Two abstract formulations to Electrical Impedance Tomography

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Resumo

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A Tomografia por Impedância Elétrica (EIT) é uma modalidade de tomografia que busca recuperar a distribuição da condutividade dentro de um corpo a partir de medições elétricas realizadas na superfície do corpo. Nesta tese, são apresentadas duas contribuições para o EIT. A primeira é um marco teórico que unifica a formulação e análise dos modelos para a EIT. Lá, as formulações fracas dos modelos e os mapas de corrente-voltagem são generalizados, e propriedades duais que revelam a ligação entre as formulações em termos de potenciais elétricos e campos de corrente são encontradas. A segunda é a regularização de uma formulação *all-at-once* do problema inverso na EIT. Três problemas regularizados são formulados com base nas regularizações clássicas de Tikhonov, Ivanov e Morozov, e a existência, estabilidade e convergência de soluções regularizadas são provadas. Em ambos os desenvolvimentos, vários modelos para a EIT são admitidos e exemplos numéricos para ilustrar nossos resultados teóricos são fornecidos.

Palavras-chave: tomografia por impedância elétrica, dualidade, abordagem all-at-once, modelo contínuo, modelo de electrodo completo.

Abstract

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Electrical Impedance Tomography (EIT) is an imaging modality that seeks to recover the conductivity distribution inside a physical body from electrical measurements taken on the body surface. In this thesis, two contributions to EIT are presented. The first is a theoretical framework that unifies the formulation and analysis of EIT models. There, the weak formulations of the EIT models and the current-voltage maps are generalized, and dual properties that reveal the link between the formulations in terms of electric potentials and current fields are found. The second is the regularization of an all-at-once formulation of the EIT inverse problem. Three regularized problems are formulated based on the classic Tikhonov, Ivanov, and Morozov regularizations, and the existence, stability, and convergence of regularized solutions are proved. In both developments, several EIT models are admitted and numerical examples to illustrate our theoretical results are provided.

Keywords: electrical impedance tomography, duality, all-at-once approach, continuum model, complete electrode model.

Contents

1 Introduction

Electrical Impedance Tomography (EIT) is a technique to recover images of the internal conductivity of a body based on electrical measurements at electrodes attached on the body surface. The Argentinian engineer and mathematician Alberto Pedro Calderon wrote the first mathematical formulation of this problem in his paper *On an inverse boundary value problem* (1980) [22], which was published by the Brazilian Mathematical Society (SBM). To perform EIT, current is sent through electrodes placed on the surface of the body and the resulting voltage on these same electrodes is measured. Then, EIT aims to recover the conductivity distribution from this knowledge of current and voltage. An alternative approach is to measure the current caused by voltage applied to electrodes. Due to its potential advantages over other imaging techniques, e.g. low cost, rapid response, high contrast, non-intrusiveness, portability, and absence of ionizing radiation, EIT has applications in fields such as medical imaging [15], geophysics [60], industrial process tomography [90], and non-destructive testing [68]. For a recent account of the applications we refer the reader to [2].

A number of EIT models have been proposed for modeling the electric potential induced inside a conducting body by boundary current injection (or voltage excitation). In all of them, the electric potential u in the body Ω is governed by the elliptic partial differential equation

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega,$$

where σ is the internal conductivity of Ω. In the general case, σ is replaced by the complex admittivity $\sigma + i\omega\varepsilon$, where ε is the permittivity and ω is the frequency. This equation can be obtained from Maxwell's equations. Starting with this equation, each EIT model proposes a different set of boundary conditions to model the electrodes attached to the body surface $\partial\Omega$ and the current-voltage application through these electrodes. Next, an account of existing EIT models is provided. The reconstruction problem in EIT was originally formulated using the equations of the *continuum model*. This model idealizes a unique continuous electrode over the entire body surface and assumes feasible the injection of current densities. Thus, applications of current and voltage are modeled by imposing Neumman and Dirichlet boundary conditions, respectively. However, in real experiments, one can only apply currents and voltages through discrete electrodes. A better model is the *gap model*. It approximates the current density by a constant over each electrode and zeroes in the gaps between electrodes, while assumes the voltages to be the mean of the electric potential over the electrodes. A disadvantage of the gap model is that it ignores that the highly conductive electrode material (metal) shunts some currents through the electrodes instead of through the body. The *shunt model* address this effect by assuming perfectly conducting electrodes. It imposes that the electric potential is always constant along each electrode. This accounts for the fact that the metal precludes any voltage difference across its surface. Moreover, this model integrates the electric flux over each electrode to model the current instead of the current density. Unfortunately, these models do not reproduce experimental data. They do not take into consideration the extra resistance produced by the electrochemical effect that takes place at the contact interface between each electrode and the body. A thin, highly resistive layer there form at this interface and is characterized by an effective contact impedance, which is modeled as a positive constant. The *complete electrode model* improves the shunt model by considering the voltage drops across the layer between the electrode and the body as the product of the contact impedance times the current flux. According to experiments, the complete electrode model is the most accurate model. It is capable of predicting experimental measurements more accurately than 0.1 percent [92]. For a complete description of these EIT models we refer the reader to [30, 92, 29, 46]. The complete electrode model has been considered with non-constant contact impedances [56, 103]. In [57] was proposed the *smoothened complete electrode model*, which replaces the contact impedances with contact admittance functions capable to vanish on some subsets of the electrodes. Other proposed models are the *point electrode models* [33, 47, 5], where the electrodes are replaced with point electrodes, eliminating the contact impedances.

The EIT models are weakly formulated in appropriate Sobolev spaces to analyze their existence and uniqueness of a solution (alternatively, an equivalent distributional formulation of them may be considered). Then, assuming the conductivity is known, the problem of finding the electric potential from the application of current (or voltage) is categorized as a *direct* problem. Moreover, if such a problem is formulated in a Hilbert space, an equivalent extremal formulation of it can be obtained. For instance, the continuum model has two well-known extremal formulations that have been interpreted as the *Dirichlet* and *Thomson variational principles* for this model [17, 71, 19]. It is well-known that the direct problem in EIT is *well-posed* in the sense of *Hadamard* [42]: it has a unique solution and is linear and stable with respect to the applied current (or voltage). As a consequence of the well-posedness, there exist continuous current-voltage maps (voltage-to-current and current-to-voltage maps), which are linear and symmetric in a certain sense. These maps are called *Dirichlet-to-Neumann* and *Neumann-to-Dirichlet* maps in the case of the continuum model and may not be well-defined in point electrode models [47]. Moreover, the power dissipated during current injection (or voltage excitation) can be expressed using these maps. Many interesting results combine the extremal formulations, current-voltage maps, and the power dissipated, namely *duality relations* [36, 19], *feasibility constraints* and *feasible sets* [17, 16, 19, 20], the *Kohn–Vogelius functional approach* [72, 71, 64], and *monotonicity estimates* [67, 58, 49, 50, 38]. It is worth noting that, perhaps because of its simple formulation, the continuum model was most often considered in these results.

Due to the modeling of electrodes, the EIT models are elliptic problems with non-standard boundary conditions (with the exception of the continuum model), where the conductivity distribution and contact impendances (or admittances) are coefficients. In general, the coefficients of a partial differential equation that models a physical phenomenon represent the physical properties of the materials involved. In practice, they are obtained through experiments with a certain degree of uncertainty. To assess the influence of this coefficient uncertainty in physical phenomena that are modeled as elliptic problems, error estimates for solutions of *idealized* problems have been derivated in [43, 44, 45] by using the duality theory of convex analysis [36][45, Ch. 2][11, Ch. 9], which allowed to obtain error bounds for coefficient, boundary condition, and domain idealizations. On the other hand, as mentioned above, in some cases the EIT models can be formulated in terms of *variational principles.* A variational principle for a problem in applied mathematics is a formulation of it as an optimization problem with an objective functional that exhibits a stationary behaviour at its optimal solution, that is, the functional derivative vanishes if the problem equations are fulfilled. For instance, problems in electrostatics (whose equations are also derived from Maxwell's equations), elasticity, and diffusion are proved to have two interrelated variational principles: one principle is a minimization problem, the other principle is a maximization problem, and the respective minimum and maximum values are the same [32, 10, 82]. In this situation, it is said that the principles are *complementary* [86]. It was proved that complementary principles can be used to obtain upper and lower bounds on the maximum-minimum value (which usually represents a relevant physical quantity) and to measure the accuracy of approximate solutions [86, 8, 9, 99]. For instance, in the continuum model, this maximum-minimum value is the power dissipated into heat [17]. In conclusion, in order to obtain error estimates, the EIT models might be analyzed from the point of view of the duality theory of convex analysis and the theory of complementary variational principles.

Consider the following abstract formulation of the inverse problem in EIT. Let x be the generated electric potential and let σ be the conductivity to be recover. The equations of the EIT model under consideration can be written as the *model equation*

$$A\left(\sigma, x\right) = 0. \tag{1.1}$$

Typically, (1.1) is expressed in a weak form and A is a map with values in a dual space. Moreover, there exists a conductivity-to-potencial map S that satisfies $A\left(\sigma, S\left(\sigma\right)\right) = 0$ for all admissible conductivity σ. A proof of the continuity of S for the case of the complete electrode model can be found in [76]. On the other hand, our hability to realize indirect observations of the conductivity can be modeled by the *observation operator* $C = C\left(\sigma, x\right)$. In general, C is a non-linear map with values in a Banach space. For instance, $C\left(\sigma, x\right)$ is a L^1-function in [53]. With these maps, and given an observed data y (e.g. voltage measurements), the classical formulation of the EIT inverse problem is written as the operator equation

$$F\left(\sigma\right) = y, \tag{1.2}$$

where F is the nonlinear operator defined as $F\left(\sigma\right) = C\left(\sigma, S\left(\sigma\right)\right)$. F is called the *forward operator*, and its continuity and differentiability are usually proved at the time of applying methods to solve (1.2) [1, 14]. Alternatively, we can avoid the conductivity-to-potential operator S and consider the system (1.1) and the *observation equation*:

$$\begin{array}{r} A\left(\sigma, x\right) = 0 \\ C\left(\sigma, x\right) = y \end{array},$$

where both x and σ are the unknowns. This is the *all-at-once* approach [63, 64] applied to the inverse problem in EIT. Below are some comments about the observation operator. C depends on σ and x because there are cases where the observed data consists of information from inside the body. Since EIT images suffer from low spatial resolution and image accuracy, several hybrid imaging techniques have been introduced that combine two or more physical modalities in order achieve high resolution while preserving high contrast [102]. For example, one of such imaging modalities, known as *acousto-electric tomography* (AET), combines EIT experiments with perturbations of the conductivity using ultrasound waves. This method furnishes observations of the *interior power density* [6, 24, 13, 1, 53]. Another modality is *current density impedance imaging* (CDII), which is an EIT technique integrated with magnetic resonance imaging to obtain observations of the *internal current density* [70, 80, 93, 97, 94]. In both cases, the observed data are internal to the body. We conclude this paragraph citing some works related to the uniqueness of solution of the EIT inverse problem: [79, 74, 25, 48, 4].

The direct problem in EIT has a unique solution, and is linear and stable with respect to the current (or voltage) applied, that is, it is *well-posed* in the sense of *Hadamard* [42]. In contrast, the inverse problem in EIT is *non-linear* and severely *ill-posed* in the sense of Hadamard. The non-linearity is a consequence of the fact that the current-voltage operators, which are the only things we can observe from EIT experiments, depend on the conductivity in a non-linear way. The ill-posedness means that the unknown conductivity does not depend continuously on the boundary measurements. Then, slightly different measurements can correspond to completely different conductivities. A more detailed explanation of these attributes is provided in [78, Subsec. 12.4-5] using the continuum model. Moreover, only noisy observations are available due to instrument errors. As a consequence, numerical methods applied to the equations of the EIT inverse problem produce unstable results. A cure is to solve the EIT inverse problem approximately and in a stable manner. Some type of *regularization* must be employed to achieve this. The idea of regularization is to consider a stable auxiliary problem to overcome the ill-posedness [91]. This new problem, called *regularized problem*, is stable in the sense that depends continuously on the noisy observations, and its solution, called *regularized solution*, approximates the solution of the inverse problem. To obtain stability, the regularized problem incorporates additional information about the solution. A *regularization parameter* controls the importance of this information on the regularized solution. Since values of the regularization parameter near zero represent regularized problems close to the inverse problem, it is desirable that the corresponding regularized solutions are also close to a solution of the inverse problem. To summarize, the *existence, estability,* and *convergence* of regularized solutions must be proved. In practice, the regularized problem is approximately solved for a fixed regularization parameter, for which some technique to find an *optimal regularization parameter* may be employed [52, 7, 35]. In [64, 55, 66, 28] the all-at-once formulation of the inverse problem in EIT was considered with two potentials at the same time, one corresponding to the problem with applied current and the other one corresponding to the problem with applied voltage. There regularized problems were proposed and the existence and convergence of regularized solutions was proved. In

[54] similar results were presented with the continuum model, but considering only the electric potential and voltage measurements as observations. Stability results were not presented in these papers.

1.1 Contributions

Two contributions to EIT are presented in this thesis. The first contribution is a step towards a better understanding of the EIT models. It is presented in Chapter 2 and is related to the direct problem in EIT. The second contribution is presented in Chapter 3 and is related to the inverse problem in EIT and its regularization.

The purpose of Chapter 2 is to provide an abstract framework that unifies the formulation and analysis of EIT models. We begin by establishing three assumptions in the context of Hilbert spaces, which attempt to generalize the mathematical objects involved in the weak formulation of EIT models and the relations between them. Based on these assumptions, abstract problems that represent the weak formulations of EIT models in terms of electric potentials and current fields are proposed. The existence of solutions to the abstract problems allows us to define current-voltage maps. Using these maps, we define two non-negative functions that model the power dissipated during current and voltage injection. Then, dual properties that connect all the abstract problems are obtained. With all these results, a posteriori error estimates are derived and the idea of feasibility constraints and feasible sets is applied to our framework. Five well-known models fit into this abstraction: the continuum model, the gap model, the shunt model, the complete electrode model, and the smoothened complete electrode model. Numerical tests that apply the results to the complete electrode model are presented. Moreover, functionals of Kohn-Vogelius type and a monotonicity principle are deduced, and the connection of our framework with the duality theory of convex analysis and the theory of complementary variational principles is shown.

In Chapter 3, an all-at-once formulation of the EIT inverse problem is proposed and three regularizations of it are analyzed. We begin by establishing a set of assumptions in the context of Banach spaces, which allows us to formulate an abstract problem that aims to generalize the all-at-once formulation of the EIT inverse problem in terms of electric potentials and with several types of observations, namely voltage measurements, current measurements, magnitudes of current density field, and interior power densities. For this abstract problem, three regularized problems are formulated, which are based on the classic Tikhonov, Ivanov, and Morozov regularizations. The existence, stability, and convergence of regularized solutions are proved. Additionally, it is proved that there exists an optimal regularization parameter by means of considering a learning problem. We emphasize that the stability of regularized solutions allowed us to prove this result. The continuum model, the shunt model, the gap model, the complete electrode model, and the smoothened complete electrode model fit into this abstraction. It turns out that the all-at-once approach allows an alternative formulation of the EIT inverse problem, where the model equation does not represent the EIT experiment. Numerical tests with the complete electrode model and the previously mentioned observations are performed.

1.2 Outline

Chapter 2 is organized as follows. In Section 2.1, the assumptions that we will use during this chapter are presented and a brief explanation of their meaning in the context of EIT is provided. In Section 2.2, the abstract problems are formulated and analyzed. In Section 2.3, the current-voltage maps are defined and their properties are described. In Section 2.4, the power functions that model the power dissipated in an EIT experiment are defined. Here are obtained the dual properties. In Section 2.5, a posteriori error estimates for approximate solutions of the abstract problems are derived from the previous results. In Section 2.6, the idea of feasibility constraints and feasible sets is applied to our framework. In Section 2.7, we show that the complete electrode model fits into the assumptions and apply the results of the preceding sections to this model. Here, numerical tests are performed to determine the error of approximate solutions and to draw the feasible sets. In all sections, with the exception of the latter, examples related to the continuum model are provided for a better understanding of the topic developed in each section. An appendix is included, in which other EIT models that fit into the assumptions are presented, and interpretations of the abstract problems from the viewpoint of the duality theory of convex analysis and the theory of complementary variational principles are provided.

Chapter 3 is organized as follows. In Section 3.1, the main assumptions are presented and their meaning in the context of EIT is explained. The abstract inverse problem is formulated here. In Section 3.2, the three regularizations are presented and preliminary results are provided. In Section 3.3, the existence, stability, and convergence of regularized solutions are proved. At the end, we formulate the learning problem and prove that it has a solution. In Section 3.4, we consider the equations of the complete electrode model to provide three instances of the assumptions; the abstract inverse problem and their regularizations are formulated for each of them. Finally, in Section 3.5, numerical tests are performed to aproximate regularized solutions. An appendix with examples of other EIT models that fit into the assumptions is included.

2 A framework for EIT models

In this chapter, we provide an abstract framework that unifies the formulation and analysis of EIT models. We begin by formulating three assumptions in the context of real Hilbert spaces, which aim to be abstract representations of the spaces, the bilinear forms, and other mathematical objects involved in the weak formulation of the EIT models. There, two bilinear forms (the first defined in the assumptions and the second derived from the first), a gradient-like operator, and a domain-to-boundary potential operator are considered. Based on these assumptions, four abstract problems are proposed and their solvability is obtained by applying the Lax-Milgram theorem. The first two problems represent the weak formulations of the EIT models in terms of electric potentials, with applied current and applied voltage, respectively. The last two problems represent the same EIT problems, but in terms of current fields. We shall point out that the abstract problems are well-posed and that can be formulated as optimization problems. Using the solutions of the abstract problems, two *current-voltage* maps are defined. We prove that these maps are linear, continuous, symmetric, and, in a certain sense, inverses of each other. One can see that all these properties are inherited from the mathematical objects defined in the assumptions. It turns out that the current-to–voltage (resp. voltage-to-current) map induces an inner (resp. semi-inner) product. The corresponding norms lead to the introduction of two non-negative functions that model the power dissipated during current and voltage injection, which take the same value when a current-voltage pair corresponds to the same EIT experiment. These *power* functions are used to obtain *dual* properties that link all the abstract problems. The extremal formulations play an important role here. In fact, new extremal formulations are obtained in this part, whose functionals may be interpred as ones of Kohn-Vogelius type. It is worth pointing out that the dual properties hold for current-voltage pairs not necessarily corresponding to the same EIT experiment. From the preceding results we derive a posteriori error estimates for approximate solutions of the abstract problems. We begin by proving general error estimates for pairs of abstract problems and then error estimates for each abstract problem are obtained as particular cases of the first ones. The error is measured with “energy norms” and is expressed in terms of “energy functionals”. At the end, we extend the idea of feasibility constraints and feasible sets to our framework. We begin by defining the set $\mathscr{F}$ of pairs of bilinear forms for which the preceding results can be applied. Then, a lemma inspired in the well-know monotocity estimates is proved. We use it to provide a description of $\mathscr{F}$ by considering subsets of pair of bilinear forms defined by a power constraint. These subsets are called feasibles sets and are proved to have monotonic properties. In the situation that the feasible sets correspond to a current-voltage pair from a EIT experiment, a interesting partition of $\mathscr{F}$ is obtained.

2.1 Assumptions

Our framework is based on the following three assumptions:

A1. Let X and Z be real Hilbert spaces equipped with the inner products $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Z$ respectively. Let $G : X \to Z$ be a linear continuous operator. Assume that there exists a non-zero vector $1_X \in X$ such that $1_X \in \ker G$. Denote by $X/\mathbb{R}$ the quotient space of X with respect to $\operatorname{span}\{1_X\}$, which is also a real Hilbert space equipped with the inner product that induces the norm $[x] \in X/\mathbb{R} \to \|[x]\|_{X/\mathbb{R}} \coloneqq \min_{\lambda\in\mathbb{R}} \|x + \lambda 1_X\|_X$. In addition, assume that there exists $C > 0$ such that $C\,\|[x]\|_{X/\mathbb{R}} \leq \|Gx\|_Z$ for all $x \in X$.

A2. Let $b : Z \times Z \to \mathbb{R}$ be a symmetric continuous coercive bilinear form.

A3. Let Y be a real vector space and let $P : X \to Y$ be a surjective linear map. Assume that $\ker P$ is closed and that $1_X \notin \ker P$.

By Assumption A2 we know that there exists a self-adjoint isomorphism $B : Z \to Z$ defined by $b\,(z_1, z_2) = \langle z_1, Bz_2\rangle_Z$ for all $z_1, z_2 \in Z$. Define the operator $T : X \to Z$ by $T \coloneqq B^{-1} \circ G$ and $a : X \times X \to \mathbb{R}$ by

$$a\,(x_1, x_2) \coloneqq b\,(Tx_1, Tx_2) \quad \text{for all } x_1, x_2 \in X. \tag{2.1}$$

It is easy to check that a is a symmetric continuous bilinear form. Also, it can be deduced from the assumptions on G that $a\,(1_X, x) = 0$ and that there exists $C > 0$ such that $a\,(x, x) \geq C\,\|[x]\|^2_{X/\mathbb{R}}$ for all $x \in X$. The bilinear forms a and b play an important role in this work. In [99] a similar relation to (2.1) was assumed to obtain a general complementary variational principle for elliptic problems.

In the context of Electrical Impedance Tomography, the mathematical objects that were defined in Assumptions A1-A3 have the following interpretation. X and Z represent the spaces of electric potentials and current fields, respectively. G is a gradient-like operator. Y represents the space of electrode voltages. P can be viewed as a projection of the electric potential on the domain to that on the electrode regions. a and b represent the bilinear forms that arise from the weak formulation of an EIT model in terms of electric potentials and current fields, respectively. Finally, T models the correspondence between an electric potential in X and its current field in Z.

The following are direct consequences of Assumptions A1-A3:

1. Since $a\,(x, 1_X) = a\,(1_X, x) = 0$ for all $x \in X$, a can be defined on $X/\mathbb{R} \times X/\mathbb{R}$, namely

$$\begin{aligned} &X/\mathbb{R} \times X/\mathbb{R} \to \mathbb{R} \\ &\quad([x_1], [x_2]) \mapsto a\,(x_1, x_2) = a\,(x_1 + \lambda_1 1_X, x_2 + \lambda_2 1_X) \end{aligned}$$

   for all $\lambda_1, \lambda_2 \in \mathbb{R}$. Also, since $1_X \in \ker G$, we can define

$$\begin{aligned} &X/\mathbb{R} \times X/\mathbb{R} \to \mathbb{R} \\ &\quad([x_1], [x_2]) \mapsto \langle Gx_1, Gx_2\rangle_Z = \langle G\,(x_1 + \lambda_1 1_X), G\,(x_2 + \lambda_2 1_X)\rangle_Z . \end{aligned} \tag{2.2}$$

for all $\lambda_1, \lambda_2 \in \mathbb{R}$. It easy to check these maps are symmetric continuous coercive bilinear forms.

2. We claim that $G(X)$ is closed in Z. Indeed, from the assumptions on G one can deduce that there exist $C, C' > 0$ such that

$$C\,\|[x]\|_{X/\mathbb{R}} \leq \|Gx\|_Z \leq C'\,\|[x]\|_{X/\mathbb{R}} \quad \text{for all } x \in X.$$

On the other hand, observe that since $1_X \in \ker G$, G can be defined on $X/\mathbb{R}$ as follows: given $[x] \in X/\mathbb{R}$, $G[x] := Gx = G(x + \lambda 1_X)$ for all $\lambda \in \mathbb{R}$. This map is linear and continuous. It follows that $X/\mathbb{R}$ is isomorphic to $G(X/\mathbb{R})$, and hence $G(X/\mathbb{R})$ is closed in Z. As $G(X/\mathbb{R}) = G(X)$, the conclusion follows.

3. Using Assumptions A1 and A3 one can show that X and P determine on Y the norm

$$\|\cdot\|_Y : Y \to \mathbb{R} \quad \text{with} \quad \|y\|_Y := \min_{\substack{x\in X \\ Px=y}} \|x\|_X . \tag{2.3}$$

Since the map $P|_{(\ker P)^\perp} : (\ker P)^\perp \to Y$ defined by $x^\perp \mapsto Px^\perp = y$ is a linear bijective isometry and $(\ker P)^\perp$ is a closed subspace, it follows that Y equipped with $\|\cdot\|_Y$ is complete. Moreover, $x^\perp \in (\ker P)^\perp$ is the unique minimizer of the minimization problem in (2.3) with $y = Px^\perp$. It can be proved that $\|\cdot\|_Y$ satisfies the paralelogram law. Thus, Y becomes a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_Y$ that induces $\|\cdot\|_Y$ (for a detailed proof of this result, see Lemma 2.34 in Appendix).

4. It follows from (2.3) that $\|Px\|_Y \leq \|x\|_X$ for all $x \in X$. Therefore P is continuous.

5. The topological dual of Y is denoted by $Y^\star$. Given $f \in Y^\star$ and $y \in Y$ we write $\langle f, y \rangle$ instead of $f(y)$. Let 1_Y be the image of 1_X under P and let $Y/\mathbb{R}$ be the quotient space of Y with respect to $\operatorname{span}\{1_Y\}$.

6. Denote by 1_Y the image of 1_X under P. Let $Z_\perp \subset Z$ and $Y_\diamond^\star \subset Y^\star$ be the closed subspaces defined by

$$Z_\perp := \left\{ z \in Z \,\middle|\, G^\star z \in (\ker P)^\perp \right\} = (G(\ker P))^\perp \quad \text{and}$$
$$Y_\diamond^\star := \{ f \in Y^\star \,|\, \langle f, 1_Y \rangle = 0 \},$$

where $G^\star$ denotes the adjoint of G. Let $R : Z_\perp \to Y_\diamond^\star$ be the linear map defined by

$$\langle Rz, y \rangle := \left\langle G^\star z, P^{-1} y \right\rangle_X \quad \text{for all } y \in Y,$$

where $P^{-1}y$ denotes some element in the inverse image of y under P. Since $\langle G^\star z, x \rangle_X = 0$ for all $x \in \ker P$ provided $z \in Z_\perp$, R does not depend on the choice of $P^{-1}y$. Also, since $P(1_X) = 1_Y$ and $1_X \in \ker G$, it follows that $\langle Rz, 1_Y \rangle = \langle z, G(1_X) \rangle_Z = 0$ for all $z \in Z_\perp$. Hence $R(Z_\perp) \subseteq Y_\diamond^\star$. Therefore R is well-defined. Moreover, we have:

a) The continuity of $G^\star$ implies that of R.

b) $\ker R = \ker G^\star$.

c) R is surjective. Indeed, consider the problem:

$$\text{given } f \in Y_\diamond^\star \text{ find } \hat{z} \in Z \,:\, \langle \hat{z}, Gx \rangle_Z = \langle f, Px \rangle \quad \text{for all } x \in X \tag{2.4}$$

It is clear that if $\hat{z}$ solves (2.4) then $\hat{z} \in Z_\perp$ and $R\hat{z} = f$ by the surjectivity of P. Since $1_X \in \ker G$ and $\langle f, P(1_X) \rangle = 0$, (2.4) can be rewritten in $X/\mathbb{R}$. Therefore, it suffices to apply the Lax-Milgram theorem to find a solution $\hat{z} = G\hat{x}$ with $\hat{x} \in X$.

7. We denote by $Y/\mathbb{R}$ the quotient space of Y with respect to $\operatorname{span}\{1_Y\}$.
8. It is well-known that $(\ker G^\star)^\perp = \overline{\operatorname{ran} G}$ [27, Thm. 4.4]. As $G(X)$ is closed in Z and $\ker R = \ker G^\star$, we have $Z = \operatorname{ran} G \oplus \ker R$.
9. By the Hahn-Banach theorem, given any closed subspace $X' \subset X$ such that $1_X \notin X'$, the bilinear form a is continuous and coercive on $X' \times X'$ (for a proof of this assertion, see Lemma 2.35 in Appendix). In particular, a is continuous and coercive on $\ker P \times \ker P$ and on any subspace of the form $\{x \in X \,|\, \Gamma(x) = 0\}$, where Γ is a linear continuous functional on X satisfying $\Gamma(1_X) \neq 0$.
10. Since $b\left(z, \left(B^{-1} \circ G\right) x\right) = \left\langle z, BB^{-1}Gx \right\rangle_Z = \langle Gx, z \rangle_Z$ for all $x \in X$ and all $z \in Z$, it follows that
$$b(Tx, z) = \langle Rz, Px \rangle \quad \text{for all } x \in X \text{ and all } z \in Z_\perp. \tag{2.5}$$
11. It follows directly from the assumptions on G that $\ker G = \operatorname{span}\{1_X\}$.

$Z_\perp$ and $Y_\diamond^\star$ represent the subspaces of appropriate current fields and applied currents, respectively, while R is a normal component-like operator (this interpretation is best illustrated in the example at the end of this section).

To provide the examples, we introduce the following notations.

Notation. Let Ω be an open, connected, bounded, and Lipschitz domain in $\mathbb{R}^d$ $(d = 2, 3)$ with boundary $\partial\Omega$. $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$. Let M be an integer and let $\mathcal{E}_1, \ldots, \mathcal{E}_M$ be open connected subsets of $\partial\Omega$ such that $\overline{\mathcal{E}_i} \cap \overline{\mathcal{E}_j} = \emptyset$ for $i \neq j$, and if $d = 3$, the boundary of each $\mathcal{E}_m$ is a smooth curve on $\partial\Omega$. $|\mathcal{E}_m|$ denotes the area of $\mathcal{E}_m$. $L^2\left(\Omega, \mathbb{R}^d\right)$ denotes the space of square integrable vector-valued functions from Ω into $\mathbb{R}^d$. $L^2(\mathcal{E}_m)$ denotes the space of square integrable functions from $\mathcal{E}_m$ into $\mathbb{R}$, for $m = 1, \ldots, M$. $H^1(\Omega)$ denotes the usual Sobolev space on Ω. $C_c^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω and its closure with respect to the $H^1(\Omega)$-norm is denoted by $H_0^1(\Omega)$. Let $\mathbb{R}_\diamond^M$ be the subspace of vectors with zero mean value $\left\{U \in \mathbb{R}^M \,\middle|\, \sum_{m=1}^M U_m = 0\right\}$. $H^{1/2}(\partial\Omega)$ denotes the space of traces on $\partial\Omega$ and $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ denotes the trace operator on $\partial\Omega$. $H^{1/2}(\mathcal{E}_m)$ denotes the space of traces on $\mathcal{E}_m$ and $\gamma_m : H^1(\Omega) \to H^{1/2}(\mathcal{E}_m)$ denotes the trace operator on $\mathcal{E}_m$, for $m = 1, \ldots, M$. The dual space of $H^{1/2}(\partial\Omega)$ is denoted by $H^{-1/2}(\partial\Omega)$. $L^\infty(\Omega)$ denotes the space of bounded measurable functions. For a measurable function σ, the essential infimum of σ is denoted by $\operatorname{ess\,inf}_{\mathbf{x}\in\Omega}\sigma(\mathbf{x})$. $\mathbf{1}$ denotes the constant function

$\mathbf{1}(\mathbf{x}) = 1$ for all $\mathbf{x} \in \Omega$, $\overrightarrow{1}$ denotes the all-ones vector $\left(1_{(1)}, \ldots, 1_{(M)}\right) \in \mathbb{R}^M$, and $\overrightarrow{0}$ denotes the zero vector of $\mathbb{R}^M$. In all the examples the domain Ω represents a body with a internal conductivity $\sigma \in L^\infty(\Omega)$ satisfying $\sigma_- = \operatorname{ess\,inf}_{\mathbf{x}\in\Omega}\sigma(\mathbf{x}) > 0$, and the subsets $\mathcal{E}_1, \ldots, \mathcal{E}_M$ represent M electrodes attached on the surface $\partial\Omega$.

Below, the first example of an EIT model that fits into our framework.

Example 2.1. The *continuum model* for the electric potential u consists of the equation

$$\nabla\cdot(\sigma\nabla u) = 0 \quad \text{in } \Omega \tag{2.6}$$

with the Neumann boundary condition

$$\sigma\nabla u \cdot \mathbf{n} = f \quad \text{on } \partial\Omega \tag{2.7}$$

if a current density f is applied, or with the Dirichlet boundary condition

$$u = g \quad \text{on } \partial\Omega \tag{2.8}$$

if a boundary voltage g is applied. For this model, Assumptions A1-A3 are verified with:

A1. $X := H^1(\Omega)$ and $Z := L^2\left(\Omega, \mathbb{R}^d\right)$ equipped with their usual inner products, $G : X \to Z$ defined by $Gu := -\nabla u$ (gradient operator), and $1_X := \mathbf{1} \in X$.

A2. $b : Z \times Z \to \mathbb{R}$ defined by $b(\mathbf{p}_1, \mathbf{p}_2) := \int_\Omega \frac{1}{\sigma}\mathbf{p}_1 \cdot \mathbf{p}_2 \, \mathrm{d}\mathbf{x}$.

A3. $Y := H^{1/2}(\partial\Omega)$ and $P : X \to Y$ defined by $Pu := \gamma u$ (trace operator).

It is easy to check that $B : Z \to Z$ is given by $B\mathbf{p} = \sigma^{-1}\mathbf{p}$. Thus, the linear operator $T : X \to Z$ and the bilinear form $a : X \times X \to \mathbb{R}$ are given by

$$\begin{aligned} Tu &= \left(B^{-1} \circ G\right) u = -\sigma\nabla u \quad \text{and} \\ a(u_1, u_2) &= b(Tu_1, Tu_2) = \int_\Omega \sigma\nabla u_1 \cdot \nabla u_2 \, \mathrm{d}\mathbf{x}. \end{aligned}$$

The norm in $X/\mathbb{R} = H^1(\Omega)/\mathbb{R}$ is given by $\|[u]\|_{X/\mathbb{R}} = \min_{\lambda\in\mathbb{R}} \|u + \lambda\mathbf{1}\|_{H^1(\Omega)}$. The coercive property of G follows from the estimate

$$\|[u]\|_{H^1(\Omega)/\mathbb{R}} \leq C \|\nabla u\|_{L^2\left(\Omega,\mathbb{R}^d\right)} \quad \text{for all } u \in H^1(\Omega),$$

which is a direct consequence of Poincaré's inequality [88, Cor. 7.3]. The remaining assumptions are easily verified. From (2.3), the norm $\|\cdot\|_Y$ is defined by $\|g\|_Y = \min\left\{\|u\|_{H^1(\Omega)} \,\middle|\, u \in H^1(\Omega), \gamma u = g\right\}$, which is the usual norm of $H^{1/2}(\partial\Omega)$. The dual of Y is $Y^\star = H^{-1/2}(\partial\Omega)$. So, $Y^\star_\diamond$ is given by

$$H^{-1/2}_\diamond(\partial\Omega) = \left\{f \in H^{-1/2}(\partial\Omega) \,\middle|\, \langle f, \gamma\mathbf{1}\rangle_{H^{-1/2}\times H^{1/2}} = 0\right\}.$$

The closed subspace $Z_\perp$ is given by

$$Z_\perp = \left\{\mathbf{p} \in Z \,\middle|\, \int_\Omega \mathbf{p} \cdot \nabla u \, \mathrm{d}\mathbf{x} = 0 \text{ for all } u \in H^1_0(\Omega) = \ker P\right\}.$$

Considering that the distributional divergence of a vector-valued function $\mathbf{p} \in L^2\left(\Omega, \mathbb{R}^d\right)$ is defined as $(\nabla \cdot \mathbf{p}, \varphi) = -\int_\Omega \mathbf{p} \cdot \nabla\varphi \,\mathrm{d}\mathbf{x}$ for all test function $\varphi \in C_c^\infty(\Omega)$, $Z_\perp$ can be written as

$$Z_\perp = \left\{\mathbf{p} \in L^2\left(\Omega, \mathbb{R}^d\right) \,|\, \nabla \cdot \mathbf{p} = 0\right\}$$

by the density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$, where the equality $\nabla \cdot \mathbf{p} = 0$ has to be understood in the sense of distributions. By definition, given $\mathbf{p} \in Z_\perp$, the action of $R\mathbf{p}$ on $g \in Y$ is expressed by

$$\langle R\mathbf{p}, g\rangle = \langle \mathbf{p}, Gu\rangle_Z = -\int_\Omega \mathbf{p} \cdot \nabla u \,\mathrm{d}\mathbf{x} \quad \text{with } u \in H^1(\Omega) \text{ such that } \gamma u = g.$$

As the distributional divergence of any $\mathbf{p} \in Z_\perp$ is the zero function, the normal component of $\mathbf{p} \in Z_\perp$ is well-defined [88, Sec. 6.2], namely $\mathbf{p} \cdot \mathbf{n} \in H^{-1/2}(\partial\Omega)$ is defined as

$$\langle \mathbf{p} \cdot \mathbf{n}, \gamma u\rangle = \int_\Omega u \underbrace{(\nabla \cdot \mathbf{p})}_{=0} \mathrm{d}\mathbf{x} + \int_\Omega \mathbf{p} \cdot \nabla u \,\mathrm{d}\mathbf{x} \quad \text{for all } u \in H^1(\Omega). \tag{2.9}$$

From this we deduce that $R\mathbf{p} = -\mathbf{p} \cdot \mathbf{n}$ for all $\mathbf{p} \in Z_\perp$ and that the kernel of R can be written as

$$\ker R = \left\{\mathbf{p} \in L^2\left(\Omega, \mathbb{R}^d\right) \,|\, \nabla \cdot \mathbf{p} = 0,\ \mathbf{p} \cdot \mathbf{n} = 0\right\}.$$

Finally, observe that the orthogonal decomposition of Z reads as

$$\begin{aligned} Z &= \operatorname{ran} G \oplus \ker R \\ &= \left\{\nabla u \,\middle|\, u \in H^1(\Omega)\right\} \oplus \left\{\mathbf{p} \in L^2\left(\Omega, \mathbb{R}^d\right) \,|\, \nabla \cdot \mathbf{p} = 0,\ \mathbf{p} \cdot \mathbf{n} = 0\right\}. \end{aligned}$$

This was proved in [23, Prop. 1] for smooth vector fields in order to prove the Hodge Decomposition Theorem.

2.2 Abstract problems

Definition 2.2. Consider the following abstract problems.

$\mathcal{C}$. Given $f \in Y_\diamond^\star$, find a $\bar{x} \in X$ satisfying

$$a(\bar{x}, x) = \langle f, Px\rangle \quad \text{for all } x \in X. \tag{2.10}$$

$\mathcal{V}$. Given $g \in Y$, find a $\tilde{x} \in X$ satisfying

$$a(\tilde{x}, x) = 0 \quad \text{for all } x \in \ker P, \tag{2.11}$$

$$P\tilde{x} = g \quad \text{in } Y. \tag{2.12}$$

$\mathcal{C}'$. Given $f \in Y_\diamond^\star$, find a $\bar{z} \in Z_\perp$ satisfying

$$b(\bar{z}, z) = 0 \quad \text{for all } z \in \ker R, \tag{2.13}$$

$$R\bar{z} = f \quad \text{in } Y^\star. \tag{2.14}$$

$\mathcal{V}'$. Given $g \in Y$, find a $\tilde{z} \in Z_\perp$ satisfying

$$b(\tilde{z}, z) = \langle Rz, g\rangle \quad \text{for all } z \in Z_\perp. \tag{2.15}$$

The equations of $\mathcal{C}$ and $\mathcal{V}$ are abstractions of the weak formulations of the EIT models in terms of electric potentials (x's). $\mathcal{C}$ represents the problem with applied current (f), while $\mathcal{V}$ represents the problem with applied voltage (g). The problems $\mathcal{C}'$ and $\mathcal{V}'$ have the same purpose that the problems $\mathcal{C}$ and $\mathcal{V}$, respectively, but considering current fields (z's). $\mathcal{C}$ and $\mathcal{V}'$ share the same structure. The same happens with $\mathcal{V}$ and $\mathcal{C}'$. Observe that $\mathcal{C}$ and $\mathcal{V}$ are defined on the entire space X, whereas $\mathcal{C}'$ and $\mathcal{V}'$ are defined on the closed subspace $Z_\perp$ because R is not well-defined on Z. On the other hand, Y and its dual $Y^\star$ are related to the boundary condition spaces of the EIT models. In $\mathcal{V}$, the equation $Px = g$ in Y can be interpreted as a *voltage* boundary condition, while in $\mathcal{C}'$ the equation $R\bar{z} = f$ in $Y^\star$ can be interpreted as a *current* boundary condition.

Let us make some quick comments about the abstract problems.

Compatibility condition. In $\mathcal{C}$ and $\mathcal{C}'$, f must belong to $Y^\star_\diamond$ to ensure that a solution exists. Indeed, from (2.10) it follows that $\langle f, 1_Y\rangle = \langle f, P(1_X)\rangle = a(\bar{x}, 1_X) = 0$ and from (2.14) it follows that $\langle f, 1_Y\rangle = \langle R\bar{z}, P(1_X)\rangle = \langle \bar{z}, G(1_X)\rangle_X = 0$. This is similar to the compatibility condition that arises in elliptic problems with Neumann boundary conditions (see for instance [88, Prop. 7.7]). For the EIT models, this represents the inclusion of charge conservation [92].

Well-definiteness. Clearly, $\mathcal{C}$ and $\mathcal{V}'$ are well-defined. Since $P : X \to Y$ and $R : Z_\perp \to Y^\star_\diamond$ are surjective maps, equation (2.12) and (2.14) make sense, and hence $\mathcal{V}$ and $\mathcal{C}'$ are also well-defined.

Extremal equivalences. Since a and b are symmetric and positive semidefinite, $P(X) = Y$, and $R(Z_\perp) = Y^\star_\diamond$, the abstract problems can be equivalently formulated as optimization problems [88, Lem. 2.2, 4.3], namely

- $\mathcal{C}$. $\max\limits_{x \in X} J(x)$ with $J(x) := \langle f, Px\rangle - (1/2)\, a(x, x)$,
- $\mathcal{V}$. $\min\limits_{\substack{x \in X \\ Px = g}} K(x)$ with $K(x) := (1/2)\, a(x, x)$,
- $\mathcal{C}'$. $\min\limits_{\substack{z \in Z_\perp \\ Rz = f}} J'(z)$ with $J'(z) := (1/2)\, b(z, z)$,
- $\mathcal{V}'$. $\max\limits_{z \in Z_\perp} K'(z)$ with $K'(z) := \langle Rz, g\rangle - (1/2)\, b(z, z)$.

Linearity. The abstract problems are linear problems due to the linearity of the objects used to define them.

The following proposition establishes the existence and uniqueness of solutions to the abstract problems.

Proposition 2.3.

(i) *For every $f \in Y^\star_\diamond$ there exists $\bar{x} \in X$ so that $[\bar{x}]$ is the set of solutions to $\mathcal{C}$. Furthermore, there exists $C > 0$ such that $\|[\bar{x}]\|_{X/\mathbb{R}} \le C\, \|f\|_{Y^\star}$ and*

$$[\bar{x}] = \operatorname{span}\{1_X\} \Leftrightarrow f = 0_{Y^\star}.$$

(*ii*) *For every* $g \in Y$ *there exists a unique solution* $\tilde{x} \in X$ *to* $\mathcal{V}$*. Furthermore, there exists* $C > 0$ *such that* $\|\tilde{x}\|_X \leq C \|g\|_Y$ *and*

$$\tilde{x} = \lambda 1_X \Leftrightarrow g = \lambda 1_Y \quad \textit{for all } \lambda \in \mathbb{R}.$$

(*iii*) *For every* $f \in Y_\diamond^\star$ *there exists a unique solution* $\bar{z} \in Z_\perp$ *to* $\mathcal{C}'$*. Furthermore, there exists* $C > 0$ *such that* $\|\bar{z}\|_Z \leq C \|f\|_{Y^\star}$ *and*

$$\bar{z} = 0_Z \Leftrightarrow f = 0_{Y^\star}.$$

(*iv*) *For every* $g \in Y$ *there exists a unique solution* $\tilde{z} \in Z_\perp$ *to* $\mathcal{V}'$*. Furthermore, there exists* $C > 0$ *such that* $\|\tilde{z}\|_Z \leq C \|g\|_Y$ *and*

$$\tilde{z} = 0_Z \Leftrightarrow [g] = \operatorname{span}\{1_Y\}.$$

The proofs are based on the application of the Lax-Milgram theorem.

Proof of (i). Define $f \circ P$ on $X/\mathbb{R}$ as $(f \circ P)[x] := \langle f, Px \rangle$. It is well-defined since $(f \circ P)[1_X] = \langle f, 1_Y \rangle = 0$. From the continuity of f and P, it follows that $|(f \circ P)[x]| \leq \|f\|_{Y^\star} \|[x]\|_{X/\mathbb{R}}$ for all $x \in X$. Thus, by the Lax-Milgram lemma [88, Prop. 5.1], there exists $\bar{x} \in X$ such that $[\bar{x}] \in X/\mathbb{R}$ is the unique solution to

$$a([\bar{x}], [x]) = (f \circ P)[x] \quad \text{for all } [x] \in X/\mathbb{R},$$

and $\|[\bar{x}]\|_{X/\mathbb{R}} \leq C \|f\|_{Y^\star}$. From this it is easy to check that the solutions to $\mathcal{C}$ are the elements of $[\bar{x}]$. If $\bar{x} = \lambda 1_X$ for some $\lambda \in \mathbb{R}$, it follows that $\langle f, Px \rangle = 0$ for all $x \in X$ by (2.10). Since P is surjective, f is the zero functional. The converse follows from the estimate for $[\bar{x}]$. □

Proof of (ii). Since a is continuous and coercive on $\ker P \times \ker P$ and P is continuous and surjective, there exists a unique solution $\tilde{x} \in X$ to $\mathcal{V}$ and $\|\tilde{x}\|_X \leq C \|g\|_Y$ by the Generalized Lax-Milgram lemma [88, Prop. 5.2]. Suppose that $g = \lambda 1_Y$. Then $a(\tilde{x}, \tilde{x}) = a(\tilde{x}, \tilde{x} - \lambda 1_X) = 0$ by (2.11) and (2.12). Thus $\tilde{x} \in \operatorname{span}\{1_Y\}$. Therefore $\tilde{x} = \lambda 1_X$ since $P\tilde{x} = \lambda 1_Y$. The converse is obvious. □

Proof of (iii). Since R is surjective and b is continuous and coercive, by the Generalized Lax-Milgram lemma [88, Prop. 5.2], $\mathcal{C}'$ has a unique solution $\bar{z} \in Z_\perp$ and $\|\bar{z}\|_X \leq C \|f\|_{Y^\star}$. From this $f = 0_{Y^\star}$ implies $\bar{z} = 0_Z$. The converse follows from (2.14). □

Proof of (iv). By the continuity of R the map $z \in Z_\perp \mapsto \langle Rz, g \rangle \in \mathbb{R}$ is continuous on $Z_\perp$. Moreover, b is continuous and coercive. Thus, by the Lax-Milgram lemma [88, Prop. 5.1], there exists a unique solution $\tilde{z} \in Z_\perp$ to $\mathcal{V}'$ and $\|\tilde{z}\|_Z \leq C \|g\|_Y$. By (2.15), the coercivity of b, and the surjectivity of R, $\tilde{z} = 0_Z$ iff $\langle f, g \rangle = 0$ for all $f \in Y_\diamond^\star$. But, since $Y_\diamond^\star$ is the annihilator of $\{1_Y\}$, g belongs to the annihilator of $Y_\diamond^\star$ iff $g \in \operatorname{span}\{1_Y\}$ (see for instance [27, Thm. 4.24]). The conclusion follows. □

Remark 2.4. Using (2.5) we obtain that if $\bar{x}$ is a solution to $\mathcal{C}$ then $T\bar{x}$ is a solution to $\mathcal{C}'$ and that if $\tilde{x}$ is a solution to $\mathcal{V}$ then $T\tilde{x}$ is a solution to $\mathcal{V}'$. Thus, T maps solutions in terms of electric potentials to solutions in terms of current fields.

Remark 2.5. Uniqueness of a solution was obtained for $\mathcal{C}'$, $\mathcal{V}'$, and $\mathcal{V}$, whereas it is possible to find a unique solution for $\mathcal{C}$ in the kernels of linear functionals that do not contain 1_X. Indeed, let Γ be a linear continuous functional on X such that $\Gamma(1_X) \neq 0$ and let $X_\diamond := \ker \Gamma$. It is easy to check that, given $x \in X$, the set $[x] \cap X_\diamond$ consists of single element, namely

$$[x] \cap X_\diamond = \left\{ x - \frac{\Gamma(x)}{\Gamma(1_X)} 1_X \right\}.$$

On the other hand, since $X_\diamond$ is a closed subspace and $1_X \notin X_\diamond$, a is coercive on $X_\diamond \times X_\diamond$ and there exists $C_\Gamma > 0$ such that $C_\Gamma^{-1} \|x\|_X^2 \leq a(x,x)$ for all $x \in X_\diamond$. Suppose that $x \in X_\diamond$ is a solution to $\mathcal{C}$. Then, from (2.10) and the continuity of f and P, it can be deduced that $\|x\|_X \leq C_\Gamma \|f\|_{Y^\star}$. Therefore, if $\bar{x}$ is a solution to $\mathcal{C}$ with f, there exists a unique element $\bar{x}_\Gamma := \bar{x} - (\Gamma(\bar{x})/\Gamma(1_X)) 1_X$ of $[\bar{x}]$ in $X_\diamond$ and $\|\bar{x}_\Gamma\|_X \leq C_\Gamma \|f\|_{Y^\star}$. Conversely, since any element $x \in X$ can be written as $x = (x - (\Gamma(x)/\Gamma(1_X)) 1_X) + (\Gamma(x)/\Gamma(1_X)) 1_X$, with $x - (\Gamma(x)/\Gamma(1_X)) 1_X \in X_\diamond$ and $(\Gamma(x)/\Gamma(1_X)) 1_X \in \operatorname{span}\{1_X\}$, it follows that if $\bar{x} \in X_\diamond$ satisfies

$$a(\bar{x}, x) = \langle f, Px \rangle \quad \text{for all } x \in X_\diamond,$$

then $\bar{x}$ satisfies (2.10).

Remark 2.6. In the light of Proposition 2.3 and Remark 2.5, we can assert that $\mathcal{C}$, $\mathcal{V}$, $\mathcal{C}'$, and $\mathcal{V}'$ are well-posed in the sense of *Hadamard* [42]: they have a unique solution, which depend continuously on the input (f in $\mathcal{C}$ and $\mathcal{C}'$, g in $\mathcal{V}$ and $\mathcal{V}'$). This stable character of the solutions is a consequence of the solution estimates and the linearity of the abstract problems.

Example 2.7. The abstract problems for the continuum model are presented below.

$\mathcal{C}$. Given $f \in Y_\diamond^\star = H_\diamond^{-1/2}(\partial\Omega)$, find a function $\bar{u} \in X = H^1(\Omega)$ satisfying

$$a(\bar{u}, u) = \int_\Omega \sigma \nabla \bar{u} \cdot \nabla u \, \mathrm{d}\mathbf{x} = \langle f, \gamma u \rangle \quad \text{for all } u \in X = H^1(\Omega).$$

This is the weak formulation of the continuum model written in terms of electric potentials u's and with applied current f. Choosing the linear functional $\Gamma(u) = \int_{\partial\Omega} \gamma u \, \mathrm{d}\mathbf{s}$, we have there exists a unique solution in $X_\diamond = \left\{ u \in H^1(\Omega) \,\middle|\, \int_{\partial\Omega} \gamma u \, \mathrm{d}\mathbf{s} = 0 \right\}$ by Remark 2.5.

$\mathcal{V}$. Given $g \in Y = H^{1/2}(\partial\Omega)$, find a function $\tilde{u} \in X = H^1(\Omega)$ satisfying

$$\begin{aligned} a(\tilde{u}, u) = \int_\Omega \sigma \nabla \tilde{u} \cdot \nabla u \, \mathrm{d}\mathbf{x} = 0 \quad & \text{for all } u \in \ker P = H_0^1(\Omega) \\ \gamma \tilde{u} = g \quad & \text{in } Y = H^{1/2}(\partial\Omega) \end{aligned}.$$

This is the weak formulation of the continuum model written in terms of electric potentials u's with applied voltage g. By Proposition 2.3(*ii*), constant voltages yield constant electric potentials.

$\mathcal{C}'$. Given $f \in Y_\diamond^\star = H_\diamond^{-1/2}(\partial\Omega)$, find a vector-valued function $\bar{\mathbf{p}} \in Z_\perp$ (that is, $\bar{\mathbf{p}} \in L^2\left(\Omega, \mathbb{R}^d\right)$ and $\nabla \cdot \bar{\mathbf{p}} = 0$) satisfying

$$\begin{aligned} b\left(\bar{\mathbf{p}}, \mathbf{p}\right) = \int_\Omega \frac{1}{\sigma} \bar{\mathbf{p}} \cdot \mathbf{p} \, \mathrm{d}\mathbf{x} = 0 \quad & \text{for all } \mathbf{p} \in \ker R \\ -\bar{\mathbf{p}} \cdot \mathbf{n} = f \quad & \text{in } H_\diamond^{-1/2}(\partial\Omega) \end{aligned}.$$

This is the weak formulation of the continuum model written in terms of current fields $\mathbf{p}$'s and with applied current f. Recall that $\mathbf{p} \in \ker R$ means that $\mathbf{p} \in L^2\left(\Omega, \mathbb{R}^d\right)$, $\nabla \cdot \mathbf{p} = 0$, and $\mathbf{p} \cdot \mathbf{n} = 0$. Observe that the first equation is equivalent to say that $\frac{1}{\sigma}\bar{\mathbf{p}}$ is orthogonal to $\ker R$. Hence, there exists $u \in H^1(\Omega)$ such that $\frac{1}{\sigma}\bar{\mathbf{p}} = \nabla u$ by the decomposition of $Z = L^2\left(\Omega, \mathbb{R}^d\right)$ (see Example 2.1). It implies that $\nabla \times \frac{1}{\sigma}\bar{\mathbf{p}} = 0$ if $\Omega \subset \mathbb{R}^3$ and $\nabla^\perp \cdot \frac{1}{\sigma}\bar{\mathbf{p}} = 0$ if $\Omega \subset \mathbb{R}^2$ in the sense of distributions (the first is the *curl* operator and the second is the *2-d rotation* operator). The converse is true when the domain Ω is simply connected.

$\mathcal{V}'$. Given $g \in Y = H^{1/2}(\partial\Omega)$, find a vector-valued function $\tilde{\mathbf{p}} \in Z_\perp$ satisfying

$$b\left(\tilde{\mathbf{p}}, \mathbf{p}\right) = \int_\Omega \frac{1}{\sigma} \tilde{\mathbf{p}} \cdot \mathbf{p} \, \mathrm{d}\mathbf{x} = \langle -\mathbf{p} \cdot \mathbf{n}, g \rangle \quad \text{for all } \mathbf{p} \in Z_\perp \ \left(\nabla \cdot \mathbf{p} = 0\right).$$

This is the weak formulation of the continuum model written in terms of current fields $\mathbf{p}$'s and with applied voltage g.

To conclude this example, note that the solutions of the above problems are related as $\bar{\mathbf{p}} = T\bar{u} = -\sigma\nabla\bar{u}$ and $\tilde{\mathbf{p}} = T\tilde{u} = -\sigma\nabla\tilde{u}$.

2.3 Current-Voltage maps

Two maps associated with the abstract problems are discussed in this section. These maps attempt to represent the current-to-voltage and voltage-to-current maps in EIT, which are respectively called *Neumann-to–Dirichlet* and *Dirichlet-to–Neumann* maps in the case of the continuum model.

Let us begin by defining them.

Definition 2.8. The *current-to-voltage* map is defined as

$$\Phi : Y_\diamond^\star \to Y/\mathbb{R} \quad \text{with} \quad \Phi f := [P\bar{x}], \tag{2.16}$$

where $\bar{x}$ is a solution to $\mathcal{C}$ with f.

Definition 2.9. The *voltage-to-current* map is defined as

$$\Psi : Y \to Y_\diamond^\star \quad \text{with} \quad \Psi g := R\tilde{z}, \tag{2.17}$$

where $\tilde{z}$ is the unique solution to $\mathcal{V}'$ with g.

Φ and Ψ represent the current-to-voltage and voltage-to-current maps in EIT, respectively. In (2.16), the applied current f generates the electrode voltage Φf, while in (2.17), the applied voltage g generates the electrode current Ψg.

Let us write some details to clarify the well-definiteness of the current-voltage maps. Since $P(1_X) = 1_Y$, we can deduce that $[P(x + \lambda 1_X)] = [Px]$ for all $x \in X$ and all $\lambda \in \mathbb{R}$, and hence Φ does not depend on the choice of $\bar{x}$. It is also true that $[P\bar{x}] = P([\bar{x}])$. The continuity and linearity of Ψg follows from those of R, the linearity of $\mathcal{V}'$, and the estimate for $\tilde{z}$ given in Proposition (2.3)(*iv*). Since $R(Z_\perp) = Y_\diamond^\star$, Ψg belongs to $Y_\diamond^\star$ for all $g \in Y$.

Notation. Observe that given $f \in Y_\diamond^\star$ and $g \in Y$ the set $f([g]) \subset \mathbb{R}$ has exactly one element, namely

$$f([g]) = \{\langle f, g\rangle + \lambda \langle f, 1_Y\rangle \mid \lambda \in \mathbb{R}\} = \{\langle f, g\rangle\}.$$

So, from now on $\langle f, [g]\rangle$ denotes the value $\langle f, g\rangle$.

Here are some elementary properties of the current-voltage maps.

Proposition 2.10.

(*i*) *Φ and Ψ are linear and continuous.*

(*ii*) *Φ and Ψ are symmetric in the sense that*

$$\langle f_1, \Phi f_2\rangle = \langle f_2, \Phi f_1\rangle \quad \textit{and} \quad \langle \Psi g_1, g_2\rangle = \langle \Psi g_2, g_1\rangle$$

for all $f_1, f_2 \in Y_\diamond^\star$ and all $g_1, g_2 \in Y$.

