to give a description of the set of real continuous functions on $U = \Sigma_A \sqcup F_A$ using Stone-Weierstrass' theorem. In order to do so, we need to present a family of functions that separates the points in U.

To construct a family of real continuous functions separating the points of U, we will need an positive summable sequence $b : \mathcal{S} \to \mathbb{R}$ such that

$$\sum_{s \in B} b(s) = \sum_{s \in B'} b(s) \iff B = B'.$$

Note that this sequence induces a finite measure $H : \mathcal{P}(\mathcal{S}) \to [0, \infty)$ via

$$H(B) = \sum_{s \in B} b(s) \tag{2.38}$$

for any $B \in \mathcal{P}(\mathcal{S})$. Since this measure is finite, we have that

$$H\left(\lim_{n \to \infty} B_n\right) = \lim_{n \to \infty} H(B_n) \tag{2.39}$$

if $\lim_{n \to \infty} B_n$ exists. Let $a : \mathcal{S} \to \mathbb{R}$ be a positive function bounded by 1 whose image has only 0 as an accumulation point. Furthermore, suppose that $a(s)H(B) = a(s')H(B')$ if, and only if, $(s, B) = (s', B')$.

Then, we may define $h_n : U \to \mathbb{R}$ by

$$h_n(y) = \begin{cases} \prod_{k=1}^{n-1} a(\omega_k)H(R), & \text{if } y = \omega_{\mathcal{R}} \in \sigma^{-n}e_{\mathcal{R}}, \\ \prod_{k=1}^{n-1} a(x_k)H(\mathfrak{s}(x_n)), & \text{if } y = x \in \Sigma_A, \\ \prod_{k=1}^{n-1} a(\omega_k)H(\mathfrak{s}(\omega_n)), & \text{if } y = \omega_{\mathcal{R}} \in F_A \text{ with } |\omega| > n, \\ 0, & \text{otherwise}, \end{cases} \tag{2.40}$$

for $n \in \mathbb{N}$, where $R \doteq \cup_{F \in \mathcal{R}} F$ with the convention $\prod_{k=1}^{0} a(x_k) \doteq 1$.

Let us first prove that h_m is continuous. We begin by noting that $\{y^n\}_{n\in\mathbb{N}}$ converges to $x \in \Sigma_A$ if, and only if, for all $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that $y^n[0,k) = x[0,k)$ if $n \geq N_k$. The sequence $n \mapsto h_m(y^n)$ is eventually constant and equal to

$$\prod_{k=1}^{m-1} a(x_k)H(\mathfrak{s}(x_m)) = h_m(x).$$

In the case of finite words, we need to break the problem in two. Let us first deal with the case $|\omega| \geq m$. Take $\{y^n\}_{n\in\mathbb{N}}$ converges to $\omega_{\mathcal{R}} \in F_A$ with $|\omega| \geq m$, there are two cases we need to consider. In the first, the stem of y^n is eventually equal to ω, i.e., $y^n = \omega_{\mathcal{R}_n}$, in which case the series converges to $\omega_{\mathcal{R}}$ if, and only if, $\lim_{n\to\infty} \mathcal{R}_n = \mathcal{R}$. We get that

$$h_m(\omega_{\mathcal{R}_n}) = \prod_{k=1}^{m-1} a(\omega_k)H(R_n) \xrightarrow{n\to\infty} \prod_{k=1}^{m-1} a(\omega_k)H(R) = h_m(\omega_{\mathcal{R}})$$

In the second case, we may suppose that $|y^n| > \omega$ with $y^n[0,|\omega|) = \omega$ and $\{y^n_{|\omega|}\}_{n\in\mathbb{N}}$ injective and compatible with $\mathcal{R}$. Hence

$$h_m(\omega_{\mathcal{R}_n}) = \prod_{k=1}^{m-1} a(\omega_k)H\left(\mathfrak{s}\left(y^n_{|\omega|}\right)\right) \xrightarrow{n\to\infty} \prod_{k=1}^{m-1} a(\omega_k)H(R) = h_m(\omega_{\mathcal{R}}).$$

Finally, suppose that $|\omega| \leq m$ and $\{y^n\}$ converges to $\omega_{\mathcal{R}}$. The non-evident case is precisely when $|y^n| > m$ for all $n \in \mathbb{N}$. We may suppose that $n \mapsto y^n_{|\omega|}$ is injective. Since $H(\mathcal{S}) < \infty$ and $a(\mathcal{S})$ is bounded with 0 being its only accumulation point, we get that

$$\lim_{n\to\infty} a(y^n_{|\omega|}) = 0$$

and we have thus proved the continuity of h_m.

The fact that it separates points comes from the hypothesis $H(B) = H(B')$ if, and only if, $B = B'$ and $a(s)H(B) = a(s')H(B')$ if, and only if, $(s,B) = (s',B')$. We conclude, by the Stone-Weierstrass theorem, that the set of continuous functions on the generalized cyllinder $[s]$ is the completion of the algebra

$$A = \mathbb{R}[\{1\} \cup \{h_n : n \in \mathbb{N}\}]. \tag{2.41}$$

endowed with the usual algebraic operations of real functions and the usual normal. Therefore, if $\phi : U \to \mathbb{R}$ is continuous, there exists $\{c(n,s) : n \in \mathbb{N}, s \in \mathcal{S}\} \subset \mathbb{R}$ such that

$$\sum_{n\in\mathbb{N}} |c(n,s)| \|h_n\|_\infty < \infty$$

for all $s \in \mathcal{S}$ and

$$\phi(x) = \sum_{s\in\mathcal{S}} \sum_{n\in\mathbb{N}} c(n,s)\chi_{[s]}(x)h_n(x). \tag{2.42}$$

Corollary 6. *Let $\phi : U \to \mathbb{R}$ be a continuous function. Define the n-th variation of ϕ on the cyllinder $[s]$ by*

$$\mathrm{Var}_n(\phi, s) = \sup\{|\phi(x) - \phi(y)| : x, y \in [s] \text{ with } |x| \geq n, |y| \geq n \text{ and } x[0,n) = y[0,n)\}. \tag{2.43}$$

Then, for all $s \in \mathcal{S}$,

$$\lim_{n\to\infty} \mathrm{Var}_n(\phi, s) = 0. \tag{2.44}$$

Remark 12. *Note that it is not necessary that $\mathrm{Var}_n\phi \doteq \sup_{s\in\mathcal{S}} \mathrm{Var}_n(\phi, s) < \infty$ for any $n \in \mathbb{N}$.*

Definition 19. *We say that $\phi : U \to \mathbb{R}$ is uniformly continuous if there exists $m \in \mathbb{N}$ such that*

$$\mathrm{Var}_m\phi \doteq \sup_{s\in\mathcal{S}} \mathrm{Var}_m(\phi, s) < \infty \tag{2.45}$$

and

$$\lim_{n\to\infty} Var_n\phi = 0. \tag{2.46}$$

We call $\mathrm{Var}_n\phi$ the generalized n-th variation ofϕ.

Remark 13. *It should be easy to see that if the restriction of $\phi : U \to \mathbb{R}$ is Walter's in the usual sense, then ϕ is uniformly continuous and the generalized variation satisfies the same bounds as the usual variation.*

2.4 Examples

In this section we will always deal with the case $\mathcal{S} = \mathbb{N}$, and write $\mathcal{S}$ or $\mathbb{N}$ interchangeably whenever no confusion is possible.

2.4.1 The Generalized Renewal Shift

Let $A \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ be given by

$$A(i,j) = \begin{cases} 1, & \text{if } i = j+1, \\ 1, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.47}$$

Note that for all $x \in \Sigma_A$ and $n \in \mathbb{N}_0$, we have that

$$x[n, n+x_n) = x_n(x_n - 1)...21. \tag{2.48}$$

In this case the only infinite emitter is the symbol 1 and the only accumulation point of the sources is given by δ_1. Hence, the only root is given by $\mathcal{R} = \mathcal{P}^*(\{1\})$ and an infinite subset of symbols compatible with $\mathcal{R}$ is $J = \mathcal{S}$.

Let us describe F_A with the help of the notion of $\mathcal{R}$-family. We have that

$$F_A = \bigsqcup_{k\geq 1} \sigma^{-k}(\{e_{\mathcal{R}}\}).$$

On the other hand, we have that

$$\sigma^{-1}(\{e_{\mathcal{R}}\}) = \{1_{\mathcal{R}}\}. \tag{2.49}$$

Suppose $\omega_{\mathcal{R}} \in \sigma^{-k}(\{e_{\mathcal{R}}\})$, it should be easy to see that

$$\sigma^{-1}(\{\omega_{\mathcal{R}}\}) = \{1\omega_{\mathcal{R}}, (\omega_0+1)\omega_{\mathcal{R}}\}. \tag{2.50}$$

This implies by an argument of induction that

$$\#\sigma^{-k}(\{e_{\mathcal{R}}\}) = 2^{k-1}. \tag{2.51}$$

Futhermore, note that if $\omega_n = m$, then $|\omega| \geq n + m$ and

$$\omega[n, n+m) = m(m-1)...21. \tag{2.52}$$

We note that

$$\omega_{\mathcal{R}} = \lim_{n\to\infty} \omega n(n-1)...21x^n \in \Sigma_A. \tag{2.53}$$

Furthermore, if $\{\omega^n_{\mathcal{R}}\}_{n\in\mathbb{N}}$ converges to $\omega_{\mathcal{R}}$ and $\{|\omega^n|\}_{n\in\mathbb{N}}$ is bounded, then there exists $N \in \mathbb{N}$ such that $\omega^n_{\mathcal{R}} = \omega_{\mathcal{R}}$ for $n \geq N$. This follows from the last observation of the previous paragraph.

