# Sheaves on semicartesian monoidal categories and applications in the quantalic case

## *A significant change in the Grothendieck pretopologies*

Ana Luiza da Conceição Tenório

Thesis presented to the
Institute of Mathematics and Statistics
of the University of São Paulo
in partial fulfillment
of the requirements
for the degree of
Doctor of Science

Program: Mathematics
Advisor: Prof. Dr. Hugo Luiz Mariano

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001

São Paulo
July 13, 2023

# Sheaves on semicartesian monoidal categories and applications in the quantalic case

***A significant change in the Grothendieck pretopologies***

Ana Luiza da Conceição Tenório

This version of the thesis includes the corrections and modifications suggested by the Examining Committee during the defense of the original version of the work, which took place on July 13, 2023.

A copy of the original version is available at the Institute of Mathematics and Statistics of the University of São Paulo.

Examining Committee:

Prof. Dr. Hugo Luiz Mariano (advisor) – IME-USP

Prof. Dr. Kostiantyn Iusenko – IME-USP

Prof. Dr. Peter Arndt – Universität Düsseldorf

Prof. Dr. Valeria Correa Vaz de Paiva – Topos Institute

Prof. Dr. Ciro Russo – UFBA

# Acknowledgment

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. Also, I had the opportunity to stay in Germany at Heinrich Heine University Düsseldorf for six months to improve my research thanks to the program PrInt-USP, idealized by CAPES. Grant Number 88887.694529/2022-00. The financial support is essential for the development of scientific research.

I am thankful for the suggestions from the examining committee. I am sure this thesis has improved thanks to their guidance.

This work was only possible thanks to a series of situations and people who contributed to my personal and academic development, which goes far beyond my mathematical abilities. There are numerous people I am grateful to, but some have had a more direct impact on this thesis and my journey in the doctoral program and academic career.

In particular, I am grateful for the fortunate opportunity to be mentored by Professor Hugo Luiz Mariano, for the time he dedicated to guiding this work; for caring about my future in times when academic careers are increasingly competitive, with few scholarship and job opportunities; for understanding the difficulties imposed by life, especially in times of pandemic; for respecting my doubts and even the most beginner mistakes; and for treating me not only as a student but as a collaborator.

I am immensely thankful to Peter Arndt for agreeing to be my supervisor in Germany and promptly taking care of the necessary paperwork and spending long hours discussing mathematics with me. It was a fantastic experience to witness how another experienced mathematician works. I also thank Peter for being so welcoming and empathetic to the challenges of being in another country. Part of the results in this thesis were obtained thanks to my time with Peter, his expertise in mathematics, and his support. I cherish the moments with Frano and Ole!

I also want to thank Jonathan, my flatmate whose company made my stay in the German winter warmer. Thanks for the kindness and for the great meals! And I could not fail to mention Silke and Ralph, for the welcome and farewell dinners and for being the best landlady/lord an exchange student could wish for.

I appreciate the support from my family, who always respected and supported my decisions, even when they were so far from their own reality, such as pursuing a doctoral degree and striving to continue in academia. I also appreciate the company and affection of my canine siblings Bebê and Brown!

I acknowledge the importance of my classmates, course instructors, and academic peers in my education. I highlight João Schwarz and Kaique Matias for our mathematical discussions, and Thiago Dourado, Fernando Lima, and Michel Smykalla for our conversations about our political role as young researchers and our struggles as student representatives. Thank you for the partnership.

I am grateful for making incredible friends with whom I spent good years living in the same house while pursuing (the master's and) the doctoral degree: Bartira Maués, Marisa Cantarino, Marli Cantarino, and Rafael Bordoni. I am also grateful for the friends with whom I did not have the opportunity to live together but who made all the difference in emotional support during these past years: Gladys Chalom, Luciana Bonatto, Simone Harnik, and Viviana Giampaoli. We had numerous conversations about mathematics, academic careers, politics, emotional support, and practical advice. I have no words to express my gratitude.

During the doctoral program, my participation in feminist collectives decreased significantly. I wanted to focus on the growth of my research area, category theory, in Brazil. However, I still want to emphasize that the collectives and the interaction with experienced women mathematicians, whether professors or not, were factors in my permanence and completion of this thesis. The support network is important not only from the perspective of representation and the power to be inspired by these women but also for the exchange of knowledge, practicing how to prepare for a presentation, how to write a project, how to prepare for a job competition, and so on.

Lastly, I would like to thank my life partner, Patricia Cezar. Patricia came into my life more towards the final stages of the doctoral program, and she has made me a better person. Besides being grateful because life is better with her by my side, she has always been willing to help me by reviewing my texts and watching my practice sessions for important presentations. Even though she is not from the field of mathematics, it was through mathematics that we found each other, and I am grateful for that. However, I believe destiny would have found a way to bring us together sooner or later.

# Resumo

Ana Luiza da Conceição Tenório. **Feixes em categorias monoidais semicartesianas e aplicações no caso quantálico: *Uma mudança significativa nas pré-topologias de Grothendieck***. Tese (Doutorado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2023.

Nessa tese de doutorado nós apresentamos a noção de pré-lopologias de Grothendieck, que é uma noção de cobertura para categorias monoidais semicartesianas que generaliza as pré-topologias de Grothendieck. Mais do que isso, tal generalização engloba uma certa noção feixes em quantales semicartesianos, $Q$, introduzida nessa tese, a qual é mais geral que a definição usual de feixes em locales $L$. Verificamos que as respectivas categorias de feixes, $Sh(Q)$ e $Sh(L)$, possuem propriedades em comum, contudo, $Sh(Q)$ nem sempre forma um topos de Grothendieck. A análise do reticulado dos subobjetos do feixe terminal em $Sh(Q)$ sugere que a noção de feixes para as prelopologias de Grothendieck possui uma lógica interna linear em vez de intuicionista. Ainda, desenvolvemos uma cohomologia de Čech na qual os coeficientes são feixes em um quantale e encontramos um morfismo entre o locale dos abertos de um espaço topológico $X$ e o quantale dos ideais do anel $C(X)$ das funções contínuas em $X$ que permite relacionar a cohomologia de Čech de $X$ e a cohomologia (expandida) de Čech de $C(X)$.

**Palavras-chave:** Feixes. Pré-topologia de Grothendieck. Quantales. Categorias monoidais. Cohomologia de Čech.

# Abstract

Ana Luiza da Conceição Tenório. **Sheaves on semicartesian monoidal categories and applications in the quantalic case: *A significant change in the Grothendieck pretopologies*.** Thesis (Doctorate). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2023.

In this doctoral thesis, we introduce the notion of Grothendieck prelopologies, which is a notion of covering for semicartesian monoidal categories that generalizes Grothendieck pretopologies. Moreover, this generalization encompasses a certain notion of sheaves in semicartesian quantales $Q$, introduced in this thesis, which is more general than the usual definition of sheaves on locales $L$. We observe that the respective sheaf categories, $Sh(Q)$ and $Sh(L)$, share certain properties; however, $Sh(Q)$ does not always form a Grothendieck topos. The analysis of the lattice of subobjects of the terminal sheaf in $Sh(Q)$ suggests that the notion of sheaves for Grothendieck prelopologies has a linear internal logic rather than an intuitionistic one. Furthermore, we develop a Čech cohomology in which the coefficients are sheaves on a quantale, and we find a morphism between the locale of open sets of a topological space $X$ and the quantale of ideals of the ring $C(X)$ of continuous functions on $X$ that allows us to relate the Čech cohomology of $X$ and the (expanded) Čech cohomology of $C(X)$.

**Keywords:** Sheaves. Grothendieck pretopology. Quantales. Monoidal categories. Čech cohomology.

# Contents

# Chapter 1

# Introduction

Section 1.3 in this introduction is dedicated to pointing out the major technical contributions of this thesis, which I do believe were substantial by itself but, perhaps, the main contribution of our work consists of the endless questions that arise after the birth of a new theory. In my experience, there are two groups of people that work with toposes: on one side we have a group that is more interested in the study of Grothendieck toposes (generalized sheaves) and on the other side there is a group more focused on the study of elementary toposes (a category with an intuitionistic internal logic that behaves like the category of all sets and functions), and there is a small intersection between those groups. This thesis may interest the first group a bit more, but the second should be interested that we generalized the notion of a Grothendieck topos with the future goal to also generalize elementary toposes, obtaining a category with a linear internal logic. Moreover, we achieved cohomological methods that may interest geometers and algebraists too. Therefore, it is important to give a panoramic view of topos theory to motivate the reading for those distinct groups.

In this introduction, we also explain the original goal of the project and then summarize the main contributions contained in this text. Finally, I explain how this thesis is organized.

## 1.1 Topos Theory

The origin of sheaf theory is attributed to J. Leray, more specifically, to his paper [Ler45]. Leray was interested in solving partial differential equations using a tool that could track local properties that under gluing conditions also hold globally. The theory spread quickly: in Cartan's seminars, between the late 1940s and early 1950s two versions of sheaves were studied. One is given by local homeomorphisms (= étale maps) into a topological space $X$ and the other is given by a "coherent family" of structures indexed on the lattice of open subsets of $X$. Denote such lattice by $\mathcal{O}(X)$. Both versions are intimately related by an equivalence of categories, as described in [MM92], for instance.

Later we will talk about a third concept that is related to the notion of a sheaf (a $L$-set) by an equivalence of categories. Then, what is a sheaf? For us, a sheaf of sets (on a topological space $X$) is given by a coherent family, which we describe using categorical terms: it is a functor $F : \mathcal{O}(X)^{op} \to Set$ such that for all $U \in \mathcal{O}(X)$ and all $U = \bigcup_{i \in I} U_i$ open cover of $U$ the diagram below is an equalizer in $Set$

$$F(U) \xrightarrow{e} \prod_{i \in I} F(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(U_i \cap U_j)$$

where:

1. $e(t) = \{t_{|U_i} \mid i \in I\}, \quad t \in F(U)$
2. $p((t_k)_{k \in I}) = (t_{i_{|U_i \cap U_j}})_{(i,j) \in I \times I}$
   $q((t_k)_{k \in I}) = (t_{j_{|U_i \cap U_j}})_{(i,j) \in I \times I}, \quad (t_k)_{k \in I} \in \prod_{k \in I} F(U_k)$

Here we want to focus on the shape of the diagram that defines a sheaf on a topological space. In particular, we highlight that the cover of $U$ and the intersection operation are the main characters in the notion of a sheaf. The generalization of sheaves on topological spaces to sheaves on categories is motivated by problems in algebraic geometry but the topological case already gives a fundamental class of sheaves: consider $R$ a commutative ring with unity and the spectrum $Spec(R)$ of $R$ formed by all prime ideals of $R$. Then $Spec(R)$ is a topological space under the Zariski topology, where the closed sets are of the form $V(I) = \{P \text{ prime ideal } : I \subseteq P\}$. Taking complements of $V(I)$ we have open sets and then we may consider sheaves of rings[1] of the form $\mathcal{O}(Spec(R))^{op} \to CRing$. In particular, there is a canonical sheaf associated to $R$ that is determined on a basis of the Zariski topology of $Spec(R)$ by taking adequate localizations of the ring $R$; the stalk of this sheaf at a proper prime ideal $P \in Spec(R)$ is isomorphic to the local ring $R_P = R[R \setminus P]^{-1}$.

This construction appears in the concept of schemes, which are central in modern algebraic geometry because the pairs of the form $(Spec(R), \mathcal{O}_{Spec(R)})$ are called affine schemes – more precisely, affine schemes are locally ringed spaces isomorphic to $Spec(R)$ – and the gluing of ringed spaces of the form $(Spec(R), \mathcal{O}_R)$ results in the notion of a scheme – more precisely, a scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that $X = \bigcup_{i \in I} U_i$ and each $U_i$ is a locally ringed space. In turn, schemes are used to prove the Grothendieck-Riemann–Roch theorem, a generalization of the Riemann–Roch theorem [Gro71]. Moreover, there is a notion of morphism between schemes called *étale maps*. Grothendieck envisioned, based on Jean-Pierre Serre's ideas, that if he replaced the usual notion of open covering with one that uses étale coverings, then he would be able to construct a Weil cohomology theory, and so prove the Weil conjectures. This motivates Grothendieck and his school's efforts to pursue a more general notion of covering, the Grothendieck pretopologies.

We properly define it in 2.3.1 but roughly speaking, given $\mathcal{C}$ a category with pullbacks and $U$ an object in $\mathcal{C}$, a Grothendieck pretopology is a map $K$ that assigns to $U$ a family of morphisms $K(U) = \{U_i \to U\}_{i \in I}$ satisfying three properties, including a property of stability under pullbacks. The families $K(U)$ are called covering families. Once one known that in the poset category $\mathcal{O}(X)$ pullbacks are given by intersections, it is expected that a sheaf on a category equipped with a Grothendieck pretopology should be a functor $F : \mathcal{C}^{op} \to Set$

[1] A functor $\mathcal{O}(X)^{op} \to CRing$ such that the composition with the forgetful functor $\mathcal{O}(X)^{op} \to CRing \to Set$ is a sheaf.

such that the following diagram is an equalizer (see details in Definition 2.3.4)

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

A sheaf cohomology in this general framework was developed and étale cohomology is a particular case of sheaf cohomology that Grothendieck, together with Artin and Verdier, used to prove three of the four Weil conjectures, by the end of 1964. Later, Deligne proves the remaining conjecture. In this context, Grothendieck pretopologies are enough but they have the issue that different pretopologies may provide the same class of sheaves. To solve this, there are *Grothendieck topologies* and a respective notion of sheaves. It is possible to obtain a Grothendieck pretopology from a Grothendieck topology and vice-versa, see [MM92, Chapter III.2]. Actually, some authors refer to Grothendieck pretopologoies as a basis for a Grothendieck topology. Now, we can say that a *Grothendieck topos* is any category that is equivalent to a category of sheaves on a certain category equipped with a Grothendieck topology.

In the early 1970s, Grothendieck's work reached W. Lawvere and M. Tierney and they realized that a Grothendieck topos have categorical properties that make it close to the category *Set* of all sets and functions. For example, sheaves admit exponential objects that are analogs of the set $A^B$ of all functions from $B$ to $A$, and there is an object of truth-values (subobject classifier) that, in the category *Set*, is the set $\{true, false\}$. Eventually, they defined an *elementary topos* as a category that is cartesian closed, has a subobject classifier, and has all finite limits. Since every Grothendieck topos is an elementary topos but the converse does not hold, sheaves are an important object of study in topos theory, where we use "topos" to refer to an elementary topos.

One of the striking features of topos theory is that every topos has an (intuitionistic) internal language, known as *Mitchell-Bénabou language*, and a canonical *interpretation* - a procedure to give a meaning to the symbols introduced in the canonical language. We recommend [Bor94c] and [McL92] as an introduction to those logical aspects, for now we just highlight that the lattice structure of the subobjects of a topos, in particular of the subobject classifier, is crucial for the behavior of its logic. The point is that such considerations are useful for the working mathematician: due to the *Soundness Theorem* [McL92, Chap 15], sometimes, we can pretend that a given topos is just *Set*. For instance, to develop a sheaf cohomology theory for a Grothendieck topos $\mathcal{E}$ an important step is to prove that the category $Ab(\mathcal{E})$ of abelian group objects in $\mathcal{E}$ is an abelian category. This can be done by brute force or by pretending that $\mathcal{E}$ is *Set*. Since there is an equivalence $Ab(Set) \simeq Ab$, the proof follows from the fact that $Ab$ is an abelian category, see [Joh77] or [TM21]). Thus, the Soundness Theorem exempts us from 10 pages of calculations, which are done in [Şte81]. The dictionary between the external and the internal point of view is extensive: objects in a topos are, internally, sets; monomorphisms are injections; sheaves of rings are rings; etc., and we recommend [Ble17] to see the use of the internal language of toposes in algebraic geometry. Unfortunately, there is a cost to pretend that a topos is *Set*. If you want to do this then our reasoning needs to be constructive/intuitionistic, because the law of excluded middle ( i.e., $\varphi \vee \neg\varphi$) does not hold for all toposes.

The above was a panoramic view of well-established concepts in sheaf and topos theory that will be relevant to this thesis. However, in this thesis, we also address the problem of achieving a notion of a sheaf on a quantale. First, note that the definition of a sheaf on a topological space does not use the points of the space. Indeed, the lattice $\mathcal{O}(X)$ forms a structure called *locale*, a complete lattice where the meet distributes over arbitrary joins and we define a sheaf on a locale $L$ by replicating the definition of a sheaf on a topological space $X$ where instead of taking open subsets of $X$ we take elements in $L$ and we replace unions and intersections with joins and meets, respectively. Locales admit a generalization in which we have an additional binary operation $\odot$ and then is the new operation $\odot$ that has to distribute over arbitrary joins. Therefore, it is natural to wonder how to define sheaves on quantales and, in fact, different authors under different approaches answer this question [BB86], [BC94], [MS98], [FS79], [ASV08], [HS12], [Res12]. We have the following remarks about the currently available notions:

- Most of them are concerned with the class of idempotent quantales. In this work, we focus on semicartesian quantales, which has a natural notion of *projection*, in the sense they always have arrows of the form $u \odot v \to u$ and $u \odot v \to v$. Since a quantale that is simultaneously idempotent and semicartesian is, necessarily, a locale, the notions only compare in the already well-known case of sheaves on locales. Therefore, we can say that the theory we developed is orthogonal to the one usually developed in the literature;

- Most of them are not exactly about sheaves in the traditional functorial sense. By this, we mean they are not defined as contravariant functor that forms an equalizer diagram, as we introduced above. In this work, the objects under analysis are functorial in the traditional sense. In some cases (as in [BC94]), the literature uses the equivalence between the category of sheaves on a locale $L$ and the category of $L$-sets (for a suitable notion of morphism) and they define sheaves on a quantale $Q$ as a structure that is generalizing an $L$-set. In the preliminaries, we will introduce the concept of $L$-sets for the comfort of the reader.

- To the best of our knowledge, previous work either deal with a notion of a sheaf on $Q$ that forms a Grothendieck topos or do not discuss this matter at all. In the opposite direction, we pursue a notion of a sheaf on a quantale that since the beginning we expected to not be a Grothendieck topos.

The results we will state below explain the relevance of studying sheaves on commutative semicartesian quantales but since the literature is significantly focused on sheaves on idempotent quantales, it is important to highlight the interest in quantales in general and also in this different class of quantales. In [Mul86], C. Mulvey introduced quantales as a non-commutative version of locales. Since locales are used to study point-free topology, quantales are a natural candidates to study non-commutative topology. Besides it, C. Mulvey was interested in foundations of quantum mechanics. The point is that the algebra of observables in quantum mechanics is an algebra of operators defined on a Hilbert space and $C^*$-algebras provide an abstract framework to describe operators on a Hilbert space. Since closed right (left) ideals of a $C^*$-algebra form an idempotent and right-sided (left-sided) quantale, the study of idempotent right-sided quantales also improves investigations regarding foundations of quantum mechanics. A successful example of application in this direction was obtained by Francisco Miraglia and Marcelo Coniglio: they proved that, in their category of sheaves $Sh(Q)$, every finitely generated projective module over a local ring is free with a finite basis [CM01, Theorem 7.2.] (this is an analogous version of Kaplansky's theorem on projective modules, but in $Sh(Q)$ instead of $Set$). Note that a consequence of such result could be a characterization of finitely generated projective $A$-modules, where $A$ is a $C^*$-algebra, if we obtain a representation of $A$ in terms of (global sections) of a sheaf on the quantale of closed right ideals of $A$. Observe that a representation theorem as that is not unexpected since sheaf theory has examples of representation theorems in the same vein. For instance, in the classical notion of sheaves on locales, every commutative ring with unit is isomorphic to the ring of global sections of a corresponding structural space [Bor94c, Theorem 2.11.15]. Similarly, in [BC94] there is an analogous statement for non-commutative rings (and non-commutative quantales).

## 1.2 About this thesis

The first question we wanted to answer was: how to define a sheaf on a quantale so that it is as close as possible to the sheaf on a locale definition? Bearing this in mind, we concluded that the definition should be the following:

**(Definition 3.2.1)** A presheaf $F : Q^{op} \to Set$ is a **sheaf on** $Q$ when for all $u \in Q$ and all $u = \bigvee_{i\in I} u_i$ cover of $u$ the following diagram is an equalizer in $Set$

$$F(u) \xrightarrow{e} \prod_{i\in I} F(u_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j)\in I\times I} F(u_i \odot u_j)$$

where:

1. $e(t) = \{t_{|u_i} \mid i \in I\}, \quad t \in F(u)$
2. $p((t_k)_{k\in I}) = (t_{i_{|u_i\odot u_j}})_{(i,j)\in I\times I}$
   $q((t_k)_{k\in I}) = (t_{j_{|u_i\odot u_j}})_{(i,j)\in I\times I}, \quad (t_k)_{k\in I} \in \prod_{k\in I} F(u_k)$

We use $Q$ semicartesian so that $u_i \odot u_j$ always is less or equal to $u_i$ and $u_j$, for all $i, j \in I$ and then we the maps $F(u_i) \to F(u_i \odot u_j)$ exist.

Then we started to investigate the categorical properties of $Sh(Q)$, where the objects are sheaves on $Q$ and the morphisms are natural transformations. Some categorical properties of the localic case can be checked almost verbatim for the quantalic case, as we show in Section 3.2, but some that are the core of topos theory (being cartesian closed and having subobject classifier) seemed to no longer hold, which leads to the challenge of

*How to expand the definition of sheaf on a site in a way that encompasses both the classic notion of a sheaf and our sheaves on quantales?*

Note that the first step is to interpret the quantale multiplication $\odot$ in categorical terms. In the same way the infimum is the pullback in the poset category given by a locale, we will have that the multiplication is the *pseudo-pullback* in the poset category given by a locale. Moreover, we construct such limit in a way that the pseudo-pullback of a cartesian category (i.e., where the monoidal tensor is the categorical product) is the pullback. In other words, pseudo-pullbacks are generalizations of pullbacks that encompass the quantale multiplication.

Now, recall that for a locale $L$, a cover of $u \in L$ is a family of $\{u_i \leq u\}_{i \in I}$ such that $u = \bigvee_{i \in I} u_i$ and such a family actually satisfies the axioms of a Grothendieck pretopology. For a quantale $Q$, we also have that a cover of $u \in Q$ is a family of $\{u_i \leq u\}_{i \in I}$ such that $u = \bigvee_{i \in I} u_i$ but such a family does not satisfy the axioms of a Grothendieck pretopology, even if we replace the pullbacks with pseudo-pullbacks. So we had to non-trivially adapt the notion of a Grothendieck pretopology, in what we call a Grothendieck prelopology (Definition 4.1). This will give rise to a notion of a sheaf on a site – for a semicartesian monoidal category with pseudo-pullbacks equipped with a Grothendieck prelopology – that encompass both the notion of sheaves on quantales and sheaves on a category with pullbacks equipped with a Grothendieck pretopology.

The general framework above helped us to show that the inclusion functor from sheaves to presheaves has a left right adjoint, that is, the *sheafification functor* exists. Besides it, in the quantalic case we were able to prove that such sheafification preserves the monoidal structure in the category of presheaves $PSh(Q)$, which is inherited by the monoidal structure on $Q$, by Day convolution. In particular, we obtain that $Sh(Q)$ is monoidal closed. Using the monoidal structure in $Sh(Q)$ we were able to finally prove that our sheaves, indeed, do not form an elementary topos and, therefore, they do not form a Grothendieck topos. Actually, in the process we proved more: we showed that the lattice of subterminal objects in $Sh(Q)$ is quantalic isomorphic to $Q$. This suggests that the internal logic of $Sh(Q)$ may be affine[2], to match with the structure of semicartesian quantales.

The existence of the sheafification functor also allowed to obtain sheaves on a quantale $Q$ from a given sheaf on a quantale $Q'$, if there is a *strong morphism* $Q' \to Q$. This is particularly crucial as it is challenging to find concrete examples of sheaves within our theory, but if we have a strong morphism $L \to Q$ from a locale to a quantales, then we can obtain sheaves on quantales from sheaves on locales. Surprisingly or not, this is useful to construct bridges between geometry and algebra. On the geometric side, we can consider, for example, the locale of open subsets of a topological space $X$, and on the algebraic side we have the quantale of ideals of the ring $C(X)$ of continuous real valued functions on $X$. During my time at the University of Düsseldorf, under the supervision of Peter Arndt, we constructed a strong morphism between those quantales, if $X$ admits partition of unity subordinate to a cover. Furthermore, we realized that the Čech cohomology of $X$ with coefficients in a sheaf $F$ on the respective locale is isomorphic to an expanded Čech cohomology of $C(X)$ – developed in this thesis – with coefficients in a sheaf in the respective quantale that arises from $F$. Indicating a cohomological method to relate geometrical/topological properties of $X$ and algebraic properties of $C(X)$. The most exciting part about this result is that the isomorphism between the cohomological groups does not depend on properties of the topological spaces $X$ or the ring $C(X)$, the important steps rely on the properties of the morphisms between the quantales. So the same phenomenon

[2] In this thesis, we are not explicit about the logical aspects of the theory. A reader used to categorical logic may infer the consequences of the results regarding subobjects but we only make more explicit considerations about it in the final chapter, where we explore future topics of research.

should appear in other contexts.

If we consider that "classic" topos theory follows a course akin to the below:

$$\text{sheaves on locales} \rightsquigarrow \text{Grothendieck toposes} \rightsquigarrow \text{toposes.}$$

Then this thesis is a step to the construction of a *lopos* theory that follows a course

$$\text{sheaves on (semicartesian) quantales} \rightsquigarrow \text{Grothendieck loposes} \rightsquigarrow \text{loposes.}$$

In a way that generalizes the classic and well-established topos theory.

## 1.3 Main Contributions

By what was said before, in a panoramic view, this thesis contributes to a research that goes towards non-intuitionistic versions of topos theory, as in Höhle's work [Höh91]. More concretely, we achieved two major contributions. The first is more related to categorical aspects of the theory and states that $Sh(Q)$ is not a topos, showing that our sheaves for a Grothendieck pretolopogy are actually describing a structure that the current theory does not describe. The more skeptical reader may wonder: why we should care about $Sh(Q)$ in the first place? We have two answers for it: from the perspective of categorical logic, it is interesting because it suggests we have a notion of sheaves that may lead to non-intuitionistic versions of topos theory. From the perspective of mathematical practice, we argue that expanded notions of sheaves lead, for instance, to more applications of sheaf cohomology. In particular, we provided an expanded Čech cohomology and an isomorphism between the $q$th Čech cohomology group of a topological space $X$ and the $q$th Čech cohomology group of the ring $C(X)$ of continuous real valued functions on $X$.

Going straight to the point, we list below the main results of the thesis:

**On the main theoretical constructions and results of the theory**

- A definition of sheaves on quantales that generalizes the definition of sheaves on locales. The category of sheaves on a fixed quantale that is not a locale has some of the categorical properties of the category of sheaves on a fixed locale.
- We proved a base change Theorem (3.4.8) that allows to creat sheaves on quantales from sheaves on other quantales and, in particular, from sheaves on locales. We use this to show that the structure sheaf $\mathcal{O}_{Spec(R)}$ defined by $\mathcal{O}_{SpecR}(D(a)) \cong R_a$ – where $R$ is commutative ring with unity, $D(a)$ is the principal open[3] for $a \in R$ and $R_a$ is the localization $R[a^{-1}]$ – coincides with a sheaf on a basis of the quantale of ideals of $R$, which is defined without the spectrum of $R$. See Example 3.3.5 and the discussion that follows it.
- We defined a notion of a Grothendieck prelopology such that the notion of a cover we used to define sheaves on quantales is a cover in the prelopology sense 4.1.10

[3] The principal open $D(a)$ forms a basis under the Zariski topology.

and such that every Grothendieck pretopology is a Grothendieck prelopology 4.1.11. Conversely, the Grothendieck prelopology of a cartesian category with equalizers (i.e., a category with finite limits) is a Grothendieck pretopology 4.1.12.

- There is sheafification functor $a$ (considering sheaves for Grothendieck prelopologies), which is adjoint to the inclusion functor $i$ and, in the quantalic case, the monoidal structure is preserved in a way such that $Sh(Q)$ is monoidal closed and we have $F \otimes G = a(i(F) \otimes_{Day} i(G))$, for $F, G$ sheaves on $Q$

**On categorical logic**

- Let $Q$ be a semicartesian quantale, then the lattice of subobjetcs of the terminal sheaf in $Sh(Q)$ is quantalic isomorphic to $Q$ 4.3.6. This result implies that:
- In general, $Sh(Q)$ is not a topos 4.3.8, and $Sh(Q)$ does not have subobject classifier if the meet in $Q$ does not distribute over the arbitrary joins 4.3.10.

**On cohomological methods**

- We developed an extended Čech cohomology to be able to talk about the Čech cohomology of a ring, where the covers are sums of ideals.
- We construct a pair of adjoint functors between the quantale $\mathcal{I}(C(X))$ of ideals of the ring $C(X)$ of continuous real valued functions on $X$ and the locale $\mathcal{O}(X)$ of open subsets of $X$ 3.4.14 such that the constant sheaf on $\mathcal{I}(C(X))$ is the composition of the constant sheaf on $\mathcal{O}(X)$ with the left adjoint functor of the pair 3.4.16. Moreover, we use this result to obtain an isomorphism between Čech cohomology groups of $X$ with coefficients in the constant sheaf on $\mathcal{O}(X)$ and the Čech cohomology groups of $C(X)$ with coefficients in the constant sheaf on $\mathcal{I}(C(X))$ 4.5.7. An analogous result holds for other coefficients 4.5.8.

We observe that part of this thesis (sections 3.1, 3.2, and 4.3) was submitted to a Journal and a preliminary version of the paper is available on ArXiv (everytime we mention this fact we are referring to the same paper). We – Ana Luiza Tenório, Hugo Luiz Mariano and Peter Arndt – are also preparing another paper with the sections about base change and cohomological methods. We hope to write a paper about sheaves on semicartesian monoidal categories in the near future.

## 1.4 Organisation of the thesis

Chapter 2 introduces technical details regarding monoidal categories and sheaf theory, providing preliminaries to the development of our sheaf theory on semicartesian monoidal categories. In Chapter 3 we introduce our novel notion of sheaves on semicartesian quantales and study the very first properties of the category $Sh(Q)$ formed by sheaves on a quantale $Q$ and natural transformations. Section 3.4 contains a base change theorem and examples of it that will be used to develop applications of our cohomological methods later. However, to prove the base change theorem and explore the examples we need a functor that is left adjoint to the inclusion functor $Sh(Q) \to PSh(Q)$. We choose to give those more applied results before providing the proof of the existence of such a left adjoint functor aiming to have some motivation first and then developing more technical constructions. We end the chapter addressing the problem of sheaves on quantales with algebraic structure, which will be necessary to a rigorous development of an expanded sheaf cohomology.

In Chapter 4, we develop the notion of a Grothendieck prelopology in a way that the respective notion of sheaves for a Grothendieck prelopology describes sheaves for a Grothendieck pretopology and also our sheaves on quantales. In Section 4.2, we prove that the sheafification exists, both as a localization functor (where we invert a certain class of arrows in the base category $(C, \otimes 1)$) and as the left adjoint of the inclusion functor from sheaves for Grothendieck prelopologies to presheaves, and prove that the sheafification preserves terminal object and, in the quantalic case, it also preserves the monoidal tensor, which, in $PSh(Q)$ is inherited by the monoidal structure on the base category $C$, by Day convolution. Once we presented the sheafification we use it, in Section 4.3, to show that $Sh(Q)$ is not a topos in general and that, unless the meet in $Q$ also distributes with arbitrary joins, $Sh(Q)$ does not have subobject classifier. Nevertheless, we dedicate a part of this work to discuss a subobject classifier candidate in $Sh(Q)$ that actually essentially classifies a certain class of monomorphisms. In Section 4.4 Since Theorem 4.3.6, states that the lattice of subobjetcs of the terminal sheaf in $Sh(Q)$ is quantalic isomorphic to $Q$, holds for (commutative and semicartesian) non-unital quantales – more specifically, for interval quantales of the form $[0, a]$, where $a \in Q$ – In Section 4.4 we construct sheaves on semicartesian categories that do not have a monoidal unity, those are known as *semigroupal categories*. This is motivated by Theorem 4.3.6, which states that the lattice of subobjetcs of the terminal sheaf in $Sh(Q)$ is quantalic isomorphic to $Q$, holds for (commutative and semicartesian) non-unital quantales – more specifically, for interval quantales of the form $[0, a]$, where $a \in Q$. We conclude the chapter with the part of the thesis that points out to the potential of our theory in other areas of Mathematics by using cohomological methods on Section 4.2. We develop Čech cohomology for quantales and evoke results presented in Section 3.4 about the change of base to show that the Čech cohomology groups of a topological spaces $X$ is isomorphic to the Čech cohomology groups of the ring $C(X)$ of continuous real valued functions on $X$, with the appropriated coefficients.

Finally, we present the conclusion of our work and propose a list of further developments with brief ideas of how to pursue each one of them. They are about non-commutative version of our theory, that is, for non-commutative quantales and non-symmetric monoidal semicartesian categories; thoughts into a sheaf cohomology theory, using right derived functors; paths towards a non-intuitionistic version of elementary toposes; possible applications in Quantum Mechanics; and the properties of the sheafification functor that we were not able to prove and how our approach to a broader version of sheaves provides a bird's-eye into sheaf categories as special cases of localization functors.

*In the thesis, we assume that the reader is familiar with category theory. Also, we are not careful about size issues in our theory, but to avoid problems assume the categories in the domain of a functor are small categories.*

# Chapter 2

# Preliminaries

This thesis is mainly about a generalization of sheaves of sets on (semicartesian) monoidal categories. Therefore, we present key concepts and results about monoidal categories and sheaf theory that will help us to expand sheaf theory. There is no novelty in this Chapter.

## 2.1 Monoidal categories

We start this section by recalling the definition of a monoidal category and providing basic examples. A reference for details is [Eti+16], for example.

**Definition 2.1.1.** *A **monoidal category** consists of:*

- *A category $C$;*
- *A bifunctor $\otimes : C \times C \to C$ called the **tensor product**;*
- *A natural isomorphism $a : (- \otimes -) \otimes - \xrightarrow{\simeq} - \otimes (- \otimes -)$ with components*

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\simeq} X \otimes (Y \otimes Z)$$

  *called the **associator (or associativity isomorphism)**;*
- *An object $1$ of $C$, called **tensor unit**.*
- *A natural isomorphism $\lambda : (1 \otimes (-)) \xrightarrow{\simeq} (-)$ with components*

$$\lambda_X : 1 \otimes X \to X$$

  *called the **left unitor**;*
- *A natural isomorphism $\rho : (-) \otimes 1 \xrightarrow{\simeq} (-)$ with components*

$$\rho_X : X \otimes 1 \to X$$

  *called the **right unitor**.*

*Such that the following two axioms hold:*

– ***The pentagon axiom:*** *For all $W, X, Y, Z$ objects in $C$, the diagram below commutes*

$$
\begin{tikzcd}
 & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "a_{W,X,Y\otimes Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "a_{W\otimes X,Y,Z}"] \arrow[d, "a_{W,X,Y}\otimes id_z"'] & & (W \otimes (X \otimes (Y \otimes Z))) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "a_{W,X\otimes Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "id_W \otimes a_{X,Y,Z}"']
\end{tikzcd}
$$

– ***The triangle axiom:*** *For all $X, Y$ objects in $C$, the diagram below commutes*

$$
\begin{tikzcd}
X \otimes (1 \otimes Y) \arrow[rr, "a_{X,1,Y}"] \arrow[dr, "id_X \otimes \lambda_Y"'] & & (X \otimes 1) \otimes Y \arrow[dl, "\rho_X \otimes id_Y"] \\
 & X \otimes Y &
\end{tikzcd}
$$

The definition is quite abstract, but there are simple examples of monoidal categories.

**Example 2.1.2.**

- *The category $K$-$Vect$ of vector spaces over a field $K$ has the well-known tensor of vector spaces as its (associative) tensor product and the field $K$ is the unit;*
- *The category of abelian groups and, more generally, $R$-Mod, the category of modules over a commutative ring $R$, with $R$ as the unit object, are monoidal categories;*
- *If $R$ is a commutative ring with unity, the category of $R$-algebras has the tensor product of algebras as the tensor product and $R$ as the unity;*
- *The category of pointed spaces $Top_*$, restricted to compactly generated spaces, has as tensor functor the smash product and the unity is the pointed $0$-sphere;*
- *The category Set of sets has the cartesian product as the tensor product and the unity is the singleton set (any one-element set).*
- *A lattice is a monoidal (posetal) category with tensor product given by the infimum.*
- *A monoid is a monoidal category with a single object and with tensor product given by the multiplication of the monoid.*

If a monoidal category is equipped with a natural isomorphism $B_{X,Y} : X \otimes Y \to Y \otimes X$ called **braiding** such that for all objects $X, Y, Z$ the following diagrams commute

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "B_{X,Y} \otimes id_Z"'] & X \otimes (Y \otimes Z) \arrow[r, "B_{X,Y\otimes Z}"] & (Y \otimes Z) \otimes X \arrow[d, "a_{y,z,x}"] \\
(Y \otimes X) \otimes Z \arrow[r, "a_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[r, "id_Y \otimes B_{X,Z}"'] & Y \otimes (Z \otimes X)
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "a^{-1}_{X,Y,Z}"] \arrow[d, "id_Z \otimes B_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[r, "B_{X\otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[d, "a^{-1}_{Z,X,Y}"] \\
X \otimes (Z \otimes Y) \arrow[r, "a^{-1}_{X,Z,Y}"] & (X \otimes Z) \otimes Y \arrow[r, "B_{X,Z} \otimes id_Y"'] & (Z \otimes X) \otimes Y
\end{tikzcd}$$

Then the monoidal category is called **braided**. Moreover, if the braiding satisfies that $B_{Y,X} \circ B_{X,Y} = id_{X \otimes Y}$, then we have what is called a **symmetric monoidal category**. So the symmetry in monoidal categories is almost given a notion of commutativity for the tensor product.

**Definition 2.1.3.**

- *A monoidal category is **cartesian monoidal** if the tensor functor is the categorical product.*
- *A monoidal category is **cocartesian monoidal** if the tensor functor is the categorical coproduct.*

**Definition 2.1.4.** *We say that a category is **cartesian closed** if it has cartesian products and if every object $X$ and $Y$ admits an exponential object $X^Y$.*

An exponential object $X^Y$ in a cartesian category $C$ is precisely the object that provides a natural isomorphism $Hom_C(X \times Y, Z) \cong Hom_C(X, Z^Y)$ for all $X, Y, Z$ objects in $C$. In other words, the functor $- \times Y$ has a right adjoint functor $(-)^Y$. The main example of a cartesian closed category is the $Set$, where the exponential $X^Y$ is the set of functions $\{Y \to X\}$. A locale $L$, which is a special kind of lattice such that the infimum distributes over arbitrary supremum (Definition 2.2.4), is a cartesian closed category where $z^y = \bigvee\{x \in L \ : \ x \wedge y \leq z\}$ and we have $(x \wedge y) \leq z \iff x \leq z^y$. Every topos is a cartesian closed category, by definition.

Analogously,

**Definition 2.1.5.** *We say that a category is **monoidal closed** if it has tensor products and if every two objecta $X$ and $Y$ admits an internal hom $[Y, X]$.*

In the case of monoidal closed categories, we have a natural isomorphism $Hom_C(X \otimes Y, Z) \cong Hom_C(X, [Y, Z])$ for all $X, Y, Z$ objects in $C$. In other words, the functor $- \otimes Y$ has a right adjoint functor $[Y, -]$. The category of $R$-modules is a monoidal closed category. A unital commutative quantale $Q$, which is a special kind of lattice such that a monoidal operation $\odot$ distributes over arbitrary supremum (Definition 3.1.1), is a (posetal and symmetric) monoidal closed category where $[y, z] = \bigvee\{x \in Q \ : \ x \odot y \leq z\}$ and we have $(x \odot y) \leq z \iff x \leq [y, z]$. Every cartesian closed category is a monoidal closed category where the tensor is given by the cartesian product.

Cartesian monoidal categories have special properties, we highlight two: they have a well-behaved diagonal map $\triangle_X : X \to X \times X$, and the unit object with respect to the tensor, which is the product, is a terminal object. As we will see, if a monoidal category simultaneously satisfies such conditions then it necessarily is cartesian monoidal. Therefore, not having one of those two properties weakens the notion of a cartesian monoidal category. In this Ph.D. thesis we work with the following weakening:

**Definition 2.1.6.** *A monoidal category is* ***semicartesian*** *if the unit for the tensor functor is a terminal object.*

In general, monoidal categories do not have projections since their tensor may be different from the categorical product, but every semicartesian monoidal category admits projections in the sense that for all $X, Y$ objects in a monoidal category there are "good" morphisms $\pi_1 : X \otimes Y \to X$ and $\pi_2 : X \otimes Y \to Y$. Under an appropriate notion of projection, we may go even further and say that semicartesian monoidal categories are precisely categories with projections.

**Definition 2.1.7.** *A* ***monoidal category with projections*** *is a monoidal category* $(C, \otimes, 1)$ *equipped with two natural transformations*

- *A natural transformation* $\pi^1 : (- \otimes -) \longrightarrow (-)$ *with components*

  $$\pi^1_{X \otimes Y} : X \otimes Y \to X$$

  *called the* ***projection onto the first coordinate****;*

- *A natural transformation* $\pi^2 : (- \otimes -) \longrightarrow (-)$ *with components*

  $$\pi^2_{X \otimes Y} : X \otimes Y \to Y$$

  *called the* ***projection onto the second coordinate****.*

*Such that*

1. *The following diagrams commute*

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "\pi^1_{(X \otimes Y),Z}"'] & X \otimes (Y \otimes Z) \arrow[d, "\pi^1_{X,Y \otimes Z}"] \\
X \otimes Y \arrow[r, "\pi^1_{X,Y}"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "\pi^2_{(X \otimes Y),Z}"'] & X \otimes (Y \otimes Z) \arrow[d, "\pi^2_{X,Y \otimes Z}"] \\
Z & Y \otimes Z \arrow[l, "\pi^2_{Y,Z}"]
\end{tikzcd}$$

2. $\pi^1_{X \otimes 1} : X \otimes 1 \to X$ *and* $\pi^2_{1 \otimes Y} : 1 \otimes Y \to Y$ *are, respectively, the right and the left unitor.*

Now we can formally state the following result.

**Proposition 2.1.8.** *A monoidal category is semicartesian if and only if it has projections.*

*Proof.* It is dually proved in [GLS+22, Theorem 3.5]. Here we just note that the projections are the compositions

$$\begin{tikzcd} X \otimes Y \arrow[r, "id_X \otimes !_Y"] \arrow[rr, bend left, "\pi_1"] & X \otimes 1 \arrow[r, "\rho_X"] & X \end{tikzcd}$$

$$\begin{tikzcd} X \otimes Y \arrow[r, "!_X \otimes id_Y"] \arrow[rr, bend right, "\pi_2"'] & 1 \otimes Y \arrow[r, "\lambda_Y"] & u' \end{tikzcd}$$

$\square$

Through this thesis we will work with a specific semicartesian category, the poset category given by an integral quantale (3.1.1). We define a category of sheaves on quantales and then developed a notion of sheaves on semicartesian monoidal categories to establish a general framework that encompasses the well-known notion of sheaves on categories with pullbacks and our notion of sheaves on quantales. We envision that such a framework may be interesting to study sheaves on other semicartesian categories, such as the category of Poisson manifolds; the opposite of the category of associative algebras over a given base field; the category of convex spaces (see [nLa22c] ); and Markov categories — categories that relates to probability theory, for example, the category of finite sets with stochastic matrices as morphisms and the category of measurable spaces with Markov kernels as morphisms. See [nLa23a].

When developing our category of sheaves on semicartesian categories, sometimes it is more convenient to use that the terminal object is the monoidal unity but sometimes is better to use the existence of projections, so in Chapter 4 we constantly use Proposition 2.1.8, even though we do not say it explicitly.

Finally, if the reader is interested in this relationship between the property of being semicartesian and the structure of having "good" projections, we recommend the following Blog post[1], especially the comments.

### 2.1.1 The monoidal structure in $PSh(C)$

Given a monoidal category $C = (C, \otimes, 1)$, the category $PSh(C)$ (also denoted by $Set^{C^{op}}$ and formed by functors $F : C^{op} \to Set$ and natural transformations) inherits the monoidal structure of $C$ through a quite complicated construction call Day convolution. We present the formulas here and recommend [Day70] and [Lor21] for a proper introduction to the topic.

