Chaos and Turing Machines on Bidimensional Models at Zero Temperature

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Resumo

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Em mecânica estatística de equilíbrio ou formalismo termodinâmico um dos principais objetivos é descrever o comportamento das famílias de medidas de equilíbrio para um dado potencial parametrizado pelo inverso da temperatura β. Entendemos aqui por medidas de equilíbrio as medidas shift invariantes que mazimizam a pressão. Diversas construções já demonstraram um comportamento caótico destas medidas quando o sistema congela, ou seja, $\beta \to +\infty$. Um dos principais exemplos é o construído por Chazottes e Hochman [11] onde eles conseguem provar a não convergência de uma família de medidas de equilíbrio para um dado potential localmente constante nos casos onde a dimensão é maior ou igual a 3. Neste trabalho apresentaremos a construção de um exemplo no caso bidimensional sobre um alfabeto finito e um potencial localmente constante tal que existe uma sequencia $(\beta_k)_{k\geqslant 0}$ onde não ocorre a convergência para qualquer sequência de medidas de equilíbrio ao inverso da temperatura β_k quando $\beta_k \to +\infty$. Para tal, usaremos a construção descrita por Aubrun e Sablik em [2] que melhora o resultado de Hochman [19] usado na construção de Chazottes e Hochman [11].

Palavras-chave: formalismo termodinâmico, medida de equilíbrio, subshift.

Abstract

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In equilibrium statistical mechanics or thermodynamics formalism one of the main objectives is to describe the behavior of families of equilibrium measures for a potential parametrized by the inverse temperature β. Here we consider equilibrium measures as the shift invariant measures that maximizes the pressure. Other constructions already prove the chaotic behavior of these measures when the system freezes, that is, when $\beta \to +\infty$. One of the most important examples was given by Chazottes and Hochman [11] where they prove the non-convergence of the equilibrium measures for a locally constant potential when the dimension is bigger than or equal to 3. In this work we present a construction of a bidimensional example described by a finite alphabet and a locally constant potential in which there exists a subsequence $(\beta_k)_{k\geqslant 0}$ where the non-convergence occurs for any sequence of equilibrium measures at inverse temperatures β_k when $\beta_k \to +\infty$. In order to describe such an example, we use the construction described by Aubrun and Sablik [2] which improves the result of Hochman [19] used in the construction of Chazottes and Hochman [11].

Keywords: thermodynamic formalism, equilibrium measure, subshift.

Résumé

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En mécanique statistique d'équilibre ou formalisme thermodynamique un des objectifs est de décrire le comportement des familles de mesures d'équilibre pour un potentiel paramétré par la température inverse β. Nous considérons ici une mesure d'équilibre comme une mesure shift invariante qui maximise la pression. Il existe d'autres constructions qui prouvent le comportement chaotique de ces mesures lorsque le système se fige, c'est-à-dire lorsque $\beta \to +\infty$. Un des exemples les plus importants a été donné par Chazottes et Hochman [11] où ils prouvent la non-convergence des mesures d'équilibre pour un potentiel localement constant lorsque la dimension est supérieure à 3. Dans ce travail, nous présentons une construction et un exemple potentiel localement constant tel qu'il existe une suite $(\beta_k)_{k\geqslant 0}$ où la non-convergence est assurée pour toute choix suite de mesures d'équilibre à l'inverse de la température β_k lorsque $\beta_k \to +\infty$. Pour cela nous utilisons la construction décrite par Aubrun et Sablik [2] qui améliore le résultat de Hochman [19] utilisé dans la construction de Chazottes et Hochman [11].

Mots clés: formalisme thermodynamique, measure d'équilibre, décalage.

Contents

Chapter 1

Introduction

One of the most important problems in equilibrium statistical mechanics consists in describing families of Gibbs states for a given potential or an interaction family. We work with classical lattice systems, which means that our configuration space will be

$$\Sigma^d(\mathcal{A}) := \mathcal{A}^{\mathbb{Z}^d}$$

where $\mathcal{A}$ is a finite set and $d \in \mathbb{N}$ is the dimension of our lattice. Let us introduce the function

$$\varphi : \Sigma^d(\mathcal{A}) \to \mathbb{R}$$

which is called *per site potential* and can be physically interpreted as the energy contribution of the origin of the lattice for each configuration $x \in \Sigma^d(\mathcal{A})$, since we are only considering only translation invariant measures.

Given these elements we denote for every $\beta > 0$ the set $\mathcal{G}(\beta\varphi)$ which is the set of *Gibbs measures* associated to $\beta\varphi$ at the inverse temperature β. The are several definitions we could consider as a Gibbs measure, using conformal measures, DLR equations, thermodynamic limits etc. See Georgii [17], the classical book about Gibbs measures and [25] for the equivalence of several of these definitions. By compactness we know that this set has at least one shift translation invariant Gibbs measure. In the present thesis we are interested on the behavior of the set of Gibbs measures which are translational-invariant probability measures, called equilibrium measures, when the temperature goes to zero, that is, when $\beta \to +\infty$.

A probability measure μ_β over $\Sigma^d(\mathcal{A})$ is an *equilibrium measure* (or *equilibrium state*) at inverse temperature $\beta > 0$ for a potential $\beta\varphi$ if it is a shift invariant (or translation invariant) measure which maximizes the pressure, that is if

$$P(\beta\varphi) := \sup_{\mu \in \mathcal{M}_\sigma(\Sigma^d(\mathcal{A}))} \left\{ h(\mu) - \int \beta\varphi \, d\mu \right\} = h(\mu_\beta) - \int \beta\varphi \, d\mu_\beta.$$

We will consider later the whole set of equilibrium measures μ_β which maximize the pressure $P(\beta\varphi)$ above over all shift invariant probability measures on $\Sigma^d(\mathcal{A})$. The function $h(\nu)$ in the expression of $P(\beta\varphi)$ is the Kolmogorov-Sinai entropy of ν.

In the one-dimensional case if a potential φ is Hölder continuous we always have a unique Gibbs measure which is also the only equilibrium measure. For a dimension $d > 1$ the situation is dramatically different and we can have multiple Gibbs states even for a potential with finite range, the most famous example is the Ising model.

The zero-temperature equilibrium states (*ground states*) are the shift invariant probability measures which minimize

$$\int \varphi d\nu$$

over all shift-invariant probability measures ν. In other words, given a potential, we have that the weak* accumulation points of equilibrium states as $\beta \to +\infty$ are necessarily minimizing measures for the potential φ. A more detailed study on the limit when the system freezes and how it is related with the configurations with minimal energy can be found in [36].

Chazottes and Hochman [11] showed in the one-dimensional case an example of a Lipschitz potential φ (but long-range) where the sequence $\mu_{\beta\varphi}$ does not converge when $\beta \to +\infty$. Here $\mu_{\beta\varphi}$ is the unique shift-invariant Gibbs measure (or the unique Gibbs measure) at the inverse temperature $\beta > 0$ (which is also the unique equilibrium measure). On the other hand, [8, 10, 16, 27] showed that an interaction of finite-range in the one-dimensional case over a finite alphabet implies the convergence of $\mu_{\beta\varphi}$. The case when $\mathcal{A}$ is a countable set was also studied in [23]. The breakthrough for the construction of examples of the non-convergence was given by van Enter and W. Ruszel [37], where an example of finite range potential on a continuous state space and chaotic behavior was constructed. Recently the argument of van Enter and Ruszel was implemented for the case where $\mathcal{A}$ is a finite set in [7, 3, 12].

Chazottes and Hochman [11] also showed that the same kind of non-convergence may occur when the dimension is $d \geqslant 3$ even for a locally constant potential. The construction of their example is possible only for $d \geqslant 3$ because they rely heavily on the theory of multidimensional subshifts of finite type and Turing Machines, developed by Hochman [19] that provides a method to transfer a one-dimensional construction to a higher-dimensional subshift of finite type. Thanks to Hochman's theorem, Chazottes and Hochman could construct an example for $d = 3$ with a potential φ locally constant on a finite state space. Their construction can be easily extended to any dimension $d \geqslant 3$. These results led us to believe that the statement is also true for $d = 2$. Our main result is two-fold: we extend Chazottes-Hochman's theorem of chaotic behavior to dimension 2 using a different approach involving the space-time diagram of a Turing machine developed by Aubrun-Sablik and we clarify the role of the reconstruction and relative complexity function of the extension by a subshift of finite type that is missing in Chazottes-Hochman's arguments.

The main result of Aubrun and Sablik [2], called simulation theorem, asserts that any d-dimensional subshift defined by a set of forbidden patterns that is enumerated by a Turing machine is a subaction of a $(d+1)$-dimensional subshift of finite type. There are other works in which the simulation results obtained so far in this theory have been improved [14, 15]. In these works they improve the results obtained so far by decreasing the dimension of the subshift of finite type which generates the effective subshift, but they are based on Kleene's fixed point theorem and they do not uses geometric arguments.

The construction of Aubrun and Sablik [2] improves the method of Hochman [19], because they increase the dimension by 1 and this leads us to improve the Chazottes and Hochman [11] construction for the dimension 2.

In the second chapter we present the main definitions of thermodynamic formalism and computability, classical results and standard notations. We begin with the definition of subshifts and define a special class of subshifts based on the concatenation of blocks of the same size in order to form each possible configuration. In the second section of this chapter we provide a brief review of entropy dealing with partitions, entropy of a partition, metric and topological entropy and the concepts of pressure, equilibrium measure and Gibbs measure. In the third section we give a general idea of operations transforming a subshift into another one based on [1] in order to comprehend the notion of simulating a subshift by another one. Finally, we present a formal definition of a Turing machine, how to represent the work of a Turing machine in a space-time diagram and also an idea of the construction of Aubrun and Sablik [2].

The third chapter is dedicated to define and construct our example that is inspired by the construction presented in the work of Chazottes and Hochman [11]. First we define a one-dimensional subshift based on an iteration process that gives us at each step blocks of the same length that are concatenated to form a subshift as defined in Chapter 2. We prove that the control we have obtained over the set of forbidden words of this subshift, implies there exists a Turing machine that lists all of the forbidden words, that is, our subshift is an effectively closed subshift. From there we are able to use the simulation theorem of Aubrun-Sablik [2] and obtain a bidimensional subshift of finite type that simulates our previous one-dimensional effectively closed subshift. Also in the second section of this chapter, we prove some important results that explain how to deconstruct a configuration in the 2-dimensional subshift as concatenated patterns in a given dictionary. In the third and last part of this chapter, we define a new coloring for the bidimensional subshift, as in Chazottes and Hochman [11], that consists in duplicating a distinguished symbol, in order to transfer the entropy of the initial effective subshift to the simulated subshift of finite type.

After all these constructions, we end up with a bidimensional SFT X defined over a finite alphabet $\mathcal{A}$, an integer $D \geqslant 1$ and a finite set of forbidden patterns $\mathcal{F} \subset \mathcal{A}^{[\![1,D]\!]^2}$.

We then define the following locally constant per site potential

$$\begin{array}{rcl}\varphi : \mathcal{A}^{\mathbb{Z}^2} = \Sigma^2(\mathcal{A}) & \to & \mathbb{R} \\ x & \mapsto & \varphi(x) = \mathbb{1}_F(x)\end{array}$$

where F is the clopen set equal to the union of cylinders generated by every pattern in $\mathcal{F}$.

The last chapter is dedicated to prove the main result which is the following.

Theorem 1. There exists a locally constant potential $\varphi : \Sigma^2(\mathcal{A}) \to \mathbb{R}$, there exists a subsequence $(\beta_k)_{k\geqslant 0}$ going to infinity and two disjoint non-empty compact invariant sets X_A, X_B of $\Sigma^2(\mathcal{A})$, such that if μ_{β_k} is an equilibrium measure at inverse temperature β_k associated to the potential $\beta_k\varphi$, the support of any weak* accumulation point of $(\mu_{\beta_{2k}})_{k\geqslant 0}$ is included in X_B, the support of any weak* accumulation point of $(\mu_{\beta_{2k+1}})_{k\geqslant 0}$ is included in X_A.

The previous theorem asserts that there exists a subsequence $(\beta_k)_{k\in\mathbb{N}}$ with $\beta_k \to +\infty$ such that any choice of equilibrium measure associated with the potential $\beta_k\varphi$ alternates between two disjoint compact sets of probability measures. That is there exists a locally constant per site potential that exhibits a zero-temperature chaotic convergence.

We compute in the appendix an upper bound of the relative complexity and reconstruction functions of the SFT given in [2]; we thank S.B. for many discussions on this topic.

Chapter 2

Subshifts

2.1 Forbidden words

In this chapter we establish the basic definitions, notations and main results of the objects that we use in this work. We begin by two definitions of a subshift: one topological and one combinatorial. These two definitions coincide.

We will always work with a finite set of letters that we call *alphabet* and we will denote it with a cursive letter $\mathcal{A}$. With this alphabet we construct the set of configurations defined over $\mathbb{Z}^d$ where $d \geqslant 1$ is the dimension.

Definition 1. Let $\mathcal{A}$ be a finite alphabet, and $d \geqslant 1$. Let $S \subseteq \mathbb{Z}^d$ be a subset. A *pattern with support* S is an element of p of $\mathcal{A}^S$. We write $S = \operatorname{supp}(p)$ for the *support* of the pattern p. If $S' \subseteq S$, the pattern $p' = p|_{S'}$ denotes the restriction of p to S'. A *configuration* is a pattern with full support $S = \mathbb{Z}^d$.

When $d = 1$ a one-dimensional finite pattern is called a *word*.

The set of all possible $\mathbb{Z}^d$-configurations defined over an alphabet $\mathcal{A}$ is denoted by $\Sigma^d(\mathcal{A}) := \mathcal{A}^{\mathbb{Z}^d}$. On this set we define the shift action as follows.

Definition 2. The shift action on a configuration space $\Sigma^d(\mathcal{A})$ is a collection $\sigma = (\sigma^u)_{u \in \mathbb{Z}^d}$ such that

$$\begin{array}{rcl} \sigma^u : \Sigma^d(\mathcal{A}) & \to & \Sigma^d(\mathcal{A}) \\ x & \mapsto & \sigma^u(x) = y, \text{where } \ \forall\, v \in \mathbb{Z}^d,\ y_v = x_{u+v}. \end{array}$$

We will use the same notation for the shift acting on a finite pattern, that is, if $S \subset \mathbb{Z}^d$ is a finite set and $p \in \mathcal{A}^S$ is a pattern, then we can write for all $u \in \mathbb{Z}^d$ the shift acting on the pattern p as

$$\sigma^u(p) = w \in \mathcal{A}^{S-u} \text{ where } w_v = u_{v+u},\ \forall v \in S - u$$

Remark 1. Sometimes we will use the term shift invariant patterns for a class of patterns

$p \sim q$ if and only if $q = \sigma^u(p)$, for some $u \in \mathbb{Z}^d$. In that sense, the **shape** of the support of the pattern is fixed, but the form can be located in any translate of this support.

Let $S, T \subset \mathbb{Z}^d$ are two subsets, and p, q be two patterns with support S and T, respectively. We say that p is *a sub-pattern* of q, if $S \subseteq T$ and $p = q|_S$. Similarly we say that p is a sub-pattern of a configuration $x \in \mathcal{A}^{\mathbb{Z}^d}$, if $p = x|_S$. We can also say that a pattern $p \in \mathcal{A}^S$ *appears* in another pattern $q \in \mathcal{A}^T$ (respectively, in a configuration $x \in \mathcal{A}^{\mathbb{Z}^d}$) if there exists a vector $u \in \mathbb{Z}^d$ such that $\sigma^u(p)$ is a sub-pattern of q (respectively, $\sigma^u(p)$ is a sub-pattern of x). In that case we write $p \sqsubset q$ (respectively, $p \sqsubset x$).

Definition 3. If $p \in \mathcal{A}^S$ is a pattern with support S, the *cylinder generated* by p, denoted by $[p]$, is the subset of configurations defined by

$$[p] := \{x \in \Sigma^d(\mathcal{A}) : x|_S = p\}.$$

For $a \in \mathcal{A}$ and $i \in \mathbb{Z}^d$ we denote the cylinder

$$[a]_i = \{x \in \Sigma^d(\mathcal{A}) : x_i = a\}.$$

Definition 4. Let $P \subseteq \mathcal{A}^S$ be a subset of patterns of support S. The cylinder generated by P is the subset,

$$[P] := \bigcup_{p \in P} [p].$$

The following is the topological definition of one of the most important objects that we work with.

Definition 5. A subshift X is a closed subset of $\Sigma^d(\mathcal{A})$ which is invariant under $\sigma^u : \Sigma^d(\mathcal{A}) \to \Sigma^d(\mathcal{A})$ for all $u \in \mathbb{Z}^d$, that is, $\sigma^u(X) = X$.

As said before, there is a combinatorial definition of a subshift, which is given by the set of forbidden patterns as presented below.

Definition 6. Let X be a subset of $\Sigma^d(\mathcal{A})$. We say that X is a *subshift generated by a set $\mathcal{F}$ of forbidden patterns* if $\mathcal{F} \subseteq \bigsqcup_{R \geqslant 1} \mathcal{A}^{[\![1,R]\!]^d}$ is a subset of patterns with finite support and

$$X = \Sigma^d(\mathcal{A}, \mathcal{F}) := \{x \in \Sigma^d(\mathcal{A}) : \forall\, p \in \mathcal{F},\ p \not\sqsubset x\}.$$

The following proposition assures that every subshift is generated by a set of forbidden patterns.

Proposition 1. The two definitions of subshift (Definition 5 and Definition 6) coincide.

The entire configuration space $\Sigma^d(\mathcal{A}) = \mathcal{A}^{\mathbb{Z}^d}$ is a subshift, and we call it the *full shift*. We will denote by $(\Sigma^d(\mathcal{A}), \mathcal{B})$ the measurable space where $\mathcal{B}$ is the Borel σ-algebra generated by the cylinder sets in $\Sigma^d(\mathcal{A})$. We will describe a classification for the subshifts based on the set of forbidden patterns. For the full shift the set of forbidden patterns is empty. If the set of forbidden patterns is finite we will say that subshift is a *subshift of finite type* or SFT. When the set of forbidden patterns can be enumerated by a Turing machine, then we say that the subshift is an *effectively closed subshift* (we explain what we are considering as a set enumerated by a Turing machine in Section 2.4).

Another way of describing a subshift is by its language, that we define next.

Definition 7. Let $\mathcal{A}$ be a finite alphabet, and $d \geqslant 1$. Let X be a subshift of $\mathcal{A}^{\mathbb{Z}^d}$. The *language of* X, denoted $\mathcal{L}(X)$, is the set of square patterns that appear in X, or more formally,

$$\mathcal{L}(X) := \bigsqcup_{\ell \geqslant 1} \left\{ p \in \mathcal{A}^{[\![1,\ell]\!]^d} : \exists x \in X, \text{ s.t. } p \sqsubset x \right\}. \tag{2.1}$$

We will denote the set of square patterns of a fixed length ℓ as

$$\mathcal{L}(X, \ell) := \left\{ p \in \mathcal{A}^{[\![1,\ell]\!]^d} : \exists x \in X, \text{ s.t. } p = x|_{[\![1,\ell]\!]^2} \right\}. \tag{2.2}$$

A *dictionary* L *of size* ℓ *and dimension* d over the alphabet $\mathcal{A}$ is a subset of $\mathcal{A}^{[\![1,\ell]\!]^d}$. A dictionary is a specialized subset of patterns. We say that a dictionary L of size ℓ is a *sub-dictionary* of L' of size ℓ' (where both have the same dimension d), if every pattern of L is a sub-pattern of a pattern of L'. Given a dictionary we can define the set of all configurations obtained by the infinite concatenation of patterns of this dictionary. In fact, this subset is a subshift as described below.

Definition 8. The *concatenated subshift* of a dictionary L of size ℓ and dimension d is the subshift of the form

$$\begin{aligned} \langle L \rangle &= \bigcup_{u \in [\![1,\ell]\!]^d} \bigcap_{v \in \mathbb{Z}^d} \sigma^{-(u+v\ell)}[L], \\ &= \left\{ x \in \Sigma^d(\mathcal{A}) : \exists u \in [\![1,\ell]\!]^d, \ \forall\, v \in \mathbb{Z}^d, \ (\sigma^{u+\ell v}(x))|_{[\![1,\ell]\!]^d} \in L \right\}. \end{aligned}$$

Another important concept concerns the admissibility of a pattern. Given a set of forbidden patterns, we define local and global admissibility.

Definition 9. Let $\mathcal{F} \subseteq \mathcal{A}^{[\![1,D]\!]^d}$ for a fixed $D \geqslant 2$. We say that a pattern $w \in \mathcal{A}^{[\![1,R]\!]^d}$ where $R \geqslant D$ is *locally* $\mathcal{F}$*-admissible* if

$$\sigma^u(x)|_{[\![1,D]\!]^d} \notin \mathcal{F}, \ \forall\, u \in [\![0, R-D]\!]^d,$$

that is, we do not find a pattern of $\mathcal{F}$ inside the pattern w. We say that a pattern

$w \in \mathcal{A}^{[\![1,R]\!]^d}$ is *globally $\mathcal{F}$-admissible* if there exists $x \in \Sigma^d(\mathcal{A}, \mathcal{F})$ such that

$$x|_{[\![1,R]\!]^d} = w.$$

It is clear that if a pattern is globally admissible, then it is locally admissible, but the reverse it not always true. The next proposition assures that for every d-dimensional subshift, every really large pattern that is locally admissible has a central block that is globally admissible.

Proposition 2. Let $X = \Sigma^d(\mathcal{A}, \mathcal{F})$ be a subshift given by a set of forbidden patterns $\mathcal{F}$. There exists a function $R\colon \mathbb{N} \to \mathbb{N}$ so that if $q \in \mathcal{A}^{[\![-R(n),R(n)]\!]^d}$ is locally admissible, then $p = q|_{[\![-n,n]\!]^d}$, the restriction of q to $\mathcal{A}^{[\![-n,n]\!]^d}$, is globally admissible.

Proof. The proof follows from a standard compactness argument as described in Lemma 4.3 of [5] in a more general setting.

Suppose such a function does not exist, then there exists $n \in \mathbb{N}$ such that for every $m \geqslant n$ there exists a locally admissible pattern q_m of size m such that $p_m = q_m|_{[\![-n,n]\!]^d}$ is not globally admissible. Let $x_m \in \Sigma^d(\mathcal{A})$ be a configuration such that $x_m|_{[\![-m,m]\!]^d} = q_m$. By compactness of $\Sigma^d(\mathcal{A})$, we may extract a converging subsequence $x_{m(k)}$ which converges to some $\bar{x} \in A^{\mathbb{Z}^d}$.

We claim $\bar{x} \in X$. Indeed, if not, there is a forbidden pattern which occurs somewhere in $\bar{x}$. In particular, there is $k \in \mathbb{N}$ such that the pattern is completely contained in $[\![-m(k), m(k)]\!]^d$. It follows by convergence of the sequence $\{x_{m(k)}\}_{k\in\mathbb{N}}$ that eventually every pattern $q_{m(k)}$ contains the forbidden pattern. This is a contradiction because q_m is locally admissible. Hence $\bar{x} \in X$.

As $\bar{x} \in X$, then $\bar{x}|_{[\![-n,n]\!]^d}$ is globally admissible, but this is equal to p_m for some $m \in \mathbb{N}$ and thus not globally admissible. This is again a contradiction. Therefore the function R must exist. It is non-decreasing as subpatterns of globally admissible patterns are themselves globally admissible. ☐

2.2 Entropy and variational principle

We establish here some of the most important results about entropy of subshifts. The results here were developed by several authors in different approaches and they were able to generalize these results even for amenable group actions and non-compact configuration spaces. Here we focus on the $\mathbb{Z}^d$-action over a compact configuration space $\Sigma^d(\mathcal{A}) = \mathcal{A}^{\mathbb{Z}^d}$.

We always consider $\Sigma^d(\mathcal{A}) = \mathcal{A}^{\mathbb{Z}^d}$ and $\sigma = (\sigma^u)_{u\in\mathbb{Z}^d}$ the shift action. We will denote by $\mathcal{M}_1(\Sigma^d(\mathcal{A}))$ the set of all probability measures defined on $\Sigma^d(\mathcal{A})$ and by $\mathcal{M}_\sigma(\Sigma^d(\mathcal{A}))$ the set of shift-invariant probability measures. Here we always consider $(\Sigma^d(\mathcal{A}), \mathcal{B}, \mu)$ as a *probability space* where $\mathcal{B}$ is the sigma algebra generated by the cylinder sets and $\mu \in \mathcal{M}_\sigma(\Sigma^d(\mathcal{A}))$.

Definition 10. A collection $\mathcal{P} = \{P_1, P_2, ..., P_n\}$ of measurable sets is a *finite partition* of $\Sigma^d(\mathcal{A})$ if

- $P_i \cap P_j = \varnothing$ for $i \neq j$; and
- $\bigcup_i P_i = \Sigma^d(\mathcal{A})$.

For a probability space $(\Sigma^d(\mathcal{A}), \mathcal{B}, \mu)$ we call a collection of measurable sets $\mathcal{P} = \{P_1, P_2, ..., P_n\}$ a *μ-partition* if

- $\mu(P_i) > 0$, $\forall i$;
- $\mu(P_i \cap P_j) = 0$, for $i \neq j$; and
- $\mu\left(\Sigma^d(\mathcal{A}) \backslash \bigcup_{i=1}^{n} P_i\right) = 0$.

One of the most important concepts in thermodynamics is the entropy of a system. Here we present the definition of Shannon entropy and some useful properties that we use in this text. The definitions and results can be found in Keller [22] and Kerr-Li [24].

Definition 11. The *information* of a μ-partition $\mathcal{P} = \{P_1, P_2, ..., P_n\}$ is the function $I_{\mathcal{P}} : \Sigma^d(\mathcal{A}) \to \mathbb{R}$ defined as

$$I_{\mathcal{P}}(x) := -\sum_{P \in \mathcal{P}} \log(\mu(P)) \cdot \mathbb{1}_P(x).$$

The *entropy of a partition* with respect a measure μ is given by

$$H(\mathcal{P}, \mu) := \int I_{\mathcal{P}}(x) d\mu = -\sum_{i=1}^{n} \mu(P_i) \log(\mu(P_i))$$

We will use the notation $H(\mathcal{P}) = H(\mathcal{P}, \mu)$ when there is no confusion over which measure we are considering in order to not overload the notation.

Given two μ-partitions $\mathcal{P} = \{P_1, P_2, ..., P_n\}$ and $\mathcal{Q} = \{Q_1, ..., Q_m\}$ of a configuration space $\Sigma^d(\mathcal{A})$, we can define the *conditional information of $\mathcal{P}$ given $\mathcal{Q}$* as the function $I_{\mathcal{P}|\mathcal{Q}} : \Sigma^d(\mathcal{A}) \to \mathbb{R}$ defined as

$$I_{\mathcal{P}|\mathcal{Q}}(x) := -\sum_{i=1}^{n} \sum_{j=1}^{m} \log\left(\frac{\mu(P_i \cap Q_j)}{\mu(Q_j)}\right) \cdot \mathbb{1}_{P_i \cap Q_j}(x).$$

In the same fashion we can define the *conditional entropy of $\mathcal{P}$ given $\mathcal{Q}$* with respect to a measure μ as the value

$$H(\mathcal{P}|\mathcal{Q}, \mu) := \int I_{\mathcal{P}|\mathcal{Q}} d\mu = \int H(\mathcal{P}, \mu_x^{\mathcal{Q}}) d\mu(x) \tag{2.3}$$

where $(\mu_x^{\mathcal{Q}})_{x\in\Sigma^d(\mathcal{A})}$ is the family of conditional probabilities with respect to $\mathcal{Q}$. We can also express the conditional entropy as the sum

$$H(\mathcal{P}|\mathcal{Q},\mu) = -\sum_{i=1}^{n}\sum_{j=1}^{m}\mu(P_i \cap Q_j)\log\left(\frac{\mu(P_i \cap Q_j)}{\mu(Q_j)}\right).$$

As before we will use the notation $H(\mathcal{P}|\mathcal{Q}) = H(\mathcal{P}|\mathcal{Q},\mu)$ when there is no confusion over which measure we are considering in order to not overload the notation.

