Parabolic Equations on Conic Manifolds

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Equações Parabólicas em Variedades Cônicas

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Resumo

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Apresentaremos resultados sobre equações parabólicas em variedades cônicas usando funções contínuas e L_p. Inicialmente, mostraremos a existência de soluções globais para uma típica equação de reação-difusão. Consideramos espaços de Mellin Sobolev, que são espaços de funções construídos usando funções L_p.

Em segundo lugar, mostramos a sectorialidade de operadores diferenciais e pseudodiferenciais elípticos agindo sobre funções contínuas em variedades compactas sem bordo e sem singularidades cônicas.

Por fim, para uma variedade cônica, estendemos as funções contínuas e C^1 para $\mathcal{C}^{0,\gamma}(\mathbb{B})$ e $\mathcal{C}^{1,\gamma}(\mathbb{B})$, e mostramos que operadores diferenciais elípticos definem operadores quase sectoriais nesses espaços.

Palavras-chave: Variedades cônicas, espaços Mellin-Sobolev, operadores pseudodiferenciais, operadores setoriais e quase setoriais.

Abstract

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We are going to present results about parabolic equations on conic manifolds using L_p and continuous functions. First, we show existence of global solutions for a typical Reaction-Diffusion equation on a conic manifold $\mathbb{B}$. More precisely, we study the equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta_{\mathbb{B}} u + u - u^q \quad \text{on} \quad \mathbb{B}^\circ, \\ u(0,x) &= u_0(x) \quad x \in \mathbb{B}^\circ. \end{aligned}$$

where q is odd. We consider Mellin-Sobolev spaces $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$. These are function spaces built using L_p norms.

Second, we show sectoriality for elliptic differential and pseudodifferential operators acting on continuous spaces on compact manifolds without boundary and without conical points.

Third, for a conic manifold $\mathbb{B}$, we extend the continuous and C^1 spaces to $\mathcal{C}^{0,\gamma}(\mathbb{B})$ and $\mathcal{C}^{1,\gamma}(\mathbb{B})$ and we show that elliptic differential operators define almost sectorial operators on these spaces.

Keywords: Conic manifolds, Mellin-Sobolev spaces, Pseudodifferential operators, Sectorial and almost sectorial operator.

Contents

Chapter 1

Introduction

The study of partial differential equations is very important, since many phenomena in science can be described by this mathematical tool, which tries to give solutions whenever they exist and their respective properties. There is a lot of material about such equations and their solutions by different methods. We want to contribute to parabolic equations on manifolds with conical points by a combination of semigroup theory and pseudodifferential operators that we will describe with more details later.

We recall that the parabolic equations are essential, because they model physical phenomena as heat transfer, diffusion and it has many applications in real life. Several authors had studied different ways to solve this type of problems on non-smooth manifolds and domains such as Kondratiev, Kozlov, Grisvard, Dauge, among others. In the last decades, important contributions have been given by schools such as the one led by B-W Schulze (see [11]), mainly using L_p spaces to solve elliptic and parabolic equations on manifolds with singular points or corners, which are the inspiration for this work.

Our main aim is to study solutions in different spaces based on L_p norms, Hölder and continuous spaces using techniques developed recently by [4] and [2] on conic manifolds. First, we show how to use analytic semigroup theory and results provided by pseudodifferential operators to obtain global solutions of a typical Reaction-Diffusion equation on conic manifolds using L_p spaces, more precisely Mellin-Sobolev spaces. Second, we show how to use techniques of pseudodifferential operators and estimates for integral operators in order to study the behaviour of the operators that compose the resolvent operators in the case of a manifold without singularities and a conic manifold.

Analytic semigroups have important applications to parabolic equations. Using mild solutions defined over the domain of fractional power of sectorial operators, Dlotko and Cholewa showed existence of global attractors under some conditions in the nonlinear term and their domains. For more details, see [6] and [10]. In order to use their technique we have used many tools or pre requisites as for example pseudodifferential calculus that is continuously giving important new results to partial differential equations and geometry. Relevant material that we used can be found in [1], [5], [14], [20], [21], [29], [31].

Inspired by Dore and Venni's work, which is based on the behaviour of imaginary powers and their use to regularity of evolution equations, E. Schrohe and J. Seiler have investigated the resolvent for Cone Differential operators which appears in a natural way on Manifolds with conical singularities. Their work was based on Sobolev spaces with weight, which they call Mellin-Sobolev spaces. One of our aims is to extend their results to Hölder and continuous spaces. For this purpose, we need to study the Pseudo-Differential Calculus as presented, for instance, by Y. V. Egorov and B-W. Schulze, see [11]. The pseudodifferential calculus is based on the Fourier transform. The idea is to built an approximation of the inverse operator called parametrix using the symbols that we will introduce later. Similarly and connected with Fourier transform, we have used the Mellin transform near to the conical points to describe a similar calculus with other types of symbols and parameters that depend on a sector domain on the complex plane. This calculus was developed by B.W. Schulze.

For simplicity, we emphasize that our computations are done using local coordinates on the manifold. We have tried to give more information and to explain easily details that can be difficult to read and understand in the papers. Besides, we give our contributions with new definitions and build concepts that perhaps can be used in future by us or others interested in this area. In a way, some of our results improve the ones obtained by Schrohe and Roidos ([23]) who have proved existence of local solutions of the Allen-Cahn equation, but have neither given conditions to obtain global solutions nor studied the dynamics of them. Next, we show how this material is organized.

In Chapter 2, we state our objectives and some important equations that we will study. We want to show the importance of this type of problems and look at them from different angles and to give directions that can be taken in the future. In particular, we present Sectorial operators, Linear Semigroups, some aspects about interpolation spaces, nonlinear semigroups and Pseudodifferential operators. Chapter 3 is concerned with the L_p theory and we give our first contribution. Here, we define the Conic Manifolds, we present results on extensions of unbounded operators defined by differential operators and a typical Reaction-Diffusion equation. Moreover, we prove existence of global solutions for that problem. In chapter 4, we work with Λ-Elliptic operators on BUC and BUC^1 spaces and we give applications for the case of differential and pseudodifferential operators on manifolds without singularities.

Finally, in Chapter 5 we define the typical continuous and Hölder spaces on conic manifolds and give our second main contribution. We describe the operators that appear in the structure of the resolvent of elliptic operators. Besides, we present the Mellin Differential and Pseudo-Differential Operators. Next, we show known results in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ spaces and the behaviour for the operators that compose the resolvent operator in $\mathcal{C}^{0,\gamma}(\mathbb{B})$ and $\mathcal{C}^{1,\gamma}(\mathbb{B})$. We prove that certain elliptic operators are almost sectorial in a conic manifold and we give an example for an equation with the Laplace operator in such spaces.

In a parallel work during the development of this thesis, I studied symmetric tensor power of graphs under the sponsorship of The American Institute of Mathematics (AIM). As a result, we have submited the pre print that can be found in `https://arxiv.org/pdf/2309.13741v1.pdf`.

Chapter 2

Objectives and Preliminary results

The main aim of this work is to study parabolic equations on manifolds with conical singularities. We introduce new tools to show global well-posedness for this type of equation in $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ spaces and we also prove well posedness in $\mathcal{C}^{0,\gamma}(\mathbb{B})$ spaces, where $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ and $\mathcal{C}^{0,\gamma}(\mathbb{B})$ are conical versions of L_p and BUC (Bounded Uniformly Continuous) spaces. In this work we:

1) Study semilinear parabolic equations of the type

$$u_t + Au = F(u),$$

where A can be a sectorial or almost sectorial operator and F is a nonlinear operator.

2) Introduce manifolds with a conic point and provide the most important information about them that are relevant to this work.

3) Introduce the Mellin Sobolev spaces, $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$.

4) Study semilinear parabolic equations using these spaces under suitable conditions.

5) Recall the definitions of BUC and Hölder spaces and we show that certain elliptic differential and pseudodifferential operators are sectorial in $C(M)$, where M is a compact manifold without singularities.

6) Introduce our new spaces $\mathcal{C}^{s,\gamma}(\mathbb{B})$ and we show that certain elliptic conic operators are almost sectorial in $\mathcal{C}^{0,\gamma}(\mathbb{B})$ and $\mathcal{C}^{1,\gamma}(\mathbb{B})$.

7) Apply our theory to parabolic equations to the new Hölder and continuous spaces $\mathcal{C}^{s,\gamma}(\mathbb{B})$.

Let $\mathbb{B}$ be a conic manifold that we will define in following chapter. We will consider the following semilinear heat equation on $\mathbb{B}$:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta_{\mathbb{B}} u + u - u^q, \quad x \in \mathbb{B}, t > 0, \\ u(0,x) &= u_0(x), \qquad\qquad x \in \mathbb{B}. \end{aligned} \tag{2.1}$$

In order to motivate our work, we first recall that a simple cone C on $\mathbb{R}^3$ can be defined by

$$C := \{(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi), r \geq 0, 0 < \theta \leq 2\pi\},$$

where $\phi \in (0, \frac{\pi}{2})$ is fixed. Let us see how is the solution for the equation given by

$$\begin{aligned} \Delta u(r,\theta) &= 0 \quad on\ C, \\ u(r,0) &= u(r,2\pi). \end{aligned} \tag{2.2}$$

Therefore, we are looking for periodic solutions in θ and we will proceed as follows. First, we express the Laplace-Beltrami operator in spherical coordinates. Recalling that the angle ϕ is constant, we see that

$$\Delta u(r,\theta) = \frac{1}{r^2}\left[\partial_r(r^2\partial_r u) + \frac{1}{\sin^2\phi}\partial_\theta^2 u\right].$$

If we note that $\partial_r(r^2\partial_r) = (r\partial_r)^2 + r\partial_r$ then (2.2) is equivalent to

$$\begin{aligned} \left[(r\partial_r)^2 + r\partial_r + \frac{1}{\sin^2\phi}\partial_\theta^2\right] u(r,\theta) &= 0, \\ u(r,0) &= u(r,2\pi). \end{aligned}$$

This problem can be transformed into an ODE of second order applying the Mellin transform (see Definition 5.2.5 in Section 5.2) at the variable r

$$\begin{aligned} \left[z^2 - z + \frac{1}{\sin^2\phi}\partial_\theta^2\right] \mathcal{M}u(z,\theta) &= 0, \\ \mathcal{M}u(z,0) &= \mathcal{M}u(z,2\pi). \end{aligned} \tag{2.3}$$

Later we will define the Mellin transform and show some of its properties. After some computations we find that the solutions of (2.3) are given by $\mathcal{M}u(z,\theta) = c_1\cos(k\theta) + c_2\sin(k\theta)$ with $k \in \mathbb{Z}$ and $z = \frac{1}{2} \mp \sqrt{\frac{1}{4} - \left(\frac{k}{\sin\phi}\right)^2}$. They can be expressed also as

$$\mathcal{M}u(z,\theta) = c_1\cos(|\sin\phi|\sqrt{z-z^2}\theta) + c_2\sin(|\sin\phi|\sqrt{z-z^2}\theta)\delta\left(z - \frac{1}{2} \mp \sqrt{\frac{1}{4} - \left(\frac{k}{\sin\phi}\right)^2}\right).$$

Therefore, by linearity the solution can be written as

$$\begin{aligned} &\mathcal{M}^{-1}\left[\sum_{k=1}^{l}\left(c_{1k}\cos(|\sin\phi|\sqrt{z-z^2}\theta) + c_{2k}\sin(|\sin\phi|\sqrt{z-z^2}\theta)\right)\delta\left(z - \frac{1}{2} \mp \sqrt{\frac{1}{4} - \left(\frac{k}{\sin\phi}\right)^2}\right)\right] \\ &= \sum_{k=1}^{l}\frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{2}}} t^{-z}c_{1k}\cos(|\sin\phi|\sqrt{z-z^2}\theta) + c_{2k}\sin(|\sin\phi|\sqrt{z-z^2}\theta)\delta\left(z - \frac{1}{2} \mp \sqrt{\frac{1}{4} - \left(\frac{k}{\sin\phi}\right)^2}\right) dz, \end{aligned}$$

where $\Gamma_{\frac{1}{2}} = \{z \in \mathbb{C} : \mathfrak{Re} z = \frac{1}{2}\}$. Then, after replacing $z = \frac{1}{2} + i\beta$, with $\beta = \mp i\sqrt{\frac{1}{4} - \left(\frac{k}{\sin\phi}\right)^2}$, we have that $u(r,\theta)$ is equal to

$$\sum_{k=1}^{l}\left(c_{1k}r^{\frac{1}{2}+i\sqrt{\frac{1}{4}-(\frac{k}{\sin\phi})^2}} + \tilde{c}_{1k}r^{\frac{1}{2}-i\sqrt{\frac{1}{4}-(\frac{k}{\sin\phi})^2}}\right)\cos(k\theta) + \left(c_{2k}r^{\frac{1}{2}+i\sqrt{\frac{1}{4}-(\frac{k}{\sin\phi})^2}} + \tilde{c}_{2k}r^{\frac{1}{2}-i\sqrt{\frac{1}{4}-(\frac{k}{\sin\phi})^2}}\right)\sin(k\theta). \tag{2.4}$$

The behaviour of this solution near the conical point depends on the space where we are looking for it. For example, the conic metric on this manifold is $dr^2 + r^2 \sin^2 \phi d^2\theta$ and if we want the solution with $r^{\frac{1}{2} \pm i\sqrt{\frac{1}{4} - (\frac{k}{\sin \phi})^2}}$ in $L_p(C \cap B(0,1))$, then we have to note that for k larger enough, $\frac{1}{4} - (\frac{k}{\sin\phi})^2 < 0$. In this case, the integral is

$$\int_0^1 \int_0^{2\pi} \left| r^{\frac{1}{2} \pm i\sqrt{\frac{1}{4} - (\frac{k}{\sin \phi})^2}} \cos\theta \right|^p r \sin\phi dr d\theta \leq 2\pi \int_0^1 \left| r^{1 + \frac{p}{2} \pm p\sqrt{-\frac{1}{4} + (\frac{k}{\sin\phi})^2}} \right| dr d\theta.$$

The last integral is finite, if and only if, $1 + \frac{p}{2} \pm p\sqrt{-\frac{1}{4} + \left(\frac{k}{\sin\phi}\right)^2} > -1$, which is equivalent to

$$2 + \frac{p}{2} \pm p\sqrt{-\frac{1}{4} + \left(\frac{k}{\sin\phi}\right)^2} > 0,$$

or equivalently,

$$k < \sin\phi \sqrt{\left(\frac{p+4}{2p}\right)^2 + \frac{1}{4}},$$

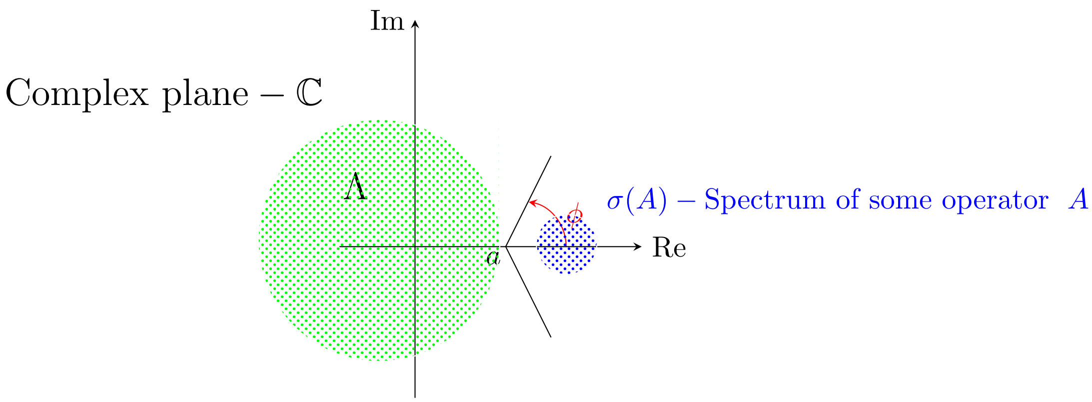
since $\sin\phi > 0$. The previous example is just a motivation without rigorous arguments to study the behaviour of solutions for equations on conic manifolds.

2.1 Sectorial and Almost Sectorial Operators

In this section, we define an important class of operators, the sectorial operators. From now on, for $a \in \mathbb{R}$, $\phi \in [0, 2\pi]$, we will denote by $\Lambda_a(\phi)$ the closed set in $\mathbb{C}$ defined by

$$\Lambda_a(\phi) := \{\lambda \in \mathbb{C} : \phi \leq |\arg(\lambda - a)|\}. \quad (2.5)$$

In principle, the sector depends on ϕ and a, but we just write Λ in order to simplify the notation.



In this section, X denotes a Banach space unless stated otherwise.

Definition 2.1.1. *For an operator $A : D(A) \subset X \to X$ the set*

$$\{\lambda \in \mathbb{C} : \overline{\mathfrak{Im}(\lambda I - A)}^{||.||_X} = X,\ (\lambda I - A)^{-1} \quad \textit{exist and is bounded on} \quad \mathfrak{Im}(\lambda I - A)\}$$

is called the resolvent of the operator A and is denoted by $\rho(A)$. The set $\sigma(A) := \mathbb{C}/\rho(A)$ is called the spectrum of A. For λ in $\rho(A)$, we write the operator $R(\lambda, A) := (\lambda I - A)^{-1}$, also called the resolvent operator. Here, $\mathfrak{Im}B$ denotes the range of the operator B.

Next, we define the class of operators that we need for our work. They are very important for the theory of parabolic equations.

Definition 2.1.2. *For a linear and closed defined operator $A : D(A) \subset X \to X$ we say that A is sectorial if there exist $a \in \mathbb{R}$, $\phi \in (0, \frac{\pi}{2})$ and M such that*

i) The resolvent $\rho(A)$ contains the sector Λ.

ii) $||(\lambda I - A)^{-1}|| \leq \frac{M}{|\lambda - a|}$ for each $\lambda \in \Lambda$.

Remark 2.1.1. *In this work we need to consider two cases: The first case is when the operator A satisfies that $\overline{D(A)}^{||\cdot||_X} = X$. This is the case for $X = L_p$ with $1 < p < \infty$. The second case is when $\overline{D(A)}^{||\cdot||_X} \subsetneq X$ as for example for $X = BUC, L_\infty$. Therefore, as we will see later we will work with analytic C^0 semigroups (see Definition 2.2.1) for the first case and with just analytic semigroups in the second case. In both cases, we can define the sectorial operators but we should be careful in which case we are working. In particular, we can also define sectorial operators without requiring that A is densely defined.*

Example 1. *If $A : D(A) \subset X \to X$ is a bounded operator, then A is sectorial.*

Proof. In fact, we have that $\{\lambda \in \mathbb{C} : |\lambda| > ||A||\} \subset \rho(A)$, because $\lambda I - A = \lambda(I - \frac{1}{\lambda}A)$ and $||I - (I - \frac{1}{\lambda}A)|| = ||\frac{1}{\lambda}A|| = \frac{1}{|\lambda|}||A|| < 1$. Then, the inverse of $(I - \frac{1}{\lambda}A)$ exists and, when $|\lambda| > ||A||$, it is given by

$$(\lambda I - A)^{-1} = \frac{1}{\lambda}(I - \frac{1}{\lambda}A)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{A^n}{\lambda^n}.$$

In particular, if $\Lambda \subset \{\lambda \in \mathbb{C} : |\lambda| > 2||A||\}$, then $\frac{||A||}{|\lambda|} < \frac{1}{2}$. Hence,

$$||\lambda R(\lambda, A)|| \leq \sum_{n=0}^{\infty}\left(\frac{||A||}{|\lambda|}\right)^n \leq \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 2$$

and A is a sectorial operator. □

Definition 2.1.3. *Let X be a Banach space, $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ and $\gamma \in (-1, 0)$. If*

$$||(\lambda I - \mathcal{A})^{-1}||_{\mathcal{L}(X)} \leq C_\phi |\lambda|^\gamma$$

for all $\lambda \in \Lambda_0(\phi)$, then we say that the operator $\mathcal{A}$ is almost sectorial.

2.2 C^0 Semigroups

From now on, V is a metric space and X is a Banach space.

Definition 2.2.1. *A one parameter family of maps $T(t) : V \to V$, $t \geq 0$, is called a C^0 semigroup if :*

i) $T(0)$ is the identity map of V.

ii) $T(t + s) = T(t)T(s)$, for all $t, s \geq 0$.

iii) The function

$$[0,\infty) \times V \ni (t,x) \to T(t)x \in V$$

is continuous at each point $(t,x) \in [0,\infty) \times V$*.*

The existence of a certain limit involving the semigroup $\{T(t)\}$ is important for the definition of the generator of a linear C^0 semigroup. So, we state the following definition.

Definition 2.2.2. *Let* $T(t) : X \to X$ *be a linear* C^0 *semigroup, that is, a* C^0 *semigroup such that* $T(t)$ *is a bounded linear operator for each* $t \geq 0$*. The linear operator* $A : D(A) \subset X \to X$ *defined by*

$$D(A) := \{x \in X : \lim_{t \to 0+} \frac{T(t)x - x}{t} \quad exists\}$$

and

$$Ax := \lim_{t \to 0+} \frac{T(t)x - x}{t} \quad for \quad x \in D(A)$$

is called the infinitesimal generator of a linear C^0 *semigroup* $\{T(t) : t \geq 0\}$*.*

We define the analytic semigroups complementing the definition 2.2.1 given before.

Definition 2.2.3. *Let* $\{T(t)\}$ *be a linear semigroup in* X*. We say that* $\{T(t)\}$ *is a* C^0 *analytic semigroup if there exists a sector* $\Lambda = \Lambda_a(\phi)$*, here* $a = 0$*, as we defined in* (2.5) *of the complex plane and linear operators* $T(z)$ *that match with* $T(t)$ *for* $t \geq 0$*, such that*

i) $z \to T(z)$ *is analytic in* Λ^c*.*

ii) $\lim_{z \to 0, z \in \Lambda^c} T(z)x = x$ *for every* $x \in X$*.*

iii) $T(z_1 + z_2) = T(z_1)T(z_2)$ *with* $z_1, z_2 \in \Lambda^c$*.*

The following important theorem characterize the generator of C^0 analytic semigroups and is very important for this work. Before, we introduce the definition of a curve γ on the sector $\Lambda_a(\phi)$ which will depend on this sector. Then, for $r > 0$ and $\eta \in (\theta, \pi/2)$, where θ is given by the sector Λ, we define

$$\gamma = \{\lambda \in \mathbb{C} : |arg(\lambda - a)| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |arg(\lambda - a)| \geq \eta, |\lambda| = r\} \tag{2.6}$$

oriented counterclockwise.

Theorem 2.2.1. *A densely defined linear operator* A *is a negative generator of an analytic semigroup* $\{T(t)\}$ *of bounded operators* $T(t) : X \to X$*,* $t \geq 0$*, if and only if* $-A$ *is a sectorial operator in* X *with sector* $\Lambda_a(\phi)$ *for* $a \in \mathbb{R}$ *and* $0 \leq \phi < \frac{\pi}{2}$*.*

Proof. See [16]. □

Remark 2.2.1. *Given a sectorial operator* $A : D(A) \to X$ *with sector* $\Lambda_a(\phi)$ *for* $a \in \mathbb{R}$ *and* $0 \leq \phi < \frac{\pi}{2}$*, we define* $T(t) : X \to X$ *by*

$$T(t)x = \frac{1}{2\pi i} \int_\gamma e^{-\lambda t} (\lambda - A)^{-1} x d\lambda, \tag{2.7}$$

where γ *is a curve on the sector* Λ *as* (2.6)*.* $T(t)$ *is also denoted by* $T(t) = e^{-tA}$*.*

Remark 2.2.2. *As noted before, it is possible to define analytic semigroups that are not C^0 analytic semigroups. In fact, if $A : D(A) \to X$ is a sectorial operator with sector $\Lambda_a(\phi)$ for $a \in \mathbb{R}$ and $0 \le \phi < \frac{\pi}{2}$, but $D(A)$ is not dense in X, then the operator defined by (2.7) satisfies:*

i) $z \in \tilde{\Lambda}^c \to T(z)$ *is analytic for some sector* $\tilde{\Lambda}$.

ii) $\lim_{z\to 0, z\in\tilde{\Lambda}^c} T(z)x = x \quad \leftrightarrow \quad x \in \overline{D(A)}^{||\cdot||_X}$.

iii) $T(z_1 + z_2) = T(z_1)T(z_2) \quad$ *for* $\quad z_1, z_2 \in \tilde{\Lambda}^c$.

2.3 Intermediate Spaces: Real and Complex interpolation and fractional powers

The interpolation theory tries to construct suitable families of intermediate interpolation spaces and to study their properties. The most well known families are the real and complex interpolation spaces. We present their most relevant information to study our problems in this work.

Let X, Y be two Banach spaces. The couple of Banach spaces (X, Y) is said to be an interpolation couple if both X and Y are continuously embedded in a Hausdorff topological vector space $\mathcal{V}$. In this case, $X \cap Y$ and $X + Y$ are linear subspaces of $\mathcal{V}$ endowed with the norms $||v||_{X\cap Y} = \max\{||v||_X, ||v||_Y\}$ and $||v||_{X+Y} = \inf_{x\in X, y\in Y: v=x+y}\{||x||_X + ||y||_Y\}$ respectively. If (X, Y) is an interpolation couple, an intermediate space is any Banach space E such that

$$X \cap Y \subset E \subset X + Y.$$

For the rest of this section, let (X, Y) be an interpolation couple and let $L(X)$ denote all bounded operators $T : X \to X$. We say that $T \in L(X) \cap L(Y)$ if $T : X + Y \to X + Y$ and $T_{|_X} \in L(X)$ and $T_{|_Y} \in L(Y)$.

Definition 2.3.1. *Let (X, Y) be an interpolation couple. An interpolation space is any intermediate space E such that for all $T \in L(X) \cap L(Y)$, the restriction of T to E belongs to $L(E)$.*

Definition 2.3.2. *For every $x \in X + Y$ and $t > 0$, set*

$$K(t, x, X, Y) := \inf_{x=a+b, a\in X, b\in Y} ||a||_X + t||b||_Y.$$

Remark 2.3.1. *We note that $K(1, x, X, Y) = ||x||_{X+Y}$ and for every $t > 0$, $K(t, \cdot, X, Y)$ defines an equivalent norm to $||\cdot||_{X+Y}$.*

Definition 2.3.3. *Let $0 < \theta < 1$, $1 \le p \le \infty$, and set*

$$\begin{cases} (X, Y)_{\theta,p} &= \{x \in X + Y : t \to t^{-\theta}K(t, x, X, Y) \in L_p((0, \infty), \frac{dt}{t})\}, \\ ||x||_{(X,Y)_{\theta,p}} &= ||t^{-\theta}K(t, x, X, Y)||_{L_p((0,\infty), \frac{dt}{t})} \end{cases} \tag{2.8}$$

Such spaces are called real interpolation spaces and are Banach spaces. There is a well established theory and many properties about these spaces are known, but we do not need to study all the theory for our project. For more details, we can see [21]. Another important definition, when $Y \subset X$ is (we refer to [3], Section I.2)

$$(X, Y)^0_{\theta,p} := \overline{Y}^{(X,Y)_{\theta,p}}$$

and it is known that, if $Y \subset X$, then by Proposition 1.17 of [21] we have

$$(X,Y)^0_{\theta,\infty} = \{x \in X : \lim_{t\to 0} t^{-\theta} K(t,x,X,Y) = 0\}.$$

Now we consider (X,Y) an interpolation couple of complex Banach spaces.

Definition 2.3.4. *Let S be the strip $\{z \in \mathbb{C} : 0 \le Re(z) \le 1\}$. We define $\mathcal{F}(X,Y)$ as the space of all functions $f : S \to X+Y$ such that:*

i) f is holomorphic in the interior of S, continuous and bounded up to its boundary, with values in $X+Y$.

ii) $t \to f(it) \in BC(\mathbb{R},X)$, $t \to f(1+it) \in BC(\mathbb{R},Y)$, and

$$||f||_{\mathcal{F}(X,Y)} = \max\left\{\sup_{t\in\mathbb{R}} ||f(it)||_X, \sup_{t\in\mathbb{R}} ||f(1+it)||_Y\right\} < \infty,$$

where $BC(\mathbb{R},X)$ is the set of bounded continuous functions.

Now, we define the complex interpolation spaces.

Definition 2.3.5. *Let $\theta \in [0,1]$. The space defined by*

$$\begin{cases} [X,Y]_\theta &= \{f(\theta), f \in \mathcal{F}(X,Y)\}, \\ ||a||_{[X,Y]_\theta} &= \inf_{f\in\mathcal{F}(X,Y), f(\theta)=a} ||f||_{\mathcal{F}(X,Y)}. \end{cases} \tag{2.9}$$

is the complex interpolation space between X and Y.

Remark 2.3.2. *For $\theta \in (0,1)$ we have that $X \cap Y \subset [X,Y]_\theta \subset X+Y$. The same for $(X,Y)_{\theta,p}$.*

For more properties and other important facts we can see [21]. Other intermediate spaces can be defined through fractional powers of sectorial operators as follows.

Definition 2.3.6. *Let $\alpha \in (0,+\infty)$ and $A : D(A) \to X$ be a sectorial operator with $\mathfrak{Re}\sigma(A) > 0$, is that, $\sigma(A) \subset \{z \in \mathbb{C} : \mathfrak{Re}(z) > 0\}$. Then, for all $v \in X$, we define the operators $A^{-\alpha} : X \to X$ by*

$$A^{-\alpha}v = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{-At} v dt,$$

where $\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx$ denotes the gamma function.

Under the condition of sectoriality and $\mathfrak{Re}\sigma(A) > 0$ it can be shown that $A^{-\alpha}$ is well defined and these operators are bounded linear operators in X which have inverses denoted by $A^\alpha := (A^{-\alpha})^{-1}$ and satisfy $A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)}$ with $\alpha, \beta > 0$. We will denote by X^α the domain for the operator A^α. Note that $X^\alpha = Rank(A^{-\alpha})$. We have an important relation for those spaces: if $\alpha \ge \beta$ then $X^\alpha \subset X^\beta$ is a dense and continuous inclusion. For more details, see [6].

2.4 Abstract Cauchy problems and Non-Linear Semigroups

In this work, we will deal with some problems that can be studied using the semigroups defined in Section 2.3. Next, we define the type of solutions that we will consider for parabolic equations, which is the subject of this work. For more details, one can consult [6]. In general, for a Banach space X, we call an abstract parabolic equation an expression of the form

$$\begin{aligned} u_t + Au &= F(u), \quad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{2.10}$$

where $A : D(A) \to X$ is a sectorial operator, $\mathfrak{Re}\sigma(A) > 0$ and $F : X^\alpha \to X$ is a Lipschitz continuous function on bounded subsets of X^α for some $\alpha \in [0, 1)$. Now, we present some important definitions. They will be important for the proof of the existence of a global attractor, the definition of a nonlinear semigroup and problems that we will study in the next sections.

Definition 2.4.1. *If $u_0 \in X^\alpha$ and for some real $\tau > 0$, $u \in C([0,\tau), X^\alpha) \cap C^1((0,\tau), X) \cap C((0,\tau), D(A))$ is such that (2.10) holds for all $t \in (0,\tau)$, then u is called a local X^α- solution of (2.10).*

Another type of solution that we will use later for the initial valued problem

$$\begin{aligned} u'(t) &= f(t, u(t)), \quad t \geq 0 \\ u(0) &= u_0 \end{aligned} \tag{2.11}$$

where $f : [0, +\infty) \times D \to X$ is a nonlinear function and D is a continuously embedded subspace of X is stated in the following definition.

Definition 2.4.2. *A function $u \in C^1([0,\tau); X) \cap C([0,\tau); D)$ that satisfies (2.11) for $0 \leq t < \tau$ is said to be a strict solution in $[0,\tau)$.*

Proposition 2.4.1. *Under the assumptions of (2.10), there exists always a local X^α- solution. We can always find a maximal local X^α-solution $u : [0, \tau_{u_0}) \to X^\alpha$ such all local solution with $u(0) = u_0$ is a restriction of the maximal solution. Finally, if $\sup_{t\in[0,\tau_{u_0})} ||u(t)||_{X^\alpha} < \infty$, then $\tau_{u_0} = \infty$.*

Proof. See Theorem 2.1.1 in [6].

Definition 2.4.3. *A solution u is called global X^α-solution if it fulfills all the requirements of Definition 2.4.1 with $\tau = +\infty$.*

If for each initial value $u_0 \in X^\alpha$ in (2.10) there is a global X^α - solution $u(t, u_0)$, we can define a nonlinear C^0 semigroup $\{T(t)\}$ by

$$T(t)u_0 = u(t, u_0) \quad \text{for} \quad t \geq 0. \tag{2.12}$$

Proposition 2.4.2. *If there exists a global X^α - solution of (2.10) for each $u_0 \in X^\alpha$, then the relation $T(t)u_0 = u(t, u_0)$ defines a nonlinear C^0 semigroup for $t \geq 0$.*

Proof. In effect, $T(0)u_0 = u(0, u_0) = u_0$ by the initial condition of (2.10) . For $t, s \geq 0$ we have that

$$T(t)T(s)u_0 = T(t)u(s, u_0).$$

On the other hand, if we consider

$$\begin{aligned} u_t + Au &= F(u), \quad t > 0, \\ u(0) &= u(s, u_0), \end{aligned} \tag{2.13}$$

then $u(t+s, u_0)$ is the unique X^α solution that in $t = 0$ is equal to $u(s, u_0)$. So, by the uniqueness of solutions we have that $T(t)u(s, u_0) = u(t+s, u_0) = T(t+s)u_0$, or, equivalently, $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$. The continuity of the function $[0, +\infty) \times X^\alpha \to X^\alpha$ is a consequence of Proposition 2.3.2 in [6]. □

Definition 2.4.4. *Let (V, d) be a metric space and $\mathcal{A} \subset V$ be a nonempty set. We say that $\mathcal{A}$ is a global attractor for a C^0 semigroup $\{T(t)\}$ if:*

i) $\mathcal{A}$ is compact.

ii) $T(t)\mathcal{A} = \mathcal{A}$. Here, $T(t)\mathcal{A} = \{T(t)x : x \in \mathcal{A}\}$.

iii) $\lim_{t\to\infty}(\sup_{b\in B} d(T(t)b, \mathcal{A})) = 0, \quad$ for all $B \subset V$ bounded, where $d(T(t)b, \mathcal{A}) = \inf_{a\in\mathcal{A}} d(T(t)b, a)$.