**Definition 2.1.9.** *Let $(C, \otimes, 1)$ be a monoidal category. The* ***Day convolution*** *in $Set^C$ is a tensor product $\otimes_{Day} : Set^C \times Set^C \to Set^C$ defined by the coend*

$$X \otimes_{Day} Y : c \mapsto \int^{(c_1,c_2):C\times C} C(c_1 \otimes c_2, c) \times X(c_1) \times Y(c_2)$$

[1] If you are reading a printed version of the thesis, the link is https://golem.ph.utexas.edu/category/2016/08/monoidal_categories_with_proje.html

**Remark 2.1.10.** *Equivalently, the Day convolution is a left Kan extension (the left Kan extension of their external tensor product* $(X\bar{\otimes}Y)(c_1, c_2) := X(c_1) \times Y(c_2)$ *along the tensor product* $\otimes$ *in* $C$*). In other words, there is a natural isomorphism* $X \otimes_{Day} Y \cong Lan_{\otimes_C}(X\bar{\otimes}Y)$.

In the case of presheaves, we have:

Consider $\mathcal{C} = (\mathcal{C}, \otimes, 1)$ monoidal and $F, G : \mathcal{C}^{op} \to Set$ presheaves. Then the (Day) tensor product of presheaves is given by the coend

$$F \otimes_{Day} G : c \mapsto \int^{(c_1,c_2):C\times C} Hom_C(c, c_1 \otimes c_2) \times F(c_1) \times G(c_2)$$

The Day convolution is a tensor product for presheaves in which $PSh(\mathcal{C})$ is a monoidal closed category. Moreover, if the monoidal structure in $\mathcal{C}$ is given by the cartesian product, then $PSh(\mathcal{C})$ is cartesian closed.

## 2.2 Sheaves on Locales

We begin this section by introducing the notion of sheaves on topological spaces. Let $X$ be a topological space and consider the poset $\mathcal{O}(X)$ of all open sets of $X$ with order given by inclusion. As we will see, sheaves are a special kind of functors so, actually, we are interested in the posetal category $\mathcal{O}(X)$ where the objects are open sets of $X$ and the unique arrow $U \to V$ is given by the inclusion $U \subseteq V$. Here we use the same notation for the poset and the posetal category, as usual.

**Definition 2.2.1.** *A* ***presheaf of sets on a topological space*** $X$ *is a (covariant) functor* $F : \mathcal{O}(X)^{op} \to Set$.

Given inclusions $U \subseteq V$, we use $s_{|^V_U}$ (or just $s_{|_U}$) to denote the **restriction map** from $F(V)$ to $F(U)$. Arrows between presheaves are natural transformations and we denote the category of presheaves by $PSh(X)$.

A sheaf is a presheaf that satisfies a certain gluing property:

**Definition 2.2.2.** *A presheaf* $\mathcal{O}(X)^{op} \to Set$ *is a* ***sheaf*** *if for all* $U \in \mathcal{O}(X)$ *and all* $U = \bigcup_{i\in I} U_i$ *open cover of* $U$ *the following is satisfied:*

1. ***(Gluing)*** *Given* $s_i \in F(U_i)$ *a compatible family, i.e.,* $s_{i_{|U_i\cap U_j}} = s_{j_{|U_i\cap U_j}}$ *for all* $i, j \in I$*, there is some* $s \in F(U)$ *such that* $s_{|U_i} = s_i, i \in I$*. We say* $s$ *is the gluing of the compatible family.*
2. ***(Separability)*** *Given* $s, s' \in F(U)$ *such that* $s_{|U_i} = s'_{|U_i}$*, for all* $i \in I$*,* $s = s'$*. This states that the gluing is unique.*

We call the second condition "separability" because a presheaf is called **separated** when it admits at most one gluing.

For more abstract contexts — other categories than $\mathcal{O}(X)$ — we will be interested in a more categorical way of defining sheaves. The following definition is equivalent to Definition 2.2.2.

**Definition 2.2.3.** *A presheaf $\mathcal{O}(X)^{op} \to Set$ is a* ***sheaf*** *if for all $U \in \mathcal{O}(X)$ and all $U = \bigcup_{i \in I} U_i$ open cover of $U$ the diagram below is an equalizer in Set*

$$F(U) \xrightarrow{e} \prod_{i \in I} F(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(U_i \cap U_j)$$

*where:*

1. $e(t) = \{t_{|U_i} \mid i \in I\}, \;\; t \in F(U)$
2. $p((t_k)_{k \in I}) = (t_{i_{|U_i \cap U_j}})_{(i,j) \in I \times I}$
   $q((t_k)_{k \in I}) = (t_{j_{|U_i \cap U_j}})_{(i,j) \in I \times I}, \;\; (t_k)_{k \in I} \in \prod_{k \in I} F(U_k)$

The essence of this thesis is to analyze this definition in-depth, specifically, what is the role of the intersection and what we have to change if we replace it by more general operations.

Note that in the above definition, we did not use the points of the topological space, that is, only their *localic* structure was necessary:

**Definition 2.2.4.** *A* ***locale*** *$(L, \leq)$ is a complete lattice such that*

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i), \forall a, b_i \in L.$$

Locales coincide with complete Heyting algebras[2]. The poset of all open sets of a topological space $X$ is a locale where the supremum is the union and the infimum in the intersection.

So for any locale $L$, viewed as poset category, we have

**Definition 2.2.5.** *A presheaf $F : L^{op} \to Set$ is a* ***sheaf on $\mathcal{L}$*** *if for all $U \in L$ and all $U = \bigvee_{i \in I} U_i$ a cover of $U$ the diagram below is an equalizer in Set*

$$F(U) \xrightarrow{e} \prod_{i \in I} F(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(U_i \wedge U_j)$$

*where:*

1. $e(t) = \{t_{|U_i} \mid i \in I\}, \;\; t \in F(U)$
2. $p((t_k)_{k \in I}) = (t_{i_{|U_i \wedge U_j}})_{(i,j) \in I \times I}$
   $q((t_k)_{k \in I}) = (t_{j_{|U_i \wedge U_j}})_{(i,j) \in I \times I}, \;\; (t_k)_{k \in I} \in \prod_{k \in I} F(U_k)$

A morphism between sheaves is a natural transformation and then we obtain the category of sheaves on $L$, denoted by $Sh(L)$. The category of sheaves on a topological space is denoted by $Sh(X)$ as an abbreviation of $Sh(\mathcal{O}(X))$.

In the above, we replaced the union with the join $\bigvee$ and the intersection with the meet operation $\wedge$, which probably does not generate any discomfort. However, it is not clear

---

[2] The class of all Heyting algebras provides the natural algebraic semantics for the intuitionistic propositional logic, that is the "constructive fragment" of the classical propositional logic.

what is the meaning of a cover in this localic context. The cover of an element $U$ in $L$ is a collection $\{U_i \in L \ : \ \bigvee_{i\in I} U_i = U\}$. In the next section, we will see that this is a cover in the sense of a Grothendieck (pre)topology.

When we finish the preliminaries our first step will be to introduce sheaves on quantales — a complete lattice that generalizes locales by considering an associative binary operation (that is not necessarily the meet operation) that distributes over the join. In the equalizer diagram we will replace the meet by such more general associative operation and use the same notion of a cover, however, for quantales, this will no longer be a cover in the sense of a Grothendieck pretopology.

Now, we present a simple but instructive example of a sheaf. The power of sheaf theory is due to its ability in providing machinery to solve global problems by resolving them locally. This is more easily understood for the locale of open subsets of a topological space $X$ and the classical example of the sheaf of continuous functions: consider the functor $C_{\mathbb{R}}$ that takes open subsets $U$ of $X$ and sends it to the set $C_{\mathbb{R}}(U) = \{f : U \to \mathbb{R} \,|\, f$ is a continuous function$\}$. The restriction maps are given by restrictions, that is, if $V \subseteq U$ is an open subset, the restriction map takes a continuous function $f : U \to \mathbb{R}$ and sends it to the restriction $f_{|V} : V \to \mathbb{R}$, which still is a continuous function. It is straightforward to check that $C_{\mathbb{R}}$ is a presheaf. Besides, since $f_i(x) = f_j(x), \forall x \in U_i \cap U_j$, there is a unique function $f$ such that $f_{|U_i} = f_i$, given by $f(x) = f_i(x)$, for all $x \in U_i$. The continuity of the $f_i$'s implies the continuity of the gluing $f$, so $f \in C_{\mathbb{R}}(U)$. Analogously, the presheaves of differential, smooth, or analytic functions are sheaves [Ten75]. By this, we observe that sheaves are tools to track when local properties (as continuity) still holds globally.

Next, we will introduce the objects of a category that is equivalent to the category of sheaves.

**Definition 2.2.6.** *A $L$-**set** is a pair $(A, \delta)$ where $A$ is a set and $\delta : A \times A \to L$ is a function satisfying*

$$\delta(a, b) = \delta(b, a)$$
$$\delta(a, b) \wedge \delta(b, c) \leq \delta(a, c), \forall a, b, c \in A$$

The idea is that an $L$-set is a fuzzy set with values in a locale [Bar86], where $\delta(a, a)$ measures the degree of existence of $a$ and $\delta(a, b)$ measure to which extent $a$ and $b$ are equal.

**Example 2.2.7.** *Let $X$ be a topological space. Consider $A = \{f : U \to \mathbb{R} \,|\, U \in \mathcal{O}(X)\}$. Define $\delta(f, f) = dom(f)$ and $\delta(f, g) = int\{W \in \delta(f, f) \cap \delta(g, g) \,|\, f_{|W} = g_{|W}\}$, where $int(S)$ is the interior operator of $S$, that is, $int(S) = \bigcup\{U \in \mathcal{O}(X) \,|\, \forall V \in \mathcal{O}(X), U \subseteq V$ implies $V \in S\}$.*

There are two notions of morphisms between $L$-set: relational and functional, forming the categories $L - set^{rel}$ and $L - set^{func}$, respectively. On one hand, relational morphisms are more difficult to describe but they have the interesting feature of providing an equivalence $Sh(L) \simeq L - set^{rel}$. On the other hand, functional morphisms are easier to describe but if we want an equivalence with the category of sheaves, then we need to consider *complete* $L$-sets. Denoting the category of complete $L - sets$ with functional morphisms by $L - set^{func}_{compl}$, we have equivalences

$$Sh(L) \simeq L - set^{rel} \simeq L - set^{rel}_{compl} \simeq L - set^{func}_{compl}$$

In light of such equivalences, some authors may refer to $L$-sets as sheaves. This occurs especially in the literature about sheaves on quantales: the first step usually goes towards a generalization of $L$-sets rather than sheaves on $L$. We introduced the notion of $L$-sets to clarify that in this thesis we are more concerned with a generalization of sheaves on $L$, and the reader may find much more about the $L$-sets in [Bor94c], [FS79], and [Hig84].

## 2.3 Grothendieck Pretopologies and Sheaves on a Site

In 1955, Jean-Pierre Serre introduced sheaf theory in algebraic geometry with coherent sheaves [Ser55], but the outstanding applications of sheaf theory into algebraic geometry emerged with a generalization of sheaves proposed by Grothendieck. Roughly speaking, the idea was that schemes should admit a cover in a way similar to the cover of topological spaces. In this section, we briefly introduce Grothendieck pretopologies and the respective generalized notion of a sheaf.

Suppose $C$ is a small category with finite limits (or just with pullbacks).

**Definition 2.3.1.** *A **Grothendieck pretopology** on $C$ associates to each object $U$ of $C$ a set $K(U)$ of families of morphisms $\{U_i \to U\}_{i\in I}$ satisfying some rules:*

1. *The singleton family $\{U' \xrightarrow{f} U\}$, formed by an isomorphism $f : U' \xrightarrow{\cong} U$, is in $K(U)$;*
2. *If $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ is in $K(U)$ and $\{V_{ij} \xrightarrow{g_{ij}} U_i\}_{j\in J_i}$ is in $K(U_i)$ for all $i \in I$, then $\{V_{ij} \xrightarrow{f_i \circ g_{ij}} U\}_{i\in I, j\in J_i}$ is in $K(U)$;*
3. *If $\{U_i \to U\}_{i\in I}$ is in $K(U)$, and $V \to U$ is any morphism in $C$, then the family of pullbacks $\{V \times_U U_i \to V\}$ is in $K(V)$.*

**Remark 2.3.2.** *Some authors also call the above of **a basis for a Grothendieck topology**. We believe such nomenclature is better in a context where the Grothendieck topology has a fundamental role. In this thesis we focus on the basis, i.e., on the Grothendieck pretopologies, and the word "basis" will be used in a different context.*

The families $K(U)$ are called covering families and the pair $(C, K)$ is called **site**. Grothendieck pretopologies provide a notion of covering in non-topological contexts, and covers of topological spaces are an example of a Grothendieck pretopology:

**Example 2.3.3.** *An object in $\mathcal{O}(X)$ is an open set $U$ in $X$ and the morphisms in $\mathcal{O}(X)$ are inclusions of open subsets of $X$, the pullback in this category is given by the intersection of open subsets. Thus we define a Grothendieck pretopology $K$ in $\mathcal{O}(X)$ by*

$$\{U_i \overset{f_i}{\hookrightarrow} U\}_{i\in I} \in K(U) \iff U = \bigcup_{i\in I} U_i.$$

*An isomorphism $V \to U$ in $\mathcal{O}(X)$ means that $V = U$, so the first axiom is satisfied. For the second, notice that $U = \bigcup_{i\in I} U_i = \bigcup_{i\in I} \bigcup_{j\in J_i} V_{ij} = \bigcup_{i\in I, j\in J_i} V_{ij}$. Finally, the third axiom holds because for any $V \subseteq U$, $V = V \cap U = V \cap (\bigcup_{i\in I} U_i) = \bigcup_{i\in I}(V \cap U_i)$. The verification is the same for any locale $L$.*

Now we have all the ingredients that we need to define sheaves on a category with pullbacks:

**Definition 2.3.4.** *A presheaf $F : C^{op} \to Set$ is **a sheaf for the Grothendieck pretopology** $K(U) = \{f_i : U_i \to U\}_{i\in I}$ if the following diagram is an equalizer in Set:*

$$F(U) \xrightarrow{\;e\;} \prod_{i\in I} F(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j)\in I\times I} F(U_i \times_U U_j)$$

*where*

1. $e(f) = \{F(f_i)(t) \mid i \in I\}, \quad f \in F(U)$
2. $p((f_k)_{k\in I}) = (F(\pi^1_{ij})(f_i))_{(i,j)\in I\times I}$
   $q((f_k)_{k\in I}) = (F(\pi^2_{ij})(f_j))_{(i,j)\in I\times I}, \quad (f_k)_{k\in I} \in \prod_{k\in I} F(U_k)$

*with $\pi^1_{i,j} : U_i \times_U U_j \to U_i$ is the projection in the first coordinate and $\pi^2_{i,j} : U_i \times_U U_j \to U_j$ is the projection in the second coordinate.*

The problem with this definition is that distinct pretopologies can provide the same class of sheaves. For instance, if $\bigcup_{i\in I} U_i = U$ is an open cover of the open subset $U \subseteq X$, for any $V \subseteq U_j$, for some $j \in I$, we have $V \cup \bigcup_{i\in I} U_i = U$. To solve this problem, *covering sieves* are used to replace the role of the pullback and then *Grothendieck topologies* are defined.

**Definition 2.3.5.** *Let $U$ be an object in a small category $C$, a **sieve** on $U$ is a collection $S$ of morphisms $f$ with codomain $U$ such that $f \circ g \in S$, for all morphism $g$ with $dom(f) = cod(g)$.*

Now, we define the collection that plays the same role as the pullback in the third axiom of a Grothendieck pretopology so that $C$ does not need to actually have pullbacks. Given $h : V \to U$, define

$$h^*(S) = \{g \mid cod(g) = V,\ h \circ g \in S\}.$$

**Definition 2.3.6.** *A Grothendieck topology in $C$ associates each object $U$ of $C$ to a collection $J(U)$ of sieves on $U$ such that:*

1. *The maximal sieve on $U$, $\{f \mid cod(f) = U\}$, is in $J(U)$;*
2. *If $R$ and $S$ are sieves on $U$, $S$ is in $J(U)$ and $h^*(R)$ is in $J(V)$ for all $h : V \to U$ in $S$, then $R$ is in $J(U)$;*
3. *If $S$ is in $J(C)$, then $h^*(S)$ is in $J(V)$ for all $h : V \to U$.*

The pair $(C, J)$ formed by a category $C$ and a Grothendieck topology $J$ is also called *site*. We can define sheaves for Grothendieck topologies and obtain a sheaf category $Sh(C, J)$, where the morphisms are given by natural transformations. Moreover, pretopologies and topologies are intimately related: If $K$ is a Grothendieck pretopology, then $K$ *generates a topology* $J$ by

$$S \in J(U) \iff \exists R \in K(U) \text{ s.t. } R \subseteq S$$

Then a presheaf $F : C^{op} \to Set$ is a sheaf for the Grothendieck pretopology $K$ iff it is a sheaf for the generated Grothendieck topology $J$ [MM92, Chapter III.4, Proposition 1].

**Definition 2.3.7.** *A Grothendieck topos is a category that is equivalent to $Sh(C, J)$, for some site.*

Then $Sh(L) = Sh(C, J)$ is a Grothendieck topos where $C = L$ is a locale and $J$ is the Grothendieck topology generated by the pretopology

$$\{U_i \overset{f_i}{\hookrightarrow} U\}_{i \in I} \in K(U) \iff U = \bigvee_{i \in I} U_i.$$

Nevertheless, pretopologies are good enough to develop sheaf cohomology, and, in this thesis, we only generalize the Grothendieck pretopologies. Later, in Chapter 5, we indicate the steps for future work towards a generalization of Grothendieck toposes.

**Remark 2.3.8.** *Why "toposes" and not "topoi"? To answer this we quote Colin McLarty in [McL90]:*

> *"Notice as a point of orthography that 'topos' is a French word, formed from 'topologie' and not a Greek word. In writing, Grothendieck always forms the plural according to the French rule for words ending in 's', so it is invariant – 'les topos'. So the English plural ought to follow the English rule – 'toposes' ".*

*We add that the Portuguese rule for words ending in 's' is also invariant and so I advocate that in Portuguese we should also use 'topos' even in the plural form.*

We end this section with considerations about the sheafification functor, which is the process of taking a presheaf and making it become a sheaf.

**Definition 2.3.9.** *Let $P$ be a presheaf and $R$ a sieve of $C$ in $\mathcal{C}$. A* ***compatible family*** *for $S$ of elements in $P$ is a map that sends each element $f : D \to C$ of $R$ into $x_f$ in $P(D)$ such that:*

$$P(g)(x_f) = x_{f \circ g}, \text{ for all } g : E \to D \text{ in } \mathcal{C}.$$

Consider $Comp(R, P)$ the set of compatible families of a sieve $R$.

**Definition 2.3.10.** *Let $P$ be a presheaf on a category $\mathcal{C}$. Define*

$$P^+ = \varinjlim_{R \in J(C)} Comp(R, P)$$

Note that a morphism of presheaves $\phi : P \to Q$ induces a map $\phi^+ : P^+ \to Q^+$, where $Q^+ = \varinjlim_{R \in J(C)} Comp(R, Q)$, since it induces a morphism on the compatible families $\phi' : Comp(R, P) \to Comp(R, Q)$, $\{x_f\}_f \mapsto \{\phi(x_f)\}_f$. It is possible to prove that $(-)^+$ is a functor, called *plus construction* or *semi-sheafification functor*. The sheafification is defined by applying $(-)^+$ twice.

**Definition 2.3.11.** *The* ***sheafification functor*** $a : PSh(C) \to Sh(C, J)$ *is defined by*

$$a(P) = (P^+)^+$$

It is known that $a : PSh(C) \to Sh(C, J)$ is a left adjoint functor of the inclusion $i : Sh(C, J) \to PSh(C)$ that preserves all finite limits. In other words, $a$ if the left exact reflector of $i$. Moreover, Grothendieck toposes are precisely the left exact reflective subcategories of a presheaf category [Bor94c, Corollary 3.5.5].

We will see that the plus construction will not work in our case and that it does not preserve all finite limits since our sheaves may not provide a Grothendieck topos.

## 2.4 Sheaf Cohomology

To talk about sheaf cohomology we do not work with sheaves as presented above but in sheaves of abelian groups or sheaves of rings, that is, sheaves with values in $Ab$ or $CRing$, instead of $Set$. It is known that such sheaves form abelian categories with enough injective objects. The reader may consult the definitions in [Wei94] and [Bor94b], here we only observe that having an abelian category with enough injective objects gives enough structure to define (co)homology as right/left derived functors of a left/right exact functor. This is true even for the "abelian form" of Grothendieck toposes, as one may check in [Gro63], [Joh77], on in our survey [TM21] about sheaf cohomology.

In this section, we only briefly introduce sheaf cohomology for sheaves on a topological space.

For every sheaf $F$ in $Sh_{Ab}(X)$ and $U$ open set of $X$, we have the abelian group of **sections of** $F$ **over** $U$ defined by

$$\Gamma(U, F) = F(U).$$

Sections over $X$ are called *global sections.*

**Definition 2.4.1.** *The* ***global section functor*** *is a functor* $\Gamma(X, -) : Sh_{Ab}(X) \to Ab$ *that sends an abelian sheaf to its global section abelian group*

Since the global section functor is a left exact functor, we define the $q$-th cohomology group of $X$ with coefficients in $F$ by the $q$-th right derived functor of $\Gamma(X, F)$. In other words, given an injective resolution $F \to I^{\bullet}$, we have

$$H^q(X, F) = R^q\Gamma(X, I^{\bullet}).$$

The above definition is not the best option if we want to calculate the cohomology groups but there is a technique to do so, which is called Čech cohomology: fix an abelian sheaf $F$ on $X$ and consider $\mathcal{U} = (U_i)_{i \in I}$ an open cover of $X$. For each $q \in \mathbb{N}$, we consider the Čech nerve, which is $U_{i_0,\ldots,i_q} = U_{i_0} \cap \ldots \cap U_{i_q}$ for $i_0, \ldots, i_q \in I$. Then

**Definition 2.4.2.** *The Čech cochain complex is*

$$C^q(\mathcal{U}, F) = \prod_{i_0,\ldots,i_q} F(U_{i_0,\ldots,i_q}), \forall q \geq 0,$$

*and its coboundary morphisms* $d^q : C^q(\mathcal{U}, F) \to C^{q+1}(\mathcal{U}, F)$ *are*

$$(d^q\alpha) = \sum_{k=0}^{q+1} (-1)^k \alpha(\delta_k)\Big|_{U_{i_0,\ldots,i_{q+1}}}$$

*where* $\delta_k$ *is used to indicate that we are removing* $i_k$*, i.e.,* $\alpha(\delta_k) = \alpha_{i_0,\ldots,\widehat{i_k},\ldots,i_{q+1}}$.

A straightforward verification shows that $d^{q+1} \circ d^q = 0$ so, indeed, this is a cochain complex and we can define the $q$-th Čech cohomology group of $F$ with respect to the covering $\mathcal{U}$ by

$$\check{\mathrm{H}}^q(\mathcal{U}, F) = Ker(d^q)/Im(d^{q-1}).$$

**Proposition 2.4.3.** *Let $F$ be an abelian sheaf on $X$, and $\mathcal{U} = (U_i)_{i\in I}$ a covering of $X$. There is a canonical morphism $k^q_{\mathcal{U}} : \check{\mathrm{H}}^q(\mathcal{U}, F) \to \mathrm{H}^q(X, F)$ natural and functorial in $F$ for each $q \in \mathbb{N}$.*

*Proof.* [Har77, Lemma III 4.4]. □

So the sheaf cohomology we defined using derived functors is connected with Čech cohomology of a covering with coefficients in a sheaf. Now, we want to improve the Čech cohomology groups so that they are defined for $X$ and not only for a given covering in $X$. To this end we introduce the idea of refinement of coverings: Let $\mathcal{V} = (V_j)_{j\in J}$ be another covering of $X$. We say that $\mathcal{U} = (U_i)_{I\in I}$ is a **refinement** of $\mathcal{V}$ if there is a function $r : I \to J$ and a morphisms $U_i \to V_{r(i)}$, for all $i \in I$. Choose any function $r : I \to J$ such that $U_i \subseteq V_{r(i)}, i \in I$; then there is a induced morphism of cochain complexes $m_r : C^\bullet(\mathcal{V}, F) \to C^\bullet(\mathcal{U}, F)$ and a corresponding morphism of Čech cohomology groups w.r.t. the coverings $\mathcal{U}$ and $\mathcal{V}$, $\check{m}_r : \check{\mathrm{H}}^\bullet(\mathcal{V}, F) \to \check{\mathrm{H}}^\bullet(\mathcal{U}, F)$. Moreover, if $s : I \to J$ is another chosen function w.r.t. the refinement of $\mathcal{V}$ by $\mathcal{U}$, then the induced morphisms of complexes $m_r, m_s$ are homotopic, thus there is a unique induced morphism of cohomology groups[3] $\check{m}_{\mathcal{U},\mathcal{V}} : \check{\mathrm{H}}^\bullet(\mathcal{V}, F) \to \check{\mathrm{H}}^\bullet(\mathcal{U}, F)$.

The class $Ref(X)$ of all coverings of $X$ is partially ordered with order relation given by the refinement, and the construction above is functorial:

- $\check{m}_{\mathcal{U},\mathcal{U}} = id : \check{\mathrm{H}}^\bullet(\mathcal{U}, F) \to \check{\mathrm{H}}^\bullet(\mathcal{U}, F)$;
- If $\mathcal{W} = (W_k)_{k\in K}$ is a covering of $X$ such that $\mathcal{V}$ is a refinement of $\mathcal{W}$, then $\check{m}_{\mathcal{U},\mathcal{W}} = \check{m}_{\mathcal{U},\mathcal{V}} \circ \check{m}_{\mathcal{V},\mathcal{W}} : \check{\mathrm{H}}^\bullet(\mathcal{W}, F) \to \check{\mathrm{H}}^\bullet(\mathcal{U}, F)$.

**Definition 2.4.4.** *The* ***Čech cohomology group of an element*** *of $X$ with coefficient in a*

[3] It is a classic result in homological algebra that (co)chain maps which are homotopic induce equal maps on (co)homology.

*sheaf $F$ is the directed (co)limit*

$$\check{\mathrm{H}}^{\bullet}(X,F) := \varinjlim_{\mathcal{U}\in Ref(X)} \check{\mathrm{H}}^{\bullet}(\mathcal{U},F).$$

The main general result concerning Čech cohomology provides the following relation between the Čech cohomology of $X$ and the sheaf cohomology of $X$:

**Theorem 2.4.5.** *The canonical morphisms $k^q_{\mathcal{U}} : \check{\mathrm{H}}^q(\mathcal{U},F) \to \mathrm{H}^q(X,F), q \in \mathbb{N}$, according notation in Proposition 2.4.3, are compatible under refinement. Moreover, the induced morphism on colimit*

$$k^q : \check{\mathrm{H}}^q(X,F) \to \mathrm{H}^q(X,F), q \in \mathbb{N},$$

*is an isomorphism if $q \leq 1$ and a monomorphism if $q = 2$.*

*Proof.* Proved in [Joh77, Theorem 8.27] in the more general case of Grothendieck toposes. □

Furthermore, under mild geometrical hypothesis on the topological space $X$ (for instance, if $X$ is a Hausdorff paracompact space[4]), then the canonical morphisms $k^q$ are isomorphisms for all $q \geq 0$. Thus, there is a large class of topological spaces $X$ in which the Čech cohomology and sheaf cohomology coincide. Now, we present two examples of how sheaf theory connects with other areas of Mathematics through the use of cohomological methods.

Given a topological space $X$, and a set $K$, the constant presheaf with values in $K$ can be transformed into a constant sheaf with values in $K$ by the sheafification process. If $K$ is the underlying set of an abelian group such as $\mathbb{R}$, the additive group of real numbers, and the topological space is a compact manifold $M$ of dimension $m$ and class at least $C^{m+1}$, there is an isomorphism $H^q_{dR}(M) \cong \check{H}^q(M,\mathbb{R})$, for all $q \leq m$, where $H^q_{dR}$ denotes the de Rham cohomology groups [Pet06, Appendix]. In the same vein, Čech cohomology and singular cohomology are isomorphic for any topological space $X$ that is homotopically equivalent to a CW-complex, and with the constant sheaf of an abelian group $K$ as coefficient. For the reader that is not used with such cohomology theories, we note that both measures some kind of *obstruction*. The de Rham cohomology measures the extent to which the fundamental theorem of calculus fails in higher dimensions and on general manifolds [Tao07], and singular homology measures the number of holes of $X$ and it is related to singular cohomology by the universal coefficient theorem for cohomology.

As we mentioned at the beginning of this section, the above ideas can be replicated in the contexts of Grothendieck toposes. Again, we will obtain interesting applications in other areas of Mathematics: Let $C$ be the slice category of schemes over a scheme $X$, where the objects are étale morphisms $Spec(R) \xrightarrow{f} X$, and, by abuse of notation, the morphism $f \xrightarrow{\varphi} g$ are morphisms of schemes $Spec(R) \xrightarrow{\varphi} Spec(R')$ such that $g \circ \varphi = f$. The étale covers provide a Grothendieck pretopology and then we have sheaves on the *étale site*. Étale cohomology is the sheaf cohomology for the étale site. Originally, it was

[4] This holds for any CW-complex or any topological manifold.

constructed to study algebraic geometry but in fields different of $\mathbb{C}$ and $\mathbb{R}$. Currently, étale cohomology has applications in representation theory of finite groups, number theory, and $K$-theory.

Other sites provide other cohomologies such as crystalline, Deligne, and flat cohomologies, and all are examples of Grothendieck topos cohomology. Therefore, sheaf cohomology is a general framework in which we can talk about distinct cohomology theories.

# Chapter 3

# Sheaves on Semicartesian Quantales

Quantales are complete lattices that generalize locales, thus it is natural to wonder how to define sheaves on them and we are not the first to approach such subject. For *idempotent* quantales we have the following: in [BB86], sheaves on quantales are defined with the goal of forming Grothendieck toposes. In [MS98], the sheaf definition preserves an intimate relation with $Q$-sets, an object introduced in the paper as a generalization of $\Omega$-sets, defined in [FS79], for $\Omega$ a complete Heyting algebra[1]. More recently, in [ASV08], sheaves on idempotent quantales are functors that make a certain diagram an equalizer. Besides it, an extensive work about sheaves on *involutive quantales* was recently studied by Hans Heymans, Isar Stubbe [HS12], and Pedro Resende [Res12], for instance.

We will study sheaves on *semicartesian* quantales. Our approach is similar to the one in [ASV08], in the sense that our sheaves are also described in terms of equalizers but our sheaves and theirs are orthogonal in the sense we are dealing with orthogonal kinds of quantales: every quantale that is simultaneously idempotent and semicartesian is a locale (Proposition 3.1.6). Thus, the notions of sheaves coincide only in the localic case, which is already well known (see Definition 2.2.5). As far as we know, there is only one paper regarding sheaves on semicartesian/integral quantales, however, the definition resembles more what should be a $Q$-set [BC94].

## 3.1 Quantales

We begin this Chapter by introducing the definition of quantales and constructions that will be useful to us in the following sections. In Section 4.3.2 we will investigate a subobject classifier for the category $Sh(Q)$ that we are constructing. As we will see, there is class of commutative semicartesian quantales $Q$ (called geometric quantales) that contains the subclass of locales and such that $Sh(Q)$ admits subobject classifier not for all but for a

[1] Given a proper notion of morphisms of $\Omega$-sets, the category of $\Omega$-sets is equivalent to the category of sheaves on $\Omega$. We briefly explain this in 2.2.

class of monomorphisms. In the current section, we also present examples of geometric quantales.

**Definition 3.1.1.** *A* ***quantale*** *$Q$ is a complete lattice structure $(Q, \leq)$ with a semigroup structure*[2] *$(Q, \odot)$ such that for all $a \in Q$ and $\{b_i\}_{i \in I} \subseteq Q$ the following distributive laws hold:*

1. $a \odot (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \odot b_i)$
2. $(\bigvee_{i \in I} b_i) \odot a = \bigvee_{i \in I} (b_i \odot a)$

**Remark 3.1.2.** *Note that:*

1. *In any quantale $Q$ the multiplication is increasing in both entries, where increasing in the second entry means that given $a, b, b' \in Q$ such that $b \leq b'$, we have $a \odot b \leq a \odot b'$. Indeed $a \odot b' = a \odot (b \vee b') = (a \odot b) \vee (a \odot b')$, thus $a \odot b \leq a \odot b'$.*
2. *Since the least element of the quantale $Q$, here denoted by $0$ (or $\perp$), is also the supremum of the emptyset, note that $a \odot 0 = 0 = 0 \odot a, \forall a \in Q$.*

Similarly, a *unital* quantale is a structure $(Q, \leq, \odot, 1)$, where $(Q, \odot, 1)$ is a *monoid.* Note that the associativity of $\odot$ and the identity element provide a (strict) monoidal structure to $(Q, \leq, \odot, 1)$.

**Example 3.1.3.** *(Quantales)*

1. *The extended half-line $[0, \infty]$ with order $\geq$, and the usual sum of real numbers as the multiplication. Since the order relation is $\geq$, the top element is $0$ and the bottom element is $\infty$. This is the famous Lawvere quantale;*
2. *The extended natural numbers $\mathbb{N} \cup \{\infty\}$, with the same quantalic structure of $[0, \infty]$;*
3. *The set $\mathcal{I}(R)$ of ideals of a commutative and unital ring $R$ with order $\subseteq$, the inclusion of ideals, and the multiplication as the multiplication of ideals. The supremum is the sum of ideals, the top element is $R$ and the trivial ideal is the bottom;*
4. *The set $\mathcal{RI}(R)$ of right (or left) ideals of an unital ring $R$ with the same order and multiplication of the above example. Then the supremum and the top and the bottom elements are also the same as $\mathcal{I}(R)$.*
5. *The set of closed right (or left) ideals of a unital $C^*$-algebra, the order is the inclusion of closed right (or left) ideals, and the multiplication is the topological closure of the multiplication of the ideals.*

For more details and examples we recommend [Ros90b].

**Definition 3.1.4.** *A quantale $Q = (Q, \leq, \odot)$ is*

1. ***commutative*** *when $(Q, \odot)$ is a commutative semigroup;*
2. ***idempotent*** *when $a \odot a = a$, for $a \in Q$;*
3. ***right-sided*** *when $a \odot \top = a$, for all $a \in Q$, where $\top$ is the top member of the poset;*

[2] That is, the binary operation $\odot : Q \times Q \to Q$ (called multiplication) is associative

4. ***semicartesian*** *when* $a \odot b \leq a, b$, *for all* $a, b \in Q$;

5. ***integral*** *when* $Q$ *is unital and* $1 = \top$.

The quantales $[0, \infty]$, $\mathbb{N} \cup \{\infty\}$, and $\mathcal{I}(R)$ are commutative and integral unital quantales. The last example is neither commutative nor semicartesian but it is a right-sided quantale. [Ros90b].

**Remark 3.1.5.** *Note that:*

1. *A quantale* $(Q, \leq, \odot)$ *is semicartesian iff* $a \odot b \leq a \wedge b$, *for all* $a, b \in Q$.

2. *Let* $Q$ *be a unital quantale, then it is integral iff it is semicartesian. Indeed: suppose that* $Q$ *is integral, since* $b \leq \top$ *we have* $a \odot b \leq a \odot \top = a \odot 1 = a$, *then* $Q$ *is semicartesian; conversely, suppose that* $Q$ *is semicartesian, since* $\top = \top \odot 1 \leq 1$, *then* $\top = 1$.

The following result explains why we claim that our sheaves are orthogonal to notions of sheaves on idempotent quantales:

**Proposition 3.1.6.** *Let* $Q$ *be a semicartesian quantale, if* $Q$ *is idempotent then the multiplication is the infimum operation, in other words,* $Q$ *is a locale.*

*Proof.* Since $Q$ is semicartesian, for any $b, c$ in $Q$, $b \odot c \leq b, c$. If $Q$ is idempotent, for any $a \in Q$ such that $a \leq b$ and $a \leq c$ we have $a = a \odot a \leq b \odot c$, because the multiplication is increasing in both entries. So if $Q$ is semicartesian and idempotent, by the transitivity of the order relation:

$$a \leq b \odot c \iff a \leq b \text{ and } a \leq c$$

Thus the multiplication satisfies the definition of the meet operation. □

The above proposition is just a particular case of [nLa22b, Proposition 2.1]. Observing that any notion of a sheaf on an idempotent and semicartesian quantale is equivalent to a sheaf on a locale, which is a well-established object of study, it becomes clear that the study of sheaves on semicartesian quantales is orthogonal to the study of sheaves on idempotent quantales.

**Construction of quantales:**

1. Notice that given a family of quantales $\{Q_i : i \in I\}$ the cartesian product $\prod_{i \in I} Q_i$ with component-wise order (i.e., $(a_i)_i \leq (b_i)_i \iff a_i \leq b_i, \forall i \in I$) is a quantale. Define $\bigvee_{j \in I}(a_{ij})_i = (\bigvee_{j \in I} a_{ij})_i$, $\bigwedge_{j \in I}(a_{ij})_i = (\bigwedge_{j \in I} a_{ij})_i$ and $(a_i)_i \odot (b_i)_i = (a_i \odot b_i)_i$. The verifications are straightforward but we will check one of the distributive laws:

$$\bigvee_{j \in I}(a_i \odot b_{ij})_i = (\bigvee_{j \in I} a_i \odot b_{ij})_i = (a_i \odot \bigvee_{j \in I} b_{ij})_i = (a_i)_i \odot \bigvee_{j \in I}(b_{ij})_i$$

   It is easy to see that $\prod_{i \in I} Q_i$ is a semicartesian/commutative quantale whenever each $Q_i$ is a semicartesian/commutative quantale.

2. If $Q$ is a commutative semicartesian quantale, it is straightforward to check that given $e \in Idem(Q)$ and $u \in Q$ such that $e \leq u$, then the subset $[e, u] = \{x \in Q : e \leq x \leq u\}$

is closed under $\bigvee$ and $\odot$, thus it determines an **interval subquantale** that is also semicartesian and commutative.

An example of a semicartesian quantale that is not integral is constructed as follow: let $Q$ be a integral and not idempotent quantale. Given $a \in Q \setminus Idem(Q)$, then the interval $[\bot, a]$ is a non-unital semicartesian quantale.

**Remark 3.1.7.** *Every commutative unital quantale $Q$ can be associated to a closed monoidal symmetric poset category $Q$, where exists a unique arrow in $Hom(a,b)$ iff $a \leq b$. Note that the product $\prod$, coproduct $\coprod$, and tensor $\otimes$ are defined, respectively, by the infimum, $\bigwedge$, the supreme $\bigvee$, and the dot $\odot$. The "exponential" is given by $b^a = \bigvee\{c \in Q \,:\, a \odot c \leq b\}$, where $b^a$ is an alternative notation for $a \to b$ or $a \backslash b$. This was mentioned in Section 2.1 with a slightly different notation.*

Now, we introduce an operation that sends elements of a commutative and semicartesian quantale $Q$ into an idempotent element in the locale $Idem(Q)$.

**Definition 3.1.8.** *Let $Q$ be a commutative and semicartesian quantale. We define*

$$q^- := \bigvee\{p \in Idem(Q) \,:\, p \leq q \odot p\}$$

Since $Q$ is semicartesian and commutative, note that $p \leq q \odot p$ iff $p = q \odot p = p \odot q$.

Now, we list properties of $(-)^- : Q \to Idem(Q)$.

**Proposition 3.1.9.** *If $Q$ is a commutative and semicartesian quantale, and $\{q_i \,:\, i \in I\} \subseteq Q$, then*

1. $0^- = 0$ *and* $1^- = 1$ *(if $Q$ is unital)*
2. $q^- \leq q$
3. $q^- \odot q = q^-$
4. $q = q^- \Leftrightarrow q \odot q = q$
5. $q^- \odot q^- = q^-$
6. $q^- = max\{e \in Idem(Q) \,:\, e \leq q\}$
7. $q^{-^-} = q^-$
8. $p \leq q$ *and* $x \odot p = x$, *then* $x \odot q = x$
9. $p \leq q \Rightarrow p^- \leq q^- \Leftrightarrow p^- \odot q^- = p^-$
10. $(a \odot b)^- = a^- \odot b^-$
11. $q_j^- \odot \bigvee_{i \in I} q_i = q_j^-$
12. $\bigvee_i q_i^- \leq (\bigvee_i q_i)^-$

*Proof.*
1. Straightforward.
2. If $e \in Idem(Q)$ is such that $e \leq q \odot e$, then $e \leq q \odot e \leq q$, since $Q$ is semicartesian. Thus, by the definition of $q^-$ as a least upper bound, $q^- \leq q$.
3. Since multiplication distributes over arbitrary joins,

$$q^- \odot q = \bigvee\{p \odot q \,:\, p = q \odot p, p \in Idem(Q)\} = \bigvee\{p \in Idem(Q) \,:\, p = q \odot p\} = q^-$$

4. $(\Rightarrow)$ From the previous item.

   $(\Leftarrow)$ By maximality, $q \leq q^-$. Thus the result follows from item (2).

5. Since multiplication distributes over arbitrary joins,

$$\begin{aligned} q^{-} \geq q^{-} \odot q^{-} &= \bigvee \{p \odot q^{-} Idem(Q) \,:\, p \in Idem(Q), p = q \odot p\} \\ &= \bigvee \{p \odot p' \,:\, p, p' \in Idem(Q), p = q \odot p, p' = q \odot p'\} \\ &\geq \bigvee \{p \odot p \,:\, p \in Idem(Q), p = q \odot p\} \\ &= \bigvee \{p \,:\, p \in Idem(Q), p = q \odot p\} \\ &= q^{-} \end{aligned}$$

6. By items (2) and (5), $q^{-} \in \{e \in Idem(Q) \,:\, e \leq q\}$. If $e \in Idem(Q)$ is such that $e \leq q$, then $e = e \odot e \leq q \odot e \leq e$, thus $e = e \odot q$; then $e \leq q^{-}$, by the definition of $q^{-}$ as a l.u.b.

7. By item (2), $q^{-^{-}} \leq q^{-}$. On the other hand, by items (4) and (5) and the maximality of $q^{--}$, we have $q^{-} \leq q^{-^{-}}$.

8. Since $x = x \odot p \leq x \odot q \leq x$.

9. Suppose $p \leq q$. Then by items (3) and (8), $p^{-} \odot q = p^{-}$.

   By item (5), $p^{-} \in Idem(Q)$ and, by maximality of $q^{-}$, $p^{-} \leq q^{-}$.

   Since $p^{-}, q^{-} \in Idem(Q)$ (item (5)) then, by the argument in the proof of item (6), we have $p^{-} \leq q^{-}$ iff $p^{-} \odot q^{-} = p^{-}$.

10. Note that $a^{-} \odot b^{-}$ is an idempotent such that $a^{-} \odot b^{-} \odot a \odot b = a^{-} \odot b^{-}$. So $(a \odot b)^{-} \geq a^{-} \odot b^{-}$.

    On the other hand, by item (9), $(a \odot b)^{-} \odot a^{-} = (a \odot b)^{-} = (a \odot b)^{-} \odot b^{-}$. Then, $(a \odot b)^{-} \odot (a^{-} \odot b^{-}) = ((a \odot b)^{-} \odot a^{-}) \odot b^{-} = (a \odot b)^{-} \odot b^{-} = (a \odot b)^{-}$. Thus $(a \odot b)^{-} \leq a^{-} \odot b^{-}$.

11. Since $q_j^{-} = q_j^{-} \odot q_j \leq q_j^{-} \odot \bigvee_{i \in I} q_i \leq q_j^{-}$.

12. Since $q_j \leq \bigvee_i q_i$, from item (9) we obtain $q_j^{-} \leq (\bigvee_i q_i)^{-}$, and then $\bigvee_j q_j^{-} \leq (\bigvee_i q_i)^{-}$, by sup definition.