We say that a partition $\mathcal{P}'$ is a *refinement* of another partition $\mathcal{P}$ if every element of $\mathcal{P}'$ is contained in an element of $\mathcal{P}$. We denote as $\mathcal{P}' \succeq \mathcal{P}$.

We denote the *common refinement* of two partitions denoted by $\mathcal{P} \vee \mathcal{Q}$ as the partition generated by

$$\mathcal{P} \vee \mathcal{Q} := \{P_i \cap Q_j : P_i \in \mathcal{P},\ Q_j \in \mathcal{Q}\}.$$

For a subset $S \subseteq \mathbb{Z}^d$ we denote by

$$\mathcal{P}^S := \bigvee_{u\in S}\sigma^{-u}\mathcal{P}$$

the common refinement of the partitions $\sigma^{-u}\mathcal{P}$ where $u \in S$. A partition $\mathcal{P}$ is a μ-*generated partition* of $(\Sigma^d(\mathcal{A}),\mathcal{B},\mu)$ if the sigma algebra generated by $\mathcal{P}^S$ for every finite subset $S \subset \mathbb{Z}^d$ is equal to $\mathcal{B} \mod \mu$.

The next lemma gives us the Jensen inequality that will be used many times.

Lemma 1 (Jensen's Inequality)**.** Consider $I \subset \mathbb{R}$ an open interval and $\psi : I \to \mathbb{R}$ a concave function. If $f : \Sigma^d(\mathcal{A}) \to I$ a μ-integrable function, then the integral of $\psi \circ f$ is well defined and

$$\psi\left(\int f d\mu\right) \geqslant \int \psi \circ f d\mu.$$

If we consider $\psi : [0,1] \to \mathbb{R}$ defined as

$$\psi(x) = \begin{cases} -x\log(x), & 0 < x \leqslant 1 \\ 0, & x = 0, \end{cases} \tag{2.4}$$

then ψ is a strictly concave function and therefore we obtain

$$\psi\left(\sum_{i=1}^{n}\lambda_i x_i\right) \geqslant \sum_{i=1}^{n}\lambda_i\psi(x_i), \tag{2.5}$$

where $x_i \in [0,1]$ and $\lambda_i > 0$ for each $i \in [\![1,n]\!]$ with $\sum_{i=1}^{n}\lambda_i = 1$. We will use this inequality for the proof of the next lemma which presents some important properties of the entropy.

Lemma 2. Consider $\mathcal{P} = \{P_1, ..., P_n\}$ and $\mathcal{Q} = \{Q_1, Q_2, ..., Q_m\}$ two μ-partitions of $\Sigma^d(\mathcal{A})$. Then

(*i*) $0 \leqslant H(\mathcal{P}|\mathcal{Q}) \leqslant H(\mathcal{P}) \leqslant \log |\mathcal{P}|$;

(*ii*) $H(\mathcal{P} \vee \mathcal{Q}) = H(\mathcal{P}) + H(\mathcal{Q}|\mathcal{P})$;

(*iii*) $H(\mathcal{P}) \leqslant H(\mathcal{Q}) + H(\mathcal{P}|\mathcal{Q})$;

(*iv*) if $\mathcal{Q} \succeq \mathcal{P}$, then $H(\mathcal{P}|\mathcal{Q}) = 0$.

(*v*) if $\mathcal{Q} \succeq \mathcal{P}$, then $H(\mathcal{P} \vee \mathcal{Q}) = H(\mathcal{Q}) \geqslant H(\mathcal{P})$;

Proof. (*i*) The inequality $0 \leqslant H(\mathcal{P}|\mathcal{Q})$ follows from the definition of the entropy of a partition. Now we will prove that if $\mathcal{R} = \{C_1, ..., C_l\}$ is a partition such that $\mathcal{Q} \succeq \mathcal{R}$ we have that

$$H(\mathcal{P}|\mathcal{Q}) \leqslant H(\mathcal{P}|\mathcal{R}). \tag{2.6}$$

Denote

$$\lambda_{k,j} := \frac{\mu(B_j \cap C_k)}{\mu(C_k)} \quad \text{and} \quad x_{j,i} = \frac{\mu(A_i \cap B_j)}{\mu(B_j)}.$$

As we are considering $\mathcal{Q} \succeq \mathcal{R}$, $\mu(B_j \cap C_k)$ is equal to $\mu(B_j)$ or 0, because either $B_j \subseteq C_k$ or $B_j \cap C_k = \varnothing$. Thus for a fixed i and k

$$\sum_{j=1}^{m} \lambda_{k,j} x_{j,i} = \sum_{\substack{j \in [\![1,m]\!] \\ B_j \subseteq C_k}} \frac{\mu(A_i \cap B_j)}{\mu(C_k)} = \frac{\mu(A_i \cap C_k)}{\mu(C_k)}.$$

$$\begin{aligned}
H(\mathcal{P}|\mathcal{Q}) &= \sum_{i=1}^{n} \sum_{j=1}^{m} -\mu(P_i \cap Q_j) \log\left(\frac{\mu(P_i \cap Q_j)}{\mu(Q_j)}\right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \mu(Q_j) \psi(x_{j,i}) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\sum_{k=1}^{l} \mu(C_k) \lambda_{k,j}\right) \psi(x_{j,i}) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{l} \mu(C_k) \sum_{j=1}^{m} \lambda_{k,j} \psi(x_{j,i}) \\
&\leqslant \sum_{i=1}^{n} \sum_{k=1}^{l} \mu(C_k) \psi\left(\sum_{j=1}^{m} \lambda_{k,j} x_{j,i}\right) \\
&= H(\mathcal{P}|\mathcal{R}).
\end{aligned}$$

If we take $\mathcal{R} = \{\Sigma^d(\mathcal{A})\}$ the trivial partition, we obtain $H(\mathcal{P}|\mathcal{Q}) \leqslant H(\mathcal{P})$.

In (2.5) if we consider $x_i = \mu(P_i)$ and $\lambda_i = 1/n$ we obtain that

$$\begin{aligned} -\frac{1}{n}\log\left(\frac{1}{n}\right) &= \psi\left(\frac{1}{n}\right) \\ &= \psi\left(\frac{1}{n}\sum_{i=1}^{n}\mu(P_i)\right) \\ &\geqslant \frac{1}{n}\sum_{i=1}^{n}\psi(\mu(P_i)) \\ &= \frac{1}{n}H(\mathcal{P}), \end{aligned}$$

and therefore $H(\mathcal{P}) \leqslant \log(n) = \log|\mathcal{P}|$.

(ii) Each element of the partition $\mathcal{P} \vee \mathcal{Q}$ is of the form $P \cap Q$ where $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$. Then

$$\begin{aligned} I_{\mathcal{P}\vee\mathcal{Q}}(x) &= -\sum_{P\in\mathcal{P}}\sum_{Q\in\mathcal{Q}}\log(\mu(P\cap Q))\cdot \mathbb{1}_{P\cap Q}(x) \\ &= -\sum_{P\in\mathcal{P}}\sum_{Q\in\mathcal{Q}}\log\left(\frac{\mu(P\cap Q)}{\mu(P)}\cdot\mu(P)\right)\cdot \mathbb{1}_{P\cap Q}(x) \\ &= -\sum_{P\in\mathcal{P}}\sum_{Q\in\mathcal{Q}}\log\left(\frac{\mu(P\cap Q)}{\mu(P)}\right)\cdot \mathbb{1}_{P\cap Q}(x) - \sum_{P\in\mathcal{P}}\sum_{Q\in\mathcal{Q}}\log(\mu(P))\cdot \mathbb{1}_{P\cap Q}(x) \\ &= -\sum_{P\in\mathcal{P}}\sum_{Q\in\mathcal{Q}}\log\left(\frac{\mu(P\cap Q)}{\mu(P)}\right)\cdot \mathbb{1}_{P\cap Q}(x) - \sum_{P\in\mathcal{P}}\log(\mu(P))\cdot \mathbb{1}_{P}(x) \\ &= I_{\mathcal{P}|\mathcal{Q}}(x) + I_{\mathcal{P}}(x). \end{aligned}$$

By integrating with respect to a measure μ we obtain that

$$H(\mathcal{P}\vee\mathcal{Q}) = H(\mathcal{P}) + H(\mathcal{Q}|\mathcal{P}).$$

(iii) By the previous items we obtain that

$$\begin{aligned} H(\mathcal{P}) &= H(\mathcal{P}\vee\mathcal{Q}) - H(\mathcal{Q}|\mathcal{P}) \\ &\leqslant H(\mathcal{P}\vee\mathcal{Q}) \\ &= H(\mathcal{Q}) + H(\mathcal{P}|\mathcal{Q}). \end{aligned}$$

(iv) For any two partitions $\mathcal{P}$ and $\mathcal{Q}$, we have

$$\begin{aligned} H(\mathcal{P}|\mathcal{Q}) &= \sum_{P\in\mathcal{P}}\sum_{Q\in\mathcal{Q}} -\mu(P\cap Q)\log\left(\frac{\mu(p\cap Q)}{\mu(Q)}\right) \\ &= \sum_{P\in\mathcal{P}}\sum_{Q\in\mathcal{Q}} \mu(Q)\cdot\psi\left(\frac{\mu(P\cap Q)}{\mu(Q)}\right). \end{aligned}$$

If we consider that $\mathcal{Q} \succeq \mathcal{P}$ each $Q \in \mathcal{Q}$ is completely contained in an element $P \in \mathcal{P}$.

Hence each term of the sum above is equal to zero because either $\frac{\mu(P\cap Q)}{\mu(Q)} = 0$ or $\frac{\mu(P\cap Q)}{\mu(Q)} = 1$, and in both cases we have that

$$H(\mathcal{P}|\mathcal{Q}) = \sum_{P\in\mathcal{P}}\sum_{Q\in\mathcal{Q}} \mu(Q)\cdot\psi\left(\frac{\mu(P\cap Q)}{\mu(Q)}\right) = 0.$$

(v) It follows from the items (iii) and (iv).

□

Lemma 3. Consider $(\Sigma^d(\mathcal{A}), \mathcal{B}, \mu)$ a shift-invariant probability space and $\mathcal{P}$ a finite partition of $\Sigma^d(\mathcal{A})$. The *dynamical entropy relative to the partition* $\mathcal{P}$ is given by

$$h(\mathcal{P},\mu) := \inf_{n\geqslant 0}\frac{1}{|\Lambda_n|}H(\mathcal{P}^{\Lambda_n}) = \lim_{n\to+\infty}\frac{1}{|\Lambda_n|}H(\mathcal{P}^{\Lambda_n})$$

which is well defined, where $\Lambda_n := [\![-n,n]\!]^d$ for $n \geqslant 1$.

Proof. For each $n\geqslant 1$ we will consider $\Lambda_n := [\![-n,n]\!]^d \subset \mathbb{Z}^d$. For a fixed $m\geqslant 1$ we denote $\Lambda_m = [\![-m,m]\!]^d$ and $l_m = 2m+1$. Consider the set

$$V_n := \left\{p \in (l_m\mathbb{Z})^2 : (p+\Lambda_m)\cap\Lambda_n \neq \varnothing\right\}$$

Then

$$\Lambda_n \subseteq \tilde{\Lambda}_n := \bigcup_{u\in V_n}(\Lambda_m + u).$$

Note that $|\tilde{\Lambda}_n| = |V_n|\cdot|\Lambda_m| \leqslant |\Lambda_{n+m}|$. We obtain that

$$\begin{aligned} H(\mathcal{P}^{\Lambda_n}) &\leqslant H(\mathcal{P}^{\tilde{\Lambda}_n}) \\ &\leqslant \sum_{u\in V_n} H(\sigma^{-u}\mathcal{P}^{\Lambda_m}) \\ &= |V_n|H(\mathcal{P}^{\Lambda_m}) \\ &\leqslant \frac{|\Lambda_{n+m}|}{|\Lambda_m|}H(\mathcal{P}^{\Lambda_m}), \end{aligned}$$

and therefore

$$\limsup_{n\to+\infty}\frac{1}{|\Lambda_n|}H(\mathcal{P}^{\Lambda_n}) \leqslant \limsup_{n\to+\infty}\frac{|\Lambda_{n+m}|}{|\Lambda_m|}\frac{1}{|\Lambda_m|}H(\mathcal{P}^{\Lambda_m}) = \frac{1}{|\Lambda_m|}H(\mathcal{P}^{\Lambda_m}).$$

The last estimate holds for every fixed m, thus we conclude that

$$\limsup_{n\to+\infty}\frac{1}{|\Lambda_n|}H(\mathcal{P}^{\Lambda_n}) \leqslant \inf_{m>0}\frac{1}{|\Lambda_m|}H(\mathcal{P}^{\Lambda_m}) \leqslant \liminf_{m\to+\infty}\frac{1}{|\Lambda_m|}H(\mathcal{P}^{\Lambda_m}).$$

□

Theorem 2 (Shannon-McMillan-Breiman)**.** Let $(\Sigma^d(\mathcal{A}), \mathcal{B}, \mu)$ a shift-invariant probability space and $\mathcal{P}$ a finite partition of $\Sigma^d(\mathcal{A})$. Then

$$\lim_{n \to +\infty} -\frac{1}{|\Lambda_n|} \log(\mu(\mathcal{P}^{\Lambda_n})) = h(\mathcal{P}, \mu)$$

pointwise a.e. and in L_1.

The previous theorem has already been proved for a larger class of group actions only with the assumptions that the group is amenable [29, 24, 35]. The proof for Theorem 2 as stated here can be found in Krengel [26].

Now we define the Kolmogorov-Sinai entropy also called dynamical entropy of a measure.

Definition 12. The *entropy* of the space $(\Sigma^d(\mathcal{A}), \mathcal{B}, \mu)$, also known as the *dynamical entropy of* μ is given by

$$h(\mu) = \sup_{\mathcal{P}} \{h(\mathcal{P}, \mu) : \mathcal{P} \text{ is a finite partition}\}.$$

Definition 13. The *topological entropy* of a subshift $X \subseteq \Sigma^d(\mathcal{A})$ is given by

$$h_{top}(\Sigma^d(\mathcal{A})) = \lim_{n \to +\infty} \frac{1}{|\Lambda_n|} \log(|\mathcal{L}(X, 2n+1)|).$$

In Chazottes-Meyerovitch [20] they establish important results about the characterization of the entropy for multidimensional SFT. Next we present the variational principle for the entropy.

Theorem 3 (Variational Principle)**.** Let $X \subseteq \Sigma^d(\mathcal{A})$ be a subshift, then

$$h_{top}(X) = \sup_{\mu} h(\mu)$$

where the supremum is taken over the set of shift-invariant probability measures $\mathcal{M}_\sigma(\Sigma^d(\mathcal{A}))$.

The Variational Principle as stated above has already been proved for amenable group actions in [24]. One important result for the characterization of the dynamical entropy of a measure is given by the following theorem.

Theorem 4 (Kolmogorov-Sinai)**.** If $\mathcal{P}$ is μ-generated partition for $(\Sigma^d(\mathcal{A}), \mathcal{B}, \mu)$ and $H(\mathcal{P}) < +\infty$, then

$$h(\mu) = h(\mathcal{P}, \mu).$$

Proof. For any finite subset we have that

$$h(\mathcal{P}^{\Lambda}, \mu) = h(\mathcal{P}, \mu). \tag{2.7}$$

Indeed, consider a fixed $N > 0$ such that $\Lambda \subset \Lambda_N$, then we have that

$$\begin{aligned} h(\mathcal{P}^\Lambda, \mu) &= \lim_{n\to+\infty} \frac{1}{|\Lambda_n|} H\big((\mathcal{P}^\Lambda)^{\Lambda_n}\big) \\ &\leqslant \lim_{n\to+\infty} \frac{1}{|\Lambda_n|} H\big(\mathcal{P}^{\Lambda_{n+N}}\big) \\ &\leqslant \lim_{n\to+\infty} \frac{|\Lambda_{n+N}|}{|\Lambda_n|} \frac{1}{|\Lambda_{n+N}|} H\big(\mathcal{P}^{\Lambda_{n+N}}\big) \\ &= h(\mathcal{P}, \mu) \\ &\leqslant h(\mathcal{P}^\Lambda, \mu) \end{aligned}$$

since $\mathcal{P}^\Lambda \geq \mathcal{P}$.

Now consider $\mathcal{P}$ a finite μ-generated partition with finite entropy and $\mathcal{Q}$ a finite partition. From 2.7 and Lemma 2 we obtain that

$$\begin{aligned} h(\mathcal{Q}, \mu) &\leqslant h(\mathcal{P}^{\Lambda_n}, \mu) + H(\mathcal{Q}|\mathcal{P}^{\Lambda_n}) \\ &= h(\mathcal{P}, \mu) + H(\mathcal{Q}|\mathcal{P}^{\Lambda_n}). \end{aligned}$$

As $\lim_{n\to+\infty} H(\mathcal{Q}|\mathcal{P}^{\Lambda_n}) = H(\mathcal{Q}|\mathcal{B}) = 0$, it follows that for an arbitrary partition $\mathcal{Q}$, is true that $h(\mathcal{Q}, \mu) \leqslant h(\mathcal{P}, \mu)$, and therefore the result follows. □

2.3 Potential

A function $f : \Sigma^d(\mathcal{A}) \to \mathbb{R}$ is *upper semi-continuous* if the set $\{x \in \Sigma^d(\mathcal{A}) : f(x) < c\}$ is an open set for every $c \in \mathbb{R}$.

Definition 14. A potential $\varphi : \Sigma^d(\mathcal{A}) \to \mathbb{R}$ is *regular* if

$$\sum_{n=1}^{+\infty} n^{d-1} \delta_n(\varphi) < +\infty,$$

where $\delta_n(\varphi) := \sup\{|\varphi(w) - \varphi(v)| : w, v \in \Sigma^d(\mathcal{A}),\ w|_{\Lambda_n} = v|_{\Lambda_n}\}$.

We say that a potential ψ has *finite range* if there exists $n_0 \in \mathbb{N}$ such that $\delta_n(\psi) = 0$, for all $n \geqslant n_0$. If a potential has finite range, then it is regular.

Next we define the pressure of an upper semi-continuous potential, the notion of an equilibrium measure and recall several results that characterize the equilibrium measures for a certain class of potentials.

Definition 15. The *pressure* of a upper semi-continuous potential $\varphi : \Sigma^d(\mathcal{A}) \to \mathbb{R}$ at inverse temperature β is the value

$$P(\beta\varphi) := \sup_{\mu\in\mathcal{M}_\sigma(\Sigma^d(\mathcal{A}))} \left\{ h(\mu) - \int \beta\varphi d\mu \right\}.$$

Definition 16. An *equilibrium measure* for a potential φ at inverse temperature β is a measure $\mu_{\beta\varphi} \in \mathcal{M}_\sigma(\Sigma^d(\mathcal{A}))$ such that

$$P(\beta\varphi) = h(\mu_{\beta\varphi}) - \int \beta\varphi d\mu_{\beta\varphi}.$$

An important characterization for the set of equilibrium measures for a regular local potential is that it is exactly the set of invariant Gibbs measures. In order to state this result, we present one possible definition of Gibbs measures based on [22].

Remark 2. Here we will define all these notions and results for the full shift over a finite alphabet, but these definitions and results are also valid for a more general class of subshifts, for instance Muir [31] works with a countable alphabet in multidimensional subshifts and Israel [21] extended to general compact spin spaces and quantum systems for the full shift.

Consider φ a regular potential on $\Sigma^d(\mathcal{A})$ and denote

$$\varphi_n := \sum_{g\in\Lambda_n} \varphi \circ \sigma^g$$

where $\Lambda_n = [\![-n, n]\!]^d$. We are interested in how $\psi_n(w)$ will change if we alter finitely many sites. For that, we will introduce, as in Keller [22], a class of local homeomorphisms on $\Sigma^d(\mathcal{A})$.

Definition 17. Let φ be a regular potential defined over $\Sigma^d(\mathcal{A})$. We denote by ε_n the set of all maps $\tau : \Sigma^d(\mathcal{A}) \to \Sigma^d(\mathcal{A})$ such that

$$(\tau(w))_i = \begin{cases} \tau_i(w_i), & i \in \Lambda_n \\ w_i, & i \notin \Lambda_n \end{cases}$$

where $\tau_i : \mathcal{A} \to \mathcal{A}$ are permutations in the state space. We denote by $\varepsilon := \bigcup_{n>0} \varepsilon_n$ the set of all homeomorphisms in $\Sigma^d(\mathcal{A})$ that change only finitely many coordinates.

Lemma 4. (Keller [22]) Let φ be a regular potential and $\tau \in \varepsilon$. For $n > 0$ define

$$\Psi_\tau^n : \Sigma^d(\mathcal{A}) \to \mathbb{R}, \quad \Psi_\tau^n := \varphi_n \circ \tau^{-1} - \varphi_n.$$

Then the limit

$$\Psi_\tau := \lim_{n\to+\infty} \Psi_\tau^n$$

exists uniformly on $\Sigma^d(\mathcal{A})$.

Definition 18. Let φ be a regular local potential. We say that a probability measure

$\mu \in \mathcal{M}_1(\Sigma^d(\mathcal{A}))$ is a *Gibbs measure for the potential* φ if

$$\tau_* \mu = \mu \cdot e^{\Psi_\tau}$$

for each $\tau \in \varepsilon$.

The previous definition goes back to Capocaccia [9] and does not involve conditional measures as in a more classical definition of Gibbs measure [17, 32].

As said before, there are several characterizations for a Gibbs measure (see Georgii [17] and Ruelle [32]) and several results for the equivalence between these definitions (see Kimura [25] and Keller [22]) even for potentials defined over more general subshifts.

The next theorem from Keller [22] gives a important characterization of the set of invariant Gibbs measures for a regular local potential.

Theorem 5. Let $\Sigma^d(\mathcal{A}) = \mathcal{A}^{\mathbb{Z}^d}$ be the full shift and $\varphi : \Sigma^d(\mathcal{A}) \to \mathbb{R}$ be a regular local potential. The set of equilibrium measures for φ is nonempty, compact, convex subset of $\mathcal{M}_\sigma(\Sigma^d(\mathcal{A}))$ and every equilibrium measure is also a Gibbs invariant probability measure.

Given a potential $\beta\varphi$ at inverse temperature β and φ a regular local potential, the set of equilibrium measures is exactly the set of Gibbs invariant measures for $\beta\varphi$.

2.4 Turing Machines and the Simulation Theorem

We present here the basic concepts of a Turing machine and how we can characterize a language based on its computability. The automaton that we call Turing machine was first introduced by Alan Turing in 1936 and is similar to a finite automaton but with unlimited and unrestricted memory. This model works on an infinite tape and therefore has unlimited memory. There is a head of calculation which can read and write symbols on the tape and move over the tape, both forward and backward. We will introduce a formal definition of a Turing machine as in Sipser [34].

Definition 19. A Turing machine $\mathcal{M}$ is a 7−tuple $(Q, \mathcal{A}, \mathcal{T}, \delta, q_0, q_a, q_r)$, where

- Q is a finite set of states of the head of calculation;
- $\mathcal{A}$ is the input alphabet which does not contain the blank symbol $\sharp$;
- $\mathcal{T}$ is the tape alphabet which contains the blank symbol $\sharp$ and $\mathcal{A} \subseteq \mathcal{T}$;
- $\delta : Q \times \mathcal{T} \to Q \times \mathcal{T} \times \{-1, +1\}$ is the transition function;
- q_0 is the initial state of the head of calculation;
- $q_a \in Q$ is the accept state; and

- $q_r \in Q$ is the reject state.

The machine works on an infinite tape divided into discrete boxes on which the head will act. If we think of $\mathbb{Z}$ as a bi-infinite tape filled with symbols of $\mathcal{T}$, we can express the Turing machine $\mathcal{M}$ by describing the state of the head and in which box the head is.

We always start the calculation over a word defined on the alphabet $\mathcal{A}$ that will be written on the tape of the machine. The other boxes of the infinite tape are filled with the blank symbols $\sharp$. The head will start on the leftmost symbol of the word with the initial state q_0. At each step of its calculation the head acts (read/write) only on the box where the head is located. Based on the symbol that the head reads and the state of the head, the transition function will give us which symbol the head must write in the box, the new state of the head and in which direction the head should move, -1 if it should move for the left box or $+1$ if it should move for the right box. It is possible to define the transition function with the possibility of the head staying in the same box after a calculation, but the definitions are equivalent.

One way of representing the transition function is by a *directed graph* where each node represents a state of the head of calculation and the arrows are tagged with the rules of the transition function. See the transition represented below.

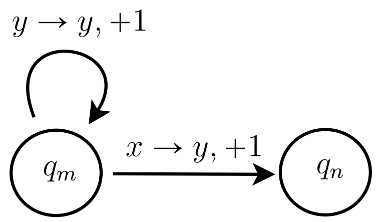

Figure 2.1: Directed graph representing two rules of some transition function δ.

If the head of calculation is in the state q_m and it reads the symbol x, then the head replaces this symbol by y, change of state to q_n and move to the box to the right. If instead the head is in the state q_m and reads the symbol y, then the head keeps the symbol y in that box, does not change the state and moves to the box on the right.

The calculation of a Turing machine stops when the head reaches the accept state q_a or the reject state q_r. If the machine never reaches one of these states the calculation will never stop. As said before, the calculation of a Turing machine starts over a finite word w defined over the alphabet $\mathcal{A}$ that is written over the tape. If the machine reaches the accept state after a number of valid transitions, we say that the initial word is accepted by this Turing machine. A set of words L, also called *language*, is recognized by a Turing machine if the machine reaches the accept state for each word in this set and never reaches the accept state if the word is not in L (the machine can reach a reject state or go into a infinite loop).

Definition 20. A set L of words over an alphabet $\mathcal{A}$ is called *recursive* if there is a Turing machine that recognizes it. A set L of words over an alphabet $\mathcal{A}$ is called *recursively enumerable* if there is a Turing machine that stops its calculation only on words of L.

As said before the machine can also reach the reject state or enter in an infinite loop that never stops. There is a special classification for the set of words for which it is possible to define a Turing machine that never enters in a infinite loop, that is, for each finite initial word the machine always reaches q_a or q_r. In this case we say that this Turing machine *decides* or, most popularly found in the literature, *recognizes* the language L.

These two concepts of recognizability and recursive enumerability, although seemingly equivalent, are two different notions. There are certain languages that only can be enumerate by a Turing machine. Now we present an example presented in [4] of a Turing machine that recognizes (and also enumerates) a language defined over the alphabet $\mathcal{A} = \{a, b\}$.

Example 1. This machine stops for every word that we write on the tape and it tells us whether such word belongs or not to the language $\mathcal{L} = \{a^n b^n ; n \in \mathbb{N}\}$. The input alphabet is $\mathcal{A} = \{a, b\}$ and the tape alphabet is $\mathcal{T} = \{a, b, \sharp\}$, where $\sharp$ is the blank symbol. We start with the word to be evaluated written on a bi-infinite tape filled with black symbols $\sharp$ and we set the head of calculation on the state q_0 on the leftmost symbol of the word. This Turing machine has 9 states $Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_a, q_r\}$ and the transition function $\delta : Q \times \mathcal{T} \to Q \times \mathcal{T} \times \{-1, +1\}$ is represented by the directed graph in Figure 2.2.

We are representing the accept state by q_a and the reject state by q_r. Note that the transition function is not defined for every possible pair in $Q \times \mathcal{T}$ because this configuration never occurs in the calculation process. Another important aspect is that when the transition function goes to q_a or q_r, we are not defining the symbol substitution or the move that the head should do, because it is irrelevant since the calculation will stop after this iteration.