Let us consider two important conditions to show existence of global attractor that we need in this work:

($\mathbf{A}_1$) The relation (2.12) defines on X^α, corresponding to (2.10), a C^0 semigroup $\{T(t)\}$ of global X^α solutions having orbits of bounded sets bounded. This means that, if $B \subset X^\alpha$ is bounded then $\bigcup_{t\geq 0} T(t)B$ is bounded.

($\mathbf{A}_2$) It is possible to choose

- A Banach space Y, with $D(A) \subset Y$,
- A locally bounded function $c : [0, +\infty) \to [0, +\infty)$
- A nondecreasing function $g : [0, +\infty) \to [0, +\infty)$
- A certain number $\theta \in [0, 1)$, such that, for every $u_0 \in X^\alpha$, both conditions

$$||u(t, u_0)||_Y \leq c(||u_0||_{X^\alpha}), \quad \forall t \in (0, \tau_{u_0}),$$

and

$$||F(u(t, u_0))||_X \leq g(||u(t, u_0)||_Y)(1 + ||u(t, u_0)||^\theta_{X^\alpha}), \quad \forall t \in (0, \tau_{u_0})$$

hold, where τ_{μ_0} is the maximal interval of existence of the solution.

Theorem 2.4.1. *Under the assumptions of* (2.10) *the conditions* ($\boldsymbol{A}_1$) *and* ($\boldsymbol{A}_2$) *are equivalent.*

Proof. See Theorem 3.1.1 [6]. □

2.5 Operator and Function Theory in an open set $\Omega \subset \mathbb{R}^n$

Next, we introduce the basic spaces and operators necessary to study the main problems in this work. More information can be found in [20] or [6].

Definition 2.5.1. *Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ be an open set. The $L_p(\Omega)$ space is the set of all measurable functions $f : \Omega \to \mathbb{C}$ such that the integral $\int_\Omega |f(x)|^p d\mu(x) < \infty$, where $d\mu(x)$ denotes the lebesgue measure.*

Remark 2.5.1. *Actually the elements in $L_p(\Omega)$ are equivalence classes, where $f \sim g$ if and only if $f = g$ almost everywhere. As usual, we just write f and understand that we work with the equivalence class of f.*

Definition 2.5.2. *Let $1 \leq p < \infty$ and $m \in \mathbb{N}_0$. We define the Sobolev spaces by*

$$H_p^m(\Omega) = \{u \in L_p(\Omega) : D^\alpha u \in L_p(\Omega) \quad \forall \alpha \quad \textit{multi-index} : |\alpha| \leq m\}$$

More generally, we define

Definition 2.5.3. *Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then,*

$$H_p^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \mathfrak{F}^{-1}[1 + |\xi|^2]^{\frac{s}{2}} \mathfrak{F} f \in L_p(\mathbb{R}^n)\},$$

where $\mathcal{S}'(\mathbb{R}^n)$ denote the usual dual of the Schwartz space, called tempered distributions, and $\mathfrak{F}$ the Fourier transform.

In this work, we use the following convention:

$$\mathfrak{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} u(x)dx \quad \textit{and} \quad \mathfrak{F}^{-1}u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi} u(\xi)d\xi.$$

Below we show a result that we will use later.

For $-\infty < s_0, s_1 < +\infty$, $1 < p_0, p_1 < +\infty$, $0 < \theta < 1$, $s_0 \neq s_1$,

$$s = (1 - \theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}$$

we have by Theorem 1, section 2.4.2 (Page 185) from [31] that

$$[H_{p_0}^{s_0}(\mathbb{R}^n), H_{p_1}^{s_1}(\mathbb{R}^n)]_\theta = H_p^s(\mathbb{R}^n).$$

2.6 Pseudodifferential Operators

In this section, we are going to present some tools that are necessary to the study of the differential operators that appear in this work. We state the most important part without going into great detail. For more details, see for example [1], [5], [21].

Let us start with the subject of pseudodifferential operators on $\mathbb{R}^n$. In this section, we use $D_{x_j} = \frac{1}{i}\partial_{x_j}$ with i the complex number such that $i^2 = -1$. Moreover, we say that $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index. Here, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$.

Definition 2.6.1. *Let $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ be a multi-index. For a linear operator $P = \sum_{|\alpha| \leq m} c_\alpha D_x^\alpha$ with constant coefficients, the function $p(\xi) = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha$ is called the symbol of P.*

Example 2. *Let* Δ *be the laplacian operator. If we recall that* $\partial_{x_j} = iD_{x_j}$ *and define*

$$\alpha_i = (0, \ldots, 0, \underset{\uparrow_{i^{th}\text{-entry}}}{2}, 0, \ldots, 0),$$

then

$$\Delta = \sum_{i=1}^{n} \partial_{x_i^2} = \sum_{i=1}^{n} (iD_{x_i})^2 = -\sum_{i=1}^{n} D_x^{\alpha_i}$$

Therefore, its symbol is $-|\xi|^2$.

Definition 2.6.2. *Let* $m \in \mathbb{R}$ *and* $n \in \mathbb{N}$. *Then* $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ *is the vector-space of all smooth functions* $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ *such that*

$$|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq c_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|},$$

for all $\alpha, \beta \in \mathbb{N}^n$, *where* $c_{\alpha,\beta}$ *is independent of* $x, \xi \in \mathbb{R}^n$. *The function* p *is called a pseudo-differential symbol and* m *is called the order of* p. *Moreover,*

$$S^\infty(\mathbb{R}^n \times \mathbb{R}^n) = \cup_{m\in\mathbb{R}} S^m(\mathbb{R}^n \times \mathbb{R}^n) \text{ and}$$

$$S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n) = \cap_{m\in\mathbb{R}} S^m(\mathbb{R}^n \times \mathbb{R}^n).$$

Remark 2.6.1. *If* $p \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ *is a symbol, then*

$$p(x, D_x)f(x) := op(p)f(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x, \xi)\mathfrak{F}(f)(\xi)d\xi, \text{ for all } x \in \mathbb{R}^n,$$

defines the associate pseudodifferential operator, where $f \in \mathcal{S}(\mathbb{R}^n)$ *(Schwartz space).*

The pseudo differential operators are continuous on Sobolev spaces, as can be seen below.

Theorem 2.6.1. *Let* $p \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ *be a symbol,* $1 < q < \infty$ *and* $s \in \mathbb{R}$. *Then* $p(x, D_x)$ *extends to a bounded linear operator*

$$p(x, D_x) : H_q^{s+m}(\mathbb{R}^n) \to H_q^s(\mathbb{R}^n).$$

Proof. See Theorem 5.20 [1]. □

Definition 2.6.3. *Let* $p \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ *and* $m \in \mathbb{R}$. *A symbol* p *is called elliptic if there are* $C, R > 0$ *such that*

$$|p(x, \xi)| \geq C|\xi|^m,$$

for all $|\xi| \geq R$ *and* $x \in \mathbb{R}^n$.

Example 3. *We have already seen that* $p(\xi) = -|\xi|^2$ *is the symbol of the Laplacian operator. Therefore, for* $C = 1$ *we have that* $|p(\xi)| \geq |\xi|^2$, *for all* $\xi \in \mathbb{R}^n$. *Hence, we can choose any* $R > 0$. *We conclude that* p *is an elliptic symbol of order 2.*

The following two results are classical and necessary to this work.

Theorem 2.6.2. *Let $p \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$. Then the following conditions are equivalent.*

1. *p is elliptic.*

2. *There is some $q \in S^{-m}(\mathbb{R}^n \times \mathbb{R}^n)$ such that*

$$p(x, D_x)q(x, D_x) = I + r_1(x, D_x)$$

and

$$q(x, D_x)p(x, D_x) = I + r_2(x, D_x)$$

where $r_1, r_2 \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$.

Proof. See Theorem 3.24 in [1]. □

Theorem 2.6.3. *Let $p \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ be elliptic, $m \in \mathbb{R}$, and let $1 < q < \infty$. Moreover, let $u \in H^r_q(\mathbb{R}^n)$ be a solution of*

$$p(x, D_x)u = f,$$

for some $f \in H^s_q(\mathbb{R}^n)$, where $r, s \in \mathbb{R}$. Then $u \in H^{s+m}_q(\mathbb{R}^n)$. Moreover, there is some constant $C_{r,s,q} > 0$ independent of u and f such that

$$||u||_{H^{s+m}_q(\mathbb{R}^n)} \leq C_{r,s,q} \left(||f||_{H^s_q(\mathbb{R}^n)} + ||u||_{H^r_q(\mathbb{R}^n)} \right).$$

Proof. See Theorem 7.13 in [1]. □

Another important result for two symbols is stated below.

Proposition 2.6.1. *Let $p_j \in S^{m_j}(\mathbb{R}^n \times \mathbb{R}^n)$ be two pseudo-differential symbols, with $j = 1, 2$. Then there is some $r \in S^{m_1+m_2-1}(\mathbb{R}^n \times \mathbb{R}^n)$ such that*

$$[p_1(x, D_x), p_2(x, D_x)] = r(x, D_x),$$

where $[A, B] := AB - BA$ denotes the commutator of two operators.

Proof. See Corollary 3.17 [1]. □

The following result is showed in Theorem 7.16 [1]. Here, we enunciate and provide its proof. Later we will use this result for an important remark. (see Remark 3.2.1)

Theorem 2.6.4. *Let $p \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, be elliptic, $s \in \mathbb{R}$ and let $1 < q < \infty$. Moreover, let $u \in \cup_{r\in\mathbb{R}} H^r_q(\mathbb{R}^n)$ be a solution of*

$$p(x, D_x)u = f,$$

for some $f \in \bigcup_{r\in\mathbb{R}} H^r_q(\mathbb{R}^n)$. Moreover, assume that there is some $g \in H^s_q(\mathbb{R}^n)$ such that f and g coincides on some open set $U \subseteq \mathbb{R}^n$. Then for every open bounded set V with $\overline{V} \subset U$ there is some $v \in H^{s+m}_q(\mathbb{R}^n)$ such that u and v coincide on V.

Proof. Let us suppose that $f \in H^r_q(\mathbb{R}^n)$ for some $r \in \mathbb{R}$. By Theorem 2.6.3, we know that $u \in H^{r+m}_q(\mathbb{R}^n) \hookrightarrow H^r_q(\mathbb{R}^n)$. The statement of the theorem follows from: For all $k \in \mathbb{N}_0$ and every bounded open set V with $\overline{V} \subset U$ there is some $v \in H^{\min\{s+m,r+k\}}_q(\mathbb{R}^n)$ such that u and v coincide on V. We note

that if the previous claim is true, is enough to find some $k \in \mathbb{N}_0$ such that $\min\{s+m, r+k\} = s+m$ hence $v \in H_q^{s+m}(\mathbb{R}^n)$ in that case.
The proof of the claim can be showed by induction. In effect, for $k = 0$, with $v := u \in H_q^r(\mathbb{R}^n) \subset H_q^{\min\{s+m,r\}}(\mathbb{R}^n)$. Let us suppose that the claim is valid for some $k \neq 0$. We will prove for $k+1$. We have two options: If $r+k \geq s+m$ then $\min\{s+m, r+k+1\} = \min\{s+m, r+k\} = s+m$ which implies that $v \in H_q^{\min\{s+m,r+k+1\}}(\mathbb{R}^n) = H_q^{\min\{s+m,r+k\}}(\mathbb{R}^n)$ and we are done. If $s+m > r+k$ we take $\psi \in C_c^\infty(\mathbb{R}^n)$ a function such that $\psi = 1$ on $\overline{V}$ and $\operatorname{supp}\psi \subset U$ and $v = \psi u$. Then, we notice that for $v = \psi u$, we have

$$p(x,D)\psi u = \psi p(x,D)u + p(x,D)\psi u - \psi p(x,D)u = \psi f + [p(x,D),\psi](u).$$

By Proposition 2.6.1, $[p(x,D),\psi] \in S^{m-1}(\mathbb{R}^n \times \mathbb{R}^n)$, because $p(x,D) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $\psi \in C_c^\infty(\mathbb{R}^n) \subset S^0(\mathbb{R}^n \times \mathbb{R}^n)$ (ψ do not depend of $\xi \in \mathbb{R}^n$). Hence, since $g \in H_q^s(\mathbb{R}^n)$ coincides with f on U and $u \in H_q^{r+k}(\mathbb{R}^n)$ then by Theorem 2.6.1 we have that $\psi f + [p(x,D),\psi](u) = \psi g + [p(x,D),\psi](u) \in H_q^s(\mathbb{R}^n) + H_q^{r+k-m+1}(\mathbb{R}^n) \subset H_q^{\min\{s,r+k-m+1\}}(\mathbb{R}^n)$. Then, if we apply Theorem 2.6.3 for v then $v \in H_q^{\min\{s+m,r+k+1\}}(\mathbb{R}^n)$. Hence, the statement is valid for $k+1$ and, by induction, we conclude the claim. Finally, for some k we have that $\min\{s+m, r+k\} = s+m$. □

Remark 2.6.2. *The previous facts can be used to obtain the analogous results on manifolds using a partition of unity and an atlas.*

There are many more general symbol classes. Below, we show two of them, which we need for this work.

Definition 2.6.4. *Let $\mu, d \in \mathbb{R}$. The space of symbols of order μ and anisotropy d,*

$$S^{\mu,d}(\mathbb{R}_y^m \times \mathbb{R}_\eta^n \times \Lambda),$$

consists of all functions $a \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \Lambda)$, which fulfill the estimates

$$|\partial_y^\beta \partial_\eta^\alpha \partial_\lambda^\gamma a(y,\eta,\lambda)| \leq C_{\alpha\beta\gamma}(1 + |\eta|^2 + |\lambda|^{\frac{2}{d}})^{\frac{\mu-|\alpha|-d|\gamma|}{2}},$$

for all multi-indices $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}_0^m$ and $\gamma \in \mathbb{N}_0^2$, where Λ denotes a complex sector in $\mathbb{C}$.

Definition 2.6.5. *Let $\gamma, \mu \in \mathbb{R}$. The space $MS^\mu(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{n+1}{2}-\gamma} \times \mathbb{R}^n)$ consists of all functions $a \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n \times \Gamma_{\frac{n+1}{2}-\gamma} \times \mathbb{R}^n)$ which satisfy the estimates*

$$|\partial_\tau^l (x\partial_x)^k \partial_\xi^\alpha \partial_y^\beta a(x,y,\frac{n+1}{2} - \gamma + i\tau, \xi)| \leq C_{kl\alpha\beta}(1 + \tau^2 + |\xi|^2)^{\frac{-\mu-l-|\alpha|}{2}}$$

for all $l, k \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^n$, where $\Gamma_\sigma = \{z \in \mathbb{C} : Re(z) = \sigma\}$.

Chapter 3

L_p Theory

In this chapter we study parabolic equations on a conic manifold. In order to do this, we first define manifolds with conical singularities (see definition below). Then the important function spaces for the applications are studied. We will use the theory presented in the following chapter.

3.1 Conic Manifolds

In our work, a conic manifold is defined as a pair $\mathbb{B} = (\mathcal{B}, g)$, where:

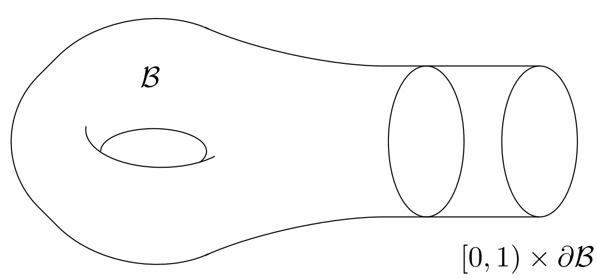
1) $\mathcal{B}$ is a smooth $(n+1)$ dimensional, compact manifold with boundary $\partial\mathcal{B}$.

2) g is a Riemannian metric on $\mathcal{B}\backslash\partial\mathcal{B}$.

3) In some collar neighborhood of $\partial\mathcal{B}$, $[0,1) \times \partial\mathcal{B}$,

$$g = dx^2 + x^2 h(x),$$

where $h(x)$, with $x \in [0,1)$ is a family of Riemannian metrics on $\partial\mathcal{B}$ that is smooth and does not degenerate in $x = 0$.



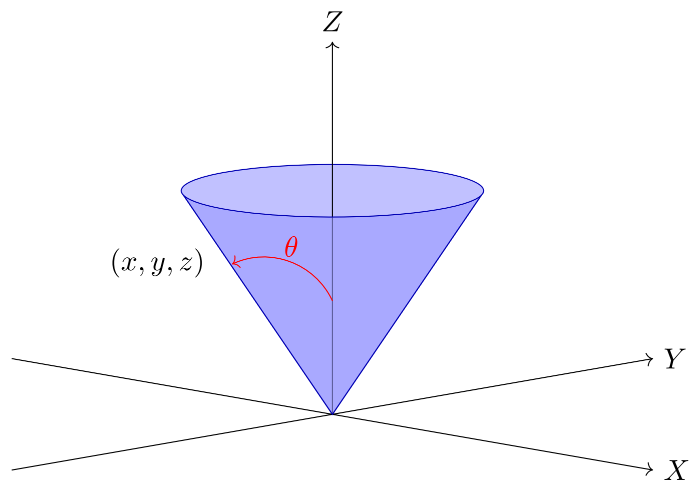
The set $\mathbb{B} \setminus \partial\mathbb{B}$ with the metric g is also denoted by $\mathbb{B}^\circ$.

The particular metric structure of these manifolds appears frequently. For instance, consider the following easy example.

Example 4. *Consider the cone given by*

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2 \sin^2 \theta, \quad z \geq 0\},$$

where θ is fixed.



Let us consider the vector (x, y, z) with size r, θ the angle between Z-axis and the vector (x, y, z) and ϕ the angle between X-axis and the projection of the vector (x, y, z) on the plane XY. Hence, we have the following relations:

$x^2 + y^2 + z^2 = r^2$,

$\sin \theta = \frac{\sqrt{x^2+y^2}}{r}$,

$\cos \theta = \frac{z}{r}$,

$\cos \phi = \frac{x}{\sqrt{x^2+y^2}}$,

$\sin \phi = \frac{y}{\sqrt{x^2+y^2}}$.

We note that if $G(r, \phi) = (x(r, \phi), y(r, \phi), z(r, \phi))$ represents the parametrization of the cone, then

$$g = \left\langle \frac{\partial G}{\partial r}, \frac{\partial G}{\partial r} \right\rangle dr^2 + \left\langle \frac{\partial G}{\partial r}, \frac{\partial G}{\partial \phi} \right\rangle dr d\phi + \left\langle \frac{\partial G}{\partial \phi}, \frac{\partial G}{\partial r} \right\rangle d\phi dr + \left\langle \frac{\partial G}{\partial \phi}, \frac{\partial G}{\partial \phi} \right\rangle d\phi^2.$$

From our relations above, we have that $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$. Moreover, we have

$$dx = \sin \theta (dr \cos \phi - r \sin \phi d\phi), \quad dy = \sin \theta (dr \sin \phi + r \cos \phi d\phi), \quad dz = dr \cos \theta.$$

Finally, we get that the metric on the cone is given by the relation

$$g = dx^2 + dy^2 + dz^2 = dr^2 + r^2 \sin^2 \theta d\phi^2,$$

and we can see that $\langle \frac{\partial G}{\partial r}, \frac{\partial G}{\partial r} \rangle = 1$, $\langle \frac{\partial G}{\partial r}, \frac{\partial G}{\partial \phi} \rangle = 0 = \langle \frac{\partial G}{\partial \phi}, \frac{\partial G}{\partial r} \rangle$ and $\langle \frac{\partial G}{\partial \phi}, \frac{\partial G}{\partial \phi} \rangle = r^2 \sin^2 \theta$.

We could associate a cone to a noncompact conic manifold. In this case, the conic manifold would be $\mathbb{B} = [0,1) \times S^1$, where $S^1 \subset \mathbb{R}^2$ is the unit circle. Here, $\partial\mathbb{B}$ is $\{0\} \times S^1$ and the metric is $g = dr^2 + r^2 \sin^2 \theta d\phi^2$, where $r \in [0, \infty)$ and $\phi \in S^1$. The family of metrics on S^1 is $h(r) = \sin^2 \theta d\phi^2$. Note that the conical tip of the cone is associate to $\partial\mathbb{B}$.

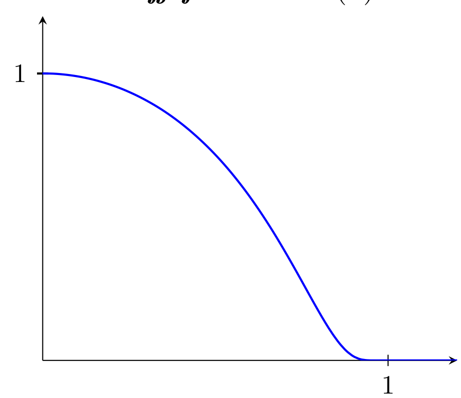
Definition 3.1.1. *Let $\mathbb{B} = (\mathcal{B}, g)$ be a conic manifold, $s \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$ and $1 < p < \infty$. Then $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ denotes the space of all distributions $u \in H^s_{p,loc}(\mathbb{B}^\circ)$ such that*

$$x^{\frac{n+1}{2}-\gamma}(x\partial_x)^k \partial_y^\alpha (\omega u)(x,y) \in L_p\left([0,1) \times \partial\mathcal{B}, \sqrt{\det[h(x)]}\frac{dx}{x}dy\right) \quad \forall k \in \mathbb{N}_0, \forall \alpha \in \mathbb{N}_0^n : k + |\alpha| \le s$$

for some cutoff function $\omega \in C^\infty([0,1))$. Here y belongs to a local chart of $\partial\mathcal{B}$ and $x \in [0,1)$. The space $[0,1) \times \partial\mathcal{B}$ is identified with a collar neighborhood of $\mathcal{B}$. Moreover, the definition of $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ is independent of ω.

$$\textit{Here, } \omega(x) = \begin{cases} 1 \quad if \quad x \quad \textit{is close to} \quad 0, \\ \\ 0 \quad if \quad |x| \ge 1 - \epsilon, \epsilon \in (0,1). \end{cases}$$

cutoff function $\omega(x)$



To extend the definition for $s \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, we consider

$$M_\gamma : C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^n) \to C_c^\infty(\mathbb{R}^{n+1})$$

defined by

$$M_\gamma u(x,y) = e^{(\gamma - \frac{n+1}{2})x} u(e^{-x}, y).$$

Furthermore, we take a covering $k_i : U_i \subset \partial\mathcal{B} \to \mathbb{R}^n$ with $i = 1, \ldots, N$ and $N \in \mathbb{N}\backslash\{0\}$ of $\partial\mathcal{B}$ by coordinate charts and let $\{\phi_i\}_{i=1\ldots,N}$ be a subordinated partition of unity. For any $s \in \mathbb{R}$ and $p \in (1, \infty)$, $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ is the space of all distributions u on $\mathbb{B}^o$ such that

$$||u||_{\mathcal{H}_p^{s,\gamma}(\mathbb{B})} = \sum_{i=1}^N ||M_\gamma(1 \otimes k_i)_*(\omega\phi_i u)||_{H_p^s(\mathbb{R}^{n+1})} + ||(1-\omega)u||_{H_p^s(\mathbb{B})}$$

is defined and finite, where $*$ refers to the push-forward of distributions. The space $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ is called Mellin-Sobolev space and is independent of the choice of the cutoff function, the covering $\{k_i\}_{i=1,\ldots,N}$

and the partition of unity $\{\phi_i\}_{i=1,\ldots,N}$. From now on, we will not use the term $\sqrt{\det[h(x)]}$ because our neighborhood is compact, so those terms are bounded by above and below. Hence the norms are equivalent.
In the particular case when $s=0$, we have

$$\mathcal{H}_p^{0,\gamma}(\mathbb{B}) = \left\{ u \in L_{p,loc}(\mathbb{B}^\circ) : x^{\frac{n+1}{2}-\gamma}(\omega u)(x,y) \in L_p\left([0,1)\times\partial\mathcal{B}, \frac{dx}{x}dy\right)\right\}.$$

Note that

$$x^{\frac{n+1}{2}-\gamma}(\omega u)(x,y) \in L_p\left([0,1)\times\partial\mathcal{B}, \frac{dx}{x}dy\right) \leftrightarrow \int_{\partial\mathcal{B}}\int_0^1 |x^{\frac{n+1}{2}-\gamma}(\omega u)(x,y)|^p \frac{dx}{x}dy < \infty.$$

The norm of $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ is given by

$$||u||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} = \sum_{i=1}^{N} ||M_\gamma(1\otimes k_i)_*(\omega\phi_i u)||_{L_p(\mathbb{R}^{n+1})} + ||(1-\omega)u||_{L_p(\mathbb{B})}.$$

For the above definitions, we have followed [6] where we can find more information. In particular, using the measure induced by g we have

$$L_2(\mathbb{B}) = \mathcal{H}_2^{0,0}(\mathbb{B}) \quad \text{and} \quad L_p(\mathbb{B}) = \mathcal{H}_p^{0,\gamma_p}(\mathbb{B})$$

with $\gamma_p = (n+1)(\frac{1}{2}-\frac{1}{p})$.

Remark 3.1.1. *For clarity we remark that the push forward is evaluate as below:*

$$\begin{aligned} M_\gamma\left((1\otimes k_i)_*(\omega\phi_i u)(x,y)\right) &= M_\gamma(\omega\phi_i(u(1\otimes k_i)(x,y))) \\ &= M_\gamma(\omega\phi_i(k_i^{-1}(y))u(x,k_i^{-1}(y))) \\ &= e^{(\gamma-\frac{n+1}{2})x}\omega(e^{-x})\phi_i(k_i(y))u(e^{-x},k_i^{-1}(y)). \end{aligned}$$

We collect some important properties of Mellin-Sobolev spaces that we will use later. Their proof can be found in [19]. For $1<p<\infty$, we have that:

i) If $q\geq p$, $s \geq t+(n+1)(\frac{1}{p}-\frac{1}{q})$ and $\gamma_1\geq\gamma_2$, then $\mathcal{H}_p^{s,\gamma_1}(\mathbb{B}) \hookrightarrow \mathcal{H}_q^{t,\gamma_2}(\mathbb{B})$.

ii) If $q\leq p$, $s\geq t\geq 0$ and $\gamma_1>\gamma_2$, then $\mathcal{H}_p^{s,\gamma_1}(\mathbb{B}) \hookrightarrow \mathcal{H}_q^{t,\gamma_2}(\mathbb{B})$.

iii) (Green's identity) If w and v belong to $\mathcal{H}_2^{1,1}(\mathbb{B})\oplus\mathbb{C}_\omega$ and $\Delta v\in\mathcal{H}_2^{0,\gamma}(\mathbb{B})$ for some $\gamma>-1$. Then,

$$\int_{\mathbb{B}} \langle\nabla w,\nabla v\rangle_g d\mu_g = -\int_{\mathbb{B}} w\Delta v d\mu_g.$$

Where $\mathbb{C}_\omega$ is a finite dimensional space of functions that are locally constants close to singularities. They are of the form $\sum_{i=1}^M c_i\omega_i$, where $c_i\in\mathbb{C}$, M is the number of connect components of $\partial\mathcal{B}$ and ω_i is the restriction of ω to each of those connect components. Recall that $\omega:[0,1)\times\partial\mathcal{B}\to\mathbb{R}$ is the cut off function described before. In particular, when we work with only one singularity, that is, $\partial\mathcal{B}$ is connected, then we denote this space as $\mathbb{C}$. Besides, $\langle\cdot,\cdot\rangle_g$ and $d\mu_g$ denote, respectively, the Riemannian scalar product and the Riemannian measure with respect to the metric g. The gradient associate to the metric g is denoted by ∇.

Remark 3.1.2. *If $\gamma < \frac{n+1}{2} - 1$ then $\mathbb{C} \hookrightarrow \mathcal{H}_p^{1,1+\gamma}(\mathbb{B})$. In fact, if u is a constant function different from 0, then it belongs to $\mathcal{H}_p^{1,1+\gamma}(\mathbb{B})$ if and only if*

$$\int_{\partial\mathcal{B}}\int_0^1 |x^{\frac{n+1}{2}-\gamma-1}|^p \frac{dx}{x} dy = \int_{\partial\mathcal{B}}\int_0^1 x^{(\frac{n+1}{2}-\gamma-1)p-1} dxdy < \infty,$$

and this happens if, and only if, $\frac{n+1}{2} - \gamma - 1 > 0$. Hence, those values of γ imply $\mathcal{H}_p^{1,1+\gamma}(\mathbb{B}) \oplus \mathbb{C} = \mathcal{H}_p^{1,1+\gamma}(\mathbb{B})$, for all $p \in (1,\infty)$.

The following important result will be needed later.

Theorem 3.1.1. *Let $\mathbb{B}$ be a conic manifold of dimension $n+1$. If $s > \frac{n+1}{p}$ and $\gamma > \frac{n+1}{2}$ then $\mathcal{H}_p^{s,\gamma}(\mathbb{B}) \subset C(\mathbb{B})$. Moreover, $|u(x,y)| \leq Cx^{\gamma-\frac{n+1}{2}}$.*

Proof. By the Collar Theorem (see Theorem 3.42. [15]), we can identify the neighborhood of $\partial\mathcal{B}$ with $[0,1) \times \partial\mathcal{B}$ and let

$$\omega : \mathbb{B} \to [0,1],$$

be the scalar C^∞ function such that $\omega = 1$ on an open set that contains $\partial\mathcal{B}$ and 0 outside of the collar neighborhood. We recall the function

$$M_\gamma : C_c^\infty(\mathbb{R}_+^{n+1}) \to C_c^\infty(\mathbb{R}^{n+1}),$$

where $\mathbb{R}_+^{n+1} = \mathbb{R}_+ \times \mathbb{R}^n$, defined by

$$M_\gamma u(x,y) = e^{(\gamma-\frac{n+1}{2})x} u(e^{-x}, y).$$

The norm is given by

$$||u||_{\mathcal{H}_p^{s,\gamma}(\mathbb{B})} = \sum_{i=1}^N ||M_\gamma(1\otimes k_i)_*(\omega\phi_i u)||_{H_p^s(\mathbb{R}^{n+1})} + ||(1-\omega)u||_{H_p^s(\mathbb{B})}.$$

Note that, $(1-\omega)u$ is smooth with compact support, and by the Sobolev Embedding Theorem, (see Proposition 1.2.1, [6]) if $s > \frac{n+1}{p}$, then $H_p^s(\mathbb{R}^{n+1}) \subset BUC(\mathbb{R}^{n+1})$. The only problem is the factor ωu. However, by the push-forward for distributions, we have

$$\begin{aligned}
|M_\gamma\left[(1\otimes k_i)_*(\omega\phi_i u)(x,y)\right]| &= |M_\gamma(\omega\phi_i(u(1\otimes k_i)(x,y)))| \\
&= |M_\gamma(\omega\phi_i u(x, k_i^{-1}(y)))| \\
&= |e^{(\gamma-\frac{n+1}{2})x}\omega(e^{-x})\phi_i(k_i^{-1}(y))u(e^{-x}, k_i^{-1}(y))| < C.
\end{aligned}$$

Therefore, $M_\gamma\left[(1\otimes k_i)_*(\omega\phi_i u)(x,y)\right]$ is continuous and bounded. If x is big enough then e^{-x} is small (<1). This implies that $\omega(e^{-x}) = 1$. So, inserting the term $e^{(\gamma-\frac{n+1}{2})x}$ on the right side, we obtain

$$|\phi_i(k_i^{-1}(y))u(e^{-x}, k_i^{-1}(y))| < Ce^{(\gamma-\frac{n+1}{2})(-x)}.$$

In this case, when $(x,y) \in [0,1) \times \partial\mathcal{B}$, the above inequality implies

$$|\phi_i(k_i^{-1}(y))u(x, k_i^{-1}(y))| < Cx^{\gamma-\frac{n+1}{2}}.$$

Finally, if x is positive, we can see that

$$\lim_{x\to 0^+} x^{\gamma-\frac{n+1}{2}} = 0 \quad \text{when} \quad \gamma - \frac{n+1}{2} > 0. \quad \left(\text{or} \quad \gamma > \frac{n+1}{2}\right)$$

That is, $\lim_{x\to 0} u(x,y) = 0$, or, equivalently, u is continuous on $\mathbb{B}$. □

The natural differential operators acting on the above spaces are known as conical operators.