□

**Proposition 3.1.10.** *Let $Q$ be a commutative and integral (unital) quantale. Consider the inclusion map $i \,:\, Idem(Q) \hookrightarrow Q$ and the map $(-)^{-} \,:\, Q \to Idem(Q)$ defined in 3.1.8, then:*

1. *$(Idem(Q), \bigvee, \odot, 1)$ is a locale and the inclusion map $i \,:\, Idem(Q) \hookrightarrow Q$ preserves $\odot$, sups and $\top$.*

2. *The map $(-)^{-} \,:\, Q \to Idem(Q)$ preserves $\odot$ and $\top$.*

3. *The adjunction relation (for posets) holds for each $e \in idem(Q)$ and $q \in Q$*

$$Hom_Q(i(e), q) \cong Hom_{idem(Q)}(e, q^{-})$$

*Proof.* 1. The sup of a set of idempotents is idempotent (in the same vein of the proof of item (5) in the previous proposition). If $f, e, e' \in Idem(Q)$, then $e \odot e' \leq e, e'$, since $Q$ is semicartesian. Moreover, if $f \leq e, e'$, then $f = f \odot f \leq e \odot e'$. Thus $e \odot e'$ is the greatest lower bound of $e, e'$ in $Idem(Q)$. The other claims are straightforward.

2. This is contained in items (1) and (10) of the previous proposition.

3. Since we are dealing with posets, it is enough to show that, for each $q \in Q, e \in Idem(Q)$,
$$i(e) \leq q \iff e \leq q^-$$
If $i(e) \leq q$, then by item (6) in the previous proposition $e \leq q^-$.

On the other hand, if $e \leq q^-$, then by item (4) and the equivalence in the (9) in the previous proposition $e = e \odot q^-$. Then, by item (2), $e \leq e \odot q \leq q$.

□

**Definition 3.1.11.** *Let Loc be the category of locales with morphisms that preserves finitary infs and arbitrary sups, and CSQ the category of commutative semicartesian quantales with morphisms that preserves sups and $\top$ satisfying that $f(a \odot b) \geq f(a) \odot f(b)$.*

The next proposition is analogous to the previous one, but for the category $CSQ$ instead of the poset category of commutative and semicartesian quantales.

**Proposition 3.1.12.** *Consider inclusion functor $\iota : Loc \hookrightarrow CSQ$. Then:*

1. *The inclusion functor $\iota : Loc \hookrightarrow CSQ$ is full and faithful.*
2. *$Q \mapsto Idem(Q)$ determines the right adjoint of the inclusion functor $\iota : Loc \hookrightarrow CSQ$, where the inclusion $i_Q : \iota(Idem(Q)) \hookrightarrow Q$, is a component of the co-unity of the adjunction.*

*Proof.* 1. Recall that a locale is a commutative semicartesian quantale where $\odot = \wedge$. It is clear that $\iota$ is a well-defined and faithful functor. Let $L, L'$ be locales and $f : \iota(L) \to \iota(L')$ be a $CSQ$-morphism: we must show that it preserves finitary infs. By hypothesis, $f(a \wedge b) \geq' f(a) \wedge' f(b)$. On the other hand, since $f$ preserves sups it is increasing, and then $f(a \wedge b) \leq' f(a) \wedge' f(b)$. Thus $f$ preserves binary infs and $\top$ (by hypothesis). Thus it preserves finitary infs.

2. By item (1) in the previous proposition, $i_Q : \iota(Idem(Q)) \hookrightarrow Q$ is a morphism that preserves sups, $\top$ and $\odot$. Let $H$ be a locale and $f : \iota(H) \to Q$ be a $CSQ$-morphism. Let $a \in H$, then $f(a) = f(a \wedge a) = f(a) \odot f(a)$, thus $f(a) \in Idem(Q)$.

   Moreover, $f_\restriction : H \to Idem(Q)$ is a locale morphism, by item (1). Since $i_Q$ is injective $f_\restriction$ is the unique locale morphism $H \to Idem(Q)$ such that $i_Q \circ \iota(f_\restriction) = f$.

□

Observe that the last item follows the same idea of Lemma 2.2 in [nLa22b].

Next, we explore other properties of the construction $q \mapsto q^-$ in a more specific class of quantales.

**Definition 3.1.13.** *We say that a (commutative, semicartesian) quantale $Q$ is:*

1. *An **Artinian quantale** if each infinite descending chain $q_0 \geq q_1 \geq q_2 \geq q_3 \geq \dots$, stabilizes for some natural number $n \in \mathbb{N}$, which may vary according to the chain.*

2. *A **p-Artinian quantale** if for each $q \in Q$, the infinite descending chain of powers of $q$, $q^1 \geq q^2 \geq q^3 \geq \dots$, stabilizes for some natural number $n \in \mathbb{N} \setminus \{0\}$, which may vary according to the chain.*

3. *If there is a natural number $n \geq 1$ such that for all $q \in Q$ we have $q^n = q^{n+1}$, then we say that $Q$ is **uniformly p-Artinian**. The least $n \in \mathbb{N}$ such that, for each $q \in Q$, $q^{n+1} = q^n$ is called the* **degree of Q**.

The following results are straightforward.

**Remark 3.1.14.** *Let $Q$ be a commutative and semicartesian quantale.*

1. *If $Q$ is Artinian or uniformly p-Artinian, then $Q$ is p-Artinian.*

2. *If $Q$ is a p-Artinian quantale, $q \in Q$ and $q^{n+1} = q^n$, then $q^- = q^n$.*

The example that motivates such terminology is the set of ideals of an Artinian commutative unitary ring. Concerning this example, we add the following

**Proposition 3.1.15.** *Let $A$ be a commutative unitary ring and consider $Q = \mathcal{I}(A)$ be its quantale of all ideals. Consider:*

1. *$\mathcal{I}(A)$ is p-Artinian;*

2. *For each $a \in A$, there is $n \in \mathbb{N}$ such that $(a)^n = (a)^{n+1}$;*

3. *For each $a \in A$, there is $n \in \mathbb{N}$ and $b \in A$ that $a^n = b.a^{n+1}$. This means that $A$ is strongly $\pi$-regular.*

4. *Each prime proper ideal of $A$ is maximal;*

5. *$A/nil(A)$ is a von Neumann regular ring;*

   *Then we have the following implications*

$$1 \implies 2 \iff 3 \implies 4 \iff 5.$$

   *Moreover, if $A$ is a reduced ring (i.e. $nil(A) = \{0\}$), then all items above are equivalent between them and also are equivalent to*

6. *$\mathcal{I}(A)$ is a uniformly p-Artinian quantale of degree 2.*

*Proof.* $1 \implies 2$ : By the definition of p-Artinian.

$2 \iff 3$ : Straightforward.

$3 \implies 4$ : Let $P$ be a prime proper ideal and take $a \notin P$. By 3, there is $n \in \mathbf{N}$ and $b \in A$ such that $a^n - ba^{n+1} = 0$. So $a^n(1 - ba) = 0$. Since $a^n \notin P$, and $P$ is prime, we have that

$a^n(1-ba) \in P$. Then $1 \in P + Ra$ and we obtain $A = P + Aa$. In other words, every non-zero element in $A/P$ is invertible, which means that $A/P$ is field and therefore $P$ is maximal.

$4 \iff 5$ This is stated in [Lam06, Exercise 4.15], where Krull dimension 0 means precisely that all prime ideals are maximal ideals.

Now, suppose that $\mathcal{I}(A)$ is uniformly $p$-Artinian. This gives that $\mathcal{I}(A)$ is $p$-Artinian and so we do not have to verify 1. We conclude the sequence of implications by showing that 5 implies 2: we use that a ring is von Neumann regular iff every principal left ideal is generated by an idempotent element. Since $A$ is commutative and $A/nil(A) = A$ is von Neumann regular, for each $a \in A = A$, $(a) = (e)$ for some idempotent $e \in A$. Therefore, $(a)^n = (e)^n = (e)^{n+1} = (a)^{n+1}$

□

**Example 3.1.16.** *Since $A$ is a strongly (von Neumann) regular ring if and only if $A$ is a reduced regular ring [Reg86, Remark 2.13], any reduced regular ring satisfies condition* 3 *(every regular ring is $\pi$-regular) so $\mathcal{I}(A)$ is an example of a uniformly p-Artinian quantale of degree 2.*

**Remark 3.1.17.** *For commutative rings, strongly von Neumann regular is equivalent to von Neumann regular.*

**Proposition 3.1.18.** *If $Q$ is a uniformly p-Artinian quantale, and $(I, \leq)$ is an upward directed poset, then $(\bigvee_{i\in I} q_i^-) = (\bigvee_{i\in I} q_i)^-$.*

*Proof.* The relation $(\bigvee_{i\in I} q_i^-) \leq (\bigvee_{i\in I} q_i)^-$ holds in general.

Now suppose that the degree of $Q$ is $n \in \mathbb{N}$. Then

$$(\bigvee_{i\in I} q_i)^- = (\bigvee_{i\in I} q_i)^n = \bigvee_{i_1,\cdots i_n\in I} q_{i_1} \odot \cdots \odot q_{i_n}.$$

But, since $(\bigvee_{i\in I} q_i)$ is an upward directed sup, for each $i_1, \cdots, i_n \in I$, there is $j \in I$ such that $q_{i_1}, \cdots, q_{i_n} \leq q_j$ then

$$\bigvee_{i_1,\cdots,i_n\in I} q_{i_1} \odot \cdots \odot q_{i_n} \leq \bigvee_{j\in I} q_j^n = (\bigvee_{j\in I} q_j^-).$$

□

Now, observe that the equality $\bigvee_{i\in I} q_i^- = (\bigvee_{i\in I} q_i)^-$ holds, in general (i.e. for each sup) for any locale, but not for any quantale.

**Example 3.1.19.** *If $Q = \mathbb{R}_+ \cup \{\infty\}$ is the extended half-line presented in Example 3.1.3(1), then all elements of $Q$ are in the interval $[0, \infty]$. There are only two idempotent elements in this quantales, $0$ and $\infty$. Since, in this case, the supremum is the infimum, and $0 \geq 0 + q$ if and only if $q = 0$, we have*

$$q^- = \begin{cases} 0, & \textit{if } q = 0, \\ \infty, & \textit{if } q \in (0, \infty] \end{cases}$$

*So, for a subset* $\{q_i : q_i \neq 0, \forall i \in I\} \subseteq Q$ *then* $\bigvee_{i\in I}(q_i^-) = \infty$ *but* $(\bigvee_{i\in I} q_i)^-$ *may be zero or* $\infty$ *depending if the supremum (which is the infimum in the usual ordering) of* $q_i$*'s is zero or not.*

Since some but not all quantales satisfies such equality, we provide a name for it.

**Definition 3.1.20.** *Let* $Q$ *be a commutative semicartesian quantale, we say* $Q$ *is a* ***geometric quantale*** *whenever* $\bigvee_{i\in I} q_i^- = (\bigvee_{i\in I} q_i)^-$*, for each* $\{q_i : i \in I\} \subseteq Q$*.*

Locales satisfy, trivially, this geometric condition. Moreover

**Example 3.1.21.** *The extended natural numbers presented in 3.1.3(2) is a geometric quantale. We argue in 3.1.19 that the extended positive real numbers is not a geometric quantale because* $(\bigvee_{i\in I} q_i)^-$ *could be zero or* $\infty$*, for a subset* $\{q_i : q_i \neq 0, \forall i \in I\} \subseteq \mathbb{R}_+ \cup \{\infty\}$*. However, we only have* $(\bigvee_{i\in I} q_i)^- = \infty$ *when considering the subset* $\{q_i : q_i \neq 0, \forall i \in I\} \subseteq \mathbb{N} \cup \{\infty\}$*. Therefore,* $\mathbb{N} \cup \{\infty\}$ *is a geometric quantale.*

Note that the poset of all ideals of a principal ideal domain *is not* a geometric quantale. In particular, $(\mathbb{N}, \cdot, \sqsubseteq)$, where $a \sqsubseteq b$ iff $b \mid a$, is not a geometric quantale.

We choose such terminology to indicate that under those conditions the operation $(-)^-$ is a *strong geometric morphism*, whose definition we provide in Section 3.4 and it coincides with the notion of a *quantale (homo)morphism* as defined for example in [Ros90a].

Moreover, we may construct geometric quantales from other geometric quantales:

**Proposition 3.1.22.** *The subclass of geometric quantales is closed under arbitrary products and interval construction.*

*Proof.* Given a family of quantales $Q = \{Q_i : i \in I\}$ the cartesian product $\prod_{i\in I} Q_i$ with component-wise order is a geometric quantale.

It follows from the fact that $(-)^-$ is component-wise. Indeed,

$$\begin{aligned}
(q_i)_i^- &= \bigvee\{(p_i)_i \in Idem(Q) \ : \ (p_i)_i \leq (p_i)_i \odot (q_i)_i\} \\
&= \bigvee\{(p_i)_i \in Idem(Q) \ : \ p_i \leq p_i \odot q_i, \forall i \in I\} \\
&= (\bigvee\{(p_i \in Idem(Q) \ : \ p_i \leq p_i \odot q_i, \forall i \in I\})_i \\
&= (q_i^-)_i
\end{aligned}$$

Then

$$\bigvee_{j\in I}(q_{ij})_i^- = \bigvee_{j\in I}(q_{ij}^-)_i = (\bigvee_{j\in I}(q_{ij}^-)_i = ((\bigvee_{j\in I} q_{ij})^-)_i$$

□

We defined $(-)^-$ as a supremum and verified it is the best lower idempotent approximation, in the sense that $q^-$ is the maximum of idempotents $e$ such that $e \leq q$ (Proposition 3.1.9.6), or equivalently $e \preceq q$ (Remark 3.1.5.3) . Analogously, we are tempted to define an operation $(-)^+$ as an infimum and obtain that $q^+$ is the minimum of idempotents $e$ such that $q \leq e$ (or, possibly, $q \preceq e$). To achieve this, we need **double-distributive quantales**, which are quantales that satisfy the following additional distributive law, for $I \neq \emptyset$:

$$a \odot (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \odot b_i)$$

Examples of double-distributive quantales are: locales; the extended half-line $[0, \infty]$ and the extended natural numbers $\mathbb{N} \cup \{\infty\}$; a subclass of the quantales of ideals of a commutative and unital ring that is closed under quotients and finite products and contains the principal ideal domains. Besides, double-distributive quantales are closed under arbitrary products and interval construction.

For members $u \in Q$ of those quantales $Q$, there is a more explicit construction of $u^-$ as a transfinite power $u^\alpha$, where $\alpha$ is an ordinal with cardinality $\leq$ cardinality of $Q$.

**Proposition 3.1.23.** *Let $Q$ be a unital double distributive commutative and semicartesian quantale. Given $q \in Q$, consider the transfinite chain of powers $(q^\alpha)_{\alpha \geq 1 \text{ ordinal}}$: $q^1 := q$; $q^{\alpha+1} := q^\alpha \odot q$; if $\gamma \neq 0$ is a limit ordinal, then $q^\gamma := \bigwedge_{\beta < \gamma} q^\beta$*

1. *It is a descending chain;*
2. *It stabilizes for some ordinal $\alpha$, $q^\beta = q^\alpha$ for each $\beta \geq \alpha$, and $\alpha <$ successor$(card(Q))$;*
3. *Moreover, if $q^\alpha = q^{\alpha+1}$, then $q^- = q^\alpha$.*

*Proof.* 1. This follows directly by induction.

2. Suppose that the restriction of the descending chain to all ordinal $\gamma$ with $1 \leq \delta \leq \alpha$ is a strictly descending chain in $Q$. Thus we have an injective function $[1, \alpha] \to Q$, $\delta \mapsto q^\delta$. Since $card(\alpha) = card([1, \alpha])$, we must have $card(\alpha) \leq card(Q)$, then $\alpha <$ successor$(card(Q))$. Thus, in particular, there is a largest ordinal $\alpha$ such that $(q^\delta)_{1 \leq \delta \leq \alpha}$ is a strictly descending chain. Thus $q^{\alpha+1} = q^\alpha$ and, by induction, $q^\beta = q^\alpha$ for each $\beta \geq \alpha$.

3. Suppose that the transfinite descending chain stabilizes at $\alpha$ (i.e. $q^{\alpha+1} = q^\alpha$). So $q^\alpha = q^\alpha \odot q^\alpha$ and $q^\alpha = q^{\alpha+1} = q^\alpha \odot q$ and $q^\alpha = q^\alpha \odot q^\alpha$. Thus $q^\alpha$ is an idempotent element such that $q^\alpha \preceq q$ (in particular, $q^\alpha \leq q^-$). On the other hand, for any idempotent $p \in Q$ such that $p \preceq q$ (i.e., $p = p \odot q$) we have, by induction $p \preceq q^\beta$, for all ordinal $\beta \geq 1$: in the induction step for ordinal limits we have to use the hypothesis that $Q$ is double-distributive. So $p = p \odot q^\alpha \leq q^\alpha$. Thus $q^\alpha$ is the largest idempotent (in the orders $\leq$ and $\preceq$) such that $q^\alpha \preceq q$. Then, by Proposition 3.1.9(6), $q^- = q^\alpha$.

□

Moreover, we can define a "best upper approximation":

**Definition 3.1.24.** *Let $Q$ be a commutative and semicartesian quantale that is also unital and "double-distributive". For each for $q \in Q$, define:*

$$q^+ := \bigwedge \{p \in Q : q \leq q \odot p\} = \bigwedge \{p \in Q : q \preceq p\}.$$

In this thesis we show the importance of a best lower idempotent approximation in Section 4.3.2, to analyze a candidate of subobject classifier that actually only (essentially) classifies a certain subclass of monomorphisms. In a recently submitted paper[3] with Caio de Andrade Mendes and Hugo Luiz Mariano as co-authors, we use a best upper idempotent approximation to discuss another candidate of a subobject classifier and this one actually classifies all monomorphisms. The drawback was the need to impose extra conditions on the quantale to make the construction possible.

## 3.2 Sheaves on Quantales

The current most general and well-accepted form of a sheaf is a *Grothendieck topos*, which arise from Grothendieck topologies but in this section we want to forget about Grothendieck toposes for a moment and think only about sheaves on locales. Then, how should we define sheaves on quantales? We propose the following answer for commutative semicartesian quantales:

**Definition 3.2.1.** *A presheaf $F : Q^{op} \to Set$ is a **sheaf on** $Q$ when for all $u \in Q$ and all $u = \bigvee_{i \in I} u_i$ cover of $u$ the following diagram is an equalizer in Set*

$$F(u) \xrightarrow{e} \prod_{i \in I} F(u_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(u_i \odot u_j)$$

*where:*

1. $e(t) = \{t_{|u_i} \mid i \in I\}, \quad t \in F(u)$
2. $p((t_k)_{k \in I}) = (t_{i_{|u_i \odot u_j}})_{(i,j) \in I \times I}$
   $q((t_k)_{k \in I}) = (t_{j_{|u_i \odot u_j}})_{(i,j) \in I \times I}, \quad (t_k)_{k \in I} \in \prod_{k \in I} F(u_k)$

Morphisms between sheaves on quantales are natural transformations and we denote the category of sheaves on quantales by $Sh(Q)$. Notice that the maps $F(u_i) \to F(u_i \odot u_j)$ exist because $u_i \odot u_j$ always is less or equal to $u_i$ and $u_j$, for all $i, j \in I$. This is where we use the semicartesianity.

**Remark 3.2.2.** *The category of sheaves $Sh(Q)$ is a full subcategory of the category of presheaves $PSh(Q)$, that is, the inclusion functor $i : Sh(Q) \to PSh(Q)$ is full.*

We highlight how natural this definition is: in the same way the intersection is a particular kind of an infimum, the infimum is a particular case of an associative binary operation. Besides, we still have the distributive law being satisfied. Therefore, why this is not the standard way of defining sheaves on quantales? We never saw this discussion in the literature, but one possible reason is that the cover $\{u_i \in Q : \bigvee_{i \in I} u_i = u\}$ is not a cover in the sense of a Grothendieck pretopology. It could exist a Grothendieck topology $J$ such that the correspondent category $Sh(Q, J)$ would be equivalent to the correspondent category $Sh(Q)$ of sheaves on quantales that we are proposing. We prove in Theorem 4.3.8 that $Sh(Q)$ may not be a Grothendieck topos and so such equivalence can not exist. On one hand, this is unfortunate since we are not able to use the vast and powerful theory

[3] There is a preliminary version of such paper available on https://arxiv.org/abs/2204.08351.

of sheaves. On the other hand, we may explore to which extent we can generalize sheaf theory starting from this new concept. Moreover, we want to apply this expanded sheaf theory to problems in Mathematics that were not yet reached in its current form. Now, we focus on sheaves on quantales, present examples, and prove basic properties about them, constantly comparing them with sheaves on locales. Deeper properties about $Sh(Q)$ will appear only after a more general treatment of sheaves on monoidal categories, with a more suitable notion of cover, which we present in the next Chapter.

We begin to construct a sheaf theory on quantales mimicking the presentation of sheaves on locales in [Bor94c]. Let $F : Q^{op} \to Set$ be a presheaf.

**Definition 3.2.3.** *Let $(u_i)_{i\in I}$ be a family of elements of $Q$. We say a family $(s_i \in F(u_i))_{i\in I}$ of elements of $F$ is **compatible** if for all $i, j \in I$ we have*

$$s_{i|_{u_i \odot u_j}} = s_{j|_{u_i \odot u_j}}$$

**Definition 3.2.4.** *We say a presheaf $F$ is **separated** if, given $u = \bigvee_{i\in I} u_i$ in $Q$ and $s, s' \in F(u)$, we have*

$$\left(\forall i \in I\ s_{|_{u_i}} = s'_{|_{u_i}}\right) \implies (s = s')$$

Using compatible families we equivalently define:

**Definition 3.2.5.** *Let $u = \bigvee_{i\in I} u_i$ in $Q$ and $(s_i \in F(u_i))_{i\in I}$ a compatible family in $F$, we say the presheaf $F$ is a **sheaf** if exists a unique element $s \in F(u)$ (called the gluing of the family) such that $s_{|_{u_i}} = s_i$, for all $i \in I$.*

It is a straightforward exercise in category theory to show that definitions 3.2.1 and 3.2.5 are equivalent.

**Lemma 3.2.6.** *If $F$ is presheaf on $Q$, the following conditions are equivalent:*

1. *$F$ is a sheaf.*
2. *$F$ is a separated presheaf and given $u = \bigvee_{i\in I} u_i$ in $Q$, every compatible family $(s_i \in F(u_i))_{i\in I}$ can be glued into an element $s \in F(u)$ such that $s_{|_{u_i}} = s_i$, for all $i \in I$.*

*Proof.* Note that the separated condition $\left(\forall i \in I\ s_{|_{u_i}} = s'_{|_{u_i}}\right) \implies (s = s')$ is the same as the uniqueness condition of the gluing in the sheaf definition. □

**Lemma 3.2.7.** *Let $F$ be a presheaf on a quantale $Q$. Then:*

1. *If $F$ is separated, $F(0)$ has at most one element.*
2. *If $F$ is a sheaf, $F(0)$ has exactly one element.*

*Proof.* Consider the empty cover $0 = \bigvee_{i\in\varnothing} u_i$ in $Q$. So every family of elements of $F$ is of the form $(s_i \in F(u_i))_{i\in\varnothing}$, so it is immediately compatible. Suppose $F$ is separated, then for every $s$ and $s'$ in $F(0)$, the condition $s_{|_{u_i}} = s'_{|_{u_i}}$ holds. Since $F$ is separable, $s = s'$. So, if there is any element in $F(0)$, it is unique.

Suppose $F$ is a sheaf, then there is a unique $s \in F(0)$ such that $s_{|_{u_i}} = s_i, \forall i \in \emptyset$. Since, in this context, every family is compatible, there is a unique element in $F(0)$, because $s_{|_{u_i}} = s_i$ always holds. $\square$

**Lemma 3.2.8.** *Let $F$ be presheaf on $Q$. If $u = \bigvee_{i\in I} u_i$ in $Q$ and $s \in F(u)$, then the family $\left(s_{|_{u_i}}\right)_{i\in I}$ is compatible.*

*Proof.* Let $i, j \in I$, then

$$\left(s_{|_{u_i}}\right)_{|_{u_i \odot u_j}} = s_{|_{u_i \odot u_j}} = \left(s_{|_{u_j}}\right)_{|_{u_i \odot u_j}}$$

$\square$

**Definition 3.2.9.** *A set of elements $\{q_i \mid i \in I\}$ of $Q$ is a **partition of** $q \in Q$ if $\bigvee_{i\in I} q_i = q$ and $q_i \odot q_j = 0$, for each $i \neq j$.*

It is clear that if $\{q_i \mid i \in I\}$ is a partition of $q$, then $\{q_i \odot a \mid i \in I\}$ is a partition of $a \odot q$, for any $a \in Q$. Thus, every partition of unity determines a partition for any $q \in Q$.

**Example 3.2.10.** *For a commutative ring $A$, any ideal $I$ has a partition: Take an idempotent $e \in A$ and observe that $1 = e + 1 - e$. We have $(1) = A$ (the unity), $(e + 1 - e) = (e) + (1 - e)$ and $(e) \odot (1 - e) = (e) \cap (1 - e) = 0$. So $\{(e), (1 - e)\}$ is a partition of $(1)$ and from it we obtain a partition $\{(e) \odot I, (1 - e) \odot I\}$ for any ideal $I$. If $A$ only have trivial idempotents then the ideals admit trivial partition.*

**Lemma 3.2.11.** *Let $F$ be a sheaf on $Q$ and $\{u_i \in Q \ : \ i \in I\}$ a partition of $u$. Then $F(u) \cong \prod_{i\in I} F(u_i)$.*

*Proof.* Since $0 \leq u_i$, for all $i \in I$, there is $s \in F(u)$ such that

$$(s_{|_{u_i}})_{|_{u_i \odot u_j}} = (s_{|_{u_i}})_{|_0} = s_{|_0}$$

and

$$(s_{|_{u_j}})_{|_{u_i \odot u_j}} = (s_{|_{u_j}})_{|_0} = s_{|_0}$$

Besides that, if $i = j$, $(s_{|_{u_i}})_{|_{u_i \odot u_j}} = (s_{|_{u_j}})_{|_{u_i \odot u_j}}$ immediately. So, for each $s \in F(u)$ we have a correspondent compatible family $(s_{u_i} \in F(u_i))_{i\in I}$. Since $F$ is a sheaf, this correspondence is bijective.

It is clear that every compatible family $(s_i \in F(u_i))_{i\in I}$ is an element of $\prod_{i\in I} F(u_i)$. On the other hand, every element $(s_i \in F(u_i))_{i\in I}$ of $\prod_{i\in I} F(u_i)$ has a correspondent compatible family, by what we reasoned above. $\square$

The following constructions provide sheaves on a quantale from another sheaf on the same or a different quantale.

**Proposition 3.2.12.** *Let $F$ be a sheaf on a quantale $Q$ and $u \in Q$. For each $w \leq v$, consider:*

$$F_{|_u}(v) = \begin{cases} F(v), & \text{if } v \leq u \\ \emptyset, & \text{otherwise.} \end{cases}$$

$$F_{|_u}(w \to v) = \begin{cases} F(w \to v) : F(v) \to F(w), & \text{if } w \leq v \leq u \\ ! : \emptyset \to F_{|_u}(w), & \text{if } w \leq v \nleq u. \end{cases}$$

*is a sheaf.* [4]

*Proof.* It is clear that $F_{|_u}$ is a presheaf. Consider $v = \bigvee_{i \in I} v_i$ in $Q$, and $s, s' \in F_{|_u}(v)$ such that $s_{|_{v_i}} = s'_{|_{v_i}}, \forall i \in I$

If $v \leq u$, then $s,' s \in F(v) = F_{|_u}(v)$. Since $F$ is a sheaf, it is separated so $s = s'$. If $v \nleq u$, then $s,' s \in \emptyset$ and there is nothing to do. Thus $F_{|_u}$ is a separated presheaf.

Now consider $(s_i \in F_{|_u}(v_i))_{i \in I}$ a compatible family. Suppose $F_{|_u}(v_i) = \emptyset$ for some $i \in I$. For such $i \in I$, there is no $s_i$ in $F_{|_u}(v_i)$, then, there is $j \in I$ such that $s_{i_{|u_i \odot u_j}} \neq s_{j_{|u_i \odot u_j}}$. In other words, the family is not compatible. This implies $F_{|_u}(v_i) = F(v_i)$, for all $i \in I$. So $v_i \leq u$, which means $\bigvee_{i \in I} v_i = v \leq u$. Therefore $F_{|_u}(v) = F(v)$.

Since $F$ is a sheaf, we that conclude the compatible family $(s_i \in F_{|_u}(v_i))_{i \in I}$ can be glued into $s \in F_{|_u}(v_i)$ such that $s_{|_{v_i}}$, $\forall i \in I$. By 3.2.6, $F_{|_u}$ is a sheaf. □

**Proposition 3.2.13.**

1. *Let $(Q_j)_{j \in J}$ be a family of commutative and semicartesian quantales and $(F_j)_{j \in J}$ be a family of sheaves, $F_j : Q_j^{op} \to Set$, for each $j \in J$. Then: $\prod_{j \in J} Q_j$ is a commutative semicartesian quantale; a family $\{(u^i_j)_{j \in J} : i \in I\}$ is a cover of $(u_j)_{j \in J} \in \prod_{j \in J} Q_j$ iff for each $j \in J$, $\{u^i_j : i \in I\}$ is a cover of $u_j \in Q_j$; and $\prod_{j \in J} F_j : (\prod_{j \in J} Q_j)^{op} \to Set$ given by $(\prod_{j \in J} F_j)(u_j)_{j \in J} := \prod_{j \in J} F_j(u_j)$ is a sheaf with the restriction maps defined component-wise from each $F_j$.*

2. *Let $F : Q^{op} \to Set$ be a sheaf on the commutative and semicartesian quantale $Q$. Let $e, a \in Q$, $e \leq a$, $e^2 = e$ and consider $Q' = [e, a]$, the (commutative and semicartesian) "subquantale" of $Q$. Then $F' : Q'^{op} \to Set$ defined by $F'(u) = F(u)$, if $u \neq e$ and $F'(e) = \{*\}$, with non-trivial restriction maps $F'(v) \to F'(u) = F(v) \to F(u)$, if $e < v \leq u$, is a sheaf.*

*Proof.*

1. Let $(u_j)_{j \in J} = \bigvee_{i \in I} (u^i_j)_{j \in J} = (\bigvee_{i \in I} u^i_j)_{j \in J}$ be a cover, and $(s_i)_{i \in I} \in (\prod_{j \in J} F_j)(u_j)_{j \in J}$ a compatible family. So $s_i = (s^i_j)_{j \in J}$, where $x_i \in F(u_i)$ and $y \in G(v_i)$ are compatible families. Since $F$ and $G$ are sheaves, there are gluing $x \in F(u)$ and $y \in G(v)$ for the compatible families $x_i$ and $y_i$, respectively. Take $s = (x, y) \in (F \times G)(u, v)$. Thus

$$s_{\restriction_{(u_i, v_i)}} = (x_{\restriction_{u_i}}, y_{\restriction_{v_i}}) = (x_i, y_i) = s_i, \; \forall i \in I$$

[4] Note that $F_{|_1} = F$ and if $u' \leq u \in Q$, then $F_{|_u|_{u'}} = F_{|_{u'}}$.

So $s$ is a gluing of $s_i$ and it is unique by the uniqueness of $x$ and $y$.

2. Let $u = \bigvee_{i \in I} u_i$ be a cover in $Q'$, and $(s_i \in F'(u_i))_{i \in I}$ a compatible family. Since $F'(u_i) = F(u_i)$ and the restriction maps for $F'$ are restriction maps for $F$, we have that $(s_i \in F(u_i))_{i \in I}$ is a compatible family. Since $F$ is a sheaf, there is a unique gluing.

□

Next we introduce a concrete example of a sheaf, showing a way in which balls centered in a fixed point of a given extended metric space may be interpreted as sheaves:

**Example 3.2.14.** *Take $Q = ([0, \infty], +, \geq)$ the extended half-line quantale. Let $(X, d)$ be an (extended) metric space. For each $A \subseteq X$ and each $r \in [0, \infty]$ consider balls $F_A(r) = B_r(A) = \{x \in X : d(x, A) \leq r\}$. Note that $s \geq r$ entails $B_r(A) \subseteq B_s(A)$ and, in the obvious way $F_A : [0, \infty] \to Set$ became a presheaf over the quantale $Q$ where $F_A((s \geq r)) : F_A(r) \hookrightarrow F_A(s)$ is the inclusion. Moreover, this is a sheaf, since if $r = \bigwedge_{i \in I} s_i$ in $[0, \infty]$, then the diagram below is an equalizer*

$$B_r(A) \to \prod_i B_{s_i}(A) \rightrightarrows \prod_{i,j} B_{s_i + s_j}(A)$$

*for non-empty coverings. However, if $I = \emptyset$, then $r = \bigwedge_{i \in I} s_i = \infty$. Therefore, $B_\infty(A)$ is not a single element (i.e., is not the terminal object in Set). This means that the sheaf condition fails when $I = \emptyset$. To surpass this, we maintain our definition $B_r(A)$ for all $r \in [0, \infty)$ but for $r = \infty$ we define $B_\infty(A) = \{*\}$. For any $s \geq r$, the restrictions map is the identity map on $\{*\}$.*

The next result is simple but it is fundamental for our argument to show that $Sh(Q)$ is a monoidal closed category.

**Proposition 3.2.15.** *Let $F$ be a sheaf on a quantale $Q$ and $u \in Q$. For each $w \leq v$, consider:*

$$F^{(u)}(v) := F(v \odot u)$$

$$F^{(u)}(w \to v) := F(w \odot u \to v \odot u).$$

*Then $F^{(u)}$ is a sheaf.* [5]

*Proof.* It is clear that $F^{(u)}$ is a presheaf since $F$ is a sheaf and $w \leq v$ in $Q$ implies that $(w \odot u) \leq (v \odot u)$.

We want to show that for a given cover $v = \bigvee_i v_i$, the following diagram is an equalizer:

$$F^{(u)}(v) \xrightarrow{\ e\ } \prod_{i \in I} F^{(u)}(v_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F^{(u)}(v_i \odot v_j)$$

Note that $v \odot u = \bigvee_i (v_i \odot u)$ is a cover, since $v = \bigvee_i v_i$ is a cover.

[5] Note that $F^{(1)} = F$ and if $u', u \in Q$, then $(F^{(u)})^{(u')} = F^{(u \odot u')}$.

Take a family $(s_i) \in F^{(u)}(v_i) = F(v_i \odot u)$, such that

$$F(v_i \odot v_j \odot u \to v_i \odot u)(s_i) = F(v_i \odot v_j \odot u \to v_j \odot u)(s_j) \in F(v_i \odot v_j \odot u) \, \forall i \in I$$

Since $v_i \odot u \odot v_j \odot u \leq v_i \odot v_j \odot u$, we have that $s_i \in F(v_i \odot u)$ is a compatible family for $F$. Since $F$ is a sheaf, there is a unique gluing $s \in F(v \odot u) = F^{(u)}(v)$ for the family $(s_i)_{i \in I}$. □

**Proposition 3.2.16.** *The functor $Q(-, v)$ is a sheaf, for every fixed $v \in Q$.*

*Proof.* Recall that $Q(-, v)$ is the functor $Hom_Q(-, v)$ so it is a presheaf, where if $w \leq u$, then we send the unique element $\{(u \to v)\}$ in $Q(u, v)$ to the unique element $\{(w \to v)\}$ in $Q(w, v)$.

Observe that we have two cases:

1. Suppose $u \leq v$: since $u_i \leq u$, for all $i \in I$, we have that $u_i \leq v$, for all $i \in I$. Take $s_i = (u_i \to v) \in Q(u_i, v)$, since $u_i \odot u_j \leq u_i, u_j$, for all $i, j \in I$,

$$s_{i|_{u_i \odot u_j}} = (u_i \odot u_j \to v) = s_{j|_{u_i \odot u_j}}$$

   So $(s_i)_{i \in I}$ is a compatible family. To conclude $Q(u, v)$ is a sheaf, take the only element $s = (u \to v) \in Q(u, v)$ and observe that $s_{|_{u_i}} = (u_i \to v) = s_i$, for all $i \in I$.

2. Suppose $u \nleq v$: if $u_i \leq v$, for all $i \in I$, by definition of supremum, $\bigvee_{i \in I} u_i \leq v$, which is not possible. So there is at least one $i \in I$ (if $I \neq \emptyset$) such that $u_i \nleq v$. Thus, $Q(u, v)$ and $Q(u_i, v)$ are empty sets, for such an $i \in I$. Then the sheaf condition is vacuously true.

   If $I = \emptyset$, then $\bigvee\limits_{i \in \emptyset} u_i = 0$ and $Q(0, v)$ fits in the first case since $0 \leq v$.

□

**Proposition 3.2.17.** *The subcategory $Sh(Q) \hookrightarrow Set^{Q^{op}}$ is closed under limits.*

*Proof.* Consider a small (index) category $\mathcal{J}$ and a functor $F : \mathcal{J} \to PSh(Q)$ with limits[6] $(L, p_J : L \to F(J))_{J \in \mathcal{J}_0}$. To show that $Sh(Q) \hookrightarrow Set^{Q^{op}}$ is closed under limits we have to prove that if $F(J)$ is a sheaf for all objects $J$ in $\mathcal{J}$, then the limit $L$ is a sheaf. Now, the argument is the verbatim copy of the argument used in the proof of [Bor94c, Proposition 2.2.1], but replacing $J$ and $\mathcal{J}$ by $I$ and $\mathcal{I}$, respectively.

□

**Corollary 3.2.18.** *$Sh(Q)$ has a terminal object, the (essentially unique) presheaf such that $card(\mathbf{1}(u)) = 1$, for each $u \in Q$. Moreover, $Hom_Q(-, 1) \cong \mathbf{1}$*

[6] All presheaf categories are complete and the limits are computed pointwise.

*Proof.* Since $Sh(Q) \hookrightarrow Set^{Q^{op}}$ is closed under limit, a terminal object in $Sh(Q)$ must be the terminal presheaf $Hom_Q(-,1) \cong 1$ ($Hom_Q(u,1) = 1, \forall u \in Q$ ) in $PSh(Q)$. Since 1 is the top element of $Q$, there is an arrow from $u \to 1$, for all $u \in Q$. So $card(\mathbf{1}(u)) = 1$. $\square$

**Proposition 3.2.19.** *A monomorphism between sheaves $\eta : F \rightarrowtail G$ is just a monomorphism between their underlying presheaves (and they are monomorphisms if and only if $\eta_u : F(u) \to G(u)$ is injective, for each $u \in Q$).*

*Proof.* Again, the argument is the same as in the case of sheaves on locales: $\eta : F \rightarrowtail G$ is a mono if and only if the pullback of $\alpha$ with itself is

$$\begin{tikzcd} G \arrow[r, "id_G"] \arrow[d, "id_G"'] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "\alpha"] \\ G \arrow[r, "\alpha"'] & F \end{tikzcd}$$

In other words, $\eta : F \rightarrowtail G$ is a mono if and only $(G, id_G, id_G)$ is the kernel pair of $\alpha$. This holds in any category. So $\eta$ is a mono of sheaves iff the kernel pair of $\alpha$ is $(G, id_G, id_G)$ iff $\eta$ is a mono of the underlying presheaves. Next, we use another general fact about categories. For a small category $\mathcal{A}$ and a category with pullbacks $\mathcal{B}$, a morphism $\beta : F \to G$ in the functor category $Func(\mathcal{A}, \mathcal{B})$ is mono iff all the components $\beta_a : F(a) \to G(a)$ are injective for all object $a$ of $\mathcal{A}$ [Bor94a, Corollary 2.15.3]. Take $\mathcal{A} = Q^{op}$ and $\mathcal{B} = Set$, then we obtain that $\eta$ is mono iff $\eta_u$ is injective for all $u \in Q$. $\square$

**Corollary 3.2.20.** *Every morphism $\eta : Q(-,v) \to F$, where $F$ is a (pre)sheaf is, automatically, a monomorphism.*

*Proof.* By Proposition 3.2.19, is enough to show that $\eta_u : Q(u,v) \to F(u)$ is injective for all $u \in Q$. This always holds because $Q(u,v)$ has at most one element. $\square$

**Proposition 3.2.21.** *The family of representable sheaves $Q(-,u)$, indexed by elements of $Q$, is a set of generators for $Sh(Q)$.*

*Proof.* Take $\eta, \eta' : H \to F$ two distinct morphisms of sheaves. Consider the index $u$ as the infimum element of $Q$. Observe the following composition

$$\begin{tikzcd} Q(-,u) \arrow[r, "\zeta"] & H \arrow[r, shift left, "\eta"] \arrow[r, shift right, "\eta'"'] & F \end{tikzcd}$$

Since there is no element smaller than $u$, the only component in which makes sense to calculate $\zeta$ is the element $u$ himself. So we only have

$$\begin{tikzcd} Q(u,u) \arrow[r, "\zeta_u"] & H \arrow[r, shift left, "\eta_u"] \arrow[r, shift right, "\eta'_u"'] & F \end{tikzcd}$$

where $Q(-,u)$ is (naturally isomorphic to) the identity map $u \to u$. Since $\eta \neq \eta'$, we conclude $\eta \circ \zeta \neq \eta' \circ \zeta$, as desired. $\square$

**Proposition 3.2.22.** *Note that:*

1. *For each $v, v' \in Q$, there is at most one (mono)morphism $Q(-, v) \to Q(-, v')$ and this exists precisely when $v \leq v'$.*
2. *If $H$ is a sheaf and $\epsilon : H \rightarrowtail Q(-, v)$ is a monomorphism, then $H \cong Q(-, h)$ where $h = \bigvee\{u \leq v : H(u) \neq \emptyset\}$.*

*Proof.* 1. For each $u, v \in Q$, note that $card(Q(u, v)) \in \{0, 1\}$.
Suppose there is a morphism $\eta : Q(-, v) \to Q(-, v')$. So, for all $u \in Q$ we have $\eta_u : Q(u, v) \to Q(u, v')$. If $Q(u, v') = \emptyset$, then $Q(u, v) = \emptyset$. Thus if $u \leq v$, then $u \leq v'$. In particular, for $u = v$, we obtain $v \leq v'$.

Conversely, if $v \leq v'$, consider $i_{v,v'} : Q(-, v) \to Q(-, v')$. For all $u \in Q$ we have $i_{v,v'}(u) : Q(u, v) \to Q(u, v')$.

If $u \not\leq v$, then $Q(u, v) = \emptyset$ and $i_{v,v'}(u) : \emptyset \to Q(u, v')$ the unique function from the $\emptyset$, since the $\emptyset$ is an initial object in *Set*.

If $u \leq v$, since $v \leq v'$, $u \leq v'$ and then $i_{v,v'}(u)(u \to v) = (u \to v')$. For any other morphism $j_{v,v'} : Q(-, v) \to Q(-, v')$ we obtain that $j_{v,v'}(u) : \emptyset \to Q(u, v')$ the unique function from the $\emptyset$, whenever $u \not\leq v$ and $j_{v,v'}(u)(u \to v) = (u \to v')$, whenever $u \leq v$. So $i_{v,v'} = j_{v,v'}$.

2. Since $\epsilon$ is a monomorphism, $\epsilon_u$ is injective and then $card(H(u)) \in \{0, 1\}$ for each $u \in Q$ with $H(u) = \emptyset$ whenever $u \not\leq v$. So let

$$h = \bigvee\{u \leq v : H(u) \neq \emptyset\} = \bigvee\{u \in Q : H(u) \neq \emptyset\}.$$

We will show that $H(u)$ is non-empty only when $u \leq v$. Note that:

- If $q \leq p$ and $H(p) \neq \emptyset$, then $H(q) \neq \emptyset$ (since $H$ is a presheaf);

- Since $card(H(h)) = 1$, we have $H(h) \neq \emptyset$. Once $H(p), H(q) \neq \emptyset$ entails $H(p \odot q) \neq \emptyset$, by the sheaf condition we have an equalizer diagram between two parallel arrows where the source and target are both singletons.

Therefore, $H(u) \neq \emptyset$ iff $u \leq h$. Now, we will show that $H(u) \to Q(u, h)$ is a (unique) bijection, for each $u \in Q$.

If $u \not\leq h$, then $\emptyset = H(u) \to Q(u, h) = \emptyset$. If $u \leq h$, then $H(u)$ and $Q(u, h)$ are both singletons. So $\epsilon_u$ is an injection and a surjection in *Set*, therefore, a bijection for all $u \in Q$ and then $\epsilon$ is an isomorphism.