Now we give a summary of the role played by each of the eight states that the machine can reach:

q_0: This state marks the beginning of the calculation. The head of the machine begins the calculation on the leftmost letter of the word written on the tape. If the head reads the symbol a then the head replaces the symbol by a blank symbol, moves to the right and also changes the state. If the head reads a symbol b then the head of the machine goes to the reject state and the computation stops, which means that the word written on the tape does not belongs to the language.

q_1: In this state the head of the machine goes to the rightmost symbol a of the word without changing the symbols or the state of the machine. When the machine finds the first symbol b the head of the machine does not change the letter, but changes the state and moves to the right. In this state the machine goes to the reject state

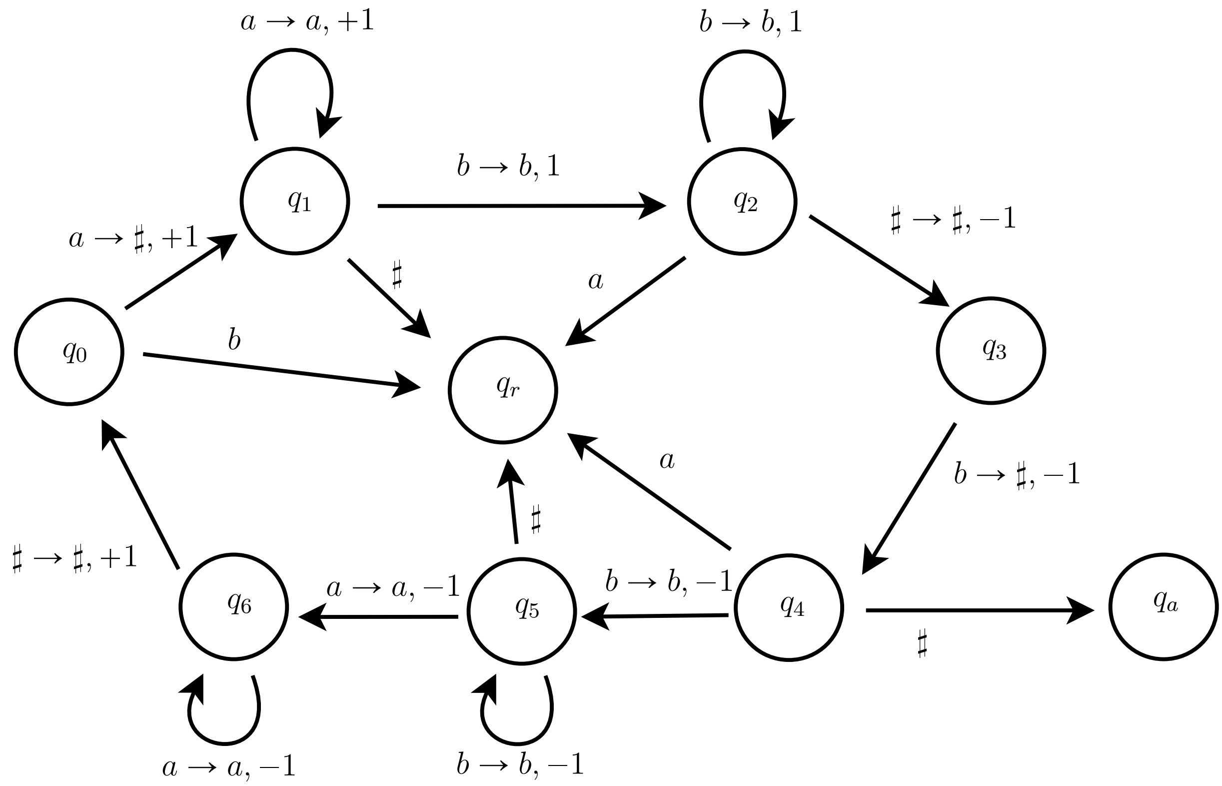

Figure 2.2: Directed graph representing the transition function for the Turing machine that decides the language $a^n b^n$.

if the head reads the blank symbol, which means that the word written on the tape has only the symbol a.

q_2**:** This state makes the head of the machine goes to the end of the word without changing the symbols b's that are written on the tape. The head goes to the last symbol b and then when it finds the first blank symbol this state makes the head go to the left, but not replace the blank symbol. If the head is in this state and finds a symbol a, it means that in the word written on the tape exists the subword ba which is forbidden in the language $\mathcal{L}$, so the head goes to the reject state and the calculation stops.

q_3**:** This state always appears on the head when it is on the last symbol b of the finite word written on the tape of calculation. The symbol b is replaced by a blank symbol and the head of calculation moves to the box on the left. The symbol b is the only possibility for the head to read.

q_4**:** In this state if the head of the machine reads the symbol b it means that there exists still symbols written on the tape of calculation that are different from the blank symbol, then the head of the machine does not replace the symbol b, but moves to the left and changes the state. If the head of the machine in this state reads the blank symbol it means that now, on the tape of calculation, there are only blank symbols, which means that the machine has replaced all of the symbols a's and b's in the initial finite word written and the number of a's and b's are the same. In this case the machine changes to the accept state which means that the initial word written on the tape belongs to the language $\mathcal{L}$. The other possibility is that the head of the machine in this state reads the symbol a which means that the number of symbol a's is bigger than the number of symbol b's and then the machine changes to the reject state.

q_5: This state makes the head of the machine reach the symbol a most to the right on the word written on the tape. The head on this state when placed on the symbol b, does not replace the symbol b and only moves to the left without changing the state. When the head reaches one symbol a the machine still moves to the left without replacing the letter, but it changes the state. If the head in this state reaches a blank symbol this means that on the tape of calculation there are only letters b's which means that the number of symbol b's on the initial word is bigger than the number of letters a's. In this case the machine changes to the reject state which means that the machine recognizes that the initial word written on the tape does not belong to the language $\mathcal{L}$.

q_6: This state makes that the head of the calculation go to the leftmost symbol not blank on the tape. If the head in this state reads the letter a, the head does not change the state but moves to the left. When the head reaches a blank symbol this means that the head reaches the beginning of the word that is now written on the tape. In this case the head does not replace the blank symbol, changes the state and moves to the right leaving the head on the leftmost symbol on the word that is written on the tape. In this state it is not possible that the head reads the letter b because of the construction and the way that the previous calculations occur.

q_a: This is the accept state, which means that if the head of the machine reaches this state then the initial word written on the tape belongs to the language $\mathcal{L}$.

q_r: This is the reject state, which means that if the head of the machine reaches this state then the initial word written on the tape does not belong to the language $\mathcal{L}$

The name 'recursively enumerable' comes from a variation of the Turing machine presented that is called *enumerator*. We can think of it as a general Turing machine attached to a printer that prints some output words that the machine has written on its tape. An enumerator starts with a infinite tape filled with blank symbols. Each word that this machine prints belongs to a language, that is why we say that this machine enumerates.

Proposition 3. Given a set of words L defined over an alphabet $\mathcal{A}$. The set L is recursively enumerable if and only if there is a Turing machine that enumerates it.

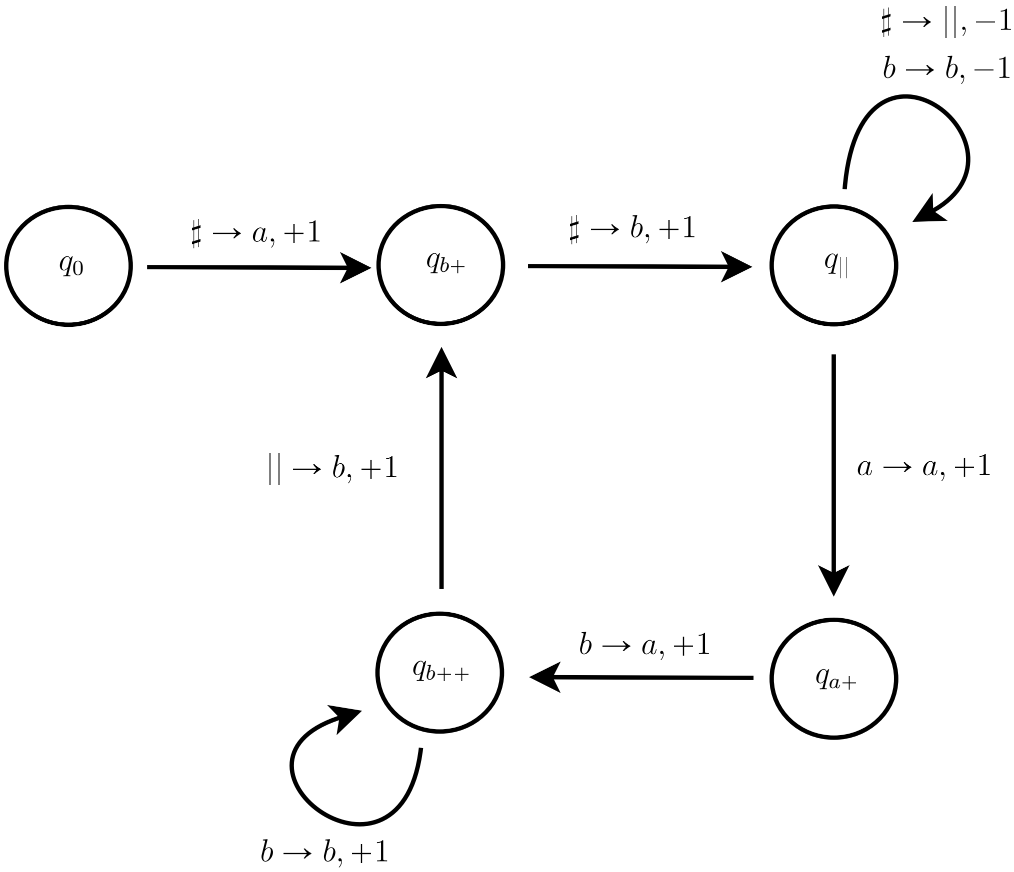

Figure 2.3: Directed graph of the transition function δ of the enumerator for the language $a^n b^n$.

The next example from [1] shows a Turing machine that enumerates the language described in the previous example.

Example 2. We describe an example of a Turing machine that enumerates the language $L = \{a^n, b^n, n \in \mathbb{N}\}$. The input alphabet is $\mathcal{A} = \{a, b\}$ and the tape alphabet is $\mathcal{T} = \{a, b, \sharp, ||\}$. This machine has five possible states $Q = \{q_0, q_{a+}, q_{b+}, q_{b++}, q_{||}\}$ and it never stops its calculation. The symbol $||$ helps the machine to know when it must print the word written on the tape. The transition function will be $\delta : Q \times \mathcal{T} \to Q \times \mathcal{T} \times \{-1, +1\}$ given by Figure 2.3.

The following is a summary of the role played by each of the five states that the machine can reach:

q_0**:** This state begins the work of the machine. In our case it always occurs in the bi-infinite tape filled with the blank symbol. It marks the start of the calculation of the machine by replacing the blank symbol by a and moving the head to the right.

q_{b+}**:** In this state the machine replaces the blank symbol by a letter b. This occurs after the head of the machine arrives at the end of the word that is written on the tape of calculation. This symbol b will be the rightmost b required to achieve the same number of letters b's and letters a's in the word written on the tape.

$q_{||}$: When the machine has this state and reads the blank symbol, that is $(q_{||}, \sharp)$, the machine prints the word written on the tape because it will be of the form $a^n b^n$. Besides that, this states is also responsible to return the head of calculation to the rightmost symbol a on the tape. The head changes the blank symbol by a marker $||$ and moves to the left. The head goes to the left without making any changes until it achieves the rightmost symbol a on the tape. The machine does not replace the symbol a, but it changes the state and moves to the right, leaving the head over the leftmost symbol b written on the tape.

q_{a+}: This state is responsible for adding a new symbol a into the word written on the tape. It is the beginning of several changes to achieve the next word in the language $a^n b^n$. The head in this state always reads the symbol b. It changes to an a, it changes the state and it moves to the right.

q_{b++}: In this state the head of the machine goes to the end of the word written on the tape without making any changes, that is, the head goes to the marker $||$ after all the symbols b's that compose the word on the tape. The head replaces it by a symbol b, it moves to the right and it changes the state.

The action of this Turing machine can also be described by a space-time diagram. The horizontal direction stands for the tape on which the machine works and the vertical direction for the time evolution of the machine.

The calculation of a Turing machine, that is, the set of rules defined by the transition function can be represented by a set of bidimensional patterns as proposed in [6]. For example, consider the Turing machine presented in the last example and the transition function when the head of the machine is in the state $q_{||}$ and reads the symbol a. In this case the head of the machine does not change the symbol a written on the tape, it changes its state to q_{a+} and it moves to the right. This action can be represented by the following set of 3×2 blocks or tiles described as below

s_1	a	(q_{a+}, s_3)
s_1	$(q_{\|\|}, a)$	s_3

a	(q_{a+}, s_2)	s_3
$(q_{\|\|}, a)$	s_2	s_3

(q_{a+}, s_1)	s_2	s_3
s_1	s_2	s_3

s_1	s_2	a
s_1	s_2	$(q_{\|\|}, a)$

where $s_1, s_2, s_3 \in \mathcal{T}$ are the symbols that have previously been written on the tape. These four patterns describe all the possible 3×2 patterns that can be found in a bidimensional representation of this Turing machine for the rule $\delta(q_{||}, a) = (q_{a+}, a, +1)$. We can do this representation for each rule of the transition function. Since there is a finite number of rules, the set that describes all the possible 3×2 patterns is also finite. Note that we

$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\sharp$	a	a	a	a	b	(q_{b++}, b)	$\Vert$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	a	a	(q_{b++}, b)	b	$\Vert$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	a	(q_{a+}, b)	b	b	$\Vert$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	$(q_{\Vert}, a)$	b	b	b	$\Vert$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	a	$(q_{\Vert}, b)$	b	b	$\Vert$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	a	b	$(q_{\Vert}, b)$	b	$\Vert$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	a	b	b	$(q_{\Vert}, b)$	$\Vert$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	a	b	b	b	$(q_{\Vert}, \sharp)$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	a	b	b	$(q_{b+}, \sharp)$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	a	b	$(q_{b++}, \Vert)$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	a	(q_{b++}, b)	$\Vert$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	(q_{a+}, b)	b	$\Vert$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	$(q_{\Vert}, a)$	b	b	$\Vert$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	$(q_{\Vert}, b)$	b	$\Vert$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	b	$(q_{\Vert}, b)$	$\Vert$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	b	b	$(q_{\Vert}, \sharp)$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	b	$(q_{b+}, \sharp)$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	a	$(q_{b++}, \Vert)$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	(q_{a+}, b)	$\Vert$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$(q_{\Vert}, a)$	b	$\Vert$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	$(q_{\Vert}, b)$	$\Vert$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	b	$(q_{\Vert}, \sharp)$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	a	$(q_{b+}, \sharp)$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$
$\cdots$	$\sharp$	$(q_0, \sharp)$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\sharp$	$\cdots$

have to include the pattern

s_1	s_2	s_3
s_1	s_2	s_3

where the head of the Turing machine does not appear in this window that we are considering.

The set of all possible patterns 3×2 in the alphabet

$$\mathcal{T} \cup (Q \times \mathcal{T}) \cup (Q \times \mathcal{T} \times \{-1, +1\})$$

is finite. Since we are able to describe the language with patterns of the form 3×2, we can take the complementary set from all the possible 3×2 patterns and denote it as the set of forbidden patterns. Therefore, it is always possible to describe the calculation of a Turing machine by a SFT.

Based on the computability of a set of forbidden words, we can define another important class of subshifts.

Definition 21. We say that a subshift $X \subset \mathcal{A}^{\mathbb{Z}}$ is an *effectively closed subshift* if there exists a recursively enumerable set of words $\mathcal{F}$ such that $X = \Sigma^d(\mathcal{A}, \mathcal{F})$, that is, the set of forbidden words for the subshift X can be recognized by a Turing machine.

Here we define this class of subshifts only for one-dimensional subshifts, but it is possible to define the same class for multidimensional subshifts. In our main construction we describe a one-dimensional effectively closed subshift by an iteration process that builds the language of the subshift.

2.5 The Aubrun-Sablik simulation theorem

The simulation theorem in Aubrun-Sablik [2] allows us to represent a one-dimensional effectively closed subshift as a subaction of a bidimensional SFT. We introduce some operations in subshifts as defined in [1] so that we can give an idea of the construction proposed by Aubrun-Sablik [2].

Let $\mathcal{A}$ and $\mathcal{B}$ be two finite alphabets and $X_1 \subseteq \Sigma^d(\mathcal{A})$ and $X_2 \subseteq \Sigma^d(\mathcal{B})$ be two subshifts of the same dimension d. If we consider $x_1 \in X_1$ and $x_2 \in X_2$ two configurations in each subshift we define

$$x_1 \times x_2 = y \in \Sigma^d(\mathcal{A} \times \mathcal{B})$$

such that

$$y = (y_j)_{j \in \mathbb{Z}^d} \text{ where } y_j = ((x_1)_j, (x_2)_j) \in \mathcal{A} \times \mathcal{B}.$$

Definition 22. Let be $X_1 \subseteq \Sigma^d(\mathcal{A})$ and $X_2 \subseteq \Sigma^d(\mathcal{B})$. We define the *product of* X_1 *and* X_2 as the subshift $(X_1 \times X_2) \subseteq \Sigma^d(\mathcal{A} \times \mathcal{B})$

$$X_1 \times X_2 = \{x_1 \times x_2 : x_i \in X_i, i = 1, 2\}.$$

Note that the new alphabet is a product alphabet $\mathcal{A} \times \mathcal{B}$ of the two previous alphabets but the dimension of the subshift remains the same.

Definition 23. A *morphism* $\pi : \Sigma^d(\mathcal{A}) \to \Sigma^d(\mathcal{B})$ is a continuous function which commutes with the shift action, that is,

$$\sigma^u \circ \pi = \pi \circ \sigma^u, \quad \forall u \in \mathbb{Z}^d.$$

Hedlund [18] proved that such morphisms are block factors, that is, there exists a finite $U \subset \mathbb{Z}^d$ that we call *neighborhood* and there exists a function $\overline{\pi}$ such that

$$\begin{array}{rcl} \overline{\pi} : \mathcal{A}^U & \to & \mathcal{B} \\ (w_i)_{i \in \mathbb{Z}^d} & \mapsto & \overline{\pi}(w)_i = \overline{\pi}(\sigma^i(x)|_U), \quad \forall i \in \mathbb{Z}^d. \end{array}$$

Definition 24. Let $\pi : \Sigma^d(\mathcal{A}) \to \Sigma^d(\mathcal{B})$ be a morphism and $X \subseteq \Sigma^d(\mathcal{A})$ be a subshift. We define the *topological factor of the subshift* X *by* π as the subshift $X_\pi \subseteq \Sigma^d(\mathcal{B})$ such that

$$X_\pi = \left\{y \in \Sigma^d(\mathcal{B}) : \exists x \in X \text{ such that } \pi(x) = y\right\}.$$

Example 3. Consider two alphabets $\mathcal{A} = \{0,1,2\}$ and $\mathcal{B} = \{0,2\}$ and define $X = \Sigma^1(\mathcal{A},\mathcal{F})$ where $\mathcal{F} = \{00, 11, 02, 21\}$. Let $\overline{\pi} : \mathcal{A} \to \mathcal{B}$ be a one-to-one block defined as

$$\overline{\pi}(0) = \overline{\pi}(1) = 0 \quad \text{and} \quad \overline{\pi}(2) = 2.$$

We can define a morphism π as

$$\begin{array}{rcl} \pi : \Sigma^1(\mathcal{A}) & \to & \Sigma^1(\mathcal{B}) \\ (x_i)_{i\in\mathbb{Z}} & \mapsto & (y_i)_{i\in\mathbb{Z}} = (\overline{\pi}(x_i))_{i\in\mathbb{Z}}. \end{array}$$

Thus the topological factor of the subshift X by π is

$$X_\pi = \left\{x \in \Sigma^1(\mathcal{B}) : \text{ finite blocks of consecutive 0's are of even length } \right\}$$

which is called the even shift. This subshift is not a subshift of finite type because we cannot represent the set of forbidden patterns by a finite number of patterns, since one needs to exclude all arbitrarily large blocks of consecutive 0's of odd lengths to describe it.

Remark 3. A *sofic subshift* is a factor of a subshift of finite type. The class of sofic subshifts is bigger than the class of subshifts of finite type and there exists several representations for a sofic subshift, see [28].

The following definitions of a projective subaction and extension can be generalized for any subgroup as in [1, 19], but for the purpose of our construction the projective $\mathbb{Z}$-subaction and extension by duplication are enough.

Definition 25. Let $X \subseteq \Sigma^2(\mathcal{A})$ be a bidimensional subshift defined over the alphabet $\mathcal{A}$. We define the *projective $\mathbb{Z}$-subaction* as the one-dimensional subshift Y given by

$$Y = \{y \in \Sigma^1(\mathcal{A}) : \exists x \in X, s.t.\, x|_{\mathbb{Z}\times\{0\}} = y\},$$

that is, we are only considering the $e_1 = (1,0)$-action on the subshift X.

Definition 26. Let $X \subseteq \Sigma^1(\mathcal{A})$ be a subshift. We define the *extension by duplication* of the subshift X to be the bidimensional subshift $\overline{X} \subseteq \Sigma^2(\mathcal{A})$ given as

$$\overline{X} := \left\{\overline{x} \in \Sigma^2(\mathcal{A}) : x|_{\mathbb{Z}\times\{0\}} \in X \,\text{and}\, \overline{x}|_{(i,j)} = \overline{x}_{(i,j+1)}, \forall (i,j) \in \mathbb{Z}^2\right\}.$$

Theorem 6 (Aubrun and Sablik [2], Durand Romaschenko and Shen [14])**.** For every effectively closed $\mathbb{Z}$-subshift $Z \subseteq \Sigma^1(\mathcal{A})$ there exists an alphabet $\mathcal{B}$, a $\mathbb{Z}^2$-subshift of finite type $X \subseteq \Sigma^2(\mathcal{B})$ and a morphism $\pi : \Sigma^2(\mathcal{B}) \to \Sigma^2(\mathcal{A})$ so that

1. The topological entropy of X is zero.

2. The action of $e_2 = (0,1)$ on $X_\pi \subseteq \Sigma^2(\mathcal{A})$ is trivial, that is, the restriction of the action of the subgroup $\{0\} \times \mathbb{Z}$ is the identity on X_π.

3. The projective $\mathbb{Z}$-subaction of X_π is equal to Z, that is, the one-dimensional effectively closed subshift Z can be seen as a $\mathbb{Z}$-subaction of the topological projection of a bidimensional SFT X.

The proof of this theorem is constructive and it uses several different elements to construct the final subshift. Among the techniques that they use are the representation of Turing machines via a space-time diagram as in the Example 2 as proposed by Berger [6] and the substitution theorem by Mozes [30]. The final subshift is built as four different layers with four different alphabets that are combined in order to form a really large alphabet in which it is possible to describe a finite set of forbidden patterns that defines a subshift that simulates our first subshift.

As said before, the subshift of finite type X in the Aubrun-Sablik construction [2] is composed of four layers, that is, it is a subshift of a product of four subshifts of finite type given by a finite number of forbidden patterns which impose conditions on how the layers superpose. See Figure 14 of [2]. The layers are:

1. `Layer` 1: The set of all configurations $x \in \mathcal{A}^{\mathbb{Z}^2}$ obtained by the extension by duplication as in Definition 26.

2. `Layer` 2: $\mathbf{T}_{\texttt{Grid}}$ A subshift of finite type extension of a sofic subshift which is generated by the substitution given in Figure 3 of [2]. The sofic subshift induces infinite vertical "strips" of computation which are of width 2^n for every $n \in \mathbb{N}$ and occur with bounded gaps (horizontally) in any configuration.

3. `Layer` 3: $\mathcal{M}_{\texttt{Forbid}}$ A subshift of finite type given by Wang tiles which replicates the space-time diagram of a Turing machine which enumerates all forbidden patterns of X and communicates this information to the fourth layer.

4. `Layer` 4: $\mathcal{M}_{\texttt{Search}}$ A subshift of finite type given by Wang tiles which simulates a Turing machine which serves the purpose of checking whether the patterns enumerated by the third layer appear in the first layer. "responsibility zone" which is determined by the hierarchical structure of `Layer` 2.

The rules between the four layers described in [2] force the Turing machine space-time diagrams to occur in every strip, and to restart their computation after an exponential number of steps. This ensures that every configuration restarts the computation everywhere, and that every forbidden pattern is written on the tape by the Turing machine in every large enough strip. The fourth layer searches for occurrences of the forbidden patterns in the first layer and thus discards any configuration in the first layer where one of these patterns occurs.

Based on their construction and the objects that we will define later, it will be possible to have some important estimates.

Chapter 3

Main Construction

In this chapter we present the main construction that allows us to define our locally constant potential. First we define a one-dimensional effectively closed subshift generated by an iteration process that defines the language of this subshift. We prove that this subshift is in fact effectively closed. We prove also some important properties. Next we apply the simulation theorem of Aubrun-Sablik [2] in order to get a bidimensional SFT that simulates our initial subshift. We also prove some properties for this subshift and define a new coloring of this subshift.

3.1 One-dimensional effectively closed subshift

Now we present a general lemma that we use in our construction. It gives us certain properties based on how we define the iteration process that defines our one-dimensional subshift. See Definition 8 for concatenated subshifts.

Lemma 5. Let $\mathcal{A}$ be a finite alphabet. Let $(\ell_k)_{k\geqslant 0}$ be a strictly increasing sequence of integers, and $(L_k)_{k\geqslant 0}$ be a sequence of dictionaries of size $(\ell_k)_{k\geqslant 0}$ over the alphabet $\mathcal{A}$, say $L_k \subseteq \mathcal{A}^{[\![1,\ell_k]\!]}$. We assume that, for every $k \geqslant 0$, every word in L_{k+1} is the concatenation of words of L_k. Then

1. $\forall\, k \geqslant 0$, $\langle L_{k+1}\rangle \subseteq \langle L_k\rangle$,

2. $X := \bigcap_{k\geqslant 0}\langle L_k\rangle = \Sigma^1(\mathcal{A},\mathcal{F})$ where $\mathcal{F} = \bigsqcup_{k\geqslant 0}\mathcal{F}_k$ and $\mathcal{F}_k$ is the set of words of length ℓ_k that are not subwords of the concatenation of two words of L_k.

If we assume in addition that every concatenation of two words in L_k is a subword of the concatenation of two words of L_{k+1}, then

3. for every $n \geqslant 0$, the concatenation of two words of L_n is a word of the language of X.

Proof. For this proof we use the following notation: for each $k \geqslant 0$ and $i \in \mathbb{Z}$ we denote $E_k(i) \subset \mathbb{Z}$ as the set

$$E_k(i) := [\![i, i + \ell_k - 1]\!] \subset \mathbb{Z}.$$

Consider $x \in \langle L_{k+1} \rangle$. By definition there exists $j \in [1, \ell_{k+1}]$ such that

$$x|_{E_{k+1}(j+1+i\ell_{k+1})} \in L_{k+1}, \quad \forall i \in \mathbb{Z}$$

that is, x can be seen as an infinite concatenation of words in L_{k+1}. By our assumptions every word in L_{k+1} is a concatenation of words in L_k. Then $x \in \langle L_k \rangle$ and that means $\langle L_{k+1} \rangle \subseteq \langle L_k \rangle$.