(*iii*) *The inverse properties*

$$\Psi(\Phi f) = \{f\} \quad \textit{for all } f \in Y_\diamond^\star \quad \textit{and} \quad \Phi(\Psi g) = [g] \quad \textit{for all } g \in Y$$

hold. Moreover,

$$\begin{aligned} \ker \Phi &= \{0_{Y^\star}\}, & \ker \Psi &= \operatorname{span}\{1_Y\}, \\ \operatorname{ran} \Phi &= Y/\mathbb{R}, & \operatorname{ran} \Psi &= Y_\diamond^\star. \end{aligned}$$

(*iv*) *Φ induces on $Y_\diamond^\star$ the inner product*

$$\langle \cdot, \cdot\rangle_\Phi : Y_\diamond^\star \times Y_\diamond^\star \to \mathbb{R} \quad \textit{with} \quad \langle f_1, f_2\rangle_\Phi := \langle f_1, \Phi f_2\rangle.$$

There exists $C > 0$ such that $\|f\|_\Phi \le C\|f\|_{Y^\star}$ for all $f \in Y_\diamond^\star$, where $\|\cdot\|_\Phi$ is the norm induced by $\langle \cdot, \cdot\rangle_\Phi$.

(*v*) *Ψ induces on Y the semi-inner product*

$$\langle \cdot, \cdot\rangle_\Psi : Y \times Y \to \mathbb{R} \quad \textit{with} \quad \langle g_1, g_2\rangle_\Psi := \langle \Psi g_1, g_2\rangle.$$

There exists $C > 0$ such that $\|g\|_\Psi \le C\|g\|_Y$ for all $g \in Y$, where $\|\cdot\|_\Psi$ is the semi-norm induced by $\langle \cdot, \cdot\rangle_\Psi$.

Proof of (i). The linearity of Φ (resp. Ψ) follows from the linearity of P (resp. R) and the fact that $\mathcal{C}$ (resp. $\mathcal{V}'$) is a linear problem. The continuity of Φ (resp. Ψ) follows from the continuity of P (resp. R) and the solution estimate of $\mathcal{C}$ (resp. $\mathcal{V}'$) given in Proposition 2.3. □

Proof of (ii). Let $f_1, f_2 \in Y^\star_\diamond$ and $g_1, g_2 \in Y$. Observe the following:

(1) Let $\bar{x}_1, \bar{x}_2$ be solutions to $\mathcal{C}$ with f_1, f_2 respectively. Then $\langle f_1, P\bar{x}_2\rangle = a(\bar{x}_1, \bar{x}_2)$ by (2.10).

(2) Let $\tilde{z}_1, \tilde{z}_2$ be solutions to $\mathcal{V}'$ with g_1, g_2 respectively. Then $\langle R\tilde{z}_1, g_2\rangle = b(\tilde{z}_2, \tilde{z}_1)$ by (2.15).

Combining the above with the symmetry of a and b we obtain the result. □

Proof of (iii). Let $f \in Y^\star_\diamond$ and $g \in Y$. Let $\bar{x}$ be a solution to $\mathcal{C}$ with f and let $\tilde{z}$ be the unique solution to $\mathcal{V}'$ with g. Since $T\tilde{x} = \tilde{z}$, we have

$$\langle \Psi g, \Phi f\rangle = \langle R\tilde{z}, P\bar{x}\rangle = b(T\bar{x}, \tilde{z}) = a(\bar{x}, \tilde{x}) = \langle f, P\tilde{x}\rangle = \langle f, g\rangle$$

by (2.5) and (2.1). Hence $\langle \Psi g, \Phi f\rangle = \langle f, g + \lambda 1_Y\rangle$ for all $\lambda \in \mathbb{R}$. To conclude, it suffices to use the symmetry of Φ and Ψ showed in (ii): by the symmetry of Ψ, $\langle \Psi y, g\rangle = \langle \Psi g, \Phi f\rangle = \langle f, g\rangle$ for all $g \in Y$ and all $y \in \Phi f$, and hence $\Psi(\Phi f) = \{f\}$; by the symmetry of Φ, $\langle f, \Phi(\Psi g)\rangle = \langle \Psi g, \Phi f\rangle = \langle f, [g]\rangle$ for all $f \in Y^\star_\diamond$, and since $Y^\star_\diamond$ is isometric to $Y/\mathbb{R}$ [21, Prop. 11.9], we have $\Phi(\Psi g) = [g]$. The kernel and range statements are derived directly from the inverse properties. □

Proof of (iv). Let $f \in Y^\star_\diamond$ and $\bar{x}$ be a solution to $\mathcal{C}$ with f. By (1) in the proof of (ii),

$$\langle f, f\rangle_\Phi = \langle f, P\bar{x}\rangle = a(\bar{x}, \bar{x}). \tag{2.18}$$

Hence $\langle f, f\rangle_\Phi \geq 0$. We prove that $\langle f, f\rangle_\Phi = 0$ iff $f = 0_{Y^\star}$. If $\langle f, f\rangle_\Phi = 0$ then $[\bar{x}] = \operatorname{span}\{1_X\}$ by (2.18). Hence $f = 0_{Y^\star}$ by Proposition 2.3(i). The converse is obvious. Moreover, $\langle\cdot,\cdot\rangle_\Phi$ is symmetric by (ii) and is bilinear by the linearity of P and $\mathcal{C}$. Therefore $\langle\cdot,\cdot\rangle_\Phi$ is an inner product. On the other hand, by (2.18) and the properties of a, we have $\langle f, f\rangle_\Phi \leq C\,\|\bar{x} + \lambda 1_X\|^2$ for all $\lambda \in \mathbb{R}$. Combining this inequality with the solution estimate given in Proposition 2.3(i) we deduce that $\langle f, f\rangle_\Phi \leq C\,\|f\|^2_{Y^\star}$. □

Proof of (v). Let $g \in Y$ and let $\tilde{z}$ be the solution to $\mathcal{V}'$ with g. By (2) in the proof of (ii),

$$\langle g, g\rangle_\Psi = \langle R\tilde{z}, g\rangle = b(\tilde{z}, \tilde{z}). \tag{2.19}$$

Hence $\langle g, g\rangle_\Psi \geq 0$. We prove that $\langle g, g\rangle_\Psi = 0$ iff $g \in \operatorname{span}\{1_Y\}$. If $\langle g, g\rangle_\Psi = 0$ then and $\tilde{z} = 0_Z$ by (2.19). Hence $g \in \operatorname{span}\{1_Y\}$ by Proposition 2.3(iv). If $g \in \operatorname{span}\{1_Y\}$ then $\langle g, g\rangle_\Psi = 0$ since $R\tilde{z} \in Y^\star_\diamond$. On the other hand, $\langle\cdot,\cdot\rangle_\Psi$ is symmetric by (ii) and is bilinear by the linearity of R and $\mathcal{V}'$. Therefore $\langle\cdot,\cdot\rangle_\Psi$ is a semi-inner product. On the other hand, by (2.19) and the continuity of b, we have $\langle g, g\rangle_\Psi \leq C\,\|\tilde{z}\|^2$. Combining this inequality with the solution estimate given in Proposition 2.3(iv) we deduce that $\|g\|_\Psi \leq C\,\|g\|_Y$. □

Remark 2.11 (Are Φ and Ψ inverses of each other?). Since $\ker\Psi = \operatorname{span}\{1_Y\}$, Ψ can be defined on $Y/\mathbb{R}$. In this situation, Φ and Ψ are inverses of each other and the inverse properties read as $\Psi(\Phi f) = f \in Y_\diamond^\star$ and $\Phi(\Psi g) = [g] \in Y/\mathbb{R}$. An alternative is the following. Let $\Upsilon \in Y^\star$ be such that $\Upsilon(1_Y) \neq 0$. Let $Y_\diamond := \ker\Upsilon$. It is easy to check that, given $g \in Y$, the set $[g] \cap Y_\diamond$ consists of a single element, namely

$$[g] \cap Y_\diamond = \left\{ g - \frac{\Upsilon(g)}{\Upsilon(1_Y)} 1_Y \right\}.$$

Define the map $\Phi_\Upsilon : Y_\diamond^\star \to Y_\diamond \subset Y$ by

$$\Phi_\Upsilon f := \text{the unique element of } \Phi f \cap Y_\diamond.$$

One can prove that Φ_Υ and $\Psi|_{Y_\diamond}$ are inverses of each other using the inverse properties given in Proposition 2.10(*iii*). If, in addition, 1_Y is orthogonal to $Y_\diamond$, it turns out that Φ_Υ is the generalized inverse of Ψ by the *Desoer-Whalen* equivalence [101, Def. 11.1.5]. Moreover, observe that if Γ in (2.5) is chosen so that $\Gamma = \Upsilon \circ P$, then $\Phi_\Upsilon f = P\bar{x}_\Gamma$.

Example 2.12. The current-voltage maps for the continuum model are

- (*current-to-voltage* map) $\Phi : Y_\diamond^\star = H_\diamond^{-1/2}(\partial\Omega) \to Y/\mathbb{R} = H^{1/2}(\partial\Omega)/\mathbb{R}$ given by $\Phi f := [P\bar{u}] = \{\gamma\bar{u} + \lambda\gamma\mathbf{1} \mid \lambda \in \mathbb{R}\}$,
- (*voltage-to-current* map) $\Psi : Y = H^{1/2}(\partial\Omega) \to Y_\diamond^\star = H_\diamond^{-1/2}(\partial\Omega)$ given by $\Psi g = R(T\tilde{u}) = \sigma\nabla\tilde{u}\cdot\mathbf{n}$,

where $\bar{u} \in H^1(\Omega)$ is a solution to the weak formulation of continuum model with current f (problem $\mathcal{C}$) and $\tilde{u} \in H^1(\Omega)$ is the unique solution to the weak formulation of continuum model with voltage g (problem $\mathcal{V}$). We have used the fact $T\tilde{u}$ is the unique solution to $\mathcal{V}'$ with g. Thus, Φ maps the applied current f to the potential generated on $\partial\Omega$ (up to additive constants), while Ψ maps the applied voltage g to the current density generated on $\partial\Omega$. Hence, Φ and Ψ are the well-known *Neumann-to-Dirichlet* (NtD) and *Dirichlet-to-Neumann* (DtN) maps of the continuum model, respectively. If we consider $\Gamma(u) = \int_{\partial\Omega}\gamma u\,\mathrm{d}\mathbf{s}$ and $\Upsilon(g) = \int_{\partial\Omega} g\,\mathrm{d}\mathbf{s}$ linear continuous functionals on $X = H^1(\Omega)$ and $Y = H^{1/2}(\partial\Omega)$, respectively, by Remarks 2.5 and 2.11 (note that $\Gamma = \Upsilon\circ\gamma$) we have $\Phi_\Upsilon : Y_\diamond^\star \to Y_\diamond$ defined by

$$\Phi_\Upsilon f = \text{the unique element of } \Phi f \cap Y_\diamond = \gamma\bar{u}_\Gamma$$

and $\Psi|_{Y_\diamond}$ are inverses of each other, with $\bar{u}_\Gamma = \bar{u} - \left(\frac{1}{|\partial\Omega|}\int_{\partial\Omega}\gamma\bar{u}\,\mathrm{d}\mathbf{s}\right)\mathbf{1}$,

$$\bar{u}_\Gamma \in X_\diamond = \ker\Gamma = \left\{ u \in H^1(\Omega) \,\middle|\, \int_{\partial\Omega}\gamma u\,\mathrm{d}\mathbf{s} = 0 \right\},$$

$$\gamma\bar{u}_\Gamma \in Y_\diamond = \ker\Upsilon = \left\{ g \in H^{1/2}(\partial\Omega) \,\middle|\, \int_{\partial\Omega} g\,\mathrm{d}\mathbf{s} = 0 \right\}.$$

	current f	voltage g
abstract problems	$\mathcal{C}$, $\mathcal{C}'$	$\mathcal{V}$, $\mathcal{V}'$
current-voltage maps	Φ	Ψ
power functions	ϕ	ψ

Table 2.1: Problems, maps, and functions classified according their input.

2.4 Dual properties

Here two non-negative functions that model the power dissipated during current and voltage injections are analyzed. Their properties reveal the links between all the abstract problems.

Definition 2.13. The *power* function associated to current-to-voltage map Φ is defined as

$$\phi : Y_\diamond^\star \to \mathbb{R} \quad \text{with} \quad \phi(f) \coloneqq \frac{1}{2} \langle f, f \rangle_\Phi ,$$

where $\langle \cdot, \cdot \rangle_\Phi$ is the inner product induced by Φ on $Y_\diamond^\star$.

Definition 2.14. The *power* function associated to voltage-to-current map Ψ is defined as

$$\psi : Y \to \mathbb{R} \quad \text{with} \quad \psi(g) \coloneqq \frac{1}{2} \langle g, g \rangle_\Psi ,$$

where $\langle \cdot, \cdot \rangle_\Psi$ is the semi-inner product induced by Ψ on Y.

Let us start with some observations.

1. Since $\langle \cdot, \cdot \rangle_\Phi$ and $\langle \cdot, \cdot \rangle_\Psi$ are inner and semi-inner products respectively (see Proposition 2.10(*iv*)(*v*)), it follows that

$$\phi(f) = \phi(-f) \geq 0, \quad \psi(g) = \psi(-g) \geq 0,$$

$$\phi(f) = 0 \Leftrightarrow f = 0_{Y^\star}, \quad \phi(g) = 0 \Leftrightarrow g \in \operatorname{span}\{1_Y\},$$

$$\phi(f_1 - f_2) = \phi(f_1) - \langle f_1, f_2 \rangle_\Phi + \phi(f_2), \quad \text{and}$$

$$\psi(g_1 - g_2) = \psi(g_1) - \langle g_1, g_2 \rangle_\Psi + \psi(g_2)$$

   for all $f, f_1, f_2 \in Y_\diamond^\star$ and all $g, g_1, g_2 \in Y$.
2. In the case that f and g are related to each other as $f = \Psi g$ or $g \in \Phi f$, Proposition 2.10(*iii*) yields

$$\phi(f) = \psi(g) = \frac{1}{2} \langle f, g \rangle . \tag{2.20}$$

In the context of EIT, $\phi(f)$ and $\psi(g)$ represent the power dissipated into heat during current and voltage injection, respectively. Thus, (2.20) says that when the current f and voltage g are associated with the same EIT experiment, then the powers $\phi(f)$ and

$\psi(g)$ are equal. In the past, this physical quantity was used in the study of the EIT inverse problem [17, 20].

The following theorem is our main result. It shows that ϕ and ψ connect all the abstract problems by means of several properties.

Theorem 2.15.

(i) *ϕ is continuous, convex, and satisfies*

$$\phi(f) = \max_{x\in X}\left\{\langle f, Px\rangle - \frac{1}{2}a(x,x)\right\} = \min_{\substack{z\in Z_\perp\\ Rz=f}}\left\{\frac{1}{2}b(z,z)\right\} \tag{2.21}$$

for all $f \in Y_\diamond^\star$.

(ii) *ψ is continuous, convex, and satisfies*

$$\psi(g) = \min_{\substack{x\in X\\ Px=g}}\left\{\frac{1}{2}a(x,x)\right\} = \max_{z\in Z_\perp}\left\{\langle Rz, g\rangle - \frac{1}{2}b(z,z)\right\} \tag{2.22}$$

for all $g \in Y$.

(iii) *The functions ϕ and ψ are related to each other as*

$$\phi(f) = \max_{g\in Y}\{\langle f, g\rangle - \psi(g)\} \quad \text{for all } f \in Y_\diamond^\star \tag{2.23}$$

and

$$\psi(g) = \max_{f\in Y_\diamond^\star}\{\langle f, g\rangle - \phi(f)\} \quad \text{for all } g \in Y, \tag{2.24}$$

where the maximums are attained at all $g \in \Phi f$ and $f = \Psi g$, respectively. In other words, the inequality

$$\langle f, g\rangle \le \phi(f) + \psi(g) \quad \text{for all } f \in Y_\diamond^\star \text{ and all } g \in Y$$

holds, and equality holds iff $f = \Psi g$ or $g \in \Phi f$.

(iv) *Given $f \in Y_\diamond^\star$ and $g \in Y$, we have the identity*

$$\phi(f) - \langle f, g\rangle + \psi(g) = \frac{1}{2}a(\bar{x}-\tilde{x}, \bar{x}-\tilde{x}) = \frac{1}{2}b(\bar{z}-\tilde{z}, \bar{z}-\tilde{z}), \tag{2.25}$$

where $\bar{x}, \tilde{x}, \bar{z}, \tilde{z}$ are the corresponding solutions of the abstract problems with f and g. In the case that $f = \Psi g$ or $g \in \Phi f$,

$$\bar{x} - \tilde{x} \in \operatorname{span}\{1_X\} \quad \text{and} \quad \bar{z} = \tilde{z}. \tag{2.26}$$

(v) *We have*

$$\begin{aligned} &\phi(f) - \langle f, g\rangle + \psi(g) \\ &\quad = \min_{\substack{(x,z)\in X\times Z_\perp\\ Px=g,\, Rz=f}} \frac{1}{2}b(z-Tx, z-Tx) && (2.27)\\ &\quad = \max_{(x,z)\in X\times Z_\perp} \langle f - Rz, Px - g\rangle - \frac{1}{2}b(z-Tx, z-Tx) && (2.28)\end{aligned}$$

for all $f \in Y_\diamond^\star$ and all $g \in Y$.

1. *The minimum is attained at* $(x_{min}, z_{min}) = (\tilde{x}, \bar{z})$, *where* $\tilde{x}$ *is the unique solution to* $\mathcal{V}$ *with* g *and* $\bar{z}$ *is the unique solution to* $\mathcal{C}'$ *with* f. *Moreover,*

$$z_{min} = Tx_{min} \Leftrightarrow \textit{the minimum value is } 0 \Leftrightarrow f = \Psi g \vee g \in \Phi f.$$

2. *The maximum is attained at all* $(x_{max}, z_{max}) \in [\bar{x}] \times \{\tilde{z}\}$, *where* $\bar{x}$ *is a solution to* $\mathcal{C}$ *with* f *and* $\tilde{z}$ *is the unique solution to* $\mathcal{V}'$ *with* g. *Moreover,*

$$z_{max} = Tx_{max} \Leftrightarrow \textit{the maximum value is } 0 \Leftrightarrow f = \Psi g \vee g \in \Phi f.$$

Proof of (i). The continuity and convexity of ϕ follow from the fact that $\phi(\cdot) = (1/2)\|\cdot\|_\Phi^2$ and the estimate $\|\cdot\|_\Phi \le C\|\cdot\|_{Y^\star}$ given in Proposition 2.10(*iv*). Let $f \in Y_\diamond^\star$. Let $\bar{x}$, $\bar{z}$ be solutions to $\mathcal{C}$ and $\mathcal{C}'$ respectively, both with f. From (2.18) and (2.1) it follows that

$$\phi(f) = \begin{cases} \frac{1}{2}a(\bar{x}, \bar{x}) = \langle f, P\bar{x}\rangle - \frac{1}{2}a(\bar{x}, \bar{x}) \\ \frac{1}{2}b(\bar{z}, \bar{z}) \end{cases} \tag{2.29}$$

since $T\bar{x} = \bar{z}$. Thus, (2.21) follows by combining (2.29) with fact that $\bar{x}$ and $\bar{z}$ are also solutions to $\max_{x\in X} J(x)$ and $\min_{\substack{z\in Z_\perp \\ Rz=f}} J'(z)$ respectively. ☐

Proof of (ii). The continuity and convexity of ψ follow from the fact that $\psi(\cdot) = (1/2)\|\cdot\|_\Psi^2$ and the estimate $\|\cdot\|_\Psi \le C\|\cdot\|_Y$ given in Proposition 2.10(*v*). Let $g \in Y$ and $\tilde{x}$ be the unique solution to $\mathcal{V}$ with g. Let $g \in Y$. Let $\tilde{x}$, $\tilde{z}$ be solutions to $\mathcal{V}$ and $\mathcal{V}'$ respectively, both with g. From (2.19) and (2.1) it follows that

$$\psi(g) = \frac{1}{2}a(\tilde{x}, \tilde{x}) = \frac{1}{2}b(\tilde{z}, \tilde{z}) = \langle R\tilde{z}, g\rangle - \frac{1}{2}b(\tilde{z}, \tilde{z})$$

$$\psi(g) = \frac{1}{2}a(\tilde{x}, \tilde{x}) = \frac{1}{2}b(\tilde{z}, \tilde{z}) = \langle R\tilde{z}, g\rangle - \frac{1}{2}b(\tilde{z}, \tilde{z}) \tag{2.30}$$

since $T\tilde{x} = \tilde{z}$. Thus, (2.22) follows by combining (2.30) with the fact that $\tilde{x}$ and $\tilde{z}$ are also solutions to $\min_{\substack{x\in X \\ Px=g}} K(x)$ and $\max_{z\in Z_\perp} K'(z)$ respectively. ☐

Proof of (iii). Let $f \in Y_\diamond^\star$ and $g \in Y$. From (*i*) and (*ii*) it follows that

$$\begin{aligned} -\phi(f) \le \min_{\substack{x\in X \\ Px=g}} \left\{\frac{1}{2}a(x,x) - \langle f, Px\rangle\right\} &= \min_{\substack{x\in X \\ Px=g}} \left\{\frac{1}{2}a(x,x)\right\} - \langle f, g\rangle \\ &= \psi(g) - \langle f, g\rangle . \end{aligned}$$

Therefore $\langle f, g\rangle \le \phi(f) + \psi(g)$. Fix a $f \in Y_\diamond^\star$. Then $\langle f, g\rangle - \psi(g) \le \phi(f)$ for all $g \in Y$, and taking the supremum over g yields

$$\sup_{g\in Y} \{\langle f, g\rangle - \psi(g)\} \le \phi(f).$$

Let $g' \in \Phi f$. Since $\phi(f) = \psi(g') = (1/2)\langle f, g'\rangle$, it follows that g' is a maximizer. Suposse that $g'' \in Y$ is another maximizer. Then $\langle f, g''\rangle - \psi(g'') = \psi(g')$. Moreover, $\Psi g' = f$ since $\Psi(\Phi f) = \{f\}$ (see Proposition 2.10(*iii*)). Thus $\langle f, g''\rangle = \langle g', g''\rangle_\Psi$, which implies $\psi(g' - g'') = 0$.

Now fix a $g \in Y$. Then $\langle f, g\rangle - \phi(f) \le \psi(g)$ for all $f \in Y_\diamond^\star$, and taking the supremum over f yields

$$\sup_{f \in Y_\diamond^\star} \{\langle f, g\rangle - \phi(f)\} \le \psi(g).$$

Let $f' = \Psi g$. Since $\phi(f') = \psi(g) = (1/2)\langle f', g\rangle$, it follows that f' is a maximizer. Suposse that $f'' \in Y_\diamond^\star$ is another maximizer. Then $\langle f'', g\rangle - \phi(f'') = \phi(f')$. Moreover, $\Phi f' = [g]$ since $\Phi(\Psi g) = [g]$ (see Proposition 2.10(*iii*)). Thus $\langle f'', g\rangle = \langle f'', f\rangle_\Phi$, which implies $\phi(f' - f'') = 0$. □

Proof of (iv). First, observe that since $\bar{x}$ solves (2.10) with f and $P\tilde{x} = g$, it follows that $a(\bar{x}, \tilde{x}) = \langle f, P\tilde{x}\rangle = \langle f, g\rangle$; also, since $\tilde{z}$ solves (2.15) with g and $R\bar{z} = f$, it follows that $b(\tilde{z}, \bar{z}) = \langle R\bar{z}, g\rangle = \langle f, g\rangle$. Combining these equalities with (2.29) and (2.30) we obtain the identity. On the other hand, if $f = \Psi g$ or $g \in \Phi f$ then $\phi(f) - \langle f, g\rangle + \psi(g) = 0$ by (*iv*). Thus, (2.26) is a consequence of (2.25) and of the coercivity of a and b. □

Proof of (v). Let $f \in Y_\diamond^\star$ and $g \in Y$. From (*i*) and (*ii*), it can be deduced that

$$\phi(f) - \langle f, g\rangle + \psi(g) \le \frac{1}{2} b(z, z) - \langle f, g\rangle + \frac{1}{2} a(x, x) \tag{2.31}$$

for all $(x, z) \in X \times Z_\perp$ such that $Px = g$ and $Rz = f$, and

$$\phi(f) - \langle f, g\rangle + \psi(g) \ge \langle f, Px\rangle - \frac{1}{2} a(x, x) - \langle f, g\rangle + \langle Rz, g\rangle - \frac{1}{2} b(z, z) \tag{2.32}$$

for all $(x, z) \in X \times Z_\perp$. Using (2.1) and (2.5), the right-hand side of (2.31) can be written as

$$\frac{1}{2} b(z - Tx, z - Tx)$$

and the right-hand side of (2.32) can be written as

$$\langle f - Rz, Px - g\rangle - \frac{1}{2} b(z - Tx, z - Tx).$$

If $(x, z) = (\tilde{x}, \bar{z})$ in (2.31), it is clear that equality holds. Hence $(\tilde{x}, \bar{z})$ is a minimizer of (2.27). Let $E(x, z) := \frac{1}{2} b(z - Tx, z - Tx)$ and let (x', z') be another minimizer. Then $E(x', z') \le E(x, z')$, i.e., $(1/2)\, a(x', x') \le (1/2)\, a(x, x)$, for all $x \in X$ such that $Px = g$. Since $\mathcal{V}$ is equivalent to $\min_{\substack{x \in X \\ Px = g}} K(x)$, $x' = \tilde{x}$ by uniqueness of solution. Similarly, from $E(x', z') \le E(x', z)$ for all $z \in Z_\perp$ such that $Rz = f$ we deduce that $z' = \bar{z}$. Therefore the minimum in (2.27) is attained at $(\tilde{x}, \bar{z})$. Similar arguments show that the maximum in (2.28) is attainted at all $(x, z) \in [\bar{x}] \times \{\tilde{z}\}$. From (*iii*) and (*iv*) we obtain the equivalences. □

Remark 2.16. From Theorem 2.15(*iii*), ϕ and ψ are the *Legendre-Fenchel conjugate* of each other [11, Def. 9.3.1].

Remark 2.17. Let us point out that (*i*) and (*ii*) imply the following inequalities:

- Given $f \in Y_\diamond^\star$,

$$J(x) = \langle f, Px\rangle - \frac{1}{2}a(x,x) \leq \phi(f) \leq \frac{1}{2}b(z,z) = J'(z)$$

  for all $x \in X$ and all $z \in Z_\perp$ such that $Rz = f$.

- Given $g \in Y$,

$$K'(z) = \langle Rz, g\rangle - \frac{1}{2}b(z,z) \leq \psi(g) \leq \frac{1}{2}a(x,x) = K(x)$$

  for all $x \in X$ such that $Px = g$ and all $z \in Z_\perp$.

These inequalities provide upper and lower bounds for the values of the power functions.

Example 2.18. By definition, the power functions for the continuum model are given by $\phi(f) = \frac{1}{2}\langle f, \gamma\bar{u}\rangle$ and $\psi(g) = \frac{1}{2}\langle \sigma\nabla\tilde{u}\cdot\mathbf{n}, g\rangle$, where $\bar{u}$ is a solution to $\mathcal{C}$ with f and $\tilde{u}$ is the unique solution to $\mathcal{V}$ with g. From Theorem 2.15(*i*)(*ii*) it follows that the power functions can be expressed as the minimization problems

$$\phi(f) = \min_{\substack{\mathbf{p}\in L^2(\Omega,\mathbb{R}^d)\\ \nabla\cdot\mathbf{p}=0\\ -\mathbf{p}\cdot\mathbf{n}=f}} \left\{\frac{1}{2}\int_\Omega \frac{1}{\sigma}|\mathbf{p}|^2\,\mathrm{d}\mathbf{x}\right\} \quad f \in Y_\diamond^* = H_\diamond^{-1/2}(\partial\Omega)$$

and

$$\psi(g) = \min_{\substack{u\in H^1(\Omega)\\ \gamma u = g}} \left\{\frac{1}{2}\int_\Omega \sigma|\nabla u|^2\,\mathrm{d}\mathbf{x}\right\} \quad g \in Y = H^{1/2}(\partial\Omega).$$

(recall that given $\mathbf{p} \in L^2(\Omega,\mathbb{R}^d)$ such that $\nabla\cdot\mathbf{p} \in L^2(\Omega)$, $\mathbf{p}\cdot\mathbf{n} \in H^{-1/2}(\partial\Omega)$ [88, Thm. 6.1]). These are the well-known *Thomson* and *Dirichlet variational principles* for the continuum model [32, Ch. 4][72, 17][31, Subs. 2.1.3]. From Theorem 2.15(*i*)(*ii*) and Remark 2.17 one can also see that upper and lower bounds for the power dissipated arise in a complementary manner when f and g are related to each other as $f = \Psi g$ or $g \in \Phi f$, namely (recall that $\phi(f) = \psi(g) = \frac{1}{2}\langle f, g\rangle$ in this case)

$$\langle f, \gamma u_1\rangle - \frac{1}{2}\int_\Omega \sigma|\nabla u_1|^2\,\mathrm{d}\mathbf{x} \leq \frac{1}{2}\langle f, g\rangle \leq \frac{1}{2}\int_\Omega \sigma|\nabla u_2|^2\,\mathrm{d}\mathbf{x}$$

for all $u_1, u_2 \in H^1(\Omega)$ such that $\gamma u_2 = g$, and

$$\langle -\mathbf{p}_1\cdot\mathbf{n}, g\rangle - \frac{1}{2}\int_\Omega \frac{1}{\sigma}|\mathbf{p}_1|^2\,\mathrm{d}\mathbf{x} \leq \frac{1}{2}\langle f, g\rangle \leq \frac{1}{2}\int_\Omega \frac{1}{\sigma}|\mathbf{p}_2|^2\,\mathrm{d}\mathbf{x}$$

for all $\mathbf{p}_1, \mathbf{p}_2 \in L^2(\Omega,\mathbb{R}^d)$ such that $\nabla\cdot\mathbf{p}_1 = \nabla\cdot\mathbf{p}_2 = 0$ and $-\mathbf{p}_2\cdot\mathbf{n} = f$. In [10] was obtained a similar inequality to the first one for the electrostatic energy generated by a charge density. By Theorem 2.15(*iii*), the relations

$$\frac{1}{2}\langle f, \Phi f\rangle = \max_{g\in H^{1/2}(\partial\Omega)} \left\{\langle f, g\rangle - \frac{1}{2}\langle \Psi g, g\rangle\right\} \quad \text{for all } f \in Y_\diamond^* = H_\diamond^{-1/2}(\partial\Omega)$$

and

$$\frac{1}{2}\langle \Psi g, g\rangle = \max_{f\in H_\diamond^{-1/2}(\partial\Omega)} \left\{\langle f, g\rangle - \frac{1}{2}\langle f, \Phi f\rangle\right\} \quad \text{for all } g \in Y = H^{1/2}(\partial\Omega)$$

hold, which are called *convex duality relations* [19, Lem. 4]. In fact, this shows that ϕ and ψ are *conjugate* functions [36, Ch. 1 Def. 4.1]. On the other hand, the minimization problem (2.27) read as

$$\min_{\substack{(u,\mathbf{p})\in H^1(\Omega)\times L^2(\Omega,\mathbb{R}^d)\\ \nabla\cdot\mathbf{p}=0\\ \gamma u=g,\, -\mathbf{p}\cdot\mathbf{n}=f}} \frac{1}{2}\int_\Omega \left|\sigma^{-1/2}\mathbf{p} + \sigma^{1/2}\nabla u\right|^2 \,\mathrm{d}\mathbf{x} \tag{2.33}$$

and the maximization problem (2.28) reads as

$$\max_{\substack{(u,\mathbf{p})\in H^1(\Omega)\times L^2(\Omega,\mathbb{R}^d)\\ \nabla\cdot\mathbf{p}=0}} \langle f + \mathbf{p}\cdot\mathbf{n}, \gamma u - g\rangle - \frac{1}{2}\int_\Omega \left|\sigma^{-1/2}\mathbf{p} + \sigma^{1/2}\nabla u\right|^2 \,\mathrm{d}\mathbf{x}.$$

The minimization problem (2.27) is closely related to the *Kohn–Vogelius functional approach* [72, 71], which aims to recover the internal conductivity from Dirichlet and Neumman data (g, f) (in our context, this means $f = \Psi g$ or $g \in \Phi f$) by minimizing the functional $(\sigma, u, \mathbf{p}) \mapsto \frac{1}{2}\int_\Omega \left|\sigma^{-1/2}\mathbf{p} + \sigma^{1/2}\nabla u\right|^2 \mathrm{d}\mathbf{x}$ subject to $\sigma_- \le \sigma \le \sigma_+$, $\gamma u = g$, $\nabla\cdot\mathbf{p} = 0$, and $-\mathbf{p}\cdot\mathbf{n} = f$, with σ_-, σ_+ being two positive bounds. This problem is (2.27) with σ as a variable. Recall that the above optimization problems admit unrelated f and g.

2.5 Error estimates

In this section, a posteriori error estimates for approximate solutions of the abstract problems are obtained. These results are inspired by [99], where a certain class of linear boundary value problems for elliptic partial differential equations was considered and a posteriori error estimates were obtained by using the associated complementary extremal principles.

Notation. In this section, we denote $E(x, z) = \frac{1}{2}b(z - Tx, z - Tx)$ for all $x \in X$ and all $z \in Z$. This functional refers to the global error functional considered in the *constitutive error approach* [85, 3].

The following proposition provides general error estimates for pairs of approximate solutions $(x, z) \in X \times Z_\perp$, where one corresponds to an abstract problem with f (i.e. $\mathcal{C}$ or $\mathcal{C}'$) and the other one corresponds to an abstract problem with g (i.e. $\mathcal{V}$ or $\mathcal{V}'$).

Proposition 2.19. *Let $f \in Y_\diamond^\star$ and $g \in Y$.*

(i) Let $\tilde{x}$ and $\bar{z}$ be the solutions of $\mathcal{V}$ with g and $\mathcal{C}'$ with f, respectively. Then

$$\begin{aligned}&\frac{1}{2}a(\tilde{x}-x,\tilde{x}-x)+\frac{1}{2}b(\bar{z}-z,\bar{z}-z)\\&\quad=E(x,z)-\langle f-Rz,Px-g\rangle+\langle f-Rz,\Phi f-g\rangle+\langle f-\Psi g,Px-g\rangle\\&\qquad-(\phi(f)-\langle f,g\rangle+\psi(g))\qquad\textit{for all }(x,z)\in X\times Z_{\perp}.\end{aligned}\tag{2.34}$$

Consequently,

$$\frac{1}{2}a(\tilde{x}-x,\tilde{x}-x)+\frac{1}{2}b(\bar{z}-z,\bar{z}-z)\le E(x,z)$$
$$\textit{for all }(x,z)\in X\times Z_{\perp}\textit{ such that }Px=g\textit{ and }Rz=f.$$

(ii) Let $\bar{x}$ and $\tilde{z}$ be the solutions of $\mathcal{C}$ with f and $\mathcal{V}'$ with g, respectively. Then

$$\begin{aligned}&\frac{1}{2}a(\bar{x}-x,\bar{x}-x)+\frac{1}{2}b(\tilde{z}-z,\tilde{z}-z)\\&\quad=E(x,z)-\langle f-Rz,Px-g\rangle\\&\qquad+(\phi(f)-\langle f,g\rangle+\psi(g))\qquad\textit{for all }(x,z)\in X\times Z_{\perp}.\end{aligned}\tag{2.35}$$

Consequently,

$$\frac{1}{2}a(\bar{x}-x,\bar{x}-x)+\frac{1}{2}b(\tilde{z}-z,\tilde{z}-z)\ge E(x,z)-\langle f-Rz,Px-g\rangle$$
$$\textit{for all }(x,z)\in X\times Z_{\perp}.$$

Therefore, when x and z are considered as approximate solutions, (i) and (ii) represent a posteriori error estimates. The term $E(x,z)$ represents the error on the domain and the duality pairing terms represent the error on the boundary. The last estimates in (i) and (ii) are consequences of the fact that $\phi(f)-\langle f,g\rangle+\psi(g)$ is always non-negative (see Theorem 2.15(iii)); in these estimates the error bound functions are the objective functionals of the optimization problems (2.27) and (2.28), respectively.

Proof of (i). By (2.1), the left-hand side of (2.34) can be rewritten as

$$\begin{aligned}&=\tfrac{1}{2}b(z-Tx,z-Tx)-(\phi(f)-\langle f,g\rangle+\psi(g))\\&\quad+(b(z,Tx)+a(\tilde{x},\tilde{x}-x)+b(\bar{z},\bar{z}-z)-\langle f,g\rangle)\end{aligned}\tag{2.36}$$

since $(1/2)\,a(\tilde{x},\tilde{x})=\psi(g)$ and $(1/2)\,b(\bar{z},\bar{z})=\phi(f)$. By (2.5), $b(z,Tx)=\langle Rz,Px\rangle$. By (2.1), (2.5), and the fact that $T\tilde{x}=\tilde{z}$ and $T\bar{x}=\bar{z}$, we obtain

$$a(\tilde{x},\tilde{x}-x)=b(T\tilde{x},T(\tilde{x}-x))=\langle R\tilde{z},P\tilde{x}-Px\rangle=\langle\Psi g,g-Px\rangle\quad x\in X,$$

$$b(\bar{z},\bar{z}-z)=a(\bar{x},\bar{x})-b(T\bar{x},z)=\langle f,P\bar{x}\rangle-\langle Rz,P\bar{x}\rangle=\langle f-Rz,\Phi f\rangle\quad z\in Z_{\perp}.$$

Substituting into the last term of (2.36), the estimate (2.34) follows. □

Proof of (ii). By (2.1), the left-hand side of (2.35) can be rewritten as

$$\begin{aligned} &= \tfrac{1}{2}b\,(z - Tx, z - Tx) + (\phi\,(f) - \langle f, g\rangle + \psi\,(g)) \\ &\quad + (b\,(z, Tx) - a\,(\bar{x}, x) - b\,(\tilde{z}, z) + \langle f, g\rangle) \end{aligned} \tag{2.37}$$

since $(1/2)\,a\,(\bar{x}, \bar{x}) = \phi\,(f)$ and $(1/2)\,b\,(\tilde{z}, \tilde{z}) = \psi\,(g)$. By (2.5) and since $\bar{x}$ solves (2.10) and $\tilde{z}$ solves (2.15), the last term of (2.37) becomes $\langle Rz, Px\rangle - \langle f, Px\rangle - \langle Rz, g\rangle + \langle f, g\rangle$. The estimate (2.35) follows. □

Here are error estimates for pairs of approximate solutions $(x, z) \in X \times Z_\perp$ of abstract problems with the same input, either f or g.

Corollary 2.20. *Let $f \in Y_\diamond^\star$ and $g \in Y$.*

(i) Let $\bar{x}$ and $\bar{z}$ be the solutions of $\mathcal{C}$ and $\mathcal{C}'$, respectively, both with f. Then

$$\frac{1}{2}a\,(\bar{x} - x, \bar{x} - x) + \frac{1}{2}b\,(\bar{z} - z, \bar{z} - z) = E\,(x, z) - \langle f - Rz, Px - \Phi f\rangle$$
for all $(x, z) \in X \times Z_\perp$. (2.38)

Consequently,

$$\frac{1}{2}a\,(\bar{x} - x, \bar{x} - x) + \frac{1}{2}b\,(\bar{z} - z, \bar{z} - z) = E\,(x, z) = J'\,(z) - J\,(x)$$
for all $(x, z) \in X \times Z_\perp$ *such that* $Rz = f$. (2.39)

(ii) Let $\tilde{x}$ and $\tilde{z}$ be the solutions of $\mathcal{V}$ and $\mathcal{V}'$, respectively, both with g. Then

$$\frac{1}{2}a\,(\tilde{x} - x, \tilde{x} - x) + \frac{1}{2}b\,(\tilde{z} - z, \tilde{z} - z) = E\,(x, z) - \langle \Psi g - Rz, Px - g\rangle$$
for all $(x, z) \in X \times Z_\perp$. (2.40)

Consequently,

$$\frac{1}{2}a\,(\tilde{x} - x, \tilde{x} - x) + \frac{1}{2}b\,(\tilde{z} - z, \tilde{z} - z) = E\,(x, z) = K\,(x) - K'\,(z)$$
for all $(x, z) \in X \times Z_\perp$ *such that* $Px = g$. (2.41)

The last estimates in (i) and (ii) are direct consequences of the constraints imposed on each of them; the error term in both estimates is $E\,(x, z)$, which is the objective functional of the minimization problem (2.27).

Proof. The estimates (2.38) and (2.40) are obtained by taking $g \in \Phi f$ and $f = \Psi g$ in (2.34) (or (2.35)), respectively, and by noting that $\phi\,(f) - \langle f, g\rangle + \psi\,(g) = 0$ in this case (see Theorem 2.15(*iii*)). The last equalities in (2.39) and (2.41) follow from the fact that

$$E\,(x, z) = \frac{1}{2}a\,(x, x) - \langle Rz, Px\rangle + \frac{1}{2}b\,(z, z) \quad \text{for all } x \in X \text{ and all } z \in Z_\perp. \tag{2.42}$$

□

Finally, error estimates for approximate solutions of each abstract problem are provided below.

Corollary 2.21. *YLet $f \in Y_\diamond^\star$ and $g \in Y$.*

(i) Let $\bar{x}$ and $\bar{z}$ be the solutions of $\mathcal{C}$ and $\mathcal{C}'$, respectively, both with f. Then

$$\frac{1}{2}a\left(\bar{x}-x,\bar{x}-x\right) = \phi\left(f\right) - J\left(x\right) \leq J'\left(z\right) - J\left(x\right) = E\left(x,z\right)$$
for all $x \in X$ and all $z \in Z_\perp$ such that $Rz = f$. (2.43)

$$\frac{1}{2}b\left(\bar{z}-z,\bar{z}-z\right) = J'\left(z\right) - \phi\left(f\right) \leq J'\left(z\right) - J\left(x\right) = E\left(x,z\right)$$
for all $z \in Z_\perp$ such that $Rz = f$ and all $x \in X$. (2.44)

(ii) Let $\tilde{x}$ and $\tilde{z}$ be the solutions of $\mathcal{V}$ and $\mathcal{V}'$, respectively, both with g. Then

$$\frac{1}{2}a\left(\tilde{x}-x,\tilde{x}-x\right) = K\left(x\right) - \psi\left(g\right) \leq K\left(x\right) - K'\left(z\right) = E\left(x,z\right)$$
for all $x \in X$ such that $Px = g$ and all $z \in Z_\perp$. (2.45)

$$\frac{1}{2}b\left(\tilde{z}-z,\tilde{z}-z\right) = \psi\left(g\right) - K'\left(z\right) \leq K\left(x\right) - K'\left(z\right) = E\left(x,z\right)$$
for all $z \in Z_\perp$ and all $x \in X$ such that $Px = g$. (2.46)

Thus, in the case that the dissipated power is available, to compare the errors of two approximate solutions it suffices to evaluate the corresponding objective functional at the approximate solutions.

Proof. Since $E\left(x,z\right) = J'\left(z\right) - J\left(x\right)$ for all $x \in X$ and all $z \in Z_\perp$ such that $Rz = f$ and $E\left(x,z\right) = K\left(x\right) - K'\left(z\right)$ for all $x \in X$ and all $z \in Z_\perp$ such that $Px = g$ (see (2.42)), the estimates (2.43) and (2.44) follow by setting $z = \bar{z}$ and $x = \bar{x}$ in (2.39), respectively, and the estimates (2.45) and (2.46) follow by setting $z = \tilde{z}$ and $x = \tilde{x}$ in (2.41), respectively. □

Remark 2.22. Combining (2.42) with ideas from [83, 84] we obtain an estimate in the case that z does not belong to $Z_\perp$:

$$\frac{1}{2}a\left(\bar{x}-x,\bar{x}-x\right) \leq \left(1+\gamma\right)E\left(x,z\right) + \left(1+\frac{1}{\gamma}\right)\frac{1}{2}b\left(G\hat{x},G\hat{x}\right)$$

for all $\gamma > 0$, all $x \in X$, and all $z \in Z$ such that $\hat{x} \in X$ is a solution to

$$\left\langle G\hat{x}, Gx'\right\rangle_Z = \left\langle G^\star z, x'\right\rangle_X - \left\langle f, Px'\right\rangle \quad \text{for all } x' \in X.$$

As in problem $\mathcal{C}$ we can replace X by $X_\diamond$ to find a unique solution $\hat{x}$.

Example 2.23. Corollary 2.20 gives the following a posteriori error estimates for pairs of approximate solutions of the continuum model. Let $f \in H_{\diamond}^{-1/2}(\partial\Omega)$ be a current and $g \in H^{1/2}(\partial\Omega)$ be a voltage. Let

$$E(u,\mathbf{p}) = \frac{1}{2}\int_\Omega \left|\sigma^{-1/2}\mathbf{p} + \sigma^{1/2}\nabla u\right|^2 \mathrm{d}\mathbf{x} \quad \text{for all } (u,\mathbf{p}) \in H^1(\Omega) \times L^2\left(\Omega, \mathbb{R}^d\right).$$

- Let $\bar{u}$ and $\bar{\mathbf{p}}$ be the solutions of $\mathcal{C}$ and $\mathcal{C}'$, respectively, both with f. Then
$$\frac{1}{2}\int_\Omega \sigma\left|\nabla(\bar{u}-u)\right|^2 \mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega \frac{1}{\sigma}\left|\bar{\mathbf{p}}-\mathbf{p}\right|^2 \mathrm{d}\mathbf{x} = E(u,\mathbf{p}) \tag{2.47}$$
for all $(u,\mathbf{p}) \in H^1(\Omega) \times L^2\left(\Omega,\mathbb{R}^d\right)$ such that $\nabla\cdot\mathbf{p} = 0$ and $-\mathbf{p}\cdot\mathbf{n} = f$.
- Let $\tilde{u}$ and $\tilde{\mathbf{p}}$ be the solutions of $\mathcal{V}$ and $\mathcal{V}'$, respectively, both with g. Then
$$\frac{1}{2}\int_\Omega \sigma\left|\nabla(\tilde{u}-u)\right|^2 \mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega \frac{1}{\sigma}\left|\tilde{\mathbf{p}}-\mathbf{p}\right|^2 \mathrm{d}\mathbf{x} = E(u,\mathbf{p}) \tag{2.48}$$
for all $(u,\mathbf{p}) \in H^1(\Omega) \times L^2\left(\Omega,\mathbb{R}^d\right)$ such that $\nabla\cdot\mathbf{p} = 0$ and $\gamma u = g$.

Now we use these estimates to assess the error of the exact solution to an approximate problem (also called *idealization* [45]) of the original one. Suppose that $\sigma_- \le \sigma$ a.e. on Ω, with $\sigma_- > 0$, and that σ_0 is a approximation of σ such that $|\sigma - \sigma_0| \le \varepsilon$ a.e. on Ω. Let $\bar{u}_0$ and $\tilde{u}_0$ be the solutions corresponding to σ_0, with f and g respectively. That is

$$\nabla\cdot(\sigma_0\nabla\bar{u}_0) = \nabla\cdot(\sigma_0\nabla\tilde{u}_0) = 0 \text{ in the sense of distributions,}$$

$$(\sigma_0\nabla\bar{u}_0)\cdot\mathbf{n} = f, \text{ and } \gamma\tilde{u}_0 = g.$$

Setting $(u,\mathbf{p}) = (\bar{u}_0, -\sigma_0\nabla\bar{u}_0)$ in (2.47) and $(u,\mathbf{p}) = (\tilde{u}_0, -\sigma_0\nabla\tilde{u}_0)$ in (2.48), it can be deduced that

$$\|\nabla(\bar{u}-\bar{u}_0)\|_{L^2(\Omega,\mathbb{R}^d)} \le \frac{\varepsilon}{\sigma_-}\|\nabla\bar{u}_0\|_{L^2(\Omega,\mathbb{R}^d)},$$
$$\|\nabla(\tilde{u}-\tilde{u}_0)\|_{L^2(\Omega,\mathbb{R}^d)} \le \frac{\varepsilon}{\sigma_-}\|\nabla\tilde{u}_0\|_{L^2(\Omega,\mathbb{R}^d)}.$$

Similar estimates have been obtained for elliptic boundary value problems in [43, Thm. 2.2] using complementary variational principles and in [45, Thm. 3.1] via the duality theory of convex analysis.

2.6 Feasible sets

In this section the set of all the pairs (a,b) of bilinear forms satisfying Assumption A2 and (2.1) is explored. The result is a description of this set in terms of *feasible* subsets. The term "feasible" refers to the methodology that will be employed: a pair will be fixed and any other pair will be classified according to its possibility of being the fixed pair. This possibility will be obtained by using the values and the extremal expressions of the power functions corresponding to the fixed pair. This idea is inspired by the works [17, 16, 20].

Definition 2.24. Let $\mathscr{F}$ be the set of all pairs of bilinear forms (a_0, b_0) such that b_0 satisfies Assumption A2 and a_0 is defined by $a_0(x_1, x_2) = b_0(T_0 x_1, T_0 x_2)$ for all $x_1, x_2 \in X$, where $T_0 = B_0^{-1} \circ G$ and $B_0 : Z \to Z$ is the isomorphism associated to b_0. Observe that X, Z, Y, G, and P remain fixed.

The following lemma, which will be used to prove the proposition that describes $\mathscr{F}$, takes inspiration from the monotonicity estimates used to solve the shape reconstruction problem in EIT [95, 50, 39], which is a particular case of the EIT inverse problem.

Lemma 2.25. *Let (a, b) and (a_0, b_0) be two pairs in $\mathscr{F}$. Let Φ, Ψ, ϕ, ψ and $\Phi_0, \Psi_0, \phi_0, \psi_0$ their corresponding current-voltage maps and power functions.*

(i) *Let $f \in Y_\diamond^\star$. Let $\bar{x}_0$ and $\bar{z}_0$ be solutions to $\mathcal{C}$ formulated with a_0 and $\mathcal{C}'$ formulated with b_0, respectively, both with f. Then*

$$\frac{1}{2}(b_0 - b)(\bar{z}_0, \bar{z}_0) \leq \phi_0(f) - \phi(f) \leq \frac{1}{2}(a - a_0)(\bar{x}_0, \bar{x}_0). \tag{2.49}$$

From this is deduced that

1. *$a(x, x) \leq a_0(x, x)$ for all $x \in X \Rightarrow \phi_0 \leq \phi$ on $Y_\diamond^\star \Leftrightarrow \Phi_0 \leq \Phi$,*
2. *$b(z, z) \leq b_0(z, z)$ for all $z \in Z_\perp \Rightarrow \phi \leq \phi_0$ on $Y_\diamond^\star \Leftrightarrow \Phi \leq \Phi_0$.*

(ii) *Let $g \in Y$. Let $\tilde{x}_0$ and $\tilde{z}_0$ be solutions to $\mathcal{V}$ formulated with a_0 and $\mathcal{V}'$ formulated with b_0, respectively, both with g. Then*

$$\frac{1}{2}(a_0 - a)(\tilde{x}_0, \tilde{x}_0) \leq \psi_0(g) - \psi(g) \leq \frac{1}{2}(b - b_0)(\tilde{z}_0, \tilde{z}_0). \tag{2.50}$$

From this is deduced that

1. *$a(x, x) \leq a_0(x, x)$ for all $x \in X \Rightarrow \psi \leq \psi_0$ on $Y \Leftrightarrow \Psi \leq \Psi_0$,*
2. *$b(z, z) \leq b_0(z, z)$ for all $z \in Z_\perp \Rightarrow \psi_0 \leq \psi$ on $Y \Leftrightarrow \Psi_0 \leq \Psi$.*

The inequalities of the current-voltage maps are in the sense of semidefiniteness.

Proof. Let $g_0 \in \Phi_0 f$. Applying Theorem 2.15(*iii*) to ϕ and ψ, and adding $\phi_0(f)$, we have

$$\phi_0(f) - \phi(f) \leq \phi_0(f) + \psi(g_0) - \langle f, g_0 \rangle.$$

Since $\phi_0(f) = (1/2)\langle f, g_0 \rangle$ it follows that

$$\phi_0(f) - \phi(f) \leq \psi(g_0) - \phi_0(f).$$

Replacing $\phi_0(f)$ by $(1/2)\, a_0(\bar{x}_0, \bar{x}_0)$ and $(1/2)\, b_0(\bar{z}_0, \bar{z}_0)$ yields

$$\frac{1}{2} b_0(\bar{z}_0, \bar{z}_0) - \phi(f) = \phi_0(f) - \phi(f) \leq \psi(g_0) - \frac{1}{2} a_0(\bar{x}_0, \bar{x}_0).$$

Using the representations of ϕ and ψ as minimization problems given in Theorem 2.15(*i*)(*ii*), it follows that

$$\frac{1}{2}(b_0(\bar{z}_0, \bar{z}_0) - b(z, z)) \leq \phi_0(f) - \phi(f) \leq \frac{1}{2}(a(x, x) - a_0(\bar{x}_0, \bar{x}_0))$$

for all $(x,z) \in X \times Z_\perp$ such that $Px = g_0$ and $Rz = f$. Since $g_0 \in \Phi_0 f$, there exists $\lambda \in \mathbb{R}$ such that $P\bar{x}_0 + \lambda 1_Y = g_0$. Therefore, (2.49) follows by choosing $(x,z) = (\bar{x}_0 + \lambda 1_X, \bar{z}_0)$. Similar arguments apply to (2.50). $\square$

Motivation for feasible sets. Consider the following observation. Given a pair $(a,b) \in \mathscr{F}$ and $f \in Y_\diamond^\star$, Theorem 2.15(*i*) says that

$$\begin{cases} \langle f, Px\rangle - \dfrac{1}{2} a(x,x) \le \phi(f) \le \dfrac{1}{2} b(z,z) \\ \text{for all } x \in X \text{ and all } z \in Z_\perp \text{ such that } Rz = f \end{cases}, \tag{2.51}$$

where ϕ is the power function associated to (a,b). Suppose now that (a,b) is unknown and that only ϕ is available. Let $(a_0,b_0) \in \mathscr{F}$ be another pair. If there is at least one $x \in X$ such that

$$\phi(f) < \langle f, Px\rangle - \frac{1}{2} a_0(x,x),$$

or if there is at least one $z \in Z_\perp$, with $Rz = f$, such that

$$\phi(f) > \frac{1}{2} b_0(z,z),$$

then (a_0,b_0) cannot be (a,b) by (2.51). Similar conclusions could be drawn from Theorem 2.15(*ii*) when the power function ψ associated to (a,b) is available. Based on this observation, the following definition is provided.

Definition 2.26. Let (a,b) be a fixed pair in $\mathscr{F}$ and let ϕ, ψ be its power functions. Given $f \in Y_\diamond^\star$ and $g \in Y$, the sets

$$\mathcal{C}(f) := \left\{ (a_0,b_0) \in \mathscr{F} \,\middle|\, \phi(f) \ge \langle f, Px\rangle - \frac{1}{2} a_0(x,x) \text{ for all } x \in X \right\},$$

$$\mathcal{V}(g) := \left\{ (a_0,b_0) \in \mathscr{F} \,\middle|\, \psi(g) \le \frac{1}{2} a_0(x,x) \text{ for all } x \in X,\ Px = g \right\},$$

$$\mathcal{C}'(f) := \left\{ (a_0,b_0) \in \mathscr{F} \,\middle|\, \phi(f) \le \frac{1}{2} b_0(z,z) \text{ for all } z \in Z_\perp,\ Rz = f \right\},$$

and

$$\mathcal{V}'(g) := \left\{ (a_0,b_0) \in \mathscr{F} \,\middle|\, \psi(g) \ge \langle Rz, g\rangle - \frac{1}{2} b_0(z,z) \text{ for all } z \in Z_\perp \right\}$$

are called *feasible sets* of $\mathscr{F}$ with respect to (a,b). A pair $(a_0,b_0) \in \mathscr{F}$ is said to be $\mathcal{C}$*-feasible* for f if $(a_0,b_0) \in \mathcal{C}(f)$. The same notation applies to the elements of the other feasible sets. Note that the intersection of the feasible sets is not empty since (a,b) belongs to the "boundary" of these.

Remark 2.27 (Complementarity). The feasible sets can equivalently be defined as

$$\mathcal{C}(f) := \{(a_0,b_0) \in \mathscr{F} \mid \phi(f) \ge \phi_0(f)\},\ \mathcal{V}(g) := \{(a_0,b_0) \in \mathscr{F} \mid \psi(g) \le \psi_0(g)\},$$

$$\mathcal{C}'(f) := \{(a_0, b_0) \in \mathscr{F} \mid \phi(f) \leq \phi_0(f)\}, \quad \mathcal{V}'(g) := \{(a_0, b_0) \in \mathscr{F} \mid \psi(g) \geq \psi_0(g)\},$$

where ϕ_0 and ψ_0 are the power functions associated to (a_0, b_0). Hence, it is obvious that

$$(\mathcal{C}(f))^{\mathrm{c}} = \operatorname{int}\mathcal{C}'(f), \quad (\mathcal{C}'(f))^{\mathrm{c}} = \operatorname{int}\mathcal{C}(f),$$

$$(\mathcal{V}(g))^{\mathrm{c}} = \operatorname{int}\mathcal{V}'(g), \quad (\mathcal{V}'(g))^{\mathrm{c}} = \operatorname{int}\mathcal{V}(g),$$

where strict inequality defines the interior of a feasible set.

It may be said that the feasible sets contain candidates to be (a, b) when only the knowledge of $\phi(f)$ or $\psi(g)$ is available. This is connected with the fact that in the EIT inverse problem the power dissipated during current or voltage injection can be used to restrict the set of possible conductivities [17].

Here are some elementary properties of the feasible sets.

Proposition 2.28.

(*i*) *Monotonic properties I: Let* $(a_1, b_1), (a_2, b_2) \in \mathscr{F}$.

- *If* $(a_1, b_1) \in \mathcal{C}(f)$ *and*

$$a_1(x, x) \leq a_2(x, x) \quad \text{for all } x \in X,$$

then $(a_2, b_2) \in \mathcal{C}(f)$. *The same holds with* $\mathcal{V}(g)$.

- *If* $(a_1, b_1) \in \mathcal{C}'(f)$ *and*

$$b_1(z, z) \leq b_2(z, z) \quad \text{for all } z \in Z_\perp,$$

then $(a_2, b_2) \in \mathcal{C}'(f)$. *The same holds with* $\mathcal{V}'(g)$.