□

## 3.3 Sheaves on a basis

Now, recall that given a topological space $X$, a basis is a family $B$ of open sets such that every open set of $X$ can be represented as the union of some subfamily of $B$. This definition can be written using quantalic terminology.

**Definition 3.3.1.** *If $Q$ is a commutative semicartesian quantale, a subset $B \subseteq Q$ is a* ***basis of*** *$Q$ if it is a submonoid of $Q$ such that for each $u \in Q$ there exists a covering $\{u_i : u_i \in B, i \in I\}$ of $u$.*

Observe that a basis of the locale $\mathcal{O}(X)$ of all open sets of $X$ gives the usual notion of basis in topology.

**Example 3.3.2.** *If $Q = \mathcal{I}(R)$ is the quantale of ideals of a commutative ring with unity $R$, then the set of principal ideals of $R$ is a basis of $\mathcal{I}(R)$. We denote this base by $\mathcal{PI}(R)$.*

In the case of sheaves on a topological space $X$ it is known that it is enough to define sheaves on the basis of $X$, [MM92, Theorem 3, Chapter II.2]. Next, we prove the same for sheaves on quantales.

**Definition 3.3.3.** *A presheaf $F : B^{op} \to Set$ is a* ***sheaf on a basis*** *$B$ if, given any $u \in B$ and any covering $u = \bigvee_{i \in I} u_i$ with $u_i \in B$ for all $i \in I$, it holds that the following is an equalizer diagram*

$$F(u) \xrightarrow{e} \prod_{i \in I} F(u_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(u_i \odot u_j)$$

*where*

1. $e(t) = \{t_{|_{u_i}} \mid i \in I\}, \quad t \in F(u)$
2. $p((t_k)_{k \in I}) = (t_{i_{|_{u_i \odot u_j}}})_{(i,j) \in I \times I}$
   $q((t_k)_{k \in I}) = (t_{j_{|_{u_i \odot u_j}}})_{(i,j) \in I \times I}, \quad (t_k)_{k \in I} \in \prod_{k \in I} F(u_k)$

Morphisms of sheaves on a basis are natural transformations.

Clearly, any sheaf $F : Q^{op} \to Set$ restricts to a sheaf on its basis $B$. This process yields a restriction functor $r : Sh(Q) \to Sh(B)$. Moreover,

**Theorem 3.3.4.** *There is an equivalence $Sh(Q) \to Sh(B)$.*

*Proof.* The idea is the same as proposed in [MM92, Chapter II, Exercise 4]. We define a functor $s : Sh(B) \to Sh(Q)$ in the following way: for any $u \in Q$ with a cover $u = \bigvee_{i \in I} b_i$ such that $b_i \in B, \forall i \in I$, we define $s(F)$ as the equalizer

$$s(F)(u) \xrightarrow{e} \prod_{i \in I} F(b_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(b_i \odot b_j)$$

where $F$ is a sheaf on the basis $B$.

Given a morphism $F \to G$ of sheaves on $B$, the universal property of the equalizer induces a morphism $s(F) \to s(G)$ of sheaves on $Q$.

Given $F$ sheaf on $Q$ and a covering on the base $u = \bigvee_{i \in I} b_i$, by definition of $s$ we have that the following diagram is an equalizer:

$$s(r(F))(u) \xrightarrow{e} \prod_{i\in I} r(F)(b_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{(i,j)\in I\times I} r(F)(b_i \odot b_j)$$

Since $r$ is the restriction, $r(F)(b_i) = F(b_i)$ and $r(F)(b_i \odot b_j) = F(b_i \odot b_j)$, for all $i, j \in I$. So $s(r(F))(u)$ is an equalizer of $\prod_{i\in I} F(b_i) \rightrightarrows \prod_{(i,j)\in I\times I} F(b_i \odot b_j)$ . Since $F$ is sheaf, $F(u)$ is the equalizer of the same pair of arrows. By the uniquiness (up to isomorphism) of the equalizer, $s(r(F))(u)$ is isomorphic to $F(u)$, for all $u \in Q$ and $F$ sheaf on $Q$. Therefore, we have a natural isomorphism $s \circ r \Rightarrow id_{Sh(Q)}$. By a similar reasoning we conclude that exists a natural isomorphism $r \circ s \Rightarrow id_{Sh(B)}$.

□

The above theorem says that to describe a sheaf on a quantale it is enough to describe it on its basis. Then we may, for example, define a sheaf on the quantale of ideals of an integral domain $R$ by defining it on its base. In the following example, we used the hypothesis that $R$ is an integral domain, but after the verification, we provide a different argument, showing that the defined sheaf still is a sheaf for any commutative ring with unity.

**Example 3.3.5.** *Take $B = \mathcal{PI}(R)$ the set of principal ideals of $R$. The functor*

$$\begin{aligned} L_R : \ B^{op} &\to CRing \\ nR &\mapsto R[n^{-1}] \end{aligned}$$

*is a sheaf.*

First, we recall the notation and some basic facts from algebra: the functor takes principal ideals from $R$ and sends it to the localization of $R$ at $n$, that is, to the ring where all elements are of the form $\frac{z}{n^k}$, for some $z \in R$ and some $k \in \mathbb{N}$.

We also have to describe the restriction maps of the sheaf. Note that $nR \subseteq mR$ if and only if $m$ divides $n$. In other words, if and only if there is an $l \in R$ such that $nR = mlR$. So we define $L_R$ to send morphisms $nR \subseteq mR$ in $B$ to ring homomorphisms

$$\begin{aligned} \varphi : \ R[m^{-1}] &\to R[n^{-1}] \\ \frac{z}{m^k} &\mapsto \frac{zl^k}{n^k} \end{aligned}$$

We denote $\varphi(\frac{z}{m^k})$ by $\frac{z}{m^k}\restriction_{nR}$. Now we show that $L_R$ is a sheaf.

Take an ideal $nR$ of $R$ and a covering $nR = \Sigma_{i\in I} n_i R$.

**Locality:** Take $s = \frac{z}{n^k}$ and $t = \frac{z'}{n^{k'}}$ in $R[n^{-1}]$ such that $s\restriction_{n_iR} = t\restriction_{n_iR}$, for all $i \in I$.

$$s\restriction_{n_iR} = t\restriction_{n_iR} \implies \frac{zl_i^k}{n_i^k} = \frac{z'l_i^{k'}}{n_i^{k'}} \in R[n_i^{-1}] \implies \exists p \in \mathbb{N} \text{ such that } zl_i^k n_i^{k'} n_i^p = z'l_i^{k'} n_i^k n_i^p$$

So

$$zl_i^k n_i^{k'+p} = z' l_i^{k'} n_i^{k+p}$$

We substitute $n_i = nl_i$ in the above equation and obtain

$$zl_i^{k+k'+p} n^{k'+p} = z' l_i^{k+k'+p} n^{k+p}$$

Thus, since we are in an integral domain,

$$zn^{k'+p} = zn^{k+p}$$

(This passage holds because $\frac{zl_i^{k+k'+p} n^{k'+p}}{l_i^{k+k'+p}} = \frac{zn^{k'+p}}{1}$, since $zl_i^{k+k'+p} n^{k'+p} 1 l_i^0 = zn^{k'+p} l_i^{k+k'+p} l_i^0$).

Then $zn^k n^p = zn^{k'} n^p$, i.e.,

$$s = \frac{z}{n^k} = \frac{z'}{n^{k'}} = t$$

**Gluing:**

Let $s_i \in R[n_i^{-1}]$ of the form $s_i = \frac{z_i'}{n_i^{k_i}}$ for some $z_i' \in R$ and some $k_i \in \mathbb{N}$. Take $k = \max_i k_i$, call $z_i = z_i' n_i^{k-k_i}$, and then

$$s_i = \frac{z_i'}{n_i^{k_i}} \left(\frac{n_i}{n_i}\right)^{k-k_i}$$

Using this trick, $s_i \restriction_{n_i n_j R} = s_j \restriction_{n_i n_j R}$ means that

$$\frac{z_i}{n_i^k} \restriction_{n_i n_j R} = \frac{z_j}{n_j^k} \restriction_{n_i n_j R} \implies \frac{z_i n_j^k}{n_i^k n_j^k} = \frac{z_j n_i^k}{n_j^k n_i^k} \text{ in } R[(n_i n_j)^{-1}]$$

So, for each $i, j \in I$, there is a $q_{ij} \in \mathbb{N}$ such that

$$(n_i n_j)^{q_{ij}} (n_i n_j)^k z_i n_j^k = (n_i n_j)^{q_{ij}} (n_i n_j)^k z_j n_i^k$$

If all elements were in $\mathbb{Z}$ we could use $n_j^k z_i = n_i^k z_j$, but we will proceed with the calculations using the above equation so the proof remains valid for any integral domain.

If we take $q = \max_{i,j} q_{ij}$ then

$$n_i^{q+k} n_j^{q+2k} z_i = n_i^{q+2k} n_j^{q+k} z_j \tag{3.1}$$

Given $s = \frac{z}{n^p} \in R[n^{-1}]$, we have $s\restriction_{n_i R} = \frac{zl_i^p}{n_i^p}$ (using that trick). We want to show that

$$\frac{zl_i^p}{n_i^p} = \frac{z_i}{n_i^k} \text{ in } R[n_i^{-1}]$$

Since $nR = n_1R + ... + n_sR$ and $n$ is the greatest common divisor of $n_i$'s, we have that $n^r$ is the greatest common divisor of $n_i^r$'s, for any $r \in \mathbb{N}$. By the Bézout's identity, there are $b_1, ..., b_t \in R$ such that

$$n^r = b_1n_1^r + ... + b_tn_t^r \tag{3.2}$$

In the following, we assume $t = 2$ to help in the visualization of the calculations. Multiplying 3.2 by $n^{m+p-r}l_1^{m+p}z_1$ in both sides we have:

$$\begin{aligned} n^{m+p}l_1^{m+p}z_1 &= (b_1n_1^r + b_2n_2^r)n^m l_1^m n^{p-r}l_1^p z_1 \\ &= (b_1n_1^{r+m}z_1 + b_2n_2^r n_1^m z_1)n^{p-r}l_1^p && \text{Since } n_i = nl_i \end{aligned}$$

We have to guarantee that $p - r \geq 0$ and use equation 3.1. Take $r = q + 2k = m$ and $p \geq r$ to obtain

$$\begin{aligned} n^{m+p}l_1^{m+p}z_1 &= (b_1n_1^{2(q+2k)}z_1 + b_2n_2^{q+2k}n_1^{q+2k}z_1)n^{p-r}l_1^p \\ &= (b_1n_1^{2(q+2k)}z_1 + b_2n_2^{q+2k}n_1^{q+k}n_1^k z_1)n^{p-r}l_1^p \\ &= (b_1n_1^{2(q+2k)}z_1 + b_2n_2^{q+k}n_1^{q+2k}n_1^k z_2)n^{p-r}l_1^p && \text{By 3.1} \\ &= n_1^{m+k}(b_1n_1^q z_1 + b_2n_2^{q+k}z_2)n^{p-r}l_1^p \end{aligned}$$

Call $z = (b_1n_1^q z_1 + b_2n_2^{q+k}z_2)n^{p-r}$. Then, for such a $z \in R$ we have that $n^{m+p}l_1^{m+p}z_1 = n_1^{m+k}zl_1^p$. Using $n_i = nl_i$ and a simple manipulation this is

$$n_1^m n_1^k z l_i^p = n_i^m n_i^p z_i.$$

So, $m = q + 2k$ is the natural number that makes

$$\frac{zl_1^p}{n_1^p} = \frac{z_1}{n_1^k} \text{ in } R[n_1^{-1}]$$

This illustrates what we must do to conclude the desired result for any $i \in I$ and $t \in \mathbb{N}$. Fix an $i$, take $r = q + 2k = m$ and $p \geq r$, and multiply 3.2 by $n^{m+p-r}l_i^{m+p}z_1$ on both sides.

Now we have:

$$
\begin{aligned}
n^{m+p}l_i^{m+p}z_i &= (\sum_{j=1}^{t} b_j n_j^{q+2k}) n^{q+2k} l_i^{q+2k} n^{p-r} l_i^p z_i \\
&= (b_i n_i^{2(q+2k)} z_i + \sum_{\substack{j=1 \\ j\neq i}}^{t} b_j n_j^{q+2k} n_i^{q+k} n_i^k z_i) n^{p-r} l_i^p \\
&= (b_i n_i^{2(q+2k)} z_i + \sum_{\substack{j=1 \\ j\neq i}}^{t} b_j n_j^{q+k} n_i^{q+2k} z_j) n^{p-r} l_i^p && \text{By 3.1} \\
&= n_i^{m+k} (b_i n_i^q z_i + \sum_{\substack{j=1 \\ j\neq i}}^{t} b_j n_j^{q+k} z_j) n^{p-r} l_i^p
\end{aligned}
$$

So, for $z = n^{p-q-2k} l_i^p$, $p \geq q+2k$, and $m = q+2k$, we have

$$\frac{z l_i^p}{n_i^p} = \frac{z_i}{n_i^k} \text{ in } R[n_i^{-1}]$$

This holds for all $i \in I$, thus we proved the gluing property and concluded that $L_R$ is a sheaf.

By Theorem 3.3.4, the above example provides a more sophisticated example of a sheaf on a quantale.

The reader that already studied Algebraic Geometry may recognize that the above sheaf is similar to the structure sheaf attached to $SpecA$, i.e., the sheaf $\mathcal{O}_{SpecA}(D(f)) \cong A_f$, where $A$ is a commutative ring with unity, $D(f)$ is the principal open for $f \in A$ and $A_f$ is the localization $A[f^{-1}]$. The principal open $D(f)$ forms a basis under the Zariski topology. So our approach is completely analogous to the one largely used in Algebraic Geometry. Moreover, both sheaves are deeply connected by a general process of "change of base" that we address in the next Section. For now, we observe the following:

Since the structure sheaf is a sheaf we have that the following diagram is an equalizer:

$$\mathcal{O}_{SpecA}(D(f)) \longrightarrow \prod_{i\in I} \mathcal{O}_{SpecA}(D(f_i)) \rightrightarrows \prod_{(i,j)\in I\times I} \mathcal{O}_{SpecA}(D(f_i) \cap D(f_j))$$

By the definition of the structure sheaf and since $D(f_i) \cap D(f_j) = D(f_i.f_j)$, the next diagram is also an equalizer

$$A[f^{-1}] \longrightarrow \prod_{i\in I} A[f^{-1}] \rightrightarrows \prod_{(i,j)\in I\times I} A[(f_i.f_j)^{-1}]$$

If $A$ is an integral domain then we may use the functor $L_A$ defined in Example 3.3.5 to obtain the following equalizer diagram.

$$L_A(fA) \longrightarrow \prod_{i\in I} L_A(f_iA) \rightrightarrows \prod_{(i,j)\in I\times I} L_A(f_iA.f_jA)$$

where we used $f_iA.f_jA = (f_i.f_j)A$ in the final expression. So $L_A$ is a sheaf. Note that using this argument, we do not have to suppose that the ring is an integral domain. Therefore, actually, $L_A(n) \cong R[n^{-1}]$ defines a sheaf for any commutative ring with unity.

The above observation is interesting because:

**Remark 3.3.6.** *If we have a sheaf on a topological space and such topological space arises from a ring, it is natural to wonder how to think about the sheaf on the correspondent ring. Those calculations show that our definition of a sheaf on a quantale appears almost spontaneously. In the next section, we will see a way of constructing sheaves on quantales by taking sheaves on locales (Theorem 3.4.8). The example we presented is a sheaf on the quantale of ideals of a ring that comes from a sheaf on the locale of open subsets of the topological space $Spec(A)$.*

## 3.4 Change of Base

We recognize that the nomenclature is not fortunate but for us "base" and "basis" are distinct notions. The basis of a quantale somehow is a subset of the quantale that describes it as a whole, as we defined in 3.3.1. A "base quantale" is the quantale that is the category in the domain of a sheaf functor. In this Section we study a kind of functor $Q \to Q'$ between quantales that will give interesting (adjoint) functors between the correspondent sheaf categories.

**Definition 3.4.1.** *A geometric morphism is a pair of adjoint functors* $Q \underset{f_*}{\overset{f^*}{\leftrightarrows}} Q'$ *such that*

1. *$f^*$ preserves arbitrary sups and 1;*
2. *$f^*$ weakly preserves the multiplication, i.e., $f^*(p) \odot f^*(q) \leq f^*(p \odot' q), \forall p, q \in Q$.*

**Remark 3.4.2.** *In [Ros90b] morphisms of quantales are defined as maps that preserve arbitrary sup and the multiplication.*

**Remark 3.4.3.** *The right adjoint of $f^*$ comes from the Adjoint Functor Theorem: since $f^*$ preserves sups, it has a right adjoint $f_* : Q \to Q'$. We call $f_*$ direct image, and $f^*$ is called inverse image.*

Since $f_*$ has a left adjoint, the Adjoint Functor Theorem guarantees that $f_*$ preserves all limits. In particular, $f_*$ preserves meets and 1.

Even though we are considering semicartesian quantales instead of idempotent quantales, the above definition is exactly the same as a morphism of quantales given in [BB86]. As stated there, if the quantale is a locale this definition coincides with the classical definition of a morphism of locales: since $f^*$ preserves arbitrary sups it is a functor, consequently, we have $f^*(p \wedge q) \leq f^*(p)$ and $f^*(p \wedge q) \leq f^*(q)$. Therefore, $f^*(p \wedge q) \leq f^*(p) \wedge f^*(q)$. The definition provides the other side of the inequality.

**Example 3.4.4.** *The inclusion $Idem(Q) \to Q$ is a geometric morphism with $(-)^{-} : Q \to Idem(Q)$ as right adjoint (see Proposition 3.1.10).*

Next, we define what is called a *strict morphism of quantales* in [BB86].

**Definition 3.4.5.** *A **strong geometric morphism of quantales** is a geometric morphism of quantales where $f^*$ preserves the multiplication. In other words, the other inequality holds.*

**Example 3.4.6.** *(Strong geometric morphisms)*

1. *The inclusion $Idem(Q) \to Q$.*
2. *The idempotent approximation $(-)^{-} : Q \to Idem(Q)$ is a strong geometric morphism if $Q$ is a geometric quantale. By 3.1.10 and the definition of geometric quantale.*
3. *A projection map $p_i : \prod_{i\in I} Q_i \to Q_i$ preserves sups, unit, and multiplication.*
4. *The inclusions $[a^{-}, \top] \to Q$ preserves sups, unit and multiplication, for all $a \in Q$.*
5. *Every surjective homomorphism $f : R \to S$ of commutative and unital rings induces a strong geometric morphism $f^* : \mathcal{I}(R) \to \mathcal{I}(S)$ given by $f^*(J) = f(J)$. Indeed, notice that $1 = R$ in $\mathcal{I}(R)$ and the surjectivity gives that $f^*(R) = S$, so $f^*$ preserves $1$. To show it preserves arbitrary sups and the multiplication, we do not need the surjective condition. In fact, this is stated in [Ros90b, 2.3(3)], with right adjoint $f_*$ given by the pre-image.*

**Remark 3.4.7.** *Since idempotent approximation is pointwise, the projections also preserve idempotent approximation. Then the following diagram commutes:*

$$\begin{tikzcd} \prod_{i\in I} Q_i \arrow[r, "(-)^{-}"] \arrow[d, "p_i"'] & \prod_{i\in I} Idem(Q_i) \arrow[d, "p_i"] \\ Q_i \arrow[r, "(-)^{-}"'] & Idem(Q_i) \end{tikzcd}$$

**Theorem 3.4.8.** *A strong geometric morphism $f : Q \to Q'$ induces an adjunction in the respective category of sheaves. More precisely, the pair of adjoint functors $Q \underset{f_*}{\overset{f^*}{\leftrightarrows}} Q'$ induces a pair $Sh(Q) \underset{\phi_*}{\overset{\phi^*}{\leftrightarrows}} Sh(Q')$ where $\phi^*$ is left adjoint to $\phi_*$.*

*Proof.* *First step: find a $\phi_*$.*

Define $\phi_* : Sh(Q) \to Sh(Q')$ by $\phi_*(F) = F \circ f^*$ and $\phi_*(\eta_u) = \eta_{f^*_u}$ for all $u \in Q'$. We have to show that $F \circ f^*$ is a sheaf in $Q'$. Take $u = \bigvee_{i\in I} u_i$ a cover in $Q'$ and a compatible family $(s_i \in F \circ f^*(u_i))_{i\in I}$ in $F \circ f^*$. This compatible family can be written as $(s_i \in F(f^*(u_i)))_{i\in I}$ and it remains a compatible family in $F$ because

1. $f^*(u) = f^*(\bigvee_{i\in I} u_i) = \bigvee_{i\in I} f^*(u_i)$ is a cover;

2. $f^*$ preserves the multiplication so $s_{i|_{u_i \odot u_j}} = s_{j|_{u_i \odot u_j}}$ implies that

$$s_{i|_{f^*(u_i)\odot f^*(u_j)}} = s_{i|_{f^*(u_i \odot u_j)}} = s_{j|_{f^*(u_i \odot u_j)}} = s_{j|_{f^*(u_i)\odot f^*(u_j)}}.$$

By the sheaf condition on $F$, there is a unique $s \in F(f^*(u))$ such that $s_{|_{f^*(u_i)}} = s_i$, for all $i \in I$ in $F$. So, in $F \circ f^*$, $s_{|_{u_i}} = s_i$, for all $i \in I$, as desired.

*Second step: find $\phi^*$ left adjoint to $\phi_*$.*

Have the following diagram in mind:

$$\begin{tikzcd}
PSh(Q') \arrow[r, shift left, "\phi^-"] \arrow[d, shift left, "b"] & PSh(Q) \arrow[l, shift left, "\phi^+"] \arrow[d, shift left, "a"] \\
Sh(Q') \arrow[u, shift left, "j"] \arrow[r, shift left, "\phi^*"] & Sh(Q) \arrow[u, shift left, "i"] \arrow[l, shift left, "\phi_*"]
\end{tikzcd}$$

where $i$ and $j$ are inclusions with left adjoints (the sheafifications, which we construct in Section 4.2) $a$ and $b$, respectively. The functor $\phi^+$ is the precomposition with $f^*$, as we did to define $\phi_*$, and we take $\phi^- = Lan_{f^*}P$, i.e., $\phi^-$ is the left Kan extension of a presheaf $P : Q' \to Set$ along $f^*$. Since $\phi^+$ is precomposition with $f^*$, $\phi^-$ is left adjoint to $\phi^+$.

Define $\phi^* = a \circ \phi^- \circ j$. For any $F$ sheaf in $Sh(Q)$ and any $G$ sheaf in $Sh(Q')$ we have:

$$\begin{aligned}
Hom_{Sh(Q)}(a \circ \phi^- \circ j(G), F) &\cong Hom_{PSh(Q)}(\phi^- \circ j(G), i(F)) \\
&\cong Hom_{PSh(Q')}(j(G), \phi^+ \circ i(F)) \\
&\cong Hom_{PSh(Q')}(j(G), j \circ \phi_*(F)) \\
&\cong Hom_{Sh(Q')}(G, \phi_*(F))
\end{aligned}$$

So $\phi^*$ is left adjoint to $\phi_*$.

□

We have sufficient conditions for an equivalence:

**Theorem 3.4.9.** *Consider a pair of adjoint functors* $Q \underset{f_*}{\overset{f^*}{\leftrightarrows}} Q'$ *that induces the adjunction* $Sh(Q) \underset{\phi_*}{\overset{\phi^*}{\leftrightarrows}} Sh(Q')$ *where $\phi^*$ is left adjoint to $\phi_*$. Suppose that any $u \in Q$ is of the form $u = f^*(u')$ for some $u' \in Q'$, then $\phi_*$ is a full and faithful functor. Moreover, if $f_*$ preserves sup and multiplication, and $f_* \circ f^* = id_{Q'}$, then $\phi_*$ is dense and therefore it gives an equivalence between $Sh(Q)$ and $Sh(Q')$.*

*Proof.* We begin by showing that $\phi_{*_{F,G}} : Hom_{Sh(Q)}(F, G) \to Hom_{Sh(Q')}(F \circ f^*, G \circ f^*)$ is surjective: Let $\psi : F \circ f^* \to G \circ f^*$ be a natural transformation. For each $u' \in Q'$, $\psi_{u'} : F(f^*(u')) \to G(f^*(u'))$, which is a natural transformation $\varphi_{f^*(u)} : F(f^*(u')) \to G(f^*(u'))$. So $\psi = \phi_*(\varphi)$.

Now, whe check that $\phi_*$ is faithful: Take $\varphi, \psi \in Hom_{Sh(Q)}(F,G)$ such that $\phi_{*_{F,G}}(\varphi)_{u'} = \phi_{*_{F,G}}(\psi)_{u'}$, for all $u' \in Q'$. Then $\varphi_{f^*(u')} = \psi_{f^*(u')}$. Therefore, $\varphi_u = \psi_u$, for all $u \in Q$, as desired.

To prove that $\phi_*$ is dense we need to use $f_*$ and suppose that it preserves sup and multiplication.

Let $F$ be a sheaf on $Q'$. Note that $F \circ f_*$ is a sheaf on $Q$: take $u = \bigvee_{i \in I} u_i$ a covering in $Q$. Then $\bigvee f_*(u_i) = f_*(u)$ is a covering. Let $(s_i \in F \circ f_*(u_i))_{i \in I}$ be a compatible family in $F \circ f_*$. So $s_{i_{|u_i \odot u_j}} = s_{j_{|u_i \odot u_j}}, \forall i, j \in I$. This implies that

$$s_{i_{|f_*(u_i) \odot f_*(u_j)}} = s_{i_{|f_*(u_i \odot u_j)}} = s_{j_{|f_*(u_i \odot u_j)}} = s_{j_{|f_*(u_i) \odot f_*(u_j)}}$$

Therefore, this is a compatible family in $F$. Since $F$ is a sheaf, there is a unique $s \in F((f_*)(u))$ such that $s_{|f_*(u_i)} = s_i$, for all $i \in I$. So, in $F \circ f_*$, $s_{|u_i} = s_i$, for all $i \in I$.

Finally, observe that $\phi_*(F \circ f_*)(u) = (F \circ f_* \circ f^*)(u') \cong F(u')$, for all $u' \in Q'$. So $\phi_*$ is dense. □

**Remark 3.4.10.** *The extra conditions provide that $f^*$ is a homomorphism of quantales (preserve sups and multiplication) and then $f_* \circ f^* = id_{Q'}$ implies that $Q$ and $Q'$ are isomorphic quantales. In other words, in the conditions of the above theorem, it is expected to obtain an equivalence between $Sh(Q)$ and $Sh(Q')$.*

The importance of the above theorem is shown by the following applications – the first and the last are already known:

- If $f : X \to Y$ is a homeomorphism of topological spaces, then $f^*(U) = f(U)$ and $f_*(V) = f^{-1}(V)$ satisfy all the the required conditions. So $Sh(X)$ is equivalent to $Sh(Y)$.
- If $f : R \to S$ is an isomorphism of commutative rings, then $f^*(I) = f(I)$ and $f_*(J) = f^{-1}(J)$ satisfy all the the required conditions. So $Sh(R)$ is equivalent to $Sh(S)$.
- Any isomorphism $f^* : Q' \to Q$ between quantales satisfies the hypothesis. In particular, since there is an isomorphism $\mathbb{L} = ([0,\infty], +, \geq) \to ([0,1,.,\leq]) = \mathbb{I}$ via the map $x \mapsto e^{-x}$, the categories $Sh(\mathbb{L})$ and $S(\mathbb{I})$ are equivalent.
- Another interesting case of quantalic isomorphism is the localic isomorphism $D : Rad(R) \to \mathcal{O}(Spec(R)), I \mapsto D(I) = \{J : J \text{ prime ideal of } R, I \nsubseteq J\} = \bigcup_{a \in I} D(a)$, where $D(a) = \{J : J \text{ prime ideal of } R, a \notin J\}$ [Bor94c, Propostion 2.11.2]. So $Sh(\mathcal{O}(Spec(R)))$ and $Sh(Rad(R))$ are equivalent categories.

We remind the reader of Example 3.3.5 and the discussion about how it looked like a structure sheaf that appears in Algebraic Geometry. Now we can use Theorem 3.4.8 to show that such sheaf actually comes from the usual structure sheaf. Observe that if $f^* : Q \to L$ is a strong geometric morphism, where $Q$ is a semicartesian commutative quantale and $L$ is a locale, then $\phi_* : Sh(L) \to Sh(Q)$ gives a way to construct sheaves on quantales from sheaves on locales. By [BC94, Section 6], we have that the inclusion functor $i : Rad(R) \to \mathcal{I}(R)$ is right adjoint to $\sqrt{-} : \mathcal{I}(R) \to Rad(R)$. By [BC94, Lemma 1.4 (8)], $\sqrt{-}$ preserves the multiplication and thus is a strong geometric morphism. Since the map $Rad(R) \to \mathcal{O}(Spec(R))$ is an isomorphism it is, in particular, a left adjoint. So the composition $\mathcal{I}(R) \xrightarrow{\sqrt{-}} Rad(R) \xrightarrow{D} \mathcal{O}(Spec(R))$ is a left adjoint that preserves multiplication, that is, $D(\sqrt{IJ}) = D(\sqrt{I} \cap \sqrt{J}) = D(\sqrt{I}) \cap D(\sqrt{I})$. By Theorem 3.4.8, the structure sheaf $\mathcal{O}_{SpecR} : \mathcal{O}(Spec(R))^{op} \to CRing$ gives rise to a sheaf $\mathcal{O}_{SpecR} \circ D(\sqrt{-}) : \mathcal{I}(R)^{op} \to CRing$. Note that if we restrict for principal ideals of $R$, then $\mathcal{O}_{SpecR} \circ D(\sqrt{-})$ is precisely the functor $L_R$ defined in Example 3.3.5, since $D(a) = D(\sqrt{aR})$ (see the proof of [Bor94c, Proposition 2.11.2] for details) we have $\mathcal{O}_{SpecR} \circ D(\sqrt{aR}) = \mathcal{O}_{SpecR}(D(a)) \cong R[a^{-1}] \cong L_R(aR)$.

**Remark 3.4.11.** *In [BC94], $\sqrt{-}$ is actually defined for any quantale $Q$ but it coincides with the usual notion of radical of an ideal if $Q = \mathcal{I}(R)$.*

**Remark 3.4.12.** *The previous Theorems provide enough information for the applications considered in this thesis, but it could be interesting to carry out a more systematic analysis of the mapping between (strong) geometric morphisms of quantales and (certain) adjoint pairs of functors between the categories of sheaves over the corresponding quantales.*

In Section 3.1, we introduced a way of obtaining locales from a given semicartesian quantale by applying the functor $(-)^- : Q \to Idem(Q)$. In Example 3.4.6 we mentioned that this functor is a strong geometric morphism if $Q$ is geometric. We argue that our notion of sheaves on quantales is "the best approximation" of the notion of sheaves on locales because given a sheaf $F$ on $Q$, we obtain that the left Kan extension of $F$ a sheaf along $(-)^- : Q \to Idem(Q)$ is a sheaf on $Idem(Q)$.

**Proposition 3.4.13.** *If $F : Q^{op} \to Set$ is a sheaf on a geometric quantale, then $Lan\_F : Idem(Q)^{op} \to Set$ is a sheaf on a locale.*

*Proof.* First note that for any semicartesian quantale, $F : Q^{op} \to Set$ presheaf, and $p \in Idem(Q)$,

$$\begin{aligned}(Lan\_F)(p) &= \varinjlim((-)^-/ct_p \to Q^{op} \xrightarrow{F} Set)\\ &= \varinjlim((a^- \to b^- \text{ such that } p \to a^-) \xrightarrow{F} Set)\\ &= \varinjlim_{a^- \geq p} F(a)\end{aligned}$$

Since $p \in Idem(Q)$, we have $p^- = p$, so $p$ itself is the smallest element such that $p^- \geq p$: if $q \in Q$ is such that $q \leq p$ then $q^- \leq p^- = p$. So $(Lan\_F)(p) \cong F(p)$.

Now, suppose that $Q$ is a geometric quantale and $F$ is a sheaf on the base $B = \{u_i\}_i$ of $Q$. Since $Q$ is geometric, $B^- = \{u_i^-\}_i$ is a base for the locale $Q^- = \{q^- : q \in Q\} = Idem(Q)$ ($Idem(Q) \subseteq Q^-$ since every idempotent is $e = e^-$). We want to show that $Lan\_F$ is a sheaf on $B^-$: take a cover $u^- = \bigvee_{j \in J} u_j^-$ in $B^-$. This is also a cover in $B$. Since $F$ is a sheaf on $B$, the following diagram is an equalizer:

$$F(u^-) \longrightarrow \prod_j F(u_j^-) \rightrightarrows \prod_{j,k} F(u_j^- \odot u_k^-)$$

Recall that $(u_j \odot u_k)^- = u_j^- \odot u_k^- = u_j^- \wedge' u_k^-$, since $u_i^-$ and $u_k^-$ are elements in the locale $Idem(Q)$. Then

$$F(u^-) \longrightarrow \prod_j F(u_j^-) \rightrightarrows \prod_{j,k} F((u_j \odot u_k)^-)$$

is an equalizer and so it is

$$Lan\_F(u^-) \longrightarrow \prod_j Lan\_F(u_j^-) \rightrightarrows \prod_{j,k} Lan\_F(u_j^- \wedge' u_k^-)$$

□

Note that it is expected that if $F : Q^{op} \to Set$ is a sheaf on a quantale then a restriction of it to an appropriated locale returns a sheaf on a locale, but is not expected that the Kan extension of a sheaf is a also a sheaf.

Next, we introduce a more specific example of base change that will be used to apply on expanded Čech cohomology in 2.4.

**Proposition 3.4.14.** *Let $X$ be a topological space that admits partition of unity subordinate to a cover (for example, paracompact Hausdorff spaces), and $C(X)$ the ring of all real-valued continuous functions on $X$. Then*

1. *The functor*

$$\begin{aligned}\tau : \mathcal{I}(C(X)) &\to \mathcal{O}(X)\\ I &\mapsto \bigcup_{f \in I} f^{-1}(\mathbb{R} - \{0\})\end{aligned}$$

*preserves arbitrary supremum, multiplication, and unity.*

2. *The functor*

$$\begin{aligned}\theta : \mathcal{O}(X) &\to \mathcal{I}(C(X))\\ U &\mapsto \langle \{f \ : \ f{\restriction}_{X-U} \equiv 0\}\rangle\end{aligned}$$

*preserves arbitrary supremum, and unity.*

3. *The functor $\tau$ is left adjoint to $\theta$.*

*Proof.* 1. To check that $\tau$ preserves supremum, observe that if $x \in (f_1 + ... + f_n)^{-1}(\mathbb{R} - \{0\})$ then $f_j(x) \neq 0$ for some $j \in \{1, ..., n\}$ and so $x \in \bigcup_{j \in \{1,...,n\}} f_j^{-1}(\mathbb{R} - \{0\})$. Therefore:

$$\tau(\Sigma_{j \in J} I_j) = \bigcup_{f \in \Sigma_{j \in J} I_j} f^{-1}(\mathbb{R} - \{0\}) \subseteq \bigcup_{j \in J} \bigcup_{f_j \in I_j} f_j^{-1}(\mathbb{R} - \{0\}) = \bigcup_{j \in J} \tau(I_j)$$

Since $\tau$ is increasing (if $I \subseteq K$ and $x \in \bigcup_{f \in I} f^{-1}(\mathbb{R} - \{0\}) = \tau(I)$ then $f \in K$ and $x \in \tau(J)$), we have that $\tau(I_j) \subseteq \tau(\Sigma_{j \in J} I_j)$ for all $j \in J$. By definition of supremum, $\bigcup_{j \in J} \tau(I_j) \subseteq \tau(\Sigma_{j \in J} I_j)$.

The fact that $\tau$ is increasing also gives that given ideals $I$ and $J$, $\tau(IJ) \subseteq \tau(I) \cap \tau(J)$. On the other side, if $x \in \tau(I) \cap \tau(J)$ then $f(x) \neq 0$ and $g(x) \neq 0$, for $f \in I$ and $g \in J$. Then $fg(x) \neq 0$. Since $fg \in IJ$, we obtain that $x \in \tau(IJ)$. Therefore, $\tau$ preserves multiplication.

We also have $\tau(C(X)) = \bigcup_{f \in C(X)} f^{-1}(\mathbb{R} - \{0\}) = X$, since for all $x \in X$ there is some continuous function $f$ such that $f(x) \neq 0$, so $\tau$ preserves unity.

2. Observe that

$$\theta(\bigcup_{i \in I} U_i) = \langle \{ f \ : \ f\restriction_{X - \bigcup_{i \in I} U_i} \equiv 0 \} \rangle$$

Let $g \in \Sigma_{i \in I} \langle \{ f \ : \ f\restriction_{X-U_i} \equiv 0 \} \rangle$. So $g = \Sigma_{i \in I} f_i$ where $f_i(x) = 0$ for all $x \notin U_i$. So $g(x) = \Sigma_{i \in I} f_i(x) = 0$ for all $x \notin U$. Actually, since we are dealing with an ideal generated by a set, each $f_i$ is a sum $\Sigma_{j \in \mathbb{N}} (\phi_{ij}.h_{ij})(x)$, where $\phi_{ij} \in C(X)$ and $h_{ij}(x) = 0$ for all $x \notin U_i$, but we write as above since it does not change the verification and it is easier to follow the argument.

For the other inclusion, if $U = \bigcup_{i \in I} U_i$ is a covering and $g \in \langle \{ f \ : \ f\restriction_{X-U} \equiv 0 \} \rangle$, define $g_i = g.\sigma_i$ where $\{\sigma_i\}_{i \in I}$ is partition of unity subordinate to $\{U_i\}_{i \in I}$. Then $\sigma_i(x) = 0$ for all $x \notin U_i$, and thus $g_i(x) = 0$ for all $x \notin U_i$ and

$$g(x) = g(x).1 = g(x).\Sigma_{i \in I} \sigma_i(x) = \Sigma_{i \in I} g_i(x) \in \Sigma_{i \in I} \langle \{ f \ : \ f\restriction_{X-U_i} \equiv 0 \} \rangle.$$

Note that $\theta$ preserves unity because every function in $C(X)$ when restricted to the empty-set is the empty function, and $f\restriction_{\emptyset} = 0$ vacuously holds. Thus, $\theta(X) = C(X)$

3. Suppose $\tau(I) \subseteq U$. Let $g \in I$, then $g^{-1}(\mathbb{R} - \{0\}) \subseteq \bigcup_{g \in I} g^{-1}(\mathbb{R} - \{0\}) \subseteq U$.

If $x \notin U$, then $x \notin g^{-1}(\mathbb{R} - \{0\})$, thus $g(x) = 0$, for all $x \notin U$. So $g \in \theta(U)$.

Suppose $I \subseteq \theta(U)$. Let $x \in \tau(I)$, then $f(x) \neq 0$ for some $f \in I \subseteq \theta(U)$, which implies that $f(x) = 0$, for all $x \notin U$, since $f(x) = \Sigma_{j \in J} \phi_j h_j(x)$ where $\phi_j \in C(X)$ and $h_j(x) = 0$ for all $x \notin U$. To avoid the contradiction $0 = f(x) \neq 0$, we have that $x \in U$.

□

**Proposition 3.4.15.** *In the same conditions and notation of Proposition 3.4.14, we have that* $- \circ \tau$ *and* $- \circ \theta$ *preserve sheaves.*

*Proof.* In the above proposition we proved that $\tau$ is the left adjoint in a geometric morphism given by the pair $\tau$ and $\theta$. By Theorem 3.4.8, we have that if $F$ is a sheaf on $\mathcal{O}(X)$, then $F \circ \tau$ is a sheaf on $\mathcal{I}(C(X))$.

Now, note that $\theta(U_i) \odot \theta(U_j) \subseteq \theta(U_i \cap U_j)$, since $g \in \theta(U_i) \odot \theta(U_j)$ is of the form $(\sum_k \phi_k p_k)(\sum_l \psi_l q_l)$ where $p_k(x) = 0, \forall x \notin U_i$ and $q_l(x) = 0, \forall x \notin U_j$. So $p_k q_l(x) = 0, \forall x \in (X \setminus U_i) \cup (X \setminus U_j) = X \setminus (U_i \cap U_j)$.

Thus, if $F$ is a sheaf on $\mathcal{I}(C(X))$ and $s_i \in F \circ \theta(U_i)$ is such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, for all $i, j \in I$, then $s_i|_{\theta(U_i) \odot \theta(U_j)} = s_j|_{\theta(U_i) \odot \theta(U_j)}$. So $s_i$ is a compatible family in $F$. Since $F$ is a sheaf, it admits unique gluing. Thus $F \circ \theta$ is a sheaf on $\mathcal{O}(X)$. $\square$

Take $P$ a presheaf on $\mathcal{O}(X)$, the left Kan extension of $P$ along $\theta$ is the following colimit (we are using the pointwise Kan extension formula but applied to $\mathcal{O}(X)^{op}$ and $\mathcal{I}(C(X))^{op}$, since presheaves are contravariant functors)

$$(Lan_\theta P)(I) = \varinjlim_{\theta(U) \supseteq I} P(U) = P(\tau(I)) = (P \circ \tau)(I) \tag{3.3}$$

because $\tau$ is left adjoint to $\theta$. Since $\tau$ is a geometric morphism (Proposition 3.4.14), by Theorem 3.4.8 we have that $P \circ \tau$ is a sheaf. So $Lan_\theta$ also preserves sheaves.

Since the left Kan extension along $\theta$ is left adjoint to the precomposition with $\theta$, we have that $(- \circ \tau)$ is left adjoint to $(- \circ \theta)$. Let $P$ presheaf in $PSh(X)$ and $F$ sheaf in $Sh(C(X))$

$$\begin{aligned} Hom_{Sh(C(X))}(a_{C(X)} \circ (- \circ \tau)(P), F) &\cong Hom_{PSh(C(X))}((- \circ \tau)(P), j(F)) \\ &\cong Hom_{PSh(X)}(P, (- \circ \theta) \circ j(F)) \end{aligned}$$

By Proposition 3.4.15, $(- \circ \theta)$ preserves sheaves. Then we have

$$\begin{aligned} Hom_{Sh(C(X))}((- \circ \tau) \circ a_X(P), F) &\cong Hom_{Sh(C(X))}(Lan_\theta \circ a_X(P), F) \\ &\cong Hom_{Sh(X)}(a_X(P), (- \circ \theta)(F)) \\ &\cong Hom_{PSh(X)}(P, i \circ (- \circ \theta)(F)) \end{aligned}$$

Notice that $i \circ (- \circ \theta)(F) = (- \circ \theta) \circ j(F)$. So $(- \circ \tau) \circ a_X$ and $a_{C(X)} \circ (- \circ \tau)$ are both left adjoint functors of the same functor. Therefore, $(- \circ \tau) \circ a_X \cong a_{C(X)} \circ (- \circ \tau)$ and we obtain that the following diagram commutes (up to natural isomorphism)

$$\begin{tikzcd}
Sh(X) \arrow[rr, shift right, "Lan_\theta = - \circ \tau"'] \arrow[d, shift left, "i"] & & Sh(C(X)) \arrow[ll, shift right, "- \circ \theta"'] \arrow[d, shift left, "j"] \\
PSh(X) \arrow[u, shift left, "a_X"] \arrow[rr, shift right, "Lan_\theta = - \circ \tau"'] & & PSh(C(X)) \arrow[ll, shift right, "- \circ \theta"'] \arrow[u, shift left, "a_{C(X)}"] \\
& Set \arrow[ul, "const_X"] \arrow[ur, "const_{C(X)}"'] &
\end{tikzcd}$$

Therefore, we proved:

**Corollary 3.4.16.** *The constant sheaf in $Sh(C(X))$, denoted $\underline{K}^a_{C(X)}$, is naturally isomorphic to the composition $\underline{K}^a_X \circ \tau$ where $\underline{K}^a_X$ is a constant sheaf in $Sh(X)$.*

*Proof.* We are using $\underline{K}^a_X$ to denote the sheafification of a constant presheaf $\underline{K}_X$. Explicitly, the commutativity of the diagram reads as follows:

$$\begin{aligned}\underline{K}^a_{C(X)} &= a_{C(X)} \circ const_{C(X)}(K) \\ &= a_{C(X)} \circ (- \circ \tau) \circ const_X(K) \\ &\cong (- \circ \tau) \circ a_X \circ const_X(K) \\ &= (- \circ \tau)(a_X(K)) \\ &= a_X(K) \circ \tau \\ &= \underline{K}^a_X \circ \tau\end{aligned}$$

□

We will use this Corollary in Section 4.5.