Now we prove that $X = \Sigma^1(\mathcal{A}, \mathcal{F})$ where $\mathcal{F}$ is the set of words of length ℓ_k, $k \geqslant 0$, that are not subwords of the concatenation of two words of L_k. For a fixed $k \geqslant 0$, denote $\mathcal{F}_k$ the set of words of length ℓ_k that are not subwords of the concatenation of two words of L_k. In this case the set $\mathcal{F}_k$ is finite and if $\Sigma^1(\mathcal{A}, \mathcal{F}_k)$ is the SFT generated by the set of forbidden words $\mathcal{F}_k$ it is clear that $\langle L_k \rangle \subseteq \Sigma^1(\mathcal{A}, \mathcal{F}_k)$. By our assumptions $\langle L_{k+1} \rangle \subseteq \langle L_k \rangle$ for every $k \geqslant 0$, thus

$$\bigcap_{i \geqslant k} \langle L_i \rangle \subset \Sigma^d(\mathcal{A}, \mathcal{F}_k).$$

Therefore

$$X = \bigcap_{k \geqslant 0} \langle L_k \rangle \subseteq \bigcap_{k \geqslant 0} \Sigma^1(\mathcal{A}, \mathcal{F}_k) = \Sigma^1(\mathcal{A}, \mathcal{F}).$$

For every $k \geqslant 0$, define the interval

$$I_k := \left[\!\!\left[1 - \left\lfloor \frac{\ell_k}{2} \right\rfloor, \ell_k - \left\lfloor \frac{\ell_k}{2} \right\rfloor \right]\!\!\right].$$

If we consider $x \in \Sigma^1(\mathcal{A}, \mathcal{F})$, then $x|_{I_k}$ is a subword of length ℓ_k of the concatenation of two words of L_k. For every $k \in \mathbb{N}$ we can assure that there exists a configuration $y^k \in \langle L_k \rangle$ such that $x|_{I_k} = y^k|_{I_k}$. We may take a subsequence of indices k such that $(y^k)_{k \geqslant 0}$ converges to some $y \in \mathcal{A}^{\mathbb{Z}}$. Since $y^k \in \langle L_j \rangle$ for every $k \geqslant j$, by taking the limit in k we obtain $y \in \langle L_j \rangle$, for every $j \geqslant 0$, thus $y \in X$. For every $k \geqslant j$, as $I_j \subseteq I_k$, we have $x|_{I_j} = y^k|_{I_j}$. Since $(y^k)_{k \geqslant 0}$ converges to y, $x|_{I_j} = y|_{I_j}$ for every $j \geqslant 0$, thus $x = y \in X$. Therefore $X = \Sigma^1(\mathcal{A}, \mathcal{F})$.

Consider two words $u_k, v_k \in L_k$. There exists a configuration $x^k \in \langle L_k \rangle$ such that

$$x^k|_{[\![-\ell_k, \ell_k - 1]\!]} = u_k v_k.$$

If the concatenation $u_k v_k$ can be found in a word of $u_{k+1} \in L_{k+1}$, then it is enough to assure there exists a configuration $x \in X$ that $x|_{[\![-\ell_k, \ell_k-1]\!]} = u_k v_k$ and therefore $u_k v_k \in \mathcal{L}(X)$.

If $u_k v_k$ is not a subword of a word in L_{k+1}, then by our assumptions the concatenation $u_k v_k$ can be seen as a subword of a concatenation of two words in L_{k+1}, that is, there exists $u_{k+1}, v_{k+1} \in L_{k+1}$ such that $u_k v_k \sqsubset u_{k+1}v_{k+1}$. We can assure again there exists $x^{k+1} \in \langle L_{k+1} \rangle$ such that

$$x^{k+1}|_{[\![-\ell_{k+1},\ell_{k+1}-1]\!]} = u_{k+1}v_{k+1},$$

and therefore the word $u_k v_k$ appears in the configuration x^{k+1}. Hence we assure that for every $j \geqslant k$ we can find a configuration $x^j \in \langle L_j \rangle$ and two words $u_j, v_j \in L_j$ such that $x^j|_{[\![-\ell_j,\ell_j-1]\!]} = u_j v_j$ and $u_k v_k \sqsubset u_j v_j$. We may take a subsequence of indexes j such that x^j converges to some $x \in X$. As we have $\lim\limits_{k\to+\infty} \ell_k = +\infty$ we obtain a configuration $x \in X$ such that $u_k v_k \sqsubset x \in X$ and therefore $u_k v_k \in \mathcal{L}(X)$.

□

First we describe a one-dimensional construction that satisfies all of our previous hypotheses and from there we describe our bidimensional elements. We use the notation with a marker $\sim$ for the one-dimensional elements. Consider an alphabet $\tilde{A} = \{0,1,2\}$, a sequence of integers ℓ_k, sets of blocks $\tilde{A}_k, \tilde{B}_k \subset \tilde{\mathcal{A}}^{\ell_k}$ (or $\tilde{\mathcal{A}}^{[\![1,\ell_k]\!]}$) and two auxiliary sequences of integers $(N_k)_{k\geqslant 0}$ and $(N'_k)_{k\geqslant 0}$. We impose assumptions on these sequences in order to properly build our example. We assume that $N'_k \geqslant 4$ and N_k is a multiple of N'_k for each $k \geqslant 0$.

Notation 1. For each $k \geqslant 0$ the sets $\tilde{A}_k$ and $\tilde{B}_k$ will be

$$\tilde{A}_k = \{a_k, 1^{\ell_k}\} \quad \tilde{B}_k = \{b_k, 2^{\ell_k}\},$$

where $a_k, b_k \in \tilde{\mathcal{A}}^{[\![1,\ell_k]\!]}$. We define these blocks by an iteration process described below.

Start with $\ell_0 = 2$, $a_0 = 01$ and $b_0 = 02$, then we have

$$\tilde{A}_0 = \{01, 11\} \quad \text{and} \quad \tilde{B}_0 = \{02, 22\}.$$

If $k \geqslant 1$ is odd we define

$$\begin{aligned} a_k &= \underbrace{a_{k-1}a_{k-1}\cdots a_{k-1}}_{N_k\text{-times}} \text{ and} \\ b_k &= b_{k-1}2^{(N_k-2)\ell_{k-1}}b_{k-1}; \end{aligned} \tag{3.1}$$

and if $k \geqslant 2$ is even we define

$$\begin{aligned} a_k &= a_{k-1}1^{(N_k-2)\ell_{k-1}}a_{k-1} \text{ and} \\ b_k &= \underbrace{b_{k-1}b_{k-1}\cdots b_{k-1}}_{N_k\text{-times}}. \end{aligned} \tag{3.2}$$

In our iteration process, for every $k \geqslant 0$, the sets $\tilde{A}_k$ and $\tilde{B}_k$ are formed by two blocks of length ℓ_k and we always have $1^{\ell_k} \in \tilde{A}_k$ and $2^{\ell_k} \in \tilde{B}_k$. The length of the blocks at each stage is given by

$$\ell_k = N_k \ell_{k-1}.$$

Notation 2. Now we define the sub-dictionaries $\tilde{A}'_k$ and $\tilde{B}'_k$ which are made of subwords of length $\ell'_k = N'_k \cdot \ell_{k-1}$ that are either initial or terminal words of a word in $\tilde{A}_k$ and $\tilde{B}_k$. Formally,

1. if k is odd, $\tilde{A}'_k = \{a'_k, 1^{\ell'_k}\}$, $\tilde{B}'_k = \{b'_k, b''_k, 2^{\ell'_k}\}$,

$$\begin{aligned} a'_k &:= a_{k-1}a_{k-1}\cdots a_{k-1}, \quad N'_k \text{ times},\\ b'_k &:= b_{k-1}2^{(N'_k-1)\ell_{k-1}} \text{ and}\\ b''_k &:= 2^{(N'_k-1)\ell_{k-1}}b_{k-1}; \end{aligned} \tag{3.3}$$

2. if k is even, $\tilde{A}'_k = \{a'_k, a''_k, 1^{\ell'_k}\}$, $\tilde{B}'_k = \{b'_k, 2^{\ell'_k}\}$,

$$\begin{aligned} a'_k &:= a_{k-1}1^{(N'_k-1)\ell_{k-1}},\\ a''_k &= 1^{(N'_k-1)\ell_{k-1}}a_{k-1} \text{ and}\\ b_k &:= b_{k-1}b_{k-1}\cdots b_{k-1}, \quad N'_k \text{ times}. \end{aligned} \tag{3.4}$$

Notice that, as N_k is a multiple of N'_k, we have $\langle \tilde{A}_k \rangle \subset \langle \tilde{A}'_k \rangle$ and $\langle \tilde{B}_k \rangle \subset \langle \tilde{B}'_k \rangle$.

Remark 4. For each $k \in \mathbb{N}$, we denote the block of ℓ_k consecutive 1's by $1_k := 1^{\ell_k}$ and, in a similar fashion $2_k := 2^{\ell_k}$.

The frequency of the symbol 0 in any word $\tilde{w} \in \tilde{\mathcal{A}}^{[\![1,\ell_k]\!]}$ of length ℓ_k is denoted by

$$f_k(\tilde{w}) := \frac{1}{\ell_k}\mathrm{card}\big(\{i \in [\![1, \ell_k]\!] : \tilde{w}(i) = 0\}\big). \tag{3.5}$$

We denote in the same fashion the frequency of the symbol 0 in words $\tilde{w} \in \tilde{\mathcal{A}}^{[\![1,\ell'_k]\!]}$ as

$$f'_k(\tilde{w}) := \frac{1}{\ell'_k}\mathrm{card}\big(\{i \in [\![1, \ell'_k]\!] : \tilde{w}(i) = 0\}\big).$$

Let f_k^A, f_k^B (resp. ${f'_k}^A$, ${f'_k}^B$) be the largest frequency of the symbol 0 in the words of $\tilde{A}_k$, $\tilde{B}_k$ (resp. $\tilde{A}'_k$, $\tilde{B}'_k$).

Lemma 6. Let $\tilde{A}_k$ and $\tilde{B}_k$ be the two languages defined in Notation 1, $\tilde{A}'_k$ and $\tilde{B}'_k$ those defined in Notation 2. Then

1. if $k \geqslant 1$ is odd, then

$$\begin{cases} f_k'^A = f_k^A = f_{k-1}^A, \quad f_k^B = \dfrac{2}{N_k} f_{k-1}^B, \quad f_k'^B = \dfrac{1}{N_k'} f_{k-1}^B, \\ f_k^A = \displaystyle\prod_{i=1}^{(k+1)/2} \left(\frac{2}{N_{2i-2}}\right) f_0^A, \quad f_k^B = \displaystyle\prod_{i=1}^{(k+1)/2} \left(\frac{2}{N_{2i-1}}\right) f_0^B, \end{cases}$$

   with $N_0 = 2$;

2. if $k \geqslant$ is even, then

$$\begin{cases} f_k^A = \dfrac{2}{N_k} f_{k-1}^A, \quad f_k'^A = \dfrac{1}{N_k'} f_{k-1}^A, \quad f_k'^B = f_k^B = f_{k-1}^B, \\ f_k^A = \displaystyle\prod_{i=1}^{k/2} \left(\frac{2}{N_{2i}}\right) f_0^A, \quad f_k^B = \displaystyle\prod_{i=1}^{k/2} \left(\frac{2}{N_{2i-1}}\right) f_0^B. \end{cases}$$

Consider $\tilde{L}_k := \tilde{A}_k \bigsqcup \tilde{B}_k$ (resp. $\tilde{L}_k' := \tilde{A}_k' \bigsqcup \tilde{B}_k'$). We will say that two words $a, b \in \tilde{\mathcal{A}}^\ell$ overlap if there exists a non-trivial shift $0 < s < \ell$ such that the terminal segment of length s of the word a coincides with the initial segment of the word b of the same length, or vice-versa by permuting a and b. Note that we exclude the overlapping where a and b coincide.

The next three lemmas are technical lemmas that concern some important properties about the possible types of overlapping in the objects that we described before. The first one ensures that there is no possible overlapping between two words one of $\tilde{A}_k$ and the other one from $\tilde{B}_k$ (resp. $\tilde{A}_k'$ and $\tilde{B}_k'$). The next two lemmas characterize the possible overlaps between any two words at each stage k of the iteration process.

Lemma 7. In our construction described above, a word from $\tilde{A}_k'$ and a word from $\tilde{B}_k'$ never overlap, neither can a word from $\tilde{A}_k$ and a word from $\tilde{B}_k$ overlap.

Proof. Every word in $\tilde{A}_k'$ ends with the symbol 1 which does not appear in any word in $\tilde{B}_k'$. Conversely, every word in $\tilde{B}_k'$ ends with the symbol 2 that does not appear in any word in $\tilde{A}_k'$. The same argument is valid for the words in $\tilde{A}_k$ and $\tilde{B}_k$. □

The next lemma is formulated for the case k even, but a similar lemma holds for the case k odd. First we need to fix some notations. Consider $k \geqslant 1$ an even integer and the even rules described in (3.2) and (3.4). We denote the **initial segment** of length ℓ_{k-1} of a_k and a_k' by a_{k-1}^I; the **terminal segment** of length ℓ_{k-1} of a_k and a_k'' by a_{k-1}^T; and the remaining segment $1^{(N_k'-1)\ell_{k-1}}$ that we call **marker**. We can represent

$$a_k = \underbrace{a_{k-1}}_{a_{k-1}^I} 1^{(N_k-2)\ell_{k-1}} \underbrace{a_{k-1}}_{a_{k-1}^T},$$

$$a'_k = \underbrace{a_{k-1}}_{a^I_{k-1}} \underbrace{1^{(N'_k-1)\ell_{k-1}}}_{\text{marker}} \quad \text{and} \quad a''_k = \underbrace{1^{(N'_k-1)\ell_{k-1}}}_{\text{marker}} \underbrace{a_{k-1}}_{a^T_{k-1}}.$$

We define similarly the initial and terminal segments of b'_k and denoted as b^I_{k-1} and b^T_{k-1}, respectively, as shown below

$$b'_k = \underbrace{b_{k-1}}_{b^I_{k-1}} b_{k-1}^{(N'_k-2)} \underbrace{b_{k-1}}_{b^T_{k-1}}.$$

Note that $a^I_{k-1} = a^T_{k-1} = a_{k-1}$ and $b^I_{k-1} = b^T_{k-1} = b_{k-1}$.

Lemma 8. Let $k \geqslant 1$ be even, $a_k \in \tilde{A}_k$ and $b_k \in \tilde{B}_k$ as described in (3.2). Then

1. two words of the same type a_k can only overlap on their initial and terminal segment, that is, a^I_{k-1} of one of the two words overlaps a^T_{k-1} of the other word a_k;

2. on the other hand, two words of the same type b_k can overlap exactly on a multiple of b_{k-1} or they have an overlap of length ℓ_{k-2} between b^I_{k-1} and b^T_{k-1}.

Proof. 1. We consider a non-trivial shift $0 < s < \ell_k$ and a word $w \in \tilde{\mathcal{A}}^{[\![1,s+\ell_k]\!]}$ made of two overlapping a_k:

$$a_k = w|_{[\![1,\ell_k[\![}, \quad \tilde{a}_k := w|_{s+[\![1,\ell_k]\!]}, \quad \forall\, i \in [\![1,\ell_k]\!],\ \tilde{a}_k(s+i) = a_k(i).$$

We assume first that $0 < s < \ell_{k-1}$. Then on the one hand a^T_{k-1} of a_k starts with the symbol 0 at the index $i = (N_k-1)\ell_{k-1}+1$. On the other hand the symbol 1 appears in $\tilde{a}_k$ at the indices in the range $[\![\tilde{i},\tilde{j}]\!] := [\![s+\ell_{k-1}+1, s+(N_k-1)\ell_{k-1}]\!]$. Since $i \in [\![\tilde{i},\tilde{j}]\!]$ we obtain a contradiction.

We assume next that $\ell_{k-1} \leqslant s < (N_k-1)\ell_{k-1}$. Then on the one hand the symbol 1 appears in a_k at the indices in the range $[\![\tilde{i},\tilde{j}]\!] := [\![\ell_{k-1}+1, (N_k-1)\ell_{k-1}]\!]$. On the other hand $\tilde{a}_k$ starts with the symbol 0 at the index $i = s+1$. We obtain a contradiction.

We conclude that s should satisfy $s \geqslant (N_k-1)\ell_{k-1}$: two words of the form a_k can only overlap on their initial and terminal segments.

2. We notice that $k-1$ is odd and b_{k-1} has the same structure as a_k in the first item. Two words of the form b_{k-1} only overlap on their initial and terminal segments. Then b_{k-1} cannot be a subword of the concatenation $c = b_{k-1}b_{k-1}$ of two words b_{k-1} unless b_{k-1} coincides with the first or the last b_{k-1} in c. If b_k and $\tilde{b}_k$ overlap, either $\tilde{b}_k$ has been shifted by a multiple of ℓ_{k-1}, $s \in \{\ell_{k-1}, 2\ell_{k-1}, \dots, (N'_k-1)\ell_{k-1}\}$. Note that $k-1$ is an odd number, then b_{k-1} has the same behavior as a_k described in

the previous item. Therefore, it is only possible to have an overlap of a word b_{k-2} of length ℓ_{k-2} between b^T_{k-1} and $\tilde{b}^I_{k-1}$.

□

Lemma 9. Let $k \geqslant 1$ be an even integer and a'_k and a''_k as described in (3.4). Then the following holds:

1. two words of the same form a'_k never overlap; the same is true for two words of the same form a''_k;

2. two words a'_k and a''_k overlap if and only if they overlap either partially on their marker or partially on their initial and terminal segments, respectively.

Proof. 1. We consider a non trivial shift $0 < s < \ell'_k$ and two overlapping words of the form a'_k shifted by s. Let be $w \in \tilde{\mathcal{A}}^{[\![1,s+\ell'_k]\!]}$ such that

$$a'_k = w|_{[\![1,\ell'_k]\!]}, \quad \tilde{a}'_k := w|_{s+[\![1,\ell'_k]\!]}, \quad \forall\, i \in [\![1,\ell'_k]\!],\ \tilde{a}'_k(s+i) = a'_k(i).$$

We assume first that $\ell_{k-1} \leqslant s < \ell'_k$. On the one hand, $\tilde{a}'_k$ starts with the symbol 0, $w(s+1) = 0$; on the other hand, $w|_{[\![\ell_{k-1}+1,\ell'_k]\!]}$ contains only the symbol 1. Since $s+1 \in [\![\ell_{k-1}+1,\ell'_k]\!]$ we obtain a contradiction.

We assume next that $0 < s < \ell_{k-1}$. We observe that $k-1$ is odd and the two initial segments a^I_{k-1} of a'_k and $\tilde{a}'_k$ are of the same form as b_k in the second item. They overlap on a multiple of words of the form a_{k-2} or at their initial and terminal segments. Necessarily $s \geqslant l_{k-2} \geqslant 2$. On the one hand, the initial segment of $\tilde{a}'_k$ ends with the symbols 01, $w(s+\ell_{k-1}-1) = 0$, on the other hand, $w|_{[\![\ell_{k-1}+1,\ell'_k]\!]}$ contains only the symbol 1. Since $s+\ell_{k-1}-1 \in [\![\ell_{k-1}+1,\ell'_k]\!]$ we obtain a contradiction.

A similar proof works for a''_k instead of a'_k.

2. We divided our discussion in two cases. We consider first the case,

$$a'_k = w|_{[\![1,\ell'_k]\!]}, \quad \tilde{a}''_k := w|_{s+[\![1,\ell'_k]\!]}, \quad \forall\, i \in [\![1,\ell'_k]\!],\ \tilde{a}''_k(s+i) = a''_k(i).$$

We assume that $0 < s < \ell_{k-1}$. The terminal segment of $\tilde{a}''_k$ is a word like a_{k-1} and then it starts with the symbol 0 which appears in w at the index $s+(N'_k-1)\ell_{k-1} \in [\![\ell_{k-1},\ell'_k]\!]$. On the other hand $w|_{[\![\ell_{k-1},\ell'_k]\!]}$ contains only the symbol 1. We obtain a contradiction, then necessarily $\ell_k \leqslant s$ and the two words a'_k and a''_k overlap (partially or completely) on their markers.

We consider next the case,

$$a''_k = w|_{[\![1,\ell'_k]\!]}, \quad \tilde{a}'_k := w|_{s+[\![1,\ell'_k]\!]}, \quad \forall\, i \in [\![1,\ell'_k]\!],\ \tilde{a}'_k(s+i) = a'_k(i).$$

Assume that $0 < s < (N'_k - 1)\ell_{k-1}$. The initial segment of $\tilde{a}'_k$ starts with the symbol 0 which is located at the index $s + 1 \in [\![1, (N'_k - 1)\ell_{k-1}]\!]$ in w. On the other hand $w|_{[\![1,(N'_k-1)\ell_{k-1}]\!]}$ is the marker of a''_k and contains only the symbol 1. We obtain a contradiction, then it is only possible to have $s \geqslant (N'_k - 1)\ell_{k-1}$, which means that the terminal segment of a''_k overlaps with the initial segment of a'_k. Both segments are copies of a_{k-1} and as we consider $k \geqslant 2$ even, $k - 1$ is odd and a_{k-1} has the same behavior described in Lemma 8 item 2. Therefore the possible overlap can occur (partially or completely) on their initial and terminal segments by the rules described as in Lemma 8 item 2.

□

As defined in (3.24) we consider for each $k \geqslant 0$ the concatenated subshifts generated by the sets $\tilde{L}_k$, $\tilde{A}_k$ and $\tilde{B}_k$ that are denoted as $\langle\tilde{L}_k\rangle$, $\langle\tilde{A}_k\rangle$ and $\langle\tilde{B}_k\rangle$, respectively.

By the definition of these subshifts we have that for each $k \geqslant 0$

$$\langle\tilde{A}_k\rangle \subseteq \langle\tilde{A}_{k+1}\rangle, \quad \langle\tilde{B}_k\rangle \subseteq \langle\tilde{B}_{k+1}\rangle$$

and

$$\langle\tilde{L}_{k+1}\rangle \subseteq \langle\tilde{L}_k\rangle.$$

Lemma 10. Consider the iteration process described in Notation 1 and Notation 2. If we denote $\tilde{L}_k = \tilde{A}_k \bigsqcup \tilde{B}_k$ and $\tilde{L}'_k = \tilde{A}'_k \bigsqcup \tilde{B}'_k$ for each $k \in \mathbb{R}$, then

$$\langle\tilde{L}_k\rangle \subseteq \langle\tilde{L}'_k\rangle.$$

Proof. If we consider the iteration process described in Notation 1 and Notation 2, then N'_k divides N_k. More than that, every word of $\tilde{A}_k$, $\tilde{B}_k$ is obtained as concatenation of words of $\tilde{A}'_k$, $\tilde{B}'_k$ respectively. Therefore, the concatenated subshift $\langle\tilde{L}_k\rangle$ is a subset of $\langle\tilde{L}'_k\rangle$, since every pattern in $\tilde{L}'_k$ is a subpattern in $\tilde{L}_k$. □

We consider

$$\tilde{X} := \bigcap_{k\in\mathbb{N}} \langle\tilde{L}_k\rangle. \tag{3.6}$$

The construction presented here satisfies all the hypotheses of Lemma 5, therefore $\tilde{X} = \Sigma^1(\tilde{\mathcal{A}}, \overline{\mathcal{F}})$ is the subshift generated by the set of forbidden words $\overline{\mathcal{F}} = \bigsqcup_{k\geqslant 0} \tilde{\mathcal{F}}(\ell_k)$, where $\tilde{\mathcal{F}}(\ell_k)$ is the set of words of length ℓ_k that are not subwords of the concatenation of two words of $\tilde{L}_k$.

From now on we give a specialized algorithm which produces our auxiliary sequences (N_k, ℓ_k, N'_k and ℓ'_k) and also the choice of β_k for each k. We introduce two integer numbers ρ^A_k and ρ^B_k that count the number of symbols 0 in the words a_k and b_k

$$\rho^A_k := \ell_k f^A_k, \quad \rho^B_k := \ell_k f^B_k.$$

Definition 27 (The recursive sequences)**.** We define the partial recursive function $S : \mathbb{N}^4 \to \mathbb{N}^4$

$$(\ell_k, \beta_k, \rho_k^A, \rho_k^B) = S(\ell_{k-1}, \beta_{k-1}, \rho_{k-1}^A, \rho_{k-1}^B).$$

satisfying $\ell_0 = 2$, $\beta_0 = 0$, $\rho_0^A = \rho_0^B = 1$ and defined such that the following holds:

In the case k is even:

1. $N'_k := \left\lceil \dfrac{k\rho_{k-1}^A}{\rho_{k-1}^B} \right\rceil$, $\ell'_k = N'_k \ell_{k-1}$,

2. $\beta_k := \left\lceil \dfrac{\ell_{k-1}^2 2^{k\ell'_k}}{(\rho_{k-1}^B)^2} \right\rceil$,

3. $N_k := N'_k \left\lceil \dfrac{k\beta_k}{N'_k \rho_{k-1}^B} \right\rceil$, $\ell_k = N_k \ell_{k-1}$,

4. $\rho_k^A = 2\rho_{k-1}^A$, $\rho_k^B = N_k \rho_{k-1}^B$,

In the case k is odd:

5. $(\ell_k, \beta_k, \rho_k^A, \rho_k^B)$ are computed as before with A and B permuted.

The following proposition assures there exists a Turing machine that enumerates all the forbidden patterns of $\tilde{X}$, which means that $\tilde{X}$ is an effectively closed subshift. More than that, this Turing machine can be constructed such that it enumerates the forbidden words in increasing length, it gives an exponential upper bound for the number of steps to enumerate every forbidden word up to a given length and it also gives a trivial reconstruction function ($R(n) = n$) that will be defined later (Definition 30).

Proposition 4. Let $\tilde{X}$ be the subshift defined as in (3.6). Let $\tilde{\mathcal{F}} := \bigsqcup_{n\in\mathbb{N}} \tilde{\mathcal{F}}(n)$ where $\tilde{\mathcal{F}}(n)$ is the set of words of length n that are not sub-words of the concatenation of two words of $\tilde{L}_k$ for some $k \geqslant 0$ such that $\ell_k \geqslant n$.

Then the following holds:

1. $\tilde{X} = \Sigma^1(\tilde{\mathcal{A}}, \tilde{\mathcal{F}})$.

2. For every $n \geqslant 0$, there exist unique integers $k \geqslant 1$ and $p \geqslant 2$ satisfying

$$\ell_{k-1} < n \leqslant \ell_k \text{ and } (p-1)\ell_{k-1} < n \leqslant p\ell_{k-1}.$$

We denote $\tilde{\mathcal{F}}'(n)$ as the set of words of length n that are not sub-words of any word of the form $\overrightarrow{w_1}\overleftarrow{w_2}$ where $\overrightarrow{w_1}$ is a terminal segment of w_1 of length $(p+1)\ell_{k-1}$, $\overleftarrow{w_2}$ is an initial segment of w_2 of length $(p+1)\ell_{k-1}$, and w_1 or w_2 are either one of the words $a_k, b_k, 1_k, 2_k$. Then

$$\tilde{\mathcal{F}}'(n) = \tilde{\mathcal{F}}(n).$$

3. There exists a Turing machine $\mathcal{M}$ that enumerates all patterns of $\tilde{\mathcal{F}}$ in increasing order (words of $\tilde{\mathcal{F}}(n)$ are enumerated before those in $\tilde{\mathcal{F}}(n+1)$). If we denote by $\tau\colon \mathbb{N} \to \mathbb{N}$ the function $\tau(n)$ that counts the number of steps that $\mathcal{M}$ takes to enumerate all patterns of $\tilde{\mathcal{F}}$ up to length n, then $\tau(n) \leqslant P(n)|\tilde{\mathcal{A}}|^n$, for some polynomial $P(n)$.

The proof for the previous proposition is in Appendix A.

The next lemma gives that the sets $\mathcal{L}(\langle \tilde{A}_k \rangle, \ell_k)$ and $\mathcal{L}(\langle \tilde{B}_k \rangle, \ell_k)$ can be seen as the set of all possible words of length ℓ_k that can be seen as a subword of a concatenation of two words of $\tilde{A}_k$ and $\tilde{B}_k$, respectively.