(*ii*) *Let* $f \in Y_\diamond^\star$ *and* $g \in Y$. *If* $f = \Psi g$ *or* $g \in \Phi f$ *then*

(1) $\mathcal{C}(f) \subseteq \mathcal{V}(g)$,

(2) $\mathcal{V}'(g) \subseteq \mathcal{C}'(f)$,

3) $\operatorname{int}\mathcal{C}(f) \cap \operatorname{int}\mathcal{V}'(g) = \emptyset$,

(4) $(\operatorname{int}\mathcal{C}(f) \cup \operatorname{int}\mathcal{V}'(g))^c = \mathcal{V}(g) \cap \mathcal{C}'(f)$.

It follows that $\mathscr{F}$ *can be partitioned as*

$$\mathscr{F} = \mathcal{C}(f) \cup \mathcal{V}'(g) \cup \left(\mathcal{V}(g) \cap \mathcal{C}'(f)\right). \tag{2.52}$$

Note that only if $(a_0, b_0) \in \mathcal{V}(g) \cap \mathcal{C}'(f)$ *both bilinear forms are constrained.*

(*iii*) *The following relations hold:*

(1) $\bigcap_{f \in Y_\diamond^\star} \mathcal{C}(f) = \bigcap_{g \in Y} \mathcal{V}(g) =: \mathscr{D}$.

(2) $\bigcap_{g \in Y} \mathcal{V}'(g) = \bigcap_{f \in Y_\diamond^\star} \mathcal{C}'(f) =: \mathscr{T}$.

(3) $\operatorname{int}\mathscr{D} \cap \operatorname{int}\mathscr{T} = \emptyset$, *that is,* $\mathscr{D}$ *and* $\mathscr{T}$ *intersect just at the boundary.*

(*iv*) *Monotonic properties II: Let* $(a_0, b_0) \in \mathscr{F}$.

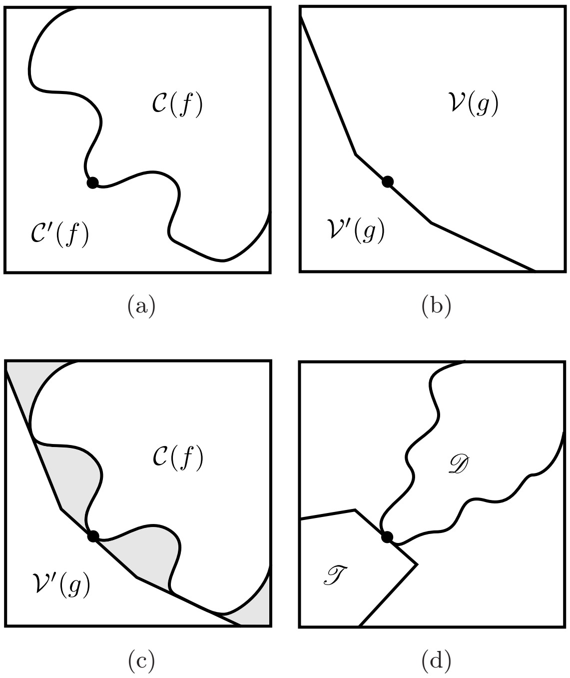


Figure 2.1: Each square above represents the set $\mathscr{F}$. The bullet symbol $\bullet$ represents the fixed pair (a, b), which belongs always to the "boundary" of the feasible sets. (a) The feasible sets $\mathcal{C}(f)$ and $\mathcal{C}'(f)$ are complementary. (b) The same happens with $\mathcal{V}(g)$ and $\mathcal{V}'(g)$. (c) When f and g are related to each other as $f = \Psi g$ or $g \in \Phi f$, the boundary of $\mathcal{C}(f)$ touches the boundary of $\mathcal{V}'(g)$. The gray area is the intersection $\mathcal{V}(g) \cap \mathcal{C}'(f)$. (d) If all f and all g are considered, this gray area is reduced to the pairs having the same power functions as (a, b). In the figure, $\mathscr{D}$ and $\mathscr{T}$ are the intersections of all possible feasible sets constraining a_0 and b_0, respectively, and there is not other pair with the same power functions. See Proposition 2.28$(ii)(iii)$.

- *If $a(x,x) \le a_0(x,x)$ for all $x \in X$, then $(a_0, b_0) \in \mathscr{D}$.*
- *If $b(z,z) \le b_0(z,z)$ for all $z \in Z_\perp$, then $(a_0, b_0) \in \mathscr{T}$.*

Proof of (i). The properties are direct consequences of Theorem 2.15(*i*)(*ii*) and Lemma 2.25. □

Proof of (ii). In this case, $\phi(f) = \psi(g) = (1/2)\langle f, g\rangle =: \mathcal{P}$. (1) If $(a_0, b_0) \in \mathcal{C}(f)$ then $\phi(f) \ge \phi_0(f)$. Applying the inequality given in Theorem 2.15(*iii*) to ϕ_0 and ψ_0, it follows that $\psi_0(g) \ge \mathcal{P}$. Hence $(a_0, b_0) \in \mathcal{V}(g)$. The same reasoning applies to (2). (3) To obtain a contradiction, suppose that $(a_0, b_0) \in \operatorname{int}\mathcal{C}(f) \cap \operatorname{int}\mathcal{V}'(g)$. Then $\phi_0(f) < \phi(f)$ and $\psi_0(g) < \psi(g)$. It follows that $\langle f, g\rangle - \psi(g) < \langle f, g\rangle - \psi_0(g) \le \phi_0(f) < \phi(f)$ by Theorem 2.15(*iii*), which is a contradiction. (4) The result follows from Remark 2.27. □

Proof of (iii). Let (a_0, b_0) be a pair in $\mathscr{F}$ and let ϕ_0, ψ_0 be its corresponding power functions. (1) Suppose $(a_0, b_0) \in \bigcap_{g\in Y} \mathcal{V}(g)$. Then $\psi(g) \le \psi_0(g)$ for all $g \in Y$. It follows that

$$\langle f, g\rangle - \psi_0(g) \le \langle f, g\rangle - \psi(g) \quad \text{for all } f \in Y_\diamond^\star \text{ and all } g \in Y.$$

By Theorem 2.15(*iii*), we have

$$\phi_0(f) = \max_{g\in Y}\{\langle f, g\rangle - \psi_0(g)\} \le \max_{g\in Y}\{\langle f, g\rangle - \psi(g)\} = \phi(f) \quad \text{for all } f \in Y_\diamond^\star.$$

That is, $(a_0, b_0) \in \mathcal{C}(f)$ for all $f \in Y_\diamond^\star$. Therefore $\bigcap_{g\in Y} \mathcal{V}(g) \subseteq \bigcap_{f\in Y_\diamond^\star} \mathcal{C}(f)$. The reverse inclusion follows from (*iii*). Indeed, for any $g \in Y$ we have $\mathcal{C}(\Psi g) \subseteq \mathcal{V}(g)$ by **??**(1). So, if (a_0, b_0) is a pair in $\bigcap_{f\in Y_\diamond^\star} \mathcal{C}(f)$, then $(a_0, b_0) \in \mathcal{C}(\Psi g)$ for all $g \in Y$. Consequently $(a_0, b_0) \in \mathcal{V}(g)$ for all $g \in Y$. Therefore $\bigcap_{f\in Y_\diamond^\star} \mathcal{C}(f) \subseteq \bigcap_{g\in Y} \mathcal{V}(g)$. The same reasoning applies to (2). Since $\operatorname{int}\mathscr{D} \cap \operatorname{int}\mathscr{T} \subseteq \operatorname{int}\mathcal{C}(f) \cap \operatorname{int}\mathcal{V}'(g)$ for all $f \in Y_\diamond^\star$ and all $g \in Y$, to prove (3) it suffices to use **??**(3). □

Proof of (iv). Since (a, b) belongs to all the feasible sets, the result follows by applying (*ii*) to (a, b) and (a_0, b_0). □

Example 2.29. From the instance of Assumptions A1-A3 given in Example 2.1, we can assert that the pairs (a_0, b_0) of the form

$$\begin{cases} a_0(u_1, u_2) := b_0(T_0 u_1, T_0 u_2) & \text{for all } u_1, u_2 \in X \\ b_0(\mathbf{p}_1, \mathbf{p}_2) := \int_\Omega \frac{1}{\sigma_0} \mathbf{p}_1 \cdot \mathbf{p}_2 \,\mathrm{d}\mathbf{x} & \text{for all } \mathbf{p}_1, \mathbf{p}_2 \in Z \end{cases} \tag{2.53}$$

belong to $\mathscr{F}$, where $\sigma_0 \in L^\infty(\Omega)$ is such that $\operatorname{ess\,inf}_{\mathbf{x}\in\Omega} \sigma_0(\mathbf{x}) > 0$. Note that $T_0 : X \to Z$ is given by $T_0 u = -\sigma_0 \nabla u$. It is also easy to verify that the pairs (a_0, b_0) of the form

$$\begin{cases} a_0(u_1, u_2) := b_0(T_0 u_1, T_0 u_2) & \text{for all } u_1, u_2 \in X \\ b_0(\mathbf{p}_1, \mathbf{p}_2) := \int_\Omega \mathbf{p}_1 \cdot \Sigma_0^{-1} \mathbf{p}_2 \,\mathrm{d}\mathbf{x} & \text{for all } \mathbf{p}_1, \mathbf{p}_2 \in Z \end{cases} \tag{2.54}$$

belong to $\mathscr{F}$, where $\Sigma_0 \in L^\infty\left(\Omega, \mathbb{R}^{d\times d}\right)$ (space of $d \times d$ matrices of $L^\infty(\Omega)$-functions) is symmetric and positive-semidefinite. Here $T_0 : X \to Z$ is given by $T_0 u = -\Sigma_0 \nabla u$.

Therefore, a pair (a_0, b_0) is completely determined by a conductivity in the case of the continuum model. Considering pairs of the form (2.53), given a current $f \in H_\diamond^{-1/2}(\partial\Omega)$ and a voltage $g \in H^{1/2}(\partial\Omega)$, from Lemma 2.25 the monotonicity properties

$$\int_\Omega \left(\frac{1}{\sigma_0} - \frac{1}{\sigma}\right) |\bar{\mathbf{p}}_0|^2 \, \mathrm{d}\mathbf{x} \leq \langle f, (\Phi_0 - \Phi) f \rangle \leq \int_\Omega (\sigma - \sigma_0) |\nabla \bar{u}_0|^2 \, \mathrm{d}\mathbf{x} \tag{2.55}$$

and

$$\int_\Omega (\sigma_0 - \sigma) |\nabla \tilde{u}_0|^2 \, \mathrm{d}\mathbf{x} \leq \langle (\Psi_0 - \Psi) g, g \rangle \leq \int_\Omega \left(\frac{1}{\sigma} - \frac{1}{\sigma_0}\right) |\tilde{\mathbf{p}}_0|^2 \, \mathrm{d}\mathbf{x} \tag{2.56}$$

hold, where $\bar{u}_0$ and $\bar{\mathbf{p}}_0$ (resp. $\tilde{u}_0$ and $\tilde{\mathbf{p}}_0$) are the solutions corresponding to σ_0 and f (resp. g). Φ and Φ_0 are the Neumman-to-Dirichlet maps associated to σ and σ_0, respectively. Ψ and Ψ_0 are the Dirichlet-to-Neumman maps associated to σ and σ_0, respectively. Since $\bar{\mathbf{p}}_0 = T_0 \bar{u}_0 = -\sigma_0 \nabla \bar{u}_0$, the estimate (2.55) can be expressed in terms of potentials

$$\int_\Omega \frac{\sigma_0}{\sigma} (\sigma - \sigma_0) |\nabla \bar{u}_0|^2 \, \mathrm{d}\mathbf{x} \leq \langle f, (\Phi_0 - \Phi) f \rangle \leq \int_\Omega (\sigma - \sigma_0) |\nabla \bar{u}_0|^2 \, \mathrm{d}\mathbf{x} \tag{2.57}$$

or in terms of current fields

$$\int_\Omega \left(\frac{1}{\sigma_0} - \frac{1}{\sigma}\right) |\bar{\mathbf{p}}_0|^2 \, \mathrm{d}\mathbf{x} \leq \langle f, (\Phi_0 - \Phi) f \rangle \leq \int_\Omega \frac{\sigma}{\sigma_0} \left(\frac{1}{\sigma_0} - \frac{1}{\sigma}\right) |\bar{\mathbf{p}}_0|^2 \, \mathrm{d}\mathbf{x}.$$

Similar expressions can be obtained from (2.56), where the solutions are related by $\tilde{\mathbf{p}}_0 = T_0 \tilde{u}_0 = -\sigma_0 \nabla \tilde{u}_0$. The estimate (2.57) is called the *monotonicity principle* and is the basis of the so-called *monotonicity method* [50, 40]. On the other hand, if a current $f \in H_\diamond^{-1/2}(\partial\Omega)$ and a voltage $g \in H^{1/2}(\partial\Omega)$ are associated to a same experiment, then the feasible subset $\mathcal{V}(g) \cap \mathcal{C}'(f)$ can be interpreted as the set that contains conductivities σ_0 satisfying the so-called *variational constraints* [17, 16, 20]:

$$\mathcal{P} \leq \min_{\substack{u \in H^1(\Omega) \\ \gamma u = g}} \left\{ \frac{1}{2} \int_\Omega \sigma_0 |\nabla u|^2 \, \mathrm{d}\mathbf{x} \right\} \tag{2.58}$$

and

$$\mathcal{P} \leq \min_{\substack{\mathbf{p} \in L^2(\Omega, \mathbb{R}^d) \\ \nabla \cdot \mathbf{p} = 0 \\ -\mathbf{p} \cdot \mathbf{n} = f}} \left\{ \frac{1}{2} \int_\Omega \frac{1}{\sigma_0} |\mathbf{p}|^2 \, \mathrm{d}\mathbf{x} \right\}, \tag{2.59}$$

where $\mathcal{P} = \frac{1}{2} \langle f, g \rangle$ is the power dissipated into heat. These constraints were introduced by [17] for the study of the EIT inverse problem and derived from the Dirichlet and Thomson variational principles associated to the continuum model. According to [20], a conductivity σ_0 is *Dirichlet* (resp. *Thomson*) *feasible* if it satisfies (2.58) (resp. (2.59)) and is *feasible* if it is Dirichlet and Thomson feasible. Thus, the feasible sets $\mathcal{V}(g)$ and $\mathcal{C}'(f)$ contain Dirichlet and Thomson feasible conductivities, respectively. Therefore, the formulation of $\mathscr{F}$ and its description given in Proposition 2.28 provide a framework to consider these ideas for other EIT models.

2.7 The complete electrode model

The equations of the *complete electrode model* [92] for the electric potential (u, U) are

$$\nabla\cdot(\sigma\nabla u) = 0 \qquad \text{in } \Omega \tag{2.60}$$

$$\sigma\nabla u\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega\setminus\bigcup_{m=1}^{M}\mathcal{E}_m \tag{2.61}$$

$$u + z_m\sigma\nabla u\cdot\mathbf{n} = U_m \qquad \text{on } \mathcal{E}_m,\ m = 1,\dots,M \tag{2.62}$$

with

$$\int_{\mathcal{E}_m}\sigma\nabla u\cdot\mathbf{n}\,\mathrm{d}\mathbf{s} = I_m \quad m = 1,\dots,M \tag{2.63}$$

if a current pattern $I = (I_1,\dots,I_M)$ is applied, or with

$$U_m = V_m \quad m = 1,\dots,M \tag{2.64}$$

if a voltage pattern $V = (V_1,\dots,V_M)$ is applied. Each z_m is a positive real number that represents the effective contact impedance associated to the electrode $\mathcal{E}_m$ and the current and voltage patterns are vectors in $\mathbb{R}^M$. The complete electrode model (2.60)-(2.64) fits into Assumptions A1-A3 with

A1. $X \coloneqq H^1(\Omega)\times\mathbb{R}^M$ and $Z \coloneqq L^2(\Omega,\mathbb{R}^d)\times\left(L^2(\mathcal{E}_1)\times\ldots\times L^2(\mathcal{E}_M)\right)$ equipped with the inner products induced by the direct sum operation (considering $H^1(\Omega)$, $\mathbb{R}^M$, $L^2(\Omega,\mathbb{R}^d)$, and $L^2(\mathcal{E}_m)$ with their usual inner products), $1_X \coloneqq \left(\mathbf{1},\overrightarrow{1}\right)\in X$, and $G : X\to Z$ defined by

$$G(u,U) \coloneqq -\left(\nabla u, (\gamma_m u - U_m)_{m=1}^M\right).$$

Recall that $\gamma_m : H^1(\Omega)\to H^{1/2}(\mathcal{E}_m)$ is the trace operator on $\mathcal{E}_M$.

A2. $b : Z\times Z\to\mathbb{R}$ defined by

$$b\left((\mathbf{p}_1,\mathbf{P}_1),(\mathbf{p}_2,\mathbf{P}_2)\right) \coloneqq \int_\Omega \frac{1}{\sigma}\mathbf{p}_1\cdot\mathbf{p}_2\,\mathrm{d}\mathbf{x} + \sum_{m=1}^{M}\int_{\mathcal{E}_m} z_m\mathbf{P}_{1,m}\mathbf{P}_{2,m}\,\mathrm{d}\mathbf{s}.$$

A3. $Y \coloneqq \mathbb{R}^M$ and $P : X\to Y$ defined by $P(u,U) \coloneqq U$.

It is easy to check that $B : Z\to Z$ is given by $B(\mathbf{p},\mathbf{P}) \coloneqq \left(\sigma^{-1}\mathbf{p}, (z_m\mathbf{P}_m)_{m=1}^M\right)$. Thus, the linear operator $T : X\to Z$ and the bilinear form $a : X\times X\to\mathbb{R}$ are given by

$$T(u,U) = \left(B^{-1}\circ G\right)(u,U) = -\left(\sigma\nabla u, \left(\frac{\gamma_m u - U_m}{z_m}\right)_{m=1}^{M}\right) \quad \text{and}$$

$$\begin{aligned} a&\left((u_1,U_1),(u_2,U_2)\right)\\ &= b\left(T(u_1,U_1), T(u_2,U_2)\right)\\ &= \int_\Omega \sigma\nabla u_1\cdot\nabla u_2\,\mathrm{d}\mathbf{x} + \sum_{m=1}^{M}\int_{\mathcal{E}_m}\frac{(\gamma_m u_1 - U_{1,m})(\gamma_m u_2 - U_{2,m})}{z_m}\,\mathrm{d}\mathbf{s}. \end{aligned}$$

respectively. The norm in $X/\mathbb{R} := \left(H^1(\Omega) \times \mathbb{R}^M\right)/\mathbb{R}$ is given by

$$\|[(u,U)]\|_{X/\mathbb{R}} = \min_{\lambda\in\mathbb{R}} \left(\|u+\lambda\mathbf{1}\|^2_{H^1(\Omega)} + \left\| U + \lambda \overrightarrow{1} \right\|^2_{\mathbb{R}^M} \right)^{1/2}.$$

To prove the coercive property of G we use the estimate

$$\left\| U + \lambda \overrightarrow{1} \right\|^2_{\mathbb{R}^M} \leq 2 \left(\max_{m=1,M} \frac{1}{|\mathcal{E}_m|} \right) \sum_{m=1}^{M} \left(\|\gamma_m\|^2 \|u+\lambda\mathbf{1}\|^2_{H^1(\Omega)} + \|\gamma_m u - U_m\|^2_{L(\mathcal{E}_m)} \right)$$

for all $u \in H^1(\Omega)$, all $U \in \mathbb{R}^M$, and all $\lambda \in \mathbb{R}$, which is derived using the continuity of the trace operators. Indeed, from the above we can find a $C > 0$ such that

$$\|[(u,U)]\|^2_{X/\mathbb{R}} \leq C \left(\|u+\lambda\mathbf{1}\|^2_{H^1(\Omega)} + \sum_{m=1}^{M} \|\gamma_m u - U_m\|^2_{L(\mathcal{E}_m)} \right) \quad \text{for all } \lambda \in \mathbb{R}.$$

Since $\|u+\lambda\mathbf{1}\|_{H^1(\Omega)} \leq C' \|\nabla u\|_{L^2\left(\Omega,\mathbb{R}^d\right)}$ for all $u \in H^1(\Omega)$ by the Poincaré's inequality [88, Cor. 7.3] (with $\lambda = -\frac{1}{|\Omega|}\int_\Omega u\, d\mathbf{x}$), the conclusion follows. The remaining assumptions are easily verified. Observe that the norm on $Y = \mathbb{R}^M$ induced by X and P is just the 2-norm of $\mathbb{R}^M$. Indeed,

$$\|U_0\|_Y := \min_{\substack{(u,U)\in X \\ P(u,U)=U_0}} \|(u,U)\|_X = \min_{u\in H^1(\Omega)} \left(\|u\|^2_{H^1(\Omega)} + \|U_0\|^2_{\mathbb{R}^M} \right)^{1/2} = \|U_0\|_{\mathbb{R}^M},$$

where the minimum is attained at $(0,U_0)$. So, $P^{-1}\big|_{(\ker P)^\perp} U = (0,U)$ and the inner product of Y is the usual inner product of $\mathbb{R}^M$. Thus, the closed subspace $Y^\star_\diamond$ can be identified with $\mathbb{R}^M_\diamond$ by the Riesz-Fréchet representation theorem. Since $\ker P = H^1(\Omega) \times \left\{ \overrightarrow{0} \right\}$, we deduce that

$$Z_\perp = \left\{ (\mathbf{p},\mathbf{P}) \in Z \;\middle|\; \begin{array}{l} \nabla\cdot\mathbf{p} = 0, \\ \langle \mathbf{p}\cdot\mathbf{n}, \gamma u\rangle + \sum_{m=1}^{M} \int_{\mathcal{E}_m} \mathbf{P}_m \gamma_m u \,\mathrm{d}\mathbf{s} = 0 \\ \text{for all } u \in H^1(\Omega) \end{array} \right\}$$

by (2.9), where the equality $\nabla\cdot\mathbf{p} = 0$ has to be understood in the sense of distributions. By definition, given $(\mathbf{p},\mathbf{P}) \in Z_\perp$, the action of $R(\mathbf{p},\mathbf{P})$ on $U \in \mathbb{R}^M$ is expressed by

$$\begin{aligned} \langle R(\mathbf{p},\mathbf{P}), U\rangle &= \langle (\mathbf{p},\mathbf{P}), G(0,U)\rangle_Z \\ &= -\int_\Omega \mathbf{p}\cdot\nabla(0)\,\mathrm{d}\mathbf{x} - \sum_{m=1}^{M} \int_{\mathcal{E}_m} \mathbf{P}_m (\gamma_m(0) - U_m)\,\mathrm{d}\mathbf{s} \\ &= \sum_{m=1}^{M} \left(\int_{\mathcal{E}_m} \mathbf{P}_m \,d\mathbf{s} \right) U_m. \end{aligned}$$

Hence $R(\mathbf{p}, \mathbf{P})$ can be identified with the vector $\left(\int_{\mathcal{E}_m} \mathbf{P}_m \, d\mathbf{s}\right)_{m=1}^{M}$. Thus, the kernel of R is given by

$$\ker R = \left\{ (\mathbf{p}, \mathbf{P}) \in Z_\perp \,\middle|\, \int_{\mathcal{E}_m} \mathbf{P}_m \, d\mathbf{s} = 0 \text{ for } m = 1, \dots, M \right\}.$$

Remark 2.30. In [37] the electrode voltages are determined by $\left(\int_{\mathcal{E}_m} \gamma_m u \, d\mathbf{s}\right)_{m=1}^{M}$. This choice corresponds to $P(u, U) := \left(\int_{\mathcal{E}_m} \gamma_m u \, d\mathbf{s}\right)_{m=1}^{M}$. It is easy to check that such a P satisfies A2 and that the vector 1_Y is given by $(|\mathcal{E}_1|, \dots, |\mathcal{E}_M|)$ in this case. This P does not yield the 2-norm of $\mathbb{R}^M$.

Remark 2.31. The closed subspace $Z_\perp$ can be expressed as

$$Z_\perp = \left\{ (\mathbf{p}, \mathbf{P}) \in Z \,\middle|\, \begin{array}{l} \nabla \cdot \mathbf{p} = 0 \text{ (in the sense of distributions)}, \\ \mathbf{p} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \setminus \cup_{m=1}^{M} \mathcal{E}_m, \\ (\mathbf{p} \cdot \mathbf{n})|_{\mathcal{E}_m} = -\mathbf{P}_m \text{ for } m = 1, \dots, M \end{array} \right\}.$$

2.7.1 Abstract problems

The abstract problems for the complete electrode model are presented below.

$\mathcal{C}$. Given $f \in Y_\diamond^\star$ (that is, $I \in \mathbb{R}_\diamond^M$), find $(\bar{u}, \bar{U}) \in X = H^1(\Omega) \times \mathbb{R}^M$ satisfying

$$\int_\Omega \sigma \nabla \bar{u} \cdot \nabla u \, d\mathbf{x} + \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{(\gamma_m \bar{u} - \bar{U}_m)(\gamma_m u - U_m)}{z_m} \, d\mathbf{s} = \sum_{m=1}^{M} I_m U_m$$

for all $(u, U) \in X$. This is the weak formulation of the complete electrode model written in terms of electric potentials (u, U)'s and with applied current I. Choosing the linear continuous functional $\Gamma(u, U) = \sum_{m=1}^{M} U_m$, we have that there exists a unique solution in the closed subspace $X_\diamond = \ker \Gamma = H^1(\Omega) \times \mathbb{R}_\diamond^M$.

$\mathcal{V}$. Given $V \in Y = \mathbb{R}^M$, find $(\tilde{u}, \tilde{U}) \in X$ satisfying

$$\begin{array}{rl} a\left((\tilde{u}, \tilde{U}), (u, U)\right) = 0 & \text{for all } (u, U) \in \ker P = H^1(\Omega) \times \left\{\overrightarrow{0}\right\} \\ P(\tilde{u}, \tilde{U}) = V & \text{in } Y = \mathbb{R}^M \end{array}.$$

Replacing $\tilde{U} = P(\tilde{u}, \tilde{U})$ by V in the first equation and noting that U is equal to $\overrightarrow{0}$ in every test pair (u, U), we obtain

$$\int_\Omega \sigma \nabla \tilde{u} \cdot \nabla u \, d\mathbf{x} + \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{\gamma_m \tilde{u} \gamma_m u}{z_m} \, d\mathbf{s} = \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{V_m \gamma_m u}{z_m} \, d\mathbf{s}$$

for all $u \in H^1(\Omega)$. This is the weak formulation of the complete electrode model written in terms of electric potentials u's and with applied voltage V. This formulation was considered in [98].

$\mathcal{C}'$. Given $f \in Y_\diamond^\star$ (that is, $I \in \mathbb{R}_\diamond^M$), find $\left(\bar{\mathbf{p}}, \bar{\mathbf{P}}\right) \in Z_\perp$ satisfying

$$\begin{array}{r} \displaystyle\int_\Omega \frac{1}{\sigma}\bar{\mathbf{p}}\cdot\mathbf{p}\, d\mathbf{x} + \sum_{m=1}^{M}\int_{\mathcal{E}_m} z_m \bar{\mathbf{P}}_m \mathbf{P}_m \, \mathrm{d}\mathbf{s} = 0 \qquad \text{for all } (\mathbf{p},\mathbf{P}) \in \ker R \\ \displaystyle\int_{\mathcal{E}_m} \bar{\mathbf{P}}_m \, \mathrm{d}\mathbf{s} = I_m \quad \text{for } m = 1,\dots,M \end{array}.$$

This is the weak formulation of the complete electrode model written in terms of current fields $(\mathbf{p},\mathbf{P})$'s and with applied current I. In [75] was obtained also a formulation of this model in terms of current fields, which can be viewed as a particular case of another model (see Appendix).

$\mathcal{V}'$. Given $V \in Y = \mathbb{R}^M$, find $\left(\tilde{\mathbf{p}}, \tilde{\mathbf{P}}\right) \in Z_\perp$ satisfying

$$\int_\Omega \frac{1}{\sigma}\tilde{\mathbf{p}}\cdot\mathbf{p}\, \mathrm{d}\mathbf{x} + \sum_{m=1}^{M}\int_{\mathcal{E}_m} z_m \tilde{\mathbf{P}}_m \mathbf{P}_m \, \mathrm{d}\mathbf{s} = \sum_{m=1}^{M}\left(\int_{\mathcal{E}_m} \mathbf{P}_m \, d\mathbf{s}\right) V_m.$$

for all $(\mathbf{p},\mathbf{P}) \in Z_\perp$. This is the weak formulation of the complete electrode model written in terms of current fields $(\mathbf{p},\mathbf{P})$'s and with applied voltage V.

Remark 2.32. To our knowledge, the problems $\mathcal{C}'$ and $\mathcal{V}'$ are novel formulations of the complete electrode model. In [75], a version of the complete electrode model in terms of current fields was also proposed. It turns out that formulation can be deduced from the assumptions that we made for the *shunt* model (see Appendix).

2.7.2 Current-Voltage maps

The current-voltage maps for the complete electrode model are

- (*current-to-voltage* map) $\Phi : Y_\diamond^\star \equiv \mathbb{R}_\diamond^M \to Y/\mathbb{R} = \mathbb{R}^M/\mathbb{R}$ given by $\Phi I := \left[P\left(\bar{u},\bar{U}\right)\right] = \left[\bar{U}\right]$ and
- (*voltage-to-current* map) $\Psi : Y = \mathbb{R}^M \to Y_\diamond^\star \equiv \mathbb{R}_\diamond^M$ given by $\Psi V = R\left(\tilde{\mathbf{p}},\tilde{\mathbf{P}}\right) \equiv \left(\int_{\mathcal{E}_m} \tilde{\mathbf{P}}_m \, \mathrm{d}\mathbf{s}\right)_{m=1}^{M}$.

Here $\left(\bar{u},\bar{U}\right)$ is a solution to the weak formulation of complete electrode model with current I (problem $\mathcal{C}$) and $\left(\tilde{\mathbf{p}},\tilde{\mathbf{P}}\right)$ is the unique solution to the weak formulation of complete electrode model with voltage V (problem $\mathcal{V}'$). Since $\left(\tilde{\mathbf{p}},\tilde{\mathbf{P}}\right) = T\left(\tilde{u},V\right)$, where $\tilde{u}$ is the unique solution to $\mathcal{V}$ with V, it follows that

$$\Psi V \equiv \left(\int_{\mathcal{E}_m} \frac{(V-\gamma_m\tilde{u})}{z_m}\,\mathrm{d}\mathbf{s}\right)_{m=1}^{M}.$$

Choosing $\Gamma(u,U) = \sum_{m=1}^M U_m$ and $\Upsilon(U) = \sum_{m=1}^M U_m$ linear continuous functionals on X and Y, respectively, by Remarks (2.5) and (2.11) we have that $\Phi_\Upsilon : Y_\diamond^\star \equiv \mathbb{R}_\diamond^M \to Y_\diamond$

defined as $\Phi_\Upsilon I = P\left(\bar{u}_\Gamma, \bar{U}_\Gamma\right) = \bar{U}_\Gamma$ and $\Psi|_{Y_\diamond}$ are inverses of each other, with

$$\left(\bar{u}_\Gamma, \bar{U}_\Gamma\right) = \left(\bar{u}, \bar{U}\right) - \left(\frac{1}{M}\sum_{m=1}^{M} \bar{U}_m\right)\left(\mathbf{1}, \overrightarrow{1}\right),$$

$$\left(\bar{u}_\Gamma, \bar{U}_\Gamma\right) \in X_\diamond = \ker\Gamma = H^1\left(\Omega\right) \times \mathbb{R}^M_\diamond, \quad \text{and} \quad Y_\diamond = \ker\Upsilon = \mathbb{R}^M_\diamond.$$

Φ and Ψ are the *current-to-voltage* and *voltage-to-current* operators of the complete electrode model [48, 98]. In fact, it is usual to consider Φ_Υ and $\Psi|_{Y_\diamond}$. It is well-known that Φ satisfies the symmetry property $\langle \Phi I_1, I_2\rangle_{\mathbb{R}^M} = \langle I_1, \Phi I_2\rangle_{\mathbb{R}^M}$ for all $I_1, I_2 \in \mathbb{R}^M_\diamond$ [92, Sec. 4]. This is equivalent to the symmetry property given in Proposition 2.10(*ii*), which Ψ also holds.

2.7.3 Dual properties

With the identification of $Y^\star_\diamond$ with $\mathbb{R}^M_\diamond$, and of ΨV with a vector in $\mathbb{R}^M_\diamond$, the power functions ϕ and ψ can be written as

$$\phi\left(I\right) = \frac{1}{2}\left\langle I, \Phi I\right\rangle_{\mathbb{R}^M} \quad I \in \mathbb{R}^M_\diamond \quad \text{and} \quad \psi\left(V\right) = \frac{1}{2}\left\langle \Psi V, V\right\rangle_{\mathbb{R}^M} \quad V \in \mathbb{R}^M.$$

By Theorem 2.15(*i*)-(*ii*), we also have

$$\phi\left(I\right) = \min_{\substack{(\mathbf{p},\mathbf{P})\in Z_\perp \\ \left(\int_{\mathcal{E}_m} \mathbf{P}_m\, \mathrm{d}\mathbf{s}\right)_{m=1}^M = I}} \frac{1}{2}\left(\int_\Omega \frac{1}{\sigma}\left|\mathbf{p}\right|^2 \mathrm{d}\mathbf{x} + \sum_{m=1}^{M}\int_{\mathcal{E}_m} z_m \mathbf{P}^2\, \mathrm{d}\mathbf{s}\right) \quad I \in \mathbb{R}^M_\diamond$$

and

$$\psi\left(V\right) = \min_{\substack{(u,U)\in X \\ U=V}} \frac{1}{2}\left(\int_\Omega \sigma\left|\nabla u\right|^2 \mathrm{d}\mathbf{x} + \sum_{m=1}^{M}\int_{\mathcal{E}_m} \frac{\left(\gamma_m u - U_m\right)^2}{z_m}\, \mathrm{d}\mathbf{s}\right) \quad V \in \mathbb{R}^M,$$

which can be interpreted as the Thomson and Dirichlet variational principles for the complete electrode model, respectively. On the other hand, the optimization problems given in Theorem 2.15(*v*) read as

$$\min_{\substack{(u,U)\in X,\, U=V \\ (\mathbf{p},\mathbf{P})\in Z_\perp,\, \left(\int_{\mathcal{E}_m} \mathbf{P}_m\, d\mathbf{s}\right)_{m=1}^M = I}} E\left(\left(u, U\right), \left(\mathbf{p}, \mathbf{P}\right)\right)$$

and

$$\max_{\substack{(u,U)\in X \\ (\mathbf{p},\mathbf{P})\in Z_\perp}} \left\langle I - \left(\int_{\mathcal{E}_m} \mathbf{P}_m\, d\mathbf{s}\right)_{m=1}^M, U - V\right\rangle_{\mathbb{R}^M} - E\left(\left(u, U\right), \left(\mathbf{p}, \mathbf{P}\right)\right) \quad ,$$

where $E : X \times Z \to [0,\infty[$ is defined as

$$E\left(\left(u,U\right),\left(\mathbf{p},\mathbf{P}\right)\right) \coloneqq \frac{1}{2}\int_{\Omega}\left|\sigma^{-1/2}\mathbf{p}+\sigma^{1/2}\nabla u\right|^{2}\,\mathrm{d}\mathbf{x} + \frac{1}{2}\sum_{m=1}^{M}\int_{\mathcal{E}_m}\left(z_m^{1/2}\mathbf{P}_m + z_m^{-1/2}\left(\gamma_m u - U_m\right)\right)^{2}\,\mathrm{d}\mathbf{s}.$$

Recall that these optimization problems attain their optimal values at the solutions of the abstract problems. It is worth pointing out that the functional E can be viewed as the complete electrode model counterpart of the Kohn-Vogelius functional.

2.7.4 Error estimates

Corollary 2.20 provides a posteriori error estimates for pairs of approximate solutions of the complete electrode model. Let $I \in \mathbb{R}^M_\diamond$ be a current pattern. Let $\left(\bar{u},\bar{U}\right)$ and $\left(\bar{\mathbf{p}},\bar{\mathbf{P}}\right)$ be the solutions of $\mathcal{C}$ and $\mathcal{C}'$, respectively, both with I. From Corollary 2.20(i) we have the estimate

$$\int_{\Omega}\sigma\left|\nabla\left(\bar{u}-u\right)\right|^{2}\,\mathrm{d}\mathbf{x} + \sum_{m=1}^{M}\int_{\mathcal{E}_m}\frac{\left(\gamma_m\left(\bar{u}-u\right)-\left(\bar{U}_m-U_m\right)\right)^{2}}{z_m}\,\mathrm{d}\mathbf{s} + \int_{\Omega}\frac{1}{\sigma}\left|\bar{\mathbf{p}}-\mathbf{p}\right|^{2}\,\mathrm{d}\mathbf{x} + \sum_{m=1}^{M}\int_{\mathcal{E}_m} z_m\left(\bar{\mathbf{P}}-\mathbf{P}\right)^{2}\,\mathrm{d}\mathbf{s} = \mathsf{KV}\left(\left(u,U\right),\left(\mathbf{p},\mathbf{P}\right)\right) \tag{2.65}$$

for all $(u,U) \in X$ and all $(\mathbf{p},\mathbf{P}) \in Z_\perp$ such that $\left(\int_{\mathcal{E}_m}\mathbf{P}_m\,d\mathbf{s}\right)_{m=1}^{M} = I$. Consider the following application. Suppose that σ_0 and $z_{1,0},\dots,z_{M,0}$ are approximations of σ and $z_1,\dots,z_M$ and let $\left(\bar{u}_0,\bar{U}_0\right)$ be a solution to $\mathcal{C}$ formulated with I, σ_0, and $z_{1,0},\dots,z_{M,0}$. Then $\left(\bar{\mathbf{p}}_0,\bar{\mathbf{P}}_0\right) \in Z$ defined by

$$\left(\bar{\mathbf{p}}_0,\bar{\mathbf{P}}_0\right) \coloneqq -\left(\sigma_0\nabla\bar{u}_0, \left(\frac{\gamma_m\bar{u}_0-\bar{U}_{0,m}}{z_{0,m}}\right)_{m=1}^{M}\right)$$

belongs to $Z_\perp$ and satisfies $\left(\int_{\mathcal{E}_m}\bar{\mathbf{P}}_{0,m}\,\mathrm{d}\mathbf{s}\right)_{m=1}^{M} = I$. Setting $(u,U) = \left(\bar{u}_0,\bar{U}_0\right)$ and $(\mathbf{p},\mathbf{P}) = \left(\bar{\mathbf{p}}_0,\bar{\mathbf{P}}_0\right)$ in (2.65), and leaving the non-negative term

$$\int_{\Omega}\frac{1}{\sigma}\left|\bar{\mathbf{p}}-\mathbf{p}\right|^{2}\,\mathrm{d}\mathbf{x} + \sum_{m=1}^{M}\int_{\mathcal{E}_m} z_m\left(\bar{\mathbf{P}}-\mathbf{P}\right)^{2}\,\mathrm{d}\mathbf{s}$$

out, we have the estimate

$$\int_{\Omega}\sigma\left|\nabla\left(\bar{u}-\bar{u}_0\right)\right|^{2}\,\mathrm{d}\mathbf{x} + \sum_{m=1}^{M}\int_{\mathcal{E}_m}\frac{\left(\gamma_m\left(\bar{u}-\bar{u}_0\right)-\left(\bar{U}_m-\bar{U}_{0,m}\right)\right)^{2}}{z_m}\,\mathrm{d}\mathbf{s} \le \int_{\Omega}\frac{\left(\sigma-\sigma_0\right)^{2}}{\sigma}\left|\nabla\bar{u}_0\right|^{2}\,\mathrm{d}\mathbf{x} + \sum_{m=1}^{M}\int_{\mathcal{E}_m} z_m\left(\frac{1}{z_m}-\frac{1}{z_{0,m}}\right)^{2}\left(\gamma_m\bar{u}_0-\bar{U}_{0,m}\right)^{2}\,\mathrm{d}\mathbf{s}.$$

If, in addition, there are positive constants σ_-, ε_σ, z_+, and ε_z such that

$$\sigma_- \leq \sigma,\ |\sigma - \sigma_0| \leq \varepsilon_\sigma, \qquad z_m \leq z_+,\ \left|\frac{1}{z_m} - \frac{1}{z_{0,m}}\right| \leq \frac{1}{\varepsilon_z} \text{ for } m = 1, \ldots, M,$$

it follows that

$$\left\|\left(\bar{u}, \bar{U}\right) - \left(\bar{u}_0, \bar{U}_0\right)\right\|_\square \leq \frac{\max\left\{\varepsilon_\sigma, \frac{1}{\varepsilon_z}\right\}}{\min\left\{\sigma_-, \frac{1}{z_+}\right\}} \left\|\left(\bar{u}_0, \bar{U}_0\right)\right\|_\square,$$

where $\|\cdot\|_\square : H^1(\Omega) \times \mathbb{R}^M \to \mathbb{R}$ is defined as

$$\|(u, U)\|_\square \coloneqq \left(\int_\Omega |\nabla u|^2 \,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} (\gamma_m u - U_m)^2 \,\mathrm{d}\mathbf{s}\right)^{1/2}.$$

Proceeding in the same way, given a voltage pattern V and $\tilde{u}$ the solution to $\mathcal{V}$ with V, from Corollary 2.20(*ii*) it can be deduced that the estimate

$$\|\tilde{u} - \tilde{u}_0\|_\triangle \leq \frac{\max\left\{\varepsilon_\sigma, \varepsilon_z^{-1}\right\}}{\min\left\{\sigma_-, z_+^{-1}\right\}} \|(\tilde{u}_0, V)\|_\square$$

holds, where $\tilde{u}_0$ is the unique solution to $\mathcal{V}$ formulated with V, σ_0, and $z_{1,0}, \ldots, z_{M,0}$, and $\|\cdot\|_\triangle : H^1(\Omega) \to \mathbb{R}$ is defined as

$$\|u\|_\triangle \coloneqq \left(\int_\Omega |\nabla u|^2 \,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} (\gamma_m u)^2 \,\mathrm{d}\mathbf{s}\right)^{1/2} \quad \left(= \left\|\left(u, \vec{0}\right)\right\|_\square\right).$$

Observe that $\|\cdot\|_\square$ is a norm on $X = \left(H^1(\Omega) \times \mathbb{R}^M\right) / \mathbb{R}$ equivalent to $\|\cdot\|_{X/\mathbb{R}}$ (see, for instance, [92, Lem. 3.2]). Moreover, it can be proved that $\|\cdot\|_\square$ is a norm on $H^1(\Omega) \times \mathbb{R}^M_\diamond$ equivalent to the norm induced by the direct sum operation of $H^1(\Omega)$ and $\mathbb{R}^M_\diamond$ (considering both spaces with their usual norms) and that $\|\cdot\|_\triangle$ is a norm on $H^1(\Omega)$ equivalent to its usual norm.

To conclude this part, we provide a numerical example where Corollary 2.21 is applied to obtain the error of approximate solutions of the problems $\mathcal{C}$ and $\mathcal{V}$. First, the a posteriori error estimates are presented. Let $I \in \mathbb{R}^M_\diamond$ be a current and $V \in \mathbb{R}^M_\diamond$ be a voltage. Applying the estimates (2.43) and (2.45) to the problems $\mathcal{C}$ and $\mathcal{V}$ we deduce that

$$\frac{1}{2}\left(\int_\Omega \sigma \left|\nabla\left(\bar{u}_\Gamma - u\right)\right|^2 \,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{\left(\gamma_m\left(\bar{u}_\Gamma - u\right) - \left(\bar{U}_{\Gamma,m} - U_m\right)\right)^2}{z_m} \,\mathrm{d}\mathbf{s}\right) = \phi(I) - J(u, U)$$

for all $(u, U) \in H^1(\Omega) \times \mathbb{R}^M_\diamond$ and

$$\frac{1}{2}\left(\int_\Omega \sigma \left|\nabla\left(\tilde{u} - u\right)\right|^2 \,d\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{\left(\gamma_m\left(\tilde{u} - u\right)\right)^2}{z_m} \,d\mathbf{s}\right) = K(u, V) - \psi(V)$$

for all $u \in H^1(\Omega)$, where $\left(\bar{u}_\Gamma, \bar{U}_\Gamma\right)$ is the unique solution to $\mathcal{C}$ (with I) in the subspace $\ker\Gamma = H^1(\Omega) \times \mathbb{R}^M_\diamond$ ($\Gamma(u, U) = \sum_{m=1}^M U_m$ was chosen) and $(\tilde{u}, V)$ is the unique solution to $\mathcal{V}$ (with V). Recall that J and K are the objective functionals of the extremal formulations of $\mathcal{C}$ and $\mathcal{V}$, respectively. The above estimates are used to compute the error of approximate solutions obtained by the finite element method. Consider the domain $\Omega =]0,1[\times]0,1[\subset \mathbb{R}^2$ with $M = 12$ electrodes, contact impedances $z_1, \ldots, z_M = 0.5$, and a conductivity $\sigma : \Omega \to \mathbb{R}$ defined as

$$\sigma(\mathbf{x}) \coloneqq 1 + 4 \times \mathbb{1}_{\text{square}}(\mathbf{x}) + 2 \times \mathbb{1}_{\text{rectangle}}(\mathbf{x}),$$

where $\mathbb{1}_{\text{square}}$ and $\mathbb{1}_{\text{rectangle}}$ are indicator functions (see Figure 2.2). Here, an approximate solution consists of a piecewise linear function defined over an admissible triangulation of Ω and, in the case of $\mathcal{C}$, of a second component represented by its coordinates in a basis of $\mathbb{R}^M_\diamond$. Numerical tests with triangulations of $T_i = 2 \times (7 \times i)^2$ triangles and $N_i = (1 + 7 \times i)^2$ nodes, for $i = 1, \ldots, 13$, were performed. In the $i-$th test the area of each triangle is $h_i = 1/T_i$. One random current $I \in \mathbb{R}^M_\diamond$ of norm 1 is applied to obtain the approximate solutions $(u_i, U_i) \in C^0\left(\bar{\Omega}\right) \times \mathbb{R}^M_\diamond$ of $\mathcal{C}$ and the errors $\phi(I) - J(u_i, U_i)$ are calculated. Similarly, one random voltage $V \in \mathbb{R}^M$ of norm 1 is applied to obtain the approximate solutions $u_i \in C^0\left(\bar{\Omega}\right)$ of $\mathcal{V}$ and the errors $K(u_i, V) - \psi(V)$ are calculated. The power values $\phi(I) = (1/2)\langle I, \Phi I\rangle_{\mathbb{R}^M}$ and $\psi(V) = (1/2)\langle \Psi V, V\rangle_{\mathbb{R}^M}$ are calculated from an "exact" solution generated by choosing $i = 25$. Since $1 \leq \sigma, 1/z_1, \ldots, 1/z_M$, it follows that the relative errors in norms $\|\cdot\|_\square$ and $\|\cdot\|_\triangle$ are bounded as follows

$$\begin{cases} \frac{\left\|\left(\bar{u}_\Gamma, \bar{U}_\Gamma\right) - (u_i, U_i)\right\|_\square}{\left\|\left(\bar{u}_\Gamma, \bar{U}_\Gamma\right)\right\|_\square} \leq \sqrt{2} \frac{k_i}{\left\|\left(\bar{u}_\Gamma, \bar{U}_\Gamma\right)\right\|_\square} \text{ with } k_i \coloneqq (\phi(I) - J(u_i, U_i))^{1/2} & \text{case } \mathcal{C} \\ \frac{\|\tilde{u} - u_i\|_\triangle}{\|\tilde{u}\|_\triangle} \leq \sqrt{2} \frac{l_i}{\|\tilde{u}\|_\triangle} \text{ with } l_i \coloneqq (K(u_i, V) - \phi(V))^{1/2} & \text{case } \mathcal{V} \end{cases}.$$

The numerical results are presented in Tables 2.2 and 2.3. The 4th and 5th columns of the tables displays the convergence rate and the above bounds. Observe that the sequences $J(u_i, U_i)/J(u_{i-1}, U_{i-1})$ and $K(u_i, V)/K(u_{i-1}, V)$ are decreasing and increasing, respectively, and both converge to 1. It is worth pointing out that it is not neccesary to discretize the error estimates since the conductivity is a piecewise constant function. Finally, note that to compare the errors of two approximate solutions (u_i, U_i) and (u_j, U_j) of $\mathcal{C}$, it suffices to compute the values $J(u_i, U_i)$ and $J(u_j, U_j)$; similarly, to compare the errors of two approximate solutions u_i and u_j of $\mathcal{V}$, it suffices to compute the values $K(u_i, V)$ and $K(u_j, V)$.

2.7.5 Feasible sets

From the instance of Assumptions A1-A3 given at the beginning, we can assert that the pairs (a_0, b_0) of the form

$$a_0((u_1, U_1), (u_2, U_2)) \coloneqq b_0(T_0(u_1, U_1), T_0(u_2, U_2)),$$

$$b_0((\mathbf{p}_1, \mathbf{P}_1), (\mathbf{p}_2, \mathbf{P}_2)) \coloneqq \int_\Omega \frac{1}{\sigma_0} \mathbf{p}_1 \cdot \mathbf{p}_2 \, \mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} z_{0,m} \mathbf{P}_{1,m} \mathbf{P}_{2,m} \, \mathrm{d}s$$

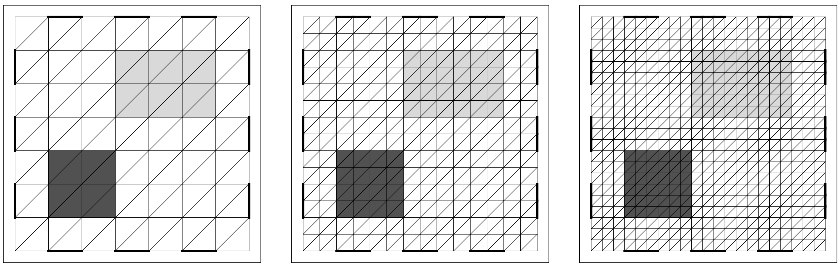


Figure 2.2: Triangulations of Ω corresponding to $i = 1$ (left), $i = 2$ (center), and $i = 3$ (right). The test conductivity has value 4 in the square, 2 in the rectangle, and 1 in the background. The thick lines on the boundary represent the positions of the $M = 12$ electrodes.

i	T_i	$\phi(I) - J(u_i, U_i)$	$\frac{\log(k_i/k_{i-1})}{\log(h_i/h_{i-1})}$	$\sqrt{2}\frac{k_i}{\left\|(\bar{u}_\Gamma, \bar{U}_\Gamma)\right\|_\square} \times 100\%$	$\frac{J(u_i,U_i)}{J(u_{i-1},U_{i-1})}$
1	98	0.1076	0.1676	29.33%	-
2	392	0.0417	0.3416	18.26%	1.0324
3	882	0.0222	0.3901	13.31%	1.0093
4	1568	0.0137	0.4147	10.49%	1.0040
5	2450	0.0093	0.4326	08.64%	1.0021
6	3528	0.0067	0.4489	07.34%	1.0012
7	4802	0.0051	0.4655	06.36%	1.0008
8	6272	0.0039	0.4833	05.59%	1.0005
9	7938	0.0031	0.5032	04.96%	1.0004
10	9800	0.0025	0.5257	04.44%	1.0003
11	11858	0.0020	0.5514	04.00%	1.0002
12	14112	0.0016	0.5812	03.61%	1.0002
13	16562	0.0013	0.6162	03.28%	1.0001

Table 2.2: Errors of the approximate solutions (u_i, U_i) of the problem $\mathcal{C}$.

i	T_i	$K(u_i, V) - \phi(V)$	$\frac{\log(l_i/l_{i-1})}{\log(h_i/h_{i-1})}$	$\sqrt{2}\frac{l_i}{\|\tilde{u}\|_\triangle} \times 100\%$	$\frac{K(u_i,V)}{K(u_{i-1},V)}$
1	98	0.0055	0.4925	43.13%	-
2	392	0.0020	0.3549	26.37%	0.9689
3	882	0.0011	0.3966	19.12%	0.9909
4	1568	0.0007	0.4187	15.02%	0.9961
5	2450	0.0004	0.4353	12.37%	0.9980
6	3528	0.0003	0.4509	10.50%	0.9988
7	4802	0.0002	0.4670	09.09%	0.9992
8	6272	0.0002	0.4845	07.99%	0.9995
9	7938	0.0001	0.5041	07.09%	0.9996
10	9800	0.0001	0.5264	06.35%	0.9997
11	11858	0.0001	0.5520	05.71%	0.9998
12	14112	0.0001	0.5817	05.16%	0.9998
13	16562	0.0001	0.6166	04.68%	0.9999

Table 2.3: Errors of the approximate solutions u_i of the problem $\mathcal{V}$.

belong to $\mathscr{F}$, where $\sigma_0 \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{\mathbf{x}\in\Omega}\sigma_0(\mathbf{x}) > 0$ is a conductivity and $z_{0,1},\dots,z_{0,M}$ are positive contact impedances. Here the linear operator $T_0 : X \to Z$ is given by

$$T_0(u,U) \coloneqq -\left(\sigma_0\nabla u, \left(\frac{\gamma_m u - U_m}{z_{0,m}}\right)_{m=1}^M\right).$$

It is also easy to verify that the pairs (a,b) of the form

$$a_0((u_1,U_1),(u_2,U_2)) \coloneqq b_0(T_0(u_1,U_1),T_0(u_2,U_2)),$$

$$b_0((\mathbf{p}_1,\mathbf{P}_1),(\mathbf{p}_2,\mathbf{P}_2)) \coloneqq \int_\Omega \mathbf{p}_1\cdot\Sigma_0^{-1}\mathbf{p}_2\,\mathrm{d}\mathbf{x} + \sum_{m=1}^M\int_{\mathcal{E}_m}\zeta_{0,m}^{-1}\mathbf{P}_{1,m}\mathbf{P}_{2,m}\,\mathrm{d}\mathbf{s}$$

belong to $\mathscr{F}$, where $\Sigma_0 \in L^\infty\left(\Omega,\mathbb{R}^{d\times d}\right)$ is a symmetric positive-semidefinite conductivity matrix with and $\zeta_{0,1},\dots,\zeta_{0,M}$ are bounded functions satisfying $\zeta_m \in L^\infty(\mathcal{E}_m)$ and $\operatorname{ess\,inf}_{\mathbf{x}\in\mathcal{E}_m}\zeta_m(\mathbf{x}) > 0$. In this case,

$$T_0(u,U) = -\left(\Sigma_0\nabla u, \left(\zeta_{0,m}(\gamma_m u - U_m)\right)_{m=1}^M\right).$$

Therefore, in the case of the complete electrode model, a pair is completely determined by a conductivity and M contact impedances. Considering pairs composed of real-valued conductivities and constant contact impedances, given a current pattern $I \in \mathbb{R}^M_\diamond$, from Lemma 2.25(*i*) the monotonicity property

$$\begin{aligned}\int_\Omega\left(\frac{1}{\sigma_0}-\frac{1}{\sigma}\right)|\bar{\mathbf{p}}_0|^2\,\mathrm{d}\mathbf{x} &+ \sum_{m=1}^M\int_{\mathcal{E}_m}(z_{m,0}-z_m)\left(\bar{\mathbf{P}}_{0,m}\right)^2\,\mathrm{d}\mathbf{s}\\ &\le \langle I,(\Phi_0-\Phi)I\rangle_{\mathbb{R}^M}\\ &\le \int_\Omega(\sigma-\sigma_0)|\nabla\bar{u}_0|^2\,\mathrm{d}\mathbf{x} + \sum_{m=1}^M\int_{\mathcal{E}_m}\left(\frac{1}{z_m}-\frac{1}{z_{m,0}}\right)\left(\gamma_m\bar{u}_0-\bar{U}_{0,m}\right)^2\,\mathrm{d}\mathbf{s}\end{aligned} \tag{2.66}$$

holds, where $\left(\bar{u}_0,\bar{U}_0\right)$ and $\left(\bar{\mathbf{p}}_0,\bar{\mathbf{P}}_0\right)$ are the solutions corresponding to the problems $\mathcal{C}$ and $\mathcal{C}'$ formulated with σ_0, $z_{1,0},\dots,z_{M,0}$, and I. In (2.66), Φ and Φ_0 are the current-to-voltage maps associated to σ, $z_1,\dots,z_M$ and σ_0, $z_{1,0},\dots,z_{M,0}$, respectively. Since the solutions are related as

$$\left(\bar{\mathbf{p}}_0,\bar{\mathbf{P}}_0\right) \coloneqq -\left(\sigma_0\nabla\bar{u}_0, \left(\frac{\gamma_m\bar{u}_0-\bar{U}_{0,m}}{z_{m,0}}\right)_{m=1}^M\right),$$

the estimate (2.66) can be expressed in terms of potentials

$$\begin{aligned}\int_\Omega\frac{\sigma_0}{\sigma}(\sigma-\sigma_0)|\nabla\bar{u}_0|^2\,\mathrm{d}\mathbf{x} &+ \sum_{m=1}^M\int_{\mathcal{E}_m}\frac{z_m}{z_{m,0}}\left(\frac{1}{z_m}-\frac{1}{z_{m,0}}\right)\left(\gamma_m\bar{u}_0-\bar{U}_{0,m}\right)^2\,\mathrm{d}\mathbf{s}\\ &\le \langle I,(\Phi_0-\Phi)I\rangle_{\mathbb{R}^M}\\ &\le \int_\Omega(\sigma-\sigma_0)|\nabla\bar{u}_0|^2\,\mathrm{d}\mathbf{x} + \sum_{m=1}^M\int_{\mathcal{E}_m}\left(\frac{1}{z_m}-\frac{1}{z_{m,0}}\right)\left(\gamma_m\bar{u}_0-\bar{U}_{0,m}\right)^2\,\mathrm{d}\mathbf{s}.\end{aligned} \tag{2.67}$$

Similar expressions can be obtained from Lemma 2.25(*ii*) when a voltage pattern $V \in \mathbb{R}^M$ is considered. The monotonicity estimate (2.67) was obtained in [51, Th. 2]. There, (2.67) was used to characterize the achievable resolution in the shape reconstruction problem. The monotonicity method based on the complete electrode model also uses this estimate [39].