2.4.2 The Super Renewal Shift

Let $A \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ be given by

$$A(i,j) = \begin{cases} 1, & \text{if } i = j+1, \\ 1, & \text{if } i = 2^k \text{ and } 2^k \mid j \text{ for some } k \in \mathbb{N}_0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.54}$$

Note that for all $x \in \Sigma_A$ and $n \in \mathbb{N}_0$, we have that

$$x\left[n, n+1+x_n - 2^{\lfloor \log_2 x_n \rfloor}\right) = x_n(x_n-1)...(2^{\lfloor \log_2 x_n \rfloor}+1)2^{\lfloor \log_2 x_n \rfloor}. \tag{2.55}$$

In this case there is an infinite number of infinite emitters, more specifically every power of 2, including 1, is an infinite emitter and there are no others. It is also true that there is an infinite number of accumulation points of the sources, more precisely

$$\mathbf{S}_k \doteq \sum_{j=0}^{k} \delta_{2^j}, \tag{2.56}$$

where $k \in \mathbb{N}_0 \cup \{\infty\}$, is an accumulation point, and every accumulation point is of this form. The root $\mathcal{R}_k$ associated to $\mathbf{S}_k$ is given by

$$\mathcal{R}_k = \mathcal{P}_f^*(\{2^j : 0 \leq j \leq k\}). \tag{2.57}$$

Furthermore,

$$\mathcal{R}_\infty = \lim_{n \to \infty} \mathcal{R}_n. \tag{2.58}$$

For $k \in \mathbb{N}_0$, a set of symbols compatible with $\mathcal{R}_k$ is

$$J_k = \{2^k j : j \in \mathbb{N} \text{ and } j \text{ odd}\} \tag{2.59}$$

and every set J' compatible with $\mathcal{R}_k$ is up to a finite number of elements a subset of J_k. A set of symbols compatible with $\mathcal{R}_\infty$ is

$$J_\infty = \{2^j : j \in \mathbb{N}\} \tag{2.60}$$

and, similarly, every set J' compatible with $\mathcal{R}_\infty$ is up to a finite number of elements a subset of J_∞.

2.4.3 The Generalized Full Shift

Let $A \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ be given by

$$A(i,j) = 1. \tag{2.61}$$

In this case every symbol is an infinite emitter but there is only one accumulation point of the sources given by $\sum_{n\in\mathbb{N}} \delta_n$. Hence the only root is given by $\mathcal{R} = \mathcal{P}_f^*(\mathbb{N})$ and an infinite subset of symbols compatible with $\mathcal{R}$ is $J = \mathcal{S} = \mathbb{N}$. It should be easy to see that there is a natural bijection between the set of all finite admssible words and the set of finite words on X.

Chapter 3

Conformal Measures on Generalized Shifts

Since the shift application is not defined on the whole space, we need to make some tiny modifications to the definitions present in the usual theory of conformal measure on countable Markov shifts. If we omit a proof in this section, it is very likely that it is identical to the case of an usual countable Markov shift. In this section we will write U instead of $\Sigma_A \sqcup F_A$ and, in general, we will omit the subscript A. Furthermore, unless stated otherwise, we assume that every function $\phi : \sigma \to \mathbb{R}$ may be extended continuously to U. For more equivalent definitions of conformality in the generalized we recommend [RAS20].

3.1 Ruelle's Operator on Generalized Markov Shifts

As in the usual case, we say that a σ-finite Borel measure μ on X is not said to be *singular* if $\sigma_*\mu \sim \mu$, i.e., given $B \subset X$ we have that

$$\sigma_*\mu(B) = \mu(\sigma^{-1}B) = 0 \iff \mu(B) = 0. \tag{3.1}$$

We note that if a measure μ is not singular and $\mu(e_{\mathcal{R}}) = 0$ for some root $\mathcal{R}$, then $\mu(Y_{\mathcal{R}}) = 0$, and similarly if $\mu(\omega_{\mathcal{R}}) > 0$ for some $\omega_{\mathcal{R}} \in Y_{\mathcal{R}}$, then $\mu(e_{\mathcal{R}}) > 0$. Furthermore, if μ on Σ is non-singular in the usual sense, then $\tilde{\mu}$ defined by

$$\tilde{\mu}(B) \doteq \mu(B \cap \Sigma)$$

is also non-singular. .

Suppose now that μ is a non-singular measure on X, then we define the measure $\mu \circ \sigma$ on U by

$$\mu \circ \sigma(B) \doteq \sum_{s \in \mathcal{S}} \mu(\sigma(B \cap [s])), \tag{3.2}$$

where $B \subset U$ is Borel measurable. Furthermore, noting that

$$\chi_{\sigma(B \cap [s])}(x) = \chi_{\sigma[s]}(x)\chi_B(sx)$$

for all $x \in X$, $s \in \mathcal{S}$ and $B \subset U$ Borel measurable, we get that

$$\begin{aligned}\int_U \chi_E \mathrm{d}(\mu \circ \sigma) &= \sum_{s \in \mathcal{S}} \mu[\sigma(E \cap [s])] = \sum_{s \in \mathcal{S}} \int_X \chi_{\sigma(E \cap [s])}(x) \mathrm{d}\mu \\ &= \sum_{s \in \mathcal{S}} \int_X \chi_{\sigma[s]}(x) \chi_E(sx) \mathrm{d}\mu(x) = \sum_{s \in \mathcal{S}} \int_{\sigma[s]} \chi_E(sx) \mathrm{d}\mu(x)\end{aligned}$$

and therefore

$$\int_U f \mathrm{d}(\mu \circ \sigma) = \sum_{s \in \mathcal{S}} \int_X f(sx) \mathrm{d}\mu(x) \tag{3.3}$$

for all $f : U \to \mathbb{R}_{\geq 0}$ Borel measurable.

Let $\phi : U \to \mathbb{R}$ be an continuous function such that

$$\mathrm{S}(\phi) \doteq \sum_{s \in \mathcal{S}} e^{\sup\{\phi(x) : x \in [s]\}} < \infty. \tag{3.4}$$

We define the *Ruelle operator* $L_\phi : L^\infty(U) \to L^\infty(X)$ associated to ϕ by

$$L_\phi f(x) = \sum_{\sigma y = x} e^{\phi(y)} f(y) = \sum_{s \in \mathcal{S}} \chi_{\sigma[s]}(x) e^{\phi(sx)} f(sx). \tag{3.5}$$

This operator is evidently continuous: let $x \in X$ and $f, g \in L^\infty(U)$, then

$$|L_\phi f(x) - L_\phi g(x)| \leq \sum_{s \in \mathcal{S}} e^{\phi(sx)} |f(x) - g(x)| \leq \mathrm{S}(\phi) \|f - g\|_\infty.$$

We may also define recursively $L_\phi^n : L^\infty(U) \to L^\infty(X)$ by

$$L_\phi^{n+1} f(x) \doteq L_\phi \left(L_\phi^n f\right)_U (x).$$

Then,

$$L_\phi^n f(x) = \sum_{\sigma^n y} e^{\phi_n(y)} f(y) \tag{3.6}$$

for all $n \in \mathbb{N}$, where

$$\phi_n(y) = \sum_{k=0}^{n-1} \phi(\sigma^k y)$$

denotes the usual Birkhoff sum.

As usual it induces an operator $L_\phi^* : \mathcal{M}_\sigma(X) \to \mathcal{M}_\sigma(U)$ taking σ-finite Borel measures on X_A to σ-finite Borel measures on U via

$$L_\phi^* \mu(B) \doteq \int_X L_\phi \chi_B(x) \mathrm{d}\mu(x), \tag{3.7}$$

where $B \subset U$ is a Borel measurable set. On the other hand,

$$\begin{aligned} L_\phi^*\mu(B) &= \int_X \sum_{s\in\mathcal{S}} \chi_{\sigma[s]}(x)e^{\phi(sx)}\chi_B(sx)\mathrm{d}\mu(x) \\ &= \sum_{s\in\mathcal{S}} \int_{\sigma[s]} e^{\phi(sx)}\chi_B(sx)\mathrm{d}\mu(x) \\ &= \int_U e^{\phi(x)}\chi_B(x)\mathrm{d}(\mu\circ\sigma)(x) = \int_B e^{\phi(x)}\mathrm{d}(\mu\circ\sigma)(x). \end{aligned}$$

We conclude that $L_\phi^*\mu \sim \mu\circ\sigma$.