Definition 3.1.2. *A cone differential operator of order $\mu \in \mathbb{N}_0$ is an μ-th order differential operator A with smooth coefficients in the interior $\mathbb{B}^{\circ}$ of $\mathbb{B}$ such that, when it is restricted to the collar part $[0,1) \times \partial\mathcal{B}$, it admits the following local form*

$$A = x^{-\mu} \sum_{k=0}^{\mu} a_k(x)(-x\partial_x)^k, \quad \text{where} \quad a_k \in C^{\infty}([0,1); \mathit{Diff}^{\,\mu-k}(\partial\mathcal{B})).$$

We note that, locally, $a_k(\cdot) = \sum_{|\alpha| \leq \mu-k} a_{k\alpha}(\cdot, y)\partial_y^{\alpha}$. Hence, $A = x^{-\mu} \sum_{k=0}^{\mu} \sum_{|\alpha|\leq \mu-k} a_{k\alpha}(x,y)(-x\partial_x)^k \partial_y^{\alpha}$, with $x \in [0,1)$ and y in a local chart of $\partial\mathcal{B}$. These operators are also known as Operators of Fuchs type.

Example 5. *Let us see how is the Laplace-Beltrami on a collar neighborhood $[0,1) \times \partial\mathcal{B}$ with a metric given by*

$$g = dx^2 + x^2 h(x),$$

where $x \in [0,1) \to h(x)$ is a family of Riemannian metrics. We know that $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & x^2(h_{ij}) \end{pmatrix}$ with inverse matrix given by $(g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & x^{-2}(h^{ij}) \end{pmatrix}$.

We denoted $G = |\det(g_{ij})|^{\frac{1}{2}} = |\det x^2(h_{ij})|^{\frac{1}{2}} = x^n H$, where $H := |\det(h_{ij})|^{\frac{1}{2}}$.

Now, let $k, j = 0, 1, \ldots, n$, where $k = j = 0$ is reference for the variable $x_0 = x$, and $x_i = y_i$ if $i \geq 1$. Then, we have

$$\begin{aligned}
\Delta &= G^{-1} \sum_{k=0}^{n} \frac{\partial}{\partial x_k} \left\{ \sum_{i=0}^{n} g^{ik} G \frac{\partial}{\partial x_i} \right\} \\
&= G^{-1} \frac{\partial}{\partial x}(G \frac{\partial}{\partial x}) + G^{-1} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left\{ \sum_{i=1}^{n} g^{ik} G \frac{\partial}{\partial x_i} \right\} \\
&= x^{-n} H^{-1} \frac{\partial}{\partial x}(x^n H \frac{\partial}{\partial x}) + x^{-n} H^{-1} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left\{ \sum_{i=1}^{n} x^{-2} h^{ik} x^n H \frac{\partial}{\partial x_i} \right\} \qquad (3.1) \\
&= x^{-n} H^{-1} \left(n x^{n-1} H \frac{\partial}{\partial x} + x^n \left(\frac{\partial H}{\partial x} \frac{\partial}{\partial x} + H \frac{\partial^2}{\partial x^2} \right) \right) + x^{-2} H^{-1} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left\{ \sum_{i=1}^{n} h^{ik} H \frac{\partial}{\partial x_i} \right\} \\
&= n x^{-1} \frac{\partial}{\partial x} + H^{-1} \frac{\partial H}{\partial x} \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2} + x^{-2} H^{-1} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left\{ \sum_{i=1}^{n} h^{ik} H \frac{\partial}{\partial x_i} \right\} = \star.
\end{aligned}$$

Note that $\frac{1}{x^2}(x\frac{\partial}{\partial x})(x\frac{\partial}{\partial x}) = \frac{1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2}$ or, equivalently, $\frac{\partial^2}{\partial x^2} = \frac{1}{x^2}(x\frac{\partial}{\partial x})^2 - \frac{1}{x}\frac{\partial}{\partial x}$. Besides,

$$\Delta_h = H^{-1} \sum_{k=0}^{n} \frac{\partial}{\partial x_k} \left\{ \sum_{i=0}^{n} h^{ik} H \frac{\partial}{\partial x_i} \right\}.$$

Therefore,

$$
\begin{aligned}
\star &= nx^{-1}\frac{\partial}{\partial x} + H^{-1}\frac{\partial H}{\partial x}\frac{\partial}{\partial x} + \frac{1}{x^2}(x\frac{\partial}{\partial x})^2 - \frac{1}{x}\frac{\partial}{\partial x} + x^{-2}\Delta_h \\
&= \frac{1}{x^2}\left(\left(x\frac{\partial}{\partial x}\right)^2 + (n-1)\left(x\frac{\partial}{\partial x}\right) + H^{-1}x\frac{\partial H}{\partial x}\left(x\frac{\partial}{\partial x}\right) + \Delta_h\right) \\
&= \frac{1}{x^2}\left(\left(x\frac{\partial}{\partial x}\right)^2 + \left[(n-1) + H^{-1}x\frac{\partial H}{\partial x}\right]\left(x\frac{\partial}{\partial x}\right) + \Delta_h\right).
\end{aligned} \tag{3.2}
$$

There exists more advanced material with examples of Riemannian metrics and conical operators, but we believe that the above examples are enough for our presentation.

3.2 Unbounded operators and closed extensions

This short section prepares us for applications using the Mellin-Sobolev spaces studied before. We present some considerations about the domain of the Laplacian operator as an example of a sectorial operator and we give some results without proofs for a general conic operator A. For the rest of this chapter, we will consider only the case when $\partial\mathbb{B}$ is connected.
Consider the operator

$$\Delta : \mathcal{H}_p^{2,2+\gamma}(\mathbb{B}) \oplus \mathbb{C} \to \mathcal{H}_p^{0,\gamma}(\mathbb{B}),$$

where $\mathbb{C}$ is the space of all constant functions in $C^\infty(\mathbb{B}^\circ)$.

For the rest of our work, we will always assume that γ is such that

$$\frac{n-3}{2} < \gamma < \min\left\{-1 + \sqrt{\left(\frac{n-1}{2}\right)^2 - \lambda_1}, \frac{n+1}{2}\right\}, \tag{3.3}$$

where λ_1 is the greatest non-zero eigenvalue of the boundary Laplacian $\Delta_{h(0)}$. We recall that $h(x)$ with $x \in [0,1)$ is a family of Riemannian metrics on $\partial\mathcal{B}$ that is smooth and does not degenerate up to $x = 0$.

Remark 3.2.1. *It is important to notice that when $\mathbb{B}$ is a compact manifold without boundary and without a conical singularity, then $u \in L_p(\mathbb{B})$ and $\Delta u \in L_p(\mathbb{B})$ imply that $u \in H_p^2(\mathbb{B})$, by elliptic regularity. Here, $1 < p < \infty$. In fact, by Theorem 2.6.3, the equation*

$$\Delta u = f$$

with $u, f \in H_p^0(\mathbb{B}) = L_p(\mathbb{B})$ has local principal symbol $\sum_{i,j} g_{ij}\xi^i\xi^j \in S^2(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ that corresponds to the laplacian operator. Hence, the solution u belongs to $H_p^{0+2}(\mathbb{B}) = H_p^2(\mathbb{B})$. When we have a conical point, then $u \in \mathcal{H}_p^{0,\gamma}(\mathbb{B})$ and $\Delta u \in \mathcal{H}_p^{0,\gamma}(\mathbb{B})$ do not imply that $u \in \mathcal{H}_p^{2,2+\gamma}(\mathbb{B})$. In fact, for almost every γ, by a result for M. Lesch ([17]), we have that $u \in \mathcal{H}_p^{2,2+\gamma}(\mathbb{B}) \oplus \mathcal{E}$, where $\mathcal{E}$ is a finite dimensional space of functions of the form $\omega(x)x^{-\alpha}\ln^h(x)v(y)$, $v \in C^\infty(\partial\mathcal{B})$, $\alpha \in [\frac{n-3}{2} - \gamma, \frac{n+1}{2} - \gamma)$ and $h \in \mathbb{N}_0$.

If we choose γ as in (3.3), then $\Delta : \mathcal{H}_p^{2,2+\gamma}(\mathbb{B}) \oplus \mathbb{C} \to \mathcal{H}_p^{0,\gamma}(\mathbb{B})$ is such that $c - \Delta$ is sectorial with angle 0 for every $c > 0$. Moreover,

$$[\mathcal{H}_p^{0,\gamma}(\mathbb{B}), \mathcal{H}_p^{2,2+\gamma}(\mathbb{B}) \oplus \mathbb{C}]_\alpha = D((c - \Delta)^\alpha),$$

see [27].

Now we give some facts about closed extensions of conic operators. We follow [8] due to its importance for our work. We remark that cone operators have many closed extensions but the resolvent will have good properties only on few of them, see for example [17]. An important result is that for a conic operator A of order μ, we have that

$$A : \mathcal{H}_p^{s+\mu,\gamma+\mu}(\mathbb{B}) \to \mathcal{H}_p^{s,\gamma}(\mathbb{B})$$

is continuous for any s and p, as we can see in Lemma 3.2, [19]. Next, we define three symbols for a Fuchs differential operator that are important for our work.

Definition 3.2.1. *Let* $A = x^{-\mu}\sum_{k=0}^{\mu}\sum_{|\alpha|\le\mu-k} a_{k\alpha}(x,y)(-x\partial_x)^k\partial_y^\alpha$ *be a Fuchs type operator, with* $x \in [0,1)$ *and* $y \in \partial\mathcal{B}$. *Then,*

i) The homogeneous principal symbol is $\sigma_\psi^\mu(A)(x,y,\eta,\xi) := x^{-\mu}\sum_{k+|\alpha|=\mu} a_{k\alpha}(x,y)(-ix)^k\eta^k\xi^\alpha$ *for* $(\eta,\xi) \in \mathbb{R}^{n+1}$, $(\eta,\xi) \neq (0,0)$ *and* $x > 0$, *in* $T^*(\mathbb{B}^\circ \setminus \{0\})$.

ii) The rescaled symbol is $\tilde{\sigma}_\psi^\mu(A)(y,\eta,\xi) := \sum_{k+|\alpha|=\mu} a_{k\alpha}(0,y)(-i\eta)^k\xi^\alpha$, *for* $(\eta,\xi) \neq (0,0)$, *in* $T^*(\partial\mathcal{B} \times \mathbb{R} \setminus \{0\})$.

iii) The conormal symbol is $\sigma_M^\mu(A)(z) := \sum_{k+|\alpha|\le\mu} a_{k\alpha}(0,y)\partial_y^\alpha z^k$ *with* $z \in \mathbb{C}$.

Where T^* *denote the cotangent bundle space.*

Remark 3.2.2. *For all* $s \in \mathbb{R}$ *the conormal symbol defines a family of continuous operators*

$$\sigma_M^\mu(A)(z) : H_p^s(\partial\mathcal{B}) \to H_p^{s-\mu}(\partial\mathcal{B}).$$

Definition 3.2.2. *A is called elliptic with respect to* $\gamma + \mu$ *if*

i) The principal symbol of A on $\mathbb{B}^\circ$ is invertible (see Remark 3.2.3). In particular the homogeneous principal symbol is invertible.

ii) The conormal symbol is an isomorphism for all z in the line $Re(z) = \frac{n+1}{2} - \gamma - \mu$.

Finally, if we consider A as the unbounded operator in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ with domain $C_c^\infty(\mathbb{B}^\circ)$, then there exists a countable set $\mathcal{C} \subset \mathbb{C}$ without accumulation points such that when $\gamma \in \mathcal{C}$, then its closure is given by

$$D(A_{\min}) = \mathcal{H}_p^{\mu,\gamma+\mu}(\mathbb{B}),$$

and

$$D(A_{\max}) = \left\{u \in \mathcal{H}_p^{0,\gamma}(\mathbb{B}) : Au \in \mathcal{H}_p^{0,\gamma}(\mathbb{B})\right\} = D(A_{\min}) \oplus \mathcal{E},$$

where $\mathcal{E}$ is a finite dimensional space of smooth functions of the form $\omega(x)x^k\ln^l(x)v(y)$ that belongs to $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$, where $v \in C^\infty(\partial\mathcal{B})$.

Example 6. *Let us recall the operator given in Example 5.*

$$\Delta = \frac{1}{x^2}\left(\left(x\frac{\partial}{\partial x}\right)^2 + \left[(n-1) + H^{-1}x\frac{\partial H}{\partial x}\right]\left(x\frac{\partial}{\partial x}\right) + \Delta_h\right).$$

Then,

$$\sigma_\psi^\mu(\Delta) = \frac{1}{x^2}\left((-ix)^2\eta^2 - |\xi|^2_{h(0)}\right)$$
$$= \frac{1}{x^2}\left(-x^2\eta^2 - |\xi|^2_{h(0)}\right).$$

$$\tilde{\sigma}_\psi^\mu(\Delta) = -\eta^2 - |\xi|^2_{h(0)},$$

and

$$\sigma_M^\mu(\Delta)(z) = z^2 + (n-1)z + \Delta_h$$

where $z \in \mathbb{C}$ define the homogeneous principal, rescaled and conormal symbols respectively for the Laplace-Beltrami operator Δ.

Remark 3.2.3. *We recall the definition of ellipticity on an open set $\Omega \subset \mathbb{R}^n$ and on a manifold without conical point $\mathbb{B}$. Let $\Omega \subset \mathbb{R}^n$ and $m \in \mathbb{N}_0$. Given an operator $A = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha$ defined over Ω, we say that A is an elliptic operator if for every $x \in \Omega$ and every non zero $\xi \in \mathbb{R}^n$, we have $\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \neq 0$. For a conic manifold $\mathbb{B}$ we say that A is elliptic over $\mathbb{B}$ if A is locally elliptic. That means that for any local chart (U, ϕ) such that $x \in U$, we have*

$$\sum_{|\alpha|=m} a_\alpha(\phi^{-1}(x))\xi^\alpha \neq 0,$$

where locally, $\sum_{|\alpha|\leq m} a_\alpha(\phi^{-1}(x))\partial^\alpha(u \circ \phi^{-1}(x))$ is the form of A.

3.3 Reaction-Diffusion equation on conic manifolds

In this section, we work following the ideas of the book **Global Attractors in Abstract Parabolic Problems** by W. Cholewa & Tomasz Dlokto. In particular, our considerations are based on Chapter 6 of [6].

Let $\mathbb{B}$ be a conic manifold. We consider the following equation.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta_{\mathbb{B}} u + u - u^q \quad \text{on} \quad \mathbb{B}^\circ, \\ u(0,x) &= u_0(x) \quad x \in \mathbb{B}^\circ. \end{aligned} \tag{3.4}$$

Theorem 3.3.1. *If (3.3) holds and $u_0 \in X^\alpha$, $2\alpha + \gamma \geq \frac{n+1}{2}$, $2\alpha > \frac{n+1}{p}$. Then, there exist an unique global solution u of (3.4). Moreover, $u \in C^1([0,\infty), \mathcal{H}^{0,\gamma}(\mathbb{B})) \cap C^1((0,\infty), \mathcal{H}^{2,2+\gamma}(\mathbb{B}) \oplus \mathbb{C})$.*

Proof. We use the following interpolation result that can be found in Lemma 4.5 ii) in [19]. For $\alpha \in [0,1]$ such that $\gamma + 2\alpha - 1 \notin \left\{\mp\sqrt{(\frac{n-1}{2})^2 - \lambda_j} : j \in \mathbb{N}\right\}$ then

$$X^\alpha := D((I-\Delta)^\alpha) = [\mathcal{H}_p^{0,\gamma}(\mathbb{B}), \mathcal{H}_p^{2,2+\gamma}(\mathbb{B}) \oplus \mathbb{C}]_\alpha = \begin{cases} \mathcal{H}_p^{2\alpha,2\alpha+\gamma}(\mathbb{B}) \oplus \mathbb{C} & \text{if} \quad \gamma + 2\alpha \geq \frac{n+1}{2}, \\ \mathcal{H}_p^{2\alpha,2\alpha+\gamma}(\mathbb{B}) & \text{if} \quad \gamma + 2\alpha < \frac{n+1}{2}. \end{cases}$$

Suppose that $2\alpha + \gamma \geq \frac{n+1}{2}$ and $2\alpha > \frac{n+1}{p}$. Then

$$X^\alpha = [\mathcal{H}_p^{0,\gamma}(\mathbb{B}), \mathcal{H}_p^{2,2+\gamma}(\mathbb{B}) \oplus \mathbb{C}]_\alpha = \mathcal{H}_p^{2\alpha,2\alpha+\gamma}(\mathbb{B}) \oplus \mathbb{C} \subset L_\infty(\mathbb{B}).$$

By Theorem 3.1.1, let us choose $2\alpha > \frac{n+1}{p}$, $\gamma + 2\alpha \geq \frac{n+1}{2}$ and consider

$$F : \mathcal{H}_p^{2\alpha,2\alpha+\gamma}(\mathbb{B}) \oplus \mathbb{C} \to \mathcal{H}_p^{0,\gamma}(\mathbb{B})$$

defined by

$$F(\phi) = \phi - \phi^q, \quad \forall \phi \in \mathcal{H}_p^{2\alpha,2\alpha+\gamma}(\mathbb{B}) \oplus \mathbb{C}.$$

We can show that F is Lipschitz continuous on bounded subsets of $X^\alpha = \mathcal{H}_p^{2\alpha,2\alpha+\gamma}(\mathbb{B}) \oplus \mathbb{C}$. In effect:

If $\tilde{\phi} \in \mathcal{H}_p^{2\alpha,2\alpha+\gamma}(\mathbb{B}) \oplus \mathbb{C}$, then $\tilde{\phi} = \phi + c$ with $\phi \in \mathcal{H}_p^{2\alpha,2\alpha+\gamma}(\mathbb{B})$ and $c \in \mathbb{C}$, with norm equal to

$$||\tilde{\phi}||_{\mathcal{H}_p^{2\alpha,2\alpha+\gamma}(\mathbb{B})\oplus\mathbb{C}} = ||\phi||_{\mathcal{H}_p^{2\alpha,2\alpha+\gamma}(\mathbb{B})} + |c|.$$

Taking this into account, we consider two functions $\tilde{\phi}, \tilde{\psi} \in B$ where $B \subset X^\alpha$ is a bounded set, with $\tilde{\phi} = \phi + c_1$, $\tilde{\psi} = \psi + c_2$, where $\phi, \psi \in \mathcal{H}^{2\alpha,2\alpha+\gamma}(\mathbb{B})$ and $c_1, c_2 \in \mathbb{C}$. Hence, with $2\alpha > \frac{n+1}{p}$, $\gamma + 2\alpha > \frac{n+1}{2}$ and recalling that $X^\alpha \hookrightarrow L^\infty(\mathbb{B})$, we have

$$\begin{aligned}
&||F(\tilde{\phi}) - F(\tilde{\psi})||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \\
&= ||\tilde{\phi} - \tilde{\phi}^q - (\tilde{\psi} - \tilde{\psi}^q)||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \\
&= ||\tilde{\phi} - \tilde{\psi} - (\tilde{\phi}^q - \tilde{\psi}^q)||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \\
&\leq ||\tilde{\phi} - \tilde{\psi}||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} + ||\tilde{\phi}^q - \tilde{\psi}^q||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \\
&= ||\tilde{\phi} - \tilde{\psi}||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} + ||(\tilde{\phi} - \tilde{\psi})(\tilde{\phi}^{q-1} + \ldots + \tilde{\psi}^{q-1})||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \\
&\leq ||\tilde{\phi} - \tilde{\psi}||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} + ||\tilde{\phi} - \tilde{\psi}||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} ||\tilde{\phi}^{q-1} + \ldots + \tilde{\psi}^{q-1}||_{L_\infty(\mathbb{B})} \\
&\leq ||\tilde{\phi} - \tilde{\psi}||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} + ||\tilde{\phi} - \tilde{\psi}||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \left(qC \sup_{\tau \in B} ||\tau||_{X^\alpha}^{q-1} \right) \\
&= \left(1 + qC \sup_{\tau \in B} ||\tau||_{X^\alpha}^{q-1}\right) ||\tilde{\phi} - \tilde{\psi}||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \\
&= \left(1 + qC \sup_{\tau \in B} ||\tau||_{X^\alpha}^{q-1}\right) ||\phi + c_1 - (\psi + c_2)||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \\
&= \left(1 + qC \sup_{\tau \in B} ||\tau||_{X^\alpha}^{q-1}\right) ||\phi - \psi + (c_1 - c_2)||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \\
&\leq \left(1 + qC \sup_{\tau \in B} ||\tau||_{X^\alpha}^{q-1}\right) \left(||\phi - \psi||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} + |c_1 - c_2| \right) \\
&\leq \left(1 + qC \sup_{\tau \in B} ||\tau||_{X^\alpha}^{q-1}\right) \left(||\phi - \psi||_{\mathcal{H}_p^{2\alpha,\gamma+2\alpha}(\mathbb{B})} + |c_1 - c_2| \right) := (\star) \\
&= \tilde{C} ||\tilde{\phi} - \tilde{\psi}||_{\mathcal{H}_p^{2\alpha,\gamma+2\alpha}(\mathbb{B})\oplus\mathbb{C}},
\end{aligned}$$

where, in $(\star)$ we have used that $\mathcal{H}_p^{2\alpha,\gamma+2\alpha}(\mathbb{B}) \hookrightarrow \mathcal{H}_p^{0,\gamma}(\mathbb{B})$, (see Lemma 3.2, [19]). Finally, we have proved that F is locally Lipschitz on bounded subsets and this implies the existence and uniqueness of a local solution, see Section 2.4.

In order to show global solution of (3.4), we need to use Green's identity on Mellin-Sobolev Spaces (see Remark 9 and Lemma 4.3 in [19])

$$\int_{\mathbb{B}} \langle \nabla u, \nabla v \rangle_g d\mu_g = - \int_{\mathbb{B}} u \Delta v d\mu_g$$

with $u, v \in \mathcal{H}_2^{1,1+\gamma}(\mathbb{B}) \oplus \mathbb{C}$, $\Delta v \in \mathcal{H}_2^{0,\gamma}(\mathbb{B})$ and $\gamma > -1$. In our case, we use that $\mathcal{H}_p^{1,1+\gamma}(\mathbb{B}) \hookrightarrow \mathcal{H}_2^{1,1+\gamma}(\mathbb{B})$ and $\mathcal{H}_p^{0,\gamma}(\mathbb{B}) \hookrightarrow \mathcal{H}_2^{0,\gamma}(\mathbb{B})$ (see Remark 9 and Lemma 3.2 in [19]). So, in order to apply the Green's identity it is enough consider $\gamma > -1$. In this case, the identity holds for all $u, v \in \mathcal{H}_p^{1,1+\gamma}(\mathbb{B}) \oplus \mathbb{C}$ such that $\Delta v \in \mathcal{H}_p^{0,\gamma}(\mathbb{B})$.

We are considering $u \in \mathcal{H}_p^{1,1+\gamma}(\mathbb{B}) \oplus \mathbb{C}$ with $\Delta_{\mathbb{B}} u \in \mathcal{H}_p^{0,\gamma}(\mathbb{B})$ with $\gamma > -1$. Hence, multiplying by u^{2m-1} and integrating to the equation (3.4), we have the following

$$\begin{aligned}
\int_{\mathbb{B}} u^{2m-1} u_t &= \int_{\mathbb{B}} u^{2m-1}(\Delta_{\mathbb{B}} u + u - u^q) \\
&= \int_{\mathbb{B}} u^{2m-1} \Delta_{\mathbb{B}} u + \int_{\mathbb{B}} u^{2m-1}(u - u^q) \\
&= -\int_{\mathbb{B}} \nabla_{\mathbb{B}} u^{2m-1} \nabla_{\mathbb{B}} u + \int_{\mathbb{B}} u^{2m} - \int_{\mathbb{B}} u^{2m+q-1} \\
&= -(2m-1) \int_{\mathbb{B}} u^{2m-2} |\nabla_{\mathbb{B}} u|^2 + \int_{\mathbb{B}} u^{2m} - \int_{\mathbb{B}} u^{2m+q-1} \\
&\leq \int_{\mathbb{B}} u^{2m} - \int_{\mathbb{B}} u^{2m+q-1} \cdot 1 \\
&\leq -|\mathbb{B}|^{-\frac{q-1}{2m}} \left(\int_{\mathbb{B}} u^{2m} \right)^{\frac{2m+q-1}{2m}} + \int_{\mathbb{B}} u^{2m},
\end{aligned}$$

where in last inequality we used Hölder inequality for integrals with $p' = \frac{2m+q-1}{2m}$ and $q' = 1 - p'$. Equivalently, we have proved that

$$\frac{d}{dt} y(t) \leq -2m|\mathbb{B}|^{-\frac{q-1}{2m}} (y(t))^{\frac{2m+q-1}{2m}} + 2my(t),$$

where $y(t) = \int_{\mathbb{B}} u^{2m}(t, u_0) dx$. By Bernoulli inequality (Lemma 1.2.4, [6]), we have

$$y(t) \leq \max\{y(0), |\mathbb{B}|\},$$

$$\lim_{t \to \infty} \sup y(t) \leq |\mathbb{B}|.$$

Taking $m \to \infty$, we conclude that $||u||_{L_\infty} \leq \max\{||u_0||_{L_\infty}, 1\} \leq \max\{C||u_0||_{X^\alpha}, 1\} \leq C_1 c(||u_0||_{X^\alpha})$, where $c := \max\{||u_0||_{X^\alpha}, 1\}$ is a locally bounded function and $C_1 = \max\{C, 1\}$.

Hence,

$$\begin{aligned}
||F(u(t, u_0))||^p_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} &= ||-u^q + u||^p_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \\
&= ||1(-u^q + u)||^p_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \\
&= \int_{\mathbb{B}^\circ \setminus [0,1) \times \partial\mathcal{B}} |-u^q + u|^p dz + \int_{\partial\mathcal{B}} \int_0^1 |x^{\frac{n+1}{2} - \gamma} \omega(x)(-u^q + u)|^p \frac{dx}{x} dy \\
&\leq \int_{\mathbb{B}^\circ \setminus [0,1) \times \partial\mathcal{B}} |-u^q + u|^p dz + \int_{\partial\mathcal{B}} \int_0^1 |-u^q + u|^p x^{(\frac{n+1}{2} - \gamma)p - 1} dx dy \\
&\leq |\mathbb{B}^\circ \setminus [0,1) \times \partial\mathcal{B}| ||-u^q + u||^p_{L_\infty(\mathbb{B}^\circ \setminus [0,1) \times \partial\mathcal{B})} + \\
&\quad ||x^{(\frac{n+1}{2} - \gamma)p - 1}||_{L_1([0,1) \times \partial\mathcal{B})} ||-u^q + u||^p_{L_\infty([0,1) \times \partial\mathcal{B})} \\
&\leq C_1 ||-u^q + u||^p_{L_\infty(\mathbb{B}^\circ \setminus [0,1) \times \partial\mathcal{B})} + C_2 ||-u^q + u||^p_{L_\infty([0,1) \times \partial\mathcal{B})} \\
&\leq C \left(||u||^q_{L_\infty(\mathbb{B})} + ||u||_{L_\infty(\mathbb{B})} \right)^p.
\end{aligned}$$

Next, if we define $g : [0, \infty) \to [0, \infty)$ by $g(t) = C(t^q + t)^p$ then

$$||F(u(t, u_0))||_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \leq g(||u(t, u_0)||_{L_\infty(\mathbb{B})})$$

where g is a non-decreasing function and we conclude by the Theorem 2.4.1 that there exists global solution for (3.4). Note that we used the space $Y = L_\infty(\mathbb{B})$ such that $X^\alpha \subset Y$. $\square$

Chapter 4

Sectorial operators on BUC generated by Λ - Elliptic Operators

In this chapter, we present the concept Λ- Ellipticity (see [12]) which we will use for our application. Here we consider a sector $\Lambda_a(\theta) := \Lambda(\theta)$ with $a = 0$ in the complex plane as we did in Section 2.1 and an operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ for some Banach space X such that that $\Lambda(\theta) \setminus \{0\}$ is contained in the resolvent set of $\mathcal{A}$, that $||\lambda(\lambda - \mathcal{A})||_{\mathcal{L}(X)}$ is uniformly bounded in $0 \neq \lambda \in \Lambda(\theta)$, and that $\mathcal{A}$ is injective with dense range, $\sigma(\mathcal{A})$ and $\rho(\mathcal{A})$ represent the spectrum and the resolvent of the operator $\mathcal{A}$ respectively. From now on, we fix the angle θ and only use Λ.

4.1 Λ - Ellipticity and continuity of pseudos with parameters in BUC

Definition 4.1.1. *A symbol $a \in \mathcal{S}^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m > 0$ is called Λ-elliptic if there exists constants $0 < m$, $C_0 \geq 1$, and $R \geq 0$ such that:*

(H1) For all $x \in \mathbb{R}^n$ and all $|\xi| \geq R$ we have $a(x,\xi) \in \Omega_\xi := \{z \in \mathbb{C} : \frac{1}{C_0}\langle\xi\rangle^m < |z| < C_0\langle\xi\rangle^m, z \notin \Lambda\}$.

(H2) Given $\alpha, \beta \in \mathbb{N}_0^n$, there exists a $C \geq 0$ such that for all $x \in \mathbb{R}^n$, $|\xi| \geq R$, and $\lambda \in \Lambda$,

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)(\lambda - a(x,\xi))^{-1}| \leq C\langle\xi\rangle^{-|\alpha|}.$$

4.2 Estimates in BUC

Here and below for $t \in \mathbb{R}$, $\lfloor t \rfloor$ denotes the largest integer smaller or equal to t.

Definition 4.2.1. *For $s \in \mathbb{R}_+ \backslash \mathbb{N}$, BUC^s denotes all the functions $f : \mathbb{R}^n \to \mathbb{C}$ such that its derivatives smaller or equal to $\lfloor s \rfloor$ are bounded and uniformly continuous, and whose derivatives of order $\lfloor s \rfloor$ are uniformly $(s - \lfloor s \rfloor)$-Hölder continuous. The norm is given by two cases. For $s \in (0,1)$, then*

$$||u||_{BUC^s} := ||u||_{L_\infty} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^s}.$$

And, for $s > 1$ then there exists $s' \in (0,1)$ and $k \in \mathbb{N}_0$ such that $s = k + s'$. Then,

$$||u||_{BUC^s} := \sum_{|\alpha|\leq k} ||D^\alpha u||_{L_\infty} + \sum_{|\alpha|=k} \sup_{x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^s}.$$

For $s \in \mathbb{N}$ we have a similar definition for BUC^s.

Definition 4.2.2. *Let $s \in \mathbb{N}_0$. We define*

$$BUC^s = \{f : \mathbb{R}^n \to \mathbb{C} : f \quad \text{and} \quad \partial^\alpha f \quad \text{are uniformly continuous and bounded for all} \quad |\alpha| \leq s\}$$

with norm given by

$$||f||_{BUC^s} = \sum_{|\alpha|\leq s} ||\partial^\alpha f||_{L_\infty}.$$

First we present some important results due to Amann (see [2]). Those results were the inspiration for the results of Section 4.3 that will allow us to prove almost sectoriality on $\mathcal{C}^{0,\gamma}(\mathbb{B})$ spaces. Let $m \in \mathbb{N}_0$ and $a_\alpha : \mathbb{R}^n \to \mathbb{C}$ be bounded C^∞ functions with bounded derivatives of any order for all α multi index in $\mathbb{N}_0^n$ with $|\alpha| \leq m$, that is $a_\alpha \in BUC^\infty(\mathbb{R}^n) = \bigcap_{s\geq 0} BUC^s(\mathbb{R}^n)$. Moreover,

$$\mathcal{A} = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$$

denotes a linear differential operator on $\mathbb{R}^n$ with values in $\mathbb{C}$. Its symbol and principal symbol are denoted by $a(x,\xi)$ and $\sigma\mathcal{A}$. They are given by

$$\sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha \quad \text{and} \quad \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha,$$

respectively.

Definition 4.2.3. *The operator $\mu^m + \mathcal{A}$ is said elliptic by parameters if there exist $C, R > 0$ such that*

$$|a(x,\xi,\mu)| \geq C\langle \xi, \mu\rangle^m$$

for all $||(\xi,\mu)|| \geq R$ and $x \in \mathbb{R}^n$, where $\langle \xi,\mu\rangle^m = (1 + |\xi|^2 + |\mu|^2)^{\frac{m}{2}}$ and $a(x,\xi,\mu) = \mu^m + a(x,\xi)$.

We defined the sector $\Lambda_0(\phi)$ in Section 2.1. We will also use the complement of this set denoted by $\Lambda^c(\phi) := \Lambda^c$.

Definition 4.2.4. *Let $\kappa \geq 1$ and $\phi \in [0,\pi)$. The operator $\mathcal{A}$ is (κ,ϕ) elliptic if $-\sigma\mathcal{A}(x,\xi) \notin \Lambda^c$ for all $x \in \mathbb{R}^n$ and $\xi \neq 0$ and*

$$|(\lambda + \sigma\mathcal{A})^{-1}(x,\xi)| \leq \frac{\kappa}{1+|\lambda|}$$

for $\lambda \in \Lambda^c$, $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, $|\xi| = 1$. The operator $\mathcal{A}$ is ϕ-elliptic if it is (κ,ϕ) elliptic for some $\kappa \geq 1$ and it is normally elliptic if it is $\frac{\pi}{2}$-elliptic.

