Problemas de evolução com acoplamento local/não local

Evolution problems with local/nonlocal coupling

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Resumo

DOS SANTOS, B. C. **Problemas de evolução com acoplamento local/não local**. 2021. 127 f. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2021.

Modelos clássicos, como Equações Diferenciais Parciais (EDPs), são amplamente usados para fazer aproximações locais, mesmo tendo algumas limitações para capturar efeitos de longo alcance. Por outro lado, a modelagem de efeitos não locais está recebendo atenção em muitas áreas aplicadas, como ecologia, epidemiologia, física e engenharia. O desenvolvimento de uma estrutura teórica e computacional rigorosa para modelos não locais ainda está em desenvolvimento em contrapartida a teoria local.

Neste trabalho, propomos e estudamos um problema de evolução que acopla equações locais e não locais. A parte local é classicamente representada pelo operador Laplaciano, enquanto a parte não local é representada pelo operador de difusão com um núcleo integrável em forma de convolução, $J(x-y)$. Como uma primeira aproximação, estudamos as propriedades do modelo no caso unidimensional. Resultados de existência, unicidade, conservação de massa e decaimento assintótico das soluções foram verificados. A seguir, estendemos esses resultados para dimensões mais altas. Para o caso unidimensional, com o reescalonamento adequado do núcleo não local, é possível recuperar a equação do calor em todo o domínio. Em seguida, continuando nossa análise e, aproveitando as vantagens da estrutura de acoplamento particular, usamos o método Operador de Divisão para fornecer uma prova diferente de existência e unicidade de soluções. Além disso, desenvolvemos alguns experimentos numéricos para ilustrar os resultados teóricos obtidos. Usando métodos numéricos clássicos para EDPs, verificamos que a solução do modelo discreto converge para o valor médio da condição inicial (quando assumimos condições de contorno do tipo Neumann), como mostramos teoricamente. Finalmente, estudamos as propriedades do problema de evolução em um domínio fino. Consideramos o caso limite quando o subdomínio não local é estreitado em uma direção, fazendo com que o domínio não local se concentre em um conjunto de dimensão mais baixa. Dessa forma, obtemos um modelo no qual as partes locais e não locais do problema são definidas em subdomínios de dimensões distintas. Também mostramos que o problema limite compartilha as mesmas propriedades obtidas no caso unidimensional; existência e unicidade, conservação de massa, comparação e decaimento assintótico de soluções, para t suficientemente grande.

Palavras-chave: Difusão não local, equação do calor, comportamento assintótico, métodos numéricos, domínios finos.

Abstract

DOS SANTOS, B. C. **Evolution problems with local/nonlocal coupling**. 2021. 127 f. Thesis (Doctorate) - Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2021.

Classical models, such as Partial Differential Equations (PDE), are widely used for making local approximations even if they have some limitations for capturing long-range effects. On the other side, the modeling of nonlocal effects is getting attention in many applied areas, like ecology, epidemiology, physics, and engineering. The development of a rigorous theoretical and computational framework for nonlocal models is far less developed than its local counterpart.

In this work, we propose and study an evolution problem that couple local and nonlocal equations. The local part is classically represented by the Laplacian operator, while the nonlocal part is represented by a diffusion operator with an integrable kernel in convolution form, $J(x-y)$. As a first approximation, we study the properties of the model in the one-dimensional case. Results of existence, uniqueness, mass conservation, and asymptotic decay of solutions were verified. Next, we extend these results to higher dimensions. For the one-dimensional case, with the appropriate rescale of the nonlocal kernel, it is possible to recover the heat equation in the whole domain. Next, we continue our analysis of this coupled problem and, taking advantage of the particular coupling structure, we use the Splitting Operator method to provide a different proof of existence and uniqueness. We also develop some numerical experiments to illustrate the obtained theoretical results. Using classical numerical methods for PDE, we check that the solution of the discrete model converges to the mean value of the initial condition (when we assume Neumann type boundary conditions), as we have shown theoretically. Finally, we study the properties of the evolution problem in a thin domain. We consider the limit case when the nonlocal subdomain is narrowed in one direction, making the nonlocal domain concentrates in a set of smaller dimension. In this way, we obtain a model in which the local and nonlocal parts of the problem are defined in subdomains of different dimensions. We also show that the limit problem shares the same properties obtained in the one-dimensional case; existence and uniqueness, mass conservation, comparison, and asymptotic decay of solutions for large times.

Keywords: Nonlocal diffusion, heat equation, asymptotic behavior, numerical methods, thin domains.

Contents

Chapter 1

Introduction

Diffusion problems appear frequently, as they represent a natural process that occurs in various situations of our daily life, such as, for example, the uniform spreading of a drop of ink in a container with water, or even the disperse of species within an ecological environment. In mathematical epidemiology, for example, diffusion models are used to map the dynamics of the spreading diseases. This type of modeling allows the introduction of relevant information that will help us to understand the patterns of the disease, like how the vector mobility affects the spreading.

In this sense, parabolic equations are widely used to model diffusion phenomena in applied sciences. Probably the most studied diffusion equation is the classical heat equation,

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t),$$

that is associated with a stochastic process, the Brownian motion, that describes the random movement of a particle. This equation is linear and also a local PDE, since to verify that a function u is a solution at a certain point (x,t) one only needs to know u in any neighborhood of (x,t). For a general reference we refer to Evans (1998).

On the other hand, most of the problems in nature have more complex structures and behaviors, which makes difficult to capture their dynamics in all its richness of details using linear and local evolution PDEs like the heat equation. More precisely, long-range interactions cannot be captured by classical diffusion. One way of considering this lack of local behaviour in the environment is by using nonlocal kernels.

Concerning nonlocal evolution equations, one popular choice is given by a convolution type equation in space, that is,

$$\frac{\partial u}{\partial t}(x,t) = \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy,$$

where $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative kernel (this kind of equation include the widely studied fractional Laplacians). Here the equation at a point x and time t depends on the values of the unknown u at all points in the set $x + \operatorname{supp} J$, which is what makes the equation nonlocal. This kind of evolution equation is associated to jump processes. Evolution equations of this form and variations of it have been recently used to model diffusion processes; see for instance Bates e Chmaj (1999); Carrillo e Fife (2005); Cortazar *et al.* (2007); D'Elia *et al.* (2017); Fife (2003); Sastre Gómez (2014); Wang (2002); Zhang (2004).

For example, in population models used in biology, if $u(x,t)$ is thought of as the density of a population at the point x at time t, and $J(x-y)$ is regarded as the probability distribution of jumping from location y to location x, then the rate at which individuals are arriving to position x from all other places is given by $\int_{\mathbb{R}^N} J(y-x)u(y,t)\,\mathrm{d}y$, while the rate at which they are leaving location x to travel to all other sites is given by $-\int_{\mathbb{R}^N} J(y-x)u(x,t)\,\mathrm{d}y$. Therefore, in the absence of external or internal sources, the density u satisfies the nonlocal diffusion equation, see Fife (2003).

Besides of applications in ecology, this kind of equation is getting attention in other fields in natural sciences as biology, physics, and engineering, due to its flexibility to accurately capture effects that are not easily obtained from classical local models. Biological mobility models of animals and plants are examples of how distinct patterns of mobility can affect the success of invasions Berestycki *et al.* (2015); Strickland *et al.* (2014). In epidemiology, the effects of long-range interactions are responsible for the spreading of diseases around the world Wang e Zhao (2011). Nonlocal patterns also play an important role in molecular interactions in dissimilar interfaces, continuum mechanics, Han e Lubineau (2012); Seleson *et al.* (2013), and peridynamics (a model of elasticity and mechanics), Silling (2000); Silling e Lehoucq (2010).

In this work, we focus on nonlocal equations with smooth kernels (therefore, fractional Laplacians are not considered). For applications of nonlocal equations with non-singular kernels we refer to nonlocal continuum theories such as peridynamics, Silling e Lehoucq (2010), physics-based nonlocal elasticity, Di Paola *et al.* (2009), and nonlocal descriptions resulting from homogenization of nonlinear damage models Han e Lubineau (2012). For general references concerning nonlocal evolution equations we refer the book Andreu-Vaillo *et al.* (2010) and references therein.

The two previous models (local or nonlocal) are well suited for homogeneous environments. When one deals with an inhomogeneous diffusion process one possibility is to add a diffusion coefficient and consider equations like

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\Big(a(x)\frac{\partial u}{\partial x}\Big)(x,t),$$

or

$$\frac{\partial u}{\partial t}(x,t) = \int_{\mathbb{R}^N} a(x,y)J(x-y)(u(y,t)-u(x,t))\,dy.$$

However, by adding a diffusion coefficient one can not deal with media that combine local and nonlocal diffusions in different regions. These coupling strategies are interesting since they can include a transition region such that the local and nonlocal equations are superposed or a lower-dimensional interface that separates the two regimes.

There are different strategies for couplings between local and nonlocal models. Let us briefly summarize previous results in (D'Elia *et al.*, 2016a,b; Du *et al.*, 2018; Gal e Warma, 2017; Gárriz *et al.*, 2020; Kriventsov, 2015), see also the review D'Elia *et al.* (2019). In D'Elia *et al.* (2016a), local and nonlocal problems are coupled through a prescribed solid region in which both kinds of equations overlap (the value of the solution in the nonlocal part of the domain is used as a Dirichlet boundary condition for the local part and vice-versa). This kind of coupling gives continuity of the solution in the overlapping region but does not preserve the total mass. In Gárriz *et al.* (2020) (see also Gal e Warma (2017); Kriventsov (2015)), an energy and its associated gradient flow provide an equation that combines local and nonlocal operators. For this model in the local region, the coupling with the nonlocal part appears as an external source in the heat equation (that is complemented with zero flux boundary conditions in the whole boundary of the local region). In probabilistic terms, in the model described in Gárriz *et al.* (2020), particles may jump across the interface between the two regions but can not pass coming from the local side unless they jump.

In the same direction, in Berestycki *et al.* (2015) the authors study the effects of network transportation on enhancing a biological invasion. The proposed mathematical model consists of one equation with nonlocal diffusion in a one-dimensional domain coupled via boundary condition with a standard reaction-diffusion, in a two-dimensional domain. The results suggested that the fast diffusion enhances the spread inside the domain, in which the local diffusion takes place.

Motivated by the range of applications and because it is an area with a lot of potential for development in what concerns its associated mathematical theory, our aim in this thesis is the study of a strategy for coupling local and nonlocal diffusion equations. In particular, we combine a local diffusion equation, the classical heat equation,

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) \tag{1.1}$$

with a nonlocal diffusion equation with an integrable kernel

$$\frac{\partial u}{\partial t}(x,t) = \int J(x,y)(u(y,t) - u(x,t))dy. \tag{1.2}$$

The kernel J is assumed to be a function $J \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})$ nonnegative, with $J(x,x) > 0$ and symmetric $J(x,y) = J(y,x)$ (in some cases we will assume that $J(x,y) = J(x-y)$).

The coupling of the problems (1.1) and (1.2) was thought in such a way that the following features (that are the usual ones when one deals with a diffusion problem) hold:

- The complete coupled problem is well-posed in the sense that there are existence and uniqueness of solutions. Besides, a comparison principle holds.
- There is an energy functional such that the evolution problem can be obtained as the gradient flow associated with this energy.
- When Neumann boundary conditions are imposed, the total mass of the initial condition is preserved along the evolution.
- When Neumann boundary conditions are imposed, solutions converge exponentially fast to the mean value of the initial condition.
- When homogeneous Dirichlet boundary conditions are imposed, solutions converge exponentially fast to zero.
- By taking the limit on the rescaled nonlocal kernel we can recover the heat equation in the whole domain.

For a first approach, we restrict ourselves to a simple configuration in one space dimension. We will split the domain $\Omega = (-1,1)$ into two subdomains $(-1,0)$ and $(0,1)$. If we think the evolution problem as a particle system, in the interval $(-1,0)$ particles move by Brownian motion (this gives the equation $\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$, $x \in (-1,0)$) with a reflexion at $x = -1$ (then $\frac{\partial u}{\partial x}(-1,0) = 0$) and when the particle arrives to $x = 0$ it passes trough to the other subdomain, $(0,1)$ (this will give a flux boundary condition at $x = 0$). On the other hand, in $(0,1)$ particles obey a pure jump process with jumping probability given by $J(x-y)$ (this gives an equation of the form (1.2) in $(0,1)$, when a particle that is at $x \in (0,1)$ wants to jump to a location $y \in (-1,0)$ it enters the domain $(-1,0)$ at the point $x = 0$ (particles are stuck there, giving the counterpart to the flux coming from $(-1,0)$). At this point, it is important to notice that we do not impose any continuity of the densities at the interface $x = 0$, but instead, we can ensure continuity of the densities inside the local and nonlocal subdomains $(-1,0)$ and $(0,1)$ by assuming continuity of the initial data.

Moreover, we can extend our results to higher dimensions. Take Ω, as a bounded smooth domain in $\mathbb{R}^N$ and split it into two subdomains Ω_l and Ω_{nl}, $\Omega = \Omega_l \cup \Omega_{nl}$. Let us call Σ, the interface between Ω_l and Ω_{nl} inside Ω, that is,

$$\Sigma = \overline{\Omega_l} \cap \overline{\Omega_{nl}} \cap \Omega.$$

We will assume that Ω_l has a Lipschitz boundary (in order to solve a heat equation with Newman boundary conditions, we need some regularity of the boundary).

Let us state the energies that will be considered in this work. We split $w \in L^2(\Omega)$ as $w = u + v$, with $u = w\chi_{\Omega_l}$ and $v = w\chi_{\Omega_{nl}}$. Fix a nonnegative continuous kernel $G : \Omega_{nl} \times \Sigma \mapsto \mathbb{R}$. For any

$$w = (u,v) \in \mathcal{B} := \left\{ w \in L^2(\Omega) : u|_{\Omega_l} \in H^1(\Omega_l), v \in L^2(\Omega_{nl}) \right\}.$$

The energy associated to our evolution problem will be composed by three positive terms given by

$$\begin{aligned} E(u,v) := {} & \frac{1}{2}\int_{\Omega_l} |\nabla u|^2 dx + \frac{C_{J,1}}{4}\int_{\Omega_{nl}}\int_{\Omega_{nl}} J(x-y)\left(v(y) - v(x)\right)^2 dydx \\ & + \frac{C_{J,2}}{2}\int_{\Omega_{nl}}\int_{\Sigma} G(x,z)\left(v(x) - u(z)\right)^2 d\sigma(z)dx. \end{aligned} \tag{1.3}$$

Remark that in this energy we have

$$\int_{\Omega_{nl}} \int_{\Sigma} G(x,z)\,(v(x)-u(z))^2\, d\sigma(z)dx \tag{1.4}$$

as coupling term. This integral can be obtained from an integral of the form

$$\iint_A J(x-y)\,(v(x)-u(z))^2\, dydx$$

assuming the following geometric condition on the interface Σ; for every $x \in \Omega_l$ and every $y \in \Omega_{nl}$ with $x-y \in \text{supp}(J)$ there exists a unique $z \in \Sigma$ that belongs to the segment that joins x with y (hence $z = z(x,y)$). To provide examples, notice that this geometric condition holds if Σ is almost flat. This assumption is useful since, from a probabilistic viewpoint, when a particle wants to jump from $y \in \Omega_{nl}$ to $x \in \Omega_l$ we want that it gets stuck at the interface (and then we want that there exist a unique point on Σ that belongs to the segment $[x,y]$, otherwise, some selection principle has to be assumed and, the selected point on the interface will not depend continuously on x and y, in general). This assumption is used to make the change of variables

$$z = ax + (1-a)y$$

in

$$\iint_A J(x-y)\,(v(x)-u(z))^2\, dydx$$

with $A = \{(x,y) : x \in \Omega_{nl}, y \in \Omega_l,\ \text{with } z \in \Sigma, z = ax+(1-a)y\}$ to obtain the coupling term in our energy, (1.4). The kernel G is nonnegative and comes from the change of variables that involves a jacobian $D(x,z)$.

For the $N-$dimensional evolution model, we can also prove existence and uniqueness following the same steps that we made for the one-dimensional case. In fact, the strategy of building a solution as a fixed point of the composition of the maps that solves the problem for u (given v) and for v (fixing u) also works here. Remark that we obtain a solution $u(x,t)$ that is in $H^1(\Omega_l)$ for $t > 0$ and hence $u(z,t)$ is defined on Σ in the sense of traces (and belongs to $L^2(\Sigma)$ for $t > 0$).

In the following, we explore some numerical aspects of the coupled evolution problem. In particular, our coupling takes advantage of the fact that we can show a splitting structure for our evolution equation allowing us to deal with the local and nonlocal parts of the equation separately. This particular structure is quite flexible, allowing, for example, to consider different meshes in the local and in the nonlocal region. We also perform some numerical experiments by using classical methods to show the qualitative features of the model.

Finally, we propose to investigate the same coupled evolution problem acting in subdomains with different dimensions. For this case, we consider the limit case when one of the subdomains is thin in one direction (it is concentrated to a domain of smaller dimension) and as a limit problem we obtain coupling between local and nonlocal equations acting in domains of different dimension. The same qualitative properties (like existence, uniqueness of solutions and a comparison principle) were verified for this limit model.

Concerning references for equations in thin domains, we refer to the references Arrieta e Pereira (2011); Arrieta *et al.* (2006, 2009a,b); Pereira e Rossi (2018); Shuichi e Yoshihisa (1992) that develop some techniques and methods to understand the effects of the geometry of the thin domain on the solutions of elliptic and parabolic singular problems. We can find some applications in elastic beam theories (as torsion and warping functions), see Rodríguez e Viaño (1998), fluid flows as ocean dynamics, geophysical fluid dynamics, and fluid flows in cell membranes, see for instance Iftimie *et al.* (2007).

In Arrieta *et al.* (2006), the authors investigate the dynamics of a reaction-diffusion equation with homogeneous boundary conditions in a dumbbell domain. The type of domain is composed of two disconnected regions joined by a thin channel, that depends on a thickness parameter ε

and degenerates to a line segment as the parameter $\varepsilon \to 0$. As part of a series of articles (see Arrieta *et al.* (2009a,b)) the authors also prove some properties about the continuity of the set of equilibria. On the other hand, in Pereira e Rossi (2018) the authors deal with nonlocal evolution problems with nonsingular kernels in thin domains obtaining a limit problem when the thickness of the domain goes to zero, but without considering any coupling with a local part of the problem. Passing to the limit in these coupling terms is one of the main contributions of this work.

Among the references raised in this work, that deals with coupling local/nonlocal models and, also considering a small parameter, we emphasize that our approach takes into account that the evolution problem is the gradient flow associated with an energy functional and also we do not impose any kind of continuity at the coupling interface. Moreover, we can preserve the same qualitative properties that both local and nonlocal problems satisfy separately.

1.1 Contributions

The main contributions of this work were collected in three articles. One of them has already been published and the other two have already been submitted and are under review.

(i) DOS SANTOS, Bruna C.; OLIVA, Sergio M.; ROSSI, Julio D. A local/nonlocal diffusion model. Applicable Analysis, p. 1-34, 2021.

In this work, we verified the existence and uniqueness of the solutions to the evolution problem through two different proofs: using a fixed point argument and abstract semigroup theory. The total mass of the initial condition is preserved and the solutions converge exponentially to the average value of the initial data when t goes to infinity. Finally, we recovered the heat equation in the whole domain by taking the limit on the rescaled nonlocal kernel. Besides, this results were generalized to higher dimensions.

(ii) DOS SANTOS, Bruna C.; OLIVA, Sergio M.; ROSSI, Julio D. Splitting methods and numerical approximations for a coupled local/nonlocal diffusion model. Submitted for the Journal Computational and Applied Mathematics, SBMAC.

In this work, we provided the third proof for the existence and uniqueness of the solution for the evolution problem by using the idea of splitting operators. Here the splitting idea was used in a different way than usual and we prove that the splitting method converges to the unique solution of the evolution problem as the time step goes to zero. Also, we developed some numerical experiments using classical techniques for partial differential equations to verify the theoretical results proved in the first paper.

(iii) DOS SANTOS, Bruna C.; OLIVA, Sergio M.; ROSSI, Julio D. Coupled local/nonlocal models in thin domains. Submitted for the Journal Asymptotic Analysis.

In this work we investigate our evolution model to be defined in a thin domain $\Omega_\varepsilon = \Omega \cup R_\varepsilon \subset \mathbb{R}^N$ by to considering two different approaches for coupling: via source term and at the boundary. We have a particular interest in the limit as the nonlocal region, R_ε, gets thinner, that is, to study the limit as $\varepsilon \to 0$. In this case we have the evolution problem defined in a domain with different dimensions. After obtaining the limit equations, we will also prove some qualitative properties of this limit problem (like conservation of the total mass and study the asymptotic behavior of the solutions). Finally, we design some numerical experiments to check the theoretical results.

1.2 Organization of the thesis

In Chapter 2, we present the results developed in the first paper, entitled A local/nonlocal diffusion model. In this article we explore some properties of a problem of local evolution (classical heat equation) defined in the interval $(-1, 0)$ coupled to a problem of nonlocal evolution, defined in the interval $(0,1)$. The coupling was designed in such a way that the resulting evolution problem is the gradient flow associated with the functional energy of this problem. For the resulting problem, we verified the uniqueness and existence of the solution, mass conservation, asymptotic decay, and we propose a strategy to recover the heat equation in the entire $(-1,1)$ domain by re-scaling the non-local core.

In Chapter 3 we deal with the numerical study of the evolution problem described in Chapter 2. Given the construction of the evolution problem, it was possible to verify and apply the properties of the Splitting Operator method. We prove the convergence of the numerical method and also include some numerical experiments that show the convergence properties of the solution for the average value of the initial data.

In Chapter 4 we analyze an evolution problem with local/nonlocal coupling acting in domains with different dimensions. We prove qualitative properties such as existence, uniqueness, mass conservation, and asymptotic decay of solutions. Finally, we prove that solutions can be obtained when the limit on the nonlocal domain is concentrated in a smaller domain.

In Chapter 5 we recall the results obtained in this work and point some possibilities of applications. We also list some possible extensions of our results for future work.

Finally, for completeness, in the Appendix section we enunciate the main results about L^p spaces and nonlocal equations used along all the three papers included in this thesis.

Chapter 2

A local/nonlocal diffusion model

The Chapter 2 is composed by the first paper entitled **A local/nonlocal diffusion model** that was published by the journal Applicable Analysis.

C. DOS SANTOS, Bruna; OLIVA, Sergio M.; ROSSI, Julio D. A local/nonlocal diffusion model. **Applicable Analysis**, p. 1-34, 2021.

A local/nonlocal diffusion model

Bruna C. dos Santos, Sergio M. Oliva and Julio D. Rossi

August 3, 2021

Abstract: In this paper, we study some qualitative properties for solutions to an evolution problem that combines local and nonlocal diffusion operators acting in two different subdomains. The coupling takes place at the interface between these two domains in such a way that the resulting evolution problem is the gradient flow of an energy functional. We prove existence and uniqueness results, as well as that the model preserves the total mass of the initial condition. We also study the asymptotic behavior of the solutions. Besides, we show a suitable way to recover the heat equation at the whole domain from taking the limit at the nonlocal rescaled kernel. Finally, we propose a brief discussion about the extension of the problem to higher dimensions.

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1 Introduction and main results

Numerous records along the past decades provide examples of how the spreading and establishment of worldwide transport networks have contributed to the development of global pandemics. The Black Death, in the middle of the 14th century, is a classical example of one of the most aggressive and devastating epidemics in history. By killing more than 200 million people in Europe, the disease spread fastly across Europe through the silk routes [37]. Recently, another communicable disease takes place and surprise the whole world for its rapid spread and fatality becoming a global public health concern. First observed in Wuhan, a Hubei province, China, the virus spreads to many countries in a few months by air network, and once at a new place, the diffusion occurs slower [27].

Another example of fast diffusion appears in ecology. Studies have corroborated the hypothesis of successive invasion waves of mosquitoes of the species Culex pipiens, Aedes aegypti, and more recently, Aedes albopictus, facilitated by the worldwide shipping [3, 24, 18]. Due to their ability to survive and develop in artificial containers (such as tires and bamboo), mosquitoes have obtained a high success rate in invasions to new regions. Besides, the effect of climate change, in particular, the increase of the temperature has played a fundamental role in creating favorable conditions for the establishment and local propagation of these invasive species across their common geographical boundaries [3, 31]. Further interesting behavior of propagation lines occurs on the wolves population [30]. The authors observed in this study that the wolves concentrate and move faster along seismic lines formed in areas of oil and gas exploration in the Western Canada Forest. Nonlocal patterns also play an important role in molecular interactions in dissimilar interfaces [16], continuum mechanics [15, 25, 33], peridynamics applied to elasticity and mechanics [34, 35].

From a modeling perspective, empirical studies have shown that the spreading effect involves much more complex characteristics than the classical models have suggested, as the classical heat equation, $\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t)$, which is associated with a process (Brownian motion) and describes the random movement of a particle [10]. This type of modeling has largely ignored long-range dispersion.

On the other hand, an alternative to capture these features is the nonlocal diffusion equations. One popular choice is $\frac{\partial u}{\partial t}(x,t) = \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))\,dy$, where $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative kernel (these kind of equations included the widely studied fractional Laplacians). Here the equation at a point x and time t depends on the values of the unknown u at all points in the set $x + \operatorname{supp} J$, which is what makes the diffusion nonlocal. These kinds of problems are associated to jump processes. Evolution equations of this form and variations of it have been recently widely used to model diffusion processes; see for instance [2, 6, 8, 11, 20, 21, 26, 28, 38, 39]. For example in biology, if $u(x,t)$ is thought of as the density of a population at the point x at time t, and $J(x-y)$ is regarded as the probability distribution of jumping from location y to location x, then the rate at which individuals are arriving to position x from all other places is given by $\int_{\mathbb{R}^N} J(y-x)u(y,t)\,\mathrm{d}y$, while the rate at which they are leaving location x to travel to all other sites is given by $-\int_{\mathbb{R}^N} J(y-x)u(x,t)\,\mathrm{d}y = -u(x,t)$. Therefore, in the absence of external or internal sources, the density u satisfies the nonlocal diffusion equation, see [20].

The two previous models (local or nonlocal) are well suited for homogeneous environments. When one deals with an inhomogeneous diffusion process one possibility is to add a diffusion coefficient and consider equations like $\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}(a(x,t)\frac{\partial u}{\partial x})(x,t)$, or $\frac{\partial u}{\partial t}(x,t) = \int_{\mathbb{R}^N} a(x,y)J(x-y)(u(y,t)-u(x,t))\,dy$. However, by adding a diffusion coefficient one can not deal with media that combine local and nonlocal diffusions in different regions. Therefore, to provide good models for inhomogeneous media we need to study couplings between local and nonlocal diffusion equations. These coupling strategies can include a transition region such that the local and nonlocal equations are superposed or a lower-dimensional interface separates the two regimes (we will describe these previous results below).

In [4] the effects of network transportation on enhancing biological invasion is studied. The proposed mathematical model consists of one equation with nonlocal diffusion in a one-dimensional domain coupled via boundary condition with a standard reaction-diffusion, in a two-dimensional domain. The results suggested that the fast diffusion enhances the spread in the domain in which the local diffusion takes place.

From a mathematical point of view, interesting properties arise from coupling local and nonlocal models. See for instance [12, 13, 17, 22, 23, 29] and references therein. In [12], local and nonlocal problems were coupled through a prescribed region in which both kinds of equations overlap (the value of the solution in the nonlocal part of the domain is used as a Dirichlet boundary condition for the local part and vice-versa). This kind of coupling gives continuity of the solution in the overlapping region but does not preserve the total mass. In [12] and [17], numerical schemes using local and nonlocal equations were developed and used to improve the computational accuracy when approximating a purely nonlocal problem. In [23] (see also [22, 29]), energy closely related to ours was studied, but the gradient flow of this energy (that it has all the nice properties listed above) gives an equation in the local region in which the coupling with the nonlocal part appears as an external source in the heat equation (that is complemented with zero flux boundary conditions in the whole boundary of the local region). In probabilistic terms, in the model described in [23], particles may jump across the interface between the two regions but can not pass coming from the local side unless they jump.

Here, our aim is the study of coupling local and nonlocal diffusion equations and propose a model in which there is a sharp interface between the two regimes, and the coupling is done via the fluxes at the interface. In particular, we combine a local diffusion equation, the classical heat equation,

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \tag{1.1}$$

with a nonlocal diffusion equation with an integrable kernel

$$\frac{\partial u}{\partial t}(x,t) = \int J(x-y)(u(y,t)-u(x,t))dy. \tag{1.2}$$

The kernel $J(z)$ is assumed to be nonnegative, continuous, symmetric, compactly supported with $supp(J) = [-R,R]$ and $\int J(z)\,dz = 1$ (these hypotheses on J will be assumed from now on).

The coupling of the problems (1.1) and (1.2) was thought in such a way that the following features (that are the usual ones when one deals with a diffusion problem) hold:

- The problem is well-posed in the sense that there are existence and uniqueness of solutions. Besides, a comparison principle holds.
- There is an energy functional such that the evolution problem can be obtained as the gradient flow associated with this energy.
- The total mass of the initial condition is preserved along with the evolution, naturally obtained by the Neumann boundary condition.
- Solutions converge exponentially fast to the mean value of the initial condition.

Effectively, we can think of our model in terms of a particle system. To simplify the exposition we will restrict ourselves to a one-dimensional problem and comment on the extension to higher dimensions at the end of the paper. We split the domain $\Omega = (-1,1)$ into two subdomains $(-1,0)$ and $(0,1)$ (to simplify we will restrict ourselves to this simple configuration). In $(-1,0)$ particles move by Brownian motion (this gives the equation $\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$, $x \in (-1,0)$) with a reflexion at $x=-1$ (then $\frac{\partial u}{\partial x}(-1,0) = 0$) and when the particle arrives to $x=0$ it passes trough to the other subdomain, $(0,1)$ (this will give a flux boundary condition at $x=0$). On the other hand, in $(0,1)$ particles obey a pure jump process with jumping probability given by $J(x-y)$ (this gives an equation of the form (1.2) in $(0,1)$, when a particle that is at $x \in (0,1)$ wants to jump to a location $y \in (-1,0)$ it enters the domain $(-1,0)$ at the point $x=0$ (particles are stuck there, giving the counterpart to the flux coming from $(-1,0)$). This process has a density $w(x,t)$, which obeys an evolution equation associated with the gradient flow of a local/nonlocal energy that we describe in the next section. At this point, it is important to notice that we do not impose any continuity of the densities at the interface $x=0$, but instead, we can ensure continuity of the densities inside the local and nonlocal subdomains $(-1,0)$ and $(0,1)$ by assuming continuity of the initial data.

1.1 A local/nonlocal diffusion model

As we mentioned, let us consider as the reference domain $\Omega = (-1,1) \subset \mathbb{R}$ that is divided in two disjoint regions, the intervals $\Omega_l = (-1,0)$ and $\Omega_{nl} = (0,1)$, the local and nonlocal domains, respectively. We split a function $w \in L^2(-1,1)$ as $w = u + v$, with $u = w\chi_{(-1,0)}$ and $v = w\chi_{(0,1)}$. For any

$$w = (u,v) \in \mathcal{B} := \left\{ w \in L^2(-1,1) : u \in H^1(-1,0), v \in L^2(0,1) \right\}$$

we define the energy

$$\begin{aligned} E(u,v) := {} & \frac{1}{2}\int_{-1}^{0} \left|\frac{\partial u}{\partial x}\right|^2 dx + \frac{C_{J,1}}{4}\int_0^1\int_0^1 J(x-y)\left(v(y)-v(x)\right)^2 dydx \\ & + \frac{C_{J,2}}{2}\int_0^1\int_{-1}^0 J(x-y)\left(v(x)-u(0)\right)^2 dydx, \end{aligned}$$

where $C_{J,1}$ and $C_{J,2}$ are fixed positive constants. Notice that, in this energy functional we have two terms

$$\frac{1}{2}\int_{-1}^{0} \left|\frac{\partial u}{\partial x}\right|^2 dx \qquad \text{and} \qquad \frac{C_{J,1}}{4}\int_0^1\int_0^1 J(x-y)\left(v(y)-v(x)\right)^2 dydx$$

that are naturally associated with the equations (1.1) and (1.2), plus a coupling term

$$\frac{C_{J,2}}{2}\int_0^1\int_{-1}^0 J(x-y)\left(v(x)-u(0)\right)^2 dydx$$

that involves only the value of u at $x=0$.

We aim to write our model as the gradient flow associated with this energy, that is, (u,v) will be the solution of the abstract ODE problem

$$(u,v)'(t) = -\partial E\left[(u,v)(t)\right], \qquad t \geq 0,$$

with $u(0) = u_0$, $v(0) = v_0$ and, $\partial E\left[(u,v)\right]$ denotes the subdifferential of E at the point (u,v). Let us compute the derivative of E at (u,v), in the direction of $\varphi \in C_0^\infty(-1,1)$,

$$\begin{aligned} \partial_\varphi E(u,v) &= \lim_{h\to 0} \frac{E(u+h\varphi, v+h\varphi) - E(u,v)}{h} \\ &= \int_{-1}^0 \frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x} dx + \frac{C_{J,1}}{2}\int_0^1\int_0^1 J(x-y)(v(y)-v(x))(\varphi(y)-\varphi(x))dydx \\ &\quad + \frac{C_{J,2}}{2}\int_0^1\int_{-1}^0 J(x-y)(v(x)-u(0))(\varphi(x)-\varphi(0))dydx. \end{aligned}$$

Thus, if u is smooth, we would have

$$\partial_\varphi E(u,v) = \left\{ \frac{\partial u}{\partial x}(0) - C_{J,2} \int_0^1 \int_{-1}^0 J(x-y)(v(x)-u(0))dydx \right\} \varphi(0) - \frac{\partial u}{\partial x}(-1)\varphi(-1)$$

$$- \int_{-1}^0 \frac{\partial^2 u}{\partial x^2} \varphi dx - C_{J,1} \int_0^1 \left\{ \int_0^1 J(x-y)(v(y)-v(x))dy \right\} \varphi(x)dx$$

$$+ C_{J,2} \int_0^1 \left\{ \int_{-1}^0 J(x-y)(v(x)-u(0))dy \right\} \varphi(x)dx.$$

Since $\langle \partial E[u,v], \varphi \rangle = \partial_\varphi E(u,v)$, we can derive the local/nonlocal problem associated to this gradient flow. The evolution problem consists of two parts. A local part, composed of a heat equation with Neumann/Robin type boundary conditions,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t), \\ \dfrac{\partial u}{\partial x}(-1,t) = 0, \\ \dfrac{\partial u}{\partial x}(0,t) = C_{J,2} \displaystyle\int_{-1}^0 \int_0^1 J(x-y)(v(y,t)-u(0,t))dydx, \\ u(x,0) = u_0(x), \end{cases} \tag{1.3}$$

for $x \in (-1,0)$, $t>0$. Notice that we have a Robin type boundary condition at $x=0$ that encodes the coupling with the nonlocal part of the problem.

We complete the system with the nonlocal part,

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) = C_{J,1} \displaystyle\int_0^1 J(x-y)\left(v(y,t)-v(x,t)\right)dy - C_{J,2} \int_{-1}^0 J(x-y)dy(v(x,t)-u(0,t)), \\ v(x,0) = v_0(x), \end{cases} \tag{1.4}$$

for $x \in (0,1)$, $t>0$. Here we have a nonlocal diffusion problem for v, where the coupling with the local part u appears as a source term in the equation, while the value of u appears only at the interface $x=0$.

The complete problem can be summarized as follows: we look for w defined by

$$w(x,t) = \begin{cases} u(x,t), & \text{if} \quad x \in (-1,0), \\ v(x,t), & \text{if} \quad x \in (0,1), \end{cases} \tag{1.5}$$

where (u,v) are the solutions to (1.3)–(1.4).

For this problem we have the following result:

Theorem 1.1. *Given $w_0 = (u_0,v_0) \in L^2(-1,1)$, there exists an unique mild solution*

$$w(\cdot,t) \in \mathcal{B} := \left\{ w \in L^2(-1,1) : u \in H^1(-1,0), v \in L^2(0,1) \right\}$$

to the local/nonlocal problem (1.5) *with* (u,v) *satisfying* (1.3)*–*(1.4) *that is globally defined. If, $w_0 = (u_0,v_0)$, with $u_0 \in C([-1,0])$ and $v_0 \in C([0,1])$ then, the solution (u,v) is such that $u(\cdot,t) \in C([-1,0])$ and $v(\cdot,t) \in C([0,1])$ for every $t>0$.*

A comparison principle holds: if the initial data are ordered, $w_0 \geq z_0$, then the corresponding solutions are also ordered, they verify $w \geq z$ in $(-1,1) \times \mathbb{R}_+$.

Moreover, the total mass of the solution is preserved along the evolution, that is,

$$\int_{-1}^1 w(x,t)dx = \int_{-1}^1 w_0(x)dx = \int_{-1}^0 u_0(x)dx + \int_0^1 v_0(x)dx.$$

Remark 1. Notice that, for a continuous initial datum u_0 and v_0, we obtain a solution (u,v) such that $u(\cdot,t) \in C([-1,0])$ and $v(\cdot,t) \in C([0,1])$ for every $t>0$, but we are not imposing (nor obtaining) continuity across the interface, that is, we do not necessarily have $u(0,t) = v(0,t)$.

1.2 Asymptotic behavior

Once we proved the existence and uniqueness of a global solution, our next goal is to look for its asymptotic behavior as $t \to \infty$. We start by observing that the constants are stationary solutions of (1.3)–(1.4).

For the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

with Neumann boundary conditions, it is well known that solutions have an exponential time decay to the mean value of the initial condition, that is,

$$\left\| u(\cdot, t) - \fint u_0 \right\|_{L^2} \leq C(u_0) e^{-\beta t}.$$

The same is valid (with a different β) for solutions to the nonlocal heat equation

$$\frac{\partial v}{\partial t}(x,t) = \int_0^1 J(x-y)(v(y,t) - v(x,t))dy,$$

with the additional assumption on the kernel, $M(J) := \int_{\mathbb{R}} J(z)|z|^2 dz < \infty$, see [1, 7].

The problem (1.5) shares of the same behavior as each, local and nonlocal equation have, individually, that is, the solution of the coupled local/nonlocal problem converges exponentially to the mean value of the initial condition.

Theorem 1.2. *Given $w_0 \in L^2(-1,1)$, the solution to* (1.5) *with initial condition w_0 converges to its mean value as $t \to \infty$ with an exponential rate.*

$$\left\| w(\cdot, t) - \fint w_0 \right\|_{L^2(-1,1)} \leq C e^{-2\beta_1 t}, \qquad t > 0,$$

where $\beta_1 > 0$ depends only on J and Ω and, the constant C depends on the initial condition, w_0.

1.3 Rescaling the kernel

In the following, we will show that the solutions of the evolution problem (1.3)-(1.4), with the kernel J rescaled suitably, converges to the classical local problem (given by the heat equation) at the whole domain. The idea consists of to rescale the kernel J by a $\varepsilon > 0$ parameter

$$J^\varepsilon(x) := \frac{1}{\varepsilon^3} J\left(\frac{x}{\varepsilon}\right). \tag{1.6}$$

From now on, we choose (and fix) the constants $C_{J,1}$ and $C_{J,2}$ that appears before the nonlocal terms as

$$C_{J,1} := \frac{2}{M(J)} \text{ with } M(J) = \int_{\mathbb{R}} J(z) z^2 \, dz \qquad \text{and} \qquad C_{J,2} := 1. \tag{1.7}$$

Summarizing, our goal is to show that, the solutions of the local heat equation with Neumann boundary conditions,

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), \quad x \in (-1,1), \quad t > 0, \\ \dfrac{\partial w}{\partial x}(-1,t) = \dfrac{\partial w}{\partial x}(1,t) = 0, \quad t > 0, \\ w(x,0) = w_0(x), \quad x \in (-1,1), \end{cases} \tag{1.8}$$

can be obtained as the limit as $\varepsilon \to 0$ of the solution w^ε to our local/nonlocal problem with J replaced by J^ε, given by (1.6). We will call $w^\varepsilon = (u^\varepsilon, v^\varepsilon)$ the solution to (1.3)–(1.4) with the rescaled kernel and a fixed initial condition $w(x,0) = u_0(x)\chi_{(-1,0)}(x) + v_0(x)\chi_{(0,1)}(x)$.

We have the following result:

Theorem 1.3. *Let $w_0 \in L^2(-1,1)$. For each $\varepsilon > 0$, let w^ε be the solution to* (1.3)-(1.4) *with J replaced by J^ε ((1.6)) and, initial condition w_0. Then, it holds the following*

$$\lim_{\varepsilon\to 0}\left(\max_{t\in[0,T]} \| w^\varepsilon(\cdot,t) - w(\cdot,t) \|_{L^2(-1,1)}\right) = 0,$$

where w is the solution to (1.8).

ORGANIZATION OF THE PAPER: The paper is organized as follows: In Section 2, we prove a key result concerning the control of the pure nonlocal energy by our local/nonlocal energy. In Section 3, we prove the existence and uniqueness of the problem, the total mass conservation property, and the asymptotic behavior of the solutions for large times. In Section 4, we deal with the rescaling of the kernel. Finally, in the final section (Section 5), we explain how to extend our results to higher dimensions.

2 Preliminaries

2.1 Control of the nonlocal energy

In this section, we prove the first important lemma that ensures the domination of the energy for the complete problem over the pure nonlocal energy.

Lemma 2.1. *Let*

$$(u,v) \in \mathcal{B} := \left\{u \in H^1(-1,0), v \in L^2(0,1)\right\}.$$

Then, there exists a constant $k := k(J,\Omega) > 0$ such that

$$\begin{aligned}\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u}{\partial x}\right|^2 dx + \frac{C_{J,1}}{4}\int_0^1\int_0^1 J(x-y)\left(v(y)-v(x)\right)^2 dydx + \frac{C_{J,2}}{2}\int_0^1\int_{-1}^0 J(x-y)\left(v(x)-u(0)\right)^2 dydx \\ \geq k\int_{-1}^1\int_{-1}^1 J(x-y)\left(w(y)-w(x)\right)^2 dydx.\end{aligned} \tag{2.1}$$

Proof. Assume that the conclusion does not hold. This implies that, there exists a sequence $\{w_n\} \in L^2(-1,1)$, with $\{u_n\} \in H^1(-1,0)$ and $\{v_n\} \in L^2(0,1)$, such that it satisfies

$$\int_{-1}^1\int_{-1}^1 J(x-y)(w_n(y)-w_n(x))^2 dydx = 1, \tag{2.2}$$

and satisfying,

$$\int_{-1}^1 w_n = \int_{-1}^0 u_n + \int_0^1 v_n = 0, \tag{2.3}$$

and

$$1 \geq n\left(\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u_n}{\partial x}\right|^2 + \frac{C_{J,1}}{4}\int_0^1\int_0^1 J(x-y)(v_n(y)-v_n(x))^2 dydx + \frac{C_{J,2}}{2}\int_{-1}^0\int_0^1 J(x-y)(u_n(0)-v_n(x))^2 dxdy\right), \tag{2.4}$$

for every $n \in \mathbb{N}$.

Taking the limit in n, in (2.4), we obtain

$$\lim_n\left(\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u_n}{\partial x}\right|^2\right) = 0, \tag{2.5}$$

$$\lim_n\left(\frac{C_{J,1}}{4}\int_0^1\int_0^1 J(x-y)(v_n(y)-v_n(x))^2 dydx\right) = 0, \tag{2.6}$$

and

$$\lim_n \left(\frac{C_{J,2}}{2} \int_{-1}^{0} \int_{0}^{1} J(x-y)(u_n(0)-v_n(x))^2 dx dy \right) = 0. \tag{2.7}$$

From equations (2.2)–(2.3) and (2.5), it implies a bound on the L^2-norm of w_n, so we can take a subsequence, also denoted $\{u_n\}$, which weakly converge for some limit in $H^1(-1,0)$. This limit is given by a constant A,

$$\begin{aligned} u_n \to A \quad &\text{in} \quad L^2(-1,0) \quad \text{and,} \\ u_n \to A \quad &\text{uniformly} \quad \text{in} \quad (-1,0). \end{aligned}$$

Note that, in particular, $u_n(0) \to A$.

Since the sequence $\{w_n\}$ is bounded in $L^2(-1,1)$ we have that $\{v_n\}$ is also bounded in $L^2(0,1)$ and then extracting a subsequence if necessary we can assume that their mean values converge, $\int_0^1 v_n(x)dx := B_n \to B$. Now, from [1] we know that there exists a constant c such that

$$\int_0^1 \int_0^1 J(x-y)(v_n(y)-v_n(x))^2 dx dy \geq c \int_0^1 (v_n(x)-B_n)^2 dx$$

Therefore, from equation (2.6) we also can take a subsequence, also denoted as $\{v_n\}$, which strongly converges for some limit in $L^2(0,1)$ that is given by the constant B.

From the (2.7), we obtain $A = B$. Moreover, from equation (2.3) we get that $A + B = 0$, which leads to $A = B = 0$. On the other hand, we have

$$\int_{-1}^{1} \int_{-1}^{1} J(x-y)(w_n(y)-w_n(x))^2 dy dx = 1,$$

which implies

$$\int_{-1}^{1} \int_{-1}^{1} J(x-y)(A-B)^2 dy dx = 1,$$

that give us the contradiction. □

The main advantage of this estimate is to observe that the constant obtained from (2.1) can be taken independent of ε, when we consider the rescaled kernel J^ε, as we will prove by Lemma 2.3.

2.2 A Poincaré-type inequality

Let us consider w_ε as in the introduction, that is,

$$w_\varepsilon(x) = \begin{cases} u_\varepsilon(x), & \text{if} \quad x \in (-1,0) \\ v_\varepsilon(x), & \text{if} \quad x \in (0,1). \end{cases}$$

From [1] we have that

Lemma 2.2. *There exists a constant $C > 0$ (independent of ϵ) such that, for every $\{w_{\varepsilon_n}\} \in L^2(-1,1)$ it holds*

$$\int_{-1}^{1} \left| w_{\varepsilon_n}(x) - \fint_{-1}^{1} w_{\varepsilon_n}(x)dx \right|^2 dx \leq C \frac{1}{\varepsilon_n^3} \int_{-1}^{1} \int_{-1}^{1} J\left(\frac{x-y}{\varepsilon_n}\right) (w_{\varepsilon_n}(y) - w_{\varepsilon_n}(x))^2 dy dx. \tag{2.8}$$

As a consequence of (2.8) and the control of the nonlocal energy given by (2.1) we have the following Poincaré-type inequality [32].