Lemma 11. Given our construction of $\tilde{A}_k$ and $\tilde{B}_k$ we have that for each $k \geqslant 0$

$$\mathcal{L}(\langle \tilde{A}_k \rangle, \ell_k) = \left\{ w \in \tilde{\mathcal{A}}^{[\![1,\ell_k]\!]} : \exists a_1, a_2 \in \tilde{A}_k \text{ such that } w \sqsubset a_1 a_2 \right\} \tag{3.7}$$

and

$$\mathcal{L}(\langle \tilde{B}_k \rangle, \ell_k) = \left\{ w \in \tilde{\mathcal{A}}^{[\![1,\ell_k]\!]} : \exists b_1, b_2 \in \tilde{B}_k \text{ such that } w \sqsubset b_1 b_2 \right\}. \tag{3.8}$$

3.2 Bidimensional SFT

We can apply the construction of Aubrun-Sablik to our one-dimensional effectively closed subshift $\tilde{X} = \Sigma^1(\tilde{\mathcal{A}}, \tilde{\mathcal{F}})$ and obtain a bidimensional SFT $\hat{X} \subseteq \Sigma^2(\hat{\mathcal{A}})$ defined over an alphabet $\hat{\mathcal{A}} = \tilde{\mathcal{A}} \times \mathcal{C}$. We are using the symbol $\wedge$ over the objects that are defined for the SFT generated by the Theorem 6. Let $\hat{\mathcal{F}} \subseteq \mathcal{A}^{[\![1,D]\!]^2}$ be a finite set of forbidden patterns such that

$$\hat{X} := \Sigma^2(\hat{\mathcal{A}}, \hat{\mathcal{F}}) \tag{3.9}$$

as the corresponding subshift generated by $\hat{\mathcal{F}}$.

Definition 28. Let $\mathcal{V}_*$ be the set of forbidden patterns in $\Sigma^2(\tilde{\mathcal{A}})$ that are not vertically aligned, that is,

$$\mathcal{V}_* := \{p \in \tilde{\mathcal{A}}^{\{1\}\times[\![1,2]\!]} : p(1,1) \neq p(1,2)\}.$$

Let $\overline{\pi} : \hat{\mathcal{A}} \to \tilde{\mathcal{A}}$ defined as

$$\begin{cases} \overline{\pi} : \hat{\mathcal{A}} = \tilde{\mathcal{A}} \times \mathcal{C} & \to & \tilde{\mathcal{A}} \\ \qquad (a,c) & \mapsto & \overline{\pi}(a,c) = a; \end{cases} \tag{3.10}$$

and let $\pi : \Sigma^2(\hat{\mathcal{A}}, \hat{\mathcal{F}}) \to \Sigma^2(\tilde{\mathcal{A}})$ be the projection defined as

$$\begin{cases} \pi : \hat{X} = \Sigma^2(\tilde{\mathcal{A}}, \hat{\mathcal{F}}) & \to & \Sigma^2(\tilde{\mathcal{A}}) \\ \qquad x & \mapsto & \pi(x) = \left(\overline{\pi}(x_{(i,j)})\right)_{(i,j)\in\mathbb{Z}^2}. \end{cases} \tag{3.11}$$

We denote

$$\hat{X}_\pi := \left\{\pi(x) : x \in \hat{X}\right\}.$$

Note that $\tilde{X}_\pi \subseteq \Sigma^2(\tilde{\mathcal{A}}, \mathcal{V}_*)$ since $\hat{\mathcal{F}}$ contains all the patterns that are not vertically aligned.

Remark 5. Here we always use the expression "vertically aligned" to express the vertical alignment over the the first coordinate of $\hat{\mathcal{A}}$, that is, over the one-dimensional alphabet $\tilde{A}$.

By Theorem 6, the projective $\mathbb{Z}$-subaction of $\hat{X}_\pi$ is equal to $\tilde{X}$, which means that

$$\hat{X}_\pi = \{x \in \Sigma^2(\tilde{\mathcal{A}}, \mathcal{V}_*) : x|_{\mathbb{Z}\times\{0\}} \in \tilde{X}\}.$$

Definition 29. We define $\tilde{A}'_{k*} \subseteq \tilde{\mathcal{A}}^{[\![1,\ell'_k]\!]^2}$ as the bidimensional dictionary of linear size ℓ'_k of vertically aligned patterns that project onto $\tilde{A}'_k$, formally defined as

$$\tilde{A}'_{k*} := \left\{p \in \tilde{\mathcal{A}}^{[\![1,\ell'_k]\!]^2} : \exists \tilde{p} \in \tilde{A}'_k, \text{ s.t. } \forall, (i,j) \in [\![1,\ell'_k]\!]^2,\ p(i,j) = \tilde{p}(i)\right\}.$$

$\tilde{B}'_{k*} \subseteq \tilde{\mathcal{A}}^{[\![1,\ell'_k]\!]^2}$ is defined similarly. We use the notation $\pi_* : \tilde{A}'_{k*} \to \tilde{A}'_k$ (resp. $\pi_* : \tilde{B}'_{k*} \to \tilde{B}'_k$) to represent the projection of a square pattern $p \in \tilde{A}'_{k*}$ (resp. $\tilde{B}'_{k*}$) to its word $\tilde{p} \in \tilde{A}'_k$ (resp. $\tilde{B}'_k$) that defines it.

We consider a large pattern $p \in \tilde{\mathcal{A}}^{[\![1,n]\!]^2}$ and translates u of small squares of size $2\ell'_k$ inside this pattern that are labeled by vertically aligned words of $\tilde{A}'_k$ or $\tilde{B}'_k$. Let $k \geqslant 2$, $n > 2\ell'_k$, and $p \in \tilde{\mathcal{A}}^{[\![1,n]\!]^2}$. We denote

$$I(p,\ell'_k) := \left\{u \in [\![0, n-2\ell'_k]\!]^2 : \sigma^u(p)|_{[\![1,2\ell'_k]\!]^2} \in \mathcal{L}(\hat{X}_\pi, 2\ell'_k)\right\}, \tag{3.12}$$

$$I^A(p,\ell'_k) := \left\{u \in [\![0, n-\ell'_k]\!]^2 : \sigma^u(p)|_{[\![1,\ell'_k]\!]^2} \in \tilde{A}'_{k*}\right\} \tag{3.13}$$

and

$$J^A(p,\ell'_k) := \bigcup_{u\in I^A(p,\ell'_k)} \left(u + [\![1,\ell'_k]\!]^2\right). \tag{3.14}$$

We define $I^B(p,\ell'_k)$ and $J^B(p,\ell'_k)$ similarly with replacing $\tilde{A}'_{k*}$ for $\tilde{B}'_{k*}$ in (3.13) and (3.14), respectively.

Lemma 12. Let $k \geqslant 2$, $n > 2\ell'_k$, $p \in \tilde{\mathcal{A}}^{[\![1,n]\!]^2}$ and the sets defined above. We will denote $\tau'_k =: (\ell'_k, \ell'_k) \in \mathbb{N}^2$. Then $J^A(p,\ell'_k) \cap J^B(p,\ell'_k) = \varnothing$ and for each $u \in I(p,\ell'_k)$

$$u + \tau'_k \in J^A(p,\ell'_k) \bigsqcup J^B(p,\ell'_k).$$

Proof. The fact that $J^A(p,\ell'_k)$ and $J^B(p,\ell'_k)$ do not intersect is a consequence of Lemma 7. Let be $u \in I(p,\ell'_k)$ and $w_* = \sigma^u(p)|_{[\![1,2\ell'_k]\!]^2}$. There exists $w \in \mathcal{L}(\langle \tilde{L}_k \rangle, 2\ell'_k)$ such that

$w_*(i,j) = w(i)$ for all $(i,j) \in [\![1, 2\ell'_k]\!]^2$. By definition of $\langle \tilde{L}_k \rangle$, $w \sqsubset w_1 w_2$ is a subword of the concatenation of two words of $\tilde{L}_k$. Note that, by Lemma 10 $\langle \tilde{L}_k \rangle \subseteq \langle \tilde{L}'_k \rangle$. Hence $\mathcal{L}(\langle \tilde{L}_k \rangle, 2\ell'_k) \subseteq \mathcal{L}(\langle \tilde{L}'_k \rangle, 2\ell'_k)$

On the other hand, a word in $\tilde{L}_k$ is either a word of $\tilde{A}_k$ or a word of $\tilde{B}_k$. As $\langle \tilde{A}_k \rangle \subset \langle \tilde{A}'_k \rangle$ and $\langle \tilde{B}_k \rangle \subset \langle \tilde{B}'_k \rangle$, w_1 and w_2 are obtained as a concatenation of words of $\tilde{A}'_k$ or $\tilde{B}'_k$. There exists $0 \leqslant s < \ell'_k$ such that

$$\sigma^s(w)|_{[\![1,\ell'_k]\!]} \in \tilde{A}'_k \bigsqcup \tilde{B}'_k.$$

Then

$$u + (s,s) \in I^A(p, \ell'_k) \bigsqcup I^B(p, \ell'_k),$$

and therefore

$$u + \tau'_k \in J^A(p, \ell'_k) \bigsqcup J^B(p, \ell'_k).$$

□

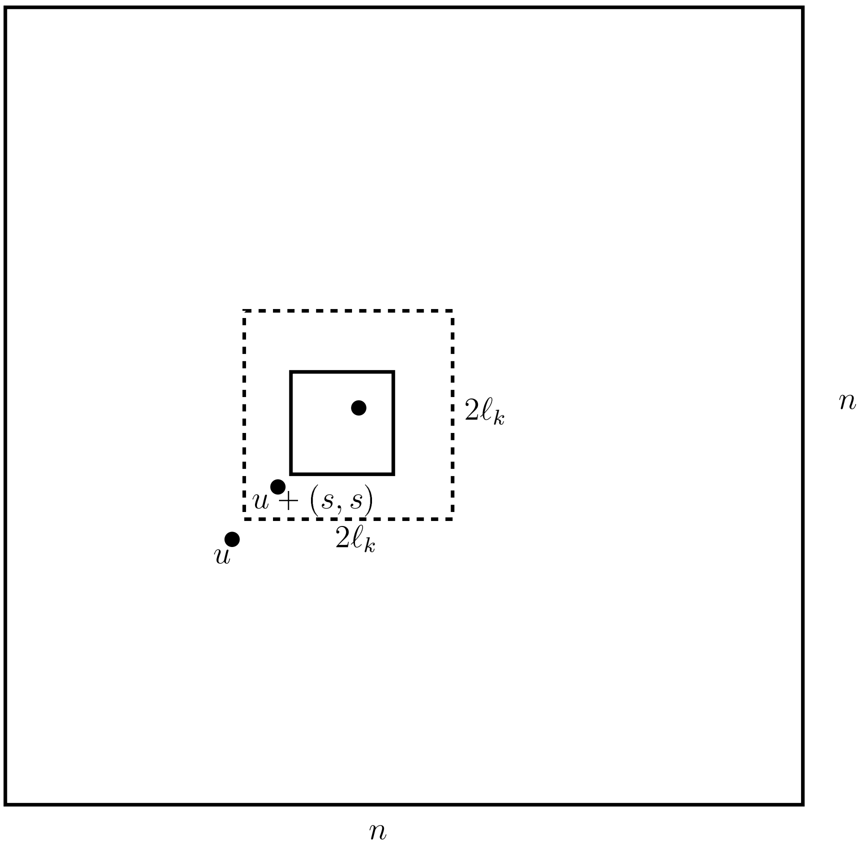

Figure 3.1: In the figure we are taking a square pattern $p \in \tilde{\mathcal{A}}^{[\![0,n]\!]^2}$ shown as the biggest square. We are considering that $u \in I(p, \ell'_k)$ and therefore the patterns located in the dashed square of size $2\ell'_k$ belong to $\mathcal{L}(\hat{X}_\pi, 2\ell'_k)$. We know that the pattern located in the most inner box of size ℓ'_k belongs to $\tilde{A}'_k \bigsqcup \tilde{B}'_k$. The most inner dot represents $u + \tau'_k$.

Lemma 13. Let $k \geqslant 2$ be an even integer, $n > 2\ell'_k$, and $p \in \tilde{\mathcal{A}}^{[\![1,n]\!]^2}$. Let $I^A(p, \ell'_k)$,

$J^A(p,\ell'_k)$, $I^B(p,\ell'_k)$, $J^B(p,\ell'_k)$ be the sets defined in Lemma 12. Define

$$K^A(p,\ell'_k) = \{v \in J^A(p,\ell'_k) : p(v) = 0\}, \quad K^B(p,\ell'_k) = \{v \in J^B(p,\ell'_k) : p(v) = 0\}. \tag{3.15}$$

Then

1. $\mathrm{card}(K^B(p,\ell'_k)) \leqslant \left(1 - N_{k-1}^{-1}\right)^{-1}\mathrm{card}(J^B(p,\ell'_k))f^B_{k-1}$,
2. $\mathrm{card}(K^A(p,\ell'_k)) \leqslant \dfrac{2}{N'_k}\mathrm{card}(J^A(p,\ell'_k))f^A_{k-1}$.

Proof. Let $k \geqslant 2$ even, $n > 2\ell'_k$ and a fixed $p \in \tilde{\mathcal{A}}^{[\![1,n]\!]^2}$. To simplify the notations, we write $I^A = I^A(p,\ell'_k)$, $J^A = J^A(p,\ell'_k)$ and so on. As the symbol 0 does not appear in the markers $1^{N'_k\ell_{k-1}} \in \tilde{A}'_k$ and $2^{N'_k\ell_{k-1}} \in \tilde{B}'_k$, we only need to consider the subset of I^A (resp. I^B) that corresponds to the translates $u \in [\![0, n-\ell'_k]\!]^2$ and the subwords $w_* = \sigma^u(p)|_{[\![1,\ell'_k]\!]^2}$ satisfying $\pi_*(w_*) \in \{a'_k, a''_k\}$ (resp. $\pi_*(w_*) = b'_k$).

Item 1. We first enumerate $I^B = \{u_1, u_2, \ldots, u_H\}$. Let be $u_h = (u^x_h, u^y_h) \in \mathbb{Z}^2$. Let

$$J^B := \bigcup_{h=1}^{H} J_h \quad \text{where} \quad J_h := u_h + [\![1,\ell'_k]\!]^2, \quad \pi_*(\sigma^{u_h}(p))|_{[\![1,\ell'_k]\!]^2} = b'_k,$$

that is, we are only considering the J_h elements of $J^B(p,\ell'_k)$ such that the one-dimensional projection is the block b'_k. For each box J_h we divide into N'_k vertical strips of length ℓ_{k-1}. Formally we have

$$J_h = \bigcup_{i=1}^{N'_k} J_{h,i} \quad \text{where} \quad J_{h,i} := u_h + [\![1+(i-1)\ell_{k-1}, i\ell_{k-1}]\!] \times [\![1,\ell'_k]\!].$$

We construct a partition of J^B inductively by,

$$J^B = \bigsqcup_{h=1}^{H} J^*_h, \quad J^*_1 = J_1, \quad \forall\, h \geqslant 2,\ J^*_h := J_h \backslash \left(J_1 \cup \cdots \cup J_{h-1}\right).$$

Let

$$K^*_h := \{v \in J^*_h : p(v) = 0\}.$$

It will be enough to show that for every $h \in [\![1,H]\!]$

$$\mathrm{card}(K^*_h) \leqslant \left(1 - N_{k-1}^{-1}\right)^{-1}\mathrm{card}(J^*_h)f^B_k, \tag{3.16}$$

By definition of u_h, $\tilde{w}_h = \pi_*(p|_{(u_h+[\![1,\ell'_k]\!]^2)})$ is a translate of $b'_k \in \tilde{\mathcal{A}}^{\ell'_k}$,

$$\forall\, i,j \in [\![1,\ell_k]\!]^2,\ \tilde{w}_h(u^x_h + i) = b'_k(i).$$

Since b'_k is made of N'_k subwords of the form b_{k-1}, we denote by $\tilde{w}_{h,i} \in \tilde{\mathcal{A}}^{\ell_{k-1}}$, the successive subwords, $\forall\, 1 \leqslant i \leqslant N'_k$,

$$\tilde{w}_{h,i} := \tilde{w}_h|_{(u_h^x + [\![1+(i-1)\ell_{k-1}, i\ell_{k-1}]\!])} \text{ and } \sigma^{u_h^x+(i-1)\ell_{k-1}}(\tilde{w}_{h,i}) = b_{k-1}.$$

We are considering a fixed h and we show that J_h^* is equal to a disjoint union of N'_k vertical strips $(J_{h,i}^*)_{i=1}^{N'_k}$ of the following forms:

- the initial strip $J_{j,1}^*$,

$$u_h + ([\![1+\ell_{k-2}, \ell_{k-1}]\!] \times [\![c_{h,1}, d_{h,1}]\!]) \subseteq J_{j,1}^* \subseteq (u_h + [\![1, \ell_{k-1}]\!]) \times [\![a_{h,1}, b_{h,1}]\!];$$

- the intermediate strips, $J_{h,i}^*$, $1 < i < N'_k$,

$$J_{h,i}^* = u_h + ([\![(i-1)\ell_{k-1}+1, i\ell_{k-1}]\!] \times [\![c_{h,i}, d_{h,i}]\!])\ ; \text{and}$$

- the terminal strip J_{h,N'_k}^*,

$$u_h + ([\![1+(N_k-1)\ell_{k-1}, \ell_k - \ell_{k-2}]\!] \times [\![c_{h,N_k}, d_{h,N_k}]\!]) \subseteq$$
$$\subseteq J_{h,N'_k}^* \subseteq u_h + \left([\![1+(N'_k-1)\ell_{k-1}, \ell'_k]\!] \times [\![a_{h,N'_k}, b_{h,N'_k}]\!]\right).$$

Here for each $i \in [\![1, N'_k]\!]$, the values $1 \leqslant c_{h,i}, d_{h,i} \leqslant \ell_k$ are integers that represent the vertical extent of each strip and it will be possible that $c_{h,i} < d_{h,i}$ to denote an empty strip $J_{h,i}^*$.

Indeed, for a fixed $1 \leqslant i \leqslant N'_k$, we first consider the previous J_g, $1 \leqslant g < h$, that intersects the strip $J_{h,i}$ so that the word $\tilde{w}_g$ overlaps $\tilde{w}_h$ on a multiple of b_{k-1} (see item 2 of Lemma 8). Then $c_{h,i}$ is the largest upper level of those $J_g \cap J_{h,i}$, more precisely,

$$c_{h,i} = \max_g \left\{u_g^y + \ell'_k + 1 : u_g^y \leqslant u_h^y, \left(u_h^x + (i-1)\ell_{k-1} + [\![1, \ell_{k-1}]\!]\right) \subseteq \left(u_g^x + [\![1, \ell'_k]\!]\right)\right\}. \quad (3.17)$$

and similarly $d_{h,i}$ is the smallest lower level of those $J_g \cap J_{h,i}$, formally we have

$$d_{h,i} = \min_g \left\{u_g^y + 1 : u_g^y \geqslant u_h^y, \left(u_h^x + (i-1)\ell_{k-1} + [\![1, \ell_{k-1}]\!]\right) \subseteq \left(u_g^x + [\![1, \ell'_k]\!]\right)\right\}. \quad (3.18)$$

We have just constructed the intermediate strips $J_{h,i}^*$ for $1 < i < N_k$.

We now construct the initial strip (the terminal strip is constructed similarly). We intersect the remaining J_g with $J_{h,1}$. The terminal segment b_{k-1}^T of $\tilde{w}_g$ overlaps the initial segment b_{k-1}^I of $\tilde{w}_h$. Thanks to item 1 of Lemma 8, as $k-1$ is odd, b_{k-1} has the same structure as a_k, the overlapping can only happen at their end segments of the form b_{k-2}. We have just proved that $J_{h,1}^*$ contains a small strip $\left(u_h + [\![1+\ell_{k-2}, \ell_{k-1}]\!]\right) \times [\![c_{h,1}, d_{h,1}]\!]$ of base $b_{k-1}^I \backslash b_{k-2}$ and is included in a larger strip $\left(u_h + [\![1, \ell_{k-1}]\!]\right) \times [\![c_{h,1}, d_{h,1}]\!]$ of base b_{k-1}.

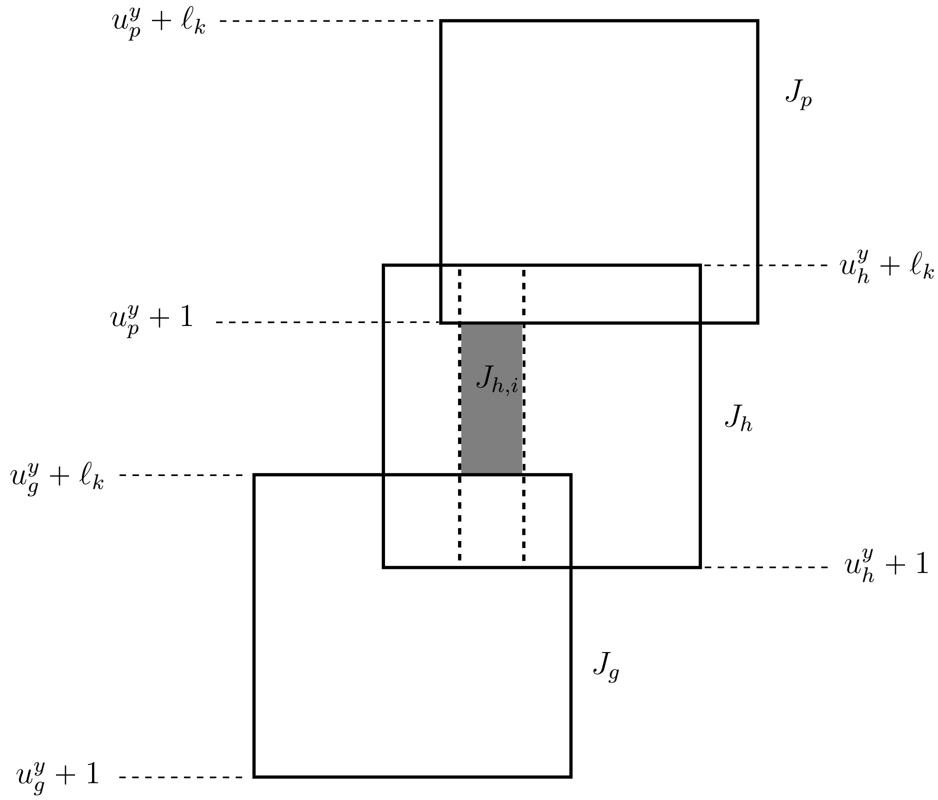

Figure 3.2: We are representing here the case where there is an intersection but the strip $J_{h,i}$ is not completely covered by the previous squares J_g. The squares J_g and J_p are already in the partition, then $J^*_{h,i}$ is only the highlighted gray area.

For the initial and terminal strip the vertical extension ($[\![c_{h,1}, d_{h,1}]\!]$ and $[\![c_{h,N'_k}, d_{h,N'_k}]\!]$) of the elements $J^*_{h,1}$ and J^*_{h,N'_k} are defined as in (3.17) and (3.18).

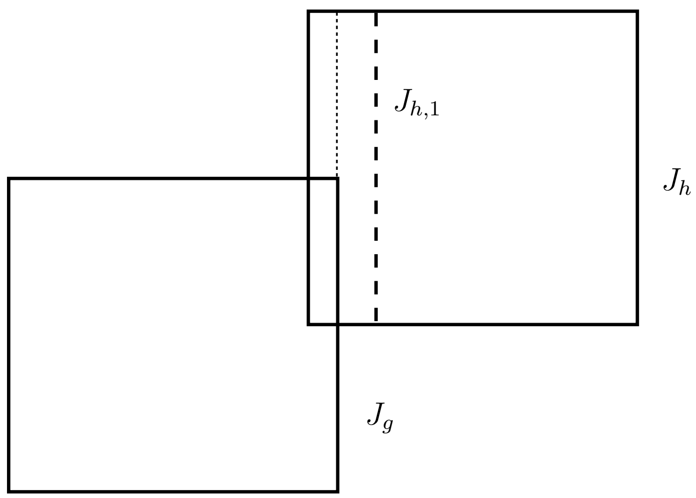

Figure 3.3: The strip of length $\ell_{k-1} - \ell_{k-2}$ is always contained in $J^*_{h,1}$.

Let be $K^*_{h,i} := \{v \in J^*_{h,i} : p_v = 0\}$. We show that

$$\forall\, 1 \leqslant i \leqslant N_k,\ \mathrm{card}(K^*_{h,i}) \leqslant \left(1 - N_{k-1}^{-1}\right)^{-1} \mathrm{card}(J^*_{h,i}) f_k^B. \tag{3.19}$$

For the intermediate strips $J^*_{h,i}$, where $1 < i < N'_k$, we use the fact that $J^*_{h,i}$ is a square strip of base b_{k-1}, and the fact that the frequency f^B_{k-1} of the symbol 0 in the word b_{k-1} is identical to the frequency f^B_k of the symbol 0 in b_k. We have,

$$\mathrm{card}(K^*_{h,i}) = \ell_{k-1}(d_{h,i} - c_{h,i} + 1) f^B_k = \mathrm{card}(J^*_{h,i}) f^B_k .$$

For the initial strip $J^*_{h,1}$, we use the fact that $J^*_{h,1}$ resembles largely a square strip of base b_{k-1}. We have,

$$\begin{aligned}
\mathrm{card}(\ K^*_{h,i}) &\leqslant \ell_{k-1}(d_{h,1} - c_{h,1} + 1) f^B_k \\
&\leqslant \frac{\ell_{k-1}}{\ell_{k-1} - \ell_{k-2}} (\ell_{k-1} - \ell_{k-2})(d_{h,1} - c_{h,1} + 1) f^B_k \\
&\leqslant \left(1 - N^{-1}_{k-1}\right)^{-1} \mathrm{card}(J^*_{h,i}) f^B_k .
\end{aligned}$$

We have proved (3.19) and by summing over $i \in [\![1, N'_k]\!]$ we have proved (3.16).

Item 2. As before we will consider I^A (defined in (3.13), but only consider the translates $u \in [\![0, n - \ell'_k]\!]^2$ such that $\pi_*(\sigma^u(p)|_{[\![1,\ell'_k]\!]^2}) \in \{a'_k, a''_k\}$. If $J_g \cap J_h \neq \varnothing$, the two projected words $\tilde{w}_g = \pi_*(\sigma^{u_g}(p)|_{[\![1,\ell'_k]\!]^2})$ and $\tilde{w}_h = \pi_*(\sigma^{u_h}(p)|_{[\![1,\ell'_k]\!]^2})$ may either coincide in three ways: $\tilde{w}_g = \tilde{w}_h$, so $u^x_g = u^x_h$; overlap partially on their markers or overlap on their initial and terminal segments as proved in Lemma 9.

We redefine again I^A by clustering into a unique rectangle adjacent squares where the overlap occurs in the whole word, that is, we group the squares J_g and J_h that pairwise satisfy $J_g \cap J_h \neq \varnothing$, $u^x_g = u^x_h$, $\tilde{w}_g = \tilde{w}_h$, $|u^y_g - u^y_h| < \ell'_k$. Then, after reindexing I^A, one obtains,

$$J^A = \bigcup_{h=1}^{H} J_h, \quad J_h = u_h + \left([\![1, \ell_{k-1}]\!] \times [\![1, d_h]\!]\right),$$

where d_h is the final height of each rectangle obtained after the clustering. Thus $w^*_h = \sigma^{u_h}(p)|_{[\![1,\ell'_k]\!] \times [\![1,d_h]\!]}$ is a vertically aligned pattern whose projection $\tilde{w}_h = \pi_*(w^*_h)$ is a word of the form a'_k or a''_k, and so that $\tilde{w}_g$, $\tilde{w}_h$ never entirely coincide if $J_g \cap J_h \neq \varnothing$.