Definition 20. *We say that a measure μ on X is an eigenmeasure of the Ruelle operator associated to ϕ with eigenvalue $\lambda > 0$ if*

$$L_\phi^*\mu = \lambda\mu_U. \tag{3.8}$$

Proposition 11. *A measure μ on X is an eigenmeasure of the Ruelle Operator associated to ϕ with eigenvalue $\lambda > 0$ if, and only if,*

$$\frac{\mathrm{d}\mu_U}{\mathrm{d}(\mu\circ\sigma)}(x) = \lambda^{-1}e^{\phi(x)} \quad \mu_U\text{-}a.e. \tag{3.9}$$

Proof. The proposition follows from the computation above the definition of eigenmeasure. □

3.2 Eigenmeasures supported on the set of finite words

Let us investigate what are the eigenmeasures μ of the Ruelle operator which give positive mass to a $Y_\mathcal{R}$ family. We know that such measure must satisfy $\mu(e_\mathcal{R}) > 0$ if it is non-singular. Let $\tilde{s}_\mathcal{R} \in \sigma^{-1}(e_\mathcal{R})$, then

$$\begin{aligned} \lambda\mu(\tilde{s}_\mathcal{R}) = L_\phi^*\mu(\tilde{s}_\mathcal{R}) &= \sum_{s\in\mathcal{S}} \int_{\sigma[s]} e^{\phi(sx)}\chi_{\tilde{s}_\mathcal{R}}(sx)\mathrm{d}\mu(x) \\ &= \int_{\sigma[\tilde{s}]} e^{\phi(\tilde{s}x)}\chi_{e_\mathcal{R}}(x)\mathrm{d}\mu(x) = e^{\phi(\tilde{s}_\mathcal{R})}\mu(e_\mathcal{R}) \end{aligned}$$

or rearranging the terms

$$\mu(\tilde{s}_\mathcal{R}) = \lambda^{-1}e^{\phi(\tilde{s}_R)}\mu(e_\mathcal{R}).$$

This implies that

$$\mu(\sigma^{-1}e_\mathcal{R}) = \sum_{s:\{s\}\in\mathcal{R}} \mu(s_\mathcal{R}) = \lambda^{-1}\sum_{\sigma y = e_\mathcal{R}} e^{\phi(y)}\mu(e_\mathcal{R}) = \lambda^{-1}L_\phi 1(e_\mathcal{R})\mu(e_\mathcal{R}).$$

Proposition 12. *Suppose μ is a λ-eigenmeasure of the Ruelle operator, then*

$$\mu(\sigma^{-n}e_\mathcal{R}) = \lambda^{-n}L_\phi^n 1(e_\mathcal{R})\mu(e_\mathcal{R}) \tag{3.10}$$

for all $n \in \mathbb{N}$ and $e_{\mathcal{R}} \in E$. In particular, if $\mu(e_{\mathcal{R}}) > 0$, then $\mu(Y_{\mathcal{R}}) < \infty$ if, and only if,

$$\sum_{n=1}^{\infty} \lambda^{-n} L_{\phi}^{n} 1(e_{\mathcal{R}}) = \sum_{n=1}^{\infty} \lambda^{-n} \sum_{\sigma^n y = e_{\mathcal{R}}} e^{\phi_n(y)} < \infty. \tag{3.11}$$

Proof. Let us begin the proof by showing by induction that if $\omega_{\mathcal{R}} \in \sigma^{-n} e_{\mathcal{R}}$, then

$$\mu(\omega_{\mathcal{R}}) = \lambda^{-n} e^{\phi_n(\omega_{\mathcal{R}})} \mu(e_{\mathcal{R}}). \tag{3.12}$$

The base case is already done. Suppose now the result true from $n = k$ and take $\omega_{\mathcal{R}} \in \sigma^{-(k+1)} e_{\mathcal{R}}$, then

$$\begin{aligned} \lambda \mu(\omega_{\mathcal{R}}) = L_{\phi}^{*}(\omega_{\mathcal{R}}) &= \sum_{s \in \mathcal{S}} \int_{\sigma[s]} e^{\phi(sx)} \chi_{\omega_{\mathcal{R}}}(sx) \mathrm{d}\mu(x) = \int_{\sigma[\omega_0]} e^{\phi(\omega_0 x)} \chi_{\omega_{\mathcal{R}}}(\omega_0 x) \mathrm{d}\mu(x) \\ &= \int_{\sigma[\omega_0]} e^{\phi(\omega_0 x)} \chi_{\sigma(\omega)_{\mathcal{R}}}(x) \mathrm{d}\mu(x) = e^{\phi(\omega_{\mathcal{R}})} \mu(\sigma(\omega)_{\mathcal{R}}). \end{aligned}$$

Therefore,

$$\mu(\omega_{\mathcal{R}}) = \lambda^{-1} e^{\phi(\omega_{\mathcal{R}})} \mu(\sigma(\omega)_{\mathcal{R}}) = \lambda^{-1} e^{\phi(\omega_{\mathcal{R}})} \lambda^{-k} e^{\phi_k(\sigma(\omega)_{\mathcal{R}})} \mu(e_{\mathcal{R}}) = \lambda^{-(k+1)} e^{\phi_{k+1}(\omega_{\mathcal{R}})} \mu(e_{\mathcal{R}})$$

and we conclude the induction.

Let us now sum $\mu(\omega_{\mathcal{R}})$ over $\sigma^{-n} e_{\mathcal{R}}$, we get that

$$\mu(\sigma^{-n} e_{\mathcal{R}}) = \sum_{\sigma^n y = e_{\mathcal{R}}} \mu(y) = \sum_{\sigma^n y = e_{\mathcal{R}}} \lambda^{-n} e^{\phi_n(y)} \mu(e_{\mathcal{R}}) = \lambda^{-n} L_{\phi}^{n} 1(e_{\mathcal{R}}) \mu(e_{\mathcal{R}})$$

and therefore

$$\mu(Y_{\mathcal{R}}) = \sum_{n=1}^{\infty} \mu(\sigma^{-n} e_{\mathcal{R}}) = \mu(e_{\mathcal{R}}) \sum_{n=1}^{\infty} \lambda^{-n} L_{\phi}^{n} 1(e_{\mathcal{R}})$$

and we conclude. □

Corollary 7. *There is a finite λ-eigenmeasure μ of the Ruelle operator associated to ϕ supported on $Y_{\mathcal{R}}$ if*

$$\lambda > \limsup_{n \to \infty} \frac{1}{n} \log L_{\phi}^{n} 1(e_{\mathcal{R}}) \doteq P(\phi, e_{\mathcal{R}}).$$

There is no finite λ-eigenmeasure μ of the Ruelle operator supported on $Y_{\mathcal{R}}$ if $\lambda < P(\phi, e_{\mathcal{R}})$.

Definition 21. *Let $x \in X$, we define the n-th partition function at x by*

$$Z_n(\phi, x) \doteq L_{\phi}^{n} 1(x) = \sum_{\sigma^n y = x} e^{\phi_n(y)} \tag{3.13}$$

and the pressure of ϕ at x by

$$P(\phi, x) \doteq \limsup_{n \to \infty} \frac{1}{n} \log Z_n(\phi, x). \tag{3.14}$$

For all $n \geq 1$ and $x \in X$, note that

$$\begin{aligned} Z_{n+1}(\phi, x) &= \sum_{\sigma^{n+1} y = x} e^{\phi_{n+1}(y)} = \sum_{\sigma w = x} \sum_{\sigma^n y = \omega} e^{\phi_{n+1}(y)} \\ &= \sum_{\sigma w = x} e^{\phi(w)} \sum_{\sigma^n y = w} e^{\phi_n(w)} = \sum_{\sigma w = x} e^{\phi(w)} Z_n(\phi, w). \end{aligned}$$

In particular, we have that

$$Z_{n+1}(\phi, e_{\mathcal{R}}) = \sum_{s : \{s\} \in \mathcal{R}} e^{\phi(s_{\mathcal{R}})} Z_n(\phi, s_{\mathcal{R}}). \tag{3.15}$$

Therefore, in general,

$$P(\phi, e_{\mathcal{R}}) \geq \sup_{s : \{s\} \in \mathcal{R}} P(\phi, s_{\mathcal{R}}) \tag{3.16}$$

and in the case $\# \mathcal{R} < \infty$

$$P(\phi, e_{\mathcal{R}}) = \sup_{s : \{s\} \in \mathcal{R}} P(\phi, s_{\mathcal{R}}). \tag{3.17}$$

Let us show that, in the case that $\phi : U \to \mathbb{R}$ is uniformly continuous and $\mathrm{Var}_2 \phi < \infty$, the pressure function is constant on the generalized cyllinders. Let $x, x' \in U$ such that $x_0 = x_0'$, then there is a natural bijection between $\sigma^{-n} x$ and $\sigma^{-n} x'$ for all $n \in \mathbb{N}$, more specifically,

$$\sigma^{-n} x \ni y \mapsto y[0, n) x' \in \sigma^{-n} x'.$$

On the other hand,

$$|\phi_n(y) - \phi_n(y[0, n) x')| \leq \mathrm{Var}_{n+1} \phi_n$$

and therefore

$$\left| \frac{1}{n} \log Z_n(\phi, x) - \frac{1}{n} \log Z_n(\phi, x') \right| \leq \frac{1}{n} \mathrm{Var}_{n+1} \phi_n \xrightarrow{n \to \infty} 0.$$

Proposition 13. *Suppose that $\phi : U \to \mathbb{R}$ is uniformly continuous and $\mathrm{Var}_2 \phi < \infty$, then $P(\phi, \cdot)$ is constant on generalized cyllinders.*

Let us return to the usual theory of countable Markov shifts for a moment. For Walter's potentials it has been proved that the Gurevich pressure given by

$$P_g(\phi, s) \doteq \limsup_{n \to \infty} \frac{1}{n} \log Z_n(F, [s]) < \infty \tag{3.18}$$

exists and does not depend on $s \in \mathcal{S}$, where

$$Z_n(\phi, s) = \sum_{\substack{\sigma^n x = x \\ x_0 = s}} e^{\phi_n(x)} \tag{3.19}$$

is Sarig's n-th partition function on the symbol s.