Lemma 2.3. *Let $w_\varepsilon \in \mathcal{B} := \{u_\varepsilon \in H^1(-1,0), v_\varepsilon \in L^2(0,1)\}$. Then there exists a constant $k := k(J,\Omega) > 0$, independent of ε, such that*

$$\begin{aligned}\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial w_\varepsilon}{\partial x}\right|^2 dx + \frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J\left(\frac{x-y}{\varepsilon}\right)(w_\varepsilon(y)-w_\varepsilon(x))^2\,dydx \\ +\frac{C_{J,2}}{2\varepsilon^3}\int_0^1\int_{-1}^0 J\left(\frac{x-y}{\varepsilon}\right)(w_\varepsilon(x)-w_\varepsilon(0))^2\,dydx \\ \geq k\frac{1}{\varepsilon^3}\int_{-1}^1\int_{-1}^1 J\left(\frac{x-y}{\varepsilon}\right)(w_\varepsilon(y)-w_\varepsilon(x))^2\,dydx.\end{aligned} \tag{2.9}$$

Proof. Let us argue by contradiction. Suppose that (2.9) is false. Then, for every $n \in \mathbb{N}$, there exists a subsequence $\varepsilon_n \to 0$, and $\{w_{\varepsilon_n}\} \in L^2(-1,1) \cap H^1(-1,0)$, such that

$$\int_{-1}^1 w_{\varepsilon_n} = \int_{-1}^0 u_{\varepsilon_n} + \int_0^1 v_{\varepsilon_n} = 0, \tag{2.10}$$

$$\frac{1}{\varepsilon_n^3}\int_{-1}^1\int_{-1}^1 J\left(\frac{x-y}{\varepsilon_n}\right)(w_{\varepsilon_n}(y)-w_{\varepsilon_n}(x))^2 dydx = 1, \tag{2.11}$$

and,

$$\begin{aligned}\frac{1}{n} \geq \Bigg(\frac{1}{2}\int_{-1}^0\left|\frac{\partial w_{\varepsilon_n}}{\partial x}\right|^2 + \frac{C_{J,1}}{4\varepsilon_n^3}\int_0^1\int_0^1 J\left(\frac{x-y}{\varepsilon_n}\right)(w_{\varepsilon_n}(y)-w_{\varepsilon_n}(x))^2 dydx \\ + \frac{C_{J,2}}{2\varepsilon_n^3}\int_{-1}^0\int_0^1 J\left(\frac{x-y}{\varepsilon_n}\right)(w_{\varepsilon_n}(0)-w_{\varepsilon_n}(x))^2 dxdy\Bigg), \quad \forall n \in \mathbb{N}.\end{aligned} \tag{2.12}$$

Taking the limit in n in (2.12), we obtain

$$\lim_n\left(\frac{1}{2}\int_{-1}^0\left|\frac{\partial w_{\varepsilon_n}}{\partial x}\right|^2\right) = 0, \tag{2.13}$$

$$\lim_n\left(\frac{C_{J,1}}{4\varepsilon_n^3}\int_0^1\int_0^1 J\left(\frac{x-y}{\varepsilon_n}\right)(w_{\varepsilon_n}(y)-w_{\varepsilon_n}(x))^2 dydx\right) = 0, \tag{2.14}$$

and

$$\lim_n\left(\frac{C_{J,2}}{2\varepsilon_n^3}\int_{-1}^0\int_0^1 J\left(\frac{x-y}{\varepsilon_n}\right)(w_{\varepsilon_n}(0)-w_{\varepsilon_n}(x))^2 dxdy\right) = 0. \tag{2.15}$$

From Lemma 2.2 and the limit (2.13) we have that w_{ε_n} is bounded in $H^1(-1,0)$, so passing to a subsequence, also denoted $\{w_{\varepsilon_n}\}$, such that $\varepsilon_n \to 0$, we have

$$\begin{aligned}&w_{\varepsilon_n} \rightharpoonup w \quad \text{in} \quad H^1(-1,0),\\ &w_{\varepsilon_n} \to w \quad \text{in} \quad L^2(-1,0) \quad \text{and}\\ &w_{\varepsilon_n} \quad \text{converges} \quad \text{uniformly} \quad \text{in} \quad (-1,0).\end{aligned}$$

Thanks to (2.13), and by the weak lower semicontinuity of the norm, we also know that

$$\frac{1}{2}\int_{-1}^0\left|\frac{\partial w}{\partial x}\right|^2 \leq \liminf_{\varepsilon_n\to 0}\frac{1}{2}\int_{-1}^0\left|\frac{\partial w_{\varepsilon_n}}{\partial x}\right|^2 = 0.$$

Hence, the limit w is a constant, let us call $w = A_1 \in H^1(-1,0)$.

Now, we shall see that, $\{w_{\varepsilon_n}\}$ is also bounded in $L^2(0,1)$, and by taking a subsequence $\{w_{\varepsilon_n}\}$, as $\varepsilon_n \to 0$, $\{w_{\varepsilon_n}\}$ weakly converges in $L^2(0,1)$ to some limit w (that will be a constant A_2).

Thanks to (2.8) and, by the assumption (2.11), we have

$$\int_{-1}^1\left|w_{\varepsilon_n}(x) - ⨍_{-1}^1 w_{\varepsilon_n}(x)dx\right|^2 dx \leq C,$$

which implies

$$\int_{-1}^{1} |w_{\varepsilon_n}(x)dx|^2 dx = \int_{-1}^{0} |u_{\varepsilon_n}(x)dx|^2 dx + \int_{0}^{1} |v_{\varepsilon_n}(x)dx|^2 dx \leq C. \tag{2.16}$$

Besides, from (2.16) we have that $\{v_{\varepsilon_n}\}$ is bounded in $L^2(0,1)$ and then, there exists a subsequence, also denoted by $\{v_{\varepsilon_n}\}$, which weakly converges for some limit $w \in L^2(0,1)$.

Performing a variable change in (2.14), $x = y + \varepsilon_n z$, we obtain

$$\frac{C_{J,1}}{4\varepsilon_n^3} \int_0^1 \int_0^1 J\left(\frac{x-y}{\varepsilon_n}\right) (w_{\varepsilon_n}(y) - w_{\varepsilon_n}(x))^2 dy dx = \frac{C_{J,1}}{4} \int_0^1 \int_{\frac{-y}{\varepsilon_n}}^{\frac{1-y}{\varepsilon_n}} J(z) \frac{(w_{\varepsilon_n}(y) - w_{\varepsilon_n}(y+\varepsilon_n z))^2}{\varepsilon_n^2} dz dy. \tag{2.17}$$

As the limit in (2.14) is zero, it follows that

$$\frac{C_{J,1}}{4} \int_0^1 \int_{\frac{-y}{\varepsilon_n}}^{\frac{1-y}{\varepsilon_n}} J(z) \frac{(w_{\varepsilon_n}(y) - w_{\varepsilon_n}(y+\varepsilon_n z))^2}{\varepsilon_n^2} dz dy \leq C. \tag{2.18}$$

So, as a consequence of (2.18) and the weak convergence of $\{v_{\varepsilon_n}\}$ in $L^2(0,1)$, by [[1], Theorem 6.11] we have that, the limit $w \in H^1(0,1)$ and, moreover

$$\left(\frac{C_{J,1}}{4} J(z)\right)^{1/2} \frac{(w_{\varepsilon_n}(y) - w_{\varepsilon_n}(y+\varepsilon_n z))}{\varepsilon_n} \rightharpoonup \left(\frac{C_{J,1}}{4} J(z)\right)^{1/2} z \cdot \frac{\partial w}{\partial x}(y)$$

weakly in $L^2(0,1) \times L^2(\mathbb{R})$. Therefore, taking the limit $\varepsilon_n \to 0$ in (2.17) we get,

$$\frac{1}{2} \int_0^1 \left|\frac{\partial w}{\partial x}\right|^2 = 0.$$

Hence, $w = A_2$ is just a constant.

Besides, from (2.14) we ensure the strong convergence of a subsequence in $L^2(0,1)$. Finally, from (2.15), taking $\varepsilon_n \to 0$ and by the Monotone Convergence Theorem, we obtain that $A_1 = A_2$. On the other hand, from equation (2.10) we get that $A_1 + A_2 = 0$, which contradicts (2.10). □

3 The local/nonlocal problem

3.1 Existence and uniqueness

Now, our goal is to show the existence and uniqueness of solutions. The main idea to prove this result is, given a function u defined for $x \in [-1,0]$ we will use it as an initial input for the equation (1.4) in $[0,1]$. The solution v of this problem is then used to solve the equation (1.3) in $[-1,0]$, which yields a function z. This procedure in two steps can be regarded as an operator H given by $H(u) = z$. Now our task is to look for a fixed point of H via contraction in an adequate norm, meaning that, there must exists $u = H(u)$, solving the equation for $x \in [-1,0]$ with its corresponding v solving the equation for $x \in [0,1]$.

Fix $T > 0$ and consider the Banach spaces

$$X_T = \{u \in C([-1,0] \times [0,T])\} \quad \text{and} \quad Y_T = \{v \in C([0,1] \times [0,T])\},$$

with the respective norms

$$\| u \|_l = \max_{t\in[0,T]} \max_{x\in[-1,0]} |u| \quad \text{and} \quad \| v \|_{nl} = \max_{t\in[0,T]} \max_{x\in[0,1]} |v|.$$

Given $T > 0$, we define the operator $H_1 : X_T \to Y_T$ as $H_1(u) = v$, where v is the unique solution of

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) = C_{J,1} \displaystyle\int_0^1 J(x-y)\,(v(y,t) - v(x,t))\,dy - C_{J,2} \int_{-1}^0 J(x-y)(v(x,t) - u(t,0))dy, \\ v(0,x) = v_0(x), \end{cases}$$

for $x \in (0,1)$ and $t \in (0,T)$.

In the next lemma we will show that this problem has an unique solution (that means that H_1 is well defined). In addition, we show continuous dependence on u.

Lemma 3.1. *There are constants $C_{J,i}$, $i = 1,2$, depending only on J, such that for $T \in \left(0, \frac{1}{2C_{J,1}+C_{J,2}}\right)$, given $u(x,t) \in C([-1,0]\times[0,T])$ and $v_0 \in C([0,1])$, there exists an unique $v(x,t) \in C([0,1]\times[0,T])$, solution to* (1.4)*. Moreover, if v_1 and v_2 are the solutions corresponding to u_1 and u_2 then*

$$\| v_1 - v_2 \|_{nl} \le \frac{C_{J,2}T}{1-(2C_{J,1}+C_{J,2})T} \| u_1 - u_2 \|_l \, . \tag{3.1}$$

Proof. To show the existence and uniqueness we will use a fixed point argument. Let us define an operator $A_u(v) : Y_T \to Y_T$ as

$$A_u(v)(t,x) := v_0(x) + C_{J,1}\int_0^t\int_0^1 J(x-y)(v(y,s)-v(x,s))dyds$$

$$- C_{J,2}\int_0^t\int_{-1}^0 J(x-y)v(x,s)dyds + C_{J,2}\int_0^t\int_{-1}^0 J(x-y)u(0,s)dyds.$$

Taking the difference $A_u(v_1) - A_u(v_2)$ we get

$$\| A_u(v_1) - A_u(v_2) \|_{nl} \le C_{J,1} \max_{t\in[0,T]}\max_{x\in[0,1]}\int_0^t\int_0^1 J(x-y)|v_1(y,s)-v_2(y,s)|dyds$$

$$+ C_{J,1} \max_{t\in[0,T]}\max_{x\in[0,1]}\int_0^t\int_0^1 J(x-y)|v_2(x,s)-v_1(x,s)|dyds$$

$$+ C_{J,2} \max_{t\in[0,T]}\max_{x\in[0,1]}\int_0^t\int_{-1}^0 J(x-y)|v_2(x,s)-v_1(x,s)|dyds.$$

Since $J \ge 0$ and $\int_{\mathbb{R}} J = 1$, applying Fubini's theorem, we obtain

$$\| A_u(v_1) - A_u(v_2) \|_{nl} \le (2C_{J,1}+C_{J,2})T \| v_1 - v_2 \|_{nl} \, .$$

Choosing $T < \frac{1}{2C_{J,1}+C_{J,2}}$, A_u is a strict contraction, and hence it has an unique fix point.

To check the dependence on the data, let v_1 and v_2 be defined as $v_1 = A_{u_1}(v_1)$ and $v_2 = A_{u_2}(v_2)$. Indeed, following the same idea as before we will get

$$\| v_1 - v_2 \|_{nl} \le (2C_{J,1}+C_{J,2})T \| v_1 - v_2 \|_{nl} + C_{J,2}T \| u_1 - u_2 \|_l,$$

which yields (3.1) and it completes the proof. □

Remark 2. *In particular, any positive constants C_i, $i = 1,2$ will ensure the statement* (3.1)*. More specifically, we specify these constants, as in* (1.7)*, to recover the classical heat equation at the whole domain from rescaling the nonlocal kernel.*

Remark 3. We also have existence and uniqueness in L^2, that is, given $u(x,t) \in L^2([-1,0]\times[0,T])$ and $v_0 \in L^2([0,1])$, there exists an unique $v(t,x) \in L^2([0,1]\times[0,T])$, solution to (1.4). The proof is analogous and hence we omit the details.

In addition, we have a comparison principle, if we have two ordered functions $u \ge \tilde{u}$ and two initial conditions $v_0 \ge \tilde{v_0}$ then the corresponding solutions verify $v(x,t) \ge \tilde{v}(x,t)$.

Now, we need to look back to the local part. Given $v \in C([0,1]\times[0,T])$, we will show that there exists a unique solution $u \in C([-1,0]\times[0,T])$ to (1.3), with u_0 as initial condition. We define $H_2 : Y_T \to X_T$ as the solution operator $H_2(v) = u$ and once again we will prove continuity of this operator.

Lemma 3.2. *Fix $T > 0$. Given $v(x,t) \in C([0,1]\times[0,T])$ and $u_0 \in C([-1,0])$, there exists an unique $u(x,t) \in C([-1,0]\times[0,T])$, solution to (1.3). Moreover, if u_1 and u_2 are the solutions corresponding to v_1 and v_2 then*

$$\| u_1 - u_2 \|_l \leq C_2 \| v_1 - v_2 \|_{nl} . \tag{3.2}$$

Proof. It is well known, see [19], that given $v(t,x) \in C([0,1]\times[0,T])$ and $u_0 \in C([-1,0])$, the problem (1.3) has an unique solution $u(t,x) \in C([-1,0]\times[0,T])$. Therefore, the operator H_2 is well defined.

To show the bound (3.2), we will use a comparison argument.

Before we start the proof, we will make some observations that can simplify our problem. First, note that due to the symmetry of the kernel and the fact that $\int_{-1}^{1} J(r)dr = 1$, it is reasonably to assume

$$C_2 = C_{J,2}\int_{-1}^{0}\int_{0}^{1} J(x-y)dydx = \frac{C_{J,2}}{2}. \tag{3.3}$$

To obtain the estimate (3.2), let us consider $z = u_1 - u_2$, where both u_1 and u_2 satisfy (1.3) with the same initial condition $u_0(x)$ and two different functions v_1 and v_2, respectively. Then $z(x,t)$ is a solution to the following problem,

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \dfrac{\partial^2 z}{\partial x^2}(x,t), \\ \dfrac{\partial z}{\partial x}(-1,t) = 0, \\ \dfrac{\partial z}{\partial x}(0,t) = c_{J,2}\displaystyle\int_{-1}^{0}\int_{0}^{1} J(x-y)\left[v_1(y,t) - v_2(y,t) - (u_1(0,t) - u_2(0,t))\right] dydx, \\ z(x,0) = 0. \end{cases}$$

Using (3.3), we can get the following estimate

$$\begin{aligned} &\left| C_{J,2}\int_{-1}^{0}\int_{0}^{1} J(x-y)(v_1(y,t) - v_2(y,t))dydx \right| \\ &\quad \leq C_{J,2}\int_{-1}^{0}\int_{0}^{1} J(x-y)|v_1(y,t) - v_2(y,t)|dydx \\ &\quad \leq C_{J,2}\int_{-1}^{0}\int_{0}^{1} J(x-y)\max_{t\in[0,T]}\max_{y\in[0,1]}\left|v_1(y,t) - v_2(y,t)\right|dydx \\ &\quad = C_2 \| v_1 - v_2 \|_{nl} . \end{aligned}$$

Hence, we can define

$$w(t,x) = \frac{z(t,x)}{C_2 \| v_1 - v_2 \|_{nl}},$$

and w satisfies the following problem

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), \\ \dfrac{\partial w}{\partial x}(-1,t) = 0, \\ \dfrac{\partial w}{\partial x}(0,t) \leq -C_2 w(0,t) + 1 \quad \text{and} \quad \dfrac{\partial w}{\partial x}(0,t) \geq -C_2 w(0,t) - 1, \\ w(x,0) = 0. \end{cases} \tag{3.4}$$

Now, we aim to verify that the problem (3.4) possess a pair of sub and supersolutions, $\underline{w}, \overline{w}$, respectively. Proving the existence of sub and supersolution we can verify that the solutions satify the comparison principle and then we obtain (3.2).

For this, we will recall that a function $\overline{w}(x,t)$ is called a supersolution for the problem (3.4) if it satisfies

$$\begin{cases} \dfrac{\partial \overline{w}}{\partial t}(x,t) \geq \dfrac{\partial^2 w}{\partial x^2}(x,t), \\ \dfrac{\partial \overline{w}}{\partial x}(-1,t) \leq 0, \\ \dfrac{\partial \overline{w}}{\partial x}(0,t) \geq -C_2 w(0,t) + 1, \\ \overline{w}(x,0) = 0. \end{cases}$$

Respectively, a function $\underline{w}(x,t)$ is called a subsolution if, it satisfies the reverse inequalities.

Let us introduce an auxiliary function. Given $\xi < 0$ and $0 < a < 1$ we can define

$$g(\xi) = \frac{1}{a} f(\xi a), \quad \text{with} \quad g'(0) = f'(0) = 1. \tag{3.5}$$

Here, the function f is chosen such that the following conditions hold:

Given $\xi_0 > 1$, f is increasing in $(-\xi_0, 0]$, $C^2(-\xi_0, 0)$, and $f \equiv 1$ in $(-\infty, -\xi_0]$.

Let us fix $T < \frac{a^2}{2\xi_0^2}$. For each $t \in [0,T]$ and $x \in [-1,0]$ we define

$$\overline{w}(x,t) = (T+t)^{1/2} g\left(\frac{x}{(T+t)^{1/2}}\right),$$

with g given by (3.5).

Claim: $\overline{w}$ is a supersolution for (3.4). Let us check this claim in the following steps.

i) We want to prove that $\frac{\partial \overline{w}}{\partial t} \geq \frac{\partial^2 \overline{w}}{\partial x^2}$. Differentiating $\overline{w}$ with respect to t and x, we would like to verify

$$\frac{1}{2} g\left(\frac{x}{(T+t)^{1/2}}\right) - \frac{1}{2} x (T+t)^{-1/2} g'\left(\frac{x}{(T+t)^{1/2}}\right) \geq g''\left(\frac{x}{(T+t)^{1/2}}\right).$$

Observe that, since $x \in [-1,0]$ and $g'\left(\frac{x}{(T+t)^{1/2}}\right) > 0$, we only need to check that

$$\frac{1}{2} g\left(\frac{x}{(T+t)^{1/2}}\right) \geq g''\left(\frac{x}{(T+t)^{1/2}}\right). \tag{3.6}$$

To deal with this, let us call $\eta = \frac{x}{(T+t)^{1/2}}$. According to the definition of g, to prove (3.6) is equivalent to prove

$$\frac{1}{2a} f(a\eta) \geq a f''(a\eta). \tag{3.7}$$

We know that, for each $\xi \leq 0$ and $0 < a < 1$,

$$\frac{f(\xi a)}{2} = \begin{cases} 1/2, & \text{if } \xi a < -\xi_0, \\ \dfrac{f(\xi a)}{2}, & \text{if } -\xi_0 \leq \xi a < 0. \end{cases}$$

Moreover, as $f \in C^2(-\xi_0, 0)$ and increasing in the same interval, we obtain

$$f''(\xi a) \leq \begin{cases} 0, & \text{if } \xi a < -\xi_0, \\ M, & \text{if } -\xi_0 \leq \xi a < 0, \end{cases}$$

where $M = \max_{-\xi_0 \leq \xi \leq 0} |f''(\xi)|$.

Hence, given M, we can choose $0 < a < 1$ in order to have the estimate $\frac{1}{2} \geq Ma^2$. With this in mind we are able to verify (3.7). Indeed,

a) If $-\xi_0 \leq \xi a < 0$, it follows that

$$\frac{f(\xi a)}{2} \geq \frac{1}{2} \geq Ma^2 \geq f''(\xi a)a^2.$$

b) If $\xi a < -\xi_0$, we have that

$$\frac{f(\xi a)}{2} \geq \frac{1}{2} \geq 0 \geq f''(\xi a)a^2.$$

ii) We want to verify that $\overline{w}$ satisfies

$$\frac{\partial \overline{w}}{\partial x}(-1, t) \leq 0.$$

At $x = -1$ we have,

$$\frac{\partial \overline{w}}{\partial x}(-1, t) = g'\left(\frac{-1}{(T+t)^{1/2}}\right) = f'\left(\frac{-a}{(T+t)^{1/2}}\right). \tag{3.8}$$

We know that $f \equiv 1$ in $(-\infty, -\xi_0)$. Then, taking $T < \frac{a^2}{2\xi_0^2}$, we obtain that $\frac{-a}{(T+t)^{1/2}} < -\xi_0$ and therefore

$$f'\left(\frac{-a}{(T+t)^{1/2}}\right) = 0,$$

which it proves (3.8).

iii) We want to check that

$$\frac{\partial \overline{w}}{\partial x}(0, t) \geq -C_2 \overline{w}(0, t) + 1.$$

Differentiating $\overline{w}$ with respect to x, we aim to check if,

$$g'(0) \geq -C_2(T+t)^{1/2} g(0) + 1.$$

Since we assume $g'(0) = f'(0) = 1$ and $g(0) \geq 0$, we get

$$1 \geq -C_2(T+t)^{1/2} g(0) + 1,$$

which it proves the item *iii*).

iv) Finally, we aim to verify that $\overline{w}(x, 0) \geq 0$.

Indeed, we have

$$\overline{w}(x, 0) = (T)^{1/2} \underbrace{g\left(\frac{x}{T^{1/2}}\right)}_{>0} > 0.$$

With these four items we proved that $\overline{w}$ is a supersolution of (3.4).
The same analysis can be done to check that

$$\underline{w}(t, x) = -(T+t)^{1/2} g\left(\frac{x}{(T+t)^{1/2}}\right)$$

is a subsolution for the problem (3.4).

So, by the comparison principle, the solution $w(x, t)$ of the problem (3.4), verifies

$$\underline{w}(x, t) \leq w(x, t) \leq \overline{w}(x, t).$$

Hence, we get the estimate

$$|\overline{w}(x,t)| \leq \max_{x\in[-1,0]} \max_{t\in[0,T]} \left| (T+t)^{1/2} g\left(\frac{x}{(T+t)^{1/2}}\right)\right| = \frac{1}{a}(2T)^{1/2} < \frac{1}{\xi_o} < 1.$$

Therefore, going back to our original variable, z, we obtain the following

$$\frac{|z(x,t)|}{C_2 \parallel v_1 - v_2 \parallel_{nl}} = \frac{|u_1 - u_2|}{C_2 \parallel v_1 - v_2 \parallel_{nl}} = w(x,t) \leq \overline{w}(x,t) \leq 1,$$

which implies that

$$||u_1 - u_2||_l \leq C_2 \parallel v_1 - v_2 \parallel_{nl},$$

and then the proof is complete. □

Remark 4. In this case, we also have existence and uniqueness in L^2, as in Remark 3. Given $v(x,t) \in L^2([0,1]\times[0,T])$ and $u_0 \in L^2([-1,0])$, there exists an unique $u(x,t) \in C^1([-1,0]; L^2[0,T])$, solution to (1.3).

Again, we have a comparison principle, if we have two ordered functions $v \geq \tilde{v}$ and two initial conditions $u_0 \geq \tilde{u_0}$ then the corresponding solutions verify $u(x,t) \geq \tilde{u}(x,t)$.

Finally, combining the two lemmas, we get the following theorem.

Theorem 3.3. *Given $w_0 \in C([-1,1])$ (or given $w_0 \in L^2([-1,1])$), there exists a unique solution to problem* (1.3)–(1.4)*, which has w_0 as initial condition.*

Proof. Let $T \in \left(0, \frac{1}{2C_{J,1}+C_{J,2}}\right)$. We consider the operator $H : X_T \mapsto X_T$ given by

$$H(u) := H_2(H_1(u)) = H_2(v),$$

and we obtain, from our previous results,

$$\begin{aligned}\parallel H_2(H_1(u_1)) - H_2(H_1(u_2)) \parallel_l = \parallel H_2(v_1) - H_2(v_2) \parallel_l &\leq C_2 \parallel v_1 - v_2 \parallel_{nl} \\ &\leq C_2 \frac{C_{J,2}T}{1-(2C_{J,1}+C_{J,2})T} \parallel u_1 - u_2 \parallel_l,\end{aligned}$$

which proves that H is a strict contraction for T small enough. Therefore, there is a fixed point

$$u = H(u)$$

that gives us a unique solution $(u, v = H_1(u))$ in $(0,T)$. Since T can be chosen independently of the initial condition, the fixed point argument can be iterated to obtain a global solution for our problem. □

3.2 Conservation of mass

As we expected, the model (1.5) preserves the total mass of the solution.

Theorem 3.4. *The solution w of the problem* (1.5)*, with initial condition $w_0 \in C([-1,1])$ satisfies*

$$\int_{-1}^{1} w(x,t)dx = \int_{-1}^{1} w_0(x)dx, \quad \textit{for every } t \geq 0.$$

Proof. Notice that

$$\int_{-1}^{1} w(x,t)dx = \int_{-1}^{0} u(x,t)dx + \int_{0}^{1} v(x,t)dx,$$

and, the same is valid for w_0

$$\int_{-1}^{1} w_0 = \int_{-1}^{0} u_0 + \int_{0}^{1} v_0.$$

From the kernel's symmetry, and by Fubbini's Theorem, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_{-1}^{1} w(x,t)dx\right)\\
&= \int_{-1}^{0} \frac{\partial^2 u}{\partial x^2}(x,t)dx + C_{J,1}\int_0^1\int_0^1 J(x-y)(v(y,t)-v(x,t))dydx\\
&\qquad - C_{J,2}\int_0^1\int_{-1}^0 J(x-y)v(x,t)dydx + C_{J,2}u(0,t)\int_0^1\int_{-1}^0 J(x-y)dydx\\
&= \frac{\partial u}{\partial x}(0,t) - \frac{\partial u}{\partial x}(-1,t) - C_{J,2}\int_0^1\int_{-1}^0 J(x-y)v(x,t)dydx + C_{J,2}u(0,t)\int_0^1\int_{-1}^0 J(x-y)dydx\\
&= 0.
\end{aligned}$$

This shows that the total mass is independent of t. □

3.3 Comparison principle

Thanks to the linearity of the operator, if we have two solutions to the local/nonlocal problem (1.3)-(1.4) then, the difference is also a solution. Besides, given a nonnegative initial data u_0, v_0, the solution is nonnegative for every positive time (this follows from our construction of the solutions as a fixed point and Remarks 3 and 4). Therefore, we have the following result:

Proposition 3.5. *Let $u_0 \geq \tilde{u_0}$ and $v_0 \geq \tilde{v_0}$ then the corresponding solutions to the local/nonlocal problem* (1.3)–(1.4) *verify*

$$u(x,t) \geq \tilde{u}(x,t), \qquad v(x,t) \geq \tilde{v}(x,t),$$

for every $t \geq 0$.

To go one step further, let us define what we understand by sub and supersolutions.

Definition 3.6. The functions $\overline{u}$ and $\overline{v}$ are called supersolutions of the problem (1.3)-(1.4), in $[-1,1]\times[0,T]$ if, $\overline{u}, \overline{v}$ verify

$$\begin{cases}
\dfrac{\partial \overline{u}}{\partial t}(t,x) \geq \dfrac{\partial^2 \overline{u}}{\partial x^2}(t,x), \quad x \in (-1,0), \quad t>0,\\[2mm]
\dfrac{\partial \overline{u}}{\partial x}(t,-1) \leq 0, \quad t>0,\\[2mm]
\dfrac{\partial \overline{u}}{\partial x}(t,0) \geq C_{J,2}\displaystyle\int_{-1}^0\int_0^1 J(x-y)(\overline{v}(y,t)-\overline{u}(t,0))dydx, \quad t>0,\\[2mm]
\dfrac{\partial \overline{v}}{\partial t}(t,x) \geq C_{J,1}\displaystyle\int_0^1 J(x-y)(\overline{v}(y,t)-\overline{v}(x,t))dy - C_{J,2}\int_{-1}^0 J(x-y)(\overline{v}(x,t)-\overline{u}(t,0))dy,\\
\hfill x\in(0,1),\ t>0,\\[2mm]
\overline{u}(0,x) \geq u_0(x), \quad x\in(-1,0), \qquad \overline{v}(0,x)\geq v_0(x), \quad x\in(0,1).
\end{cases}$$

As usual, subsolutions, $\underline{u}, \underline{v}$, are defined analogously by reversing the inequalities.

Theorem 3.7. *Let $\overline{u}$ and $\overline{v}$ be supersolutions of the problem* (1.3)-(1.4)*. If $(\overline{u}_0, \overline{v}_0) \geq 0$ then*

$$\overline{u}(x,t) \geq 0, \qquad \text{and} \qquad \overline{v}(x,t) \geq 0,$$

for every $t > 0$.

Moreover, given $(\overline{u}, \overline{v})$, a supersolution and, $(\underline{u}, \underline{v})$, a subsolution, with $\overline{u}_0 \geq \underline{u}_0$ and $\overline{v}_0 \geq \underline{v}_0$, then,

$$\overline{u}(x,t) \geq \underline{u}(x,t) \qquad \text{and} \qquad \overline{v}(x,t) \geq \underline{v}(x,t),$$

for every $t > 0$.

Proof. Let us define

$$\begin{cases} w = \overline{u} - \underline{u}, \\ z = \overline{v} - \underline{v}. \end{cases}$$

They are supersolutions of the problem, (1.3)–(1.4), respectively, with $w(x,0) \geq 0$ and $z(x,0) \geq 0$. In fact, we have that

i) $\dfrac{\partial w}{\partial t} = \dfrac{\partial \overline{u}}{\partial t} - \dfrac{\partial \underline{u}}{\partial t} \geq \dfrac{\partial^2 \overline{u}}{\partial x^2} - \dfrac{\partial^2 \underline{u}}{\partial x^2} = \dfrac{\partial^2 w}{\partial x^2};$

ii) $\dfrac{\partial w}{\partial x}(-1,t) = \dfrac{\partial \overline{u}}{\partial x}(-1,t) - \dfrac{\partial \underline{u}}{\partial x}(-1,t) \leq 0;$

iii)

$$\begin{aligned} \frac{\partial w}{\partial x}(0,t) &= \frac{\partial \overline{u}}{\partial x}(0,t) - \frac{\partial \underline{u}}{\partial x}(0,t) \\ &\leq C_{J,2} \int_{-1}^{0} \int_{0}^{1} J(x-y)(\overline{v}(y,t) - \overline{u}(0,t))dydx - C_{J,2} \int_{-1}^{0} \int_{0}^{1} J(x-y)(\underline{v}(y,t) - \underline{u}(0,t))dydx \\ &= C_{J,2} \int_{-1}^{0} \int_{0}^{1} J(x-y)(z(y,t) - w(0,t))dydx; \end{aligned}$$

iv)

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial \overline{v}}{\partial t} - \frac{\partial \underline{v}}{\partial t} \\ &\geq C_{J,1} \int_{0}^{1} J(x-y)(\overline{v}(y,t) - \overline{v}(x,t))dy - C_{J,2} \int_{-1}^{0} J(x-y)(\overline{v}(x,t) - \overline{u}(0,t))dy \\ &\quad - \left(C_{J,1} \int_{0}^{1} J(x-y)(\underline{v}(y,t) - \underline{v}(x,t))dy - C_{J,2} \int_{-1}^{0} J(x-y)(\underline{v}(x,t) - \underline{u}(0,t))dy \right) \\ &= C_{J,1} \int_{0}^{1} J(x-y)(z(y,t) - z(x,t))dy - C_{J,2} \int_{-1}^{0} J(x-y)(z(x,t) - w(0,t))dy. \end{aligned}$$

Once we have check that the pair (w,z) is a supersolution, we need to prove that $w \geq 0$ and $z \geq 0$, for all $t > 0$, which implies $\overline{u} \geq \underline{u}$ and $\overline{v} \geq \underline{v}$. To perform this task we need to show that its negative parts are identically zero, $w_- \equiv 0$ and $z_- \equiv 0$. Take $\varphi = w_- \geq 0$ and $\psi = z_- \geq 0$ as our test functions. Multiplying φ and ψ by w_t, z_t and integrating by parts, we obtain,

$$\begin{aligned} 0 &\leq \left(\int_{-1}^{0} \frac{\partial w}{\partial t}\varphi + \int_{0}^{1} \frac{\partial z}{\partial t}\psi \right) \\ &= -\int_{-1}^{0} \frac{\partial w}{\partial x}\frac{\partial \varphi}{\partial x}dx - \frac{C_{J,1}}{2} \int_{0}^{1} \int_{0}^{1} J(x-y)(z(y) - z(x))(\psi(y) - \psi(x))dydx \\ &\quad - C_{J,2} \int_{0}^{1} \int_{-1}^{0} J(x-y)(z(x) - w(0))(\psi(x) - \varphi(0))dydx \\ &= -2E(w_-, z_-) \leq 0. \end{aligned}$$

Thus $E(w_-, z_-) = 0$, which implies $w_- \equiv 0$ and $z_- \equiv 0$. This complete the proof. $\square$

3.4 Asymptotic decay

As we mentioned before, the coupled local/nonlocal diffusion problem shares the same property about asymptotic behavior than, the local and nonlocal problems, individually. In this section, we will derive the asymptotic behavior of the solution as $t \to \infty$. To perform this, we need to introduce an estimate that it was inspired by classical Poincare's inequality [1, 32].

We start by analyzing the corresponding stationary problem. First, observe that for any constant k, $u = v = k$, is a solution to the problem (1.3)–(1.4). Besides, this constant solution is a minimizer of the energy (a simple inspection of the energy shows more, every minimizer is constant in the whole domain $(-1, 1)$).

Let us take β_1 as the first nontrivial eigenvalue of the problem (1.3)–(1.4) given by

$$\beta_1 = \inf_{u,v:\int_{-1}^0 u+\int_0^1 v=0} \frac{E(u,v)}{\displaystyle\int_{-1}^0 (u(x))^2dx + \int_0^1 (v(x))^2dx}. \tag{3.9}$$

Before we prove the asymptotic decay of the solution, we need an extra result.

Lemma 3.8. *Let β_1 be given by* (3.9)*, then*

$$\beta_1 > 0,$$

and moreover

$$E(u,v) \geq \beta_1 \left(\int_{-1}^0 u(x)^2dx + \int_0^1 v(x)^2dx \right), \tag{3.10}$$

for every (u, v), solution of (1.3)–(1.4)*, such that it satisfies $\int_{-1}^0 u + \int_0^1 v = 0$.*

Proof. For the positivity of β_1 we refer to [1]. Let us prove (3.10). To verify this estimate we will argue by contradiction. Suppose that (3.10) is false. Then, there exists sequences $\{u_n\} \in H^1(-1, 0)$ and $\{v_n\} \in L^2(0, 1)$ such that

i) $\displaystyle\int_{-1}^0 u_n + \int_0^1 v_n = 0,$

ii) $\displaystyle\int_{-1}^0 (u_n)^2 + \int_0^1 (v_n)^2 = 1$ and,

iii)

$$\frac{1}{2}\int_{-1}^0 \left|\frac{\partial u_n}{\partial x}\right|^2 dx+\frac{C_{J,1}}{4}\int_0^1\int_0^1 J(x-y)\left(v_n(y)-v_n(x)\right)^2 dydx+\frac{C_{J,2}}{2}\int_0^1\int_{-1}^0 J(x-y)dy\left(v_n(x)-u_n(0)\right)^2 dx \leq \frac{1}{n}.$$

Consequently, taking the limit in n, in item *iii*), we obtain

$$\lim_n \left(\frac{1}{2} \int_{-1}^0 \left|\frac{\partial u_n}{\partial x}\right|^2 \right) = 0, \tag{3.11}$$

$$\lim_n \left(\frac{C_{J,1}}{4} \int_0^1\int_0^1 J(x-y)(v_n(y) - v_n(x))^2dydx \right) = 0, \tag{3.12}$$

and

$$\lim_n \left(\frac{C_{J,2}}{2} \int_{-1}^0\int_0^1 J(x-y)(u_n(0) - v_n(x))^2dxdy \right) = 0. \tag{3.13}$$

From *ii*), we have that $\int_{-1}^0 (u_n)^2 \leq 1$. Then, by (3.11), u_n is bounded in $H^1(-1, 0)$, and hence there exists a subsequence $\{u_{nj}\} \in H^1(-1, 0)$, which weakly converges for a limit $u \in H^1(-1, 0)$.

From the weak convergence in $H^1(-1,0)$, it follows the convergence of $\{u_{nj}\} \to u$ in $L^2(-1,0)$ and the uniform convergence in $[-1,0]$ to $u \in H^1(-1,0)$. Moreover, as $\frac{1}{2}\int_{-1}^0 ((u_n)_x)^2 \to 0$ we have that the limit u is a constant, $u = k_1$. In particular, $u_{nj}(0) \to k_1$.

Also, from *ii*) and by Cauchy-Schwarz inequality, we obtain that $\int_0^1 |v_n| \le \left(\int_0^1 |v_n|^2\right)^{1/2} \le 1$. Let $\{k_n\}$, be $\{k_n\} = \int_0^1 v_n$, with $|k_n| \le 1$. We observe that, since $\{k_n\}$ is bounded in $L^2(0,1)$, we can extract a convergent subsequence k_{n_j}, which converges for some limit k_2, as $n_j \to \infty$.

Consider $\{z_{n_j}\} = \{v_n\} - \{k_{n_j}\}$, such that $\int_0^1 z_{n_j} = 0$. By [8], there exists a constant $c > 0$, such that

$$\int_0^1 \int_0^1 J(x-y)(z_{n_j}(y) - z_{n_j}(x))^2 dydx \ge c \int_0^1 (z_{n_j}(x))^2 dx.$$

From (3.12) we get the following

$$\int_0^1 \int_0^1 J(x-y)(z_{n_j}(y) - z_{n_j}(x))^2 dydx = \int_0^1 \int_0^1 J(x-y)(v_n(y) - v_n(x))^2 dydx \to 0,$$

which yields

$$0 \ge \lim_{n_j \to \infty} c \int_0^1 (z_{n_j}(x))^2 dx.$$

By the last inequality, we conclude that $z_{n_j} \to 0$ in $L^2(0,1)$, which leads to $(v_n - k_{n_j}) \to 0$ in $L^2(0,1)$, and thus we get that $v_n \to k_2$ in $L^2(0,1)$.

Taking the limit in (3.13), we obtain

$$\frac{C_{J,2}}{2} \int_{-1}^0 \int_0^1 J(x-y)(k_1 - k_2)^2 dxdy \to 0.$$

Hence $k_1 = k_2$. On the other hand, by *i*) we have $\int_{-1}^0 u_n + \int_0^1 v_n = 0$, so $k_1 = 0$ and $k_2 = 0$, then $k_1 = k_2 = 0$. But this is impossible since, by item *ii*) we have $\int_{-1}^0 (k_1)^2 + \int_0^1 (k_2)^2 = 1$. $\square$

Remark 5. The value β_1 should be the first nontrivial eigenvalue for our problem (notice that $\beta = 0$ is an eigenvalue with $u = v = cte$ as eigenfunctions). However, due to the lack of compactness of the nonlocal part, it is not clear that the infimum defining β_1 is attained.

Now, we are ready to prove the exponential convergence of the solutions to the mean value of the initial datum as $t \to +\infty$.

Theorem 3.9. *Given $w_0 \in L^2(-1,1)$, the solution to* (1.5)*, with initial condition w_0, converges to its mean value as $t \to \infty$, with an exponential rate,*

$$\left\| w(\cdot,t) - \fint w_0 \right\|_{L^2(-1,1)} \le C(\|w_0\|_{L^2(-1,1)}) e^{-2\beta_1 t}, \qquad t > 0,$$

where β_1 is given by (3.9) *and $C(w_0) > 0$.*

Proof. As we know, $u = v = k$, k constant, is a solution of (1.3)–(1.4). In particular $h(x,t) = u(x,t) - k$ and $z(x,t) = v(x,t) - k$ is also a solution. If $k = \int_{-1}^0 u_0 + \int_0^1 v_0$, then h and z satisfy

$$\int_{-1}^0 h(x,t)dx + \int_0^1 z(x,t)dx = 0.$$

Let

$$f(t) = \frac{1}{2}\int_{-1}^0 h(x,t)^2 dx + \frac{1}{2}\int_0^1 z(x,t)^2 dx.$$

Differentiating f with respect to t, we obtain

$$f'(t) = \int_{-1}^{0} h\frac{\partial h}{\partial t}dx + \int_{0}^{1} z\frac{\partial z}{\partial t}dx$$

$$= \int_{-1}^{0} h\frac{\partial^2 h}{\partial x^2}dx + C_{J,1}\int_{0}^{1} z(x,t)\int_{0}^{1} J(x-y)(z(y,t)-z(x,t))dydx$$

$$- C_{J,2}\int_{0}^{1} z(x,t)\int_{-1}^{0} J(x-y)z(x,t)dydx + C_{J,2}\int_{0}^{1} z(x,t)h(0,t)\int_{-1}^{0} J(x-y)dydx$$

$$= h(0,t)\frac{\partial h}{\partial x}(0,t) - h(-1,t)\frac{\partial h}{\partial x}(-1,t) - \int_{-1}^{0}\left|\frac{\partial h}{\partial x}\right|^2 dx$$

$$+ C_{J,1}\int_{0}^{1}\int_{0}^{1} J(x-y)(z(y,t)-z(x,t))z(x,t)dydx$$

$$- C_{J,2}\int_{0}^{1}\int_{-1}^{0} J(x-y)(z(x,t)-h(0,t)dyz(x,t)dx.$$

Applying Fubini's Theorem, and using the symmetry of the kernel, we obtain

$$f'(t) = -2E(h,z)(t).$$

Finally, by Lemma 3.8, we obtain the following

$$2E(h,z) \geq 2\beta_1\int_{-1}^{0} h^2 + \int_{0}^{1} z^2 \geq 2\beta_1 f(t),$$

which implies

$$f'(t) \leq -2\beta_1 f(t).$$

Hence,

$$f(t) \leq e^{-2\beta_1 t} f(0),$$

where

$$f(0) = \frac{1}{2}\left(\int_{-1}^{0} h_0^2 dx + \int_{0}^{1} z_0^2 dx\right) = C(\| w_0 \|_{L^2(-1,1)}).$$

From this follows that

$$\int_{-1}^{0} |u(t,x)-k|^2 dx + \int_{0}^{1} |v(t,x)-k|^2 dx \leq C(\| w_0 \|_{L^2(-1,1)})e^{-2\beta_1 t} \to 0,$$

as $t \to \infty$. In particular, we have that $u \to k$ in $L^2(-1,0)$ and $v \to k$ in $L^2(0,1)$. □

4 Rescaling the kernel. Convergence to the local problem

We derive a strong convergence in $L^2(-1,1)$, uniformly or bounded times, of the solutions of the rescaled problem (with J as in (1.6)) to the solution of the local problem (1.8) (the heat equation in the whole domain with homogeneous Newman boundary conditions) using the Brezis-Pazy Theorem with Mosco's convergence result. To perform this task we need to provide another existence and uniqueness result for the problem (1.3)-(1.4), based on semigroup theory for m-accretive operators.

4.1 Existence and uniqueness of a mild solution

On the concept of solution. We will introduce now the concept of solution for the complete problem (1.5). We rely on serigroup theory and introduce the operator

$$B_J u(x) = \begin{cases} -\dfrac{\partial^2 u}{\partial x^2}(x) & \text{for} \quad x \in (-1,0), \\ -C_{J,1} \displaystyle\int_0^1 J(x-y)(v(y)-v(x))dy + C_{J,2} \int_{-1}^0 J(x-y)(v(y)-u(0))dy & \text{for} \quad x \in (0,1). \end{cases}$$

Let

$$D(B_J) := \left\{ (u,v) : u \in H^2(-1,0), v \in L^2(0,1) \text{ with } \frac{\partial u}{\partial x}(-1) = 0 \text{ and } \frac{\partial u}{\partial x}(0) = -C_{J,2} \int_{-1}^0 J(x-y)(v(y)-u(0))dy \right\}$$

be the domain of the operator, and

$$B_J : D(B_J) \subset L^2(-1,1) \mapsto L^2(-1,1).$$

Now, according to [1], we can define a mild solution in $L^2(-1,1)$, of the abstract Cauchy problem by:

$$\begin{cases} u'(t) = B_J(u(t)), & t > 0 \\ u(0) = u_0. \end{cases}$$

Moreover, given an initial condition in the domain of the operator, there exists a unique strong solution for this problem, provided by the semigroup related to B_J operator, see [1, 5] for more details.

Following the ideas presented in [1], we will prove that, the operator B_J is completely accretive in $L^2(-1,1)$ and satisfies the range condition, $L^2(-1,1) \subset R(I+B_J)$. Once the B_J operator satisfies these two conditions, we can conclude that B_J is $m-$completely accretive in $L^2(-1,1)$. The range condition implies that for any $f \in L^2(-1,1)$ there exists $u \in D(B_J)$ such that, $u + B_J(u) = f$, and the resolvent, $(I+B_J)^{-1}$, is a contraction in $L^2(-1,1)$. With this in mind, by the Crandall-Ligget's Theorem we will obtain the existence and uniqueness of a mild solution for the coupled local/nonlocal evolution problem.

Theorem 4.1. *Given and initial condition $w_0 \in L^2(-1,1)$, there exists a mild solution w of the problem* (1.5) *that is a contraction in the L^2-norm.*

Proof. According to [1], it is enough to show that the operator B_J is completely accretive in $L^2(-1,1)$ and satisfies the range condition, $L^2(-1,1) \subset R(I+B_J)$. Consider the set

$$P_0 = \left\{ q \in C^\infty(-1,1) : 0 \leq q \leq 1, \quad \text{supp}(q') \text{ is compact and } 0 \notin \text{supp}(q) \right\}.$$

To prove the operator B_J is completely accretive, is equivalent to show that, given $w_1, w_2 \in D(B_J)$, and $q(w_1 - w_2)$, as a test function, we have that

$$\int_{-1}^1 (B_J(w_1(x)) - B_J(w_2(x)))q(w_1(x) - w_2(x))dx \geq 0. \tag{4.1}$$

Using the weak form of the operator we get

$$\int_{-1}^{1}(B_J(w_1(x)) - B_J(w_2(x)))q(w_1(x) - w_2(x))dx$$

$$= \int_{-1}^{0} \frac{\partial(w_1 - w_2)}{\partial x} \frac{\partial[q(w_1 - w_2(x))]}{\partial x} dx$$

$$+ \frac{C_{J,1}}{2} \int_{0}^{1} \int_{0}^{1} J(x-y)[(w_1 - w_2)(y) - (w_1 - w_2)(x)][q(w_1(y) - w_2(y)) - q(w_1(x) - w_2(x))]dydx$$

$$+ C_{J,2} \int_{0}^{1} \int_{-1}^{0} J(x-y)[(w_1 - w_2)(x) - (w_1 - w_2)(0)][q(w_1(x) - w_2(x)) - q(w_1(0) - w_2(0))]dydx.$$

Since $J \geq 0$, using the Mean Value Theorem, we obtain that the inequality (4.1) is holds.

To derive that, B_J is m–completely accretive in $L^2(-1,1)$ we need to show that it satisfies the range condition

$$L^2(-1,1) \subset R(I + B_J).$$

Given $f \in L^2(-1,1)$, we consider the variational problem

$$I[u] = \min_{u \in L^2(-1,1)} \left\{ \frac{1}{2} \int_{-1}^{1} u^2 + E(u) - \int_{-1}^{1} fu \right\}. \tag{4.2}$$

The existence of a unique minimizer u, of the variational problem (4.2), is proved using the direct method in the calculus of variations. This operator is continuous, monotone, and coercive in $L^2(-1,1)$. Indeed, using Young's inequality, we obtain

$$\frac{1}{2} \int_{-1}^{1} u^2 + E(u) - \int_{-1}^{1} fu \geq \frac{1}{2} \int_{-1}^{1} u^2 + E(u) - \left(\int_{-1}^{1} f^2 \right)^{1/2} \left(\int_{-1}^{1} u^2 \right)^{1/2} \geq \frac{3}{8} \int_{-1}^{1} u^2 + E(u) - C, \tag{4.3}$$

and then

$$\lim_{\|u\|_{L^2(-1,1)} \to \infty} \frac{I(u)}{\| u \|_{L^2(-1,1)}} \geq \lim_{\|u\|_{L^2(-1,1)} \to \infty} \frac{\left(\frac{3}{8} \| u \|_{L^2(-1,1)} + E(u) - C \right)}{\| u \|_{L^2(-1,1)}} = +\infty.$$

Then, from [19], there exists a minimizing sequence $\{u_n\}$ in $H^1(-1,0) \cap L^2(-1,1)$, with $n \in \mathbb{N}$, such that

$$\frac{1}{2} \int_{-1}^{1} u_n^2 + E(u_n) - \int_{-1}^{1} fu_n \leq C, \quad \forall \quad n \in \mathbb{N}.$$

Therefore $\| u_n \|_{L^2(-1,1)} \leq M$ and $\| u_n \|_{H^1(-1,0)} \leq M$, for all $n \in \mathbb{N}$. Hence, by the compact embedding theorem [[19], Rellich-Kondrachov Compactness Theorem], we can assume, taking a subsequence if necessary, that $u_n \rightharpoonup u$ in $L^2(-1,1)$, $u_n \to u$ in $L^2(-1,0)$, and by the reflexivity of $H^1(-1,0)$, we get that $u \in H^1(-1,0)$.