We now show that an index $v = (v^x, v^y) \in J^A$ may belong to at most two rectangles J_g and J_h. Indeed, by construction, $u^x_g \neq u^x_h$, if v^x belongs to two overlapping words of the form a'_k, a''_k, then v^x belongs to either the intersection of the two markers $1^{(N'_k-1)\ell_{k-1}}$ or the intersection of the terminal segment a^T_{k-1} of a''_k and the initial segment a^I_{k-1} of a'_k. In both cases described in Lemma 9 we exclude the overlapping of a third word of the form a'_k, a''_k, thus we exclude the fact that v may belong to a third rectangle J_k with $u^x_k \neq u^x_g$

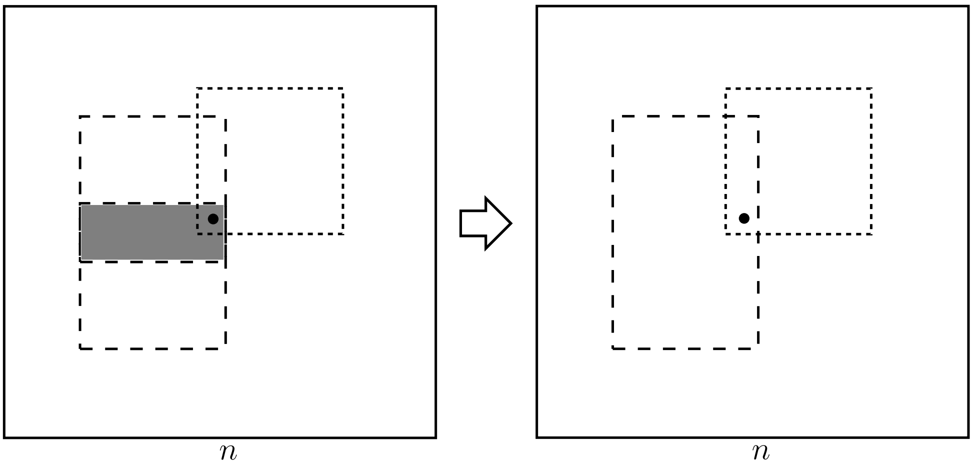

Figure 3.4: The biggest square has size n. On the left side the three squares of size ℓ'_k intersect each other and the black dot belongs to each of these squares. The highlighted gray area belongs to both of the vertically aligned squares. After the clustering, on the right side, the two previous dashed squares emerge into one box of size $\ell'_k \times d_h$ and thus the point represented in the figure only belong to two boxes.

and $u^x_k \neq u^x_h$. Then

$$\begin{aligned}
\operatorname{card}(K^A) &= \sum_{v \in J^A} \mathbb{1}_{(p(v)=0)} \\
&\leqslant \sum_{h=1}^{H} \sum_{v \in (u_h + [\![1,\ell'_k]\!] \times [\![1,d_h]\!])} \mathbb{1}_{(p(v)=0)} \leqslant \sum_{h=1}^{H} f^A_{k-1} \ell_{k-1} d_h \\
&\leqslant \frac{f^A_{k-1}\ell_{k-1}}{\ell'_k} \sum_{h=1}^{H} \sum_{v \in J^A} \mathbb{1}_{v \in (u_h + [\![1,\ell'_k]\!] \times [\![1,d_h]\!])} = \frac{f^A_{k-1}}{N'_k} \sum_{v \in J^A} \sum_{h=1}^{H} \mathbb{1}_{(v \in J_h)} \\
&\leqslant \frac{2 f^A_{k-1}}{N'_k} \operatorname{card}(J^A).
\end{aligned}$$

□

3.3 The new coloring

Based on our previous construction we define a new coloring for the SFT generated by the Aubrun-Sablik construction. This new subshift will be defined using the alphabet $\mathcal{A} = \mathcal{B} \times \tilde{\mathcal{C}}$, where $\mathcal{B} = \{0', 0'', 1, 2\}$. Consider $\mathcal{A} = \mathcal{B} \times \tilde{\mathcal{C}}$, $\gamma : \mathcal{A} \to \hat{\mathcal{A}}$ obtained by collapsing the two symbols $0', 0''$ to 0, that is,

$$\forall\, c \in \mathcal{C}, \; \begin{cases} \gamma(0', c) = (0, c), & \gamma(0'', c) = (0, c), \\ \gamma(1, c) = (1, c), & \gamma(2, c) = (2, c), \end{cases}$$

and

$$\Gamma : \Sigma^2(\mathcal{A}) \to \Sigma^2(\hat{\mathcal{A}}) \tag{3.20}$$

be the 1-block canonical projection.

Remember that we are denoting $\hat{\mathcal{A}} = \tilde{\mathcal{A}} \times \mathcal{C}$ and $\tilde{\mathcal{A}} = \{0, 1, 2\}$. Let $\overline{\pi} : \hat{\mathcal{A}} \to \tilde{\mathcal{A}}$ be the first projection over the alphabet $\tilde{\mathcal{A}}$ as defined in (3.10). We set $\hat{\Pi} : \Sigma^2(\hat{\mathcal{A}}) \to \Sigma^1(\tilde{\mathcal{A}})$ defined as

$$\begin{cases} \hat{\Pi} : \Sigma^2(\hat{\mathcal{A}}) & \to & \Sigma^1(\tilde{\mathcal{A}}) \\ \qquad x & \mapsto & y = (\overline{\pi}(x_{(i,0)}))_{i \in \mathbb{Z}}. \end{cases} \tag{3.21}$$

We will always apply $\hat{\Pi}$ for configurations that are vertically aligned for the symbols in $\tilde{\mathcal{A}}$ and therefore there is no problem in selecting the zero row with indices $(i, 0)$ where $i \in \mathbb{Z}$.

Let $\mathcal{F}$ be the pullback of $\hat{\mathcal{F}}$ by Γ and X be the subshift generated by $\mathcal{F}$,

$$\mathcal{F} := \{p \in \mathcal{A}^{[\![1,D]\!]^2} : \Gamma(p) \in \hat{\mathcal{F}}\}, \quad X := \Gamma^{-1}(\hat{X}) = \Sigma^2(\mathcal{A}, \mathcal{F}).$$

Let be

$$\pi = \overline{\pi} \circ \gamma \quad \text{and} \quad \Pi = \hat{\Pi} \circ \Gamma. \tag{3.22}$$

Observation 1. We will also use the projection Π as defined before for finite patterns without any distinction. Note that the extended set of forbidden patterns $\mathcal{F}$ forces every locally admissible configuration to be vertically aligned with respect to the initial alphabet $\tilde{\mathcal{A}}$ provided we identify the two duplicated symbols $0'$ and $0''$.

We can define the bidimensional subshifts generated by each step of the iteration process. Consider k large enough such that we have $\ell_k > D$ where $D \geqslant 2$ is defined by the set of forbidden patterns $\mathcal{F} \subset \mathcal{A}^{[\![1,D]\!]^2}$. We will denote

$$L_k := \mathcal{L}(X, \ell_k) \tag{3.23}$$

that is, the language of X of size ℓ_k as defined in (2.2). We say that a pattern w belongs to L_k if and only if it is globally admissible with respect to X. Let $\langle L_k \rangle$ be the corresponding concatenated subshift as defined in Definition 8, that is,

$$\langle L_k \rangle := \bigcup_{u \in [\![1,\ell_k]\!]^2} \bigcap_{v \in \mathbb{Z}^2} \sigma^{-(u+v\ell_k)}[L_k]. \tag{3.24}$$

Note that every pattern in L_{k+1} is obtained by concatenating N_k^2 patterns of L_k and the subshifts satisfy $\langle L_{k+1} \rangle \subset \langle L_k \rangle$.

We define two intermediate sub-languages of $\hat{X}$ of size ℓ_k by,

$$\forall\, k \geqslant 0, \quad \begin{cases} \hat{A}_k := \{w \in \mathcal{L}(\hat{X}, \ell_k) : \Pi(w) \in \mathcal{L}(\langle \tilde{A}_k \rangle, \ell_k)\}, \\ \hat{B}_k := \{w \in \mathcal{L}(\hat{X}, \ell_k) : \Pi(w) \in \mathcal{L}(\langle \tilde{B}_k \rangle, \ell_k)\}, \end{cases} \tag{3.25}$$

and two sub-languages of X,

$$\forall\, k \geqslant 0, \quad \begin{cases} A_k := \{w \in \mathcal{A}^{[\![1,\ell_k]\!]^2} : \Gamma(w) \in \hat{A}_k\}, \\ B_k := \{w \in \mathcal{A}^{[\![1,\ell_k]\!]^2} : \Gamma(w) \in \hat{B}_k\} \end{cases} \tag{3.26}$$

Every pattern of A_{k+1} (respectively B_{k+1}) is made of N_k^2 patterns of A_k (respectively B_k). In particular $\langle A_{k+1}\rangle \subseteq \langle A_k\rangle$, $\langle B_{k+1}\rangle \subseteq \langle B_k\rangle$.

We recall two definitions. The reconstruction function is associated to a subshift generated by a set of forbidden words which was also described in [13, 33] on a different context. The relative complexity function is associated to a shift equivariant extension of a dynamical system. The role of the reconstruction function is clearly put forward in Chazottes-Hochman [11]. The fact that the subshift of finite type obtained in Aubrun-Sablik [2] or [11] has zero entropy is relatively easy to prove. We actually need a more precise estimate of the growth of the complexity. An exponential growth proportional to the boundary of a square (not proportional to the volume of a square) is enough for instance. This issue seems to be missing in [11].

Definition 30. Let $\hat{\mathcal{A}}$ be a finite alphabet, $D \geqslant 1$, $\hat{\mathcal{F}} \subseteq \hat{\mathcal{A}}^{[\![1,D]\!]^2}$, and $\hat{X} = \Sigma^2(\hat{\mathcal{A}}, \hat{\mathcal{F}})$ be the subshift generated by the forbidden patterns $\hat{\mathcal{F}}$, as defined before. We define the *reconstruction function* of the subshift $\hat{X}$ as the function $R^{\hat{X}} : \mathbb{N}^* \to \mathbb{N}^*$ which associates to every ℓ the smallest R such that every locally $\hat{\mathcal{F}}$-admissible word in $\mathcal{A}^{[\![1,2R]\!]^2}$ admits a globally $\hat{\mathcal{F}}$-admissible restriction in its central block of length ℓ.

We will denote by $M(\hat{\mathcal{F}}, R) \subseteq \hat{\mathcal{A}}^{[\![1,R]\!]^2}$ the set of all square patterns of size R in $\hat{\mathcal{A}}$ such that no pattern of $\hat{\mathcal{F}}$ appears inside, that is,

$$M(\hat{\mathcal{F}}, R) := \{w \in \hat{\mathcal{A}}^{[\![1,R]\!]^2} : \forall\, p \in \hat{\mathcal{F}},\ \forall\, u \in [\![0, R-D]\!]^2,\ p \not\sqsubseteq \sigma^u(w)\} \tag{3.27}$$

We will use the reconstruction function for the subshift $\hat{X}$ and the sequence $(R'_k)_{k\geqslant 0}$ defined as

$$R'_k := R^{\hat{X}}(2\ell'_k) = \inf\{R > 2\ell'_k : \forall\, w \in M(\hat{\mathcal{F}}, R),\ \exists\, x \in X,\ w|_{Q(2\ell'_k,R)} = x|_{Q(2\ell'_k,R)}\}, \tag{3.28}$$

where $Q(2\ell'_k, R)$ is the central block of length $2\ell'_k$, formally defined as

$$Q(2\ell'_k, R) := T(2\ell'_k, R) + [\![1, 2\ell'_k]\!]^2, \tag{3.29}$$

where $T(2\ell'_k, R) = \left(\left\lfloor \frac{R}{2} - \ell'_k \right\rfloor, \left\lfloor \frac{R}{2} - \ell'_k \right\rfloor\right) \in \mathbb{Z}^2$.

Remark 6. The reconstruction function exists for every subshift as stated in Proposition 2, but establishing its growth or computability is not always possible.

Definition 31. Let $\tilde{X} \subset \Sigma^1(\tilde{\mathcal{A}})$ be the effectively closed subshift described before and $\hat{X} \subset \Sigma^2(\hat{\mathcal{A}})$ be the SFT given by the simulation Theorem 6 that simulates $\tilde{X}$. The *relative complexity function* of the simulation is the function $C^{\hat{X}} : \mathbb{N}^* \to \mathbb{N}^*$ defined by

$$C^{\hat{X}}(\ell) := \sup_{\tilde{w} \in \mathcal{L}(\tilde{X}, \ell)} \operatorname{card}\big(\{\hat{w} \in \mathcal{L}(\hat{X}, \ell) : \hat{\Pi}(\hat{w}) = \tilde{w}\}\big).$$

The two following propositions give us an idea of the growth of each of the functions (reconstruction and relative complexity). The proofs of these two results are in Appendix A. They are very technical proofs that are based on the construction described by Aubrun-Sablik [2] and the iteration process described previously.

Proposition 5. Let $\tilde{X}$ be the one-dimensional effectively closed subshift defined before and $\hat{X}$ be the bidimensional SFT from the Aubrun-Sablik theorem. There is a constant $K > 0$ and a polynomial $P(n)$ such that

$$R^{\hat{X}}(n) = P(n)K^n.$$

Proposition 6. Let $\hat{X}$ be the $\mathbb{Z}^2$-SFT in the Aubrun-Sablik construction. There is a constant $K > 0$ and a polynomial $P(n)$ such that

$$C^{\hat{X}}(n) = P(n)K^n.$$

As a result of these two propositions, we have the next lemma that gives us important bounds for the reconstruction function and the relative complexity function that will be necessary in our final proof.

Lemma 14 (A priori estimates)**.** Let $R^{\hat{X}}$ and $C^{\hat{X}}$ be the reconstruction and relative complexity function of the SFT given by Aubrun-Sablik, then

1. $\limsup_{n \to +\infty} \frac{1}{n} \ln(C^{\hat{X}}(n)) < +\infty$,
2. $\limsup_{n \to +\infty} \frac{1}{n} \ln(R^{\hat{X}}(n)) < +\infty$.

The demonstration of these properties is more technical and uses computability theory and Turing machines. These proofs can be found in Appendix A but for now on we will assume that they are true.

To simplify the notations, we write

$$\begin{aligned} R'_k &:= R^{\hat{X}}(2\ell'_k), & C'_k &:= C^{\hat{X}}(\ell'_k), \\ Q'_k &:= \mathcal{Q}(2\ell'_k 2, R'_k) \subset \mathbb{Z}^2, & T'_k &:= T(2\ell'_k, R'_k) \in \mathbb{Z}^2, \\ \hat{M}'_k &= M(\hat{\mathcal{F}}, R'_k) \subset \hat{\mathcal{A}}^{[\![1,R'_k]\!]^2}, & M'_k &= \Gamma^{-1}(\hat{M}'_k) \subset \mathcal{A}^{[\![1,R'_k]\!]^2}. \end{aligned} \tag{3.30}$$

We denote by $[M'_k]$ the cylinder generated by the set M'_k, which consists of the configurations that are $\mathcal{F}$-locally admissible in $[\![1, R'_k]\!]^2$. We compute the topological entropy of patterns that are most of the time (in terms of translations of $\mathbb{Z}^2$) globally admissible with respect to $\hat{\mathcal{F}}$. We naturally point out the relative complexity function. Notice that the relative entropy is computed using the volume of the square.

Lemma 15. Let $n > 2\ell > 2$ be some integers, $\varepsilon \in (0,1)$ be some real number, and $S \subseteq [\![0, n-2\ell]\!]^2$ be a subset satisfying $\mathrm{card}(S) \geqslant n^2(1-\varepsilon)$. Let $\hat{E}$ be the set

$$\hat{E} := \left\{ w \in \hat{\mathcal{A}}^{[\![1,n]\!]^2} : \forall\, u \in S,\ \sigma^u(w)|_{[\![1,2\ell]\!]^2} \in \mathcal{L}(\hat{X}, 2\ell) \right\}.$$

Then

$$\frac{1}{n^2}\ln(\mathrm{card}(\hat{E})) \leqslant \frac{1}{\ell}\ln(\mathrm{card}(\tilde{\mathcal{A}})) + \frac{1}{\ell^2}\ln(C^{\hat{X}}(\ell)) + \varepsilon\ln(\mathrm{card}(\hat{\mathcal{A}})).$$

Proof. Here we consider n as a multiple of ℓ in order to simplify the notations since we are interested in the limit when $n \to +\infty$ there is no problem. We decompose the square $[\![1, n]\!]^2$ into a disjoint union of squares of size ℓ,

$$[\![1, n]\!]^2 = \bigcup_{v \in [\![0, \frac{n}{\ell}-1]\!]^2} \left(\ell v + [\![1, \ell]\!]^2\right).$$

We define the set of indices v that intersect S, more precisely, we have

$$V := \left\{ v \in \left[\!\!\left[0, \frac{n}{\ell} - 2\right]\!\!\right]^2 : \left(\ell v + [\![0, \ell-1]\!]^2\right) \bigcap S \neq \varnothing \right\}.$$

Then for every $w \in \hat{E}$, $v \in V$, and $u \in \left(\ell v + [\![0, \ell-1]\!]^2\right) \bigcap S$, therefore

$$\left(\ell v + [\![1+\ell, 2\ell]\!]^2\right) \subseteq \left(u + [\![1, 2\ell]\!]^2\right).$$

Since we are taking $u \in S$ we have that

$$\sigma^u(w)|_{[\![1,2\ell]\!]^2} \in \mathcal{L}(\hat{X}, 2\ell),$$

and then

$$\sigma^{\ell v + (\ell,\ell)}(w)|_{[\![1,\ell]\!]^2} \in \mathcal{L}(\hat{X}, \ell).$$

The restriction of w on every square $\left(\ell v + [\![1+\ell, 2\ell]\!]^2\right)$ is globally admissible with respect to $\hat{\mathcal{F}}$. Note that these squares are pairwise disjoint and the cardinality of their union is at least $n^2(1-\varepsilon)$, since

$$\mathrm{card}\left(\bigcup_{v\in V} \left(\ell v + [\![1+\ell, 2\ell]\!]^2\right)\right) = \mathrm{card}\left(\bigcup_{v\in V} \left(\ell v + [\![0, \ell-1]\!]^2\right)\right) \geqslant \mathrm{card}(S).$$

Hence we have proved that $\hat{E}$ is a subset of the set of patterns w made of independent and disjoint words $(w_v)_{v \in V}$, with $w_v \in \mathcal{L}(\hat{X}, \ell)$, and of arbitrary symbols on $[\![0, n-2\ell]\!]^2 \backslash S$ of size at most εn^2. Using the trivial bound $\mathrm{card}(\mathcal{L}(\tilde{X}, \ell)) \leqslant \mathrm{card}(\tilde{\mathcal{A}})^\ell$, we have

$$\mathrm{card}(\hat{E}) \leqslant \left(\mathrm{card}(\tilde{\mathcal{A}})^\ell \cdot C^{\hat{X}}(\ell)\right)^{(n/\ell)^2} \cdot \mathrm{card}(\hat{\mathcal{A}})^{\varepsilon n^2}$$

and therefore

$$\frac{1}{n^2} \ln(\mathrm{card}(\hat{E})) \leqslant \frac{1}{\ell} \ln(\mathrm{card}(\tilde{\mathcal{A}})) + \frac{1}{\ell^2} \ln(C^{\hat{X}}(\ell)) + \varepsilon \ln(\mathrm{card}(\hat{\mathcal{A}})).$$

$\square$

Chapter 4

Analysis of the zero-temperature limit

Consider the full shift $\Sigma^2(\mathcal{A})$ and the finite set of forbidden patterns $\mathcal{F}$ for the subshift X. We denote by F the cylinder defined by

$$F := [\mathcal{F}]. \tag{4.1}$$

We consider

$$\begin{cases} \varphi : \Sigma^2(\mathcal{A}) & \to & \mathbb{R} \\ x & \mapsto & \varphi(x) = \mathbb{1}_F(x). \end{cases} \tag{4.2}$$

We consider $(\beta_k)_{k\geqslant 0}$ as in Definition 27. We denote by $\mathcal{G}(\beta_k\varphi) \subset \mathcal{M}_1(\Sigma^2(\mathcal{A}))$ the set of the equilibrium measures for the potential φ at inverse temperature β_k.

Since our potential φ has finite range, it is regular and as in Theorem 5 the set of equilibrium measures for $\beta\varphi$ is equal to the set of shift invariant Gibbs measures. Our main goal is to prove that for such a sequence $(\beta_k)_{k\geqslant 0}$ when $\beta_k \to +\infty$ any sequence of equilibrium measures μ_{β_k} does not converge when $k \to +\infty$.

An invariant measure that has support inside X gives zero mass to F. We quantify in the following lemma this estimate when the support of the measure is close to X, that is inside $\langle L_k \rangle$.

Lemma 16. Let be $k \geqslant 0$ and ν be an ergodic probability measure on $\Sigma^2(\mathcal{A})$ such that $\mathrm{supp}(\nu) \subseteq \langle L_k \rangle$. Then

$$\nu(F) \leqslant \frac{2D}{\ell_k}$$

Proof. We assume that $\mathrm{supp}(\nu) \subseteq \langle L_k \rangle$ where $L_k = \mathcal{L}(X, \ell_k)$ the language of size ℓ_k of the subshift $X = \Sigma^2(\mathcal{A}, \mathcal{F})$. By Birkhoff's ergodic theorem, for ν-almost every point x

$$\nu(F) = \lim_{n\to+\infty} \frac{\mathrm{card}(\{u \in \Lambda_n : \sigma^u(x) \in F\})}{\mathrm{card}(\Lambda_n)}.$$

We choose such a point $x \in \langle L_k \rangle$ and $s \in [\![1, \ell_k]\!]^2$ such that $\sigma^s(x)$ and all its translates $y_t = \sigma^{s+t\ell_k}(x)$, $t \in \mathbb{Z}^2$, satisfy $y_t|_{[\![1,\ell_k]\!]^2} \in L_k$. We take a sub-sequence $\tilde{\Lambda}_n$ of Λ_n with size a multiple of ℓ_k defined as

$$\tilde{\Lambda}_n := [\![-n\ell_k, n\ell_k - 1]\!].$$

Note that

$$\begin{aligned}\nu(F) &= \lim_{n\to+\infty} \frac{\operatorname{card}(\{u \in \tilde{\Lambda}_n - s : \sigma^u(x) \in F\})}{\operatorname{card}(\tilde{\Lambda}_n)}, \quad y = \sigma^s(x).\\ &= \lim_{n\to+\infty} \frac{\operatorname{card}(\{u \in \tilde{\Lambda}_n : \sigma^u(y) \in F\})}{\operatorname{card}(\tilde{\Lambda}_n)}.\end{aligned}$$

By definition of L_k as described in (3.23) we have that

$$x \in \langle L_k \rangle \Rightarrow \forall\, w \in \mathbb{Z}^2,\ \sigma^{s+w\ell_k}(x) = \sigma^{w\ell_k}(y)|_{[\![1,\ell_k]\!]^2} \in L_k$$

and

$$\forall v \in [\![0, \ell_k - D]\!]^2, \forall w \in \mathbb{Z}^2, \sigma^{v+w\ell_k}(y)|_{[\![1,D]\!]^2} \notin \mathcal{F}.$$

Thus for a fixed $w \in \mathbb{Z}^2$ we have that the number of possible $v \in [\![0, \ell_k - 1]\!]^2$ such that $\sigma(v + w\ell_k)(y) \in \mathcal{F}$ is bounded by

$$\operatorname{card}\left([\![0, \ell_k - 1]\!]^2 \backslash [\![0, \ell_k - D]\!]^2\right) \leqslant 2D\ell_k.$$

Therefore if we calculate this bound in the box $\tilde{\Lambda}_n$ we obtain that

$$\operatorname{card}(\{u \in \tilde{\Lambda}_n : \sigma^u(y) \in F\}) \leqslant (2n)^2 2D\ell_k.$$

Since $\operatorname{card}(\tilde{\Lambda}_n) = (2n)^2\ell_k^2$, we take the quotient on each side and take the limit with $n \to +\infty$ we obtain $\nu(F) \leqslant 2D/\ell_k$. □

We show in the following lemma that an equilibrium measure at low temperature have most of its support close to the largest compact invariant set on which the potential is zero. We quantify more precisely the speed of convergence of the measure on the set of locally admissible patterns as the size of the box goes to infinity.

Lemma 17. For every k and every equilibrium measure μ_{β_k},

$$\mu_{\beta_k}\left(\Sigma^2(\mathcal{A}) \backslash [M'_k]\right) \leqslant \frac{R'^2_k}{\beta_k} \ln(\operatorname{card}(\mathcal{A})) =: \varepsilon_k \tag{4.3}$$

where R'_k as defined in (3.28) and M'_k as defined in (3.30).

Proof. If $x \notin [M'_k]$, there exists $u \in [\![1, R'_k - D]\!]^2$ such that $\sigma^u(x) \in F$ and therefore

$\varphi(\sigma^u(x)) = 1$. Thus we obtain

$$\begin{aligned}\int \beta_k \varphi d\mu_{\beta_k} &= \int \beta_k \mathbb{1}_F(y) d\mu_{\beta_k}(y) \\ &\geqslant \beta_k R_k'^2 \cdot \mu_{\beta_k}\left(\Sigma \backslash [M_k']\right),\end{aligned}$$

and therefore

$$-\beta_k \int \varphi d\mu_{\beta_k} \leqslant -\beta_k R_k'^2 \cdot \mu_{\beta_k}\left(\Sigma \backslash [M_k']\right).$$

We have that $P(\beta_k\varphi) \geqslant 0$ and also by the variational principle we obtain

$$0 \leqslant P(\beta_k\varphi) = h(\mu_{\beta_k}) - \beta_k \int \varphi d\mu_{\beta_k} \leqslant h_{top}(\Sigma) - \beta_k R_k'^2 \cdot \mu_{\beta_k}\left(\Sigma \backslash [M_k']\right).$$

Since $h_{top}(\Sigma) \leqslant \ln(\text{card}(\mathcal{A}))$ we have

$$\mu_{\beta_k}\left(\Sigma \backslash [M_k']\right) \leqslant \frac{R_k'^2}{\beta_k} \ln(\text{card}(\mathcal{A})).$$

$\square$

The following lemma shows that the topological entropy of the extension depends on the frequency of the symbol 0 and not on the topological entropy of the base dynamics. By lifting patterns of the 1D subshift we can only expect an exponential growth proportional to the size of the boundary of a box. As the Aubrun-Sablik extension has zero entropy, we use, as in Chazottes-Hochman [11], the idea of duplicating the zero symbol in the vertical direction of $\mathbb{Z}^2$ in order to obtain an exponential growth proportional to the size of the volume of a box.

Lemma 18. For every $k \geqslant 0$,

$$\ln(2) f_k^B \leqslant h_{top}\big(\langle B_k \rangle\big)$$

A similar estimate holds for $\langle A_k \rangle$.

Proof. Since $\langle B_k \rangle$ is the concatenated subshift generated by the dictionary B_k as defined in (8), we have

$$h_{top}(\langle B_k \rangle) = \frac{1}{\ell_k^2} \ln(\text{card}(B_k)).$$

Let be $\tilde{w} \in \mathcal{L}(\langle \tilde{B}_k \rangle, \ell_k)$ such that $f_k(\tilde{w}) = f_k^B$. $\tilde{w}$ can be seen as a subword of a concatenation bb' of two words of $\tilde{B}_k$. By Lemma 5, bb' is a subword of some configuration $\tilde{x} \in \tilde{X}$.

By our construction there exists $\hat{x} \in \hat{X}$ such that $\tilde{x} = \hat{\Pi}(\hat{x})$ and $\tilde{w} = \hat{\Pi}(\hat{w})$ where

$\hat{w} = \hat{x}|_{[\![1,\ell_k]\!]^2} \in \hat{B}_k$. Thus we obtain

$$\operatorname{card}(B_k) \geqslant \operatorname{card}(\{w \in \mathcal{A}^{[\![1,\ell_k]\!]^2} : \Gamma(w) = \hat{w}\}) = 2^{\ell_k^2 f_k(\tilde{w})},$$

and therefore

$$h_{top}(\langle B_k \rangle) \geqslant \ln(2) f_k^B.$$

□

The following corollary is our first main estimate of the pressure. We bound from below the pressure by taking the pressure of the maximal entropy measure of the concatenated subshifts $\langle A_k \rangle$ or $\langle B_k \rangle$. We use here the large scale ℓ_k because β_k has already been defined using the small scale ℓ_k' (see Definition 27).

