3-2-1 foliations for Reeb flows on S^3

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Resumo

Folheações do tipo 3-2-1 para fluxos de Reeb em S^3

Neste trabalho, estudamos sistemas globais de seções transversais para fluxos de Reeb associados a formas de contato *tight* na 3-esfera. Tais fluxos incluem, em particular, fluxos Hamiltonianos em $\mathbb{R}^4$ restritos a níveis de energia regulares estrelados. Um sistema global de seções transversais adaptado a um fluxo em S^3 é uma folheação singular de S^3 cujo conjunto singular, chamado de amarração, consiste de um número finito de órbitas periódicas e as folhas regulares são transversais ao fluxo. Como demonstrado por H. Hofer, K. Wysocki e E. Zehnder em [HWZ03], fluxos de Reeb associados a formas de contato *tight* não degeneradas em S^3 admitem um sistema global de seções transversais, cujas folhas regulares são esferas furadas. Tais sistemas são construídos como a projeção em S^3 de uma folheação de $\mathbb{R} \times S^3$ por curvas pseudo-holomorfas.

Utilizando a teoria de curvas pseudo-holomorfas em simplectizações, estudamos a existência de um tipo de sistema de seções transversais, que chamamos de folheação 3-2-1, possuindo exatamente três órbitas na amarração, com índices de Conley-Zehnder respectivamente 3, 2 e 1. Mais precisamente, apresentamos condições suficientes sob as quais três órbitas de Reeb formam a amarração de uma folheação 3-2-1.

Palavras-chave: Dinâmica Hamiltoniana, Fluxos de Reeb, Curvas pseudo-holomorfas, Folheações de energia finita

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Abstract

In this work, we study global systems of transversal sections for Reeb flows associated with tight contact forms on the 3-sphere. These flows include, in particular, Hamiltonian flows on $\mathbb{R}^4$ restricted to star-shaped regular energy levels. A global system of transversal sections naturally generalizes the concept of global surface of section. It is a singular foliation of S^3 whose singular set consists of finitely many periodic orbits, called binding orbits, and the regular leaves are transverse to the flow. As proved by H. Hofer, K. Wysocki and E. Zehnder in [HWZ03], Reeb flows associated with non-degenerate tight contact forms on S^3 admit a global system of transversal sections whose regular leaves are punctured spheres. Such system is the projection to S^3 of a foliation of $\mathbb{R} \times S^3$ by pseudo-holomorphic curves.

The aim of this work is to use the theory of pseudo-holomorphic curves in symplectizations to study the existence of a particular type of system of transversal sections, called 3-2-1 foliation, which has exactly three binding orbits with Conley-Zehnder indices respectively 3, 2 and 1. More precisely, we give sufficient conditions under which three Reeb orbits are the binding orbits of a 3-2-1 foliation.
Keywords: Reeb Flows, Hamiltonian dynamics, finite energy foliations, pseudoholomorphic curves

Contents

Introduction

In this thesis we use the theory of pseudoholomorphic curves in symplectizations to study the existence of finite energy foliations for Reeb flows on S^3.

We consider S^3 equipped with the contact form $\lambda = f\lambda_0|_{S^3}$, where $f : S^3 \to \mathbb{R} \setminus \{0\}$ is smooth and λ_0 is the standard contact form $\lambda_0 = \sum_{i=1}^{2} x_i dy_i - y_i dx_i$ on $\mathbb{R}^4$ with coordinates (x_1, x_2, y_1, y_2). The *Reeb vector field* R_λ associated with the contact form λ is uniquely determined by the equations

$$i_{R_\lambda} d\lambda = 0, \quad \lambda(R_\lambda) = 1.$$

The flow of R_λ is called *Reeb flow*. The *symplectization* of (S^3, λ) is the symplectic manifold $(\mathbb{R} \times S^3, d(e^a\lambda))$, where a is the coordinate on $\mathbb{R}$. We consider pseudoholomorphic maps $\tilde{u} : (S \setminus \Gamma, j) \to (\mathbb{R} \times S^3, \tilde{J})$, where S is a closed Riemann surface, $\Gamma \subset S$ is a finite set, and the almost complex structure $\tilde{J}$ is defined as