Observe that, from Proposition 2.28(*ii*), given a current $I \in \mathbb{R}^M_\diamond$ and a voltage $V \in \mathbb{R}^M$ associated to a same experiment with conductivity σ and contact impedances $z_1, \ldots, z_M$, the feasible subset $\mathcal{V}(V) \cap \mathcal{C}'(I)$ can be interpreted as the set that contains conductivities σ_0 and impedances $z_{1,0}, \ldots, z_{M,0}$ satisfying

$$\mathcal{P} \leq \min_{\substack{(u,U)\in X \\ U=V}} \frac{1}{2}\left(\int_\Omega \sigma_0 \left|\nabla u\right|^2 \, \mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{\left(\gamma_m u - U_m\right)^2}{z_{0,m}} \, \mathrm{d}\mathbf{s}\right)$$

and

$$\mathcal{P} \leq \min_{\substack{(\mathbf{p},\mathbf{P})\in Z_\perp \\ \left(\int_{\mathcal{E}_m} \mathbf{P}_m \, d\mathbf{s}\right)_{m=1}^M = I}} \frac{1}{2}\left(\int_\Omega \frac{1}{\sigma_0} \left|\mathbf{p}\right|^2 \, \mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} z_{0,m} \mathbf{P}^2 \, \mathrm{d}\mathbf{s}\right),$$

where $\mathcal{P} = (1/2)\left\langle I, V\right\rangle_{\mathbb{R}^M}$ is the power dissipated into heat. These inequalities are the complete electrode model counterparts of the variational constraints introduced by [17] for the continuum model.

Now, the feasible sets are numerically calculated for a simple example. Consider the domain $\Omega =]0,1[\times]0,1[$ with conductivity and contact impedances

$$\sigma(\mathbf{x}) = 1 + \lambda_1 \cdot \mathbb{1}_{\text{rectangle}}(\mathbf{x}) \quad \text{and} \quad z_1 = \ldots = z_M = \lambda_2,$$

which are parametrized by $(\lambda_1, \lambda_2) \in \Lambda$, where

$$\Lambda = \left\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \,|\, 1 \leq \lambda_1 \leq 5,\, 0.01 \leq \lambda_2 \leq 0.10\right\}.$$

See Figure 2.3. On the boundary of Ω, $M = 16$ electrodes are attached. We consider a discretization of Λ. For each (λ_1, λ_2) in this discretization the four current patterns

$$\begin{aligned}
I_1 &= (1,1,0,0,0,0,0,0,-1,-1,0,0,0,0,0,0)\,, \\
I_2 &= (0,0,1,1,0,0,0,0,0,0,-1,-1,0,0,0,0)\,, \\
I_3 &= (0,0,0,0,1,1,0,0,0,0,0,0,-1,-1,0,0)\,, \\
I_4 &= (0,0,0,0,0,0,1,1,0,0,0,0,0,0,-1,-1) \in \mathbb{R}^{16}_\diamond
\end{aligned}$$

are applied. The resulting voltage patterns are $V_1 := \Phi I_1, \ldots, V_4 := \Phi I_4$. A pair in $\mathscr{F}$ is fixed by choosing a conductivity and contact impedances corresponding to $\left(\hat{\lambda}_1, \hat{\lambda}_2\right) = (2, 0.05)$. Then, by Proposition 2.28(*ii*), $\mathscr{F}$ can be partitionated with respect to $\left(\hat{\lambda}_1, \hat{\lambda}_2\right)$ as

$$\mathscr{F} = \mathcal{C}(I_i) \cup \mathcal{V}'(V_i) \cup \left(\mathcal{V}(V_i) \cap \mathcal{C}'(I_i)\right) \quad \text{for } i = 1,2,3,4.$$

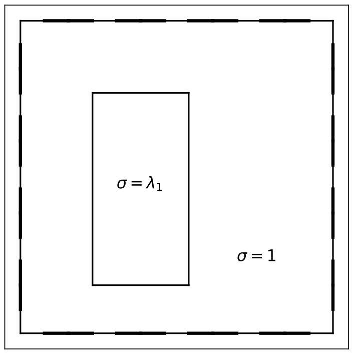


Figure 2.3: The test is performed in an square domain with $M = 16$ electrodes on its boundary and containing an inhomogeneous conductivity of background value 1. It is assumed that all the electrodes have the same contact impedance.

In Figure 2.4, for each current I_i, the part of these partitions contained in Λ is drawn. In Figure 2.5, the part of the feasible intersections $\bigcap_{i=1,2,3,4} \mathcal{C}(I_i)$ (colored in ▢) and $\bigcap_{i=1,2,3,4} \mathcal{V}'(V_i)$ (colored in ■) contained in Λ are drawn. It was used the finite element method with piecewise linear functions for the potential and an admissible triangulation of 5408 triangles. The monotonic properties given in Proposition 2.28(i) were helpful for the calculation of the feasible sets.

2.8 Conclusions

In this work, we have proposed a framework for the analysis of EIT models in terms of electric potentials and current fields. Abstract problems that generalize the weak formulation of known EIT models have been analyzed and properties of current-voltage maps associated with them have been proved. We have exploited the extremal formulation of the abstract problems to obtain dual properties that link them by means of functions that model the power dissipated in an EIT experiment. These results have led to a posteriori error estimates and a generalization of the ideas of feasibility constraints and feasible sets. Furthermore, functionals of Kohn-Vogelius type and a generalization of the well-known monotonicity principle have been deduced.

The examples showed that the error estimates obtained here may be used to assess the error of exact solutions to idealized problems and approximate solutions obtained by numerical methods. Also, it is remarkable the continued presence of the error term $E(x, z)$ in the estimates. In fact, this error is closely related to the Kohn-Vogelius functional.

The extension of the idea of feasible constrains allowed to consider the contact impedances of the complete electrode model as unknown parameters; also conductivity tensors are proved to be admitted.

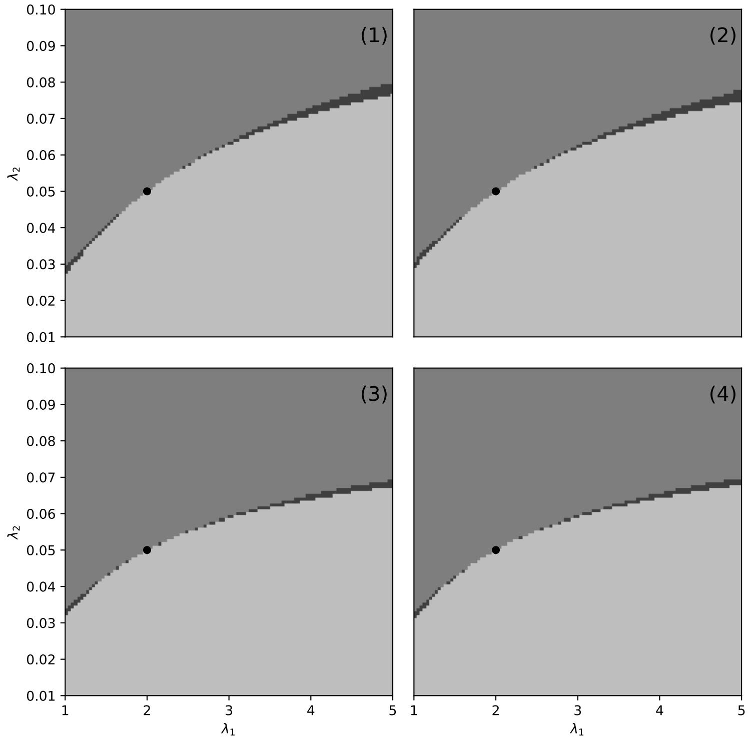


Figure 2.4: For each current pattern I_i $(i = 1, 2, 3, 4)$, the intersections of the feasible sets $\mathcal{C}(I_i)$ (colored in), $\mathcal{V}'(V_i)$ (colored in), and $\mathcal{V}(V_i) \cap \mathcal{C}'(I_i)$ (colored in) with the region Λ are ploted. In each case, the points (λ_1, λ_2) that belongs to the intersection of all the feasible boundaries represent the candidates to be the true conductivity-impedance, which is parametrized by $\left(\hat{\lambda}_1, \hat{\lambda}_2\right) = (2, 0.05)$ (bullet symbol).

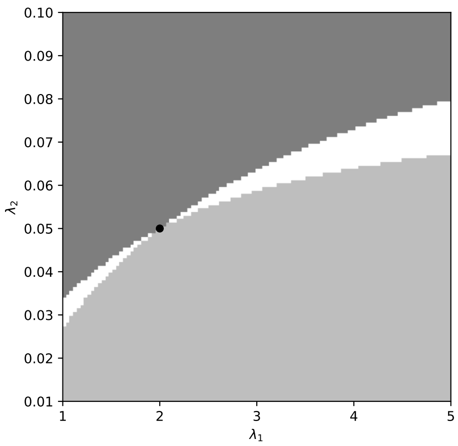


Figure 2.5: The feasible intersections $\bigcap_{i=1,2,3,4} \mathcal{C}(I_i)$ (colored in) and $\bigcap_{i=1,2,3,4} \mathcal{V}'(V_i)$ (colored in) are identified in the region Λ. Note that the point $\left(\hat{\lambda}_1, \hat{\lambda}_2\right) = (2, 0.05)$ (bullet symbol), which represents the true conductivity-contact impedances, lies in the intersection of the feasible boundaries.

Future work might be concerned with a framework for EIT models with a complex-value conductivity as well as in terms of electric fields and current potentials (in this context the gradient-like operator must be replaced by a curl-like operator). Also the optimization problems with functionals of Kohn-Vogelius type would be of interest in the study of the EIT inverse problem, particularly when it is formulated with the equations of the complete electrode model.

According to our knowledge, all existing EIT models verify our assumptions, except the so-called *point model* [47], about which we do not assert anything.

Since the Lax-Milgram Theorem is not necessarily true in Banach spaces, the extension of our framework to EIT models in Banach spaces is not guaranteed.

Finally, it is worth noting that our abstract problems can also be interpreted as dual and complementary problems from the point of view of the duality theory of convex analysis and the theory of complementary variational principles, respectively.

Appendix

Shunt model

The equations of the *shunt model* [92, 29, 46, 69] for the electric potential (u, U) are

$$\nabla\cdot(\sigma\nabla u) = 0 \qquad \text{in } \Omega \tag{2.68}$$

$$\sigma\nabla u\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega\setminus\bigcup_{m=1}^{M}\mathcal{E}_m \tag{2.69}$$

$$u = U_m \qquad \text{on } \mathcal{E}_m,\ m = 1,\dots,M \tag{2.70}$$

with

$$\int_{\mathcal{E}_m}\nabla u\cdot\mathbf{n}\,\mathrm{d}\mathbf{s} = I_m \quad m = 1,\dots,M \tag{2.71}$$

if a current pattern $I = (I_1,\dots,I_M)$ is applied, or with

$$U_m = V_m \quad m = 1,\dots,M \tag{2.72}$$

if a voltage pattern $V = (V_1,\dots,V_M)$ is applied. The shunt model (2.68)-(2.72) fits into Assumptions A1-A3 with

A1. $X \coloneqq \left\{(u,U)\in H^1(\Omega)\times\mathbb{R}^M \,|\, \gamma_m u = U_m\right\}$ equipped with the inner product induced by the direct sum of $H^1(\Omega)$ and $\mathbb{R}^M$, $Z \coloneqq L^2\left(\Omega,\mathbb{R}^d\right)$ equipped with its usual inner product, $1_X \coloneqq \left(\mathbf{1},\overrightarrow{1}\right)\in X$, and $G : X\to Z$ defined by $G(u,U) \coloneqq -\nabla u$.

A2. $b : Z\times Z\to\mathbb{R}$ defined by $b(\mathbf{p}_1,\mathbf{p}_2) \coloneqq \int_\Omega \frac{1}{\sigma}\mathbf{p}_1\cdot\mathbf{p}_2\,\mathrm{d}\mathbf{x}$.

A3. $Y \coloneqq \mathbb{R}^M$ and $P : X\to Y$ defined by $P(u,U) \coloneqq U$.

It is easy to check that the isomorphism $B : Z\to Z$ associated to b is given by $B\mathbf{p} \coloneqq \sigma^{-1}\mathbf{p}$. Thus, the linear operator $T : X\to Z$ and the bilinear form $a : X\times X\to\mathbb{R}$ are given by

$$T(u,U) = \left(B^{-1}\circ G\right)(u,U) = -\sigma\nabla u \quad \text{and}$$

$$a\left((u_1,U_1),(u_2,U_2)\right) = b\left(T(u_1,U_1),T(u_2,U_2)\right) = \int_\Omega \sigma\nabla u_1\cdot\nabla u_2\,\mathrm{d}\mathbf{x}.$$

Here, the norm on $Y = \mathbb{R}^M$ induced by X and P is

$$\|U_0\|_Y \coloneqq \min_{\substack{(u,U)\in X\\ P(u,U)=U_0}} \|(u,U)\|_X = \min_{\substack{u\in H^1(\Omega)\\ \gamma_m u = U_{0,m}}} \left(\|u\|^2_{H^1(\Omega)} + \|U_0\|^2_{\mathbb{R}^M}\right)^{1/2} \geq \|U_0\|_{\mathbb{R}^M}.$$

The subspace $\mathbb{R}^M_\diamond$ is contained in $Y^\star_\diamond$. Indeed, given $I\in\mathbb{R}^M_\diamond$, the linear functional $\langle f,U\rangle = \sum_{m=1}^M I_m U_M$ satisfies $\langle f,1_Y\rangle = 0$ ($1_Y = P(1_X) = \overrightarrow{1}$) and is continuous on $\mathbb{R}^M$ equipped with the norm $\|\cdot\|_Y$. Since

$$\ker P = \left\{\left(u,\overrightarrow{0}\right) \,|\, u\in H^1(\Omega),\ \gamma_m u = 0 \text{ for } m = 1,\dots,M\right\},$$

we deduce that

$$Z_{\perp} = \left\{ \mathbf{p} \in Z \,\middle|\, \nabla \cdot \mathbf{p} = 0, \begin{array}{l} \langle \mathbf{p} \cdot \mathbf{n}, \gamma u \rangle = 0 \\ \text{for all } u \in H^1(\Omega) \text{ such that } (\gamma_m u)_{m=1}^M = \overrightarrow{0} \end{array} \right\}$$

by (2.9), where the equality $\nabla \cdot \mathbf{p} = 0$ has to be understood in the sense of distributions. Given $\mathbf{p} \in Z_{\perp}$, it can be deduced that the action of $R\mathbf{p}$ on $U \in Y = \mathbb{R}^M$ is expressed by

$$\langle R\mathbf{p}, U \rangle = -\langle \mathbf{p} \cdot \mathbf{n}, \gamma u \rangle \quad \text{with } (u, U) \in X,$$

that is, u is any function in $H^1(\Omega)$ satisfying $\gamma_m u = U_m$ for $m = 1, \ldots, M$. From this, we deduce that the kernel of R is the set

$$\ker R = \left\{ \mathbf{p} \in Z_{\perp} \,\middle|\, \begin{array}{l} \langle \mathbf{p} \cdot \mathbf{n}, \gamma u \rangle = 0 \\ \text{for all } u \in H^1(\Omega) \text{ such that } (\gamma_m u)_{m=1}^M \in \mathbb{R}^M \end{array} \right\}.$$

Now we show that given $\mathbf{p} \in Z_{\perp}$, $R\mathbf{p}$ can be identified with a vector of $\mathbb{R}^M_{\diamond}$. Let $e_1, \ldots, e_M$ be functions in $H^1(\Omega)$ satisfying

$$\gamma_{m'} e_m = \begin{cases} 1 & m = m' \\ 0 & m \neq m' \end{cases} \quad \text{for all } m, m'.$$

So, given $U \in \mathbb{R}^M$, the element $\left(\sum_{m=1}^M U_m e_m, U\right)$ belongs to X and the action of $R\mathbf{p}$ on U can be written as

$$\langle R\mathbf{p}, U \rangle = \sum_{m=1}^M U_m \langle -\mathbf{p} \cdot \mathbf{n}, \gamma e_m \rangle .$$

Therefore, $R\mathbf{p}$ can identified with the vector $(\langle -\mathbf{p} \cdot \mathbf{n}, \gamma e_1 \rangle, \ldots, \langle -\mathbf{p} \cdot \mathbf{n}, \gamma e_M \rangle)$, which belongs to $\mathbb{R}^M_{\diamond}$ since $\langle R\mathbf{p}, 1_Y \rangle = 0$. Observe that this identification does not come from the Riesz-Fréchet representation theorem because $\|\cdot\|_Y$ and $\|\cdot\|_{\mathbb{R}^M}$ are not equivalent.

Abstract problems. The abstract problems for the shunt model are presented below.

$\mathcal{C}$. Given $f \in Y^{\star}_{\diamond}$, find a $(\bar{u}, \bar{U}) \in X$ satisfying

$$\int_{\Omega} \sigma \nabla \bar{u} \cdot \nabla u \, \mathrm{d}\mathbf{x} = \langle f, P(u, U) \rangle = \langle f, U \rangle \quad \text{for all } (u, U) \in X.$$

Choosing f defined by $\langle f, U \rangle = \sum_{m=1}^M I_m U_M$, with $I \in \mathbb{R}^M_{\diamond}$, this problem is the weak formulation of the shunt model written in terms of electric potentials (u, U)'s and with applied current I. A unique solution of this problem can be obtained in the subpace $X_{\diamond} = \left\{ (u, U) \in X \,\middle|\, \sum_{m=1}^M U_m = 0 \right\}$, which corresponds to the linear continuous functional $\Gamma(u, U) = \sum_{m=1}^M U_m$.

$\mathcal{V}$. Given $V \in Y = \mathbb{R}^M$, find a $\left(\tilde{u}, \tilde{U}\right) \in X$ satisfying

$$\begin{aligned} \int_\Omega \sigma \nabla \tilde{u} \cdot \nabla u \, \mathrm{d}\mathbf{x} = 0 \quad & \text{for all } u \in H^1(\Omega) \text{ with } (\gamma_m u)_{m=1}^M = \overrightarrow{0} \\ \tilde{U} = V \quad & \text{in } Y = \mathbb{R}^M \end{aligned} .$$

This is the weak formulation of the shunt model written in terms of electric potentials (u, U)'s with applied voltage V.

$\mathcal{C}'$. Given $f \in Y_\diamond^\star$, find a $\bar{\mathbf{p}} \in Z_\perp$ satisfying

$$\begin{aligned} b(\bar{\mathbf{p}}, \mathbf{p}) = \int_\Omega \frac{1}{\sigma} \bar{\mathbf{p}} \cdot \mathbf{p} \, \mathrm{d}\mathbf{x} = 0 \qquad & \text{for all } \mathbf{p} \in \ker R \\ -\langle \bar{\mathbf{p}} \cdot \mathbf{n}, \gamma u \rangle = \langle f, U \rangle \quad & \text{for all } (u, U) \in X \end{aligned} .$$

Choosing f defined by $\langle f, U \rangle = \sum_{m=1}^M I_m U_M$, with $I \in \mathbb{R}_\diamond^M$, the second equation becomes

$$-\langle \bar{\mathbf{p}} \cdot \mathbf{n}, \gamma e_m \rangle = I_m \quad \text{for } m = 1, \ldots, M.$$

This problem can be viewed as the weak formulation of the shunt model written in terms of current fields $\mathbf{p}$'s and with applied current I.

$\mathcal{V}'$. Given $V \in Y = \mathbb{R}^M$, find a $\tilde{\mathbf{p}} \in Z_\perp$ satisfying

$$b(\tilde{\mathbf{p}}, \mathbf{p}) = \int_\Omega \frac{1}{\sigma} \tilde{\mathbf{p}} \cdot \mathbf{p} \, \mathrm{d}\mathbf{x} = \langle -\mathbf{p} \cdot \mathbf{n}, \gamma u_V \rangle \quad \text{for all } \mathbf{p} \in Z_\perp,$$

where u_V is some function in $H^1(\Omega)$ satisfying $(\gamma_m u_V)_{m=1}^M = V$. This is the weak formulation of the shunt model written in terms of current fields $\mathbf{p}$'s and with applied voltage V. Alternatively, we can write the above equation as

$$\int_\Omega \frac{1}{\sigma} \tilde{\mathbf{p}} \cdot \mathbf{p} \, \mathrm{d}\mathbf{x} = \sum_{m=1}^M V_m \langle -\mathbf{p} \cdot \mathbf{n}, \gamma e_m \rangle \quad \text{for all } \mathbf{p} \in Z_\perp.$$

Example 2.33. Here we show that the EIT model proposed in [75] fits in the assumptions made for the shunt model. Consider the domain Ω as $\Omega = \Omega_0 \cup \left(\cup_{m=1}^M \Omega_m\right)$ and the conductivity σ as

$$\sigma(\mathbf{x}) = \begin{cases} \sigma_0(\mathbf{x}) & \mathbf{x} \in \Omega_0 \\ \sigma_m(\mathbf{x}) & \mathbf{x} \in \Omega_m, \, m = 1, \ldots, M \end{cases} .$$

See Figure (2.6). $\mathcal{E}_m$ and $\mathcal{D}_m$ are the outer and inner boundaries of Ω_m, respectively. Then, the problem $\mathcal{C}'$ becomes

$$\begin{aligned} \int_{\Omega_0} \frac{1}{\sigma_0} \bar{\mathbf{p}} \cdot \mathbf{p} \, \mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\Omega_m} \frac{1}{\sigma_m} \bar{\mathbf{p}} \cdot \mathbf{p} \, \mathrm{d}\mathbf{x} = 0 \qquad & \text{for all } \mathbf{p} \in \ker R \\ -\langle \bar{\mathbf{p}} \cdot \mathbf{n}, \gamma e_m \rangle = I_m \quad & \text{for } m = 1, \ldots, M \end{aligned} ,$$

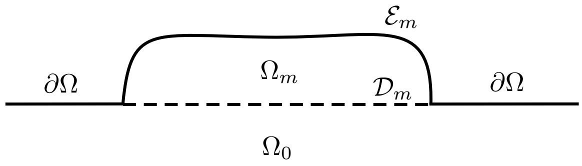


Figure 2.6: Subdomains and boundaries.

or equivalently

$$\min_{\substack{\mathbf{p}\in Z_\perp \\ (-\langle \mathbf{p}\cdot\mathbf{n},\gamma e_m\rangle)_{m=1}^M = I}} \frac{1}{2}\left(\int_{\Omega_0}\frac{1}{\sigma_0}|\mathbf{p}|^2\,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\Omega_m}\frac{1}{\sigma_m}|\mathbf{p}|^2\,\mathrm{d}\mathbf{x}\right) \tag{2.73}$$

where the closed subspace $Z_\perp$ can be written as the set

$$Z_\perp = \left\{\mathbf{p}\in L^2\left(\Omega,\mathbb{R}^d\right) \mid \nabla\cdot\mathbf{p} = 0 \text{ in } \Omega,\ \mathbf{p}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\setminus\cup_{m=1}^M\mathcal{E}_m\right\}.$$

Clearly, if $\mathbf{p}\in Z_\perp$ then $\langle\mathbf{p}\cdot\mathbf{n},\gamma\mathbf{1}\rangle = \int_\Omega \mathbf{p}\cdot\nabla\mathbf{1}\,\mathrm{d}\mathbf{x} = 0$. It turns out that the minimization problem (2.73) is the integral formulation of EIT proposed in [75]. There, a version of the complete electrode model is derived from (2.73). Now, we provide a derivation of that version. It is known that $\bar{\mathbf{p}} = -\sigma\nabla\bar{u}$, where $\bar{u}$ is a solution of $\mathcal{C}$. Considering the subdomains and their conductivities, the equation for $\bar{u}$ can be written as

$$\int_{\Omega_0}\sigma_0\nabla\bar{u}\cdot\nabla u\,\mathrm{d}\mathbf{x} + \sum_{m=1}^M\int_{\Omega_m}\sigma_m\nabla\bar{u}\cdot\nabla u\,\mathrm{d}\mathbf{x} = \sum_{m=1}^M I_m U_M \quad \text{for all } (u,U)\in X. \tag{2.74}$$

Since $\nabla\cdot(\sigma\nabla\bar{u}) = 0$ in Ω,

$$\int_{\Omega_m}\sigma_m\nabla\bar{u}\cdot\nabla u\,\mathrm{d}\mathbf{x} = \langle\sigma_m\nabla\bar{u}\cdot\mathbf{n}_{\partial\Omega_m},\gamma u\rangle_{H^{-1/2}(\partial\Omega_m)\times H^{1/2}(\partial\Omega_m)} \quad \text{for all } u\in H^1(\Omega).$$

Assume that $\sigma_m\nabla\bar{u}\cdot\mathbf{n}\in L^2(\partial\Omega_m)$. Then, given $(u,U)\in X$,

$$\begin{aligned}\int_{\Omega_m}\sigma_m\nabla\bar{u}\cdot\nabla u\,\mathrm{d}\mathbf{x} &= \int_{\partial\Omega_m}(\sigma_m\nabla\bar{u}\cdot\mathbf{n}_{\partial\Omega_m})\,\gamma u\,\mathrm{d}\mathbf{s} \\ &= \int_{\mathcal{D}_m}(\sigma_m\nabla\bar{u}\cdot\mathbf{n}_{\partial\Omega_m})\,\gamma_{\mathcal{D}_m}u\,\mathrm{d}\mathbf{s} + \int_{\mathcal{E}_m}(\sigma_m\nabla\bar{u}\cdot\mathbf{n}_{\partial\Omega_m})\,\gamma_m u\,\mathrm{d}\mathbf{s}\end{aligned}$$

Since $\gamma_m u = U_m$ on $\mathcal{E}_m$ and $\int_{\mathcal{D}_m}\sigma_m\nabla\bar{u}\cdot\mathbf{n}_{\partial\Omega_m}\,\mathrm{d}\mathbf{s} + \int_{\mathcal{E}_m}\sigma_m\nabla\bar{u}\cdot\mathbf{n}_{\partial\Omega_m}\,\mathrm{d}\mathbf{s} = 0$ (when $u=\mathbf{1}$) it follows that

$$\begin{aligned}\int_{\Omega_m}\sigma_m\nabla\bar{u}\cdot\nabla u\,\mathrm{d}\mathbf{x} &= \int_{\mathcal{D}_m}(\sigma_m\nabla\bar{u}\cdot\mathbf{n}_{\partial\Omega_m})\,\gamma_{\mathcal{D}_m}u\,\mathrm{d}\mathbf{s} - U_m\int_{\mathcal{D}_m}\sigma_m\nabla\bar{u}\cdot\mathbf{n}_{\partial\Omega_m}\,\mathrm{d}\mathbf{s} \\ &= \int_{\mathcal{D}_m}(\sigma_m\nabla\bar{u}\cdot\mathbf{n}_{\partial\Omega_m})\left(\gamma_{\mathcal{D}_m}u - U_m\right)\,\mathrm{d}\mathbf{s} \\ &= \int_{\mathcal{D}_m}\frac{-\left(\sigma_m\nabla\bar{u}\cdot\mathbf{n}_{\partial\Omega_0}\right)}{\left(\gamma_{\mathcal{D}_m}\bar{u}-\bar{U}_m\right)}\left(\gamma_{\mathcal{D}_m}\bar{u}-\bar{U}_m\right)\left(\gamma_{\mathcal{D}_m}u - U_m\right)\,\mathrm{d}\mathbf{s}.\end{aligned}$$

Thus, we define the contact impedances $z_1, \ldots, z_M$ as the functions

$$z_m(\mathbf{x}) := \frac{\left(\gamma_{\mathcal{D}_m}\bar{u} - \bar{U}_m\right)}{-\sigma_m(\mathbf{x})\nabla\bar{u}(\mathbf{x})\cdot\mathbf{n}_{\partial\Omega_0}(\mathbf{x})} = \frac{\int_\Gamma (\bar{\mathbf{p}}/\sigma_m)\,\mathrm{d}\mathbf{s}}{\bar{\mathbf{p}}(\mathbf{x})\cdot\mathbf{n}_{\partial\Omega_0}(\mathbf{x})} \quad \mathbf{x}\in\mathcal{D}_m.$$

where Γ is a path within Ω_m that goes from the point $\mathbf{x}$ to an arbitrary point on $\mathcal{E}_m$. Therefore, (2.74) becomes the weak formulation of the complete electrode model on the domain Ω_0, namely

$$\int_{\Omega_0}\sigma_0\nabla\bar{u}\cdot\nabla u\,\mathrm{d}\mathbf{x} + \int_{\mathcal{D}_m}\frac{\left(\gamma_{\mathcal{D}_m}\bar{u} - \bar{U}_m\right)\left(\gamma_{\mathcal{D}_m}u - U_m\right)}{z_m}\,\mathrm{d}\mathbf{s} = \sum_{m=1}^M I_m U_M$$

for all $(u, U) \in H^1(\Omega_0)\times\mathbb{R}^M$.

Gap model

The equations of the *gap model* [92, 29, 46, 69] for the electric potential (u, U) are

$$\nabla\cdot(\sigma\nabla u) = 0 \qquad \text{in } \Omega \tag{2.75}$$

$$\sigma\nabla u\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega\setminus\bigcup_{m=1}^M\mathcal{E}_m \tag{2.76}$$

$$\sigma\nabla u\cdot\mathbf{n} = \text{const.} \qquad \text{on } \mathcal{E}_m,\ m = 1,\ldots,M \tag{2.77}$$

$$\frac{1}{|\mathcal{E}_m|}\int_{\mathcal{E}_m} u\,\mathrm{d}\mathbf{s} = U_m \qquad \text{on } \mathcal{E}_m,\ m = 1,\ldots,M \tag{2.78}$$

with

$$\sigma\nabla u\cdot\mathbf{n}|_{\mathcal{E}_m} = \frac{I_m}{|\mathcal{E}_m|} \quad m = 1,\ldots,M \tag{2.79}$$

if a current pattern $I = (I_1, \ldots, I_M)$ is applied, or with

$$U_m = V_m \quad m = 1,\ldots,M \tag{2.80}$$

if a voltage pattern $V = (V_1, \ldots, V_M)$ is applied. The gap model (2.75)-(2.80) fits into Assumptions A1-A3 with

A1. $X := \left\{(u, U)\in H^1(\Omega)\times\mathbb{R}^M \;\middle|\; \frac{1}{|\mathcal{E}_m|}\int_{\mathcal{E}_m}\gamma_m u\,\mathrm{d}\mathbf{s} = U_m\right\}$ equipped with the inner product induced by the direct sum of $H^1(\Omega)$ and $\mathbb{R}^M$, $Z := L^2(\Omega, \mathbb{R}^d)$ equipped with its usual inner product, $1_X := \left(\mathbf{1}, \overrightarrow{1}\right)\in X$, and $G : X\to Z$ defined by $G(u, U) := -\nabla u$.

A2. $b : Z\times Z\to\mathbb{R}$ defined by $b(\mathbf{p}_1, \mathbf{p}_2) := \int_\Omega\frac{1}{\sigma}\mathbf{p}_1\cdot\mathbf{p}_2\,\mathrm{d}\mathbf{x}$.

A3. $Y := \mathbb{R}^M$ and $P : X\to Y$ defined by $P(u, U) := U$.

The norm on $Y = \mathbb{R}^M$ induced by X and P is

$$\|U_0\|_Y = \min_{\substack{u \in H^1(\Omega) \\ \frac{1}{|\mathcal{E}_m|}\int_{\mathcal{E}_m} \gamma_m u \, d\mathbf{s} = U_{0,m}}} \left(\|u\|_{H^1(\Omega)}^2 + \|U_0\|_{\mathbb{R}^M}^2 \right)^{1/2} \geq \|U_0\|_{\mathbb{R}^M} .$$

As in the shunt model, the subspace $\mathbb{R}^M_\diamond$ is contained in $Y^\star_\diamond$. Since

$$\ker P = \left\{ (u, 0) \,\middle|\, \frac{1}{|\mathcal{E}_m|} \int_{\mathcal{E}_m} \gamma_m u \, d\mathbf{s} = 0, \; m = 1, \ldots, M \right\}$$

we deduce that

$$Z_\perp = \left\{ \mathbf{p} \in Z \,\middle|\, \begin{array}{l} \nabla \cdot \mathbf{p} = 0, \\ \langle \mathbf{p} \cdot \mathbf{n}, \gamma u \rangle = 0 \\ \text{for all } u \in H^1(\Omega) \text{ such that } \left(\frac{1}{|\mathcal{E}_m|} \int_{\mathcal{E}_m} \gamma_m u \, d\mathbf{s} \right)_{m=1}^M = \overrightarrow{0} \end{array} \right\}$$

by (2.9), where the equality $\nabla \cdot \mathbf{p} = 0$ has to be understood in the sense of distributions. Given $\mathbf{p} \in Z_\perp$, it can be deduced that the action of $R\mathbf{p}$ on $U \in \mathbb{R}^M$ is given by

$$\langle R\mathbf{p}, U \rangle = - \langle \mathbf{p} \cdot \mathbf{n}, \gamma u \rangle \quad \text{with } (u, U) \in X,$$

that is, u is any function in $H^1(\Omega)$ satisfying $\frac{1}{|\mathcal{E}_m|} \int_{\mathcal{E}_m} \gamma_m u \, d\mathbf{s} = U_m$ for $m = 1, \ldots, M$. From this, the kernel of R is

$$\ker R = \left\{ \mathbf{p} \in Z_\perp \,\middle|\, \begin{array}{l} \langle \mathbf{p} \cdot \mathbf{n}, \gamma u \rangle = 0 \\ \text{for all } u \in H^1(\Omega) \text{ such that } \left(\frac{1}{|\mathcal{E}_m|} \int_{\mathcal{E}_m} \gamma_m u \, d\mathbf{s} \right)_{m=1}^M \in \mathbb{R}^M \end{array} \right\}.$$

Given $\mathbf{p} \in Z_\perp$, we can identified $R\mathbf{p}$ with a vector of $\mathbb{R}^M_\diamond$. Indeed, let $e_1, \ldots, e_M$ be functions in $H^1(\Omega)$ satisfying

$$\int_{\mathcal{E}_{m'}} \gamma_m e_m \, d\mathbf{s} = \begin{cases} |\mathcal{E}_m| & m = m' \\ 0 & m \neq m' \end{cases} \quad \text{for all } m, m'.$$

So, given $U \in \mathbb{R}^M$, the element $\left(\sum_{m=1}^M U_m e_m, U \right)$ belongs to X and the action of $R\mathbf{p}$ on U can be written as

$$\langle R\mathbf{p}, U \rangle = \sum_{m=1}^M U_m \langle -\mathbf{p} \cdot \mathbf{n}, \gamma e_m \rangle .$$

Therefore, $R\mathbf{p}$ can identified with the vector $(\langle -\mathbf{p} \cdot \mathbf{n}, \gamma e_1 \rangle, \ldots, \langle -\mathbf{p} \cdot \mathbf{n}, \gamma e_M \rangle)$, which belongs to $\mathbb{R}^M_\diamond$ since $\langle R\mathbf{p}, 1_Y \rangle = 0$. As in the shunt model, this identification does not come from the Riesz-Fréchet representation theorem because $\|\cdot\|_Y$ and $\|\cdot\|_{\mathbb{R}^M}$ are not equivalent.

The abstract problems of the gap model are as those of the shunt model (a, b, and T are the same). To formulate them, in addition to the above closed subspaces, we must consider a function u_V satisfying $\frac{1}{|\mathcal{E}_m|} \int_{\mathcal{E}_m} \gamma_m u_V \, d\mathbf{s} = V_m$ for $m = 1, \ldots, M$.

Smoothened complete electrode model

In [57] was proposed the *smoothened complete electrode model*:

$$\nabla\cdot(\sigma\nabla u)=0 \qquad \text{in } \Omega \tag{2.81}$$

$$\sigma\nabla u\cdot\mathbf{n}=0 \qquad \text{on } \partial\Omega\backslash\bigcup_{m=1}^{M}\mathcal{E}_m \tag{2.82}$$

$$\sigma\nabla u\cdot\mathbf{n}+\zeta_m\left(u-U_m\right)=0 \qquad \text{on } \mathcal{E}_m,\ m=1,\ldots,M \tag{2.83}$$

with

$$\int_{\mathcal{E}_m}\sigma\nabla u\cdot\mathbf{n}\,\mathrm{d}\mathbf{s}=I_m \quad m=1,\ldots,M \tag{2.84}$$

if a current pattern $I=(I_1,\ldots,I_M)$ is applied, or with

$$U_m=V_m \quad m=1,\ldots,M \tag{2.85}$$

if a voltage pattern $V=(V_1,\ldots,V_M)$ is applied. It replaces the contact impedances of the complete electrode model with contact admittance functions $\zeta_1,\ldots,\zeta_M$ capable to vanish on some subsets of the electrodes. The contact admittances satisfy

$$\zeta_m\in L^\infty\left(\mathcal{E}_m\right),\quad \zeta_m\geq 0 \text{ a.e. on } \mathcal{E}_m,\quad \text{and}\quad \zeta_m\not\equiv 0 \quad \text{for } m=1,\ldots,M. \tag{2.86}$$

Considering the complete electrode model assumptions with $\zeta_1,\ldots,\zeta_M$ instead of $z_1,\ldots,z_M$, this model satisfies partially Assumptions A1-A3 since the bilinear form b, given by

$$b\left(\left(\mathbf{p}_1,\mathbf{P}_1\right),\left(\mathbf{p}_2,\mathbf{P}_2\right)\right):=\int_\Omega\frac{1}{\sigma}\mathbf{p}_1\cdot\mathbf{p}_2\,\mathrm{d}\mathbf{x}+\sum_{m=1}^{M}\int_{\mathcal{E}_m}\frac{1}{\zeta_m}\mathbf{P}_{1,m}\mathbf{P}_{2,m}\,\mathrm{d}\mathbf{s}$$

in this case, is not defined for contact admittances that vanish on open subsets of the electrodes. However, applying fact that, for each m, there exists a open subset $e_m\subseteq\mathcal{E}_m$ such that $\operatorname{ess\,inf}_{\mathbf{x}\in e_m}\zeta_m(\mathbf{x})>0$ and $\zeta_m\equiv 0$ in $\mathcal{E}_m\backslash e_m$, it suffices to replace $\mathcal{E}_1,\ldots,\mathcal{E}_M$ with $e_1,\ldots,e_M$ to fit this model entirely.

Connection with the duality theory of convex analysis

An interpretation of the abstract problems from the point of view of the *duality theory* of convex analysis is provided here. Our references in this subject are [36, Ch. 3], [45, Ch. 2], and [11, Ch. 9]. Let us start by noting that the extremal formulations of $\mathcal{C}$ and $\mathcal{V}$, that is,

$$\max_{x\in X}\left\{\langle f,Px\rangle-\frac{1}{2}a\left(x,x\right)\right\} \qquad \text{and} \qquad \min_{\substack{x\in X\\ Px=g}}\left\{\frac{1}{2}a\left(x,x\right)\right\},$$

respectively, can be written in the form

$$\inf_{x\in X}\left\{\mathbf{F}\left(x\right)+\mathbf{G}\left(\Lambda x\right)\right\}, \tag{2.87}$$

where the functionals $\mathbf{F} : X \cup \{+\infty\} \to \mathbb{R}$ and $\mathbf{G} : Z \to \mathbb{R}$ are defined as

$$\mathbf{F}(x) := -\langle f, Px\rangle \quad \text{in the case of } \mathcal{C},$$

$$\mathbf{F}(x) := \begin{cases} 0 & \text{if } Px = g \\ +\infty & \text{otherwise} \end{cases} \quad \text{in the case of } \mathcal{V},$$

$$\text{and} \quad \mathbf{G}(z) := \frac{1}{2} b\left(B^{-1}z, B^{-1}z\right) \quad \text{in both cases,}$$

and the operator $\Lambda : X \to Z$ is $\Lambda x := -Gx$. Using the properties of b and P, it is easy to see the following: $\mathbf{F}$ is linear and continuous in the case of $\mathcal{C}$, and convex, lower semicontinuous, and proper in the case of $\mathcal{V}$; $\mathbf{G}$ is convex and continuous. All the above make evident that the optimization problem (2.87) fits into the duality theory of convex analysis [36, Ch. 3 Rem. 4.2][11, Ch. 9]. There, (2.87) is called the *primal* problem, and its corresponding *dual* problem is defined as

$$\sup_{z\in Z} \{-\mathbf{F}^\star(\Lambda^\star z) - \mathbf{G}^\star(-z)\}, \tag{2.88}$$

where $\mathbf{F}^\star : X \to \mathbb{R} \cup \{+\infty\}$ and $\mathbf{G}^\star : Z \to \mathbb{R}$ are the *polar* or *conjugate* functions of $\mathbf{F}$ and $\mathbf{G}$, respectively. By definition (see [36, Ch. 1 Def. 4.1]),

$$\mathbf{F}^\star(x) := \sup_{x'\in X} \left\{\left\langle x, x'\right\rangle_X - \mathbf{F}\left(x'\right)\right\} \quad \text{and} \quad \mathbf{G}^\star(z) := \sup_{z'\in Z} \left\{\left\langle z, z'\right\rangle_Z - \mathbf{G}\left(z'\right)\right\}.$$

We have

$$\begin{aligned} \mathbf{F}^\star(x) &= \sup_{x'\in X} \left\{\left\langle x, x'\right\rangle_X + \left\langle f, Px'\right\rangle\right\} \\ &= \begin{cases} 0 & \text{if } \langle -x, x'\rangle_X = \langle f, Px'\rangle \text{ for all } x' \in X \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

in the case of $\mathcal{C}$,

$$\begin{aligned} \mathbf{F}^\star(x) &= \sup_{x'\in X,\, Px'=g} \left\langle x, x'\right\rangle_X \\ &= \sup_{x'\in X,\, Px'=g} \left\langle x_\perp, x'\right\rangle_X + \left\langle x_0, x'\right\rangle_X \quad \left(x_\perp \in (\ker P)^\perp,\, x_0 \in \ker P\right) \\ &= \left\langle x_\perp, P^{-1}g\right\rangle_X + \sup_{x'\in X,\, Px'=g} \left\langle x_0, x'\right\rangle_X \\ &\Rightarrow \quad \mathbf{F}^\star(x) = \begin{cases} \left\langle x, P^{-1}g\right\rangle_X & x \in (\ker P)^\perp \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

in the case of $\mathcal{V}$, andd

$$\begin{aligned}\mathbf{G}^\star(z) &= \sup_{z'\in Z}\left\{\left\langle z,z'\right\rangle_Z - \frac{1}{2}b\left(B^{-1}z',B^{-1}z'\right)\right\}\\ &= \sup_{z'\in Z}\left\{\left\langle z,z'\right\rangle_Z - \frac{1}{2}\left\langle B^{-1}z',z'\right\rangle_Z\right\}\\ &= \frac{1}{2}\left\langle z,\hat{z}\right\rangle_Z \quad (\left\langle B^{-1}\hat{z},z'\right\rangle = \left\langle z,z'\right\rangle_Z \quad \text{for all } z'\in Z)\\ &= \frac{1}{2}\left\langle z,Bz\right\rangle_Z = \frac{1}{2}b(z,z).\end{aligned}$$

Hence, the objective functional of the dual problem (2.88) is

$$-\mathbf{F}^\star(\Lambda^\star z) - \mathbf{G}^\star(-z) = \begin{cases} -\frac{1}{2}b(z,z) & \text{if } \left\langle G^\star z,x'\right\rangle_X = \left\langle f,Px'\right\rangle \text{ for all } x'\in X\\ -\infty & \text{otherwise}\end{cases}$$

in the case of $\mathcal{C}$ and

$$-\mathbf{F}^\star(\Lambda^\star z) - \mathbf{G}^\star(-z) = \begin{cases} \left\langle G^\star z,P^{-1}g\right\rangle_X - \frac{1}{2}b(z,z) & G^\star z\in(\ker P)^\perp\\ -\infty & \text{otherwise}\end{cases}$$

in the case of $\mathcal{V}$. Thus, the dual problem (2.88) in the case of $\mathcal{C}$ reads as

$$\sup_{\substack{z\in Z\\ \left\langle G^\star z,\cdot\right\rangle_X = \left\langle f,P(\cdot)\right\rangle}} -\frac{1}{2}b(z,z). \tag{2.89}$$

Since the constraint $\left\langle G^\star z,\cdot\right\rangle_X = \left\langle f,P(\cdot)\right\rangle$ can be written as $Rz=f$, (2.89) is equivalent to the extremal formulation of $\mathcal{C}'$. On the other hand, the dual problem (2.88) in the case of $\mathcal{V}$ reads as

$$\sup_{\substack{z\in Z\\ G^\star z\in(\ker P)^\perp}} \left\langle G^\star z,P^{-1}g\right\rangle_X - \frac{1}{2}b(z,z). \tag{2.90}$$

Since $\left\langle Rz,g\right\rangle = \left\langle G^\star z,P^{-1}g\right\rangle_X$ and $G^\star z\in(\ker P)^\perp$ iff $z\in Z_\perp$, (2.90) is the extremal formulation of $\mathcal{V}'$. In light of all of the above, we interpret that

$\rightsquigarrow$ $\mathcal{C}'$ is the *dual* of $\mathcal{C}$ and $\mathcal{V}'$ is the *dual* of $\mathcal{V}$.

It is worth pointing out that the primal and dual problems are linked to each other by the (strong) dual relation [36, Ch. 3 Th. 4.2]

$$\inf_{x\in X}\left\{\mathbf{F}(x)+\mathbf{G}(\Lambda x)\right\} = \sup_{z\in Z}\left\{-\mathbf{F}^\star(\Lambda^\star z)-\mathbf{G}^\star(-z)\right\}, \tag{2.91}$$

which is equivalent to the dual properties (*i*) and (*ii*) given in Theorem 2.15, where ϕ links the problems $\mathcal{C}$ and $\mathcal{C}'$, and ψ links the problems $\mathcal{V}$ and $\mathcal{V}'$. Furthermore, Theorem 2.15(*iii*) shows that the power functions ϕ and ψ are conjugate of each other (in the sense of [36, Ch. 1 Def. 4.1]), that is, $\phi^\star=\psi$ and $\psi^\star=\phi$. Thus, transferring the conjugate relation of the power functions to the abstract problems, we interpret that

$\rightsquigarrow$ $\mathcal{C}/\mathcal{C}'$ and $\mathcal{V}/\mathcal{V}'$ are *conjugate* problems.

To conclude, we mention that there is a *Lagrangian* function $\mathbf{L} : X \times Z \to \overline{\mathbb{R}}$ associated to the primal and dual problems

$$\begin{aligned}\mathbf{L}(x,z) &:= \inf_{z' \in Z} \left\{ -\left\langle z, z' \right\rangle_Z + \mathbf{F}(x) + \mathbf{G}\left(\Lambda x - z'\right) \right\} \\ &= \mathbf{F}(x) - \sup_{z' \in Z} \left\{ \left\langle z, z' \right\rangle_Z - \mathbf{G}\left(\Lambda x - z'\right) \right\} \\ &= \mathbf{F}(x) - \langle z, \Lambda x \rangle_Z - \sup_{z' \in Z} \left\{ \left\langle -z, \Lambda x - z' \right\rangle_Z - \mathbf{G}\left(\Lambda x - z'\right) \right\} \\ &= \mathbf{F}(x) - \langle z, \Lambda x \rangle_Z - \mathbf{G}^\star(-z),\end{aligned}$$

which satisfies the relations

$$\inf_{x \in X} \{ \mathbf{F}(x) + \mathbf{G}(\Lambda x) \} = \inf_{x \in X} \left\{ \sup_{z \in Z} \mathbf{L}(x,z) \right\}$$

and

$$\sup_{z \in Z} \{ -\mathbf{F}^\star(\Lambda^\star z) - \mathbf{G}^\star(-z) \} = \sup_{z \in Z} \left\{ \inf_{x \in X} \mathbf{L}(x,z) \right\}.$$

It turns out that $\mathbf{L}$ has a *saddle point* at the solutions of the primal and dual problems, that is,

$$\mathbf{L}(x_{\min}, z) \le \mathbf{L}(x_{\min}, z_{\max}) \le \mathbf{L}(x, z_{\max}) \quad \text{for all } x \in X \text{ and all } z \in Z,$$

where $x_{\min}$ is a solution to (2.87) and $z_{\max}$ is a solution to (2.88). This Lagrangian reads as

$$\mathbf{L}(x,z) = -\langle f, Px \rangle + \langle Gx, z \rangle_Z - \frac{1}{2} b(z,z)$$

in the case of $\mathcal{C}$ and

$$\mathbf{L}(x,z) = \begin{cases} 0 & Px = g \\ +\infty & \text{otherwise} \end{cases} + \langle Gx, z \rangle_Z - \frac{1}{2} b(z,z)$$

in the case of $\mathcal{V}$.

Connection with the theory of complementary variational principles

Here it is shown that the abstract problems fit into the setting of the theory of *complementary variational principles*. Our references in this subject are [10, 86, 8]. Let us start by proposing the following systems of equations: given $f \in Y_\diamond^\star$ and $g \in Y$, let

$$\begin{cases} G\bar{x} = B\bar{z} & \text{in } Z \\ G^\star \bar{z} = \tau^{-1}(f \circ P) & \text{in } X \end{cases} \tag{2.92}$$

and

$$\begin{cases} (G\tilde{x}, P\tilde{x}) = (B\tilde{z}, g) & \text{in } Z \times Y \\ G^\star \tilde{z} + P^\star \tilde{y} = 0_X & \text{in } X \end{cases}, \tag{2.93}$$

where $(\bar{x}, \bar{z})$ and $(\tilde{x}, \tilde{z}, \tilde{y})$ are the unknowns, respectively. Here τ is the Riesz-Fréchet isomorphism associated to X and $P^\star$ is the adjoint of P. It is easy to verify that $(\bar{x}, \bar{z})$ solves (2.92) if and only if $\bar{x}$ solves $\mathcal{C}$ and $\bar{z}$ solves $\mathcal{C}'$, and that $(\tilde{x}, \tilde{z}, \tilde{y})$ solves (2.93) if and only if $\tilde{x}$ solves $\mathcal{V}$ and $\tilde{z}$ solves $\mathcal{V}'$.

The above systems can be written in a canonical form (also called *Euler-Hamilton* equations), namely

$$\begin{cases} \mathbf{T}\bar{x} = \dfrac{\partial \mathbf{W}}{\partial z}(\bar{x}, \bar{z}) \\ \mathbf{T}^\star \bar{z} = \dfrac{\partial \mathbf{W}}{\partial x}(\bar{x}, \bar{z}) \end{cases} \quad \text{with} \quad \begin{array}{l} \mathbf{T} : X \to Z \quad \mathbf{T}x := Gx \\ \text{and} \\ \mathbf{W} : X \times Z \to \mathbb{R} \\ \mathbf{W}(x, z) := \frac{1}{2} b(z, z) + \langle f, Px \rangle \end{array} \tag{2.94}$$

and

$$\begin{cases} \mathbf{T}\tilde{x} = \dfrac{\partial \mathbf{W}}{\partial (z, y)}(\tilde{x}, (\tilde{z}, \tilde{y})) \\ \mathbf{T}^\star (\tilde{z}, \tilde{y}) = \dfrac{\partial \mathbf{W}}{\partial x}(\tilde{x}, (\tilde{z}, \tilde{y})) \end{cases} \quad \text{with} \quad \begin{array}{l} \mathbf{T} : X \to Z \times Y \quad \mathbf{T}x := (Gx, Px) \\ \text{and} \\ \mathbf{W} : X \times (Z \times Y) \to \mathbb{R} \\ \mathbf{W}(x, (z, y)) := \frac{1}{2} b(z, z) + \langle g, y \rangle_Y \end{array} . \tag{2.95}$$

In system (2.95), $Z \times Y$ is equipped with the inner product induced by the direct sum operation. $\mathbf{T}$ is a linear operator, $\mathbf{T}^\star$ is the adjoint of $\mathbf{T}$, and $\mathbf{W}$ is a real-valued function called *Hamiltonian* functional. Here, the partial derivatives of $\mathbf{W}$ are Fréchet derivatives. In the general case, the canonical system admits additional equations that represent "boundary conditions". Here, our formalism allows us to avoid additional equations.

The basic problem in the theory of complementary variational principles is to find an *action* functional whose stationary behaviour will coincide with the solution of the canonical system under consideration. The action functional $\mathbf{I} : X \times Z \to \mathbb{R}$ associated to system (2.94) is given by

$$\begin{aligned} \mathbf{I}(x, z) &= \langle \mathbf{T}x, z \rangle_Z - \mathbf{W}(x, z) \\ &= \langle Gx, z \rangle_Z - \langle f, Px \rangle - \frac{1}{2} b(z, z) \end{aligned}$$

and satisfies $\frac{\partial \mathbf{I}}{\partial x} = \mathbf{T}^\star - \frac{\partial \mathbf{W}}{\partial x}$ and $\frac{\partial \mathbf{I}}{\partial z} = \mathbf{T} - \frac{\partial \mathbf{W}}{\partial z}$. Thus, $\mathbf{I}$ is stationary at the solution $(\bar{x}, \bar{z})$. The action functional $\mathbf{I} : X \times (Z \times Y) \to \mathbb{R}$ associated to system (2.95) is given by

$$\begin{aligned} \mathbf{I}(x, (z, y)) &= \langle \mathbf{T}x, (z, y) \rangle_{Z \times Y} - \mathbf{W}(x, (z, y)) \\ &= \langle Gx, z \rangle_Z + \langle Px - g, y \rangle_Y - \frac{1}{2} b(z, z) \end{aligned}$$

and satisfies $\frac{\partial \mathbf{I}}{\partial x} = \mathbf{T}^\star - \frac{\partial \mathbf{W}}{\partial x}$ and $\frac{\partial \mathbf{I}}{\partial (z,y)} = \mathbf{T} - \frac{\partial \mathbf{W}}{\partial (z,y)}$. Thus, $\mathbf{I}$ is stationary at the solution $(\tilde{x}, (\tilde{z}, \tilde{y}))$. See that in both cases $\mathbf{W}$ is concave (linear) in the first variable and convex in the second variable, that is, $\mathbf{W}$ is a *saddle* function. This property of $\mathbf{W}$ leads to upper and lower bounds for the functional $\mathbf{I}$ in the form of complementary variational principles.

Complementary principles for system (2.94). Let $\mathbf{J}$ and $\mathbf{G}$ be functionals defined by

$$\begin{aligned}\mathbf{J}(x) &\coloneqq \mathbf{I}(x,z) \quad \text{such that } \partial_z \mathbf{I}(x,z) = 0_Z,\\ \mathbf{G}(z) &\coloneqq \mathbf{I}(x,z) \quad \text{such that } \partial_x \mathbf{I}(x,z) = 0_X.\end{aligned}$$

It is easy to check that $\partial_z \mathbf{I}(x,z) = 0_Z$ iff $z = Tx$ and that $\partial_x \mathbf{I}(x,z) = 0_X$ iff $Rz = f$. Substituting $z = Tx$ in the definition of $\mathbf{J}$ it follows that

$$\mathbf{J}(x) = \langle Gx, Tx\rangle_Z - \langle f, Px\rangle - \frac{1}{2} b(Tx,Tx) = \frac{1}{2} a(x,x) - \langle f, Px\rangle .$$

Using the constraint $Rz = f$ the functional $\mathbf{G}$ can be written as

$$\mathbf{G}(z) = \langle G^\star z, x\rangle_X - \langle Rz, Px\rangle - \frac{1}{2} b(z,z) = -\frac{1}{2} b(z,z) .$$

The saddle property of $\mathbf{W}$ implies that the minimum principle

$$\mathbf{J}(\bar{x}) \leq \mathbf{J}(x) \quad \text{for all } x \in X \tag{2.96}$$

and the maximum principle

$$\mathbf{G}(z) \leq \mathbf{G}(\bar{z}) \quad \text{for all } z \in Z \text{ such that } Rz = f \tag{2.97}$$

hold [9, Th. 2.6.1]. Combining (2.96) and (2.97), it follows that these are complementary extremum principles, namely

$$\mathbf{G}(z) \leq \mathbf{G}(\bar{z}) = \mathbf{I}(\bar{x},\bar{z}) = \mathbf{J}(\bar{x}) \leq \mathbf{J}(x) \tag{2.98}$$

for all $x \in X$ and all $z \in Z$ such that $Rz = f$. From this, upper and lower bounds for $\mathbf{I}(\bar{x},\bar{z})$ can be obtained. One can check that $\mathbf{I}(\bar{x},\bar{z}) = -\phi(f)$, where ϕ is the power function defined in Section 2.4, and that (2.98) is equivalent to the dual property (2.21) given in Theorem (2.15)(i). Therefore, we interpreted that

$\rightsquigarrow$ $\mathcal{C}$ and $\mathcal{C}'$ are *complementary* problems.

Complementary principles for system (2.95). Let $\mathbf{J}$ and $\mathbf{G}$ be functionals defined by

$$\begin{aligned}\mathbf{J}(x) &\coloneqq \mathbf{I}(x,(z,y)) \quad \text{such that } \partial_{(z,y)} \mathbf{I}(x,(z,y)) = (0_Z, 0_Y),\\ \mathbf{G}(z,y) &\coloneqq \mathbf{I}(x,(z,y)) \quad \text{such that } \partial_x \mathbf{I}(x,(z,y)) = 0_X.\end{aligned}$$

It is easy to check that $\partial_{(z,y)} \mathbf{I}(x,(z,y)) = (0_Z, 0_Y)$ iff $z = Tx$ and $Px = g$, and that $\partial_x \mathbf{I}(x,(z,y)) = 0_X$ iff $G^\star z = -P^\star y$. Substituting $z = Tx$ and imposing the constraint $Px = g$ in the definition of $\mathbf{J}$ it follows that

$$\mathbf{J}(x) = \langle Gx, Tx\rangle_Z - \langle Px - g, y\rangle_Y - \frac{1}{2} b(Tx,Tx) = \frac{1}{2} a(x,x) .$$

Substituting $G^\star z = -P^\star y$ (which is equivalent to $-Rz = \tau(y)$) in the definition of $\mathbf{G}$ we have

$$\mathbf{G}(z,y) = \langle G^\star z + P^\star y, x\rangle_X - \langle g, y\rangle_Y - \frac{1}{2} b(z,z) = \langle Rz, g\rangle - \frac{1}{2} b(z,z) .$$

Hence $\mathbf{G}$ does not depend on y. The saddle property of $\mathbf{W}$ implies that the minimum principle

$$\mathbf{J}(\bar{x}) \leq \mathbf{J}(x) \quad \text{for all } x \in X \tag{2.99}$$

and the maximum principle

$$\mathbf{G}(z) \leq \mathbf{G}(\bar{z}) \quad \text{for all } (z, y) \in Z \times Y \text{ such that } G^{\star}z = -P^{\star}y \tag{2.100}$$

hold [9, Th. 2.6.1]. Combining (2.99) and (2.100), it follows that these are complementary extremum principles, namely

$$\mathbf{G}(z) \leq \mathbf{G}(\bar{z}) = \mathbf{I}(\bar{x}, (\bar{z}, \bar{y})) = \mathbf{J}(\bar{x}) \leq \mathbf{J}(x) \tag{2.101}$$

for all $x \in X$ and all $(z, y) \in (Z \times Y)$ such that $G^{\star}z = -P^{\star}y$, which is equivalent to impose that $z \in Z_{\perp}$. From this, upper and lower bounds for $\mathbf{I}(\bar{x}, (\bar{z}, \bar{y}))$ can be obtained. One can check that $\mathbf{I}(\bar{x}, (\bar{z}, \bar{y})) = \psi(g)$, where ψ is the power function defined in Section 2.4, and that (2.101) is equivalent to the dual property (2.22) given in Theorem (2.15)(*ii*). Therefore, we interpreted that

$\rightsquigarrow$ $\mathcal{V}$ and $\mathcal{V}'$ are *complementary* problems.