Corollary 1. For every $k \geqslant 1$,

$$P(\beta_k \varphi) \geqslant \max(f_k^A, f_k^B) \ln(2) - 2D\frac{\beta_k}{\ell_k}.$$

Proof. Follows from Lemma 18 and Lemma 16. □

Next, we will need to define some notations for standard definitions. Consider $\Sigma^2(\mathcal{A})$ and μ be a σ-invariant probability measure. The *canonical generating partition of* $\Sigma^2(\mathcal{A})$ is the partition

$$\mathcal{G} := \{[a]_0 : a \in \mathcal{A}\}. \tag{4.4}$$

We will denote the *base generating partition* as the partition

$$\mathcal{G}_* := \{G_0^*, G_1^*, G_2^*\} \quad \text{where} \quad G_{\tilde{a}}^* := \big\{x \in \Sigma^2(\mathcal{A}) : \pi(x(0)) = \tilde{a}\big\},\ \tilde{a} \in \tilde{\mathcal{A}}.$$

For each $k \in \mathbb{N}$, we will denote by $\mathcal{U}_k$ the partition

$$\mathcal{U}_k := \Big\{ [M_k'],\ \Sigma^2(\mathcal{A}) \backslash [M_k'] \Big\}. \tag{4.5}$$

For each $\varepsilon \in (0,1)$ we will define

$$H(\varepsilon) := -\varepsilon \ln(\varepsilon) - (1-\varepsilon)\ln(1-\varepsilon). \tag{4.6}$$

We introduce a notion of relative entropy which measures the dynamical entropy of a measure conditioned to be close to X.

Definition 32. The *relative dynamical entropy of size k* of an invariant probability measure μ is the quantity

$$h_{rel}(\mu) := \sup_{\mathcal{P}} \left\{ \lim_{n\to+\infty} \frac{1}{n^2} H\Big(\mathcal{P}^{[\![1,n]\!]^2} \mid \mathcal{G}_*^{[\![1,n]\!]^2} \bigvee \mathcal{U}_k^{[\![0,n-R_k]\!]^2}, \mu \Big) \right\}$$

where the supremum is taken over every finite partition $\mathcal{P}$.

The relative dynamical entropy is well defined for each k and we can use a version of the Kolmogov-Sinai Theorem (Theorem 4) for h_{rel}. This theorem gives us that the supremum of the definition is attained by a generating partition of the σ-algebra of $\Sigma^2(\mathcal{A})$.

For each $n \in \mathbb{N}$ consider the set $V_n \subset \Sigma$ defined as

$$V_n := \left\{x \in \Sigma^2(\mathcal{A}) : \pi(x_{(i,j_1)}) = \pi(x_{(i,j_2)}), \forall i, j_1, j_2 \in [\![1,n]\!]\right\},$$

that is, the set of configurations that are vertically aligned over the projection π on the alphabet $\tilde{\mathcal{A}}$ in the box $[\![1,n]\!]^2$. If we consider μ_{β_k} some equilibrium measure at inverse temperature β we have that

$$\lim_{n\to+\infty} \mu_{\beta_k}\left(\Sigma^2(\mathcal{A})\backslash V_n\right) = 0.$$

Note that

$$\begin{aligned} h_{rel}(\mu_{\beta_k}) &= \sup_{\mathcal{P}}\left\{\lim_{n\to+\infty}\frac{1}{n^2}H\left(\mathcal{P}^{[\![1,n]\!]^2} \mid \mathcal{G}_*^{[\![1,n]\!]^2}\bigvee\mathcal{U}_k^{[\![0,n-R_k]\!]^2}, \mu_{\beta_k}\right)\right\} \\ &= \lim_{n\to+\infty}\frac{1}{n^2}H\left(\mathcal{G}^{[\![1,n]\!]^2} \mid \mathcal{G}_*^{[\![1,n]\!]^2}\bigvee\mathcal{U}_k^{[\![0,n-R_k]\!]^2}, \mu_{\beta_k}\right) \\ &= \lim_{n\to+\infty}\left[\int_{V_n} H(\mathcal{G}^{[\![1,n]\!]^2}, \mu_x)d\mu_{\beta_k}(x) + \int_{\Sigma^2(\mathcal{A})\backslash V_n} H(\mathcal{G}^{[\![1,n]\!]^2}, \mu_x)d\mu_{\beta_k}(x)\right], \end{aligned}$$

where $(\mu_x)_{x\in\Sigma}$ is a family of conditional measures with respect to $\mathcal{G}_*^{[\![1,n]\!]^2}\bigvee\mathcal{U}_k^{[\![0,n-R_k]\!]^2}$.

Hence if we consider a configuration $x \in V_n$, the number of possible configurations in $\mathcal{G}^{[\![1,n]\!]^2}$ is bounded by $\mathrm{card}(\mathcal{A})^n$. Therefore

$$H(\mathcal{G}^{[\![1,n]\!]^2}, \mu_x) \leqslant n \cdot \ln(\mathrm{card}(\mathcal{A})),$$

and then $h_{rel}(\mu_{\beta_k}) < +\infty$.

The next lemma gives us an upper bound of the entropy of the equilibrium measure μ_{β_k} for each $k \in \mathbb{N}$.

Lemma 19. For every k and every equilibrium measure μ_{β_k}

$$h(\mu_{\beta_k}) \leqslant h_{rel}(\mu_{\beta_k}) + \left(\frac{8}{R'_k} + \varepsilon_k\right)\ln(\mathrm{card}(\tilde{\mathcal{A}})) + H(\varepsilon_k).$$

Proof. We take the supremum over all finite partitions of $\Sigma^2(\mathcal{A})$, so we can always consider that we are taking $\mathcal{P} \geq \mathcal{G}_*$ and $\mathcal{P} \geq \mathcal{U}_k$ and therefore $\mathcal{P} \geq \tilde{\mathcal{G}}\bigvee\mathcal{U}_k$. For consequence we obtain

$$\mathcal{P}^{[\![1,n]\!]^2} \succeq \mathcal{G}_*^{[\![1,n]\!]^2} \bigvee \mathcal{U}_k^{[\![0,n-R'_k]\!]^2}.$$

By the definition of relative entropy

$$\begin{aligned} H\Big(\mathcal{P}^{[\![1,n]\!]^2}, \mu_{\beta_k}\Big) &= H\Big(\mathcal{P}^{[\![1,n]\!]^2} \bigvee \mathcal{G}_*^{[\![1,n]\!]^2} \bigvee \mathcal{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\Big) \\ &= H\Big(\mathcal{P}^{[\![1,n]\!]^2} \mid \tilde{\mathcal{G}}^{[\![1,n]\!]^2} \bigvee \mathcal{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\Big) + \\ &\quad + H\Big(\mathcal{G}_*^{[\![1,n]\!]^2} \bigvee \mathcal{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\Big) \\ &= H\Big(\mathcal{P}^{[\![1,n]\!]^2} \mid \mathcal{G}_*^{[\![1,n]\!]^2} \bigvee \mathcal{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\Big) + \\ &\quad + H\Big(\mathcal{G}_*^{[\![1,n]\!]^2} \mid \mathcal{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\Big) + H\Big(\mathcal{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\Big). \end{aligned}$$

The first term of the right hand side is computed using the relative dynamical entropy (Definition 32). The third term is bounded from above using Lemma 17 (provided $\varepsilon_k < e^{-1}$),

$$\begin{aligned} H\big(\mathcal{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\big) &= \sum_{P \in \mathcal{U}_k^{[\![0,n-R'_k]\!]^2}} -\mu_{\beta_k}(P)\ln(\mu_{\beta_k}(P)) \\ &\leqslant n^2 H(\mathcal{U}_k, \mu_{\beta_k}) \\ &\leqslant n^2 H(\varepsilon_k). \end{aligned}$$

We now compute the term in the middle. We choose $\varepsilon'_k > \varepsilon_k$ and define

$$U_n := \Big\{x \in \Sigma^2(\mathcal{A}) : \mathrm{card}\,\big\{u \in [\![0, n-R'_k]\!]^2 : \sigma^u(x) \in [M'_k]\big\} \geqslant n^2(1-\varepsilon'_k)\Big\}.$$

By Birkhoff ergodic theorem we have that

$$\lim_{n\to+\infty} \mu_{\beta_k}(U_n) = 1.$$

Note that

$$\begin{aligned} H\left(\mathcal{G}_*^{[\![1,n]\!]^2} \mid \mathcal{U}_k^{[\![0,n-R_k]\!]^2}, \mu_{\beta_k}\right) &= \int H\Big(\mathcal{G}_*^{[\![1,n]\!]^2}, \mu_x\Big) d\mu_{\beta_k}(x) \\ &= \int_{U_n} H\Big(\mathcal{G}_*^{[\![1,n]\!]^2}, \mu_x\Big) d\mu_{\beta_k}(x) + \\ &\quad + \int_{\Sigma^2(\mathcal{A})\backslash U_n} H\Big(\mathcal{G}_*^{[\![1,n]\!]^2}, \mu_x\Big) d\mu_{\beta_k}(x) \\ &\leqslant \int_{U_n} H\Big(\mathcal{G}_*^{[\![1,n]\!]^2}, \mu_x\Big) d\mu_{\beta_k}(x) + \\ &\quad + n^2 \mu_{\beta_k}\left(\Sigma^2(\mathcal{A})\backslash U_n\right) \ln(\mathrm{card}(\mathcal{A})), \end{aligned}$$

and therefore

$$\limsup_{n\to+\infty} \frac{1}{n^2} H\left(\mathcal{G}_*^{[\![1,n]\!]^2} \mid \mathcal{U}_k^{[\![0,n-R'_k]\!]^2}, \mu_{\beta_k}\right) \leqslant \limsup_{n\to+\infty} \int_{U_n} \frac{1}{n^2} H\left(\mathcal{G}_*^{[\![1,n]\!]^2}, \mu_x\right) d\mu_{\beta_k}(x),$$

where $(\mu_x)_{x\in\Sigma}$ is the family of conditional measures with respect to $\mathcal{U}_k^{[\![0,n-R'_k]\!]}$.

Now consider a fixed $x \in U_n$. We compute the cardinality of elements in $\mathcal{G}_*^{[\![1,n]\!]^2}$ that are compatible with the constraint

$$\mathrm{card}\{u \in [\![0, n - R'_k]\!]^2 : \sigma^u(x) \in [M'_k]\} \geqslant n^2(1 - \varepsilon'_k).$$

Note that

$$\mathcal{G}_*^{[\![1,n]\!]^2} = \bigvee_{u\in[\![1,n]\!]^2} \sigma^{-u}\left(\mathcal{G}_*\right)$$

where $\mathcal{G}_* = \{G_0^*, G_1^*, G_2^*\}$ and here we refer to the elements of this partition as patterns defined in $\tilde{\mathcal{A}}^{[\![1,n]\!]^2}$ because there is a unique equivalence between these objects.

We denote by $I(x) = I \subset [\![0, n - R'_k]\!]^2$ such that

$$I := \left\{u \in [\![0, n - R'_k]\!]^2 : \sigma^u(x) \in [M'_k]\right\}.$$

Since $x \in U_n$, then

$$\frac{\mathrm{card}(I)}{n^2} \geqslant 1 - \varepsilon'_k.$$

Let $J \subseteq I$ be a maximal subset satisfying for every $u, v \in J$,

$$\|u - v\|_\infty \geqslant \frac{1}{2} R'_k.$$

For every $u \in J$, consider

$$I_u := \{v \in I : \|u - v\|_\infty < \frac{1}{2} R'_k\}$$

Then $I = \bigcup_{u\in J} I_u$. We first observe that the sets $\left(u + [\![1, \lceil R'_k/2\rceil]\!]^2\right)_{u\in J}$ are pairwise disjoint. Then

$$\mathrm{card}(J) \leqslant \frac{4n^2}{R_k'^2}.$$

We also observe that for every $v_1, v_2 \in I_u$, $\|v_1 - v_2\|_\infty < R'_k$ and

$$\left(v_1 + [\![1, R'_k]\!]^2\right) \bigcap \left(v_2 + [\![1, R'_k]\!]^2\right) \neq \varnothing.$$

For each $u \in I$ let be

$$K_u := \bigcup_{v\in I_u} \left(v + [\![1, R'_k]\!]^2\right) \subset [\![1, n]\!]^2.$$

For $v \in I_u$, we have that

$$x|_{v+[\![1,R'_k]\!]^2} \in [M'_k]$$

and therefore this pattern is locally $\mathcal{F}$-admissible and also satisfies the constraint that all the $\tilde{\mathcal{A}}$-symbols are vertically aligned in $v + [\![1, R_k]\!]^2$ and also in K_u.

The width of K_u is less than $2R'_k$, so the cardinality of possible patterns $p \in \tilde{\mathcal{A}}^{K_u}$ satisfying the constraint of vertically aligning of $\tilde{\mathcal{A}}$-symbols is bounded by $\mathrm{card}(\tilde{\mathcal{A}})^{2R'_k}$. The cardinality of possible patterns over the support $\bigcup_{u\in J} K_u$ is thus bounded by

$$\left(\mathrm{card}(\tilde{\mathcal{A}})^{2R'_k}\right)^{4n^2/R'^2_k} = \exp\left(\left[2R'_k \cdot \frac{4n^2}{R'^2_k}\right] \ln(\mathrm{card}(\tilde{\mathcal{A}}))\right) = \exp\left(\frac{8n^2}{R'_k} \ln(\mathrm{card}(\tilde{\mathcal{A}}))\right).$$

Since $\bigcup_{u\in J} K_u$ covers I, the cardinality of the set of possible patterns over the support $[\![1, n]\!]^2 \setminus \bigcup_{u\in J} K_u$ is bounded by $\mathrm{card}(\tilde{\mathcal{A}})^{n^2\varepsilon'_k}$. We have proved that, for every $x \in U_n$,

$$H\left(\mathcal{G}_*^{[\![1,n]\!]^2}, \mu_x\right) \leqslant \left(2R'_k \cdot \frac{4n^2}{R'^2_k} + n^2\varepsilon'_k\right) \ln(\mathrm{card}(\tilde{\mathcal{A}})).$$

We conclude by letting $n \to +\infty$ and $\varepsilon'_k \to \varepsilon_k$. □

The following lemma is the second main estimate on the pressure. We bound from above the pressure assuming that the generic patterns of the equilibrium measure exhibit a positive frequency (here 1/4) of the symbol 1. Since the potential is non-negative, it is enough to bound from above the pressure by the entropy of μ_{β_k}.

We denote as $\overline{\Pi} : \Sigma^2(\hat{\mathcal{A}}) \to \Sigma^2(\tilde{\mathcal{A}})$ the projection on the first coordinate. Using (3.20) we set

$$\Pi_* = \Gamma \circ \overline{\Pi} : \Sigma^2(\mathcal{A}) \to \Sigma^2(\tilde{\mathcal{A}}) \tag{4.7}$$

the projection on the bidimensional configurations over the alphabet $\tilde{\mathcal{A}}$.

Lemma 20. Let $k \geqslant 2$ be an integer and μ_{β_k} be any equilibrium measure. Then

1. $\mu_{\beta_k}([0]) \leqslant \dfrac{2}{N'_k} f^A_{k-1} + (1 - N^{-1}_{k-1})^{-1} f^B_{k-1} + \varepsilon_k$,
2. if k is even and $\mu_{\beta_k}([1]) > \frac{1}{4}$,

$$h_{rel}(\mu_{\beta_k}) \leqslant \left(\frac{2}{N'_k} f^A_{k-1} + (1 - N^{-1}_{k-1})^{-1}\left(\frac{3}{4} + \varepsilon_k\right) f^B_{k-1}\right) \ln(2) \\ + \frac{1}{\ell'_k} \ln(\mathrm{card}(\tilde{\mathcal{A}})) + \frac{1}{\ell'^2_k} \ln(C'_k) + \varepsilon_k \ln(2\mathrm{card}(\hat{\mathcal{A}})),$$

3. if k is odd and $\mu_{\beta_k}([2]) > \frac{1}{4}$, the previous estimate is valid with f^A_{k-1} and f^B_{k-1} permuted,

where for each $\tilde{a} \in \tilde{\mathcal{A}}$, $\mu_{\beta_k}([\tilde{a}])$ is the measure μ_{β_k} of the cylinder $\Pi_*^{-1}([\tilde{a}]_{(0,0)}) =: \Pi_*^{-1}[\tilde{a}] \subset \Sigma^2(\mathcal{A})$.

Proof. Let be $\Pi_* : \Sigma^2(\mathcal{A}) \to \Sigma^2(\tilde{\mathcal{A}})$ the projection over the first letter on the $\tilde{A}$-alphabet. By Birkhoff ergodic theorem and Lemma 17, for almost every $x \in \Sigma^2(\mathcal{A})$,

$$\lim_{n\to+\infty} \frac{1}{n^2}\mathrm{card}\big(\{u \in [\![0, n - R'_k]\!]^2 : \sigma^u(x) \in [M'_k]\}\big) = \mu_{\beta_k}([M'_k])$$

and

$$\lim_{n\to+\infty} \frac{1}{n^2}\mathrm{card}\big(\{u \in [\![1, n]\!]^2 : \pi(x(u)) = \tilde{a}\}\big) = \mu_{\beta_k}([\tilde{a}]), \quad \forall\, \tilde{a} \in \tilde{\mathcal{A}}.$$

Here we are denoting $\mu_{\beta_k}([\tilde{a}])$ for the measure μ_{β_k} of the cylinder $\Pi_*^{-1}[\tilde{a}]$, but we suppress the pre-image of the projection π to simplify our notation.

We choose $n > R'_k$. An element of the partition $\mathcal{G}_*^{[\![1,n]\!]^2} \bigvee \mathcal{U}^{[\![0,n-R'_k[\![^2}$ is of the form $G_p^* \cap U_S$ where $p \in \tilde{\mathcal{A}}^{[\![1,n]\!]^2}$ is a pattern and $S \subseteq [\![0, n - R'_k]\!]^2$ is a subset, that satisfies

$$U_S := \left\{x \in \Sigma^2(\mathcal{A}) : \forall\, u \in S,\ \sigma^u(x) \in [M'_k],\ \forall\, u \in [\![0, n - R'_k]\!]^2 \backslash S,\ \sigma^u(x) \notin [M'_k]\right\},$$

$$G_p^* := \left\{x \in \Sigma^2(\mathcal{A}) : (\Pi_*(x))\,|_{[\![1,n]\!]^2} = p\right\}.$$

We set $\varepsilon > \varepsilon_k$ and $\eta < \mu_{\beta_k}([0])$. By the Lemma 17 we have that $\mu_{\beta_k}\big(\Sigma^2(\mathcal{A})\backslash[M'_k]\big) \leqslant \varepsilon_k$ and then

$$\lim_{n\to+\infty} \mu_{\beta_k}\left(\bigcup_S \{U_S : \mathrm{card}(S) \geqslant n^2(1-\varepsilon)\}\right) = 1.$$

For n large enough, we choose $S \subseteq [\![0, n-R'_k]\!]^2$ such that $U_S \neq \varnothing$ and $\mathrm{card}(S) \geqslant n^2(1-\varepsilon)$. By definition of M'_k and T'_k, if $x \in U_S$, then for every $u \in S$, $\sigma^u(x)|_{[\![1,R'_k]\!]^2}$ is a locally admissible pattern with respect to $\mathcal{F}$ and

$$\sigma^{u+T'_k}(x)|_{[\![1,2\ell'_k]\!]^2} \in \mathcal{L}(X, 2\ell'_k).$$

Define for every $n > R'_k$ and every pattern $p \in \tilde{\mathcal{A}}^{[\![1,n]\!]^2}$ the set

$$K_n(p) := \{u \in [\![1, n]\!]^2 : p(u) = 0\}.$$

As we are considering $\mu_{\beta_k}([0]) > \eta$

$$\lim_{n\to+\infty} \mu_{\beta_k}\left(\bigcup_p \{G_p^* : \mathrm{card}(K_n(p)) > n^2 \cdot \eta\}\right) = 1.$$

We may choose p such that $U_S \cap G_p^* \neq \varnothing$ and $\mathrm{card}(K_p) > n^2\eta$. Using the objects as defined in (3.12), (3.13), (3.14) and (3.15), one obtains

$$T'_k + S \subseteq I(p, \ell'_k) \quad \text{and} \quad \tau'_k + I(p, \ell'_k) \subseteq J^A(p, \ell'_k) \bigsqcup J^B(p, \ell'_k) =: J^A \bigsqcup J^B,$$

therefore by our choice of S we obtain

$$n^2(1-\varepsilon) \leqslant \mathrm{card}(S) = \mathrm{card}(\tau'_k + T'_k + S) \leqslant \mathrm{card}\big(J^A \bigsqcup J^B\big) \leqslant n^2. \tag{4.8}$$

Besides that we have

$$n^2\eta \leqslant \mathrm{card}(K_n(p)) \leqslant \mathrm{card}(K^A \bigsqcup K^B) + n^2\varepsilon$$

and by the Lemma 13 we have

$$\mathrm{card}(K_n(p)) \leqslant \frac{2}{N'_k}\mathrm{card}(J^A)f^A_{k-1} + \big(1 - N^{-1}_{k-1}\big)^{-1}\mathrm{card}(J^B)f^B_{k-1} + n^2\varepsilon.$$

We divide each term by n^2 and take the limit with $n \to +\infty$, $\varepsilon \to \varepsilon_k$, and $\eta \to \mu_{\beta_k}([0])$. Thus we proved the first item of this lemma.

We now assume that k is even and $\mu_{\beta_k}([1]) > \frac{1}{4}$. We choose $p \in \tilde{\mathcal{A}}^{[\![1,n]\!]^2}$ such that $G^*_p \cap U_S \neq \varnothing$ and

$$\mathrm{card}\left(\big\{u \in [\![1,n]\!]^2 : p(u) = 1\big\}\right) > \frac{n^2}{4}. \tag{4.9}$$

Let be $x \in G^*_p \cap U_S$ and $(\mu_x)_{x\in\Sigma}$ be the family of conditional measures with respect to the partition $\mathcal{G}^{[\![1,n]\!]^2}_* \bigvee \mathcal{U}^{[\![0,n-R_k[\![^2}$. We use the trivial upper bound of the entropy, so

$$H(\mathcal{G}^{[\![1,n]\!]^2}, \mu_x) \leqslant \ln(\mathrm{card}(E_{p,S})) \tag{4.10}$$

where

$$E_{p,S} := \big\{w \in \mathcal{A}^{[\![1,n]\!]^2} : \pi(w) = p \text{ and } \ \forall u \in S,\ \sigma^{u+T'_k}(w)|_{[\![1,2\ell'_k]\!]^2} \in \mathcal{L}(X, 2\ell'_k)\big\}.$$

Also consider

$$\hat{E}_{p,S} := \Gamma(E_{p,S}).$$

Note that every word in $E_{p,S}$ is obtained from a word in $\hat{E}_{p,S}$ by duplicating twice a symbol 0 and by Lemma 15 we can conclude that

$$\ln(\mathrm{card}(E_{p,S})) \leqslant \ln(\mathrm{card}(\hat{E}_{p,S})) + \mathrm{card}(K_p)\ln(2) \text{ and}$$
$$\frac{1}{n^2}\ln(\mathrm{card}(\hat{E}_{p,S})) \leqslant \frac{1}{\ell'_k}\ln(\mathrm{card}(\tilde{\mathcal{A}})) + \frac{1}{{\ell'_k}^2}\ln(C'_k) + \varepsilon_k\ln(\mathrm{card}(\hat{\mathcal{A}})),$$

thus

$$\begin{aligned}\frac{1}{n^2}\ln(\mathrm{card}(E_{p,S})) \;&\leqslant\; \Big(\frac{2}{N'_k}\mathrm{card}(J^A)f^A_{k-1} + (1-N^{-1}_{k-1})^{-1}\mathrm{card}(J^B)f^B_{k-1} + n^2\varepsilon_k\Big)\frac{\ln(2)}{n^2}+ \\ &+\frac{1}{\ell'_k}\ln(\mathrm{card}(\tilde{\mathcal{A}})) + \frac{1}{{\ell'_k}^2}\ln(C'_k) + \varepsilon_k\ln(\mathrm{card}(\hat{\mathcal{A}})).\end{aligned} \tag{4.11}$$

The symbol 1 does not appear in $J^B = J^B(p,\ell'_k)$, so we can affirm

$$\big\{u\in[\![1,n]\!]^2 : p(u)=1\big\} \subset J^A \bigsqcup \Big([\![1,n]\!]^2\backslash(J^A\bigsqcup J^B)\Big).$$

Since we are assuming (4.9) and using (4.8) we obtain that

$$\mathrm{card}(J^A) \geqslant n^2\Big(\frac{1}{4}-\varepsilon_k\Big) \quad \text{and} \quad \mathrm{card}(J^B) \leqslant n^2\Big(\frac{3}{4}+\varepsilon_k\Big). \tag{4.12}$$

By replacing the upper bound for $\mathrm{card}(J^B)$ given in (4.12) and $\mathrm{card}(J^A)\leqslant n^2$ in (4.11) we obtain that

$$\begin{aligned}\frac{1}{n^2}\ln(\mathrm{card}(E_{p,S})) \;&\leqslant\; \left(\frac{2}{N'_k}f^A_{k-1} + (1-N^{-1}_{k-1})^{-1}\left(\frac{3}{4}+\varepsilon_k\right)f^B_{k-1}+\varepsilon_k\right)\ln(2)+ \\ &+\frac{1}{\ell'_k}\ln(\mathrm{card}(\tilde{\mathcal{A}})) + \frac{1}{{\ell'_k}^2}\ln(C'_k) + \varepsilon_k\ln(\mathrm{card}(\hat{\mathcal{A}})).\end{aligned} \tag{4.13}$$

By integrating with respect to μ_{β_k} in both sides and taking the limit when $n\to+\infty$ we obtain item 2 of this lemma. Item 3 has an analogous proof. ☐

Theorem 7. Let $X=\Sigma^2(\mathcal{A},\mathcal{F})$ be the bidimensional SFT described before, which is generated by the finite set of forbidden patterns $\mathcal{F}\subset\mathcal{A}^{[\![1,D]\!]^2}$ defined over the alphabet $\mathcal{A}$. Let F be the cylinder generated by the set $\mathcal{F}$ as described in (4.1) and $\varphi:\Sigma^2(\mathcal{A})\to\mathbb{R}$ be the locally constant potential defined as $\varphi=\mathbb{1}_F$. Let X_A, respectively X_B, be the compact sets of configurations in X that have only the symbol 1, respectively 2, in terms of the $\tilde{\mathcal{A}}$ alphabet, therefore, X_A and X_B are two disjoint invariant compact sets. Then there exists a sequence of inverse temperatures $(\beta_k)_{k\geqslant0}$ such that for every equilibrium measure μ_{β_k} associated to the potential $\beta_k\varphi$, the support of every accumulation point μ^A_∞ or μ^B_∞, of the subsequence $(\mu_{\beta_{2k+1}})_{k\geqslant0}$ or $(\mu_{\beta_{2k}})_{k\geqslant0}$, is included in X_A or X_B.

Proof. We consider $X=\Sigma^2(\mathcal{A},\mathcal{F})$ the SFT as described before, F as in (4.1) and $\varphi=\mathbb{1}_F$. We denote by μ_{β_k} an equilibrium measure at inverse temperature β_k. We will prove that as $\beta_k\to+\infty$ the sequence $(\mu_{\beta_k})_{k>0}$ does not converge.