Lemma 4.2.1. *If the operator $\mu^m + \mathcal{A}$ is elliptic by parameters then $\mathcal{A}$ is (κ,ϕ)-elliptic.*

Proof. Let us suppose that $\mathcal{A}$ is elliptic by parameters. Then, there exists $C, R > 0$ such that

$$|a(x,\xi,\mu)| \geq \langle \xi, \mu \rangle^m$$

for $||(\xi,\mu)|| \geq R$. Now, we see that

$$C\langle \xi,\mu\rangle^m \leq |\mu^m + a(x,\xi)| = |\mu^m + \sum_{|\alpha|<m} a_\alpha(x)\xi^\alpha + \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha| = |\mu^m + \sigma\mathcal{A} + \sum_{|\alpha|<m} a_\alpha(x)\xi^\alpha|. \quad (4.1)$$

Therefore,

$$\begin{aligned} |\mu^m + \sigma\mathcal{A}| &\geq C\langle \xi,\mu\rangle^m - |\sum_{|\alpha|<m} a_\alpha(x)\xi^\alpha| \\ &\geq C\langle \xi,\mu\rangle^m - \tilde{C}\langle \xi,\mu\rangle^{m-1} \\ &= \langle \xi,\mu\rangle^m \left(C - \tilde{C}\langle \xi,\mu\rangle^{-1}\right) \\ &\geq C_2\langle \xi,\mu\rangle^m \geq R, \end{aligned} \quad (4.2)$$

where R is a sufficiently large constant. Therefore, in particular with $|\xi| = 1$ in $\mathbb{R}^n$ we have that $|\mu^m + \sigma\mathcal{A}| \geq C_2\langle \xi,\mu\rangle^m$ which implies

$$|(\mu^m + \sigma\mathcal{A})^{-1}| \leq C_2\langle \xi,\mu\rangle^{-m} \leq \frac{C_3}{1+|\mu^m|} \quad \text{for} \quad \mu \quad \text{large.}$$

Hence, if we take $\lambda = \mu^m$, the homogeneity of $\mu^m + \sigma\mathcal{A}$ implies that

$$|(\lambda + \sigma\mathcal{A})^{-1}| \leq C_2\langle \xi,\mu\rangle^{-m} \leq \frac{C_3}{1+|\lambda|},$$

which completes the proof. □

We set an important result where we use the previous definitions and find a relation with sectorial operators.

Theorem 4.2.2. *Let $m \in \mathbb{N}_0$ and $\mathcal{A} = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$ be an operator such that $\mu^m + \mathcal{A}$ is an elliptic operator with parameters. Then, there exists two operators $K_\infty(\mu), T_\infty(\mu) \in \mathcal{L}(BUC(\mathbb{R}^n))$ such that*

$$\left(\mu^m + \sum_{|\alpha|=m} a_\alpha(x)D^\alpha\right) K_\infty(\mu) = Id + T_\infty(\mu)$$

with

$$||K_\infty(\mu)||_{\mathcal{L}(BUC)} \leq C|\mu|^{-m}$$

and

$$||T_\infty(\mu)||_{\mathcal{L}(BUC)} \leq C|\mu|^{-r},$$

for some $r > 0$ with $|\eta| \geq \eta_0 > 0$ and $\eta \in \Lambda^c$. As a consequence, we have that for $\lambda := \mu^m$

$$||(\lambda + \mathcal{A})^{-1}||_{\mathcal{L}(BUC)} \leq \frac{C}{|\lambda|}.$$

for large λ. In particular, $\mathcal{A}$ is a sectorial operator, see Definition 2.1.2.

Proof. See Theorem 5.10 in [2]. □

4.3 Estimates of parameter dependent symbols on BUC

In this section, we obtain new estimates for the norms of pseudodifferential operators that depend on parameters acting on $BUC(\mathbb{R}^n)$ spaces. First, we present a result due to Amann. It will be used in Theorem 4.3.1. Finally, we apply our results to prove that some elliptic operators define analytic semigroups in $BUC(\mathbb{R}^n)$ spaces.

Definition 4.3.1. *We say that* $k \in BUC(\mathbb{R}^n, L_1(\mathbb{R}^n))$ *if* $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ *is a measurable function such that*

1) The function k *is integrable in the second variable and* $\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |k(x,y)|dy < \infty$.

2) For each $\varepsilon > 0$, *there is an* $\delta > 0$ *such that if* $|x-\tilde{x}| < \delta$, *then* $\int_{\mathbb{R}^n} |k(x,y) - k(\tilde{x},y)|dy < \varepsilon$.

The set of functions $BUC(\mathbb{R}^n, L_1(\mathbb{R}^n))$ is a Banach space with norm

$$\|k\|_{BUC(L_1)} := \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |k(x,y)|dy < \infty.$$

Proposition 4.3.1. *If* $k \in BUC(\mathbb{R}^n, L_1(\mathbb{R}^n))$, *then we can define a continuous linear map* $K_\infty : BUC(\mathbb{R}^n) \to BUC(\mathbb{R}^n)$ *by the formula below:*

$$K_\infty u(x) = \int_{\mathbb{R}^n} k(x, x-y)u(y)dy.$$

Moreover, $\|K_\infty\|_{\mathcal{L}(BUC(\mathbb{R}^n))} \leq \|k\|_{BUC(L_1)}$.

Proof. Let us prove by steps:

Step 1: $\|K_\infty u(x)\|_{L_\infty(\mathbb{R}^n)} \leq \|k\|_{BUC(L^1)}\|u\|_{L_\infty(\mathbb{R}^n)}$.

To prove this, we note that

$$\begin{aligned}|K_\infty u(x)| &\leq \int_{\mathbb{R}^n} |k(x,x-y)||u(y)|dy \leq \left(\int_{\mathbb{R}^n} |k(x,x-y)|dy\right) \|u\|_{L_\infty(\mathbb{R}^n)} \\ &\leq \left(\int_{\mathbb{R}^n} |k(x,z)|dz\right) \|u\|_{L_\infty(\mathbb{R}^n)} \leq \|k\|_{BUC(L_1)}\|u\|_{L_\infty(\mathbb{R}^n)}.\end{aligned}$$

Step 2: $K_\infty u$ *is uniformly continuous.*

We note that

$$\begin{aligned}&|K_\infty u(x) - K_\infty u(\tilde{x})| = |\int_{\mathbb{R}^n} (k(x,x-y)u(y) - k(\tilde{x},\tilde{x}-y)u(y))dy| \\ &= |\int_{\mathbb{R}^n} (k(x,y)u(x-y) - k(\tilde{x},y)u(\tilde{x}-y))\, dy| \\ &= |\int_{\mathbb{R}^n} (k(x,y) - k(\tilde{x},y))u(x-y) + k(\tilde{x},y)(u(x-y) - u(\tilde{x}-y))dy| \\ &\leq \int_{\mathbb{R}^n} |k(x,y) - k(\tilde{x},y)||u(x-y)|dy + \int_{\mathbb{R}^n} |k(\tilde{x},y)||u(x-y) - u(\tilde{x}-y)|dy \\ &\leq \int_{\mathbb{R}^n} |k(x,y) - k(\tilde{x},y)|dy\|u\|_{L_\infty(\mathbb{R}^n)} + \|k\|_{BUC(L_1)} \sup_{y\in\mathbb{R}^n} |u(x-y) - u(\tilde{x}-y)|.\end{aligned}$$

The result now follows from the uniform continuity of k and u.

Corollary 4.3.1. *Let $\Lambda \subset \mathbb{C}$ be a sector of the complex plane and $k : \mathbb{R}^n \times \mathbb{R}^n \times \Lambda \to \mathbb{C}$ be a measurable function. If for each $\eta \in \Lambda$, the function $k(\eta) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ defined by $k(\eta)(x, y) = k(x, y, \eta)$ belongs to $BUC(\mathbb{R}^n, L_1(\mathbb{R}^n))$ and $\|k(\eta)\|_{BUC(L_1)} \leq C\langle\eta\rangle^r$ for some $r > 0$, then the function $K_\infty(\eta)$ defined by*

$$K_\infty(\eta)u(x) = \int_{\mathbb{R}^n} k(x, x - y, \eta)u(y)dy$$

belongs to $\mathcal{L}(BUC(\mathbb{R}^n))$ and $\|K_\infty(\eta)\|_{\mathcal{L}(BUC)} \leq C\langle\eta\rangle^r$.

Theorem 4.3.1. *Let $p : \mathbb{R}^n \times \mathbb{R}^n \times \Lambda \to \mathbb{C}$ be a C^∞ function.*

i. Case 1) If $\mu > 0$ and $|\partial_x^\beta \partial_\xi^\alpha p(x, \xi, \eta)| \leq C_{\alpha\beta}\langle\xi, \eta\rangle^{-\mu-|\alpha|}$, then there exists a measurable function $k : \mathbb{R}^n \times \mathbb{R}^n \times \Lambda \to \mathbb{C}$ such that

$$op(p(\eta))u(x) = \int_{\mathbb{R}^n} k(x, x - y, \eta)u(y)dy,$$

where $k(\eta) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ defined by $k(\eta)(x, y) = k(x, y, \eta)$ belongs to $BUC(\mathbb{R}^n, L_1(\mathbb{R}^n))$ and $\|k(\eta)\|_{BUC(L_1)} \leq C\langle\eta\rangle^{-\mu}$. In particular, $op(p) \in \mathcal{L}(BUC)$ and $||op(p)||_{\mathcal{L}(BUC)} \leq C\langle\eta\rangle^{-\mu}$.

ii. Case 2) If $\mu \geq 2$, $|\partial_x^\beta p(x, \xi, \eta)| \leq C_{\alpha\beta}\langle\xi, \eta\rangle^{-\mu}$, $\forall\beta \in \mathbb{N}_0^n$, and $|\partial_x^\beta \partial_\xi^\alpha p(x, \xi, \eta)| \leq C_{\alpha\beta}\langle\xi, \eta\rangle^{-2\mu}\langle\xi\rangle^{\mu-|\alpha|}$, $\forall\alpha, \beta \in \mathbb{N}_0^n$, $\alpha \neq 0$, then there exists a measurable function $k : \mathbb{R}^n \times \mathbb{R}^n \times \Lambda \to \mathbb{C}$ such that

$$op(p(\eta))u(x) = \int_{\mathbb{R}^n} k(x, x - y, \eta)u(y)dy,$$

where $k(\eta) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ defined by $k(\eta)(x, y) = k(x, y, \eta)$ belongs to $BUC(\mathbb{R}^n, L_1(\mathbb{R}^n))$ and $\|k(\eta)\|_{BUC(L_1)} \leq C\langle\eta\rangle^{-\mu}$. In particular, $op(p) \in \mathcal{L}(BUC)$ and $||op(p)||_{\mathcal{L}(BUC)} \leq C\langle\eta\rangle^{-\mu}$.

iii. Case 3) If $\mu \geq 2$ and $|\partial_x^\beta \partial_\xi^\alpha p(x, \xi, \eta)| \leq C_{\alpha\beta}\langle\xi, \eta\rangle^{-\mu}\langle\xi\rangle^{-|\alpha|}$, then there exists a measurable function $k : \mathbb{R}^n \times \mathbb{R}^n \times \Lambda \to \mathbb{C}$ such that

$$op(p(\eta))u(x) = \int_{\mathbb{R}^n} k(x, x - y, \eta)u(y)dy,$$

where $k(\eta) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ defined by $k(\eta)(x, y) = k(x, y, \eta)$ belongs to $BUC(\mathbb{R}^n, L_1(\mathbb{R}^n))$ and $\|k(\eta)\|_{BUC(L_1)} \leq C\langle\eta\rangle^{-\mu+\varepsilon}$. In particular, $op(p) \in \mathcal{L}(BUC)$ and $||op(p)||_{\mathcal{L}(BUC)} \leq C\langle\eta\rangle^{-\mu+\varepsilon}$.

For the proof, we fix a function $\varphi_0 \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \varphi_0 \leq 1$, $\varphi_0(\xi) = 1$ for $|\xi| \leq 1$, and $\varphi_0(\xi) = 0$ for $|\xi| \geq 2$. We also define $\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi)$, for $j \geq 1$. We note that $\text{supp}\varphi_j \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. Notice that there exist constants $c, C > 0$ such that, inside the support of φ_j, we have that $c2^j \leq \langle\xi\rangle \leq C2^j$. Moreover, $\lim_{N\to\infty} \sum_{j=0}^N \varphi_j(\xi) \to 1$ and $|D_\xi^\gamma \varphi_j| \leq 2^{-j|\gamma|}$.

Let us prove by steps. First notice that

$$op(p(\eta))u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} p(x, \xi, \eta)\hat{u}(\xi)d\xi.$$

Note that $\hat{u} = \lim_{N\to\infty} \sum_{j=0}^N \varphi_j(\langle\eta\rangle^{-1}\xi)\hat{u}(\xi)$ in $\mathcal{S}(\mathbb{R}^n)$. Since $op(p(\eta)) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous, we conclude that

$$\begin{aligned} op(p(\eta))u(x) &= \sum_{j=0}^\infty \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi\cdot x} p(x, \xi, \eta)\varphi_j(\langle\eta\rangle^{-1}\xi)\hat{u}(\xi)d\xi \\ &= \sum_{j=0}^\infty \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)} p(x, \xi, \eta)\varphi_j(\langle\eta\rangle^{-1}\xi)d\xi \right) u(y)dy \\ &= \sum_{j=0}^\infty \int_{\mathbb{R}^n} k_j(x, x - y, \eta)u(y)dy, \end{aligned}$$

where

$$k_j(x,z,\eta) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\xi\cdot z} p(x,\xi,\eta)\varphi_j(\langle\eta\rangle^{-1}\xi)d\xi.$$

Lemma 4.3.2. *The function k_j satisfies:*

i. Case 1) We have, for all $N \in \mathbb{N}$, that

$$|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| \le c_\beta |z|^{-N}\langle\eta\rangle^{n-\mu}2^{j(-\mu+n-N)}.$$

ii. Case 2) We have, for $N \in \mathbb{N}$ such that, $N \le n+1$ that

$$|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| \le c_\beta |z|^{-N}\langle\eta\rangle^{n-\mu}2^{j(-\mu+n-N)}.$$

iii. Case 3) We have, for some $M \in (n,n+1)$ and $\varepsilon > \tilde{\varepsilon} > 0$, that

$$|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| \le c_\beta |z|^{-M}\langle\eta\rangle^{n-\mu}\langle\eta\rangle^{\varepsilon}2^{j(-\mu-\tilde{\varepsilon})}.$$

And for $N = 0$, we have

$$|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| \le c_\beta \langle\eta\rangle^{n-\mu}2^{j(n-\mu)}.$$

Proof. Note that, for $p_j(x,\xi,\eta) = p(x,\xi,\eta)\varphi_j(\langle\eta\rangle^{-1}\xi)$, we have

$$\begin{aligned}
&(2\pi)^n z^\gamma \partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta) = z^\gamma \partial_x^\beta \int_{\mathbb{R}^n} e^{i\xi\cdot\langle\eta\rangle^{-1}z} p_j(x,\xi,\eta)d\xi \\
&= \int_{\mathbb{R}^n} z^\gamma e^{i\xi\cdot\langle\eta\rangle^{-1}z}\partial_x^\beta p_j(x,\xi,\eta)d\xi = \int_{\mathbb{R}^n} \langle\eta\rangle^{|\gamma|} D_\xi^\gamma(e^{i\xi\cdot\langle\eta\rangle^{-1}z})\partial_x^\beta p_j(x,\xi,\eta)d\xi \\
&= \langle\eta\rangle^{|\gamma|}\int_{\mathbb{R}^n} e^{i\xi\cdot\langle\eta\rangle^{-1}z} D_\xi^\gamma\left(\partial_x^\beta p_j(x,\xi,\eta)\right)d\xi \\
&= \langle\eta\rangle^{|\gamma|}\langle\eta\rangle^{-|\gamma_2|}\sum_{\gamma_1+\gamma_2=\gamma} c_{\gamma_1,\gamma_2}\int_{\mathbb{R}^n} e^{i\xi\cdot\langle\eta\rangle^{-1}z} \\
&\quad\times \partial_x^\beta D_\xi^{\gamma_1} p(x,\xi,\eta) D_\xi^{\gamma_2}\varphi_j(\langle\eta\rangle^{-1}\xi)d\xi \\
&= \langle\eta\rangle^{|\gamma|-|\gamma_2|+n}\sum_{\gamma_1+\gamma_2=\gamma} c_{\gamma_1,\gamma_2}\int_{\mathbb{R}^n} e^{i\xi\cdot z}\partial_x^\beta D_\xi^{\gamma_1} p(x,\langle\eta\rangle\xi,\eta) D_\xi^{\gamma_2}\varphi_j(\xi)d\xi.
\end{aligned}$$

Now we note that

$$\langle\langle\eta\rangle\xi,\eta\rangle^2 = 1 + \langle\eta\rangle^2|\xi|^2 + |\eta|^2 = \langle\eta\rangle^2\langle\xi\rangle^2.$$

For the first case, we need that

$$\left|\int_{\mathbb{R}^n} e^{i\xi\cdot z}\partial_x^\beta D_\xi^{\gamma_1} p(x,\langle\eta\rangle\xi,\eta) D_\xi^{\gamma_2}\varphi_j(\xi)d\xi\right| \le C\langle\eta\rangle^{-|\gamma_1|-\mu}2^{j(-\mu+n-|\gamma|)},$$

for all $\gamma \in \mathbb{N}_0^n$.

For the second case, we need that

$$\left|\int_{\mathbb{R}^n} e^{i\xi\cdot z}\partial_x^\beta D_\xi^{\gamma_1} p(x,\langle\eta\rangle\xi,\eta) D_\xi^{\gamma_2}\varphi_j(\xi)d\xi\right| \le C\langle\eta\rangle^{-|\gamma_1|-\mu}2^{j(-\mu+n-|\gamma|)},$$

for all $|\gamma| \le n+1$.

For the third case, we need that

$$\left|\int_{\mathbb{R}^n} e^{i\xi\cdot z}\partial_x^\beta D_\xi^{\gamma_1} p(x,\langle\eta\rangle\xi,\eta) D_\xi^{\gamma_2}\varphi_j(\xi)d\xi\right| \le C\langle\eta\rangle^{-|\gamma_1|-\mu}\ln(\langle\eta\rangle)2^{j(-\mu)}, \quad |\gamma|=n,$$

and

$$\left|\int_{\mathbb{R}^n} e^{i\xi\cdot z}\partial_x^\beta D_\xi^{\gamma_1} p(x,\langle\eta\rangle\xi,\eta) D_\xi^{\gamma_2}\varphi_j(\xi)d\xi\right| \le C\langle\eta\rangle^{-n-|\gamma_1|}\langle\eta\rangle 2^{j(-\mu-1)}, \quad |\gamma|=n+1.$$

In fact, we have

$$|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| \le c_{\alpha\beta\gamma}|z|^{-n}\langle\eta\rangle^{n-\mu}\ln\langle\eta\rangle 2^{j(-\mu)}.$$
$$|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| \le c_{\alpha\beta\gamma}|z|^{-n-1}\langle\eta\rangle^{n-\mu+1}2^{j(-\mu-1)}.$$

Hence,

$$\begin{aligned}|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| &= |\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)|^\theta|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)|^{1-\theta}\\ &= c_{\alpha\beta\gamma}\left(|z|^{-n}\langle\eta\rangle^{n-\mu}\ln\langle\eta\rangle 2^{j(-\mu)}\right)^\theta\left(|z|^{-n-1}\langle\eta\rangle^{n-\mu+1}2^{j(-\mu-1)}\right)^{1-\theta}\\ &= c_{\alpha\beta\gamma}\langle\eta\rangle^{n-\mu}|z|^{-n-(1-\theta)}2^{j(-\mu+\theta-1)}\left(\ln\langle\eta\rangle\right)^\theta\langle\eta\rangle^{1-\theta}\\ &\le c_{\alpha\beta\gamma}\langle\eta\rangle^{n-\mu}|z|^{-n-\tilde\varepsilon}2^{j(-\mu-\tilde\varepsilon)}\langle\eta\rangle^{\varepsilon},\end{aligned}$$

where $0<\tilde\varepsilon:=1-\theta<\varepsilon$.

Case 1)

In this case, we have

$$\begin{aligned}&|\int_{\mathbb{R}^n} e^{i\xi\cdot z}\partial_x^\beta D_\xi^{\gamma_1} p(x,\langle\eta\rangle\xi,\eta) D_\xi^{\gamma_2}\varphi_j(\xi)d\xi|\\ &\le c_{\beta\gamma_2}\int_{\mathbb{R}^n}\langle\langle\eta\rangle\xi,\eta\rangle^{-\mu-|\gamma_1|}|D_\xi^{\gamma_2}\varphi_j(\xi)|d\xi\\ &\le c_{\beta\gamma_2}\int_{\mathbb{R}^n}\langle\eta\rangle^{-\mu-|\gamma_1|}\langle\xi\rangle^{-\mu-|\gamma_1|}|D_\xi^{\gamma_2}\varphi_j(\xi)|d\xi\\ &\le C\langle\eta\rangle^{-\mu-|\gamma_1|}2^{jn}2^{j(-\mu-|\gamma_1|)}2^{-j|\gamma_2|},\end{aligned}$$

where 2^{jn} comes from $\int_{\operatorname{supp}\varphi_j} d\varphi$.

Case 2)

Suppose $j\neq 0$ *or* $\gamma_2\neq 0$. In this case, $\operatorname{supp} D_\xi^{\gamma_2}\varphi_j(\xi)\subset\mathbb{R}^n\setminus B_1(0)$, where $B_1(0)$ denotes the unit ball. Then,

$$\begin{aligned}&|\int_{\mathbb{R}^n} e^{i\xi\cdot z}\partial_x^\beta D_\xi^{\gamma_1} p(x,\langle\eta\rangle\xi,\eta) D_\xi^{\gamma_2}\varphi_j(\xi)d\xi|\\ &\le c_{\beta\gamma_2}\int_{\mathbb{R}^n}\langle\langle\eta\rangle\xi,\eta\rangle^{-2\mu}\langle\langle\eta\rangle\xi\rangle^{\mu-|\gamma_1|}|D_\xi^{\gamma_2}\varphi_j(\xi)|d\xi\\ &\le c_{\beta\gamma_2}\int_{\mathbb{R}^n}\langle\eta\rangle^{-2\mu}\langle\xi\rangle^{-2\mu}\langle\eta\rangle^{\mu-|\gamma_1|}\langle\xi\rangle^{\mu-|\gamma_1|}|D_\xi^{\gamma_2}\varphi_j(\xi)|d\xi\\ &\le c_{\beta\gamma_2}\int_{\mathbb{R}^n}\langle\eta\rangle^{-\mu-|\gamma_1|}\langle\xi\rangle^{-\mu-|\gamma_1|}|D_\xi^{\gamma_2}\varphi_j(\xi)|d\xi\\ &\le C\langle\eta\rangle^{-\mu-|\gamma_1|}2^{jn}2^{j(-\mu-|\gamma_1|)}2^{-j|\gamma_2|},\end{aligned}$$

where we have used that, for $|\xi| \geq 1$, we have $\langle\langle\eta\rangle\xi\rangle^{-1} \leq C\langle\eta\rangle^{-1}\langle\xi\rangle^{-1}$, for some constant $C > 0$. This can be seen from the following estimate: For $|\xi| \geq 1$

$$\begin{aligned}\langle\langle\eta\rangle\xi\rangle^2 = 1 + \langle\eta\rangle^2|\xi|^2 = 1 + \langle\eta\rangle^2(\frac{1}{2}|\xi|^2 + \frac{1}{2}|\xi|^2) &\geq 1 + \langle\eta\rangle^2(\frac{1}{2} + \frac{1}{2}|\xi|^2)\\ &= 1 + \frac{1}{2}\langle\eta\rangle^2(1 + |\xi|^2)\\ &\geq \frac{1}{2}\langle\eta\rangle^2\langle\xi\rangle^2.\end{aligned} \tag{4.3}$$

Suppose $j = 0$ and $|\gamma_2| = 0$.

We must prove that

$$\left|\int_{\mathbb{R}^n} e^{i\xi\cdot z}\partial_x^\beta D_\xi^{\gamma_1} p(x, \langle\eta\rangle\xi, \eta)\varphi_0(\xi)d\xi\right| \leq C\langle\eta\rangle^{-|\gamma_1|-\mu}.$$

We notice that for $\gamma_1 = 0$ then

$$\begin{aligned}|\int_{\mathbb{R}^n} e^{i\xi\cdot z}\partial_x^\beta p(x, \langle\eta\rangle\xi, \eta)\varphi_0(\xi)d\xi| &\leq \int_{|\xi|\leq 2} |\langle\langle\eta, \xi\rangle, \eta\rangle^{-\mu}|d\xi\\ &\leq \int_{|\xi|\leq 2} \langle\eta\rangle^{-\mu}\langle\xi\rangle^{-\mu}d\xi \leq C\langle\eta\rangle^{-\mu}.\end{aligned} \tag{4.4}$$

For $\gamma_1 \neq 0$, we have

$$\begin{aligned}&|\int_{\mathbb{R}^n} e^{i\xi\cdot z}\partial_x^\beta D_\xi^{\gamma_1} p(x, \langle\eta\rangle\xi, \eta)\varphi_0(\xi)d\xi|\\ &\leq \int_{|\xi|\leq 2} |\langle\langle\eta\rangle\xi, \eta\rangle^{-2\mu}\langle\langle\eta\rangle\xi\rangle^{\mu-|\gamma_1|}|d\xi\\ &\leq \langle\eta\rangle^{-2\mu}\int_{|\xi|\leq 2} |\langle\xi\rangle^{-2\mu}\langle\langle\eta\rangle\xi\rangle^{\mu-|\gamma_1|}|d\xi\\ \text{(Using } \zeta = \langle\eta\rangle\xi\text{)} &\leq \langle\eta\rangle^{-2\mu}\int_{|\zeta|\leq 2\langle\eta\rangle} |\langle\langle\eta\rangle^{-1}\zeta\rangle^{-2\mu}\langle\zeta\rangle^{\mu-|\gamma_1|}\langle\eta\rangle^{-n}|d\zeta\\ \text{(Using polar coordinates)} &\leq C\langle\eta\rangle^{-2\mu-n}\left[\int_0^1 |\langle\rho\rangle^{\mu-|\gamma_1|}|\rho^{n-1}d\rho + \int_1^{2\langle\eta\rangle} \langle\rho\rangle^{\mu-|\gamma_1|}\rho^{n-1}|d\rho\right]\\ &\leq \langle\eta\rangle^{-2\mu-n}|C + C\langle\eta\rangle^{\mu-|\gamma_1|+n}| \leq C_1\langle\eta\rangle^{-\mu-|\gamma_1|},\end{aligned}$$

as long as $\mu + n > |\gamma|$. Since, $\mu \geq 2$, this is the case for $|\gamma| \leq n + 1$.

Case 3)

Suppose $j \neq 0$ or $\gamma_2 \neq 0$.

We use again that $\langle\langle\eta\rangle\xi\rangle \leq C\langle\eta\rangle\langle\xi\rangle$. Then, as we did above

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} e^{i\xi\cdot z}\partial_x^\beta D_\xi^{\gamma_1} p(x, \langle\eta\rangle\xi, \eta)D_\xi^{\gamma_2}\varphi_j(\xi)d\xi\right|\\ &\leq \int_{\mathbb{R}^n} |\langle\langle\eta\rangle\xi, \eta\rangle^{-\mu}\langle\langle\eta\rangle\xi\rangle^{-|\gamma_1|}D_\xi^{\gamma_2}\varphi_j(\xi)|d\xi\\ &\leq \int_{\mathbb{R}^n} |\langle\eta\rangle^{-\mu-|\gamma_1|}\langle\xi\rangle^{-\mu-|\gamma_1|}D_\xi^{\gamma_2}\varphi_j(\xi)|d\xi\\ &\leq C\langle\eta\rangle^{-\mu-|\gamma_1|}2^{j(-\mu-|\gamma_1|)}2^{-j|\gamma_2|}2^{jn}.\end{aligned}$$

Suppose $j=0$ and $|\gamma_2|=0$.
First we prove that for $|\gamma_1|<n$, we have

$$\left|\int_{\mathbb{R}^n} e^{i\xi\cdot z}\partial_x^\beta D_\xi^{\gamma_1} p(x,\langle\eta\rangle\xi,\eta)\varphi_0(\xi)d\xi\right| \leq C\langle\eta\rangle^{-|\gamma_1|-\mu}.$$

In fact, if $\zeta=\langle\eta\rangle\xi$ and $|\xi|\leq 2$, we have $1\geq\langle\langle\eta\rangle^{-1}\zeta\rangle^{-\mu}\geq\langle\langle\eta\rangle^{-1}2\langle\eta\rangle\rangle^{-\mu}=\langle 2\rangle^{-\mu}$ which is bounded. Then

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} e^{i\xi\cdot z}\partial_x^\beta D_\xi^{\gamma_1} p(x,\langle\eta\rangle\xi,\eta)\varphi_0(\xi)d\xi\right| \\
&\leq \left|\int_{|\xi|\leq 2}\langle\langle\eta\rangle\xi,\eta\rangle^{-\mu}\langle\langle\eta\rangle\xi\rangle^{-|\gamma_1|}d\xi\right| \\
&\leq \langle\eta\rangle^{-\mu}\int_{|\xi|\leq 2}|\langle\xi\rangle^{-\mu}\langle\langle\eta\rangle\xi\rangle^{-|\gamma_1|}|d\xi \\
(\text{Using } \zeta=\langle\eta\rangle\xi)\ &\leq \langle\eta\rangle^{-\mu}\int_{|\zeta|\leq 2\langle\eta\rangle}|\langle\langle\eta\rangle^{-1}\zeta\rangle^{-\mu}\langle\zeta\rangle^{-|\gamma_1|}\langle\eta\rangle^{-n}|d\zeta \\
(\text{Using polar coordinates})\ &\leq C\langle\eta\rangle^{-\mu-n}\left[\int_0^1\langle\rho\rangle^{-|\gamma_1|}\rho^{n-1}d\rho+\int_1^{2\langle\eta\rangle}\langle\rho\rangle^{-|\gamma_1|}\rho^{n-1}d\rho\right] \\
&\leq \langle\eta\rangle^{-\mu-n}|C+C\langle\eta\rangle^{-|\gamma_1|+n}|\leq C_1\langle\eta\rangle^{-\mu-|\gamma_1|},
\end{aligned}$$

as long as $n>|\gamma_1|$.
If $|\gamma_1|=n$, then

$$\begin{aligned}
&\langle\eta\rangle^{-\mu-n}\left[\int_0^1\langle\rho\rangle^{-|\gamma_1|}\rho^{n-1}d\rho+\int_1^{2\langle\eta\rangle}\langle\rho\rangle^{-|\gamma_1|}\rho^{n-1}d\rho\right] \\
&\leq \langle\eta\rangle^{-\mu-n}\left[\int_0^1\langle\rho\rangle^{-|\gamma_1|}\rho^{n-1}d\rho+\int_1^{2\langle\eta\rangle}\rho^{-1}d\rho\right] \\
&\leq \langle\eta\rangle^{-\mu-n}|C+\ln\langle\eta\rangle|\leq C_1\langle\eta\rangle^{-\mu-|\gamma_1|}\ln\langle\eta\rangle,
\end{aligned}$$

for $\langle\eta\rangle\geq R>1$.
If $|\gamma_1|=n+1$, then

$$\begin{aligned}
&\langle\eta\rangle^{-\mu-n}\left[\int_0^1\langle\rho\rangle^{-|\gamma_1|}\rho^{n-1}d\rho+\int_1^{2\langle\eta\rangle}\langle\rho\rangle^{-|\gamma_1|}\rho^{n-1}d\rho\right] \\
&\leq \langle\eta\rangle^{-\mu-n}\left[\int_0^1\langle\rho\rangle^{-|\gamma_1|}\rho^{n-1}d\rho+\int_1^{2\langle\eta\rangle}\rho^{-2}d\rho\right] \\
&\leq \langle\eta\rangle^{-\mu-n}|C+C\langle\eta\rangle^{-1}|\leq C\langle\eta\rangle^{-\mu-n}=C\langle\eta\rangle^{-\mu-|\gamma_1|}\langle\eta\rangle.
\end{aligned}$$

The proof of Lemma 4.3.2 is now complete. □

Proposition 4.3.2. *The function $k=\sum_{j=0}^{\infty}k_j$ satisfies:*

Case 1: Let $\beta\in\mathbb{N}^n$.

$$|\partial_x^\beta k(x,\langle\eta\rangle^{-1}z,\eta)|\leq C\phi(z)\langle\eta\rangle^{-\mu+n}$$

for some $\phi\in L_1(\mathbb{R}^n)$.

Case 2: Let $\beta \in \mathbb{N}^n$. *Then,*

$$|\partial_x^\beta k(x, \langle\eta\rangle^{-1}z, \eta)| \leq C\phi(z)\langle\eta\rangle^{-\mu+n},$$

for some $\phi \in L_1(\mathbb{R}^n)$.

Case 3: Let $\beta \in \mathbb{N}^n$. *Then,*

$$|\partial_x^\beta k(x, \langle\eta\rangle^{-1}z, \eta)| \leq C\phi(z)\langle\eta\rangle^{-\mu+n+\varepsilon}$$

for some $\phi \in L_1(\mathbb{R}^n)$.