Lemmas

Lemma 2.34. *Given any $y \in Y$, there exists a unique solution $\hat{x} \in X$ of the minimization problem*

$$\min_{\substack{x \in X \\ Px = y}} \|x\|_X . \tag{2.102}$$

Furthermore, the map $\|\cdot\|_Y : Y \to \mathbb{R}$ defined by $\|y\|_Y := \|\hat{x}\|_X$ is a norm on Y, which is induced by an inner product, and Y equipped with this inner product becomes a Hilbert space.

Proof. Let $y \in Y$. We claim that $P^{-1}\{y\}$ is closed, convex, and nonempty. Since P is linear and surjective, $P^{-1}\{y\}$ is convex and nonempty. Let (x_i) be a convergent sequence in $P^{-1}\{y\}$ and $x \in X$ be its limit. Let $x_y \in P^{-1}\{y\}$. Then $P(x_i - x_y) = 0$ for all i and $(x_i - x_y)$ converges to $x - x_y$. Since $\ker P$ is closed, $x - x_y \in \ker P$ and $x \in P^{-1}\{y\}$ ($Px = Px_y = y$). Hence $P^{-1}\{y\}$ is closed. Therefore, since X is a Hilbert space, there exists a unique element $\hat{x} \in P^{-1}\{y\}$ such that

$$\operatorname{dist}\left(0_X, P^{-1}\{y\}\right) = \min_{\substack{x \in X \\ Px = y}} \|x\|_X = \|\hat{x}\|_X .$$

$\|\cdot\|_Y : Y \to \mathbb{R}$ defined by $y \mapsto \|\hat{x}\|_X$ is a norm on Y. Indeed, we have that:

- $\|y\|_Y = 0 \Leftrightarrow y = 0$. If $\|y\|_Y = \|\hat{x}\|_X = 0$ then $\hat{x} = 0$ and consequently $y = P\hat{x} = 0$. If $y = 0$ then $\|\hat{x}\|_X \leq \|x\|_X$ for all x such that $Px = 0$. Set $x = 0$, so that $\hat{x} = 0$. Hence $\|y\|_Y = \|\hat{x}\|_X = 0$.

- $\|\lambda y\|_Y = |\lambda| \, \|y\|_Y$. Let $\hat{x}'$ be the minimizer corresponding to λy. Then $P\hat{x} = y$ and $P\hat{x}' = \lambda y$. From the optimality of $\hat{x}$ and $\hat{x}'$, it follows that $\|\hat{x}\|_X \leq \|\hat{x}'/\lambda\|_X$ and $\|\hat{x}'\|_X \leq \|\lambda \hat{x}\|_X$. Hence $\|\hat{x}'\|_X = \|\lambda \hat{x}\|_X$ and the conclusion follows.
- $\|y + y_0\|_Y \leq \|y\|_Y + \|y_0\|_Y$. Let $\hat{x}'$ and $\hat{x}''$ be the minimizers corresponding to y_0 and $y + y_0$, respectively. Since $P(\hat{x} + \hat{x}') = y + y_0$, the optimality of $\hat{x}''$ implies that $\|\hat{x}''\|_X \leq \|\hat{x} + \hat{x}'\|_X \leq \|\hat{x}\|_X + \|\hat{x}'\|_X$ and the conclusion follows.

We claim that Y is a Hilbert space. First note that since $\ker P$ is closed we have $X = \ker P \oplus (\ker P)^{\perp}$. Thus, the minimizer $\hat{x} = x^0 + x^{\perp}$, with $x^0 \in \ker P$ and $x^{\perp} \in (\ker P)^{\perp}$, and then $P\hat{x} = Px^{\perp}$. Moreover, from the optimality of $\hat{x}$ ($\|\hat{x}\|_X \leq \left\|x^{\perp}\right\|_X$) and the Pithagorean relation $\|\hat{x}\|_X^2 = \left\|x^0\right\|_X^2 + \left\|x^{\perp}\right\|_X^2$ we deduce that $x^0 = 0$. Therefore $\hat{x} = x^{\perp}$. Hence, the operator $P|_{(\ker P)^{\perp}} : (\ker P)^{\perp} \to Y$ defined by $\hat{x} \mapsto P\hat{x} = y$ is a linear bijective isometry, that is, $P|_{(\ker P)^{\perp}}$ is bijective and $\|\hat{x}\|_X = \|P\hat{x}\|_Y$ (the bijectivity follows from the uniqueness of $\hat{x}$ and the surjectivity of P). Thus $(\ker P)^{\perp}$ and Y are isometric. As $(\ker P)^{\perp}$ is closed, Y is complete for the norm $\|\cdot\|_Y$. Moreover, using the fact that $\|\cdot\|_X$ satisfies the parallelogram law, we can prove that $\|\cdot\|_Y$ satisfies the parallelogram law. Hence $\|\cdot\|_Y$ is induced by an inner product $\langle\cdot,\cdot\rangle_Y$. More precisely, given $y_1, y_2 \in Y$, $\langle y_1, y_2\rangle_Y = \langle \hat{x}_1, \hat{x}_2\rangle_X$, where $\hat{x}_1 = P^{-1}\big|_{(\ker P)^{\perp}} y_1$ and $\hat{x}_2 = P^{-1}\big|_{(\ker P)^{\perp}} y_1$. Therefore Y equipped with $\langle\cdot,\cdot\rangle_Y$ is a Hilbert space. □

Lemma 2.35. *Let $X' \subset X$ be a closed subspace such that $1_X \notin X'$. Then the bilinear form a is continuous and coercive on $X' \times X'$.*

Proof. It suffices to prove the coercivity of a. Since $1_X \in X \backslash X'$, the Hahn-Banach theorem says that there exists a linear continuous functional $\Gamma : X \to \mathbb{R}$ such that $\Gamma(1_X) > 0$ and $\Gamma(x) = 0$ for all $x \in X'$. We define the operator $S : X \to \ker \Gamma$ as $Sx \coloneqq x - (\Gamma(x)/\Gamma(1_X))\, 1_X$, where $\ker\Gamma$, which is closed in X, is equipped with the norm of X. This operator is linear. The continuity of S follows from that of Γ. Moreover, since $Sx = x$ for all $x \in \ker\Gamma$, S is surjective. Then, by a classical result of functional analysis, the new operator $\tilde{S} : X/\ker S \to \ker\Gamma$ defined by $\tilde{S}[x] = Sx$, where $[x] = x + \ker S$, is also linear and continuous. But

$$\ker S = \left\{ x \in X \,\middle|\, x = \frac{\Gamma(x)}{\Gamma(1_X)} 1_X \right\} = \operatorname{span}\{1_X\}.$$

Thus $X/\ker S = X/\mathbb{R}$ and the continuity of $\tilde{S}$ yields

$$\left\| x - \frac{\Gamma(x)}{\Gamma(1_X)} 1_X \right\|_X = \left\| \tilde{S}[x] \right\|_X \leq C \, \|[x]\|_{X/\mathbb{R}} \quad \text{for all } x \in X.$$

In particular, if $x \in X' \subset \ker\Gamma$, then $\|x\|_X \leq C\,\|[x]\|_{X/\mathbb{R}}$. The desired conclusion follows by combining this with the coercivity a on $X/\mathbb{R} \times X/\mathbb{R}$. □

3 Regularization of an all-at-once formulation of the EIT inverse problem

The aim of Chapter 3 is to contribute to the study of the all-at-once formulation of the EIT inverse problem and its regularization. To that end, we construct an abstract setting in the context of Banach spaces, which admits most of the EIT models and several types of observations. This setting is based on a set of assumptions that generalizes the mathematical objects appearing in the model and observation equations of the EIT inverse problem. There, the conductivity is considered as a L^{∞}-function and the electric potential is modeled as an element of a Banach space. Using the assumptions we formulate an abstract inverse problem where the conductivity and the electric potential are the unknowns. This problem consists of a model equation defined in a dual space, an observation equation that depends on both unknowns, and a constraint to have positive conductivities. Next, three regularized problems are formulated, which take inspiration from the Tikhonov, Ivanov, and Morozov regularizations [59, 96, 77]. The regularization funtional that was used applies total variation regularization [12, 28]. In this regularization stage, it is assumed the existence of a compact, reflexive subspace of the space of electric potentials. In fact, the regularized problems and regularization functional are defined on this subspace. In order to prove the well-posedness of the regularized problems, they are rewritten in a unified way as an unconstrained minimization problem defined on a topological space and a few properties regarding lower semicontinuity and compactness are provided. Then, the existence, stability, and convergence of regularized solutions are proved. In addition, a basic learning problem is formulated to select a regularization parameter based on their performance on a set of data that consists of a number of exact observations and of the corresponding solutions of the abstract inverse problem. The existence of a solution to this problem is proved by using the results of existence and stability of regularized solutions. Thus, our abstract inverse problem and its regularizations are analized in the context of Banach spaces [89, 91], while the sequences of conductivities are analized under the weak* convergence of L^{∞} [28, 54]. The continuum model, the gap model, the shunt model, and the complete electrode model are admitted . The continuum model with voltage point measurements [5] and the smoothened complete electrode model fit also into our assumptions. With respect to the point models, we do not assert anything. Moreover, the assumptions over the observation operator allow observations of voltage measurements, current measurements, magnitudes of current density field, and interior power densities.

Numerical solutions with the complete electrode model are performed. All the types of observations mentioned earlier are taken into account in the tests. Three instances of the assumptions are provided. The first and second ones yield as model equations the weak formulations of the complete electrode model with applied current and applied voltage, respectively. The third one yields as model equation the weak form of the equations of the complete electrode model without the equations involving current and voltage application. Thus, in this case, the model equation does not model the EIT experiment.

3.1 All-at-once formulation

To formulate the abstract problem that represents the all-at-once formulation of the EIT inverse problem, four assumptions are established, followed by some notations and definitions.

We start with the assumptions.

Let Ω be an open, connected, bounded, and Lipschitz domain in $\mathbb{R}^d$ $(d = 2, 3)$ with boundary $\partial\Omega$. Let $L^1(\Omega)$ and $L^\infty(\Omega)$ be the classical Lebesgue spaces over Ω. It is well-known that $L^\infty(\Omega)$ can be identified with the dual of $L^1(\Omega)$ [21, Thm. 4.14] and that the convergence in the weak* topology defined on $L^\infty(\Omega) \equiv \left(L^1(\Omega)\right)^\star$ can be expressed as: a sequence (σ_i) in $L^\infty(\Omega)$ converges to a some $\sigma \in L^\infty(\Omega)$ in the weak* topology defined on $L^\infty(\Omega)$, or $\sigma_i \overset{\star}{\rightharpoonup} \sigma$, if and only if

$$\int_\Omega \sigma_i \phi \,\mathrm{d}\mathbf{x} \to \int_\Omega \sigma \phi \,\mathrm{d}\mathbf{x} \quad \text{for all } \phi \in L^1(\Omega).$$

Considering $L^\infty(\Omega)$ and the weak* convergence defined on it, fix an integer N and assume the following:

A1. Let X and Z be two Banach spaces with norms denoted by $\|\cdot\|_X$ and $\|\cdot\|_Z$ respectively. The dual of Z is denoted by $Z^\star$ and is equipped with the usual dual norm $z^\star \mapsto \|z^\star\|_{Z^\star} = \sup_{z\in Z}\{z^\star(z) \mid \|z\|_Z \leq 1\}$. Let $b^1,\ldots,b^N \in Z^\star$ be fixed.

A2. Let $a : L^\infty(\Omega) \times X \to Z^\star$ be a map satisfying

$$\sigma_i \overset{\star}{\rightharpoonup} \sigma \text{ in } L^\infty(\Omega),\ x_i \to x \text{ in } X \Rightarrow a(\sigma_i, x_i) \overset{\star}{\rightharpoonup} a(\sigma, x) \text{ in } Z^\star. \tag{3.1}$$

A3. Let Y be a Banach space with a norm denoted by $\|\cdot\|_Y$ and let $\bar{y}^1, \ldots, \bar{y}^N \in Y$ be fixed. The dual of Y is denoted by $Y^\star$ and is equipped with the usual dual norm $y^\star \mapsto \|y^\star\|_{Y^\star} = \sup_{y\in Y}\{y^\star(y) \mid \|y\|_Y \leq 1\}$.

A4. Let $c^1, \ldots, c^N : L^\infty(\Omega) \times X \to Y$ be maps satisfying

$$\sigma_i \overset{\star}{\rightharpoonup} \sigma \text{ in } L^\infty(\Omega),\ x_i \to x \text{ in } X \Rightarrow c^n(\sigma_i, x_i) \rightharpoonup c^n(\sigma, x) \text{ in } Y \tag{3.2}$$

for $n = 1, \ldots, N$.

Remark 3.1. The weak* convergence in (3.1) means that the sequence $(a(\sigma_i, x_i)(z))$ converges to $a(\sigma, x)(z)$ for all $z \in Z$. The weak convergence in (3.2) means that the sequence $(y^\star(c^n(\sigma_i, x_i)))$ converges to $y^\star(c^n(\sigma, x))$ for all $y^\star \in Y^\star$.

In the context of Electrical Impedance Tomography, the mathematical objects considered in Assumptions A1-A4 have the following interpretation:

- The domain Ω represents a body with an internal conductivity, which is usually modeled as a function from Ω into $]0,\infty[$. Here, a conductivity is a function σ in $L^\infty(\Omega)$ not necessarily positive. This is because the weak formulation of the EIT models is well-defined for this type of conductivities. The positivity of σ will be imposed separately from the model equations.
- X represents the spaces of electric potentials. Z represents the space of test functions used in the weak formulation of the EIT models.
- With N it is indicated how many experiments are performed.
- a and b^n generalize the components that arise when an EIT model is written in its weak form. More precisely, given a conductivty $\sigma \in L^\infty(\Omega)$,

  $$a(\sigma, x)(z) = b^n(z) \quad \text{for all } z \in Z$$

  is an abstraction of the weak formulation of the EIT models, where x represents the electric potential and b^n contains the information from the applied current (or voltage). Observe that the property (3.1) of a is a sort of continuity in both variables, which mimics the fact that

  $$\sigma_i \overset{\star}{\rightharpoonup} \sigma \text{ in } L^\infty(\Omega) \quad \text{and} \quad u_i \to u \text{ in } W^{1,p}(\Omega)$$

  imply

  $$\int_\Omega \sigma_i \nabla u_i \cdot \nabla w \,\mathrm{d}\mathbf{x} \to \int_\Omega \sigma \nabla u \cdot \nabla w \,\mathrm{d}\mathbf{x} \quad \text{for all } w \in W^{1,q}(\Omega),$$

  where $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ are the classical Sobolev spaces and $p^{-1} + q^{-1} = 1$. This convergence is important for the all-at-once aproach because σ and u are considered as variables in $\int_\Omega \sigma \nabla u \cdot \nabla w \,\mathrm{d}\mathbf{x}$, which is the main term in the weak formulation of the EIT models.
- Y meaning the space of observable data from experiments and $\bar{y}^n$ represents the exact data corresponding to the n-th experiment. Here, this data can be either voltage measurements, current measurements [98], interior power densities [6], or magnitudes of current density field [70]. Actually, these are all the possible types of data that can be observed.
- c^n models the observation in the n-th experiment. It is a map from $L^\infty(\Omega) \times X$ into the space of observations Y and is in general non-linear. The property (3.2) of c^n is also a sort of continuity in both variables. It is remarkable that the observation maps associated with all possible types of observable data satisfy this requirement.

Based on the above assumptions, some notations and definitions are given below. We will work on the product spaces X^N, Y^N, and $(Z^\star)^N$. The elements in these spaces are denoted by $x = (x^1, \ldots, x^N)$, $y = (y^1, \ldots, y^N)$, and $z^\star = (z^{\star,1}, \ldots, z^{\star,N})$, respectively. Given a sequence (x_i) in X^N and some $x \in X^N$, (x_i) converges to x if and only if (x_i) converges component-wise to x. The same for sequences in Y^N and $(Z^\star)^N$. Given $z^\star \in (Z^\star)^N$ and $y \in Y^N$, the p-norm of $z^\star$ and the q-norm of y are defined by

$$\|z^\star\|_p := \begin{cases} \left(\|z^{\star,1}\|_{Z^\star}^p + \ldots + \|z^{\star,N}\|_{Z^\star}^p\right)^{1/p} & \text{if } 1 \le p < \infty \\ \max\left\{\|z^{\star,1}\|_{Z^\star}, \ldots, \|z^{\star,N}\|_{Z^\star}\right\} & \text{if } p = \infty \end{cases}$$

	Meaning
σ	conductivity
x	electric potential
$\bar{y}$	exact observation
$A(\sigma, x) = 0_{(Z^\star)^N}$	model equation
$C(\sigma, x) = \bar{y}$	observation equation

Table 3.1: Elements in the abstract inverse problem.

$$\text{and} \quad \|y\|_q := \begin{cases} \left(\|y^1\|_Y^q + \ldots + \|y^N\|_Y^q\right)^{1/q} & \text{if } 1 \le q < \infty \\ \max\left\{\|y^1\|_Y, \ldots, \|y^N\|_Y\right\} & \text{if } q = \infty \end{cases},$$

respectively (recall that all these norms are equivalents on their respective spaces). The map $A : L^\infty(\Omega) \times X^N \to (Z^\star)^N$ defined by

$$A(\sigma, x) := \left(a\left(\sigma, x^1\right) - b^1, \ldots, a\left(\sigma, x^N\right) - b^N\right)$$

is called the *model map*, the map $C : L^\infty(\Omega) \times X^N \to Y^N$ defined by

$$C(\sigma, x) := \left(c^1\left(\sigma, x^1\right), \ldots, c^N\left(\sigma, x^N\right)\right)$$

is called the *observation map*, and $\bar{y} := \left(\bar{y}^1, \ldots, \bar{y}^N\right) \in Y^N$ is called the *exact observation.* We are now ready to consider the following all-at-once formulation of the EIT inverse problem.

Abstract inverse problem. Find $(\sigma, x) \in L^\infty(\Omega) \times X^N$ satisfying[1]

$$\operatorname*{ess\,inf}_{\mathbf{x}\in\Omega} \sigma(\mathbf{x}) > 0 \qquad \text{and} \qquad \begin{aligned} A(\sigma, x) &= 0_{(Z^\star)^N} && \text{in } (Z^\star)^N \\ C(\sigma, x) &= \bar{y} && \text{in } Y^N \end{aligned}. \tag{$\boldsymbol{I}$}$$

The abstract inverse problem $\boldsymbol{I}$ represents the all-at-once formulation of the EIT inverse problem. $A(\sigma, x) = 0$ is called the *model equation* and represents the weak formulation of the EIT model under consideration. $C(\sigma, x) = \bar{y}$ is called the *observation equation* and represents the correspondence between the predicted and observed data.

We conclude this section with a direct consequence of the assumptions.

Lemma 3.2. *Let (σ_i) and (x_i) be sequences in $L^\infty(\Omega)$ and X^N, respectively. Let $\sigma \in L^\infty(\Omega)$ and $x \in X^N$. Suppose that $\sigma_i \overset{\star}{\rightharpoonup} \sigma$ in $L^\infty(\Omega)$ and $x_i \to x$ in X^N.*

(i) Then

$$\|A(\sigma, x)\|_p \le \liminf_{i\to\infty} \|A(\sigma_i, x_i)\|_p \quad 1 \le p \le \infty.$$

Moreover, if $\|A(\sigma_i, x_i)\|_p \le \varepsilon_i$ for all i, where (ε_i) is a convergent sequence in $\mathbb{R}$, then

[1] Recall that $\operatorname{ess\,inf}_{\mathbf{x}\in\Omega}\sigma(\mathbf{x}) = \sup\{c > 0 \mid \mu(\{\mathbf{x} \mid \sigma(\mathbf{x}) < c\}) = 0\}$ is the essential infimum of σ, where μ is the Lebesgue measure on $\mathbb{R}^d$.

1. $\|A(\sigma, x)\|_p \leq \lim_{i\to\infty} \varepsilon_i$,
2. and if, in addition, $\lim_{i\to\infty} \varepsilon_i = 0$, then (σ, x) solves the model equation and $A(\sigma_i, x_i) \to 0_{(Z^\star)^N}$ in $(Z^\star)^N$.

(ii) Let (y_i) be a sequence in Y and let $y \in Y$ be such that $y_i \rightharpoonup y$ in Y^N. Then

$$\|C(\sigma, x) - y\|_q \leq \liminf_{i\to\infty} \|C(\sigma_i, x_i) - y_i\|_q \quad 1 \leq q \leq \infty.$$

Moreover, if $\|C(\sigma_i, x_i) - y_i\|_q \leq \varepsilon_i$ *for all* i, *where* (ε_i) *is a convergent sequence in* $\mathbb{R}$, *then*

1. $\|C(\sigma, x) - y\|_q \leq \lim_{i\to\infty} \varepsilon_i$,
2. and if, in addition, $\lim_{i\to\infty} \varepsilon_i = 0$ and $y = \bar{y}$, then (σ, x) solves the observation equation and $C(\sigma_i, x_i) - y_j \to 0_{Y^N}$ in Y^N.

Remark 3.3 (Lower and upper limits [87, Sec. 1.E]). Given a sequence of numbers (u_i) in $\overline{\mathbb{R}} = [-\infty, \infty]$, the lower and upper limits of (u_i) are defined by $\liminf_{i\to\infty} u_i = \lim_{i\to\infty} \{\inf_{j\geq i} u_i\}$ and $\limsup_{i\to\infty} u_i = \lim_{i\to\infty} \{\sup_{j\geq i} u_i\}$, respectively.

Proof. The proofs are based on well-known results in weak topologies. See for instance [21, Prop. 3.5, 3.13]. By Assumption A2, for each n, $a(\sigma_i, x_i^n) - b^n$ converges to $a(\sigma, x^n) - b^n$ in the weak* topology defined on $Z^\star$; then the sequence $\left(\|a(\sigma_i, x_i^n) - b^n\|_{Z^\star}\right)$ is bounded and

$$\|a(\sigma, x^n) - b^n\|_{Z^\star} \leq \liminf_{i\to\infty} \|a(\sigma_i, x_i^n) - b^n\|_{Z^\star}.$$

Thus, applying lower limit properties, *(ii)*(*1.*) follows. If there exists a sequence (ε_i) in $\mathbb{R}$ such that $\|A(\sigma_i, x_i)\|_p \leq \varepsilon_i$ for all i, then

$$\|A(\sigma, x)\|_p \leq \liminf_{i\to\infty} \|A(\sigma_i, x_i)\|_p \leq \limsup_{i\to\infty} \|A(\sigma_i, x_i)\|_p \leq \limsup_{i\to\infty} \varepsilon_i.$$

From this, *(ii)*(*2.*) follows immediately. Using similar arguments, *(ii)* follows from Assumption A4. ☐

3.2 Regularized problems

Here, a "compactness" assumption and a regularization functional allow us to build three regularizations of ***I***, which will be appropriately formulated in a topological vector space.

In addition to Assumptions A1-A4, from now on assume the following.

A5. Let $\tilde{X}$ be a linear subspace of X. Assume that $\tilde{X}$ has its own norm $\|\cdot\|_{\tilde{X}}$ and that it is a reflexive Banach space with $\|\cdot\|_{\tilde{X}}$. In addition, assume that the inclusion operator $\left(\tilde{X}, \|\cdot\|_{\tilde{X}}\right) \to (X, \|\cdot\|_X)$ is compact. The dual of $\tilde{X}$ is denoted by $\tilde{X}^\star$ and is equipped with the usual dual norm $\|\cdot\|_{\tilde{X}^\star} : x^\star \mapsto \|x^\star\|_{\tilde{X}^\star} = \sup_{x\in\tilde{X}} \left\{|x^\star(x)| \mid \|x\|_{\tilde{X}} \leq 1\right\}$.

Remark 3.4. $\left(\tilde{X}, \|\cdot\|_{\tilde{X}}\right) \to (X, \|\cdot\|_X)$ is compact means that every bounded sequence in $\tilde{X}$ has a convergent subsequence in X. Recall that compact linear operators maps weakly convergent sequences into strongly convergent sequences [73, Thm. 8.1-7].

We use the reflexive Banach space $\tilde{X}$ to define the following functional.

Definition 3.5. Let $R : L^\infty(\Omega) \times \tilde{X}^N \to [0,\infty]^4$ be the functional defined by

$$R(\sigma, x) := \begin{pmatrix} \frac{1}{r}\left(\|x^1\|^r_{\tilde{X}} + \ldots + \|x^N\|^r_{\tilde{X}}\right) \\ \int_\Omega |D(\sigma - \sigma')| \\ \frac{1}{s_1}\sum_{t=1}^{T}\left|\int_\Omega (\sigma - \sigma'')\,\phi_t\,\mathrm{d}\mathbf{x}\right|^{s_1} \\ \frac{1}{s_2}\sum_{t=1}^{T}\left|\int_\Omega (\sigma - \sigma_-)\,\psi_t\,\mathrm{d}\mathbf{x}\right|^{s_2}\left|\int_\Omega (\sigma_+ - \sigma)\,\psi_t\,\mathrm{d}\mathbf{x}\right|^{s_2} \end{pmatrix}, \tag{3.3}$$

where r, s_1, and s_2 are fixed exponents such that $1 \le r, s_1, s_2 < \infty$, $\phi_1, \ldots, \phi_T$ and $\psi_1, \ldots, \psi_T$ are weight functions in $L^1(\Omega)$, $\sigma' \in BV(\Omega)$ and $\sigma'' \in L^\infty(\Omega)$ are reference functions, and σ_-, σ_+ are two positive constants such that $\sigma_- < \sigma_+$. R is called the *regularization functional.*

Very often R will be multiplied by a vector of non-negative numbers α. This multiplication is denoted by $\alpha \cdot R(\sigma, x)$. The zero and all-ones vectors are denoted by $\mathbf{0}$ and $\mathbf{1}$ respectively.

Remark 3.6 (Space of functions of bounded variation). Given $\sigma \in L^1(\Omega)$, the total variation of σ is defined by

$$\int_\Omega |D\sigma| = \sup\left\{ \int_\Omega \sigma\,(\nabla\cdot f)\,\mathrm{d}\mathbf{x} \;\middle|\; \begin{array}{c} f = (f_1, \ldots, f_d) \in C_c^1\left(\Omega, \mathbb{R}^d\right), \\ |f| = \sup_{\mathbf{x}\in\Omega}\left(\sum_{i=1}^d f_i^2(\mathbf{x})\right)^{1/2} \le 1 \end{array} \right\},$$

where $\nabla\cdot f = \sum_{i=1}^d \partial_{x_i} f_i$ and $C_c^1\left(\Omega, \mathbb{R}^d\right)$ denotes the space of continuously differentiable vector fields with values in $\mathbb{R}^d$ and compact support in Ω [12, Def. 2.2.2]. A function $\sigma \in L^1(\Omega)$ is said to have bounded variation if $\int_\Omega |D\sigma| < \infty$. The space of functions of bounded variation is denoted by $BV(\Omega) = \left\{\sigma \in L^1(\Omega) \,\middle|\, \int_\Omega |D\sigma| < \infty\right\}$. This regularization functional is popular in image processing due to its ability to preserve edges and has been used for numerical reconstruction of piecewise constant conductivities. [100, 26, 76, 62].

Now, consider the following regularized problems:

1. Fix $1 \le p, q < \infty$. Let $\Lambda = \Lambda(\bar{y})$ be the parameter space defined by

$$\Lambda(\bar{y}) := \left\{(\alpha, \delta, y) \in [0,\infty[^4 \times [0,\infty[\times Y^N \;\middle|\; \|\bar{y} - y\|_q \le \delta\right\}.$$

	Meaning
α	regularization parameter
β	maximum model error level
δ	noise level
y	noisy observation
γ	safety factor for the error level in the observation equation

Table 3.2: Components of a parameter λ.

Given a parameter $\lambda = (\alpha, \delta, y) \in \Lambda$, Tikhonov regularization [96] of the model and observation equations combined with Ivanov regularization [59] of σ yield the box constrained minimization problem

$$\begin{array}{ll} \min\limits_{(\sigma,x)\in L^\infty(\Omega)\times\tilde{X}^N} & \frac{1}{p}\left\|A(\sigma,x)\right\|_p^p + \frac{1}{q}\left\|C(\sigma,x)-y\right\|_q^q + \alpha\cdot R(\sigma,x) \\ \text{s.t.} & \sigma_- \le \sigma \le \sigma_+ \text{ a.e. on } \Omega \end{array}. \tag{$\boldsymbol{I}_\lambda^1$}$$

2. Fix $1 \le p < \infty$ and set $q = \infty$. Fix $\gamma \in [1,\infty[$. Let $\Lambda = \Lambda(\bar{y})$ be the parameter space defined by

$$\Lambda(\bar{y}) := \left\{(\alpha,\delta,y) \in [0,\infty[^4 \times [0,\infty[\times Y^N \mid \left\|\bar{y}-y\right\|_\infty \le \delta\right\}.$$

Given a parameter $\lambda = (\alpha, \delta, y) \in \Lambda$, the minimization problem

$$\begin{array}{ll} \min\limits_{(\sigma,x)\in L^\infty(\Omega)\times\tilde{X}^N} & \frac{1}{p}\left\|A(\sigma,x)\right\|_p^p + \alpha\cdot R(\sigma,x) \\ \text{s.t.} & \sigma_- \le \sigma \le \sigma_+ \text{ a.e. on } \Omega \text{ and } \left\|C(\sigma,x)-y\right\|_\infty \le \gamma\delta \end{array}, \tag{$\boldsymbol{I}_\lambda^2$}$$

is a Tikhonov regularization of the model equation, where Morozov regularization [77] of the observation equation and Ivanov regularization of σ have been imposed.

3. Set $p = \infty$ and fix $1 \le q < \infty$. Let $\Lambda = \Lambda(\bar{y})$ be the parameter space defined by

$$\Lambda(\bar{y}) := \left\{(\alpha,\beta,\delta,y) \in [0,\infty[^4 \times [0,\infty[\times [0,\infty[\times Y^N \,\middle|\, \left\|\bar{y}-y\right\|_q \le \delta\right\}.$$

Given a parameter $\lambda = (\alpha, \beta, \delta, y) \in \Lambda$, the minimization problem

$$\begin{array}{ll} \min\limits_{(\sigma,x)\in L^\infty(\Omega)\times\tilde{X}^N} & \frac{1}{q}\left\|C(\sigma,x)-y\right\|_q^q + \alpha\cdot R(\sigma,x) \\ \text{s.t.} & \sigma_- \le \sigma \le \sigma_+ \text{ a.e. on } \Omega \text{ and } \left\|A(\sigma,x)\right\|_\infty \le \beta, \end{array} \tag{$\boldsymbol{I}_\lambda^3$}$$

is a Tikhonov regularization of the observation equation, where Morozov regularization of the model equation and Ivanov regularization of σ have been imposed.

Remark 3.7. Observe that the constraint $\sigma_- \le \sigma \le \sigma_+$ a.e. Ω can be written in the form $\left\|\sigma - \frac{\sigma_+ + \sigma_-}{2}\right\|_{L^\infty(\Omega)} \le \frac{\sigma_+ - \sigma_-}{2}$.

Before formulating the regularized problems $\boldsymbol{I}_\lambda^1$, $\boldsymbol{I}_\lambda^2$, and $\boldsymbol{I}_\lambda^3$ in an appropiate topological space, consider the following lemma, which related the weak* convergence in $L^\infty(\Omega)$ and the weak convergence of $\tilde{X}$ to the Ivanov regularization of σ and the regularization functional R.

Lemma 3.8. *Let (σ_i) and (x_i) be sequences in $L^\infty(\Omega)$ and $\tilde{X}^N$, respectively. Let $\sigma \in L^\infty(\Omega)$ and $x \in \tilde{X}^N$. Suppose that $\sigma_i \overset{\star}{\rightharpoonup} \sigma$ in $L^\infty(\Omega)$ and $x_i \rightharpoonup x$ in $\tilde{X}^N$.*

(i) If $\sigma_- \le \sigma_i \le \sigma_+$ a.e. on Ω for all i, then $\sigma_- \le \sigma \le \sigma_+$ a.e. on Ω.

(ii) Let (α_i) be a sequence in $]0,\infty[^4$ and $\alpha \in]0,\infty[^4$ be such that $\alpha_i \to \alpha$. Then

$$\alpha \cdot R(\sigma, x) \le \liminf_{i\to\infty} \alpha_i \cdot R(\sigma_i, x_i).$$

The same holds if $\alpha = \mathbf{0}$ and if all the components of $R(\sigma, x)$ are finite numbers.

(iii) Items (i) and (ii) of Lemma 3.2 hold.

Remark 3.9. Recall that the weak convergence $x_i \rightharpoonup x$ in $\tilde{X}$ is equivalent means that the sequence $(x^\star(x_i))$ converges to $x^\star(x)$ for all $x^\star \in \tilde{X}^\star$.

Proof of (i). By Remark 3.7, it suffices to use the weak* lower sequential semicontinuity of the norm $\|\cdot\|_{L^\infty(\Omega)}$ (see for instance [21, Prop. 3.13]). □

Proof of (ii). Each component of R is analized separately.

- *Norm function.* For each n, the weak convergence of (x_i^n) to x^n implies that the sequence $\left(\|x_i^n\|_{\tilde{X}}\right)$ is bounded and that $\|x^n\|_{\tilde{X}} \le \liminf_{i\to\infty} \|x_i^n\|_{\tilde{X}}$ (see for instance [21, Prop. 3.5]). Using lower limit properties we obtain

$$\frac{1}{r}\left(\|x^1\|_{\tilde{X}}^r + \ldots + \|x^N\|_{\tilde{X}}^r\right) \le \liminf_{i\to\infty} \frac{1}{r}\left(\|x_i^1\|_{\tilde{X}}^r + \ldots + \|x_i^N\|_{\tilde{X}}^r\right).$$

- *Total variation function.* Let $f \in C_c^1(\Omega, \mathbb{R}^d)$ such that $|f| \le 1$. Then $\nabla \cdot f$ belongs to $L^1(\Omega)$ and the weak* convergence of (σ_i) to σ implies

$$\int_\Omega (\sigma_i - \sigma')\,(\nabla \cdot f)\,\mathrm{d}\mathbf{x} \to \int_\Omega (\sigma - \sigma')\,(\nabla \cdot f)\,\mathrm{d}\mathbf{x}$$

(recall that $\sigma' \in BV(\Omega)$ was fixed in R). As $\int_\Omega (\sigma_i - \sigma')\,(\nabla \cdot f)\,\mathrm{d}\mathbf{x}$ is less than or equal to $\int_\Omega |D(\sigma_i - \sigma')|$ (which belongs to $\overline{\mathbb{R}}$ since $\sigma_i \in L^\infty(\Omega)$), by applying lower limit it follows that

$$\int_\Omega (\sigma - \sigma')\,(\nabla \cdot f)\,\mathrm{d}\mathbf{x} \le \liminf_{i\to\infty} \int_\Omega |D(\sigma_i - \sigma')|$$

and therefore

$$\int_\Omega |D(\sigma - \sigma')| \le \liminf_{i\to\infty} \int_\Omega |D(\sigma_i - \sigma')|.$$

- *Difference and boundary fitting functions.* It follows easily from the weak* convergence of (σ_i) that

$$\lim_{i\to\infty} \frac{1}{s_1} \sum_{t=1}^{T} \left| \int_\Omega \left(\sigma_i - \sigma''\right) \phi_t \, \mathrm{d}\mathbf{x} \right|^{s_1} = \frac{1}{s_1} \sum_{t=1}^{T} \left| \int_\Omega \left(\sigma - \sigma''\right) \phi_t \, \mathrm{d}\mathbf{x} \right|^{s_1}$$

and

$$\lim_{i\to\infty} \frac{1}{s_2} \sum_{t=1}^{T} \left| \int_\Omega \left(\sigma_i - \sigma_-\right) \psi_t \, \mathrm{d}\mathbf{x} \right|^{s_2} \left| \int_\Omega \left(\sigma_+ - \sigma_i\right) \psi_t \, \mathrm{d}\mathbf{x} \right|^{s_2}$$
$$= \frac{1}{s_2} \sum_{t=1}^{T} \left| \int_\Omega \left(\sigma - \sigma_-\right) \psi_t \, \mathrm{d}\mathbf{x} \right|^{s_2} \left| \int_\Omega \left(\sigma_+ - \sigma\right) \psi_t \, \mathrm{d}\mathbf{x} \right|^{s_2}.$$

Therefore, *(ii)* follows by applying lower limit properties to the above limits (observe that the addends of $\alpha_i \cdot R(\sigma_i, x_i)$ and $\alpha \cdot R(\sigma, x)$ are never $0 \cdot \infty$). ☐

Proof of (iii). By Assumption A5, $\left(\tilde{X}, \|\cdot\|_{\tilde{X}}\right) \to (X, \|\cdot\|_X)$ is compact, and therefore $x_i \to x$ in X^N. With this, the assumptions of Lemma 3.2 are satisfied. ☐

3.2.1 Unified formulation

A unified formulation of the regularized problems is provided below. Then, a preliminary result that will be a consequence of Lemmas 3.2 and 3.8 is proved.

From now on, $\boldsymbol{I}_\lambda$ stands for any regularized problem, that is, $\boldsymbol{I}_\lambda \in \left\{\boldsymbol{I}_\lambda^1, \boldsymbol{I}_\lambda^2, \boldsymbol{I}_\lambda^3\right\}$. Given a parameter $\lambda \in \Lambda$, the regularized problems can be written as the unconstrained minimization problem

$$\min_{(\sigma,x)\in L^\infty(\Omega)\times\tilde{X}^N} \mathscr{F}_\lambda(\sigma, x) := \mathscr{T}_\lambda(\sigma, x) + \mathscr{A}_\lambda(\sigma, x)$$

where $\mathscr{T}_\lambda$ is the cost functional of the regularized problem $\boldsymbol{I}_\lambda$ and $\mathscr{A}_\lambda$ is the indicator function of the corresponding admissible set, that is,

$$\mathscr{A}_\lambda(\sigma, x) := \begin{cases} 0 & \text{if } (\sigma, x) \text{ is admissible for } \boldsymbol{I}_\lambda \\ \infty & \text{otherwise} \end{cases}.$$

The Cartesian product $L^\infty(\Omega) \times \tilde{X}^N$ is equipped with the product topology that is induced by the weak* topology of $L^\infty(\Omega)$ and the weak topology of $\tilde{X}$. This topology is denoted by τ. Thus, a sequence (σ_i, x_i) in the topological space $L^\infty(\Omega) \times \tilde{X}^N$ converges to a element $(\sigma, x) \in L^\infty(\Omega) \times \tilde{X}^N$, if and only if

$$\sigma_i \overset{\star}{\rightharpoonup} \sigma \text{ in } L^\infty(\Omega) \quad \text{and} \quad x_i^n \rightharpoonup x^n \text{ in } \tilde{X} \text{ for } n = 1, \ldots, N.$$

We say that (σ_i, x_i) is τ-convergent to (σ, x) or write $(\sigma_i, x_i) \overset{\tau}{\rightharpoonup} (\sigma, x)$. Observe that, given $\varepsilon > 0$, the closed ball

$$\mathcal{B}_\varepsilon := \left\{\sigma \in L^\infty(\Omega) \,\middle|\, \|\sigma\|_{L^\infty(\Omega)} \leq \varepsilon\right\} \times \left\{x \in \tilde{X} \,\middle|\, \|x\|_{\tilde{X}} \leq \varepsilon\right\}^N$$

is sequentially τ-compact. Indeed, by Alaoglu's Theorem (see for instance [21, Thm. 3.16]), the closed unit ball $\mathcal{B}_{L^\infty(\Omega)}$ of $L^\infty(\Omega)$ is compact in the weak* topology of $L^\infty(\Omega) \equiv \left(L^1(\Omega)\right)^\star$. Moreover, since $L^1(\Omega)$ is separable, $\mathcal{B}_{L^\infty(\Omega)}$ is also metrizable (see for instance [27, Thm. 5.5]), that is, there exists a metric on $\mathcal{B}_{L^\infty(\Omega)}$ that induces the weak* topology restricted to $\mathcal{B}_{L^\infty(\Omega)}$. Thus, as a metric space is compact if and only if it is sequentially compact, $\mathcal{B}_{L^\infty(\Omega)}$ is also sequentially compact in the weak* topology of $L^\infty(\Omega)$. On the other hand, by Kukatani's Theorem (see for instance [21, Thm. 3.17]), the closed unit ball $\mathcal{B}_{\tilde{X}}$ of $\tilde{X}$ is compact in the weak topology of $\tilde{X}$ (recall that $\tilde{X}$ was assumed to be reflexive in Assumption A5). Using this, it can be proved that any bounded sequence in $\tilde{X}$ has a weakly convergent subsequence (the closed unit ball in the space generated by the elements of any bounded sequence is compact and metrizable in the weak topology). Thus, $\mathcal{B}_{\tilde{X}}$ is also sequentially compact in the weak topology of $\tilde{X}$. Therefore, given a sequence (σ_i, x_i) in $\mathcal{B}_\varepsilon$, we can find a subsequence $\left(\sigma_{i_j}, x_{i_j}\right)$ and some $(\sigma, x) \in \mathcal{B}_\varepsilon$ such that $\sigma_{i_j} \overset{\star}{\rightharpoonup} \sigma$ and $x_{i_j}^n \rightharpoonup x^n$ for $n = 1, \ldots, N$, that is, the subsequence $\left(\sigma_{i_j}, x_{i_j}\right)$ is τ-convergent to (σ, x).

The following result will help us to prove the well-posedness of the regularized problems. But first, consider the following notation.

$\Lambda_+ \subset \Lambda$ denotes the subset of all parameters $\lambda = (\alpha, \ldots) \in \Lambda$ such that all the components of α are positive. $\bar{\lambda} \in \Lambda$ denotes the parameter

$$\bar{\lambda} := \begin{cases} (\mathbf{0}, 0, \bar{y}) & \text{if } \boldsymbol{I}_\lambda = \boldsymbol{I}_\lambda^1, \boldsymbol{I}_\lambda^2 \\ (\mathbf{0}, 0, 0, \bar{y}) & \text{if } \boldsymbol{I}_\lambda = \boldsymbol{I}_\lambda^3 \end{cases}.$$

Note that $\Lambda_+ \cup \left\{\bar{\lambda}\right\}$ is strictly contained in Λ. A sequence (λ_i) in Λ converges weakly (resp. strongly) to some $\lambda \in \Lambda$ if it converges weakly (resp. strongly) componentwise, and we write $\lambda_i \rightharpoonup \lambda$ (resp. $\lambda_i \to \lambda$). Since only the component y (noisy observation) of $\lambda = (\ldots, y)$ belongs to an abstract space, the type of convergence refers to that in Y^N.

Corollary 3.10.

(i) *Let* $\mathscr{R} : L^\infty(\Omega) \times \tilde{X}^N \to [0, \infty]^4$ *be the functional defined by*

$$\mathscr{R}(\sigma, x) := \mathbf{1} \cdot R(\sigma, x) + \begin{cases} 0 & \text{if } \sigma_- \leq \sigma \leq \sigma_+ \text{ a.e. on } \Omega \\ \infty & \text{otherwise} \end{cases}.$$

Given $\varepsilon > 0$, *the level set*

$$\mathcal{L}_\varepsilon := \left\{(\sigma, x) \in L^\infty(\Omega) \times \tilde{X}^N \,|\, \mathscr{R}(\sigma, x) \leq \varepsilon\right\}$$

is sequentially τ*-compact.*

(ii) *Let* (λ_i) *and* (σ_i, x_i) *be sequences in* Λ_+ *and* $L^\infty(\Omega) \times \tilde{X}^N$*, respectively. If* $\lambda_i \rightharpoonup \lambda \in \Lambda_+$ *and* $(\sigma_i, x_i) \overset{\tau}{\rightharpoonup} (\sigma, x)$ *then*

$$\mathscr{F}_\lambda(\sigma, x) \leq \liminf_{i\to\infty} \mathscr{F}_{\lambda_i}(\sigma_i, x_i).$$

The same holds if $\lambda = \bar{\lambda}$ *and* $(\sigma, x) \in \mathcal{L}_\varepsilon$ *for some* $\varepsilon > 0$.

(iii) *Let* (λ_i) *and* (σ_i, x_i) *be sequences in* Λ_+ *and* $L^\infty(\Omega) \times \tilde{X}^N$*, respectively. Suppose that* $\lambda_i \rightharpoonup \lambda$, $(\sigma_i, x_i) \overset{\tau}{\rightharpoonup} (\sigma, x)$*, and* $\lim_{i\to\infty} \mathscr{F}_{\lambda_i}(\sigma_i, x_i) = \mathscr{F}_\lambda(\sigma, x)$*, with* $\mathscr{F}_{\lambda_i}(\sigma_i, x_i), \mathscr{F}_\lambda(\sigma, x) < \infty$ *for all* i.

- *If* $\lambda \in \Lambda_+$ *then*
  - *Regularized problem 1* ($\lambda_i = (\alpha_i, \delta_i, y_i) \rightharpoonup \lambda = (\alpha, \delta, y)$):

$$\lim_{i\to\infty} \|A(\sigma_i, x_i)\|_p = \|A(\sigma, x)\|_p$$
$$\lim_{i\to\infty} \|C(\sigma_i, x_i) - y_i\|_q = \|C(\sigma, x) - y\|_q$$

  - *Regularized problem 2* ($\lambda_i = (\alpha_i, \delta_i, y_i) \rightharpoonup \lambda = (\alpha, \delta, y)$):

$$\lim_{i\to\infty} \|A(\sigma_i, x_i)\|_p = \|A(\sigma, x)\|_p$$
$$\liminf_{i\to\infty} \|C(\sigma_i, x_i) - y_i\|_\infty \in \left[\|C(\sigma, x) - y\|_\infty, \gamma\delta\right]$$

  - *Regularized problem 3* ($\lambda_i = (\alpha_i, \beta_i, \delta_i, y_i) \rightharpoonup \lambda = (\alpha, \beta, \delta, y)$):

$$\liminf_{i\to\infty} \|A(\sigma_i, x_i)\|_\infty \in \left[\|A(\sigma, x)\|_\infty, \beta\right]$$
$$\lim_{i\to\infty} \|C(\sigma_i, x_i) - y_i\|_q = \|C(\sigma, x) - y\|_q$$

  - *All the regularized problems:*

$$R(\sigma_i, x_i) \to R(\sigma, x)$$

- *If* $\lambda = \bar{\lambda}$ *and* $\mathscr{F}_\lambda(\sigma, x) = 0$*, then* (σ, x) *solves* $\boldsymbol{I}$, $A(\sigma_i, x_i) \to 0_{(Z^\star)^N}$ *in* $(Z^\star)^N$*, and* $C(\sigma_i, x_i) \to \bar{y}$ *in* Y^N *(strong convergences).*

Proof of (i). Since $\mathcal{L}_\varepsilon \subseteq \mathcal{B}_{\max\{\sigma_+, (r\varepsilon)^{1/r}\}}$ (which is sequentially τ-compact), it suffices to prove that $\mathcal{L}_\varepsilon$ is τ-closed. Let (σ_i, x_i) be a sequence in $\mathcal{L}_\varepsilon$ (that is, $\mathbf{1} \cdot R(\sigma_i, x_i) \leq \varepsilon$ and $\sigma_- \leq \sigma_i \leq \sigma_+$ a.e. on Ω for all i) τ-convergent to some $(\sigma, x) \in L^\infty(\Omega) \times \tilde{X}^N$. By Lemma 3.8*(i)(ii)*, $\sigma_- \leq \sigma \leq \sigma_+$ a.e. on Ω and $\mathbf{1} \cdot R(\sigma, x) \leq \liminf_{i\to\infty} \mathbf{1} \cdot R(\sigma_i, x_i) \leq \varepsilon$. Therefore $(\sigma, x) \in \mathcal{L}_\varepsilon$. □

Proof of (ii). First note that by Lemma 3.8, the implication

$$\left.\begin{cases} \lambda_i \rightharpoonup \lambda \\ (\sigma_i, x_i) \overset{\tau}{\rightharpoonup} (\sigma, x) \\ (\sigma_i, x_i) \text{ is admissible for } \boldsymbol{I}_{\lambda_i} \end{cases}\right\} \Rightarrow (\sigma, x) \text{ is admissible for } \boldsymbol{I}_\lambda \tag{3.4}$$

is true. Indeed, since

$$(\sigma_i, x_i) \text{ is admissible for } \boldsymbol{I}_{\lambda_i} \Leftrightarrow \begin{cases} \sigma_- \leq \sigma_i \leq \sigma_+ & \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{1,2,3}_{\lambda_i} \\ \|C(\sigma_i, x_i) - y_i\|_\infty \leq \gamma\delta_i & \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{2}_{\lambda_i} \\ \|A(\sigma_i, x_i)\|_\infty \leq \beta_i & \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{3}_{\lambda_i} \end{cases},$$

it suffices to apply Lemma 3.8(*i*)(*iii*). Now, we can affirm that

$$\left\{ \begin{matrix} \lambda_i \rightharpoonup \lambda \\ (\sigma_i, x_i) \overset{\mathcal{T}}{\rightharpoonup} (\sigma, x) \end{matrix} \right\} \Rightarrow \mathscr{A}_\lambda(\sigma, x) \leq \liminf_{i\to\infty} \mathscr{A}_{\lambda_i}(\sigma_i, x_i). \tag{3.5}$$

If not, there exists a subsequence (σ_{i_j}, x_{i_j}) and a constant $0 < c < \infty$ such that

$$\infty = \mathscr{A}_\lambda(\sigma, x) > c > \mathscr{A}_{\lambda_{i_j}}(\sigma_{i_j}, x_{i_j}) = 0 \quad \text{for all } j$$

(use the definition of lower limit to obtain it). Therefore (σ_{i_j}, x_{i_j}) is admissible for $\boldsymbol{I}_{\lambda_{i_j}}$ and (σ, x) is not admissible for $\boldsymbol{I}_\lambda$, which contradicts (3.4). On the other hand, by applying Lemma 3.8(*ii*)(*iii*) and lower limit properties, we obtain (recall that $\mathscr{T}_\lambda$ and $\mathscr{T}_{\lambda_i}$ are the cost functionals of the regularized problems)

$$\left\{ \begin{matrix} \lambda_i \rightharpoonup \lambda \\ (\sigma_i, x_i) \overset{\mathcal{T}}{\rightharpoonup} (\sigma, x) \end{matrix} \right\} \Rightarrow \mathscr{T}_\lambda(\sigma, x) \leq \liminf_{i\to\infty} \mathscr{T}_{\lambda_i}(\sigma_i, x_i), \tag{3.6}$$

where (λ_i) is a sequence in Λ_+ and either $\lambda \in \Lambda_+$ or $\{\lambda = \bar{\lambda}$ and $(\sigma, x) \in \mathcal{L}_\varepsilon\}$ (that is, $\alpha = \mathbf{0}$ and all the components of $R(\bar{\sigma}, \bar{x})$ are finite numbers). Therefore, from (3.5) and (3.6),

$$\begin{aligned} \mathscr{T}_\lambda(\sigma, x) + \mathscr{A}_\lambda(\sigma, x) &\leq \liminf_{i\to\infty} \mathscr{A}_{\lambda_i}(\sigma_i, x_i) + \liminf_{i\to\infty} \mathscr{T}_{\lambda_i}(\sigma_i, x_i) \\ &\leq \liminf_{i\to\infty} \mathscr{A}_{\lambda_i}(\sigma_i, x_i) + \mathscr{T}_{\lambda_i}(\sigma_i, x_i) \end{aligned}$$

and the desired conclusion follows. □

Proof of (iii). First note that as $\lim_{i\to\infty} \mathscr{F}_{\lambda_i}(\sigma_i, x_i) < \infty$ and $\mathscr{F}_{\lambda_i}(\sigma_i, x_i) < \infty$ for all i, the sequence $(\mathscr{F}_{\lambda_i}(\sigma_i, x_i))$ is bounded, which implies that $\mathscr{A}_{\lambda_i}(\sigma_i, x_i) = 0$ for all i. Hence, for each i, (σ_i, x_i) is admissible for $\boldsymbol{I}_{\lambda_i}$. Since $\lambda_i \rightharpoonup \lambda$ and $(\sigma_i, x_i) \overset{\mathcal{T}}{\rightharpoonup} (\sigma, x)$, by Lemma 3.8(*i*)(*iii*) we obtain the following:

- If $\lambda \in \Lambda_+$ then

$$\begin{matrix} \sigma_- \leq \sigma \leq \sigma_+ & \text{if } \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{1,2,3}_{\lambda_i} \\ \|C(\sigma, x) - y\|_\infty \leq \liminf_{i\to\infty} \|C(\sigma_i, x_i) - y_i\|_\infty \leq \gamma\delta & \text{if } \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{2}_{\lambda_i} \\ \|A(\sigma, x)\|_\infty \leq \liminf_{i\to\infty} \|A(\sigma_i, x_i)\|_\infty \leq \beta & \text{if } \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{3}_{\lambda_i} \end{matrix}. \tag{3.7}$$

- If $\lambda = \bar{\lambda}$ then

$$\begin{array}{ll} \sigma_- \leq \sigma \leq \sigma_+ & \text{if } \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{1,2,3}_{\lambda_i} \\ C(\sigma,x) = \bar{y} \text{ and } C(\sigma_i,x_i) - y_j \to 0_{Y^N} & \text{if } \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{2}_{\lambda_i} \\ A(\sigma,x) = 0 \text{ and } A(\sigma_i,x_i) \to 0_{(Z^\star)^N} & \text{if } \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{3}_{\lambda_i} \end{array} . \tag{3.8}$$

On the other hand, since $\mathscr{F}_\lambda(\sigma,x) < \infty$, we also have that $\mathscr{A}_\lambda(\sigma,x) = 0$. So,

$$\lim_{i\to\infty} \mathscr{T}_{\lambda_i}(\sigma_i,x_i) = \lim_{i\to\infty} \mathscr{F}_{\lambda_i}(\sigma_i,x_i) = \mathscr{F}_\lambda(\sigma,x) = \mathscr{T}_\lambda(\sigma,x) < \infty.$$

From this, it can be deduced that

$$\begin{array}{ll} \lim\limits_{i\to\infty} \|A(\sigma_i,x_i)\|_p = \|A(\sigma,x)\|_p \leq (p\mathscr{T}_\lambda(\sigma,x))^{1/p} & \text{if } \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{1,2}_{\lambda_i} \\ \lim\limits_{i\to\infty} \|C(\sigma_i,x_i) - y_i\|_q = \|C(\sigma,x) - y\|_q \leq (q\mathscr{T}_\lambda(\sigma,x))^{1/q} & \text{if } \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{1,3}_{\lambda_i} \\ R(\sigma_i,x_i) \to R(\sigma,x) & \text{if } \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{1,2,3}_{\lambda_i} \end{array} , \tag{3.9}$$

and if $\mathscr{T}_\lambda(\sigma,x) = 0$ and $\lambda = \bar{\lambda}$, it follows immediately that

$$\begin{array}{ll} A(\sigma,x) = 0 \text{ and } A(\sigma_i,x_i) \to 0_{(Z^\star)^N} & \text{if } \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{1}_{\lambda_i}, \boldsymbol{I}^{2}_{\lambda_i} \\ C(\sigma,x) = \bar{y} \text{ and } C(\sigma_i,x_i) - y_i \to 0_{Y^N} & \text{if } \boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^{1}_{\lambda_i}, \boldsymbol{I}^{3}_{\lambda_i} \end{array} . \tag{3.10}$$

From (3.7) and (3.9), the results for $\lambda \in \Lambda_+$ follows. From (3.8) and (3.10) ($\lambda = \bar{\lambda}$ and $\mathscr{F}_\lambda(\sigma,x) = 0$), the limit (σ,x) solves $\boldsymbol{I}$, $A(\sigma_i,x_i) \to 0_{(Z^\star)^N}$, and $C(\sigma_i,x_i) - y_i \to 0_{Y^N}$. Actually $C(\sigma_i,x_i) \to \bar{y}$ since $y_i \to \bar{y}$ (because $\|\bar{y} - y_i\|_q \leq \delta_i$ and $\delta_i \to 0$ in this case).