$$\tilde{J}\frac{\partial}{\partial_a} = R_\lambda, \quad \tilde{J}|_{\ker \lambda} = J,$$

for any complex structure J on $\ker \lambda$ compatible with $d\lambda$. Due to results of [Hof93] and [HWZ96], if a pseudoholomorphic curve has *finite energy*, its projection onto S^3 converges near the ends to periodic orbits of the Reeb flow.

A *finite energy foliation* $\mathcal{F}$ for the contact manifold (S^3, λ) is a collection of $\tilde{J}$-holomorphic curves with uniformly bounded energy, whose images form a smooth foliation of $\mathbb{R} \times S^3$. In [HWZ03], H. Hofer, K. Wysocki and E. Zehnder proved that for any nondegenerate contact form $\lambda = f\lambda_0$ on S^3, there exists a finite energy foliation for (S^3, λ). The projection of $\mathcal{F}$ onto S^3 is a singular foliation of S^3 satisfying the following properties. The singular set consists of finitely many periodic orbits of the Reeb flow, called bindings, having Conley-Zehnder indices in $\{1, 2, 3\}$. The regular leaves are punctured spheres transverse to the Reeb vector field. Each component of the boundary of the regular leaves is one of the bindings.

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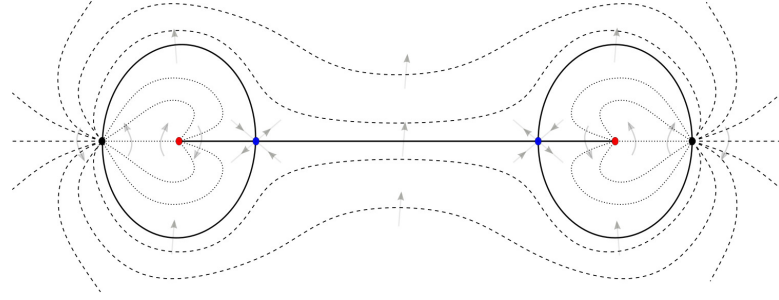
Figure 1: A $3 - 2 - 1$ foliation cut by a plane. The dots represent periodic orbits P_1, P_2 and P_3 with Conley-Zehnder indices respectively 1, 2 and 3. The bold curves represent two rigid cilinders connecting P_2 and P_3, a rigid cylinder connecting P_1 and P_2, and a rigid disk with boundary P_2. The dashed curves represent a family of planes with boundary P_3. The dotted curves represent a family of cylinders connecting P_1 and P_3. The arrows indicate the Reeb flow. The 3-sphere is viewed as $\mathbb{R}^3 \cup \{\infty\}$.

If the singular set of the finite energy foliation $\mathcal{F}$ consists of a unique orbit P, then P has Conley-Zehnder index 3 and the projection of $\mathcal{F}$ onto S^3 is an open book decomposition with pages diffeomorphic to disks, where every page is a global surface of section for the Reeb flow. This kind of foliation was obtained in [HWZ98] for Hamiltonian flows on $\mathbb{R}^4$ restricted to strictly convex energy levels. Such foliations were also studied in [HSa11, Hry12, Hry14].

The first result on the existence of a finite energy foliation for S^3 other than the one that projects onto an open book decomposition is due to N. de Paulo e P. Salomão. In [dPSa18], the authors study the existence of a finite energy foliation that projects to a singular foliation of S^3, called 3-2-3 foliation, having an orbit with Conley-Zehnder index 2 and two orbits with Conley-Zehnder index 3 as bindings.