We believe that it is possible to use the above construction in other contexts. For instance, we could replace the ring of real-valued continuous functions with the ring of real-valued smooth functions or replace real-valued with complex-valued (continuous/smooth) functions. We plan to check such changes in a future work.

## 3.5 Sheaves with algebraic structure

The pair of adjoint functors between the open subsets of $X$ and the ideal of $C(X)$ introduced in the previous section was constructed with the goal of relating cohomological groups of $X$ with coefficient in $\underline{K}_X$ and cohomological groups of $C(X)$ with coefficient in $\underline{K}_{C(X)}$. We accomplish such a goal in Section 4.5 by extending the well-known Čech cohomology in a framework that encompasses both topological spaces and commutative rings with unit.

However, Čech cohomology is more like a technique to compute other cohomology theories, such as Singular and De Rham, and less like a cohomology theory. For a Grothendieck topos $\mathcal{E} \cong Sh(C, J)$, it is known that the category $Ab(\mathcal{E})$ of abelian group objects in $\mathcal{E}$ is an abelian category with enough injective objects. This basically means that we can apply homological algebra techniques[7] in $Ab(\mathcal{E})$, in particular, it gives that the functor of global sections has right derived functors (because it is a left exact functor). The right derived functors define the cohomological groups of a sheaf in $Ab(\mathcal{E})$, and this is the beginning of sheaf cohomology.

In our case, we have already mentioned and will prove in Theorem 4.3.8, that $Sh(Q)$ is not a Grothendieck topos. Nevertheless, is $Ab(Sh(Q))$ an abelian category? Does it have enough injectives? We believe the answer is no, $Ab(Sh(Q))$ is not an abelian category. In this thesis, we do not explore if $Ab(Sh(Q))$ has enough injectives. Luckily, to define Čech cohomology we will only need the Čech cochains to be abelian (groups), and they will if the coefficients are sheaves of abelian groups. This Section is devoted to talking about sheaves with algebraic structures and so have a well-behaved theory to define Čech cochains for

[7] Homological algebra can be done for categories that are not abelian, like triangulated or semi-abelian categories.

quantales. The idea is once again to replicate the case of sheaves on topological spaces [MM92, Chapter II.7].

First, we define abelian group objects in a category.

**Definition 3.5.1.** *Let $C$ be a category with binary products and terminal object* $1$, *we can define the notion of* **group object in** $C$ *as an object* $G$ *in* $C$ *equipped with morphisms*

$$e : 1 \longrightarrow G \qquad i : G \longrightarrow G \qquad m : G \times G \longrightarrow G$$

*in* $C$*, such the following diagrams commute*

$$\begin{tikzcd} G\times G\times G \arrow[r, "id_G\times m"] \arrow[d, "m\times id_G"'] & G\times G \arrow[d, "m"] \\ G\times G \arrow[r, "m"'] & G \end{tikzcd} \qquad \begin{tikzcd} 1\times G \arrow[r, "e\times id_G"] \arrow[dr, "\cong"'] & G\times G \arrow[d, "m"] & G\times 1 \arrow[l, "id_G\times e"'] \arrow[dl, "\cong"] \\ & G & \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "\triangle"] \arrow[d, "!"] & G\times G \arrow[r, "i\times id_G"] & G\times G \arrow[d, "m"] \\ 1 \arrow[rr, "e"'] & & G \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "\triangle"] \arrow[d, "!"] & G\times G \arrow[r, "id_G\times i"] & G\times G \arrow[d, "m"] \\ 1 \arrow[rr, "e"'] & & G \end{tikzcd}$$

*where* $\triangle = (id_G, id_G) : G \to G \times G$ *is the diagonal morphism.*

The above diagrams express the group axioms. So, if we want an **abelian group object** we must add the following commutative diagram

$$\begin{tikzcd} G\times G \arrow[r, "\tau"] \arrow[dr, "m"'] & G\times G \arrow[d, "m"] \\ & G \end{tikzcd}$$

where $\tau = (\pi_2, \pi_1) : G \times G \to G \times G$ is the twist morphism.

In short, an *abelian group object* is a quadruple $(G, e, i, m)$, where the diagrams above commute. They are the objects in the category $Ab(C)$ of abelian groups object in $C$, and the morphisms are arrows in $C$ that commute with the corresponding morphisms $e, i$, and $m$.

Explicitly, a morphism $h : \mathcal{G} = (G, e, i, m) \to (G', e', i', m') = \mathcal{G}'$ in $Ab(C)$ corresponds to a morphism $h : G \to G'$ in $C$ such that the following diagrams commute:

$$\begin{tikzcd} G\times G \arrow[r, "h\times h"] \arrow[d, "m"'] & G'\times G' \arrow[d, "m'"] \\ G \arrow[r, "h"'] & G' \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "h"] & G' \\ 1 \arrow[u, "e"] \arrow[ur, "e'"'] & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "h"] \arrow[d, "i"'] & G' \arrow[d, "i'"] \\ G \arrow[r, "h"'] & G' \end{tikzcd}$$

Observe that the forgetful functor $U : Ab(\mathcal{C}) \to \mathcal{C}$ creates limits since abelian group objects use only products and commutative diagrams in the category $\mathcal{C}$.

An abelian sheaf on $Q$ is an abelian group object in $Sh(Q)$. Equivalently,

**Definition 3.5.2.** *An* ***abelian sheaf*** *(or a sheaf of abelian groups) is a functor $F : Q^{op} \to Ab$ such that the composite $U \circ F$ with the forgetful functor $U : Ab \to Set$ is a sheaf of sets.*

In this case, we denote the correspondent category by $Sh_{Ab}(Q)$.

We can replace $Ab$ with any category $D$ that admits a forgetful functor $D \to Set$. Note that Example 3.3.5 is actually an example of a sheaf of rings, that is, $D = CRing$. Similarly, one can talk about ring objects (and other algebraic structure objects) in a category $\mathcal{C}$ depending on categorical properties of $\mathcal{C}$[8]. From now and until the end of this section, we only work with sheaves of abelian groups.

It is known that $Sh_{Ab}(L)$ is equivalent $Ab(Sh(L))$. The argument to show that $Sh_{Ab}(Q)$ is equivalent to $Sh(Q)$ is the same, but we sketch it here. Consider an abelian presheaf $Q^{op} \to Ab$. Then, $F(u)$ is an abelian group for every $u \in Q$. As so, we have operations $m_u : (F \times F)(u) \cong F(u) \times F(u) \to F(u)$, $i_u : F(u) \to F(u)$, and $e_u : 1 \to F(u)$ that are components of natural transformations $m, i, u$ that satisfy the diagrammatic rules of an abelian group object. Thus, $F$ is an abelian group object of $PSh(Q)$. On the other side, if $G \in Ab(PSh(Q)^{op})$, then $G : Q^{op} \to Set$ and we have natural transformations $m, i$, and $e$ as in the definition of an abelian group object. For every $u \in Q$ we consider $m_U, i_U$, and $e_U$ such that the diagrammatic rules still hold, then, $G(U)$ is an abelian group. Therefore, $G$ is a functor from $Q^{op}$ to $Ab$. Those correspondences describe an equivalence of categories $Ab(Set^{\mathcal{O}(X)^{op}}) \simeq Ab^{\mathcal{O}(X)^{op}}$.

Since we have inclusions $Sh(Q) \to PSh(Q)$ and $Sh_{Ab}(Q) \to PSh_{Ab}(Q)$, the equivalence $Ab(Set^{\mathcal{O}(X)^{op}}) \simeq Ab^{\mathcal{O}(X)^{op}}$ induces an equivalence $Ab(Sh(X)) \simeq Sh_{Ab}(X)$, because the subcategories of sheaves, over $Set$ and over $Ab$, are closed under all small limits.

Since $Ab(Sh(L))$ is an abelian category, this equivalence says that $Sh_{Ab}(L)$ also is. The following is a sequence of categorical results that make $Sh_{Ab}(Q)$ close to an abelian category.

**Proposition 3.5.3.** *$Sh_{Ab}(Q)$ has a zero object.*

*Proof.* Since $Sh_{Ab}(Q) \to PSh_{Ab}(Q)$ is closed under limit, a terminal object in $Sh_{Ab}(Q)$ must be the terminal abelian presheaf, that is, the constant functor with values in the zero object in $Ab$, which makes $Hom_Q(-, 1) \cong 0$ a zero object in $Sh_{Ab}(Q)$. □

**Proposition 3.5.4.** *$Sh_{Ab}(Q)$ has biproducts.*

*Proof.* Define $(F \oplus G)(u) = F(u) \oplus G(u)$, for $F, G \in Sh_{Ab}(Q)$ Let $u = \bigvee_{i \in I} u_i$ and consider $(s_i \in (F \oplus G)(u_i))_{i \in I}$ a compatible family. We have $s_i = f_i + g_i$, where $f_i \in F(u_i)$ and $g_i \in G(u_i)$

[8] There are at least two approaches to this, via models of a Lawvere theory [Law63] or via the microcosm principle [BD98].

are unique for each $i \in I$. By the unicity of the decomposition of the biproduct $F(u_i) \oplus G(u_i)$ in $Ab$, we obtain that $(f_i \in F(u_i))_{i\in I}$ and $(g_i \in G(u_i))_{i\in I}$ are compatible families.

Since $F$ and $G$ are sheaves, there are unique $f \in F(u)$ and $g \in G(u)$ such that $f_{\restriction_{u_i}} = f_i$ and $g_{\restriction_{u_i}} = g_i$. Define $s = f + g$. So,

$$s_{\restriction_{u_i}} = (f+g)_{\restriction_{u_i}} = f_{\restriction_{u_i}} + g_{\restriction_{u_i}} = f_i + g_i = s_i, \ \forall i \in I$$

Therefore, the gluing exists. If $s' = f' + g'$ is another gluing with $f' \in F(u)$ and $g' \in G(u)$. We have that $s_{\restriction_{u_i}} = s'_{\restriction_{u_i}}$ implies $f_{\restriction_{u_i}} + g_{\restriction_{u_i}} = f'_{\restriction_{u_i}} + g'_{\restriction_{u_i}}$. So $f_{\restriction_{u_i}} = f'_{\restriction_{u_i}}$ and $g_{\restriction_{u_i}} = g'_{\restriction_{u_i}}$. Since $F$ and $G$ are sheaves, $f = f'$ and $g = g'$. Therefore, $s = s'$. □

**Proposition 3.5.5.** *$Sh_{Ab}(Q)$ has kernels.*

*Proof.* Let $\varphi : F \to G$ be a morphism of abelian sheaves and $u = \bigvee_{i\in I} u_i$. For all $v \in Q$, we define $(Ker\varphi)(v) = Ker(\varphi_v)$. Let $s, s' \in (Ker\varphi)(u)$ such that $s_{\restriction_{u_i}} = s'_{\restriction u_i}$. As $Ker\varphi$ is a presheaf, the following diagram commutes:

$$\begin{array}{ccc} Ker(\varphi_{u_i}) & \xrightarrow{ker(\varphi_{u_i})} & F(u_i) \\ \uparrow & & \uparrow \\ Ker(\varphi_u) & \xrightarrow{ker(\varphi_u)} & F(u) \end{array}$$

So $((Ker\varphi_u)(s))_{\restriction u_i} = ((Ker\varphi_u)(s'))_{\restriction u_i}$ Since $F$ is a sheaf, we obtain that $(Ker\varphi_u)(s) = (Ker\varphi_u(s')$. Since every kernel is mono, we have $s = s'$.

Now let $s_i \in (Ker\varphi)(u_i)$ such that $s_{i_{\restriction_{u_i \odot u_j}}} = s_{j_{\restriction_{u_i \odot u_j}}}$. Then $Ker(\varphi_{u_i})(s_i) \in F(u_i)$. Since $F$ is a sheaf, there exists $s \in F(u)$ such that $s_{\restriction_{u_i}} = Ker(\varphi_{u_i})(s_i)$.

We will see that $s \in (Ker\varphi)(u) = Ker(\varphi_u)$, that is, $\varphi_u(s) = 0$. By the commutativity of

$$\begin{array}{ccc} F(u_i) & \xrightarrow{\varphi_{u_i}} & G(u_i) \\ \uparrow & & \uparrow \\ F(u) & \xrightarrow[\varphi_u]{} & G(u) \end{array}$$

we have that $(\varphi_u(s))_{\restriction_{u_i}} = \varphi_{u_i}(s_{\restriction_{u_i}})$. But

$$\varphi_{u_i}(s_{\restriction_{u_i}}) = \varphi_{u_i}(Ker\varphi_{\restriction_{u_i}}(s_i)) = (\varphi_{\restriction_{u_i}} \circ Ker\varphi_{\restriction_{u_i}})(s_i) = 0, \ \forall i \in I$$

Therefore, $(\varphi_u(s))_{\restriction u_i} = 0, \forall i \in I$. Since $G$ is a sheaf, $\varphi_u(s) = 0$, as desired. □

After this, we would like to prove that $Sh_{Ab}(Q)$ has cokernels. We can make the cokernel presheaf into a sheaf by applying the sheafification functor that we will introduce in the next chapter. However, this process may produce a sheaf that is not a cokernel anymore. In the standard case, the sheafification has a universal property that guarantees that the sheafification of the cokernel still is a cokernel.In our case, we were not able to prove such a universal property yet. It also remains to check if $Sh_{Ab}(Q)$ has an incarnation of the isomorphism theorem, which is necessary to make the category abelian. To prove the incarnation of the isomorphism theorem in $Sh_{Ab}(L)$ it is used that the sheafification is left exact. It is also known that Grothendieck toposes are the left exact reflective subcategories of a presheaf category. However, $Sh(Q)$ is a reflective subcategory of $PSh(Q)$ that is not a Grothendieck topos in general (Theorem 4.3.8), and thus our sheafification cannot be left exact. Of course, maybe there is another argument to show that $Sh_{Ab}(L)$ is abelian, but all the proofs that we know use such property of the sheafification at some point so we do not expect $Sh(Q)$ to be an abelian category. Nevertheless, the existence of enough injectives in $Sh_{Ab}(Q)$ may be true: by [Gro57, Theorem 1.10.1], when a category is AB5 (abelian categories possessing arbitrary coproducts and in which filtered colimits of exact sequences are exact) and has a generator – $Sh(Q)$ has generator by Proposition 3.2.21 – then if has enough injectives.

There is another class of categories that is famous in standard sheaf cohomology: triangulated categories. Therefore in the perspective of founding a cohomology theory for our sheaves we plan to check if $Sh_{Ab}(Q)$ is triangulated. Fortunately, it is possible to apply cohomological methods in our $Sh_{Ab}(Q)$ if we use Čech cohomology. While sheaf cohomology is about the right derived functor of the global section functor, Čech cohomology of a sheaf constructs a cochain complex and the cohomology is the quotient of the appropriate kernel by the appropriate image. Under mild conditions, cohomology of a sheaf defined by right derived functors and Čech cohomology coincide. For the comfort of the reader, we stated it here 2.4.5. This means that even we do not have a sheaf cohomology in its most general framework, in Section 4.5 we expanded Čech cohomology of sheaves on locales to encompass sheaves on quantales.

# Chapter 4

# Sheaves on Monoidal Categories

Alexander Grothendieck created what we currently call Grothendieck pretopologies by investigating the behavior of coverings for a fixed topological space. The stability axiom in the definition of Grothendieck pretopologies says that if we have a family $\{U_i \to U\}_{i\in I}$ of objects in a fixed category and $\{U_i \to U\}_{i\in I}$ is covering of an object $U$ – whatever this means – and a morphism $V \to U$, then the family of pullbacks $\{V \times_U U_i \to V\}_{i\in I}$ covers $V$. Note that in the category $\mathcal{O}(X)$ of open subsets of a topological space $X$, once we know that intersections are the pullbacks in $\mathcal{O}(X)$, such property arises naturally from the distributivity $\bigcup\limits_{i\in I}(V \cap U_i) = V \cap (\bigcup\limits_{i\in I} U_i) = V \cap U = V$. In this thesis, when we defined sheaves on quantales we persisted in maintaining that a cover should be given by the joins/union but, for quantales, joins distribute over the quantalic product, not the infimum/intersection. This already suggests that we need a general notion of covering that considers the extra monoidal structure in its axioms to be able to encompass $Sh(Q)$ into some more general notion of Grothendieck topos. Indeed, we will see that in $Q$, $U = \bigvee_{i\in I} U_i$ is not a covering in the sense of a Grothendieck pretolopogy. However, this is not enough to say that $Sh(Q)$ is not, in general, a Grothendieck topos (or even an elementary topos) because a Grothendieck topos is any category that is equivalent to a category of sheaves defined for a category equipped with a *Grothendieck topology*. So, there could exist some category $C$ and some Grothendieck topology $J$ such that $Sh(Q)$ is equivalent to $Sh(C, J)$.

We start this Chapter "ignoring" if $Sh(Q)$ is a Grothendieck topos or not: since a commutative unital and semicartesian quantale $Q$ is a thin symmetric monoidal semicartesian category, we construct a notion of covering, *Grothendieck prelopologies*, that encompass both the covering in $Q$ and the covering in the Grothendieck pretopologies if the monoidal tensor is the cartesian product. Then we can consider a quite natural notion of sheaves for (symmetric) monoidal semicartesian categories. We do not developed a correspondent notion of Grothendieck lopologies since prelopologies are enough for our applications in Cohomology. We do however develop a mechanism to obtain a sheaf, in our sense, given a presheaf – the *sheafification*. The general framework provided techniques to finally prove that $Sh(Q)$ is not, in general, a topos (Theorem 4.3.8). Moreover, we proved that the lattice of subterminal objects in $Sh(Q)$ is isomorphic to $Q$ (Theorem 4.3.6). Since $Sh(Q)$ is not a Grothendiek topos, the development of the theory of sheaves we construct in this Chapter is justified: we are going toward the development of an elementary topos theory that has a different internal logic, probably, an affine linear logic[1]. A good understanding of subobjects classifiers in $Sh(Q)$ could guide us to a better understanding of the internal logic of $Sh(Q)$, so we approach this issue in Section 4.3.2.

We also discuss how to further generalize our theory for semigroupal categories instead of monoidal categories and in the end we focus on the application of our theory in cohomology. In this final part, we show that the Čech cohomology of a topological space $X$ with coefficients in a constant sheaf in $Sh(X)$ is isomorphic to the (adapted) Čech cohomology of the ring of real-valued continuous functions $C(X)$ with coefficients in a constant sheaf in $Sh(C(X))$ 4.5.7. We prove an analogous result where the coefficient in the "topological side" may be any sheaf in $Sh(X)$ and the coefficient in the "algebraic side" will be induced by the sheaf in $Sh(X)$ 4.5.8.

## 4.1 Grothendieck prelopologies

In this section we generalize the notion of a pullback, and carefully modify the definition of a Grothendieck pretopology. This will lead to a presentation of sheaves on semicartesian categories as a generalization of sheaves on categories with pullbacks.

We want to define *Grothendieck prelopologies*[2] for (semicartesian symmetric monoidal) categories with a limit analogous to a pullback. Recall that if the category $\mathcal{C}$ has products then the pullback $A \times_C B$ of arrows $f : A \to C$ and $g : B \to C$ is the equalizer of $A \times B \underset{g\circ\pi_2}{\overset{f\circ\pi_1}{\rightrightarrows}} C$, where $\pi_1$ and $\pi_2$ are the projections of the product.

Instead of a product, we have the tensor of a monoidal category. Since we are considering semicartesian monoidal categories, by Proposition 2.1.8, we have projections

$$\begin{tikzcd} A \otimes B \arrow[r, "id_A \otimes !_B"] \arrow[rr, bend left, "\pi_1"] & A \otimes 1 \arrow[r, "\rho_A"] & A \end{tikzcd}$$

$$\begin{tikzcd} A \otimes B \arrow[r, "!_A \otimes id_B"] \arrow[rr, bend right, "\pi_2"'] & 1 \otimes B \arrow[r, "\lambda_B"] & B \end{tikzcd}$$

Where the diagram

$$\begin{tikzcd} A \otimes B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

[1] This may have interesting consequences for Quantum Mechanics and Non-commutative Topology. We will address this discussion in the last Chapter

[2] We expect that sheaves under this cover will have some kind of linear logic as its internal language, which motivated us to choose this terminology.

is not always commutative, but we do have parallel arrows $A \otimes B \underset{g\circ\pi_2}{\overset{f\circ\pi_1}{\rightrightarrows}} C$ so the monoidal (non-cartesian) analogous version of a pullback is to take the equalizer of this parallel arrows.

**Definition 4.1.1.** *Let $(C, \otimes, 1)$ be a semicartesian (symmetric) monoidal category with equalizers. The* ***pseudo-pullback*** *of morphisms $f : A \to C$ and $g : B \to C$ is the equalizer of the parallel arrows $A \otimes B \underset{g\circ\pi_2}{\overset{f\circ\pi_1}{\rightrightarrows}} C$ where $\pi_1 = \rho_A \circ (id_A \otimes !_B)$ and $\pi_2 = \lambda_B \circ (id_A \otimes id_B)$*

Diagrammatically, the pseudo-pullback is the object $A\ {}_f\!\otimes_g B$ with the equalizer arrow in the following equalizer diagram

$$\begin{tikzcd}
A\ {}_f\!\otimes_g B \arrow[dr, "e'"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& A \otimes B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

The arrows $p_1$ and $p_2$ are just the compositions $\pi_1 \circ e'$ and $\pi_2 \circ e'$, respectively.

**Example 4.1.2.** *If $(C, \otimes, 1) = (Q, \odot, 1)$, then the pseudo-pullback is the multiplication $\odot$.*

**Remark 4.1.3.** *The multiplication $\odot$ of a quantale $Q$ is not necessarily a pullback in the poset category $Q$. This justifies the need for a generalization of pullbacks in our research.*

**Example 4.1.4.** *If $(C, \otimes, 1) = (C, \times, 1)$, i.e., the monoidal tensor in $(C, \otimes, 1)$ is the cartesian product with unity the terminal object 1, then the pseudo-pullback is the pullback. This shows that we are indeed generalizing pullbacks.*

**Example 4.1.5.** *Let $(C, \otimes, I)$ be a monoidal category. The slice category $C/I$ is a semicartesian monoidal category with tensor defined by*

$$(A \xrightarrow{\phi} I) \otimes_{C/I} (B \xrightarrow{\psi} I) = (A \otimes B \xrightarrow{\phi\otimes\psi} I \otimes I \cong I).$$

*If $E \xrightarrow{e} A \underset{q}{\overset{p}{\rightrightarrows}} B$ is an equalizer diagram in $(C, \otimes, I)$ then*

$$\begin{tikzcd}
E \arrow[r, "e"] \arrow[dr, "\phi\circ e"'] & A \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] \arrow[d, "\phi"] & B \arrow[dl, "\psi"] \\
& I &
\end{tikzcd}$$

*is an equalizer diagram in $(C/I, \otimes_{C/I}, I)$. In particular, suppose that $(C, \otimes, I)$ is the category of $R$-modules, where $R$ is a commutative ring with unity, the monoidal structure is given by the tensor product of $R$-modules and the monoidal unity is the ring $R$. Then the objects in $(C/I, \otimes_{C/I}, I)$ are module homomorphisms $M \to R$, where $M$ is an $R$-module. Now, $R$ is not* *only the monoidal unity but also the terminal object in* $(C/I, \otimes_{C/I}, I)$. *So we have "projections"* $\pi_1 : M \otimes_R N \to M \otimes_R R \to M$ *and* $\pi_2 : M \otimes_R N \to R \otimes_R N \to N$. *Then the pseudo-pullback of* $f : M \to O$ *and* $g : N \to O$ *is obtained from the equalizer of* $M \otimes_R N \underset{f \circ \pi_1}{\overset{g \circ \pi_2}{\rightrightarrows}} O$ *in the category of R-modules and module homomorphisms.*

**Example 4.1.6.** *Consider two categories* $(C, \otimes, 1_C)$ *and* $(\mathcal{D}, \star, 1_{\mathcal{D}})$ *both with pseudo-pullbacks. Then the product category* $C \times \mathcal{D}$ *has pseudo-pullbacks: given* $C, C'$ *objects in* $C$ *and* $D, D'$ *objects in* $\mathcal{D}$. *Define*

$$(C, D) \otimes_{C \times \mathcal{D}} (C', D') = (C \otimes C', D \star D')$$

*If* $C\ {}_f\!\otimes_g C'$ *is the pseudo-pullback of* $f : C \to E$ *and* $g : C' \to E$, *and* $D\ {}_\phi\!\star_\psi D'$ *is the pseudo-pullback of* $\phi : D \to F$ *and* $\psi : D' \to F$, *then the pseudo-pullback of* $(f, \phi) : (C, D) \to (E, F)$ *and* $(g, \psi) : (C', D') \to (E, F)$ *is* $(C\ {}_f\!\otimes_g C', D\ {}_\phi\!\star_\psi D')$. *In particular, we can take* $(C, \otimes, 1_C) = (C, \times, 1)$ *and* $(\mathcal{D}, \star, 1_{\mathcal{D}}) = (Q, \odot, 1)$.

Next, following the naive way of thinking, we introduce the first candidate to a Grothendieck prelopology (as we will see, it will not work).

Let $(C, \otimes, 1)$ be a semicartesian monoidal category with pseudo-pullbacks. A Grothendieck prelopology on $C$ associates to each object $U$ of $C$ a set $L(U)$ of families of morphisms $\{U_i \to U \mid i \in I\}$ such that:

1. The singleton family $\{U' \xrightarrow{f} U\}$, formed by an isomorphism $f : U' \xrightarrow{\cong} U$, is in $L(U)$;

2. If $\{U_i \xrightarrow{f_i} U\}_{i \in I}$ is in $L(U)$ and $\{V_{ij} \xrightarrow{g_{ij}} U_i\}_{j \in J_i}$ is in $L(U_i)$ for all $i \in I$, then $\{V_{ij} \xrightarrow{f_i \circ g_{ij}} U\}_{i \in I, j \in J_i}$ is in $L(U)$;

3. If $\{U_i \xrightarrow{f_i} U\}_{i \in I}$ is in $L(U)$, and $V \to U$ is any morphism in $C$, then the family of pseudo-pullbacks $\{g_i : U_i \otimes_U V \to V\}_{i \in I}$ is in $L(V)$.

In the above, we just replaced the pullback with the pseudo-pullback. This definition is not good for us because in the category $(Q, \odot, 1)$ we want $U = \bigvee_{i \in I} U_i$ to be a cover of $U$, in the sense of a Grothendieck prelopology. However, if $V \leq U$, we have

$$V \neq U \odot V = \bigvee_{i \in I} U_i \odot V$$

So if we define

$$\{f_i : U_i \to U\} \in L(U) \iff U = \bigvee_{i \in I} U_i$$

we obtain that $\{U_i \otimes V \to V\} \notin L(V)$. Therefore, $U = \bigvee_{i \in I} U_i$ is not a cover in the sense of the proposed prelopology because of the third axiom (it is easy to check that the other axioms are satisfied). But notice that for all $W \in Q$

$$U \odot W = \bigvee_{i \in I} U_i \odot W$$

This suggests that we should replace the third axiom by something of the form

3' If $\{f_i : U_i \to U\}_{i \in I} \in L(U)$, then $\{f_i \otimes id_V : U_i \otimes V \to U \otimes V\}_{i \in I}$ is in $L(U \otimes V)$

This is well-suited for the quantalic case but is not enough to obtain the stability axiom in the definition of a Grothendieck pretopology 2.3.1 when the monoidal product is the categorical product. So we add a fourth axiom to solve this problem.

**Definition 4.1.7.** *Let $(C, \otimes, 1)$ be a semicartesian symmetric monoidal category with pseudo-pullbacks. A* ***Grothendieck prelopology*** *on $C$ associates to each object $U$ of $C$ a set $L(U)$ of families of morphisms $\{U_i \to U\}_{i\in I}$ such that:*

1. *The singleton family $\{U' \xrightarrow{f} U\}$, formed by an isomorphism $f : U' \xrightarrow{\cong} U$, is in $L(U)$;*
2. *If $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ is in $L(U)$ and $\{V_{ij} \xrightarrow{g_{ij}} U_i\}_{j\in J_i}$ is in $L(U_i)$ for all $i \in I$, then $\{V_{ij} \xrightarrow{f_i \circ g_{ij}} U\}_{i\in I, j\in J_i}$ is in $L(U)$;*
3. *If $\{f_i : U_i \to U\}_{i\in I} \in L(U)$, then $\{f_i \otimes id_V : U_i \otimes V \to U \otimes V\}_{i\in I}$ is in $L(U \otimes V)$ and $\{id_V \otimes f_i : V \otimes U_i \to V \otimes U\}_{i\in I}$ is in $L(V \otimes U)$, for any $V$ object in $C$;*
4. *If $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ is in $L(U)$ and $g : V \to U$ is any morphism in $C$, then $\{\phi_i : U_i \,{}_{f_i}\!\otimes_g V \to Eq(\pi_1, g \circ \pi_2)\}_{i\in I}$ is in $L(Eq(\pi_1, g \circ \pi_2))$ and $\{\phi_i : V \,{}_g\!\otimes_{f_i} U \to Eq(\pi_2, g \circ \pi_1)\}_{i\in I}$ is in $L(Eq(\pi_2, g \circ \pi_1))$.*

**Remark 4.1.8.** *For each $i \in I$, the arrow $\phi_i : U_i \,{}_{f_i}\!\otimes_g V \to Eq(\pi_1, g \circ \pi_2)$ is unique because of the universal property of the equalizer:*

$$\begin{tikzcd}
U_i \,{}_{f_i}\!\otimes_g V \arrow[r, tail, "e_i"] \arrow[d, dashed, "\phi_i"'] & U_i \otimes V \arrow[r, "\pi_1^i"] \arrow[d, "f_i \otimes id_V"'] & U_i \arrow[d, "f_i"] \\
Eq(\pi_1, g \circ \pi_2) \arrow[r, tail, "e"'] & U \otimes V \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & U \\
& & V \arrow[u, "g"']
\end{tikzcd}$$

**Remark 4.1.9.** *If you are interested in noncommutative geometry/topology, it may be interesting to study semicartesian non-symmetric categories where just one side of the third and the fourth axioms hold but not the other.*

**Example 4.1.10.** *If $(C, \otimes, 1)$ is a semicartesian quantale $(Q, \odot, 1)$, then*

$$\{f_i : U_i \to U\} \in L(U) \iff U = \bigvee_{i\in I} U_i$$

*determines a Grothendieck prelopology: Axioms 1 and 2 are immediate, axiom 3 was proved in the previous discussion. The last axiom follows from observing that $U_i \,{}_{f_i}\!\otimes_g V = U_i \otimes V$ (see 4.1.2 ) and then realizing that $\phi_i = f_i \otimes id_V$ for all $i \in I$. In other words, in the quantalic case the third axiom, implies the fourth one.*

*Of course, if $(Q, \odot, 1)$ is a locale, then the above prelopology coincides with the usual Grothendieck pretopology in $Q$ – because $v \leq u$ iff $v \wedge u = v = u \wedge v$.*

Now, note that if the tensor product is given by the cartesian product then we have a diagram of the following form

$$\begin{tikzcd}
U_i \times_U V \arrow[r, tail, "e_i"] \arrow[d, dashed, "\phi_i"'] & U_i \times V \arrow[rr, "\pi_1^i"] \arrow[d, "f_i \times id_V"'] & & U_i \arrow[d, "f_i"] \\
U \times_U V \arrow[r, tail, "e"'] & U \times V \arrow[rr, "\pi_1"] \arrow[dr, "\pi_2"'] & & U \\
& & V \arrow[ur, "g"'] &
\end{tikzcd}$$

In Proposition 4.1.12, we prove that in such case the fourth axiom is the stability (under pullbacks) axiom of the definition of a Grothendieck pretopology. Since, the stability under pullback implies that $\{U_i \times V \xrightarrow{f \times id_V} U \times V\}$ cover $U \times V$ (proved in Proposition 4.1.11 bellow) we can say that, in the cartesian case, the fourth axiom implies the third one.

We may say that the following proposition shows that we are generalizing Grothendieck pretopologies. In fact, we say that Grothendieck pretopologies are an example of Grothendieck prelopologies, but this fact is overly relevant to be stated as a simple example.

**Proposition 4.1.11.** *If $C$ is a cartesian category with equalizers and $K$ is a Grothendieck pretopology in $C$, then $K$ is a Grothendieck prelopology.*

*Proof.* In this case, the pseudo-pullback is the pullback. The first two axioms are automatically satisfied. It remains to prove the last two. Let $\{f_i : U_i \to U\} \in K(U)$

3. It holds because

$$\begin{tikzcd}
U_i \times V \arrow[r, "f_i \times id_V"] \arrow[d, "\pi_1^i"'] & U \times V \arrow[d, "\pi_1"] \\
U_i \arrow[r, "f_i"'] & U
\end{tikzcd}$$

is a pullback diagram.

4. Observe the following diagram, where we use the same notation of 4.1.8.

$$\begin{tikzcd}
U_i \times_U V \arrow[r, "\pi_i^1 \circ e_i"] \arrow[d, "\phi_i"'] & U_i \arrow[d, "f_i"] \\
U \times_U V \arrow[r, "\pi^1 \circ e"] \arrow[d, "\pi_2 \circ e"'] & U \arrow[d, "id_U"] \\
V \arrow[r, "g"'] & U
\end{tikzcd}$$

The outer rectangle is a pullback, and the square on the bottom is a pullback. So, by the pullback lemma, the square on the top is a pullback. Since $K$ is a Grothendieck topology, $\phi_i \in L(U \times_U V)$

□

Conversely,

**Proposition 4.1.12.** *If $L$ is a Grothendieck prelopology on a cartesian category with equalizers $\mathcal{C}$, then $L$ is a Grothendieck pretopology*

*Proof.* The only axiom that we have to prove is the stability (under pullback) axiom. Consider $\{f_i : U_i \to U\} \in L(U)$ and $g : V \to U$ a morphism in $\mathcal{C}$. Using the same notion as in Definition 4.1, we know that $\{\phi_i : U_i \,{}_{f_i}\!\otimes_g V \to Eq(\pi_1, g \circ \pi_2)\}_{i\in I} \in L(Eq(\pi_1, g \circ \pi_2))$. By Example 4.1.4, $U_i \,{}_{f_i}\!\otimes_g V$ is the pullback $U_i \times_U V$ and since $\otimes = \times$ it is clear that $Eq(\pi_1, g \circ \pi_2) = U \times_U V$. Now, note that the following is a pullback diagram

$$\begin{tikzcd}
U \times_U V \arrow[r, "\pi_2 \circ e"] \arrow[d, "\pi_1 \circ e"'] & V \arrow[d, "g"] \\
U \arrow[r, "id_U"'] & U
\end{tikzcd}$$

Then $U \times_U V \to V$ is an isomorphism, which implies that $\{\pi_2 \circ e : U \times_U V \to V\} \in L(V)$, by the first axiom of a Grothendieck prelopology. Thus, the composition $\{U_i \times_U V \xrightarrow{\phi_i} U \times_U V \xrightarrow{e} U \times V \xrightarrow{\pi_2} V\}_{i\in I} \in L(V)$, proving that $L$ satisfies the stability axiom. □

**Example 4.1.13.** ***Prelopology for product category:*** *Consider two categories $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ and $(\mathcal{D}, \star, 1_{\mathcal{D}})$ both with pseudo-pullbacks and equipped, respectively, with Grothendieck prelopologies $L_{\mathcal{C}}$ and $L_{\mathcal{D}}$. Define a Grothendieck prelopology in $\mathcal{C} \times \mathcal{D}$ by $\{(\gamma_i, \delta_i) : (C_i, D_i) \to (C, D)\} \in L_{\mathcal{C}\times\mathcal{D}}(C, D)$ iff $\{\gamma_i : C_i \to C\} \in L_{\mathcal{C}}(C)$ and $\{\delta_i : D_i \to D\} \in L_{\mathcal{D}}(D)$.*

*The verification is straightforward and we are going to show the calculations only for the fourth axiom. Since $\{\gamma_i : C_i \to C\} \in L_{\mathcal{C}}(C)$, for any $\epsilon : E \to C$ in $\mathcal{C}$ we have*

$$\begin{tikzcd}
C_i \,{}_{\gamma_i}\!\otimes_\epsilon E \arrow[r, tail, "e_i"] \arrow[d, dashed, "\phi_i"'] & C_i \otimes E \arrow[rr, "\pi_1^i"] \arrow[d, "\gamma_i \otimes id_E"'] & & C_i \arrow[d, "\gamma_i"] \\
Eq(\pi_1, \epsilon \circ \pi_2) \arrow[r, tail, "e"'] & C \otimes E \arrow[rr, "\pi_1"] \arrow[dr, "\pi_2"'] & & C \\
& & E \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

*with $\{C_i \,{}_{\gamma_i}\!\otimes_\epsilon E \to Eq(\pi_1, \epsilon \circ \pi_2)\}_{i\in I} \in L_{\mathcal{C}}(Eq(\pi_1, \epsilon \circ \pi_2))$. Analogously, for any $\zeta : F \to D$ in $\mathcal{D}$ we have*

$$\begin{tikzcd}
D_i \,{}_{\gamma_i}\!\otimes_\epsilon F \arrow[r, tail, "e_i'"] \arrow[d, dashed, "\phi_i'"'] & D_i \otimes F \arrow[rr, "\pi_1^i"] \arrow[d, "\delta_i \otimes id_F"'] & & D_i \arrow[d, "\delta_i"] \\
Eq(\pi_1, \zeta \circ \pi_2) \arrow[r, tail, "e'"'] & D \otimes F \arrow[rr, "\pi_1"] \arrow[dr, "\pi_2"'] & & D \\
& & F \arrow[ur, "\zeta"'] &
\end{tikzcd}$$

*with* $\{D_i \,{}_{\gamma_i}\!\otimes_\epsilon F \to Eq(\pi_1, \zeta \circ \pi_2)\}_{i\in I} \in L_D(Eq(\pi_1, \zeta \circ \pi_2))$ *By Example 4.1.6, for each* $i \in I$*, the pseudo-pullback of* $(\gamma_i, \delta_i) : (C_i, D_i) \to (C, D)$ *and* $(\epsilon, \zeta) : (E, F) \to (C, D)$ *is* $(C_i \,{}_{\gamma_i}\!\otimes_\epsilon E, D_i \,{}_{\delta_i}\!\star_\zeta F)$*. Thus,* $\{(C_i \,{}_{\gamma_i}\!\otimes_\epsilon E, D_i \,{}_{\delta_i}\!\star_\zeta F) \to (Eq(\pi_1, \epsilon \circ \pi_2), Eq(\pi_1, \zeta \circ \pi_2))\} \in L_{C\times D}((Eq(\pi_1, \epsilon \circ \pi_2), Eq(\pi_1, \zeta \circ \pi_2)))$*.*

**Remark 4.1.14.** *Observe that if* $L_C$ *is a Grothendieck pretopology and* $L_D$ *is a quantalic covering, this construction provides an example of prelopology that is not quantalic neither is a Grothendieck pretopology.*

Next, mimicking the definition 2.3 of a sheaf for a Grothendieck pretopology we define

**Definition 4.1.15.** *Let* $C = (C, \otimes, 1)$ *be a monoidal semicartesian category with equalizers. A presheaf* $F : C^{op} \to Set$ *is* ***a sheaf for the Grothendieck prelopology*** $L(U) = \{f_i : U_i \to U\}_{i\in I}$ *if the following diagram is an equalizer in Set:*

$$F(U) \xrightarrow{e} \prod_{i\in I} F(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{(i,j)\in I\times I} F(U_i \,{}_{f_i}\!\otimes_{f_j} U_j)$$

*where*

1. $e(f) = \{F(f_i)(t) \mid i \in I\}, \quad f \in F(U)$
2. $p((f_k)_{k\in I}) = (F(p^1_{ij})(f_i))_{(i,j)\in I\times I}$
   $q((f_k)_{k\in I}) = (F(p^2_{ij})(f_j))_{(i,j)\in I\times I}, \quad (f_k)_{k\in I} \in \prod_{k\in I} F(U_k)$

*with* $p^1_{i,j} = e' \circ \pi^1_{i,j}$ *and* $p^2_{i,j} = e' \circ \pi^2_{i,j}$

Since there is a considerable amount of information in the above definition, it may be useful to provide diagrams to guide the reader. In $C$, we have the following pseudo-pullback diagram

$$\begin{tikzcd}
U_i \,{}_{f_i}\!\otimes_{f_j} U_j \arrow[dr, "e'"] \arrow[drr, bend left, "p^2_{ij}"] \arrow[ddr, bend right, "p^1_{ij}"'] & & \\
& U_i \otimes U_j \arrow[r, "\pi^2_{ij}"] \arrow[d, "\pi^1_{ij}"'] & U_j \arrow[d, "f_j"] \\
& U_i \arrow[r, "f_i"'] & U
\end{tikzcd}$$

Applying the presheaf $F$ and omitting the not necessarily commutative square we obtain

$$\begin{tikzcd}
F(U_i \,{}_{f_i}\!\otimes_{f_j} U_j) & F(U_j) \arrow[l, "F(p^2_{ij})"'] \\
F(U_i) \arrow[u, "F(p^1_{ij})"] & F(U) \arrow[l, "F(f_i)"] \arrow[u, "F(f_j)"']
\end{tikzcd}$$

By the universal property of the product $\prod F(U_i\ {}_{f_i}\!\otimes_{f_j} U_j)$, for each $i \in I$ there is a unique dashed arrow such that the triangles commute in the following diagram

$$
\begin{tikzcd}
F(U_i\ {}_{f_i}\!\otimes_{f_j} U_j) & & F(U_j) \arrow[ll, "F(p^2_{ij})"'] \arrow[dl, dashed] \\
& \prod F(U_i\ {}_{f_i}\!\otimes_{f_j} U_j) \arrow[ul, "proj"'] & \\
F(U_i) \arrow[uu, "F(p^1_{ij})"] \arrow[ur, dashed] & & F(U) \arrow[uu, "F(f_j)"'] \arrow[ll, "F(f_i)"]
\end{tikzcd}
$$

Now, using the universal property of the product $\prod F(U_i)$,

$$
\begin{tikzcd}
F(U_i\ {}_{f_i}\!\otimes_{f_j} U_j) & & & F(U_j) \arrow[lll, "F(p^2_{ij})"'] \arrow[dll, dashed, blue] \\
& \prod F(U_i\ {}_{f_i}\!\otimes_{f_j} U_j) \arrow[ul, "proj"'] & & \\
& & \prod F(U_i) \arrow[ur, blue, "proj_j"'] \arrow[dll, red, "proj_i"] & \\
F(U_i) \arrow[uuu, "F(p^1_{ij})"] \arrow[uur, dashed, red] & & & F(U) \arrow[uuu, "F(U_j)"'] \arrow[lll, "F(U_i)"] \arrow[ul, dashed, "e'"']
\end{tikzcd}
$$

So $F$ is sheaf if the unique arrow $e'$ is the equalizer of the pair of compositon of the red arrows and the composition of blue arrows.