According to [19], as the functional $I(u)$ is bounded and convex, it follows that $I(u)$ is weakly lower semicontinuous,

$$I(u) \leq \liminf_{n \to \infty} I(u_n). \tag{4.4}$$

Thanks to (4.4) and (4.3), we can conclude that u is actually a minimizer of the variational problem (4.2). The uniqueness follows by the strict convexity of the functional. □

Remark 6. One can also show existence and uniqueness using Hille-Yosida Theorem. In fact, one can show that B_J is closed, its domain $D(B_J)$ is dense in $L^2(-1,1)$ and it holds that for every $\lambda > 0$,

$$\|(\lambda - B_J)^{-1}\|_{L^2(-1,1)} \leq \frac{1}{\lambda}.$$

Since we prove the existence and uniqueness of a mild solution to the local/nonlocal problem, we are ready to show that we can recover the local heat equation at the whole domain, (1.8), from a suitable rescaling of the kernel J. The convergence result proved here will be given at the Mosco sense. For more details, see [1].

Before we prove the main result of this section, we need to define the energy functional associated with the rescaled problem

$$\begin{aligned} E^\varepsilon(w^\varepsilon) := & \frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u^\varepsilon}{\partial x}\right|^2 + \frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^\varepsilon(x-y)\left(v^\varepsilon(y)-v^\varepsilon(x)\right)^2 dydx \\ & + \frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^\varepsilon(x-y)\left(u^\varepsilon(0)-v^\varepsilon(x)\right)^2 dxdy, \end{aligned}$$

if $w^\varepsilon \in D(E^\varepsilon) := H^1(-1,0)\times L^2(0,1)$, and $E^\varepsilon(w) := \infty$, otherwise. Analogously, we define the limit energy functional as

$$E(w) := \frac{1}{2}\int_{-1}^{1}\left|\frac{\partial w}{\partial x}\right|^2 dx,$$

if $w \in D(E) := H^1(-1,1)$, and $E^\varepsilon(w) := \infty$, otherwise.

Given $w_0 \in L^2(-1,1)$, for each $\varepsilon > 0$, let w^ε be the solution to the evolution problem associated with the energy E^ε, and w be the solution associated to the functional E, considering the same initial condition.

Theorem 4.2. *Under the above assumptions, the solutions to the rescaled problem, w^ε, converge to w, the solution of* (1.8)*. For any finite $T > 0$ we have*

$$\lim_{\varepsilon\to 0}\left(\max_{t\in[0,T]} \| w^\varepsilon(\cdot,t) - w(\cdot,t) \|_{L^2(-1,1)}\right) = 0.$$

Proof. To prove this result we will make use of the Brezis-Pazy Theorem (Theorem A.37, see [1]), for a sequence of m-accretive operators $B_{J^\varepsilon} \in L^2(-1,1)$ defined in the beginning of the section. To apply this result we would like to show the convergence of the resolvents, that is

$$\lim_{\varepsilon\to 0}\left(I + B_{J^\varepsilon}\right)^{-1}\phi = \left(I + A\right)^{-1}\phi, \tag{4.5}$$

where $A(w) := -w_{xx}$ is the classic operator for the heat equation, and for every $\phi \in L^2(-1,1)$. If we can prove (4.5) then, by the Brezis-Pazy Theorem, we get the convergence of the solutions w^ε to w in $L^2(-1,1)$ uniformly in $[0,T]$. To prove the convergence of resolvents, we will use a convergence result given by Mosco, checking the following statements:

1) For every $w \in D(E)$, there exists a sequence $\{w^\varepsilon\} \in D(E^\varepsilon)$, such that $w^\varepsilon \to w$ in $L^2(-1,1)$ and
$$E(w) \geq \limsup_{\varepsilon\to 0} E^\varepsilon(w^\varepsilon).$$

2) If, $w^\varepsilon \to w$ weakly in $L^2(-1,1)$ and,
$$E(w) \leq \liminf_{\varepsilon\to 0} E^\varepsilon(w^\varepsilon).$$

Let us start the proof by the assertion 2). We can suppose that the inferior limit is finite, otherwise, there is nothing to prove. Hence, we can assume that $E^\varepsilon(w^\varepsilon) \leq C$. With this in mind, and because all the terms involved in the energy are positive, we have

i) $\displaystyle \frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u^\varepsilon}{\partial x}\right|^2 dx \leq C;$

ii) $\displaystyle \frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^\varepsilon(x-y)(v^\varepsilon(y)-v^\varepsilon(x))^2 dydx \leq C;$

iii) $\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_{0}^{1} J^\varepsilon(x-y)(u^\varepsilon(0)-v^\varepsilon(x))^2dxdy \leq C.$

From *i*), it follows that, there exists a subsequence, also denoted by $\{u^\varepsilon\}$, such that

$$u^\varepsilon \rightharpoonup u \quad \in H^1(-1,0),$$

which implies

$$u^\varepsilon \to u \quad \text{in } L^2(-1,0), \qquad \text{and} \qquad u^\varepsilon \to u \quad \text{uniformly in } (-1,0).$$

Consider the following

$$\begin{aligned}
&\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_{0}^{1} J^\varepsilon(x-y)(u^\varepsilon(y)-v^\varepsilon(x))^2dxdy \\
&\leq \frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_{0}^{1} J^\varepsilon(x-y)(u^\varepsilon(y)-u^\varepsilon(0))^2dxdy \\
&\quad + \underbrace{\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_{0}^{1} J^\varepsilon(x-y)(u^\varepsilon(0)-v^\varepsilon(x))^2dxdy.}_{\leq C}
\end{aligned}$$

Let us show that

$$\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_{0}^{1} J^\varepsilon(x-y)(u^\varepsilon(y)-u^\varepsilon(0))^2dxdy$$

is bounded. Performing a change of variables, $z=\frac{x-y}{\varepsilon}$ and, observing that the $supp(J)=B(0,R)$, we get

$$\begin{aligned}
&\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_{0}^{1} J^\varepsilon(x-y)(u^\varepsilon(y)-u^\varepsilon(0))^2dxdy \\
&= \frac{C_{J,2}}{2\varepsilon^2}\int_{-1}^{0}\int_{\frac{-y}{\varepsilon}}^{\frac{1-y}{\varepsilon}} J(z)dz(u^\varepsilon(y)-u^\varepsilon(0))^2dy.
\end{aligned}$$

Now, taking $\int_{\frac{-y}{\varepsilon}}^{\frac{1-y}{\varepsilon}} J(z)dz = f_\varepsilon(y)$ and, analysing the limit in y we get

$$\frac{C_{J,2}}{2\varepsilon^2}\int_{-R\varepsilon}^{0} f_\varepsilon(y)\frac{u^\varepsilon(y)-u^\varepsilon(0))^2}{\varepsilon}\frac{dy}{\varepsilon}.$$

Changing variables again, taking $w=\frac{y}{\varepsilon}$, it follows that

$$\frac{C_{J,2}}{2}\int_{-R}^{0} f_\varepsilon(\varepsilon w)\left(\frac{(u^\varepsilon(0)-u^\varepsilon(\varepsilon w)}{\varepsilon}\right)^2\varepsilon dw$$

Now, looking for the integrand by applying Holder's inequality, and by the arithmetic-geometric inequality we have

$$\left(\frac{(u^\varepsilon(0)-u^\varepsilon(\varepsilon w)}{\varepsilon}\right)^2 \leq \frac{1}{2}\frac{(-w)}{\varepsilon}\left(\int_{\varepsilon w}^{0}[u_x^\varepsilon]^2(s)ds\right).$$

Then

$$\begin{aligned}
\frac{C_{J,2}}{2}\int_{-R}^{0} f_\varepsilon(\varepsilon w)\left(\frac{(u^\varepsilon(0)-u^\varepsilon(\varepsilon w)}{\varepsilon}\right)^2\varepsilon dw &\leq \frac{C_{J,2}}{2}\int_{-R}^{0}\varepsilon f_\varepsilon(\varepsilon w)\left[\frac{1}{2}\frac{(-w)}{\varepsilon}+\frac{1}{2}\int_{\varepsilon w}^{0}(u_x^\varepsilon(s))^2ds\right]dw \\
&\leq \frac{C_{J,2}}{8}\int_{-R}^{0}(-w)dw + \frac{C_2}{8}\varepsilon\int_{-R}^{0}\left[\int_{\varepsilon w}^{0}(u_x^\varepsilon(s))^2ds\right]dw \\
&= \tilde{C} + \frac{C_{J,2}}{8}\varepsilon\int_{-R}^{0}\underbrace{\left[\int_{-1}^{0}(u_x^\varepsilon(s))^2ds\right]}_{\leq C}dw. \qquad (4.6)
\end{aligned}$$

Therefore, we conclude that

$$\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_{0}^{1} J^\varepsilon(x-y)(u^\varepsilon(y)-v^\varepsilon(x))^2 dxdy$$

is bounded. By (4.6) we can write a new bounded energy functional,

$$\overline{E}(w^\varepsilon) := \frac{1}{2}\int_{-1}^{0}\Big|\frac{\partial u^\varepsilon}{\partial x}\Big|^2 dx + \frac{C_{J,1}}{4\varepsilon^3}\int_{0}^{1}\int_{0}^{1} J^\varepsilon(x-y)\,(v^\varepsilon(y)-v^\varepsilon(x))^2\,dydx$$

$$+\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_{0}^{1} J^\varepsilon(x-y)\,(u^\varepsilon(y)-v^\varepsilon(x))^2\,dxdy \le C.$$

By Lemma 2.1, there exists $k>0$ (independent of ε) such that

$$C \ge \overline{E}(w^\varepsilon) \ge k\frac{1}{\varepsilon^3}\int_{-1}^{1}\int_{-1}^{1} J^\varepsilon(x-y)\,(w^\varepsilon(y)-w^\varepsilon(x))^2\,dydx. \tag{4.7}$$

Using (4.7) it follows from [1] that there exists a subsequence, also denoted $\{w^\varepsilon\}$, which converges in $L^2(-1,1)$ to a limit $w \in H^1(-1,1)$.

Moreover, taking the inferior limit at the first term of the energy $\overline{E}(w^\varepsilon)$,

$$\liminf_{\varepsilon\to 0}\frac{1}{2}\int_{-1}^{0}\Big|\frac{\partial u^\varepsilon}{\partial x}\Big|^2 dx \ge \frac{1}{2}\int_{-1}^{0}\Big|\frac{\partial u}{\partial x}\Big|^2 dx. \tag{4.8}$$

Now, using the fact that

$$\frac{C_{J,1}}{2\varepsilon^3}\int_{0}^{1}\int_{0}^{1} J^\varepsilon(x-y)(v^\varepsilon(y)-v^\varepsilon(x))^2 dxdy$$

is bounded, by Theorem 6.11, in [1], there exists a subsequence, also denoted by $\{v^\varepsilon\}$, such that

$$v^\varepsilon \to v \quad \text{in} \quad L^2(0,1),$$

the limit $v \in H^1(0,1)$ and, moreover

$$\left(\frac{C_{J,1}}{4}J(z)\right)^{1/2}\frac{\overline{v}^\varepsilon(x+\varepsilon z)-v^\varepsilon(x)}{\varepsilon} \rightharpoonup \left(\frac{C_{J,1}}{4}J(z)\right)^{1/2} z\cdot\frac{\partial v}{\partial x}, \tag{4.9}$$

weakly in $L^2(0,1)\times L^2(\mathbb{R})$. Then, taking the limit in the equation (4.9) and, after a change of variables, we have that

$$\liminf_{\varepsilon\to 0}\frac{C_{J,1}}{4\varepsilon^3}\int_{0}^{1}\int_{0}^{1} J^\varepsilon(x-y)\,(v^\varepsilon(y)-v^\varepsilon(x))^2\,dydx \ge \frac{1}{2}\int_{0}^{1}\Big|\frac{\partial v}{\partial x}\Big|^2 dx.$$

Moreover, we have

$$\liminf_{\varepsilon\to 0}\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_{0}^{1} J^\varepsilon(x-y)\,(u^\varepsilon(y)-v^\varepsilon(x))^2\,dxdy \ge 0. \tag{4.10}$$

Therefore, from (4.8)-(4.10) we conclude that

$$\liminf_{\varepsilon\to 0} E^\varepsilon(w^\varepsilon) \ge \frac{1}{2}\int_{-1}^{0}\Big|\frac{\partial u}{\partial x}\Big|^2 dx + \frac{1}{2}\int_{0}^{1}\Big|\frac{\partial v}{\partial x}\Big|^2 dx = E(w).$$

Now let us prove 1). Given $w \in H^1(-1,1)$ we choose as the approximating sequence $w_n^\varepsilon \equiv w^\varepsilon$. In fact, here we are saying that any sequence w_n^ε that converges to w in H^1 would satisfy 1). We have,

$$E^\varepsilon(w^\varepsilon) := \frac{1}{2}\int_{-1}^{0}\Big|\frac{\partial w^\varepsilon}{\partial x}\Big|^2 dx + \frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(y)-w^\varepsilon(x)\right)^2 dydx$$

$$+\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(0)-w^\varepsilon(x)\right)^2 dxdy$$

and we want to show that

$$\limsup_{\varepsilon\to 0} E^\varepsilon(w^\varepsilon) \le E(w). \tag{4.11}$$

The inequality (4.11) is holds if , we verify the following

$$\limsup_{\varepsilon\to 0}\left(\frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(y)-w^\varepsilon(x)\right)^2 dydx\right) = \frac{1}{2}\int_0^1\Big|\frac{\partial w^\varepsilon}{\partial x}\Big|^2 dx.$$

and

$$\limsup_{\varepsilon\to 0}\left(\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(0)-w^\varepsilon(x)\right)^2 dxdy\right) = 0. \tag{4.12}$$

Let us first show (4.12). Performing a change of variables and using Holder's inequality, the equation (4.12) can be written as

$$\begin{aligned}
&\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(0)-w^\varepsilon(x)\right)^2 dxdy\\
&= \frac{C_{J,2}}{2\varepsilon^2}\int_{-1}^{0}\int_{\frac{-y}{\varepsilon}}^{\frac{1-y}{\varepsilon}} J(z)\left(w^\varepsilon(y+\varepsilon z)-w^\varepsilon(0)\right)^2 dzdy\\
&= \frac{C_{J,2}}{2\varepsilon^2}\int_{-R\varepsilon}^{0}\int_{\frac{-y}{\varepsilon}}^{\frac{1-y}{\varepsilon}} J(z)\left(w^\varepsilon(y+\varepsilon z)-w^\varepsilon(0)\right)^2 dzdy\\
&= \frac{C_{J,2}}{2}\int_{-R\varepsilon}^{0}\int_{\frac{-y}{\varepsilon}}^{R} J(z)\left[\int_0^{y+\varepsilon z}\frac{\frac{\partial w^\varepsilon}{\partial x}(s)}{\varepsilon}ds\right]^2 dzdy\\
&\le \frac{C_{J,2}}{2}\int_{-R\varepsilon}^{0}\int_{\frac{-y}{\varepsilon}}^{R} J(z)\left[\int_0^{y+\varepsilon z}\Big|\frac{\partial w^\varepsilon}{\partial x}(s)\Big|^2 ds\right] dz\frac{dy}{\varepsilon}.
\end{aligned}$$

Changing variables again and since $\int_{-R}^{R} J(z)dz = 1$, we obtain

$$\begin{aligned}
&\frac{C_{J,2}}{2}\int_{-R}^{0}\int_{-t}^{R} J(z)\left[\int_0^{\varepsilon(t+z)}\Big|\frac{\partial w^\varepsilon}{\partial x}\Big|^2 ds\right] dzdt\\
&\le \frac{C_{J,2}}{2}\int_{-R}^{0}\int_{-R}^{R} J(z)dz\left[\int_0^{2R\varepsilon}\Big|\frac{\partial w^\varepsilon}{\partial x}\Big|^2 ds\right] dt\\
&\le \frac{C_{J,2}}{2}\int_{-R}^{0}\left[\int_0^{2R\varepsilon}\Big|\frac{\partial w^\varepsilon}{\partial x}\Big|^2 ds\right] dt.
\end{aligned}$$

Now, we observe that, as $\frac{\partial w}{\partial x} \in L^2(-1,1)$ then $|\frac{\partial w}{\partial x}|^2 \in L^1(-1,1)$. Then, we have

$$\int_0^{2R\varepsilon}\Big|\frac{\partial w^\varepsilon}{\partial x}\Big|^2 dz \to 0$$

as $\varepsilon \to 0$, which yields (4.12).

Now, it remains to derive the following

$$\limsup_{\varepsilon\to 0}\left(\frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(y)-w^\varepsilon(x)\right)^2 dydx\right)=\frac{1}{2}\int_0^1\left|\frac{\partial w^\varepsilon}{\partial x}\right|^2 dx.$$

Changing variables, and using Taylor's expansion, it follows that

$$\begin{aligned}
&\left|\frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(y)-w^\varepsilon(x)\right)^2 dydx\right| \\
&=\left|\frac{C_{J,1}}{4\varepsilon^2}\int_0^1\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}} J(z)\left(w^\varepsilon(x+\varepsilon z)-w^\varepsilon(x)\right)^2 dzdx\right| \\
&\leq \frac{C_{J,1}}{4\varepsilon^2}\int_0^1\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}} J(z)\left|w^\varepsilon(x+\varepsilon z)-w^\varepsilon(x)\right|^2 dzdx \\
&=\frac{C_{J,1}}{4\varepsilon^2}\int_0^1\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}} J(z)\left|\frac{\partial w^\varepsilon}{\partial x}(x)\varepsilon z+\frac{1}{2}\frac{\partial^2 w^\varepsilon}{\partial x^2}(\xi)\varepsilon^2 z^2\right|^2 dzdx \\
&\leq \frac{C_{J,1}}{4\varepsilon^2}\int_0^1\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}} J(z)\left(\left|\frac{\partial w^\varepsilon}{\partial x}(x)\varepsilon z\right|+\frac{1}{2}\left|\frac{\partial^2 w^\varepsilon}{\partial x^2}(\xi)\varepsilon^2 z^2\right|\right)^2 dzdx.
\end{aligned}$$

Now, using Minkowski's inequality

$$\begin{aligned}
&\frac{C_{J,1}}{4\varepsilon^2}\int_0^1\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}} J(z)\left(\left|\frac{\partial w^\varepsilon}{\partial x}(x)\varepsilon z\right|+\frac{1}{2}\left|\frac{\partial^2 w^\varepsilon}{\partial x^2}(\xi)\varepsilon^2 z^2\right|\right)^2 dzdx \\
&\leq \frac{C_{J,1}}{4\varepsilon^2}\int_0^1\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}} J(z)\left|\frac{\partial w^\varepsilon}{\partial x}(x)\varepsilon z\right|^2 dzdx+\frac{C_{J,1}}{4\varepsilon^2}\int_0^1\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}} J(z)\left|\frac{1}{2}\frac{\partial^2 w^\varepsilon}{\partial x^2}(\xi)\varepsilon^2 z^2\right| dzdx \\
&\leq \frac{C_{J,1}}{4}\int_0^1\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}} J(z)\left|\frac{\partial w^\varepsilon}{\partial x}(x)\right|^2 |z|^2 dzdx+\varepsilon^2\frac{C_{J,1}}{16}\int_0^1\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}} J(z)\left|\frac{\partial^2 w^\varepsilon}{\partial x^2}(\xi)\right|^2 |z|^4 dzdx \\
&\leq \frac{C_{J,1}}{4}\int_0^1\int_{-R}^{R} J(z)\left|\frac{\partial w^\varepsilon}{\partial x}(x)\right|^2 |z|^2 dzdx+\varepsilon^2\frac{C_{J,1}}{16}\int_0^1\int_{-R}^{R} J(z)\left|\frac{\partial^2 w^\varepsilon}{\partial x^2}(\xi)\right|^2 |z|^4 dzdx \\
&\leq \frac{C_{J,1}}{4}\int_0^1\int_{\mathbb{R}} J(z)|z|^2 dz\left|\frac{\partial w^\varepsilon}{\partial x}(x)\right|^2 dx+\varepsilon^2\frac{C_{J,1}}{16}\int_0^1\int_{\mathbb{R}} J(z)\left|\frac{\partial^2 w^\varepsilon}{\partial x^2}(\xi)\right|^2 |z|^4 dzdx.
\end{aligned}$$

Since $\int_{\mathbb{R}} J(z)|z|^2dz=M(J)$, $\frac{\partial^2 w^\varepsilon}{\partial x^2}$ is bounded, and $\int_{\mathbb{R}} J(z)|z|^4dz$ is finite, we can conclude that

$$\limsup_{\varepsilon\to 0}\left(\varepsilon^2\frac{C_{J,1}}{16}\int_0^1\int_{\mathbb{R}} J(z)|z|^4 dz\left|\frac{\partial^2 w^\varepsilon}{\partial x^2}(\xi)\right|^2 dx\right)=0,$$

and

$$\limsup_{\varepsilon\to 0}\left(\frac{C_{J,1}.M(J)}{4}\int_0^1\left|\frac{\partial w^\varepsilon}{\partial x}(x)\right|^2 |z|^2 dx\right)=\frac{1}{2}\int_0^1\left|\frac{\partial w^\varepsilon}{\partial x}\right|^2 dx.$$

Finally, we have

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} E^\varepsilon(w^\varepsilon)\\
&= \limsup_{\varepsilon\to 0}\left(\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial w^\varepsilon}{\partial x}\right|^2 dx + \frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(y)-w^\varepsilon(x)\right)^2 dydx\right.\\
&\qquad \left. + \frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(0)-w^\varepsilon(x)\right)^2 dxdy\right)\\
&= \limsup_{\varepsilon\to 0}\left(\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial w^\varepsilon}{\partial x}\right|^2 dx\right)\\
&\qquad + \limsup_{\varepsilon\to 0}\left(\frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(y)-w^\varepsilon(x)\right)^2 dydx\right)\\
&\qquad + \limsup_{\varepsilon\to 0}\left(\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(0)-w^\varepsilon(x)\right)^2 dxdy\right)\\
&\le \frac{1}{2}\int_{-1}^{0}\left|\frac{\partial w}{\partial x}\right|^2 dx + \frac{1}{2}\int_0^1\left|\frac{\partial w}{\partial x}\right|^2 dx\\
&= E(w),
\end{aligned}
$$

as we wanted to show. □

Remark 7. Our convergence result can be also read as: take, as before, $w^\varepsilon = (u^\varepsilon, v^\varepsilon)$. Then, for any finite $T > 0$ we have

$$\lim_{\varepsilon\to 0}\left(\max_{t\in[0,T]} \| u^\varepsilon(\cdot,t) - u(\cdot,t) \|_{L^2(-1,0)}\right) = 0$$

and

$$\lim_{\varepsilon\to 0}\left(\max_{t\in[0,T]} \| v^\varepsilon(\cdot,t) - v(\cdot,t) \|_{L^2(0,1)}\right) = 0.$$

The limit pair (u, v) is the unique solution to two heat equations

$$
\begin{cases}
\dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t), & x\in(-1,0), t\in(0,T),\\
\dfrac{\partial u}{\partial x}(-1,t) = 0,
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial v}{\partial t}(x,t) = \dfrac{\partial^2 v}{\partial x^2}(x,t), & x\in(0,1), t\in(0,T),\\
\dfrac{\partial v}{\partial x}(1,t) = 0,
\end{cases}
$$

with the coupling

$$u(0,t) = v(0,t), \qquad \frac{\partial u}{\partial x}(0,t) = \frac{\partial v}{\partial x}(0,t)$$

and initial conditions

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x).$$

Notice that the coupling gives continuity, and continuity of the derivative of the function

$$w(x,t) = \begin{cases} u(x,t), & \text{if} \quad x \in (-1,0) \\ v(x,t), & \text{if} \quad x \in (0,1) \end{cases}$$

that therefore, turns out to be a solution to

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), \quad x \in (-1,1),\, t > 0, \\ \dfrac{\partial w}{\partial x}(-1,t) = \dfrac{\partial w}{\partial x}(1,t) = 0, \quad t > 0, \\ w(x,0) = w_0(x), \quad x \in (-1,1). \end{cases}$$

5 Extension to higher dimensions.

In this final section, we will briefly describe how our results can be extended to higher dimensions. Take Ω, as a bounded smooth domain in $\mathbb{R}^N$ and split it into two subdomains Ω_l and Ω_{nl}, $\Omega = \Omega_l \cup \Omega_{nl}$. Let us call Σ, the interface between Ω_l and Ω_{nl} inside Ω, that is,

$$\Sigma = \overline{\Omega_l} \cap \overline{\Omega_{nl}} \cap \Omega.$$

We will assume that Ω_l has a Lipschitz boundary (in order to solve a heat equation with Newman boundary conditions, we need some regularity of the boundary).

As before, we split $w \in L^2(\Omega)$ as $w = u + v$, with $u = w\chi_{\Omega_l}$ and $v = w\chi_{\Omega_{nl}}$. Fix a nonnegative continuous kernel $G : \Omega_{nl} \times \Sigma \mapsto \mathbb{R}$. For any

$$w = (u,v) \in \mathcal{B} := \left\{ w \in L^2(\Omega) : u|_{\Omega_l} \in H^1(\Omega_l), v \in L^2(\Omega_{nl}) \right\}$$

we define the energy

$$E(u,v) := \frac{1}{2}\int_{\Omega_l} |\nabla u|^2 dx + \frac{C_{J,1}}{4}\int_{\Omega_{nl}}\int_{\Omega_{nl}} J(x-y)\,(v(y)-v(x))^2\, dydx$$
$$+\frac{C_{J,2}}{2}\int_{\Omega_{nl}}\int_{\Sigma} G(x,z)\,(v(x)-u(z))^2\, d\sigma(z)dx.$$

Remark that in this energy we have

$$\int_{\Omega_{nl}}\int_{\Sigma} G(x,z)\,(v(x)-u(z))^2\, d\sigma(z)dx \tag{5.1}$$

as coupling term. This integral can be obtained from an integral of the form

$$\iint_A J(x-y)\,(v(x)-u(z))^2\, dydx$$

assuming the following geometric condition on the interface Σ; for every $x \in \Omega_l$ and every $y \in \Omega_{nl}$ with $x - y \in \operatorname{supp}(J)$ there exists a unique $z \in \Sigma$ that belongs to the segment that joins x with y (hence $z = z(x,y)$). To provide examples, notice that this geometric condition holds if Σ is almost flat. This assumption is useful since, from a probabilistic viewpoint, when a particle wants to jump from $y \in \Omega_{nl}$ to $x \in \Omega_l$ we want that it gets stuck at the interface (and then we want that there exist a unique point on Σ that belongs to the segment $[x,y]$, otherwise, some selection principle has to be assumed and, the selected point on the interface will not depend continuously on x and y, in general). This assumption is used to make the change of variables

$$z = ax + (1-a)y$$

in

$$\iint_A J(x-y)\,(v(x)-u(z))^2\,dydx$$

with $A = \{(x,y) : x \in \Omega_{nl}, y \in \Omega_l,$ with $z \in \Sigma, z = ax + (1-a)y\}$ to obtain the coupling term in our energy, (5.1). The kernel G is nonnegative and comes from the change of variables that involves a jacobian $D(x,z)$.

With this energy, $E(u,v)$, the associated evolution problems reads as,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t), \\ \dfrac{\partial u}{\partial \eta}(z,t) = 0, \qquad z \in \partial\Omega_l \cap \partial\Omega, \\ \dfrac{\partial u}{\partial \eta}(z,t) = C_{J,2} \displaystyle\int_{\Omega_{nl}} G(x,z)(v(y,t) - u(z,t))dx, \qquad z \in \Sigma, \\ u(x,0) = u_0(x). \end{cases} \tag{5.2}$$

for $x \in \Omega_l$, $t > 0$, and

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) = C_{J,1} \displaystyle\int_{\Omega_{nl}} J(x-y)\,(v(y,t) - v(x,t))\,dy - C_{J,2} \int_\Sigma G(x,z)(v(x,t) - u(z,t))d\sigma(z), \\ v(x,0) = v_0(x), \end{cases} \tag{5.3}$$

for $x \in \Omega_{nl}$, $t > 0$.

For this problem (5.2)–(5.3), we can also prove existence and uniqueness following the same steps that we made for the one-dimensional case. In fact, the strategy of building a solution as a fixed point of the composition of the maps that solves the problem for u (given v) and for v (fixing u) also works here. Remark that we obtain a solution $u(x,t)$ that is in $H^1(\Omega_l)$ for $t > 0$ and hence $u(z,t)$ is defined on Σ in the sense of traces (and belongs to $L^2(\Sigma)$ for $t > 0$). The more abstract approach using semigroup theory also works here. Consider the operator

$$B_J(u,v) = \begin{cases} -\Delta u \qquad \text{for} \quad x \in \Omega_l, \\ -C_{J,1} \displaystyle\int_{\Omega_{nl}} J(x-y)(v(y) - v(x))dy + C_{J,2} \int_\Sigma G(x,z)(v(x) - u(z))d\sigma(z) \quad \text{for} \quad x \in \Omega_{nl}, \end{cases}$$

with domain

$$\begin{aligned} D(B_J) := \Big\{ (u,v) : u \in H^2(\Omega_l), v \in L^2(\Omega), \text{ with } \frac{\partial u}{\partial \eta}(z) = 0 \text{ on } \partial\Omega \cap \partial\Omega_l \\ \text{and } \frac{\partial u}{\partial \eta}(z) = -C_{J,2} \int_{\Omega_{nl}} G(x,z)(v(x) - u(z))dx \text{ on } \Sigma, \Big\} \end{aligned}$$

and proceed as we did previously.

The total mass is preserved. In fact, we have

$$\begin{aligned} \frac{\partial}{\partial t} \left(\int_\Omega w(x,t)dx \right) &= \int_{\Omega_l} \Delta u(x,t)dx + C_{J,1} \int_{\Omega_{nl}} \int_{\Omega_{nl}} J(x-y)(v(y,t) - v(x,t))dydx \\ &\quad - C_{J,2} \int_{\Omega_{nl}} \int_\Sigma G(x,z)(v(x,t) - u(z,t)d\sigma(z)dx \\ &= \int_{\partial\Omega_l} \frac{\partial u}{\partial \eta}(x,t)dx - C_{J,2} \int_{\Omega_{nl}} \int_\Sigma G(x,z)(v(x,t) - u(z,t)d\sigma(z)dx \\ &= 0. \end{aligned}$$

The key control of the nonlocal energy,

$$\frac{1}{2}\int_{\Omega_l} |\nabla u|^2 dx + \frac{C_{J,1}}{4}\int_{\Omega_{nl}}\int_{\Omega_{nl}} J(x-y)\,(v(y)-v(x))^2\,dydx + \frac{C_{J,2}}{2}\int_{\Omega_{nl}}\int_{\Sigma} J(x,z)\,(v(x)-u(z))^2\,d\sigma(z)dx$$
$$\geq k\int_{\Omega}\int_{\Omega} J(x-y)\,(w(y)-w(x))^2\,dydx. \tag{5.4}$$

can be proved, as before, arguing by contradiction.

With the key inequality (5.4), we can show that solutions converge to the mean value of the initial condition, as $t\to\infty$ with an exponential rate.

$$\left\| w(\cdot,t) - \fint w_0 \right\|_{L^2(\Omega)} \leq Ce^{-\beta_1 t}, \qquad t>0.$$

In fact, we have that

$$0<\beta_1 = \inf_{w:\int_\Omega w=0} \frac{E(w)}{\displaystyle\int_\Omega (w(x))^2 dx}$$

is strictly positive. This fact can be proved by contradiction as we did before, but it also follows from (5.4) and the results in [1] since we have

$$\beta_1 = \inf_{w:\int_\Omega w=0} \frac{E(w)}{\displaystyle\int_\Omega (w(x))^2 dx} \geq \inf_{w:\int_\Omega w=0} \frac{k\displaystyle\int_\Omega\int_\Omega J(x-y)\,(w(y)-w(x))^2\,dydx}{\displaystyle\int_\Omega (w(x))^2 dx} > 0.$$

The approximation of the heat equation with Neumann boundary conditions under rescales of the kernel is left open. We believe that the result holds with extra assumptions on the coupling kernel G.

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Chapter 3

Splitting methods and numerical approximations for a coupled local/nonlocal diffusion model

The Chapter 3 is composed by the second paper entitled **Splitting methods and numerical approximations for a coupled local/nonlocal diffusion model** that was submitted to the journal Computational & Applied Mathematics.

Splitting methods and numerical approximations for a coupled local/nonlocal diffusion model

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Abstract: In this paper we study a numerical method to approximate solutions to an evolution problem that couples local and nonlocal diffusion operators. The method proposed here takes advantage of the fact that we can show a splitting structure for our evolution equation allowing us to deal with the local and nonlocal parts of the equation separately. This has the capability of being quite flexible, allowing, for example, to consider different meshes in the local and in the nonlocal region. We prove convergence of the method and include some numerical experiments that show some qualitative features of the model.

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1 Introduction

Our main goal in this paper is to perform a splitting strategy and to approximate numerically the local/nonlocal diffusion problem studied in [17].

Local and nonlocal diffusion equations are used in a wide spectrum of applications. Besides the well known and widely studied local diffusion equations (whose prototype is the classical heat equation, see [6, 21]), nonlocal models are also of interest to deal with applied situations. Here we focus on nonlocal equations with smooth kernels (therefore, fractional Laplacians are not considered). For applications of nonlocal equations with non-singular kernels we refer to nonlocal continuum theories such as peridynamics, [30], physics-based nonlocal elasticity, [16], and nonlocal descriptions resulting from homogenization of nonlinear damage models [24]. For general references concerning nonlocal evolution equations we refer to [5, 7, 8, 9, 13, 14, 15, 18, 19], the book [3] and references therein.

In composite materials or heterogeneous environments there are different spacial zones in which different kinds of diffusion take place. In order to model this kind of situation one needs to couple local and nonlocal diffusion equations. There are different strategies for couplings between local and nonlocal models. Let us briefly summarize previous results in [14, 15, 17, 20, 22, 23, 27], see also the review [12]. In [14], local and nonlocal problems are coupled trough a prescribed solid region in which both kinds of equations overlap (the value of the solution in the nonlocal part of the domain is used as a Dirichlet boundary condition for the local part and vice-versa). This kind of coupling gives continuity of the solution in the overlapping region but does not preserve the total mass. In [23] (see also [22, 27]), an energy and its associated gradient flow provides an equation that combines local and nonlocal operators. In this model in the local region the coupling with the nonlocal part appears as an external source in the heat equation (that is complemented with zero flux boundary conditions in the whole boundary of the local region). In probabilistic terms, in the model described in [23], particles may jump across the interface between the two regions but can not pass coming from the local side unless they jump. Finally, in [17], the authors studied local and nonlocal diffusion models in different zones coupled via the fluxes across the surface that separates the two regions. This model is well posed, preserves the total mass of the solutions and converges exponentially fast to the mean value of the initial conditions (see below for a better description of these properties). However, there is no continuity of the solution across the interface between the local and the nonlocal zones.

For numerical approximations of nonlocal problems we refer to [1, 2]. In [14] and [20], numerical schemes using local and nonlocal equations were developed and used to improve the computational accuracy when approximating a purely nonlocal problem. The reference [10] (see also [11]) contains a detailed study of a numerical approximation of a different local/nonlocal model.

As we have mentioned our aim here is to continue the study of the coupling between local and nonlocal evolution problems in [17] and approximate numerically the solutions. In this problem we have two components of the domain in which local and nonlocal diffusion take place. These operators are coupled at the common boundary by a Neumann/Robin-type boundary condition and Neumann boundary conditions on the exterior boundary. We deal with the one-dimensional setting and consider the following problem:

$$
\begin{cases}
\dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t), \quad x \in (-1,0),\ t>0,\\
\dfrac{\partial u}{\partial x}(-1,t) = 0,\ t>0,\\
\dfrac{\partial u}{\partial x}(0,t) = \displaystyle\int_{-1}^{0}\int_{0}^{1} J(x-y)(v(y,t)-u(0,t))dydx,\ t>0,\\
u(x,0) = u_0(x),\ x\in(-1,0),\\
\dfrac{\partial v}{\partial t}(x,t) = \displaystyle\int_{0}^{1} J(x-y)\,(v(y,t)-v(x,t))\,dy - \int_{-1}^{0} J(x-y)dy(v(x,t)-u(0,t)),\ x\in(0,1),\ t>0,\\
v(x,0) = v_0(x),\ x\in(0,1),
\end{cases}
\tag{1.1}
$$

where $J:\mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous kernel with compact support, $supp(J) = [-1,1]$, and $J(0)>0$. Notice that for $x \in (-1,0)$ we have the usual heat equation (the simplest local diffusion equation), while for $x\in(0,1)$ we have a nonlocal diffusion equation with the convolution kernel J. The coupling between the two parts of the domain is done via the fluxes (local and nonlocal) at $x=0$ (the common boundary that separates the local and nonlocal regions), in fact, the local flux $\frac{\partial u}{\partial x}(0,t)$ is equal to the nonlocal one $\int_{-1}^{0}\int_0^1 J(x-y)(v(y,t)-u(0,t))dydx$. At the external boundary of the domain we consider Neumann boundary conditions, and hence the total mass of the initial condition is preserved as time evolves.

As we describe in [17], the model has a probabilistic interpretation as a particle system. As we have mentioned, the whole domain $(-1,1)$ is partitioned in two subdomains, a local, $(-1,0)$ and, a nonlocal domain, $(0,1)$. At the local domain particles can move as a Brownian motion described by the classical Laplacian as infinitesimal generator with zero flux of particles at $x=-1$ (this is represented by a Neumann boundary condition). If a particle reaches $x=0$ it enters into the nonlocal subdomain, which give us the boundary condition at $x=0$. On the other hand, in $(0,1)$ particles follow a pure jump process given by the kernel $J(x-y)$, which means that the probability of a particle, located at $x\in(0,1)$ to jump to a location y is given by $J(x-y)$. Notice that particles may try to jump from $x\in(0,1)$ to $y\in(-1,0)$ trough the point $x=0$, and in this case the particle enters the local domain at $x=0$. Hence, we have a coupling of the local and nonlocal fluxes at $x=0$.

One key feature of the model (1.1) is that it has a gradient flow structure. The associated energy functional is given by

$$
E(u,v) := \frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u}{\partial x}\right|^2 dx + \frac{1}{4}\int_0^1\int_0^1 J(x-y)\,(v(y)-v(x))^2\,dydx + \frac{1}{2}\int_0^1\int_{-1}^{0} J(x-y)\,(v(x)-u(0))^2\,dydx.
\tag{1.2}
$$

Indeed, (1.1) can be written as

$$
(u,v)'(t) = -\partial E\left[(u,v)(t)\right], \qquad t\geq 0,
$$

with $u(0)=u_0$, $v(0)=v_0$. Here $\partial E\left[(u,v)\right]$ denotes the subdifferential of the energy E at the point (u,v). See [17].

Let us summarize some properties of solutions to the local/nonlocal equation under consideration (proved in [17]),

- The problem is well-posed in the sense that there are existence and uniqueness of global solutions. For $u_0\in C(-1,0)$, $v_0\in C(0,1)$ there exists a unique solution $u\in C^1(0,\infty;C(-1,0))$, $v\in C^1(0,\infty;C(0,1))$.

  Moreover, a comparison principle holds. If $u_0 \geq \widetilde{u}_0$ and $v_0\geq \widetilde{v}_0$ then the corresponding solutions verify $u\geq\widetilde{u}$ and $v\geq\widetilde{v}$ for every $t\in[0,\infty)$.

- There is an energy functional, given by (1.2), such that the evolution problem can be view as the gradient flow associated with this energy.

- The total mass of the initial condition is preserved along with the evolution. That is,

$$\int_{-1}^{0} u(x,t)dx + \int_{0}^{1} v(x,t)dx = \int_{-1}^{0} u_0(x)dx + \int_{0}^{1} v_0(x)dx.$$

- Solutions converge exponentially fast to the mean value of the initial condition as $t \to \infty$.

Our first goal in this paper is to apply a splitting technique to solve (1.1). Many problems in nature become increasingly complicated making the models used hard to analyze both theoretically and numerically. In the context of evolution equations, a strategy to deal with this problems is called Operator Splitting Methods. The basic idea of this method is to split the model into a set of subproblems, where each subproblem is simpler and easier to solve using standard tools.

Splitting procedures are widely used for problems that has the form

$$\frac{\partial u}{\partial t} = A(u) + F(u)$$

and work as follows: divide the time interval $[0,T]$, in which you want to solve the problem, into small subintervals $[t_i, t_{i+1})$. Now, in each subinterval we will solve $\frac{\partial u}{\partial t} = A(u)$ or $\frac{\partial u}{\partial t} = F(u)$ (alternating) with initial condition $u(t_i) = u(t_{i-})$. Under some conditions it it proved that this procedure converge to the solution to $\frac{\partial u}{\partial t} = A(u) + F(u)$ as the length of the time subintervals goes to zero, see [25, 26]. The main advantage of splitting techniques is that $\frac{\partial u}{\partial t} = A(u)$ and $\frac{\partial u}{\partial t} = F(u)$ are simpler to solve than the original problem $\frac{\partial u}{\partial t} = A(u) + F(u)$.

For our local/nonlocal problem (1.1) we propose a splitting method taking advantage that the problem has two regions. The main idea consist of to divide the time interval of the problem, $[0,T]$, into small subintervals $[t_i, t_{i+1})$. In each of these subintervals we solve the local evolution equation (freezing the nonlocal part) or we solve the nonlocal problem (freezing the local part). Hence we solve iteratively the two problems,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t), \\ \dfrac{\partial u}{\partial x}(-1,t) = 0, \\ \dfrac{\partial u}{\partial x}(0,t) = \displaystyle\int_{-1}^{0}\int_{0}^{1} J(x-y)(v(y,t_i-) - u(0,t))dydx, \\ u(x,t_i) = u(x,t_i-), \end{cases} \tag{1.3}$$

keeping $v(x,t) \equiv v(x,t_i-)$ for $t \in [t_i, t_{i+1})$; and,

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) = \displaystyle\int_{0}^{1} J(x-y)\left(v(y,t) - v(x,t)\right)dy - \int_{-1}^{0} J(x-y)dy(v(x,t) - u(0,t_j-)), \\ v(x,t_j) = v(x,t_j-), \end{cases} \tag{1.4}$$

keeping $u(x,t) \equiv u(x,t_j-)$ for $t \in [t_j, t_{j+1})$. See Section 2 for more details.

Our first result shows that this procedure converges to a solution to our original problem (1.1).

- The splitting method converges to the unique solution of (1.1) as the length of the time intervals $\Delta t = t_{i+1} - t_i$ goes to zero.

Notice that, here the splitting idea has a different flavor than usual. In the classical case, the splitting is based on a partition of the right hand side of the evolution equation into two simpler parts. Here, the splitting is based on a spacial partition of the domain (taking into account the two different regions in which different diffusions take place). We alternate the procedure of advancing in time with the local equation while freezing the nonlocal part (that is used in the boundary condition), solving (1.3), and, alternatively, we advance with the nonlocal equation, keeping the local part fixed (the local part appears in the nonlocal equation), solving (1.4).

Also remark that this result provides an alternative proof of existence for solutions to (1.1) (the two previous proofs contained in [17] use a fixed point or abstract semigroup theory).

Now, we are ready to perform a discretization of (1.1) in such a way that the discrete problem share the same properties of the continuous problem listed above (well-posedness, comparison principle, conservation of mass and convergence to the mean value of the initial datum). First, we introduce a semi discrete scheme (discretizing only in space). We approximate the continuous solution $u(x,t)$, for $(x,t) \in (-1,0)\times\mathbb{R}$ and $v(x,t)$, for $(x,t)\in(0,1)\times\mathbb{R}$, by discrete values u_j and v_i, respectively, with $i,j\in\mathbb{Z}$. In order to do so we recall that the support of J is assumed to be the interval $\mathcal{T}=[-1,1]$. For simplicity, let us consider a uniform mesh $x_1,\cdots,x_N,z_1,\cdots,z_N$ of the interval $\mathcal{T}$ of size $h=\frac{2}{N}$. Here, we call $x_1,\cdots,x_N$ the points of the mesh in $[-1,0]$ and $z_1,\cdots,z_N$ those in $[0,1]$. In order to obtain a nontrivial scheme we assume that $h \ll 1$.

Then, the numerical approximation of the problem (1.1), considering a spatial discretization provides the following ODE system, for each node x_j and z_i,

$$\begin{cases} u_1'(t) = \dfrac{u_2(t)-u_1(t)}{h^2}, \\ u_j'(t) = \dfrac{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}{h^2}, \quad j=2,\cdots,N-1 \\ u_N'(t) = \dfrac{u_{N-1}(t)-u_N(t)}{h^2} + h\displaystyle\sum_{j=1}^N\sum_{i=1}^N J(z_i-x_j)(v_i(t)-u_N(t)), \\ v_i'(t) = h\displaystyle\sum_{k=1}^N J(z_i-z_k)(v_k(t)-v_i(t)) - h\sum_{j=1}^N J(z_i-x_j)(v_i(t)-u_N(t)), \quad i=1,\cdots,N \\ u_j(0)=u_{j0}, \quad j=1,\cdots,N \\ v_i(0)=v_{i0}, \quad i=1,\cdots,N. \end{cases} \tag{1.5}$$

Next, we discretize the time variable using the explicit Euler method to obtain a fully discrete scheme

$$\begin{cases} \dfrac{u_1^{l+1}-u_1^l}{\tau_l} = \dfrac{u_2^l-u_1^l}{h^2}, \\ \dfrac{u_j^{l+1}-u_j^l}{\tau_l} = \dfrac{u_{j+1}^l-2u_j^l+u_{j-1}^l}{h^2}, \quad j=2,\cdots,N-1 \\ \dfrac{u_N^{l+1}-u_N^l}{\tau_l} = \dfrac{u_{N-1}^l-u_N^l}{h^2} + h\displaystyle\sum_{j=1}^N\sum_{i=1}^N J(z_i-x_j)(v_i^l-u_N^l), \\ \dfrac{v_i^{l+1}-v_i^l}{\tau_l} = h\displaystyle\sum_{k=1}^N J(z_i-z_k)(v_k^l-v_i^l) - h\sum_{j=1}^N J(z_i-x_j)(v_i^l-u_N^l), \quad i=1,\cdots,N \\ u_j^0=u_{j0}, \quad j=1,\cdots,N \\ v_i^0=v_{i0}, \quad i=1,\cdots,N, \end{cases} \tag{1.6}$$

for $l>0$.

We will describe in more detail how these schemes are obtained in Section 3.

Our results concerning the semidiscrete and fully discrete schemes will be described in detail in the next sections. To summarize them, we will verify that both schemes preserve the main properties of the continuous problem, namely:

- The problem is well-posed and a comparison principle holds.

- The solutions of the numerical schemes converge uniformly to the continuous solution for mesh size h and time step τ small enough.
- There is an energy functional such that the semidiscrete problem is its corresponding the gradient flow associated.
- The total mass of the initial condition is preserved along the evolution.
- Solutions converge exponentially fast to the mean value of the initial condition as $t \to \infty$.

We remark that our results also hold when we deal with approximations in a multidimensional domain. The proofs are quite similar to the one-dimensional case. In fact, we can just use finite elements with mass lumping to discretize the local part and finite differences for the nonlocal part (approximating integrals with sums over the nodes inside the support of $J(\cdot - x_j)$).

ORGANIZATION OF THE PAPER: The paper is organized as follows: In Section 2, we prove that the splitting strategy applied to the continuous problem works. In Section 3, we study the semidiscrete and the fully discrete approximations. Finally, in Section 4, we show some numerical experiments that illustrate our results.