The results of [HWZ98] and [dPSa18] cited above apply to weakly convex contact forms, that is, such that every closed Reeb orbit has Conley-Zehnder index ≥ 2. Hence, it is natural to ask about the existence of energy levels of Hamiltonians on $\mathbb{R}^4$ that admit transverse foliations with index 1 orbits as bindings. The results in this thesis go in the direction of the solution of this question. From assumptions about the Reeb flow and three prescribed Reeb orbits P_3, P_2 and P_1, with Conley-Zehnder index 3, 2 and 1 respectively, we study the existence of a finite energy foliation that projects onto S^3 as a singular foliation, that we call $3 - 2 - 1$-*foliation*, having the orbits P_3, P_2 and P_1 as bindings. See figure 1.

Our main theorem is the following:

Theorem. *Let λ be a nondegenerate tight contact form on S^3 Let $P_1 = (x_1, T_1)$, $P_2 = (x_2, T_2)$ and $P_3 = (x_3, T_3)$ be simply covered closed Reeb orbits with Conley-Zehnder indices respectively 1, 2 and 3. Assume that the orbits P_1, P_2 and P_3 are unknotted, P_i and P_j are not linked for $i \neq j$, $i, j \in \{1, 2, 3\}$ and the following conditions hold:*

   (i) P_3 spans an embedded disk whose interior is transverse to the Reeb flow;

   (ii) $T_1 < T_2 < T_3 < 2T_1$;

   (iii) P_2 is the only Reeb orbit with Conley-Zehnder index 2 not linked to P_3 with period $< T_3$;

   (iv) P_1 is the only Reeb orbit with Conley-Zehnder index 1 not linked to P_2 with period $< T_2$;

   (v) There is no C^1-embedding $\Psi : S^2 \to S^3$ such that $x_2(T_2\cdot) = \Psi|_{S^1 \times \{0\}}$ and each hemisphere is a strong transverse section.

Then P_1, P_2 and P_3 are the binding orbits of a $3 - 2 - 1$ foliation.

We also prove that the hypothesis *(v)* is necessary to the existence of a $3 - 2 - 1$ foliation.

The proof of the main theorem depends on a result about *gluing* of pseudoholomorphic curves that is stated in Chapter 5, but is not proved in this thesis.

Outline of the Thesis

Chapters 1 and 2 consist of preliminary results. In Chapter 1 we recall some relevant definitions and results about contact geometry and Reeb flows and present the definition and important properties of the asymptotic operator. The subject of Chapter 2 is the theory of pseudoholomorphic curves in symplectizations. We recall results of [HWZ96, HWZ95b, HWZ99b] about asymptotic behavior, algebraic invariants and Fredholm theory. We finish the chapter recalling facts about finite energy foliations.

In chapter 3 we define $3 - 2 - 1$ foliations, state the main theorem and give the first steps of the proof. Following the ideas of [Hof93, HWZ95a, HWZ99a, HSa11], we consider a family of pseudoholomorphic disks with boundary in a special embedded disk with spanning P_3. From a bubbling-off process and exploring the non-compactness of the space of such disks and the properties of its compactification, one can prove the existence of finite energy pseudoholomorphic curves. By our linking hypotheses we have to consider

two possibilities: either 1. The boundaries of the family of disks converge to the orbit P_3 or 2. The family of disks breaks before approaching P_3.

In chapter 4 we conclude the proof of case 1. In chapter 5 we present an idea of the proof of case 2.

Chapter 1

Contact geometry and Reeb dynamics

1.1 Contact structures

Let M be a smooth manifold of dimension $2n + 1$. A *contact structure* ξ on M is a maximally non-integrable hyperplane distribution on M. This is equivalent to the following: ξ can be locally written as the kernel of a 1-form λ such that $\lambda \wedge (d\lambda)^n$ is a non-vanishing $(2n + 1)$-form. If ξ is a contact structure on M, the pair (M, ξ) is called a contact manifold.