**Example 4.1.16.** ***Sheaves on quantales:*** *In this case, $(C, \otimes, 1) = (Q, \odot, 1)$ and the Grothendieck prelopology is given by $\{f_i : U_i \to U\} \in L(U) \iff U = \bigvee_{i \in I} U_i$ (see Example 4.1.10). Since in $Q$ the pseudo-pullback is given by the quantalic product, the definition of sheaves on quantales (3.2.1) fits perfectly in the above definition.*

**Example 4.1.17.** ***Sheaves on the product category:*** *Consider two categories $(C, \otimes, 1_C)$ and $(\mathcal{D}, \star, 1_{\mathcal{D}})$ both with pseudo-pullbacks and equipped, with Grothendieck prelopologies $L_C$ and $L_{\mathcal{D}}$, respectively. Let $F : C^{op} \to Set$ be a sheaf for $L_C$ and $G : \mathcal{D}^{op} \to Set$ be a sheaf for $L_{\mathcal{D}}$. In Example 4.1.13 we described a Grothendieck prelopology $L_{C \times \mathcal{D}}$ for the product category $C \otimes \mathcal{D}$. Then $F \times G : (C \times \mathcal{D})^{op} \to Set$ defined by $(F \times G)((C, D)) = F(C) \times G(D)$ is a sheaf.*

Given the previously discussion about pseudo-pullbacks and Grothendieck pretopologies the next result may be seen as a corollary, but its importance for this thesis justifies that we are calling it a theorem:

**Theorem 4.1.18.** *Let $C = (C, \times, 1)$ be a cartesian category with equalizers. If $F : C^{op} \to Set$ is a sheaf for a Grothendieck pretopology, then $F$ is a sheaf for a Grothendieck prelopology.*

*Conversely, a sheaf* $F : \mathcal{C}^{op} \to Set$ *for a given Grothendieck prelopology is a sheaf for a Grothendieck pretopology.*

*Proof.* Assume $\mathcal{C} = (\mathcal{C}, \times, 1)$ and consider a sheaf $F$ for a Grothendieck pretopology. By Proposition 4.1.11, every Grothendieck pretopology is a Grothendieck prelopology. Besides, in such conditions, the pullback in $\mathcal{C}$ is the pseudo-pullback in $\mathcal{C}$. Therefore, $F$ is a sheaf for a Grothendieck prelopology. Conversely, we use that the pseudo-pullback in a cartesian category with equalizers is precisely the pullback, and that Grothendieck prelopologies in cartesian categories with equalizers are Grothendieck pretopologies, by Proposition 4.1.12. Thus, if $F : \mathcal{C}^{op} \to Set$ is a sheaf for a Grothendieck prelopology then it is a sheaf for a Grothendieck pretopology. □

This is not a generalization by the sake of generalization, because this notion of a sheaf includes both our sheaves on quantales and the standard notion of sheaves for Grothendieck pretopologies. Besides, Example 4.1.17 says that we if we have a sheaf $F$ for a Grothendieck pretopology and a sheaf $G$ on quantale, then we can obtain a sheaf $F \times G$ for the Grothendieck prelopology of the product category.

One of the major difficulties of this thesis is to find (natural) examples of sheaves with Grothendieck prelopologies that are not Grothendieck pretopologies, that is, beyond the standard case. The reasons for that are: $(i)$ we were not able to find many semicartesian categories with equalizers that are not cartesian, and $(ii)$ explicitly calculate the pseudo-pullback is not an easy task. Thus, the problem begins in finding examples of Grothendieck prelopologies. Therefore, we also have a weaker version of covering where we omit the fourth axiom in Definition 4.1.

**Definition 4.1.19.** *Let* $(\mathcal{C}, \otimes, 1)$ *be a monoidal category. A* ***weak Grothendieck prelopology*** *on* $\mathcal{C}$ *associates to each object* $U$ *of* $\mathcal{C}$ *a set* $L(U)$ *of families of morphisms* $\{U_i \to U\}_{i \in I}$ *such that:*

1. *The singleton family* $\{U' \xrightarrow{f} U\}$*, formed by an isomorphism* $f : U' \xrightarrow{\cong} U$*, is in* $L(U)$*;*
2. *If* $\{U_i \xrightarrow{f_i} U\}_{i \in I}$ *is in* $L(U)$ *and* $\{V_{ij} \xrightarrow{g_{ij}} U_i\}_{j \in J_i}$ *is in* $L(U_i)$ *for all* $i \in I$*, then* $\{V_{ij} \xrightarrow{f_i \circ g_{ij}} U\}_{i \in I, j \in J_i}$ *is in* $L(U)$*;*
3. *If* $\{f_i : U_i \to U\}_{i \in I} \in L(U)$*, then* $\{f_i \otimes id_V : U_i \otimes V \to U \otimes V\}_{i \in I}$ *is in* $L(U \otimes V)$ *and* $\{id_V \otimes f_i : V \otimes U_i \to V \otimes U\}_{i \in I}$ *is in* $L(V \otimes U)$*, for any* $V$ *object in* $\mathcal{C}$

**Example 4.1.20.** 1. *Let* $R$ *be a commutative ring with unity. The category of* $R$*-algebras is monoidal with the tensor product of algebras as the product and* $R$ *as the unit. Consider* $M$ *an* $R$*-module and* $I$ *an index set. Define*

$$\{M_i \to M\}_{i \in I} \in L(M) \iff M \cong \oplus_{i \in I} M_i$$

*The first axiom clearly holds. The second is also simple: if* $M = \oplus_{i \in I} M_i$ *and* $M_i = \oplus_{j \in J_i} M_{ij}$ *for all* $i \in I$*, then* $M \cong \oplus_{i \in I} \oplus_{j \in J} M_{ij} \cong \oplus_{i \in I, j \in J_i} M_{ij}$*.*

*The third axiom holds because the tensor product distributes over direct sum, i.e., if* $N$ *in a* $R$*-module, that is an isomorphism* $N \otimes (\oplus_{i \in I} M_i) \cong \oplus_{i \in I}(N \otimes M_i)$*.*

2. *In $Top$, the category of topological spaces with continuous functions, define:*

$$\{X_i \to X\} \in L(X) \iff X \approx \coprod X_i,$$

*where $\coprod X_i$ is the disjoint union of topological spaces $X_i$. Similarly to the above case we have $X \approx \coprod_{i\in I, j\in J_i} X_{ij}$ and $Y \times (\coprod_{i\in I} X_i) \approx \coprod_{i\in I}(Y \times X_i)$*

3. *In $Top_{\{*\}}$, the category of pointed topological spaces, also has a similar behavior. Replace the product of topological spaces with the smash product and the disjoint union with the wedge sum.*

There is an alternative way to define sheaves (Definition 4.2.5) that does not require $(\mathcal{C}, \otimes, 1)$ to be semicartesian and have pseudo-pullbacks and if it has, then such alternative definition is equivalent to Definition 4.1.15. In this work, we refrain from looking for examples of sheaves for weak Grothendieck prelopologies or even investigate the properties of the correspondent category that could arise. We only observe that weak Grothendieck prelopologies also are more general than Grothendieck pretopologies but it is not a *strict* generalization since we do not have a converse result like Proposition 4.1.12, that is, even that $(\mathcal{C}, \times, 1)$ is a cartesian categories with equalizers, we do not obtain that the weak Grothendieck prelopology is a Grothendieck pretopology.

In the next section we use the alternative definition of sheaves to help us in the task of constructing the sheafification functor.

## 4.2 Sheafification

In classic sheaf theory, the full subcategory inclusion from sheaves (Grothendieck topos) to presheaves has a left adjoint functor that preserves finite limits. The sheafification is that left adjoint functor. In the case of Grothendieck topos, there are at least two ways to construct/find the sheafification. One of them consists of considering a semi-sheafification, also know as the *plus construction*, that sends presheaves into separated presheaves by taking the colimit over all coverings (in the sense of a Grothendieck topology) for a fixed object in the category. If the presheaf already was separated, then the semi-sheafification gives a sheaf. So sheafification is the process of applying the semi-sheafification twice. Finally, it may be shown that this is left adjoint to the inclusion functor from sheaves to presheaves, and preserves finite limits. See [MM92] for details or consult the formulas in Section 2.3. The idea of applying the semi-sheafification twice, may be used in the case of Grothendieck pretopologies, as in [MR77, Chapter 1]. Unfortunately, the obvious way to replicate it for Grothendieck prelopologies does not work. Actually, the first application of the semi-sheafification, even in the quantalic case, does not work: Let $P$ be a presheaf and $\{V_i\}_{i\in I}$ a covering of $V$. In the localic case, for each $V \leq U$ we define the map $P^+(U) \to P^+(V)$ by $\{x_i \in P(U_i)\} \mapsto \eta_{\mathcal{V}}(\{x_{i|_{V\wedge U_i}} \in P(V)\})$, where $\eta_{\mathcal{V}}$ is the map from the set of compatibles families $Comp(\mathcal{V}, P)$ to the colimit $P^+(V) = \varinjlim_{\mathcal{V}\in K(V)} Comp(\mathcal{V}, P)$. Since $V \wedge U_i$ is a covering of $V$ whenever $U = \bigvee_{i\in I} U_i$, we have that $x_{i|_{V\wedge U_i}} \in P(V)$ and the semi-sheafification is a functor. However, in the quantalic case, $\odot$ is not idempotent and this implies that $V \odot U_i$ is **not** a covering of $V$ and then we are not able to define a map $P^+(U) \to P^+(V)$ as expected.

The other way consists of looking at sheafification in terms of local isomorphisms. The standard reference for this is [KS06, Section 16], but we believe the steps are clearer in [nLa22d], since the latter also includes the relation with the sheafification in terms of the localization of a certain class of morphisms in $PSh(\mathcal{C})$. We will follow this approach, but we need to reintroduce sheaves. Recall that an $L$-cover is a cover in the sense of a Grothendieck prelopology.

**Definition 4.2.1.** *Let $\{f_i : U_i \to U\}_{i\in I}$ be an L-cover of $U$. The* ***sieve*** *$S(\{U_i\})$ of $\{f_i : U_i \to U\}_{i\in I}$ is defined as the following coequalizer in $PSh(\mathcal{C})$:*

$$\coprod_{i,j} y(U_i)\ {}_{y(f_i)}\!\star_{\ y(f_j)} y(U_j) \rightrightarrows \coprod_i y(U_i) \longrightarrow S(\{U_i\})$$

*where $\star$ is the Day convolution $PSh(\mathcal{C})$, $y$ is the Yoneda embedding, and the coproduct on the left is over the pseudo-pullbacks $y(U_j)\ {}_{y(f_i)}\!\star_{\ y(f_j)} y(U_j)$.*

**Remark 4.2.2.** *This is the correspondent generalization of the sieve of a covering family in the sense of a Grothendieck pretopology.*

**Remark 4.2.3.** *The sieve $S(\{U_i\})$ of $\{f_i : U_i \to U\}_{i\in I} \in L(U)$ is a presheaf since it is as colimit of presheaves.*

We are not concerned, at this moment, with finding examples for this definition. All we want is to show that the inclusion $i : Sh(\mathcal{C}, L) \to PSh(\mathcal{C})$ has a left adjoint, and work with this abstract setting will allow us to do it.

**Remark 4.2.4.** *On one hand, we have a coequalizer diagram. By the commutativity of the pseudo-pullback, on the other hand, we have that*

$$y(U_i)\ {}_{y(f_i)}\!\star_{y(f_j)}\ y(U_j) \rightrightarrows y(U_i) \longrightarrow y(U)$$

*coequalizes. So, by the universal property of the coequalizer, we obtain a canonical morphism*

$$i_{\{U_i\}} : S(\{U_i\}) \to y(U)$$

*for all L-cover $\{U_i\}_{i\in I}$ of $U$.*

With the above notion of sieves we can say that a presheaf $P$ is a sheaf if it is a local object with respect to all $i_{\{U_i\}}$. In other words:

**Definition 4.2.5.** *A* ***sheaf*** *in $(\mathcal{C}, L)$ is a presheaf $P \in PSh(\mathcal{C})$ such that for all L-cover $\{f_i : U_i \to U\}_{i\in I}$ the hom-functor $Hom_{PSh(\mathcal{C})}(-, P)$ sends the canonical morphisms $i_{\{U_i\}} : S(\{U_i\}) \to y(U)$ to isomorphisms.*

$$Hom_{PSh(\mathcal{C})}(i_{\{U_i\}}, P) : Hom_{PSh(\mathcal{C})}(y(U), P) \xrightarrow{\cong} Hom_{PSh(\mathcal{C})}(S(\{U_i\}), P)$$

The above definition is saying that sheaves are *local objects* with respect to the class of morphisms $S(\{U_i\}) \to y(U)$.

Next, we want to show that when the base category $\mathcal{C}$ is semicartesian and admits pseudo-pullbacks then the above definition coincides with the one we introduced before (Definition 4.1.15). First we prove a useful lemma.

**Lemma 4.2.6.** *If $(C, \otimes, 1)$ is a symmetric semicartesian category that admits pseudo-pullbacks then*

$$y(U_i \,{}_{y(f_i)}\!\otimes_{y(f_j)} U_j) \cong y(U_i) \,{}_{y(f_i)}\!\star_{y(f_j)}\, y(U_j)$$

*Proof.* Notice that applying the Yoneda embedding in the pseudo-pullback we have the following, since $y$ preserves limits:

$$
\begin{tikzcd}
U_i \,{}_{f_i}\!\otimes_{f_j} U_j \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & & & y(U_i \,{}_{y(f_i)}\!\otimes_{y(f_j)} U_j) \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& U_i \otimes U_j \arrow[r] \arrow[d] & U_j \arrow[d] & \xRightarrow{y} & & y(U_i \otimes U_j) \arrow[r] \arrow[d] & y(U_j) \arrow[d] \\
& U_i \arrow[r] & U & & & y(U_i) \arrow[r] & y(U)
\end{tikzcd}
$$

In other words, if $U_i \,{}_{f_i}\!\otimes_{f_j} U_j$ is the equalizer of the commutative square on the right, then $y(U_i \,{}_{y(f_i)}\!\otimes_{y(f_j)} U_j)$ is the equalizer of the commutative square on the left.

Since the Yoneda embedding is a strong monoidal functor $(C, \otimes, 1) \to (PSh(C), \star, y(1))$, we have $y(U_i \otimes U_j) \cong y(U_i) \star y(U_j)$. Then we have the following pseudo-pullback diagram

$$
\begin{tikzcd}
y(U_i) \,{}_{y(f_i)}\!\star_{y(f_j)}\, y(U_j) \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& y(U_i) \star y(U_j) \arrow[r] \arrow[d] & y(U_j) \arrow[d] \\
& y(U_i) \arrow[r] & y(U)
\end{tikzcd}
$$

Since the equalizer is unique, up to isomorphism, this implies that

$$y(U_i \,{}_{y(f_i)}\!\otimes_{y(f_j)} U_j) \cong y(U_i) \,{}_{y(f_i)}\!\star_{y(f_j)}\, y(U_j)$$

$\square$

If pseudo-pullback in the definition of a sieve is a pullback, then the $L$-cover is a Grothendieck pretopology and so we obtain a sheaf equipped with a Grothendieck pretopology. Moreover, under pullbacks, if $\{f_i : U_i \to U\}_{i \in I}$ is a Grothendieck pretopology covering of $U$ and $W$ is an object in $C$, $S(\{U_i\})(W)$ is described as the set of morphisms $h : W \to U$ such that each $h$ factors through one of the $U_i$. Why? Since $S(\{U_i\})(W)$ is a coequalizer in *Set* we have to describe the proper equivalence relation in $\coprod_i y(U_i)$: given $\phi_i : W \to U_i$, we say that $\phi_i \sim \phi_j$ if there is $\phi_{ij} : W \to U_i \otimes_U U_j$ such that $p^1_{ij} \circ \phi_{ij} = \phi_i$ and

$p^2_{ij} \circ \phi_{ij} = \phi_j$. When the pseudo-pullback $\otimes_U$ is a weak[3] pullback $U_i \times_U U_j$ such equivalence relation is equivalent to saying that each $h : W \to U$ factors through $U_i$'s (and $W \to U_i \to U$ coincides with $W \to U_j \to U$ for all $i, j \in I$) because of the definition of the weak pullback, as we see below:

$$\begin{tikzcd}
W \arrow[rr, "\phi_j"] \arrow[dd, "\phi_i"'] \arrow[dr, dashed, "\phi_{ij}"] \arrow[ddrr, bend right, "h"] & & U_j \arrow[dd, "f_j"] \\
& U_i \times_U U_j \arrow[ur, "p^2_{ij}"'] \arrow[dl, "p^1_{ij}"] & \\
U_i \arrow[rr, "f_i"'] & & U
\end{tikzcd}$$

However, the universal property of the pseudo-pullback (which is an equalizer) is not enough to give us that the existence of $\phi_{ij} : W \to U_i \otimes_U U_j$ implies that $W \to U_i \to U$ and $W \to U_j \to U$ coincide. In other words, we do not have the outer square in the following diagram commutes

$$\begin{tikzcd}
W \arrow[rr, "\phi_j"] \arrow[dd, "\phi_i"'] \arrow[dr, dashed, "\phi_{ij}"] & & U_j \arrow[dd, "f_j"] \\
& U_i \otimes_U U_j \arrow[ur, "p^2_{ij}"'] \arrow[dl, "p^1_{ij}"] & \\
U_i \arrow[rr, "f_i"'] & & U
\end{tikzcd}$$

The consequence is that in general the canonical arrow $i_{\{U_i\}} : S(\{U_i\}) \to y(U)$ is a monomorphism only when the pseudo-pullback is a weak pullback. This distinction between our notion of sheaves and the usual one may be the central reason why we faced difficulties to construct the sheafification or establish a Grothendieck lopology.

**Proposition 4.2.7.** *If $(C, \otimes, 1)$ is a semicartesian category that admits pseudo-pullbacks, then the definition of sheaf as a local object (4.2.5) coincides with the first definition of a sheaf as a functor that makes a certain diagram an equalizer (4.1).*

*Proof.* If we apply $Hom_{PSh(C)}(-, P)$ in the coequalizer that defines the notion of sieves, we have

$$Hom_{PSh(C)}\Big(\coprod_{i,j} y(U_i) \;{}_{y(f_i)}\!\star_{\,y(f_j)} y(U_j), P\Big) \rightrightarrows Hom_{PSh(C)}\Big(\coprod_i y(U_i), P\Big) \longrightarrow Hom_{PSh(C)}(S(\{U_i\}), P)$$

Since such $Hom_{PSh(C)}(-, P)$ sends colimits to limits, we have the following equalizer diagram

$$Hom_{PSh(C)}(S(\{U_i\}), P) \longrightarrow \prod_i Hom_{PSh(C)}(y(U_i), P) \rightrightarrows \prod_{i,j} Hom_{PSh(C)}(y(U_i) \;{}_{y(f_i)}\!\star_{\,y(f_j)} y(U_j), P)$$

[3] The universal arrow exists but is not unique.

Applying the Yoneda Lemma:

$$Hom_{PSh(C)}(S(\{U_i\}), P) \longrightarrow \prod_i P(U_i) \rightrightarrows \prod_{i,j} Hom_{PSh(C)}(y(U_i)\ {}_{y(f_i)}\!\star_{y(f_j)}\ y(U_j), P)$$

So, $P$ is a sheaf if and only if the following diagram is an equalizer (for each $L$-cover $\{U_i \to U\}_{i \in I}$)

$$Hom_{PSh(C)}(y(U), P) \longrightarrow \prod_i P(U_i) \rightrightarrows \prod_{i,j} Hom_{PSh(C)}(y(U_i)\ {}_{y(f_i)}\!\star_{y(f_j)}\ y(U_j), P)$$

Applying the Yoneda Lemma, $P$ is sheaf iff the following diagram is an equalizer

$$P(U) \longrightarrow \prod_i P(U_i) \rightrightarrows \prod_{i,j} Hom_{PSh(C)}(y(U_i)\ {}_{y(f_i)}\!\star_{y(f_j)}\ y(U_j), P)$$

Since $(C, L)$ is a $L$-site, $C$ is a category with pseudo-pullbacks and so the pseudo-pullbacks $y(U_j)\ {}_{y(f_i)}\!\star_{y(f_j)}\ y(U_j)$ are representable functors. Then we apply Lemma 4.2.6 and the Yoneda Lemma again to obtain

$$Hom_{PSh(C)}(y(U_j)\ {}_{y(f_i)}\!\star_{y(f_j)}\ y(U_j), P) \cong P(U_j\ {}_{f_i}\!\otimes_{f_j} U_j)$$

So the sheaf condition is equivalent to requiring that

$$P(U) \longrightarrow \prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i\ {}_{f_i}\!\otimes_{f_j} U_j)$$

is an equalizer diagram for all coverings. □

**Definition 4.2.8.** *A **morphism of sheaves** is just a morphism of the underlying presheaves.*

**Remark 4.2.9.** *The category of sheaves $Sh(C, L)$ is a full subcategory of the category of presheaves $PSh(C)$.*

Now we recall some definitions as given in 1.32 and 1.35 of [AR94]:

**Definition 4.2.10.**

1. *An object $K$ is said to be **orthogonal** to a morphism $m : A \to A'$ provided that for each morphism $f : A \to K$ there exists a unique morphism $f' : A' \to K$ such that the following triangle commutes*

$$\begin{tikzcd} A \arrow[rr, "m"] \arrow[dr, "f"'] & & A' \arrow[dl, "f'"] \\ & K & \end{tikzcd}$$

2. *For each class $\mathcal{M}$ of morphisms in a category $\mathcal{K}$ we denote by $\mathcal{M}^\perp$ the full subcategory of $\mathcal{K}$ of all objects orthogonal to each $m : A \to A'$ in $\mathcal{M}$.*

**Definition 4.2.11.** *Let $\lambda$ be a regular cardinal. A **$\lambda$-orthogonality** class is a class of the* *form $\mathcal{M}^{\perp}$ such that every morphism in $\mathcal{M}$ has a $\lambda$-presentable[4] domain and a $\lambda$-presentable codomain.*

**Theorem 4.2.12.** ***(Theorem** 1.39**, [AR94])*** *Let $\mathcal{K}$ be a locally $\lambda$-presentable category. The following conditions on a full subcategory $\mathcal{A}$ of $\mathcal{K}$ are equivalent:*

*(i) $\mathcal{A}$ is a $\lambda$-orthogonality class in $\mathcal{K}$;*

*(ii) $\mathcal{A}$ is a reflective subcategory of $\mathcal{K}$ closed under $\lambda$-directed colimits.*

*Furthermore, they imply that $\mathcal{A}$ is locally $\lambda$-presentable.*

We know that $Sh(C, L)$ is a full subcategory of $PSh(C)$, and $PSh(C)$ is a $\lambda$-presentable category, for every regular cardinal $\lambda$ that is sufficiently big. So, if we prove that $Sh(C, L)$ is a $\lambda$-orthogonality class in $PSh(C)$, we apply the above theorem to obtain that $Sh(C, L)$ is a reflective subcategory of $PSh(C)$.

**Proposition 4.2.13.** *$Sh(C, L)$ is a $\lambda$-orthogonality class in $PSh(C)$.*

*Proof.* By definition, $F$ is a sheaf if and only if

$$Hom_{PSh(C)}(i_{U_i}, F) \,:\, Hom_{PSh(C)}(y(U), F) \to Hom_{PSh(C)}(S(\{U_i\}), F)$$

is an isomorphism in $Set$, so $Hom_{PSh(C)}(i_{U_i}, F)$ is a bijection. This means that for all $\varphi \in Hom_{PSh(C)}(S(\{U_i\}), F)$ there is a unique $\psi \in Hom_{PSh(C)}(y(U), F)$ such that $\psi \circ i_{U_i} = \varphi$. In other words, the desired triangle commutes:

$$\begin{tikzcd} S(\{U_i\}) \arrow[rr, "i_{U_i}"] \arrow[dr, "\forall\varphi"'] & & y(U) \arrow[dl, dashed, "\exists!\psi"] \\ & F & \end{tikzcd}$$

Thus, all sheaves are orthogonal to $i_{U_i}$. Then $Sh_L(C) = \mathcal{M}^{\perp}$, where $\mathcal{M} = \{i_{U_i} \,:\, S(\{U_i\}) \to y(U) \,:\, \{U_i \to U\}_{i \in I} \in L(U)\}$ is a class of morphism in $PSh(C)$. Observe that $S(\{U_i\})$ and $y(U)$ are presheaves and that they are $\lambda$-presentable. Indeed,

Since $C$ is a small category, the covering families of each $U$ are sets and a cardinal for each one of those sets. The supremum of all those cardinals is a cardinal again and then there is an even bigger regular cardinal $\lambda$ so that $PSh(C)$ is locally $\lambda$-presentable. Since $S(\{U_i\})$ and $y(U)$ are objects in $PSh(C)$, we conclude that $\mathcal{M}$ is a $\lambda$-orthogonality class $\lambda$-orthogonality class in $PSh(C)$. □

By definition of reflective subcategory:

**Corollary 4.2.14.** *The inclusion functor $i \,:\, Sh(C, L) \to PSh(C)$ has a left adjoint functor $a \,:\, PSh(C) \to Sh(C, L)$. Thus, $a$ preserves colimits.*

---

[4] An object $C$ in a category $\mathcal{C}$ is $\lambda$-presentable, for $\lambda$ a regular cardinal, when the representable functor $Hom_{\mathcal{C}}(C, -)$ preserves $\lambda$-filtered limits.

At this point we should say why this sheafification is not exactly the same as the one already available in the literature since they look the same. More precisely, the construction is the same but the fact that we are considering a weaker notion of covering (Grothendieck prelopologies instead of Grothendieck pretopologies) leads to a weaker sheafification: it is a left adjoint functor of the inclusion that does not preserve all finite limits. It is known that Grothendieck toposes are the left exact reflective subcategories of a presheaf category. However, while $Sh(C, L)$ is a reflective subcategory of $PSh(C)$, in Theorem 4.3.8 we prove that $Sh(Q)$ is not a Grothendieck topos in general, and thus our sheafification cannot be left exact/preserve all finite limits. Another reasonable question that the reader may have is if we can present the sheafification through a formula. To prove Theorem 4.2.12, the authors use an "Orthogonal-reflection Construction" that relies on a transfinite induction. Thus, theoretically, we can find a formula to describe the sheafification using transfinite induction, but in practice it does not seem to have an elucidating structure.

Next, note that Theorem 4.2.12 also provides that $Sh(C, L)$ is a locally $\lambda$-presentable category. We remember that:

**Definition 4.2.15.** *[Bor94b, Definition 5.2.1] A category $\mathcal{M}$ is locally $\lambda$-presentable, for a regular cardinal $\lambda$, when*

1. *$\mathcal{M}$ is cocomplete;*
2. *$\mathcal{M}$ has a set $(G_i)_{i\in I}$ of strong generators;*
3. *Each generator $G_i$ is $\lambda$-presentable.*

So, by Theorem 4.2.12:

**Corollary 4.2.16.** *$Sh(C, L)$ is cocomplete, has a set of strong generators with each one of then being $\lambda$-presentable.*

Now, we discuss what our sheafification preserves. We start showing that $a : PSh(C) \to Sh(C, L)$ preserves the monoidal closed structure of $PSh(C)$ in the quantalic case.

In [Day73], B. Day proves that if a class $\Sigma$ of morphisms in a symmetric monoidal category $(C, \otimes, 1)$ with the property that $f \in \Sigma$ implies $id_A \otimes f \in \Sigma$ then the category of fractions $C[\Sigma^{-1}]$ is a monoidal category. The result is even better than this but we will use a minor result from that paper. First, a definition:

**Definition 4.2.17.** *Let $\psi \dashv \phi : D \to B$ be an adjoint pair.*

1. *$\psi \dashv \phi$ is a **reflective embedding** if $\phi$ is full and faithful on morphisms.*
2. *When $B$ has a fixed monoidal closed structure the reflective embedding is called **normal** if there exists a monoidal closed structure on $D$ and monoidal functor structures on $\psi$ and $\phi$ for which $\phi$ is a normal closed functor and the unit and counit of the adjunction are monoidal natural transformations.*

The definition of "closed functor" is in [EK66] and the definition of "normal closed functor" is available from [Bar+69]. We will not copy those definitions here since what is interesting for us is Day's observation that normal enrichment is unique (up to monoidal isomorphism) and it exists if and only if one condition of the following equivalent conditions is satisfied:

$$\eta[b,\phi d] : [b,\phi d] \cong \phi\psi[b,\phi d]; \tag{4.1}$$
$$[\eta,1] : [\phi\psi b,\phi d] \cong [b,\phi d]; \tag{4.2}$$
$$\psi(\eta\otimes 1) : \psi(b\otimes b') \cong \psi(\phi\psi b\otimes b'); \tag{4.3}$$
$$\psi(\eta\otimes\eta) : \psi(b\otimes b') \cong \psi(\phi\psi b\otimes \phi\psi b') \tag{4.4}$$

where $\eta$ is the unit of the adjunction $\psi\dashv\phi : D\to B$, $d$ is any object of $D$, $b$ is any object of $B$ and $[-,-]$ is the internal hom. In particular, the components $\tilde{\phi} : \phi(b)\otimes_D\phi(b')\to\phi(b\otimes_B b')$ and $\phi^0 : I_D\to\phi(I_B)$ are isomorphisms.

A small remark: The list in [Day73] is larger but we are using a shorter version that is more than enough for us and it is available in nLab.

The next result is Proposition 1.1 in [Day73], with a small notation change.

**Proposition 4.2.18.** *Let $C=(C,\otimes,1)$ be a small monoidal category and $S$ the cartesian closed category of small*[5] *sets and set maps. Denote by $S^C$ the functor category from $C$ to $S$. A reflective embedding $\psi\dashv\phi : D\to S^C$ admits normal enrichment if and only if the functor $F(U\otimes -)$ is isomorphic to some object in $D$ whenever $F$ is a object of $D$ and $U$ is an object of $C$.*

**Proposition 4.2.19.** *If $C=Q$ is the posetal category of (unital semicartesian) quantales, then $Sh(Q)$ is monoidal closed.*

*Proof.* By Propositon 3.2.15, the functor $F(u\odot -)$ is a sheaf for every $u\in Q$, whenever $F$ is a sheaf. By 4.2.18, the reflective embedding $a\dashv i$ admits a normal enrichment. □

Stated in other words:

**Proposition 4.2.20.** *The functor $a : PSh(Q)\to Sh(Q)$ preserves the monoidal closed structure.*

Therefore, we have a **monoidal structure in** $Sh(Q)$. The above result gives that $Sh(Q)$ has a monoidal closed structure where, by Equation 4.4, $F\otimes G := a(i(F)\otimes_{Day} i(G))$, for $F,G$ sheaves on $Q$.

We were not able to prove that if $F$ is a sheaf in $Sh(C,L)$ then $F(U\otimes -)$ is a sheaf in $Sh(C,L)$ and let it to future works. Nevertheless, finding the monoidal structure of $Sh(Q)$ was essential to prove that in the quantalic case we may already have a non Grothendieck topos.

**Remark 4.2.21.** *Note that $PSh(Q)$ have cartesian products. In a certain way, our sheafification is ignoring the cartesian products and preserving the monoidal products that arise from the Day convolution.*

The next result holds for $Sh(C,L)$ in general.

**Proposition 4.2.22.** *The functor $a : PSh(C)\to Sh(C,L)$ preserves terminal objects.*

[5] The smallness condition for sets and for the monoidal category is to avoid size issues.

*Proof.* The inclusion of sheaves into presheaves is a right adjoint, so preserves limits; hence the terminal sheaf must be the terminal object in the category of presheaves. Thus, it is the terminal presheaf, which is the constant presheaf with value the terminal object in $Set$. □

The following discussion does not provide any new results but we believe it is worth including it in this section since it explains another approach to talk about the sheafification process.

Note that the class $\mathcal{M} = \{i_{U_i} : S(\{U_i\}) \to y(U) : \{U_i \to U\}_{i\in I} \in L(u)\}$ is used both to provide the arrows orthogonal to sheaves and to provide the arrows in $PSh(\mathcal{C})$ that we want to invert such that $PSh(\mathcal{C})[\mathcal{M}^{-1}] \cong Sh(\mathcal{C})$. Indeed, the orthogonal subcategory problem is closely related to the problem of localization, see [Bor94a, Chapter 5].

The following proposition is proved in Proposition 5.3.1 of [Bor94a] with a slightly different notation.

**Proposition 4.2.23.** *Let $\mathcal{K}$ be a category and $\mathcal{K}'$ be a reflective subcategory $r \dashv i : \mathcal{K}' \rightleftarrows \mathcal{K}$. Consider the class $\Sigma$ of all morphisms $f$ in $\mathcal{K}$ such that $r(f)$ is an isomorphism in $\mathcal{K}'$. In this case:*

1. *the category of fractions $\varphi : \mathcal{K} \to \mathcal{K}[\Sigma^{-1}]$ exists;*
2. *the category of fractions $\varphi : \mathcal{K} \to \mathcal{K}[\Sigma^{-1}]$ is equivalent to $r : \mathcal{K} \to \mathcal{K}'$;*
3. *the class $\Sigma$ admits a left calculus of fractions.*

We proved that $Sh_L(\mathcal{C}) = \mathcal{M}^{\perp}$ is a reflective subcategory of $PSh(\mathcal{C})$. Now we want to find the class $\Sigma$ of morphisms inverted by $r$. Observe that such $\Sigma$ contains $\mathcal{M}$:

**Lemma 4.2.24.** *Let $\mathcal{C}$ be any category and $\Gamma$ a class of morphisms $\Gamma$ in $\mathcal{C}$ such that $i : \Gamma^{\perp} \to \mathcal{C}$ is the inclusion functor and $r : \mathcal{C} \to \Gamma^{\perp}$ the reflector. Then $f \in \Gamma$ implies $r(f)$ is an isomorphism in $\Gamma^{\perp}$.*

*Proof.* In the following, we just replicate the proof in [nLa22a]. By definition of orthogonality, for every object $K$ in $\Gamma^{\perp}$, $f : A \to B$ induces an isomorphism of hom-sets

$$Hom_{\mathcal{C}}(f, i(K)) : Hom_{\mathcal{C}}(B, i(K)) \to Hom_{\mathcal{C}}(A, i(X))$$

Since $r$ is the left adjoint of $i$, for all $K$ in $\Gamma^{\perp}$ the following map is an isomorphism:

$$Hom_{\Gamma^{\perp}}(r(f), K) : Hom_{\Gamma^{\perp}}(r(B), K) \to Hom_{\Gamma^{\perp}}(r(A), K)$$

So $Hom_{\Gamma^{\perp}}(r(f), -)$ is a natural isomorphism between representables. By the Yoneda lemma, this means $r(f)$ is an isomorphism. □

The point here is that we want all morphisms that are inverted by $r$, but the class $\mathcal{M}$ provides just a few. To solve this, we state the Proposition 5.4.10 in [Bor94a].

**Proposition 4.2.25.** *Let $\mathcal{K}$ be a cocomplete category in which every object is presentable. Consider a set $\mathcal{M} \subseteq mor(\mathcal{K})$ and the corresponding reflective subcategory $r \dashv i : \mathcal{K}_{\mathcal{M}} \rightleftarrows \mathcal{K}$*

*associated with the orthogonality problem. Then the class* $\Sigma = \{f \in mor(\mathcal{K}) : r(f)$ *is an isomorphism*$\}$ *is the smallest subclass* $\Sigma \subseteq mor(\mathcal{K})$ *such that:*

1. $\mathcal{M} \subseteq \Sigma$;
2. *every isomorphism of* $\mathcal{K}$ *is in* $\Sigma$;
3. *"two of three" holds for* $\Sigma$, *i.e., if two sides of a commutative diagram are in* $\Sigma$, *so is the third;*
4. $\Sigma$ *is closed under colimits.*

By Propositions 4.2.23 and 4.2.25 we conclude and define:

**Corollary 4.2.26.** *Let* $\mathcal{M} = \{i_{U_i} : S(\{U_i\}) \to y(U) : \{U_i \to U\}_{i \in I} \in L(u)\}$ *and* $\Sigma$ *as in the Proposition 4.2.25. Then*

1. *The category of fractions* $l : PSh(C) \to PSh(C)[\Sigma^{-1}]$ *exists;*
2. *The category of fractions* $l : PSh(C) \to PSh(C)[\Sigma^{-1}]$ *is equivalent to* $r : Sh_L(C) \to PSh(C)$;
3. *the class* $\Sigma$ *admits a* <u>*left*</u> *calculus of fractions.*

Both the localization and the reflection are candidates to be called sheafification functor, so it is a relief that they are equivalent. Again we highlight a difference between sheaves with a Grothendieck prelopology and sheaves with a Grothendieck topology: we obtained that the class $\Sigma$ admits a <u>left</u> calculus of fractions by consequence of a sequence of general results. In the case of sheafification of presheaves with Grothendieck (pre)topologies, the resulting class $\Sigma$ also admits a <u>right</u> calculus of fractions. In particular, the stability axiom in definition of a Grothendieck pretopology (Definition 2.3.1) provides that $\Sigma$ satisfies the right Ore condition.

In the standard case, we also use that $\Sigma$ admits right calculus of fractions to prove that the sheafification/localization preserves finite limits. This is not our case: a sheafification that preserves finite limits has as it target a Grothendieck topos and our categories of sheaves enjoy different categorical properties, as we will prove in Theorem 4.3.8.

**Remark 4.2.27.** *If we understand our sheafification better, we may define Grothendieck loposes as those accessible reflective subcategories of presheaf categories for which the reflector preserves the (Day) monoidal structure of* $PSh(C)$ *and possibly something else. Bearing in mind that geometric morphisms of toposes consists of a pair of adjoint functors where the left adjoint preserves finite limits and this allow us to think in the* 2*-category of toposes, we may also want to explore the* 2*-category of loposes, where the morphism would be a pair of adjoint functors here the left adjoint preserves the same structure as the sheafification. Alternatively, we can analyze 3.4.8 to obtain a good morphism between categories of sheaves.*

## 4.3 Sheaves on quantales revisited

This Section, together with pieces of Sections 3.1 and 3.2, are part of a paper recently submitted to a Journal (there is a preliminary version of such paper available on ArXiv).

### 4.3.1 $Sh(Q)$ is not a topos

Given the sheafification process that we constructed, we are now able to properly present an argument to prove that $Sh(Q)$ is not a topos. The main result of this section says that the lattice of subobjects of the terminal object in $Sh(Q)$ is isomorphic to the quantale $Q$. Therefore, if $Q$ is a non-idempotent semicartisian quantale, we have that such lattice is not a locale. Since it is known that the lattice of subobjects of any objects of a topos is a locale, we conclude that $Sh(Q)$ is not a topos. This is one of the most important conclusions of this thesis.

First, we check that we will at least have a complete lattice structure:

**Proposition 4.3.1.** *$Sh(Q)$ is a complete and well-powered category, and for all $F$ sheaf on $Q$, $Sub(F)$ has all infima/intersections and suprema/unions.*

*Proof.* It is a direct consequence of the following two general results:

If $\mathcal{C}$ has finite limits and possesses a strong set of generators, so $\mathcal{C}$ is well-powered (i,e., for all $C$ object of $\mathcal{C}$, the subobjects $Sub(C)$ of $C$ forms a set)[Bor94a, Proposition 4.5.15].

In a complete and well-powered category, $Sub(C)$ has all infima/intersections and suprema/unions [Bor94a, Corollary 4.2.5]. □

As a corollary we obtain the following result about the factorization of morphisms in $Sh(Q)$.

**Corollary 4.3.2.** *For each morphism $\phi : F \to G$ in $Sh(Q)$, there exists the least subobject of $G$, represented by $\iota : G' \rightarrowtail G$, such that $\phi = \iota \circ \phi'$ for some (and thus, unique) morphism $\phi' : F \to G'$. Moreover, $\phi'$ is an* epimorphism.

*Proof.* By the previous results, there exists the extremal factorization above $\phi = \iota \circ \phi'$, such that $\iota : G' \rightarrowtail G$ is a mono. To show that $\phi' : F \to G'$ is an epi, consider $\eta, \epsilon : G' \to H$ such that $\eta \circ \phi' = \epsilon \circ \phi'$ and let $\gamma : H' \rightarrowtail G'$ be the equalizer of $\eta, \epsilon$. Then, by the universal property of $\gamma$, there exists a unique $\phi'' : F \to H'$ such that $\gamma \circ \phi'' = \phi'$. On the other hand, by the extremality of $\iota$, there exists a unique $\gamma' : G' \to H'$ such that $\iota = \iota \circ \gamma \circ \gamma'$. Since $\iota$ is a mono, we obtain that $\gamma \circ \gamma' = id_{G'}$, Thus $\gamma$ is a mono that is a retraction: this means that $\gamma = eq(\eta, \epsilon)$ is an iso, i.e., $\epsilon = \eta$. Thus $\phi'$ is an epi. □

**Remark 4.3.3.** *Keeping the notation above, if $\phi : F \to G$ is already a mono then, by the extremality of $\iota : G' \rightarrowtail G$, $\phi \cong \iota$ and thus $\phi' : F \to G'$ is an isomorphism. It is natural to ask ourselves if the converse holds in general. Conversely, does it hold that any morphism that is mono and epi is an iso? This would mean that the category $Sh(Q)$ is* balanced.

*Any category with factorization (extremal mono, epi) and where all the monos are regular (i.e., monos are equalizers) is balanced. A "topos-theoretic" way to show that all monos are regular is to show that there exists a "universal mono" $true : 1 \rightarrowtail \Omega$ that is a subobjects classifier, which is the topic of the next section.*

We want to show that the lattice of subobjects of the terminal object in $Sh(Q)$ is isomorphic to the quantale $Q$. The next theorem gives an isomorphism of complete lattices:

**Theorem 4.3.4.** *Assume that $Q$ is a unital commutative and semicartesian quantale. We have the following isomorphisms of complete lattices:*

$$\begin{aligned} h_Q : Q &\to Represented(Sh(Q)) \\ q &\mapsto Q(-,q) \end{aligned}$$

$$\begin{aligned} i_Q : Represented(Sh(Q)) &\to Representable(Sh(Q))/isos \\ Q(-,q) &\mapsto [Q(-,q)]_{iso} \end{aligned}$$

$$\begin{aligned} j_Q : Representable(Sh(Q))/isos &\to Sub(\mathbf{1}) \\ [R]_{iso} &\mapsto [R \cong Q(-,q) \rightarrowtail Q(-,1) \cong \mathbf{1}]_{iso} \end{aligned}$$

*Thus $k_Q = j_Q \circ i_Q \circ h_Q : Q \to Sub(\mathbf{1})$ is an isomorphism of complete lattices.*

*More generally, take any $a \in Q$, we may amend the map $k_Q$ in a way that it sends $b \in [0,a]$ to $[Q(-,b) \rightarrowtail Q(-,a)]_{iso}$, then we obtain a quantalic isomorphism $k_a : [0,a] \to Sub(Q(-,a))$.*

*Proof.* We will just show that $h_Q, i_Q, j_Q$ are isomorphisms of posets, and, since $Q$ is a complete lattice, then $h_Q, i_Q, j_Q$ are complete lattices isomorphisms.

*$h_Q$ is isomorphism:* By the very definition of represented functor, the map $h_Q$ is surjective. For injectivity see that $Q(-,q) = Q(-,p)$ implies that $Q(u,q) = Q(u,p)$, for all $u \in Q$, and so $p = q$. Yoneda's lemma and Proposition 3.2.22.1 establishes that it preserves and reflects order since $p \leq q$ iff there is some (unique) (mono)morphism $\eta : Q(-,p) \to Q(-,q)$.

*$i_Q$ is isomorphism:* Since it is a quotient map, it is surjective. $i_Q$ is injective: by Proposition 3.2.22.2, $Q(-,p) \cong Q(-,q)$ implies $p = q$ and thus $Q(-,p) = Q(-,q)$. The map preserves and reflects order: this is a direct consequence of Proposition 3.2.22.1.

*$j_Q$ is isomorphism:* Since $! : Q(-,1) \to \mathbf{1}$ is an isomorphism, we will just prove that $j'_Q : Representable(Sh(Q))/isos \to Sub(Q(-,1))$ $[R]_{iso} \mapsto [R \cong Q(-,q) \rightarrowtail Q(-,1)]_{iso}$ is an isomorphism. By the very definition of $Sub(F) = Mono(F)/isos$, it is clearly injective. Take $\eta : R \rightarrowtail Q(-,1)$, by Proposition 3.2.22.1, $R \cong Q(-,q)$, thus $j'_Q$ is surjective. Now let $R$ and $R'$ be representable functors, there is a morphism $\eta : R \to R'$ iff this morphism is unique and it is a monomorphism, thus $j'_Q$ preserves and reflects order. □

The next step is to equip $Sub(\mathbf{1})$ with a product that gives it a quantalic structure.

**Definition 4.3.5.** *For each $F$ sheaf on $Q$, we define the following binary operation on $Sub(F)$: Given $\phi_i : F_i \rightarrowtail F$, $i = 0,1$ define $\phi_0 * \phi_1 : F_0 * F_1 \rightarrowtail F$ as the mono in the extremal factorization of $F_0 \otimes_F F_1 \rightarrowtail F_0 \otimes F_1 \rightrightarrows F$.*

As we will see in the next result, such definition gives that the poset $Sub(Q(-,a))$ is a quantale. In a future work we would like to check if $Sub(F)$ is a quantale for any sheaf $F$ on a quantale.