Now, the proof of the case $\boldsymbol{I}_{\lambda_i} = \boldsymbol{I}^3_{\lambda_i}$ in (3.9) is provided (the other cases are similar). In this case $\lim_{i\to\infty} \mathscr{T}_{\lambda_i}(\sigma_i,x_i) = \mathscr{T}_\lambda(\sigma,x)$ reads as

$$\lim_{i\to\infty} \frac{1}{q}\|C(\sigma_i,x_i) - y_i\|_q^q + \alpha_i \cdot R(\sigma_i,x_i) = \frac{1}{q}\|C(\sigma,x) - y\|_q^q + \alpha \cdot R(\sigma,x). \tag{3.11}$$

Suppose that

$$\varepsilon := \limsup_{i\to\infty} \alpha_i \cdot R(\sigma_i,x_i) > \alpha \cdot R(\sigma,x).$$

Let $\left(\alpha_{i_j} \cdot R\left(\sigma_{i_j},x_{i_j}\right)\right)$ be a subsequence such that $\lim_{i\to\infty} \alpha_{i_j} \cdot R\left(\sigma_{i_j},x_{i_j}\right) = \varepsilon$, which always exists since ε is the highest cluster point of $(\alpha_i \cdot R(\sigma_i,x_i))$. By (3.11) it follows that

$$\begin{aligned} \lim_{j\to\infty} \frac{1}{q}\left\|C\left(\sigma_{i_j},x_{i_j}\right) - y_{i_j}\right\|_q^q &= \frac{1}{q}\|C(\sigma,x) - y\|_q^q + \alpha \cdot R(\sigma,x) - \varepsilon \\ &< \frac{1}{q}\|C(\sigma,x) - y\|_q^q, \end{aligned}$$

which contradicts Lemma (3.2)(*ii*). Hence $\lim_{i\to\infty} \alpha_i \cdot R(\sigma_i,x_i) = \alpha \cdot R(\sigma,x)$, which implies that $\lim_{i\to\infty} \frac{1}{q}\|C(\sigma_i,x_i) - y_i\|_q^q = \frac{1}{q}\|C(\sigma,x) - y\|_q^q$. To prove that $R(\sigma_i,x_i) \to R(\sigma,x)$ it suffices to apply the same arguments since each component of R also has the lower semicontinuity property. □

Remark 3.11. It is worth noting the following:

- $\mathcal{L}_\varepsilon$ is sequentially τ-closed.
- $\mathscr{T}_\lambda$ is sequentially τ-lower semicontinuous for all $\lambda \in \Lambda_+$.
- If $(\sigma, x) \in L^\infty(\Omega) \times \tilde{X}^N$ is a solution to $\boldsymbol{I}$ then

$$\mathscr{F}_\lambda(\sigma, x) \leq \begin{cases} \frac{\delta^q}{q} + \alpha \cdot R(\sigma, x) & \text{if } \boldsymbol{I}_\lambda = \boldsymbol{I}_\lambda^1, \boldsymbol{I}_\lambda^3 \\ \alpha \cdot R(\sigma, x) & \text{if } \boldsymbol{I}_\lambda = \boldsymbol{I}_\lambda^2 \end{cases} \quad \text{for all } \lambda \in \Lambda. \tag{3.12}$$

3.3 Results

The existence, stability, and convergence of regularized solutions are proved here. With the aid of Corollary 3.10, the proofs of these results are given simultaneously for the three regularized problems. At the end, a learning problem is formulated and the existence of solutions for it is proved.

First of all, a technical assumption is made.

A6. Assume that there exists a solution $(\bar{\sigma}, \bar{x}) \in L^\infty(\Omega) \times \tilde{X}^N$ to problem $\boldsymbol{I}$ such that $\sigma_- \leq \bar{\sigma} \leq \sigma_+$ a.e. on Ω and $\int_\Omega |D\bar{\sigma}| < \infty$.

Assumption A6 ensures the existence of a solution to $\boldsymbol{I}$ in a "compact" subspace of $L^\infty(\Omega) \times X^N$. Basically, this assumption is used to show that $\mathscr{F}_\lambda$ is a proper function (i.e. $\mathscr{F}_\lambda \not\equiv \infty$) and to provide easy to use estimates. In the context of EIT, Assumption A6 makes sense since the EIT inverse problem has solution if, by example, the conductivity is piecewise constant [48] (piecewise constant functions belong to $L^\infty(\Omega) \cap BV(\Omega)$, but not to any Sobolev spaces). Also, it is usual to know the conductivity bounds [72, 76, 61]. The finite bounded variation of $\bar{\sigma}$ will allows to obtain regularized solutions also with finite bounded variation. A similar assumption was considered in [54].

In what follows, $\min \alpha$ and $\max \alpha$ denote the minimum and maximum of the components of α, respectively.

3.3.1 Existence of regularized solutions

Proposition 3.12. *Given a parameter $\lambda \in \Lambda_+$, the regularized problem $\boldsymbol{I}_\lambda$ has a solution $(\sigma_\lambda, x_\lambda) \in \mathcal{L}_\varepsilon$, with ε given by* (3.13).

$(\sigma_\lambda, x_\lambda)$ is called regularized solution.

Proof. Since $(\bar{\sigma}, \bar{x})$ solves $\boldsymbol{I}$ and $\int_\Omega |D\bar{\sigma}| < \infty$ (and hence all the components of $\mathscr{R}(\bar{\sigma}, \bar{x})$ are finite numbers), there exist $0 \leq m < \infty$ and a sequence (σ_i, x_i) in $L^\infty(\Omega) \times \tilde{X}^N$ such that

$$m = \inf\left\{\mathscr{F}_\lambda(\sigma, x) \,\middle|\, (\sigma, x) \in L^\infty(\Omega) \times \tilde{X}^N\right\} \leq \mathscr{F}_\lambda(\bar{\sigma}, \bar{x}) < \infty,$$

$$m \leq \mathscr{F}_\lambda(\sigma_i, x_i) = \ldots + \alpha \cdot R(\sigma_i, x_i) \leq \mathscr{F}_\lambda(\bar{\sigma}, \bar{x}) \text{ for all } i,$$

$$\text{and} \quad \lim_{i \to \infty} \mathscr{F}_\lambda(\sigma_i, x_i) = m.$$

Considering (3.12) and the inequality $(\min\alpha)\,\mathbf{1}\cdot R(\sigma_i,x_i)\leq\alpha\cdot R(\sigma_i,x_i)$, it follows that $\mathbf{1}\cdot R(\sigma_i,x_i)\leq\varepsilon$ for all i, where

$$\varepsilon := \begin{cases} \frac{\delta^q}{(\min\alpha)q}+\frac{1}{(\min\alpha)}\alpha\cdot R(\bar\sigma,\bar x) & \text{if } \boldsymbol{I}_\lambda=\boldsymbol{I}_\lambda^1,\boldsymbol{I}_\lambda^3 \\ \frac{1}{(\min\alpha)}\alpha\cdot R(\bar\sigma,\bar x) & \text{if } \boldsymbol{I}_\lambda=\boldsymbol{I}_\lambda^2 \end{cases}. \tag{3.13}$$

From this, we can affirm that the minimizing sequence (σ_i,x_i) is contained in the level set $\mathcal{L}_\varepsilon$. Since $\mathcal{L}_\varepsilon$ sequentially τ-compact, there exist a subsequence $\left(\sigma_{i_j},x_{i_j}\right)$ and some $(\sigma_\lambda,x_\lambda)\in\mathcal{L}_\varepsilon$ such that $\left(\sigma_{i_j},x_{i_j}\right)\overset{\tau}{\rightharpoonup}(\sigma_\lambda,x_\lambda)$. Since $\mathscr{F}_\lambda$ is sequentially τ-lower semicontinuous ($\lambda\in\Lambda_+$),

$$\mathscr{F}_\lambda(\sigma_\lambda,x_\lambda)\leq\liminf_{j\to\infty}\mathscr{F}_\lambda\left(\sigma_{i_j},x_{i_j}\right)=\lim_{j\to\infty}\mathscr{F}_\lambda\left(\sigma_{i_j},x_{i_j}\right)=m$$

and the conclusion follows. □

Remark 3.13 (The case of λ scalar). If the regularization parameter λ has the form $\lambda=\alpha_0\mathbf{1}$, with $\alpha_0>0$, then the regularized solution $(\sigma_\lambda,x_\lambda)$ belongs to $\mathcal{L}_\varepsilon$, with $\varepsilon=\delta^q/(\alpha_0 q)+\mathbf{1}\cdot R(\bar\sigma,\bar x)$ if $\boldsymbol{I}_\lambda=\boldsymbol{I}_\lambda^1,\boldsymbol{I}_\lambda^3$ and $\varepsilon=\mathbf{1}\cdot R(\bar\sigma,\bar x)$ if $\boldsymbol{I}_\lambda=\boldsymbol{I}_\lambda^2$.

Remark 3.14 (σ_λ has bounded variation). Note that since $(\sigma_\lambda,x_\lambda)$ belongs to $\mathcal{L}_\varepsilon$, we have $\int_\Omega|D(\sigma_\lambda-\sigma')|\leq\varepsilon$, and as $\sigma\mapsto\int_\Omega|D\sigma|$ is a seminorm on $BV(\Omega)$, by the triangle inequality it can be deduced that $\sigma_\lambda\in BV(\Omega)$.

3.3.2 Stability of regularized solutions

Here we show that a regularized solution $(\sigma_\lambda,x_\lambda)$ depends continuously on λ. For each regularized problem, we impose a different condition on the sequence of parameters.

Proposition 3.15. *Let (λ_i) be a sequence of parameters in Λ_+ and let $\hat\lambda\in\Lambda_+$ be such that (the items refer to the regularized problems)*

(1) $\lambda_i=(\alpha_i,\delta_i,y_i)\to\hat\lambda=(\alpha,\delta,y)$,

(2) $\lambda_i=(\alpha_i,\delta_i,y_i)\rightharpoonup\hat\lambda=(\alpha,\delta,y)$ and $\gamma\delta+\|y-y_i\|_\infty\leq\gamma\delta_i$ for all $i\geq i_0$,

(3) $\lambda_i=(\alpha_i,\beta_i,\delta_i,y_i)\to\hat\lambda=(\alpha,\beta,\delta,y)$ and $\beta\leq\beta_i$ for all $i\geq i_0$,

where i_0 is some fixed index. For each i, let $(\sigma_i,x_i):=(\sigma_{\lambda_i},x_{\lambda_i})$ be a solution to $\boldsymbol{I}_{\lambda_i}$, which exists by Proposition 3.12. Then, for the sequence of regularized solutions (σ_i,x_i) we have:

(i) The sequence (σ_i,x_i) has a τ-convergent subsequence.

(ii) If $\left(\sigma_{i_j},x_{i_j}\right)$ is a τ-convergent subsequence of (σ_i,x_i), its limit $(\hat\sigma,\hat x)$ is a solution to the regularized problem $\boldsymbol{I}_{\hat\lambda}$ and

$$\begin{cases} \lim_{j\to\infty}\left\|A\left(\sigma_{i_j},x_{i_j}\right)\right\|_p=\|A(\hat\sigma,\hat x)\|_p \quad (1\leq p\leq\infty) & \textit{if (1) or (2)} \\ \lim_{j\to\infty}\left\|C\left(\sigma_{i_j},x_{i_j}\right)-y_{i_j}\right\|_q=\|C(\hat\sigma,\hat x)-y\|_q \quad (1\leq q\leq\infty) & \textit{if (1) or (3)} \\ R\left(\sigma_{i_j},x_{i_j}\right)\to R(\hat\sigma,\hat x) & \textit{all cases} \end{cases}. \tag{3.14}$$

(iii) If $\boldsymbol{I}_{\hat{\lambda}}$ *has a unique solution* $\left(\sigma_{\hat{\lambda}}, x_{\hat{\lambda}}\right)$ *then* $(\sigma_i, x_i) \xrightarrow{\tau} \left(\sigma_{\hat{\lambda}}, x_{\hat{\lambda}}\right)$.

Proof of (i). By Proposition 3.12, for each i, (σ_i, x_i) belongs to $\mathcal{L}_{\varepsilon_i}$, with

$$\varepsilon_i = \begin{cases} \frac{\delta_i^q}{(\min \alpha_i) q} + \frac{1}{(\min \alpha_i)} \alpha_i \cdot R(\bar{\sigma}, \bar{x}) & \text{if (1) or (3)} \\ \frac{1}{(\min \alpha_i)} \alpha_i \cdot R(\bar{\sigma}, \bar{x}) & \text{if (2)} \end{cases}.$$

Since (α_i) and (δ_i) are convergent sequences (here (α_i) and its limit are vectors of positive numbers), there exists an $\varepsilon > 0$ such that $\varepsilon_i \leq \varepsilon$ for all i. Thus, the sequence (σ_i, x_i) is contained in $\mathcal{L}_\varepsilon$ and the sequentially τ-compactness of $\mathcal{L}_\varepsilon$ yields the assertion. □

Proof of (ii). Let $\left(\sigma_{i_j}, x_{i_j}\right)$ be a subsequence of (σ_i, x_i) and $(\hat{\sigma}, \hat{x}) \in L^\infty(\Omega) \times \tilde{X}^N$ such that $\left(\sigma_{i_j}, x_{i_j}\right) \xrightarrow{\tau} (\hat{\sigma}, \hat{x})$. Since $\lambda_i \rightharpoonup \hat{\lambda}$ in Λ_+,

$$\mathscr{F}_{\hat{\lambda}}(\hat{\sigma}, \hat{x}) \leq \liminf_{j\to\infty} \mathscr{F}_{\lambda_{i_j}}\left(\sigma_{i_j}, x_{i_j}\right) \tag{3.15}$$

by Corollary 3.10(*ii*). Note also that by the minimality of each (σ_i, x_i),

$$\mathscr{F}_{\lambda_i}(\sigma_i, x_i) \leq \mathscr{F}_{\lambda_i}(\sigma, x) \quad \text{for all } (\sigma, x) \in L^\infty(\Omega) \times \tilde{X}^N. \tag{3.16}$$

On the other hand, observe that from the conditions on (λ_i) it can be deduced that the admissible set associated to $\hat{\lambda}$ is contained in the admissible set associated to λ_i, for $i \geq i_0$ (case (*1*) has not conditions because the admissible set in this case is independent of λ). So, given $i \geq i_0$, this implies that $\mathscr{A}_{\lambda_i}(\sigma, x) \leq \mathscr{A}_{\hat{\lambda}}(\sigma, x)$ for all $(\sigma, x) \in L^\infty(\Omega) \times \tilde{X}^N$, and hence

$$\begin{aligned} \mathscr{F}_{\lambda_i}(\sigma, x) &= \mathscr{T}_{\lambda_i}(\sigma, x) + \mathscr{A}_{\lambda_i}(\sigma, x) \\ &\leq \mathscr{T}_{\lambda_i}(\sigma, x) + \mathscr{A}_{\hat{\lambda}}(\sigma, x) \quad \text{for all } (\sigma, x) \in L^\infty(\Omega) \times \tilde{X}^N. \end{aligned} \tag{3.17}$$

Note also that the limit

$$\lim_{i\to\infty} \mathscr{T}_{\lambda_i}(\sigma, x) = \mathscr{T}_{\hat{\lambda}}(\sigma, x) \quad \text{for all } (\sigma, x) \in L^\infty(\Omega) \times \tilde{X}^N \tag{3.18}$$

holds (strong convergence of the sequence (y_i) is required in cases (1) and (3) because the cost functionals in these cases depend on y). Combining (3.15), (3.16), (3.17), and (3.18) it follows that

$$\begin{aligned} \mathscr{F}_{\hat{\lambda}}(\hat{\sigma}, \hat{x}) &\leq \liminf_{j\to\infty} \mathscr{F}_{\lambda_{i_j}}\left(\sigma_{i_j}, x_{i_j}\right) \leq \limsup_{j\to\infty} \mathscr{F}_{\lambda_{i_j}}\left(\sigma_{i_j}, x_{i_j}\right) \\ &\leq \limsup_{j\to\infty} \mathscr{F}_{\lambda_{i_j}}(\sigma, x) \leq \limsup_{j\to\infty} \mathscr{T}_{\lambda_{i_j}}(\sigma, x) + \mathscr{A}_{\hat{\lambda}}(\sigma, x) \\ &= \lim_{j\to\infty} \mathscr{T}_{\lambda_{i_j}}(\sigma, x) + \mathscr{A}_{\hat{\lambda}}(\sigma, x) = \mathscr{T}_{\hat{\lambda}}(\sigma, x) + \mathscr{A}_{\hat{\lambda}}(\sigma, x) \\ &= \mathscr{F}_{\hat{\lambda}}(\sigma, x) \quad \text{for all } (\sigma, x) \in L^\infty(\Omega) \times \tilde{X}^N. \end{aligned}$$

Therefore $(\hat{\sigma}, \hat{x})$ is a solution to $\boldsymbol{I}_{\hat{\lambda}}$. Moreover, taking $(\sigma, x) = (\hat{\sigma}, \hat{x})$,

$$\lim_{j\to\infty} \mathscr{F}_{\lambda_{i_j}}\left(\sigma_{i_j}, x_{i_j}\right) = \mathscr{F}_{\hat{\lambda}}(\hat{\sigma}, \hat{x}) < \infty,$$

and Corollary 3.10(*iii*) gives (3.14). □

Proof of (iii). Assume that there exists a unique solution $\left(\sigma_{\hat{\lambda}}, x_{\hat{\lambda}}\right)$ to $\boldsymbol{I}_{\hat{\lambda}}$. Let $\left(\sigma_{i_j}, x_{i_j}\right)$ be a arbitrary subsequence of (σ_i, x_i). By (i) there exists a τ-convergent subsequence $\left(\sigma_{i_{j_k}}, x_{i_{j_k}}\right)$. Let $(\hat{\sigma}, \hat{x})$ be its limit. By (ii) $(\hat{\sigma}, \hat{x})$ is solution to $\boldsymbol{I}_{\hat{\lambda}}$. So, $(\hat{\sigma}, \hat{x}) = \left(\sigma_{\hat{\lambda}}, x_{\hat{\lambda}}\right)$. We have proved that every subsequence of (σ_i, x_i) has a τ-convergent subsequence with limit $\left(\sigma_{\hat{\lambda}}, x_{\hat{\lambda}}\right)$. Therefore the whole sequence (σ_i, x_i) is τ-convergent to $\left(\sigma_{\hat{\lambda}}, x_{\hat{\lambda}}\right)$ [89, Lem. 8.2]. ☐

3.3.3 Convergence of regularized solutions

We introduce the concept of $\mathscr{R}$-minimizing solutions before dealing with the convergence of regularized solutions. An element $(\dot{\sigma}, \dot{x}) \in L^{\infty}(\Omega) \times \tilde{X}^N$ is called *$\mathscr{R}$-minimizing* solution if it is a solution to

$$\min_{\substack{(\sigma,x)\in L^{\infty}(\Omega)\times\tilde{X}^N \\ \text{is a solution to } \boldsymbol{I}}} \mathscr{R}(\sigma, x) \quad .$$

Lemma 3.16. *There exists at least one $\mathscr{R}$-minimizing solution.*

Proof. Clearly $(\bar{\sigma}, \bar{x})$ is admissible and $\mathscr{R}(\bar{\sigma}, \bar{x}) < \infty$. Hence, there exist a minimizing sequence (σ_i, x_i) and $0 \le m < \infty$ such that $\lim_{i\to\infty} \mathscr{R}(\sigma_i, x_i) = m$ and $\mathscr{R}(\sigma_i, x_i) \le \mathscr{R}(\bar{\sigma}, \bar{x})$. Since $\mathcal{L}_{\mathscr{R}(\bar{\sigma},\bar{x})}$ is sequentially τ-compact and (σ_i, x_i) is in $\mathcal{L}_{\mathscr{R}(\bar{\sigma},\bar{x})}$, it follows that there exists a subsequence $\left(\sigma_{i_j}, x_{i_j}\right)$ that is τ-convergent to some $(\dot{\sigma}, \dot{x})$, which also is in $\mathcal{L}_{\mathscr{R}(\bar{\sigma},\bar{x})}$. Applying Lemma 3.8(ii) the conclusion follows. ☐

Here it is proved that a sequence of regularized solutions $(\sigma_{\lambda_i}, x_{\lambda_i})$ can converge to a solution of $\boldsymbol{I}$.

Theorem 3.17. *Let (λ_i) be the sequence of parameters in Λ_+ such that (the items refer to the regularized problems)*

(1) $\lambda_i = (\alpha_i, \delta_i, y_i) \rightharpoonup \bar{\lambda}$ *and* $\left(\frac{\delta_i^q}{\min \alpha_i}\right)$, $\left(\frac{\max \alpha_i}{\min \alpha_i}\right)$ *are bounded,*

(2) $\lambda_i = (\alpha_i, \delta_i, y_i) \rightharpoonup \bar{\lambda}$ *and* $\left(\frac{\max \alpha_i}{\min \alpha_i}\right)$ *is bounded,*

(3) $\lambda_i = (\alpha_i, \beta_i, \delta_i, y_i) \rightharpoonup \bar{\lambda}$ *and* $\left(\frac{\delta_i^q}{\min \alpha_i}\right)$, $\left(\frac{\max \alpha_i}{\min \alpha_i}\right)$ *are bounded.*

For each i, let $(\sigma_i, x_i) \coloneqq (\sigma_{\lambda_i}, x_{\lambda_i})$ be a solution to $\boldsymbol{I}_{\lambda_i}$, which exists by Proposition 3.12. Then, for the sequence of regularized solutions (σ_i, x_i) we have:

(i) *The sequence (σ_i, x_i) has a τ-convergent subsequence.*

(ii) *If $\left(\sigma_{i_j}, x_{i_j}\right)$ is a τ-convergent subsequence of (σ_i, x_i), its limit is a solution to $\boldsymbol{I}$, $A\left(\sigma_{i_j}, x_{i_j}\right) \to 0_{(Z^\star)^N}$, and $C\left(\sigma_{i_j}, x_{i_j}\right) \to \bar{y}$.*

(iii) *If $(\bar{\sigma}, \bar{x})$ is the unique solution to $\boldsymbol{I}$ then $(\sigma_i, x_i) \overset{\tau}{\rightharpoonup} (\bar{\sigma}, \bar{x})$.*

Proof of (i). Let $(\dot{\sigma}, \dot{x})$ be a $\mathscr{R}$-minimizing solution. For each i, $\mathscr{F}_{\lambda_i}(\sigma_i, x_i) \leq \mathscr{F}_{\lambda_i}(\dot{\sigma}, \dot{x})$ by the minimality of (σ_i, x_i). Considering (3.12), it can be deduced that (σ_i, x_i) belongs to $\mathcal{L}_{\varepsilon_i}$, with

$$\varepsilon_i = \begin{cases} \frac{\delta_i^q}{(\min \alpha_i) q} + \frac{(\max \alpha_i)}{(\min \alpha_i)} \mathbf{1} \cdot R(\dot{\sigma}, \dot{x}) & \text{if (1) or (3)} \\ \frac{(\max \alpha_i)}{(\min \alpha_i)} \mathbf{1} \cdot R(\dot{\sigma}, \dot{x}) & \text{if (2)} \end{cases}.$$

Since the sequences $(\delta_i^q / \min \alpha_i)$ and $(\max \alpha_i / \min \alpha_i)$ are bounded, there exists an $\varepsilon > 0$ such that $\varepsilon_i \leq \varepsilon$ for all i. Thus, the sequence (σ_i, x_i) is contained in $\mathcal{L}_\varepsilon$ and the sequentially τ-compactness $\mathcal{L}_\varepsilon$ yields the assertion. ☐

Proof of (ii). Let $\left(\sigma_{i_j}, x_{i_j}\right)$ be a subsequence of (σ_i, x_i) and $(\hat{\sigma}, \hat{x}) \in L^\infty(\Omega) \times \tilde{X}^N$ such that $\left(\sigma_{i_j}, x_{i_j}\right) \xrightarrow{\tau} (\hat{\sigma}, \hat{x})$. Since $\mathcal{L}_\varepsilon$ is also sequentially τ-closed, $(\hat{\sigma}, \hat{x}) \in \mathcal{L}_\varepsilon$. Since $\lambda_i \rightharpoonup \bar{\lambda}$,

$$\mathscr{F}_{\hat{\lambda}}(\hat{\sigma}, \hat{x}) \leq \liminf_{j \to \infty} \mathscr{F}_{\lambda_{i_j}}\left(\sigma_{i_j}, x_{i_j}\right)$$

by Corollary 3.10(*ii*). Thus, by the minimality of each $\left(\sigma_{i_j}, x_{i_j}\right)$ and Remark (3.12), it follows that

$$\begin{aligned} \mathscr{F}_{\hat{\lambda}}(\hat{\sigma}, \hat{x}) &\leq \liminf_{j \to \infty} \mathscr{F}_{\lambda_{i_j}}\left(\sigma_{i_j}, x_{i_j}\right) \leq \limsup_{j \to \infty} \mathscr{F}_{\lambda_{i_j}}\left(\sigma_{i_j}, x_{i_j}\right) \\ &\leq \limsup_{j \to \infty} \mathscr{F}_{\lambda_{i_j}}(\dot{\sigma}, \dot{x}) \leq \begin{cases} \lim_{j \to \infty} \frac{\delta_{i_j}^q}{q} + \alpha_{i_j} \cdot R(\dot{\sigma}, \dot{x}) & \text{if } (1), (3) \\ \lim_{j \to \infty} \alpha_{i_j} \cdot R(\dot{\sigma}, \dot{x}) & \text{if } (2) \end{cases}. \end{aligned}$$

Therefore, since $\lambda_{i_j} \to \bar{\lambda}$, we conclude that

$$\mathscr{F}_{\bar{\lambda}}(\hat{\sigma}, \hat{x}) = \lim_{j \to \infty} \mathscr{F}_{\lambda_{i_j}}\left(\sigma_{i_j}, x_{i_j}\right) = 0.$$

The conclusion follows from Corollary 3.10(*iii*). ☐

Proof of (iii). We use the subsequence-subsequence argument. ☐

By imposing additional conditions on the sequence of parameters, it is possible to characterize the solutions of ***I*** obtained by convergence of regularized solutions.

Corollary 3.18. *Under the hypotheses of Theorem 3.17, assuming in addition that*

(1) $\lim_{i \to \infty} \left(\frac{\delta_i^q}{\min \alpha_i}\right) = 0$ *and* $\lim_{i \to \infty} \left(\frac{\max \alpha_i}{\min \alpha_i}\right) = 1$

(2) $\lim_{i \to \infty} \left(\frac{\max \alpha_i}{\min \alpha_i}\right) = 1$

(3) $\lim_{i \to \infty} \left(\frac{\delta_i^q}{\min \alpha_i}\right) = 0$ *and* $\lim_{i \to \infty} \left(\frac{\max \alpha_i}{\min \alpha_i}\right) = 1$

we have:

(i) If $\left(\sigma_{i_j}, x_{i_j}\right)$ is a τ-convergent subsequence of (σ_i, x_i), its limit $(\hat{\sigma}, \hat{x})$ is a $\mathscr{R}$-minimizing solution and $R\left(\sigma_{i_j}, x_{i_j}\right) \to R(\hat{\sigma}, \hat{x})$.

(*ii*) *If there exists a unique* $\mathscr{R}$*-minimizing solution* $(\dot\sigma, \dot x)$ *then* $(\sigma_i, x_i) \stackrel{\mathcal{T}}{\rightharpoonup} (\dot\sigma, \dot x)$.

Proof of (i). Let $\left(\sigma_{i_j}, x_{i_j}\right)$ be a subsequence of (σ_i, x_i) and $(\hat\sigma, \hat x) \in L^\infty(\Omega) \times \tilde X^N$ such that $\left(\sigma_{i_j}, x_{i_j}\right) \stackrel{\mathcal{T}}{\rightharpoonup} (\hat\sigma, \hat x)$. By Theorem 3.17(*ii*), $(\hat\sigma, \hat x)$ is a solution to $\boldsymbol{I}$ which belongs to a level set $\mathcal{L}_\varepsilon$. So, $(\hat\sigma, \hat x)$ is admissible to be a $\mathscr{R}$-minimizing solution. On the other hand, in the proof of Theorem 3.17(*i*), it was proved that

$$\mathbf{1} \cdot R\left(\sigma_{i_j}, x_{i_j}\right) \le \varepsilon_j = \begin{cases} \dfrac{\delta_{i_j}^q}{\left(\min \alpha_{i_j}\right)q} + \dfrac{\left(\max \alpha_{i_j}\right)}{\left(\min \alpha_{i_j}\right)} \mathbf{1} \cdot R(\dot\sigma, \dot x) & \text{if } (1), (3) \\ \dfrac{\left(\max \alpha_{i_j}\right)}{\left(\min \alpha_{i_j}\right)} \mathbf{1} \cdot R(\dot\sigma, \dot x) & \text{if } (2) \end{cases}$$

for all j. Applying Lemma 3.8(*ii*) to $\left(\sigma_{i_j}, x_{i_j}\right) \stackrel{\mathcal{T}}{\rightharpoonup} (\hat\sigma, \hat x)$, from the above it follows that

$$\mathbf{1} \cdot R(\hat\sigma, \hat x) \le \liminf_{i\to\infty} \mathbf{1} \cdot R\left(\sigma_{i_j}, x_{i_j}\right) \le \limsup_{i\to\infty} \mathbf{1} \cdot R\left(\sigma_{i_j}, x_{i_j}\right) \le \lim_{i\to\infty} \varepsilon_i = \mathbf{1} \cdot R(\dot\sigma, \dot x)$$

by the additional conditions imposed on $(\delta_i^q / \min \alpha_i)$ and $(\max \alpha_i / \min \alpha_i)$. Therefore $(\hat\sigma, \hat x)$ is a $\mathscr{R}$-minimizing solution. Moreover, since $\mathbf{1} \cdot R(\hat\sigma, \hat x) = \mathbf{1} \cdot R(\dot\sigma, \dot x)$, $\lim_{i\to\infty} \mathbf{1} \cdot R\left(\sigma_{i_j}, x_{i_j}\right) = \mathbf{1} \cdot R(\hat\sigma, \hat x)$, from which it can be deduced that $R\left(\sigma_{i_j}, x_{i_j}\right) \to R(\hat\sigma, \hat x)$. □

Proof of (ii). We use the subsequence-subsequence argument. □

3.3.4 Optimal parameter

Here we formulate a bilevel optimization problem to select an optimal regularization parameter α and prove that this problem has a solution. The general idea is to find a optimal parameter $\lambda \in \Lambda(\bar y)$ by minimizing some distance bewteen the corresponding regularized solution $(x_\lambda, \sigma_\lambda)$ and a known solution (σ_e, x_e) to $\boldsymbol{I}$ with $\bar y$. This is known as the *learning* approach and (σ_e, x_e) with $\bar y$ form the so-called *training data* [52]. A set of exact observations and the corresponding solutions to $\boldsymbol{I}$ are considered as training data. Due to technical reasons, our attention is focused on the component α of the parameter λ.

We begin by defining the training data. Let $\left(\sigma_e^1, x_e^1, y_e^1\right), \ldots, \left(\sigma_e^M, x_e^M, y_e^M\right)$ be M elements in $L^\infty(\Omega) \times X^N \times Y^N$ such that each (σ_e^m, x_e^m) is a solution to $\boldsymbol{I}$ with y_e^m as exact observation; in other words,

$$\operatorname*{ess\,inf}_{\mathbf{x}\in\Omega} \sigma_e^m(\mathbf{x}) > 0 \quad \text{and} \quad \begin{aligned} A(\sigma_e^m, x_e^m) &= 0_{(Z^\star)^N} && \text{in } (Z^\star)^N \\ C(\sigma_e^m, x_e^m) &= y_e^m && \text{in } Y^N \end{aligned} \qquad m = 1, \ldots, M.$$

Now, $\sigma_e^1, \ldots, \sigma_e^M$ are used to construct a distance function. Let $1 \le s < \infty$ and let $\phi_1, \ldots, \phi_T \in L^1(\Omega)$ be weigth functions. Let $\mathscr{L} : (L^\infty(\Omega))^M \to [0, \infty]$ be the distance function defined by

$$\mathscr{L}\left(\sigma^1, \ldots, \sigma^M\right) := \sum_{m=1}^M \left\{ \frac{1}{s} \sum_{t=1}^T \left| \int_\Omega (\sigma^m - \sigma_e^m) \phi_t \, d\mathbf{x} \right|^s + \int_\Omega |D(\sigma^m - \sigma_e^m)| \right\}.$$

$\mathscr{L}$ is inspired by the neighborhood definition in the weak* topology of $L^\infty(\Omega)$ [21, Prop. 3.12] and the seminorm property of the total variation function. Fix a compact set K in $]0,\infty[^4$ and consider the bilevel optimization problem

$$\begin{aligned}&\min \mathscr{L}\left(\sigma_\alpha^1,\dots,\sigma_\alpha^M\right)\\ &\text{s.t.}\quad \alpha\in K,\\ &\qquad\left(\sigma_\alpha^m,x_\alpha^m\right)\ \text{is a solution to}\ \boldsymbol{I}_{\lambda_\alpha^m}\ \text{for}\ m=1,\dots,M.\end{aligned}\tag{$\boldsymbol{J}$}$$

$\boldsymbol{J}$ is called the *learning* problem and is associated to the regularized problem $\boldsymbol{I}_\lambda$, which can be either $\boldsymbol{I}_\lambda^1$, $\boldsymbol{I}_\lambda^2$, or $\boldsymbol{I}_\lambda^3$. The regularized problems $\boldsymbol{I}_{\lambda_\alpha^1},\dots,\boldsymbol{I}_{\lambda_\alpha^M}$ are instances of $\boldsymbol{I}_\lambda$ with parameters $\lambda_\alpha^1\in\Lambda\left(y_e^1\right),\dots,\lambda_\alpha^M\in\Lambda\left(y_e^M\right)$, respectively. Specifically, each λ_α^m is given by

$$\lambda_\alpha^m:=\begin{cases}(\alpha,\delta^m,y^m) & \text{if } \boldsymbol{I}_\lambda=\boldsymbol{I}_\lambda^1,\boldsymbol{I}_\lambda^2\\ (\alpha,\beta^m,\delta^m,y^m) & \text{if } \boldsymbol{I}_\lambda=\boldsymbol{I}_\lambda^3\end{cases}.$$

where $\beta^1,\dots,\beta^M$ are fixed maximum model error levels and $\left(\delta^1,y^1\right),\dots,\left(\delta^M,y^M\right)$ are fixed noise level-noisy obsevation pairs satisfying

$$0\le\beta^1,\dots,\beta^M\quad\text{and}\quad\left\|y_e^1-y^1\right\|_q\le\delta^1,\dots,\left\|y_e^M-y^M\right\|_q\le\delta^M,$$

where $1\le q<\infty$ if $\boldsymbol{I}_\lambda=\boldsymbol{I}_\lambda^1,\boldsymbol{I}_\lambda^3$ and $q=\infty$ if $\boldsymbol{I}_\lambda=\boldsymbol{I}_\lambda^2$.

To use our previous results, assume the following.

A7. For each m assume that $(\sigma_e^m,x_e^m)\in L^\infty(\Omega)\times\tilde{X}^N$, $\sigma_-\le\sigma_e^m\le\sigma_+$ a.e. on Ω, and $\int_\Omega|D\sigma_e^m|<\infty$.

Proposition 3.19. *The learning problem* $\boldsymbol{J}$ *has a solution.*

Proof. First, the existence result (Proposition 3.12) is used to prove that there exists a minimizing sequence. Next, the stability result (Proposition 3.15) is used to prove that there exists a convergent subsequence, whose limit will be a minimizer of $\boldsymbol{J}$.

By Proposition 3.12 (existence), the admissible set of $\boldsymbol{J}$, namely

$$\mathcal{A}:=\left\{\left(\alpha,\left(\sigma_\alpha^1,x_\alpha^1\right),\dots,\left(\sigma_\alpha^M,x_\alpha^M\right)\right)\,\middle|\,\begin{array}{l}\alpha\in K\\ (\sigma_\alpha^m,x_\alpha^m)\ \text{solves}\ \boldsymbol{I}_{\lambda_\alpha^m}\ \text{for}\ m=1,\dots,M\end{array}\right\},$$

is not empty. Let $\left(\alpha,\left(\sigma_\alpha^1,x_\alpha^1\right),\dots,\left(\sigma_\alpha^M,x_\alpha^M\right)\right)$ be an element of $\mathcal{A}$. Since each σ_α^m is a $L^\infty(\Omega)$-function of bounded variation (see Remark 3.14) as well as each σ_e^m (by Assumption A7), it follows that there exist a $0\le d<\infty$ such that

$$d=\inf\left\{\mathscr{L}\left(\sigma_\alpha^1,\dots,\sigma_\alpha^M\right)\,\middle|\,\left(\alpha,\left(\sigma_\alpha^1,x_\alpha^1\right),\dots,\left(\sigma_\alpha^M,x_\alpha^M\right)\right)\in\mathcal{A}\right\}$$

and a sequence $\left(\alpha_i,\left(\sigma_{\alpha_i}^1,x_{\alpha_i}^1\right),\dots,\left(\sigma_{\alpha_i}^M,x_{\alpha_i}^M\right)\right)$ in $\mathcal{A}$ such that

$$\lim_{i\to\infty}\mathscr{L}\left(\sigma_{\alpha_i}^1,\dots,\sigma_{\alpha_i}^M\right)=d.\tag{3.19}$$

Since the sequence of regularization parameters (α_i) is in K, there exists a subsequence, still denoted by (α_i), and an $\tilde{\alpha} \in K$ such that $\alpha_i \to \tilde{\alpha}$. So, we have

$$\lambda_{\alpha_i}^m = \left(\alpha_i, \dots^{(m)}\right) \to \lambda_{\tilde{\alpha}}^m = \left(\tilde{\alpha}, \dots^{(m)}\right) \quad \text{for } m = 1, \dots, M.$$

Applying Proposition 3.15(*ii*) M times (observe that each sequence $\left(\lambda_{\alpha_i}^m\right)$ satisfies trivially the necessary conditions of this proposition), obtaining at the m time a subsequence of $\left(\lambda_{\alpha_i}^m\right)$, to form, with its indices, a subsequence of $\left(\lambda_{\alpha_i}^{m+1}\right)$, we show that there exist subsequences $\left(\sigma_{\alpha_{i_j}}^1, x_{\alpha_{i_j}}^1\right), \dots, \left(\sigma_{\alpha_{i_j}}^M, x_{\alpha_{i_j}}^M\right)$ and

$$\left(\sigma_{\tilde{\alpha}}^1, x_{\tilde{\alpha}}^1\right), \dots, \left(\sigma_{\tilde{\alpha}}^M, x_{\tilde{\alpha}}^M\right) \in L^\infty(\Omega) \times \tilde{X}^N,$$

such that

$$\begin{cases} \left(\sigma_{\alpha_{i_j}}^m, x_{\alpha_{i_j}}^m\right) \overset{\tau}{\rightharpoonup} \left(\sigma_{\tilde{\alpha}}^m, x_{\tilde{\alpha}}^m\right) \\ \left(\sigma_{\tilde{\alpha}}^m, x_{\tilde{\alpha}}^m\right) \text{ solves } \boldsymbol{I}_{\lambda_{\tilde{\alpha}}^m} \end{cases} \quad \text{for each } m = 1, \dots, M.$$

Hence $\left(\tilde{\alpha}, \left(\sigma_{\tilde{\alpha}}^1, x_{\tilde{\alpha}}^1\right), \dots, \left(\sigma_{\tilde{\alpha}}^M, x_{\tilde{\alpha}}^M\right)\right) \in \mathcal{A}$. Moreover, for each m, since $\sigma_{i_j}^m \overset{\star}{\rightharpoonup} \sigma_{\tilde{\alpha}}^m$ in the weak* topology of $L^\infty(\Omega)$, we obtain

$$\int_\Omega \left(\sigma_{\alpha_{i_j}}^m - \sigma_e^m\right) \phi_t \, d\mathbf{x} \to \int_\Omega \left(\sigma_{\tilde{\alpha}}^m - \sigma_e^m\right) \phi_t \, d\mathbf{x} \quad \text{for } t = 1, \dots, T$$

and

$$\int_\Omega \left|D\left(\sigma_{\tilde{\alpha}}^m - \sigma_e^m\right)\right| \le \liminf_{j \to \infty} \int_\Omega \left|D\left(\sigma_{\alpha_{ij}}^m - \sigma_e^m\right)\right|$$

(proceed as in the proof of Lemma 3.8(*ii*)). Thus, using (3.19) and applying limit inferior properties, we conclude that

$$\mathscr{L}\left(\sigma_{\tilde{\alpha}}^1, \dots, \sigma_{\tilde{\alpha}}^M\right) \le d.$$

Therefore, $\left(\tilde{\alpha}, \left(\sigma_{\tilde{\alpha}}^1, x_{\tilde{\alpha}}^1\right), \dots, \left(\sigma_{\tilde{\alpha}}^M, x_{\tilde{\alpha}}^M\right)\right)$ is a minimizer of $\boldsymbol{J}$. □

Remark 3.20. It is worth pointing out that the above arguments do not be apply to find optimals β and δ because the stability result requires conditions on sequences of these parameters, which are not satisfied in general by subsequences coming from compact sets such as K.

3.4 The complete electrode model

In this section, the *complete electrode model* [92] is presented and three instances of Assumptions A1-A5 are proposed using the equations of this model. Then, for each instance, the inverse problem and its regularizations are formulated.

To provide this example, we introduce the following notations. ν denotes the outward unit normal to $\partial\Omega$. Let M be an integer and $\mathcal{E}_1, \dots, \mathcal{E}_M$ be open connected subsets of $\partial\Omega$

such that $\overline{\mathcal{E}_i} \cap \overline{\mathcal{E}_j} = \emptyset$ for $i \neq j$, and if $d = 3$, the boundary of each $\mathcal{E}_m$ is a smooth curve on $\partial\Omega$ (recall that Ω is a domain in $\mathbb{R}^d$, $d = 2, 3$). $|\mathcal{E}_m|$ denotes the area of $\mathcal{E}_m$. $H^1(\Omega)$ and $H^2(\Omega)$ denote the usual Sobolev spaces over Ω. $H^{1+s}(\Omega)$ denotes the fractional Sobolev space [34, Ch. 4]

$$H^{1+s}(\Omega) = \left\{ u \in H^1(\Omega) \, \middle| \, \int_\Omega \int_\Omega \frac{\left| \frac{\partial u}{\partial x_i}(\mathbf{x}) - \frac{\partial u}{\partial x_i}(\mathbf{y}) \right|^2}{|\mathbf{x} - \mathbf{y}|^{2s+d}} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} < \infty, \, i = 1, \ldots, d \right\},$$

with $0 < s < 1$. Let $\mathbb{R}^M_\diamond$ be the subspace of vectors with zero mean value

$$\mathbb{R}^M_\diamond := \left\{ U \in \mathbb{R}^M \, \middle| \, \sum_{m=1}^M U_m = 0 \right\}.$$

$H^{1/2}(\mathcal{E}_m)$ denotes the space of traces on $\mathcal{E}_m$ and $\gamma_m : H^1(\Omega) \to H^{1/2}(\mathcal{E}_m)$ denotes the trace operator on $\mathcal{E}_m$, for $m = 1, \ldots, M$. Recall that Ω represents a conducting body. The subsets $\mathcal{E}_1, \ldots, \mathcal{E}_M$ represent M electrodes attached on the surface $\partial\Omega$.

Let $z_1, \ldots, z_M$ be positive constants which represent the contact impedances associated to the electrodes. Denoting by σ the internal conductivity of Ω, the equations of the complete electrode model for the electric potential (u, U) are

$$\nabla \cdot (\sigma \nabla u) = 0 \qquad \text{in } \Omega \tag{3.20}$$

$$\sigma \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \setminus \bigcup_{m=1}^M \mathcal{E}_m \tag{3.21}$$

$$u + z_m \sigma \frac{\partial u}{\partial \nu} = U_m \qquad \text{on } \mathcal{E}_m, \, m = 1, \ldots, M \tag{3.22}$$

with

$$\int_{\mathcal{E}_m} \sigma \frac{\partial u}{\partial \nu} \, \mathrm{d}\mathbf{s} = I_m \quad m = 1, \ldots, M \tag{3.23}$$

if a current pattern $I = (I_1, \ldots, I_M) \in \mathbb{R}^M_\diamond$ is applied, or with

$$U_m = V_m \tag{3.24}$$

if a voltage pattern $V = (V_1, \ldots, V_M) \in \mathbb{R}^M$ is applied.

Three weak formulations of this model will be considered in the following subsections, and for each of them, one instance of Assumptions A1-A5 will be proposed.

3.4.1 Formulation with applied current

We begin by considering equations (3.20)-(3.22), and (3.23), which determine the problem of finding the electric potential (u, U) when a current pattern $I \in \mathbb{R}^M_\diamond$ is applied through electrodes $\mathcal{E}_1, \ldots, \mathcal{E}_M$. The weak formulation of this problem is written as:

$$\begin{cases} \text{find } (u, U) \in H^1(\Omega) \times \mathbb{R}^M_\diamond \text{ satisfying} \\ \displaystyle\int_\Omega \sigma \nabla u \cdot \nabla w \,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{(\gamma_m u - U_m)(\gamma_m w - W_m)}{z_m} \,\mathrm{d}\mathbf{s} = \sum_{m=1}^M I_m W_m \\ \text{for all } (w, W) \in H^1(\Omega) \times \mathbb{R}^M_\diamond. \end{cases} \tag{3.25}$$

The following set of assumptions allow us to consider (3.25) as model equation. Let $\bar\sigma \in L^\infty(\Omega)$ be a conductivity such that $\operatorname{ess\,inf}_{\mathbf{x}\in\Omega} \bar\sigma(\mathbf{x}) > 0$. Suppose that N current patterns $I^1, \ldots I^N \in \mathbb{R}^M_\diamond$ are applied through the electrodes $\mathcal{E}_1, \ldots, \mathcal{E}_M$. Then, the electric potentials

$$\left(\bar u^1, \bar U^1\right), \ldots, \left(\bar u^N, \bar U^N\right) \in H^1(\Omega) \times \mathbb{R}^M_\diamond$$

are obtained, where $\bar U^1, \ldots, \bar U^N$ are the resulting voltages on the electrodes. In other words, each $\left(\bar u^n, \bar U^n\right)$ is the unique solution to (3.25) with $\sigma = \bar\sigma$ and $I = I^n$. Consider the following instance of Assumptions A1-A5:

A1. $X, Z := H^1(\Omega) \times \mathbb{R}^M_\diamond$ and $b^n \in Z^\star$ defined by $b^n(w, W) := \sum_{m=1}^M I^n_m W_m$.

A2. $a : L^\infty(\Omega) \times X \to Z^\star$ defined by

$$a(\sigma, (u, U))(w, W) := \int_\Omega \sigma \nabla u \cdot \nabla w \,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{(\gamma_m u - U_m)(\gamma_m w - W_m)}{z_m} \,\mathrm{d}\mathbf{s}.$$

A3-4. We provide three possibilities:

(i) *Voltage measurements.* $Y := \mathbb{R}^M$, $\bar y^n := \bar U^n \in \mathbb{R}^M_\diamond$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma, (u, U)) := U$.

(ii) *Magnitudes of current density field.* $Y := L^2(\Omega)$, $\bar y^n := \bar\sigma |\nabla \bar u^n| \in L^2(\Omega)$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma, (u, U)) := \sigma |\nabla u|$.

(iii) *Interior power density data.* $Y := L^1(\Omega)$, $\bar y^n := \bar\sigma |\nabla \bar u^n|^2 \in L^1(\Omega)$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma, (u, U)) := \sigma |\nabla u|^2$.

A5. $\tilde X := H^{1+s}(\Omega) \times \mathbb{R}^M_\diamond$ with some $0 < s \le 1$.

Inverse problem ($N = 1$)**.** With the above assumptions, the model equation

$$A(\sigma, (u, U)) = 0_{Z^\star} \text{ in } Z^\star = \left(H^1(\Omega) \times \mathbb{R}^M_\diamond\right)^\star$$

reads as

$$\begin{cases} \displaystyle\int_\Omega \sigma \nabla u \cdot \nabla w \,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{(\gamma_m u - U_m)(\gamma_m w - W_m)}{z_m} \,\mathrm{d}\mathbf{s} = \sum_{m=1}^M I_m W_m \\ \text{for all } (w, W) \in Z = H^1(\Omega) \times \mathbb{R}^M_\diamond \end{cases}$$

and the observation equation $C(\sigma, (u, U)) = \bar y$ in Y reads as

$$U = \bar U \quad \text{in } Y = \mathbb{R}^M \tag{i}$$

$$\sigma |\nabla u| = \bar\sigma |\nabla \bar u| \quad \text{in } Y = L^2(\Omega) \tag{ii}$$

$$\sigma |\nabla u|^2 = \bar\sigma |\nabla \bar u|^2 \quad \text{in } Y = L^1(\Omega) \tag{iii}$$

for (i) voltage measurements, (ii) magnitudes of current density field, (iii) interior power density data.

Remark 3.21. One can consider $X = H^1(\Omega) \times \mathbb{R}^M$ when voltage measurements are choosen as observations, that is, when (i) is the observation equation. This is because any solution $(\sigma, (u, U))$ of (i) satisfies $U \in \mathbb{R}^M_\diamond$ (it is known that $\bar{U} \in \mathbb{R}^M_\diamond$). Therefore, A and C need not be defined on $X = H^1(\Omega) \times \mathbb{R}^M_\diamond$ a priori. In this case, $Z \neq X$ and, although $\tilde{X} = H^{1+s}(\Omega) \times \mathbb{R}^M_\diamond \to X$ remains compact, $\tilde{X} = H^{1+s}(\Omega) \times \mathbb{R}^M$ can be choosen.

Regularizations. The regularizations $\boldsymbol{I}^1_\lambda$, $\boldsymbol{I}^2_\lambda$, $\boldsymbol{I}^3_\lambda$ are formulated below. Recall that the regularizations are defined on $L^\infty(\Omega) \times \tilde{X}^N$, with $\tilde{X} = H^{1+s}(\Omega) \times \mathbb{R}^M_\diamond$.

- $\boldsymbol{I}^1_\lambda$-regularization ($p = 2$, $1 \le q < \infty$):

$$\min_{(\sigma,x)\in L^\infty(\Omega)\times\tilde{X}^N} \frac{1}{2}\|A(\sigma,x)\|_2^2 + \frac{1}{q}\|C(\sigma,x) - y\|_q^q + \alpha \cdot R(\sigma,x)$$
$$\text{s.t.} \quad \sigma_- \le \sigma \le \sigma_+ \text{ a.e. on } \Omega$$

- $\boldsymbol{I}^2_\lambda$-regularization ($p = 2$, $q = \infty$):

$$\min_{(\sigma,x)\in L^\infty(\Omega)\times\tilde{X}^N} \frac{1}{2}\|A(\sigma,x)\|_2^2 + \alpha \cdot R(\sigma,x)$$
$$\text{s.t.}$$
$$\sigma_- \le \sigma \le \sigma_+ \text{ a.e. on } \Omega$$
$$y^n_m - \gamma\delta \le U^n_m \le y^n_m + \gamma\delta \quad \forall\, m, n$$

- $\boldsymbol{I}^3_\lambda$-regularization ($p = \infty$, $1 \le q < \infty$, $\beta = 0$):

$$\min_{(\sigma,x)\in L^\infty(\Omega)\times\tilde{X}^N} \frac{1}{q}\|C(\sigma,x) - y\|_q^q + \alpha \cdot R(\sigma,x)$$
$$\text{s.t.}$$
$$\sigma \le \sigma \le \sigma_+ \text{ a.e. on } \Omega,$$
$$\int_\Omega \sigma\nabla u^n \cdot \nabla w \, d\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{(\gamma_m u^n - U^n_m)(\gamma_m w - W_m)}{z_m} \, d\mathbf{s} = \sum_{m=1}^M I^n_m W_m$$
$$\forall (w, W) \in H^1(\Omega) \times \mathbb{R}^M_\diamond, \forall\, n$$

where $x = \left((u^1, U^1), \ldots, (u^N, U^N)\right) \in \tilde{X}^N$ and $y = \left(y^1, \ldots, y^N\right) \in Y^N$ satisfies

$$\begin{cases} y^n \in \mathbb{R}^M,\ \bar{U}^n_m - \delta \le y^n_m \le \bar{U}^n_m + \delta \quad \forall\, m, n & q = \infty,\ \text{(i)} \\ y^n \in \mathbb{R}^M,\ \left(\sum_{n=1}^N \sum_{m=1}^M \left|\bar{U}^n_m - y^n_m\right|^2\right)^{1/2} \le \delta & q = 2,\ \text{(i)} \\ y^n \in L^2(\Omega),\ \left(\sum_{n=1}^N \int_\Omega |\bar{\sigma}|\nabla\bar{u}^n| - y^n|^2 \, d\mathbf{x}\right)^{1/2} \le \delta & q = 2,\ \text{(ii)} \\ y^n \in L^1(\Omega),\ \sum_{n=1}^N \int_\Omega \left|\bar{\sigma}|\nabla\bar{u}^n|^2 - y^n\right| d\mathbf{x} \le \delta & q = 1,\ \text{(iii)} \end{cases},$$

with

$$\frac{1}{2}\left\|A\left(\sigma,x\right)\right\|_2^2=\frac{1}{2}\sum_{n=1}^{N}\left\|a\left(\sigma,\left(u^n,U^n\right)\right)-b^n\right\|^2_{\left(H^1(\Omega)\times\mathbb{R}^M_\diamond\right)^\star}$$

as the model error,

$$\frac{1}{q}\left\|C\left(\sigma,x\right)-y\right\|_q^q=\begin{cases}\frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{M}\left|U^n_m-y^n_m\right|^2 & q=2,\ \text{(i)}\\ \frac{1}{2}\sum_{n=1}^{N}\int_\Omega\left|\sigma\left|\nabla u^n\right|-y^n\right|^2\,d\mathbf{x} & q=2,\ \text{(ii)}\\ \sum_{n=1}^{N}\int_\Omega\left|\sigma\left|\nabla u^n\right|^2-y^n\right|\,d\mathbf{x} & q=1,\ \text{(iii)}\end{cases}$$

as the observation error, and

$$R\left(\sigma,x\right)=\left(\frac{1}{2}\sum_{n=1}^{N}\left\|\left(u^n,U^n\right)\right\|^2_{H^{1+s}(\Omega)\times\mathbb{R}^M_\diamond},\dots\right)$$

as the regularization functional ($r=2$). We recall that if voltage measurements are chosen as observations, then $\tilde{X}=H^{1+s}\left(\Omega\right)\times\mathbb{R}^M_\diamond$ can be replaced with $\tilde{X}=H^{1+s}\left(\Omega\right)\times\mathbb{R}^M$, as was pointed in Remark 3.21. If this is the case, the component U^n_λ of a regularized solution $\left(\sigma_\lambda,\left(u^1_\lambda,U^1_\lambda\right),\dots,\left(u^N_\lambda,U^N_\lambda\right)\right)$ does not belong to $\mathbb{R}^M_\diamond$ in general.

3.4.2 Formulation with applied voltage

On the other hand, equations (3.20)-(3.22), and (3.24) determine the problem of finding the electric potential (u,U) when a voltage pattern $V\in\mathbb{R}^M$ is applied through electrodes $\mathcal{E}_1,\dots,\mathcal{E}_M$. The weak formulation of this problem is written as:

$$\begin{cases}\text{find } u\in H^1\left(\Omega\right)\ \text{satisfying}\\ \displaystyle\int_\Omega\sigma\nabla u\cdot\nabla w\,\mathrm{d}\mathbf{x}+\sum_{m=1}^{M}\int_{\mathcal{E}_m}\frac{\gamma_m u\,\gamma_m w}{z_m}\,\mathrm{d}\mathbf{s}=\sum_{m=1}^{M}\int_{\mathcal{E}_m}\frac{V_m\,\gamma_m w}{z_m}\,\mathrm{d}\mathbf{s}\\ \text{for all } w\in H^1\left(\Omega\right).\end{cases}\tag{3.26}$$

The above is a equation for u (from (3.24) $U=V$). The following set of assumptions allow us to consider (3.26) as model equation. Let $\bar{\sigma}\in L^\infty\left(\Omega\right)$ be a conductivity such that $\operatorname{ess\,inf}_{\mathbf{x}\in\Omega}\bar{\sigma}\left(\mathbf{x}\right)>0$. Suppose that N voltage patterns $V^1,\dots V^N\in\mathbb{R}^M$ are applied through the electrodes $\mathcal{E}_1,\dots,\mathcal{E}_M$. Then, N electric potentials

$$\bar{u}^1,\dots,\bar{u}^N\in H^1\left(\Omega\right)$$

are obtained and the resulting currents on the electrodes are given by

$$\left(\int_{\mathcal{E}_m}\frac{V^1_m-\gamma_m\bar{u}^1}{z_m}\,\mathrm{d}\mathbf{s}\right)_{m=1}^{M},\dots,\left(\int_{\mathcal{E}_m}\frac{V^N_m-\gamma_m\bar{u}^N}{z_m}\,\mathrm{d}\mathbf{s}\right)_{m=1}^{M}\in\mathbb{R}^M_\diamond.$$

In other words, each $\bar{u}^n$ is the unique solution to (3.26) with $\sigma=\bar{\sigma}$ and $V=V^n$. It is easy to check the resulting currents belong to $\mathbb{R}^M_\diamond$ by setting $w=1$ in (3.26). Consider the following instance of Assumptions A1-A5:

A1. $X, Z := H^1(\Omega)$ and $b^n \in Z^\star$ defined by $b^n(w) := \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{V_m^n \gamma_m w}{z_m} \,\mathrm{d}\mathbf{s}$.

A2. $a : L^\infty(\Omega) \times X \to Z^\star$ defined by

$$a(\sigma, u)(w) := \int_\Omega \sigma \nabla u \cdot \nabla w \,\mathrm{d}\mathbf{x} + \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{\gamma_m u\, \gamma_m w}{z_m} \,\mathrm{d}\mathbf{s}.$$

A3-4. We provide three possibilities:

(i) *Current measurements.* $Y := \mathbb{R}^M$,

$$\bar{y}^n := \left(\int_{\mathcal{E}_1} \frac{V_1^n - \gamma_1 \bar{u}^n}{z_1} \,\mathrm{d}\mathbf{s}, \ldots, \int_{\mathcal{E}_M} \frac{V_M^n - \gamma_M \bar{u}^n}{z_M} \,\mathrm{d}\mathbf{s} \right) \in \mathbb{R}^M_\diamond,$$

and $c^n : L^\infty(\Omega) \times X \to Y$ defined by

$$c^n(\sigma, u) := \left(\int_{\mathcal{E}_1} \frac{V_1^n - \gamma_1 u}{z_1} \,\mathrm{d}\mathbf{s}, \ldots, \int_{\mathcal{E}_M} \frac{V_M^n - \gamma_M u}{z_M} \,\mathrm{d}\mathbf{s} \right).$$

(ii) *Magnitudes of current density field* (as in the formulation with applied current)

(iii) *Interior power density data* (as in the formulation with applied current)

A5. $\tilde{X} := H^{1+s}(\Omega)$ with some $0 < s \le 1$.