If n is odd, the sign of the local form $\lambda \wedge (d\lambda)^n$ depends only on $\xi = \ker \lambda$, not on the choice of λ. So the contact structure induces an orientation on M. Assume that M is oriented. We call the contact structure ξ *positive* if the orientation of M and the orientation given by ξ agree. Otherwise, we say that ξ is *negative.*

We say that the contact manifold (M, ξ) is *co-orientable* if the bundle $TM/\xi \to M$ is trivial. This is equivalent to the existence of a globally defined 1-form λ such that $\xi = \ker \lambda$.

A 1-form on M such that $\xi = \ker \lambda$ is a contact structure, that is, $\lambda \wedge (d\lambda)^n$ is a volume form on M, is called a *contact form*. If λ is a contact form on M, we also call the pair (M, λ) a contact manifold.

In this thesis we will only consider co-orientable contact manifolds.

The Reeb vector field Let (M, λ) be a closed contact manifold. The contact form λ defines a vector field R_λ, called *Reeb vector field*, uniquely defined by

$$\lambda(R_\lambda) \equiv 1, \qquad i_{R_\lambda} d\lambda \equiv 0 \ .$$

The Reeb vector field is transverse to the contact structure ξ, so that the tangent bundle TM naturally splits into

$$TM = \mathbb{R}R_\lambda \oplus \xi. \tag{1.1}$$

If $\{\varphi^t : M \to M\}_{t\in\mathbb{R}}$ is the flow of R_λ, then $(\varphi^t)^*\lambda = \lambda$. Indeed

$$\frac{d}{dt}(\varphi^t)^*\lambda = (\varphi^t)^*\mathcal{L}_{R_\lambda}\lambda = (\varphi^t)^*(i_{R_\lambda}d\lambda + di_{R_\lambda}\lambda) = 0 \ , \forall t \in \mathbb{R}.$$

Consequently, the linearized flow $d\varphi^t : TM \to TM$ leaves the splitting $TM = \mathbb{R}R_\lambda \oplus \xi$ invariant.

The restriction of $d\lambda$ to $\xi = \ker \lambda$ is nondegenerate, so that $(\xi, d\lambda)$ is a symplectic vector bundle over M. Moreover, the map $d\varphi^t_x : \xi_x \to \xi_{\varphi^t(x)}$ is symplectic with respect to $d\lambda$.

1.1.1   Hypersurfaces of contact type

Let (W^{2n}, ω) be a symplectic manifold. A vector field Y defined on an open set of W is called a *Liouville vector field* if it satisfies $\mathcal{L}_Y\omega = \omega$, where $\mathcal{L}_Y\omega$ is the Lie derivative of ω in the direction of Y. A compact hypersurface M in (W, ω) is said to have *contact type* if there exists a Liouville vector field on a neighborhood of M that is transverse to M. Every hypersurface of contact type is a contact manifold, by the following proposition.

Proposition 1.1. *Let $M \subset (W^{2n}, \omega)$ be a hypersurface of contact type and let Y be a Liouville vector field defined on a neighborhood U of M that is transverse to M. Then the 1-form $\lambda := i_Y\omega|_U$ satisfies $d\lambda = \omega$ and $\lambda \wedge (d\lambda)^{n-1}|_M$ is a volume form on M.*

A proof of Proposition 1.1 can be found in [Gei08, Lemma 1.4.5] or [HSa09, Prop. 1.6].

Star-shaped hypersurfaces on $\mathbb{R}^{2n}$

Definition 1.2. A regular hypersurface $S \subset \mathbb{R}^{2n}$ is said to be *star-shaped* with respect to the origin if every half-line starting at the origin intersects S transversely in exactly one point.