2 Splitting method.

In this section, we provide an alternative method to solve the problem (1.1). We refer to [25, 26] and references therein. In order to illustrate the splitting technique (recall what we have mentioned in the Introduction), let us consider the evolution abstract Cauchy problem

$$\begin{cases} \dfrac{dV}{dt} + B(V) = 0, \\ V(0) = V_0, \end{cases} \tag{2.1}$$

where B is a suitable operator. It is possible to decompose B as a sum of elementary operators, say $B = B_1 + \cdots + B_l$, which the decomposition B_j give equations that are simpler to solve

$$\begin{cases} \dfrac{dV^j}{dt} + B_j(V^j) = 0, \\ V^j(0) = V_0, \quad j = 1, \cdots, l. \end{cases} \tag{2.2}$$

Consider $V^j(t) = S_t^j V_0$ the solution of (2.2). Once the solution is know for all the subproblems, it is possible to choose a small time-step, $\triangle t$, and apply the sub-operators sequentially to construct an approximate solution of (2.1). This process can be represented by

$$V(n\triangle t) \approx [S^l_{\triangle t} \cdots S^1_{\triangle t}]^n V_0.$$

In the limit, we expect that this approximation will converge to the true solution

$$V(t) = \lim_{\triangle t \to 0, n\to\infty t = n\triangle t} [S^l_{\triangle t} \cdots S^1_{\triangle t}]^n V_0.$$

Before we describe the Operator Splitting method to our evolution problem, we first need to define precisely what is meant by a weak entropy solution of the initial value problems

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t), \\ \dfrac{\partial u}{\partial x}(-1,t) = 0, \\ \dfrac{\partial u}{\partial x}(0,t) = \displaystyle\int_{-1}^{0}\int_{0}^{1} J(x-y)(v(y,t) - u(0,t))dydx, \\ u(x,0) = u_0(x), \end{cases} \tag{2.3}$$

for $x \in (-1,0)$, $t>0$. And for the nonlocal domain,

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) = \displaystyle\int_0^1 J(x-y)\left(v(y,t)-v(x,t)\right)dy - \int_{-1}^0 J(x-y)dy(v(x,t)-u(0,t)), \\ v(x,0) = v_0(x), \end{cases} \tag{2.4}$$

for $x \in (0,1)$, $t>0$.

According to [26], a function $u(x,t)$ (respectively $v(x,t)$) is called a weak solution for (2.3) (respectively (2.4)), for time $[0,T]$ if, for all suitable test functions $\varphi \in C^\infty([-1,1]\times[0,T])$, it satisfies

$$\int_0^T\int_{-1}^0\int_0^1 J(x-y)(v(y,t)-u(0,t))dydxdt - \int_0^T\int_{-1}^0 u_x(x,t)\varphi_x(x,t)dxdt \\ + \int_0^T\int_{-1}^0 u(x,t)\varphi_t(x,t)dxdt + \int_{-1}^0 u(x,0)\varphi(x,0)dx = 0, \tag{2.5}$$

for u, with $\varphi(x,T)\equiv 0$ and

$$\int_0^T\int_0^1\int_0^1 J(x-y)(v(y,t)-v(x,t))dydxdt - \int_0^T\int_0^1\int_{-1}^0 J(x-y)(v(x,t)-u(0,t))dydxdt \\ + \int_0^T\int_0^1 v(x,t)\varphi_t(x,t)dxdt + \int_0^1 v(x,0)\varphi(x,0)dx = 0 \tag{2.6}$$

for v.

Our analysis will be divided into two steps. We first show that the splitting indeed produces a sequence of functions that converges to a solution of the equations (2.3)–(2.4). The second part consist to verify if the limit of this sequences is in fact the solution of the problem, which means, if the limit u and v satisfy (2.5) and (2.6), respectively.

Let us apply the splitting method to our evolution problem. The basic idea is to consider the split as a partition of the domain $[-1,1]$, it means that our our analysis will be carried out in the local and a nonlocal problem separately.

Fix $T>0$, $\triangle t>0$ and let $N>0$ be such that $N\triangle t = T$. Let u^n be the approximate solution of u and let v^n be the approximate solution of v at a fixed time $t_j = j\triangle t$, where $j=1,\cdots,N$ and $n\in\mathbb{N}$. If j is even we have the following subproblems

$$\begin{cases} \dfrac{\partial u^n}{\partial t}(x,t) = \dfrac{\partial^2 u^n}{\partial x^2}(x,t), \\ \dfrac{\partial u^n}{\partial x}(-1,t) = 0, \\ \dfrac{\partial u^n}{\partial x}(0,t) = \displaystyle\int_{-1}^0\int_0^1 J(x-y)(v^n(y,t_{j-1})-u^n(0,t))dydx, \\ u^0(x,0) = u_0(x), \end{cases} \tag{2.7}$$

for $x\in(-1,0)$ and $t\in(t_{j-1},t_j)$. And for the nonlocal domain,

$$\begin{cases} \dfrac{\partial v^n}{\partial t}(x,t) = \displaystyle\int_0^1 J(x-y)\left(v^n(y,t)-v^n(x,t)\right)dy - \int_{-1}^0 J(x-y)dy(v^n(x,t)-u(0,t_j)), \\ v^0(x,0) = v_0(x), \end{cases} \tag{2.8}$$

for $x\in(0,1)$ and $t\in(t_j,t_{j+1})$.

For j odd we have the following approximation

$$\begin{cases} \dfrac{\partial u^n}{\partial t}(x,t_j) = \dfrac{\partial^2 u^n}{\partial x^2}(x,t_j), \\ \dfrac{\partial u^n}{\partial x}(-1,t_j) = 0, \\ \dfrac{\partial u^n}{\partial x}(0,t_j) = \displaystyle\int_{-1}^{0}\int_{0}^{1} J(x-y)(v^n(y,t) - u^n(0,t_j))dydx, \\ u^0(x,0) = u_0(x), \end{cases}$$

for $x \in (-1,0)$ and $t \in (t_{j-1}, t_j)$. And for the nonlocal domain,

$$\begin{cases} \dfrac{\partial v^n}{\partial t}(x,t_j) = \displaystyle\int_0^1 J(x-y)\left(v^n(y,t_j) - v^n(x,t_j)\right)dy - \int_{-1}^{0} J(x-y)dy(v^n(x,t_j) - u(0,t)), \\ v^0(x,0) = v_0(x), \end{cases}$$

for $x \in (0,1)$ and $t \in (t_j, t_{j+1})$.

Assume that u_0 and v_0 are compactly supported functions in $L^\infty(-1,0) \cap BV(-1,0)$ and $L^\infty(0,1) \cap BV(0,1)$, respectively. Let $u^0(x,0) = u^0(x,t_0) = u_0(x)$ and $v^0(x,0) = v^0(x,t_0) = v_0(x)$. Define

$$u^n(x,t) = S_{\triangle t}(u_0,v_0) = (u(x,t), v_0(x)), \quad t \in [t_{j-1}, t_j],$$

$$v^n(x,t) = H_{\triangle t}(u_0,v_0) = (u_0(x), v(x,t)), \quad t \in [t_j, t_{j+1}].$$

Therefore, the complete solution of the problem (2.3)–(2.4) can be approximated by

$$w^n(x,t) \approx [H_{\triangle t} S_{\triangle t}]^n(u_o, v_o), \quad t \in [t_j, t_j + 2], \quad j = 0, \cdots, N, \tag{2.9}$$

where

$$w(x,t) = \begin{cases} u(x,t), & \text{for} \quad (x,t) \in (-1,0) \times (t_{j-1}, t_j), \quad j = 1, \cdots, N \\ v(x,t), & \text{for} \quad (x,t) \in (0,1) \times (t_j, t_{j+1}), \quad j = 1, \cdots, N. \end{cases}$$

We next establish convergence of the splitting approximations to an entropy weak solution. To this end, recall that the space $BV(\mathbb{R})$ consists of all $L^1_{loc}(\mathbb{R})$ functions $y(x)$ whose first order derivative is represented by a (locally) finite Borel measure. The total variation of y is by definition the total mass of this measure, which means

$$|y|_{BV} = \int_{\mathbb{R}} \left| \frac{dy}{dx} \right|.$$

We then have three main lemmas, which ensure the existence of a convergent subsequence.

Lemma 2.1. *There are constants $C_1, C_2 > 0$ such that we have the following maximum principle*

$$||u(\cdot,t)||_\infty \le C_1 ||(u_0,v_0)||_\infty = C_1 \left(\max_{x\in[-1,0]} \{|u_0(x)|\} + \max_{x\in[0,1]} \{|v_0(x)|\} \right), \quad t \in [0,T]. \tag{2.10}$$

$$||v(\cdot,t)||_\infty \le C_2 ||(u_0,v_0)||_\infty = C_2 \left(\max_{x\in[-1,0]} \{|u_0(x)|\} + \max_{x\in[0,1]} \{|v_0(x)|\} \right), \quad t \in [0,T]. \tag{2.11}$$

Proof. We will prove the estimates above using the comparison principle. Let us start with $||u(\cdot,t)||_\infty$. We have,

$$||u(\cdot,t)||_\infty \le ||u(\cdot,t) - \tilde{u}(\cdot,t)||_\infty + ||\tilde{u}(\cdot,t)||_\infty,$$

where $\tilde{u}$ is a solution to the following problem

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t}(x,t) = \dfrac{\partial^2 \tilde{u}}{\partial x^2}(x,t), \\ \dfrac{\partial \tilde{u}}{\partial x}(-1,t) = 0, \\ \dfrac{\partial \tilde{u}}{\partial x}(0,t) = -C\tilde{u}(0,t), \\ \tilde{u}(x,0) = u_0(x), \end{cases} \tag{2.12}$$

for $x \in (-1,0)$, $t > 0$ and $C = \int_{-1}^{0}\int_{0}^{1} J(x-y)dydx$. Since the kernel is symmetric and $\int_{-1}^{1} J(r)dr = 1$, we have that $0 < C < 1$. Now, if we define $w(x,t) = u(x,t) - \tilde{u}(x,t)$, the function w satisfies

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), \\ \dfrac{\partial w}{\partial x}(-1,t) = 0, \\ \dfrac{\partial w}{\partial x}(0,t) = -Cw(0,t) + \displaystyle\int_{-1}^{0}\int_{0}^{1} J(x-y)v_0(x)dydx, \\ w(x,0) = 0, \end{cases} \tag{2.13}$$

for $x \in (-1,0)$, $t > 0$ and $C = \int_{-1}^{0}\int_{0}^{1} J(x-y)dydx$.

Note that, we can get an estimate for $\int_{-1}^{0}\int_{0}^{1} J(x-y)v_0(x)dydx$. We have

$$\left|\int_{-1}^{0}\int_{0}^{1} J(x-y)v_0(x)dydx\right| \le \int_{-1}^{0}\int_{0}^{1} J(x-y)|v_0(x)|dydx \le \int_{-1}^{0}\int_{0}^{1} J(x-y)\max_{x\in[0,1]}\Big|v_0(x)\Big|dydx \le C\|v_0\|_\infty.$$

Hence, we consider

$$z(x,t) = \frac{w(x,t)}{C\|v_0\|_\infty},$$

and then, $z(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \dfrac{\partial^2 z}{\partial x^2}(x,t), \\ \dfrac{\partial z}{\partial x}(-1,t) = 0, \\ \dfrac{\partial z}{\partial x}(0,t) \le -Cz(0,t) + 1 \quad \textit{and} \quad \dfrac{\partial z}{\partial x}(0,t) \ge -C_1 z(0,t) - 1, \\ z(x,0) = 0, \end{cases} \tag{2.14}$$

for $x \in (-1,0)$, $t > 0$.

Let us introduce an auxiliary function. Given $\xi < 0$ and $0 < a < 1$ we can define

$$g(\xi) = \frac{1}{a}f(\xi a), \quad \text{with} \quad g'(0) = f'(0) = 1. \tag{2.15}$$

Given $\xi_0 > 1$, f is increasing in $(-\xi_0, 0]$, $C^2(-\xi_0, 0)$, and $f \equiv 1$ in $(-\infty, -\xi_0]$.

Fix $T < \frac{a^2}{2\xi_o^2}$. For each $t \in [0,T]$ and $x \in [-1,0]$ we can define

$$\overline{z^\star}(x,t) = (T+t)^{1/2} g\left(\frac{x}{(T+t)^{1/2}}\right),$$

as g given by (2.15). We aim to prove that $\overline{z^\star}$ is a supersolution for (2.14).

We have that
$$\frac{\partial \overline{z^\star}}{\partial t} \geq \frac{\partial^2 \overline{z^\star}}{\partial x^2}.$$
In fact, differentiating $\overline{z^\star}$ with respect to t and x, we obtain that it is enough to have
$$\frac{1}{2} g\left(\frac{x}{(T+t)^{1/2}}\right) - \frac{1}{2} x(T+t)^{-1/2} g'\left(\frac{x}{(T+t)^{1/2}}\right) \geq g''\left(\frac{x}{(T+t)^{1/2}}\right).$$
Observe that, since $x \in [-1,0]$ and $g'\left(\frac{x}{(T+t)^{1/2}}\right) > 0$, we only need to verify that
$$\frac{1}{2} g\left(\frac{x}{(T+t)^{1/2}}\right) \geq g''\left(\frac{x}{(T+t)^{1/2}}\right). \tag{2.16}$$
Let us call $\eta = \frac{x}{(T+t)^{1/2}}$. According to the definition of g, to prove (2.16) is equivalent to prove
$$\frac{1}{2a} f(a\eta) \geq a f''(a\eta). \tag{2.17}$$
We know that, for each $\xi \leq 0$ and $0 < a < 1$,
$$\frac{f(\xi a)}{2} = \begin{cases} 1/2, & \text{if } \xi a < -\xi_0, \\ \dfrac{f(\xi a)}{2}, & \text{if } -\xi_0 \leq \xi a < 0. \end{cases}$$
Moreover, as $f \in C^2(-\xi_0, 0)$ and increasing in the same interval, we obtain
$$f''(\xi a) \leq \begin{cases} 0, & \text{if } \xi a < -\xi_0, \\ M, & \text{if } -\xi_0 \leq \xi a < 0, \end{cases}$$
where $M = \max_{-\xi_0 \leq \xi \leq 0} |f''(\xi)|$. Hence, given M, we can choose $0 < a < 1$ in order to have the estimate $\frac{1}{2} \geq M a^2$. With the help of this inequality let us verify (2.17). Indeed,

a) If $-\xi_0 \leq \xi a < 0$, it follows that
$$\frac{f(\xi a)}{2} \geq \frac{1}{2} \geq M a^2 \geq f''(\xi a) a^2.$$

b) If $\xi a < -\xi_0$, we have that
$$\frac{f(\xi a)}{2} \geq \frac{1}{2} \geq 0 \geq f''(\xi a) a^2.$$

Now, we want to verify that $\overline{z^\star}$ satisfies
$$\frac{\partial \overline{z^\star}}{\partial x}(-1,t) \leq 0.$$
At $x = -1$ we have,
$$\frac{\partial \overline{z^\star}}{\partial x}(-1,t) = g'\left(\frac{-1}{(T+t)^{1/2}}\right) = f'\left(\frac{-a}{(T+t)^{1/2}}\right). \tag{2.18}$$
We know that $f \equiv 1$ in $(-\infty, -\xi_0)$. Then, taking $T < \frac{a^2}{2\xi_0^2}$, we obtain that $\frac{-a}{(T+t)^{1/2}} < -\xi_0$ and therefore
$$f'\left(\frac{-a}{(T+t)^{1/2}}\right) = 0$$
and we obtain (2.18) as desired.

We also need to check that

$$\frac{\partial \overline{z^\star}}{\partial x}(0,t) \geq -C_1 \overline{z^\star}(0,t) + 1.$$

Differentiating $\overline{z^\star}$ with respect to x, we need to prove that it holds that

$$g'(0) \geq -C_1 (T+t)^{1/2} g(0) + 1.$$

As $g'(0) = f'(0) = 1$ and $g(0) \geq 1$, we get

$$1 \geq -C_1 (T+t)^{1/2} g(0) + 1.$$

Finally, we aim to verify that $\overline{z^\star}(x,0) \geq 0$. Indeed, we have

$$\overline{z^\star}(x,0) = (T)^{1/2} \underbrace{g\left(\frac{x}{T^{1/2}}\right)}_{>0} > 0.$$

We proved that $\overline{z^\star}$ is a supersolution of (2.14).

The fact that

$$\underline{z^\star}(t,x) = -(T+t)^{1/2} g\left(\frac{x}{(T+t)^{1/2}}\right)$$

is a subsolution for the problem (2.14) can be proved analogously.

So, by the comparison principle, the solution $w(x,t)$ of the problem (2.13), verifies

$$\underline{z^\star}(x,t) \leq z^\star(x,t) \leq \overline{z^\star}(x,t).$$

Hence, we get the estimate

$$|\overline{z^\star}| \leq \max_{x\in[-1,0]} \max_{t\in[0,T]} \left| (T+t)^{1/2} g\left(\frac{x}{(T+t)^{1/2}}\right) \right| = \frac{1}{a}(2T)^{1/2} < \frac{1}{\xi_o} < 1.$$

Therefore, going back to our original variable w we have obtained that

$$\frac{|w(x,t)|}{C \parallel v_0 \parallel_\infty} = z(x,t) \leq z^\star(x,t) \leq \overline{z^\star} \leq 1,$$

which implies that

$$||u - \tilde{u}||_\infty \leq C \parallel v_0 \parallel_\infty . \tag{2.19}$$

Now, we need to prove that $\|\tilde{u}(\cdot,t)\|_\infty \leq C(T_0)\|u_0\|_\infty$. Recall that $\tilde{u}$ satisfies the problem (2.12). Define $h(x,t) = \frac{\tilde{u}(x,t)}{||u_0||_\infty}$. Then the solution h satisfy the following problem

$$\begin{cases} \dfrac{\partial h}{\partial t} = \dfrac{\partial^2 h}{\partial x^2}, \\ \dfrac{\partial h}{\partial x}(-1,t) = 0, \\ \dfrac{\partial h}{\partial x}(0,t) = -Ch(0,t), \\ h(x,0) \leq 1, \quad and \quad h(x,0) \geq -1, \end{cases}$$

for $x \in (-1,0)$ and $t > 0$. Consider $\overline{u^\star}(x,t) = [(x+1)^2 + 1]e^{\alpha t}$. As we have done before, one can check that $\overline{u^\star}$ is indeed a supersolution for (2.12). So, by the comparison principle, we find that

$$|\overline{u^\star}| \leq \max_{t\in[0,T_o]} \max_{x\in[-1,0]} \left| [(x+1)^2 + 1] e^{\frac{2t}{(x+1)^2+1}} \right| = 2e^{T_0} = k(T_0),$$

$k > 0$ is a constant that depends on T_0.

Hence

$$\|\tilde{u}\|_\infty \le k(T_0)\|u_0\|_\infty. \tag{2.20}$$

From (2.19) and (2.20) it follows that

$$\|u\|_\infty \le ||u-\tilde{u}||_\infty + ||\tilde{u}||_\infty \le C||v_0||_\infty + k(T_o)||u_0||_\infty \le \max\{C||v_0||_\infty + k(T_0)||u_0||_\infty\} \le C||(u_0,v_0)||_\infty,$$

where $C_1 = \max\{C + k(T_0)\}$, and hence we get the estimate (2.10).

Now, we want to deal with the estimate for v in (2.11). Let us define $v^\star(x,t) = \frac{v(\cdot,t)}{||u_0||_\infty + ||v_0||_\infty}$. Observe that $v^\star$ satisfies

$$\begin{cases} \dfrac{\partial v^\star}{\partial t}(x,t) = \displaystyle\int_0^1 J(x-y)\left(v^\star(y,t) - v^\star(x,t)\right)dy - \int_{-1}^0 J(x-y)dy\left(v^\star(x,t) - \frac{u_o(0)}{||u_0||_\infty + ||v_0||_\infty}\right), \\ v^\star(x,0) \le 1, \quad \textit{and} \quad v^\star(x,0) \ge -1, \end{cases} \tag{2.21}$$

for $x \in (0,1)$ and $t > 0$.

Fix $T_0 > 0$. For each $t \in [0,T_0]$ and $x \in [0,1]$ we will define $\overline{v}(x,t) = ||v_0||_\infty + k_1 t$, where

$$k_1 \ge \max_{x\in[0,1]}\left\{\frac{a(x)}{||u_0||_\infty + ||v_0||_\infty}u_o(0)\right\}.$$

We have that $\overline{v}$ is supersolution for the problem (2.21). Then, it follows from the comparison principle that

$$\underline{v}(x,t) \le v^\star(x,t) \le \overline{v}(x,t).$$

Moreover, we can provide an estimate for $\overline{v}$ as

$$|\overline{v}| \le \max_{x\in[0,1]}\max_{t\in[0,T_0]}\{||v_0||_\infty + k_1 t\} = ||v_0||_\infty + M_1 T_0||u_0||_\infty = C_2,$$

with $M_1 = \max_{x\in[0,1]}\left\{\frac{a(x)}{||u_0||_\infty + ||v_0||_\infty}\right\}$. Hence $||v(\cdot,t)||_\infty = ||v^\star||_\infty \le C_2\left(||u_0||_\infty + ||v_0||_\infty\right)$.

Analogously, consider $\underline{v}(x,t) = -||v_0||_\infty - k_1 t$ as a subsolution to complete the proof. □

Lemma 2.2. *We have the following bound of the total variation*

$$|u(\cdot,t)|_{BV} \le |(u_0,v_0)|_{BV}, \quad t \in [0,T].$$

Proof. By the continuous embedding $BV(-1,0) \subset L^2(-1,0)$ we only need to prove that $||u_x||^2_{L^2(-1,0)} \le C$, where $C > 0$ is a constant. Multiplying the problem (2.7) for u and integrating from -1 to 0 we get

$$\begin{aligned} \int_{-1}^0 \frac{\partial u}{\partial t} u dx &= \int_{-1}^0 \frac{\partial^2 u}{\partial x^2} u dx = u(0,t)\frac{\partial u}{\partial t}(0,t) - u(-1,t)\frac{\partial u}{\partial t}(-1,t) - \int_{-1}^0 \left(\frac{\partial u}{\partial x}\right)^2 dx \\ &= u(0,t)\int_{-1}^0\int_0^1 J(x-y)(v_0(x) - u(0,t))dydx - \int_{-1}^0 \left(\frac{\partial u}{\partial x}\right)^2 dx. \end{aligned}$$

We have the following estimate

$$\begin{aligned} \left|\int_{-1}^0 \left(\frac{\partial u}{\partial x}\right)^2 dx\right| &= \left|u(0,t)\int_{-1}^0\int_0^1 J(x-y)(v_0(x) - u(0,t))dydx - \int_{-1}^0 \frac{\partial u}{\partial t} u dx\right|, \\ &= \left|-u(0,t)^2\int_{-1}^0\int_0^1 J(x-y)dydx + \int_{-1}^0\int_0^1 J(x-y)v_0(x)u(0,t)dydx - \int_{-1}^0 \frac{\partial u}{\partial t} u dx\right|, \\ &\le \left|u(0,t)^2\int_{-1}^0\int_0^1 J(x-y)dydx\right| + \left|\int_{-1}^0\int_0^1 J(x-y)v_0(x)u(0,t)dydx\right| + \left|\int_{-1}^0 \frac{\partial u}{\partial t} u dx\right|, \\ &\le |u(0,t)|^2\int_{-1}^0\int_0^1 J(x-y)dydx + \int_{-1}^0\int_0^1 J(x-y)|v_0(x)u(0,t)|dydx + \int_{-1}^0 \left|\frac{\partial u}{\partial t}\right| |u| dx. \end{aligned}$$

Since u is Lipschitz (by Lemma 2.3), it follows that $\frac{\partial u}{\partial t}$ is bounded. By Lemma 2.1, it follows that $u(\cdot,t)$ and $v(\cdot,t)$ are bounded for each $t\in[0,T]$, which means that $\|u_x\|^2_{L^2(-1,0)}\leq M$. This concludes the proof. □

Lemma 2.3. *There are constants $M_1, M_2>0$, independent of τ_1,τ_2, such that*

$$||u(\cdot,\tau_1)-u(\cdot,\tau_2)||_1\leq M_1|\tau_1-\tau_2|,\quad \tau_1,\tau_2\in[0,T],$$
$$||v(\cdot,\tau_1)-v(\cdot,\tau_2)||_1\leq M_2|\tau_1-\tau_2|,\quad \tau_1,\tau_2\in[0,T].$$

Proof. We first want to establish the weak Lipschitz continuity in time of the L^1-norm, it means, multiplying the u equation (given by (2.7)) by a test function $\varphi(x)$ and integrating in $(-1,0)\times(\tau_1,\tau_2)$ we have

$$\left|\int_{\tau_1}^{\tau_2}\int_{-1}^{0}\frac{\partial u}{\partial t}\varphi(x)dxdt\right|=\left|\int_{-1}^{0}\int_{\tau_1}^{\tau_2}\frac{\partial u}{\partial t}\varphi(x)dtdx\right|=\left|\int_{-1}^{0}(u(x,\tau_2)-u(x,\tau_1))\varphi(x)dx\right|. \tag{2.22}$$

On the other hand, integrating the right side by parts, we get

$$\begin{aligned}
&\left|\int_{\tau_1}^{\tau_2}\int_{-1}^{0}\frac{\partial^2 u}{\partial x^2}\varphi(x)dxdt\right|=\left|\int_{\tau_1}^{\tau_2}\left[\frac{\partial u}{\partial x}\varphi(x)|_{-1}^{0}-\int_{-1}^{0}\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x}dx\right]dt\right|\\
&=\left|\int_{\tau_1}^{\tau_2}\left[\int_{-1}^{0}\int_{0}^{1}J(x-y)(v(y,t)-u(0,t))dydx\varphi(0)-\int_{-1}^{0}\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x}dx\right]dt\right|\\
&\leq\left|\int_{\tau_1}^{\tau_2}\int_{-1}^{0}\int_{0}^{1}J(x-y)(v(y,t)-u(0,t))dydx\varphi(0)dt\right|+\left|\int_{\tau_1}^{\tau_2}\int_{-1}^{0}\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x}dxdt\right|\\
&\leq\int_{\tau_1}^{\tau_2}\int_{-1}^{0}\int_{0}^{1}J(x-y)|v(y,t)-u(0,t)|dydx\varphi(0)dt+\int_{\tau_1}^{\tau_2}\int_{-1}^{0}\left|\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x}\right|dxdt\\
&\leq\int_{\tau_1}^{\tau_2}\int_{-1}^{0}\int_{0}^{1}J(x-y)|v(y,t)|dydx\varphi(0)dt+\int_{\tau_1}^{\tau_2}\int_{-1}^{0}\int_{0}^{1}J(x-y)|u(0,t)|dydx\varphi(0)dt\\
&+\int_{\tau_1}^{\tau_2}\int_{-1}^{0}\left|\frac{\partial u}{\partial x}\right|\left|\frac{\partial \varphi}{\partial x}\right|dxdt,\\
&\leq\int_{\tau_1}^{\tau_2}\int_{0}^{1}\max_y\max_t|v(y,t)||\varphi(0)|dydt+\int_{\tau_1}^{\tau_2}\max_t|u(0,t)||\varphi(0)|dt+\int_{\tau_1}^{\tau_2}\int_{-1}^{0}\max_x\left|\frac{\partial u}{\partial x}\right|\left|\frac{\partial \varphi}{\partial x}\right|dxdt,\\
&\leq|\tau_2-\tau_1|\left(||\varphi(0)||_\infty||v||_\infty+||\varphi(0)||_\infty||u_o||_\infty+||u_x||_\infty||\varphi_x||_\infty\right).
\end{aligned} \tag{2.23}$$

From (2.22) and (2.23) we obtain

$$\left|\int_{-1}^{0}(u(x,\tau_2)-u(x,\tau_1))\varphi(x)dx\right|\leq M_1|\tau_2-\tau_1|,$$

with $M_1=(||\varphi(0)||_\infty||v||_\infty+||\varphi(0)||_\infty||u_0||_\infty+||u_x||_\infty||\varphi_x||_\infty)$.

Now, we will perform the same analysis for the nonlocal part, which is given by equation (2.8). Integrating the equation (2.8) against a test function ψ with respect to $(0,1)\times(\tau_1,\tau_2)$ and, using Fubini's theorem we have

$$\left|\int_{\tau_1}^{\tau_2}\int_{0}^{1}\frac{\partial v}{\partial t}\psi(x)dxdt\right|=\left|\int_{0}^{1}\int_{\tau_1}^{\tau_2}\frac{\partial v}{\partial t}\psi(x)dtdx\right|=\left|\int_{0}^{1}(v(x,\tau_2)-v(x,\tau_1))\psi(x)dx\right|. \tag{2.24}$$

On the other hand, using the kernel symmetry and taking the maximum of u and v, we have

$$\begin{aligned}
&\left|\int_{\tau_1}^{\tau_2}\int_0^1\left[\int_0^1 J(x-y)(v(y,t)-v(x,t))\psi(x)dy-\int_{-1}^0 J(x-y)(v(x,t)-u(0,t))\psi(x)dy\right]dxdt\right|\\
&\leq\left|\int_{\tau_1}^{\tau_2}\int_0^1\int_0^1 J(x-y)v(y,t)\psi(x)dydxdt\right|+\left|\int_{\tau_1}^{\tau_2}\int_0^1\int_0^1 J(x-y)v(x,t)\psi(x)dydxdt\right|\\
&+\left|\int_{\tau_1}^{\tau_2}\int_0^1\int_{-1}^0 J(x-y)v(x,t)\psi(x)dydxdt\right|+\left|\int_{\tau_1}^{\tau_2}\int_0^1\int_{-1}^0 J(x-y)u(0,t)\psi(x)dydxdt\right|\\
&\leq\int_{\tau_1}^{\tau_2}\int_0^1\int_0^1 J(x-y)|v(y,t)||\psi(x)|dydxdt+\int_{\tau_1}^{\tau_2}\int_0^1\int_0^1 J(x-y)|v(x,t)||\psi(x)|dydxdt\\
&+\int_{\tau_1}^{\tau_2}\int_0^1\int_{-1}^0 J(x-y)|v(x,t)||\psi(x)|dydxdt+\int_{\tau_1}^{\tau_2}\int_0^1\int_{-1}^0 J(x-y)|u(0,t)||\psi(x)|dydxdt\\
&\leq|\tau_2-\tau_1|\left(3||v||_\infty||\psi||_\infty+||u_o||_\infty||\psi||_\infty\right). \qquad (2.25)
\end{aligned}$$

Hence, from (2.24) and (2.25) we prove the weak Lipschitz continuity in time

$$\left|\int_0^1(v(x,\tau_2)-v(x,\tau_1))\psi(x)dx\right|\leq M_2|\tau_2-\tau_1|,$$

where $M_2=(3||v||_\infty||\psi||_\infty+||u_0||_\infty||\psi||_\infty)$.

This two estimates ensure the weak Lipschitz continuity in time for u and v. □

Now we are ready to state and prove our main result of this section.

Theorem 2.4. *Suppose* $u_0\in L^1(-1,0)\cap L^\infty(-1,0)\cap BV$ *and* $v_0\in L^1(0,1)\cap L^\infty(0,1)\cap BV$. *Then, the semi-discrete splitting method* (2.9) *converge to an entropy weak solution of* (2.3)–(2.4).

Proof. From the estimates provided by Lemma 2.1–2.3 we are at the assumptions of the Theorem $A.8$, in [25]. It means that there exists a subsequence $\eta_j\to 0$, such that for each $t\in[0,T]$, the function $\{u_{\eta_j}(t)\}$ converges to a function $u(t)$ in $L^\infty(-1,0)$ and, the convergence is in the space $C([0,T];L^1_{loc}(-1,0))$. At the same way, there exists a subsequence $\eta_j\to 0$, such that for each $t\in[0,T]$, the function $\{v_{\eta_j}(t)\}$ converges to a function $v(t)$ in $L^\infty(0,1)$ in the space $C([0,T];L^1_{loc}(0,1))$.

Moreover, it is possible to ensure, using a diagonal argument, to obtain a subsequence, also denoted by $\{u_{\eta j}\}$, $\{v_{\eta j}\}$, which converges for all t in some dense countable subset of $[0,T]$. Then, using Lemma 2.3 we get the convergence for all t in $[0,T]$.

Now we need to prove that the limit functions $u(t)$ and $v(t)$ are the entropy solution for our problem. Recall that the entropy energy for our problem for time $[0,T]$ if, for all suitable test functions $\varphi\in C^\infty([-1,1]\times[0,T])$, it satisfies

$$\begin{aligned}
\int_0^T\int_{-1}^0\int_0^1 J(x-y)(v(y,t)-u(0,t))dydxdt-\int_0^T\int_{-1}^0 u_x(x,t)\varphi_x(x,t)dxdt\\
+\int_0^T\int_{-1}^0 u(x,t)\varphi_t(x,t)dxdt+\int_{-1}^0 u(x,0)\varphi(x,0)dx=0, \qquad (2.26)
\end{aligned}$$

for u, with $\varphi(x,T)\equiv 0$, and

$$\begin{aligned}
\int_0^T\int_0^1\int_0^1 J(x-y)(v(y,t)-v(x,t))dydxdt-\int_0^T\int_0^1\int_{-1}^0 J(x-y)(v(x,t)-u(0,t))dydxdt\\
+\int_0^T\int_0^1 v(x,t)\varphi_t(x,t)dxdt+\int_0^1 v(x,0)\varphi(x,0)dx=0, \quad (2.27)
\end{aligned}$$

for v. Consider at first the case of j is even. We want to conclude that

$$\int_0^T\int_{-1}^0 u_t^n(x,t)\varphi_(x,t)dxdt=\int_0^T\int_{-1}^0 u_{xx}^n(x,t)\varphi(x,t)dxdt, \qquad (2.28)$$

for $x \in (-1,0)$, $t \in [t_j, t_{j+1}]$, converges to a function $u \in L^\infty(-1,0)$ in a L^1-norm, given by (2.27).

At the same way, considering j even, we aim to prove that for v

$$\int_0^T \int_0^1 v_t^n(x,t)\varphi_(x,t)dxdt = \int_0^T \int_0^1 \int_0^1 J(x-y)(v^n(y,t)-v^n(x,t))\varphi(x,t)dydxdt - \int_0^T \int_0^1 \int_{-1}^0 J(x-y)(v^n(x,t)-u^n(x,t_j))\varphi(x,t)dydxdt, \quad (2.29)$$

for $x \in (0,1)$, $t \in [t_{j-1}, t_j]$, converges to a function $v \in L^\infty(0,1)$ in a L^1-norm, given by (2.26).

The same argument can be proved in case j is odd.

Let us consider the equation (2.28). Integrating by parts and considering $\varphi(x,T) \equiv 0$ we have the following

$$\int_0^T \int_{-1}^0 u_t^n(x,t)\varphi(x,t)dxdt = \int_{-1}^0 \left(u^n(x,T)\varphi(x,T) - u^n(x,0)\varphi(x,0) - \int_0^T u^n(x,t)\varphi_t(x,t)dt \right) dx.$$

On the other hand,

$$\int_0^T \int_{-1}^0 u_{xx}^n(x,t)\varphi(x,t)dxdt = \int_0^T \left(u_x^n(0,t)\varphi(0,t) - u_x^n(-1,t)\varphi(-1,t) - \int_{-1}^0 u_x^n(x,t)\varphi_x(x,t)dt \right) dx.$$

Taking

$$\int_0^T \int_{-1}^0 u_t^n(x,t)\varphi(x,t)dxdt = \int_0^T \int_{-1}^0 u_{xx}^n(x,t)\varphi(x,t)dxdt$$

it holds

$$-\int_{-1}^0 u^n(x,0)\varphi(x,0)dx = \int_{-1}^0 \int_0^T u^n(x,t)\varphi_t(x,t)dtdx + \int_0^T \int_{-1}^0 \int_0^1 J(x-y)(v^n(x,t_{j-1}) - u^n(0,t))\varphi(0,t)dydxdt - \int_0^T \int_{-1}^0 u_x^n(x,t)\varphi_x(x,t)dxdt. \quad (2.30)$$

Taking the limit in n in each term of (2.30), using Lemma 2.3 and the Convergence Dominated Theorem it follows that

i)

$$-\lim_n \int_{-1}^0 u^n(x,0)\varphi(x,0)dx = -\int_{-1}^0 \varphi(x,0)\lim_n u^n(x,0)dx = -\int_{-1}^0 \varphi(x,0)u(x,0)dx \quad (2.31)$$

ii)

$$\lim_n \int_{-1}^0 \int_0^T u^n(x,t)\varphi_t(x,t)dtdx = \int_{-1}^0 \int_0^T \lim_n u^n(x,t)\varphi_t(x,t)dtdx = \int_{-1}^0 \int_0^T u(x,t)\varphi_t(x,t)dtdx$$

iii) By Lemma 2.3 we obtain,

$$\begin{aligned}
&\int_0^T \int_{-1}^0 \int_0^1 J(x-y)v^n(x,t_{j-1})\varphi(0,t)dydxdt = -\int_0^T \int_{-1}^0 \int_0^1 J(x-y)v^n(x,t)\varphi(0,t)dydxdt \\
&\quad + \int_0^T \int_{-1}^0 \int_0^1 J(x-y)v^n(x,t)\varphi(0,t)dydxdt + \int_0^T \int_{-1}^0 \int_0^1 J(x-y)v^n(x,t_{j-1})\varphi(0,t)dydxdt \\
&= \int_0^T \int_{-1}^0 \int_0^1 J(x-y)v^n(x,t)\varphi(0,t)dydxdt + \int_0^T \int_{-1}^0 \int_0^1 J(x-y)(v^n(x,t_{j-1}) - v^n(x,t))\varphi(0,t)dydxdt \\
&\leq \int_0^T \int_{-1}^0 \int_0^1 J(x-y)v^n(x,t)\varphi(0,t)dydxdt + \int_0^T \int_{-1}^0 \int_0^1 J(x-y)M_2|t_{j-1} - t|\varphi(0,t)dydxdt.
\end{aligned}$$

Now taking the limit in n we get

$$\lim_n \left(\int_0^T \int_{-1}^0 \int_0^1 J(x-y)(v^n(x,t_j) - u^n(0,t))\varphi(0,t) dydxdt \right)$$
$$= \lim_n \left(\int_0^T \int_{-1}^0 \int_0^1 J(x-y) v^n(x,t_j)\varphi(0,t) dydxt \right) - \lim_n \left(\int_0^T \int_{-1}^0 \int_0^1 J(x-y) u^n(0,t)\varphi(0,t) dydxt \right)$$
$$\leq \lim_n \left(\int_0^T \int_{-1}^0 \int_0^1 J(x-y) v^n(x,t)\varphi(0,t) dydxt \right) + \lim_n \left(\int_0^T \int_{-1}^0 \int_0^1 J(x-y) M_2 |t_{j-1} - t| \varphi(0,t) dydxdt \right)$$
$$- \lim_n \left(\int_0^T \int_{-1}^0 \int_0^1 J(x-y) u^n(0,t)\varphi(0,t) dydxt \right)$$
$$= \left(\int_0^T \int_{-1}^0 \int_0^1 J(x-y) \lim_n v^n(x,t)\varphi(0,t) dydxt \right) - \left(\int_0^T \int_{-1}^0 \int_0^1 J(x-y) \lim_n u^n(0,t)\varphi(0,t) dydxt \right)$$
$$+ \lim_n \left(\int_0^T \int_{-1}^0 \int_0^1 J(x-y) M_2 |t_{j-1} - t| \varphi(0,t) dydxdt \right)$$
$$= \left(\int_0^T \int_{-1}^0 \int_0^1 J(x-y) v(x,t)\varphi(0,t) dydxt \right) - \left(\int_0^T \int_{-1}^0 \int_0^1 J(x-y) u(0,t)\varphi(0,t) dydxt \right),$$

as $\triangle t \to 0$.

iv)

$$- \lim_n \int_0^T \int_{-1}^0 u_x^n(x,t)\varphi_x(x,t) dxdt = - \int_0^T \int_{-1}^0 \lim_n u_x^n(x,t)\varphi_x(x,t) dxdt = - \int_0^T \int_{-1}^0 u_x(x,t)\varphi_x(x,t) dxdt. \tag{2.32}$$

Therefore, from (2.31)–(2.32) the limit equation (2.30) converges to the entropy solution (2.26) in L^1-norm.

Applying the same argument we will verify the convergence for v. Integrating by parts and taking the limit in equation (2.29) we obtain by Lemma 2.3 and the Convergence Dominated theorem the following

i)

$$- \lim_n \left(\int_0^1 v^n(0,x)\varphi(0,x) dx \right) = - \left(\int_0^1 \lim_n v^n(0,x)\varphi(0,x) dx \right) = - \int_0^1 v(0,x)\varphi(0,x) dx, \tag{2.33}$$

ii)

$$\lim_n \left(\int_0^1 \int_0^T v^n(x,t)\varphi_t(x,t) dxdt \right) = \int_0^1 \int_0^T \lim_n v^n(x,t)\varphi_t(x,t) dxdt = \int_0^1 \int_0^T v(x,t)\varphi_t(x,t) dxdt,$$

iii)

$$\lim_n \left(\int_0^T \int_0^1 \int_0^1 J(x-y)(v^n(y,t) - v^n(x,t))\varphi(x,t) dydxdt \right)$$
$$= \int_0^T \int_0^1 \int_0^1 J(x-y) \lim_n (v^n(y,t) - v^n(x,t))\varphi(x,t) dydxdt$$
$$= \int_0^T \int_0^1 \int_0^1 J(x-y)(v(y,t) - v(x,t))\varphi(x,t) dydxdt,$$

iv)

$$
\begin{aligned}
&-\lim_n\left(\int_0^T\int_0^1\int_{-1}^0 J(x-y)(v^n(x,t)-u^n(0,t_j))\varphi(x,t)\right)\\
&=-\lim_n\left(\int_0^T\int_0^1\int_{-1}^0 J(x-y)v^n(x,t)\varphi(x,t)dydxdt\right)+\lim_n\left(\int_0^T\int_0^1\int_{-1}^0 J(x-y)(u^n(0,t_j)\varphi(x,t)dydxdt\right)\\
&=-\lim_n\left(\int_0^T\int_0^1\int_{-1}^0 J(x-y)v^n(x,t)\varphi(x,t)dydxdt\right)+\lim_n\left(\int_0^T\int_0^1\int_{-1}^0 J(x-y)u^n(0,t)\varphi(x,t)dydxdt\right)\\
&\quad+\lim_n\left(\int_0^T\int_0^1\int_{-1}^0 J(x-y)(u^n(0,t_j)-u^n(0,t))\varphi(x,t)dydxdt\right)\\
&\leq-\lim_n\left(\int_0^T\int_0^1\int_{-1}^0 J(x-y)v^n(x,t)\varphi(x,t)dydxdt\right)+\lim_n\left(\int_0^T\int_0^1\int_{-1}^0 J(x-y)u^n(0,t)\varphi(x,t)dydxdt\right)\\
&\quad+\lim_n\left(\int_0^T\int_0^1\int_{-1}^0 J(x-y)M_1|t_j-t|\varphi(x,t)dydxdt\right)\\
&=-\int_0^T\int_0^1\int_{-1}^0 J(x-y)(v(x,t)-u(0,t))\varphi(x,t)dydxdt,
\end{aligned}
\tag{2.34}
$$

as $\triangle t\to 0$.

From (2.33)–(2.34) we conclude that the limit equation for v, (2.29) converges to a function v that is the entropy energy of our problem. Performing the same procedure for the approximations u^n and v^n for j odd the proof is complete. □

3 Numerical schemes

In this section we will present our results for the discretizations (1.5) and (1.6) of the continuous local/nonlocal problem (1.1). Similar studies were developed in [4] and [29].

3.1 Semidiscrete scheme

Let us divide the computational domain $[-1,1]$ in two subdomains, a local domain, $[-1,0]$ and a nonlocal domain, $[0,1]$. Each subdomain will be partitioned into finite-volume cells (or control volumes), denoted by C_j and K_i, for $j,i=1,\cdots,N$, respectively (see for instance [28]).

The interval $[-1,1]$ is partioned in $2N$ subintervals, it means, an admissible mesh of $[-1,1]$, denoted by $\mathcal{T}$ is given by the union of families $(C_j)_{j=1,\cdots,N}$ and $(K_i)_{i=1,\cdots,N}$, such that $C_j=\left(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}\right)$ and $K_i=\left(z_{i-\frac{1}{2}},z_{i+\frac{1}{2}}\right)$, centered at x_j and z_i, $j,i=1,\cdots,N$ and following the order given by,

$$x_0=x_{\frac{1}{2}}=-1<x_1<x_{\frac{3}{2}}<\cdots<x_{j-\frac{1}{2}}<x_j<x_{j+\frac{1}{2}}<\cdots<x_{N-\frac{1}{2}}<x_N<x_{N+\frac{1}{2}}=x_{N+1}=0$$

$$z_0=z_{\frac{1}{2}}=0<z_1<z_{\frac{3}{2}}<\cdots<z_{i-\frac{1}{2}}<z_i<z_{i+\frac{1}{2}}<\cdots<z_{N-\frac{1}{2}}<z_N<z_{N+\frac{1}{2}}=z_{N+1}=1.$$

The point at the center is not necessarily the midpoint of C_j and of K_i. The notations for the mesh grid are given bellow:

$$
\begin{aligned}
&x_j \in C_j = \left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right], \quad z_i \in K_i = \left[z_{i-\frac{1}{2}}, z_{i+\frac{1}{2}}\right], \\
&x_{j+\frac{1}{2}} \in C_{j+\frac{1}{2}} = [x_j, x_{j+1}], \quad z_i \in K_{i+\frac{1}{2}} = [z_i, z_{i+1}], \\
&x_{\frac{1}{2}} \in C_{\frac{1}{2}} = \left[x_{\frac{1}{2}}, x_1\right], \quad z_{\frac{1}{2}} \in K_{\frac{1}{2}} = [z_i, z_{i+1}], \\
&x_{N+\frac{1}{2}} \in C_{N+\frac{1}{2}} = \left[x_N, x_{N+\frac{1}{2}}\right], \quad z_{N+\frac{1}{2}} \in K_{N+\frac{1}{2}} = \left[z_N, z_{N+\frac{1}{2}}\right], \\
&h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, \quad h_{j+\frac{1}{2}} = x_{j+1} - x_j, \quad \sum_{j=1}^{N} h_j = 1, \\
&h_i = z_{i+\frac{1}{2}} - z_{i-\frac{1}{2}}, \quad h_{i+\frac{1}{2}} = x_{i+1} - x_i, \quad \sum_{i=1}^{N} h_i = 1.
\end{aligned}
$$

The spatial finite volume scheme is obtained by integrating (1.1) over each volume control C_j and K_i, respectively. The discrete unknowns are the average of u at the cell C_j, as well as, the average of v at the cell K_i, for $j, i = 1, \cdots, N$. Considering a piecewise constant approximation of the functions u and v on the mesh, we obtain

$$
u_j = \frac{1}{h_j}\int_{C_j} u(x,t)dx, \quad v_i = \frac{1}{h_i}\int_{K_i} v(x,t)dx,
$$

$$
J(x_j - z_i) = \frac{1}{h_j}\frac{1}{h_i}\int_{K_i}\int_{C_j} J(x-y)dxdy
$$

So considering a cell-centered scheme at the spatial variable we have:

$$
\begin{cases}
u_1'(t) = \dfrac{u_2(t) - u_1(t)}{h_1 h_{\frac{3}{2}}}, \\
u_j'(t) = \dfrac{u_{j+1}(t) - u_j(t)}{h_j h_{j+\frac{1}{2}}} - \dfrac{u_j(t) - u_{j-1}(t)}{h_j h_{j-\frac{1}{2}}}, \quad j = 2, \cdots, N-1 \\
u_N'(t) = \dfrac{u_{N-1}(t) - u_N(t)}{h_j h_{N-\frac{1}{2}}} + \displaystyle\sum_{j=1}^{N}\sum_{i=1}^{N} \frac{h_j h_i}{h_N} J(z_i - x_j)(v_i(t) - u_N(t)), \\
v_i'(t) = \displaystyle\sum_{k=1}^{N} h_k J(z_i - z_k)(v_k(t) - v_i(t)) - \sum_{j=1}^{N} h_j J(z_i - x_j)(v_i(t) - u_N(t)), \quad i = 1, \cdots, N \\
u_j(0) = u_{j0}, \quad j = 1, \cdots, N \\
v_i(0) = v_{i0}, \quad i = 1, \cdots, N,
\end{cases}
\tag{3.1}
$$

where

$$
u_{j0} = \frac{1}{h_j}\int_{C_j} u_0(x)dx, \quad v_{i0} = \frac{1}{h_i}\int_{K_i} v_0(x)dx.
$$

Energy formulation of (3.1)**.** Discretizing the integrals in (1.2) we have

$$
E_{h_{i,j}}(u_j, v_i) := \sum_{j=1}^{N-1} \frac{(u_{j+1} - u_j)^2}{2h_j} + \frac{1}{4}\sum_{i=1}^{N}\sum_{k=1}^{N} h_i h_k J(x_i - x_k)(v_i - v_k)^2 + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} h_i h_j J(z_i - x_j)(v_i - u_N)^2. \tag{3.2}
$$

As it was discussed in [17], for the continuous model, we can derive the semidiscrete scheme as the gradient flow associated with the discretized energy, that is, (u_j, v_i), $j, i = 1, \cdots, N$, will be the solution of the the ODE problem

$$
(u_j, v_i)'(t) = -\partial E_{h_{i,j}}\left[(u_j, v_i)(t)\right], \qquad t \geq 0,
$$

$u(0) = u_0$, $v(0) = v_0$, where $\partial E_{h_{i,j}}\,[(u_j, v_i)]$ denotes the subdifferential of $E_{h_{i,j}}$ at (u_j, v_i). In fact, from (3.2) we can derivate the discrete evolution problem associate to this energy. To compute the subdifferential, we obtain the derivative of $E_{h_{i,j}}$ at $(u_j, v_i) \in (\mathbb{R}^{2N})$ in the direction of $\varphi \in C_0^\infty(\mathbb{R}^{2N})$ as

$$\sum_{j=1}^{N} h_j u_j'(t)\varphi_j(t) = -\partial E_{h_j}\,|_{t=0}\,(u_j + t\varphi_j, v) \tag{3.3}$$

$$\sum_{i=1}^{N} h_i v_i'(t)\varphi_i(t) = -\partial E_{h_i}\,|_{t=0}\,(u, v_i + t\varphi_i) \tag{3.4}$$

The evolution problem, (3.3) and (3.4), consist of a system of ODEs that becomes (3.1).