Proof. *Case 1: Step 1) Let* $0 < |z| \leq 1$.
We recall that

$$\sum_{j=0}^{N} r^j \begin{cases} = N+1, & \text{if } r = 1, \\ \leq \dfrac{1}{1-r}, & \text{if } r < 1, \\ \leq Cr^N, & \text{if } r > 1. \end{cases}$$

First, we estimate the series for $2^j \leq |z|^{-1}$. In this case, for $N = 0$, we have

$$\begin{aligned} &\sum_{2^j \leq |z|^{-1}} |\partial_x^\beta k_j(x, \langle\eta\rangle^{-1}z, \eta)| \leq C\langle\eta\rangle^{-\mu+n} \left(\sum_{2^j \leq |z|^{-1}} 2^{j(-\mu+n)} \right) \\ &= C\langle\eta\rangle^{-\mu+n} \left(\sum_{j=0}^{\log_2 |z|^{-1}} (2^{-\mu+n})^j \right) \\ &\leq \left\{ \begin{array}{rl} C\langle\eta\rangle^{-\mu+n}|z|^{\mu-n}, & \text{if } -\mu+n > 0, \\ C\log_2 |z|^{-1}, & \text{if } -\mu+n = 0, \\ C\langle\eta\rangle^{-\mu+n}, & \text{if } -\mu+n < 0. \end{array} \right\} \end{aligned}$$

On the other hand, for estimate the terms $2^j \geq |z|^{-1}$, we choose $N > -\mu+n$. Then, we have

$$\begin{aligned} &\sum_{2^j > |z|^{-1}} |\partial_x^\beta k_j(x, \langle\eta\rangle^{-1}z, \eta)| \leq C|z|^{-N}\langle\eta\rangle^{-\mu+n} \left(\sum_{2^j > |z|^{-1}} 2^{j(-\mu+n-N)} \right) \\ &= C|z|^{-N}\langle\eta\rangle^{-\mu+n} \left(\sum_{j=\log_2 |z|^{-1}}^{\infty} 2^{j(-\mu+n-N)} \right) = C|z|^{-N}\langle\eta\rangle^{-\mu+n} \left(\sum_{j=0}^{\infty} (2^{-\mu+n-N})^{j+\log_2 |z|^{-1}} \right) \\ &= C|z|^{-N}\langle\eta\rangle^{-\mu+n} \left(2^{(-\mu+n-N)\log_2 |z|^{-1}} \sum_{j=0}^{\infty} 2^{j(-\mu+n-N)} \right) \\ &\leq C|z|^{-N}\langle\eta\rangle^{-\mu+n}(C_1|z|^{N+\mu-n}) = \tilde{C}|z|^{\mu-n}\langle\eta\rangle^{-\mu+n}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
|\partial_x^\beta k(x,\langle\eta\rangle^{-1}z,\eta)| &= |\sum_{j=0}^{\infty}\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| \\
&\le \sum_{2^j\le|z|^{-1}}|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| + \sum_{2^j>|z|^{-1}}|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta) \\
&\le \left\{\begin{array}{rl} \tilde{C}_1\langle\eta\rangle^{-\mu+n}|z|^{\mu-n} := \psi_1(z), & \text{if } -\mu+n>0, \\ C\log_2|z|^{-1}+\tilde{C} := \psi_2(z), & \text{if } -\mu+n=0, \\ (C+\tilde{C}|z|^{\mu-n})\langle\eta\rangle^{-\mu+n} := \psi_3(z), & \text{if } -\mu+n<0. \end{array}\right\} \in L_1(B_1(0))
\end{aligned}
\tag{4.5}
$$

Step 2) Let $|z|>1$.

We choose $M>n-\mu$ and $M>n$ to conclude that

$$
\begin{aligned}
&\sum_{j=0}^{\infty}|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| \le c_{\alpha\beta\gamma}\langle\eta\rangle^{-\mu+n}|z|^{-M}\sum_{j=0}^{\infty}2^{j(-\mu+n-M)} \\
&\le c_{\alpha\beta\gamma}\langle\eta\rangle^{-\mu+n}|z|^{-M} \in L_1(\{z\in\mathbb{R}^n : |z|>1\}).
\end{aligned}
$$

Case 2: Step 1) Let $0<|z|\le 1$ and $N=0$. We estimate the terms $2^j\le|z|^{-1}$. Then,

$$
\begin{aligned}
\sum_{2^j\le|z|^{-1}}|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| &\le C\langle\eta\rangle^{-\mu+n}\sum_{j=0}^{\log|z|^{-1}}2^{j(-\mu+n)} \\
&\le \left\{\begin{array}{rl} C\langle\eta\rangle^{-\mu+n}|z|^{\mu-n}, & \text{if } -\mu+n>0, \\ C\log_2|z|^{-1}, & \text{if } -\mu+n=0, \\ C\langle\eta\rangle^{-\mu+n}, & \text{if } -\mu+n<0, \end{array}\right\} \in L_1(B_1(0))
\end{aligned}
$$

as we did above in *Case 1, Step 1*. For the terms $2^j>|z|^{-1}$, we choose $N=n+1$. Then,

$$
\begin{aligned}
\sum_{2^j\le|z|^{-1}}|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| &\le |z|^{-n-1}\langle\eta\rangle^{n-\mu}\sum_{j=\log|z|^{-1}}^{\infty}2^{j(-\mu-1)} \\
&= |z|^{-n-1}\langle\eta\rangle^{n-\mu}2^{(-\mu-1)\log|z|^{-1}}\sum_{j=0}^{\infty}2^{j(-\mu-1)} \\
&\le C|z|^{\mu-n}\langle\eta\rangle^{n-\mu} \in L_1(B_1(0)).
\end{aligned}
\tag{4.6}
$$

Therefore,

$$
\begin{aligned}
|\partial_x^\beta k(x,\langle\eta\rangle^{-1}z,\eta)| &= |\sum_{j=0}^{\infty}\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| \\
&\le \sum_{2^j\le|z|^{-1}}|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| + \sum_{2^j>|z|^{-1}}|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta) \\
&\in L_1(B_1(0)).
\end{aligned}
\tag{4.7}
$$

Step 2). Let $|z|>1$. We use $N=n+1$ in this case, $-\mu+n-N=-\mu-1<0$. Then,

$$
\begin{aligned}
\sum_{j=0}^{\infty}|\partial_x^\beta k_j(x,\langle\eta\rangle^{-1}z,\eta)| &\le c_{\alpha\beta\gamma}\langle\eta\rangle^{-\mu+n}|z|^{-n-1}\sum_{j=0}^{\infty}2^{j(-\mu-1)} \\
&\le c_{\alpha\beta\gamma}\langle\eta\rangle^{-\mu+n}|z|^{-n-1} \in L_1(\{z\in\mathbb{R}^n : |z|>1\}).
\end{aligned}
$$

Case 3. Step 1) Let $|z| > 1$ and $M \in (n, n+1)$. Then,

$$\sum_{j=0}^{\infty} |\partial_x^\beta k_j(x, \langle\eta\rangle^{-1}z, \eta)| \le |z|^{-M}\langle\eta\rangle^{n-\mu+\varepsilon} \sum_{j=0}^{\infty} 2^{j(-\mu-\tilde{\varepsilon})} \le C|z|^{-M}\langle\eta\rangle^{n-\mu+\varepsilon} \in L_1(\{z \in \mathbb{R}^n : |z| > 1\}). \tag{4.8}$$

because using polar coordinates

$$\int_{|z|>1} |z|^{-M} dz = C \int_1^\infty \rho^{-M}\rho^{n-1} d\rho < \infty,$$

using that $n < M$. $(n - M - 1 < -1)$.

Step 2) Let $0 < |z| \le 1$. First, we estimate $2^j < |z|^{-1}$ and we choose $N = 0$. Then,

$$\begin{aligned}\sum_{2^j \le |z|^{-1}} |\partial_x^\beta k_j(x, \langle\eta\rangle^{-1}z, \eta)| &\le \langle\eta\rangle^{-\mu+n} \sum_{j=0}^{\log_2 |z|^{-1}} 2^{j(-\mu+n)} \\ &\le C\langle\eta\rangle^{-\mu+n}\phi(z) \in L_1(B_1(0)).\end{aligned} \tag{4.9}$$

For $2^j > |z|^{-1}$ we choose $M \in (n, n+1)$. Then,

$$\begin{aligned}\sum_{2^j > |z|^{-1}} |\partial_x^\beta k_j(x, \langle\eta\rangle^{-1}z, \eta)| &\le C|z|^{-M}\langle\eta\rangle^{-\mu+n+\varepsilon} \sum_{j=\log_2 |z|^{-1}}^{\infty} 2^{j(-\mu-\tilde{\varepsilon})} \\ &\le C\langle\eta\rangle^{-\mu+n+\varepsilon}|z|^{-M+\mu+\tilde{\varepsilon}} \in L_1(B_1(0)),\end{aligned} \tag{4.10}$$

because $\mu \ge 2$, which implies that $\mu + n \ge 2 + n > M$. Then, $\tilde{\varepsilon} + n + \mu > M$ if and only if, $\tilde{\varepsilon} + n + \mu - M > 0$. In conclusion, we have that

$$\begin{aligned}\sum_{j=0}^{\infty} \partial_x^\beta k_j(x, \langle\eta\rangle^{-1}z, \eta) &= \sum_{2^j \le |z|^{-1}} \partial_x^\beta k_j(x, \langle\eta\rangle^{-1}z, \eta) + \sum_{2^j > |z|^{-1}} \partial_x^\beta k_j(x, \langle\eta\rangle^{-1}z, \eta) \\ &\le \langle\eta\rangle^{-\mu+n+\varepsilon}\phi(z) \in L_1(B_1(0)).\end{aligned} \tag{4.11}$$

□

Now we prove Theorem 4.3.1.

Proof. First we will prove that k satisfies the conditions in Definition 4.3.1.

Case 1. Let us prove by steps.

1) Changing the variables $y = \langle\eta\rangle^{-1}z$ and $dy = \langle\eta\rangle^{-n}dz$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |k(x, y, \eta)| dy &= \int_{\mathbb{R}^n} |k(x, \langle\eta\rangle^{-1}z, \eta)| \langle\eta\rangle^{-n} dz \\ &\le \langle\eta\rangle^{-n} \left(\int_{|z|\le 1} \phi(z)\langle\eta\rangle^{-\mu+n+\varepsilon} dz \right),\end{aligned} \tag{4.12}$$

where $\phi(z)$ is as in Proposition 4.3.2. Notice that $\varepsilon = 0$ in *Case 1* and *Case 2*. In order to clarify the statements above about integrability for the functions that appear in estimates of k, we prove for some cases. The cases are analogous. For *Case 1,* we hold the cases, $-\mu + n > 0, -\mu + n = 0$, and $-\mu + n > 0$. For $-\mu + n > 0$, we have

$$\int_{\mathbb{R}^n} |k(x, y, \eta)| dy \le \langle\eta\rangle^{-n}\langle\eta\rangle^{n-\mu+\varepsilon} \int_{|z|\le 1} \phi(z) dz \le C\langle\eta\rangle^{-n+\varepsilon}. \tag{4.13}$$

2) Notice that for *Case 1* and *Case 2,* we have

$$\int_{\mathbb{R}^n} |k(x,y,\eta) - k(\tilde{x},y,\eta)|dy = \int_{\mathbb{R}^n} |\sum_{i=1}^{n}(\tilde{x}_i - x_i)\int_0^1 \partial_{x_i}k(x+\theta(\tilde{x}-x),y,\eta)d\theta|dy$$
$$\leq \sum_{i=1}^{n}\int_{\mathbb{R}^n}|\tilde{x}_i - x_i|\int_{\mathbb{R}^n}\int_0^1 |\partial_{x_i}k(x+\theta(\tilde{x}-x),y,\eta)|d\theta dy$$
$$\leq \sum_{i=1}^{n}\int_{\mathbb{R}^n}|\tilde{x}_i - x_i|\int_{\mathbb{R}^n}\int_0^1 |\partial_{x_i}k(x+\theta(\tilde{x}-x),\langle\eta\rangle^{-1}y,\eta)|d\theta\langle\eta\rangle^{-n}dy$$
$$\leq \sum_{i=1}^{n}\left(\int_{\mathbb{R}^n}\phi(y)dy\right)\langle\eta\rangle^{-\mu}|\tilde{x}_i - x_i| \leq C\langle\eta\rangle^{-\mu}\|x-\tilde{x}\|,$$

and the result follows easily. For *Case 3* we have $\langle\eta\rangle^{-\mu+\varepsilon}$ in place to $\langle\eta\rangle^{-\mu}$. □

4.4 Applications on compact manifolds without singularities

Let us study applications using the three cases of Theorem 4.3.1. The first and second cases of Theorem 4.3.1 will be used for compact manifold without singularities and without boundary. As for the third case, it will be applied to conic manifolds in Chapter 5. As we defined in Section 2.1, $\Lambda := \Lambda_a(\phi)$ denotes a sector in the complex plane, with $a \in \mathbb{R}$ and $\phi \in [0, 2\pi)$ an angle.
First we use an adaptation of Theorem 5.1 proved by Shubin in [29], for Λ- Elliptic operators as we saw in Definition 4.1.1. Then we study pseudodifferential operators using the approach of Seeley [28], see also Escher/Seeley [12].

Remark 4.4.1. *In [29] it was used the expression* $|\xi| + |\lambda|^{\frac{1}{\mu}}$ *in place to* $1 + |\xi| + |\lambda|^{\frac{1}{\mu}}$*. However, we recall that the two terms are equivalent. In fact, for sufficiently large* $R > 0$ *such that* $|\xi| + |\lambda|^{\frac{1}{\mu}} \geq R$ *then*

$$\begin{aligned}1 + |\xi| + |\lambda|^{\frac{1}{\mu}} \geq |\xi| + |\lambda|^{\frac{1}{\mu}} &= \frac{1}{2}\left(|\xi| + |\lambda|^{\frac{1}{\mu}}\right) + \frac{1}{2}\left(|\xi| + |\lambda|^{\frac{1}{\mu}}\right)\\ &\geq \frac{R}{2} + \frac{1}{2}\left(|\xi| + |\lambda|^{\frac{1}{\mu}}\right) \geq \min\left\{\frac{R}{2},\frac{1}{2}\right\}(1 + |\xi| + |\lambda|^{\frac{1}{\mu}}),\end{aligned} \tag{4.14}$$

which prove that the two terms are equivalent. In the same way, in [29] it was used the term $(1+|\eta| + |\lambda|^{\frac{1}{d}})$ *instead of* $(1+|\eta|^2+|\lambda|^{\frac{2}{d}})^{\frac{1}{2}}$*, which are equivalent too. Hence, we do not need to worry about the difference between the terms from the reference and the terms that we are using in this work.*

4.4.1 Differential operators

Now we consider differential operators in order to use *Case 1* from Theorem 4.3.1.

Theorem 4.4.1. *If* P *is a* Λ*-Elliptic differential operator of order* μ*, then there exists* $B_\lambda \in L^{-\mu,\mu}(M,\Lambda)$ *such that*

$$PB_\lambda = I + R_1$$

and

$$B_\lambda P = I + R_2$$

where R_1, R_2 *belong to* $L^{-\infty}(M,\Lambda)$*.*

Proof. See Theorem 5.1 in [29]. □

Consider an compact manifold M without singularities, $1 < p < \infty$, $\mu \in \mathbb{R}$ and the operator

$$P : H^{\mu}_{p}(M) \to L_p(M) \tag{4.15}$$

where $H^{\mu}_{p}(M)$ is the usual Sobolev space. Let Λ be an sector as in Section 2.1. Let us suppose that P is Λ-Elliptic as in Definition 4.1.1. That is, for any local chart, P has a symbol p that satisfies

$$|(\partial^{\alpha}_{\xi}\partial^{\beta}_{x}p(x,\xi,\lambda))(\lambda - p(x,\xi,\lambda))^{-1}| \le C\langle\xi\rangle^{-|\alpha|.}$$

By Theorem 4.4.1 for larger λ, P is invertible. In fact, $P^{-1} = B_{\lambda}(I + R_1)^{-1} = (I + R_2)^{-1}B_{\lambda}$. As consequence we have the following proposition.

Proposition 4.4.1. *For P as in* (4.15) *with $D(P) = \{u \in C(M) : Pu \in C(M)\}$. Then $P : D(P) \to C(M)$ is a sectorial operator.*

Remark 4.4.2. *We note that*

$$D(P) \subset \{u \in L_p(M) : Pu \in L_p(M)\} = H^{\mu}_{p}(M).$$

Proof. By [27] and references cited there, we know that for λ larger, $\lambda - P : H^{\mu}_{p}(M) \to L_p(M)$ is a bijection. Then, the restriction $P_C = P_{|_{D(P)}} : D(P) \to C(M)$ is a bijection. In fact, the injection of $\lambda - P_C$ is follows by injection of $\lambda - P$. For the surjection we choose $p > \frac{n}{\mu}$, which implies by Sobolev embedding theorem, that $H^{\mu}_{p}(M) \hookrightarrow C(M)$. As a consequence, for $f \in C(M) \subset L_p(M)$ there exists $u \in H^{\mu}_{p}(M)$ such that $(\lambda - P)u = f$ (Surjectivity of $\lambda - P$). Therefore, $u \in D(P)$ and $(\lambda - P_C)u = f$. Moreover by the Theorem 4.3.1, *Case 1*, we have that

$$(\lambda - P)^{-1} : C(M) \to C(M)$$

is continuous and $||(\lambda - P)^{-1}||_{\mathcal{L}(C(M))} \le C|\lambda|^{-1}$ for λ large. Finally, we conclude that $P(\lambda)$ is a sectorial operator in $C(M)$.

4.4.2 Pseudodifferential operators

Now we will use the result of Escher and Seiler [12] in order to apply the *Case 2* of Theorem 4.3.1. We consider again a pseudodifferential operator of order μ as an operator

$$P : H^{\mu}_{p}(M) \to L_p(M).$$

Let a be the symbol associate to P and let us suppose that it is $\Lambda - \mathit{Elliptic}$ as in Definition 4.1.1 with $R = 0$. In such case, we have the following result:

Theorem 4.4.2. *There exists $p \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \Lambda)$ such that*

$$|\partial^{\alpha}_{\xi}\partial^{\beta}_{x}p(x,\xi,\lambda)| \le C(\langle\xi\rangle^{\mu} + |\lambda|)^{-1}\langle\xi\rangle^{-|\alpha|} \tag{4.16}$$

uniformly in $\mathbb{R}^n \times \mathbb{R}^n \times \Lambda$ for all $\alpha, \beta \in \mathbb{N}^n_0$, as well as

$$|\partial^{\alpha}_{\xi}\partial^{\beta}_{x}(p(x,\xi,\lambda) - (\lambda - a(x,\xi))) \le C(\langle\xi\rangle^{\mu} + |\lambda|)^{-3}\langle\xi\rangle^{2\mu-1-|\alpha|}. \tag{4.17}$$

Moreover,

$$p(x,D,\lambda)(\lambda - a(x,D)) = 1 + r_0(x,D,\lambda),$$
$$(\lambda - a(x,D))p(x,D,\lambda) = 1 + r_1(x,D,\lambda)$$

with remainders satisfying

$$\sup_{(x,\xi,\lambda)\in\mathbb{R}^n\times\mathbb{R}^n\times\Lambda} |\partial^{\alpha}_{\xi}\partial^{\beta}_{x}r_j(x,\xi,\lambda)|\langle\lambda\rangle\langle\xi\rangle^{N} < \infty \quad \forall\alpha,\beta \in \mathbb{N}^n_0\ \forall N \in \mathbb{N}_0.$$

Proof. See Theorem 3.8 in [12]. □

Therefore, for λ larger, P is invertible and there exists P^{-1} with a symbol p satisfying *Case 2.* In fact, for $\eta^\mu = |\lambda|$ and (4.16) we have with $|\alpha| = 0$ the following estimate.

$$|\partial_x^\beta p(x,\xi,\eta)| \leq C(\langle\xi\rangle^\mu + \eta^\mu)^{-1} \leq C\langle\xi,\eta\rangle^{-\mu}.$$

In effect, let $a := \langle\xi\rangle$ and $b := \eta$. Then,

$$(a+b)^\mu \leq (2\max\{a,b\})^\mu \leq 2^\mu(a^\mu + b^\mu).$$

This is equivalent to $a^\mu + b^\mu \geq C(a+b)^\mu$. On the other hand, we know that for positive terms by Cauchy-Schwartz inequality

$$a + b = (a,b)\cdot(1,1) \leq \sqrt{2}(a^2+b^2)^{\frac{1}{2}}$$

then

$$(a^2+b^2)^{\frac{\mu}{2}} \geq \sqrt{2}^{\frac{-\mu}{2}}(a+b)^\mu.$$

Using the results above, we have

$$\begin{aligned}\langle\xi,\eta\rangle^{-\mu} = \left(1+|\xi|^2+\eta^2\right)^{\frac{-\mu}{2}} &= \left(\langle\xi\rangle^2+\eta^2\right)^{\frac{-\mu}{2}}\\ &\geq C(\langle\xi\rangle+\eta)^{-\mu}\\ &\geq C_2(\langle\xi\rangle^\mu+\eta^\mu)^{-1}.\end{aligned} \tag{4.18}$$

For $|\alpha| \neq 0$, we will use (4.17). Then, for $|\lambda| = \eta^\mu$,

$$\begin{aligned}|\partial_x^\beta\partial_\xi^\alpha p(x,\xi,\eta)| &= |\partial_x^\beta\partial_\xi^\alpha\left(p(x,\xi,\eta)-(\eta^\mu-a(x,\xi))^{-1}\right)+\partial_x^\beta\partial_\xi^\alpha(\eta^\mu-a(x,\xi))^{-1}|\\ &\leq |\partial_x^\beta\partial_\xi^\alpha\left(p(x,\xi,\eta)-(\eta^\mu-a(x,\xi))^{-1}\right)|+|\partial_x^\beta\partial_\xi^\alpha(\eta^\mu-a(x,\xi))^{-1}|\\ &\leq C(\langle\xi\rangle^\mu+\eta^\mu)^{-3}\langle\xi\rangle^{2\mu-1-|\alpha|}+|\partial_x^\beta\partial_\xi^\alpha(\eta^\mu-a(x,\xi))^{-1}|\\ &\leq C\langle\xi,\eta\rangle^{-3\mu}\langle\xi\rangle^{2\mu-1-|\alpha|}+I\\ &= C\langle\xi,\eta\rangle^{-2\mu}\langle\xi,\eta\rangle^{-\mu}\langle\xi\rangle^{2\mu-1-|\alpha|}+I\\ &\leq C\langle\xi,\eta\rangle^{-2\mu}\langle\xi\rangle^{-\mu}\langle\xi\rangle^{2\mu-1-|\alpha|}+I = \langle\xi,\eta\rangle^{-2\mu}\langle\xi\rangle^{\mu-1-|\alpha|}+I\\ &\leq C\langle\xi,\eta\rangle^{-2\mu}\langle\xi\rangle^{\mu-|\alpha|}+I,\end{aligned}$$

where we have used that $\langle\xi,\eta\rangle^{-1} \leq \langle\xi\rangle^{-1}$ and $I = |\partial_x^\beta\partial_\xi^\alpha(\eta^\mu - a(x,\xi))^{-1}|$. The only thing that we need to prove is that $I \leq C_1\langle\xi,\eta\rangle^{-2\mu}\langle\xi\rangle^{\mu-|\alpha|}$. For that, we use that for Λ-Elliptic operators, $R = 0$ and we describe as it was done in [12] without all details. For more information, we can see Theorem 3.8 in [12]. We notice that

$$|\partial_x^\beta\partial_\xi^\alpha(\eta^\mu-a)^{-1}| = \sum_{\text{finite}}(\eta^\mu-a)^{-1}(\partial_x^{\beta_1}\partial_\xi^{\alpha_1}a)(\eta^\mu-a)^{-1}\ldots(\eta^\mu-a)^{-1}(\partial_x^{\beta_k}\partial_\xi^{\alpha_k}a)(\eta^\mu-a)^{-1}.$$

Then, for each term in this sum, we have

$$\begin{aligned}&\leq \langle\xi,\eta\rangle^{-\mu}\langle\xi\rangle^{\mu-|\alpha_1|}\langle\xi,\eta\rangle^{-\mu}\langle\xi\rangle^{-|\alpha_2|}\langle\xi\rangle^{-|\alpha_3|}\ldots\langle\xi\rangle^{-|\alpha_n|}\\ &= \langle\xi,\eta\rangle^{-2\mu}\langle\xi\rangle^{\mu-|\alpha|},\end{aligned}$$

where we have used that, $|(\eta^\mu - a)^{-1}| \leq \langle\xi,\eta\rangle^{-\mu}$, $|\partial_x^{\beta_1}\partial_\xi^{\alpha_1}a| \leq \langle\xi\rangle^{\mu-|\alpha_1|}$ and $|(\partial_x^{\beta_j}\partial_\xi^{\alpha_j}a)(\eta^\mu-a)^{-1}| \leq \langle\xi\rangle^{-|\alpha_j|}$. Therefore, $|\partial_x^\beta\partial_\xi^\alpha(\eta^\mu-a)^{-1}| \leq C_1\langle\xi,\eta\rangle^{-2\mu}\langle\xi\rangle^{-\mu-|\alpha|}$ and we are in the conditions of *Case 2* from Theorem 4.3.1 for the symbols associated to the inverse of P. This implies that $P : D(P) \to C(M)$ is a sectorial operator by the same arguments of Proposition 4.4.1.

Chapter 5

Continuous and Hölder continuous Theory

In this chapter we will prove that elliptic operators on conic manifolds acting on a class of continuous spaces are almost sectorial operators under some conditions. All the operators that we will be describe from now on are part of the resolvent of such elliptic operators. More specifically, we will assume that such operators we are studying satisfy that

$$(\lambda - \mathcal{A})^{-1} = \omega \left\{ x^{\mu} op_M^{\gamma - \frac{n}{2}} g(\lambda) + G(\lambda) \right\} \omega_0 + (1-\omega) P(\lambda)(1-\omega_1) + G_{\infty}(\lambda),$$

where $\omega, \omega_0, \omega_1 \in C_0^{\infty}([0,1))$ are cutoff functions satisfying $\omega_1 \omega = \omega_1$ and $\omega \omega_0 = \omega$ and we will describe the spaces where such operators above belong later. The main goal is that:

1) The symbol associated to $x^{\mu} op_M^{\gamma - \frac{n}{2}} g(\lambda)$ and for $(1-\omega)P(\lambda)(1-\omega_1)$ have a boundedness as Definition 4.1.1.

2) For $G(\lambda)$ and $G_{\infty}(\lambda)$ we have a faster decay in infinity which will help us in this and future works.

5.1 $\mathcal{C}^{0,\gamma}(\mathbb{B})$ and $\mathcal{C}^{\alpha,\gamma}(\mathbb{B})$ -Spaces

Below, we give a new definition suitable to conic manifolds. Let $\mathbb{B}$ be a conic manifold. Let us define the appropriate spaces for this chapter. (compare with definition 3.1.1)

Definition 5.1.1. *Let $\alpha \geq 0$ and $\gamma \in \mathbb{R}$. The space $\mathcal{C}^{\alpha,\gamma}(\mathbb{B})$ is defined as the set of all functions $u \in C^{\alpha}(\mathbb{B}^{\circ})$ such that for any coordinates in the collar neighborhood of $[0,1) \times \partial\mathcal{B}$, where $\psi : V \subset \partial\mathcal{B} \to \mathbb{R}^n$ is a chart and $\phi \in C_c^{\infty}(V)$, we have*

$$\phi \circ \psi^{-1}(y) \omega(e^{-x}) e^{(\gamma - \frac{n+1}{2})x} u(e^{-x}, \psi^{-1}(y)) \in BUC^{\alpha}(\mathbb{R}^{n+1}),$$

for any cutoff function ω. Notice that $u \in C^{\alpha}(\mathbb{B}^{\circ})$ if and only if $\psi : U \subset \mathbb{B}^{\circ} \to \mathbb{R}^n$ and $\phi \in C_c^{\infty}(U)$ which implies that $(\phi u) \circ \psi^{-1} \in BUC^{\alpha}(\mathbb{R}^{n+1})$.

In particular, we see that

$$\mathcal{C}^{0,\gamma}(\mathbb{B}) = \{ u \in C(\mathbb{B}^{\circ}) : \omega(e^{-x}) e^{(\gamma - \frac{n+1}{2})x} u(e^{-x}, y) \in BUC(\mathbb{R}^{n+1}) \}.$$

Definition 5.1.2. *We define the space $BUC_{\ln}(\mathbb{R}_+ \times \mathbb{R}^n)$ as the set of all functions $u \in BC(\mathbb{R}_+ \times \mathbb{R}^n)$ with the following property: for all $\epsilon > 0$, there exists $\delta > 0$ such that if*

$$|\ln x - \ln x'| + |y - y'| < \delta, \quad \text{then} \quad |u(x,y) - u(x',y')| < \epsilon.$$

Remark 5.1.1. *We see that if $x, x' \in (0,1]$, then $|\ln x - \ln x'| = |\int_{x'}^{x} \frac{ds}{s}| \geq |\int_{x'}^{x} ds| = |x - x'|$. Hence, if $|\ln x - \ln x'| < \delta$, then $|x - x'| < \delta$.*

Proposition 5.1.1. *Let $u \in C(\mathbb{R}_+ \times \mathbb{R}^n)$ be a function. Then, $u \in BUC_{\ln}(\mathbb{R}_+ \times \mathbb{R}^n)$ if, and only if, $u(e^{-x}, y) \in BUC(\mathbb{R}^{n+1})$.*

Proof. If $u \in BUC_{\ln}(\mathbb{R}_+ \times \mathbb{R}^n)$, then for all $\epsilon > 0$ there exists $\delta > 0$ such that $|\ln x - \ln x'| + |y - y'| < \delta$ implies $|u(x,y) - u(x',y')| < \epsilon$. In particular, if $x = e^{-t}$ and $x' = e^{-s}$ we have

$$|s - t| + |y - y'| < \delta \Longrightarrow |u(e^{-t}, y) - u(e^{-s}, y')| < \epsilon.$$

We conclude that $u(e^{-x}, y) \in BUC(\mathbb{R}^{n+1})$. In the same way, we prove the other implication. ☐

As a consequence, we have

Corollary 5.1.1. *$u \in \mathcal{C}^{0,\gamma}(\mathbb{B})$ if, and only if, $u \in C(\mathbb{B}^\circ)$ and $\omega(x) x^{\frac{n+1}{2} - \gamma} u(x,y) \in BUC_{\ln}(\mathbb{R}_+ \times \mathbb{R}^n)$, where (x,y) are coordinates close to the collar neighborhood.*

Proof. It is a consequence of the previous proposition. In fact, if $u \in \mathcal{C}^{0,\gamma}(\mathbb{B})$, $\psi : V \subset \partial\mathcal{B} \to \mathbb{R}^n$ and $\phi \in C_c^\infty(V)$ then

$$\phi \circ \psi^{-1}(y) \omega(e^{-x}) e^{(\gamma - \frac{n+1}{2})x} u(e^{-x}, \psi^{-1}(y)) \in BUC(\mathbb{R}^{n+1})$$

if, and only if, $\phi \circ \psi^{-1}(y) \omega(x) x^{\frac{n+1}{2} - \gamma} u(x, \psi^{-1}(y)) \in BUC_{\ln}(\mathbb{R}_+ \times \mathbb{R}^n)$.

Proposition 5.1.2. *For all $\epsilon > 0$ and $1 < p < \infty$, we have that $\mathcal{C}^{0,\gamma}(\mathbb{B}) \subset \mathcal{H}_p^{0,\gamma-\epsilon}(\mathbb{B})$.*

Proof. In effect, if $u \in \mathcal{C}^{0,\gamma}(\mathbb{B})$, then for some cutoff function ω,

$$\int_{[0,1)\times\partial\mathcal{B}} |\omega(x) x^{\frac{n+1}{2} - \gamma + \epsilon} u(x,y)|^p \frac{dx}{x} dy \leq ||u||^p_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \int_{[0,1)\times\partial\mathcal{B}} x^{\epsilon p - 1} dx dy < \infty. \tag{5.1}$$

The other term $(1 - \omega(x)) x^{\frac{n+1}{2} - \gamma + \epsilon} u(x,y)$ has compact support. Therefore, its integral is finite on $\mathbb{B}$. ☐

Another important definition for us is stated below.

Definition 5.1.3. *Let $s \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}$. The space $\tilde{\mathcal{C}}^{s,\gamma}(\mathbb{B})$ is defined as the set of all the functions $u : [0,\infty) \times \partial\mathcal{B} \to \mathbb{R}$ such that*

$$\phi \circ \psi^{-1}(y) e^{(\frac{n+1}{2} - \gamma)x} u(e^{-x}, \psi^{-1} y) \in BUC^s(\mathbb{R}^{n+1}),$$

where $\psi : V \subset \partial\mathcal{B} \to \mathbb{R}^n$ is a chart and $\phi \in C_c^\infty(V)$. In particular, $\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B})$ is the set of all functions $u : [0,\infty) \times \partial\mathcal{B}$ such that

$$\phi \circ \psi^{-1}(y) x^{\frac{n+1}{2} - \gamma} u(x, \psi^{-1}(y)) \in BUC_{\ln}(\mathbb{R}_+ \times \mathbb{R}^n),$$

where $||u||_{\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B})} = ||x^{\frac{n+1}{2} - \gamma} u(x,y)||_{L_\infty(\mathbb{R}_+ \times \partial\mathcal{B})}$.