Inverse problem ($N = 1$)**.** With the above assumptions, the model equation

$$A(\sigma, u) = 0_{Z^\star} \text{ in } Z^\star = \left(H^1(\Omega)\right)^\star$$

is given by

$$\begin{cases} \displaystyle\int_\Omega \sigma \nabla u \cdot \nabla w \,\mathrm{d}\mathbf{x} + \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{\gamma_m u\, \gamma_m w}{z_m} \,\mathrm{d}\mathbf{s} = \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{V_m \gamma_m w}{z_m} \,\mathrm{d}\mathbf{s} \\ \text{for all } w \in Z = H^1(\Omega) \end{cases}$$

and the observation equation $C(\sigma, u) = \bar{y}$ in Y is given by

$$\left(\int_{\mathcal{E}_m} \frac{V_m - \gamma_m u}{z_m} \,\mathrm{d}\mathbf{s} \right)_{m=1}^{M} = \left(\int_{\mathcal{E}_m} \frac{V_m - \gamma_m \bar{u}}{z_m} \,\mathrm{d}\mathbf{s} \right)_{m=1}^{M} \quad \text{in } Y = \mathbb{R}^M \qquad \text{(i)}$$

$$\sigma |\nabla u| = \bar{\sigma} |\nabla \bar{u}| \quad \text{in } Y = L^2(\Omega) \qquad \text{(ii)}$$

$$\sigma |\nabla u|^2 = \bar{\sigma} |\nabla \bar{u}|^2 \quad \text{in } Y = L^1(\Omega) \qquad \text{(iii)}$$

for (i) current measurements, (ii) magnitudes of current density field, (iii) interior power density data.

Regularizations. The regularizations $\boldsymbol{I}^1_\lambda$, $\boldsymbol{I}^2_\lambda$, $\boldsymbol{I}^3_\lambda$ are formulated below. Recall that the regularizations are defined on $L^\infty(\Omega) \times \tilde{X}^N$, with $\tilde{X} = H^{1+s}(\Omega)$.

- $\boldsymbol{I}^1_\lambda$-regularization ($p = 2$, $1 \le q < \infty$):

$$\min_{(\sigma,x) \in L^\infty(\Omega) \times \tilde{X}^N} \frac{1}{2} \|A(\sigma, x)\|_2^2 + \frac{1}{q} \|C(\sigma, x) - y\|_q^q + \alpha \cdot R(\sigma, x)$$
$$\text{s.t.} \quad \sigma_- \le \sigma \le \sigma_+ \text{ a.e. on } \Omega$$

- $\boldsymbol{I}_\lambda^2$-regularization ($p = 2$, $q = \infty$):

$$\begin{aligned}
&\min_{(\sigma,x)\in L^\infty(\Omega)\times\tilde{X}^N} \frac{1}{2}\left\|A\left(\sigma,x\right)\right\|_2^2 + \alpha\cdot R\left(\sigma,x\right)\\
&\text{s.t.}\\
&\sigma_- \leq \sigma \leq \sigma_+ \text{ a.e. on } \Omega\\
&y_m^n - \gamma\delta \leq \int_{\mathcal{E}_m} \frac{V_m^n - \gamma_m u^n}{z_m}\,\mathrm{d}\mathbf{s} \leq y_m^n + \gamma\delta \quad \forall\, m,n
\end{aligned}$$

- $\boldsymbol{I}_\lambda^3$-regularization ($p = \infty$, $1 \leq q < \infty$, $\beta = 0$):

$$\begin{aligned}
&\min_{(\sigma,x)\in L^\infty(\Omega)\times\tilde{X}^N} \frac{1}{q}\left\|C\left(\sigma,x\right) - y\right\|_q^q + \alpha\cdot R\left(\sigma,x\right)\\
&\text{s.t.}\\
&\sigma \leq \sigma \leq \sigma_+ \text{ a.e. on } \Omega\\
&\int_\Omega \sigma\nabla u^n\cdot\nabla w\,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{\gamma_m u^n \gamma_m w}{z_m}\,\mathrm{d}\mathbf{s} = \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{V_m^n \gamma_m w}{z_m}\,\mathrm{d}\mathbf{s}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, w \in H^1\left(\Omega\right),\ \forall\, n
\end{aligned}$$

where $x = \left(u^1,\ldots,u^N\right) \in \tilde{X}^N$ and $y = \left(y^1,\ldots,y^N\right) \in Y^N$ satisfies

$$\begin{cases}
y^n \in \mathbb{R}^M,\ \int_{\mathcal{E}_m} \frac{V_m^n - \gamma_m \bar{u}^n}{z_m}\,\mathrm{d}\mathbf{s} - \delta \leq y_m^n \leq \int_{\mathcal{E}_m} \frac{V_m^n - \gamma_m \bar{u}^n}{z_m}\,\mathrm{d}\mathbf{s} + \delta \quad \forall\, m,n & q = \infty,\ \text{(i)}\\
y^n \in \mathbb{R}^M,\ \left(\sum_{n=1}^N \sum_{m=1}^M \left|\int_{\mathcal{E}_m} \frac{V_m^n - \gamma_m \bar{u}^n}{z_m}\,\mathrm{d}\mathbf{s} - y_m^n\right|^2\right)^{1/2} \leq \delta & q = 2,\ \text{(i)}\\
y^n \in L^2\left(\Omega\right),\ \left(\sum_{n=1}^N \int_\Omega \left|\bar{\sigma}\left|\nabla \bar{u}^n\right| - y^n\right|^2\,\mathrm{d}\mathbf{x}\right)^{1/2} \leq \delta & q = 2,\ \text{(ii)}\\
y^n \in L^1\left(\Omega\right),\ \sum_{n=1}^N \int_\Omega \left|\bar{\sigma}\left|\nabla \bar{u}^n\right|^2 - y^n\right|\,\mathrm{d}\mathbf{x} \leq \delta & q = 1,\ \text{(iii)}
\end{cases},$$

with

$$\frac{1}{2}\left\|A\left(\sigma,x\right)\right\|_2^2 = \frac{1}{2}\sum_{n=1}^N \left\|a\left(\sigma,u^n\right) - b^n\right\|_{(H^1(\Omega))^\star}^2$$

as the model error,

$$\frac{1}{q}\left\|C\left(\sigma,x\right) - y\right\|_q^q = \begin{cases}
\frac{1}{2}\sum_{n=1}^N \sum_{m=1}^M \left|\int_{\mathcal{E}_m} \frac{V_m^n - \gamma_m u^n}{z_m}\,\mathrm{d}\mathbf{s} - y_m^n\right|^2 & q = 2,\ \text{(i)}\\
\frac{1}{2}\sum_{n=1}^N \int_\Omega \left|\sigma\left|\nabla u^n\right| - y^n\right|^2\,\mathrm{d}\mathbf{x} & q = 2,\ \text{(ii)}\\
\sum_{n=1}^N \int_\Omega \left|\sigma\left|\nabla u^n\right|^2 - y^n\right|\,\mathrm{d}\mathbf{x} & q = 1,\ \text{(iii)}
\end{cases}$$

as the observation error, and

$$R\left(\sigma,x\right) = \left(\frac{1}{2}\sum_{n=1}^N \left\|u^n\right\|_{H^{1+s}(\Omega)}^2,\ldots\right)$$

as the regularization functional ($r = 2$). Given a regularized solution $\left(\sigma_\lambda, \left(u_\lambda^1,\ldots,u_\lambda^N\right)\right)$, note that we not have $\left(\frac{V_m^n - \gamma_m u_\lambda^n}{z_m}\,\mathrm{d}\mathbf{s}\right)_{m=1}^M \in \mathbb{R}_\diamond^M$ in general, even if $V^n \in \mathbb{R}_\diamond^M$.

3.4.3 Alternative formulation

An alternative weak formulation is proposed below. Equations (3.20)-(3.22) can be weakly formulated as the the following problem:

$$\begin{cases} \text{find } (u,U) \in H^1(\Omega) \times \mathbb{R}^M \text{ satisfying} \\ \displaystyle\int_\Omega \sigma \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x} + \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{(\gamma_m u - U_m)\,\gamma_m w}{z_m} \, \mathrm{d}\mathbf{s} = 0 \\ \text{for all } w \in H^1(\Omega). \end{cases} \tag{3.27}$$

In (3.27), the equations involving current and voltage application are left out. Note that (3.27) has infinite solutions and does not represent any physical phenomenon. Thus, to formulate adequately the EIT inverse problem using (3.27) as model equation, current and voltage data will be added as observations. Let $\bar{\sigma} \in L^\infty(\Omega)$ be a conductivity such that $\operatorname{ess\,inf}_{\mathbf{x}\in\Omega} \bar{\sigma}(\mathbf{x}) > 0$. From the previous cases, we see that there are two possibilities: apply N current patterns $I^1, \dots I^N \in \mathbb{R}^M_\diamond$ and obtain the voltage generated on the electrodes, or apply N voltage patterns $V^1, \dots V^N \in \mathbb{R}^M$ and obtain the current generated on the electrodes. In the first case, the voltage-current pairs

$$\left(\bar{U}^1, I^1\right), \dots, \left(\bar{U}^N, I^N\right)$$

are available, where each $\left(\bar{u}^n, \bar{U}^n\right)$ is the unique solution to (3.25) with $\sigma = \bar{\sigma}$ and $I = I^n$. In the second case, the voltage-current pairs

$$\left(V^1, \left(\int_{\mathcal{E}_m} \frac{V^1_m - \gamma_m \bar{u}^1}{z_m} \, \mathrm{d}\mathbf{s}\right)_{m=1}^{M}\right), \dots, \left(V^N, \left(\int_{\mathcal{E}_m} \frac{V^N_m - \gamma_m \bar{u}^N}{z_m} \, \mathrm{d}\mathbf{s}\right)_{m=1}^{M}\right)$$

are available, where each $\bar{u}^n$ is the unique solution to (3.26) formulated with $\sigma = \bar{\sigma}$ and $V = V^n$. With this, consider the following instance of Assumptions A1-A5:

A1. $X := H^1(\Omega) \times \mathbb{R}^M$, $Z := H^1(\Omega)$, and $b^1, \dots, b^N = 0_{Z^\star}$.

A2. $a : L^\infty(\Omega) \times X \to Z^\star$ defined by

$$a(\sigma, (u,U))(w) := \int_\Omega \sigma \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x} + \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{(\gamma_m u - U_m)\,\gamma_m w}{z_m} \, \mathrm{d}\mathbf{s}.$$

A3. $Y := \mathbb{R}^M \times \mathbb{R}^M$ and any of the two sets of voltage-current pairs below:

(i) *Measured voltage-applied current pairs*:

$$\bar{y}^n := \left(\bar{U}^n, I^n\right) \in \mathbb{R}^M_\diamond \times \mathbb{R}^M_\diamond.$$

(ii) *Applied voltage-measured current pairs*:

$$\bar{y}^n := \left(V^n, \left(\int_{\mathcal{E}_m} \frac{V^n_m - \gamma_m \bar{u}^n}{z_m} \, \mathrm{d}\mathbf{s}\right)_{m=1}^{M}\right) \in \mathbb{R}^M \times \mathbb{R}^M_\diamond.$$

A4. $c^n : L^\infty(\Omega) \times X \to Y$ defined by

$$c^n(\sigma,(u,U)) := \left(U, \left(\int_{\mathcal{E}_m} \frac{U_m - \gamma_m u}{z_m}\,\mathrm{d}\mathbf{s}\right)_{m=1}^{m}\right).$$

A5. $\tilde{X} := H^{1+s}(\Omega) \times \mathbb{R}^M$ with some $0 < s \le 1$.

Inverse problem ($N = 1$)**.** With the above assumptions, the model equation

$$A(\sigma,(u,U)) = 0_{Z^\star} \text{ in } Z^\star = \left(H^1(\Omega)\right)^\star$$

is given by

$$\begin{cases} \displaystyle\int_\Omega \sigma\nabla u\cdot\nabla w\,\mathrm{d}\mathbf{x} + \sum_{m=1}^{M}\int_{\mathcal{E}_m} \frac{(\gamma_m u - U_m)\,\gamma_m w}{z_m}\,\mathrm{d}\mathbf{s} = 0 \\ \text{for all } w \in Z = H^1(\Omega) \end{cases}$$

and the observation equation $C(\sigma,(u,U)) = \bar{y}$ in $Y = \mathbb{R}^M \times \mathbb{R}^M$ is given by

$$\left(U, \left(\int_{\mathcal{E}_m} \frac{U_m - \gamma_m u}{z_m}\,\mathrm{d}\mathbf{s}\right)_{m=1}^{M}\right) = \begin{cases} (\bar{U}, I) & \text{(i)} \\ \left(V, \left(\int_{\mathcal{E}_m} \frac{V_m - \gamma_m \bar{u}}{z_m}\,\mathrm{d}\mathbf{s}\right)_{m=1}^{M}\right) & \text{(ii)} \end{cases}$$

for (i) measured voltages-applied currents, (ii) applied voltages-measured currents.

Remark 3.22. If either (i) or (ii) with V beloging to $\mathbb{R}^M_\diamond$ is chosen, then the exact observation $\bar{y}$ will be in $\mathbb{R}^M_\diamond \times \mathbb{R}^M_\diamond$. In this case, one can consider $X = H^1(\Omega) \times \mathbb{R}^M_\diamond$ instead of $X = H^1(\Omega) \times \mathbb{R}^M$. Thus, the component U of a solution $(\sigma,(u,U))$ would be in $\mathbb{R}^M_\diamond$ a priori. In this case $\tilde{X} = H^{1+s}(\Omega) \times \mathbb{R}^M_\diamond$ must be choosen.

Remark 3.23. We point out that the above system is a novel formulation of the EIT inverse problem. However, the idea of an observation operator for voltage and current is not original (see for instance [64, Subsec. 2.4])

Regularizations. The regularizations $\boldsymbol{I}^1_\lambda$, $\boldsymbol{I}^2_\lambda$, $\boldsymbol{I}^3_\lambda$ are formulated below. Recall that the regularizations are defined on $L^\infty(\Omega) \times \tilde{X}^N$, with $\tilde{X} = H^{1+s}(\Omega) \times \mathbb{R}^M$.

- $\boldsymbol{I}^1_\lambda$-regularization ($p = 2$, $q = 2$):

$$\begin{aligned} &\min_{(\sigma,x)\in L^\infty(\Omega)\times(H^{1+s}(\Omega)\times\mathbb{R}^M)^N} \frac{1}{2}\|A(\sigma,x)\|_2^2 + \frac{1}{2}\|C(\sigma,x) - y\|_2^2 + \alpha\cdot R(\sigma,x) \\ &\text{s.t.}\quad \sigma_- \le \sigma \le \sigma_+ \text{ a.e. on } \Omega \end{aligned}$$

- $\boldsymbol{I}^2_\lambda$-regularization ($p = 2$, $q = \infty$):

$$\begin{aligned} &\min_{(\sigma,x)\in L^\infty(\Omega)\times(H^{1+s}(\Omega)\times\mathbb{R}^M)^N} \frac{1}{2}\|A(\sigma,x)\|_2^2 + \alpha\cdot R(\sigma,x) \\ &\text{s.t.} \\ &\sigma_- \le \sigma \le \sigma_+ \text{ a.e. on } \Omega \\ &y^n_{\mathrm{vol},m} - \gamma\delta \le U^n_m \le y^n_{\mathrm{vol},m} + \gamma\delta \quad \forall\, m,n \\ &y^n_{\mathrm{cur},m} - \gamma\delta \le \int_{\mathcal{E}_m} \frac{U^n_m - \gamma_m u^n}{z_m}\,\mathrm{d}\mathbf{s} \le y^n_{\mathrm{cur},m} + \gamma\delta \quad \forall\, m,n \end{aligned}$$

- $\boldsymbol{I}^3_\lambda$-regularization ($p = \infty$, $q = 2$, $\beta = 0$):

$$\begin{aligned}
&\min_{(\sigma,x)\in L^\infty(\Omega)\times(H^{1+s}(\Omega)\times\mathbb{R}^M)^N} \frac{1}{2}\left\|C(\sigma,x)-y\right\|_2^2 + \alpha\cdot R(\sigma,x)\\
&\text{s.t.}\\
&\sigma_- \le \sigma \le \sigma_+ \text{ a.e. on } \Omega\\
&\int_\Omega \sigma\nabla u^n\cdot\nabla w\,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{(\gamma_m u^n - U_m)\,\gamma_m w}{z_m}\,\mathrm{d}\mathbf{s} = 0\\
&\qquad\qquad \forall\, w \in H^1(\Omega),\ \forall\, n
\end{aligned}$$

where $x = \left((u^1, U^1), \ldots, (u^N, U^N)\right) \in \tilde{X}^N$ and the noisy observation

$$y = \left((y^1_{\text{vol}}, y^1_{\text{cur}}), \ldots, (y^N_{\text{vol}}, y^N_{\text{cur}})\right) \in Y^N$$

($Y = \mathbb{R}^M \times \mathbb{R}^M$) satisfies

$$\begin{cases}
\left(\sum_{n=1}^N \left(\sum_{m=1}^M \left|\bar{U}^n_m - y^n_{\text{vol},m}\right|^2 + \sum_{m=1}^M \left|I^n_m - y^n_{\text{cur},m}\right|^2\right)\right)^{1/2} \le \delta & q = 2\\
\begin{array}{l}\bar{U}^n_m - \delta \le y^n_{\text{vol},m} \le \bar{U}^n_m + \delta\\ I^n_m - \delta \le y^n_{\text{cur},m} \le I^n_m + \delta\end{array} \quad \forall\, m, n & q = \infty
\end{cases},$$

if the voltage-current pairs of type (i) are available, or

$$\begin{cases}
\left(\sum_{n=1}^N \left(\sum_{m=1}^M \left|V^N_m - y^n_{\text{vol},m}\right|^2 + \sum_{m=1}^M \left|\int_{\mathcal{E}_m} \frac{V^n_m - \gamma_m \bar{u}^n}{z_m}\,d\mathbf{s} - y^n_{\text{cur},m}\right|^2\right)\right)^{1/2} \le \delta & q = 2\\
\begin{array}{c}V^n_m - \delta \le y^n_{\text{vol},m} \le V^n_m + \delta\\ \int_{\mathcal{E}_m} \frac{V^n_m - \gamma_m \bar{u}^n}{z_m}\,\mathrm{d}\mathbf{s} - \delta \le y^n_{\text{cur},m} \le \int_{\mathcal{E}_m} \frac{V^n_m - \gamma_m \bar{u}^n}{z_m}\,\mathrm{d}\mathbf{s} + \delta\end{array} \quad \forall\, m, n & q = \infty
\end{cases}$$

if the voltage-current pairs of type (ii) are available. Here, the model error, the observation error, and the regularization functional are given by

$$\frac{1}{2}\left\|A(\sigma,x)\right\|_2^2 = \frac{1}{2}\sum_{n=1}^N \left\|a\left(\sigma,(u^n,U^n)\right)\right\|^2_{(H^1(\Omega))^\star},$$

$$\frac{1}{2}\left\|C(\sigma,x)-y\right\|_2^2 = \frac{1}{2}\sum_{n=1}^N \left(\sum_{m=1}^M \left|U^n_m - y^n_{\text{vol},m}\right|^2 + \sum_{m=1}^M \left|\int_{\mathcal{E}_m} \frac{U^n_m - \gamma_m u^n}{z_m}\,\mathrm{d}\mathbf{s} - y^n_{\text{cur},m}\right|^2\right),$$

$$\text{and}\quad R(\sigma,x) = \left(\frac{1}{2}\sum_{n=1}^N \left\|(u^n,U^n)\right\|^2_{H^{1+s}(\Omega)\times\mathbb{R}^M}, \ldots\right).$$

It was chosen $r = 2$ in the definition of R. From Remark 3.22, for either noisy observations of type (i) or (ii) with all voltages V^n belonging to $\mathbb{R}^M_\diamond$, the regularizations can be as the above minimization problems but restrited to $L^\infty(\Omega) \times \left(H^{1+s}(\Omega) \times \mathbb{R}^M_\diamond\right)^N$. In this case, the component U^n_λ of a regularized solution $\left(\sigma_\lambda, (u^1_\lambda, U^1_\lambda), \ldots, (u^N_\lambda, U^N_\lambda)\right)$ belongs to $\mathbb{R}^M_\diamond$.

Remark 3.24. Although the "natural" space for voltage and current measurements is $\mathbb{R}^M_\diamond$, it was set $Y = \mathbb{R}^M$ as the space of observations of voltage and current in Subsections 3.4.1 and 3.4.2, respectively. Also, the space of voltage-current pairs defined in Subsection 3.4.3 was set to be $Y = \mathbb{R}^M \times \mathbb{R}^M$. One can see that these choices do not prevent the adequate representation of the EIT inverse problem by the all-at-once system $\boldsymbol{I}$. For intance, in Subsection 3.4.2, the observation equation corresponding to observations of current is

$$C(\sigma, u) = \left(\int_{\mathcal{E}_m} \frac{V_m - \gamma_m u}{z_m} \, \mathrm{d}\mathbf{s} \right)_{m=1}^M = \bar{y} \in \mathbb{R}^M_\diamond \quad \text{in} \quad Y = \mathbb{R}^M.$$

Since $\mathbb{R}^M_\diamond$ is a closed subspace of $\mathbb{R}^M$, a solution $(\tilde{\sigma}, \tilde{u})$ to it must satisfy

$$\left(\int_{\mathcal{E}_m} \frac{V_m - \gamma_m \tilde{u}}{z_m} \, \mathrm{d}\mathbf{s} \right)_{m=1}^M \in \mathbb{R}^M_\diamond,$$

which is what we expected. Note that the fact that $\mathbb{R}^M_\diamond$ is closed in $\mathbb{R}^M$ is crucial. Generalizing, if we have a closed subspace $\tilde{Y}$ of Y such that $\bar{y} \in \tilde{Y}$, then $C(\sigma, x) = \bar{y}$ in Y implies $C(\sigma, x) \in \tilde{Y}$. On the other hand, these choices of Y are consistent with the observation maps. Indeed, again in the above case, since $(\sigma, u) \in L^\infty(\Omega) \times H^1(\Omega)$, $C(\sigma, u)$ will not be in $\mathbb{R}^M_\diamond$ in general. As a consequence, noisy observations of current and voltage with non-zero mean value are allowed.

Remark 3.25. It is easy to verify the instances of Assumptions A1-A5 given in this section. To check the compactness of the inclusion operator $\left(\tilde{X}, \|\cdot\|_{\tilde{X}}\right) \to (X, \|\cdot\|_X)$ consider the compact embedding of $H^{1+s}(\Omega)$ into $H^1(\Omega)$ for $0 < s \leq 1$. To check the continuity property of the maps a and c consider the continuity of the trace operators on $\mathcal{E}_1, \dots, \mathcal{E}_M$ and the following fact: if $\sigma_i \overset{\star}{\rightharpoonup} \sigma$ in $L^\infty(\Omega)$ and $u_i \to u$ in $H^1(\Omega)$ then

$$\int_\Omega \sigma_i \nabla u_i \cdot \nabla w \, \mathrm{d}\mathbf{x} \to \int_\Omega \sigma \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x} \quad \text{for all } w \in H^1(\Omega),$$

$\sigma_i |\nabla u_i| \rightharpoonup \sigma |\nabla u|$ in $L^2(\Omega)$, and $\sigma_i |\nabla u_i|^2 \rightharpoonup \sigma |\nabla u|^2$ in $L^1(\Omega)$. For a proof of these statements see Propositions 3.31 and 3.32 in Appendix. On the other hand, regarding Assumption A6, we do not prove the existence of solutions in $L^\infty(\Omega) \times \tilde{X}^N$. It represents the problem of the existence of a solution to the EIT inverse problem, which is not the purpose of this work.

Remark 3.26. Note that it was not required to specify the norms of X, $\tilde{X}$, and Z. However, in the numerical treatment of the regularizations, we will define a particular inner product and therefore a norm in these spaces. On the other hand, here and in what follows, Y is equipped with its usual norm.

3.5 Numerical tests

Here we solve numerically the $\boldsymbol{I}^1_\lambda$-regularizations of EIT inverse problem formulated with the equations of the complete electrode model (3.20)-(3.24). Noisy observations of voltage measurements, current measurements, magnitudes of current density field, and interior power density data will be considered in the tests.

3.5.1 The method

The least-square type cost functional of $\boldsymbol{I}_\lambda^1$ suggests to use the *Gauss-Newton* method [18, 81, 65, 91, 54, 61, 66]: given a parameter $\lambda = (\alpha, \delta, y) \in \Lambda$ ($\alpha > 0$, $\delta \geq 0$, and $\|\bar{y} - y\|_q \leq \delta$) and an admissible initial guess (σ_0, x_0), we successively intend to solve the approximate regularized problem

$$\begin{aligned} &\min_{(\sigma,x)\in L^\infty(\Omega)\times\tilde{X}^N} T_{\lambda,k}(\sigma, x) \\ &\text{s.t.} \quad \sigma_- \leq \sigma \leq \sigma_+ \text{ a.e. on } \Omega \end{aligned} \tag{3.28}$$

with

$$\begin{aligned} T_{\lambda,k}(\sigma, x) := &\frac{1}{p} \left\|A(\sigma_k, x_k) + A'(\sigma_k, x_k)(\sigma - \sigma_k, x - x_k)\right\|_p^p + \\ &\frac{1}{q} \left\|C(\sigma_k, x_k) + C'(\sigma_k, x_k)(\sigma - \sigma_k, x - x_k) - y\right\|_q^q + \alpha \cdot R(\sigma, x), \end{aligned}$$

where A' and C' are the Fréchet derivatives of non-linear operators A and C, and (σ_k, x_k) is the minimizer obtained in the previous step. Since the arguments in the norms are the first order approximations of A and C at (σ_k, x_k), $T_{\lambda,k}$ is an approximation of the cost functional of $\boldsymbol{I}_\lambda^1$. We hope that the sequence of minimizers obtained from (3.28) converges to a solution of $\boldsymbol{I}_\lambda^1$.

The minimization problem (3.28) is in an infinite-dimensional setting. To obtain computable solutions, we discretize σ and x with piecewise constant and piecewise linear finite elements, respectively. This leads to the following finite-dimensional minimization problem

$$\begin{aligned} &\min_{(\sigma_h,x_h)\in L_h\times\tilde{X}_h^N} T_{\lambda,k,h}(\sigma_h, x_h) \\ &\text{s.t.} \quad \sigma_- \leq \sigma_h \leq \sigma_+ \end{aligned} \tag{3.29}$$

with

$$T_{\lambda,k,h}(\sigma_h, x_h) := \frac{1}{2} \left\langle \mathbf{P}_{\lambda,k,h}(\sigma_h, x_h), (\sigma_h, x_h)\right\rangle - \left\langle \mathbf{q}_{\lambda,k,h}, (\sigma_h, x_h)\right\rangle + \mathbf{r}_{\lambda,k,h},$$

where (σ_h, x_h), $T_{\lambda,k,h}$, $L_h \times \tilde{X}_h^N$ are the discretizations of (σ, x), $T_{\lambda,k}$, and $L^\infty(\Omega) \times \tilde{X}^N$, respectively. $\mathbf{P}_{\lambda,k,h}$ is a square matrix, $\mathbf{q}_{\lambda,k,h}$ is a vector, $\mathbf{r}_{\lambda,k,h}$ is a scalar, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product. Hence $T_{\lambda,k,h}$ is a quadratic function and each step of the Gauss-Newton method consists of solving the box constrained quadratic program (3.29).

The Gauss-Newton algorithm is given in Algorithm 3.1. There the line search tries to find a stepsize θ such that

$$\begin{aligned} \|A((\sigma_k, x_k) + \theta\Delta_k)\| &< \|A(\sigma_k, x_k)\| \text{ and} \\ \|C((\sigma_k, x_k) + \theta\Delta_k) - y\| &< \|C(\sigma_k, x_k) - y\| \end{aligned}$$

hold, where Δ_k is the direction of search. In this way we guarantee that the model and observation errors decrease in each iteration. There are two stopping criteria: one for the effort realized to find a suitable stepsize in the line-search part and the other for the decrease of the cost functional. In our numerical tests, the first stopping criterion is rarely reached.

3.5.2 Discretization

We begin by definning convenient norms. Recall that $X = Z = H^1(\Omega) \times \mathbb{R}^M_\diamond$ in the formulation with applied current and $X = Z = H^1(\Omega)$ in the formulation with applied voltage. In the alternative formulation, $Z = H^1(\Omega)$ and we choose $X = H^1(\Omega) \times \mathbb{R}^M_\diamond$ (see Remark 3.22). So, equipping the spaces $H^1(\Omega) \times \mathbb{R}^M_\diamond$ and $H^1(\Omega)$ with the inner products

$$\langle (u,U), (w,W) \rangle_{H^1(\Omega)\times\mathbb{R}^M_\diamond} := \int_\Omega \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} (\gamma_m u - U_m)(\gamma_m w - W_m) \, \mathrm{d}\mathbf{s}$$

and

$$\langle u, w \rangle_{H^1(\Omega)} := \int_\Omega \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \gamma_m u \, \gamma_m w \, \mathrm{d}\mathbf{s}$$

respectively, a norm on X and a dual norm on $Z^\star$ are obtained. Note these inner products are related to the weak formulations of complete electrode model [92, 98].

Assuming that Ω is a polygon, we define an admissible triangulation $\{\Omega_t\}_{t=1}^T$ in it, with maximum size $h = \max_{1\le t\le T} |\Omega_t|$ and nodes $\{\mathbf{x}_n\}_{n=1}^N$. The conductivity is discretized as the piecewise constant function $\sigma_h(\mathbf{x}) = \sum_{t=1}^T \sigma_t \mathbb{1}_{\Omega_t}(\mathbf{x})$ for $\mathbf{x} \in \Omega$, where $\mathbb{1}_{\Omega_t}$ is the indicador function of Ω_t and $(\sigma_1, \dots, \sigma_T)$ are the coordinates of σ_h. The electric potential is discretized as the piecewise linear function $u_h(\mathbf{x}) = \sum_{n=1}^N c_n^u \phi_n(\mathbf{x})$ for for $\mathbf{x} \in \Omega$, where ϕ_n is the continuous piecewise linear basis function defined by $\phi_i(\mathbf{x}_j) = 1$ if $i = j$ and $\phi_i(\mathbf{x}_j) = 0$ if $i \neq j$, and $(c_1^u, \dots, c_N^u)$ are the coordinates of u_h. We choose $\{\eta_m\}_{m=1}^{M-1}$, with

$$\eta_m = \left(0, \dots, 0, 1_{(m)}, -1_{(m+1)}, 0, \dots, 0\right) \in \mathbb{R}^M,$$

as a basis of $\mathbb{R}^M_\diamond$. Thus, the component U is expressed as $U = \sum_{m=1}^M c_m^U \eta_m$, where $\left(c_1^U, \dots, c_M^U\right)$ are the coordinates of U. For example, in the case of the alternative formulation, we work with the following finite-dimensional subspaces $L_h \subset L^\infty(\Omega)$, $\tilde{X}_h \subset \tilde{X} \subset X$, and $Z_h \subset Z$ defined by

$$L_h = \left\{ \mu = \sum_{t=1}^T \mu_t \mathbb{1}_{\Omega_t} \,\middle|\, (\mu_1, \dots, \mu_T) \in \mathbb{R}^T \right\},$$

$$\tilde{X}_h = \left\{ (w,W) = \sum_{n=1}^N c_n^w (\phi_n, 0) + \sum_{m=1}^{M-1} c_m^W (0, \eta_m) \,\middle|\, \begin{array}{l} (c^w, C^w) \in \mathbb{R}^{N+M-1} \\ c^w = (c_1^w, \dots, c_N^w) \\ C^w = \left(c_1^W, \dots, c_{M-1}^W\right) \end{array} \right\},$$

Algorithm 3.1 Algorithm to obtain a numerical solution of regularized problem $\boldsymbol{I}_\lambda^1$ by Gauss-Newton method.

Require: (σ_0, x_0), initial guess; `niter`, maximum number of iterations of the Gauss-Newton method; `eps`, minimum cost difference; `effort`, maximum number of iterations to find a suitable stepsize.

Compute the matrices $\mathbf{P}_0$ and $\mathbf{q}_0$ using the initial guess (σ_0, x_0)

Solve the box constrained quadratic program to obtain the first iterate

$$(\sigma_1, x_1) \in \arg\min \left\{ \frac{1}{2} \langle \mathbf{P}_0 (\sigma, x), (\sigma, x) \rangle - \langle \mathbf{q}_0, (\sigma, x) \rangle \mid \sigma_- \leq \sigma \leq \sigma_+ \right\}$$

for $k = 1, \ldots, \texttt{niter} - 1$ **do**

  Compute the matrices $\mathbf{P}_k$ and $\mathbf{q}_k$ using the iterate (σ_k, x_k)

  Solve the box constrained quadratic program

$$(\sigma_{\min}, x_{\min}) \in \arg\min \left\{ \frac{1}{2} \langle \mathbf{P}_k (\sigma, x), (\sigma, x) \rangle - \langle \mathbf{q}_k, (\sigma, x) \rangle \mid \sigma_- \leq \sigma \leq \sigma_+ \right\}$$

  Set $\Delta_k = (\sigma_{\min}, x_{\min}) - (\sigma_k, x_k)$ direction of search

  Initialize $\theta = 1$, $\theta_L = 0$, and $\theta_R = 1$

  Initialize $flag =$ False

  for $l = 0, \ldots, \texttt{effort}$ **do**

    if $\left\{ \begin{array}{c} \|A((\sigma_k, x_k) + \theta\Delta_k)\| \\ < \|A(\sigma_k, x_k)\| \end{array} \text{ and } \begin{array}{c} \|C((\sigma_k, x_k) + \theta\Delta_k) - y\| \\ < \|C(\sigma_k, x_k) - y\| \end{array} \right\}$ **then**

      Set $(\sigma_{k+1}, x_{k+1}) = (\sigma_k, x_k) + \theta\Delta_k$

      $flag =$ True

      Stop

    else

      Set $\theta_R = \theta$

      Compute a new $\theta = (\theta_L + \theta_R)/2$

    end if

  end for

  if $flag =$ False **then**

    Maximum effort achieved! Stop

  end if

  if $T_\lambda^1 (\sigma_k, x_k) - T_\lambda^1 (\sigma_{k+1}, x_{k+1}) < \texttt{eps}$ **then**

    Minimum decrease achieved! Stop

  end if

end for

$$\text{and} \qquad Z_h = \left\{ w = \sum_{n=1}^{N} c_n^w \phi_n \,\middle|\, (c_1^w, \ldots, c_N^w) \in \mathbb{R}^N \right\},$$

respectively. We choose $\tilde{X} = H^2(\Omega) \times \mathbb{R}^M_\diamond$ for the formulation with applied current and for the alternative formulation, and $\tilde{X} = H^2(\Omega)$ for the formulation with applied voltage. Since u_h is a piecewise linear function, we have $\|(u_h, U)\|_{H^2(\Omega)\times\mathbb{R}^M_\diamond} = \|(u_h, U)\|_{H^1(\Omega)\times\mathbb{R}^M_\diamond}$ and $\|u_h\|_{H^2(\Omega)} = \|u_h\|_{H^1(\Omega)}$. Setting $r = 2$, $s_1, s_2 = 1$, $\phi_t = \mathbb{1}_{\Omega_t}$, $\psi_t = |\Omega_t|^{-1/2}\,\mathbb{1}_{\Omega_t}$, and $\sigma' = \sigma'' = 0$ in the regularization functional R, we have

$$R(\sigma_h, (u_h, U)) = \begin{pmatrix} \frac{1}{2}\|(u_h, U)\|^2_{H^1(\Omega)\times\mathbb{R}^M_\diamond} \\ \frac{1}{2}\sum_{t,t'=1}^{T} |\sigma_t - \sigma_{t'}|\,|\partial\Omega_t \cap \partial\Omega_{t'}| \\ \sum_{t=1}^{T} |\sigma_t|\,|\Omega_t| \\ \sum_{t=1}^{T} |(\sigma_t - \sigma_-)(\sigma_+ - \sigma_t)|\,|\Omega_t| \end{pmatrix}$$

See [41, 26] for the expresion of $\int_\Omega |D\sigma_h|$.

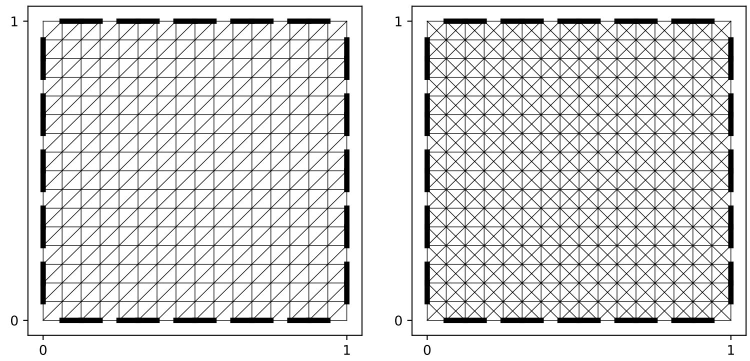
Remark 3.27 (Approximation of absolute value). Since the discrete forms of R and the L^1-norm are expressed in terms of absolute values, we must formulate these terms in a way that is more friendly to our quadratic optimization. Below is an explanation of how it was done. Consider the function $\phi(s) = s$ with $s \in [0, \infty]$. It can be written as the minimization problem

$$\phi(s) = \min_{0\le r\le\infty} \left\{ rs^2 + \psi(r) \right\},$$

where ψ is given by $\psi(0) = \infty$, $\psi(r) = 1/4r$ for $r \in]0, \infty[$, and $\psi(\infty) = 0$. It is easy to check that the minimizer $r_{\min} = r_{\min}(s)$ is given by $r_{\min}(0) = \infty$, $r_{\min}(s) = 1/2s$ for $s \in]0, \infty[$, and $r_{\min}(\infty) = 0$. Observe that $r_{\min}$ can be approximated with $r_{\min,\varepsilon}(s) = \left(s^2 + \varepsilon\right)^{-1/2}/2$, where ε is a positive small number. Moreover, by the minimization based formulation of ϕ, we have

$$s \le r_{\min,\varepsilon}(s_0) \cdot s^2 + \psi(r_{\min,\varepsilon}(s_0)) \quad \text{for all } s, s_0 \in [0, \infty].$$

In our discrete minimization problem, we deal with terms of the form $|f(v)|$, where f is a linear real-valued function and v is a vector whose components are the coordinates of the discretized variables. In each iteration of the Gauss-Newton method we choose to minimize the quadratic function $r_{\min,\varepsilon}(|f(v_k)|) \cdot (f(v))^2 + \psi(|f(v_k)|)$, which is a upper approximation of $|f(v)|$ by the above inequality, where v_k contains the coordinates of the solution obtained in the previous iteration. This approach was based on ideas from [100, 12, 1].



(a) (b)

Figure 3.1: (a) Mesh used for reconstruction from voltage and current measurements. (b) Mesh used for reconstruction from magnitudes of current density field and interior power density data. In both meshes, the thick lines represent the location of the electrodes.

3.5.3 Tests

All tests consider the domain $\Omega =]0,1[\times]0,1[\subset\mathbb{R}^2$ with 20 electrodes and all contact impedances equal to 0.1. For the cases with voltage and currents measurements, six noisy observations were used. For the cases with magnitudes of current density field and interior power density data, only two noisy observations were considered. The finite element grid was determined by subdividing the interval $[0,1]$ into 16 subintervals, leading to $(16+1)^2 = 289$ (resp. $(16+1)^2+(16)^2 = 545$) gridpoints and $2\times(16)^2 = 512$ (resp. $4\times(16)^2 = 1024$) triangles in tests with voltage and current measurements (resp. tests with magnitudes of current density field and interior power density data). The meshes are shown in Fig. 3.1.

Algorithm 3.1 was used with the following parameter values: maximum number of iterations `niter` $= 20$, minimum cost difference `eps` $= 1.10^{-4}$, and maximum effort `effort` $= 5$. In each test, three consecutive runs of Algorithm 3.1 were performed, which correspond to regularization parameters $\alpha = 10^{-3}\times\mathbf{1}$, $\alpha = 10^{-4}\times\mathbf{1}$, and $\alpha = 10^{-5}\times\mathbf{1}$, respectively. The initial guess was chosen to be $\sigma_0 = \sigma_-$ and $(u_0, U_0) = \overrightarrow{0}\in\mathbb{R}^{N+M}$ (or $u_0 = \overrightarrow{0}\in\mathbb{R}^N$) for $\alpha = 10^{-3}\times\mathbf{1}$. The next runs were initialized with the approximate solution obtained in the previous one.

The following test conductivities are considered: $\bar{\sigma}_i : \Omega\to[\sigma_-,\sigma_+]$ defined by $\bar{\sigma}_i(x,y) =$

$1 + 9 \cdot \mathbb{1}_{B_i}(x, y)$ for $i = 0, 1, 2, 3, 4$, with bounds $\sigma_- = 1$, $\sigma_+ = 10$ and sets

$$B_i = \begin{cases} B_{2h}(6,4) \cup B_{2h}(6,8) \cup B_{2h}(6,10) & i = 0 \\ B_{3h}(5,5) \cup B_{2h}(12,12) & i = 1 \\ B_{2h}(3,3) \cup B_{2h}(13,13) & i = 2 \\ B_{2h}(4,6) \cup B_{2h}(4,10) \cup B_{2h}(12,6) \cup B_{2h}(12,10) & i = 3 \\ B_{2h}(4,4) \cup B_{2h}(12,6) \cup B_{2h}(12,10) \cup B_h(3,14) \cup B_h(5,14) & i = 4 \end{cases},$$

where $h = 1/16$ and $B_r(x_0, y_0)$ is the closed ball (in ∞-norm) of radius r and center $h \cdot (x_0, y_0)$.

The test computations are performed in a Python implementation. In order to avoid an inverse crime, exact observations are generated on a finer grid. For this, the complete electrode model equations were discretized and the resulting sparse linear system was solved using the function `spsolve` from the library `scipy`. On the other hand, we use the function `qp` from the library `cvxopt` to solve the box constrained quadratic program (3.29). Observe that we do not need to solve the complete electrode model equations to obtain approximate regularized solutions.

Four tests considering the alternative formulation were performed:

- First a basic test with $\bar{\sigma}_0$ is presented. The reconstruction of $\bar{\sigma}_0$ from observations with a noise level of 5% is presented in Figure 3.2. The corresponding sequences of cost values and relative errors are plotted in Figures 3.3 and 3.4, respectively.
- Reconstructions of $\bar{\sigma}_1$ from noisy observations with three different noise levels are provided in Figure 3.5.
- Using $\bar{\sigma}_2$, a test similar the previous one is shown in Figure 3.6.
- The reconstructions of $\bar{\sigma}_3$ using different line search methods are presented in Figure 3.7. The corresponding sequences of relative errors are in Figure 3.8. The noise level was 5% in all cases.

Finally, the test conductivity $\bar{\sigma}_4$ is reconstructed using observations with noise levels of 1%, 5%, and 10% (Figures 3.10, 3.11, and 3.12, respectively). In each case, the three formulations are considered and distinguished with the following notation:

1. Formulation with applied current and observations of
   (i) voltage measurements = Mod C - Obs Vol,
   (ii) magnitudes of current density field = Mod C - Obs Mag,
   (iii) interior power density data = Mod C - Obs Pow.
2. Formulation with applied voltage and observations of
   (i) current measurements = Mod V - Obs Cur,
   (ii) magnitudes of current density field = Mod V - Obs Mag,
   (iii) interior power density data = Mod V - Obs Pow.
3. Alternative formulation and observations of
   (i) measured voltage-applied current pairs = Mod A - Obs V&C.

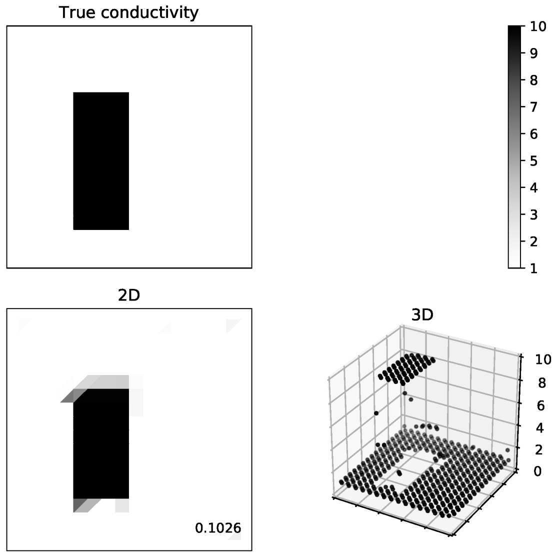


Figure 3.2: Reconstruction of the test conductivity $\bar{\sigma}_0$ from observations of voltage-current pairs with a noise level of 5%.

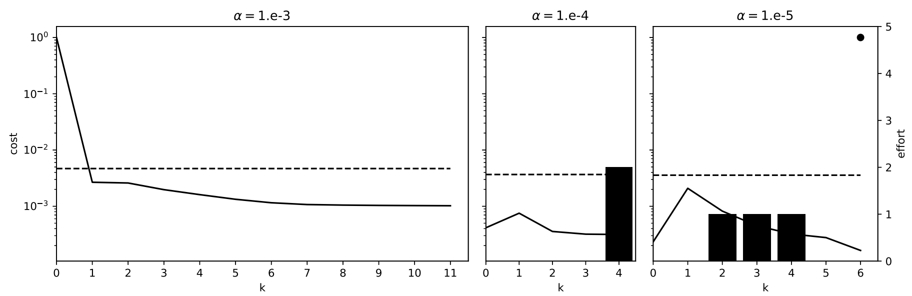


Figure 3.3: Sequence of the cost functional values at the iterated solutions corresponding to the reconstruction of $\bar{\sigma}_0$. The dashed line denotes to the cost functional value at the exact solution interpolated on the reconstruction mesh. The number of iterations corresponding to $\alpha = 10^{-3} \times \mathbf{1}$, $\alpha = 10^{-4} \times \mathbf{1}$, and $\alpha = 10^{-5} \times \mathbf{1}$ were 11, 4, and 6 respectively. The maximum effort was reached in the fourth iteration corresponding to $\alpha = 10^{-4} \times \mathbf{1}$.

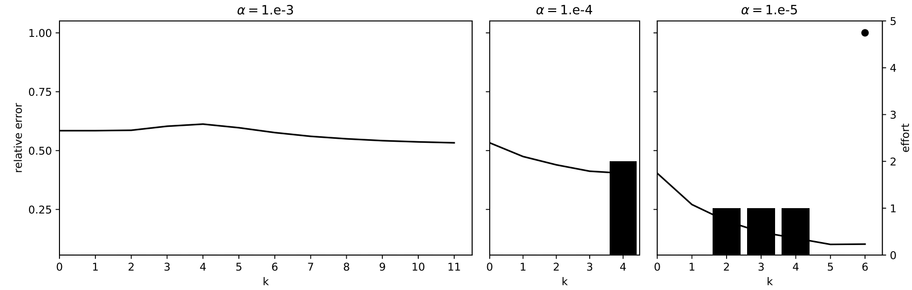


Figure 3.4: Sequence of relative errors $\left(\|\bar{\sigma}_0 - \sigma_k\|_{L^1(\Omega)} / \|\bar{\sigma}_0\|_{L^1(\Omega)}\right)$ corresponding to reconstruction of the test conductivity $\bar{\sigma}_0$. The conductivity σ_k was obtained in the iteration k. The black bars represent the effort realized to find a suitable stepsize.

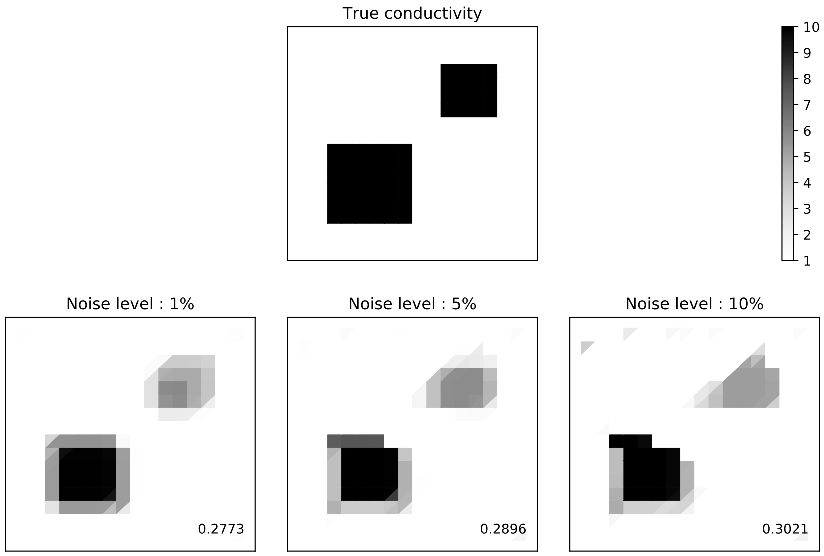


Figure 3.5: Reconstructions of the test conductivity $\bar{\sigma}_1$ from noisy observations of voltage-current pairs with different noise levels. The relative error is in the bottom right corner of the figures.

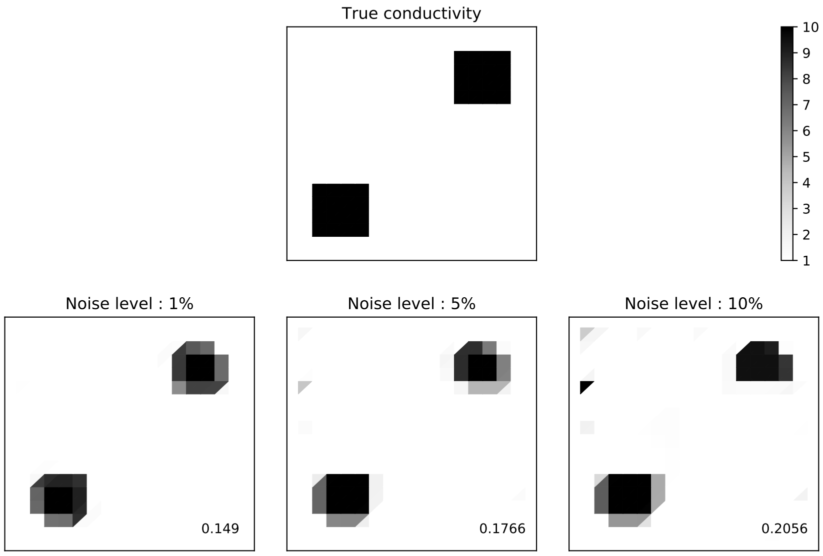


Figure 3.6: Reconstructions of the test conductivity $\bar{\sigma}_2$ from noisy observations of voltage-current pairs with different noise levels. The relative error is in the bottom right corner of the figures.

In sequence plots, the point $\bullet$ in the top right corner says that the algorithm stopped because the maximum effort was reached and the bars represent the effort (in number of iterations) realized to find a suitable stepsize. The number in the bottom right corner of reconstruction plots is the final relative error $\|\bar{\sigma} - \sigma\|_{L^1(\Omega)} / \|\bar{\sigma}\|_{L^1(\Omega)}$, where $\bar{\sigma}$ is the true conductivity and σ is the reconstructed conductivity.

Remark 3.28. Although the observations of current density field and power density considered here are taken on the entire domain, observations on a subset of Ω also fit in our assumptions.

Remark 3.29 (Exact observations). The method presented in [29] was used to generate "optimal" exact observations of current and voltage measurements. The exact observations of magnitudes of current density field and interior power density data were generated using horizontal and vertical current patters similar to those used in [6, 13, 1] with the continuum model.

Remark 3.30 (Noisy observations). Given a exact observation $\bar{y} \in Y$, a noisy observation $y \in Y$ was generated according to $y = \bar{y} + \delta\theta \in Y$, where $\theta \in Y$ is a random normalized function ($\|\theta\|_Y = 1$) and δ is a positive number satisfying $\delta \leq \|\bar{y}\|_Y$. Thus, the relative error of y with respect to $\bar{y}$ satisfies

$$\frac{\|\bar{y} - y\|_Y}{\|\bar{y}\|_Y} = \frac{\delta \, \|\theta\|_Y}{\|\bar{y}\|_Y} = \frac{\delta}{\|\bar{y}\|_Y} \leq 1.$$

See Figure 3.9. The number $l := \delta / \|\bar{y}\|_Y$ is defined as the noise level of y. So, to obtain a noisy observation y with a noise level of $l \times 100\%$, $\delta = l \times \|\bar{y}\|_Y$ must be calculated and

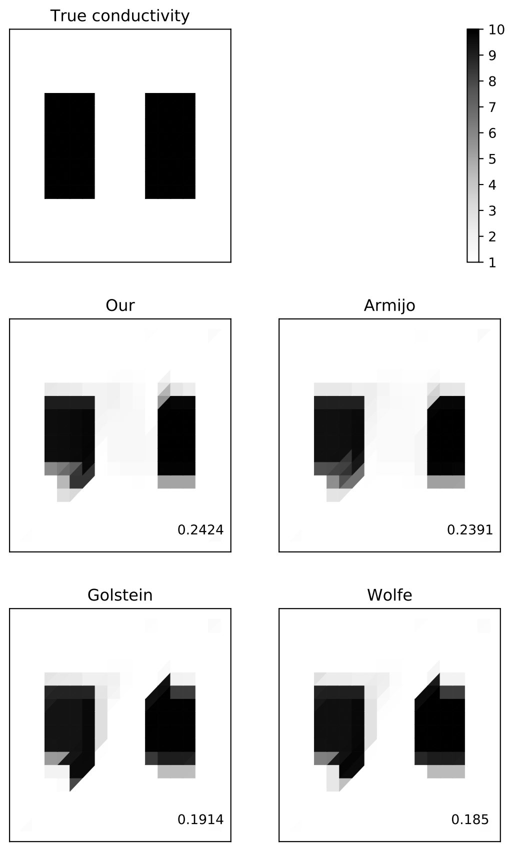


Figure 3.7: Reconstructions of the test conductivity $\bar{\sigma}_3$ from observations of voltage-current pairs with a noise level of 5% and using different line search methods. The Wolfe method yields the lowest relative error.

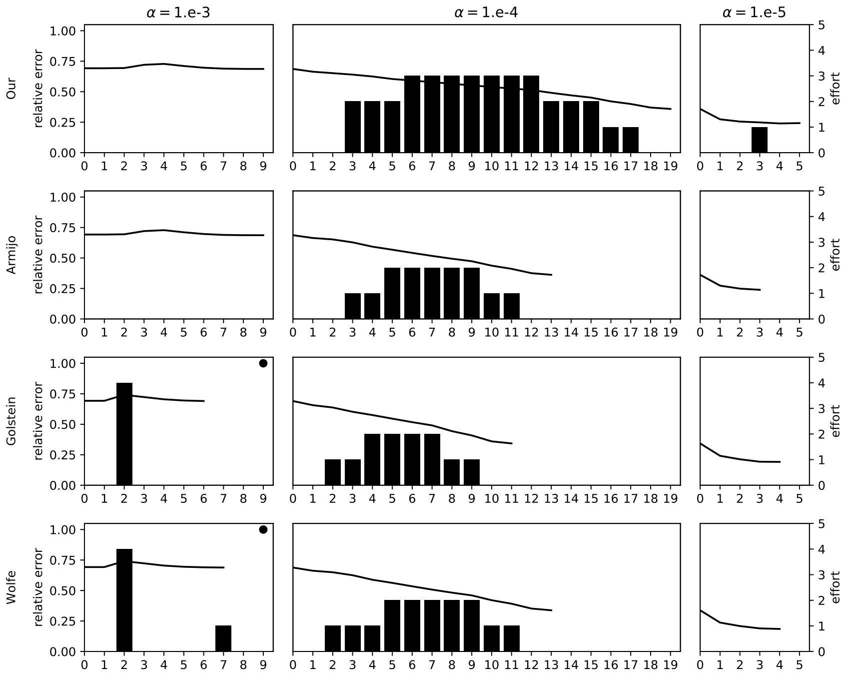


Figure 3.8: Sequences of relative errors $\left(\|\bar{\sigma}_3 - \sigma_k\|_{L^1(\Omega)} \,/\, \|\bar{\sigma}_3\|_{L^1(\Omega)}\right)$ for each of the line search methods used to reconstruct $\bar{\sigma}_3$. Our line search realizes the most effort. For $\alpha = 10^{-4} \times \mathbf{1}$, the Armijo, Holstein, and Wolfe methods perform equally well.

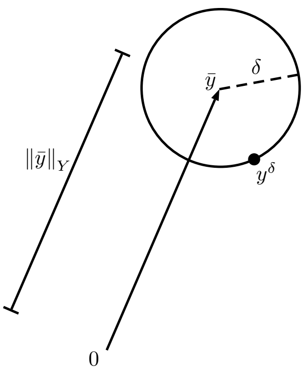


Figure 3.9: Exact observation $\bar{y}$ and noisy observation $y^\delta = \bar{y} + \delta\theta$.

θ must be generated. In our tests we added noise levels of 1% ($l = 0.01$), 5% ($l = 0.05$), and 10% ($l = 0.1$). Observe that taking into account the N noisy observations we have

$$\|\bar{y} - y\|_{Y^N} = \left(\sum_{n=1}^{N} \|\bar{y}^n - y^n\|_Y^q\right)^{1/q} = \left(\sum_{n=1}^{N} (\delta^n)^q\right)^{1/q} \qquad 1 \leq q < \infty.$$

Thus, y and $\delta \coloneqq \left(\sum_{n=1}^{N} (\delta^n)^q\right)^{1/q}$ are admissible parameters in the definition of the regularized problem $\boldsymbol{I}_\lambda^1$. For current and voltage measurements, θ is given by $\theta = \frac{r-\hat{r}}{|r-\hat{r}|_2} \in \mathbb{R}_\diamond^{20}$ with $r = (r_1, \ldots, r_{20})$, $\hat{r} = \frac{r_1+\ldots+r_{20}}{20}$. Each coordinate r_m follows a standard normal distribution with mean 0 and variance 1. For magnitudes of current density field and interior power density data, θ is given by $\theta = r/\|r\|_{L^2(\Omega)}$ and $\theta = r/\|r\|_{L^1(\Omega)}$, respectively, with $r(\mathbf{x}) = \sum_{t=1}^{T} r_t \mathbb{1}_{\Omega_t}(\mathbf{x})$. Similarly, each coordinate r_t follows a standard normal distribution with mean 0 and variance 1.