We denote V_h as the corresponding standard piecewise constant space, and $\{\chi_j\}$, $\{\chi_{j+\frac{1}{2}}\}$ and $\{\chi_i\}$, $\{\chi_{i+\frac{1}{2}}\}$, with $i, j = 0, \cdots, N$, are the characteristic functions defined on the control volume $C_j, C_{j+\frac{1}{2}}, K_i, K_{i+\frac{1}{2}}$. Then, we define the semidiscrete approximations as

$$V_h = u_h = \sum_{j=1}^{N} u_j\chi_j, \quad V_h = v_h = \sum_{i=1}^{N} v_i\chi_i, \tag{3.5}$$

$$\tilde{V_h} = \tilde{u_h} = \sum_{j=1}^{N} \tilde{u}_{j+\frac{1}{2}}\chi_{j+\frac{1}{2}}, \quad \tilde{V_h} = \tilde{v_h} = \sum_{i=1}^{N} \tilde{v}_{i+\frac{1}{2}}\chi_{i+\frac{1}{2}},$$

where $u_j(t)$ and $v_i(t)$ are the solutions of the semidiscrete problem (3.1), for every $i, j = 1, \cdots, N$.

For simplicity, we will consider a uniform mesh, it means $h = h_i = h_j$. Thus the system (3.1) becomes (1.5). Also we note that the finite volume scheme in one dimension resembles to the finite difference method. In higher dimensions the two numerical approximations are completely different.

Existence, uniqueness and mass conservation.

Lemma 3.1. *There exists a unique approximate solution $(u_h, v_h)_{h>0}$ to the semidiscrete scheme (1.5). Moreover, the semidiscrete finite volume scheme preserves the total mass.*

Proof. The system (1.5) can be write as $W'(t) = MW(t)$, where $W(t) = [u_1(t), \cdots, u_N(t), v_1(t), \cdots, v_N(t)]^T \in (\mathbb{R}^{2N})$ and $M(h)$ is an $(2N \times 2N)$ linear matrix, which does not depend explicitly on t, but only on h. The local existence and uniqueness to the system (1.5) is straightforward.

To prove the second statement, we just need to verify that

$$h\sum_{j=1}^{N} u_j'(t) + h\sum_{i=1}^{N} v_i'(t) = 0.$$

Indeed, the total mass is preserved since the kernel J is symmetric. □

The next result shows that the numerical scheme also preserves the monotonicity of the initial condition. If the initial condition is monotone decreasing in $[-1, 1]$, then the numerical solution is also monotone decreasing for positive times.

Lemma 3.2. *Let $W(t) = [u_1(t), \cdots, u_N(t), v_1(t), \cdots, v_N(t)]^T$ be a solution of (1.5), with $u_{j+1}(0) > u_j(0) > v_{i+1}(0) > v_i(0)$, for every $0 \le i, j \le N$. Then $u_{j+1}(t) > u_j(t) > v_{i+1}(t) > v_i(t)$, for every $t > 0$ and $0 \le i, j \le N$.*

Proof. We argue by contradiction. Suppose that there exists a first time t_0 and two consecutive nodes, such that the conclusion of the lemma fails. We call the two nodes $j, j+1$ if this happens for u and $i, i+1$ for v. We have that $u_{j+1}(t_0) = u_j(t_0)$ or $v_{i+1}(t_0) = v_i(t_0)$. If $j = 1$, we get

$$0 \ge u_2'(t_0) - u_1'(t_0) = \frac{u_3(t_0) - 2u_2(t_0) + u_1(t_0)}{h^2} - \frac{u_2(t_0) - u_1(t_0)}{h^2} = \frac{u_3(t_0) - u_2(t_0)}{h^2} > 0,$$

which is a contradiction. If $2 \le j \le N-1$, we get

$$0 \ge u'_{j+1}(t_0) - u'_j(t_0) = \frac{u_{j+2}(t_0) - 2u_{j+1}(t_0) + u_j(t_0)}{h^2} - \frac{u_{j+1}(t_0) - 2u_j(t_0) + u_{j-1}(t_0)}{h^2} = \frac{u_{j+2}(t_0) - u_{j-1}(t_0)}{h^2} > 0,$$

a contradiction. If $j = N$ we get

$$0 \ge u'_{N+1}(t_0) - u'_N(t_0) = \frac{u_N(t_0) - u_{N+1}(t_0)}{h^2} - \frac{u_{N-1}(t_0) - u_N(t_0)}{h^2} = \frac{u_N(t_0) - u_{N-1}(t_0)}{h^2} > 0,$$

again a contradiction. Finally, for $1 \le i \le N$ we get

$$\begin{aligned} 0 \ge v'_{i+1}(t_0) - v'_i(t_0) = h &\left[\sum_{k=1}^{N} J(z_{i+1} - z_{k+1})(v_{k+1}(t_0) - v_{i+1}(t_0)) - \sum_{j=1}^{N} J(z_{i+1} - x_{j+1})(v_{i+1}(t_0) - u_{N+1}(t_0)) \right] \\ &- h \left[\sum_{k=1}^{N} J(z_i - z_k)(v_k(t_0) - v_i(t_0)) - \sum_{j=1}^{N} J(z_i - x_j)(v_i(t_0) - u_N(t_0)) \right] \ge 0, \end{aligned}$$

and we achieve the desired contradiction for all cases. □

Comparison principle.

Our next aim is to prove the validity of the comparison principle for our numerical scheme. We need to introduce the following definition.

Definition 3.3. A continuous function $(\bar{u}, \bar{v})$ is called a supersolution of (1.5) if

$$\begin{cases} \overline{u}'_1(t) \ge \dfrac{\overline{u}_2(t) - \overline{u}_1(t)}{h^2}, \\ \overline{u}'_j(t) \ge \dfrac{\overline{u}_{j+1}(t) - 2\overline{u}_j(t) + \overline{u}_{j-1}(t)}{h^2}, \quad j = 2, \cdots, N-1 \\ \overline{u}'_N(t) \ge \dfrac{\overline{u}_{N-1}(t) - \overline{u}_N(t)}{h^2} + h \displaystyle\sum_{j=1}^{N} \sum_{i=1}^{N} J(z_i - x_j)(\overline{v}_i(t) - \overline{u}_N(t)), \\ \overline{v}'_i(t) \ge h \displaystyle\sum_{k=1}^{N} J(z_i - z_k)(\overline{v}_k(t) - \overline{v}_i(t)) - h \sum_{j=1}^{N} J(z_i - x_j)(\overline{v}_i(t) - \overline{u}_N(t)), \quad i = 1, \cdots, N \\ \overline{u}_j(0) \ge u_{j0}, \quad j = 1, \cdots, N \\ \overline{v}_i(0) \ge v_{i0}, \quad i = 1, \cdots, N. \end{cases}$$

Analogously, $(\underline{u}_j, \underline{v}_i)$ is called a subsolutions of (1.5) if the reverse inequalities hold.

Lemma 3.4 (Comparison Principle)**.** *Let* $\overline{u}_j, \overline{v}_i$ *and* $\underline{u}_j, \underline{v}_i$ *be a supersolution and a subsolution, respectively, of* (1.5)*, then*

$$\overline{u}_j(t) \ge u_j(t) \ge \underline{u}_j(t), \quad j = 1, \cdots, N, \quad t \ge 0, \qquad \text{and} \qquad \overline{v}_i(t) \ge v_i(t) \ge \underline{v}_i(t), \quad i = 1, \cdots, N, \quad t \ge 0.$$

Proof. By an approximation procedure we can consider strict inequalities for the supersolution. In fact, we can consider

$$\overline{w}_j(t) = \overline{u}_j(t) + \delta t + \delta, \quad \overline{w}_j(0) = \overline{u}_j(0) + \delta \ge u_{j0} + \delta,$$

$$\overline{y}_i(t) = \overline{v}_i(t) + \delta t + \delta, \quad \overline{y}_i(0) = \overline{v}_i(0) + \delta \ge v_{i0} + \delta,$$

where $\delta > 0$, $\forall i, j$ and $\forall t > 0$, as a strict supersolution (a supersolution with strict inequalities), and take the limit as $\delta \to 0$ at the end of the argument. Indeed, differentiating $\overline{w}_j(t)$ and $\overline{y}_i(t)$ with respect to t, we get, for $j = 1$,

$$\overline{w}'_1(t) = \overline{u}'_1(t) + \delta \ge \frac{\overline{u}_2(t) - \overline{u}'_1(t)}{h^2} + \delta = \frac{\overline{w}_2(t) - \overline{w}_1(t)}{h^2} + \delta > \frac{\overline{w}_2(t) - \overline{w}_1(t)}{h^2},$$

for $j = 2, \cdots, N-1$

$$\overline{w}_j'(t) = \overline{u}_j'(t) + \delta \geq \frac{\overline{u}_{j+1}(t) - 2\overline{u}_j(t) + \overline{u}_{j-1}'(t)}{h^2} + \delta = \frac{\overline{w}_{j+1}(t) - 2\overline{w}_j(t) + \overline{u}_{j-1}(t)}{h^2} + \delta$$
$$> \frac{\overline{w}_{j+1}(t) - 2\overline{w}_j(t) + \overline{u}_{j-1}(t)}{h^2},$$

and for $j = N$ we get

$$\overline{w}_N'(t) = \overline{u}_N'(t) + \delta \geq \frac{\overline{u}_{N-1}(t) - \overline{u}_N'(t)}{h^2} + \delta = \frac{\overline{w}_N(t) - \overline{w}_{N-1}(t)}{h^2} + \delta > \frac{\overline{w}_N(t) - \overline{w}_{N-1}(t)}{h^2}.$$

On the other hand, for $i = 1, \cdots, N$ we have

$$\begin{aligned}\overline{y}_i'(t) = \overline{v}_i'(t) + \delta &\geq h\sum_{k=1}^{N} J(z_i - z_k)(\overline{v}_k(t) - \overline{v}_i(t)) - h\sum_{j=1}^{N} J(z_i - x_j)(\overline{v}_i(t) - \overline{u}_N(t)) + \delta \\ &= h\sum_{k=1}^{N} J(z_i - z_k)(\overline{y}_k(t) - \overline{y}_i(t)) - h\sum_{j=1}^{N} J(z_i - x_j)(\overline{y}_i(t) - \overline{w}_N(t)) + \delta \\ &> h\sum_{k=1}^{N} J(z_i - z_k)(\overline{y}_k(t) - \overline{y}_i(t)) - h\sum_{j=1}^{N} J(z_i - x_j)(\overline{y}_i(t) - \overline{w}_N(t)).\end{aligned}$$

Thus, the pair $(\overline{u}_j, \overline{z}_i)$ is a strict supersolution of (1.5).

Now, our aim is to show that for all $t > 0$ and $\delta > 0$ we have

$$\begin{cases} \overline{w}_j(t) - \underline{u}_j(t) > \dfrac{\delta}{2}, \\ \overline{y}_i(t) - \underline{v}_i(t) > \dfrac{\delta}{2}. \end{cases} \tag{3.6}$$

To see this we argue by contradiction. Suppose that (3.6) is not true. Then, there exist a first time t_0 and a first node j or i such that

$$\begin{cases} \overline{w}_j(t_0) - \underline{u}_j(t_0) = \dfrac{\delta}{2}, \quad \text{and} \quad \overline{w}_j(0) - \underline{u}_j(0) \geq \delta \\ \text{or} \\ \overline{y}_i(t_0) - \underline{v}_i(t_0) = \dfrac{\delta}{2}, \quad \text{and} \quad \overline{y}_i(0) - \underline{v}_i(0) \geq \delta. \end{cases} \tag{3.7}$$

Then, using (3.7) we have, for $j = 2, \cdots, N$

$$0 \geq \overline{w}_j'(t_0) - \underline{u}_j'(t_0) \geq \frac{(\overline{w}_{j+1} - \underline{u}_{j+1})(t_0) - 2(\overline{w}_j - \underline{u}_j)(t_0) + (\overline{w}_{j-1} - \underline{u}_{j-1})(t_0)}{h^2} + \delta \geq \delta > 0,$$

since $(\overline{w}_{j+1} - \underline{u}_{j+1})(t_0) \geq \frac{\delta}{2}$ and $(\overline{w}_{j-1} - \underline{u}_{j-1})(t_0)) \geq \frac{\delta}{2}$, which is a contradiction. For $j = 1$ we obtain

$$0 \geq \overline{w}_1'(t_0) - \underline{u}_1'(t_0) \geq \frac{(\overline{w}_2 - \underline{u}_2)(t_0) - (\overline{w}_1 - \underline{u}_1)(t_0)}{h^2} + \delta \geq \delta > 0,$$

since $(\overline{w}_2 - \underline{u}_2)(t_0) \geq \frac{\delta}{2}$, which is a contradiction. For $j = N$,

$$0 \geq \overline{w}_N'(t_0) - \underline{u}_N'(t_0) \geq \frac{(\overline{w}_{N-1} - \underline{u}_{N-1})(t_0) - (\overline{w}_N - \underline{u}_N)(t_0)}{h^2} + \delta \geq \delta > 0,$$

since $(\overline{w}_{N-1}-\underline{u}_{N-1})(t_0)\geq\frac{\delta}{2}$, which is a contradiction. Finally, for $i=1,\cdots,N$ and $j=N$ we get

$$0\geq\overline{y}_i'(t_0)-\underline{v}_i'(t_0)\geq h\sum_k^N J(z_i-z_k)(\overline{y}_i(t_0)-\overline{y}_k(t_0))-h\sum_{j=1}^N J(z_i-x_j)(\overline{y}_i(t_0)-\overline{w}_N(t_0))+\delta$$
$$-\left(\sum_k^N J(z_i-z_k)(\underline{v}_i(t_0)-\underline{v}_k(t_0))-h\sum_{j=1}^N J(z_i-x_j)(\underline{v}_i(t_0)-\underline{u}_N(t_0))\right)\geq\frac{\delta}{2}>0,$$

since $(\overline{y}_i(t_0)-\underline{v}_i(t_0))=(\overline{w}_N(t_0)-\underline{u}_N(t_0))=\frac{\delta}{2}$, and $(\overline{y}_k(t_0)-\underline{v}_k(t_0))\geq\frac{\delta}{2}$, which it gives us a contradiction and the proof of claim. As δ was arbitrary, taking the limit as $\delta\to 0$ we conclude the proof of the lemma. □

Remark 1. Note that a constant function, $u_j=v_i=k$, for every $i,j=1,\cdots,N$ is a solution to (1.5). Thus, we obtain the following

$$\max\left\{\max_j u_j(0),\max_i v_i(0)\right\}\geq u_j(t),v_i(t)\geq\min\left\{\min_j u_j(0),\min_i v_i(0)\right\},\quad\forall\quad i,j=1,\cdots,N,$$

This implies that solutions are uniformly bounded.

Consistency and convergence

We are ready to prove convergence for the semidiscrete scheme (1.5) as $h\to 0$.

Theorem 3.5. *Let $u\in C^{3,1}([-1,0]\times[0,T])$ and $v\in C^{0,1}([0,1]\times[0,T])$, be a solution to* (1.1) *and u_h,v_h defined as* (3.5)*. Then, there exists a constant K (independent of h), such that for every h small enough it holds*

$$\max_{0\leq t\leq T}\left\{\max_{x_j\in[-1,0]}|u(x_j,t)-u_h(x_j,t)|,\max_{z_i\in[0,1]}|v(z_i,t)-v_h(z_i,t)|\right\}\leq Kh.$$

Proof. Let us denote by $\varepsilon_j(t)=u_j(t)-u(x_j,t)$ and $\xi_i(t)=v_i(t)-v(z_i,t)$ the error vectors. Recalling that the finite volume method approximate the average of the solution in each volume (this is a second order approximation), we have

$$u_j=\frac{1}{h}\int_{x_{j-\frac12}}^{x_{j+\frac12}}u(x,t)dx\approx u(x_j,t),\quad j=1,\cdots,N,$$
$$v_i=\frac{1}{h}\int_{z_{i-\frac12}}^{z_{i+\frac12}}v(x,t)dx\approx v(z_i,t),\quad i=1,\cdots,N.$$

Consider $j=2,\cdots,N-1$. Differentiating the error with respect to t we have

$$\begin{aligned}\varepsilon_j'(t)&=u_j'(t)-\frac{1}{h}\int_{x_{j-\frac12}}^{x_{j+\frac12}}u_t(x,t)dx=u_j'(t)-\frac{1}{h}\int_{x_{j-\frac12}}^{x_{j+\frac12}}u_{xx}(x,t)dx\\&=\frac{u_{j+1}(t)-u_j(t)}{h^2}-\frac{u_j(t)-u_{j-1}(t)}{h^2}-\frac{1}{h}\left(u_x(x_{j+\frac12},t)-u_x(x_{j-\frac12},t)\right)\\&=\frac{\varepsilon_{j+1}(t)-2\varepsilon_j(t)+\varepsilon_{j-1}(t)}{h^2}+\frac{u(x_{j+1},t)-u(x_j,t)}{h^2}-\frac{1}{h}u_x(x_{j+\frac12},t)-\frac{u(x_j,t)-u(x_{j-1},t)}{h^2}\\&\quad+\frac{1}{h}u_x(x_{j-\frac12},t).\end{aligned}\tag{3.8}$$

From Taylor's expansions of $u(x_{j+1},t)$ and $u(x_j,t)$ at $(x_{j+\frac{1}{2}},t)$, for $\xi_1 \in (x_{j+\frac{1}{2}}, x_{j+1})$ and $\xi_2 \in (x_j, x_{j+\frac{1}{2}})$ we get

$$u(x_{j+1}) = u(x_{j+\frac{1}{2}+\frac{h}{2}}) = u(x_{j+\frac{1}{2}}) + \frac{h}{2}u_x(x_{j+\frac{1}{2}}) + \frac{1}{2}\left(\frac{h}{2}\right)^2 u_{xx}(x_{j+\frac{1}{2}}) + \frac{1}{6}\left(\frac{h}{2}\right)^3 u_{xxx}(\xi_1),$$

$$u(x_j) = u(x_{j+\frac{1}{2}-\frac{h}{2}}) = u(x_{j+\frac{1}{2}}) - \frac{h}{2}u_x(x_{j+\frac{1}{2}}) + \frac{1}{2}\left(\frac{h}{2}\right)^2 u_{xx}(x_{j+\frac{1}{2}}) - \frac{1}{6}\left(\frac{h}{2}\right)^3 u_{xxx}(\xi_2),$$

which implies

$$\frac{u(x_{j+1}) - u(x_j)}{h^2} = \frac{1}{h}u_x(x_{j+\frac{1}{2}}) + \frac{1}{24}h\left(u_{xxx}(\xi_1) + u_{xxx}(\xi_2)\right).$$

By the Mean Value Theorem, there exists a $\xi_a \in (\xi_1, \xi_2)$, such that $u_{xxx}(\xi_a) = \frac{u_{xxx}(\xi_1)+u_{xxx}(\xi_2)}{2}$. Now, expanding $u(x_{j-1},t)$ and $u(x_j,t)$ at $(x_{j-\frac{1}{2}},t)$, for $\xi_3 \in (x_{j-\frac{1}{2}}, x_j)$ and $\xi_4 \in (x_{j-1}, x_{j-\frac{1}{2}})$ we get

$$u(x_j) = u(x_{j-\frac{1}{2}+\frac{h}{2}}) = u(x_{j-\frac{1}{2}}) + \frac{h}{2}u_x(x_{j-\frac{1}{2}}) + \frac{1}{2}\left(\frac{h}{2}\right)^2 u_{xx}(x_{j-\frac{1}{2}}) + \frac{1}{6}\left(\frac{h}{2}\right)^3 u_{xxx}(\xi_3),$$

$$u(x_{j-1}) = u(x_{j-\frac{1}{2}-\frac{h}{2}}) = u(x_{j-\frac{1}{2}}) - \frac{h}{2}u_x(x_{j-\frac{1}{2}}) + \frac{1}{2}\left(\frac{h}{2}\right)^2 u_{xx}(x_{j-\frac{1}{2}}) - \frac{1}{6}\left(\frac{h}{2}\right)^3 u_{xxx}(\xi_4),$$

which implies

$$\frac{u(x_j) - u(x_{j-1})}{h^2} = \frac{1}{h}u_x(x_{j-\frac{1}{2}}) + \frac{1}{24}h\left(u_{xxx}(\xi_3) + u_{xxx}(\xi_4)\right).$$

Also, by the Mean Value Theorem, there exists $\xi_b \in (\xi_3, \xi_4)$, such that $u_{xxx}(\xi_b) = \frac{u_{xxx}(\xi_3)+u_{xxx}(\xi_4)}{2}$. Then, the expression (3.8) satisfies

$$\varepsilon_j'(t) \leq \frac{\varepsilon_{j+1}(t) - 2\varepsilon_j(t) + \varepsilon_{j-1}(t)}{h^2} + C_j h, \tag{3.9}$$

where $C_j = \frac{1}{12}max|u_{xxx}(\xi_a) - u_{xxx}(\xi_b)|$. Performing the same procedure for $j = 1$, we get

$$\begin{aligned}\varepsilon_1'(t) &= u_1'(t) - \frac{1}{h}\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} u_t(x,t)dx = u_1'(t) - \frac{1}{h}\int_{x_{\frac{1}{2}}}^{x_{\frac{3}{2}}} u_{xx}(x,t)dx \\ &= \frac{u_2(t) - u_1(t)}{h^2} - \frac{1}{h}u_x(x_{\frac{3}{2}},t) = \frac{\varepsilon_2(t) - \varepsilon_1(t)}{h^2} + \frac{u(x_2,t) - u(x_1,t)}{h^2} - \frac{1}{h}u_x(x_{\frac{3}{2}},t).\end{aligned}$$

Therefore, we get

$$\varepsilon_1'(t) \leq \frac{\varepsilon_2(t) - \varepsilon_1(t)}{h^2} + C_1 h,$$

with $C_1 = \frac{1}{12}max|u_{xxx}(\xi_a) - u_{xxx}(\xi_b)|$. Now, for $j = N$, with a similar idea, we get

$$\varepsilon_N'(t) \leq \frac{\varepsilon_{N-1}(t) - \varepsilon_N(t)}{h^2} + h\sum_{j=1}^{N}\sum_{i=1}^{N} J(z_i - x_j)(\xi_i(t) - \varepsilon_N(t)) + C_N h + O(h^2),$$

with $C_N = \frac{1}{12}max|u_{xxx}(\xi_a) - u_{xxx}(\xi_b)|$. Finally, if we differentiate $\xi_i(t) = v_i(t) - v(z_i, t)$ with respect to t we get

$$\begin{aligned}
\xi_i'(t) &= v_i'(t) - \frac{1}{h}\int_{z_{i-\frac{1}{2}}}^{z_{i+\frac{1}{2}}} v_t(x,t)dx = h\sum_{k=1}^{N} J(z_i - z_k)(v_k(t) - v_i(t)) - h\sum_{j=1}^{N} J(z_i - x_j)(v_i(t) - u_N(t)) \\
&\quad -\frac{1}{h}\int_{z_{i-\frac{1}{2}}}^{z_{i+\frac{1}{2}}} \left[\int_0^1 J(x-y)(v(y,t) - v(x,t))dy - \int_{-1}^{0} J(x-y)(v(x,t) - u(0,t))dy\right] dx \\
&= h\sum_{k=1}^{N} J(z_i - z_k)(\xi_k(t) - \xi_i(t)) - h\sum_{j=1}^{N} J(z_i - x_j)(\xi_i(t) - \varepsilon_N(t)) + O(h^2).
\end{aligned} \tag{3.10}$$

Therefore, from (3.9)–(3.10), the error verifies

$$\begin{cases}
\varepsilon_1'(t) = \dfrac{\varepsilon_2(t) - \varepsilon_1(t)}{h^2} + C_1 h, \\
\varepsilon_j'(t) = \dfrac{\varepsilon_{j+1}(t) - 2\varepsilon_j(t) + \varepsilon_{j-1}(t)}{h^2} + C_j h, \quad j = 2, \cdots, N-1 \\
\varepsilon_N'(t) = \dfrac{\varepsilon_{N-1}(t) - \varepsilon_N(t)}{h^2} + h\displaystyle\sum_{j=1}^{N}\sum_{i=1}^{N} J(z_i - x_j)(\xi_i(t) - \varepsilon_N(t)) + C_N h, \\
\xi_i'(t) = h\displaystyle\sum_{k=1}^{N} J(z_i - z_k)(\xi_k(t) - \xi_i(t)) - h\sum_{j=1}^{N} J(z_i - x_j)(\xi_i(t) - \varepsilon_N(t)) + D_i(h^2), \quad i = 1, \cdots, N \\
\varepsilon_j(0) = 0, \quad j = 1, \cdots, N, \\
\xi_i(0) = 0, \quad i = 1, \cdots, N.
\end{cases} \tag{3.11}$$

Now, let us consider $w_j(t) = w(t) = Cht$ and $y_i(t) = y(t) = Cht$, for every $j, i = 1, \cdots, N$, with $C = \max_{i,j}\{C_j, D_i\}$. The pair (w, y) is a supersolution of (3.11). Indeed, we have

$$\begin{aligned}
w_1'(t) &= Ch \geq \frac{w_1 - w_2}{h^2} + C_1 h = \frac{Cht - Cht}{h^2} + C_1 h, \\
w_j'(t) &= Ch \geq \frac{w_j - 2w_j + w_{j-1}}{h^2} + C_j h = \frac{Cht - 2Cht + Cht}{h^2} + C_j h, \quad j = 2, \cdots, N-1, \\
w_N'(t) &= Ch \geq \frac{w_{N-1} - w_N}{h^2} + h\sum_{j=1}^{N}\sum_{i=1}^{N} J(z_i - x_j)(y_i - w_N) + C_N h \\
&\qquad = \frac{Cht - Cht}{h^2} + h\sum_{j=1}^{N}\sum_{i=1}^{N} J(z_i - x_j)(Cht - Cht) + C_N h, \\
y_i'(t) &= Ch \geq h\sum_{k=1}^{N} J(z_i - z_k)(y_k - y_i) - h\sum_{j}^{N} J(z_i - x_j)(y_i - w_N) + D_i h^2 \\
&\qquad = h\sum_{k=1}^{N} J(z_i - z_k)(Cht - Cht) - h\sum_{j}^{N} J(z_i - x_j)(Cht - Cht) + D_i h^2.
\end{aligned}$$

We can proceed as in Lemma 3.4 and prove that (3.11) satisfies a comparison principle. Then, we obtain $\varepsilon_j(t) \leq w_j(t) \leq Kh$ and $\xi_i(t) \leq y_i(t) \leq Kh$, for every $t \leq T$ and every $j, i = 1, \cdots, N$. Here $K = CT$.

Performing the same argument for the subsolutions of (3.11), we can conclude that

$$|\varepsilon_j(t)| \le Kh, \quad |\xi_i(t)| \le Kh, \quad j, i = 1, \cdots, N,$$

which finishes the proof. □

Asymptotic behaviour.

Let us show now that the solutions of the semidiscrete approximation to the local/nonlocal problem converge exponentially as $t \to \infty$ to the mean value of the initial condition, as it happens for the continuous counterpart. For this task, we prove the following Poincaré-type inequality.

Lemma 3.6. *Given J with $J(h) > 0$, for h sufficiently small, let*

$$\alpha_h := \inf_{\substack{u,v\in(\mathbb{R}^{2N}) \\ h\sum_{j=1}^N u_j + h\sum_{i=1}^N v_i = 0}} \frac{E_h(u_j, v_i)}{h\sum_{j=1}^N u_j^2 + h\sum_{i=1}^N v_i^2}. \tag{3.12}$$

Then, α_h is strictly positive and it holds that

$$\alpha_h \left\{ h\sum_{j=1}^N \left| u_j - \left(h\sum_{j=1}^N u_j(0) + h\sum_{i=1}^N v_i(0) \right) \right|^2 + h\sum_{i=1}^N \left| v_i - \left(h\sum_{j=1}^N u_j(0) + h\sum_{i=1}^N v_i(0) \right) \right|^2 \right\} \le E_h(u_j, v_i) \tag{3.13}$$

Proof. To prove that α_h is strictly positive let us argue by contradiction. Suppose that $\alpha_h = 0$. Then, for all $n \in \mathbb{N}$ there exist sequences $\{u_n\}, \{v_n\} \in (\mathbb{R}^{2N})$, such that

$$h\sum_{j=1}^N (u_j)_n + h\sum_{i=1}^N (v_i)_n = 0, \tag{3.14}$$

$$h\sum_{j=1}^N (u_j)_n^2 + h\sum_{i=1}^N (v_i)_n^2 = 1, \tag{3.15}$$

and

$$\frac{1}{n} \ge \left(\sum_{j=1}^{N-1} \frac{[(u_n)_{j+1} - (u_n)_j]^2}{2h} + \frac{h^2}{4}\sum_{i=1}^N\sum_{k=1}^N J(z_i - z_k)[(v_n)_i - (v_n)_k]^2 + \frac{h^2}{2}\sum_{i=1}^N\sum_{j=1}^N J(z_i - x_j)\,[(v_n)_i - (u_n)_N]^2 \right).$$

Therefore, taking the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \left(\frac{1}{2h}\sum_{j=1}^N [(u_n)_{j+1} - (u_n)_j]^2 \right) = 0, \tag{3.16}$$

$$\lim_{n\to\infty} \left(\frac{h^2}{4}\sum_{i=1}^N\sum_{k=1}^N J(z_i - z_k)[(v_n)_i - (v_n)_k]^2 \right) = 0, \tag{3.17}$$

and

$$\lim_{n\to\infty} \left(\frac{h^2}{2}\sum_{i=1}^N\sum_{j=1}^N J(z_i - x_j)[(v_n)_i - (u_n)_N]^2 \right) = 0. \tag{3.18}$$

From (3.15) we have $h\sum_{j=1}^N (u_n)_j^2 \le 1$, which implies that each term of the sum is bounded. Then, there exists a convergent subsequence $(u_{n_l})_j \to \hat{u}_j$. From (3.16), we can conclude that

$((u_{n_l})_{j+1} - (u_{n_l})_j)^2 \to 0$, and then $\hat{u}_{j+1} = \hat{u}_j$. This fact implies that all the limits are equal, $\hat{u_N} = \hat{u_{N-1}} = \cdots = \hat{u_1} = \hat{u}$.

In the same way, also from (3.15) we have $h\sum_{i=1}^{N}(v_n)_i^2 \leq 1$. Using the same argument as before, with v_i, we can extract a convergent subsequence $(v_{n_l})_i \to \hat{v}_i$. From (3.17), as $J(h) > 0$, we have that $((v_{n_l})_i - (v_{n_l})_k)^2 \to 0$, and then we conclude that all the limits are the same, $\hat{v}_N = \hat{v}_{N-1} = \cdots = \hat{v}_1 = \hat{v}$.

On the other hand, from (3.18) we have $\hat{u} = \hat{v}$. Indeed, since $J(h) > 0$ and $((v_{n_l})_i - (u_{n_l})_N)^2 \to 0$, this implies $\hat{v} = \hat{u} = 0$. Moreover, from (3.14) we get $h\sum_{j=1}^{N}\hat{u} + h\sum_{i=1}^{N}\hat{v} = 0$, which implies $h(2N)(\hat{u}+\hat{v}) = 0$. As $\hat{u} = \hat{v}$, it leads $\hat{u} \equiv 0$ and $\hat{v} \equiv 0$. This fact contradicts (3.15) and hence we obtained the proof the first statement.

Note that (3.13) follows immediately from (3.12), considering

$$u_j = w_j - \left(h\sum_{j=1}^{N} w_j(0) + h\sum_{i=1}^{N} y_i(0)\right) \qquad \text{and} \qquad v_i = y_i - \left(h\sum_{j=1}^{N} w_j(0) + h\sum_{i=1}^{N} y_i(0)\right),$$

for any $(w, y) \in \mathbb{R}^{2N}$. This concludes the proof. □

Now we are ready to prove the result about convergence of the numerical solution to the mean value of the initial condition, as $t \to \infty$. This shows that the semidiscrete scheme shares similar properties as the ones that hold for the continuous problem.

Theorem 3.7. *Let $u_h, v_h \in V_h$ the solution to problem* (1.5) *with an initial datum u_{j0}, v_{i0}, satisfies*

$$h\sum_{j=1}^{N}|u_j - k_1|^2 + h\sum_{i=1}^{N}|v_i - k_1|^2 \leq Ce^{-2\alpha_h t}, \tag{3.19}$$

where $k_1 = h\sum_{j=1}^{N} u_j(0) + h\sum_{i=1}^{N} v_i(0)$, for every $u, v \in (\mathbb{R}^{2N})$ and, a positive contant C, independent of t.

Proof. It is easy to see that the constants are solutions to the numerical scheme (1.5). In particular, $w_j(t) = \sum_{j=1}^{N} u_j - k$, for $j = 1, \cdots, N$, and $y_i(t) = \sum_{i=1}^{N} v_i - k$, $i = 1, \cdots, N$ are also solutions to (1.5). If $k = k_1$, then $w_j(t)$ and $y_i(t)$ satisfy

$$h\sum_{j=1}^{N} w_j(t) + h\sum_{i=1}^{N} y_i(t) = 0.$$

Let us define $H(t)$ as

$$H(t) = \frac{h}{2}\sum_{j=1}^{N} w_j^2(t) + \frac{h}{2}\sum_{i=1}^{N} y_i^2(t).$$

Differentiating H with respect to t, we obtain

$$H'(t) = \left[hw_1(t)w_1'(t) + h\sum_{j=2}^{N-1} w_j(t)w_j'(t) + hw_N(t)w_N'(t)\right] + \left[h\sum_{i=1}^{N} y_i(t)y_i'(t)\right] = -2E_h(w_j, y_i)(t).$$

Applying (3.13), we get

$$2E_h(w_j, z_i) \geq 2\alpha_h\left(h\sum_{j=1}^{N} w_j(t)^2 + h\sum_{i=1}^{N} y_i(t)^2\right),$$

which implies

$$H'(t) \leq -2\alpha_h H(t).$$

Hence, we obtain that

$$H(t) \leq e^{-2\alpha_h t}H(0).$$

From this follows that (3.19) holds. □

3.2 A fully discrete scheme.

We will perform a time discretization of (1.5). To deal with this we use a forward explicit Euler method. We consider a constant time step, $\tau_l > 0$, and define the discrete times by $t^l = l\tau_l$, $l \in \mathbb{N}$. The discrete variables u_j^l and v_i^l are defined to be approximations of the average of the exact solutions, u, v, over the volume control,

$$u_j^l = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t^l)dx, \quad v_i^l = \frac{1}{h}\int_{z_{i-\frac{1}{2}}}^{z_{i+\frac{1}{2}}} v(x,t^l)dx.$$

Then, a complete finite volume scheme to solve (1.1) is given by

$$\begin{cases} \dfrac{u_1^{l+1}-u_1^l}{\tau_l} = \dfrac{u_2^l - u_1^l}{h^2}, \\ \dfrac{u_j^{l+1}-u_j^l}{\tau_l} = \dfrac{u_{j+1}^l - 2u_j^l + u_{j-1}^l}{h^2}, \quad j = 2,\cdots,N-1 \\ \dfrac{u_N^{l+1}-u_N^l}{\tau_l} = \dfrac{u_{N-1}^l - u_N^l}{h^2} + h\displaystyle\sum_{j=1}^N\sum_{i=1}^N J(z_i - x_j)(v_i^l - u_N^l), \\ \dfrac{v_i^{l+1}-v_i^l}{\tau_l} = h\displaystyle\sum_{k=1}^N J(z_i - z_k)(v_k^l - v_i^l) - h\sum_{j=1}^N J(z_i - x_j)(v_i^l - u_N^l), \quad i = 1,\cdots,N \\ u_j^0 = u_{j0}, \quad j = 1,\cdots,N \\ v_i^0 = v_{i0}, \quad i = 1,\cdots,N, \end{cases} \tag{3.20}$$

for $l > 0$.

To start the analysis of the fully discrete scheme, we introduce the concept of supersolution and subsolution for the problem (3.20).

Definition 3.8. *We say that $(\overline{u}_j^l, \overline{v}_i^l)$ is a supersolution of (3.20) if, each of its components satisfy*

$$\begin{cases} \dfrac{\overline{u}_1^{l+1}-\overline{u}_1^l}{\tau_l} \geq \dfrac{\overline{u}_2^l - \overline{u}_1^l}{h^2}, \\ \dfrac{\overline{u}_j^{l+1}-\overline{u}_j^l}{\tau_l} \geq \dfrac{\overline{u}_{j+1}^l - 2\overline{u}_j^l + \overline{u}_{j-1}^l}{h^2}, \quad j = 2,\cdots,N-1 \\ \dfrac{\overline{u}_N^{l+1}-\overline{u}_N^l}{\tau_l} \geq \dfrac{\overline{u}_{N-1}^l - \overline{u}_N^l}{h^2} + h\displaystyle\sum_{j=1}^N\sum_{i=1}^N J(z_i - x_j)(\overline{v}_i^l - \overline{u}_N^l), \\ \dfrac{\overline{v}_i^{l+1}-\overline{v}_i^l}{\tau_l} \geq h\displaystyle\sum_{k=1}^N J(z_i - z_k)(\overline{v}_k^l - \overline{v}_i^l) - h\sum_{j=1}^N J(z_i - x_j)(\overline{v}_i^l - \overline{u}_N^l), \quad i = 1,\cdots,N \\ \overline{u}_j^0 \geq u_{j0}, \quad j = 1,\cdots,N \\ \overline{v}_i^0 \geq v_{i0}, \quad i = 1,\cdots,N, \end{cases} \tag{3.21}$$

for $l > 0$. Analogously, $(\underline{u}^l, \underline{v}^l)$ is a subsolution of (3.20) if, its components satisfy the reverse inequalities.

As for the semidiscrete scheme we have a comparison principle.

Proposition 3.9. *Let $(\overline{u}^l, \overline{v}^l)$ and $(\underline{u}^l, \underline{v}^l)$ be a supersolution and a subsolution of (3.20), respectively. If h is small enough and, the time step, τ_l satisfies*

$$\tau_l < \beta := \frac{h^2}{1 - h^3\sum_{j=1}^N\sum_{i=1}^N J(z_i - x_j)}, \tag{3.22}$$

then $\overline{u}^l \leq \underline{u}_l$ and $\overline{v}^l \leq \underline{v}_l$, for every $l > 0$.

Proof. Let $w^l = \overline{u}^l - \underline{u}^l$ and $y^l = \overline{v}^l - \underline{v}^l$. As we did for the semidiscrete scheme, by an approximation argument we can consider strict inequalities in (3.21). Also note that $\overline{u}^0 \le \underline{u}_0$ and $\overline{v}^0 \le \underline{v}_0$. With this in mind, w^l and y^l satisfy (3.21) with strict inequalities. To prove that they are positives, let us argue by contradiction. Suppose that there exists a first time t^{l+1} and nodes j and i, such that $w_j^{l+1} \le 0$ or $y_i^{l+1} \le 0$ while $w_j^l > 0$ and $y_i^l > 0$. Starting by $j = 1$ we get

$$0 \ge w_1^{l+1} > w_1^l + \frac{\tau_l}{h^2}(w_2^l - w_1^l) = \left(1 - \frac{\tau_l}{h^2}\right) w_1^l + \frac{\tau_l}{h^2} w_2^l > 0,$$

a contradiction. The same is valid for $j = 2, \cdots, N$. Finally, for $i = 1, \cdots, N$ we have

$$\begin{aligned}
0 \ge y_i^{l+1} &> y_i^l + \tau_l h \sum_{k=1}^{N} J(z_i - z_k)(y_k^l - y_i^l) - \tau_l h \sum_{j=1}^{N} J(z_i - x_j)(y_i^l - w_N^l) \\
&= \Big(1 - \tau_l h \sum_{k=1}^{l} J(z_i - z_k) - \tau_l h \sum_{j=1}^{l} J(z_i - x_j)\Big) y_i^l + \tau_l h \sum_{k=1}^{l} J(z_i - z_k) y_k^l + \tau_l h \sum_{j=1}^{l} J(z_i - x_j) w_N^l > 0,
\end{aligned}$$

that is a contradiction and completes the proof. ∎

Our next result proves uniform convergence of the fully discrete scheme.

Theorem 3.10. *Let $u \in C^{3,2}([-1,0] \times [0,T])$ and $v \in C^{0,2}([0,1] \times [0,T])$ be a positive solution to (1.1), and (u_h^l, v_h^l) the numerical approximation of the problem (3.20). Then, there exists a constant K, such that for every h, τ, small enough verifying (3.22), it holds that*

$$\max_{0 \le t^l \le T} \left\{ \max_{x \in [-1,0]} |u(x,t^l) - u_h^l|, \max_{x \in [0,1]} |v(x,t^l) - v_h^l| \right\} \le K(h + \tau),$$

where $\tau = \max_l \tau_l$.

Proof. Let us define the error vector at time t^{l+1} as $\varepsilon_j^{l+1} = u_j^{l+1} - u(x_j, t^{l+1})$, $j = 1, \cdots, N$, $l > 0$ and $\xi_j^{l+1} = v_i^{l+1} - v(x_i, t^{l+1})$, for every $i = 1, \cdots, N$, $l > 0$. For $j = 1$ the error vector satisfies

$$\begin{aligned}
\frac{\varepsilon_1^{l+1} - \varepsilon_1^l}{\tau_l} &= \frac{u_1^{l+1} - u_1^l}{\tau_l} - \frac{u(x_1, t^{l+1}) - u(x_1, t^l)}{\tau_l} = \frac{u_2^l - u_1^l}{h^2} - \frac{u(x_1, t^{l+1}) - u(x_1, t^l)}{\tau_l} \\
&= \frac{\varepsilon_2^l - \varepsilon_1^l}{h^2} + \frac{u(x_2, t^l) - u(x_1, t^l)}{h^2} - \frac{u(x_1, t^{l+1}) - u(x_1, t^l)}{\tau_l}
\end{aligned}$$

Expanding $u(x_1, t^{l+1})$ in Taylor series around (x_1, t^l) we get

$$\frac{\varepsilon_1^{l+1} - \varepsilon_1^l}{\tau_l} = \frac{\varepsilon_2^l - \varepsilon_1^l}{h^2} + \frac{u(x_2, t^l) - u(x_1, t^l)}{h^2} - u_t(x_1, t^l) + O(\tau_l)$$

Recalling the argument used at the proof for the semidiscrete scheme, we obtain

$$\frac{\varepsilon_1^{l+1} - \varepsilon_1^l}{\tau_l} \le \frac{\varepsilon_2^l - \varepsilon_1^l}{h^2} + C(h + \tau).$$

Performing the same procedure for $j = 2, \cdots, N$ and $i = 1, \cdots, N$, the error vector satisfies

$$\begin{cases}
\dfrac{\varepsilon_1^{l+1} - \varepsilon_1^l}{\tau_l} \le \dfrac{\varepsilon_2^l - \varepsilon_1^l}{\tau_l} + C(h + \tau), \\
\dfrac{\varepsilon_j^{l+1} - \varepsilon_j^l}{\tau_l} \le \dfrac{\varepsilon_{j+1}^l - 2\varepsilon_j^l + \varepsilon_{j-1}^l}{h^2} + C(h + \tau), \quad j = 2, \cdots, N-1 \\
\dfrac{\varepsilon_N^{l+1} - \varepsilon_N^l}{\tau_l} \le \dfrac{\varepsilon_{N-1}^l - \varepsilon_N^l}{h^2} + h \displaystyle\sum_{j=1}^{N} \sum_{i=1}^{N} J(z_i - x_j) \left(\xi_i^l - \varepsilon_N^l\right) + C(h + \tau), \\
\dfrac{\xi_i^{l+1} - \xi_i^l}{\tau_l} \le h \displaystyle\sum_{k=1}^{N} J(z_i - z_k) \left(\xi_k^l - \xi_i^l\right) - h \sum_{j=1}^{N} J(z_i - x_j) \left(\xi_i^l - \varepsilon_N^l\right) + C(h^2 + \tau), \quad i = 1, \cdots, N.
\end{cases}$$

Now, arguing as before, define $w^l_j = w^l = Ct^l(h+\tau)$, for every $j = 1, \cdots, N$ and, $y^l_i = y^l = Ct^l(h+\tau)$, for every $i = 1, \cdots, N$. For $j = 1$ we have

$$\frac{w_1^{l+1} - w_1^l}{\tau_l} = \frac{(Ct^{l+1} - Ct^l)}{\tau_l}(h+\tau) = \frac{C(t^{l+1} - t^l)}{\tau_l}(h+\tau) = C(h+\tau) \geq 0.$$

Similar inequalities are valid for $j = 2, ..., N$ and for $i = 1, ..., N$. Then w^l, y^l is a supersolution of the previous problem. Notice that there exists l_0 such that $t^{l_0} \geq T$, thus take $K = Ct^{l_0}$ we obtain the desired estimate using the comparison principle.

A similar argument using subsolutions finishes the proof. □

4 Numerical experiments.

In this section we include some numerical experiments considering the fully discrete scheme (1.6). Before we present the results it is important to remark that we do not impose any continuity of the densities at the interface $x = 0$, instead of we can guarantee continuity of the densities inside the local and nonlocal subdomains $(-1, 0)$ and $(0, 1)$, respectively, by assuming continuity of the initial data. At the simulations we will use the density probability function defined as:

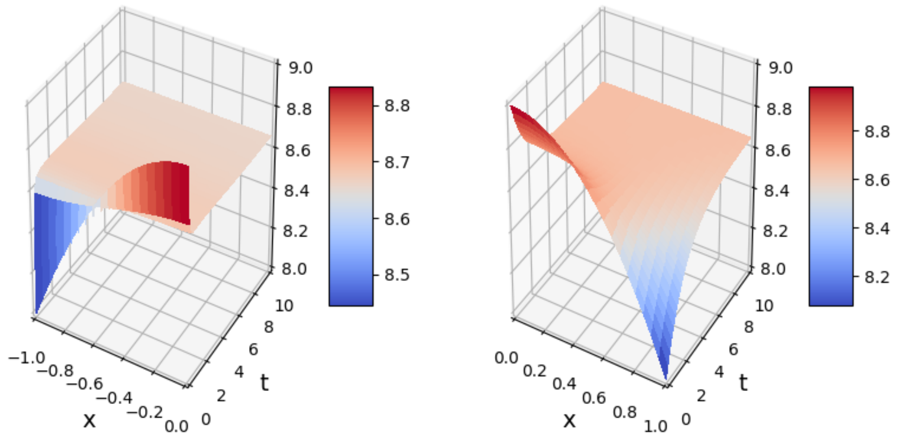
$$J(x) = \begin{cases} \frac{1}{2} cos(x), & if \quad |x| \leq \frac{\pi}{2}, \\ 0, & otherwise. \end{cases}$$

This particular kernel J satisfies the hypothesis described before, J is a nonnegative continuous function, symmetric, with compact support, $supp(J) = [-1, 1]$ and $J(0) > 0$.