**Theorem 4.3.6.** *For each $a \in Q$, the poset $Sub(Q(-,a))$, endowed with the binary operation $*$ defined above is a commutative and semicartesian quantale. Moreover, the poset isomorphism $k_a : [0,a] \to Sub(Q(-,a))$, $q \mapsto [Q(-,q)]_{iso}$, established in Theorem 4.3.4, is a quantale isomorphism.*

*Proof.* As a consequence of the proof of Theorem 4.3.4, this map is well-defined, bijective, and preserves and reflects orders. It remains to show that $Q(-,u \odot v) \cong Q(-,u) * Q(-,v)$, for all $u,v \leq a$. We have that $Q(-,u) * Q(-,v) \rightarrowtail Q(-,a)$ is the mono in the extremal factorization of the arrow

$$Q(-,u) \otimes_{Q(-,a)} Q(-,v) \xrightarrow{equ} Q(-,u) \otimes Q(-,v) \rightrightarrows Q(-,a)$$

By Day convolution,

$$Q(-,u) \otimes Q(-,v) \cong Q(-,u \odot v)$$

Since $u \odot v \leq a$, by Proposition 3.2.22, there is unique (mono)morphism $Q(-,u \odot v) \to Q(-,a)$. So $Q(-,u) \otimes Q(-,v) \rightrightarrows Q(-,a)$ corresponds to $Q(-,u \odot v) \rightrightarrows Q(-,a)$. Thus, the parallel arrows coincide and then

$$Q(-,u) \otimes_{Q(-,a)} Q(-,v) \cong Q(-,u) \otimes Q(-,v) \cong Q(-,u \odot v)$$

Hence, the arrow

$$Q(-,u) \otimes_{Q(-,a)} Q(-,v) \to Q(-,a)$$

is isomorphic with the unique mono

$$Q(-,u \odot v) \rightarrowtail Q(-,a).$$

This shows that $Q(-,u \odot v) \cong Q(-,u) * Q(-,v)$, as we wish. □

This has an interesting direct application:

**Corollary 4.3.7.** *Let $Q$ be the quantale of ideals of a ring $R$, then $Q$ is isomorphic to $Sub(Q(-,R))$.*

So we can recover any ideal of $R$ by analyzing the subobjects of the sheaf $Q(-,R)$. Of course, we also obtain that we can recover the open subsets of a topological spaces $X$ by analyzing the subobjects of the sheaf $\mathcal{O}(X)(-,X)$, which is already known from the theory of sheaves on locales.

A far more decisive consequence from the categorical logic point of view is the title of this subsection:

**Theorem 4.3.8.** *In general, the category $Sh(Q)$ is not an elementary topos.*

*Proof.* On one hand, take any commutative, semicartesian and unital quantale $Q$ whose underlying lattice is not a locale, i.e., it does not have a complete Heyting algebra structure.

Consider the sheaf $\mathbf{1} \cong Q(-,1)$. By Theorems 4.3.4 and 4.3.6, $Sub(Q(-,1)) \cong Sub(\mathbf{1})$ is isomorphic to $Q$ as a quantale. Since $Q$ is complete lattice that is not a locale, then $Sub(\mathbf{1})$ is not a Heyting algebra. On the other hand, let $\mathcal{E}$ be any elementary topos and $C$ any object in $\mathcal{E}$, then the lattice $Sub(C)$ is a Heyting algebra [MM92, Page 201]. Therefore, $Sh(Q)$ is not an elementary topos. □

**Remark 4.3.9.** *Consider the ring of integer polynomials $\mathbb{Z}[x]$. This is an example of a ring that is not a Prüfer domain, thus we do not have $I \cap (J+K) = I \cap J + I \cap K$, for nonzero ideals [Nar95][Wik23]. In other words, $\mathcal{I}(\mathbb{Z}[x])$ is an example of a quantale with no underlying localic structure.*

Another interesting consequence of Theorem 4.3.6 is that $Sh(Q)$ may not have subobject classifier if $Q$ does not have a localic structure.

**Corollary 4.3.10.** *If $Q$ is a semicartesian quantale where meets do not distribute over joins, then $Sh(Q)$ does not have a subobject classifier.*

*Proof.* Corollary 4.5 in [nLa23b] says that if $\mathcal{C}$ is a category with all finite limits with subobject classifier, then the poset $Sub(C)$ of subobjects of $C$, for every object $C$ in $\mathcal{C}$, meets distribute over any existing join.

So, suppose $Sh(Q)$ has subobject classifier. By Theorem 4.3.6, $Sub(Q(-,1)) \cong Q$. So, if meets do not distribute over joins $Q$, we obtain a contradiction. Therefore, in such circumstances, $Sh(Q)$ has no subobject classifier. □

Observe that in a quantale the multiplication distributes over joins and quantales, as complete lattices, have meets that may or may not distribute over joins. Therefore, the issue about the subobjects in $Sh(Q)$ may be more intriguing than just a matter of having it or not. If $Q$ is a quantale regarding the multiplication and a locale regarding the meet, then the subobjects may exist and also have the distributive law for both the meet and multiplication. We have not concluded the referred investigation but in the next subsection we will discuss a candidate for a subobject classifier in $Sh(Q)$.

### 4.3.2 The subobject classifier

By definition, every elementary topos has a subobject classifier and we use it to construct the internal logic of a given topos. We are also interested in the logical aspects of our Grothendieck lopos and investigating the subobject classifier of $Sh(Q)$ is a starting point.

We recall the definition:

**Definition 4.3.11.** *Let $C$ be a locally small category with all finite limits. A* ***subobjects classifier*** *consists of an object $\Omega$ and a morphism $t : \mathbf{1} \to \Omega$ that satisfies the following universal property:*

*Given any object subobject $m : U \rightarrowtail E$ in $C$, there is a unique morphism $\chi_m : E \longrightarrow \Omega$ such that the following is a pullback diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d, tail, "m"'] & 1 \arrow[d, "\top"] \\ E \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

*The object $\Omega$ is called the* ***object of truth values****, $\chi_m$ is called the* ***classifying map*** *of the subobject $m$ and $\top$ is called* ***truth morphism****.*

In the category on sheaves on a locale $L$, the subobject classifier is given by the sheaf $\Omega(u) = \{q \in L \,:\, q \leq u\}$ such that for all $v \leq u$, we map $q$ to $q \wedge v$, and $\top \,:\, \mathbf{1} \to \Omega$ is defined by $\top_u(*) \,:=\, u$. If you try to use the same application but replace $L$ by a semicartesian quantale $Q$, then $\Omega(u)$ will not be a sheaf. The major problem when we try to run the verification comes from the non-idempotence of the quantale. To overcome the issue we use the idempotent approximation that we introduced in Section 3.1 to construct the $\Omega$ below.

**Proposition 4.3.12.** *Let $Q$ be a commutative, semicartesian and* geometric *quantale. For each $u \in Q$ define $\Omega^-(u) = \{q \in Q \,:\, q \odot u^- = q\}$ then, with the restriction map*

$$\begin{aligned} \Omega^-(u) &\to \Omega^-(v) \\ q \quad &\mapsto q \odot v^- \end{aligned}$$

*for all $v \leq u$ in $Q$, $\Omega$ is a sheaf.*

*Proof.* Note that $q \odot v^- \in \Omega^-(v)$ since $q \odot v^- \odot v^- = q \odot v^-$. It is a presheaf because $q \odot u^- = q$ and, given $w \leq v \leq u$, $q \odot v^- \odot w^- = q \odot w^-$. The separability also holds: suppose $u = \bigvee_{i \in I} u_i$ and take $p, q \in \Omega^-(u)$ such that $p{\restriction_{u_i}} = q{\restriction_{u_i}}$ for all $i \in I$. Then

$$\begin{aligned} p = p \odot u^- = p \odot (\bigvee_{i \in I} u_i)^- &= p \odot \bigvee_{i \in I} u_i^- = \bigvee_{i \in I} p \odot u_i^- \\ &= \bigvee_{i \in I} q \odot u_i^- = q \odot \bigvee_{i \in I} u_i^- = q \odot (\bigvee_{i \in I} u_i)^- = q \odot u^- \\ &= q \end{aligned}$$

The gluing is $q = \bigvee_{i \in I} q_i$, where $q_i \in \Omega^-(u_i)$ for each $i \in I$. Observe that $q \in \Omega^-(u)$:

$$q \odot u^- = \bigvee_{i \in I} q_i \odot (\bigvee_{j \in I} u_j)^- = \bigvee_{i \in I} q_i \odot \bigvee_{j \in I} u_j^- = \bigvee_{i \in I} q_i \odot u_i^- \odot \bigvee_{j \in I} u_j^- = \bigvee_{i \in I} q_i = q.$$

Where we used that the quantale is geometric in the second equality, the fact $q_i \in \Omega^-(u_i)$ in the third, and the idempotence of $u_i^-$ in the fourth.

Now we check that $q$ is the gluing. On one hand

$$q_j = q_j \odot u_j^- \leq q \odot u_j^- = q{\restriction_{u_j}}$$

On the other hand, recording that $(u \odot v)^- = (u^- \odot v^-)$ by Proposition 3.1.9.10,

$$\begin{aligned} q_{\restriction_{u_j}} &= q \odot u_j^- = (\bigvee_{i\in I} q_i) \odot u_j^- = \bigvee_{i\in I}(q_i \odot u_j^-) = \bigvee_{i\in I}(q_i \odot u_i^- \odot u_j^-) \\ &= \bigvee_{i\in I}(q_i \odot (u_i \odot u_j)^-) = \bigvee_{i\in I} q_{i\restriction_{u_i\odot u_j}} = \bigvee_{i\in I} q_{j\restriction_{u_i\odot u_j}} \\ &= \bigvee_{i\in I}(q_j \odot (u_i \odot u_j)^-) = \bigvee_{i\in I}(q_j \odot u_i^- \odot u_j^-) = (\bigvee_{i\in I} u_i^-) \odot q_j \odot u_j^- \\ &= (\bigvee_{i\in I} u_i)^- \odot q_j = u^- \odot q_j \leq q_j. \end{aligned}$$

□

**Remark 4.3.13.** *1. The mapping $Q \mapsto \Omega^-$ preserves products and interval constructions (see Proposition 3.2.13).*

2. *Note that for each $v, u \in Q$, such that $v^- = u^-$, then $\Omega^-(v) = \Omega^-(u)$. In particular, if $u^- \leq v \leq u$, then $\Omega^-(v) = \Omega^-(u)$ and, moreover, $\Omega^-(u^-, v) = \Omega^-(v, u) = id_{\Omega^-(v)}$.*

3. *For each $u \in Q$, let $\perp_u, \top_u : 1(u) \to \Omega^-(u)$, where $\perp_u(*) := 0 \in \Omega^-(u)$ and $\top_u(*) := u^- \in \Omega^-(u)$. Then $\perp, \top : 1 \to \Omega^-$ are natural transformations.*

4. *For each $u \in Q$ and $v \in \Omega^-(u)$ we have $v^- \in \Omega^-(u)$: this defines a map $-_u : \Omega^-(u) \to \Omega^-(u)$, Then $(\ )^- := (-_u)_{u\in Q} : \Omega^- \to \Omega^-$ is a natural transformation and $\top^- := (\ )^- \circ \top = \top$, $\perp^- := (\ )^- \circ \perp = \perp$.*

5. *If $Q$ is a locale, then $\Omega^-(u) = \{q \in Q : q \odot u^- = q\} = \{q \in Q : q \leq u\} = \Omega_0(u)$, and $\top_u(*) = u^- = u$. Thus $\top : 1 \to \Omega^-$ coincides with the subobject classifier in the category of sheaves on locales [Bor94c, Theorem 2.3.2]. We will readdress this subject below.*

Our investigations did not lead to $\Omega^-$ being a subobjects classifier, but it does classify the dense subobjects:

**Definition 4.3.14.** *A morphism of sheaves $\eta : G \to F$ is* **dense** *whenever $\forall u \in Q, \forall y \in F(u) \exists m \in Q, u^- \leq m \leq u$ such that $F(m \leq u)(y) \in range(\eta_m)$ iff $y \in range(\eta_u)$*

**Remark 4.3.15.** *Note that in [MS98, Definition 2.6] the authors define a notion of dense. Nevertheless, we were not able to detect a discernible correlation between their definition and ours.*

Note that, since $m \leq u$, $y \in range(\eta_u) \implies F(m \leq u)(y) \in range(\eta_m)$.

It can be easily verified that a sufficient condition to a morphism of sheaves $\eta : G \xrightarrow{\cong} F$ be a dense is: $\forall u \in Q \exists m \in Q, u^- \leq m \leq u$ such that the diagram below is a pullback:

$$\begin{tikzcd} G(u) \arrow[r, "\eta_u"] \arrow[d, "G(m\leq u)"'] & F(u) \arrow[d, "F(m\leq u)"] \\ G(m) \arrow[r, "\eta_m"] & F(m) \end{tikzcd}$$

**Example 4.3.16.** *(Dense morphisms of sheaves)*

1. *Every isomorphism $\eta : G \xrightarrow{\cong} F$ is a dense (mono)morphism.*
2. *If a point $\pi : 1 \to F$ is such that $\forall u \in Q \exists m \in Q, u^- \leq m \leq u, F(m \leq u) : F(u) \to F(m)$ is bijective, then $\pi : 1 \to F$ is a dense monomorphism. In particular, every point $\pi : 1 \to \Omega^-$ is a dense monomorphism.*
3. *Let $a, b \in Q$. If $b \leq a$, let $\eta : Q(-, b) \to Q(-, a)$ be the unique monomorphism (an inclusion, in fact). Then $\eta$ is a dense monomorphism iff $\forall u \in Q \forall y \in [u, a] \exists m \in Q, u^- \leq m \leq u, (y \in [m, b] \Leftrightarrow y \in [u, b])$; therefore, taking $m = u$, we have that $Q(-, b) \hookrightarrow Q(-, a)$ is a dense inclusion.*

We register the following (straightforward) result:

**Proposition 4.3.17.** *A pullback of a dense (mono)morphism in $Sh(Q)$ is a dense (mono)morphism.*

**Theorem 4.3.18.** *Suppose that $Q$ is a (commutative, semicartesian and) geometric quantale. Then the sheaf $\Omega^-$ introduced in Proposition 4.3.12 essentially classifies the dense subobject in the category $Sh(Q)$. More precisely:*

1. *$\top : 1 \to \Omega^-$, given by $\top_u : \{*\} \to \Omega^-(u), \top_u(*) = u^-$ determines a dense monomorphism in $Sh(Q)$.*
2. *For each dense monomorphism of sheaves $m : S \rightarrowtail F$, there is a unique morphism $\chi_m : F \to \Omega^-$, such that $\chi_m^- = \chi_m$, and such the diagram below is a pullback. Moreover, for each morphisms $\phi, \phi' : F \to \Omega^-$ that determine pullback diagrams, it holds: $\phi^- = \phi'^-$.*

*Proof.* 1. Since $u^-$ is an idempotent, then $u^- \in \Omega^-(u)$ (in fact, $u^- = max\Omega^-(u)$). If $v \leq u$ then, by Proposition 3.1.9.9 $v^- \odot u^- = v^-$, thus $\top = (\top_u)_{u \in Q}$ is a natural transformation. By Proposition 3.2.19.4, it is clear that $\top$ is a monomorphism of sheaves. In Example 4.3.16.2, we argued that $\top$ is dense.

2. First note that: Since $\top$ is a dense monomorphism, it follows from Proposition 4.3.17 that the pullback of a morphism $\phi : F \to \Omega^-$ through $\top$ must be a *dense* monomorphism $m : S \rightarrowtail F$.

   Now, note that it is enough to establish the result for dense subsheaves $i_S : S \hookrightarrow F$. We will split the proof into two parts, but first we will provide some relevant definitions and calculations.

   For each $u \in Q$ and $y \in F(u)$, define:

$$\langle y, u \rangle := \{v \in \Omega^-(u) : F(v \leq u)(y) \in S(v)\};$$
$$u_y := \bigvee \langle y, u \rangle.$$

   (a) If $v, w \in \Omega^-(u)$ and $w \leq v$, then $v \in \langle y, u \rangle \Rightarrow w \in \langle y, u \rangle$: $S$ is a subpresheaf of $F$. In particular: if $v \in \langle y, u \rangle$, then $v^- \in \langle y, u \rangle$, since $(\,)^- : \Omega^- \to \Omega^-$ natural transformation, and $v^- \leq v$.

(b) If $\{v_i : i \in I\} \subseteq \langle y, u\rangle$, then $\bigvee_i v_i \in \langle y, u\rangle$: since $\Omega^-(u)$ is closed under suprema and $S$ is a subsheaf of $F$.

(c) $u_y \in \langle y, u\rangle$ (by (b)) and $u_y^- \in \langle y, u\rangle$ (by (a)). Thus:

$$u_y = max\langle y, u\rangle \text{ and } u_y^- = max(\langle y, u\rangle \cap Idem(Q)).$$

(d) $u_y^- = \bigvee\{e \in Idem(Q) : \exists v, v \preceq u^-, e = v^-, F(v \leq u)(y) \in S(v)\}$ : since $Q$ is a geometric quantale, we have

$$u_y^- = (\bigvee\{v \in \Omega^-(u) : F(v \leq u)(y) \in S(v)\})^- = \bigvee\{v^- : v \in \Omega^-(u), F(v \leq u)(y) \in S(v)\}.$$

Candidate and uniqueness: Suppose that $\phi : F \to \Omega^-$ is a natural transformation such that the diagram below is a pullback (where $S$ is dense subsheaf of $F$).

$$\begin{tikzcd} S \arrow[r, "!_S"] \arrow[d, hook, "i_S"'] & 1 \arrow[d, "\top"] \\ F \arrow[r, "\phi"'] & \Omega^- \end{tikzcd}$$

Note that if $u^- \leq m \leq u$, then, by naturality,

$$\phi_m(F(m \leq u)(y)) = \phi_u(y) \odot m^- = \phi_u(y) \odot u^- = \phi_u(y).$$

**Claim (i):** It holds: $u_y^- \leq \phi_u(y) \leq u_y$. Moreover, if $\phi_u(y) \in Idem(Q)$, then $\phi_u(y) = u_y^-$.

Since the diagram is a pullback and limits in $Sh(Q)$ are pointwise (see Proposition 3.2.17.1), then for each $w \in Q$:

$$x \in S(w) \Leftrightarrow x \in F(w) \textit{ and } \phi_w(x) = w^-$$

Thus, if $v \leq u$ is such that $F(v \leq u)(y) \in S(v)$, then by naturality:

$$v^- = \phi_v(F(v \leq u)(y)) = \phi_u(y) \odot v^-.$$

Note that $u_y \leq u^- \leq u$ and $u_y \in \langle y, u\rangle$, thus $u_y^- = \phi_u(y) \odot u_y^-$ and $u_y^- \leq \phi_u(y)$.

By naturality: $\phi_{\phi_u(y)}(F(\phi_u(y), u)(y))) = \phi_u(y) \odot \phi_u(y)^- = \phi_u(y)^-$

$\phi_u(y) \in \langle y, u\rangle$: since $\phi_u(y) \in \Omega^-(u)$ and $\phi_{\phi_u(y)}(F(\phi_u(y), u)(y))) = \phi_u(y)^-$ then, by the pullback condition, we have that $F(\phi_u(y), u)(y)) \in S(\phi_u(y))$, thus $\phi_u(y) \in \langle y, u\rangle$.

$\phi_u(y) \leq u_y$: since $\phi_u(u) \in \langle y, u\rangle$ and $u_y = max\langle y, u\rangle$.

If $\phi_u(y) \in Idem(Q)$, then $\phi_u(y) = u_y^-$: since we have established above that $\phi_u(y) \in \langle y, u\rangle$, $u_y^- \leq \phi_u(y) \leq u_y$ and because $u_y^- = max(\langle y, u\rangle \cap Idem(Q))$.

Thus, if $\phi_u(y) \in Idem(Q)$, then $\phi_{\phi_u(y)}(F(\phi_u(y), u)(y))) = \phi_u(y) = u_y^-$.

**Claim (ii):** If $\phi : F \to \Omega^-$ determines a pullback diagram, then $\phi^- = (\,)^- \circ \phi$ still determines a pullback.

Since $x \in S(w)$ iff ($x \in F(w)$ and $\phi_w(x) = w^- = (w^-)^- = \phi^-_w(x)$).

Combining Claim (ii) and Claim (i), $\phi^-_u(y) = u^-_y$ for each $u \in Q$ and $y \in F(u)$, establishing the required uniqueness assertions.

Existence: For each $u \in Q$ and $y \in F(u)$, define $\chi^S_u(y) := u^-_y$. Then $(\chi^S_u)_{u\in Q}$ is a natural transformation and it determines a pullback diagram.

Firstly, we will verify that $(\chi^S_u)_{u\in Q}$ is a natural transformation. Let $u, v \in Q$ be such that $v \leq u$ and let $y \in F(u)$. We have to show that:

$$\chi^S_v(F(v \leq u)(y)) = \chi^S_u(y) \odot v^-$$

This means:

$$max(Idem(Q) \cap \langle F(v \leq u)(y), v\rangle) = v^- \odot max(Idem(Q) \cap \langle y, u\rangle)$$

On the one hand, note that

$$\begin{aligned} v^- \odot u^-_y &= v^- \odot max(Idem(Q) \cap \langle y, u\rangle) \\ &= v^- \odot \bigvee (Idem(Q) \cap \langle y, u\rangle) \\ &= \bigvee \{v^- \odot e \,:\, e^2 = e = e \odot u^-, F(eu)(y) \in S(e)\}. \end{aligned}$$

Denoting $e' := v^- \odot u^-_y$, we have $e'^2 = e' = e' \odot v^-$ and $F(e'v)(F(v \leq u)(y)) \in S(e')$, thus $e' = v^- \odot u^-_y \in Idem(Q) \cap \langle F(v \leq u)(y), v\rangle$ and then

$$max(Idem(Q) \cap \langle F(v \leq u)(y), v\rangle) \geq v^- \odot max(Idem(Q) \cap \langle y, u\rangle).$$

On the another hand, denote $e'' = max(Idem(Q) \cap \langle F(v \leq u)(y), v\rangle)$. Then $e''^2 = e'' = e'' \odot v^-$ and $F(e''v)(F(v \leq u)(y)) \in S(e'')$. Then $e'' \in Idem(Q), e'' \leq v^- \leq u^-$ and $e'' \in \langle y, u\rangle$. Thus $e'' \in Idem(Q)$ and $e'' \leq v^-, u^-_y$. Then $e'' = e'' \odot e'' \leq v^- \odot u^-_y$, i.e., $max(Idem(Q) \cap \langle F(v \leq u)(y), v\rangle) \leq v^- \odot max(Idem(Q) \cap \langle y, u\rangle)$.
Now we show that holds the pullback condition for each $u \in Q$:

$$y \in S(u) \;\Leftrightarrow\; (y \in F(u) \; and \; u^- = \chi^S_u(y) = u^-_y).$$

On one hand, let $y \in S(u)$, then $y \in F(u)$ and $u^- \in \Omega^-(u)$ is s.t. $F(u^- \leq u)(y) \in S(u^-)$, since $S$ is a subpresheaf of $F$. Then $u^- \in Idem(Q) \cap \langle y, u\rangle$. Thus, by (b), $u^- \leq u^-_y$. On the other hand $u^-_y \in \Omega^-(u)$, thus $u^-_y \leq u^-$. Summing up: $\chi^S_u(y) = u^-_y = u^-$.

Let $y \in F(u)$ be such that $u^- = \chi^S_u(y) = u^-_y$. Then $u^- = max(\langle y, u\rangle \cap Idem(Q))$. Therefore $F(u^- \leq u)(y) \in S(u^-)$ and, since $i_S : S \hookrightarrow F$ is a dense inclusion, we have $y \in S(u)$.

□

Subobject classifiers classify monomorphism, we proved that $\Omega^-$ essentially classifies dense monomorphisms, which is coherent with Corollary 4.3.10 since $Q$ do not necessarily has that meet distributes over joins, we do not expect to obtain a subobjects classifier. On the other hand, if we find a subobjects classifier for $Sh(Q)$ for a certain $Q$, thus we may have an example of a $Q$ where the meet distributes over arbitrary joins but the meet and the multiplication operation do not coincide. In a paper[6] recently submitted, we constructed a sheaf $\Omega^+(u) := \{q \in Q : q^+ \odot u = q\} = \{q \in Q : q^+ \odot u \leq q \text{ and } q \leq u\}$ with restriction maps, for $v \leq u$ in $Q$, defined by

$$\begin{aligned} \Omega^+(u) &\to \Omega^+(v) \\ q \quad &\mapsto q^+ \odot v \end{aligned}$$

and proved that for quantales with "good" properties, $\Omega^+$ is, indeed a subobject classifier in $Sh(Q)$. By Corollary 4.3.10, this means that the complete lattice $Q$ has also localic structure but, apparently, it does not imply that $\odot = \wedge$.

## 4.4 Considerations about semigroupal categories

As already mentioned, unital semicartesian quantales may be seen as semicartesian categories. While developing the theory of sheaves on quantales we realized that the unity is not always necessary. This suggests that our notion of a sheaf for monoidal categories with projections (therefore, semicartesian by Proposition 2.1.8) could be further generalized for *semigroupal categories* with projections. Semigroupal categories, as the name suggests, look like monoidal categories except they do not have a unity for the tensor product, thus, they do not have the triangle axiom. Explicitly,

**Definition 4.4.1.** *A **semigroupal category** consists of:*

- *A category $C$;*
- *A bifunctor $\otimes : C \times C \to C$ called the **tensor product**;*
- *A natural isomorphism $a : (- \otimes -) \otimes - \xrightarrow{\simeq} - \otimes (- \otimes -)$ with components*

  $$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\simeq} X \otimes (Y \otimes Z)$$

  *called the **associator (or associativity isomorphism)**.*

  *Such that the following axiom holds:*

- ***The pentagon axiom:** For all $W, X, Y, Z$ objects in $C$, the diagram below commutes*

[6] As mentioned before, there is a preliminary version of such paper available on ArXiv.

$$\begin{tikzcd} & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "a_{W,X,Y\otimes Z}"] & \\ ((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "a_{W\otimes X,Y,Z}"] \arrow[d, "a_{W,X,Y}\otimes id_z"'] & & (W \otimes (X \otimes (Y \otimes Z))) \\ (W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "a_{W,X\otimes Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "id_W \otimes a_{X,Y,Z}"'] \end{tikzcd}$$

Our quantales may have a unit or not but they must be semicartesian, that is, we need a notion of projection. So:

**Definition 4.4.2.** *A* ***semigroupal category with projections*** *is a semigroupal category equipped with two natural transformations*

- *A natural transformation* $\pi^1 \ : \ (- \otimes -) \longrightarrow (-)$ *with components*

$$\pi^1_{X \otimes Y} : X \otimes Y \to X$$

*called the* ***projection onto the first coordinate****;*

- *A natural transformation* $\pi^2 \ : \ (- \otimes -) \longrightarrow (-)$ *with components*

$$\pi^2_{X \otimes Y} : X \otimes Y \to Y$$

*called the* ***projection onto the second coordinate****.*

*Such that*

$$\begin{tikzcd} (X \otimes Y) \otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "\pi^1_{(X\otimes Y),Z}"'] & X \otimes (Y \otimes Z) \arrow[d, "\pi^1_{X,Y\otimes Z}"] \\ X \otimes Y \arrow[r, "\pi^1_{X,Y}"'] & X \end{tikzcd} \qquad \begin{tikzcd} (X \otimes Y) \otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "\pi^2_{(X\otimes Y),Z}"'] & X \otimes (Y \otimes Z) \arrow[d, "\pi^2_{X,Y\otimes Z}"] \\ Z & Y \otimes Z \arrow[l, "\pi^2_{Y,Z}"] \end{tikzcd}$$

*commute.*

**Remark 4.4.3.** *Furthermore, if there is a unit $I$, there is no obligation to $\pi^1_{X,I} \ : \ X \otimes I \to X$ or $\pi^2_{X,I} \ : \ I \otimes X \to X$ be the unitors isomorphisms that appear in the definition of monoidal categories.*

Note that every semicartesian monoidal category (when the unit coincides with the terminal object) is a semigroupal category with projections, where the projections are compositions of the form $X \otimes Y \xrightarrow{id_X \otimes !} X \otimes I \xrightarrow[\cong]{\rho_X} X$ . According to the discussion following the Definition of semicartesian categories 2.1.6, if $\pi^1_{X,I} \ : \ X \otimes I \to X$ coincides with the unitor isomorphism then monoidal categories with projections are precisely the semicartesian monoidal categories. This is precisely the case in the poset category of quantales: the projection must coincide with the unitor isomorphism because there is a unique arrow $X \otimes I \to X$, if the quantale is unital.

We highlight that the notion of a monoidal category without unit is not unmotivated since, for example, given a unital semicartesian quantale $Q$ we may obtain a non-unital semicartesian quantale $[0, a]$ that is isomorphic to $Sub(Q(0, a))$, by Theorem 4.3.6.

Unfortunately, the study of semigroupal categories is not as well-developed as the study of monoidal categories. Furthermore, for monoidal categories with projections we know what the projections look like, which helps with calculations while for semigroupal categories with projections we can only use the commutativity of certain diagrams given in Definition 4.4.2. In the future, we would like to check if the results we found out for sheaves on semicartesian categories still hold for sheaves on other semigroupal categories with projections.

## 4.5 Sheaf Cohomology

The title of this Section corresponds to a desire: develop a cohomology theory for Grothendieck lopos that generalizes the well-known cohomology theory for Grothendieck topos and find more applications of sheaf cohomology. What we have for now is not a generalization of a sheaf cohomology theory. Instead, we generalized a standard technique in cohomology known as Čech Cohomology, which was briefly described in 2.4. In this section we:

- introduce a notion of monoidal Čech Cohomology. The idea is simple: we take the Čech Cohomology of a topological space $X$ with coefficients on a sheaf on $X$ and replace intersection by the quantalic product. In particular, we are interested in the monoidal Čech Cohomology of a commutative ring $R$ with coefficients on a sheaf on $R$.
- use base change techniques introduced in Section 3.4, to relate the Čech cohomology of a topological space $X$, with coefficients in a constant sheaf on $Sh(X)$, and the Čech Cohomology of the ring of continuous functions $C(X)$, with coefficients in a constant sheaf on $Sh(C(X))$ that arises from the chosen constant sheaf on $Sh(X)$.

**Remark 4.5.1.** *In this section, our sheaves are all sheaves of abelian groups, that is, they are of the form $F : Q^{op} \to Set$, where $Q$ is a semicartesian commutative quantale.*

Let $F : Q^{op} \to Ab$ be a sheaf and $\mathcal{U} = (u_i)_{i \in I}$ a cover in $Q$, where $I$ is a set of indices.

**Definition 4.5.2.** *The* ***monoidal Čech cochain complex*** *of $\mathcal{U}$ with coefficients in $F$ is given by*

$$C^q(\mathcal{U}, F) = \prod_{i_0 < \ldots < i_q} F(u_{i_0} \odot \cdots \odot u_{i_q})$$

*and the* ***coboundary morphism*** *$d^q : C^q(\mathcal{U}, F) \to C^{q+1}(\mathcal{U}, F)$ by*

$$(d^q \alpha) = \sum_{k=0}^{q+1} (-1)^k \alpha(\delta_k)|_{u_{i_0} \odot \cdots \odot u_{i_{q+1}}},$$

*where $\delta_k$ means that we removed the $i_k$-entry.*

Observe that if $Q = (\mathcal{O}(X), \subseteq, \cap)$, then the definition above is precisely the usual Čech cochain complex.

A straightforward verification shows that $d^{q+1} \circ d^q = 0$ so, indeed, this is a cochain complex and we define the $q$-th Čech cohomology group of $F$ with respect to the covering $\mathcal{U}$ in the following way:

**Definition 4.5.3.** *Given a cover $\mathcal{U}$ and a sheaf $F$, we define the **monoidal Čech cohomology group** of $\mathcal{U}$ with coefficients in $F$ by*

$$\check{H}^q(\mathcal{U}, F) = \frac{Ker d^q}{Im d^{q-1}}$$

This construction is about the covering and not about the actual space. When $Q = (\mathcal{O}(X), \subseteq, \cap)$, the topological space $X$ may admit many covers so this construction is not enough to talk about the cohomology of $X$, but only about the cohomology of a fixed cover of $X$. There is a way to define a Čech cohomology of $X$ and, analogously, we can define a monoidal Čech cohomology of a commutative ring with unity $R$. Before we get deeper into the theory, we explore the application we are interested in.

For convenience, we recall the pair of adjoint functors

$$\begin{aligned} \tau : \ \mathcal{I}(C(X)) &\to \mathcal{O}(X) \\ I &\mapsto \bigcup_{f \in I} f^{-1}(\mathbb{R} - \{0\}) \end{aligned}$$

and

$$\begin{aligned} \theta : \ \mathcal{O}(X) &\to \mathcal{I}(C(X)) \\ U &\mapsto \langle \{ f \ : \ f{\restriction}_{X-U} \equiv 0 \} \rangle \end{aligned}$$

that we introduced in 3.4.14.

**Proposition 4.5.4.** *Fix a cover $\mathcal{U}$ of $C(X)$. Then the Čech cohomology group of $\tau(\mathcal{U})$ with coefficients in the constant sheaf $\underline{K}_X$ is isomorphic to the monoidal Čech cohomology group of $\mathcal{U}$ with coefficients in the constant sheaf $\underline{K}_{C(X)}$.*

*Proof.* By Corollary 3.4.16,

$$\begin{aligned} C^q(\mathcal{U}, \underline{K}_{C(X)}) &= \prod_{i_0 < \ldots < i_q} \underline{K}_{C(X)}(u_{i_0} \odot \ldots \odot u_{i_q}) \\ &= \prod_{i_0 < \ldots < i_q} (\underline{K}_X \circ \tau)(u_{i_0} \odot \ldots \odot u_{i_q}) \\ &= \prod_{i_0 < \ldots < i_q} \underline{K}_X(\tau(u_{i_0}) \cap \ldots \cap \tau(u_{i_q})) \\ &= C^q(\tau(\mathcal{U}), \underline{K}_X) \end{aligned}$$

Then we have commutative squares

$$\begin{tikzcd}
\dots \arrow[r] & C^{q-1}(\mathcal{U},\mathbb{Z}_{C(X)}) \arrow[r, "d^{q-1}"] \arrow[d] & C^{q}(\mathcal{U},\mathbb{Z}_{C(X)}) \arrow[r, "d^{q}"] \arrow[d] & C^{q+1}(\mathcal{U},\mathbb{Z}_{C(X)}) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & C^{q-1}(\tau(\mathcal{U}),\mathbb{Z}_{X}) \arrow[r, "d^{q-1}"'] & C^{q}(\tau(\mathcal{U}),\mathbb{Z}_{X}) \arrow[r, "d^{q}"'] & C^{q+1}(\tau(\mathcal{U}),\mathbb{Z}_{X}) \arrow[r] & \dots
\end{tikzcd}$$

Thus, we have an isomorphism of cochain complexes and so the cohomology groups are isomorphic: $\check{\mathrm{H}}^q(\mathcal{U},\underline{K}_{C(X)}) \cong \check{\mathrm{H}}^q(\tau(\mathcal{U}),\underline{K}_X)$. □

Since the above holds for any cover $\mathcal{U}$ of $C(X)$, we expect that something at least similar will happen between the cohomology groups of $C(X)$ and $X$. First:

**Definition 4.5.5.** *Fix a quantale $Q$ and an element $u \in Q$. Let $\mathcal{U} = (u_i)_{i\in I}$ and $\mathcal{V} = (v_j)_{j\in J}$ be coverings of $u$. We say that $\mathcal{U}$ is a* ***refinement*** *of $\mathcal{V}$ if there is a function $r : I \to J$ and a morphism $u_i \to v_{r(i)}$, for all $i \in I$.*

Given $r : I \to J$ that testifies $\mathcal{U}$ as refinement of $\mathcal{V}$, we have an induced morphism of cochain complexes $m_r : C^{\bullet}(\mathcal{V},F) \to C^{\bullet}(\mathcal{U},F)$ and a corresponding morphism of Čech cohomology groups with respect to the coverings $\mathcal{U}$ and $\mathcal{V}$, $\check{m}_r : \check{\mathrm{H}}^{\bullet}(\mathcal{V},F) \to \check{\mathrm{H}}^{\bullet}(\mathcal{U},F)$. Moreover, if $s : I \to J$ is another chosen function with respect to the refinement that testifies that $\mathcal{U}$ is a refinement of $\mathcal{V}$, then the induced morphisms of complexes $m_r, m_s$ are homotopic. Therefore, there is a unique induced morphism of cohomology groups $\check{m}_{\mathcal{U},\mathcal{V}} : \check{\mathrm{H}}^{\bullet}(\mathcal{V},F) \to \check{\mathrm{H}}^{\bullet}(\mathcal{U},F)$

Besides it, the class $\mathrm{Ref}(u)$ of all coverings of $u$ is partially ordered under the refinement relation; this is a directed ordering relation. Thus, we can define

**Definition 4.5.6.** *The* ***Čech cohomology group of an element*** *$u \in Q$ with coefficient in a sheaf $F$ is the directed (co)limit*[7]

$$\check{\mathrm{H}}^q(u,F) := \varinjlim_{\mathcal{U}\in Ref(u)} \check{\mathrm{H}}^q(\mathcal{U},F).$$

Then we can prove the applied theorem of this thesis:

**Theorem 4.5.7.** *The Čech cohomology group of $C(X)$ with coefficients in $\underline{K}_{C(X)}$ is isomorphic to the Čech cohomology group of $X$ with coefficients in $\underline{K}_X$.*

*Proof.* First, observe that by Proposition 4.5.4:

$$\begin{aligned}
\check{\mathrm{H}}^q(C(X),\underline{K}_{C(X)}) &:= \varinjlim_{\mathcal{U}\in \mathrm{Ref}(C(X))} \check{\mathrm{H}}^q(\mathcal{U},\underline{K}_{C(X)}) \\
&\cong \varinjlim_{\mathcal{U}\in \mathrm{Ref}(C(X))} \check{\mathrm{H}}^q(\tau(\mathcal{U}),\underline{K}_{X})
\end{aligned}$$

[7] This (co)limit has to be taken with some set-theoretical care that we do not detail.

Now, let $\mathcal{V} = (V_i)_{i \in I}$ be a covering of $X$. It is clear that $\theta(V_i) \subseteq \theta(V_i)$. Since $\tau$ is left adjoint to $\theta$ we obtain $\tau(\theta(V_i)) \subseteq V_i$. Recall that $\theta$ and $\tau$ preserve supremum (Proposition 3.4.14), thus $\tau(\theta(V_i))$ is a covering of $X$ and then $\tau(\theta(\mathcal{V}))$ it is a refinement of $\mathcal{V}$. In other words, for every $\mathcal{V}$ covering of $X$ there is a covering $\mathcal{U}$ of $C(X)$ such that $\tau(\mathcal{U}) \subseteq \mathcal{V}$. Therefore

$$
\begin{aligned}
\check{\mathrm{H}}^q(X, \underline{K}_X) &:= \varinjlim_{\mathcal{V} \in \mathrm{Ref}(X)} \check{\mathrm{H}}^q(\mathcal{V}, \underline{K}_X) \\
&\cong \varinjlim_{\mathcal{U} \in \mathrm{Ref}(C(X))} \check{\mathrm{H}}^q(\tau(\mathcal{U}), \underline{K}_X)
\end{aligned}
$$

$\square$

Perhaps, it is not clear why the above is interesting and why it shows the potential of the theory we developed. Once we defined sheaves on quantales and desire to make the notion of sheaves on rings similar to that of sheaves on topological spaces, it was straightforward how to define the correspondent Čech cohomology groups. However, it is not simple to actually calculate the cohomology of an arbitrary ring, mainly because of two reasons: (i) we have to choose some sheaf but finding concrete examples of sheaves on a ring (or in any quantale) was not easy. We thought about some presheaves examples and then we could sheafify them, but our sheafification process is too much abstract. Thus, even the behavior of the constant sheaf was not easy to capture; (ii) given an arbitrary ring, it may be difficult to understand what to expect from the covering of its ideals and we were not able to find studies about it in the literature.

When we had almost given up on the applied part of this thesis, we realized that Theorem 3.4.8 is also a machine for producing sheaves on quantales. In particular, if we have a sheaf on a locale $L$ and a pair of adjoint functors $Q \underset{f_*}{\overset{f^*}{\rightleftarrows}} L$ then $F \circ f^*$ is sheaf on the quantale $Q$. So we can use well-known sheaves on locales to create sheaves on quantales. Moreover, we hoped that if we had a ring that come from a topological space, then we could indirectly calculate the cohomology of the ring by calculating the cohomology of the space, which probably is already known since Čech cohomology of a topological space is a topic that has been studied for a longer time. The above Theorem is an example of such a phenomenon and, surprisingly or not, the proof relied basically on finding an adjoint pair of functors between the ideals of $C(X)$ and the open subsets of $X$.

Besides it, Theorem 4.5.7 is not only about finding ways to calculate the Čech cohomology groups of $C(X)$, it is also about relating algebraic properties of $C(X)$ and topological properties of $X$: if $X$ is a compact manifold $n$ and class at least $C^{n+1}$ then there is an isomorphism $H^q_{dR}(M) \cong \check{H}^q(M, \mathbb{R})$, for all $q \leq m$, where $H^q_{dR}$ denotes the de Rham cohomology groups and $\mathbb{R}$ is the constant sheaf with values in $\mathbb{R}$ [Pet06, Appendix]. On one hand, the dimension of $H^0_{dR}(M)$ corresponds to the number of connected components of $X$. On the other hand, $X$ is connected if and only if $C(X)$ only had trivial idempotent elements (0 and 1). So, we believe that the Čech cohomology group in degree zero of a ring with coefficients in a constant sheaf is related to the number of idempotent elements of $C(X)$.

The idea of investigating algebraic properties of $C(X)$ by analyzing topological properties of $X$ (and vice-versa) is not new ([GH54], [Hew48], [GJ17]), but the cohomological approach is not usual: in fact, we only find one paper with such an approach. In [Wat65], Watts creates a cohomology theory for a commutative algebra over a fixed algebra such that the case of the algebra of continuous real-valued functions on a compact Hausdorff space coincides with the Čech cohomology of the space with real coefficients (the constant sheaf with values in $\mathbb{R}$). Our approach may have a couple of advantages:

- we do not have to construct a new cohomology theory, we actually are expanding Čech cohomology in a quite natural manner;
- our space $X$ does not have to be compact, we only need that $X$ admits partition of unity subordinate to a cover and paracompactness is enough for it;
- we provide a general framework to investigate another algebro-geometric phenomenon that relies on finding "good" functors between the locale of open subsets of a space and the quantale of the ideals of a ring that arises from such space. Conversely, it is also possible to use the exact idea but starting with a ring and establishing a space that arises from such ring, as the process of taking the spectrum of a ring.
- Watts says that "it is not easy to see how to remove the restriction to real coefficients, and we have made no attempt to do so". In our case, it is clear how to proceed to change the coefficient, as we see below.

**Theorem 4.5.8.** *The Čech cohomology group of $X$ with coefficients in $F$ is isomorphic to the Čech cohomology group of $C(X)$ with coefficients in $F \circ \tau$.*

*Proof.* Observe that

$$\begin{aligned} C^q(\tau(\mathcal{U}), F) &= \prod_{i_0<\ldots<i_q} F(\tau(u_{i_0}) \cap \ldots \cap \tau(u_{i_q})) \\ &= \prod_{i_0<\ldots<i_q} (F \circ \tau)(u_{i_0} \odot \ldots \odot u_{i_q}) \\ &= C^q(\mathcal{U}, F \circ \tau) \end{aligned}$$

Then $\check{\mathrm{H}}^q(\mathcal{U}, F \circ \tau) \cong \check{\mathrm{H}}^q(\tau(\mathcal{U}), F)$.

By the same reasoning of the proof of Theorem 4.5.7, $\check{\mathrm{H}}^q(C(X), F \circ \tau) \cong \check{\mathrm{H}}^q(X, F)$. □

**Remark 4.5.9.** *Note that we are using the unity/top elements $1$ and $1'$ because usually one talks about the Čech cohomology of a topological space $X$ and $X$ is the unity/top element of $(\mathcal{O}(X), \subseteq)$.*

This theorem gives generality but obscures how to interpret the result, since coefficients in different sheaves lead to different cohomology theories. De Rham Cohomology measures to which extent the Stokes Theorem fails; Singular homology measures the number of holes of $X$ and it is related to singular cohomology by the universal coefficient theorem for cohomology; now, without a specific sheaf in mind, is more difficult to have a first guess of what the Čech cohomology groups of a ring are measuring.