Furthermore, there is a finite conservative $e^{P_g(\phi)}$-eigenmeasure of the usual Ruelle operator if, and only if,

$$\sum_{n=1}^{\infty} e^{-nP_g(\phi)} Z_n(\phi, [s]) = \infty \tag{3.20}$$

and

$$\sum_{n=1}^{\infty} e^{-nP_g(\phi)} Z_n^*(\phi, [s]) < \infty, \tag{3.21}$$

where

$$Z_n^*(\phi, [s]) = \sum_{\substack{\sigma^n x = x \\ x_m = s \iff n | m}} e^{\phi_n(x)} \tag{3.22}$$

Note that the usual theory contrasts with the theory of eigenmeasures on finite words, since there is a finite $e^{P(F,e_{\mathcal{R}})}$-eigenmeasure supported on $Y_{\mathcal{R}}$ if, and only if,

$$\sum_{n=1}^{\infty} e^{-nP(\phi, e_{\mathcal{R}})} Z_n(\phi, e_{\mathcal{R}}) < \infty. \tag{3.23}$$

Furthermore, it is not yet evident how these two definitions of pressure and these two definitions of partition function relate to each other. Do the different definitions of pressure compute the same quantity? Do the distinct partition functions behave equally as n grows very large?

3.3 Source-compactness

If $\phi : U \to \mathbb{R}$ is uniformly continuous and $\mathrm{Var}_2\phi < \infty$, we have that $P_g(\phi, s) \leq P(\phi, x)$ for all $x \in [s]$. Indeed, given $s \in \mathcal{S}$, take $x \in U$ such that $x_0 = s$. Then, for all $z \in \Sigma$ such that $z = s$ and $\sigma^n z = z$, we have that

$$|\phi_n(z) - \phi_n(z[0, n)x)| \leq \mathrm{Var}_{n+1}\phi_n$$

and, therefore,

$$Z_n(\phi, [s]) = Z_n(\phi, [x_0]) \leq e^{\mathrm{Var}_{n+1}\phi_n} Z_n(\phi, x). \tag{3.24}$$

We conclude that

$$P_g(\phi, s) = \limsup_{n\to\infty} \frac{1}{n} \log Z_n(\phi, [s]) \leq \limsup_{n\to\infty} \frac{1}{n}\mathrm{Var}_{n+1}\phi_n + \frac{1}{n} \log Z_n(\phi, x) = P(\phi, x). \tag{3.25}$$

We shall now give a sufficient condition for the equality of the two defintions of pressure.

Definition 22. *We say that the matrix A is source-compact if for all $s \in \mathcal{S}$, there exists $m_s \in \mathbb{N}$ and $F_s \subset \mathcal{S}$ finite such that for all $\tilde{s} \in \mathcal{S}$, there exists a non-empty finite admissible word ω with*

1. $|\omega| = k \leq m_s$,

2. $\omega_0 = s$ *and* $A(\omega_{k-1}, \tilde{s}) = 1$,

3. *and* $\omega \in F_s^k$.

Remark 14. *Since A is transitive, it is sufficient to prove that the property is true for a single $s \in \mathcal{S}$.*

Proposition 14. *If there exists $F \subset \mathcal{S}$ finite such that*

$$\bigcup_{t \in F} \mathfrak{r}(t) = \mathcal{S},$$

then A is source-compact.

Proof. Given $s \in \mathcal{S}$, take for each $t \in F$, let $\omega(t)$ be a finite word starting in s and ending in t. We claim that it is sufficient to take

1. $m_s = \max_{t \in F} |\omega(t)|$,
2. and $F_s = \{\omega(t)_n : t \in F, 0 \le n < |\omega(t)|\}$.

Indeed, let $\tilde{s} \in \mathcal{S}$, there exists $t \in F$ such that $A(t, \tilde{s}) = 1$, in which case it is sufficient to take $\omega = \omega(t)$. □

Let $\phi : U \to \mathbb{R}$ be a continuous function and $x, y \in U$ such that $\sigma^n y = x$. If A is source-compact, there exist $m_{x_0} \in \mathbb{N}$ and $F_{x_0} \subset \mathcal{S}$ such that there exists $\omega(y_0)$ with

1. $|\omega(y_0)| = k_y \le m_{x_0}$,
2. $\omega(y_0)_0 = x_0$ and $\omega(y_0)y$ admissible,
3. and $\omega(y_0) \in F_{x_0}^{k_y}$.

In which case,

$$\phi_n(y) = \phi_n\left(\sigma^{k_y}\omega(y_0)y\right) = \phi_{n+k_y}(\omega(y_0)y) - \phi_{k_y}(\omega(y_0)y).$$

Note now that if $\mathrm{S}(\phi) < \infty$, then

$$\sup[-\phi_{k_y}(\omega(y_0)y)] \le \sum_{j=0}^{k_y-1} \sup|\phi|([\sigma^j\omega(y_0)]) \le \sum_{j=1}^{m_{x_0}} \sum_{\omega \in F_{x_0}^j} \sup|\phi|([\omega]) \doteq C(x_0) < \infty$$

where the finiteness of $|\phi|$ on generalized cyllinders follows from the continuity of ϕ. On the other hand, let $z(y) \in \Sigma$ be the periodic word such that

1. $\sigma^{n+k_y} z(y) = z(y)$,
2. and $z(y)[0, n+k_y) = \omega(y_0)y[0, n)$.

Note that $y \mapsto z(y)$ defines an injection from $\sigma^{-n}x$ into $\cup_{k=1}^{m_{x_0}} \{z \in \Sigma : \sigma^{n+k}z = z, z_0 = x_0\}$. Furthermore, we have by construction that

$$|\phi_{n+k_y}(\omega(y_0)y) - \phi_{n+k_y}(z_y)| \le \mathrm{Var}_{n+k_y+1}\phi_{n+k_y}.$$

Therefore,

$$Z_n(\phi,x)=\sum_{\sigma^y=x}e^{\phi_n(y)}\le e^{C_0}\sum_{k=n+1}^{n+m_{x_0}}e^{\mathrm{Var}_{k+1}\phi_k}\sum_{\substack{\sigma^{n+k}z=z\\ z_0=x_0}}e^{\phi_{n+k}(z)}=e^{C_0}\sum_{k=n+1}^{n+m_{x_0}}e^{\mathrm{Var}_{k+1}\phi_k}Z_{n+k}(\phi,[x_0]). \tag{3.26}$$

We have thus proved the following proposition.

Proposition 15. *Suppose that A is source-compact and $\phi : U \to \mathbb{R}$ is uniformly continuous with $\mathrm{Var}_2\phi < \infty$, then*

$$P(\phi,x)=P_g(\phi,[x_0]) \tag{3.27}$$

for all $x \in U$.

Corollary 8. *Suppose that A is source-compact. If ϕ is Walter's, then $P(\phi,\cdot)$ is constant on U and equal to the Gurevich pressure $P_g(\phi)$. Furthermore, if every root $\mathcal{R}$ is finite, then $P(\phi,\cdot)$ is constant on X and equal to the Gurevich pressure $P_g(\phi)$.*

Remark 15. *If every root is finite, the set of roots is countable and therefore the set of finite words F is also countable, in which case every measure on F is a sum of delta measures.*

Let us analyze the existence of non-trivial $e^{P_g(\phi)}$-eigenmeasures in the case where ϕ is Walter's and every root $\mathcal{R}$ is finite. In this case, we may rewrite the bound relating the two different definitions of partition functions in the following way

$$Z_n(\phi,x)\le M(x_0)\sum_{k=n+1}^{n+m_{x_0}}Z_{n+k}(\phi,[x_0]),$$

where $M(x_0) > 0$ does not depend on n, and in a similar manner

$$M'(x_0)Z_n(\phi,[x_0])\le Z_n(\phi,x).$$

Therefore,

$$Z_{n+1}(\phi,e_{\mathcal{R}})=\sum_{s:\{s\}\in\mathcal{R}}e^{\phi(s_{\mathcal{R}})}Z_n(\phi,s_{\mathcal{R}})\ge\sum_{s:\{s\}\in\mathcal{R}}M'(s)e^{\phi(s_{\mathcal{R}})}\sum_{k=n+1}^{n+m_s}Z_k(\phi,[s])$$

Since $P(\phi,e_{\mathcal{R}})=P_g(\phi)$,

$$\begin{aligned}\sum_{n=1}^{\infty}e^{-nP_g(\phi)}Z_{n+1}(\phi,e_{\mathcal{R}})&\ge\sum_{s:\{s\}\in\mathcal{R}}M'(s)e^{\phi(s_{\mathcal{R}})}\sum_{n=1}^{\infty}e^{-nP_g(\phi)}\sum_{k=n+1}^{n+m_s}Z_k(\phi,[s])\\&=\sum_{s:\{s\}\in\mathcal{R}}M'(s)e^{\phi(s_{\mathcal{R}})}\sum_{k=1}^{m_s}e^{kP_g(\phi)}\sum_{n=k+1}^{\infty}e^{-nP_g(\phi)}Z_n(\phi,[s]).\end{aligned}$$

We conclude that if

$$\sum_{n=1}^{\infty}e^{nP_g(\phi)}Z_n(\phi,[s])=\infty,$$

then

$$\sum_{n=1}^{\infty} e^{-nP_g(\phi)} Z_n(\phi, e_{\mathcal{R}}) = \infty.$$

We know from the usual theory of countable Markov shifts that if ϕ is Walter's and there is a non-trivial $e^{P_g(\phi)}$-eigenmeasure on Σ that is conservative and finite on cyllinders, then

$$\sum_{n=1}^{\infty} e^{-nP_g(\phi)} Z_n(\phi, [s]) = \infty$$

for all $s \in \mathcal{S}$. Therefore,

$$\sum_{n=1}^{\infty} e^{-nP_g(\phi)} Z_n(\phi, e_{\mathcal{R}}) = \infty$$

for all $e_{\mathcal{R}} \in E$. We conclude that in case ϕ is Walter's and all roots are finite, that it is impossible that there exists a finite $e^{P_g(f)}$-eigenmeasure that gives mass to Σ and F at the same time.