5.2 Operators that compose the resolvent operator

In this section, we study the operators that compose the resolvent of elliptic operators and which are important to show that they are sectorial. Most of the time, we need to use Mellin transform and tools about pseudo-operators that will be defined later. We start with definitions that involve tensorial product and a special type of symbol that we defined in Section 2.6.

Definition 5.2.1. *Let X, Y be two sets and f, g two scalar functions defined on X and Y respectively. Then, for $x \in X$ and $y \in Y$ the tensor product of f and g, $f \otimes g : X \times Y \to \mathbb{C}$, is defined by $f \otimes g(x, y) := f(x)g(y)$.*

Definition 5.2.2. *Let X, Y two topological vector spaces. We call π-topology (or projective topology) on $X \otimes Y$ the strongest locally convex topology on this vector space for which the canonical bilinear mapping $(x, y) \mapsto x \otimes y$ of $E \times F$ into $E \otimes F$ is continuous. Provided with it, the space $X \otimes Y$ will be denoted by $X \otimes_\pi Y$.*

Definition 5.2.3. *By $X \hat{\otimes}_\pi Y$, we denote the completion of $X \otimes_\pi Y$.*

We will set an important theorem that we need next. For more details, see Theorem 45.1, [30]. Before, we recall what is a Fréchet space.

Definition 5.2.4. *We say that a vectorial space E is a Fréchet space if there exist seminorms $(p_k)_{k\in\mathbb{N}}$ such that (E, d) is a complete metric space, where $d : E \times E \to \mathbb{R}$ is the function*

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{p_j(x - y)}{1 + p_j(x - y)}.$$

Example 7. *Let M be a smooth compact manifold. Then, $C^\infty(M)$ is a Fréchet space with seminorms defined by*

$$p_j(u) = \sum_{|\alpha|\leq j} \sum_{k=1}^{N} \|\phi_k \circ \psi_k^{-1} D^\alpha (u \circ \psi_k^{-1})\|_{L_\infty(\mathbb{R}^n)},$$

where $\psi_k : U_k \subset M \to \mathbb{R}^n$ are local charts such that $M = \bigcup_{k=1}^{N} U_k$ and $\{\phi_k\}$ is a partition of unity.

Example 8. *Let X be a smooth compact Riemannian manifold. The set $L^{-\infty}(X)$ consists of all operators $K : C^\infty(X) \to C^\infty(X)$ defined by*

$$Ku(x) = \int_X k(x, y)u(y)dy,$$

where dy is the measure associated to the metric of X and $k \in C^\infty(X \times X)$. Then $K \in L^{-\infty}(X) \mapsto k \in C^\infty(X \times X)$ is a bijection. The set $L^{-\infty}(X)$ is a Fréchet space with seminorms induced by $C^\infty(X \times X)$.

Theorem 5.2.1. *Let X, Y be two Fréchet spaces. Every element $\theta \in X \hat{\otimes}_\pi Y$ is the sum of an absolutely convergent series*

$$\theta = \sum_{n=0}^{\infty} \lambda_n x_n \otimes y_n,$$

where $\{\lambda_n\}$ is a sequence of complex numbers such that $\sum_{n=0}^{\infty} |\lambda_n| < \infty$ and $\{x_n\}, \{y_n\}$ are sequences converging to zero in X and Y, respectively.

Proof. See Theorem 45.1, [30]. □

From now on, we will use $\otimes$ to denote $\hat{\otimes}_\pi$.

5.2.1 Mellin Differential and Pseudodifferential Operators

We fix $[\cdot]$ as a smooth positive function $z \in \Lambda \to [z] \in [0, \infty)$ in a sector Λ over the complex plane, as we introduced in Section 2.1, such that $[\lambda] = |\lambda|$ for large λ. Besides, we recall that $\gamma_p := (n+1)(\frac{1}{2} - \frac{1}{p})$ and $\mathbb{B}$ is a compact $n+1$ dimensional manifold with a conical point. Let $\mathcal{A} : C^\infty(\mathbb{B}) \to C^\infty(\mathbb{B})$ be a conical differential operator of order μ that near the conical point has the form

$$\mathcal{A} = x^{-\mu} \sum_{|\alpha|+j \leq \mu} a_\alpha(x, y) D_y^\alpha (-x\partial_x)^j, \tag{5.2}$$

where $a_\alpha \in C^\infty(\mathbb{R}_+, \text{Diff}^{\mu-j}(\partial\mathcal{B}))$ and y is a local coordinate of $\partial\mathcal{B}$, as we have studied in Definition 3.1.2. We know the importance of the Fourier transform to find solutions on smooth manifolds. On the other hand, the Mellin transform is very useful to study the solutions in a neighbourhood of the conical point and we will use this important tool from now on.

Definition 5.2.5. *Let $u \in C_c^\infty(\mathbb{R}_+)$ be a smooth function. The Mellin transform is defined by*

$$\mathcal{M}u(z) = \int_0^\infty x^{z-1} u(x) dx, \tag{5.3}$$

where $z \in \mathbb{C}$.

Remark 5.2.1. *We recall some important properties of the Mellin transform. For more details see [11, Section 7].*

i) $\mathcal{M}x^\gamma u(z) = \mathcal{M}u(z + \gamma)$.

ii) $\mathcal{M}(-x\partial_x)u(z) = z\mathcal{M}u(z)$.

iii) The Mellin transform can be extended to an isomorphism $\mathcal{M} : L_2(\mathbb{R}_+) \to L_2(\Gamma_{\frac{1}{2}})$, where the set $\Gamma_{\frac{1}{2}} = \{z \in \mathbb{C} : Re(z) = \frac{1}{2}\}$ is a vertical line of $\mathbb{C}$.

iv) If $v(z) = \mathcal{M}u(z)$ with $u \in C_c^\infty(\mathbb{R}_+)$. Then, $u(x) = \mathcal{M}^{-1}v(x)$ where

$$\mathcal{M}^{-1}v(x) = \frac{1}{2\pi i} \int_{\Gamma_\alpha} x^{-z} v(z) dz$$

for all α real number, with $\Gamma_\alpha = \{z \in \mathbb{C} : Re(z) = \alpha\}$. If $u \in L_2(\mathbb{R}_+)$, then we must take $\alpha = \frac{1}{2}$.

In the decomposition of the resolvent operator that we will use in this work, it appears the operators: $op_M^\gamma f, G(\lambda), G_\infty(\lambda)$ and $P(\lambda)$ that we will define from here on.

Proposition 5.2.1. *Let $\mathcal{A}$ be a Fuchs type operator of order $\mu \in \mathbb{N}_0$ as (5.2). Then, $\mathcal{A}$ can be written as*

$$x^{-\mu} op_M^{\gamma+\mu-\frac{n}{2}}(f),$$

where

$$[op_M^{\gamma+\mu-\frac{n}{2}}(f)u](x) = \frac{1}{2\pi i} \int_{Re(z)=\frac{n+1}{2}-\gamma-\mu} x^{-z} f(x, z) \mathcal{M}u(z) dz \tag{5.4}$$

and $f(x, z) = \sum_{|\alpha|+j \leq \mu} a_\alpha(x, y) D_y^\alpha z^j$, with $u \in C_c^\infty((0, 1) \times \partial\mathcal{B})$ and γ is any real number.

Proof. In effect, if we recall that $\mathcal{M}^{-1}u(x) = \frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma-\mu}} x^{-z}u(z)dz$ denotes the inverse operator of $\mathcal{M}$, then we can rewrite (5.2) as

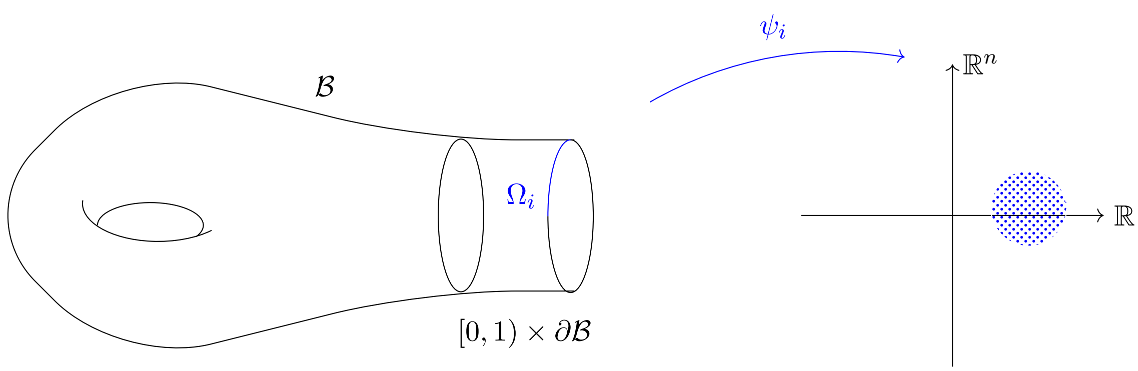
$$\begin{aligned}\mathcal{A} &= x^{-\mu}\sum_{|\alpha|+j\le\mu} a_\alpha(x,y)D_y^\alpha(-x\partial_x)^j \\ &= x^{-\mu}\sum_{|\alpha|+j\le\mu} a_\alpha(x,y)D_y^\alpha\mathcal{M}^{-1}\mathcal{M}(-x\partial_x)^j \\ &= x^{-\mu}\mathcal{M}^{-1}\sum_{|\alpha|+j\le\mu} a_\alpha(x,y)D_y^\alpha z^j\mathcal{M}.\end{aligned} \tag{5.5}$$

If we put $f(x,z) = \sum_{|\alpha|+j\le\mu} a_\alpha(x,y)D_y^\alpha z^j$, then

$$\begin{aligned}\mathcal{A} &= x^{-\mu}\mathcal{M}^{-1}f(x,z)\mathcal{M} \\ &= x^{-\mu}op_M^{\gamma+\mu-\frac{n}{2}}(f),\end{aligned} \tag{5.6}$$

with $op_M^{\gamma+\mu-\frac{n}{2}}(f)$ given by (5.4). □

Now, for the operator $\mathcal{A}$ we associate the cone operator in the Sobolev space over the infinite cylinder $\mathbb{R}_+\times\partial\mathcal{B}$, which we will denote by $\partial\mathbb{B}^\wedge$. For the next definition, let us suppose that $\partial\mathcal{B} = \bigcup\Omega_i$ is a finite covering of $\partial\mathcal{B}$ and $\psi_i : \Omega_i \subset \partial\mathcal{B} \to U_i \subset \mathbb{R}^n$ are coordinate maps and $\{\phi_i\}$ is a subordinate partition of unity.



Definition 5.2.6. *We say that $u(x,y) \in H^s_{p,cone}(\partial\mathbb{B}^\wedge)$ provided that for each i*

$$v(x,\boldsymbol{y}) = \phi_i(y)u(x,y) \in H_p^s(\mathbb{R}\times\mathbb{R}^n),$$

with $y = \psi_i^{-1}(\frac{\boldsymbol{y}}{[x]})$.

Definition 5.2.7. *Let $\mathbb{B}$ be a conic manifold, $s,\gamma\in\mathbb{R}$ and $1<p<\infty$. The spaces $\mathcal{K}_p^{s,\gamma}(\partial\mathbb{B}^\wedge)$ denote all distributions u in $\partial\mathbb{B}^\wedge$ such that for some cutoff function ω*

$$\omega u \in \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \quad \textit{and} \quad (1-\omega)u \in H^s_{p,cone}(\partial\mathbb{B}^\wedge).$$

Finally, freezing the coefficients of $\mathcal{A}$ at $t=0$, we obtain the model cone operator $\hat{\mathcal{A}}$,

$$\hat{\mathcal{A}} = x^{-\mu}\sum_{|\alpha|+j\le\mu} a_\alpha(0,y)D_y^\alpha(-x\partial_x)^j.$$

Remark 5.2.2. *Near the boundary of $\mathbb{B}$, if we work with $h(x,z,\lambda) = x^\mu \lambda - f(x,z)$ then*

$$x^{-\mu} op_M^{\gamma+\mu-\frac{n}{2}}(h) = x^{-\mu} op_M^{\gamma+\mu-\frac{n}{2}}(x^\mu \lambda - f) = \lambda - \mathcal{A}.$$

We can also define Mellin Pseudodifferential operators. First we define their symbols. The following definition is given in general for a compact manifold X. In particular, we state for the case $X = \partial\mathcal{B}$.

Definition 5.2.8. *(Pseudodifferential operators with parameters) For $\mu \in \mathbb{R}$ and $d > 0$, $L^{\mu,d}(\partial\mathcal{B},\Lambda)$ is the space of all operators $P(\lambda)$ such that, for any local chart $\psi : U \subset \partial\mathcal{B} \to V \subset \mathbb{R}^n$ and functions $\phi_1, \phi_2 \in C_c^\infty(V)$ the operator $P_{loc}(\lambda) : C_c^\infty(V) \to C_c^\infty(V)$ given by*

$$P_{loc}(\lambda)(u) = [\phi_2 P(\lambda)(\phi_1 u \circ \psi)] \circ \psi^{-1}$$

is equal to $op(p)u$, where $p \in S^{-\mu,d}(\mathbb{R}^n \times \mathbb{R}^n, \Lambda)$. In particular, we have that operators with kernel in $\mathcal{S}(\Lambda, C^\infty(\partial\mathcal{B}) \otimes C^\infty(\partial\mathcal{B}))$ belong to $L^{-\mu,\mu}(\partial\mathcal{B},\Lambda)$.

Definition 5.2.9. *For $\mu \in \mathbb{R}$ and $d > 0$, $M_O^{\mu,d}(\partial\mathcal{B},\Lambda)$ denotes all holomorphic functions $g : \mathbb{C} \to L^{\mu,d}(\partial\mathcal{B},\Lambda)$ for which*

$$g_\beta(\tau,\lambda) := g(\beta + i\tau)(\lambda) \in L^{\mu,d}(\partial\mathcal{B}, \mathbb{R}_\tau \times \Lambda)$$

and it is locally bounded as a function of β.

For $g \in M_O^{\mu,d}(\partial\mathcal{B},\Lambda)$, we can define the Mellin Pseudodifferential operator $op_M^{\gamma+\mu-\frac{n}{2}} : C_c^\infty((0,\infty) \times \partial\mathcal{B}) \to C_c^\infty((0,\infty) \times \partial\mathcal{B})$ by

$$op_M^{\gamma+\mu-\frac{n}{2}} g(\lambda)u = \frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma-\mu}} x^{-z} g(z)\mathcal{M}u(z)dz.$$

Locally, this operator can be written as

$$\frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma-\mu}} x^{-z}\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{iy\xi} g(x,y,z,\xi,\lambda)\mathfrak{F}_{x\to z}\mathcal{M}_{y\to\xi}u(z,\xi)d\xi\right)dz$$

where $x \in (0,\infty)$ and y is a local coordinate in $\partial\mathcal{B}$ and $\lambda \in \Lambda$.

5.2.2 $G_\infty(\lambda)$-Operators

In order to define the regularizing $G_\infty(\lambda)$ operators, we need a new class of function.

Definition 5.2.10. *Let $\gamma \in \mathbb{R}$. The space $C^{\infty,\gamma}(\mathbb{B})$ consist of all functions u in $C^\infty(\mathbb{B}^o)$ such that*

$$||x^{\frac{n+1}{2}-\gamma}\ln^l(x)(-x\partial_x)^j\partial_y^\alpha u(x,y)||_{L_\infty([0,1]\times K)} \le C_K, \text{ for all } l,j \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n, \tag{5.7}$$

where K is a compact subset in a coordinate neighborhood of $\partial\mathcal{B}$. Notice that $C^{\infty,\gamma}(\mathbb{B}) \neq \bigcap_{m\in\mathbb{N}} \mathcal{C}^{m,\gamma}(\mathbb{B})$.

Beware: The new class $C^{\infty,\gamma}(\mathbb{B})$ is different from the classes $\mathcal{C}^{\alpha,\gamma}(\mathbb{B})$ defined previously.

Proposition 5.2.2. *Let $u \in C^{\infty,\gamma+\epsilon}(\mathbb{B})$ be a function. Then, for all $j \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$ the integral below*

$$I = \int_0^1\int_{\partial\mathcal{B}} |x^{\frac{n+1}{2}-\gamma}(-x\partial_x)^j\partial_y^\alpha u(x,y)|^p\frac{dx}{x}dy$$

is finite. In particular, we have the inclusion $C^{\infty,\gamma+\epsilon}(\mathbb{B}) \hookrightarrow \mathcal{H}_p^{s,\gamma}(\mathbb{B})$.

Proof. In effect,

$$I = \int_0^1 \int_{\partial\mathcal{B}} |x^{\frac{n+1}{2}-\gamma-\epsilon}(x\partial_x)^j \partial_y^\alpha u(x,y)|^p x^{\epsilon p-1} dxdy \\ \leq C_K \int_0^1 \int_{\partial\mathcal{B}} |x^{\epsilon p-1}| dxdy < \infty, \tag{5.8}$$

because for $0 < x \leq 1$, we have that $\int_0^1 x^{\epsilon p-1} dx < \infty$. Therefore, since $u \in C^\infty(\mathbb{B}^\circ)$, we have $u \in \mathcal{H}_p^{s,\gamma}(\mathbb{B})$. ☐

Definition 5.2.11. *$S_0^\gamma(\partial\mathbb{B}^\wedge)$ is the space of all $u \in C^\infty(\partial\mathbb{B}^\wedge)$ which are rapidly decreasing on $x \to \infty$ and satisfies (5.7). This means that for any local chart $\psi : V \subset \partial\mathcal{B} \to \mathbb{R}^n$ and $\phi \in C_c^\infty(V)$,*

$$|x^k \partial_x^l \partial_y^\alpha (\phi u)(x, \psi^{-1}(y))| < \infty,$$

and

$$||x^{\frac{n+1}{2}-\gamma} \ln^l x (x\partial_x)^k \partial_y^\alpha (\phi u)(x, \psi^{-1}(y))||_{L_\infty([0,1)\times V)} < \infty,$$

for all $l, k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$.

Definition 5.2.12. *We say that the operator $G : C_c^\infty(\mathbb{B}^\circ) \to C^\infty(\mathbb{B}^\circ)$ has a kernel k with respect to the $\mathcal{H}_2^{0,0}(\mathbb{B})$ scalar product if*

$$(Gu)(\boldsymbol{y}) = \int_{\mathbb{B}} k(\boldsymbol{y}, y') u(y') d\mu_g(y'),$$

where locally in $[0,1] \times V$, with $V \subset \partial\mathcal{B}$, $d\mu_g(y')$ is $x^n \sqrt{\det h(x)} dxdy'$. Similarly for the $\mathcal{K}_2^{0,0}(\partial\mathbb{B}^\wedge)$ scalar product.

Definition 5.2.13. *Let E be a Fréchet space with seminorms $(p_j)_{j\in\mathbb{N}}$. We say that $u : V \to E$, $V \subset \mathbb{R}^m$ an open set, is of class C^1 if u is continuous and there exist continuous functions $\frac{\partial u}{\partial x_k} : V \to E$, for $k \in \{1, ..., m\}$, such that*

$$\lim_{h\to 0} p_j \left(\frac{u(x + he_k) - u(x)}{h} - \frac{\partial u}{\partial x_k}(x) \right) = 0, \quad \forall j \in \mathbb{N},$$

where $\{e_1, e_2, \ldots, e_n\}$ is the conical basis of $\mathbb{R}^n$. ($e_i = (0, 0, \ldots, 1, \ldots, 0)$ where 1 appears in the position i) We say that u is of class C^l, $l \geq 1$, if $\frac{\partial u}{\partial x_k}$ are of class C^{l-1}, for each k. Finally, we say that u is of class C^∞ if u is of class C^l for each l.

Definition 5.2.14. *Let E be a Frechet space with seminorms $(p_j)_{j\in\mathbb{N}}$. We say that $u \in \mathcal{S}(\Lambda, E)$, where $\Lambda \subset \mathbb{C}$ is an open set, if:*

1) $u \in C^\infty(\Lambda, E)$.

2) $\sup_{(x,y)\in\Lambda} p_j(x^l y^p \frac{\partial^{r+s} u}{\partial x^r \partial y^s}(x, y)) < \infty$, for each $j, l, p, r, s \in \mathbb{N}$.

Definition 5.2.15. *Let $\gamma \in \mathbb{R}$. Then the space $\mathcal{S}(\Lambda, C^{\infty,\gamma+\epsilon}(\mathbb{B}) \otimes C^{\infty,-\gamma+\epsilon}(\mathbb{B}))$ consists of all functions $h : \Lambda \times \mathbb{B} \times \mathbb{B} \to \mathbb{C}$ that satisfies the condition of Definition 5.2.14 with $E = C^{\infty,\gamma+\epsilon}(\mathbb{B}) \otimes C^{\infty,-\gamma+\epsilon}(\mathbb{B})$. In particular, they locally satisfy*

$$|\lambda^k \frac{\partial^m}{\partial\lambda^m} x^{\frac{n+1}{2}-\gamma-\epsilon} \ln^l(x)(-x\partial_x)^j x'^{\frac{n+1}{2}+\gamma-\epsilon} \ln^{l'}(x')(-x'\partial_{x'})^{j'} \partial_y^\alpha \partial_{y'}^{\alpha'} h(\lambda, x, y, x', y')| \leq C_{k,m,j,l,j',l',\alpha,\alpha'},$$

for all $k, l, l', j, j' \in \mathbb{N}_0$, $\alpha, \alpha' \in \mathbb{N}_0^n$.

Recalling the definition of the space $C^{\infty,\gamma+\epsilon}(\mathbb{B})$ we have the following important statement, which gives us a relationship between $C^{\infty,\gamma+\epsilon}(\mathbb{B})$ and $\mathcal{C}^{0,\gamma}(\mathbb{B})$.

Proposition 5.2.3. *The space $C^{\infty,\gamma+\epsilon}(\mathbb{B})$ is continuously embedded in $\mathcal{C}^{0,\gamma}(\mathbb{B})$.*

Proof. By definition we have that $u \in C^{\infty,\gamma+\epsilon}(\mathbb{B})$ implies that

$$||x^{\frac{n+1}{2}-\gamma-\epsilon} \ln^l(x)(-x\partial_x)^j \partial_y^\alpha u(x,y)||_{L_\infty((0,1]\times K)} \leq C,$$

where K is a compact subset of $\partial\mathcal{B}$ and y is a local coordinate. As a consequence, for $l = 0 = j$ and $\alpha = (0, \ldots, 0)$, we have

$$|x^{\frac{n+1}{2}-\gamma} u(x,y)| = |x^{\frac{n+1}{2}-\gamma-\epsilon} u(x,y) x^\epsilon| \leq C|x^\epsilon| \to 0, \text{when } x \to 0.$$

Therefore, we can extend the function $x^{\frac{n+1}{2}-\gamma}u$ by 0 in $x = 0$. Its extension $\tilde{u}$ is defined in $[0,1] \times \partial\mathcal{B}$, which is a compact set. So, our function belongs to $BUC(\mathbb{R}_+ \times \mathbb{R}^n)$. Hence, for all $\epsilon > 0$ there exists $\delta > 0$ such that, $|x - x'| + |y - y'| < \delta$ implies that $|\tilde{u}(x,y) - \tilde{u}(x',y')| < \epsilon$ and by Remark 5.1.1, we conclude that $|\ln x - \ln x'| + |y - y'| < \delta$ implies that

$$|x^{\frac{n+1}{2}-\gamma} u(x,y) - x'^{\frac{n+1}{2}-\gamma} u(x',y')| = |\tilde{u}(x,y) - \tilde{u}(x',y')| < \epsilon.$$

Therefore, $u \in \mathcal{C}^{0,\gamma}(\mathbb{B})$. □

Proposition 5.2.4. *Let $1 < p < \infty$ and s, γ be real numbers. Then, the operator $G(\lambda)$ maps $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ into $C^{\infty,\gamma}(\mathbb{B})$ if G has kernel in $\mathcal{S}(\Lambda, C^{\infty,\gamma+\epsilon}(\mathbb{B}) \otimes C^{\infty,-\gamma+\epsilon}(\mathbb{B}))$.*

Proof. In order to prove this, we see that for $u \in \mathcal{H}_p^{s,\gamma}(\mathbb{B})$ then $G(\lambda)u(\mathbf{y}) = \int_{\mathbb{B}} k(\mathbf{y}, y')u(y')d\mu_g(y')$ is well defined because $k(\mathbf{y}, y') \in C^{\infty,-\gamma+\epsilon}(\mathbb{B}) \hookrightarrow \mathcal{H}_q^{-s,-\gamma}(\mathbb{B})$ and by duality in the scalar product over $\mathcal{H}_p^{s,\gamma}(\mathbb{B}) \times \mathcal{H}_q^{-s,-\gamma}(\mathbb{B})$ with $\frac{1}{p} + \frac{1}{q} = 1$, this integral is finite. Besides, using Proposition 5.2.2 and dominate convergence theorem in local coordinates $[0,1) \times \overline{V}$, with $\overline{V}$ a compact subset in $\partial\mathcal{B}$, we have

$$x^{\frac{n+1}{2}-\gamma} \ln^l(x)(-x\partial_x)^j \partial_y^\alpha G(\lambda)u(\mathbf{y}) = \int_{[0,1)\times\overline{V}} x^{\frac{n+1}{2}-\gamma} \ln^l(x)(-x\partial_x)^j \partial_y^\alpha k(\mathbf{y}, y')u(y')d\mu_g(y'),$$

and taking sup on $[0,1) \times \overline{V}$, this integral is finite. Therefore, $G(\lambda)u(\cdot) \in C^{\infty,\gamma}(\mathbb{B})$. □

Remark 5.2.3. *Notice that $G(\lambda) : \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \to C^{\infty,\gamma+\tilde{\epsilon}}(\mathbb{B})$ is continuous for all $\tilde{\epsilon} < \epsilon$.*

Definition 5.2.16. *The space $C_G^{-\infty}(\mathbb{B}, \Lambda, \gamma)$ consists of all operators $G_\infty(\lambda)$ with kernel in $\mathcal{S}(\Lambda, C^{\infty,\gamma+\epsilon}(\mathbb{B}) \otimes C^{\infty,\gamma-\epsilon}(\mathbb{B}))$.*

5.2.3 $G(\lambda)$-Operators

Our final class of operator acts in $[0,1) \times \partial\mathcal{B}$. They are the regularizing operators close to the conical point.

Definition 5.2.17. *For $\mu \in \mathbb{R}$ and $\lambda = x + iy \in \mathbb{C}$, we say that*

$$f \in S^\mu(\Lambda) \leftrightarrow |\partial_x^j \partial_y^k f(\lambda)| \leq C\langle\lambda\rangle^{\mu-j-k} \quad \textit{for all} \quad j,k \in \mathbb{N}_0.$$

Definition 5.2.18. *Let $\gamma, \mu \in \mathbb{R}$ and $d > 0$. $R_G^{\mu,d}(\partial\mathbb{B}^\wedge, \Lambda, \gamma)$ is the space of all operator families $G(\lambda)$ that have a kernel with respect to the $\mathcal{K}_2^{0,0}(\partial\mathbb{B}^\wedge)$ scalar product of the form*

$$k(\lambda, x, y, x', y') = [\lambda]^{\frac{n+1}{d}} \tilde{k}(\lambda, [\lambda]^{\frac{1}{\mu}} x, y, [\lambda]^{\frac{1}{\mu}} x', y'),$$

where $\tilde{k} \in S^{\frac{\mu}{d}}(\Lambda) \otimes S_0^{\gamma+\epsilon}(\partial\mathbb{B}^\wedge) \otimes S_0^{-\gamma+\epsilon}(\partial\mathbb{B}^\wedge)$. In particular, this means that $\tilde{k} : \Lambda \times \partial\mathbb{B}^\wedge \times \partial\mathbb{B}^\wedge \to \mathbb{C}$ satisfies

$$\left| [\lambda]^{-\mu+|\sigma|} \frac{\partial^{|\sigma|}}{\partial\lambda^\sigma} x^{\frac{n+1}{2}-\gamma-\epsilon} \ln^l(x)(-x\partial_x)^j x'^{\frac{n+1}{2}+\gamma-\epsilon} \ln^{l'}(x')(-x'\partial_{x'})^{j'} \partial_y^\alpha \partial_{y'}^{\alpha'} \tilde{k}(\lambda, x, y, x', y') \right| \leq C_{\sigma,j,l,j',l',\alpha,\alpha'},$$

with $x, x' \in [0, 1)$. Moreover,

$$G(\lambda)u(x, y) = \int_{\partial\mathbb{B}^\wedge} [\lambda]^{\frac{n+1}{\mu}} \tilde{k}(\lambda, [\lambda]^{\frac{1}{\mu}} x, y, [\lambda]^{\frac{1}{\mu}} x', y') u(x', y') x'^n dx' dy'.$$

Here, dy' is the measure induced on $\partial\mathcal{B}$ by the metric $h(0)$. Similarly to Proposition 5.2.4, the operator families $G(\lambda)$ map $\mathcal{K}_p^{s,\gamma}(\partial\mathbb{B}^\wedge)$ into $S_0^\gamma(\partial\mathbb{B}^\wedge)$ continuously.

5.3 Known results in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$

In this section we are going to state results in the space $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ without proof. Among the references that we have used, we can mention, for example, [5],[9],[8], [27] which contain more details.

For this section and the next, we assume that our operator $\mathcal{A}$ is elliptic with respect to Λ and $\gamma + \mu$. (see Definition below). With this in mind, we enunciate two very important results in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ spaces that support the main ideas necessary for our goals in continuous and Hölder spaces.

Definition 5.3.1. *An operator $\mathcal{A}$ is said elliptic by parameters with respect to $\gamma + \mu$ and Λ if*

i) Both the homogeneous principal symbol $\sigma_\psi^\mu(\mathcal{A})$ and the rescaled symbol $\tilde{\sigma}_\psi^\mu(\mathcal{A})$, have no spectrum in Λ, pointwise on $T^*(\mathbb{B}^\circ \setminus 0)$ and $T^*(X \times \mathbb{R} \setminus 0)$, respectively.

ii) The model cone operator $\hat{\mathcal{A}}$, as in Definition 5.2.7, has no spectrum in $\Lambda \setminus 0$.

Theorem 5.3.1. *If $\mathcal{A}$ is as in Definition 3.2.2, then there exists $R > 0$ such that $\mathcal{A}$ has no spectrum in $\Lambda \cap \{|\lambda| > R\}$ and*

$$(\lambda - \mathcal{A})^{-1} = \omega \left\{ x^\mu op_M^{\gamma - \frac{n}{2}} g(\lambda) + G(\lambda) \right\} \omega_0 + (1 - \omega) P(\lambda)(1 - \omega_1) + G_\infty(\lambda),$$

where $\omega, \omega_0, \omega_1 \in C_0^\infty([0, 1))$ are cutoff functions satisfying $\omega_1\omega = \omega_1$ and $\omega\omega_0 = \omega$ and

i) $g(x, z, \lambda) = \widetilde{g}(x, z, x^\mu\lambda)$ with $\widetilde{g} \in C^\infty(\mathbb{R}_+, M_O^{-\mu,\mu}(\partial\mathcal{B}, \Lambda))$,

ii) $P(\lambda) \in L_{cl}^{-\mu,\mu}(\mathbb{B}^\circ, \Lambda)$,

iii) $G(\lambda) \in R_{\boldsymbol{G}}^{-\mu,\mu}(\partial\mathbb{B}^\wedge; \Lambda, \gamma)$,

iv) $G_\infty(\lambda) \in C_G^{-\infty}(\mathbb{B}; \Lambda, \gamma)$.

Proof. See Theorem 1 in [8]. □

Theorem 5.3.2. *Let $\mathcal{A}$ be as in Definition 3.2.2. Then, there exists a constant c_p such that*

$$||(\lambda - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B}))} \leq \frac{c_p}{|\lambda|}.$$

Proof. See Proposition 1 in [8]. □

5.4 Almost sectorial operators in $\mathcal{C}^{0,\gamma}(\mathbb{B})$

Now, we set the second main result of the thesis. In order to motivate the result, we consider $\mathcal{A} = \Delta$. We recall that, for previous results in Chapter 3, we had a conic manifold $\mathbb{B}$ with dimension $n+1$ and γ such that

$$\frac{n-3}{2} < \gamma < \min\left\{-1 + \sqrt{\left(\frac{n-1}{2}\right)^2 - \lambda_1}, \frac{n+1}{2}\right\}, \tag{5.9}$$

where λ_1 is the greatest non-zero eigenvalue of the boundary Laplacian $\Delta_{h(0)}$, where $h(x)$ with $x \in [0, 1)$ is a family of Riemannian metrics on $\partial\mathcal{B}$ that is smooth and does not degenerate up to $x = 0$. Then, we have that

$$\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}_p^{0,\gamma}(\mathbb{B})$$

with $D(\mathcal{A}) = \mathcal{H}_p^{2,2+\gamma}(\mathbb{B}) \oplus \mathbb{C}$ is sectorial. This implies that $(\lambda - \mathcal{A})^{-1} : \mathcal{H}_p^{0,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{0,\gamma}(\mathbb{B})$ is continuous. In order to show that Δ is almost sectorial in $\mathcal{C}^{0,\gamma}(\mathbb{B})$, we extend $(\lambda - \mathcal{A})^{-1}$ to a bounded operator on this set. With this in mind, we prove the result below, which is the second main result of the thesis.