Nevertheless, we have a result that generalizes the above phenomenon.

**Theorem 4.5.10.** *Consider a strong geometric morphism* $(Q, \odot, 1) \underset{f_*}{\overset{f^*}{\rightleftarrows}} (Q', \odot', 1')$ *such that* $f_*$ *preserves unity and arbitrary joins. Then* $\check{\mathrm{H}}^q(1', F \circ f^*) \cong \check{\mathrm{H}}^q(1, F)$.

*Proof.* Consider a covering $\{u_i'\}_{i\in I} = \mathcal{U}'$ in $Q'$. Then

$$\begin{aligned} C^q(\mathcal{U}', F \circ f^*) &= \prod_{i_0 < \ldots < i_q} F \circ f^*(u'_{i_0} \odot' \ldots \odot' u'_{i_q}) \\ &= \prod_{i_0 < \ldots < i_q} F(f^*(u'_{i_0}) \odot \ldots \odot f^*(u'_{i_q})) \\ &= C^q(f^*(\mathcal{U}'), F). \end{aligned}$$

Then $\check{\mathrm{H}}^q(\mathcal{U}', F \circ f^*) \cong \check{\mathrm{H}}^q(f^*(\mathcal{U}'), F)$.

Observe that $f^*(f_*(\mathcal{U}))$ is a refinement of $\mathcal{U}$, by the same argument used in Theorem 4.5.7. Thus, for every covering $\mathcal{U}$ of 1 there is a covering $\mathcal{U}'$ of $1'$ such that $f^*(\mathcal{U}') \subseteq \mathcal{U}$. Then, $\varinjlim_{\mathcal{U} \in \mathrm{Ref}(1)} \check{\mathrm{H}}^q(\mathcal{U}, F) \cong \varinjlim_{\mathcal{U}' \in \mathrm{Ref}(1')} \check{\mathrm{H}}^q(\mathcal{U}', F \circ f^*)$, as desired to obtain the result. □

Again, the conclusion is that the isomorphism between Čech cohomology group relies exclusively on the properties between the quantales involved. The pair $Q \underset{(-)^-}{\overset{i}{\rightleftarrows}} Idem(Q)$ where $i$ is the inclusion and $(-)^-$ is the idempotent approximation is another pair of adjoint functors that satisfies the hypothesis of the above theorem, if $Q$ is a geometric quantale (see Propositions 3.1.10 and 3.1.9). Recall that a surjective ring homomorphism $f : R \to S$ induces a strong geometric morphism of quantales where $f^*(J) = f(J)$ and $f_*(K) = f^{-1}(K)$ (see Example3.4.6). Consider the quotient map $q : R \to R/I$ defined by $q(r) = r + I$. So, it remains to prove that $q_*$ induced by the pre-image preserves unity and supremum to have another class of examples to apply the above theorem. Observe that for any ideal $K$ of $R/I$ we have, for some $J$ ideal of $R$,

$$q_*(K) = q_*(q(J)) = q_*(\{j + I \ : \ j \in J\}) = \bigcup_{j \in J} (j + I) = J + I.$$

Then, $q_*(R/I) = q_*(q^*(R)) = I + R = R$ and

$$q_*(\sum_{i=1}^n K_i) = q_*(\sum_{i=1}^n q(J_i)) = q_*(q \sum_{i=1}^n (J_i)) = I + \sum_{i=1}^n J_i = \sum_{i=1}^n (I + J_i) = \sum_{i=1}^n q_*(q^*(J_i)) = \sum_{i=1}^n q_*(K_i),$$

as desired.

Note that if $R$ and $S$ are Morita equivalent commutative rings, then $R$ and $S$ are isomorphic, providing another case of adjoint pair between the ideals of $R$ and the ideals of $S$, and thus an isomorphism between the cohomology of $R$ and the cohomology of $S$. Thus, our cohomology admits a Morita in the trivial case of commutative rings. We believe a next interesting application is to investigate the relation between the quantales of (bilateral) ideals of Morita equivalent non-commutative rings. However, we need to check if our constructions and theorems hold for non-commutative quantales.

# Chapter 5

# Conclusions and Future Work

We recognize that the theory of sheaves on Grothendieck prelopologies is in its very first steps. We were able to make some effective progress for sheaves on (semicartesian) quantales. We have unsuccessfully tried to construct the sheafification in a perhaps more concrete approach – as, for instance, establishing stalks for the sheaf on the ideals of a commutative ring. Nevertheless, the general framework helped us to prove the existence of the sheafification functor, permitting some of the manipulations regarding base change and Čech cohomology, showing that it preserves the (closed) monoidal structure at least for $a : PSh(Q) \to Sh(Q)$, and then leading to interesting conclusions about $Sh(Q)$ not being a topos nor having subobject classifier, in general. Therefore, even though we are still understanding the behavior of our sheafification, it already provided some interesting results. Actually, they are the main results of this thesis.

The quantalic case and the notion of Grothendieck pretopologies were our only working examples. On one side, that was the initial goal of this thesis: create a notion of sheaf general enough to encompass the well-known sheaves for a Grothendieck pretopology and our notion of sheaves on quantales. On the other side, there is room for changes in the definition of Grothendieck prelopologies and we believe that such changes, if any, should be motivated by examples since we want the theory to be as useful as possible. A clear next step in looking for monoidal categories that admit a certain natural candidate of covering but that had to be modified to fit into the axioms of Grothendieck pretolopogies. This is the case of the category of bornological coarse spaces since the first guess is to use covering families given by coarsely excisive pairs but a pullback may not preserve coarsely excisive pairs [BE16, Remark 3.1].

Even regarding sheaves on quantales, there is a long journey to understand the theory and its applications, including the task of finding more interesting examples of semicartesian sheaves – in this matter, we plan to investigate fuzzy topological spaces and possibly use our sheaves on quantales to have a theory of sheaves on certain fuzzy topological spaces in the same vein as the theory of sheaves on topological spaces.

We hope that what we accomplished can open doors for other exciting topics of research. As so, we conclude this thesis with a sequence of short sections about different aspects of our work that we want to develop in the future.

## 5.1 Noncommutative versions

When we introduced Grothendieck prelopologies 4.1 we mentioned that the reader could consider just one side of axioms 3 and 4 to reason about noncommutative versions of coverings, if the category $(C, \otimes, 1)$ was not symmetric. We had in mind, for example, noncommutative $R$-algebras and how they relate with Connes' ideas in noncommutative topology and possible applications in Hochschild and cyclic (co)homology.

Here, we just want to reinforce that if the tensor product is not symmetric then we may talk about left and right prelopologies, that is, we may break the following axioms

- If $\{f_i : U_i \to U\}_{i\in I} \in L(U)$, then $\{f_i \otimes id_V : U_i \otimes V \to U \otimes V\}_{i\in I}$ is in $L(U \otimes V)$ and $\{id_V \otimes f_i : V \otimes U_i \to V \otimes U\}_{i\in I}$ is in $L(V \otimes U)$, for any $V$ object in $C$.
- If $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ is in $L(U)$ and $g : V \to U$ is any morphism in $C$, then $\{\phi_i : U_i \ {}_{f_i}\!\otimes_g V \to Eq(\pi_1, g \circ \pi_2)\}_{i\in I}$ is in $L(Eq(\pi_1, g \circ \pi_2))$ and $\{\phi_i : V \ {}_g\!\otimes_{f_i} U_i \to Eq(\pi_2, g \circ \pi_1)\}_{i\in I}$ is in $L(Eq(\pi_2, g \circ \pi_1))$..

that appear in the definition of a Grothendieck prelopology (2.3.1) into two. One to define a right Grothendieck prelopology

- If $\{f_i : U_i \to U\}_{i\in I} \in L(U)$, then $\{f_i \otimes id_V : U_i \otimes V \to U \otimes V\}_{i\in I}$ is in $L(U \otimes V)$, for any $V$ object in $C$.
- If $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ is in $L(U)$ and $g : V \to U$ is any morphism in $C$, then $\{\phi_i : U_i \ {}_{f_i}\!\otimes_g V \to Eq(\pi_1, g \circ \pi_2)\}_{i\in I}$ is in $L(Eq(\pi_1, g \circ \pi_2))$.

and another to define a left Grothendieck prelopology:

- If $\{f_i : U_i \to U\}_{i\in I} \in L(U)$, then $\{id_V \otimes f_i : V \otimes U_i \to V \otimes U\}_{i\in I}$ is in $L(V \otimes U)$, for any $V$ object in $C$.
- If $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ is in $L(U)$ and $g : V \to U$ is any morphism in $C$, then $\{\phi_i : V \ {}_g\!\otimes_{f_i} U_i \to Eq(\pi_2, g \circ \pi_1)\}_{i\in I}$ is in $L(Eq(\pi_2, g \circ \pi_1))$.

The idea is that if $(C, \otimes, 1)$ is not symmetric then a set of families $\{U_i \to U\}_{i\in I}$ defines a right Grothendieck prelopology if and only if it defines a left Grothendieck prelopology. Moreover, note that the definition 4.1 of a sheaf for a Grothendieck prelopology can be used even if the category is not symmetric, but in practice, we have to be more careful since the terms $F(U_i \ {}_{f_i}\!\otimes_{f_j} U_j)$ and $F(U_j \ {}_{f_j}\!\otimes_{f_i} U_i)$ may be distinct.

We hope that this kind of strategy may be useful to relate a noncommutative version of the Čech cohomology we constructed and Cyclic (co)homology, and maybe help to address some questions in noncommutative algebraic geometry. The first is explained in the next section. The second we explain here: we constructed a sheaf on the quantale of ideals of a commutative ring with unity $R$ that resemble the structure sheaf usually defined on the quantale of open subsets of $Spec(R)$. Actually, we proved that such sheaf on the quantale arises from the sheaf on the locale by a change base process. Thus, we have a way to talk about a structure sheaf (on the quantale) that does not require the use of $Spec(R)$. In noncommutative algebraic geometry, the first problem is to define $Spec(R)$ for a noncommutative ring $R$. Following our approach, we do not have to use $Spec(R)$.

However, now we would be working on the noncommutative quantale of bilateral ideals of $R$. Besides the need to check if our theory still works for noncommutative quantales, we also have to choose a class of rings that can be localized and how this impact the bilateral ideals of the ring. Of course, this is just the first step, since algebraic geometry is much more than the structure sheaf. Nevertheless, we want to highlight that it looks interesting that we can "forget" the spectrum of $R$ with no apparent loss of information, and such a phenomenon deserves to be better explored.

## 5.2 Sheaves with Algebraic Structure and Cohomology

We already mentioned that we desire a cohomology theory for the sheaves we are developing. At the beginning of the Ph.D, I hoped that $Ab(Sh(Q))$, would be an abelian category with enough injectives and that the development of the cohomology theory would be quite straightforward. This hope was fading and now I believe it is unlikely that $Ab(Sh(Q))$ will be an abelian category, as explained at the end of Section 3.5. Even though it is not clear how to fully formalize a cohomological theory, that is, identify if what we have is an abelian cohomology or not, we can play with constructions that may have a central role. For instance, the functor of global sections.

**Definition 5.2.1.** *Let $F$ be a sheaf in $Sh(Q)$, we define the* **global sections functor** *of the terminal object* $1$ *of $Q$ by $\Gamma_1(F) = F(1) \cong Hom_{Sh(Q)}(\mathbf{1}, F)$ where* $\mathbf{1}$ *denotes the terminal sheaf.*

The above induces a functor $\Gamma_Q^{Ab} : Ab(Sh(Q)) \to Ab(Set)$, analogously defined by $\Gamma_1^{Ab}(F) = F(1) \cong Hom_{Sh_{Ab}(Q)}(0, F)$

**Proposition 5.2.2.** *The global sections functor $\Gamma_1^{Ab}$ is left exact.*

*Proof.* Consider a short exact sequence $0 \to F \xrightarrow{f} G \xrightarrow{g} H \to 0$ in $Sh_{Ab}(Q)$. We want to prove that

$$0 \to F(1) \xrightarrow{f_1} G(1) \xrightarrow{g_Q} H(Q)$$

is exact.

Since $Ker f = 0$ and $Ker(f_1) = (Ker f)(1)$, we have that $Ker(f_1) = 0$.

If $s \in Ker(g_1)$, then $s$ is a section of $G(1) = Hom_{Sh_{Ab}(Q)}(1, G)$ such that $g_1(s) = 0$. That is, $s$ is a sheaf morphism that maps to $G$ and satisfies $g \circ s = 0$. Therefore, for every $u \in Q$, we have $g(s|_u) = 0$. In other words, $s|_u \in Ker g = Im f$. Since $f$ is a sheaf morphism from $F$ to $G$, we conclude that $s \in F(1)$.

On the other hand, given $s \in Im(f_1)$, we have $s = f \circ r$ for some $r \in F(1)$. Thus, $s|_u = f \circ r|_u$. Then $g(s|_u) = 0$ by the exactness of the sequence. Since $G$ is a sheaf, it follows that $s \in Ker g_1$. □

Of course, $\Gamma_1^{Ab}$ is not exact since it is not exact in the localic case. Then we want to calculate how far they are from being exact by calculating the cohomology

**Definition 5.2.3.** *The cohomology groups of the terminal object* $1$ *of* $Q$ *with coefficient in* $F$ *are defined by right derived functors of the global sections functor*

$$H^q(1, F) \cong R^q\Gamma_1(I^\bullet)$$

*where* $I^\bullet$ *is an injective resolution of* $F$.

Note that

**Proposition 5.2.4.** *Let* $F$ *be an abelian sheaf, then* $\check{H}^0(1, F) = \Gamma_1(F)$

*Proof.* By definition, $\check{H}^0(1, F) = Ker(d^0 : C^0(1, F) \to C^1(1, F))$. Consider an element $\alpha \in C^0(U, F)$ such that $\alpha = \{\alpha_i \in F(u_i)\}_{i \in I}$. Thus, for each pair of indices $i < j$, we have $(d\alpha)_{ij} = \alpha_j - \alpha_i$. Then

$$\alpha \in Ker(d^0) \iff \alpha_j - \alpha_i = 0 \iff \alpha_i\restriction_{u_i \odot u_j} = \alpha_j\restriction_{u_i \odot u_j} \iff \alpha \in F(u)$$

where the last implication follows from $\alpha_i \in F(u_i)$ if $F$ is a sheaf. □

The next step is to check if given an injective resolution $F \to I^\bullet$, then $\check{H}^q(1, F) = R^q\Gamma_1(I^\bullet)$, under mild conditions. We did not adventure that far but we do have further considerations regarding applications of such ideas in other cohomology theories.

We recall that Čech cohomology of $X$ is isomorphic to Singular and de Rham cohomology of $X$, under mild conditions over $X$. More generally, sheaf cohomology (over an appropriate site) provides a framework for other cohomology theories as étale and crystalline. We expect that the generalization we are proposing will allow us to connect other (co)homology theories with sheaf cohomology. The first place to look at is cyclic homology. Cyclic homology was introduced, independently, by B. Tsygan and A. Connes in the 80's. In Connes' approach, cyclic homology is a noncommutative variation of de Rham (co)homology [Con85]. More precisely, the Hochschild-Kostant-Rosenberg theorem states that if $k$ is a field and $A$ is a finitely presented, smooth and commutative $k$-algebra, then there is an isomorphism $\Omega^n_{A/k} \cong HH_n(A)$ between the differential forms and the Hochschild homology groups of $A$. If $k$ contains $\mathbb{Q}$ then the cyclic homology of $A$ relates to de Rham cohomology by the following isomorphism:

$$HC_n(A) \cong \Omega^n_{A/k}/d\Omega^{n-1}_{A/k} \oplus H^{n-2}_{DR}(A) \oplus H^{n-4}_{DR}(A) \oplus \ldots$$

Since it is already known that, in some cases, sheaf cohomology and de Rham cohomology are isomorphic, we hope there will be interesting cases where extended sheaf cohomology and cyclic (co)homology coincide. Summing up, we are looking for a noncommutative version of the De Rham Theorem, which calls for a noncommutative version of the theory we are developing.

## 5.3 Logic

We constructed a category of sheaves on quantales that look like sheaves on locales but with a fundamental difference, $Sh(Q)$ may not be a topos. This suggests that in the same way that elementary topos generalize Grothendieck topos, we envision a notion of elementary loposes that generalize our sheaves under Grothendieck prelopologies. I hope that this thesis can provide some contribution to the ambitious project of developing a linear topos, that is, a category that looks like a topos but with a linear internal logic. Actually, we are not the first to propose a topos-like category with a different internal logic attached to it: we again cite the work of Francisco Miraglia and Marcelo Coniglio [CM01] to mention that they not only constructed a category of sheaves on (idempotent and right-sided) quantales but also developed a calculus of sequents analogous to a noncommutative linear intuitionistic calculus with equality. Futhermore. U. Höhle has extensive work in a fuzzy version of topos theory, dedicated to the foundations of fuzzy sets. For instance, in [Höh91], Höhle introduces the notion of a weak topos, which is monoidal closed instead of cartesian closed and such that only the extremal (and not all) subobjects can be classified by characteristic morphisms. We also want our lopos to be monoidal closed and to have an adapted version of subobject classifier, so lopos and weak topos shall be similar but it is not clear yet if our categories of sheaves would be a weak topos or not.

Furthermore, Höhle investigates metric spaces bounded by 1 and equipped with non-expansive maps as the main example of weak topos. Such metric space can be seen as a quantale-valued set. Recall that the category of sheaves on locales is equivalent to complete locale-valued sets. Because of such equivalence, a considerable part of the literature about sheaves on quantales is actually about $Q$-sets and it is not immediate to find the correspondent functorial notion of sheaf. The other way around is also true: we introduced a notion of a sheaf on a quantale but it is not clear for which notion of $Q$-set we would obtain an equivalence between the respective categories. We believe that a future formal definition of a lopos should encompass at least sheaves on quantales (more generally, Grothendieck loposes) and complete $Q$-sets. Luckily, my academic brothers Caio Mendes and José Alvim are studying another presentation of $Q$-sets [AAM23] so that in the future we can elaborate on the definition of a lopos.

After the above motivation, we have to explain why we believe that linear is the correspondent internal logic of our sheaves. First, a digression about propositional logic: in propositional logic, we have propositions, connectives that provide the relationships between propositions, and the values assigned to each proposition. In classical propositional logic, besides the negation $\neg$, the binary connectives are given by the conjuntion $\wedge$, the disjunction $\vee$, and the implication $\rightarrow$. Also, there are only two possible values to associate to each proposition (true $\top$ or false $\perp$) and certain inference rules that says how to obtain a conclusion with already given premises. On the other hand, a Boolean algebra is a set together with a unary operation and two binary operations satisfying certain axioms. We say that classical propositional logic corresponds to Boolean algebras because we can translate theorems of classical propositional calculus as equations of Boolean algebras and vice-versa. If we change the inference rules, for example, if we remove the law of the excluded middle, then we obtain an intuitionistic propositional logic. In this scenario, we may analogously say that intuitionistic propositional logic corresponds to Heyting algebras (recall that complete Heyting algebras coincide with locales), which are generalizations of Boolean algebras since Boolean algebras are Heyting algebras such that $a \vee \neg a = 1 = \top$ holds, that is, the excluded middle rule holds. When we replace locales with quantales we add a binary associative operation $\odot$ that plays the role of the meet but

$\odot$ is neither commutative nor idempotent, and neither has projections. Linear logic is a weakening of intuitionistic logic where we only have restricted versions of the contraction and weakening rules available. More precisely, the intuitionist conjunction splits into two binary operators: $\wedge$, the binary infimum of the lattice, which does not necessarily distributes over the supreme; and $\odot$ that does distribute with arbitrary supreme, but it does not have to be idempotent, commutative, or admit projection (be semicartesian). The absence of idempotence implies that $u$ does not prove $u \odot u$, which would be the rule of contraction. The absence of projections implies we are not allowed to use $u \odot v$ proves $u$ or $v$, which would be the weakening rule. In our case, quantales are semicartesian, so actually we are working with a linear logic that admits weakening, called **affine logic**. In a topos, the poset of subobjects[1] of any object is a Heyting algebra and, for a locale $L$, $Sub(L(-,1)) \cong L$. So when we proved that $Sub(Q(-,1)) \cong Q$, we supported the argument that the internal logic of our sheaves is related to the logic "interpreted" by quantales. Summing up, we have a result about the relation between sheaves on $Q$ and $Q$ that is completely analogous to one that holds in the localic case. We recognize this is only a first guess, and further investigations may point out that a different logic is better situated for our loposes.

We also can translate the weakening and the contractions in categorical terms: given a monoidal category, the weakening corresponds to the existence of projections/the category being semicartesian, while the contraction corresponds to the existence of a diagonal map (natural transformation) in a way that if the category has both, then the monoidal product is the cartesian product and the correspondent type theory for such category has both structural rules. The absence of the diagonal map also has an interpretation in Physics: it corresponds to the non-cloning theorem, which leads us to the next section.

## 5.4 Quantum Mechanics

In this section, I will only address aspects of quantum foundations using paradigms from category theory.

Lawvere created elementary topos interested in the foundations of Physics, but, apparently, topos-theoretic perspectives in quantum mechanics began in the late 90's with [AC95] and [IB98]. As far as I know, nowadays, the usual idea is to interpret Quantun Mechanics phenomena as Classical Mechanics phenomena internally to a certain topos: consider a noncommutative $C^*$-algebra $A$ (remind that $C^*$-algebras are used to model algebras of physical observables). Denote by $C(A)$ the poset of unital commutative $C^*$-subalgebras of $A$. In [HLS09], the authors consider a presheaf $\underline{A}$ on $C(A)$ such that $\underline{A}$ is a commutative $C^*$-algebra in the presheaf (Bohr) topos $Set^{C(A)}$. So a noncommutative $C^*$-algebra $A$ may become an (internal) commutative $C^*$-algebra $\underline{A}$. Interestingly, this makes it possible to deal with quantum states on $A$ in a similar way one deals with classical states, but internally[2] to the Bohr topos. The authors also provide a toposophic interpretation of the Kochen-Specker theorem [HLS09, Theorem 6] (this approach was also adopted in

[1] The subobject classifier and the power sets also form Heyting algebras.

[2] This is a relatively usual practice in topos theory. In algebraic geometry, it is also possible to describe complicated concepts in simpler ways but internally to some topos.

[IB98], where the Kochen-Specker theorem, developed in 1967 by Simon Kochen and Ernst Specker, is explained as a theorem that "asserts the impossibility of assigning values to quantum quantities in a way that preserves functional relations between them").

On the other hand, in quantum mechanics, we have, for example, the no-cloning theorem, that, in terms of Physics, talks about the impossibility of creating an independent and identical copy of an arbitrary unknown quantum state. In [Abr12], Abramsky states the no-cloning theorem in categorical terms: it corresponds to the absence of a natural transformation $\triangle_A : A \to A \otimes A$ which is coassociative and cocommutative (the reader may find the commutative diagrams for coassociative and cocommutative in Section 4.1 in [Abr12]). Since the monoidal structure in a topos is given by the cartesian product and the cartesian product always allows a natural transformation $\triangle_A : A \to A \times A$, in toposes, we do have cloning. I do not claim this leads to a contradiction but yet it seems that should be a categorical structure that conciliates both points of view, maybe, using a notion topos in which the non-cartesian monoidal product is more prominent than the cartesian product.

The above situation makes me intrigued to study categorical interpretations in a lopos, since we are precisely in a context suitable to the validity of the no-cloning theorem. I suppose that we do not even need to define the notion of a lopos, but providing examples focused on quantum phenomenona would be enough. There are two ideas to begin with: first, find semicartesian quantales that arise from Hilbert spaces or $C^*$-algebras and analyze the category of presheaves on such quantales. Alternatively, or maybe complementary, study (pre)sheaves on the monoidal category of Hilbert space — but first, we will need to identify a suitable (weak?) Grothendieck prelopology.

We highlight that in Marni Dee Sheppeard's thesis [She07], she introduces a version of linear topos by a more direct comparison with topos, making the category o vector spaces play the role of the category of sets. Apparently, the physics motivation came first to her. For us, our motivation came from the theory of sheaves and just recently we realized our sheaf category may be useful in quantum mechanics.

## 5.5 Sheafifications and Grothendieck loposes

In this thesis, we constructed a notion of a sheaf for Grothendieck prelopologies and tried to figure out the properties of the sheafification functor $a : PSh(C) \to Sh(C, L)$, which is a reflector. We have proved the existence of sheafification by considering sieves (Definition 4.2.1). This can also be done in the classical case of Grothendieck topologies, where we have pullbacks instead of pseudo-pullbacks. In such case, the map $S(\{U_i\}_{i\in I}) \to y(U)$ is a monomorphism but in the case of prelopologies and pseudo-pullbacks, $S(\{U_i\}_{i\in I}) \to y(U)$ is not a monomorphism. This may be the reason why our attempts to have a sheafification process using some analogous version of the plus construction failed.

In an alternative perspective, as we mentioned a few times, we may note that Grothendieck toposes are precisely those (accessible) reflective subcategories of presheaf categories for which the reflector is left exact. So, we may be interested in defining Grothendieck loposes as those accessible reflective subcategories of presheaf categories for which the reflector preserves the (Day) monoidal structure of $PSh(C)$. Actually, maybe we want the sheafification to preserve something else. We did not fully explore the proofs in Day's paper about monoidal localizations [Day73] but one of Day's results says that if a class $\Sigma$ of morphisms in a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ has the property that $f \in \Sigma$ implies $id_A \otimes f \in \Sigma$ for any object $A$ in $\mathcal{C}$ then the category of fractions of $\mathcal{C}[\Sigma^{-1}]$ is a monoidal category. Observe that such property is the third axiom of the definition of Grothendieck prelopologies 4.1, but we do not want to show that $\mathcal{C}[\Sigma^{-1}]$ is monoidal, we want to show that $PSh(\mathcal{C})[\Sigma^{-1}] \simeq Sh(\mathcal{C}, L)$. It seems that requiring the property $U_i \to U \in \Sigma \implies V \otimes U_i \to V \otimes U \in \Sigma$ in $\mathcal{C}$ implies that the same holds in the presheaf category, that is, $y(U_i) \to y(U) \in \Sigma' \implies y(V) \star y(U_i) \to y(V) \star y(U) \in \Sigma'$, which leads to the preservation of the monoidal structure. Furthermore, given the importance of the equalizers in the fourth axiom that defines Grothendieck prelopologies, I believe our sheafification also preserves equalizers or, at least, pseudo-pullbacks. In the end, we want also any Grothendieck lopos to be a category (monoidally) equivalent to $Sh(\mathcal{C}, L)$ for some Grothendieck prelopology $L$.

The above reasoning leads to *sheaf theories* where the sheaf categories change accordingly to what properties we want the sheafification to preserve and in each one we would have a respective proper notion of covering (possibly, some sheaf categories will not look like sheaves anymore) . The panoramic view is of the form:

$$
\begin{tikzcd}
 & PSh(\mathcal{C})[\mathcal{L}^{-1}] \arrow[dr] & \\
PSh(\mathcal{C}) \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & PSh(\mathcal{C})[\mathcal{L}_w^{-1}] \arrow[r] & PSh(\mathcal{C})[\mathcal{J}^{-1}] \\
 & PSh(\mathcal{C})[\mathcal{P}^{-1}] \arrow[ur] & \\
 & PSh(\mathcal{C})[\mathcal{E}^{-1}] \arrow[uur] &
\end{tikzcd}
$$

where all the arrows are sheafifications/localizations. $\mathcal{L}$ denotes the class for morphism $\{i_{U_i} : S(\{U_i\}) \to y(U) : \{U_i \to U\}_{i \in I} \in L(U)\}$, with $L$ a Grothendieck prelopology, $\mathcal{L}_w$ denotes the class for morphism $\{i_{U_i} : S(\{U_i\}) \to y(U) : \{U_i \to U\}_{i \in I} \in L_w(U)\}$ with $L_w$ a weak Grothendieck prelopology, $\mathcal{P}$ denotes the class for morphism $\{i_{U_i} : S(\{U_i\}) \to y(U) : \{U_i \to U\}_{i \in I} \in P(U)\}$ where $P$ would be a covering responsible for providing a reflector that only preserves products, $\mathcal{E}$ denotes the class for morphism $\{i_{U_i} : S(\{U_i\}) \to y(U) : \{U_i \to U\}_{i \in I} \in E(U)\}$ where $E$ would be a covering responsible to provide a reflector that only preserves equalizers, and $\mathcal{J}$ denotes the class for morphism $\{i_{U_i} : S(\{U_i\}) \to y(U) : \{U_i \to U\}_{i \in I} \in J(U)\}$ where $J$ would be a covering responsible to provide a reflector that preserves all finite limits, therefore, it would be a Grothendieck topology. In this way, $PSh(\mathcal{C})[\mathcal{J}^{-1}]$ is a Grothendieck topos and maybe, assuming that all reflectors preserve terminal objects, there is a way to have that $PSh(\mathcal{C})[\mathcal{P}^{-1}][\mathcal{E}^{-1}]$ is also a Grothendieck topos. The reader may wonder if $PSh(\mathcal{C})[\mathcal{L}^{-1}][\mathcal{E}^{-1}]$ could be enough to obtain a Grothendieck topos and I guess not because $PSh(\mathcal{C}) \to PSh(\mathcal{C})[\mathcal{L}^{-1}]$ is preserving the non-cartesian monoidal structure while $PSh(C) \to PSh(\mathcal{C})[\mathcal{P}^{-1}]$ preserves the cartesian monoidal structure, which is the one of interest in the classical sheafification. Additionally, probably $PSh(\mathcal{C})[\mathcal{L}^{-1}][\mathcal{J}^{-1}]$

is an example of a monoidal topos – it is a topos since $PSh(\mathcal{C})[\mathcal{J}^{-1}]$ is a Grothendieck topos and it is monoidal because $L_w$ should be preserving the non-cartesian structure. I highlight that for us the non-cartesian structure in $PSh(\mathcal{C})$ is always given by the Day convolution and induced by the non-cartesian product in $\mathcal{C}$, because we also want the resulting category to be monoidal closed. Of course, it is not clear if this will happen in all those sheaf categories, but the Day convolution seems the best candidate to make this happen. Definitely, there are plenty of calculations to run and check the above statements to make them rigorous and precise.

Finally, I wonder about the behavior of the subjects in each one of those sheaf categories. We know that in $PSh(\mathcal{C})[\mathcal{J}^{-1}]$ we obtain locales, in $PSh(Q)[\mathcal{L}^{-1}]$ we obtain unital semicartesian quantales (possibly, $PSh(\mathcal{C})[\mathcal{L}^{-1}]$ also provides unital semicartesian quantales), and in $PSh(\mathcal{C})$ we have Heyting algebras. Whatever the answer, I believe that this is an interesting scenario to have in mind when discussing toposes with distinct internal logics.

# References


[AAM23] José Goudet Alvim, Caio de Andrade Mendes, and Hugo Luiz Mariano. *Q-Sets and Friends: Categorical Constructions and Categorical Properties.* 2023. arXiv: 2302.03123 [math.CT] (cit. on p. 105).

[Abr12] Samson Abramsky. *No-Cloning In Categorical Quantum Mechanics.* 2012. arXiv: 0910.2401 [quant-ph] (cit. on p. 107).

[AC95] Murray Adelman and John V Corbett. "A sheaf model for intuitionistic quantum mechanics". In: *Applied categorical structures* 3.1 (1995), pp. 79–104 (cit. on p. 106).

[AR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories.* Vol. 189. Cambridge University Press, 1994 (cit. on pp. 77, 78).

[ASV08] Judit Mendoza Aguilar, Maria Victoria Reyes Sanchez, and Alain Verschoren. "Sheaves and sheafification on Q-sites". In: *Indagationes Mathematicae* 19.4 (2008), pp. 493–506 (cit. on pp. 4, 27).

[Bar+69] M Barr et al. "Adjunction for enriched categories". In: *Reports of the Midwest Category Seminar III.* Springer. 1969, pp. 166–177 (cit. on p. 79).

[Bar86] Michael Barr. "Fuzzy set theory and topos theory". In: *Canadian Mathematical Bulletin* 29.4 (1986), pp. 501–508 (cit. on p. 18).

[BB86] Francis Borceux and Gilberte Van den Bossche. "Quantales and their sheaves". In: *Order* 3.1 (1986), pp. 61–87 (cit. on pp. 4, 27, 50, 51).

[BC94] Francis Borceux and Rosanna Cruciani. "A generic representation theorem for non-commutative rings". In: *Journal of Algebra* 167.2 (1994), pp. 291–308 (cit. on pp. 4, 5, 27, 53, 54).

[BD98] John Carlos Baez and James Dolan. "Higher-dimensional algebra III. n-categories and the algebra of opetopes". In: *Advances in Mathematics* 135.2 (1998), pp. 145–206 (cit. on p. 60).

[BE16] Ulrich Bunke and Alexander Engel. "Homotopy theory with bornological coarse spaces". In: *arXiv preprint arXiv:1607.03657* (2016) (cit. on p. 101).

[Ble17] Ingo Blechschmidt. "Using the internal language of toposes in algebraic geometry". PhD thesis. Universität Augsburg, 2017 (cit. on p. 3).

[Bor94a] Francis Borceux. *Handbook of categorical algebra: Basic category theory.* Vol. 1. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994 (cit. on pp. 43, 81, 83).

[Bor94b] Francis Borceux. *Handbook of Categorical Algebra: Categories and Structures.* Vol. 2. Encyclopedia of Mathematics and its Applications. Cambridge: Cambridge University Press, 1994 (cit. on pp. 22, 79).

[Bor94c] Francis Borceux. *Handbook of Categorical Algebra: Sheaf Theory*. Vol. 3. Encyclopedia of Mathematics and its Applications. Cambridge: Cambridge University Press, 1994 (cit. on pp. 3, 5, 19, 22, 38, 42, 53, 54, 88).

[CM01] Marcelo E Coniglio and Francisco Miraglia. "Modules in the category of sheaves over quantales". In: *Annals of Pure and Applied Logic* 108.1-3 (2001), pp. 103–136 (cit. on pp. 5, 105).

[Con85] Alain Connes. "Non-commutative differential geometry". In: *Publications Mathematiques de l'IHES* 62 (1985), pp. 41–144 (cit. on p. 104).

[Day70] Brian Day. "On closed categories of functors". In: *Reports of the Midwest Category Seminar IV*. Springer. 1970, pp. 1–38 (cit. on p. 15).

[Day73] Brian Day. "Note on monoidal localisation". In: *Bulletin of the Australian Mathematical Society* 8.1 (1973), pp. 1–16 (cit. on pp. 79, 80, 108).

[EK66] Samuel Eilenberg and Gregory Maxwell Kelly. "Closed categories". In: *Proceedings of the Conference on Categorical Algebra: La Jolla 1965*. Springer. 1966, pp. 421–562 (cit. on p. 79).

[Eti+16] Pavel Etingof et al. *Tensor categories*. Vol. 205. American Mathematical Soc., 2016 (cit. on p. 11).

[FS79] Michael P Fourman and Dana S Scott. "Sheaves and logic". In: *Applications of sheaves*. Springer, 1979, pp. 302–401 (cit. on pp. 4, 19, 27).

[GH54] Leonard Gillman and Melvin Henriksen. "Concerning Rings of Continuous Functions". In: *Transactions of the American Mathematical Society* 77.2 (1954), pp. 340–362. ISSN: 00029947. URL: http://www.jstor.org/stable/1990875 (visited on 05/01/2023) (cit. on p. 98).

[GJ17] Leonard Gillman and Meyer Jerison. *Rings of continuous functions*. Courier Dover Publications, 2017 (cit. on p. 98).

[GLS+22] Malte Gerhold, Stephanie Lachs, Michael Schürmann, et al. "Categorial Independence and Lévy Processes". In: *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* 18 (2022), p. 075 (cit. on p. 14).

[Gro57] Alexander Grothendieck. "Sur quelques points d'algèbre homologique". In: *Tohoku Mathematical Journal* 9.2 (1957), pp. 119–221 (cit. on p. 62).

[Gro63] Alexander Grothendieck. "Théorie des Topos et Cohomologie Étale des Schemas (SGA4)". In: *Lecture Notes in Mathematics* 269 (1963), pp. 299–519 (cit. on p. 22).

[Gro71] Alexander. Grothendieck. *Classes de faisceaux et théorème de Riemann-Roch. (Sheaf classes and the Riemann-Roch theorem)*. French. Sem. Geom. algebrique Bois Marie 1966/67, SGA 6, Lect. Notes Math. 225, 20-77 (1971). 1971 (cit. on p. 2).

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate texts in mathematics. New York: Springer, 1977 (cit. on p. 23).

[Hew48] Edwin Hewitt. "Rings of real-valued continuous functions. I". In: *Transactions of the American Mathematical Society* 64.1 (1948), pp. 45–99 (cit. on p. 98).

[Hig84] Denis Higgs. "Injectivity in the topos of complete Heyting algebra valued sets". In: *Canadian journal of mathematics* 36.3 (1984), pp. 550–568 (cit. on p. 19).

[HLS09] Chris Heunen, Nicolaas P Landsman, and Bas Spitters. "A topos for algebraic quantum theory". In: *Communications in mathematical physics* 291.1 (2009), pp. 63–110 (cit. on p. 106).

[Höh91] Ulrich Höhle. "Monoidal closed categories, weak topoi and generalized logics". In: *Fuzzy Sets and Systems* 42.1 (1991). Mathematical Aspects of Fuzzy Sets, pp. 15–35. ISSN: 0165-0114. DOI: https://doi.org/10.1016/0165-0114(91)90086-6. URL: https://www.sciencedirect.com/science/article/pii/0165011491900866 (cit. on pp. 7, 105).

[HS12] Hans Heymans and Isar Stubbe. "Grothendieck quantaloids for allegories of enriched categories". In: *Bulletin of the Belgian Mathematical Society-Simon Stevin* 19.5 (2012), pp. 859–888 (cit. on pp. 4, 27).

[IB98] Christopher John Isham and Jeremy Butterfield. *A topos perspective on the Kochen-Specker theorem: I. Quantum States as Generalized Valuations*. 1998. arXiv: quant-ph/9803055 [quant-ph] (cit. on pp. 106, 107).

[Joh77] Peter Johnstone. *Topos Theory*. London Mathematical Society Monographs. London: Academic Press, 1977 (cit. on pp. 3, 22, 24).

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and Sheaves*. Springer Berlin Heidelberg, 2006. DOI: 10.1007/3-540-27950-4. URL: https://link.springer.com/book/10.1007/3-540-27950-4 (cit. on p. 74).

[Lam06] Tsit-Yuen Lam. *Exercises in classical ring theory*. Springer Science & Business Media, 2006 (cit. on p. 34).

[Law63] F William Lawvere. "Functorial semantics of algebraic theories". In: *Proceedings of the National Academy of Sciences* 50.5 (1963), pp. 869–872 (cit. on p. 60).

[Ler45] Jean Leray. "Sur la forme des espces topologiques et sur les points fixes des representations". In: *J. Math. Pures Appl.* 9 (1945), pp. 95–248 (cit. on p. 1).

[Lor21] Fosco Loregian. *(Co)end Calculus*. Cambridge University Press, June 2021. DOI: 10.1017/9781108778657. URL: %5Curl%7Bhttps://doi.org/10.1017%2F9781108778657%7D (cit. on p. 15).

[McL90] Colin McLarty. "The Uses and Abuses of the History of Topos Theory". In: *The British Journal for the Philosophy of Science* 41.3 (1990), pp. 351–375. ISSN: 00070882, 14643537. URL: http://www.jstor.org/stable/687825 (visited on 05/25/2023) (cit. on p. 21).

[McL92] Colin McLarty. *Elementary categories, elementary toposes*. Oxford Logic Guides. Oxford: Oxford University Press, 1992 (cit. on p. 3).

[MM92] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Universitext. New York: Springer, 1992 (cit. on pp. 1, 3, 21, 45, 59, 73, 86).

[MR77] Michael Makkai and Gonzalo E. Reyes. *First order categorical logic. Model-theoretical methods in the theory of topoi and related categories*. English. Vol. 611. Lect. Notes Math. Springer, Cham, 1977 (cit. on p. 73).

[MS98] Francisco Miraglia and Ugo Solitro. "Sheaves over right sided idempotent quantales". In: *Logic Journal of IGPL* 6.4 (1998), pp. 545–600 (cit. on pp. 4, 27, 88).

[Mul86] Christopher J. Mulvey. "&". English. In: *Suppl. Rend. Circ. Mat. Palermo, Ser. II*, 12 (1986), pp. 99–104. ISSN: 1592-9531 (cit. on p. 4).

[Nar95] Wladyslaw Narkiewicz. *Polynomial Mappings*. Springer Berlin, Heidelberg, 1995 (cit. on p. 86).

[nLa22a] nLab authors. *orthogonal subcategory problem*. https://ncatlab.org/nlab/show/orthogonal+subcategory+problem. Revision 6. Dec. 2022 (cit. on p. 81).

[nLa22b] nLab authors. *Quantale*. https://ncatlab.org/nlab/show/quantale. 2022 (cit. on pp. 29, 32).

[nLa22c] nLab authors. *Semicartesian monoidal categories*. https://ncatlab.org/nlab/show/semicartesian+monoidal+category. Feb. 2022 (cit. on p. 15).

[nLa22d] nLab authors. *Sheafification*. https://ncatlab.org/nlab/show/sheafification. 2022 (cit. on p. 74).

[nLa23a] nLab authors. *Markov category*. https://ncatlab.org/nlab/show/Markov+category. Revision 17. Apr. 2023 (cit. on p. 15).

[nLa23b] nLab authors. *subobject classifier*. https://ncatlab.org/nlab/show/subobject+classifier. Revision 59. May 2023 (cit. on p. 86).

[Pet06] Peter Petersen. *Riemannian geometry*. Graduate Texts in Mathematics. New York: Springer, 2006 (cit. on pp. 24, 97).

[Reg86] Mangesh B. Rege. "On von Neumann regular rings and SF-rings". In: *Math. Japonica* 31.6 (1986), pp. 927–936 (cit. on p. 34).

[Res12] Pedro Resende. "Groupoid sheaves as quantale sheaves". In: *Journal of Pure and Applied Algebra* 216.1 (2012), pp. 41–70 (cit. on pp. 4, 27).

[Ros90a] Kimmo I Rosenthal. *Quantales and their applications*. Vol. 234. Longman Scientific and Technical, 1990 (cit. on p. 35).

[Ros90b] Kimmo I. Rosenthal. *Quantales and their applications*. Longman Scientific & Technical, 1990. ISBN: 0470216735 (cit. on pp. 28, 29, 50, 51).

[Ser55] Jean-Pierre Serre. "Faisceaux Algebriques Coherents". In: *Annals of Mathematics* 61.2 (1955), pp. 197–278. ISSN: 0003486X (cit. on p. 19).

[She07] Marni Dee Sheppeard. "Gluon phenomenology and a linear topos". In: (2007) (cit. on p. 107).

[Ște81] Alexandru Ștefănescu. "The category of abelian groups in a topos". In: *Bulletin mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie* 25 (73).2 (1981), pp. 179–189. ISSN: 03732908. URL: http://www.jstor.org/stable/43681686 (cit. on p. 3).

[Tao07] Terence Tao. "Differential forms and integration". In: *Tech Rep Dep Math UCLA* (2007) (cit. on p. 24).

[Ten75] B. R. Tennison. *Sheaf Theory*. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 1975 (cit. on p. 18).

[TM21] Ana Luiza Tenorio and Hugo Luiz Mariano. "On sheaf cohomology and natural expansions". In: *São Paulo Journal of Mathematical Sciences* 15.2 (2021), pp. 571–614 (cit. on pp. 3, 22).

[Wat65] Charles Edward Watts. "Alexander-Spanier cohomology and rings of continuous functions". In: *Proceedings of the National Academy of Sciences* 54.4 (1965), pp. 1027–1028 (cit. on p. 98).

[Wei94] Charles Alexander Weibel. *An introduction to homological algebra*. Studies in Advanced Mathematics. Cambridge University Press, 1994 (cit. on p. 22).

[Wik23] Wikipedia. *Prüfer domain*. https://en.wikipedia.org/wiki/PrÃijfer_domain. July 2023 (cit. on p. 86).