Furthermore, given an $e^{P_g(\phi)}$-eigenmeasure μ of Ruelle's operator and $s \in \mathcal{S}$, then, for all $n \in \mathbb{N}$, we have that

$$\begin{aligned}
\mu\left(\sigma^{-(n+1)} e_{\mathcal{R}} \cap [s]\right) &= \sum_{\sigma^{n+1} y = e_{\mathcal{R}}} e^{-(n+1)P_g(\phi)} e^{\phi_{n+1}(y)} \chi_{[s]}(y) \\
&= \sum_{\sigma y = w} e^{-P_g(\phi)} e^{\phi(y)} \sum_{\sigma^n w = e_{\mathcal{R}}} e^{-nP_g(\phi)} e^{\phi_n(w)} \chi_{[s]}(y) \\
&= e^{-P_g(\phi)} \sum_{\sigma^n w = e_{\mathcal{R}}} e^{-nP_g(\phi)} e^{\phi(sw)} e^{\phi_n(w)} \sum_{t \in \mathfrak{r}(s)} \chi_{[t]}(w) \\
&= e^{-P_g(\phi)} \sum_{t \in \mathfrak{r}(s)} \sum_{\sigma^n w = e_{\mathcal{R}}} e^{\phi(sw)} e^{-nP_g(\phi)} e^{\phi_n(w)} \chi_{[s]}(w)
\end{aligned}$$

and therefore

$$e^{\inf \phi([s]) - P_g(\phi)} \sum_{t \in \mathfrak{r}(s)} \mu\left(\sigma^{-n} e_{\mathcal{R}} \cap [t]\right) \leq \mu\left(\sigma^{-(n+1)} e_{\mathcal{R}} \cap [s]\right) \leq e^{\sup \phi([s]) - P_g(\phi)} \sum_{t \in \mathfrak{r}(s)} \mu\left(\sigma^{-n} e_{\mathcal{R}} \cap [t]\right).$$

We conclude that

$$e^{\inf \phi([s]) - P_g(\phi)} \sum_{t \in \mathfrak{r}(s)} \mu\left(Y_{\mathcal{R}} \cap [t]\right) \leq \mu\left(Y_{\mathcal{R}} \cap [s]\right) - C(s, \mathcal{R}) \leq e^{\sup \phi([s]) - P_g(\phi)} \sum_{t \in \mathfrak{r}(s)} \mu\left(Y_{\mathcal{R}} \cap [t]\right), \tag{3.28}$$

where $C(s, \mathcal{R}) = \mu(\sigma^{-1} e_{\mathcal{R}} \cap [s])$. If there exists $F \subset \mathcal{S}$ finite such that $\cup_{s \in F} \mathfrak{r}(s) = \mathcal{S}$, then

$$\mu(Y_{\mathcal{R}}) \leq \sum_{s \in F} \sum_{t \in \mathfrak{r}(s)} \mu(Y_{\mathcal{R}} \cap [t]) \leq \sum_{s \in F} e^{-\inf \phi([s])} \left[\mu(Y_{\mathcal{R}} \cap [s]) + C(s, \mathcal{R})\right].$$

We conclude that in this case there is an $e^{P_g(\phi)}$-eigenmeasure μ of Ruelle's operator giving finite mass to generalized cyllinder such that $\mu(e_{\mathcal{R}}) > 0$ if, and only if, $\mu(Y_{\mathcal{R}}) < \infty$.

Proposition 16. *Let $\phi : U \to \mathbb{R}$ be a continuous function satisfying Walter's condition.*

Suppose there is $F \subset \mathcal{S}$ finite such that for all $s \in \mathcal{S}$, there is $\tilde{s} \in F$ such that $A(\tilde{s}, s) = 1$. Furthermore, suppose that every root $\mathcal{R}$ is finite, then either of the following happens

1. *There exists a conservative $e^{P_g(\phi)}$-eigenmeasure of Ruelle's operator on Σ that is finite on cyllinders, but there is no $e^{P_g(\phi)}$-eigenmeasure of Ruelle's operator giving mass to F that is finite on generalized cyllinders.*

2. *There is a finite $e^{P_g(\phi)}$-eigenmeasure of Ruelle's operator giving mass to $Y_{\mathcal{R}}$ for any root $\mathcal{R}$, but there is no conservative $e^{P_g(\phi)}$-eigenmeasure of Ruelle's operator on Σ that is finite on cyllinders.*

We have, unfortunately, that, for very well behaved functions, there is no $e^{P_g(\phi)}$-eigenmeasure finite on cyllinders that detects at the same time both the infinite words and the finite words if the matrix A is sufficiently simple (i.e. the hypothesis that all roots are finite and that there is a finite number of symbols covering the whole of $\mathcal{S}$).

3.4 An Existence Theorem for the Compact Case

In this subsection we shall prove a theorem by Denker and Yuri [DY15] which guarantees the existence of a finite $e^{P(\phi,x)}$-eigenmeasure of the Ruelle's operator in the case where X is compact, i.e., the case where 0 is not an accumulation point of $\mathbf{S}(\mathcal{S})$. This gives us a new criterion to check for the existence of a finite $e^{P_g(\phi)}$-eigenmeasure on Σ. More specifically, if

$$\sum_{n=1}^{\infty} e^{-nP_g(\phi)} \sum_{\sigma^n y = e_{\mathcal{R}}} e^{\phi_n(y)} = \infty$$

for all root $\mathcal{R}$, then there is a finite $e^{P_g(\phi)}$-eigenmeasure of the usual Ruelle's operator on Σ.

We will need the following technical lemma from [DU91].

Lemma 5. *Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers such that $\limsup_{n\to\infty} \frac{a_n}{n} = c < \infty$. Then, there exists a positive sequence $\{b_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \frac{b_n}{b_{n+1}} = 1$ and*

$$\sum_{n\in\mathbb{N}} b_n \exp a_n - nt < \infty \iff t > c.$$

Remark 16. *Note that by construction $\lim_{t\to c^+} \sum_{n\in\mathbb{N}} b_n \exp a_n - nt = \infty$.*

Suppose that $P(\phi, x) < \infty$. Note that this implies that $Z_n(\phi, x) < \infty$ for all $n \in \mathbb{N}$. Let us apply the lemma above to the sequence $\{a_n\}_{n\in\mathbb{N}}$ given by

$$a_n = \log Z_n(\phi, x). \tag{3.29}$$

There exists a positive sequence $\{b_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \frac{b_n}{b_{n+1}} = 1$ and

$$M(t, x) \doteq \sum_{n\in\mathbb{N}} b_n \exp a_n - nt = \sum_{n\in\mathbb{N}} b_n e^{-nt} Z_n(\phi, x) < \infty \iff t > P(\phi, x). \tag{3.30}$$

For $t > P(\phi, x)$, let us define the following probability measure on X

$$m(t, x) \doteq M(t, x)^{-1} \sum_{n\in\mathbb{N}} b_n e^{-nt} \sum_{\sigma^n y = x} e^{\phi_n(y)} \delta_y. \tag{3.31}$$

Then, for any $f \in L^\infty(X)$, we have that

$$\begin{aligned} m(t,x)(f) &= M(t,x)^{-1} \sum_{n\in\mathbb{N}} b_n e^{-nt} \sum_{\sigma^n y = x} e^{\phi_n(y)} \delta_y(f) \\ &= M(t,x)^{-1} \sum_{n\in\mathbb{N}} b_n e^{-nt} \sum_{\sigma^n y = x} e^{\phi_n(y)} f(y) \\ &= M(t,x)^{-1} \sum_{n\in\mathbb{N}} b_n e^{-nt} L_\phi^n f(x). \end{aligned}$$

If $g \in L^\infty(U)$, we get that

$$\begin{aligned} m(t,x)(L_\phi g) &= M(t,x)^{-1} \sum_{n\in\mathbb{N}} b_n e^{-nt} L_\phi^n (L_\phi g)(x) \\ &= M(t,x)^{-1} \sum_{n\in\mathbb{N}} (b_{n+1} - b_{n+1} + b_n) e^{-nt} L_\phi^{n+1} g(x) \\ &= M(t,x)^{-1} e^t \sum_{n\in\mathbb{N}} b_n e^{-nt} L_\phi^n g(x) - M(t,x)^{-1} b_1 L_\phi g(x) \\ &+ M(t,x)^{-1} \sum_{n\in\mathbb{N}} \left(\frac{b_n}{b_{n+1}} - 1 \right) b_{n+1} e^{-nt} L_\phi^{n+1} g(x) \\ &= M(t,x)^{-1} \sum_{n\in\mathbb{N}} \left(\frac{b_n}{b_{n+1}} - 1 \right) b_{n+1} e^{-nt} L_\phi^{n+1} g(x) \\ &+ e^t m(t,x)(g) - M(t,x)^{-1} b_1 L_\phi g(x). \end{aligned}$$