Assume k is an even number and $\mu_{\beta_k}([1])>\frac{1}{4}$. Let μ^B_k be the measure of maximal entropy of the subshift $\langle L_k\rangle$. On the one hand, from Corollary 1 we have that

$$P(\beta_k\varphi)\geqslant h(\mu^B_k)-\int\beta_k\varphi\,d\mu^B_k\geqslant f^B_k\ln(2)-2D\frac{\beta_k}{\ell_k}.$$

By the item 3 of Definition 27 we have that

$$N_k \geqslant N'_k \cdot \frac{k\beta_k}{N'_k \rho^B_{k-1}} = \frac{k\beta_k}{\ell'_k f^B_{k-1}} \Rightarrow \frac{\beta_k}{\ell_k} \leqslant \frac{\beta_k}{\ell'_k} \leqslant \frac{1}{k} f^B_{k-1}.$$

Since k is even, $f^B_{k-1} = f^B_k$, one obtains,

$$f^B_k \ln(2) - 2D\frac{\beta_k}{\ell_k} \geqslant \frac{k\beta_k}{\ell'_k} - 2D\frac{\beta_k}{\ell_k} > 0 \Rightarrow 2D\frac{\beta_k}{\ell_k} \leqslant f^B_k \ln(2).$$

On the other hand

$$\begin{aligned} P(\beta_k\varphi) \leqslant \Big(\frac{2}{N'_k} f^A_{k-1} + (1 - N^{-1}_{k-1})^{-1}\Big(\frac{3}{4} + \varepsilon_k\Big) f^B_{k-1}\Big) \ln(2) \\ + \frac{1}{\ell'_k} \ln(\mathrm{card}(\tilde{\mathcal{A}})) + \frac{1}{{\ell'_k}^2} \ln(C(\hat{X}, \ell'_k)) + \varepsilon_k \ln(2\mathrm{card}(\hat{\mathcal{A}})) \\ + \Big(\frac{8}{R(\hat{X}, \ell'_k)} + \varepsilon_k\Big) \ln(\mathrm{card}(\tilde{\mathcal{A}})) + H(\varepsilon_k). \end{aligned}$$

We have that

$$\varepsilon_k \ll f^B_{k-1} \quad \text{and} \quad H(\varepsilon_k) \ll f^B_{k-1}.$$

Indeed, from item 2 of Lemma 14 shows that there exist constants Ξ, ξ such that

$$\forall\, k \geqslant 1,\; R'_k \leqslant \Xi 2^{\xi \ell'_k}.$$

Recalling the definition of $\varepsilon_k = \frac{{R'_k}^2}{\beta_k} \ln(\mathrm{card}(\mathcal{A}))$ given in (4.3) and using item 2 of Definition 27, one gets,

$$\frac{\varepsilon_k}{(f^B_{k-1})^2} \leqslant \frac{\varepsilon_k \beta_k}{2^{k\ell'_k}} = \frac{{R'_k}^2 \ln(\mathrm{card}(\mathcal{A}))}{2^{k\ell'_k}} \leqslant \Xi^2 \ln(\mathrm{card}(\mathcal{A})) 2^{(2\xi - k)\ell'_k} \ll 1,$$

and therefore

$$\begin{aligned} \frac{\varepsilon_k}{f^B_{k-1}} \leqslant \frac{\varepsilon_k}{(f^B_{k-1})^2} \;\Rightarrow\; \varepsilon_k \ll f^B_{k-1} \text{ and} \\ H(\varepsilon_k) \leqslant 2\varepsilon_k \ln\Big(\frac{1}{\varepsilon_k}\Big) \ll \sqrt{\varepsilon_k} \ll f^B_{k-1}. \end{aligned}$$

As $\ell_k f^B_k$ counts the number of 0's in the word b_k and at each step of the construction the number is at least multiplied by 2, we have $\ell_{k-1} f^B_{k-1} \geqslant 2^{k-1}$,

$$\frac{1}{\ell'_k} = \frac{1}{N'_k \ell_{k-1}} \ll f^B_{k-1}, \quad R'_k \geqslant \ell'_k, \quad \frac{1}{R(\hat{X}, \ell'_k)} \ll f^B_{k-1}.$$

Item 1 of Lemma 14 implies

$$\frac{1}{{\ell'_k}^2}\ln(C'_k) \ll f^B_{k-1}.$$

Item 1 of Definition27 shows,

$$\frac{f^A_{k-1}}{N'_k} \leqslant \frac{f^B_{k-1}}{k}, \quad \frac{f^A_{k-1}}{N'_k} \ll f^B_{k-1}.$$

We proved that $P(\beta_k \phi)$ is bounded from below by a quantity equivalent to $f^B_k \ln(2)$ and bounded from above by a quantity equivalent to $\frac{3}{4} f^B_k \ln(2)$. We obtain a contradiction. We have proved that $\mu_{\beta_k}([1]) \leqslant \frac{1}{4}$ for every even k and every equilibrium measure μ_{β_k}. Similarly $\mu_{\beta_k}([2]) \leqslant \frac{1}{4}$ for every odd k and every equilibrium measure μ_{β_k}. As

$$\mu_{\beta_k}([0]) \leqslant \frac{2}{N'_k} f^A_{k-1} + (1 - N^{-1}_{k-1})^{-1} f^B_{k-1} + (f^B_{k-1})^2 \frac{R'_k \ln(\mathrm{card}(\mathcal{A}))}{\exp(k\ell'_k)},$$

we have proved

$$\liminf_{k\to+\infty} \inf_{\mu} \big\{ \mu([2]) : \mu \text{ is an equilibrium measure at } \beta_{2k} \big\} \geqslant \frac{3}{4},$$
$$\liminf_{k\to+\infty} \inf_{\mu} \big\{ \mu([1]) : \mu \text{ is an equilibrium measure at } \beta_{2k+1} \big\} \geqslant \frac{3}{4},$$

and therefore $(\mu_{\beta_k})_{k\geqslant 0}$ does not converge. □

Appendix A

Computability results

We thank Sebastian Barbieri for his help to compute the upper bounds for the relative complexity and for the reconstruction function. Sebastian stimulated us to prove that we can enumerate $\tilde{\mathcal{F}}$ in an increasing way and with a execution time that is at most exponential.

First we prove the upper bound for the relative complexity function given by Proposition 6.

Proof of Proposition 6. Let us denote by $C_n(\texttt{Layer}_k(\hat{X}))$ the complexity of the projection to the k-th layer. and by $C_n(\texttt{Layer}_k(\hat{X})|\texttt{Layer}_j(\hat{X}))$ the complexity of the projection to the k-th layer given that there is a fixed pattern on the j-th layer. Clearly we have that

$$C^{\hat{X}}(n) \leqslant C_n(\texttt{Layer}_1(\hat{X})) \cdot C_n(\texttt{Layer}_2(\hat{X})) \cdot C_n(\texttt{Layer}_3(\hat{X})|\texttt{Layer}_2(\hat{X})) \cdot \\ \cdot C_n(\texttt{Layer}_4(\hat{X})|\texttt{Layer}_2(\hat{X})).$$

- `Layer` 1: As this layer is given by all $x \in \tilde{\mathcal{A}}^{\mathbb{Z}^2}$ so that $x_u = x_{u+(0,1)}$ for every $u \in \mathbb{Z}^2$, a trivial upper bound for the complexity is

  $$C_n(\texttt{Layer}_1(\hat{X})) = \mathcal{O}(|\tilde{\mathcal{A}}|^n).$$

  In fact, as in the end the only configurations which are allowed are those whose horizontal projection lies in the effective subshift $\mathbb{Z}$, a better bound is given by $C_n(\texttt{Layer}_1(\hat{X})) = \mathcal{O}(\exp(n\, h_{\text{top}}(\hat{X})))$. For simplicity, we shall just keep the trivial bound.

- `Layer` 2: The complexity of every substitutive subshift in $\mathbb{Z}^2$ is $\mathcal{O}(n^2)$. To see this, suppose that the substitution sends symbols of some alphabet $\mathcal{A}_2$ to $n_1 \times n_2$ arrays of symbols. By definition, every pattern of size n occurs in a power of the substitution. If k is such that $\min\{n_1, n_2\}^{k-1} \leqslant n \leqslant \min\{n_1, n_2\}^k$, then necessarily any pattern of size n occurs in the concatenation of at most 4 k-powers of the substitution. There are $|\mathcal{A}_2|^4$ choices for the k-powers and at most $(\max\{n_1, n_2\}^k)^2 \leqslant (n \max\{n_1, n_2\})^2$ choices for the position of the pattern. It follows that there are at most $(|\mathcal{A}_2|^4 \max\{n_1, n_2\}^2)n^2 = \mathcal{O}(n^2)$ patterns of size n. We obtain,

$$C_n(\texttt{Layer}_2(\hat{X})) = \mathcal{O}(n^2)$$

- $\texttt{Layer}$ 3: It can be checked directly from the Aubrun-Sablik construction that the symbols on the third layer satisfy the following property: if the symbols on the substitution layer are fixed, then for every $u \in \mathbb{Z}^2$ the symbol at position u is uniquely determined by the symbols at positions $u-(0,1), u-(1,1)$ and $u-(-1,1)$. In consequence, it follows that knowing the symbols at positions in the "U shaped region"

$$U = (\{0\} \times [\![1, n-1]\!]) \cup ([\![0, n-1]\!] \times \{0\}) \cup (\{n-1\} \times [\![1, n-1]\!])$$

  completely determines the pattern. Therefore, if this layer has an alphabet $\mathcal{A}_3$, we have

$$C_n(\texttt{Layer}_3(\hat{X})|\texttt{Layer}_2(\hat{X})) \leqslant |\mathcal{A}_3|^{3n-2} \leqslant \mathcal{O}(K_1^n),$$

  for some positive integer K_1.

- $\texttt{Layer}$ 4: $\mathcal{M}_{\texttt{Search}}$ The same argument for Layer 3 holds for Layer 4. Therefore, if the alphabet of layer 4 is $\mathcal{A}_4$ we have that for some positive integer K_2,

$$C_n(\texttt{Layer}_4(\hat{X})|\texttt{Layer}_2(\hat{X})) \leqslant |\mathcal{A}_4|^{3n-2} \leqslant \mathcal{O}(K_2^n).$$

Putting the previous bounds together, we conclude that there is some constant $K > 0$ such that

$$C^{\hat{X}}(n) = \mathcal{O}(n^2 K^n).$$

□

Corollary 2. Let $\hat{X}$ be the $\mathbb{Z}^2$-SFT in the Aubrun-Sablik construction. There is a constant $K_C > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log(C_n(X)) \leqslant K_C.$$

Now we will work on the upper bound for the reconstruction function. We fix a Turing machine $\mathcal{M}$ that enumerates $\tilde{\mathcal{F}}$ see below the set of forbidden words that define $\tilde{X} = \Sigma(\tilde{A}, \tilde{\mathcal{F}})$. In general, the reconstruction function $R^{\hat{X}}$ as defined in (3.28) of the Aubrun-Sablik construction is not computable, but in our construction we can obtain the properties as stated in Proposition 4 that we prove below.

Proof of Proposition 4. If the integer $n \geqslant 1$ is such that $p = N_k$, then $\tilde{\mathcal{F}}'(n) = \tilde{\mathcal{F}}(n)$. We will consider now the case where the integer $n \geqslant 1$ is such that $p < N_k$. We have obviously $\tilde{\mathcal{F}}(n) \subseteq \tilde{\mathcal{F}}'(n)$. If we assume that k is even, from Notation 1 we have that

$$a_k = a_{k-1}(1^{\ell_{k-1}})^{N_k-2}a_{k-1} \quad \text{and} \quad b_k = (b_{k-1})^{N_k}.$$

We set

$$\overleftarrow{a_k}(1) = \overrightarrow{a_k}(1) = a_{k-1},$$

$$\overleftarrow{b_k}(1) = \overrightarrow{b_k}(1) = b_{k-1},$$

$$\overleftrightarrow{1_k}(1) = 1_{k-1} := 1^{\ell_{k-1}} \text{ and } \overleftrightarrow{2_k}(1) = 2_{k-1} := 2^{\ell_{k-1}}.$$

Then we define by induction if $2 \leqslant p < N_k$ then

$$\overleftarrow{a_k}(p) = \overleftarrow{a_k}(p-1)1_{k-1} = a_{k-1}(1_{k-1})^{p-1}$$

and

$$\overrightarrow{a_k}(p) = 1_{k-1}\overrightarrow{a_k}(p-1) = (1_{k-1})^{p-1}a_{k-1},$$

else $\overleftarrow{a_k}(N_k) = \overrightarrow{a_k}(N_k) = a_k$. We also define

$$\overleftarrow{b_k}(p) = \overleftarrow{b_k}(p-1)b_{k-1} = (b_{k-1})^p,$$

$$\overrightarrow{b_k}(p) = b_{k-1}\overrightarrow{b_k}(p-1) = (b_{k-1})^p,$$

$$\overleftrightarrow{1_k}(p) = \overleftrightarrow{1_k}(p-1)1_{k-1} = (1_{k-1})^p$$

and

$$\overleftrightarrow{2_k}(p) = \overleftrightarrow{2_k}(p-1)1_{k-1} = (2_{k-1})^p.$$

If w has length less than $p\ell_{k-1}$ and is a sub-word of some w_1w_2, say $w_1 = a_k$ and $w_2 = b_k$, by dragging w from the left end point of w_1w_2 to the right end point of w_1w_2, the word w appears successively as a sub-word of $\overleftarrow{a_k}(p+1)$, $\overleftrightarrow{1_k}(p+1)$, $\overrightarrow{a_k}(p+1)$, $\overrightarrow{a_k}(p+1)\overleftarrow{b_k}(p+1)$, $\overleftarrow{b_k}(p+1)$. A similar reasoning is also true for $w_1 = b_k$ and $w_2 = a_k$. We have proved $\tilde{\mathcal{F}}(n) = \tilde{\mathcal{F}}'(n)$.

We have also proved that $\tilde{X} = \Sigma^1(\tilde{\mathcal{A}}, \tilde{F})$, because we have proved that it is enough to list all the forbidden words of length n and for that it is sufficient to search in the concatenation of subwords of length $(p+1)\cdot\ell_{k-1}$ as described before. Thus

$$\Sigma^1(\tilde{\mathcal{A}}, \overline{\mathcal{F}}) = \Sigma^1(\tilde{\mathcal{A}}, \tilde{\mathcal{F}}).$$

To compute the time to enumerate successively the words of $\tilde{\mathcal{F}}(n)$ when $\ell_{k-1} < n \leqslant \ell_k$, we produce an algorithm given in Table A.1. The time to read/write on the tapes, to update the words $(\overleftarrow{a_k}(p), \overrightarrow{a_k}(p), \overleftarrow{b_k}(p), \overrightarrow{b_k}(p), \overleftrightarrow{1_k}(p), \overleftrightarrow{2_k}(p))$ by adding a word of length ℓ_{k-1}, to concatenate two words $\overrightarrow{w_1}\overleftarrow{w_2}$ from that list, and to check that a given word w of length n is a sub-word of $\overrightarrow{w_1}\overleftarrow{w_2}$ is polynomial in n. Therefore, the time to enumerate every word up to length n in an alphabet $\tilde{\mathcal{A}}$ is bounded by $P(n)|\tilde{\mathcal{A}}|^n$ where $P(n)$ is a polynomial. $\square$

Denote by $R^{\tilde{X}} \colon \mathbb{N} \to \mathbb{N}$ the reconstruction functions of $\tilde{X}$ given $\tilde{\mathcal{F}}$. From Lemma 5 we know there exists a constant $C_1 > 0$ such that $R^{\tilde{X}}(n) \leqslant C_1 n$.

For $n \in \mathbb{N}$, let $N = 2n+1$ be the length of the sides of the square $B_n := [\![-n, n]\!]^2 \subset \mathbb{Z}^2$, and let $k \in \mathbb{N}$ such that $4^{k-1} < N \leqslant 4^k$.

As before, let $\hat{X} = \Sigma(\hat{\mathcal{A}}, \hat{\mathcal{F}})$ be the $\mathbb{Z}^2$-SFT in the Aubrun-Sablik construction associated to $\tilde{X}$ and the Turing machine $\mathcal{M}$. Now we will give estimates on the reconstruction function $R^{\hat{X}} \colon \mathbb{N} \to \mathbb{N}$ of $\hat{X}$ given $\hat{\mathcal{F}}$. Of course, a formal proof of these estimates would require a restatement of the construction of Aubrun-Sablik with all its details, which is out of the scope of this thesis. Instead, we shall argue that the bounds we give suffice, making reference to the properties of the Aubrun-Sablik construction.

A description of $\hat{\mathcal{F}}$ is given in [2] in an (almost) explicit manner for all layers except the substitution layer. For the substitution layer, a description of the forbidden patterns can be extracted in an explicit way from the article of Mozes [30].

The behavior of layers 2,3 and 4 in the Aubrun-Sablik construction is mostly independent of layer 1, except for the detection of forbidden patterns which leads to the forbidden halting state of the machine in the third layer. Because of that reason the analysis of the reconstruction function $R^{\hat{X}}$ can be split into two parts:

1. **Structural:** Assuming that the contents of the first layer are globally admissible (the configuration in the first layer is an extension of a configuration from $\tilde{X}$), we give a bound that ensures that the contents of layers $2, 3$ and 4 are globally admissible, that is:

   - The contents of layer 2 correspond to a globally admissible pattern in the substitutive subshift and the clock.
   - The contents of layer 3 and 4 correspond to valid space-time diagrams of Turing machines that correctly align with the clocks.

2. **Recursive:** A bound that ensures that the contents of the first layer are globally admissible. This bound will of course depend upon $R^{\tilde{X}}$ and τ.

Finally, we are able to prove the upper bound for the reconstruction function given by Proposition 5.

Proof of Proposition 5. Let us begin with the structural part, as it is simpler and does not depend upon $\tilde{X}$. Let p be a pattern with support B_n and assume that the first layer of p is thus globally admissible.

A program enumerating the set of forbidden words

$\#$ Initialize $(\ell_0, \beta_0, \rho_0^A, \rho_0^B)$
$(\ell_-, \beta_-, \rho_-^A, \rho_-^B) \leftarrow (2, 0, 1, 1)$
$\#$ Allocate and Initialize 4 tapes $(a_k, b_k, 1_k, 2_k)$
$(a_-, b_-, 1_-, 2_-) \leftarrow (01, 02, 11, 22)$
$\#$ Allocate and Initialize 6 tapes $(\overleftarrow{a_k}(1), \overrightarrow{a_k}(1), \overleftarrow{b_k}(1), \overrightarrow{b_k}(1), \overleftrightarrow{1_k}(1), \overleftrightarrow{2_k}(1))$
$(\overleftarrow{a_+}, \overrightarrow{a_+}, \overleftarrow{b_+}, \overrightarrow{b_+}, \overleftrightarrow{1_+}, \overleftrightarrow{2_+}) \leftarrow (a_-, a_-, b_-, b_-, 1_-, 2_-)$
$\#$ Compute recursively the next length ℓ_1
$(\ell_+, \beta_+, \rho_+^A, \rho_+^B) \leftarrow S(\ell_-, \beta_-, \rho_-^A, \rho_-^B)$
$N_+ \leftarrow \ell_+/\ell_-$; $parity \leftarrow even$; $n \leftarrow 3$; $p \leftarrow 2$
$\#$ Allocate and Initialize an intermediate tape recording a possibly forbidden word
$w \leftarrow \varnothing$
while $(n \geqslant 1)$
  if $(n = \ell_+ + 1)$ then
    $\#$ Remember the previous $(\ell_{k-1}, \beta_{k-1}, \rho_{k-1}^A, \rho_{k-1}^B)$ and update the new one
    $(\ell_-, \beta_-, \rho_-^A, \rho_-^B) \leftarrow (\ell_+, \beta_+, \rho_+^A, \rho_+^B)$; $(\ell_+, \beta_+, \rho_+^A, \rho_+^B) \leftarrow S(\ell_-, \beta_-, \rho_-^A, \rho_-^B)$
    $\#$ Remember $(a_{k-1}, b_{k-1}, 1_{k-1}, 2_{k-1})$
    $(a_-, b_-, 1_-, 2_-) \leftarrow (\overleftarrow{a_+}, \overleftarrow{b_+}, \overleftrightarrow{1_+}, \overleftrightarrow{2_+})$
    $N_+ \leftarrow \ell_+/\ell_-$; $parity \leftarrow$ Permute($parity$) ; $p \leftarrow 2$
  end if
  if $(n = (p-1)\ell_- + 1)$ then
    Update $(\overleftarrow{a_+}, \overrightarrow{a_+}, \overleftrightarrow{1_+}, \overleftarrow{b_+}, \overrightarrow{b_+}, \overleftrightarrow{2_+})$ according to parity and the two particular
      cases $p = 2$ or $p = N_+$ by concatenating words from $(a_-, b_-, 1_-, 2_-)$
    $\#$ Build the set of words obtained by concatenating two words of length ℓ_{k-1}
    $W \leftarrow \{\overrightarrow{a_+}\overleftarrow{b_+},\ \overrightarrow{a_+}\overleftrightarrow{2},\ \overleftrightarrow{1}\overleftarrow{b_+},\ \overleftrightarrow{1}\overleftrightarrow{2}, \overrightarrow{b_+}\overleftarrow{a_+},\ \overrightarrow{b_+}\overleftrightarrow{1},\ \overleftrightarrow{2}\overleftarrow{a_+},\ \overleftrightarrow{2}\overleftrightarrow{1}\}$
    $p \leftarrow p + 1$
  end if
  for $(m = 0, 3^n$ excluded$)$
    $w \leftarrow$ write m in base 3 with n letters in $\{0, 1, 2\}$
    $is\_forbidden \leftarrow true$
    for $(w_1 w_2 \in W)$
      if (w is a sub-word of $w_1 w_2$) then $is\_forbidden \leftarrow false$
    end for
    if ($is\_forbidden$) then Print the word w
  end for
  $n \leftarrow n + 1$
end while

Table A.1: Algorithm that enumerates $\tilde{\mathcal{F}}$.

From Mozes's construction of SFT extensions for substitutions [30] it can be checked that any locally admissible pattern of support B_n of Mozes's SFT extension of a primitive substitution (The Aubrun-Sablik substitution is primitive) is automatically globally admissible. Let us take a support large enough such that the second layer of p occurs within four $4^k \times 2^k$ macrotiles of the substitution in any locally admissible pattern of that support.

Next, a clock runs on every strip of the Aubrun-Sablik construction. By the previous argument, the largest zone which intersects p in more than one position is of level at most k. Therefore its largest computation strip has horizontal length 2^k. In order to ensure that the clock starts on a correct configuration on every strip contained in p, we need to witness this pattern inside a locally admissible pattern which stacks $2^{2^k} + 2$ macrotiles of level k vertically. Therefore, the pattern p must occur inside four locally admissible patterns of length $4^k \times 2^k(2^{2^k} + 2)$. This ensures that the clocks in p are globally admissible.

Finally, if every clock occurring in p starts somewhere, then the contents of the third layer are automatically correct, as they are determined by clock every time it restarts. To check that the fourth layer is correct, we just need extend the horizontal length of our pattern twice, so that the responsibility zone of the largest strip is contained in it.

By the previous arguments, it would suffice to witness p inside a locally admissible pattern which contains in its center a 4×2 array of macrotiles of size $4^k \times 2^k(2^{2^k} + 2)$. As $4^{k-1} < N \leqslant 4^k$, there is a constant $C_0 > 0$ such that an estimate for this part of the reconstruction function can be written as

$$R^{\hat{X}}_{\text{Struct}}(n) = \mathcal{O}(\sqrt{n} C_0^{\sqrt{n}}).$$

Let us now deal with the recursive part. We need to find a bound such that the word of length N occurring in the first layer of p is globally admissible. By definition of $R^{\tilde{X}}$, it suffices to have p inside a pattern with support $B_{R^{\tilde{X}}(N)}$ and check that the first layer is locally admissible with respect to $\tilde{\mathcal{F}}$. In other words, we need to have the Turing machines check all forbidden words of length $R^{\tilde{X}}(N)$ in this pattern. Luckily, the number of steps in order to do this is already computed in Aubrun and Sablik's article. After Fact 4.3 of [2] they show that, if $p_0, p_1, \ldots, p_r$ are the first $r+1$ patterns enumerated by $\mathcal{M}$, then the number of steps $S(p_0, \ldots, p_r)$ needed in a computation zone to completely check whether a pattern from $\{p_0, \ldots, p_r\}$ occurs in its responsibility zone of level m satisfies the bound,

$$S(p_0, \ldots, p_r) \leqslant T(p_0, \ldots, p_r) + (r+1)\max(|p_0|, \ldots, |p_r|) m^2 2^{3m+1},$$

where $T(p_0, \ldots, p_r)$ is the number of steps needed by $\mathcal{M}$ to enumerate the patterns $p_0, p_1, \ldots, p_r$.

Specifically in our construction, we may rewrite their formula so that the number $S(R^{\tilde{X}}(N))$ of steps needed to check that all forbidden patterns of length at most $R^{\tilde{X}}(N)$

in a responsibility zone of level m satisfies the bound

$$\begin{aligned} S(R^{\tilde{X}}(N)) &\leqslant \tau(R^{\tilde{X}}(N)) + |\tilde{\mathcal{A}}|^{R^{\tilde{X}}(N)+1} R^{\tilde{X}}(N) k^2 2^{3k+1} \\ &\leqslant P(n)|\tilde{\mathcal{A}}|^N + |\tilde{\mathcal{A}}|^{C_1 N+1} C_1 N m^2 2^{3m+1} \end{aligned}$$

Simplifying the above bound, it follows that there exists constants $C_2, C_3 > 0$ such that

$$S(R^{\tilde{X}}(N)) \leqslant C_2 m^2 2^{3m+C_3 N}.$$

As N is constant, it follows that there is a smallest $\bar{m} = \bar{m}(N) \in \mathbb{N}$ such that $2^{\bar{m}} \geqslant C_4 N$ (so that the tape on the computation zone of level $\bar{m}$ can hold words of size $R_Z(N)$) and such that

$$C_2 \bar{m}^2 2^{3\bar{m}+C_3 N} \leqslant 2^{2^{\bar{m}}} + 2,$$

so that the number $2^{2^{\bar{m}}} + 2$ of computation steps in the zone of level $\bar{m}$ is enough to check all the words of size $R^{\tilde{X}}(N)$. It follows that a bound for the recursive part of $R^{\hat{X}}$ is given by

$$R^{\hat{X}}_{\text{recursive}}(n) = \mathcal{O}(2^{\bar{m}+2^{\bar{m}(N)}}).$$

In order to turn this into an explicit asymptotic expression we need to find a suitable bound for $\bar{m}(N)$. Notice that if $m \geqslant 6$ we simultaneously have that $m^2 \leqslant 2^m$ and $4m \leqslant 2^{m-1}$. We may then write for $m \geqslant 6$,

$$C_2 m^2 2^{3m} e^{C_3 N} \leqslant C_2 2^{4m+C_3 N} \leqslant C_2 2^{C_3 N} 2^{2^{m-1}}.$$

Therefore, it suffices to find $\bar{m} = \bar{m}(N)$ such that

$$C_2 2^{C_3 N} \leqslant 2^{2^{\bar{m}-1}}.$$

From here, it follows that there is a constant $C_5 > 0$ such that any value of $\bar{m}$ satisfying

$$\bar{m} \geqslant C_5 + \log_2(N),$$

satisfies the above bound. We get that

$$R^{\hat{X}}_{\text{recursive}}(n) = \mathcal{O}(N 2^{C_5 N}) = n 4^{C_5 n}.$$

Finally, putting together the structural and recursive asymptotics, we obtain that there is a constant $K > 0$ such that

$$R^{\hat{X}}(n) = \mathcal{O}(\max\{\sqrt{n} C_0^{\sqrt{n}}, \mathcal{O}(n 4^{C_5 n})\}) = \mathcal{O}(n K^n).$$

□

Corollary 3. Under the same hypotheses as in Proposition 5, there is a constant $K > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log(R_{\hat{X}}(n)) \leqslant K.$$

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