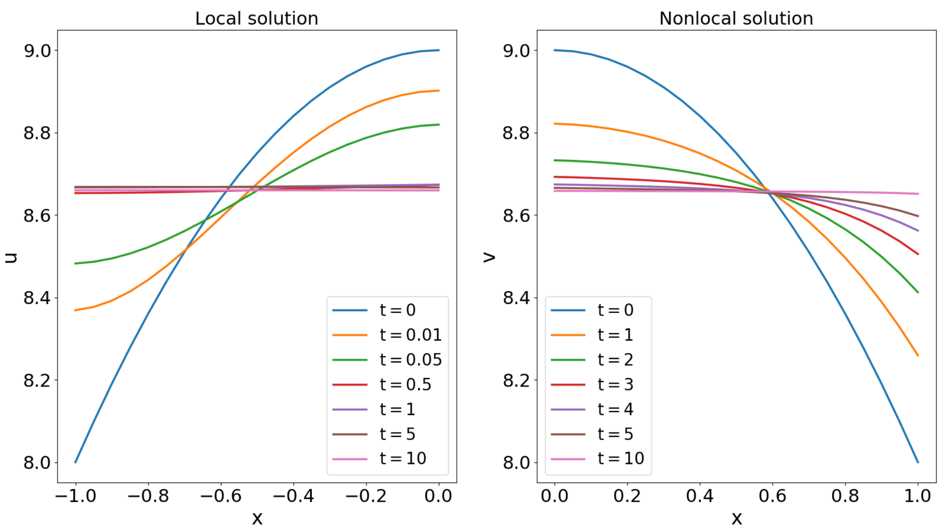
We consider for all simulations, $N = 40$ partitions of the domain $\Omega = [-1, 1]$, $h = 0.05$ and a time step 0.001, which satisfies (3.22).

Numerical experiment 1. Initial condition: $u_0(x) = 9 - x^2$, $v_0(x) = 9 - x^2$. Mean value of the initial condition $\approx 8,65$.

In Figure 1 we plot the evolution of the local and the nonlocal parts of the solution and we can observe the convergence towards the mean value of the numerical initial condition as t increases.



(a) Surface plot of the solution for the local (left) and the nonlocal part (right).



(b) Evolution of the solution for specific timesteps.

Figure 1: Numerical experiment 1.

Numerical experiment 2. Initial condition: $u_0(x) = 1 - x^2$, $v_0(x) = 9 - x^2$. Mean value of the initial condition $\approx 4,66$.

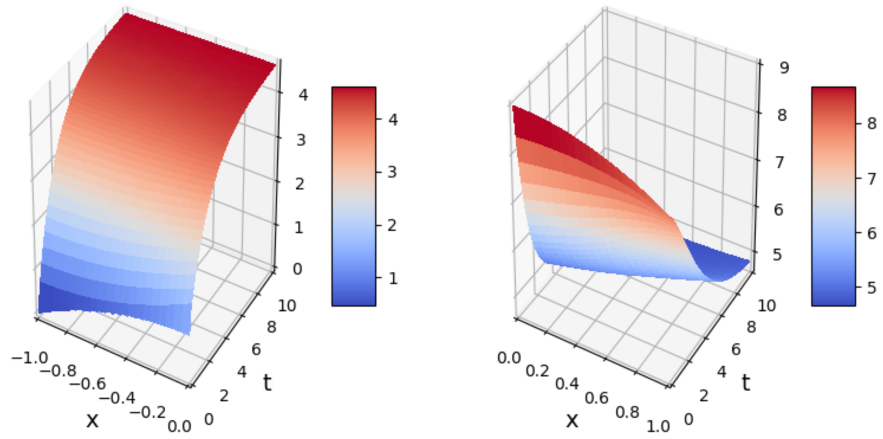
Now we are considering an initial condition that is discontinuous at the interface $x = 0$. We can observe that the solution also converges to mean value of the initial condition, $4,64$, see Figure 2.

Numerical experiment 3. Initial condition:

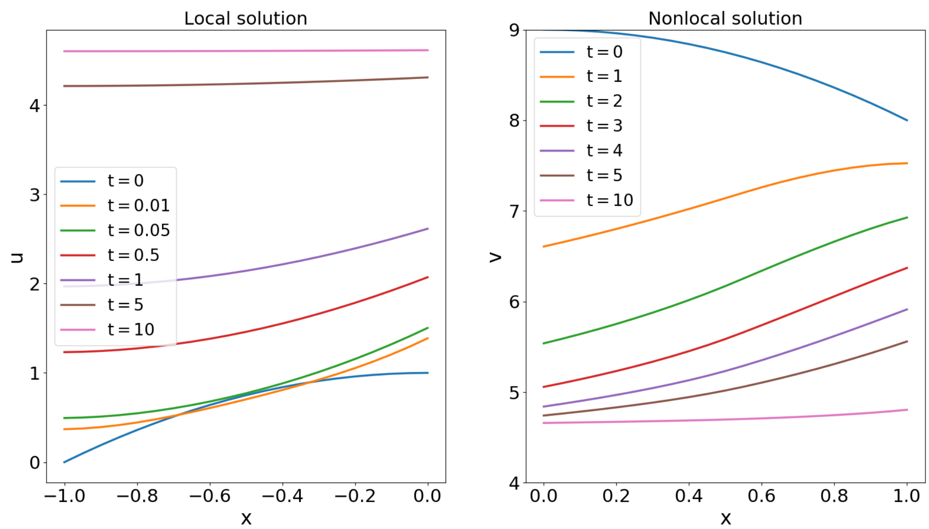
$$\begin{cases} u_0(x) = 1, \quad if \quad x \geq -0.4, \\ u_0(x) = 0, \quad otherwise, \\ v_0(x) = x^3 + 1. \end{cases}$$

Mean value of the initial condition $\approx 0,84$. Notice that the initial condition is given by two different constants in the local subdomain.

In Figure 3, we can note the regularizing effect of the local equation (the local part of the solutions is smooth for positive t) and the convergence of the solution for the mean value of the



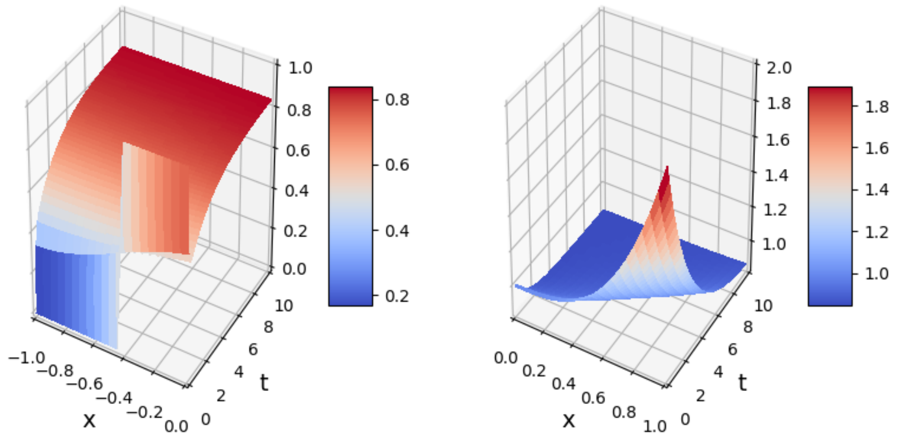
(a) Surface plot of the solution for the local (left) and the nonlocal part (right).



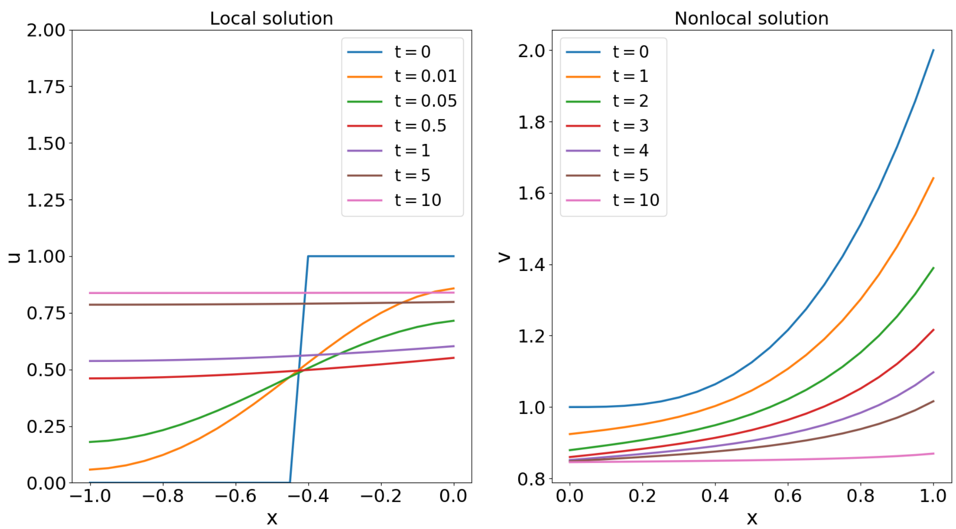
(b) Evolution of the solution for specific timesteps.

Figure 2: Numerical experiment 2.

initial condition as t increase.



(a) Surface plot of the solution for the local (left) and the nonlocal part (right).



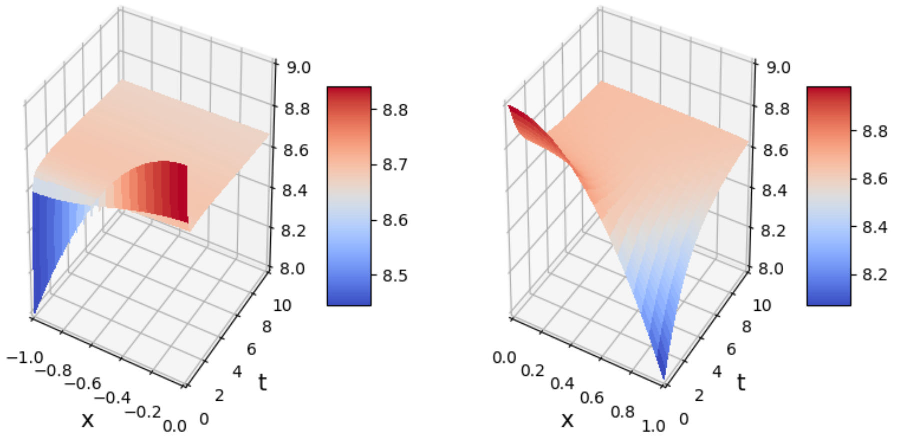
(b) Evolution of the solution for specific timesteps.

Figure 3: Numerical experiment 3.

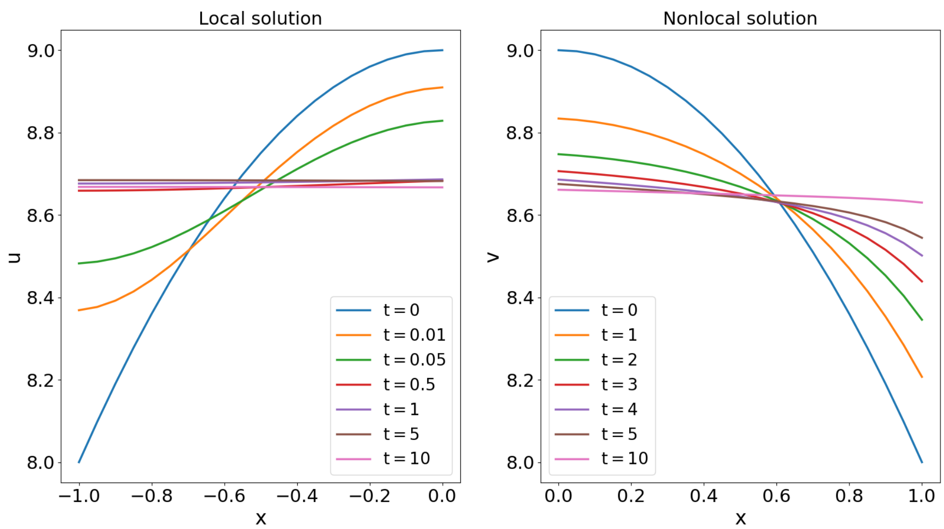
Numerical experiment 4. Now, we will also test the sensitivity of the discretization, using a discontinuous kernel

$$J(x) = \begin{cases} -|x| + 1, & if \quad |x| \leq 1, \\ 0, & otherwise. \end{cases}$$

Initial condition: $u_0(x) = 9 - x^2$ and $v_0(x) = 9 - x^2$.
Mean value of the initial condition $\approx 8,65$. Even in this case, with the a non smooth kernel we obtain the convergence of the solution for the complete problem to the mean value of the initial condition as t increases, see Figure 4.



(a) Surface plot of the solution for the local (left) and the nonlocal part (right).



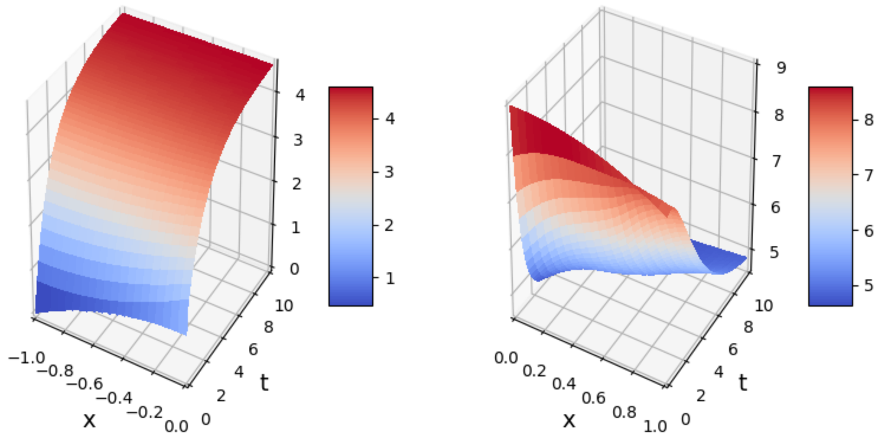
(b) Evolution of the solution for specific timesteps.

Figure 4: Numerical experiment 4.

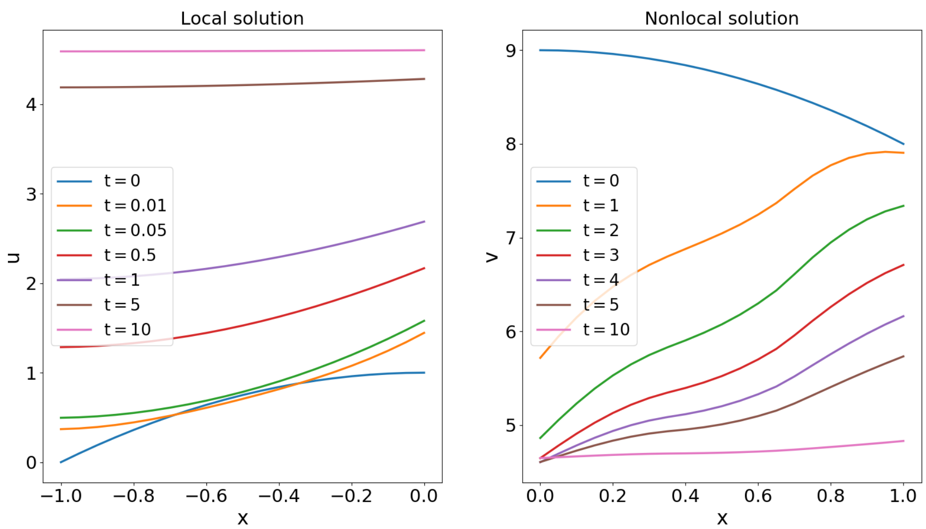
Numerical experiment 5. Finally, we consider a discontinuous initial condition at the interface and the following kernel

$$J(s) = \begin{cases} -0.75|\sin(1.5\pi s)| + 0.19(3-|s|)^2, & for \quad |s| \le 1, \\ 0, \quad otherwise. \end{cases}$$

Initial condition: $u_0(x) = 1 - x^2$ and $v_0(x) = 9 - x^2$. Mean value of the initial condition $\approx 4,66$. In Figure 5 we observe the same behavior of the example 2. The solutions for the local and nonlocal part converge for the mean value of the initial condition as t increases.



(a) Surface plot of the solution for the local (left) and the nonlocal part (right).



(b) Evolution of the solution for specific timesteps.

Figure 5: Numerical experiment 5.

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Chapter 4

Coupled local/nonlocal models in thin domains

The Chapter 4 is composed by the third paper entitled **Coupled local/nonlocal models in thin domains** that was submitted to the journal Asymptotic Analysis.

Coupled local/nonlocal models in thin domains

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Abstract: In this paper, we analyze a model composed by coupled local and nonlocal diffusion equations acting in different subdomains. We consider the limit case when one of the subdomains is thin in one direction (it is concentrated to a domain of smaller dimension) and as a limit problem we obtain coupling between local and nonlocal equations acting in domains of different dimension. We find existence and uniqueness of solutions and we prove several qualitative properties (like conservation of mass and convergence to the mean value of the initial condition as time goes to infinity).

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1 Introduction and main results

In this paper we combine a local diffusion equation, the classical heat equation,

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) \tag{1.1}$$

in a higher dimensional domain $\Omega \subset \mathbb{R}^N$, with a nonlocal diffusion equation, given by an integrable kernel

$$\frac{\partial u}{\partial t}(x,t) = \int_R J(x-y)(u(y,t)-u(x,t))dy \tag{1.2}$$

in R a different subset of $\mathbb{R}^N$. Associated with these two domains, Ω and R, in [23] and [29] the following kind of energy functional was introduced

$$E(u,v) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4}\int_R\int_R J(x-y)\,(v(y)-v(x))^2\,dydx + \frac{1}{2}\int_R\int_A G(x-y)\,(v(x)-u(y))^2\,dydx. \tag{1.3}$$

Here the set $A \subset \Omega$ is the whole Ω (and we will refer to the resulting model as having a coupling in the source terms, see the next subsection) or a part of the boundary $A = \Gamma \subset \partial\Omega$ (we refer to this case as coupling at the boundary).

Observe that, the kernels J and G do not need to be equal. We will assume that J and also G satisfy the following hypotheses that will be assumed along the whole paper without further mention,

$J \in C(\mathbb{R}^N, \mathbb{R})$ is nonnegative, with $J(0) > 0$, $J(-x) = J(x)$ for every $x \in \mathbb{R}^N$, and integrable,

$G \in C(\mathbb{R}^N, \mathbb{R})$ is nonnegative, nontrivial and integrable.

Remark 1. We can also consider kernels that are not in convolution form, that is, $J(x,y)$ and $G(x,y)$ with $J \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})$ nonnegative, with $J(x,x) > 0$, symmetric $J(x,y) = J(y,x)$ and integrable, and $G \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})$ nonnegative, nontrivial and integrable. To simplify the presentation we will deal with convolution type kernels in the proofs.

Observe that it is common to assume that the integral of J and G is equal to one. This assumption is related to the probabilistic interpretation of the model given in [23] and [29]. For example, in this interpretation, $G((x_1,x_2),y)$ is the probability of a particle (or an individual of a biological species) that is at (x_1,x_2) jumps to y in a time step). So, in this case, we have

$$\int_{R_1} G(x_1,x_2,y)dy = 1.$$

To obtain our results we only need the integrability of the kernels, hence we do not assume that they are normalized to have integral equal to one.

Associated with the energy (1.3) we have the evolution problem given by its gradient flow (with respect to $L^2(\Omega \cup R)$). This gives rise to an diffusion problem. Take (u,v) as the solution of the abstract ODE problem

$$(u,v)'(t) = -\partial E\left[(u,v)(t)\right], \qquad t \geq 0,$$

with $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ and, $\partial E[(u,v)]$ the subdifferential of E. Then, it turns out (see [23] and [29]) that (u,v) solves a system composed by a heat equation (local diffusion) of the form (1.1) in Ω and a nonlocal difusion equation in R, (1.2), coupled via source terms in the equations (when $A = \Omega$ in (1.2)) or via a boundary flux on $\Gamma \subset \partial\Omega$ (when $A = \Gamma$ in (1.2)). See Sections 1.1 and 1.2 below.

Also from [23] and [29] we know that the associated evolution problem is well-posed in the sense that there are existence and uniqueness of solutions. There are two alternative proofs of this fact. The first one uses a fixed point argument while the second relies on semigroup theory. Besides, a comparison principle holds. Also, the total mass of the initial condition is preserved along the evolution and the solutions converge exponentially fast to the mean value of the initial condition. Notice that, according to [23] and [29], we do not impose any continuity of the densities troughout the interface between the local and nonlocal domain, but we can guarantee continuity of the densities u and v inside the local and nonlocal subdomains Ω and R, respectively, by assuming continuity of the initial conditions. Also there is a probabilistic interpretation of this model (we refer one more time to [23] and [29]). In this interpretation individuals cannot diffuse neither jump from the exterior $\mathbb{R}^N \setminus \Omega$ into Ω or the other way around (the integrals accounting for jumps do not consider the complement of Ω). There is no interchange of mass between $\Omega \cup R$ and its complement. Therefore, the total mass is preserved and we can call our problem as being of Neumann type.

The study of nonlocal problems with smooth kernels has been widely considered recently, see [6, 7, 8, 9, 11, 14, 20, 21, 22] and the book [1]. This kind of equation is getting attention due to its potential applications in ecology, physics, and engineering, and to its flexibility to accurately capture effects that are not easily obtained from classical local models. Biological mobility models of animals and plants are examples of how distinct patterns of mobility can affect the success of invasions [7, 34]. In epidemiology, the effects of long-range interactions are responsible for the spreading of diseases around the world [36]. Nonlocal patterns also play an important role in molecular interactions in dissimilar interfaces, continuum mechanics, [24, 30], and peridynamics (a model of elasticity and mechanics), [31, 32].

There are different strategies for couplings between local and nonlocal models. Let us briefly summarize previous results in [15, 18, 22, 23, 26, 29], see also the review [17]. In [15], local and nonlocal problems are coupled trough a prescribed solid region in which both kinds of equations overlap (the value of the solution in the nonlocal part of the domain is used as a Dirichlet boundary condition for the local part and vice-versa). This kind of coupling gives continuity of the solution in the overlapping region but does not preserve the total mass. Here we follow [23] and [29] (see also [22, 26]). In probabilistic terms, in the model described in [23], particles may jump across the interface between the two regions but can not pass coming from the local side unless they jump. Finally, in [29], the authors studied local and nonlocal diffusion models in different zones coupled via the fluxes across the surface that separates the two regions.

Here, we take as the nonlocal region a thin domain, that is, we consider $R_\varepsilon \subset \mathbb{R}^N$ (R_ε is assumed to be open and bounded), depending on a small parameter $\varepsilon \in (0,1]$ that will go to zero and that measures the thickness of the domain. Therefore, in our model problem we have two full dimensional domains, the local domain $\Omega \subset \mathbb{R}^N$ (that is fixed) and the nonlocal domain $R_\varepsilon \subset \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. We denote $x = (x_1, x_2)$ a point in $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. The domain R_ε is assumed to be a general thin domain defined as

$$R_\varepsilon = \{(x_1, \varepsilon x_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} : (x_1, x_2) \in R\},$$

with $R \subset \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. Notice that R_ε is a domain that is thin in the x_2-variable. See Figure 1.

Our main goal here is to pass to the limit as $\varepsilon \to 0$ in the previous setting and obtain a nontrivial diffusion model in which we couple local and nonlocal diffusion equations, (1.1) and (1.2) that take place in domains of different dimension (we deal here with local diffusion in the full-dimensional domain and nonlocal diffusion in the lower-dimensional one).

For simplicity, we will concentrate in the product case and take R_ε as

$$R_\varepsilon = R_1 \times \varepsilon R_2 = \{(x_1, \varepsilon x_2) : x_1 \in R_1, x_2 \in R_2\}.$$

Our results are valid in a more general setting (see Remark 2 below) but we prefer to avoid extra notations and simplify the changes of variables that are needed in the proofs. The typical configuration under study is depicted in Figure 1.

Remark 2. Instead of a thin domain like $R_\varepsilon = \{(x_1, \varepsilon x_2) : x_1 \in R_1, x_2 \in R_2\}$, we could have a more complex domain, which could be described by some function g related to the geometry of the channel R_ε, more exactly, on the way the channel R_ε collapses to a general manifold R_1. If we want to construct a more general geometry of the channel we could, for instance, in two dimensions, consider the channel $R_\varepsilon = \{(x, y) : 0 < x_1 < 1, 0 < x_2 < \varepsilon g(x_1)\}$, although more general and complicated geometries are allowed, see [2].

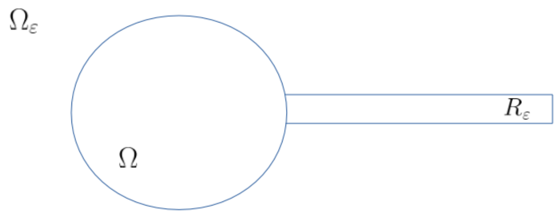

Figure 1: Perturbed domain.

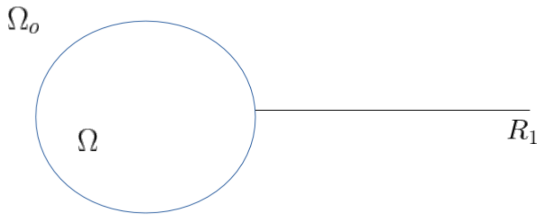

Figure 2: Limit domain.

Main goal. Let $\Omega_\varepsilon = \Omega \cup R_\varepsilon \subset \mathbb{R}^N$ and consider a local/nonlocal coupling in this domain (see subsections 1.1 and 1.2 for a precise statement of the involved equations and the obtained results). As we have mentioned, our main goal is to study the limit as the nonlocal region, R_ε, gets thinner, that is, to study the limit as $\varepsilon \to 0$. When passing the limit as $\varepsilon \to 0$, the "limit" domain, Ω_0 (see figure 2) will be the union of Ω and the lower dimensional domain R_1. In the limit of the solutions to our coupled models we will obtain solutions to a local equation in the domain Ω (with a nonlocal source) and a nonlocal equation in a domain of smaller dimension, R_1. After obtaining the limit equations, we will also prove some qualitative properties of this limit problem (like conservation of the total mass and study the asymptotic behaviour of the solutions).

Concerning references for equations in thin domains we refer to [2, 3, 4, 27, 5, 33] that develop some techniques and methods to understand the effects of the geometry of the thin domain on the solutions of elliptic and parabolic singular problems. We can find some applications in elastic beam theories (as torsion and warping functions) [28], lubrification [12], fluid flows as ocean dynamics, geophysical fluid dynamics, and fluid flows in cell membranes, see for instance [25].

Our results can be viewed as an extension of [2] and [27]. In [2], the authors investigate the dynamics of a local reaction-diffusion equation with homogeneous boundary condition in a dumbbell domain. The dumbbell domain is composed by two disconnected regions joined by a thin channel, that depends on a thickness parameter ε and degenerates to a line segment as the parameter $\varepsilon \to 0$. As part of a series of articles (see [3, 4]) the authors also prove some properties about the continuity of the set of equilibria. On the other hand, in [27] the authors deal with nonlocal evolution problems with non singular kernels in thin domains obtaining a limit problem when the thickness of the domain goes to zero, but without considering any coupling with a local part of the problem. Passing to the limit in these coupling terms is the main contribution of this work.

1.1 Coupling using source terms

We need to compare the solutions of the problem posed in the perturbed domain $\Omega_\varepsilon = \Omega \cup R_\varepsilon \subset \mathbb{R}^N$ and the solutions to the limit problem in the limit domain Ω_0. Since the solutions live in different spaces, to obtain convergence we need some care, not only in the choice of the functional space, but also with the metric chosen in this space. Decomposing a function $w \in L^2(\Omega_\varepsilon)$ as $w = u + v$, with $u = w\chi_\Omega$ and $v = w\chi_{R_\varepsilon}$, we define the metric in $L^2(\Omega_\varepsilon)$ as

$$\|w\|^2_{L^2(\Omega_\varepsilon)} = \int_\Omega |u|^2 + \frac{1}{\varepsilon^{N_2}} \int_{R_\varepsilon} |v|^2. \tag{1.4}$$

Remark that we multiply the norm of the involved functions in the thin part of the domain R_ε by a factor $\varepsilon^{-(N_2)}$. Now, we can define the energy functional

$$E_\varepsilon(u,v) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4\varepsilon^{2N_2}} \int_{R_\varepsilon}\int_{R_\varepsilon} J(x-y)|v(y)-v(x)|^2 dydx + \frac{1}{2\varepsilon^{N_2}} \int_{R_\varepsilon}\int_\Omega G(x-y)|v(x)-u(y)|^2 dydx, \tag{1.5}$$

which is finite in

$$\mathcal{B} := \left\{(u,v) \in L^2(\Omega_\varepsilon) : u \in H^1(\Omega), v \in L^2(R_\varepsilon)\right\}.$$

Notice that in this energy functional we have two terms,

$$\frac{1}{2}\int_\Omega |\nabla u|^2 dx \qquad \text{and} \qquad \frac{1}{4\varepsilon^{2N_2}} \int_{R_\varepsilon}\int_{R_\varepsilon} J(x-y)\left(v(y)-v(x)\right)^2 dydx,$$

that are naturally associated with the equations (1.1) and (1.2) plus a coupling term given by

$$\frac{1}{2\varepsilon^{N_2}} \int_{R_\varepsilon}\int_\Omega G(x-y)\left(v(x)-u(y)\right)^2 dydx.$$

Now, let us consider the evolution problem obtained as the gradient flow associated with this energy with respect to the norm previously defined in (1.4), that is, $(u(t),v(t))$ will be the solution of the abstract ODE problem

$$(u,v)'(t) = -\partial E_\varepsilon\left[(u,v)(t)\right], \qquad t \geq 0,$$

with initial data $u(x,0) = u_0(x)$, $v(x,0) = v_0^\varepsilon(x)$. Here $\partial E\left[(u,v)\right]$ denotes the subdifferential of E at the point (u,v). To see what kind of equations we are solving here, let us compute the derivative of E at (u,v) in the direction of $\varphi \in C_0^\infty(\Omega_\varepsilon)$,

$$\begin{aligned}\partial_\varphi E_\varepsilon(u,v) &= \lim_{h\to 0}\frac{E_\varepsilon(u+h\varphi, v+h\varphi) - E_\varepsilon(u,v)}{h}\\ &= \int_\Omega \nabla u \nabla\varphi dx + \frac{1}{\varepsilon^{N_2}}\int_{R_\varepsilon}\frac{1}{2\varepsilon^{N_2}}\int_{R_\varepsilon} J(x-y)(v(y)-v(x))(\varphi(y)-\varphi(x))dydx\\ &\quad + \frac{1}{\varepsilon^{N_2}}\int_{R_\varepsilon}\int_\Omega G(x-y)(v(x)-u(y))dy\varphi(x)dx\\ &\quad - \int_\Omega \frac{1}{\varepsilon^{N_2}}\int_{R_\varepsilon} G(x-y)(v(y)-u(x))dy\varphi(x)dx.\end{aligned}$$

Since $\langle \partial E[u,v], \varphi\rangle = \partial_\varphi E(u,v)$, we can derive the local/nonlocal problem associated to this gradient flow that is given by the following system of equations:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \triangle u(x,t) + \dfrac{1}{\varepsilon^{N_2}} \displaystyle\int_{R_\varepsilon} G(x-y)(v(y,t)-u(x,t))dy, \quad (x,t) \in \Omega \times (0,+\infty), \\ \dfrac{\partial u}{\partial \eta}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,+\infty), \\ \dfrac{\partial v}{\partial t}(x,t) = \dfrac{1}{\varepsilon^{N_2}} \displaystyle\int_{R_\varepsilon} J(x-y)\,(v(y,t)-v(x,t))\,dy - \int_\Omega G(x-y)(v(x,t)-u(y,t))dy, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in R_\varepsilon \times (0,+\infty), \\ u(x,0) = u_0(x), \quad x \in \Omega, \\ v(x,0) = v_0^\varepsilon(x), \quad x \in R_\varepsilon. \end{cases} \tag{1.6}$$

As we have mentioned previously, our aim is to pass to the limit as $\varepsilon \to 0$ in this evolution problem (1.6). To introduce a candidate to be a limit problem for (1.6), defined in the domain Ω_0 (see Figure 2) we will perform a change of variables (as described in [27]) in the thin domain, R_ε, in order to fix it. The change of variables is given by

$$R = R_1 \times R_2 \ni (x_1, x_2) \longmapsto (x_1, \varepsilon x_2) \in R_1 \times \varepsilon R_2 = R_\varepsilon.$$

That is, we take $\tilde{x}_2 = \frac{x_2}{\varepsilon}$ and $\tilde{y}_2 = \frac{y_2}{\varepsilon}$. With these variables we can fix the domain which allows us to analyze the asymptotic behavior as $\varepsilon \to 0$ in a fixed space of functions. To fix the initial condition for v after the change of variables, we take $v(x,0) = v_0^\varepsilon(x_1,x_2) = v_0(x_1,\tilde{x}_2)$ for some fixed function v_0. The problem (1.6) becomes after this change of variables the following equations in the fixed domain $\widehat{\Omega} = \Omega \cup R$:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(x,t) = \triangle u^\varepsilon(x,t) + \displaystyle\int_R G_\varepsilon(x-y)(v^\varepsilon(\tilde{y},t)-u(x,t))d\tilde{y}, \quad (x,t) \in \Omega \times (0,+\infty), \\ \dfrac{\partial u^\varepsilon}{\partial \eta}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,+\infty), \\ \dfrac{\partial v^\varepsilon}{\partial t}(\tilde{x},t) = \displaystyle\int_R J_\varepsilon(x-y)\,(v^\varepsilon(\tilde{y},t)-v^\varepsilon(\tilde{x},t))\,d\tilde{y} - \int_\Omega G_\varepsilon(x-y)(v^\varepsilon(\tilde{x},t)-u^\varepsilon(y,t))dy, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\tilde{x},t) \in R \times (0,+\infty), \\ u^\varepsilon(x,0) = u_0(x), \quad x \in \Omega, \\ v^\varepsilon(\tilde{x},0) = v_0(\tilde{x}), \quad \tilde{x} \in R, \end{cases} \tag{1.7}$$

where

$$J_\varepsilon(x-y) = J(x_1-y_1, \varepsilon(\tilde{x}_2-\tilde{y}_2)), \quad G_\varepsilon(x-y) = G(x_1-y_1, \varepsilon\tilde{x}_2-\tilde{y}_2) \quad \text{and} \quad v^\varepsilon(x_1,\tilde{x}_2,t) = v(x_1,\varepsilon\tilde{x}_2,t).$$

Notice that the problem (1.7) is similar to the ones obtained previously in thin domains (see for instance [27, 2]).

Now, we are ready to state our main result for this coupling.

Theorem 1.1. *Let $\{(u^\varepsilon, v^\varepsilon)\}_{\varepsilon>0}$ be a family of solutions of (1.7). Then, there exist (u^*, V^*), $u^* \in C([0,T], H^1(\Omega))$ and $V^* \in C([0,T], L^2(R_1))$, such that*

$$\begin{aligned} &u^\varepsilon \rightharpoonup u^* \quad in \quad L^\infty(0,T;L^2(\Omega)), \\ &v^\varepsilon \rightharpoonup v^* \quad in \quad L^\infty(0,T;L^2(R)) \quad and, \\ &V^\varepsilon(\cdot) = \int_{R_2} v^\varepsilon(\cdot, \varepsilon\tilde{x}_2, t)d\tilde{x}_2 \rightharpoonup V^*(\cdot) = \int_{R_2} v^*(\cdot, 0, t)d\tilde{x}_2 \quad in \quad L^\infty(0,T;L^2(R_1)). \end{aligned}$$

The pair $\{u^*, V^*\}$ satisfies the following limit problem in $\Omega_0 = \Omega \cup R_1$,

$$\begin{cases} \dfrac{\partial u^*}{\partial t}(x,t) = \Delta u^*(x,t) + \displaystyle\int_{R_1} G^*(x-y)(V^*(y,t) - |R_2|u^*(x,t))dy, \quad (x,t) \in \Omega \times (0,+\infty), \\ \dfrac{\partial u^*}{\partial \eta}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,+\infty), \\ \dfrac{\partial V^*}{\partial t}(x,t) = |R_2| \displaystyle\int_{R_1} J^*(x-y)\,(V^*(y,t) - V^*(x,t))\,dy - \int_{\Omega} G^*(x-y)(V^*(x,t) - |R_2|u^*(y,t))dy, \\ \qquad\qquad (x,t) \in R_1 \times (0,+\infty) \\ u^*(x,0) = u_0(x), \quad x \in \Omega, \\ V^*(x,0) = V_0^*(x) = \displaystyle\int_{R_2} v_0(x,0)dx_2, \quad x \in R_1, \end{cases} \tag{1.8}$$

where the limit kernels J^* and G^* are given by

$$J^*(x-y) = J(x_1 - y_1, 0), \qquad G^*(x-y) = G(x_1 - y_1, 0 - y_2).$$

We also include here some properties of the limit problem (1.8). The problem is well posed, the total mass remains constant in time, that is,

$$\int_{\Omega} u^*(x,t)dx + \int_{R_1} V^*(x,t)dx = \int_{\Omega} u_0^* dx + \int_{R_1} V_0^* dx, \quad \forall t \geq 0,$$

and solutions converge exponentially to the mean value of the initial condition as $t \to \infty$, i.e.,

$$\left\| (u^*, V^*)(\cdot,t) - ⨍(u_0^*, V_0^*) \right\|_{L^2(\Omega_0)} \leq Ce^{-\lambda_1 t}.$$

for some $C > 0$ and $\lambda_1 > 0$ (we also obtain that λ_1 can be chosen independent of the initial data).

1.2 Coupling at the boundary

Now we want to impose that an individual to pass from the nonlocal domain to the local domain, it necessarily needs to cross the boundary to then get in the local domain. As we did before, first we will define the problem in the perturbed domain (see Figure 4.2) and then derive the limit problem defined in the limit domain (see Figure 2). Let us, as before, consider the domain $\Omega_\varepsilon = \Omega \cup R_\varepsilon \subset \mathbb{R}^N$, with $\Omega \subset \mathbb{R}^N$ and $R_\varepsilon = R_1 \times \varepsilon R_2 \subset \mathbb{R}^N$, with a small parameter ε.

Let us consider the metric (1.4) and derive the evolution problem as the flux associated with the energy

$$E_\varepsilon^b(u,v) := \frac{1}{2}\int_{\Omega} |\nabla u|^2 dx + \frac{1}{4\varepsilon^{2N_2}} \int_{R_\varepsilon}\int_{R_\varepsilon} J(x-y)\,(v(y) - v(x))^2\,dydx$$
$$+ \frac{1}{2\varepsilon^{N_2}} \int_{R_\varepsilon}\int_{\Gamma} G(x-y)\,(v(x) - u(y))^2\,d\sigma(y)dx, \tag{1.9}$$

with Γ a fixed part of the boundary of Ω with $|\Gamma|_{N-1} > 0$ (then we have a well defined trace operator from $H^1(\Omega)$ into $L^2(\Gamma)$). Notice that the coupling term

$$\frac{1}{2\varepsilon^{N_2}} \int_{R_\varepsilon}\int_{\Gamma} G(x-y)\,(v(x) - u(y))^2\,d\sigma(y)dx$$

involve the values of u on $\Gamma \subset \partial\Omega$ instead of the values of u inside Ω (compare with the previous functional $E_\varepsilon(u,v)$).

Now, the evolution problem associated to the energy functional $E^b_\varepsilon(u,v)$ is given by the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t), \quad (x,t) \in \Omega \times (0,+\infty), \\ \dfrac{\partial u}{\partial \eta}(x,t) = 0, \quad (x,t) \in \partial\Omega \setminus \Gamma \times (0,+\infty), \\ \dfrac{\partial u}{\partial \eta}(x,t) = \dfrac{1}{\varepsilon^{N_2}} \displaystyle\int_{R_\varepsilon} G(x-y)(v(y,t)-u(x,t))d\sigma(y), \quad (x,t) \in \Gamma \times (0,+\infty), \\ \dfrac{\partial v}{\partial t}(x,t) = \dfrac{1}{\varepsilon^{N_2}} \displaystyle\int_{R_\varepsilon} J(x-y)\,(v(y,t)-v(x,t))\,dy - \int_\Gamma G(x-y)(v(x,t)-u(y,t))d\sigma(y), \\ \hfill (x,t) \in R_\varepsilon \times (0,+\infty) \\ u(x,0) = u_0(x), \quad x \in \Omega, \\ v(x,0) = v_0^\varepsilon(x), \quad x \in R_\varepsilon. \end{cases} \tag{1.10}$$

Notice that the nonlocal part contributes with the normal derivative of u on Γ and the local part of the problem appears as before in the source term of the equation for the nonlocal part. The coupling is balanced in such a way that the problem preserves the total mass, see [29].

After the same change of variables that we used before, $\tilde{x}_2 = \frac{x_2}{\varepsilon}$ and $\tilde{y}_2 = \frac{y_2}{\varepsilon}$, we fix the domain and then pass to the limit and obtain the limit problem. Again here we take $v(x,0) = v_0^\varepsilon(x_1,x_2) = v_0(x_1,\tilde{x}_2)$ for some fixed function v_0 as the initial condition. Notice that, as we did in the previous subsection, there exists an equivalence between the coupled local/nonlocal problem (1.10) with the following coupled local/nonlocal thin domain problem defined in $\hat{\Omega} = \Omega \cup R$, with Γ a fixed part of the boundary of Ω,

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(x,t) = \Delta u^\varepsilon(x,t), \quad (x,t) \in \Omega \times (0,+\infty), \\ \dfrac{\partial u^\varepsilon}{\partial \eta}(x,t) = 0, \quad (x,t) \in \partial\Omega \setminus \Gamma \times (0,+\infty), \\ \dfrac{\partial u^\varepsilon}{\partial \eta}(x,t) = \displaystyle\int_R G^\varepsilon(x-y)(v^\varepsilon(\tilde{y},t)-u^\varepsilon(x,t))d\tilde{y}, \quad (x,t) \in \Gamma \times (0,+\infty), \\ \dfrac{\partial v^\varepsilon}{\partial t}(\tilde{x},t) = \displaystyle\int_R J^\varepsilon(x-y)\,(v^\varepsilon(\tilde{y},t)-v^\varepsilon(\tilde{x},t))\,d\tilde{y} - \int_\Gamma G^\varepsilon(x-y)(v^\varepsilon(\tilde{x},t)-u^\varepsilon(y,t))d\sigma(\tilde{y}), \\ \hfill (\tilde{x},t) \in R \times (0,+\infty) \\ u^\varepsilon(x,0) = u_0(x), \quad x \in \Omega, \\ v^\varepsilon(\tilde{x},0) = v_0(\tilde{x}), \quad \tilde{x} \in R, \end{cases} \tag{1.11}$$

with

$$J^\varepsilon(x-y) = J(x_1-y_1, \varepsilon(\tilde{x}_2-\tilde{y}_2)) \quad G^\varepsilon(x-y) = G(x_1-y_1, x_2-\varepsilon\tilde{y}_2) \quad \text{and} \quad v^\varepsilon(\tilde{x},t) = v(x_1,\varepsilon\tilde{x}_2,t).$$

Now we can enunciate a convergence result analogous to Theorem 1.1. It says that there is a limit as $\varepsilon \to 0$ of the solutions to the problem (1.11) in the limit domain $\Omega_0 = \Omega \cup R_1$ (see Figure 2).

Theorem 1.2. *Let $\{u^\varepsilon, v^\varepsilon\}_{\varepsilon>0}$ be a family of solutions for the problem* (1.11)*. Then, there exists a solution (u^*,V^*), $u^* \in C([0,T],H^1(\Omega))$ and $V^* \in C([0,T],L^2(R_1))$, such that*

$$\begin{aligned} &u^\varepsilon \rightharpoonup u^* \quad \textit{in} \quad L^\infty(0,T;L^2(\Omega)), \\ &v^\varepsilon \rightharpoonup v^* \quad \textit{in} \quad L^\infty(0,T;L^2(R)) \quad \textit{and}, \\ &V^\varepsilon(\cdot) = \int_{R_2} v^\varepsilon(\cdot,\varepsilon\tilde{x}_2,t)d\tilde{x}_2 \rightharpoonup V^*(\cdot) = \int_{R_2} v^*(\cdot,0,t)d\tilde{x}_2 \quad \textit{in} \quad L^\infty(0,T;L^2(R_1)). \end{aligned}$$

The pair $\{u^*, V^*\}$ satisfies the following limit problem in $\Omega_0 = \Omega \cup R_1$,

$$\begin{cases} \dfrac{\partial u^*}{\partial t}(x,t) = \Delta u^*(x,t), \quad (x,t) \in \Omega \times (0,+\infty), \\ \dfrac{\partial u^*}{\partial \eta}(x,t) = 0, \quad (x,t) \in \partial\Omega \setminus \Gamma \times (0,+\infty), \\ \dfrac{\partial u^*}{\partial \eta}(x,t) = \displaystyle\int_{R_1} G^*(x-y)(V^*(y,t) - |R_2|u^*(x,t))dy, \quad (x,t) \in \Gamma \times (0,+\infty), \\ \dfrac{\partial V^*}{\partial t}(x,t) = |R_2| \displaystyle\int_{R_1} J^*(x-y)\left(V^*(y,t) - V^*(x,t)\right)dy - \int_\Gamma G^*(x-y)(V^*(x,t) - |R_2|u^*(y,t))d\sigma(y), \\ \qquad\qquad (x,t) \in R_1 \times (0,+\infty) \\ u^*(x,0) = u_0^*(x), \quad x \in \Omega, \\ V^*(x,0) = V_0^*(x) = \displaystyle\int_{R_2} v_0(x,0)dx_2, \quad x \in R_1, \end{cases} \tag{1.12}$$

where the limit kernels J^* and G^* are given by

$$J^*(x-y) = J(x_1 - y_1, 0), \quad G^*(x-y) = G(x_1 - y_1, x_2 - 0).$$

For this limit problem we also have that it is well posed, the total mass remains constant in time and solutions converge exponentially to the mean value of the initial condition as $t \to \infty$.

1.3 The local part in a thin domain

We can also consider the case in which the local part of the problem takes place in a thin domain (fixing the nonlocal domain). That is, we consider $\Omega_1 \subset \mathbb{R}^{N_1}$, $\Omega_2 \subset \mathbb{R}^{N_2}$, and can take $\Omega_\varepsilon = \Omega_1 \times \varepsilon\Omega_2 \cup R$ as our reference domain. In this case the associated energy takes the form

$$E(u,v) := \frac{1}{2\varepsilon^{N_2}} \int_{\Omega_1 \times \varepsilon\Omega_2} |\nabla u|^2 dx + \frac{1}{4}\int_R \int_R J(x-y)\left(v(y) - v(x)\right)^2 dydx + \frac{1}{2\varepsilon^{N_2}} \int_R \int_{\Omega_1 \times \varepsilon\Omega_2} G(x-y)\left(v(x) - u(y)\right)^2 dydx.$$

We can also consider the limit as $\varepsilon \to 0$ of solutions to the associated gradient flow in this case. In this case we obtain a limit problem in which the equation for u involves only the Laplacian in the first N_1-variables and the coupling kernel is given by

$$G^*(x-y) = G(x_1 - y_1, 0 - y_2)$$

(the kernel J remains unchanged since we are fixing the nonlocal domain R). The proof of this limit can be obtained following [2, 3, 4] (notice that here we are taking the limit in the local part of the problem) and hence we don't include the details in this paper.

The paper is organized as follows: in Section 2 we deal with the problem with coupling via source terms and we prove Theorem 1.1; in Section 3 we consider the coupling on the boundary and prove of Theorem 1.2; finally, in Section 4 we include some numerical experiments (based on a discretization of our models) that illustrate the behaviour of the solutions to our limit equations.

2 Coupling via source terms. Proof of Theorem 1.1

First, we introduce a result that will be important to study the large time behavior and the limit problems described in the previous Section. We state the lemma for the first problem (coupling via source terms) but the same proof can be adapted for the other evolution problem (coupling on the boundary).

Let us denote by $\hat{\Omega} = \Omega \cup R$ the fixed domain after the change of variables and by E the functional (1.5) after the change of variables, that is,

$$E(u,v) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4}\int_R\int_R J_\varepsilon(x-y)|v(y)-v(x)|^2 dydx + \frac{1}{2}\int_R\int_\Omega G_\varepsilon(x-y)|v(x)-u(y)|^2 dydx,$$

with

$$J_\varepsilon(x-y) = J(x_1-y_1, \varepsilon(\tilde{x}_2-\tilde{y}_2)), \qquad G_\varepsilon(x-y) = G(x_1-y_1, \varepsilon\tilde{x}_2-\tilde{y}_2).$$

Lemma 2.1. *Let $\{\lambda_1^\varepsilon\}_{\varepsilon>0}$ be a family of first nontrivial eigenvalues of our evolution problem that are given by*

$$\lambda_1^\varepsilon = \inf_{(u,v):\int_\Omega u^\varepsilon + \int_R v^\varepsilon = 0} \frac{E(u^\varepsilon, v^\varepsilon)}{\displaystyle\int_\Omega (u^\varepsilon)^2 + \int_R (v^\varepsilon)^2}$$

Then, there exists a constant $C>0$, that does not depends on ε such that

$$\lambda_1^\varepsilon \geq C > 0,$$

and hence we have,

$$E(u^\varepsilon, v^\varepsilon) \geq C\left(\int_\Omega (u^\varepsilon)^2 + \int_R (v^\varepsilon)^2\right), \tag{2.1}$$

for every $(u^\varepsilon, v^\varepsilon)$ solution to (1.7), such that $\int_\Omega u^\varepsilon + \int_R v^\varepsilon = 0$.