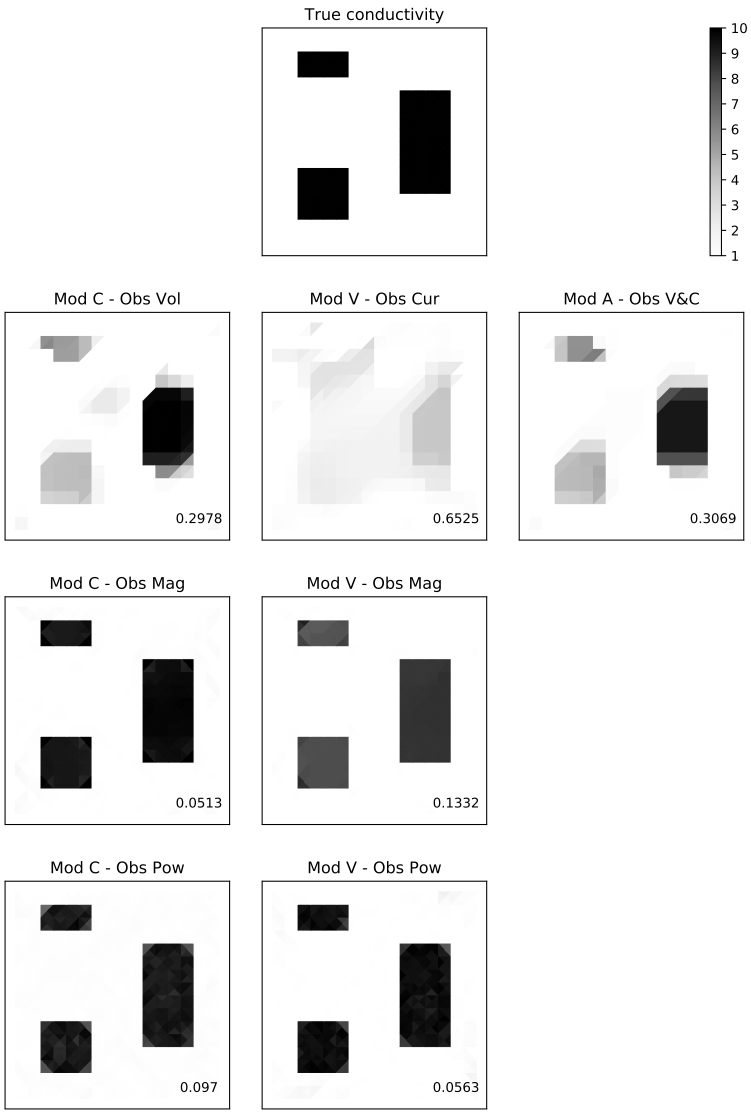


Figure 3.10: Reconstructions of $\bar{\sigma}_4$ using all the proposed formulations. All observations have a noise level of 1%. The formulation with applied current Mod C - Obs Vo and the alternative formulation Mod A - Obs V&C provide similar results.

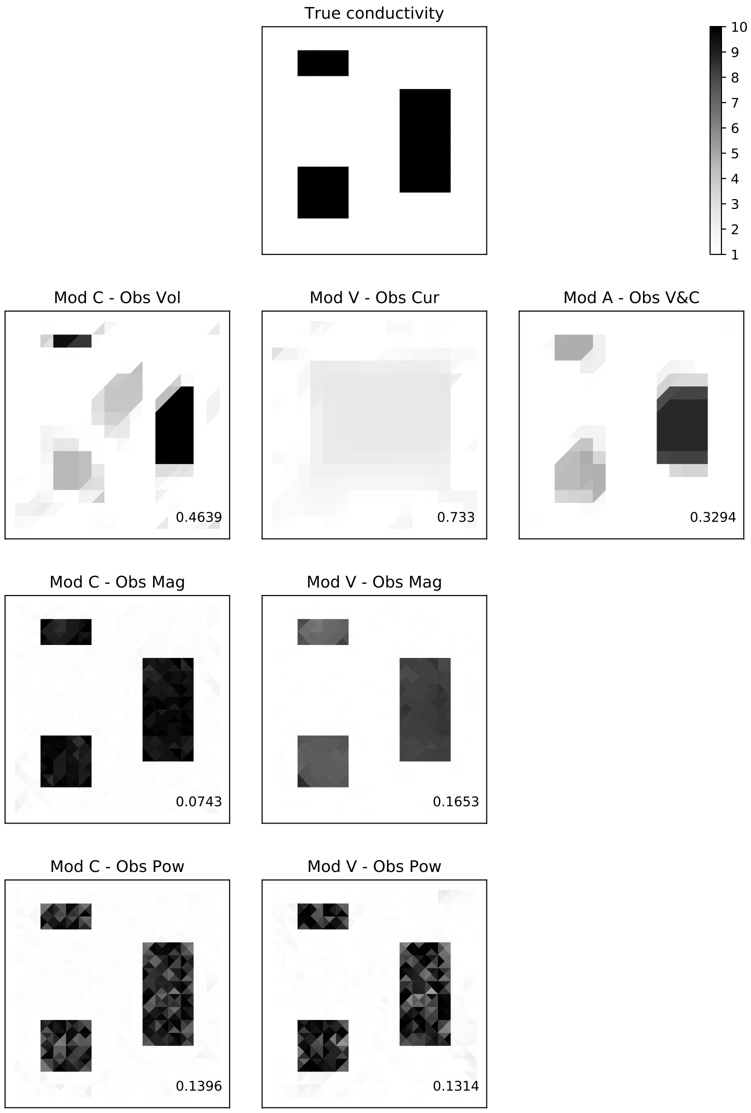


Figure 3.11: Reconstructions of $\bar{\sigma}_4$ using all the proposed formulations. All observations have a noise level of 5%. The reconstruction obtained with the alternative formulation Mod A - Obs V&C remains stable.

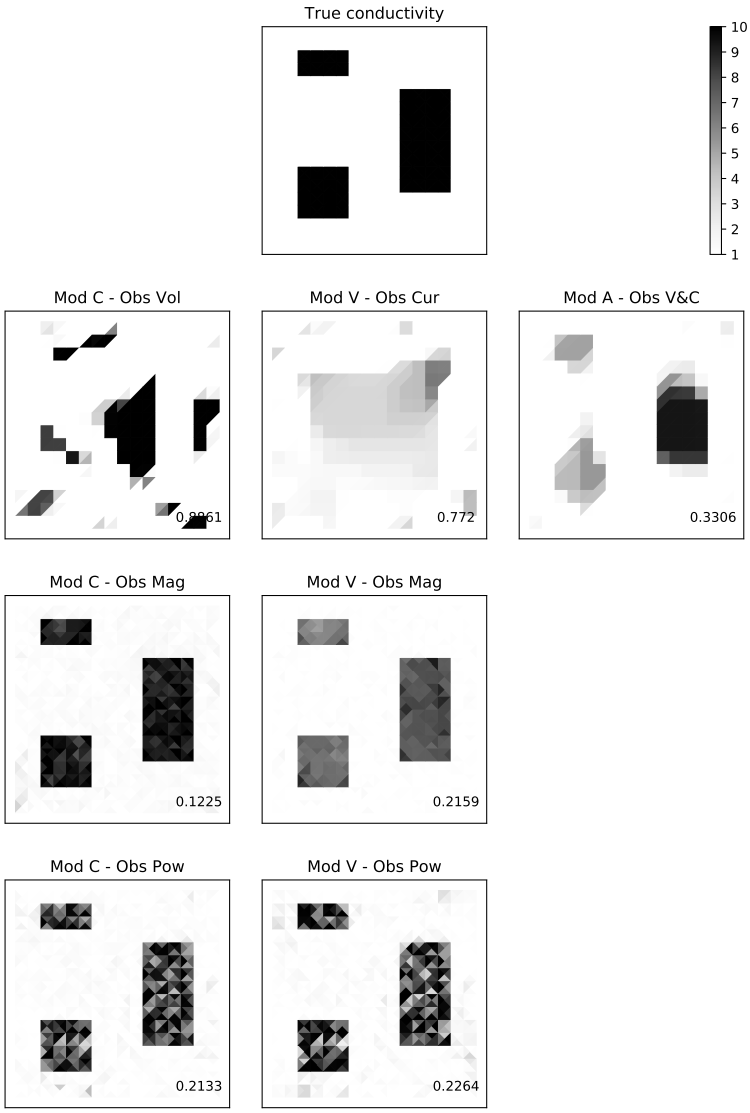


Figure 3.12: Reconstructions of $\bar{\sigma}_4$ using all proposed formulations. All observations have a noise level of 10%. The reconstruction obtained with interior power density data and magnitudes of current density field provide the best results.

3.6 Conclusions

In this work, we have studied the all-at-once formulation of the EIT inverse problem and its regularization. An abstract all-at-once formulation of the EIT inverse problem has been proposed, which admits several EIT models as well as boundary and domain observations. Three regularizations have been proposed and analyzed. We have proved the existence, stability, and convergence of regularized solutions. The existence of an optimal regularization parameter assuming a training data set has been also proved. All this was obtained in a Banach space setting. Furthermore, we applied total variation regularization to the conductivity.

As a consequence, we have obtained a novel all-at-once formulation of the EIT inverse problem based on the complete electrode model, in which current-voltage pairs are considered as observations. The regularizations of this formulation yield stable numerical reconstructions of the conductivity.

It is evident from the numerical tests that the reconstructions are better when domain measurements are used (power density data and magnitudes of current field).

The extension to the case with tensor conductivities is straightforward.

Future work might be concerned with the numerical approximation of the optimal regularization parameter, investigate the convergence rates of the regularized solutions, and apply iterative regularization methods to the abstract inverse problem formulated here.

Appendix

Here, some examples of EIT models that verify Assumptions A1-A6 are provided. These models were studied in [30, 92, 29, 57, 5]. According to our knowledge, all existing EIT models verify our assumptions, except the so-called *point model* [47], about which we do not assert anything.

Continuum model

The equations of the continuum model for the electric potential u are

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega \tag{3.30}$$

with Neumann boundary condition

$$\sigma \frac{\partial u}{\partial \nu} = f \quad \text{on } \partial\Omega \tag{3.31}$$

if a current f is applied, or with Dirichlet boundary condition

$$u = g \quad \text{on } \partial\Omega \tag{3.32}$$

if a voltage g is applied.

Notation. $H^{1/2}(\partial\Omega)$ denotes the space of traces on $\partial\Omega$. $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ denotes the trace operator on $\partial\Omega$. $H^{-1/2}(\partial\Omega)$ denotes the dual of $H^{1/2}(\partial\Omega)$. Let $H_\diamond^{1/2}(\partial\Omega)$ and $H_\diamond^{-1/2}(\partial\Omega)$ be the subspaces

$$H_\diamond^{1/2}(\partial\Omega) := \left\{ g \in H^{1/2}(\partial\Omega) \,\middle|\, \int_{\partial\Omega} g \,\mathrm{d}\mathbf{s} = 0 \right\},$$

$$H_\diamond^{-1/2}(\partial\Omega) := \left\{ f \in H^{-1/2}(\partial\Omega) \,\middle|\, \langle f, \mathbf{1} \rangle_{H^{-1/2} \times H^{1/2}} = 0 \right\}.$$

In addition, for $0 \le s \le 1$, consider the fractional subspaces

$$H_\diamond^{1+s}(\Omega) := \left\{ u \in H^{1+s}(\Omega) \,\middle|\, \int_{\partial\Omega} \gamma u \,\mathrm{d}\mathbf{s} = 0 \right\},$$

$$H_0^{1+s}(\Omega) := \left\{ u \in H^{1+s}(\Omega) \,\middle|\, \gamma u = 0 \text{ in } H^{1/2}(\partial\Omega) \right\}.$$

Formulation with applied current. Equations (3.30) and (3.31) determine the problem of finding the electric potential u when a current $f \in H_\diamond^{-1/2}(\partial\Omega)$ is applied on $\partial\Omega$. The weak formulation of this problem is written as:

$$\begin{cases} \text{find } u \in H_\diamond^1(\Omega) \text{ satisfying} \\ \displaystyle\int_\Omega \sigma \nabla u \cdot \nabla w \,\mathrm{d}\mathbf{x} = \langle f, \gamma w \rangle_{H^{-1/2} \times H^{1/2}} \text{ for all } w \in H_\diamond^1(\Omega). \end{cases} \tag{3.33}$$

Let $\bar{\sigma} \in L^\infty(\Omega)$ be a conductivity such that $\operatorname{ess\,inf}_{\mathbf{x}\in\Omega} \bar{\sigma}(\mathbf{x}) > 0$. Suppose that N currents $f^1, \dots f^N \in H_\diamond^{-1/2}(\partial\Omega)$ are applied at the boundary $\partial\Omega$. Then, the electric potentials $\bar{u}^1, \dots, \bar{u}^N \in H_\diamond^1(\Omega)$ are obtained and the resulting voltages on $\partial\Omega$ are given by $\gamma\bar{u}^1, \dots, \gamma\bar{u}^N \in H_\diamond^{1/2}(\partial\Omega)$. In other words, each $\bar{u}^n$ is the unique solution to (3.33) with $\sigma = \bar{\sigma}$ and $f = f^n$. The following instance of Assumptions A1-A5 allow us to consider (3.33) as model equation in the EIT inverse problem:

A1. $X, Z := H_\diamond^1(\Omega)$ and $b^n \in Z^\star$ defined by $b^n(w) := \langle f^n, \gamma w \rangle_{H^{-1/2} \times H^{1/2}}$.

A2. $a : L^\infty(\Omega) \times X \to Z^\star$ defined by $a(\sigma, u)(w) := \int_\Omega \sigma \nabla u \cdot \nabla w \,\mathrm{d}\mathbf{x}$.

A3-4. We provide three possibilities:

(i) *Voltage measurements.* $Y := H^{1/2}(\partial\Omega)$, $\bar{y}^n := \gamma\bar{u}^n \in H_\diamond^{1/2}(\partial\Omega)$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma, u) := \gamma u$.

(ii) *Magnitudes of current density field.* $Y := L^2(\Omega)$, $\bar{y}^n := \bar{\sigma} |\nabla \bar{u}^n| \in L^2(\Omega)$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma, u) := \sigma |\nabla u|$.

(iii) *Interior power density data.* $Y := L^1(\Omega)$, $\bar{y}^n := \bar{\sigma} |\nabla \bar{u}^n|^2 \in L^1(\Omega)$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma, u) := \sigma |\nabla u|^2$.

A5. $\tilde{X} := H_\diamond^{1+s}(\Omega)$ with some $0 < s \le 1$.

Formulation with applied voltage. Equations (3.30) and (3.32) determine the problem of finding the electric potential u when a voltage $g \in H^{1/2}(\partial\Omega)$ is applied on $\partial\Omega$.

The weak formulation of this problem is written as:

$$\begin{cases} \text{find } u \in H^1(\Omega) \text{ satisfying} \\ \displaystyle\int_\Omega \sigma \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x} = 0 \text{ for all } w \in H_0^1(\Omega) \text{ and } \gamma u = g \text{ in } H^{1/2}(\partial\Omega). \end{cases} \tag{3.34}$$

Let $\bar{\sigma} \in L^\infty(\Omega)$ be a conductivity such that $\operatorname{ess\,inf}_{\mathbf{x}\in\Omega} \bar{\sigma}(\mathbf{x}) > 0$. Suppose that N voltages $g^1, \dots g^N \in H^{1/2}(\partial\Omega)$ are applied at the boundary $\partial\Omega$. Thus, the electric potentials $\bar{u}^1, \dots, \bar{u}^N \in H^1(\Omega)$ are obtained and the resulting currents on $\partial\Omega$ are given by $\bar{\sigma}\frac{\partial \bar{u}^1}{\partial \nu}, \dots, \bar{\sigma}\frac{\partial \bar{u}^N}{\partial \nu} \in H_\diamond^{-1/2}(\partial\Omega)$. In other words, each $\bar{u}^n$ is the unique solution to (3.34) with $\sigma = \bar{\sigma}$ and $g = g^n$. Let $d^1, \dots, d^N$ be functions in $H^1(\Omega)$ with the property $\gamma d^n = g^n$ in $H^{1/2}(\partial\Omega)$. The following instance of Assumptions A1-A5 allows to consider an equivalence of (3.34) as model equation in the EIT inverse problem:

A1. $X, Z \coloneqq H_0^1(\Omega)$ and $b^n \in Z^\star$ defined by $b^n(w) \coloneqq -\int_\Omega \sigma \nabla d^n \cdot \nabla w \, \mathrm{d}\mathbf{x}$.

A2. $a : L^\infty(\Omega) \times X \to Z^\star$ defined by $a(\sigma, u)(w) \coloneqq \int_\Omega \sigma \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x}$.

A3-4. We provide three possibilities:

(i) *Current measurements.* $Y \coloneqq \left(H^1(\Omega)\right)^*$, $\bar{y}^n \coloneqq \bar{\sigma}\frac{\partial \bar{u}^n}{\partial \nu} \circ \gamma \in \{\phi \in Y \mid \phi(\mathbf{1}) = 0\}$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma, u)(w) \coloneqq \int_\Omega \sigma \nabla(u + d^n) \cdot \nabla w \, \mathrm{d}\mathbf{x}$.

(ii) *Magnitudes of current density field.* $Y \coloneqq L^2(\Omega)$, $\bar{y}^n \coloneqq \bar{\sigma} |\nabla \bar{u}^n| \in L^2(\Omega)$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma, u) \coloneqq \sigma |\nabla(u + d^n)|$.

(iii) *Interior power density data.* $Y \coloneqq L^1(\Omega)$, $\bar{y}^n \coloneqq \bar{\sigma} |\nabla \bar{u}^n|^2 \in L^1(\Omega)$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma, u) \coloneqq |\nabla(u + d^n)|^2$.

A5. $\tilde{X} \coloneqq H_0^{1+s}(\Omega)$ with some $0 < s \leq 1$.

Alternative formulation. The weak form of equation (3.30) leads to the the following problem:

$$\begin{cases} \text{find } u \in H^1(\Omega) \text{ satisfying} \\ \displaystyle\int_\Omega \sigma \nabla u \cdot \nabla w \, d\mathbf{x} = 0 \text{ for all } w \in H_0^1(\Omega). \end{cases} \tag{3.35}$$

Let $\bar{\sigma} \in L^\infty(\Omega)$ be a conductivity such that $\operatorname{ess\,inf}_{\mathbf{x}\in\Omega} \bar{\sigma}(\mathbf{x}) > 0$. From the previous formulations we know that there are two possibilities: apply the currents

$$f^1, \dots f^N \in H_\diamond^{-1/2}(\partial\Omega)$$

to obtain the voltages generated on the surface or apply the voltages

$$g^1, \dots g^N \in H^{1/2}(\partial\Omega)$$

to obtain the current generated on the surface. In the first case, the voltage-current pairs $\left(\gamma \bar{u}^1, f^1\right), \dots, \left(\gamma \bar{u}^N, f^N\right)$ are available, where each $\bar{u}^n$ is the unique solution to (3.33) formulated with $\sigma = \bar{\sigma}$ and $f = f^n$. In the second case, the voltage-current pairs $\left(g^1, \bar{\sigma}\frac{\partial \bar{u}^1}{\partial \nu}\right), \dots, \left(g^N, \bar{\sigma}\frac{\partial \bar{u}^N}{\partial \nu}\right)$ are available, where each $\bar{u}^n$ is the unique solution to (3.34) with $\sigma = \bar{\sigma}$ and $g = g^n$. The following instance of Assumptions A1-A5 allows to consider (3.35) as model equation in the EIT inverse problem:

A1. $X := H^1(\Omega)$, $Z := H_0^1(\Omega)$ and $b^1, \ldots, b^N = 0_{Z^\star}$.

A2. $a : L^\infty(\Omega) \times X \to Z^\star$ defined by $a(\sigma, u)(w) := \int_\Omega \sigma \nabla u \cdot \nabla w \, d\mathbf{x}$.

A3. $Y := H^{1/2}(\partial\Omega) \times \left(H^1(\Omega)\right)^\star$ and any of the two sets of voltage-current pairs:

(i) *Measured voltage-applied current pairs.*

$$\bar{y}^n := (\gamma \bar{u}^n, f^n \circ \gamma) \in H_\diamond^{1/2}(\partial\Omega) \times \left\{ \phi \in \left(H^1(\Omega)\right)^* \mid \phi(\mathbf{1}) = 0 \right\}.$$

(ii) *Applied voltage-measured current pairs.*

$$\bar{y}^n := \left(g^n, \bar{\sigma} \frac{\partial \bar{u}^n}{\partial \nu} \circ \gamma \right) \in H^{1/2}(\partial\Omega) \times \left\{ \phi \in \left(H^1(\Omega)\right)^* \mid \phi(\mathbf{1}) = 0 \right\}.$$

A4. $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma, u) := \left(\gamma u, \left\{ w \mapsto \int_\Omega \sigma \nabla u \cdot \nabla w \, d\mathbf{x} \right\}\right)$.

A5. $\tilde{X} := H^{1+s}(\Omega)$ with some $0 < s \leq 1$.

Continuum model with voltage point measurements

Here, the *continuum model with voltage point measurements* [5] is presented and a instance of Assumptions A1-A5 is proposed using the equations of this model.

Suppose that $\Omega \subseteq \mathbb{R}^2$. Let Γ an open subset of $\partial\Omega$ and $\Gamma_0 := \partial\Omega \backslash \Gamma$. On the surface $\partial\Omega$ we attach K electrodes, which we identify with the points $\mathbf{x}_1, \ldots, \mathbf{x}_K \in \partial\Omega$. In this case, the equations of the continuum model for the electric potential u are

$$\begin{aligned} \nabla \cdot (\sigma \nabla u) &= 0 && \text{in } \Omega \\ \sigma \frac{\partial u}{\partial \nu} &= f && \text{on } \Gamma \\ u &= 0 && \text{on } \Gamma_0 \end{aligned} \tag{3.36}$$

where f is a current applied to Γ.

Notation. Let $W^{1,p}(\Omega)$ be the classical Sobolev space of functions in $L^p(\Omega)$ with weak derivatives in $L^p(\Omega)$, with $p \geq 1$. For $0 \leq s \leq 1$, we define the Sobolev space $W_{\Gamma_0}^{1+s,p}(\Omega) := \overline{C_{\Gamma_0}^\infty(\Omega)}$, where

$$C_{\Gamma_0}^\infty(\Omega) := \left\{ w|_\Omega \,\middle|\, w \in C^\infty\left(\mathbb{R}^2\right), \operatorname{supp} w \cap \Gamma_0 = \emptyset \right\}.$$

The closure of $C_{\Gamma_0}^\infty(\Omega)$ is in the topology of $W^{1+s,p}(\Omega)$.

Consider a conductivity $\bar{\sigma} \in L^\infty(\Omega)$ such that $\operatorname{ess\,inf}_{\mathbf{x} \in \Omega} \bar{\sigma}(\mathbf{x}) > 0$ and fix a $p > 2$. Suppose that N currents $f^1, \ldots f^N \in L^\infty(\Gamma)$ are applied to Γ. Then, N electric potentials $\bar{u}^1, \ldots, \bar{u}^N \in W_{\Gamma_0}^{1,p}(\Omega)$ are obtained, which are the corresponding solutions to the weak formulation of the continuum model (3.36) with conductivity $\bar{\sigma}$. That is, each $\bar{u}^n$ solves the following problem:

$$\left\{ \begin{array}{l} \text{find } u \in W_{\Gamma_0}^{1,p}(\Omega) \text{ satisfying} \\ \displaystyle\int_\Omega \bar{\sigma} \nabla u \cdot \nabla w \, d\mathbf{x} = \int_\Gamma f^n w \, d\mathbf{s} \text{ for all } w \in W_{\Gamma_0}^{1,q}(\Omega). \end{array} \right. \tag{3.37}$$

with q such that $1/p + 1/q = 1$. Existence and uniqueness of a solution to (3.37) was proved in [5], provided $\Omega \cup \Gamma$ is regular in the sense of Groger and p is sufficiently close to 2. Note that, thanks to the continuous embedding $W^{1,p}(\Omega) \subset C\left(\overline{\Omega}\right)$ for $p > 2$ in two dimensions, given $u \in W^{1,p}(\Omega)$, the point evaluation $u(\mathbf{x})$, with $\mathbf{x} \in \Gamma$, is well-defined. In particular, given $\bar{u}$ a solution to (3.37), the voltage point measurements $\bar{u}^n(\mathbf{x}_1), \ldots, \bar{u}^n(\mathbf{x}_K)$ are available, for $n = 1, \ldots, N$. The following instance of Assumptions A1-A5 allows to consider (3.37) as model equation in the EIT inverse problem:

A1. $X \coloneqq W^{1,p}_{\Gamma_0}(\Omega)$, $Z \coloneqq W^{1,q}_{\Gamma_0}(\Omega)$, and $b^n \in Z^\star$ defined by $b^n(w) \coloneqq \int_\Gamma f^n w \, \mathrm{d}\mathbf{s}$.

A2. $a : L^\infty(\Omega) \times X \to Z^\star$ defined by $a(\sigma, u)\, w \coloneqq \int_\Omega \sigma \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x}$.

A3. $Y \coloneqq \mathbb{R}^K$ and $\bar{y}^n \coloneqq (\bar{u}^n(\mathbf{x}_1), \ldots, \bar{u}^n(\mathbf{x}_K)) \in Y$.

A4. $c : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma, u) \coloneqq (u(\mathbf{x}_1), \ldots, u(\mathbf{x}_K))$.

A5. $\tilde{X} \coloneqq W^{1+s,p}_{\Gamma_0}(\Omega)$ with some $0 < s \leq 1$.

Shunt model

The equations of the *shunt model* for the electric potential (u, U) are

$$\nabla \cdot (\sigma \nabla u) = 0 \qquad \text{in } \Omega \tag{3.38}$$

$$\sigma \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \setminus \bigcup_{m=1}^{M} \mathcal{E}_m \tag{3.39}$$

$$u = U_m \qquad \text{on } \mathcal{E}_m,\ m = 1, \ldots, M \tag{3.40}$$

with

$$\int_{\mathcal{E}_m} \sigma \frac{\partial u}{\partial \nu} \, \mathrm{d}\mathbf{s} = I_m \quad m = 1, \ldots, M \tag{3.41}$$

if a current pattern $I = (I_1, \ldots, I_M) \in \mathbb{R}^M_\diamond$ is applied, or with

$$U_m = V_m \quad m = 1, \ldots, M \tag{3.42}$$

if a voltage pattern $V = (V_1, \ldots, V_M) \in \mathbb{R}^M$ is applied.

Notation. For $0 \leq s \leq 1$, consider the subspaces

$$\mathcal{H}^{1+s} = \left\{ (u, U) \in H^{1+s}(\Omega) \times \mathbb{R}^M \,\middle|\, (\gamma_m u)_{m=1}^M = U \right\},$$

$$\mathcal{H}^{1+s}_\diamond = \left\{ (u, U) \in H^{1+s}(\Omega) \times \mathbb{R}^M_\diamond \,\middle|\, (\gamma_m u)_{m=1}^M = U \right\},$$

$$\mathcal{H}^{1+s}_0 = \left\{ u \in H^{1+s}(\Omega) \,\middle|\, (\gamma_m u)_{m=1}^M = \overrightarrow{0} \right\}.$$

Formulation with applied current. We begin by considering equations (3.38)-(3.40), and (3.41), which determine the problem of finding the electrical potential (u, U) when a current pattern $I \in \mathbb{R}^M_\diamond$ is applied through electrodes $\mathcal{E}_1, \ldots, \mathcal{E}_M$. The weak formulation of this problem is expressed as:

$$\begin{cases} \text{find } (u,U) \in \mathcal{H}^1_\diamond \text{ satisfying} \\ \displaystyle\int_\Omega \sigma \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x} = \sum_{m=1}^{M} I_m W_m \text{ for all } (w,W) \in \mathcal{H}^1_\diamond. \end{cases} \tag{3.43}$$

Let $\bar{\sigma} \in L^\infty(\Omega)$ be a conductivity such that $\operatorname{ess\,inf}_{\mathbf{x}\in\Omega} \bar{\sigma}(\mathbf{x}) > 0$. Suppose that N current patterns $I^1, \ldots I^N \in \mathbb{R}^M_\diamond$ are applied through electrodes $\mathcal{E}_1, \ldots, \mathcal{E}_M$. Then, the electric potentials $(\bar{u}^1, \bar{U}^1), \ldots, (\bar{u}^N, \bar{U}^N) \in \mathcal{H}^1_\diamond$ are obtained, where $\bar{U}^1, \ldots, \bar{U}^N$ are the resulting voltages on the electrodes. In other words, each $(\bar{u}^n, \bar{U}^n)$ is the unique solution to (3.43) with $\sigma = \bar{\sigma}$ and $I = I^n$. The following instance of Assumptions A1-A5 allows to consider (3.43) as model equation in the EIT inverse problem:

A1. $X, Z \coloneqq \mathcal{H}^1_\diamond$ and $b^n \in Z^\star$ defined by $b^n(w,W) \coloneqq \sum_{m=1}^M I^n_m W_m$.

A2. $a : L^\infty(\Omega) \times X \to Z^\star$ defined by $a(\sigma,(u,U))(w,W) \coloneqq \int_\Omega \sigma \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x}$.

A3-4. We provide three possibilities:

(i) *Voltage measurements.* $Y \coloneqq \mathbb{R}^M$, $\bar{y}^n \coloneqq \bar{U}^n \in \mathbb{R}^M_\diamond$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma,(u,U)) \coloneqq U$.

(ii) *Magnitudes of current density field.* $Y \coloneqq L^2(\Omega)$, $\bar{y}^n \coloneqq \bar{\sigma} |\nabla \bar{u}^n| \in L^2(\Omega)$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma,(u,U)) \coloneqq \sigma |\nabla u|$.

(iii) *Interior power density data.* $Y \coloneqq L^1(\Omega)$, $\bar{y}^n \coloneqq \bar{\sigma} |\nabla \bar{u}^n|^2 \in L^1(\Omega)$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma,(u,U)) \coloneqq \sigma |\nabla u|^2$.

A5. $\tilde{X} \coloneqq \mathcal{H}^{1+s}_\diamond$ with some $0 < s \leq 1$.

Formulation with applied voltage. Equations (3.38)-(3.40), and (3.42) determine the problem of finding the electric potential (u,U) when a voltage pattern $V \in \mathbb{R}^M$ is applied through electrodes $\mathcal{E}_1, \ldots, \mathcal{E}_M$. The weak formulation of this problem is written as:

$$\begin{cases} \text{find } u \in H^1(\Omega) \text{ satisfying} \\ \displaystyle\int_\Omega \sigma \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x} = 0 \text{ for all } w \in \mathcal{H}^1_0 \text{ and } (\gamma_m u)_{m=1}^M = V. \end{cases} \tag{3.44}$$

Let $\bar{\sigma} \in L^\infty(\Omega)$ be a conductivity such that $\operatorname{ess\,inf}_{\mathbf{x}\in\Omega} \bar{\sigma}(\mathbf{x}) > 0$. Suppose that N voltage patterns $V^1, \ldots V^N \in \mathbb{R}^M$ are applied through the electrodes $\mathcal{E}_1, \ldots, \mathcal{E}_M$. Thus, the electric potentials $\bar{u}^1, \ldots, \bar{u}^N \in H^1(\Omega)$ are obtained and the resulting currents on the electrodes are given by

$$\left(\left\langle \bar{\sigma} \frac{\partial \bar{u}^1}{\partial \nu}, \gamma e_m \right\rangle_{H^{-1/2} \times H^{1/2}} \right)_{m=1}^M, \ldots, \left(\left\langle \bar{\sigma} \frac{\partial \bar{u}^N}{\partial \nu}, \gamma e_m \right\rangle_{H^{-1/2} \times H^{1/2}} \right)_{m=1}^M \in \mathbb{R}^M_\diamond,$$

where $e_1, \ldots, e_M$ are any functions in $H^1(\Omega)$ satisfying $\gamma_m e_m = 1$ and $\gamma_{m'} e_m = 0$ for $m' \neq m$. In other words, each $\bar{u}^n$ is the unique solution to (3.44) with $\sigma = \bar{\sigma}$ and $V = V^n$. It is easy to check that each resulting current belongs to $\mathbb{R}^M_\diamond$ by setting $w = \sum_{m=1}^M e_m - 1$ in (3.44). Let $d^1, \ldots, d^N$ be functions in $H^1(\Omega)$ with the property

$(\gamma_m d^n)_{m=1}^M = V^n$. Clearly $\left(d^1, V^1\right), \ldots, \left(d^N, V^N\right) \in \mathcal{H}^1$. The following instance of Assumptions A1-A5 allows to consider an equivalence of (3.44) as model equation in the EIT inverse problem:

A1. $X, Z := \mathcal{H}_0^1$ and $b^n \in Z^\star$ defined by $b^n(w) := -\int_\Omega \sigma \nabla d^n \cdot \nabla w \, \mathrm{d}\mathbf{x}$.

A2. $a : L^\infty(\Omega) \times X \to Z^\star$ defined by $a(\sigma, u)(w) := \int_\Omega \sigma \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x}$.

A3-4. We provide three possibilities:

(i) *Current measurements.* $Y := \mathbb{R}^M$, $\bar{y}^n := \left(\left\langle \bar{\sigma} \frac{\partial \bar{u}^m}{\partial \nu}, \gamma e_m \right\rangle_{H^{-1/2} \times H^{1/2}}\right)_{m=1}^M \in \mathbb{R}_\diamond^M$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by

$$c^n(\sigma, u)(w) := \left(\int_\Omega \sigma \nabla (u + d^n) \cdot \nabla e_m \, \mathrm{d}\mathbf{x}\right)_{m=1}^M .$$

(ii) *Magnitudes of current density field.* $Y := L^2(\Omega)$, $\bar{y}^n := \bar{\sigma} |\nabla \bar{u}^n| \in L^2(\Omega)$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma, u) := \sigma |\nabla (u + d^n)|$.

(iii) *Interior power density data.* $Y := L^1(\Omega)$, $\bar{y}^n := \bar{\sigma} |\nabla \bar{u}^n|^2 \in L^1(\Omega)$, and $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma, u) := |\nabla (u + d^n)|^2$.

A5. $\tilde{X} := \mathcal{H}_0^{1+s}$ with some $0 < s \leq 1$.

Alternative formulation. The weak form of equations 3.38-3.40 leads to the the following problem:

$$\begin{cases} \text{find } (u, U) \in \mathcal{H}^1 \text{ satisfying} \\ \displaystyle\int_\Omega \sigma \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x} = 0 \text{ for all } w \in \mathcal{H}_0^1. \end{cases} \tag{3.45}$$

Let $\bar{\sigma} \in L^\infty(\Omega)$ be a conductivity such that $\operatorname{ess\,inf}_{\mathbf{x} \in \Omega} \bar{\sigma}(\mathbf{x}) > 0$. From the previous formulations we see that there are two possibilities: apply the current patterns $I^1, \ldots I^N \in \mathbb{R}_\diamond^M$ to obtain the voltages generated at the electrodes, or apply the voltage patterns $V^1, \ldots V^N \in \mathbb{R}^M$ to obtain the currents generated at the electrodes. In the first case, the voltage-current pairs $\left(\bar{U}^1, I^1\right), \ldots, \left(\bar{U}^N, I^N\right)$ are available, where each $\left(\bar{u}^n, \bar{U}^n\right)$ is the unique solution to (3.43) with $\sigma = \bar{\sigma}$ and $I = I^n$. In the second case, the voltage-current pairs

$$\left(V^n, \left(\left\langle \bar{\sigma} \frac{\partial \bar{u}^n}{\partial \nu}, \gamma e_m \right\rangle_{H^{-1/2} \times H^{1/2}}\right)_{m=1}^M\right) \qquad n = 1, \ldots, N$$

are available, where each $\bar{u}^n$ is the unique solution to (3.44) with $\sigma = \bar{\sigma}$ and $V = V^n$. The following instance of Assumptions A1-A5 allows to consider (3.45) as model equation in the EIT inverse problem:

A1. $X := \mathcal{H}^1$, $Z := \mathcal{H}_0^1$ and $b^1, \ldots, b^N = 0_{Z^\star}$.

A2. $a : L^\infty(\Omega) \times X \to Z^\star$ defined by $a(\sigma, (u, U))(w) := \int_\Omega \sigma \nabla u \cdot \nabla w \, \mathrm{d}\mathbf{x}$.

A3. $Y := \mathbb{R}^M \times \mathbb{R}^M$ and any of the two sets of voltage-current pairs:

(i) *Measured voltage-applied current pairs.*

$$\bar{y}^n := \left(\bar{U}^1, I^1\right) \in \mathbb{R}_\diamond^M \times \mathbb{R}_\diamond^M .$$

(ii) *Applied voltage-measured current pairs.*

$$\bar{y}^n := \left(V^n, \left(\left\langle \bar{\sigma}\frac{\partial \bar{u}^n}{\partial \nu}, \gamma e_m \right\rangle_{H^{-1/2}\times H^{1/2}}\right)_{m=1}^{M}\right) \in \mathbb{R}^M \times \mathbb{R}^M_\diamond.$$

A4. $c^n : L^\infty(\Omega) \times X \to Y$ defined by $c^n(\sigma,(u,U)) := \left(U, \left(\int_\Omega \sigma\nabla u\cdot \nabla e_m \,\mathrm{d}\mathbf{x}\right)_{m=1}^M\right)$.

A5. $\tilde{X} := \mathcal{H}^{1+s}$ with some $0 < s \leq 1$.

Gap model

The equations of the *gap model* for the electric potential (u, U) are

$$\nabla\cdot(\sigma\nabla u) = 0 \qquad \text{in } \Omega \tag{3.46}$$

$$\sigma\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega\setminus\bigcup_{m=1}^{M}\mathcal{E}_m \tag{3.47}$$

$$\sigma\frac{\partial u}{\partial \nu} = \text{const.} \qquad \text{on } \mathcal{E}_m,\ m = 1,\dots,M \tag{3.48}$$

$$\frac{1}{|\mathcal{E}_m|}\int_{\mathcal{E}_m} u\,\mathrm{d}\mathbf{s} = U_m \qquad \text{on } \mathcal{E}_m,\ m = 1,\dots,M \tag{3.49}$$

with

$$\sigma\frac{\partial u}{\partial \nu}\bigg|_{\mathcal{E}_m} = \frac{I_m}{|\mathcal{E}_m|} \quad m = 1,\dots,M \tag{3.50}$$

if a current pattern $I = (I_1,\dots,I_M) \in \mathbb{R}^M_\diamond$ is applied, or with

$$U_m = V_m \quad m = 1,\dots,M \tag{3.51}$$

if a voltage pattern $V = (V_1,\dots,V_M) \in \mathbb{R}^M$ is applied. The same instances as in the shunt model work here, but with the subspaces $(0 \leq s \leq 1)$

$$\mathcal{H}^{1+s} = \left\{(u,U) \in H^{1+s}(\Omega)\times\mathbb{R}^M \,\middle|\, \left(\frac{1}{|\mathcal{E}_m|}\int_{\mathcal{E}_m}\gamma_m u\,\mathrm{d}\mathbf{s}\right)_{m=1}^M = U\right\},$$

$$\mathcal{H}^{1+s}_\diamond = \left\{(u,U) \in H^{1+s}(\Omega)\times\mathbb{R}^M_\diamond \,\middle|\, \left(\frac{1}{|\mathcal{E}_m|}\int_{\mathcal{E}_m}\gamma_m u\,\mathrm{d}\mathbf{s}\right)_{m=1}^M = U\right\},$$

$$\mathcal{H}^{1+s}_0 = \left\{u \in H^{1+s}(\Omega) \,\middle|\, \left(\frac{1}{|\mathcal{E}_m|}\int_{\mathcal{E}_m}\gamma_m u\,\mathrm{d}\mathbf{s}\right)_{m=1}^M = \vec{0}\right\},$$

the functions $e_1,\dots,e_M$ satisfying $\int_{\mathcal{E}_m}\gamma_m e_m\,\mathrm{d}\mathbf{s} = |\mathcal{E}_m|$ and $\gamma_{m'}e_m = 0$ for $m' \neq m$, and the functions $d^1,\dots,d^N$ with the property $\left(\frac{1}{|\mathcal{E}_m|}\int_{\mathcal{E}_m}\gamma_m d^n\,\mathrm{d}\mathbf{s}\right)_{m=1}^M = V^n$.

Smoothened complete electrode model

In [57] was proposed the *smoothened complete electrode model*, which replaces the contact impedances of the complete electrode model with contact admittance functions capable to vanish on some subsets of the electrodes. It can be said that the contact admittances are represented by functions $\zeta_1, \dots, \zeta_M$ satisfying $\zeta_m \in L^\infty(\mathcal{E}_m)$, $\zeta_m \geq 0$ a.e. on $\mathcal{E}_m$, and $\zeta_m \not\equiv 0$. The equations of this model for the electric potential (u, U) are

$$\nabla\cdot(\sigma\nabla u) = 0 \qquad \text{in } \Omega \tag{3.52}$$

$$\sigma\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega\setminus\bigcup_{m=1}^{M}\mathcal{E}_m \tag{3.53}$$

$$\sigma\frac{\partial u}{\partial \nu} = \zeta_m(U_m - u) \qquad \text{on } \mathcal{E}_m,\ m = 1, \dots, M \tag{3.54}$$

with

$$\int_{\mathcal{E}_m} \sigma\frac{\partial u}{\partial \nu}\,\mathrm{d}\mathbf{s} = I_m \quad m = 1, \dots, M \tag{3.55}$$

if a current pattern $I = (I_1, \dots, I_M) \in \mathbb{R}^M_\diamond$ is applied, or with

$$U_m = V_m \quad m = 1, \dots, M \tag{3.56}$$

if a voltage pattern $V = (V_1, \dots, V_M) \in \mathbb{R}^M$ is applied. Therefore, it suffices to replace the contact impedances $z_1, \dots, z_M$ by $\zeta_1, \dots, \zeta_M$ in the instances that were proposed for the complete electrode model.

Convergence results

Proposition 3.31. *Let $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$. Let (σ_i) be a sequence in $L^\infty(\Omega)$ and $\sigma \in L^\infty(\Omega)$. Let (u_i) be a sequence in $W^{1,p}(\Omega)$ and $u \in W^{1,p}(\Omega)$. If $\sigma_i \overset{\star}{\rightharpoonup} \sigma$ in $L^\infty(\Omega)$ and $u_i \to u$ in $W^{1,p}(\Omega)$ then*

$$\int_\Omega \sigma_i\nabla u_i\cdot\nabla w\,d\mathbf{x} \to \int_\Omega \sigma\nabla u\cdot\nabla w\,d\mathbf{x} \quad \text{for all } w \in W^{1,q}(\Omega).$$

Proof. Let $a : L^\infty(\Omega)\times W^{1,p}(\Omega) \to \left(W^{1,q}(\Omega)\right)^\star$ be the bilinear map defined by $a(\sigma, u)(w) := \int_\Omega \sigma\nabla u\cdot\nabla w\,d\mathbf{x}$ for all $w \in W^{1,q}(\Omega)$. Since

$$(\sigma_1\nabla u_1 - \sigma_2\nabla u_2)\cdot\nabla w = (\sigma_1 - \sigma_2)\nabla u_1\cdot\nabla w + \sigma_2(\nabla u_1 - \nabla u_2)\cdot\nabla w \tag{3.57}$$

for all $\sigma_1, \sigma_2 \in L^\infty(\Omega)$, all $u_1, u_2 \in W^{1,p}(\Omega)$, and all $w \in W^{1,q}(\Omega)$, it follows that

$$\begin{aligned}
&|(a(\sigma, u) - a(\sigma_i, u_i))(w)| \\
&\quad\leq \left|\int_\Omega (\sigma - \sigma_i)\nabla u\cdot\nabla w\,d\mathbf{x}\right| + \left|\int_\Omega \sigma_i(\nabla u - \nabla u_i)\cdot\nabla w\,d\mathbf{x}\right| \\
&\quad\leq \left|\int_\Omega (\sigma - \sigma_i)(\nabla u\cdot\nabla w)\,d\mathbf{x}\right| + \|\sigma_i\|_{L^\infty(\Omega)}\|\nabla u - \nabla u_i\|_{L^p(\Omega,\mathbb{R}^d)}\|\nabla w\|_{L^q(\Omega,\mathbb{R}^d)}
\end{aligned}$$

for all $w \in W^{1,q}(\Omega)$ and all i. Observe the following. The weak* convergence of (σ_i) implies that $\left(\|\sigma_i\|_{L^\infty(\Omega)}\right)$ is bounded and that $\int_\Omega \sigma_i (\nabla u \cdot \nabla w)\, d\mathbf{x} \to \int_\Omega \sigma (\nabla u \cdot \nabla w)\, d\mathbf{x}$ since $\nabla u \cdot \nabla w \in L^1(\Omega)$ for all $w \in W^{1,q}(\Omega)$. The strong convergence of (u_i) implies $\nabla u_i \to \nabla u$ in $L^p\left(\Omega, \mathbb{R}^d\right)$. Therefore $a(\sigma_i, u_i)(w) \to a(\sigma, u)(w)$ for all $w \in W^{1,q}(\Omega)$. □

Proposition 3.32. *Let (σ_i) be a sequence in $L^\infty(\Omega)$ and $\sigma \in L^\infty(\Omega)$. Let (u_i) be a sequence in $H^1(\Omega)$ and $u \in H^1(\Omega)$. If $\sigma_i \overset{\star}{\rightharpoonup} \sigma$ in $L^\infty(\Omega)$ and $u_i \to u$ in $H^1(\Omega)$ then*

$$\sigma_i |\nabla u_i|^2 \rightharpoonup \sigma |\nabla u|^2 \quad \text{in } L^1(\Omega) \qquad \text{and} \qquad \sigma_i |\nabla u_i| \rightharpoonup \sigma |\nabla u| \quad \text{in } L^2(\Omega).$$

Proof. Since

$$\mu\left(\sigma_1 |\nabla u_1|^2 - \sigma_2 |\nabla u_2|^2\right) = \mu(\sigma_1 - \sigma_2)|\nabla u_1|^2 + \mu\sigma_2 (\nabla u_1 + \nabla u_2)\cdot(\nabla u_1 - \nabla u_2)$$

for all $\sigma_1, \sigma_2 \in L^\infty(\Omega)$, all $u_1, u_2 \in H^1(\Omega)$, and all $\mu \in L^\infty(\Omega)$, it follows that

$$\begin{aligned}
&\left|\int_\Omega \mu\left(\sigma |\nabla u|^2 - \sigma_i |\nabla u_i|^2\right) d\mathbf{x}\right| \\
&\leq \left|\int_\Omega \mu(\sigma - \sigma_i)|\nabla u|^2\, d\mathbf{x}\right| + \left|\int_\Omega \mu\sigma_i (\nabla u + \nabla u_i)\cdot(\nabla u - \nabla u_i)\, d\mathbf{x}\right| \\
&\leq \left|\int_\Omega (\sigma - \sigma_i)\left(\mu |\nabla u|^2\right) d\mathbf{x}\right| + \|\mu\|_{L^\infty(\Omega)} \|\sigma_i\|_{L^\infty(\Omega)} \|\nabla u + \nabla u_i\|_{L^2(\Omega,\mathbb{R}^d)} \|\nabla u - \nabla u_i\|_{L^2(\Omega,\mathbb{R}^d)}
\end{aligned}$$

for all $\mu \in L^\infty(\Omega)$ and all i. The weak* convergence of (σ_i) implies that $\left(\|\sigma_i\|_{L^\infty(\Omega)}\right)$ is bounded and that $\int_\Omega \sigma_i \left(\mu |\nabla u_i|^2\right) d\mathbf{x} \to \int_\Omega \sigma\left(\mu |\nabla u|^2\right) d\mathbf{x}$ since $\mu |\nabla u|^2 \in L^1(\Omega)$. The strong convergence of (u_i) implies that $\left(\|\nabla u + \nabla u_i\|_{L^2(\Omega,\mathbb{R}^d)}\right)$ is bounded and that $\nabla u_i \to \nabla u$ in $L^p\left(\Omega, \mathbb{R}^d\right)$. Therefore $\int_\Omega \mu\left(\sigma_i |\nabla u_i|^2\right) d\mathbf{x} \to \int_\Omega \mu\left(\sigma |\nabla u|^2\right) d\mathbf{x}$ for all $\mu \in L^\infty(\Omega)$. On the other hand, since

$$\mu(\sigma_1 |\nabla u_1| - \sigma_2 |\nabla u_2|) = \mu(\sigma_1 - \sigma_2)|\nabla u_1| + \mu\sigma_2 (|\nabla u_1| - |\nabla u_2|)$$

for all $\sigma_1, \sigma_2 \in L^\infty(\Omega)$, all $u_1, u_2 \in H^1(\Omega)$, and all $\mu \in L^2(\Omega)$, it follows that

$$\begin{aligned}
&\left|\int_\Omega \mu(\sigma |\nabla u| - \sigma_i |\nabla u_i|)\, d\mathbf{x}\right| \\
&\leq \left|\int_\Omega \mu(\sigma - \sigma_i)|\nabla u|\, d\mathbf{x}\right| + \left|\int_\Omega \mu\sigma_i (|\nabla u| - |\nabla u_i|)\, d\mathbf{x}\right| \\
&\leq \left|\int_\Omega (\sigma - \sigma_i)(\mu |\nabla u|)\, d\mathbf{x}\right| + \|\mu\|_{L^2(\Omega)} \|\sigma_i\|_{L^\infty(\Omega)} \||\nabla u| - |\nabla u_i|\|_{L^2(\Omega)}
\end{aligned}$$

for all $\mu \in L^2(\Omega)$ and all i. The weak* convergence of (σ_i) implies that $\left(\|\sigma_i\|_{L^\infty(\Omega)}\right)$ is bounded and that $\int_\Omega \sigma_i (\mu |\nabla u_i|)\, d\mathbf{x} \to \int_\Omega \sigma(\mu |\nabla u|)\, d\mathbf{x}$ since $\mu |\nabla u| \in L^1(\Omega)$. The strong convergence of (u_i) implies $|\nabla u_i| \to |\nabla u|$ in $L^2(\Omega)$ since $\||\nabla u| - |\nabla u_i|\|_{L^2(\Omega)} \leq \|\nabla u - \nabla u_i\|_{L^2(\Omega,\mathbb{R}^d)}$ for all i. The conlusion follows by the identification of $\left(L^1(\Omega)\right)^\star$ with $L^\infty(\Omega)$ and of $\left(L^2(\Omega)\right)^\star$ with $L^2(\Omega)$. □

First order approximations

In Section 3.4, three types of regularizations were formulated with the equations of complete electrode model. For each of them, its corresponding approximate cost functional $T_{\lambda,k}$ is provided below. For simplicity, we consider $N = 1$.

Formulation with applied current. In this case $x = (u, U)$ and $X = Z = H^1(\Omega) \times \mathbb{R}^M_\diamond$. The first order approximation $A_k : L^\infty(\Omega) \times X \to Z^\star$ of A at $(\sigma_k, (u_k, U_k))$ is given by

$$\begin{aligned} &A_k(\sigma, (u, U))(w, W) = \\ &\quad \int_\Omega \sigma_k \nabla u \cdot \nabla w \, d\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{(\gamma_m u - U_m)(\gamma_m w - W_m)}{z_m} \, d\mathbf{s} + \\ &\quad \int_\Omega \sigma \nabla u_k \cdot \nabla w \, d\mathbf{x} - \int_\Omega \sigma_k \nabla u_k \cdot \nabla w \, d\mathbf{x} - \sum_{m=1}^M I_m W_m. \end{aligned}$$

The first order approximations of C at $(\sigma_k, (u_k, U_k))$ are:

i. *Voltage measurements.* Here is not necessary to approximate because C is linear, namely $C(\sigma, (u, U)) = U$.

ii. *Magnitudes of current density field*:

$$(\sigma, (u, U)) \mapsto \sigma |\nabla u_k| + \frac{\sigma_k}{|\nabla u_k|} \langle \nabla u_k, \nabla u \rangle - \sigma_k |\nabla u_k|.$$

iii. *Interior power density data*:

$$(\sigma, (u, U)) \mapsto \sigma |\nabla u_k|^2 + 2\sigma_k \langle \nabla u_k, \nabla u \rangle - 2\sigma_k |\nabla u_k|^2.$$

Hence, given a noisy observation $y \in Y$, the approximate cost functional is expressed as

$$\begin{aligned} T_{\lambda,k}(\sigma, (u, U)) = &\frac{1}{2} \|A_k(\sigma, (u, U))\|^2_{\left(H^1(\Omega) \times \mathbb{R}^M_\diamond\right)^\star} \\ &+ \begin{cases} \frac{1}{2} \|U - y\|^2_{\mathbb{R}^M} & \text{(i)} \\ \frac{1}{2} \left\| \sigma |\nabla u_k| + \frac{\sigma_k}{|\nabla u_k|} \langle \nabla u_k, \nabla u \rangle - \sigma_k |\nabla u_k| - y \right\|^2_{L^2(\Omega)} & \text{(ii)} \\ \left\| \sigma |\nabla u_k|^2 + 2\sigma_k \langle \nabla u_k, \nabla u \rangle - 2\sigma_k |\nabla u_k|^2 - y \right\|_{L^1(\Omega)} & \text{(iii)} \end{cases} \\ &+ \alpha \cdot R(\sigma, (u, U)). \end{aligned}$$

Formulation with applied voltage. In this case $x = u$ and $X = Z = H^1(\Omega)$. The first order approximation $A_k : L^\infty(\Omega) \times X \to Z^\star$ of A at (σ_k, u_k) is given by

$$\begin{aligned} A_k(\sigma, u) w = &\int_\Omega \sigma_k \nabla u \cdot \nabla w \, d\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{\gamma_m u \, \gamma_m w}{z_m} \, d\mathbf{s} \\ &+ \int_\Omega \sigma \nabla u_k \cdot \nabla w \, d\mathbf{x} - \int_\Omega \sigma_k \nabla u_k \cdot \nabla w \, d\mathbf{x} - \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{V_m \gamma_m w}{z_m} \, d\mathbf{s}. \end{aligned}$$

The first order approximations of C at (σ_k, u_k) are:

i. *Current measurements.* Here is not necessary to approximate because C is affine, namely $C(\sigma, u) := \left(\int_{\mathcal{E}_m} \frac{V_m - \gamma_m u}{z_m}\, d\mathbf{s}\right)_{m=1}^{M}$.
ii. *Magnitudes of current density field*:

$$(\sigma, u) \mapsto \sigma |\nabla u_k| + \frac{\sigma_k}{|\nabla u_k|} \langle \nabla u_k, \nabla u \rangle - \sigma_k |\nabla u_k| .$$

iii. *Interior power density data*:

$$(\sigma, u) \mapsto \sigma |\nabla u_k|^2 + 2\sigma_k \langle \nabla u_k, \nabla u \rangle - 2\sigma_k |\nabla u_k|^2 .$$

Hence, given a noisy observation $y \in Y$, the approximate cost functional is expressed as

$$\begin{aligned} T_{\lambda,k}(\sigma, u) = &\frac{1}{2} \|A_k(\sigma, u)\|^2_{(H^1(\Omega))^\star} \\ &+ \begin{cases} \frac{1}{2} \left\| \left(\int_{\mathcal{E}_m} \frac{V_m - \gamma_m u}{z_m}\, d\mathbf{s}\right)_{m=1}^{M} - y \right\|^2_{\mathbb{R}^M} & \text{(i)} \\ \frac{1}{2} \left\| \sigma |\nabla u_k| + \frac{\sigma_k}{|\nabla u_k|} \langle \nabla u_k, \nabla u \rangle - \sigma_k |\nabla u_k| - y \right\|^2_{L^2(\Omega)} & \text{(ii)} \\ \left\| \sigma |\nabla u_k|^2 + 2\sigma_k \langle \nabla u_k, \nabla u \rangle - 2\sigma_k |\nabla u_k|^2 - y \right\|_{L^1(\Omega)} & \text{(iii)} \end{cases} \\ &+ \alpha \cdot R(\sigma, u) . \end{aligned}$$

Alternative model. In this case $x = (u, U)$, $X = H^1(\Omega) \times \mathbb{R}^M_\diamond$, and $Z = H^1(\Omega)$. This choice of X is allowed if the exact observation is in $\mathbb{R}^M_\diamond \times \mathbb{R}^M_\diamond$ (see Rem. 3.22). The first order approximation $A_k : L^\infty(\Omega) \times X \to Z^\star$of A at $(\sigma_k, (u_k, U_k))$ is given by

$$\begin{aligned} A_k(\sigma, (u, U))(w) = &\int_\Omega \sigma_k \nabla u \cdot \nabla w \, d\mathbf{x} + \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{(\gamma_m u - U_m)\gamma_m w}{z_m}\, d\mathbf{s} \\ &+ \int_\Omega \sigma \nabla u_k \cdot \nabla w \, d\mathbf{x} - \int_\Omega \sigma_k \nabla u_k \cdot \nabla w \, d\mathbf{x}. \end{aligned}$$

The observation map C is linear, namely $C(\sigma, (u, U)) = \left(U, \left(\int_{\mathcal{E}_m} \frac{U_m - \gamma_m u}{z_m}\, d\mathbf{s}\right)_{m=1}^{M}\right)$. Hence, the approximate cost functional is

$$\begin{aligned} T_{\lambda,k}(\sigma, (u, U)) = &\frac{1}{2} \|A_k(\sigma, (u, U))\|^2_{(H^1(\Omega))^\star} \\ &+ \frac{1}{2} \|U - y_{\text{vol}}\|^2_{\mathbb{R}^M} + \frac{1}{2} \left\| \left(\int_{\mathcal{E}_m} \frac{U_m - \gamma_m u}{z_m}\, d\mathbf{s}\right)_{m=1}^{M} - y_{\text{cur}} \right\|^2_{\mathbb{R}^M} \\ &+ \alpha \cdot R(\sigma, (u, U)) , \end{aligned}$$

where $y = (y_{\text{vol}}, y_{\text{cur}}) \in \mathbb{R}^M \times \mathbb{R}^M$ is the noisy observation.

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