Note that

$$\lim_{t\to P(\phi,x)^+} M(t,x)^{-1} b_1 L_\phi g(x) = 0$$

due to the remark under the lemma. Let us now analyze the other term, given $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|\frac{b_n}{b_{n+1}} - 1| < \varepsilon$ if $n > N_\varepsilon$. Therefore,

$$\left| \left(\frac{b_n}{b_{n+1}} - 1 \right) b_{n+1} e^{-nt} L_\phi^{n+1} g(x) \right| \le \epsilon e^t \|g\|_\infty b_{n+1} e^{-(n+1)t} Z_{n+1}(\phi,x)$$

if $n > N_\varepsilon$. On the other hand,

$$\sum_{n=1}^{N_\varepsilon} \left| \frac{b_n}{b_{n+1}} - 1 \right| b_{n+1} e^{-nt} \|g\|_\infty Z_n(\phi,x)$$

is bounded as t approaches $P(\phi,x)$ from above. Hence,

$$\limsup_{t\to P(\phi,x)^+} \left| M(t,x)^{-1} \sum_{n\in\mathbb{N}} \left(\frac{b_n}{b_{n+1}} - 1 \right) b_{n+1} e^{-nt} L_\phi^{n+1} g(x) \right| \le \varepsilon e^{P(\phi,x)} \|g\|_\infty$$

for all $\varepsilon > 0$. We conclude that

$$\lim_{t\to P(\phi,x)} m(t,x)(L_\phi g) - e^{P(\phi,x)} m(t,x)(g) = 0 \tag{3.32}$$

for all $g \in L^\infty(U)$.

Take any sequence $\{t_n\}_{n\in\mathbb{N}}$ decreasing to $P(\phi, x)$. Since X is compact, the sequence of probability measures $\{m(t_n, x)\}_{n\in\mathbb{N}}$ has as accumulation point m. The computation above shows us that m is an $e^{P(\phi,x)}$-eigenmeasure of Ruelle's operator.

Theorem 6. *Suppose that X is compact and $P(\phi, x) < \infty$ for some $x \in X$. Then, there exists an $e^{P(\phi,x)}$-eigenmeasure of Ruelle's operator.*

3.5 Relaxing the Continuity Condition

In this subsection, we will show that it is rather easy to construct potentials $\phi : U \to \infty$ that, while not continuous, satisfy

$$\mathrm{Var}_1\phi < \infty \text{ and } \lim_{n\to\infty} \mathrm{Var}_n\phi = 0$$

and such that there are $e^{P_g(\phi)}$-eigenmeasures giving mass to both finite and infinite words. We will do so by studying ways to extend product type potentials, such as those investigated in [CDLS17], to the finite words of the Generalized Full Shift that respect the conditions above.

Definition 23. *Let $\phi : \Sigma \to \mathbb{R}$, we say it is a product type potential if there exists a family of functions $(\varphi_k)_{k\in\mathbb{N}_0}$ such that*

$$\phi(x) = \sum_{k\in\mathbb{N}_0} \varphi_k(x_k). \tag{3.33}$$

For $n \geq 1$, we define $\phi^{(n)} : \Sigma \to \mathbb{R}$ by

$$\phi^{(n)}(x) = \sum_{k\in\mathbb{N}_0} \varphi_{k+n}(x_k). \tag{3.34}$$

In general, we will suppose that

$$\sum_{k=1}^{\infty} \|\varphi_k\|_\infty < \infty \tag{3.35}$$

Lemma 6. *Let $\phi : \Sigma \to \mathbb{R}$ be a product type potential. Then, ϕ admits a continuous extension to U if, and only if, there exist $\varphi_0, \varphi_1 : \mathbb{N} \to \mathbb{R}$ such that*

1. $\lim_{n\to\infty} \varphi_1(n) = 0$,
2. $F(x) = \varphi_0(x_0) + \varphi_1(x_1)$.

Proof. Suppose $\phi : \Sigma \to \mathbb{R}$ is given by

$$\phi(x) = \sum_{k\geq 0} \varphi_k(x_k),$$

where $\varphi_k : \mathbb{N} \to \mathbb{R}$ for $k \geq 0$.

Let $s \in \mathbb{N}$ and $x \in \Sigma$, then the sequence $(anx)_{n\in\mathbb{N}}$ converges to $s \in F$ in the topology of X, for any $x \in \Sigma$. Hence, for ϕ to extend continuously to U, it is necessary that

$$\lim_{n\to\infty} \phi(anx) = \lim_{n\to\infty} \left(\varphi_0(s) + \varphi_1(n) + \phi^{(2)}(x)\right) = \varphi_0(a) + \phi^{(2)}(x) + \lim_{n\to\infty} \varphi_1(n)$$

exists and does not depend on $x \in \Sigma$. Therefore, $\phi^{(2)}$ is constant and $lim_{n\to\infty}\varphi_1(n)$ exists.

Arguing in the same manner for the sequence $(abnx)_{n\in\mathbb{N}}$, where $a, b \in \mathbb{N}$ and $x \in X$, we obtain that $\phi^{(3)}$ is constant and that $lim_{n\to\infty}\varphi_2(n)$ exists. On the other hand, it is true that

$$\phi^{(2)}(nx) = \phi^{(3)}(x) + \varphi_2(n).$$

We conclude that φ_2 is constant. An argument by induction gives us that φ_n is constant for $n \geq 2$.

By absorbing a constant to the definitions of φ_0 and φ_1, we conclude that for a product type potential on Σ to extend continuously to U, it is necessary that it be given by

$$\phi(x) = \varphi_0(x_0) + \varphi_1(x_1)$$

with $\lim_{n\to\infty} \varphi_1(n) = 0$. It is indeed sufficient, $\tilde{\phi} : U \to \mathbb{R}$ given by

$$\tilde{\phi}(x) = \begin{cases} \varphi_0(x_0), & \text{if } |x| = 1 \\ \varphi_0(x_0) + \varphi_1(x_1), & \text{if } |x| \geq 2. \end{cases}$$

is the continuous extension of ϕ to U. $\square$

Proposition 17. *Let $\phi : \Sigma \to \mathbb{R}$ be a product type potential with summable variations and finite Gurevich pressure, then ϕ is positive recurrent and there is a probability measure ν that is an $e^{P_g(\phi)}$-eigenmeasure of Ruelle's operator.*

Proof. Let $s \in \mathbb{N}$, then

$$\begin{aligned} Z_n(\phi, [s]) &= \sum_{\substack{\sigma^n(x)=x \\ x_0=s}} \exp(\phi_n(x)) = \sum_{\substack{\sigma^n(x)=x \\ x_0=s}} \prod_{k=0}^{n-1} \exp\left(\sum_{m\geq 0} \varphi_m(x_k)\right) \\ &= \exp\left(\sum_{m\geq 0} \varphi_m(s)\right) \left(\sum_{l\in\mathbb{N}} \exp\left(\sum_{m\geq 0} \varphi_m(l)\right)\right)^{n-1} \end{aligned}$$

and therefore

$$P_g(\phi) = \log \sum_{l\in\mathbb{N}} \exp\left(\sum_{m\geq 0} \varphi_m(l)\right). \tag{3.36}$$

Finally,

$$\sum_{n\geq 1} e^{-nP_g(\phi)} Z_n(\phi, [s]) = \sum_{n\geq 1} e^{-nP_g(\phi)} e^{\phi(\bar{s})} e^{(n-1)P_g(\phi)} = e^{\phi(\bar{s}) - P_g(\phi)} \sum_{n\geq 1} 1 = \infty$$

and ϕ is recurrent.

A very similar calculation yields

$$Z_n^*(\phi, [s]) = \exp\left(\sum_{m\geq 0} \varphi_m(s)\right) \left(\sum_{l\in\mathbb{N}\setminus\{s\}} \exp\left(\sum_{m\geq 0} \varphi_m(l)\right)\right)^{n-1} = e^{\phi(\bar{s}) + C(n-1)}$$

and therefore

$$\sum_{n\geq 1} ne^{-nP_g(\phi)} Z_n^*(\phi,[s]) = e^{\phi(\bar{s})-C} \sum_{n\geq 1} ne^{n(C-P_g(\phi))} < \infty$$

since $C < P_g(\phi)$.