Proof. Let us argue by contradiction. Suppose that (2.1) is not hold, that means that, for every $n\in\mathbb{N}$ there exists a subsequence $\{\varepsilon_n\}\to 0$ and $\{w^{\varepsilon_n}\} = \{(u^{\varepsilon_n}, v^{\varepsilon_n})\} \in L^2(\hat{\Omega})\cap H^1(\Omega)$ such that

$$\int_\Omega u^{\varepsilon_n} + \int_R v^{\varepsilon_n} = 0,$$

$$\int_\Omega (u^{\varepsilon_n})^2 + \int_R (v^{\varepsilon_n})^2 = 1,$$

and

$$\frac{1}{2}\int_\Omega |\nabla u^{\varepsilon_n}|^2 dx + \frac{1}{4}\int_R\int_R J_{\varepsilon_n}(x-y)(v^{\varepsilon_n}(y)-v^{\varepsilon_n}(x))^2 dydx + \frac{1}{2}\int_R\int_\Omega G_{\varepsilon_n}(x-y)(v^{\varepsilon_n}(x)-u^{\varepsilon_n}(y))^2 dydx \leq \frac{1}{n}.$$

Taking the limit as $n\to\infty$ we obtain

$$\lim_{n\to\infty}\left(\frac{1}{2}\int_\Omega |\nabla u^{\varepsilon_n}|^2 dx\right) = 0,$$

$$\lim_{n\to\infty}\left(\frac{1}{4}\int_R\int_R J_{\varepsilon_n}(x-y)(v^{\varepsilon_n}(y)-v^{\varepsilon_n}(x))^2 dydx\right) = 0,$$

and

$$\lim_{n\to\infty}\left(\frac{1}{2}\int_R\int_\Omega G_{\varepsilon_n}(x-y)(v^{\varepsilon_n}(x)-u^{\varepsilon_n}(y))^2 dydx\right) = 0.$$

We have that $\int_\Omega (u^{\varepsilon_n})^2 dx \leq 1$, that is, $\{u^{\varepsilon_n}\}$ is bounded in $L^2(\Omega)$. Moreover, we get that $\{u^{\varepsilon_n}\}$ is bounded in $H^1(\Omega)$. Taking a subsequence, also denoted by $\{u^{\varepsilon_n}\}$, such that $\varepsilon_n\to 0$ we have

$$u^{\varepsilon_n} \rightharpoonup u^* \quad \text{in } H^1(\Omega)$$

$$u^{\varepsilon_n} \longrightarrow u^* \quad \text{in } L^2(\Omega).$$

Thanks to the weak lower semicontinuity of the norm we know that

$$\frac{1}{2}\int_\Omega |\nabla u^*|^2 dx \leq \liminf_{\varepsilon_n} \frac{1}{2}\int_\Omega |\nabla u^{\varepsilon_n}|^2 dx = 0.$$

Hence, the limit u^* is constant in Ω.

Also $\{v^{\varepsilon_n}\}$ is bounded in $L^2(R)$. Define $k^{\varepsilon_n} = \int_R v^{\varepsilon_n}$. From the bound in $L^2(R)$ of v^{ε_n} we obtain that there exists a constant C such that $|k^{\varepsilon_n}| \le C$ and, moreover, we can take a subsequence $\{v^{\varepsilon_{nj}}\}$ which weakly converges in $L^2(R)$ to some limit v^* as $\varepsilon_{nj} \to 0$ and such that $\{k^{\varepsilon_{nj}}\}$ also converges to a limit that we call k^*. Consider $z^{\varepsilon_{nj}} = v^{\varepsilon_{nj}} - k^{\varepsilon_{nj}}$. We have that $\int_R z^{\varepsilon_{nj}} = 0$, therefore, see [10] and [1], there exists a constant $C > 0$ independent of ε such that

$$\int_R \int_R J(x_1 - y_1, \varepsilon_{nj}(\tilde{x}_2 - \tilde{y}_2))(z^{\varepsilon_{nj}}(y) - z^{\varepsilon_{nj}}(x))^2 d\tilde{y} d\tilde{x} \ge C \int_R (z^{\varepsilon_{nj}}(x))^2 dx. \tag{2.2}$$

In fact, since J is continuous, from our hypothesis on J, we get that there exists constants $M, \delta > 0$ such that

$$J(x_1 - y_1, x_2 - y_2) \ge M, \quad \text{whenever} \quad |(x_1 - y_1, x_2 - y_2)| < \delta.$$

Then, it follows that

$$J(x_1 - y_1, \varepsilon(x_2 - y_2)) \ge \frac{M}{2}, \quad \text{whenever} \quad |x_1 - y_1| < \frac{\delta}{2}, \quad \varepsilon|x_2 - y_2| < \frac{\delta}{2},$$

for every ε small enough. Hence, the inequality (2.2) follows from Lemma 3.1 in [10] and the constant C only depends on M, δ and R but not on ε.

Note that we have

$$\lim_{n\to\infty} \left(\frac{1}{4} \int_R \int_R J_{\varepsilon_n}(x-y)(z^{\varepsilon_{nj}}(y) - z^{\varepsilon_{nj}}(x))^2 dy dx \right) = \lim_{n\to\infty} \left(\frac{1}{4} \int_R \int_R J_{\varepsilon_n}(x-y)(v^{\varepsilon_{nj}}(y) - v^{\varepsilon_{nj}}(x))^2 dy dx \right) = 0,$$

as $\varepsilon_{n_j} \to 0$, which yields

$$0 \ge \lim_{n\to\infty} C \int_R (z^{\varepsilon_{nj}}(x))^2 dx.$$

From here we conclude that $z^{\varepsilon_{nj}} \to 0$ in $L^2(R)$, which leads to $v^{\varepsilon_{nj}} \to k^*$ strongly in $L^2(R)$. Finally, as $u^{\varepsilon_n} \to u^*$ in $L^2(\Omega)$ and $v^{\varepsilon_n} \to k^*$ in $L^2(R)$, we can take the limit as $\varepsilon_n \to 0$ and obtain

$$0 = \lim_{n\to\infty} \left(\frac{1}{2} \int_R \int_\Omega G_{\varepsilon_n}(x-y)(v^{\varepsilon_n}(x) - u^{\varepsilon_n}(y))^2 dy dx \right) = \frac{1}{2} \int_R \int_\Omega G^*(x-y)(v^*(x) - u^*(y))^2 dy dx.$$

From where it follows that $k^* - u^* = 0$, that is, $k^* = u^*$. From

$$\int_\Omega u^{\varepsilon_n} + \int_R v^{\varepsilon_n} = 0,$$

it follows that

$$\int_\Omega u^* + \int_R k^* = 0,$$

and since we have $k^* = u^*$ we get

$$k^* = u^* = 0.$$

Now, from

$$\int_\Omega (u^{\varepsilon_n})^2 + \int_R (v^{\varepsilon_n})^2 = 1,$$

and the strong convergence in L^2 we obtain

$$\int_\Omega (u^*)^2 + \int_R (k^*)^2 = 1,$$

which yields a contradiction. The proof is complete. □

With this lemma, following [23] (see also [29]), we can provide an estimate for the asymptotic behavior of the solutions of the problem (1.7), that is, the solutions $\{u^\varepsilon, v^\varepsilon\}_{\varepsilon>0}$ converges to the mean value of the initial condition

$$\left\|(u^\varepsilon, v^\varepsilon)(\cdot,t) - ⨍(u_0,v_0)\right\|_{L^2(\widehat{\Omega})} \le C_1 e^{-C_2 t}, \tag{2.3}$$

with C_1, C_2 finite positive constants, independent of ε and also, C_2 independent of the initial condition. Hence, we have that the L^2-norm of $\{u^\varepsilon, v^\varepsilon\}_{\varepsilon>0}$ is bounded (independently of ε). Here

$$⨍(u_0,v_0) = \frac{\displaystyle\int_\Omega u_0 + \int_R v_0}{|\Omega| + |R|}.$$

Now we are ready to proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. First, we observe that, since J and G are continuous functions, we have

$$J_\varepsilon(x-y) = J(x_1-y_1, \varepsilon(x_2-y_2)) \longrightarrow J^*(x-y) = J(x_1-y_1,0), \quad \text{and}$$

$$G_\varepsilon(x-y) = G(x_1-y_1, \varepsilon x_2 - y_2) \longrightarrow G^*(x-y) = G(x_1-y_1, 0-y_2),$$

as $\varepsilon \to 0$, uniformly in x, y.

From Lemma 2.1, since $\{v^\varepsilon\}$ is bounded in $L^\infty(0,T;L^2(R))$ we can take a subsequence, also denoted by $\{v^\varepsilon\}$, such that

$$v^\varepsilon \rightharpoonup v^* \quad \text{weakly} \quad \text{in} \quad L^\infty(0,T;L^2(R)) \quad \text{as} \quad \varepsilon \to 0.$$

On the other hand, we have that

$$\int_\Omega |u^\varepsilon(x,t)|^2 dx \qquad \text{and} \qquad \int_\Omega |\nabla u^\varepsilon(x,t)|^2 dx,$$

are also bounded in $L^2(\Omega)$ (uniformly in $t \in [0,T]$). Hence, along a subsequence if necessary,

$$u^\varepsilon \rightharpoonup u^* \quad \text{weakly} \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)) \quad \text{as} \quad \varepsilon \to 0,$$

$$u^\varepsilon \to u^* \quad \text{strongly} \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \quad \text{as} \quad \varepsilon \to 0.$$

Now we consider the weak form of (1.7), that is, using the symmetry of the kernel J we have the following identities,

$$\begin{aligned}
\int_\Omega u^\varepsilon(x,T)\varphi(x,T)dx - \int_0^T \int_\Omega u^\varepsilon(x,t)\frac{\partial\varphi}{\partial t}(x,t)dxdt &= \int_\Omega u_0(x)\varphi(x,0)dx - \int_0^T \int_\Omega \nabla u^\varepsilon(x,t)\nabla\varphi(x,t)dxdt \\
&\quad + \int_0^T \int_\Omega \int_R G_\varepsilon(x-y)(v^\varepsilon(\tilde{y},t) - u^\varepsilon(x,t))\varphi(x,t)d\tilde{y}dxdt, \\
\int_R v^\varepsilon(x,T)\varphi(x,T)dx - \int_0^T \int_R v^\varepsilon(x,t)\frac{\partial\varphi}{\partial t}(x,t)dxdt &= \int_R v_0(x)\varphi(x,0)dx \\
&\quad - \frac{1}{2}\int_0^T \int_R \int_R J_\varepsilon(x-y)\left(v^\varepsilon(\tilde{y},t) - v^\varepsilon(\tilde{x},t)\right)\left(\varphi(y,t) - \varphi(x,t)\right)d\tilde{y}dxdt \\
&\quad - \int_0^T \int_R \int_\Omega G_\varepsilon(x-y)(v^\varepsilon(\tilde{x},t) - u^\varepsilon(y,t))\varphi(x,t)dydxdt,
\end{aligned}$$

for every $\varphi \in C^1(H^1(\Omega) \cup L^2(R))$.

Now, let us take a test function that depends only on the first variable, for $x \in R$, that is, $\varphi = \varphi(x_1)$ and us analyze the limit as $\varepsilon \to 0$ of each term in the previous equations. We have

$$\lim_{\varepsilon \longrightarrow 0} \left(\int_0^T \int_\Omega \nabla u^\varepsilon \nabla \varphi dx dt \right) = \left(\int_0^T \int_\Omega \nabla u^* \nabla \varphi dx dt \right).$$

Now, note that

$$\begin{aligned}
&\int_0^T \int_\Omega \int_{R_1} \int_{R_2} G(x_1 - y_1, x_2 - \varepsilon \tilde{y_2})(v^\varepsilon(y_1, \tilde{y_2}, t) - u^\varepsilon(x_1, x_2, t))\varphi(x_1, x_2, t) d\tilde{y_2} dy_1 dx_2 dx_1 dt \\
&= \int_0^T \int_\Omega \int_{R_1} \int_{R_2} \Big[G(x_1 - y_1, x_2 - \varepsilon \tilde{y_2}) - G(x_1 - y_1, x_2) \Big] \\
&\qquad \times (v^\varepsilon(y_1, \tilde{y_2}, t) - u^\varepsilon(x_1, x_2, t))\varphi(x_1, x_2, t) d\tilde{y_2} dy_1 dx_2 dx_1 dt \\
&\quad + \int_0^T \int_\Omega \int_{R_1} \int_{R_2} G(x_1 - y_1, x_2)(v^\varepsilon(y_1, \tilde{y_2}, t) - u^\varepsilon(x_1, x_2, t))\varphi(x_1, x_2, t) d\tilde{y_2} dy_1 dx_2 dx_1 dt.
\end{aligned}$$

Notice that the measure in Ω is the product measure and hence when we integrate we have $dx = dx_1 dx_2$.

Since

$$\Big[G(x_1 - y_1, x_2 - \varepsilon \tilde{y_2}) - G(x_1 - y_1, x_2) \Big]$$

goes to zero uniformly and u^ε and v^ε are bounded in L^2, the first term goes to zero as $\varepsilon \to 0$ and therefore we concentrate in the second. To analyze the limit of the second term, we observe that $G(x_1 - y_1, x_2)$ does not depend on y_2 and hence we can rewrite this term as follows,

$$\begin{aligned}
&\int_0^T \int_\Omega \int_{R_1} \int_{R_2} G(x_1 - y_1, x_2)(v^\varepsilon(y_1, \tilde{y_2}, t) - u^\varepsilon(x_1, x_2, t))\varphi(x_1, x_2, t) d\tilde{y_2} dy_1 dx_2 dx_1 dt \\
&= \int_0^T \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} G(x_1 - y_1, x_2) \left[\int_{R_2} v^\varepsilon(y_1, \tilde{y_2}, t) - u^\varepsilon(x_1, x_2, t) \, d\tilde{y_2} \right] dy_1 dx_2 dx_1 dt.
\end{aligned}$$

Let

$$V^\varepsilon(y_1, t) = \int_{R_2} v^\varepsilon(y_1, \tilde{y_2}, t) d\tilde{y_2}. \tag{2.7}$$

Observe that, since v^ε is bounded in $L^\infty(0, T; L^2(R))$, then V^ε is also bounded in $L^\infty(0, T; L^2(R_1))$ so, taking a subsequence if necessary

$$V^\varepsilon \rightharpoonup V^* \quad \text{weakly} \quad \text{in} \quad L^\infty(0, T; L^2(R_1)).$$

Using (2.7) we obtain

$$\int_0^T \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} G(x_1 - y_1, x_2) \left[V^\varepsilon(y_1) - |R_2| u^\varepsilon(x_1, x_2, t) \right] dy_1 dx_2 dx_1 dt.$$

Therefore, we can take the limit as $\varepsilon \to 0$ and obtain

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} G_\varepsilon(x_1 - y_1, x_2 - \varepsilon \tilde{y_2}) \left[V^\varepsilon(y_1) - |R_2| u^\varepsilon(x_1, x_2, t) \right] dy_1 dx_2 dx_1 dt \\
&= \int_0^T \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} (G(x_1 - y_1, x_2)) \lim_{\varepsilon \to 0} (V^\varepsilon(y_1)) \, dy_1 dx_2 dx_1 dt \\
&\quad - \int_0^T |R_2| \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} (G(x_1 - y_1, x_2)) \lim_{\varepsilon \to 0} (u^\varepsilon(x_1, x_2, t)) \, dy_1 dx_2 dx_1 dt \\
&= \int_0^T \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} G(x_1 - y_1, x_2 - 0) V^*(y_1) dy_1 dx_2 dx_1 dt \\
&\quad - \int_0^T |R_2| \int_\Omega \varphi(x_1, x_2, t) \int_{R_1} G(x_1 - y_1, x_2 - 0) u^*(x_1, x_2, t) dy_1 dx_2 dx_1 dt.
\end{aligned}$$

The same idea can be applied for the second integral in the weak form of the problem using the properties of the kernel G and Fubini's theorem, which leads to

$$\int_0^T \int_{R_1} \varphi(x_1,t) \int_\Omega G(x_1-y_1, 0-y_2)\left(V^*(y_1) - |R_2| u^*(x_1,x_2,t)\right) dy_1 dx_2 dx_1 dt.$$

Concerning the terms that involve time derivatives, from the $L^\infty - L^2$ convergence we obtain

$$\lim_{\varepsilon\to 0} -\int_0^T \int_\Omega u^\varepsilon(x,t)\frac{\partial\varphi}{\partial t}(x,t)dxdt = -\int_0^T \int_\Omega u^*(x,t)\frac{\partial\varphi}{\partial t}(x,t)dxdt$$

and

$$\lim_{\varepsilon\to 0} -\int_0^T \int_R v^\varepsilon(x,t)\frac{\partial\varphi}{\partial t}(x,t)dxdt = -\int_0^T \int_{R_0} V^*(x,t)\frac{\partial\varphi}{\partial t}(x,t)dxdt$$

Finally, we will deal with the pure nonlocal integral. By Fubini's theorem and (2.7) we get

$$\begin{aligned}
&\int_0^T \int_{R_1}\int_{R_2}\int_{R_1}\int_{R_2} J_\varepsilon(x-y)\left(v^\varepsilon(y_1,\tilde y_2,t) - v^\varepsilon(x_1,\tilde x_2,t)\right)\varphi(x_1,t)d\tilde y_2 dy_1 d\tilde x_2 dx_1 dt\\
&= \int_0^T \int_{R_1} \varphi(x_1,t)\int_{R_1} J(x_1-y_1, \varepsilon(\tilde x_2,\tilde y_2))\left[\int_{R_2}\int_{R_2}\left(v^\varepsilon(y_1,\tilde y_2,t) - v^\varepsilon(x_1,\tilde x_2,t)\right)d\tilde y_2 d\tilde x_2\right] dy_1 dx_1 dt\\
&= \int_0^T \int_{R_1} \varphi(x_1,t)\int_{R_1} J(x_1-y_1, \varepsilon(\tilde x_2,\tilde y_2))\left(|R_2|V^\varepsilon(y_1) - |R_2|V^\varepsilon(x_1)\right) dy_1 dx_1 dt.
\end{aligned}$$

Now, we can take the limit as $\varepsilon \to 0$, it follows that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T \int_{R_1} \varphi(x_1,t)\int_{R_1} J(x_1-y_1, \varepsilon(\tilde x_2,\tilde y_2))\left(|R_2|V^\varepsilon(y_1) - |R_2|V^\varepsilon(x_1)\right) dy_1 dx_1 dt\\
&= \int_0^T |R_2| \int_{R_1}\varphi(x_1,t)\int_{R_1}\lim_{\varepsilon\to 0}\left(J(x_1-y_1,\varepsilon(\tilde x_2,\tilde y_2))\right)\lim_{\varepsilon\to 0}(V^\varepsilon(y_1)) dy_1 dx_1 dt\\
&\quad - \int_0^T |R_2| \int_{R_1}\varphi(x_1,t)\int_{R_1}\lim_{\varepsilon\to 0}\left(J(x_1-y_1,\varepsilon(\tilde x_2,\tilde y_2))\right)\lim_{\varepsilon\to 0}(V^\varepsilon(x_1)) dy_1 dx_1 dt\\
&= \int_0^T |R_2| \int_{R_1}\varphi(x_1,t)\int_{R_1} J(x_1-y_1,0)(V^*(y_1) - V^*(x_1)) dy_1 dx_1 dt.
\end{aligned}$$

Hence, since this procedure can be carry over for every $T > 0$, the limit equation, defined in the domain $\Omega_0 = \Omega \cup R_1$, (see Figure 2) is given by the system (1.8),

$$\begin{cases}
\dfrac{\partial u^*}{\partial t}(x,t) = \Delta u^*(x,t) + \displaystyle\int_{R_1} G^*(x-y)(V^*(y,t) - |R_2|u^*(x,t))dy, \quad (x,t)\in\Omega\times(0,\infty),\\
\dfrac{\partial u^*}{\partial \eta}(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,\infty),\\
\dfrac{\partial V^*}{\partial t}(x,t) = |R_2|\displaystyle\int_{R_1} J^*(x-y)\left(V^*(y,t) - V^*(x,t)\right)dy\\
\qquad\qquad - \displaystyle\int_\Omega G^*(x-y)(V^*(x,t) - |R_2|u^*(y,t))dy, \ (x,t)\in R_1\times(0,\infty),\\
u^*(x,0) = u_0^*(x), \quad x\in\Omega,\\
V^*(x,0) = V_0^*(x), \quad x\in R_1,
\end{cases}$$

where $J^*(x-y) = J(x_1-y_1,0)$ and $G^*(x-y) = G(x_1-y_1, x_2-0)$.

To finish the proof we show existence and uniqueness of a solution of the solution to the limit problem (1.8) (notice that up to this point we have convergence along subsequences $\varepsilon_j \to 0$, proving uniqueness of the limit we obtain the existence of the full limit as $\varepsilon \to 0$).

Thanks to the limit along subsequences we ensure the existence of a solution (u^*, V^*) for the limit problem. To show the uniqueness let us suppose that there exists two solutions (u_1^*, V_1^*) and (u_2^*, V_2^*) of (1.8). Define $w^* = u_1^* - u_2^*$ and $z^* = V_1^* - V_2^*$. The pair of (w^*, z^*) satisfies the following equations

$$\begin{cases} \dfrac{\partial w^*}{\partial t}(x,t) = \triangle w^*(x,t) + \displaystyle\int_{R_1} G^*(x-y)(z^*(y,t) - |R_2| w^*(x,t))dy, \quad (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial w^*}{\partial \eta}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,\infty), \\ \dfrac{\partial z^*}{\partial t}(x,t) = |R_2| \displaystyle\int_{R_1} J^*(x-y)\,(z^*(y,t) - z^*(x,t))\,dy - \int_{\Omega} G^*(x-y)(z^*(x,t) - |R_2| w^*(y,t))dy, \\ \qquad\qquad (x,t) \in R_1 \times (0,\infty), \\ w^*(x,0) = 0, \quad x \in \Omega, \\ z^*(x,0) = 0, \quad x \in R_1. \end{cases} \tag{2.8}$$

Multiplying the first equation of the problem (2.8) by $\frac{w^*}{2}$ and integrating over Ω and, the second equation by $\frac{z^*}{2}$ and integrating over R_1, we get

$$\begin{aligned} |R_2| \int_{\Omega} \frac{\partial w^*}{\partial t} w^* dx + \int_{R_1} \frac{\partial z^*}{\partial t} z^* dx &= -|R_2| \int_{\Omega} |\nabla w^*|^2 dx - \frac{|R_2|}{2} \int_{R_1}\int_{R_1} J^*(x-y)(z^*(y,t) - z^*(x,t))^2 dydx \\ &\quad - \int_{R_1}\int_{\Omega} G^*(x-y)(z^*(y,t) - |R_2| w^*(x,t))^2 dydx \\ &= -2E(w^*, z^*) \le 0. \end{aligned}$$

Hence, if we let

$$f'(t) = |R_2| \int_{\Omega} \frac{\partial w^*}{\partial t} w^* dx + \int_{R_1} \frac{\partial z^*}{\partial t} z^* dx,$$

we have

$$f(t) = \frac{|R_2|}{2} \int_{\Omega} (w^*)^2 dx + \int_{R_1} (z^*)^2 dx.$$

Now, from Lemma 2.1, we obtain

$$2E(w^*, z^*) \ge 2\lambda_1 \left(\frac{|R_2|}{2} \int_{\Omega} (w^*)^2 dx + \int_{R_1} (z^*)^2 dx \right) = 2\lambda_1 f(t),$$

which implies $-2E(w^*, z^*) \le -2\lambda_1 f(t)$ and then we get

$$f'(t) \le -2\lambda_1 f(t).$$

Hence, Gronwall's inequality gives that

$$f(t) \le e^{-2\lambda_1 t} f(0),$$

where $f(0) = \frac{|R_2|}{2} \int_{\Omega} (w^*)^2(x,0)dx + \int_{R_1} (z^*)^2(x,0)dx$. Since $f(t) \ge 0$ and $f(0) = 0$ we have that

$$0 \le f(t) \le 0,$$

that is

$$f(t) \equiv 0$$

and hence

$$w^* = 0 \qquad \text{and} \qquad z^* = 0,$$

which means $u_1^* = u_2^*$ and $V_1^* = V_2^*$. This guarantee the uniqueness of the solution for the problem (1.8) as we wanted to show. □

Now, we include several remarks.

Remark 3. From our previous arguments, we also conclude that the limit problem (1.8) is well-posed in $L^2(\Omega_0)$ (we have existence, uniqueness and continuous dependence with respect to the initial data of the solutions).

Remark 4. We only prove weak convergence of the solution of the problem (1.6) to the solution of the problem (1.8) (we do not prove strong convergence in the L^2-norm). Moreover, we only guarantee the uniqueness of V^* and this is not enough to ensure the uniqueness of v^*.

Remark 5. Observe that, instead of the usual metric in $L^2(\Omega \cup R)$ we choose to work with the metric (1.4). This choice was made to obtain a nontrivial limit. In fact, using this metric we can observe the coupling of the local part of the problem in the domain Ω with the nonlocal part in the lower dimensional domain R_1.

Now, if we consider the usual metric in L^2 and the energy functional

$$E(u,v) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4\varepsilon^{N_2}}\int_{R_\varepsilon}\int_{R_\varepsilon} J(x-y)\left(v(y)-v(x)\right)^2 dydx + \frac{1}{2}\int_{R_\varepsilon}\int_\Omega G(x-y)\left(v(x)-u(y)\right)^2 dydx,$$

the associated evolution problem (after the change of variables) is given by

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(x,t) = \triangle u^\varepsilon(x,t) + \varepsilon \displaystyle\int_R G_\varepsilon(x-y)(v^\varepsilon(\tilde{y},t) - u(x,t))d\tilde{y}, \quad x \in \Omega, \quad t>0, \\ \dfrac{\partial u^\varepsilon}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega, \quad t>0, \\ \dfrac{\partial v^\varepsilon}{\partial t}(\tilde{x},t) = \displaystyle\int_R J_\varepsilon(x-y)\left(v^\varepsilon(\tilde{y},t) - v^\varepsilon(\tilde{x},t)\right)d\tilde{y} - \int_\Omega G_\varepsilon(x-y)(v^\varepsilon(\tilde{x},t) - u^\varepsilon(y,t))dy, \quad x \in R, \quad t>0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \\ v(\tilde{x},0) = v_0(\tilde{x}), \quad \tilde{x} \in R, \end{cases}$$

where $J_\varepsilon(x-y) = J(x_1-y_1, \varepsilon(\tilde{x}_2 - \tilde{y}_2))$, $G_\varepsilon(x-y) = G(x_1-y_1, \varepsilon\tilde{x}_2 - \tilde{y}_2)$ and $v^\varepsilon(x_1,\tilde{x}_2,t) = v(x_1, \varepsilon\tilde{x}_2, t)$. Observe that taking the limit as $\varepsilon \to 0$ the nonlocal term that appears in the equation for u^ε goes to zero and hence we will lose the coupling term in the limit (the equation for u^* will be independent of V^*). Also in this case, the limit problem will be well-posed, in the sense that we can ensure existence and uniqueness of the solution, but it is less interesting.

As we expected, the limit problem (1.8) preserves the total mass of the solution. This follows from the limit procedure and the fact that the problem (1.7) preserves the total mass for every $\varepsilon > 0$. We include below a direct proof of this fact for completeness.

Theorem 2.2. *The solution* (u^*, V^*) *of the problem* (1.8), *with initial data* $u_0^* \in H^1(\Omega)$ *and* $V_0^* \in L^2(R_1)$ *satisfies*

$$\int_\Omega u^*(x,t)dx + \int_{R_1} V^*(x,t)dx = \int_\Omega u_0^* dx + \int_{R_1} V_0^* dx, \quad \forall t \geq 0. \tag{2.9}$$

Proof. Differentiating (2.9) with respect to t we obtain

$$\begin{aligned} \int_\Omega \frac{\partial u^*}{\partial t}dx + \int_{R_1}\frac{\partial V^*}{\partial t}dx = &\int_\Omega \triangle u^*(x,t)dx + \int_\Omega\int_{R_1} G^*(x-y)(V^*(y,t) - |R_2|u^*(x,t))dydx \\ &+ \int_{R_1}|R_2|\int_{R_1} J^*(x-y)(V^*(y,t) - V^*(x,t))dydx \\ &- \int_{R_1}\int_\Omega G^*(x-y)(V^*(x,t) - |R_2|u^*(y,t))dydx \\ = &\ 0. \end{aligned}$$

Indeed, after a change of variables, due to the symmetry of G and Fubini's theorem, the second and the fourth integral cancel each other. Also, by the symmetry of J and Fubini's theorem, the second integral is zero. Finally, the first integral is zero since we have a Neumann type boundary condition for the local part.

This ends the proof. □

Finally, we include the study of the asymptotic behavior of the solutions for the limit problem (1.8).

Notice that from the fact that the constants in (2.3) do not depend on ε we obtain that the solutions for the limit problem (1.8) converge exponentially to the mean value of the initial condition. We have that

$$\left\|(u^*,V^*)(\cdot,t) - \fint (u_0,v_0)\right\|_{L^2(\Omega_0)} \leq C_1 e^{-C_2 t}.$$

However, we can obtain a better control of the constant C_2 and obtain an exponential decay in terms of the first nontrivial eigenvalue associated to the limit problem. To this end, we use the L^2-norm

$$\|(u^*,V^*)\|_{L^2(\Omega_0)} = |R_2| \int_\Omega |u^*|^2 dx + \int_{R_1} |V^*|^2 dx.$$

We can define the energy functional associated to the limit problem (1.8) by

$$\begin{aligned} E(u^*,V^*) = \frac{|R_2|}{2} \int_\Omega |\nabla u^*|^2 dx + \frac{|R_2|}{4} \int_{R_1}\int_{R_1} J^*(x-y)(V^*(y)-V^*(x))^2 dy dx \\ + \frac{1}{2} \int_{R_1}\int_\Omega G^*(x-y)(V^*(x)-|R_2|u^*(y))^2 dy dx. \end{aligned} \tag{2.10}$$

Indeed, the gradient flow associated with (2.10), is given by

$$\begin{aligned} \partial_\varphi E(u^*,V^*) &= \lim_{h\to 0} \frac{E(u^*+h\varphi, V^*+h\varphi) - E(u^*,V^*)}{h} \\ &= |R_2| \int_\Omega \nabla u^* \nabla \varphi dx + \frac{|R_2|}{2} \int_{R_1}\int_{R_1} J^*(x-y)(V^*(y)-V^*(x))(\varphi(y)-\varphi(x)) dy dx \\ &\quad + |R_2| \int_{R_1}\int_\Omega G^*(x-y)(V^*(x)-|R_2|u^*(y))(\varphi(x)dx - |R_2|\varphi(y)) dy dx. \end{aligned}$$

Hence, using that

$$|R_2| \int_\Omega \frac{\partial u^*}{\partial t}\varphi(x) dx + \int_{R_1} \frac{\partial V^*}{\partial t}\varphi(x) dx = -\partial_\varphi E[(u^*,V^*)(t)],$$

we obtain the limit problem (1.8).

With this energy at hand we can obtain the first nontrivial eigenvalue for our limit problem. Let us take λ_1 as

$$0 < \lambda_1 = \inf_{u^*,V^* \in W_0} \frac{E(u^*,V^*)}{|R_2| \displaystyle\int_\Omega (u^*)^2 + \int_{R_1} (V^*)^2}, \tag{2.11}$$

where $E(u^*,V^*)$ is given by (2.10) and

$$W_0 = \left\{ u^* \in H^1(\Omega), V^* \in L^2(R_1) : |R_2| \int_\Omega u^* + \int_{R_1} V^* = 0 \right\}.$$

Lemma 2.3. *Let λ_1 given by* (2.11)*, then $\lambda_1 > 0$ and therefore,*

$$E(u^*,V^*) \geq \lambda_1 \left(|R_2| \int_\Omega (u^*)^2 + \int_{R_1} (V^*)^2 \right),$$

for every u^*, V^* solution of (1.8)*, such that $|R_2| \int_\Omega u^* + \int_{R_1} V^* = 0$.*

Proof. The proof is similar to the one of Lemma 2.1 but we include the details for completeness. Let us suppose that $\lambda_1 = 0$. This implies that there exists a subsequence $\{u_n^*\} \in H^1(\Omega)$ and $\{v_n^*\} \in L^2(R_1)$ such that

$$|R_2| \int_\Omega u_n^* + \int_R V_n^* = 0,$$

$$|R_2| \int_\Omega (u_n^*)^2 + \int_R (V_n^*)^2 = 1,$$

and

$$\frac{|R_2|}{2} \int_\Omega |\nabla u_n^*|^2 dx + \frac{|R_2|}{4} \int_{R_1} \int_{R_1} J^*(x-y)(V_n^*(y) - V_n^*(x))^2 dydx + \frac{1}{2} \int_{R_1} \int_\Omega G^*(x-y)(V_n^*(x) - |R_2| u_n^*(y))^2 dydx \leq \frac{1}{n}.$$

Taking the limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty} \left(\frac{|R_2|}{2} \int_\Omega |\nabla u_n^*|^2 dx \right) = 0,$$

$$\lim_{n\to\infty} \frac{|R_2|}{4} \int_{R_1} \int_{R_1} J^*(x-y)(V_n^*(y) - V_n^*(x))^2 dydx = 0,$$

and

$$\lim_{n\to\infty} \left(\frac{1}{2} \int_{R_1} \int_\Omega G^*(x-y)(V_n^*(x) - |R_2| u_n^*(y))^2 dydx \right) = 0.$$

Recalling that we have

$$J^*(x-y) = J(x_1 - y_1, 0), \qquad \text{and} \qquad G^*(x-y) = G(x_1 - y_1, 0 - y_2),$$

it follows that $|R_2| \int_\Omega (u_n^*)^2 dx \leq 1$, that is, $\{u_n^*\}$ is bounded in $L^2(\Omega)$. Moreover, $\{u_n^*\}$ is also bounded in $H^1(\Omega)$. Then, we can extract a subsequence $\{u_{n_j}^*\} \in H^1(\Omega)$ which weakly converges to a limit $\hat{u} \in H^1(\Omega)$. From the weak convergence in $H^1(\Omega)$ we obtain strong convergence in $L^2(\Omega)$. Then, we have that

$$\frac{1}{2}|R_2| \int_\Omega |\nabla \hat{u}|^2 dx \leq \liminf_n \frac{1}{2}|R_2| \int_\Omega |\nabla u_n^*|^2 dx = 0.$$

Hence, the limit $\hat{u}$ is constant in Ω.

Also, it follows that $\{V_n^*\}$ is bounded in $L^2(R_1)$. Since

$$\int_R |V_n^*| dx \leq C \left(\int_\Omega (V_n^*)^2 dx \right)^{\frac{1}{2}} \leq C,$$

we let $k_n = \int_{R_1} V_n^*$, and obtain that $|k_n| \leq C$. Then, we can take a subsequence $\{V_{n_j}^*\}$ which converges in $L^2(R_1)$, to some limit $\hat{V}$ as $n_j \to \infty$. Consider $z_{n_j} = V_{n_j}^* - k_{n_j}$, this function is such that $\int_{R_1} k_{n_j} = 0$. By Lemma 3.1, in [10], there exists a constant $c_1 > 0$ such that

$$\int_{R_1} \int_{R_1} J(x_1 - y_1, 0)(z_{n_j}(y) - z_{n_j}(x))^2 dydx \geq c_1 \int_{R_1} (z_{n_j}(x))^2 dx.$$

From this inequality we have

$$\lim_{n\to\infty} \left(\frac{|R_2|}{4} \int_{R_1} \int_{R_1} J^*(x-y)(z_{n_j}(y) - z_{n_j}(x))^2 dydx \right) = \lim_{n\to\infty} \left(\frac{|R_2|}{4} \int_{R_1} \int_{R_1} J^*(x-y)(V_{n_j}^*(y) - V_{n_j}^*(x))^2 dydx \right) \to 0,$$

which yields

$$0 \geq c_1 \lim_{n\to\infty} \int_{R_1} \int_{R_1} (z_{n_j}(x))^2 dx.$$

We conclude that $z_{n_j} \to 0$ strongly in $L^2(R_1)$, which leads to $V^*_{n_j} \to \hat{V}$ strongly $\in L^2(R_1)$. Finally, as $u^*_n \to \hat{u}$ in $L^2(\Omega)$ and $V^*_n \to \hat{V}$ in $L^2(R_1)$, we can take the limit as $n \to \infty$ and obtain

$$\lim_{n\to\infty} \left(\frac{1}{2} \int_{R_1} \int_\Omega G^*(x-y)(V^*_n(x) - u^*_n(y))^2 dydx \right) = \frac{1}{2} \int_{R_1} \int_\Omega G^*(x-y)(\hat{V} - |R_2|\hat{u})^2 dydx \to 0.$$

Then, we have that $\hat{V} - |R_2|\hat{u} = 0$, that is $\hat{V} = |R_2|\hat{u}$. Hence, it follows that $\hat{V} = \hat{u} = 0$, but this is a contradiction with the fact that

$$|R_2| \int_\Omega (u^*_n)^2 + \int_R (V^*_n)^2 = 1$$

since we have strong convergence in L^2. The proof is complete. □

Thanks to Lemma 2.3 we can show that solutions to the limit problem converge exponentially fast to the mean value of their initial condition.

Theorem 2.4. *Given $u^*_0 \in H^1(\Omega)$ and $V^*_0 \in L^2(R_1)$, the solution to* (1.8)*, with initial data u^*_0, V^*_0, converges to its mean value as $t \to \infty$, with an exponential rate λ_1 (given by* (2.11)*),*

$$\left\| (u^*, V^*)(\cdot, t) - ⨍ (u^*_0, V^*_0) \right\|_{L^2(\Omega_0)} \leq C \left(\|(u^*_0, V^*_0)\|_{L^2(\Omega_0)} \right) e^{-\lambda_1 t}.$$

Proof. We know that $V^* = |R_2|u^* = k$, with k constant, is also a solution of the problem (1.8). Hence, the pair

$$(h(x,t) = |R_2|u^*(x,t) - k, z(x,t) = V^*(x,t) - k)$$

is also a solution of (1.8). If we choose

$$k = |R_2| \int_\Omega u^*_0 + \int_{R_1} V^*_0$$

then, using that the mass is preserved in time, we get that h and z satisfy

$$\int_\Omega h(x,t)dx + \int_{R_1} z(x,t)dx = 0.$$

Let

$$f(t) = \frac{|R_2|}{2} \int_\Omega h(x,t)^2 dx + \frac{1}{2} \int_{R_1} z(x,t)^2 dx.$$

Differentiating f with respect to t we obtain

$$\begin{aligned} f'(t) &= |R_2| \int_\Omega \frac{\partial h}{\partial t}(x,t)h(x,t)dx + \int_{R_1} \frac{\partial z}{\partial t}(x,t)z(x,t)dx \\ &= |R_2| \int_{\partial\Omega} \frac{\partial h}{\partial \eta}(x,t)h(x,t)dx - |R_2| \int_\Omega |\nabla h(x,t)|^2 dx - \frac{|R_2|}{2} \int_{R_1} \int_{R_1} J^*(x-y)(z(y,t) - z(x,t))^2 dydx \\ &\quad - \int_{R_1} \int_\Omega G^*(x-y)(z(x,t) - |R_2|h(y,t))z(x,t)dydx \\ &\quad + \int_{R_1} \int_\Omega G^*(x-y)(z(x,t) - |R_2|h(y,t))|R_2|h(x,t)dydx \\ &= |R_2| \int_\Omega |\nabla h(x,t)|^2 dx - \frac{|R_2|}{2} \int_{R_1} \int_{R_1} J^*(x-y)(z(y,t) - z(x,t))^2 dydx \\ &\quad - \int_{R_1} \int_\Omega G^*(x-y)(z(x,t) - |R_2|h(y,t))^2 dydx \\ &= -2E(h,z). \end{aligned}$$

From Lemma 2.3 we get

$$E(h,z) \geq \lambda_1 \left(|R_2| \int_\Omega h^2 + \int_{R_1} z^2 \right).$$

Hence, we obtain

$$f'(t) \leq -2\lambda_1 f(t)$$

so, by Gronwall's lemma we have that

$$f(t) \leq e^{-2\lambda_1 t} f(0),$$

with $f(0) = \frac{1}{2}\left(|R_2| \int_\Omega h_0^2 + \int_{R_1} z_0^2 \right)$. From this it follows that

$$|R_2| \int_\Omega ||R_2|u^*(x,t) - k|^2 dx + \int_{R_1} |V^*(x,t) - k|^2 dx \leq C \left(\|(u_0^*, V_0^*)\|_{L^2(\Omega_0)} \right) e^{-2\lambda_1 t} \longrightarrow 0,$$

as $t \to \infty$. In particular, it means that $|R_2|u^* \to k$ in $L^2(\Omega)$ and $V^* \longrightarrow k$ in $L^2(R_1)$, with k given by the mean value of the initial condition. □

3 Coupling on the boundary. Proof of Theorem 1.2

Let us first note that the existence and uniqueness of the solutions $\{u^\varepsilon, v^\varepsilon\}$, of the problem (1.11), for each $\varepsilon > 0$, was obtained in [29]. The arguments used to prove the conservation of mass and comparison principle also apply for the problem (1.11) following the ideas presented in [29].

Notice that we have an energy functional for the problem (1.11) given by (1.9),

$$E_\varepsilon^b(u,v) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4\varepsilon^{2N_2}} \int_{R_\varepsilon}\int_{R_\varepsilon} J(x-y)\left(v(y) - v(x)\right)^2 dydx + \frac{1}{2\varepsilon^{N_2}} \int_{R_\varepsilon}\int_\Gamma G(x-y)dy\left(v(x) - u(y)\right)^2 d\sigma(y)dx.$$

If we change variables as before we get

$$E^b(u,v) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4}\int_R\int_R J_\varepsilon(x-y)|v(y)-v(x)|^2 dydx + \frac{1}{2}\int_R\int_\Gamma G_\varepsilon(x-y)|v(x)-u(y)|^2 dydx,$$

with, as before,

$$J_\varepsilon(x-y) = J(x_1 - y_1, \varepsilon(\tilde{x}_2 - \tilde{y}_2)), \qquad G_\varepsilon(x-y) = G(x_1 - y_1, \varepsilon\tilde{x}_2 - \tilde{y}_2).$$

Now, we just observe that Lemma 2.1 also works here. One can define what is the analogous to the first non-zero eigenvalue for the problem (1.11) as follows:

$$\alpha_1^\varepsilon = \inf_{u^\varepsilon, v^\varepsilon \in \mathcal{A}} \frac{E^b(u^\varepsilon, v^\varepsilon)}{\displaystyle\int_\Omega (u^\varepsilon)^2 dx + \int_R (v^\varepsilon)^2 d\tilde{x}},$$

with

$$\mathcal{A} = \left\{ u^\varepsilon \in H^1(\Omega), v^\varepsilon \in L^2(R) : \int_\Omega u^\varepsilon dx + \int_R v^\varepsilon dx = 0 \right\}.$$

For the positivity of α_1^ε we refer to [1]. A uniform lower bound independent of ε can be proved as in Lemma 2.1. The large time behavior for the solutions of (1.11) can be obtained following the ideas developed in [29]. As we find in [29], the solutions of (1.11) converge exponentially to the mean value of the initial data as t goes to ∞, for each ε.

Now, we prove Theorem 1.2 taking the limit as ε goes to zero in the weak form of (1.11).

Proof of the Theorem 1.2. We proceed as in the proof of Theorem 1.1. First, we obtain convergence along subsequences. From Lemma 2.1, since

$$\int_R (v^\varepsilon(x))^2 dx$$

is bounded in $L^2(R)$ we can take a subsequence, also denoted by $\{v^\varepsilon\}$, such that

$$v^\varepsilon \rightharpoonup v^* \quad \text{weakly} \quad \text{in} \quad L^\infty(0,T;L^2(R)) \quad \text{as} \quad \varepsilon \to 0.$$

On the other hand, we have that

$$\int_\Omega (u^\varepsilon(x))^2 dx \qquad \text{and} \qquad \int_\Omega |\nabla u^\varepsilon(x)|^2 dx,$$

are also bounded, and hence u^ε is bounded in $H^1(\Omega)$. Hence, along a subsequence if necessary,

$$u^\varepsilon \rightharpoonup u^* \quad \text{weakly} \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)) \quad \text{as} \quad \varepsilon \to 0,$$

$$u^\varepsilon \to u^* \quad \text{strongly} \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \quad \text{as} \quad \varepsilon \to 0.$$

Let us consider the weak form of the problem (1.11) using for the equation for the variable v^ε (the second equation of (1.11)) a test function that depends only on the first variable, that is $\varphi = \varphi(x_1)$. We have,

$$\begin{aligned}
&\int_\Omega\int_\Omega u^\varepsilon(x_1,x_2,T)\frac{\partial\varphi}{\partial t}(x_1,x_2,T)dx_2dx_1 - \int_0^T\int_\Omega\int_\Omega u^\varepsilon(x_1,x_2,t)\frac{\partial\varphi}{\partial t}(x_1,x_2,t)dx_2dx_1\\
&= \int_\Omega\int_\Omega u_0^\varepsilon(x_1,x_2)\varphi(x_1,x_2,0)dx_2dx_1 - \int_0^T\int_\Omega\int_\Omega \nabla u^\varepsilon \nabla\varphi(x_1,x_2,t)dx_2dx_1dt\\
&\quad + \int_0^T\int_\Gamma\int_{R_1}\int_{R_2} G_\varepsilon(x-y)(v^\varepsilon(y_1,\tilde{y}_2,t) - u^\varepsilon(x_2,t))\varphi(x_1,x_2,t)d\tilde{y}_2dy_1d\sigma(x_2)dt,
\end{aligned}$$

$$\begin{aligned}
&\int_R\int_R v^\varepsilon(x_1,x_2,T)\frac{\partial\varphi}{\partial t}(x_1,x_2,T)dx_2dx_1 - \int_0^T\int_R\int_R v^\varepsilon(x_1,x_2,t)\frac{\partial\varphi}{\partial t}(x_1,x_2,t)dx_2dx_1\\
&= \int_R\int_R v_0^\varepsilon(x_1,x_2)\varphi(x_1,x_2,0)dx_2dx_1\\
&\quad + \int_0^T\int_{R_1}\int_{R_2}\int_{R_1}\int_{R_2} J_\varepsilon(x-y)\left(v^\varepsilon(y_1,\tilde{y}_2,t) - v^\varepsilon(x_1,\tilde{x}_2,t)\right)\varphi(x_1,t)d\tilde{y}_2dy_1d\tilde{x}_2dx_1dt\\
&\quad - \int_0^T\int_{R_1}\int_{R_2}\int_\Gamma G_\varepsilon(x-y)(v^\varepsilon(x_1,\tilde{x}_2,t) - u^\varepsilon(y_2,t))\varphi(x_1,t)d\sigma(y_2)d\tilde{x}_2d\tilde{x}_1dt.
\end{aligned}$$

Now we can take the limit for $\varepsilon \to 0$ in each integral on the right side of the previous equations as we did in Theorem 1.1. The only difference appears when we analyze the term

$$-\int_0^T\int_{R_1}\int_{R_2}\int_\Gamma G_\varepsilon(x-y)(v^\varepsilon(x_1,\tilde{x}_2,t) - u^\varepsilon(y_2,t))\varphi(x_1,t)d\sigma(y_2)d\tilde{x}_2d\tilde{x}_1dt.$$

In this case, we need the fact that we have a well defined and compact trace operator $Tr : H^1(\Omega) \mapsto L^2(\Gamma)$, see [19], therefore from the weak convergence

$$u^\varepsilon \rightharpoonup u^* \quad \text{weakly} \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)) \quad \text{as} \quad \varepsilon \to 0,$$

we obtain that, along a subsequence,

$$u^\varepsilon \to u^* \quad \text{strongly} \quad \text{in} \quad L^\infty(0,T;L^2(\Gamma)) \quad \text{as} \quad \varepsilon \to 0.$$

As before, since v^ε is bounded in $L^\infty(0,T;L^2(R))$, then V^ε is also bounded in $L^\infty(0,T;L^2(R_1))$ so, taking a subsequence if necessary

$$V^\varepsilon \rightharpoonup V^* \quad \text{weakly} \quad \text{in} \quad L^\infty(0,T;L^2(R_1)).$$

Using (2.7) we obtain

$$-\int_0^T \int_{R_1} \int_\Gamma G_\varepsilon(x-y)(V^\varepsilon(x_1,\tilde{x}_2,t) - |R_2| u^\varepsilon(y_2,t))\varphi(x_1,t) d\sigma(y_2) d\tilde{x}_1 dt.$$

Now we can take the limit as $\varepsilon \to 0$ to obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} &-\int_0^T \int_{R_1} \int_\Gamma G_\varepsilon(x-y)(V^\varepsilon(x_1,\tilde{x}_2,t) - |R_2| u^\varepsilon(y_2,t))\varphi(x_1,t) d\sigma(y_2) d\tilde{x}_1 dt \\ &= -\int_0^T \int_{R_1} \int_\Gamma G_\varepsilon(x-y)(V^*(x_1,\tilde{x}_2,t) - |R_2| u^*(y_2,t))\varphi(x_1,t) d\sigma(y_2) d\tilde{x}_1 dt.\end{aligned}$$

The rest of the terms can be handled as in Theorem 1.1 to obtain the weak form of the equations of the limit problem (1.12).