Consider $\mathbb{R}^{2n}$ with coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ equipped with the canonical symplectic form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. The vector field $X(z) = \frac{z}{2}$, $z = (x_1, \cdots, x_n, y_1, \cdots, y_n) \in \mathbb{R}^{2n}$ is a Liouville vector field. Indeed, $i_X\omega_0 =$

λ_0, where λ_0 is the Liouville 1-form

$$\lambda_0 = \frac{1}{2} \sum_{i=1}^{n} x_i dy_i - y_i dx_i.$$

Using the Cartan formula we get

$$\mathcal{L}_X \omega_0 = d i_X \omega_0 + i_X d\omega_0 = d\lambda_0 = \omega_0.$$

It follows that every hypersurface $S \subset \mathbb{R}^{2n}$ that is star-shaped with respect to the origin is a contact manifold equipped with the contact form $\lambda_0|_S$.

Let S be a hypersurface $S \subset \mathbb{R}^{2n}$ that is star-shaped with respect to the origin. For $x \in S^{2n-1}$, let $f(x) \in \mathbb{R}^+$ be defined by $\sqrt{f(x)}x \in S$. Define $\psi : S^{2n-1} \to S$ by $\psi(x) = \sqrt{f(x)}x$. Then ψ is a diffeomorphism.

Proposition 1.3. *The Reeb vector field $X_{\lambda_0|_S}$ on S associated with the contact form $\lambda_0|_S$ is equivalent to Reeb vector field $X_{f\lambda_0|_{S^{2n-1}}}$ on S^{2n-1} associated with the contact form $f\lambda_0|_{S^{2n-1}}$.*

Proof. For every $x \in S^{2n-1}$ and $v \in T_x S^{2n-1}$, we have

$$\begin{aligned}
(\psi^* \lambda_0|_S)_x(v) &= \lambda_0(d\psi_x v) \\
&= \omega_0 \left(\frac{1}{2} \sqrt{f(x)}x, \langle \nabla \sqrt{f(x)}, v \rangle x + \sqrt{f(x)}v \right) \\
&= f(x)\omega_0 \left(\frac{1}{2}x, v \right) \\
&= f(x)(\lambda_0)_x(v)
\end{aligned}$$

It follows that $\psi_* X_{f\lambda_0|_{S^{2n-1}}} = X_{\lambda_0|_S}$. $\qquad\square$

1.1.2 Symplectization of a contact manifold

Let (M, λ) be a contact manifold. The *symplectization* of (M, λ) is given by the manifold $\mathbb{R} \times M$ equipped with the symplectic form $d(e^a \lambda)$, where a is the coordinate on $\mathbb{R}$ and λ is interpreted as the 1-form on $\mathbb{R} \times M$ given by the pullback of λ under the natural projection $\mathbb{R} \times M \to M$.

The vector field $Y = \frac{\partial}{\partial a}$ is a Liouville vector field, so that (M, λ) can be realized as a hypersurface of contact type of its symplectization $(\mathbb{R} \times M, d(e^a \lambda))$.

More generally, if ϕ satisfies $\{\phi, \phi'\} \subset C^\infty(\mathbb{R}, \mathbb{R}^+)$ and the 1-form λ_ϕ on

$\mathbb{R} \times M$ is defined by

$$\lambda_\phi(a,x)(h,k) = \phi(a)\lambda(x)(k), \ \forall(a,x) \in \mathbb{R} \times M, \ (h,k) \in T_{(a,x)}(\mathbb{R} \times M), \ (1.2)$$

then $(\mathbb{R} \times M, d\lambda_\phi)$ is a symplectic manifold.

1.1.3   Hamiltonian dynamics

A Hamiltonian system is a triple (W, ω, H) where (W, ω) is a symplectic manifold and $H : W \to \mathbb{R}$ is a smooth function, that is called a Hamiltonian function. The Hamiltonian vector field associated with (W, ω, H) is the vector field defined on W by

$$i_{X_H}\omega = -dH \ .$$

The vector field X_H is well defined because ω is non-degenerate.