We conclude that ϕ is positive recurrent and therefore there is a σ-finite measure ν such that $L_\nu^*\nu = e^{P_g(\phi)}\nu$ and $\nu([s]) < \infty$ for all $s \in \mathbb{N}$. Let $s \in \mathbb{N}$, then

$$\infty > e^{P_g(\phi)}\nu([s]) = \int_\Sigma \exp(\phi(sx))\nu(\mathrm{d}x) = e^{\varphi_0(s)} \int_\Sigma \exp(\phi^{(1)}(x))\nu(\mathrm{d}x),$$

hence

$$\int_\Sigma \exp(\phi^{(1)}(x))\nu(\mathrm{d}x) = M < \infty$$

and

$$\nu(\Sigma) = \sum_{a\in\mathbb{N}} \nu([a]) = \sum_{a\in\mathbb{N}} Me^{\phi_0(a)-P_g(\phi)} = Me^{-P_g(\phi)} \sum_{a\in\mathbb{N}} e^{\phi_0(a)} < \infty.$$

□

Corollary 9. *Let $\phi : \Sigma \to \mathbb{R}$ be a product type potential. If ϕ admits a continuous extension to U, then there are no finite $e^{P_g(\phi)}$-eigenmeasures of the Ruelle operator giving mass to finite words.*

Proof. This corollary is a direct consequence of the observation that

$$Z_n(\phi, e) = e^{(n-1)P_g(\phi)} \sum_{k\in\mathbb{N}} \exp(\varphi_0(k)).$$

in which case $P(\phi, e) = P_g(\phi)$ and

$$\sum_{n=1}^{\infty} e^{-nP(\phi,e)} Z_n(\phi, e) = \sum_{n=1}^{\infty} e^{-nP_g(\phi)} Z_n(\phi, e) = e^{-P_g(\phi)} \sum_{n=1}^{\infty} 1 = \infty.$$

□

It should be clear that there are many different sequences of functions $\Phi = \{\varphi_n : n \in \mathbb{N}_0\}$ that yield a potential ϕ, indeed, given any absolutely summable sequence of real numbers $\mathrm{a} = (a_n)_{n\in\mathbb{N}_0}$ such that

$$\sum_{n=0}^{\infty} a_n = 0,$$

then $\Phi = (\varphi_n)_{n\in\mathbb{N}_0}$ yields ϕ if, and only if, so does $\Phi^{\mathrm{a}} = (\varphi_n^{\mathrm{a}})_{n\in\mathbb{N}_0}$, where

$$\varphi_n^{\mathrm{a}}(k) = \varphi_n(k) + a_n.$$

The converse is also true: given two sequences of functions Φ and Φ' yielding ϕ, then $\varphi_m - \varphi_m'$

is constant for all $m \in \mathbb{N}_0$,

$$\sum_{n=0}^{\infty} a_n \doteq \sum_{n=0}^{\infty} [\varphi_n(1) - \varphi'_n(1)] = \phi(\overline{1}) - \phi(\overline{1}) = 0.$$

and

$$\sum_{n=0}^{\infty} |a_n| \leq |\varphi_0(1) - \varphi'_0(1)| + \sum_{n=1}^{\infty} (\|\varphi_n\|_\infty + \|\varphi'_n\|_\infty) < \infty.$$

As a matter of convention, given any product type potential ϕ, we shall fix a sequence of functions Φ that best relates to the variations of ϕ, *i.e,*

$$\mathrm{Var}_n \phi = \sum_{m=n}^{\infty} \|\varphi_m\|_\infty. \tag{3.37}$$

We shall now consider the natural yet discontinuous extension of ϕ to U given by truncating the sum of the functions defining it. This extension shall depend on the sequence of functions Φ^{a} yielding ϕ and shall therefore be denoted by ϕ^{a}. We will show that the pressure of ϕ^{a} in the generalized setting is equal to $P_g(\phi)$ and that by carefully choosing $(a_n)_{n \in \mathbb{N}_0}$ we will be able to produce a non-trivial finite $e^{P_g(\phi)}$-eigenmeasure of Ruelle's operator giving mass to F.

Definition 24. *Let $x \in F$, we define the truncated extension of ϕ associated to Φ^a at x by*

$$\phi^a(x) \doteq \sum_{k=0}^{|x|-1} \varphi_k^a(x_k). \tag{3.38}$$

Remark 17. *It should be clear that the absolute summability of both $\{\|\varphi_n\|\}_{n\in\mathbb{N}}$ and $a = \{a_n\}_{n\in\mathbb{N}}$ implies that both $\mathrm{Var}_1\phi^a < \infty$ and $\lim_{n\to\infty} \mathrm{Var}_n\phi^a = 0$.*

Proposition 18. *Let $x \in F \sqcup E$, then*

$$P(\phi^a, x) = P_g(\phi). \tag{3.39}$$

In particular, the pressure does not depend on which truncated extension we choose.

Proof. Let $x \in F$, then

$$\begin{aligned}
Z_n\left(\phi^{\mathrm{a}}, x\right) &= \sum_{\sigma^n y = x} \exp \phi^{\mathrm{a}}_n(y) = \sum_{\sigma^n y = x} \exp\left(\sum_{j=0}^{n-1} \phi^{\mathrm{a}}\left(\sigma^j y\right)\right) \\
&= \sum_{\sigma^n y = x} \exp\left(\sum_{j=0}^{n-1} \sum_{m=0}^{|x|+n-j-1} \varphi^{\mathrm{a}}_m\left(y_{m+j}\right)\right) \\
&= \sum_{\sigma^n y = x} \exp\left[\sum_{j=0}^{n-1}\left(\sum_{m=0}^{|x|-1} \varphi^{\mathrm{a}}_{m+n-j}\left(x_m\right) + \sum_{m=0}^{n-j-1} \varphi^{\mathrm{a}}_m\left(y_{m+j}\right)\right)\right] \\
&= \left[\prod_{j=0}^{n-1} \prod_{m=0}^{|x|-1} \exp \varphi^{\mathrm{a}}_{m+n-j}\left(x_m\right)\right]\left[\sum_{\sigma^n y = x} \exp\left(\sum_{j=0}^{n-1} \sum_{m=0}^{j} \varphi^{\mathrm{a}}_m\left(y_j\right)\right)\right] \\
&= \left[\prod_{j=0}^{n-1} \prod_{m=0}^{|x|-1} \exp \varphi^{\mathrm{a}}_{m+n-j}\left(x_m\right)\right] \prod_{j=0}^{n-1}\left[\sum_{k \in \mathbb{N}} \prod_{m=0}^{j} \exp \varphi^{\mathrm{a}}_m(k)\right].
\end{aligned}$$

Let $b_n \doteq \sum_{m=0}^{|x|-1} \varphi^{\mathrm{a}}_{m+n-j}\left(x_m\right)$, then $\lim_{n\to\infty} b_n = 0$ and therefore

$$\lim_{n\to\infty} \frac{1}{n} \log \prod_{j=0}^{n-1} \prod_{m=0}^{|x|-1} \exp \varphi^{\mathrm{a}}_{m+n-j}\left(x_m\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} b_j = \lim_{n\to\infty} b_n = 0.$$

On the other hand, by Lebesgue's dominated convergence theorem

$$\lim_{n\to\infty} \sum_{k\in\mathbb{N}} \prod_{m=0}^{n} \exp \varphi^{\mathrm{a}}_m(k) = \sum_{k\in\mathbb{N}} \prod_{m=0}^{\infty} \exp \varphi^{\mathrm{a}}_m(k) = \sum_{k\in\mathbb{N}} \prod_{m=0}^{\infty} \exp \varphi_m(k) = e^{P_g(\phi)}$$

and therefore

$$\lim_{n\to\infty} \frac{1}{n} \log \prod_{j=0}^{n-1}\left[\sum_{k\in\mathbb{N}} \prod_{m=0}^{j} \exp \varphi^{\mathrm{a}}_m(k)\right] = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \sum_{k\in\mathbb{N}} \prod_{m=0}^{j} \exp \varphi^{\mathrm{a}}_m(k) = P_g(\phi).$$

Finally, we conclude that

$$P\left(\phi^{\mathrm{a}}, x\right) = \limsup_{n\to\infty} \frac{1}{n} \log Z_n\left(\phi^{\mathrm{a}}, x\right) = P_g(\phi).$$

□

We are now interested in finding a sequence of functions Φ^{a} defining ϕ for which

$$\sum_{n=1}^{\infty} e^{-nP_g(\phi)} Z_n\left(\phi^{\mathrm{a}}, e\right) < \infty. \tag{3.40}$$

In order to do so, the following simple lemma is very useful.