Uniqueness of solutions to the limit problem can be obtained as in Theorem 1.1 using the energy

$$\begin{aligned}E(u^*,V^*) = \frac{|R_2|}{2}\int_\Omega |\nabla u^*|^2 dx + \frac{|R_2|}{4}\int_{R_1}\int_{R_1} J^*(x-y)\left(V^*(y)+V^*(x)\right)^2 dy dx \\ +\frac{1}{2}\int_{R_1}\int_\Gamma G^*(x-y)(V^*(x) - |R_2| u^*(y))^2 d\sigma(y) dx\end{aligned} \tag{3.5}$$

This completes the proof. □

Now, we gather some properties of the limit problem, (1.12).

The existence and uniqueness of the limit problem (1.12) can be obtained using a fixed point argument as it was done in [29].

The mass conservation in time follows as in the previous section (see also [29]).

To deal with the large time behavior we can be proceed as we did before, since the solutions of the problem (1.12) also converges to the mean value of its initial condition as t goes to zero. Notice that from (1.12) we can define the associated eigenvalue problem. Let us consider α_1^* given by

$$\alpha_1^* = \inf_{u^*,V^*\in\mathcal{A}_0} \frac{E(u^*,V^*)}{|R_2|\displaystyle\int_\Omega (u^*)^2 dx + \int_{R_1}(V^*)^2 dx},$$

where E is given by (3.5) and

$$\mathcal{A}_0 = \Big\{u^* \in H^1(\Omega), V^* \in L^2(R_1) : |R_2|\int_\Omega u^* dx + \int_{R_1} V^* dx = 0\Big\}.$$

One can show that α_1^* is strictly positive and then, the computations of the previous rection can be adapted to prove that the solution of the limit problem (1.12) converges exponentially fast for the mean value of the initial datum as t goes to ∞.

4 Numerical experiments

In this section we propose a discrete numerical scheme for the two models, (1.8) and (1.12) described in this paper. To obtain a fully discretization of the equations in space and time we will use classical methods, centered finite differences for the interior points of the local part, forward and backward differences for the boundary points; while for the nonlocal region and the coupling terms we just approximate the involved integrals by Riemann sums. We use an explicit Euler discretization for the time variable.

As we mentioned in the Introduction, the continuous problems (1.8) and (1.12) have some properties: well-posedness, comparison principle, conservation of mass and convergence to the mean value of the initial datum. In this section we will perform numerical simulations that illustrate these properties.

We will assume that Ω is a bidimensional rectangle $\Omega = \Omega_1 \times \Omega_2 = [a,b] \times [c,d]$ and we take the mesh parameter as $h_1 = \frac{b-a}{M-1} = \frac{d-c}{N-1}$. Let h_1, be same in the two directions. For the nonlocal part, the domain R_1 will be the segment $R_1 = [b,f]$, with $h_2 = \frac{f-b}{M-1}$.The time step $\triangle_t$ is give by the difference between the final time, t_f, with the initial time, t_0.

We approximate the continuous solution $u(x,y,t)$, for $(x,y,t) \in \Omega \times \mathbb{R}$ and $V(x,t)$, for $(x,t) \in R_1 \times \mathbb{R}$, by discrete values $u^l_{i,j} \approx u(x_i, y_j, t_l)$ and $V^l_k \approx V(z_k, t_l)$, respectively, with $i,k = 1,\cdots,M$, $j = 1,\cdots,N$. For simplicity, let us consider a uniform mesh for the local and nonlocal part. The local domain Ω was discretized by the mesh (x_i, y_j), with $i = 1,\cdots,M$, $j = 1,\cdots,N$ while, the nonlocal domain, R_1 is discretized by the points z_k, $k = 1,\cdots,M$.

We will consider $h = h_1 = h_2$ (for simplicity). With this in mind, we call $x_1 = a$, $x_i = x_{i-1} + h$ and $x_M = x_{M-1} + h$, $y_1 = c$, $x_i = y_{j-1} + h$ and $y_N = y_{N-1} + h$, $z_1 = b$, $z_k = z_{k-1} + h$ and $z_M = z_{M-1} + h$.

Then, the numerical approximation of the problem (1.8), is given by the following system of equations: for the local part we have,

$$\begin{cases} u^{l+1}_{i,j} = u^l_{i,j} + \dfrac{\triangle_t}{h^2}\left(u^l_{i+1,j} + u^l_{i-1,j} + u^l_{i,j+1} + u^l_{i,j-1} - 4u^l_{i,j}\right) \\ \qquad\qquad + \triangle_t h \displaystyle\sum_{k=1}^{M} G(x_i - z_k, y_j)\left(V^l_k - |R_2| u^l_{i,j}\right), \quad i = 2,\cdots,M-1, \quad j = 2,\cdots,N-1 \\ u^{l+1}_{1,1} = u^l_{1,1} + \dfrac{\triangle_t}{h^2}\left(u^l_{1,2} + u^l_{2,1} - 2u^l_{1,1}\right) \\ u^{l+1}_{M,1} = u^l_{M,1} + \dfrac{\triangle_t}{h^2}\left(u^l_{M-1,1} + u^l_{M,2} - 2u^l_{M,1}\right) \\ u^{l+1}_{M,N} = u^l_{M,N} + \dfrac{\triangle_t}{h^2}\left(u^l_{M-1,N} + u^l_{M,N-1} - 2u^l_{M,N}\right) \\ u^{l+1}_{1,N} = u^l_{1,N} + \dfrac{\triangle_t}{h^2}\left(u^l_{2,N} + u^l_{1,N-1} - 2u^l_{1,N}\right) \\ u^{l+1}_{i,1} = u^l_{i,1} + \dfrac{\triangle_t}{h^2}\left(u^l_{i+1,1} + u^l_{i-1,1} + u^l_{i,2} - 3u^l_{i,1}\right), \quad i = 2,\cdots,M-1 \\ u^{l+1}_{M,j} = u^l_{M,j} + \dfrac{\triangle_t}{h^2}\left(u^l_{M-1,j} + u^l_{M,j+1} + u^l_{M,j-1} - 3u^l_{M,j}\right), \quad j = 2,\cdots,N-1 \\ u^{l+1}_{1,j} = u^l_{1,j} + \dfrac{\triangle_t}{h^2}\left(u^l_{2,j} + u^l_{1,j+1} + u^l_{1,j-1} - 3u^l_{1,j}\right), \quad j = 2,\cdots,N-1 \\ u^0_{i,j} = u_{ij0}, \quad i = 1,\cdots,M, \quad j = 1,\cdots,N \end{cases} \tag{4.1}$$

for $l > 0$ and, for the nonlocal part,

$$\begin{cases} V^{l+1}_k = V^l_k + \triangle_t R_2 h \displaystyle\sum_{p=1}^{N} J(z_k - z_p)(V^l_p - V^l_k) - \triangle_t h^2 \sum_{i=2}^{M-1}\sum_{j=2}^{N-1} G(x_i - z_k, y_j)(V^l_k - |R_2| u^l_{i,j}), \quad k = 1,\cdots,M \\ V^0_k = V_{k0}, \quad k = 1,\cdots,M, \end{cases} \tag{4.2}$$

for $l > 0$.

Similarly, the full discretization for the problem (1.12) is given by: for the local part

$$
\begin{cases}
u_{i,j}^{l+1} = u_{i,j}^{l} + \dfrac{\triangle_t}{h^2}\left(u_{i+1,j}^{l} + u_{i-1,j}^{l} + u_{i,j+1}^{l} + u_{i,j-1}^{l} - 4u_{i,j}^{l}\right)\\
u_{1,1}^{l+1} = u_{1,1}^{l} + \dfrac{\triangle_t}{h^2}\left(u_{1,2}^{l} + u_{2,1}^{l} - 2u_{1,1}^{l}\right)\\
u_{M,1}^{l+1} = u_{M,1}^{l} + \dfrac{\triangle_t}{h^2}\left(u_{M-1,1}^{l} + u_{M,2}^{l} - 2u_{M,1}^{l}\right)\\
u_{M,N}^{l+1} = u_{M,N}^{l} + \dfrac{\triangle_t}{h^2}\left(u_{M-1,N}^{l} + u_{M,N-1}^{l} - 2u_{M,N}^{l} + h^2\displaystyle\sum_{k=1}^{M} G(x_M - z_k, y_N)(V_k^l - |R_2|u_{M,N}^{l})\right)\\
u_{1,N}^{l+1} = u_{1,N}^{l} + \dfrac{\triangle_t}{h^2}\left(u_{2,N}^{l} + u_{1,N-1}^{l} - 2u_{1,N}^{l} + h^2\displaystyle\sum_{k=1}^{M} G(x_1 - z_k, y_N)(V_k^l - |R_2|u_{1,N}^{l})\right)\\
u_{i,1}^{l+1} = u_{i,1}^{l} + \dfrac{\triangle_t}{h^2}\left(u_{i+1,1}^{l} + u_{i-1,1}^{l} + u_{i,2}^{l} - 3u_{i,1}^{l}\right), \quad i = 2, \cdots, M-1\\
u_{i,N}^{l+1} = u_{i,N}^{l} + \dfrac{\triangle_t}{h^2}\left(u_{i+1,N}^{l} + u_{i-1,N}^{l} + u_{i,N-1}^{l} - 3u_{i,N}^{l} + h^2\displaystyle\sum_{k=1}^{M} G(x_i - z_k, y_N)(V_k^l - |R_2|u_{i,N}^{l})\right),\\
\qquad\qquad i = 2, \cdots, M-1\\
u_{M,j}^{l+1} = u_{M,j}^{l} + \dfrac{\triangle_t}{h^2}\left(u_{M-1,j}^{l} + u_{M,j+1}^{l} + u_{M,j-1}^{l} - 3u_{M,j}^{l}\right), \quad j = 2, \cdots, N-1\\
u_{1,j}^{l+1} = u_{1,j}^{l} + \dfrac{\triangle_t}{h^2}\left(u_{2,j}^{l} + u_{1,j+1}^{l} + u_{1,j-1}^{l} - 3u_{1,j}^{l}\right), \quad j = 2, \cdots, N-1,\\
u_{i,j}^{0} = u_{ij0}, \quad i = 1, \cdots, M, \quad j = 1, \cdots, N,
\end{cases}
\tag{4.3}
$$

for $l > 0$ and, for the nonlocal part

$$
\begin{cases}
V_k^{l+1} = V_k^l + \triangle_t R_2 h \displaystyle\sum_{p=1}^{N} J(z_k - z_p)(V_p^l - V_k^l) - \triangle_t h \sum_{i=1}^{M} G(x_i - z_k, y_N)(V_k^l - |R_2|u_{i,N}^{l}) \quad k = 1, \cdots, M\\
V_k^0 = V_{k0}, \quad k = 1, \cdots, M,
\end{cases}
\tag{4.4}
$$

for $l > 0$.

Notice that the main difference between the two discretizations occurs at the coupling terms, that in one case are given by

$$
\sum_{k=1}^{M} G(x_i - z_k, y_j)\left(V_k^l - |R_2|u_{i,j}^{l}\right) \qquad \text{and} \qquad \sum_{i=2}^{M-1}\sum_{j=2}^{N-1} G(x_i - z_k, y_j)(V_k^l - |R_2|u_{i,j}^{l})
$$

(these terms appear in the discretization of the model coupled via source terms, the double sums corresponds to discretizations of double integrals) and in the second discretization by

$$
\sum_{k=1}^{M} G(x_i - z_k, y_N)(V_k^l - |R_2|u_{1,N}^{l}), \qquad \text{and} \qquad \sum_{i=1}^{M} G(x_i - z_k, y_N)(V_k^l - |R_2|u_{i,N}^{l})
$$

(this corresponds to coupling on the boundary, remark that the sums here are discretizations of one dimensional integrals).

For the experiments we will consider the domain $\Omega = [-1, 1] \times [-1, 1]$, $R_1 = [1, 3]$, $R_2 = [0, 1]$ and a time step which satisfies $\triangle_t \le \frac{h^2}{4}$ (this comes from stability considerations).

At the simulations we will use the kernel J, given by the following probability density:

$$
J(x) = \begin{cases} \dfrac{1}{2}cos(x), \quad if \quad |x| \le \dfrac{\pi}{2},\\ 0, \quad otherwise. \end{cases}
\tag{4.5}
$$

This particular kernel J satisfies the hypothesis described before, J is a nonnegative continuous function, symmetric, with $J(0) > 0$ and integrable.

4.1 Numerical experiments for coupling via source terms.

Now, we will include some numerical experiments considering the fully discrete scheme for the problem (1.8) given by (4.1)–(4.2).

In this case, concerning the kernel G, as the problem (1.8) allows that particles can jump directly inside the interior of Ω, we will consider G as a function given by

$$G(x,y) = \begin{cases} \frac{1}{4} cos(x)cos(y), & if \quad |x,y| \leq \frac{\pi}{2} \\ 0, & otherwise. \end{cases} \tag{4.6}$$

The kernel G satisfies the hypothesis defined in the Introduction.

Numerical experiment 1. For this simulation we consider $M = N = 11$, $h = 0,2$, $\triangle_t = 0,005$, as initial conditions, we used $u_0(x,y) = 0$, $V_0(x) = 1$. The mean value of the initial condition is $\approx 0,083$.

In Figure 3 we plot the evolution of the local and the nonlocal parts of the solution (for the local part we have depicted the solution $u(x,y,t)$ at three different time steps, as the same for the nonlocal part of the solution, $V(x,t)$, we can observe its evolution in four time steps). Both local and nonlocal parts of the solution converge towards the mean value of the numerical initial condition as t increases.

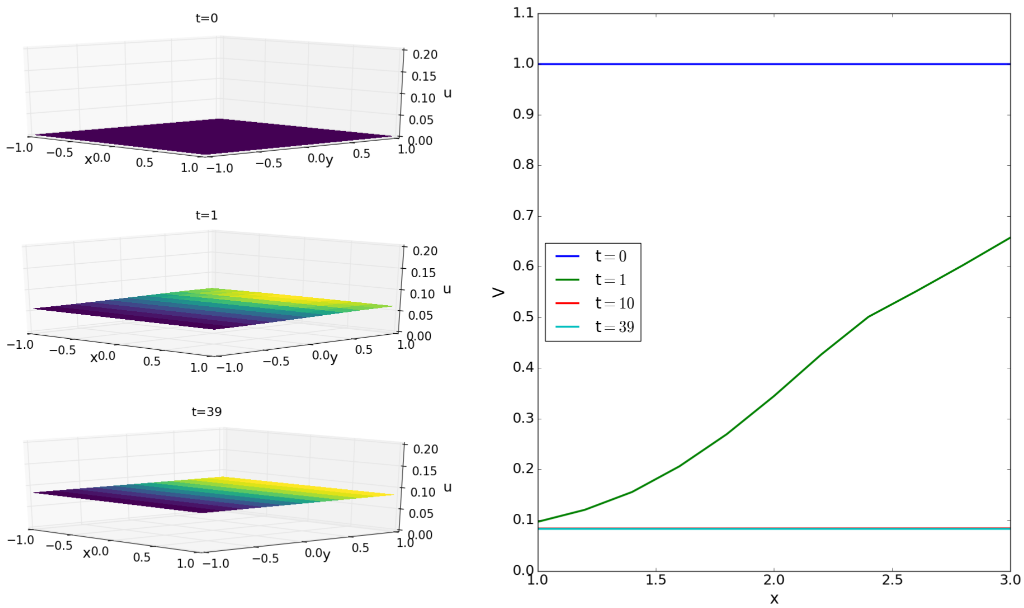

Figure 3: The local part (left) and the nonlocal part (right) with two constants as initial conditions.

Numerical experiment 2. For this simulation we consider $M = N = 11$, $h = 0,2$, $\triangle_t = 0,005$, as initial conditions, we used $u_0(x,y) = cos\left(\frac{\pi x}{2}\right) cos\left(\frac{\pi y}{2}\right)$, $V_0(x) = 1$. Now, the mean value of the initial condition $\approx 0,38$.

Figure 4 contains the plot of the local and the nonlocal parts of the solution. One can see that even with a not constant initial condition for the local part, we observe its fast convergence towards the mean value of the initial condition as t increases.

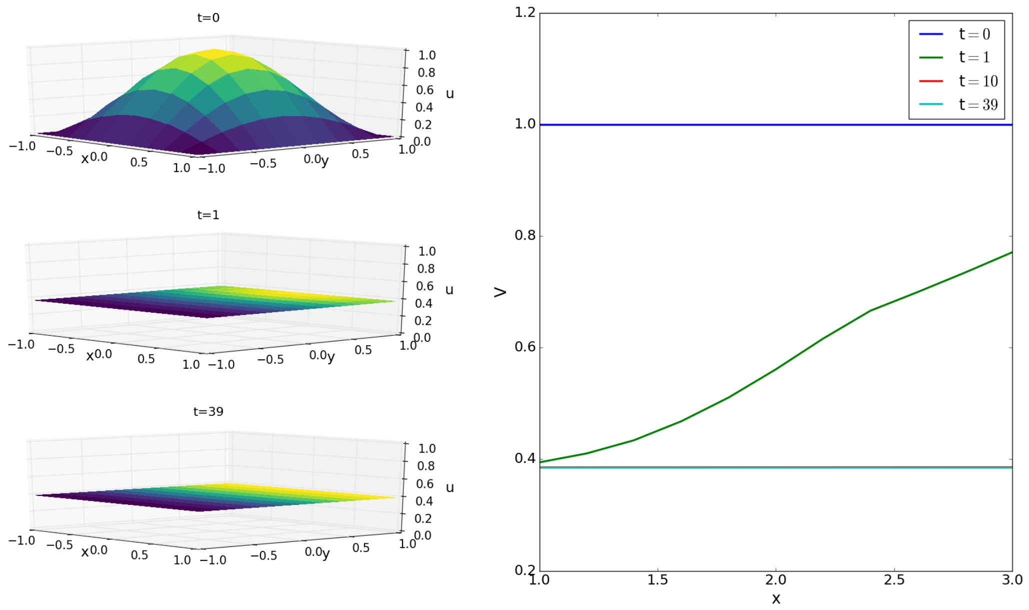

Figure 4: The local part (left) and the nonlocal part (right) with non constant initial datum for the local part.

Numerical experiment 3. For this simulation we consider $M = N = 11$, $h = 0,2$, $\triangle_t = 0,005$, as initial conditions, we used $u_0(x,y) = cos\left(\frac{\pi x}{2}\right) cos\left(\frac{\pi y}{2}\right)$, $V_0(x) = 9 - x^2$. The mean value of the initial condition is $\approx 0,68$.

In Figure 5 both local and nonlocal initial conditions are non constants and they also verify the convergence to the mean of the initial condition as t increases. Note that, even for $t = 1$ the solution of the local part is closer to the mean value of the initial condition. For the nonlocal part, as $t = 10$ the solution is very close to the mean of the initial condition that is subscribed by the last iteration.

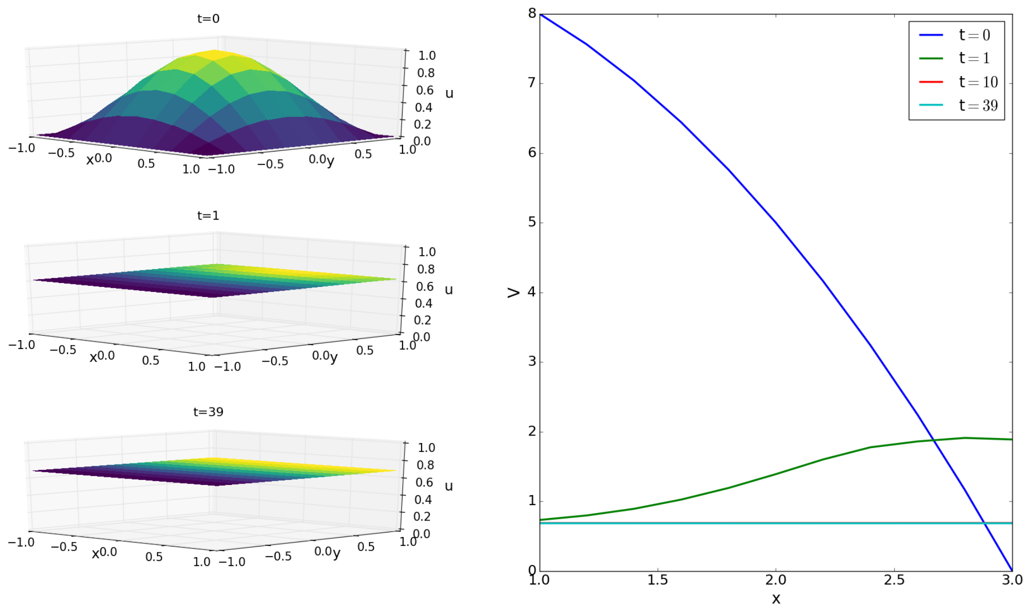

Figure 5: The local part (left) and the nonlocal part (right) for non constant initial data.

4.2 Numerical experiments for coupling via boundary terms.

Now, we will include some numerical experiments considering the fully discrete scheme for the problem (1.12) given by (4.3)–(4.4). At the simulations we will use the same kernel J, as we define in (4.5) and the kernel G as we define in (4.6). For simplicity, we have considered the local domain Ω as a square $\Omega = [-1, 1] \times [-1, 1]$, then for the coupling we will consider Γ as a whole side of the domain Ω, $\Gamma = \{1\} \times [-1, 1]$.

Numerical experiment 4. For this simulation we consider $M = N = 11$, $h = 0,2$, $\triangle_t = 0,005$, as initial conditions, we used $u_0(x, y) = 0$, $V_0(x) = 1$.Mean value of the initial condition $\approx 0,31$.

In Figure 6 we plot the evolution of the local and the nonlocal parts of the solution. Both local and nonlocal parts of the solution converge towards the mean value of the numerical initial condition as t increases.

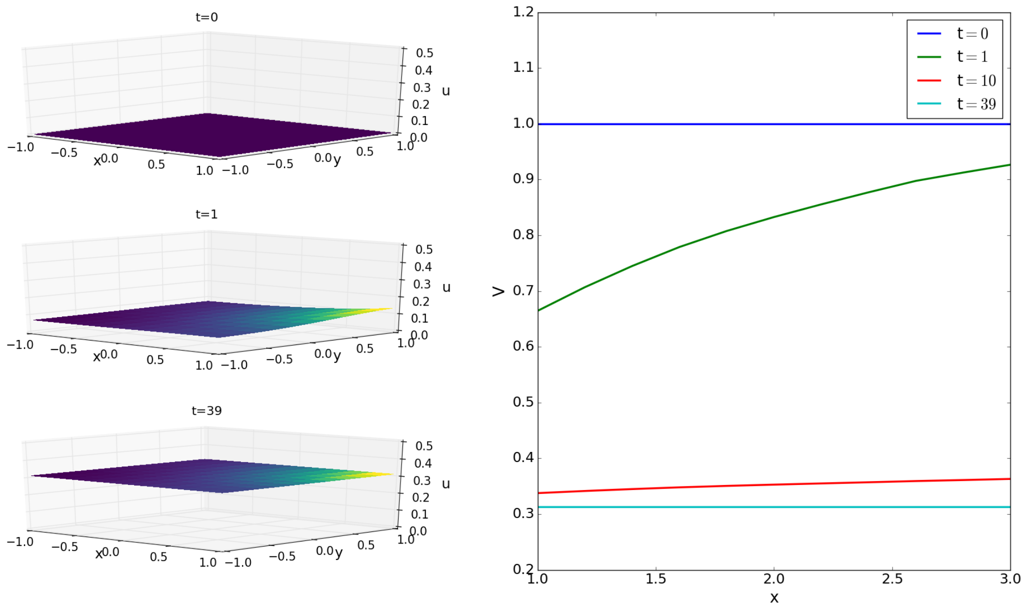

Figure 6: The local part (left) and the nonlocal part (right).

Numerical experiment 5. For this simulation we consider $M = N = 11$, $h = 0,2$, $\triangle_t = 0,005$, as initial conditions, we used $u_0(x, y) = x^2 + y^2$, $V_0(x) = 9 - x^2$.Mean value of the initial condition $\approx 1,99$.

In Figure 7, we observe that also when we take two non-constants initial conditions, the solution converges towards the mean value of the numerical initial condition as t increases. We plot the solutions for specific time steps to follow the evolution.

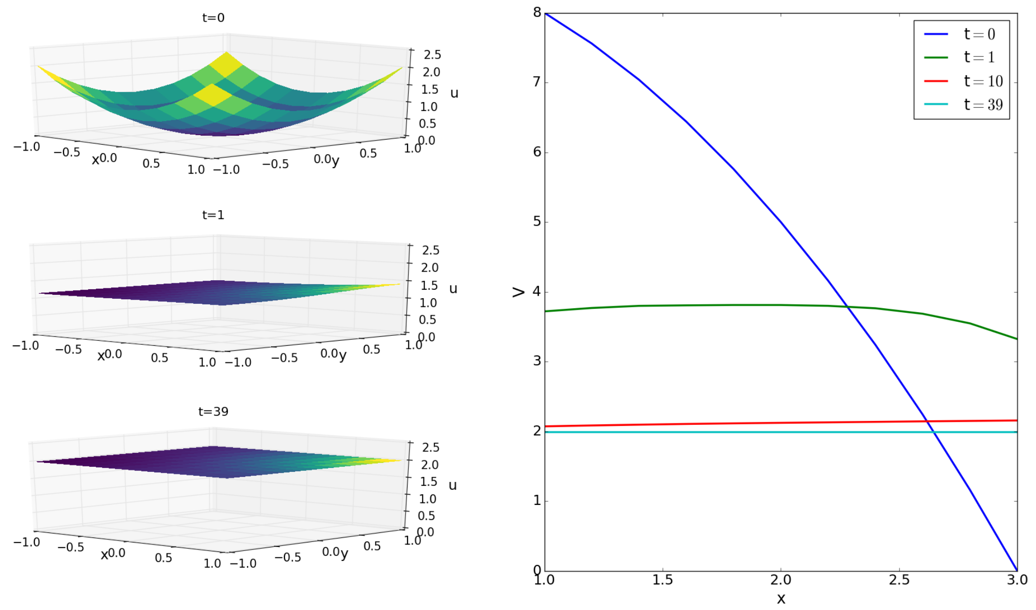

Figure 7: The local part (left) and the nonlocal part (right).

Numerical experiment 6. For this simulation we consider $M = N = 11$, $h = 0,1$, $\triangle_t = 0,005$, as initial conditions, we used $u_0(x,y) = x^2 + y^2$, $V_0(x) = x^2$.Mean value of the initial condition $\approx 1,92$. In Figure 8, we define the same initial condition for the local and nonlocal part. Note that we obtain the same behavior along the time, both local and nonlocal solution converge to the mean value of the initial condition.

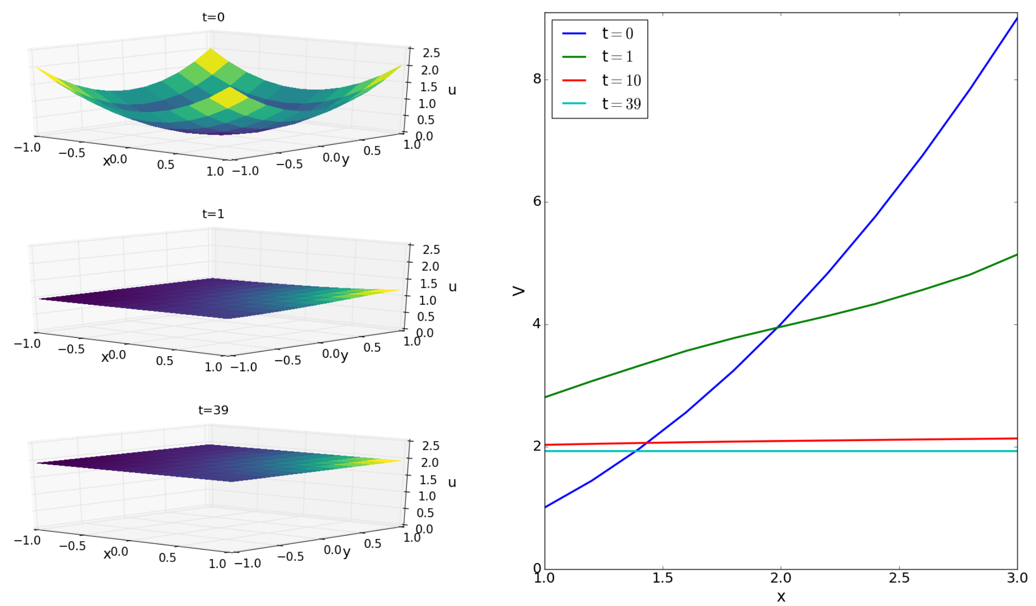

Figure 8: Surface plot of the solution for the local part (left) and the nonlocal part (right).

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Chapter 5

Conclusions

The use of nonlocal models to describe many applications in engineering, physics, ecology, and epidemiology has steadily increased over the past decade. The ability of nonlocal models to accurately capture long-range effects, not captured by classical models of PDEs, has been a driving factor in the development of mathematical theory behind these evolution problems. Besides, this type of model has a high computational cost to provide accurate solutions, which became an attractive challenge.

In this work, we studied an evolution problem with local/nonlocal coupling. As a starting point, we developed the basic theory for the one-dimensional case and then we extend it to higher dimensions. For this problem, we showed the existence and uniqueness of solutions (providing two different proofs of these properties, one using a fixed point argument and another one using semigroup theory), conservation of the total mass, the validity of the comparison principle, and study the asymptotic decay as time goes to infinity. It was also possible to show, from an appropriate rescalling of the nonlocal kernel, that one can recover the heat equation in the entire domain through a limit process. We remark that in our model, we do not impose continuity of the solution in the coupling, instead, we have continuity of the fluxes. In this model one can show that we have continuity of the solution in the local and nonlocal parts according to the continuity of the datum.

We also provide a third proof of existence and uniqueness for the proposed coupling model using the properties of the Splitting Operator method, not in the usual sense, but taking advantage of the coupling structure of the model. Besides, we developed some numerical experiments, that illustrate the theoretical results previously proved.

Finally, we propose to extend our analysis to deal with the same problem of local/nonlocal coupling defined in regions of different dimension. In particular, we study the case of a thin domain as the nonlocal region. We are interested in the study of the limit case when the small parameter that controls the width of the nonlocal region goes to zero. With this idea in mind, we can concentrate the domain of the nonlocal equation in a domain of lower dimension. For the limit problem, we proved the same properties for the solutions of the original problem: existence, uniqueness, mass conservation, and we studied the asymptotic decay of the solution. Finally, we developed some numerical experiments that illustrate the obtained theory for the problem in domains of different dimensions.

5.1 Future work

5.1.1 Population mobility. Models for infections.

Understanding human mobility patterns play a very important role in modeling infectious diseases, whether they are transmitted directly or indirectly. Usually, the description of spatial spread is approximated via diffusive dispersion or metapopulation models. In the first case, it was assumed a random walk movement of the host between different locations (this is the classic case modeled by the heat equation). The modeling developed from metapopulation models, on the other hand, considers other effects, such as bidirectional movements and distance between regions. These models are also known as network models. For these models, it is assumed that the population is distributed in patches, each patch is spatially distinct, and yet, one patch may influence another due to the dispersion of individuals between them. Using network terminology, nodes would correspond to patches, and edges represent the connections between the nodes (patches). The rate at which individuals leave a node i and move to a node j in the network is defined, in most cases, by a fixed probability rate Brockmann *et al.* (2009).

Suppose then that we want to model a disease whose transmission is direct. In this case, the spread of a disease from one city to another, for example, will only happen if there is the mobility of individuals between those cities. In this sense, the model proposed in this work could be applied in the context that simulates the interaction between cities connected by a highway. In the city, we would have a local dynamic, represented by the Laplacian operator, while on the road we could capture the effect of faster diffusion using the nonlocal operator. The nonlocal dynamics on the road can be thought of as a projection of intermediate cities on the road. We can also consider the mobility scenario by air. In this case, the individuals could jump from some point on the road or another city directly inside the domain, in which a local diffusion takes place.

The main difference between metapopulation models and the model proposed in this work is that, with the local/nonlocal coupling we are inserting not only a probability of jumping from a x position to a y position, but we are also inserting a suitable dynamic for the road (network edges).

Recent studies, with different proposals, have been developed along these lines, considering applications in ecology and epidemiology. For example, in Berestycki *et al.* (2015) the effects of network transportation on enhancing biological invasion is studied. The proposed mathematical model consists of one equation with nonlocal diffusion in a one-dimensional domain coupled via boundary condition with a standard reaction-diffusion, in a two-dimensional domain.

5.1.2 The eigenvalue problem

One interesting problem that is left open in this thesis is the existence of minimizers for the eigenvalue problem

$$\lambda_1 := \inf_{(u,v)\in L^2(\Omega)} \frac{E(u,v)}{\displaystyle\int_{\Omega_l} u^2 + \int_{\Omega_{nl}} v^2} \tag{5.1}$$

with

$$\begin{aligned} E(u,v) := \frac{1}{2}\int_{\Omega_l} |\nabla u|^2 dx + \frac{C_{J,1}}{4}\int_{\Omega_{nl}}\int_{\Omega_{nl}} J(x-y)\,|v(y)-v(x)|^2\, dydx \\ + \frac{C_{J,2}}{2}\int_{\Omega_{nl}}\int_{\Sigma} G(x,z)\,|v(x)-u(z)|^2\, d\sigma(z)dx. \end{aligned} \tag{5.2}$$

From our results we have that $\lambda_1 > 0$ but we do not known if it is attained. Here the main difficulty is the lack of regularizing effect of the nonlocal operator (recall that we have a continuous kernel J).

5.1.3 Singular kernels

One can also deal with local/nonlocal couplings involving singular kernels of the form

$$J(x-y) = \frac{C}{|x-y|^{n+2s}}$$

that are the ones that appear in the fractional Laplacian. For these kind of models one interesting problem is to look for the possible regularizing effect of the evolution for positive times.

5.1.4 Equations on manifolds or graphs

One can consider the same kind of coupled problems when the ambient space is a manifold (one may want to consider the Laplace-Beltrami operator as the local part and a nonlocal jump process as the nonlocal part) or a discrete structure (a finite or infinite graph). In this last case there are nodes in the graph in which the diffusion is local (they interact only with its neighbors) and others in which we have a nonlocal behaviour (nodes that interact with far away nodes).

5.1.5 Nonlinear problems

One may also want to consider energies like

$$\begin{aligned} E(u,v) := \frac{1}{p}\int_{\Omega_l} |\nabla u|^p dx + \frac{C_{J,1}}{2r}\int_{\Omega_{nl}}\int_{\Omega_{nl}} J(x-y)\,|v(y)-v(x)|^r\,dydx \\ +\frac{C_{J,2}}{s}\int_{\Omega_{nl}}\int_{\Sigma} G(x,z)\,|v(x)-u(z)|^s\,d\sigma(z)dx, \end{aligned} \tag{5.3}$$

which lead to nonlinear diffusion equations. Here one challenging problem is to find estimates for the long time behaviour of the solutions (that are expected to converge to the mean value of the initial datum when we deal with Neumann type boundary conditions).

5.1.6 Moving interfaces

Finally, we mention as an interesting future line of research to study these kind of couplings when the involved domains Ω_l, Ω_{nl} and the interface Γ depend on time. This is the first step in order to deal with free boundary problems in which the domains move according to the balance of the fluxes of the solution across the interface.

Appendix A

L^p Spaces

For completeness, we will list in this appendix some of the classical results about the L^p-spaces used in this thesis: Holder's and Minkowski's inequality; the Monotone Convergence Theorem and the Dominated Convergence Theorem; Fubini's Theorem and Fatou's Lemma. These notes have been written following Evans (1998); Rudin (1987); Sastre Gómez (2014).

Let (Ω, μ) be a measure space. At first, we have some classical inequalities.

Teorema A.0.1 *For $1 \leq p \leq \infty$, if p and $p^{'}$ satisfy $1/p + 1/p^{'} = 1$, and if $f_1 \in L^p(\Omega)$ and $f_2 \in L^{p^{'}}(\Omega)$, then $f_1 f_2 \in L^1(\Omega)$, and*

$$\int_\Omega |f_1 f_2| d\mu \leq \left(\int_\Omega |f_1|^p d\mu \right)^{1/p} \left(\int_\Omega |f_2|^{p^{'}} d\mu \right)^{1/p^{'}} \tag{A.1}$$

and

$$\left(\int_\Omega (f_1 + f_2)^p d\mu \right)^{1/p} \leq \left(\int_\Omega |f_1|^p d\mu \right)^{1/p} \left(\int_\Omega |f_2|^{p^{'}} d\mu \right)^{1/p^{'}}. \tag{A.2}$$

The inequality (A.1) is called Holder's inequality and (A.2) is called Minkowski's inequality.

Next, we state some results concerning the passage to the limit inside the integral.

Teorema A.0.2 (Monotone Convergence Theorem) *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions in Ω, and assume that*

i) $0 \leq f_1 \leq f_2 \leq \cdots \leq \infty$ in Ω.

ii) $f_n \longrightarrow f$ as $n \longrightarrow \infty$ almost everywhere.

Then f is measurable and

$$\int_\Omega f_n d\mu \longrightarrow \int_\Omega f d\mu$$

as $n \longrightarrow \infty$.

Lema A.0.3 (Fatou's Lemma) *If $f_n : \Omega \longrightarrow [0, \infty]$ is measurable, for each positive integer n, then*

$$\int_\Omega \left(\liminf_{n\longrightarrow\infty} f_n \right) d\mu \leq \liminf_{n\longrightarrow\infty} \int_\Omega f_n d\mu. \tag{A.3}$$

Teorema A.0.4 (Dominated Convergence Theorem) *Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions in Ω, and assume that*

$$f = \lim_{n\longrightarrow\infty} f_n,$$

almost everywhere.

If there exists a function $g \in L^1(\Omega)$ such that

$$f_n \leq g$$

for all $n \in \mathbb{N}$, then $f \in L^1(\Omega)$ and

$$\lim_{n \longrightarrow \infty} \int_\Omega |f_n - f| d\mu = 0,$$

and

$$\lim_{n \longrightarrow \infty} \int_\Omega |f_n| d\mu = \int_\Omega |f| d\mu.$$

Teorema A.0.5 (Fubini's Theorem) *Let (Ω_1, μ_1) and (Ω_2, μ_2) be $\sigma-$finite measure spaces, and let f be a $\mu_1 \times \mu_2$-measurable function on $\Omega_1 \times \Omega_2$. If $0 \leq f \leq \infty$, and if*

$$\phi(x) = \int_{\Omega_2} f(x,y) d\mu_2(y), \quad \psi(y) = \int_{\Omega_1} f(x,y) d\mu_1 dx, \quad x \in \Omega_1, y \in \Omega_2,$$

then ϕ is μ_1-measurable and ψ is μ_2-measurable, and

$$\int_{\Omega_1} \phi(x) d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f(x,y) d\mu_1(x) d\mu_2(y) = \int_{\Omega_2} \psi(y) d\mu_2(y), \quad x \in \Omega_1, \quad y \in \Omega_2.$$

Finally, we include an embedding result for Sobolev spaces.

Teorema A.0.6 (Rellich-Kondrachov Compactness Theorem) *Assume that U is a bounded open subset of $\mathbb{R}^N$, and ∂U is C^1. Suppose that $1 \leq p < N$. Then*

$$W^{1,p} \subset\subset L^q(U),$$

for each $1 \leq p < p^* = pN/(N-p)$.

Appendix B

Nonlocal Diffusion Results

In this appendix we state some important results about nonlocal diffusion equations that were used for the most of the proofs in this thesis. These notes have been written following Andreu-Vaillo *et al.* (2010); Cortazar *et al.* (2007).

Let Ω a bounded, connected and smooth domain and $J : \mathbb{R}^N \longrightarrow R$, satisfying the hyphotesis described in the Introduction.

Lema B.0.1 (Lemma 3.1, see Cortazar *et al.* (2007)) *There exists a constant $C > 0$ such that for every $u \in L^2(\Omega)$ it holds*

$$\int_\Omega (u(x) - \langle u \rangle)^2 dx \leq C \int_\Omega \int_\Omega J(x-y)(u(y) - u(x))^2 dydx,$$

where $\langle u \rangle$ is the mean value of u in Ω, that is

$$\langle u \rangle = \frac{1}{|\Omega|} \int_\Omega u(x)dx.$$

Teorema B.0.2 (Theorem 6.11, see Andreu-Vaillo *et al.* (2010)) *Let $1 \leq q < +\infty$ and $D \subset \mathbb{R}^N$ open. Let $\rho : \mathbb{R}^N \longrightarrow R$ be a nonnegative continuous radial function with compact support, non identically zero, and $\rho_n(x) := n^N \rho(nx)$. Let $\{f_n\}$ be a sequence of functions in $L^q(D)$ such that*

$$\int_D \int_D |f_n(y) - f_n(x)|^q \rho_n(y-x) dxdy \leq \frac{M}{n^q}.$$

1. *If $\{f_n\}$ is weakly convergent in $L^q(D)$ to f, then*

   (i) For $q > 1$, $f \in W^{1,q}(D)$, and moreover

   $$(\rho(z))^{\frac{1}{q}} \chi_D \left(x + \frac{1}{n}z\right) \frac{\overline{f}_n\left(x + \frac{1}{n}z\right) - f_n(x)}{1/n} \rightharpoonup (\rho(z))^{\frac{1}{q}} z.\nabla f(x)$$

   weakly in $L^q(D) \times L^q(\mathbb{R}^N)$.

   (ii) For $q = 1$, $f \in BV(D)$ (BV is the space of functions of boundary variation), and moreover

   $$\rho(z)\chi_D\left(. + \frac{1}{n}z\right) \frac{\overline{f}_n\left(. + \frac{1}{n}z\right) - f_n(.)}{1/n} \rightharpoonup \rho(z)z.Df$$

   weakly in the sense of measures.

2. *Suppose D is a smooth bounded domain in $\mathbb{R}^N$ and $\rho(x) \geq \rho(y)$ if $|x| \leq |y|$. Then $\{f_n\}$ is relatively compact in $L^q(D)$, and consequently, there exists a subsequence $\{f_{n_k}\}$ such that*

   (i) if $q > 1$, $f_{n_k} \longrightarrow f$ in $L^q(D)$ with $f \in W^{1,q}(D)$;

(ii) *if* $q = 1$, $f_{n_k} \longrightarrow f$ *in* $L^1(D)$ *with* $f \in BV(D)$.

Proposition B.0.3 (Proposition 3.4, see Andreu-Vaillo *et al.* (2010)) *Given* J *and* Ω, *the quantity*

$$\beta_1 := \beta_1(J,\Omega) = \inf_{u\in L^2(\Omega), \int_\Omega u=0} \frac{\frac{1}{2}\int_\Omega\int_\Omega J(x-y)(u(y)-u(x))^2 dydx}{\int_\Omega (u(x))^2 dx}$$

is strictly positive.

Teorema B.0.4 (Brezis-Pazy Theorem, see Andreu-Vaillo *et al.* (2010)) *Let* A_n *be m-accretive in* X, $x_n \in \overline{D(A_n)}$ *and* $f_n \in L^1(0,T;X)$ *for* $n = 1, 2, \cdots, \infty$. *Let* u_n *be the mild solution of*

$$u_n' + A_n u_n \ni f_n \quad in [0,T], \quad u_n(0) = x_n.$$

If $f_n \longrightarrow f_\infty$ *in* $L^1(0,T;X)$ *and* $x_n \longrightarrow x_\infty$ *as* $n \longrightarrow \infty$ *and*

$$\lim_{n\longrightarrow\infty} (I + \lambda A_n)^{-1} z = (I + \lambda A_\infty)^{-1} z,$$

for some $\lambda > 0$ *and all* $z \in D$, *with* D *dense in* X, *then*

$$\lim_{n\longrightarrow\infty} u_n(t) = u_\infty(t), \quad uniformly \quad on \quad [0,T].$$

In case that the operators are the subdifferentials of convex lower semicontinuous functionals in Hilbert spaces, to prove the convergence of the resolvent it is enough to show the convergence of the functionals in the following sense introduced by U. Mosco in (see Attouch (1979)). Suppose X is a metric space and $A_n \subset X$. We define

$$\liminf_{n\longrightarrow\infty} A_n = \{x \in X : \exists x_n \in A_n, x_n \longrightarrow x\}$$

and

$$\limsup_{n\longrightarrow\infty} A_n = \{x \in X : \exists x_n \in A_{n_k}, x_{n_k} \longrightarrow x\}.$$

In case X is a normed space, we denote by $s-$lim inf and $w-$lim sup the above limits associated respectively to the strong and to the weak topology of X. Given a sequence Ψ_n, $\Psi : H \longrightarrow (-\infty, +\infty]$ of convex lower semicontinuous functionals, we say that Ψ_n converges to Ψ in the sense of Mosco if

$$w - \limsup_{n\longrightarrow\infty} \mathrm{Epi}(\Psi_n) \subset \mathrm{Epi}(\Psi) \subset s - \liminf_{n\longrightarrow\infty} \mathrm{Epi}(\Psi_n).$$

This equation is equivalent to the following two conditions:

(1) $\forall u \in D(\Psi) \exists u \in D(\Psi_n) : u_n \longrightarrow u$ and $\Psi(u) \geq \limsup_{n\longrightarrow\infty} \Psi_n(u_n)$;

(1) for every subsequence n_k, as $u_k \rightharpoonup u$, we have $\Psi(u) \leq \liminf_{n_k\longrightarrow\infty} \Psi_{n_k}(u_k)$;

As a consequence of Theorem B.0.4 and using the results in Attouch (1979) we can state the following result.

Teorema B.0.5 (Theorem A.38, see Andreu-Vaillo *et al.* (2010)) *Let* $\Psi_n, \Psi : H \longrightarrow (-\infty, +\infty]$ *be convex lower semicontinuous functionals. Then the followings statements are equivalent:*

(i) Ψ_n *converges to* Ψ *in the sense of Mosco.*

(ii) $(I + \lambda\Psi_n)^{-1} u \longrightarrow (I + \lambda\Psi)^{-1} u$, $\forall \lambda > 0$, $u \in H$.

Moreover, any of these two conditions (i) or (ii) imply that *(iii)* *For every* $u_0 \in \overline{D(\partial\Psi)}$ *and* $u_{0,n} \in \overline{D(\partial\Psi_n)}$ *such that* $u_{0,n} \longrightarrow u_0$, *and every* $f_n, f \in L^2(0,T;H)$ *with* $f_n \longrightarrow f$, *if* $u_n(t), u(t)$ *are the strong solutions of the abstract Cauchy problems*

$$\begin{cases} u_n'(t) + \partial\Psi_n(u_n(t)) \ni f_n, & a.e \quad t \in (0,T) \\ u_n'(0) = u_{0,n}, \end{cases}$$

and,

$$\begin{cases} u'(t) + \partial\Psi(u(t)) \ni f, & a.e \quad t \in (0,T) \\ u'(0) = u_0, \end{cases}$$

respectively, then

$$u_n \longrightarrow u$$

in $C([0,T];H)$.

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