Theorem 5.4.1. *Let $\mathcal{A}$ be as in Definition 5.3.1. For λ large enough and $1 < p < \infty$, we have that:*

1) The operator

$$(\lambda - \mathcal{A})^{-1} : \mathcal{H}_p^{0,\gamma}(\mathbb{B}) \cap \mathcal{C}^{0,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{0,\gamma}(\mathbb{B})$$

can be extended to a continuous operator

$$(\lambda - \mathcal{A})^{-1} : \mathcal{C}^{0,\gamma}(\mathbb{B}) \to \mathcal{C}^{0,\gamma}(\mathbb{B}).$$

2) The image of $(\lambda - \mathcal{A})^{-1}$ is independent of λ, i.e, $(\lambda - \mathcal{A})^{-1}(\mathcal{C}^{0,\gamma}(\mathbb{B})) = (\mu - \mathcal{A})^{-1}(\mathcal{C}^{0,\gamma}(\mathbb{B}))$. Furthermore, if $D(\mathcal{A}) := (\lambda - \mathcal{A})^{-1}(\mathcal{C}^{0,\gamma}(\mathbb{B}))$ for large λ, then $c + \mathcal{A} : D(\mathcal{A}) \to \mathcal{C}^{0,\gamma}(\mathbb{B})$ is an almost sectorial operator for some $c \in \mathbb{R}$, that is, for all $\lambda \in \Lambda$

$$||(\lambda - c - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{C}^{0,\gamma}(\mathbb{B}))} \leq C|\lambda|^{-1+\epsilon}, \text{ for some } \epsilon \in (0, 1).$$

Proof. For the proof of 1) and the almost sectoriality of $\mathcal{A}$ in 2), we need to estimate the norm for all terms that appear in the decomposition of $(\lambda - \mathcal{A})^{-1}$. We shall do this later in this work. For the statement of independence of λ in 2), we recall the identities for resolvent operators. More explicitly, for $\mu \in \mathbb{C}$ we use that

$$R(\mathcal{A}, \lambda) - R(\mathcal{A}, \mu) = (\mu - \lambda)R(\mathcal{A}, \lambda)R(\mathcal{A}, \mu). \tag{5.10}$$

Therefore, for $f \in (\lambda - \mathcal{A})^{-1}(\mathcal{C}^{0,\gamma}(\mathbb{B}))$ there exists $w \in \mathcal{C}^{0,\gamma}(\mathbb{B})$ such that $f = (\lambda - \mathcal{A})^{-1}w$. Then, if we use (5.10), we have

$$\begin{aligned} f &= (\mu - \mathcal{A})^{-1}w - (\mu - \mathcal{A})^{-1}w + (\lambda - \mathcal{A})^{-1}w \\ &= (\mu - \mathcal{A})^{-1}w + (\mu - \lambda)(\mu - \mathcal{A})^{-1}(\lambda - \mathcal{A})^{-1}w \\ &\in (\mu - \mathcal{A})^{-1}(\mathcal{C}^{0,\gamma}(\mathbb{B})). \end{aligned}$$

In fact, the first term after the second equality belongs clearly to $(\mu - \mathcal{A})^{-1}(\mathcal{C}^{0,\gamma}(\mathbb{B}))$ and for the second term we have used that $(\lambda - \mathcal{A})^{-1} : \mathcal{C}^{0,\gamma}(\mathbb{B}) \to \mathcal{C}^{0,\gamma}(\mathbb{B})$, which implies that the composition $(\mu - \mathcal{A})^{-1}(\lambda - \mathcal{A})^{-1}$ belongs to $(\mu - \mathcal{A})^{-1}(\mathcal{C}^{0,\gamma}(\mathbb{B}))$.

□

Now, we complete the proof of Theorem 5.4.1 by studying the extension of each term of $(\lambda - \mathcal{A})^{-1}$ described by Theorem 5.3.1. Below, Λ denotes always the sector of the Theorem 5.3.1.

Proposition 5.4.1. *The operator $G_\infty(\lambda)$ satisfies*

$$||G_\infty(\lambda)||_{\mathcal{L}(\mathcal{C}^{0,\gamma}(\mathbb{B}))} \leq \frac{C}{|\lambda|},$$

where λ belongs to Λ.

Proof. First step: $G_\infty(\lambda) \in \mathcal{L}(\mathcal{C}^{0,\gamma}(\mathbb{B}))$.

For $u \in \mathcal{C}^{0,\gamma}(\mathbb{B}))$ we have by the definition of the operator $G_\infty(\lambda)$ that

$$G_\infty(\lambda)u(w) = \int_{\mathbb{B}} k(\lambda, w, w')u(w')d\mu_g(w'),$$

where $k \in \mathcal{S}(\Lambda, C^{\infty,\gamma+\epsilon}(\mathbb{B})\otimes C^{\infty,-\gamma+\epsilon}(\mathbb{B}))$. We do all the computations without the term det $\sqrt{\det|h(x)|}$ because $\mathbb{B}$ is a compact manifold and this term is bounded from above and below, so we do not need to worry about it.
By Theorem 5.2.1 , for every $\lambda \in \Lambda$,

$$k(\lambda) = \sum_{n=0}^{\infty} \alpha_n(\lambda)a_n(\lambda) \otimes b_n(\lambda),$$

where $a_n(\lambda) \in C^{\infty,\gamma+\epsilon}(\mathbb{B})$ and $b_n(\lambda) \in C^{\infty,-\gamma+\epsilon}(\mathbb{B})$, where $\sum_{n=0}^{\infty} \alpha_n(\lambda) < \infty$ and $\lim_{n\to\infty} a_n(\lambda) = 0$, $\lim_{n\to\infty} b_n(\lambda) = 0$ in $C^{\infty,\gamma+\epsilon}(\mathbb{B})$ and $C^{\infty,-\gamma+\epsilon}(\mathbb{B})$, respectively. First, we prove that $G_\infty(\lambda)u \in \mathcal{C}^{0,\gamma}(\mathbb{B})$, for all $\lambda \in \Lambda$. We divide the computations on $[0,1) \times \partial\mathcal{B}$ and $\mathbb{B} \setminus ([0,1) \times \partial\mathcal{B})$. For the first case, we have that, for dy' denoting a volume metric on $\partial\mathcal{B}$,

$$\begin{aligned}
G_\infty(\lambda)u(x,y) &= \int_{[0,1)\times\partial\mathcal{B}} k(\lambda,x,y,x',y')u(x',y')x'^n dx'dy' \\
&= \int_{[0,1)\times\partial\mathcal{B}} \sum_{n=0}^{\infty} \alpha_n(\lambda)a_n(\lambda,x,y)b_n(\lambda,x',y')u(x',y')x'^n dx'dy' \\
&= \sum_{n=0}^{\infty} \alpha_n(\lambda)a_n(\lambda,x,y)\int_{[0,1)\times\partial\mathcal{B}} b_n(\lambda,x',y')u(x',y')x'^n dx'dy' \\
&\le C\sum_{n=0}^{\infty} \alpha_n(\lambda)a_n(\lambda,x,y)||b_n||_{C^{\infty,-\gamma+\epsilon}(\mathbb{B})}\int_{[0,1)\times\partial\mathcal{B}} |x'^{-\frac{n+1}{2}-\gamma+\epsilon}u(x',y')x'^n|dx'dy' \\
&\le C_1\sum_{n=0}^{\infty} \alpha_n(\lambda)a_n(\lambda,x,y)||b_n||_{C^{\infty,-\gamma+\epsilon}(\mathbb{B})}||u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}\int_{[0,1)\times\partial\mathcal{B}} |x'^{-\frac{n+1}{2}-\gamma+\epsilon}x'^{-\frac{n+1}{2}+\gamma}x'^n|dx'dy' \\
&= C_1\sum_{n=0}^{\infty} \alpha_n(\lambda)a_n(\lambda,x,y))||b_n||_{C^{\infty,-\gamma+\epsilon}(\mathbb{B})}||u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}\int_{[0,1)\times\partial\mathcal{B}} |x'^{-1+\epsilon}|dx'dy' \\
&\le C_2\sum_{n=0}^{\infty} \alpha_n(\lambda)a_n(\lambda,x,y))||b_n||_{C^{\infty,-\gamma+\epsilon}(\mathbb{B})}||u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \\
&< \infty,
\end{aligned} \tag{5.11}$$

where we have used that $b_n(\lambda) \in C^{\infty,-\gamma+\epsilon}(\mathbb{B}) \hookrightarrow \mathcal{C}^{0,-\gamma}(\mathbb{B})$ (See Proposition 5.2.3), where we have used that $\sum_{n=0}^{\infty}\alpha_n(\lambda)a_n(\lambda) < \infty$ in $C^{\infty,\gamma+\epsilon}(\mathbb{B})$, $||b_n(\lambda)||_{C^{\infty,-\gamma+\epsilon}(\mathbb{B})}$ is bounded and $C^{\infty,\gamma+\epsilon}(\mathbb{B}) \subset \mathcal{C}^{0,\gamma}(\mathbb{B})$, we conclude the result. The integral on $\mathbb{B}\setminus\{[0,1)\times\partial\mathcal{B}\}$ is the easy part, since we just need to observe that the last set is compact and the integral as we did before is finite, because we do not have singularities.

Second step: $||G_\infty(\lambda)||_{\mathcal{L}(\mathcal{C}^{0,\gamma}(\mathbb{B}))} \le \frac{C}{|\lambda|}$.
Second, we show that $||G_\infty(\lambda)u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \le \frac{C}{|\lambda|}||u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}$. In order to complete the proof, let us choose a partition of unity $\{\phi_i\}$ of $\partial\mathcal{B}$ and a coordinate system (U_j,ψ_j) such that $\operatorname{supp}\phi_j \subset U_j$. Before, we remark that for ω a cutoff function,

$$\begin{aligned}
G_\infty(\lambda)u &= (1-\omega+\omega)G_\infty(\lambda)(1-\omega+\omega)u \\
&= (1-\omega)G_\infty(\lambda)(1-\omega)u + (1-\omega)G_\infty(\lambda)\omega u \\
&+ \omega G_\infty(\lambda)(1-\omega)u + \omega G_\infty(\lambda)\omega u.
\end{aligned} \tag{5.12}$$

Further, we recall that the expression $(\phi_i u)(x,y)$ means $\phi_i(y)u(x,y)$ for all i. We analyse two terms. The others are similar or even easier. Let us start with $\omega G_\infty(\lambda)\omega$. All the work will be done in a chart $U \subset \partial\mathcal{B}$ and, by abuse of notation, we write u in place to $\phi_i u$ to do a clear computation and to avoid the use of $u(x,y) = \sum_i(\phi_i u)(x,y)$. Therefore, there exists a compact set $K \subset\subset U$ such that

$u(x,y)=0$ if $y \notin K$, so $||\omega G_\infty(\lambda)\omega u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}$ is a finite sum of terms of the form below.

$$
\begin{aligned}
&||\omega(x)x^{\frac{n+1}{2}-\gamma}\int_{[0,1)\times U} k(\lambda,x,y,x',y')\omega(x')u(x',y')x'^n dx'dy'||_{BUC_{\ln}(\mathbb{R}_+\times\mathbb{R}^n)} \\
&\le \sup_{x\in[0,1)}\int_{[0,1)\times U}|\omega(x)x^{\frac{n+1}{2}-\gamma}\frac{1}{|\lambda|}x^{\gamma+\epsilon-\frac{n+1}{2}}x'^{-\gamma+\epsilon-\frac{n+1}{2}}u(x',y')x'^n|dx'dy' \\
&\le \sup_{x\in[0,1)}\frac{c}{|\lambda|}|x|^\epsilon\int_{[0,1)\times U}|x'^{-\gamma+\epsilon-1+\frac{n+1}{2}}u(x',y')|dx'dy' \qquad (5.13)\\
&\le \frac{c}{|\lambda|}||u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}\int_{[0,1)\times U}|x'|^{\epsilon-1}dx'dy' \\
&\le \frac{C}{|\lambda|}||u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})},
\end{aligned}
$$

and we used that the support of the function ϕ_i is contained in some U. For the term $\omega G_\infty(\lambda)(1-\omega)$, we note that $1-\omega(x')$ for $x' \in [0,1)$ is bounded, so we have the same estimate as above. For the rest of the terms, we proceed in a similar way. Therefore, $||G_\infty(\lambda)||_{\mathcal{L}(\mathcal{C}^{0,\gamma}(\mathbb{B}))} \le \frac{C}{|\lambda|}$. □

Another important estimate that we need for the operator $P(\lambda)$ is stated below.

Proposition 5.4.2. *For the operator $P(\lambda) \in L^{-\mu,\mu}(\mathbb{B}^\circ,\Lambda)$ we have that:*

$$
||(1-\omega_0)P(\lambda)(1-\omega_1)u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \le \frac{C_1}{|\lambda|}||u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}.
$$

Proof. Now we will analyze the term $(1-\omega_0)P(\lambda)(1-\omega_1)$. We note that the operator is acting outside of $\partial\mathcal{B}$ because of the terms $(1-\omega_0)$ and $(1-\omega_1)$. Let us take a partition of unity $\{\phi_i\}$ associate to $\mathrm{supp}(1-\omega_1)\cup\mathrm{supp}(1-\omega_0)$ and $\psi_i : U_i \subset \mathbb{B} \to \mathbb{R}^n$ such that $\mathrm{supp}\,\phi_i \subset U_i$ and such that $\mathrm{supp}\,\phi_j \cup \mathrm{supp}\,\phi_k$ is contained in the same U_i if $\mathrm{supp}\,\phi_j \cap \mathrm{supp}\,\phi_k \neq \emptyset$, see for example Lemma 8.4 in [14]. Hence, if we take $u \in \mathcal{C}^{0,\gamma}(\mathbb{B})$, then $(1-\omega_0)P(\lambda)(1-\omega_1)(u\circ\psi_i)\circ\psi_i^{-1} \in BUC(\mathbb{R}^{n+1})$ for all i. Let $\hat{P} : C_c^\infty(U_i) \to C_c^\infty(U_i)$ be defined by $\hat{P}v = [P(v\circ\psi)]\circ\psi^{-1}$. Then, $Pu = [\hat{P}(u\circ\psi^{-1})]\circ\psi$. If, $P = \phi_j(1-\omega_0)P(\lambda)(1-\omega_1)\phi_k$, then $\hat{P}$ is as $(\lambda+\mathcal{A})^{-1}$ in Theorem 4.2.2 and *Case 1* in Theorem 4.3.1. Therefore, $||\hat{P}||_{\mathcal{L}(BUC)} \le \frac{C}{|\lambda|}$, which implies that $||P||_{\mathcal{L}(\mathcal{C}^{0,\gamma}(\mathbb{B}))} \le \frac{C}{|\lambda|}$.

□

Lemma 5.4.2. *Let $\rho > 0$ and $\gamma \in \mathbb{R}$. Then the operator $\kappa_\rho : \tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B}) \to \tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B})$ defined by*

$$
\kappa_\rho u(x,y) = \rho^{\frac{n+1}{2}}u(\rho x,y)
$$

is continuous and satisfies

$$
||\kappa_\rho||_{\mathcal{L}(\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B}))} = \rho^\gamma.
$$

Proof. In effect, by definition we have that

$$
\begin{aligned}
||\kappa_\rho u(x,y)||_{\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B})} &= ||x^{\frac{n+1}{2}-\gamma}\rho^{\frac{n+1}{2}}u(\rho x,y)||_{L_\infty(\mathbb{R}_+\times\partial\mathcal{B})} \\
&= ||\rho^\gamma(\rho x)^{\frac{n+1}{2}-\gamma}u(\rho x,y)||_{L_\infty(\mathbb{R}_+\times\partial\mathcal{B})} \qquad (5.14)\\
&= \rho^\gamma||x^{\frac{n+1}{2}-\gamma}u(x,y)||_{L_\infty(\mathbb{R}_+\times\partial\mathcal{B})} \\
&= \rho^\gamma||u||_{\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B})},
\end{aligned}
$$

which implies that $||\kappa_\rho||_{\mathcal{L}(\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B}))} = \rho^\gamma$. $\square$

Most of the time, we use conjugation under appropriate operators in order to show necessary estimates for our goals. We can see that in the following two lemmas below. The first result will show the relation between the kernels involved with the operator $G(\lambda)$.

Lemma 5.4.3. *Let the operator* $G(\lambda)$ *be defined in* (5.2.18). *Then,*

$$\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}} G(\lambda)\kappa_{[\lambda]^{\frac{1}{\mu}}} u(x,y) = \int_{\partial\mathbb{B}^\wedge} \tilde{k}(\lambda,x,y,x',y')u(x',y')x'^n dx'dy',$$

where $\tilde{k}(\lambda,x,y,x',y') = [\lambda]^{-\frac{n+1}{\mu}} k(\lambda,[\lambda]^{-\frac{1}{\mu}}x,y,[\lambda]^{-\frac{1}{\mu}}s,y')$ *and* $k \in S^\mu(\Lambda)\otimes S_0^{\gamma+\epsilon}(\partial\mathbb{B}^\wedge)\otimes S_0^{-\gamma+\epsilon}(\partial\mathbb{B}^\wedge)$.

Proof. In fact,

$$\begin{aligned}
\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}} G(\lambda)\kappa_{[\lambda]^{\frac{1}{\mu}}} u(x,y) &= \kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}} \int_{\partial\mathbb{B}^\wedge} k(\lambda,x,y,x',y')\kappa_{[\lambda]^{\frac{1}{\mu}}} u(x',y')x'^n dx'dy' \\
&= [\lambda]^{-\frac{n+1}{2\mu}} \int_{\partial\mathbb{B}^\wedge} k(\lambda,[\lambda]^{-\frac{1}{\mu}}x,y,x',y')[\lambda]^{\frac{n+1}{2\mu}} u([\lambda]^{\frac{1}{\mu}}x',y')x'^n dx'dy' \\
&= \int_{\partial\mathbb{B}^\wedge} k(\lambda,[\lambda]^{-\frac{1}{\mu}}x,y,x',y')u([\lambda]^{\frac{1}{\mu}}x',y')x'^n dx'dy' \\
(\text{By substitution} \quad s = [\lambda]^{\frac{1}{\mu}}x') &= \int_{\partial\mathbb{B}^\wedge} k(\lambda,[\lambda]^{-\frac{1}{\mu}}x,y,[\lambda]^{-\frac{1}{\mu}}s,y')u(s,y')([\lambda]^{-\frac{1}{\mu}}s)^n[\lambda]^{-\frac{1}{\mu}} ds dy' \\
&= \int_{\partial\mathbb{B}^\wedge} [\lambda]^{-\frac{n+1}{\mu}} k(\lambda,[\lambda]^{-\frac{1}{\mu}}x,y,[\lambda]^{-\frac{1}{\mu}}s,y')u(s,y')s^n ds dy' \\
&= \int_{\partial\mathbb{B}^\wedge} \tilde{k}(\lambda,x,y,x',y')u(x',y')x'^n dx'dy'.
\end{aligned} \tag{5.15}$$

$\square$

We have another important relation for the conjugation of $G(\lambda)$ under $\kappa_{[\lambda]^{\frac{1}{\mu}}}$.

Lemma 5.4.4. *For the operator* $G(\lambda)$ *we have the following relation*

$$||G(\lambda)||_{\mathcal{L}(\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B}))} \le ||\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}} G(\lambda)\kappa_{[\lambda]^{\frac{1}{\mu}}}||_{\mathcal{L}(\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B}))}.$$

Proof. We note that

$$\begin{aligned}
||G(\lambda)||_{\mathcal{L}(\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B}))} &= ||\kappa_{[\lambda]^{\frac{1}{\mu}}}\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}} G(\lambda)\kappa_{[\lambda]^{\frac{1}{\mu}}}\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}}||_{\mathcal{L}(\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B}))} \\
(\text{By Lemma 5.4.2}) &\le ||\kappa_{[\lambda]^{\frac{1}{\mu}}}||_{\mathcal{L}(\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B}))} ||\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}} G(\lambda)\kappa_{[\lambda]^{\frac{1}{\mu}}}||_{\mathcal{L}(\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B}))} ||\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}}||_{\mathcal{L}(\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B}))} \\
&= ||\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}} G(\lambda)\kappa_{[\lambda]^{\frac{1}{\mu}}}||_{\mathcal{L}(\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B}))}.
\end{aligned} \tag{5.16}$$

$\square$

Now, we will use that $k(\lambda,x,y,x',y') \in S^\mu(\Lambda)\otimes S_0^{\gamma+\epsilon}(\partial\mathbb{B}^\wedge)\otimes S_0^{-\gamma+\epsilon}(\partial\mathbb{B}^\wedge)$, where $S_0^{\gamma+\epsilon}(\partial\mathbb{B}^\wedge)$ is defined in Definition 5.2.11 to prove the next proposition.

Proposition 5.4.3. *For the operator* $G(\lambda)$, *we have that*

$$||\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}} G(\lambda)\kappa_{[\lambda]^{\frac{1}{\mu}}}||_{\mathcal{L}(\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B}))} \le \frac{C}{|\lambda|}.$$

Proof. In this proof, we need to consider four cases. If $x, x' \in [0,1)$, $x \in [0,1)$ and $x' > 1$, $x' \in [0,1)$ and $x > 1$ and finally $x, x' > 1$.

We start with the case $x, x' \in [0,1)$. In this case, if $u \in \tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B}))$ then

$$\begin{aligned}
&\sup_{x\in[0,1)\times\partial\mathcal{B}} |x^{\frac{n+1}{2}-\gamma}\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}} G(\lambda)\kappa_{[\lambda]^{\frac{1}{\mu}}} u| \\
&= \sup_{x\in[0,1)\times\partial\mathcal{B}} |x^{\frac{n+1}{2}-\gamma}\int_{[0,1)\times\partial\mathcal{B}} \tilde{k}(\lambda,x,y,x',y')u(x',y')x'^n dx'dy'| \\
&\leq C \sup_{x\in[0,1)\times\partial\mathcal{B}} \int_{[0,1)\times\partial\mathcal{B}} |x^{\frac{n+1}{2}-\gamma}x^{\gamma+\epsilon-\frac{n+1}{2}}x'^{-\gamma-\frac{n+1}{2}+\epsilon}u(x',y')x'^n|dx'dy' \\
&= \frac{C}{|\lambda|}\int_{[0,1)\times\partial\mathcal{B}} \sup_{x\in[0,1)\times\partial\mathcal{B}} |x^\epsilon x'^{\epsilon-1}x'^{\frac{n+1}{2}-\gamma}u(x',y')|dx'dy' \\
&\leq \frac{C}{|\lambda|}\int_{[0,1)\times\partial\mathcal{B}} \sup_{x\in[0,1)} |x^\epsilon||x'^{\epsilon-1}|||u||_{\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B})}dx'dy' \\
&= \frac{C||u||_{\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B})}}{|\lambda|}\int_{[0,1)\times\partial\mathcal{B}} x'^{\epsilon-1}dx'dy' \\
&\leq \frac{\tilde{C}}{|\lambda|}||u||_{\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B})}.
\end{aligned} \tag{5.17}$$

For the second case, $x \in [0,1)$ and $x' > 1$, we need to use that $\tilde{k}$ decays to zero at infinity. First, let us consider $\tilde{k}(\lambda,x,y,x',y') = f(\lambda)a(x,y)b(x',y')$ with $a \in S_0^{\gamma+\epsilon}(\partial\mathbb{B}^\wedge)$ and $b \in S_0^{-\gamma+\epsilon}(\partial\mathbb{B}^\wedge)$ and $f \in S^\mu(\Lambda)$. Therefore, for all $m \in \mathbb{N}_0$ we have that $|b(x',y')| \leq C|x'|^{-m}$. Now, with this in mind, we have that

$$\begin{aligned}
&\sup_{(x,y)\in[0,1)\times\partial\mathcal{B}} |x^{\frac{n+1}{2}-\gamma}\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}} G(\lambda)\kappa_{[\lambda]^{\frac{1}{\mu}}} u| \\
&= \sup_{(x,y)\in[0,1)\times\partial\mathcal{B}} |\int_{[1,\infty)\times\partial\mathcal{B}} x^{\frac{n+1}{2}-\gamma}\tilde{k}(\lambda,x,y,x',y')u(x',y')x'^n|dx'dy' \\
&= \sup_{(x,y)\in[0,1)\times\partial\mathcal{B}} |\int_{[1,\infty)\times\partial\mathcal{B}} x^{\frac{n+1}{2}-\gamma}f(\lambda)a(x,y)b(x',y')u(x',y')x'^n|dx'dy' \\
&= \sup_{(x,y)\in[0,1)\times\partial\mathcal{B}} |\int_{[1,\infty)\times\partial\mathcal{B}} x^{\frac{n+1}{2}-\gamma}f(\lambda)a(x,y)b(x',y')x'^{-\frac{n+1}{2}+\gamma}x'^{\frac{n+1}{2}-\gamma}u(x',y')x'^n|dx'dy' \\
&= \sup_{(x,y)\in[0,1)\times\partial\mathcal{B}} |\int_{[1,\infty)\times\partial\mathcal{B}} x^{\frac{n+1}{2}-\gamma}f(\lambda)a(x,y)b(x',y')x'^{\frac{n-1}{2}+\gamma}x'^{\frac{n+1}{2}-\gamma}u(x',y')|dx'dy'.
\end{aligned} \tag{5.18}$$

Here, by definition of the space $S_0^{\gamma+\epsilon}(\partial\mathbb{B}^\wedge)$, we have $|x^{\frac{n+1}{2}-\gamma}a(x,y)| \leq K|x|^\epsilon$ for some $K, \epsilon > 0$. Then, $\sup_{x\in[0,1)\times\partial\mathcal{B}} |x^{\frac{n+1}{2}-\gamma}a(x,y)| < \infty$. Besides, as $u \in \tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B})$ it follows that $|x'^{\frac{n+1}{2}-\gamma}u(x',y')| \leq ||u||_{\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B})}$. Besides,

$$|x'^{\frac{n-1}{2}+\gamma}b(x',y')| \leq C_1|x'|^{-l},$$

for any $l > 1$ because b is rapidly decreasing when x' goes to ∞. Therefore, its last term is an integrable function in $[1,\infty)$. On the other hand, we recall that $f(\lambda)$ is bounded by $\frac{\tilde{C}}{|\lambda|}$ for some scalar $\tilde{C}$. Therefore, (5.17) and (5.18) imply

$$\sup_{(x,y)\in[0,1)\times\partial\mathcal{B}} |\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}} G(\lambda)\kappa_{[\lambda]^{\frac{1}{\mu}}} u(x,y)| \leq \frac{C}{|\lambda|}||u||_{\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B})}.$$

For the case $x' \in [0,1)$ and $x \in [1,\infty)$

$$\begin{aligned} &\sup_{(x,y)\in[1,\infty)\times\partial\mathcal{B}} |x^{\frac{n+1}{2}-\gamma}\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}} G(\lambda)\kappa_{[\lambda]^{\frac{1}{\mu}}} u| \\ &\leq \sup_{(x,y)\in[1,\infty)\times\partial\mathcal{B}} |\int_{[0,1)\times\partial\mathcal{B}} x^{\frac{n+1}{2}-\gamma}\tilde{k}(\lambda,x,y,x',y')u(x',y')x'^{n}|dx'dy' \\ &= \sup_{(x,y)\in[1,\infty)\times\partial\mathcal{B}} \int_{[0,1)\times\partial\mathcal{B}} |x^{\frac{n+1}{2}-\gamma} f(\lambda)a(x,y)b(x',y')u(x',y')x'^{n}|dx'dy'. \end{aligned} \tag{5.19}$$

For this case, we use that $|a(x,y)| \leq M|x|^{-l}$ for some l such that $\frac{n+1}{2}-\gamma < l$. Then,

$$\sup_{x\in[1,\infty)} |x^{\frac{n+1}{2}-\gamma}a(x,y)| \leq M.$$

Besides, we have that

$$|b(x',y')u(x',y')x'^{n}| = |x'^{\frac{n+1}{2}+\gamma-\epsilon}b(x',y')x'^{\frac{n+1}{2}+\gamma}u(x',y')x'^{-1+\epsilon}| \leq M||u||_{\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B})}|x'|^{-1+\epsilon}. \tag{5.20}$$

Since $f(\lambda)$ is bounded by $\frac{C}{|\lambda|}$, then

$$\sup_{x\in[1,\infty)\times\partial\mathcal{B}} \left| \int_{[0,1)\times\partial\mathcal{B}} x^{\frac{n+1}{2}-\gamma} f(\lambda)a(x,y)b(x',y')u(x',y')x'^{n} \right| dx'dy' \leq \frac{M_2}{|\lambda|}||u||_{\tilde{\mathcal{C}}^{0,\gamma}(\mathbb{B})}.$$

Finally, when $x, x' > 1$ we have

$$\begin{aligned} &\sup_{(x,y)\in[1,\infty)\times\partial\mathcal{B}} |x^{\frac{n+1}{2}-\gamma}\kappa^{-1}_{[\lambda]^{\frac{1}{\mu}}} G(\lambda)\kappa_{[\lambda]^{\frac{1}{\mu}}} u| \\ &= \sup_{x\in[1,\infty)\times\partial\mathcal{B}} |\int_{[1,\infty)\times\partial\mathcal{B}} x^{\frac{n+1}{2}-\gamma}\tilde{k}(\lambda,x,y,x',y')u(x',y')x'^{n}|dx'dy' \\ &= \sup_{x\in[1,\infty)\times\partial\mathcal{B}} |\int_{[1,\infty)\times\partial\mathcal{B}} x^{\frac{n+1}{2}-\gamma} f(\lambda)a(x,y)b(x',y')u(x',y')x'^{n}|dx'dy'. \end{aligned} \tag{5.21}$$

Now we use that $x^{\frac{n+1}{2}-\gamma}a(x,y)$ is bounded by $|x|^{p}$ for some $p < -1$ and $b(x',y')u(x',y')x'^{n}$ is bounded in the same way as we did above. Therefore, we can conclude by similar way the estimate that we require. We remark that in our computations we have used constantly properties of the tensorial product, see Theorem 5.2.1. The proof for general k follows by taking limits as in Theorem 5.2.1. □

The last term that we will analyse is $\omega_0 x^{\mu} op_M^{\gamma} g(\lambda)\omega_1$. Before we do, we recall that $g(x,y,z,\lambda) = \widetilde{g}(x,y,z,x^{\mu}\lambda)$ with $\widetilde{g} \in C^{\infty}(\mathbb{R}_+, M_O^{-\mu,\mu}(\partial\mathcal{B},\Lambda))$, which implies that

$$op_M^{\gamma}\widetilde{g}(\lambda)u = \int_{\mathbb{R}^n}\int_{Rez=\frac{n+1}{2}-\gamma} x^{-z}e^{ix\xi}\widetilde{g}(x,y,\frac{n+1}{2}-\gamma+i\tau,\xi,\lambda)(\mathcal{M}\mathfrak{F}u)(z,\xi)d\tau d\xi$$

represents the general form.

Lemma 5.4.5. *Let $a : \mathbb{R}_+^{n+1}\times\mathbb{R}^{n+1}\times\Lambda \to \mathbb{C}$ be an operator given by*

$$a(x,y,\tau,\xi,\lambda) := \omega(x)x^{\mu}\widetilde{g}(x,y,\tau,\xi,x^{\mu}\lambda).$$

Then,

$$|(x\partial_x)^k\partial_y^{\beta}\partial_{(\tau,\xi)}^{\alpha}a(x,y,\tau,\xi,\lambda)| \leq C(1+|\tau|+|\xi|+|\lambda|^{\frac{1}{\mu}})^{-\mu}(1+|\tau|+|\xi|)^{-|\alpha|}$$

for all $k \in \mathbb{N}_0$, $\beta, \alpha \in \mathbb{N}_0^n$.