Lemma 7. *We have for all $n \in \mathbb{N}$ that*

$$Z_n\left(\phi^a, e\right) = Z_n\left(\phi^0, e\right) \exp\left(-\sum_{j=1}^{n} \sum_{m=j}^{\infty} a_m\right) \tag{3.41}$$

and

$$\exp\left(-\sum_{j=1}^{n} \sum_{m=j}^{\infty} \|\varphi_m\|_\infty\right) \leq e^{-nP_g(\phi)} Z_n\left(\phi^0, e\right) \leq \exp\left(\sum_{j=1}^{n} \sum_{m=j}^{\infty} \|\varphi_m\|_\infty\right). \tag{3.42}$$

In particular, if ϕ has summable variations, there exists $C \geq 1$ such that

$$C^{-1} \leq e^{-nP_g(\phi)} Z_n\left(\phi^0, e\right) \leq C. \tag{3.43}$$

Proof. Let $n \in \mathbb{N}$, then

$$\begin{aligned} Z_n\left(\phi^{\mathrm{a}}, e\right) &= \sum_{\sigma^n y = e} \exp \phi^{\mathrm{a}}(y) = \prod_{j=0}^{n-1} \left[\sum_{k \in \mathbb{N}} \prod_{m=0}^{j} \exp \varphi_m^{\mathrm{a}}(k)\right] \\ &= \left\{\prod_{j=0}^{n-1} \left[\sum_{k \in \mathbb{N}} \prod_{m=0}^{j} \exp \varphi_m(k)\right]\right\} \left[\prod_{j=0}^{n-1} \prod_{m=0}^{j} \exp a_m\right] \\ &= Z_n\left(\phi^0, e\right) \exp\left(-\sum_{j=1}^{n} \sum_{m=j}^{\infty} a_m\right). \end{aligned}$$

On the other hand, for all $k \in \mathbb{N}$ we have

$$\exp\left(-\sum_{m=j+1}^{\infty} \|\varphi_m\|_\infty\right) \leq \left[\prod_{m=0}^{\infty} \exp \varphi_m(k)\right]^{-1} \left[\prod_{m=0}^{j} \exp \varphi_m(k)\right] \leq \exp\left(\sum_{m=j+1}^{\infty} \|\varphi_m\|_\infty\right)$$

and therefore

$$\exp\left(-\sum_{j=1}^{n} \sum_{m=j}^{\infty} \|\varphi_m\|_\infty\right) \leq e^{-nP_g(\phi)} Z_n\left(\phi^0, e\right) \leq \exp\left(\sum_{j=1}^{n} \sum_{m=j}^{\infty} \|\varphi_m\|_\infty\right).$$

Finally, if ϕ has summable variations, it is sufficient to take

$$C = \exp\left(\sum_{j=1}^{\infty} \sum_{m=j}^{\infty} \|\varphi_m\|_\infty\right).$$

$\square$

Using the bounds established by the lemma above, it becomes really easy to make a choice of $\mathrm{a} = \{a_n\}_{n \in \mathbb{N}_0}$ giving us a finite $e^{P_g(\phi)}$-eigenmeasure of Ruelle's operator giving mass to finite words. Indeed, let $\mathrm{a} = (a_n)_{n \in \mathbb{N}_0}$ be given by

$$a_n = \begin{cases} -1 - \sum_{m=1}^{\infty} \|\varphi_n\|_\infty, & \text{if } n = 0 \\ \frac{1}{n^{\frac{1}{2}}} - \frac{1}{(n+1)^{\frac{1}{2}}} + \|\varphi_n\|_\infty, & \text{if } n \in \mathbb{N}. \end{cases}$$

Then,

$$\sum_{j=1}^{n}\sum_{m=j}^{\infty}\|\varphi_m\|_\infty - \sum_{j=1}^{n}\sum_{m=j}^{\infty} a_m = -\sum_{j=1}^{n}\sum_{m=j}^{\infty}\left(\frac{1}{m^{\frac{1}{2}}} - \frac{1}{(m+1)^{\frac{1}{2}}}\right) = -\sum_{j=1}^{n}\frac{1}{j^{\frac{1}{2}}}$$
$$\leq -\int_1^n \frac{\mathrm{d}x}{x^{\frac{1}{2}}} = -\frac{1}{2}\left[n^{\frac{1}{2}} - 1\right] = \frac{1}{2} - \frac{1}{2}n^{\frac{1}{2}}.$$

Hence

$$e^{-nP_g(\phi)} Z_n\left(\phi^{\mathrm{a}}, e\right) \leq \exp\left(\frac{1}{2} - \frac{1}{2}n^{\frac{1}{2}}\right)$$

which is absolutely summable.

We have thus succeeded, in a rather crude manner, in finding a sequence of functions Φ^{a} defining ϕ for which there are two distinct finite $e^{P_g(\phi)}$-eigenmeasures of Ruelle's operator with disjoint supports, namely the usual one supported on Σ and a new one supported on Y. In fact, a similar construction is possible for all Walter's potentials ϕ on Σ arising from a sequence of functions $\Phi = \{\varphi_n\}_{n\in\mathbb{N}_0}$ such that $\mathrm{Var}_{n+1}\varphi_n = 0$ (this is NOT a Birkhoff sum) and such that $\{\|\varphi_n\|_\infty\}_{n\in\mathbb{N}}$ is absolutely summable.

Chapter 4

Conclusions

We have not yet given up on the idea that there might be a continuous potential $\phi : U \to \mathbb{R}$ for which both definitions of pressure are in accordance and such that there is an $e^{P_g(\phi)}$-eigenmeasure giving mass to both Σ and F that is finite on the generalized cyllinders. and we shall provide a possible route to construct such a potential in this section. We will atempt to do so on the Generalized Renewal Shift.

We begin by finding a family of functions more appropriate to tackle the problem that describes a subset (it is important to know that this choice of functions will NOT separate all points of U) of continuous functions on the Generalized Renewal Shift than the one provided some sections ago. Let us define the "maximum" function $\mathrm{M} : \cup_{n\in\mathbb{N}}\mathbb{N}^n \to \mathbb{N}$, where

$$\mathrm{M}(\omega) = \max\{\omega_k : 0 \leq k < |\omega|\}. \tag{4.1}$$

Let $a : \mathbb{N} \to \mathbb{R}$ be a strictly decreasing positive sequence converging to 0. We define $h_n : U \to \mathbb{R}$ by

$$h_n(x) = \begin{cases} a(M(x[0,n))), & \text{if } |x| \geq n, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

It should be evident at this point that such functions are continuous. We consider the closure of the following subalgebra of continuous functions on the generalized cyllinder $[s]$

$$A = \mathbb{R}[\{1\} \cup \{h_n : n \in \mathbb{N}\}]. \tag{4.3}$$

We are therefore looking at continuous potentials $\phi : U \to \mathbb{R}$ of the form

$$\phi(x) = \sum_{s\in\mathbb{N}} \sum_{n=0}^{\infty} c(n,s)\chi_{[s]}(x)h_n(x) \tag{4.4}$$

such that for all $s \in \mathbb{N}$

$$\sum_{n=1}^{\infty} |c(n,s)| < \infty.$$

We recall that there is a canonical bijection between the set of finite words of length n given by $\sigma^{-n}e$ and $\mathrm{Per}(1,n) = \{y \in \Sigma : y_0 = 1, \sigma^n = y\}$, the set of periodic words starting 1 with period n, which is given by

$$\omega \longleftrightarrow \overline{1\omega[0,n-1)}. \tag{4.5}$$

Let us estimate $\phi_n(\overline{1\omega[0,n-1)}) - \phi_n(\omega) = \phi_n(\overline{\omega}) - \phi_n(\omega)$ supposing that $c(n,s) \geq 0$ for all $n \geq 1$ and $s \in \mathbb{N}$. We have that

$$\phi_n(\omega) = \sum_{k=0}^{n-1} \left[\sum_{s\in\mathcal{S}} \sum_{l=0}^{\infty} c(l,s)\chi_{[s]}(\sigma^k\omega)h_n(\sigma^k\omega) \right] = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1-k} c(l,\omega_k)a(M(\omega[k,k+l)))$$

and

$$\begin{aligned}\phi_n(\overline{\omega}) &= \sum_{k=0}^{n-1} \left[\sum_{s\in\mathcal{S}} \sum_{l=0}^{\infty} c(l,s)\chi_{[s]}(\sigma^k\overline{\omega})h_n(\sigma^k\overline{\omega}) \right] = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1-k} c(l,\omega_k)a(M(\overline{\omega}[k,k+l))) \\ &= \phi_n(\omega) + \sum_{k=0}^{n-1} \sum_{l=n-k}^{\infty} c(l,\omega_k)a(M(\overline{\omega}[k,k+l))).\end{aligned}$$

Noting that $a(M(\overline{\omega}[k,k+l))) \geq a(M(\omega)) \geq a(n)$, where the second inequality follows from the structure of the renewal shift, we get that

$$\phi_n(\overline{\omega}) - \phi_n(\omega) \geq a(n) \sum_{k=0}^{n-1} \sum_{l=n-k}^{\infty} c(l,\omega_k) \tag{4.6}$$

and therefore

$$Z_n(\phi,[1]) \geq Z_n(\phi,e) \exp a(n) \sum_{k=0}^{n-1} \sum_{l=n-k}^{\infty} c(l,\omega_k). \tag{4.7}$$

Our hope is that by choosing $c(n,s)$ well we should be able to construct a potential with pressure 0 (on both definitions) for which

$$\sum_{n=1}^{\infty} Z_n(\phi,e) < \infty$$

and

$$\sum_{n=1}^{\infty} Z_n(\phi,e) \exp a(n) \sum_{k=0}^{n-1} \sum_{l=n-k}^{\infty} c(l,\omega_k) = \infty.$$

In general, it is not very hard to produce a sequence of measures that would converge to an $e^{P_g(\phi)}$-eigenmeasure on Σ, it is sufficient to follow a recipe similar to that present in the proof of Denker-Yuri's theorem. The problem lies in proving that there actually exists an accumulation point to that sequence. In the standard literature, for example in Sarig's Lecture Notes on the Thermodynamic Formalism for Topological Markov Shifts [SAR09], the proof that the natural sequence of measures satisfies the hypotheses of Helly-Prohorov's theorem heavily relies on the Walter's condition of the potential. Our hope is that given the very simple dynamic properties of the renewal shift and the sufficiently strict structure of the potential, it is possible to prove that the candidate sequence satisfies Helly-Prohorov's theorem and hence obtain a finite eigenmeasure giving mass to both finite and infinite words. This is most likely how our research on this type of local compactification of the usual countable Markov shifts will proceed, attempting to construct a potential whose eigenmeasures see both the infinite and finite words.

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