Proof. First, let us suppose that $k = |\alpha| = |\beta| = 0$. In this case, since we only need to consider $x \in [0, 1)$, we have

$$\begin{aligned} |\omega(x)x^\mu \widetilde{g}(x, y, \tau, \xi, x^\mu\lambda)| &\le x^\mu(1 + |\tau| + |\xi| + |x^\mu\lambda|^{\frac{1}{\mu}})^{-\mu} \\ &= \frac{x^\mu(1 + |\tau| + |\xi| + |\lambda|^{\frac{1}{\mu}})^\mu}{(1 + |\tau| + |\xi| + |x^\mu\lambda|^{\frac{1}{\mu}})^\mu}(1 + |\tau| + |\xi| + |\lambda|^{\frac{1}{\mu}})^{-\mu} \\ &= \frac{x^\mu(1 + |\tau| + |\xi| + |\lambda|^{\frac{1}{\mu}})^\mu}{x^\mu(x^{-1}(1 + |\tau| + |\xi|) + |\lambda|^{\frac{1}{\mu}})^\mu}(1 + |\tau| + |\xi| + |\lambda|^{\frac{1}{\mu}})^{-\mu} \\ &= \frac{(1 + |\tau| + |\xi| + |\lambda|^{\frac{1}{\mu}})^\mu}{(x^{-1}(1 + |\tau| + |\xi|) + |\lambda|^{\frac{1}{\mu}})^\mu}(1 + |\tau| + |\xi| + |\lambda|^{\frac{1}{\mu}})^{-\mu} \\ &\le (1 + |\tau| + |\xi| + |\lambda|^{\frac{1}{\mu}})^{-\mu}. \end{aligned} \tag{5.22}$$

For the general case, we need to observe the general form for the derivatives that appear. So, we remark that $(x\partial_x)^k x^\mu = \mu^k x^\mu$ and $(x\partial_x)^k \widetilde{g}(x, y, \tau, \xi, x^\mu\lambda) = \sum_{k_1+k_2=k} x^{k_2\mu}\lambda^{k_2}(x\partial_x)^{k_2}\partial_\lambda^{k_3}\widetilde{g}(x, y, \tau, \xi, x^\mu\lambda)$. Therefore, by the previous observation, we have that

$$\begin{aligned} &|(x\partial_x)^k \partial_y^\beta \partial_{(\tau,\xi)}^\alpha \omega(x)x^\mu \widetilde{g}(x, y, \tau, \xi, x^\mu\lambda)| = |(x\partial_x)^k \omega(x)x^\mu \partial_y^\beta \partial_{(\tau,\xi)}^\alpha \widetilde{g}(x, y, \tau, \xi, x^\mu\lambda)| \\ &\le \sum_{k_2+k_3=k} |\tilde{\omega}(x)x^{(k_2+1)\mu}\lambda^{k_2}(x\partial_x)^{k_2}\partial_\lambda^{k_3}\partial_y^\beta\partial_{(\tau,\xi)}^\alpha \widetilde{g}(x, y, \tau, \xi, x^\mu\lambda)| \\ &\le \sum_{k_2+k_3=k} C|x^\mu\lambda|^{k_2} x^\mu(1 + |\tau| + |\xi| + |x^\mu\lambda|^{\frac{1}{\mu}})^{-\mu-\mu k_2-|\alpha|} \\ &= \sum_{k_2+k_3=k} C|x^\mu\lambda|^{k_2}(1 + |\tau| + |\xi| + |x^\mu\lambda|^{\frac{1}{\mu}})^{-\mu k_2} x^\mu(1 + |\tau| + |\xi| + |x^\mu\lambda|^{\frac{1}{\mu}})^{-\mu}(1 + |\tau| + |\xi| + |x^\mu\lambda|^{\frac{1}{\mu}})^{-|\alpha|} \\ &\le \tilde{C}(1 + |\tau| + |\xi| + |\lambda|^{\frac{1}{\mu}})^{-\mu}(1 + |\tau| + |\xi|)^{-|\alpha|}, \end{aligned} \tag{5.23}$$

since we apply the same argument as we did in (5.22) to each term in last inequality of (5.23). □

Proposition 5.4.4. *Let $\widetilde{g} \in C^\infty(\mathbb{R}_+, M_O^{-\mu,\mu}(\partial\mathcal{B}, \Lambda))$. Then $\omega x^\mu op^{\gamma-\frac{n}{2}}(\widetilde{g}(x^\mu\lambda))\omega_0$ has a continuous extension to an operator in $\mathcal{L}(\mathcal{C}^{0,\gamma}(\mathbb{B}))$. Moreover,*

$$\|\omega x^\mu op^{\gamma-\frac{n}{2}}(\widetilde{g}(x^\mu\lambda))\omega_0\|_{\mathcal{L}(\mathcal{C}^{0,\gamma}(\mathbb{B}))} \le \frac{C}{|\lambda|^{1-\varepsilon}},$$

for some $\varepsilon > 0$.

Proof. We use the following notation for the variables: $x \in [0, \infty)$, $\lambda \in \Lambda$ and z is from the holomorphic map $z \mapsto L^{\mu,-\mu}(\partial\mathcal{B}, \Lambda)$. In particular, for each (x, z, λ) fixed, we have $\widetilde{g}(x, z, x^\mu\lambda) \in L^{-\mu}(\partial\mathcal{B})$. Hence,

$$\omega op^{\gamma-\frac{n}{2}}(\widetilde{g}(x^\mu\lambda))\omega_0 u(x) = \omega(x)\frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} x^{-z}\widetilde{g}(x, z, x^\mu\lambda)\mathcal{M}(\omega_0 u)(z)dz.$$

In a local coordinate, the operator $\omega op^{\gamma-\frac{n}{2}}(\widetilde{g}(x^\mu\lambda))\omega_0$ can be written, modulo a regularizing operator, in the form

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} x^{-z}\omega(x)\widetilde{g}(x,z,x^\mu\lambda)\mathcal{M}(\omega_0 u)(z)dz\\&=\frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} x^{-z}\omega(x)\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{iy\xi}\widetilde{g}(x,y,z,\xi,x^\mu\lambda)\mathcal{F}_{y\to\xi}\mathcal{M}_{x\to z}(\omega_0 u)(z,\xi)d\xi dz.\end{aligned} \tag{5.24}$$

With some abuse of notation, we denote by $\widetilde{g}(x,y,\tau,\xi,x^\mu\lambda)$ and $\mathcal{F}_{y\to\xi}\mathcal{M}_{x\to z}(\omega_0 u)u(x,\tau)$ the functions $\widetilde{g}(x,y,\frac{n+1}{2}-\gamma+i\tau,\xi,x^\mu\lambda)$ and $\mathcal{F}_{y\to\xi}\mathcal{M}_{x\to z}(\omega_0 u)u(x,\frac{n+1}{2}-\gamma+i\tau)$, respectively.

We notice also that for $z\in\Gamma_{\frac{n+1}{2}-\gamma}$, the Mellin transform can be transformed into the Fourier transform, as follows:

$$\begin{aligned}\mathcal{M}_{x\to z}v(z)&=\int_0^\infty x^{z-1}v(x)dx=\int_0^\infty x^{\frac{n+1}{2}-\gamma+i\tau-1}v(x)dx=\int_{-\infty}^\infty e^{-s(\frac{n+1}{2}-\gamma+i\tau)}v(e^{-s})ds\\&=\int_{-\infty}^\infty e^{-is\tau}e^{-s(\frac{n+1}{2}-\gamma)}v(e^{-s})ds=\mathcal{F}_{s\to\tau}\left(e^{-s(\frac{n+1}{2}-\gamma)}v(e^{-s})\right).\end{aligned}$$

Hence, if $x=e^{-s}$, then Equation (5.24) can be written as

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^\infty x^{-\frac{n+1}{2}+\gamma-i\tau}\omega(x)\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{iy\xi}\widetilde{g}(x,y,\tau,\xi,x^\mu\lambda)\\&\quad\times\mathcal{F}_{y\to\xi}\mathcal{F}_{s\to\tau}\left(e^{-s(\frac{n+1}{2}-\gamma)}\omega_0(e^{-s})u(e^{-s},y)\right)(\tau,\xi)d\xi d\tau\\&=e^{(\frac{n+1}{2}-\gamma)s}\frac{1}{(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}} e^{i\tau s+iy\xi}\omega(e^{-s})\widetilde{g}(e^{-s},y,\tau,\xi,e^{-\mu s}\lambda)\\&\quad\times\mathcal{F}_{y\to\xi}\mathcal{F}_{s\to\tau}\left(e^{-(\frac{n+1}{2}-\gamma)s}\omega_0(e^{-s})u(e^{-s},y)\right)(\tau,\xi)d\xi d\tau.\end{aligned}$$

Therefore, in order to finish the proof, we will show that $op(\omega(e^{-s})e^{-\mu s}\widetilde{g}(e^{-s},y,\tau,\xi,e^{-\mu s}\lambda))$ satisfies the condition of *Case 3* of Theorem 4.3.1. In fact, if we change x by e^{-s} we have that $a(s,y,\tau,\xi,\lambda):=\omega(e^{-s})e^{-s\mu}\widetilde{g}(e^{-s},y,\tau,\xi,e^{-s\mu}\lambda)$ satisfies the conditions of Lemma 5.4.5. This, means that

$$|\partial^\beta_{(s,y)}\partial^\alpha_{(\tau,\xi)}a(s,y,\tau,\xi,\lambda)|\le C(1+|\tau|+|\xi|+|\lambda|^{\frac{1}{\mu}})^{-\mu}(1+|\tau|+|\xi|)^{-|\alpha|},$$

which is the case 3) in Theorem 4.3.1. Therefore, for $u\in\mathcal{C}^{0,\gamma}(\mathbb{B})$ and using the estimates above, we have that

$$\|x^{\frac{n+1}{2}-\gamma}\omega x^\mu op^{\gamma-\frac{n}{2}}(\widetilde{g}(x^\mu\lambda)u)\omega_0\|_{L_\infty(\mathbb{R}_+\times\mathbb{R}^n)}\le C\langle\eta\rangle^{-\mu+\epsilon}=\frac{C}{\langle|\lambda|^{\frac{1}{\mu}}\rangle^{\mu-\epsilon}}||x^{\frac{n+1}{2}-\gamma}u||_{L_\infty(\mathbb{R}_+\times\mathbb{R}^n)},$$

where $|\lambda|^{\frac{1}{\mu}}=\eta$, or equivalently,

$$\|\omega x^\mu op^{\gamma-\frac{n}{2}}(\widetilde{g}(x^\mu\lambda)u)\omega_0\|_{\mathcal{L}(\mathcal{C}^{0,\gamma}(\mathbb{B}))}\le\frac{C}{|\lambda|^{1-\varepsilon}}.$$

Proposition 5.4.1,5.4.2,5.4.3,5.4.4 imply that $\mathcal{A}:D(A)\to\mathcal{C}^{0,\gamma}(\mathbb{B})$ is an almost sectorial operator. Therefore, we conclude the proof of Theorem 5.4.1. □

5.5 Almost sectorial operators in $\mathcal{C}^{1,\gamma}(\mathbb{B})$

In this section we will prove almost sectoriality for an elliptic operator defined on $\mathcal{C}^{1,\gamma}(\mathbb{B})$ spaces. We consider elliptic operators $\mathcal{A}$ that satisfy the condition of Definition 5.3.1 and we study each component of the resolvent of $\mathcal{A}$, which is given by

$$(\lambda-\mathcal{A})^{-1}=\omega_1(x^\mu op_M^\gamma g(\lambda)+G(\lambda))\omega_2+(1-\omega_1)P(\lambda)(1-\omega_3)+G_\infty(\lambda) \tag{5.25}$$

as we have seen in Theorem 5.3.1. Next, we state two lemmas in order to prove the almost sectoriality in $\mathcal{C}^{1,\gamma}(\mathbb{B})$.

Lemma 5.5.1. *Let $p\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n\times\Lambda)$ such that $(x,\xi)\mapsto p(x,\xi,\lambda)\in S^\mu(\mathbb{R}^n\times\mathbb{R}^n)$, $\mu\in\mathbb{R}$, for all $\lambda\in\Lambda$. Then, we have that:*

$$\partial_{x_j}op(p(x,\xi,\lambda))u=op(\partial_{x_j}p(x,\xi,\lambda))u+op(p(x,\xi,\lambda))\partial_{x_j}u$$

Proof. By definition,

$$\begin{aligned}
&\partial_{x_j}op(p(x,\xi,\lambda))u\\
&=\partial_{x_j}\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}p(x,\xi,\lambda)\mathfrak{F}u(\xi)d\xi\right)\\
&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}i\xi_je^{ix\cdot\xi}p(x,\xi,\lambda)\mathfrak{F}u(\xi)d\xi+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\partial_{x_j}p(x,\xi,\lambda)\mathfrak{F}u(\xi)d\xi\\
&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}p(x,\xi,\lambda)\mathfrak{F}[\partial_{x_j}u](\xi)d\xi+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\partial_{x_j}p(x,\xi,\lambda)\mathfrak{F}u(\xi)d\xi\\
&=op(p(x,\xi,\lambda))\partial_{x_j}u+op(\partial_{x_j}p(x,\xi,\lambda))u,
\end{aligned} \tag{5.26}$$

or equivalently, with notation of commutator operator, $[\partial_{x_j},op(p)]=op(\partial_{x_j}p)$. □

Lemma 5.5.2. *Let $p\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n\times\Lambda)$ be a function that satisfies the conditions of* Case 3 *of Theorem 4.3.1. Then, $||op(p)||_{BUC^1(\mathbb{R}^n)}\le\frac{C}{|\lambda|^{1-\varepsilon}}$ for some $\varepsilon>0$. If p satisfies the conditions of* Case 1 *of Theorem 4.3.1, then, $||op(p)||_{BUC^1(\mathbb{R}^n)}\le\frac{C}{|\lambda|}$.*

Proof. We prove for p satisfying *Case 3* of Theorem 4.3.1. The other case is analogous and will be omitted. By definition of the norm in $BUC^1(\mathbb{R}^n)$, we have that

$$\begin{aligned}
||op(p)u||_{BUC^1(\mathbb{R}^n)}&=||op(p)u||_{BUC(\mathbb{R}^n)}+\sum_{j=1}^n||\partial_{x_j}op(p)u||_{BUC(\mathbb{R}^n)}\\
&\le||op(p)u||_{BUC(\mathbb{R}^n)}+\sum_{j=1}^n\left(||op(p)\partial_{x_j}u||_{BUC(\mathbb{R}^n)}+||op(\partial_{x_j}p)u||_{BUC(\mathbb{R}^n)}\right)\\
&\le\frac{C}{|\lambda|^{1-\varepsilon}}||u||_{BUC(\mathbb{R}^n)}+\sum_{j=1}^n\left(\frac{C}{|\lambda|^{1-\varepsilon}}||\partial_{x_j}u||_{BUC(\mathbb{R}^n)}+\frac{C}{|\lambda|^{1-\varepsilon}}||u||_{BUC(\mathbb{R}^n)}\right)\\
&\le\frac{C_1}{|\lambda|^{1-\varepsilon}}||u||_{BUC^1(\mathbb{R}^n)}.
\end{aligned} \tag{5.27}$$

□

Theorem 5.5.3. *Let $\mathcal{A}$ be as in Definition 5.3.1. For some $c \in \mathbb{R}$, we have that*

$$c + \mathcal{A} : D(\mathcal{A}) \to \mathcal{C}^{1,\gamma}(\mathbb{B})$$

is an almost sectorial operator, where $D(\mathcal{A}) = (\lambda - \mathcal{A})^{-1}(\mathcal{C}^{1,\gamma}(\mathbb{B}))$ for some $\lambda \in \Lambda$.

Remark 5.5.1. *The following proof is based on Roidos and Schrohe, see Theorem 3.3 in [26].*

Proof. In order to show this result, we will estimate all the terms that appear in (5.25) and we will use that the norm in $\mathcal{C}^{1,\gamma}(\mathbb{B})$ is given in terms of the norm in $\mathcal{C}^{0,\gamma}(\mathbb{B})$. More specifically, we have $||\cdot||_{\mathcal{C}^{1,\gamma}(\mathbb{B})} = ||\cdot||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} + ||x\partial_x(\cdot)||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} + \sum_{i=1}^{n} ||\partial_{y_i}(\cdot)||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}$.

First, we study the term $op_M^{\gamma} g(\lambda)$.

For $x\partial_x$ derivative, we have

$$\begin{aligned}
&(x\partial_x)(op_M^{\gamma} g(\lambda)u(x,y))\\
&= (x\partial_x)\int_{\Gamma} x^{-z} g(x,y,z,\lambda)\mathcal{M}u(x,y)(z)dz\\
&= (x\partial_x)\int_{\Gamma} x^{-z} g(x,y,z,\lambda)\int_0^{\infty} x'^{z-1}u(x',y)dx'dz\\
&= (x\partial_x)\int_{\Gamma}\int_0^{\infty}\left(\frac{x'}{x}\right)^{z} g(x,y,z,\lambda)u(x',y)\frac{dx'}{x'}dz\\
&= (x\partial_x)\int_{\Gamma}\int_0^{\infty}\left(\frac{x'}{x}\right)^{z} \widetilde{g}(x,y,z,\lambda x^{\mu})u(x',y)\frac{dx'}{x'}dz\\
&= \int_{\Gamma}\int_0^{\infty}\left[(x\partial_x)\left(\frac{x'}{x}\right)^{z}\widetilde{g}(x,y,z,\lambda x^{\mu}) + \left(\frac{x'}{x}\right)^{z}(x\partial_x)\widetilde{g}(x,y,z,\lambda x^{\mu})\right]u(x',y)\frac{dx'}{x'}dz\\
&= \quad \int_{\Gamma}\int_0^{\infty} -z\left(\frac{x'}{x}\right)^{z}\widetilde{g}(x,y,z,\lambda x^{\mu})u(x',y)\frac{dx'}{x'}dz + \int_{\Gamma}\int_0^{\infty}\left(\frac{x'}{x}\right)^{z} x\partial_x\widetilde{g}(x,y,z,x^{\mu}\lambda)u(x',y)\frac{dx'}{x'}dz\\
&\quad + \int_{\Gamma}\int_0^{\infty}\left(\frac{x'}{x}\right)^{z}\lambda\mu x^{\mu}\partial_{\lambda}\widetilde{g}(x,y,z,x^{\mu}\lambda)u(x',y)\frac{dx'}{x'}dz\\
&= \quad -\int_{\Gamma}\int_0^{\infty}\left(x'\partial_{x'}\left(\frac{x'}{x}\right)^{z}\right)\widetilde{g}(x,y,z,\lambda x^{\mu})u(x',y)\frac{dx'}{x'}dz+\\
&\int_{\Gamma}\int_0^{\infty}\left(\frac{x'}{x}\right)^{z} x\partial_x\widetilde{g}(x,y,z,x^{\mu}\lambda)u(x',y)\frac{dx'}{x'}dz + \int_{\Gamma}\int_0^{\infty}\left(\frac{x'}{x}\right)^{z}\lambda\mu x^{\mu}\partial_{\lambda}\widetilde{g}(x,y,z,x^{\mu}\lambda)u(x',y)\frac{dx'}{x'}dz\\
&= \int_{\Gamma}\int_0^{\infty}\left(\frac{x'}{x}\right)^{z}\widetilde{g}(x,y,z,\lambda x^{\mu})\left((x'\partial_{x'})\frac{u(x',y)}{x'}\right)dx'dz+\\
&\int_{\Gamma}\int_0^{\infty}\left(\frac{x'}{x}\right)^{z} x\partial_x\widetilde{g}(x,y,z,x^{\mu}\lambda)u(x',y)\frac{dx'}{x'}dz + \int_{\Gamma}\int_0^{\infty}\left(\frac{x'}{x}\right)^{z}\lambda\mu x^{\mu}\partial_{\lambda}\widetilde{g}(x,y,z,x^{\mu}\lambda)u(x',y)\frac{dx'}{x'}dz\\
&= \int_{\Gamma}\int_0^{\infty}\left(\frac{x'}{x}\right)^{z}\widetilde{g}(x,y,z,\lambda x^{\mu})\left(x'\partial_{x'}u(x',y)\frac{1}{x'} - u(x',y)\frac{1}{x'}\right)dx'dz+\\
&\int_{\Gamma}\int_0^{\infty}\left(\frac{x'}{x}\right)^{z} x\partial_x\widetilde{g}(x,y,z,x^{\mu}\lambda)u(x',y)\frac{dx'}{x'}dz + \int_{\Gamma}\int_0^{\infty}\left(\frac{x'}{x}\right)^{z}\lambda\mu x^{\mu}\partial_{\lambda}\widetilde{g}(x,y,z,x^{\mu}\lambda)u(x',y)\frac{dx'}{x'}dz\\
&= L(\widetilde{g}),
\end{aligned} \tag{5.28}$$

where $\widetilde{g}$ is given by the relation $g(x,z,\lambda)=\widetilde{g}(x,z,x^{\mu}\lambda)$. In local charts, we have

$$\begin{aligned}
&||x^{\mu}(x\partial_x)op_M^{\gamma}g(\lambda)u(x,y)||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}\\
&=||x^{\frac{n+1}{2}-\gamma+\mu}(x\partial_x)op_M^{\gamma}g(\lambda)u(x,y)||_{L_\infty(\mathbb{R}_+\times\mathbb{R}^n)}\\
&\le ||x^{\frac{n+1}{2}-\gamma+\mu}op_M^{\gamma}\widetilde{g}(\lambda)(x\partial_x u)||_{L_\infty(\mathbb{R}_+\times\mathbb{R}^n)}+||x^{\frac{n+1}{2}-\gamma+\mu}op_M^{\gamma}\widetilde{g}(\lambda)u||_{L_\infty(\mathbb{R}_+\times\mathbb{R}^n)}\\
&\quad+||x^{\frac{n+1}{2}-\gamma+\mu}op_M^{\gamma}(x\partial_x\widetilde{g})(\lambda)(u)||_{L_\infty(\mathbb{R}_+\times\mathbb{R}^n)}+||\lambda\mu x^{\mu}x^{\frac{n+1}{2}-\gamma+\mu}op_M^{\gamma}(\partial_\lambda\widetilde{g})(\lambda)(u)||_{L_\infty(\mathbb{R}_+\times\mathbb{R}^n)}.
\end{aligned}\tag{5.29}$$

Now we notice that all the terms have the same behaviour as $op_M^{\gamma}g(\lambda)$. As a consequence, we have that

$$||x^{\mu}(x\partial_x)op_M^{\gamma}g(\lambda)u(x,y)||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}\le\frac{M||u||_{\mathcal{C}^{1,\gamma}(\mathbb{B})}}{|\lambda|^{1-\varepsilon}},$$

due to the term $x\partial_x$ and Proposition 5.4.4.

On the other hand,

$$(x\partial_x)[\omega_1(x^{\mu}op_M^{\gamma}g(\lambda))\omega_2]=(x\partial_x\omega_1)x^{\mu}op_M^{\gamma}g(\lambda)+\mu\omega_1x^{\mu}op_M^{\gamma}g(\lambda)\omega_2+\omega_1x^{\mu}op_M^{\gamma}g(\lambda)(x\partial_x\omega_2)+\omega_1x^{\mu}L(\widetilde{g})\omega_2.\tag{5.30}$$

We notice that all the terms in (5.30) are uniformly bounded in $\mathcal{C}^{0,\gamma}(\mathbb{B})$ by $\frac{M}{|\lambda|^{1-\varepsilon}}$ for larger λ, hence

$$||(x\partial_x)[\omega_1(x^{\mu}op_M^{\gamma}g(\lambda))\omega_2]u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}\le\frac{M}{|\lambda|^{1-\varepsilon}}||u||_{\mathcal{C}^{1,\gamma}(\mathbb{B})}.$$

Below we analyse the terms with ∂_{y_i}. By Lemma 5.5.1 we have that

$$\partial_{y_j}x^{\mu}op_M^{\gamma}g(\lambda)u=x^{\mu}op_M^{\gamma}g(\lambda)\partial_{y_j}u+x^{\mu}op_M^{\gamma}\partial_{y_j}g(\lambda)u,$$

which is uniformly bounded in $\mathcal{C}^{0,\gamma}(\mathbb{B})$ by $\frac{M}{|\lambda|^{1-\varepsilon}}$ for larger λ by a similar analysis as before, that is

$$||(\partial_{y_i})[\omega_1(x^{\mu}op_M^{\gamma}g(\lambda))\omega_2]u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}\le\frac{M}{|\lambda|^{1-\varepsilon}}||u||_{\mathcal{C}^{1,\gamma}(\mathbb{B})}.$$

As a consequence, by Lemma 5.5.2 we have that

$$\begin{aligned}
\|\omega x^{\mu}op^{\gamma-\frac{n}{2}}(g(\lambda))\omega_0u\|_{\mathcal{C}^{1,\gamma}(\mathbb{B})}&=||\omega x^{\mu}op^{\gamma-\frac{n}{2}}(g(\lambda))\omega_0u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}+||(x\partial_x)\omega x^{\mu}op^{\gamma-\frac{n}{2}}(g(\lambda))\omega_0u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}\\
&\quad+||\sum_{j=1}^{n}\partial_{y_j}\omega x^{\mu}op^{\gamma-\frac{n}{2}}(g(\lambda))\omega_0u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}\\
&\le\frac{C}{|\lambda|^{1-\varepsilon}}\|u\|_{\mathcal{C}^{0,\gamma}(\mathbb{B})}+\frac{M_1}{|\lambda|^{1-\varepsilon}}||x\partial_xu||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}+\sum_{j=1}^{n}\frac{M_j}{|\lambda|^{1-\varepsilon}}||\partial_ju||_{\mathcal{C}^{0,\gamma}(\mathbb{B})}\\
&\le\frac{\tilde{C}}{|\lambda|^{1-\varepsilon}}\|u\|_{\mathcal{C}^{1,\gamma}(\mathbb{B})},
\end{aligned}\tag{5.31}$$

Second, we study the behaviour of $G_\infty(\lambda),G(\lambda)$ and $P(\lambda)$.
For the term $G(\lambda)$, we will estimate the terms $(x\partial_x)G(\lambda)$ and $\partial_yG(\lambda)$. We note that since $G(\lambda)\in R_G^{-\mu,\mu}(\partial\mathbb{B}^\wedge;\Lambda,\gamma)$, it has a kernel with respect to the $\mathcal{K}_2^{0,0}(\partial\mathbb{B}^\wedge)$ scalar product of the form

$$k(\lambda,x,y,x',y')=[\lambda]^{\frac{n+1}{\mu}}\tilde{k}(\lambda,[\lambda]^{\frac{1}{\mu}}x,y,[\lambda]^{\frac{1}{\mu}}x',y')$$

where $\tilde{k} \in S^{-1}(\Lambda) \otimes S_0^{\gamma+\epsilon}(\partial\mathbb{B}^\wedge) \otimes S_0^{-\gamma+\epsilon}(\partial\mathbb{B}^\wedge)$. Then,

$$\begin{aligned}(x\partial_x)G(\lambda)u(x,y) &= (x\partial_x)\int_{\partial\mathbb{B}^\wedge}[\lambda]^{\frac{n+1}{\mu}}\tilde{k}(\lambda,[\lambda]^{\frac{1}{\mu}}x,y,[\lambda]^{\frac{1}{\mu}}x',y')u(x',y')x'^n dx'dy' \\ &= \int_{\partial\mathbb{B}^\wedge}[\lambda]^{\frac{n+1}{\mu}}(x\partial_x)\tilde{k}(\lambda,[\lambda]^{\frac{1}{\mu}}x,y,[\lambda]^{\frac{1}{\mu}}x',y')u(x',y')x'^n dx'dy',\end{aligned} \tag{5.32}$$

and $(x\partial_x)\tilde{k}(\lambda,[\lambda]^{\frac{1}{\mu}}x,y,[\lambda]^{\frac{1}{\mu}}x',y') = [\lambda]^{\frac{1}{\mu}}x\partial_x\tilde{k}(\lambda,[\lambda]^{\frac{1}{\mu}}x,y,[\lambda]^{\frac{1}{\mu}}x',y') = (x\partial_x\tilde{k})(\lambda,[\lambda]^{\frac{1}{\mu}}x,y,[\lambda]^{\frac{1}{\mu}}x',y')$ and $x\partial_x\tilde{k} \in S^{-1}(\Lambda) \otimes S_0^{\gamma+\epsilon}(\partial\mathbb{B}^\wedge) \otimes S_0^{-\gamma+\epsilon}(\partial\mathbb{B}^\wedge)$. Analogously, we have the same estimate for the term $\partial_y G(\lambda)$. Therefore, we have the same boundedness in $\mathcal{L}(\mathcal{C}^{0,\gamma}(\mathbb{B}))$. This implies that all the terms are bounded in $\mathcal{L}(\mathcal{C}^{1,\gamma}(\mathbb{B}))$ by a constant $\frac{M}{|\lambda|}$. The operator $G_\infty(\lambda)$ have behavior of order $O(|\lambda|^{-N})$ in the norm $\mathcal{C}^{1,\gamma}(\mathbb{B})$ for any $N \in \mathbb{N}_0$. For the operator $P(\lambda)$, we use Lemma 5.5.2. Therefore, using *Case 1*, $||P(\lambda)||_{\mathcal{L}(\mathcal{C}^{1,\gamma}(\mathbb{B}))}$ is bounded by $\frac{M}{|\lambda|}$ and with this, we finish the proof of the theorem. □

5.6 Application for Almost Sectorial Operators

In order to finish this work, we give an application for almost sectorial operators (see Definition 2.1.3). In this case, we will use Theorem 5.4.1. Let $\mathbb{B}$ be a conic manifold. Consider the following equation

$$\begin{aligned} u_t = \Delta u + f(t,u) \quad &\text{on} \quad \mathbb{B},\ t > 0 \\ u(0,x) = u_0(x) \quad &\text{on} \quad \mathbb{B}. \end{aligned} \tag{5.33}$$

Let X be a complex Banach space. For $\omega \in (0, \frac{\pi}{2})$ we have the following definition. For more details, see [10].

Definition 5.6.1. *Let $A : D(A) \to X$ be an almost sectorial operator in $\Lambda(\phi)$. We define the family of operators $\{T(t) : t \in \mathbb{C} \setminus \{0\}, |arg\ t| < \frac{\pi}{2} - \phi\}$ by*

$$T(t) := \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{-tz}(z-A)^{-1}dz,$$

where $\Gamma_\theta = \{re^{-i\theta} : r > 0\} \cup \{re^{i\theta} : r > 0\}$ with $\phi < \theta < \frac{\pi}{2}$ is oriented counter-clockwise.

The mild solution of (5.33) is defined by the continuous solution $u : (0,T] \to X$ of the integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds.$$

For $T > 0$ and fixed $\delta > 0$, we define the metric space

$$K(T,u_0) = \{v \in C((0,T],X) : \sup_{0<t\leq T} ||v(t) - T(t)u_0||_X \leq \delta\}.$$

The metric is defined as

$$\varrho_T(v_1,v_2) = \sup_{0<t\leq T} ||v_1(t) - v_2(t)||_X \quad \text{for } v_1, v_2 \in K(T,u_0).$$

Remark 5.6.1. *In order to show the existence of mild solution to* (5.33) *we notice that, for $u_0 \in D(A)$ we have that the function $[0,\infty) \ni t \to T(t)u_0 \in X$ is continuous and the sets where $||x - T(t)u_0|| \leq \delta$ are bounded. For more details see [10].*

We use the following proposition.

Proposition 5.6.1. *Let $A : D(A) \to X$ be an almost sectorial operator and f in (5.33) be a Lipschitz continuous function. More precisely, we require that:*

1) $\exists L > 0, \forall 0 < t \leq T, \forall x, y \in X : ||x - T(t)u_0||_X, ||y - T(t)u_0||_X \leq \delta$ then $||f(t,x) - f(t,y)||_X \leq L||x - y||_X$. And that f is bounded there.

2) $\exists N > 0, \forall 0 < t \leq T, \forall x \in X : ||x - T(t)u_0||_X \leq \delta$ then $||f(t,x)||_X \leq N$.
Then for sufficiently small positive $\tau_0 \leq T$ there is a mild solution to (5.33) in $K(T, u_0)$.

Proof. See Proposition 2 in [12]. □

Finally, we show the application to (5.33) with the non linear term,

$$f : X \to X,$$

with $f(u) := \omega u + (1-\omega)u^3$ and $X := \mathcal{C}^{0,\gamma}(\mathbb{B})$, where ω is a cutoff function defined in a neighborhood of the conic manifold $\mathbb{B}$ and $u_0 \in D(A) = (\lambda - \Delta)^{-1}(\mathcal{C}^{0,\gamma}(\mathbb{B}))$, for large $\lambda \in \Lambda$. We assume that γ is such that Δ satisfies the conditions of Definition 5.3.1, see [9]. We notice that

$$\begin{aligned}
&||f(u) - f(v)||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \\
&= ||\omega u + (1-\omega)u^3 - \omega v - (1-\omega)v^3||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \\
&= ||\omega(u-v) + (1-\omega)(u^3 - v^3)||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \\
&\leq ||u-v||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} + ||(1-\omega)(u^3 - v^3)||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \\
&= ||u-v||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} + ||(u-v)(1-\omega)(u^2 + uv + v^2)||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \\
&\leq ||u-v||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \left(1 + ||\sqrt{(1-\omega)}u||^2_{L_\infty(\mathbb{B})} + ||\sqrt{(1-\omega)}u||_{L_\infty(\mathbb{B})}||\sqrt{(1-\omega)}v||_{L_\infty(\mathbb{B})} + ||\sqrt{(1-\omega)}v||^2_{L_\infty(\mathbb{B})}\right) \\
&\leq ||u-v||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \left(1 + \frac{3}{2}(||\sqrt{(1-\omega)}u||^2_{L_\infty(\mathbb{B})} + ||\sqrt{(1-\omega)}v||^2_{L_\infty(\mathbb{B})})\right) \\
&\leq L||u-v||_{\mathcal{C}^{0,\gamma}(\mathbb{B})},
\end{aligned} \tag{5.34}$$

where we have used that far from the singularity $||\cdot||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} = ||\cdot||_{L_\infty(\mathbb{B})}$ and $u, v \in B \subset \mathcal{C}^{0,\gamma}(\mathbb{B})$, where B is a bounded subset, see Remark 5.6.1. Moreover, with the same arguments we obtain

$$\begin{aligned}
||f(u)||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} &= ||\omega u + (1-\omega)u^3||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \\
&= ||\omega u - (1-\omega)u + (1-\omega)u + (1-\omega)u^3||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \\
&\leq ||u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} + ||(1-\omega)(u^3 - u)||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \\
&\leq ||u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} + ||(1-\omega)(u^3 - u)||_{L_\infty(\mathbb{B})} \\
&\leq ||u||_{\mathcal{C}^{0,\gamma}(\mathbb{B})} \left(1 + (||u||^2_{\mathcal{C}^{0,\gamma}(\mathbb{B})} + 1)\right) \leq N,
\end{aligned} \tag{5.35}$$

for some constant N. As a consequence, by Proposition 5.6.1 we prove that (5.33) with this particular f has a mild solution in a conic manifold $\mathbb{B}$. □

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