Let $x : \mathbb{R} \to W$ be a solution of the equation

$$\dot{x} = X_H(x) \ .$$

Then the Hamiltonian function is constant along x. Indeed,

$$\frac{d}{dt}H(x(t)) = dH(X_H(x(t))) = -\omega(X_H, X_H) = 0 \ .$$

Therefore all the trajectories lie in energy levels of the Hamiltonian function.

Let $S = H^{-1}(e)$ be a regular energy level of H. Assume that $S \subset (W, \omega)$ is a hypersurface of contact type. Racall that for a Liouville field Y defined on a neighborhood of S and transverse to S, the form $\lambda = i_Y\omega|_S$ is a contact form on S. Hence $X_H|_S$ and the Reeb vector field R_λ lie in the kernel of $\omega|_S$. Since $\ker \omega|_S$ is 1-dimensional, there exists a function $a : S \to \mathbb{R} \setminus \{0\}$ such that $X_H|_S(x) = a(x)R_\lambda(x), \ \forall x \in S$. Thus, the trajectories of $X_H|_S$ and R_λ are the same, modulo reparemetrization.

1.1.4   Classification of contact structures in dimension 3

Let (M^3, ξ) be a contact manifold of dimension 3. An embedded disk $D \subset M$ satisfying

$$T\partial D \subset \xi \text{ and } T_pD \neq \xi_p, \ \forall p \in \partial D$$

is called an *overtwisted disk*.

The contact structure ξ on M is called *overtwisted* if it admits an overtwisted disk. Otherwise, ξ is called *tight*. If $\xi = \ker \lambda$ globally, we call the

contact form *overtwisted* (*tight*) if $\xi = \ker \lambda$ is overtwisted (tight).

In this thesis, we will deal with co-oriented tight contact structures on S^3.

Consider $\mathbb{R}^4$ with coordinates (x_1, x_2, y_1, y_2). The Liouville 1-form

$$\lambda_0 = \frac{1}{2} \sum_{i=1}^{2} x_i dy_i - y_i dx_i \tag{1.3}$$

restricts to a contact form on S^3. By a result of Bennequim [Ben83], the contact structure $\xi_0 = \ker(\lambda_0|_{S^3})$ is tight. By a Theorem of Eliashberg [Eli92], $\xi_0 = \ker(\lambda_0|_{S^3})$ is the only positive tight contact structure on S^3, up to diffeomorphism.

1.2 Reeb orbits

Let (M, λ) be a closed contact manifold of dimension 3. We call a pair $P = (x, T)$, where $x : \mathbb{R} \to S^3$ is a periodic trajectory of $\dot{x}(t) = R_\lambda(x(t))$ and $T > 0$ is a period of x, a *closed Reeb orbit*. We identify $P = (x, T)$ with the element of $\dfrac{C^\infty(\mathbb{R}/\mathbb{Z}, M)}{\mathbb{R}/\mathbb{Z}}$ induced by the loop

$$x_T : \frac{\mathbb{R}}{\mathbb{Z}} \to M, \quad x_T(t) = x(Tt), \tag{1.4}$$

where the quotient is relative to the translations $t \mapsto x_T(t + c)$. The set of periodic Reeb orbits will be denoted by $\mathcal{P}(\lambda)$. If T is the minimal positive period of x, we call P *simply covered* or *prime*. If $m \geq 1$ is an integer, the mth iterate of P will be denoted by $P^m := (x, mT)$.

Assume $M = S^3$. The *self-linking number* $\mathrm{sl}(L)$ of a knot $L \subset S^3$ transverse to ξ is defined as follows. Consider S^3 oriented by $\lambda \wedge d\lambda$, choose a Seifert surface [1] Σ for L and a smooth nonvanishing section Z of $\xi|_\Sigma \to \Sigma$. The section Z is used to slightly perturb L to another knot $L_\epsilon = \{\exp_x(\epsilon Z_x) | x \in L\}$ transverse to ξ and Σ. A choice of orientation for Σ induces orientations of L and L_ϵ. The self-linking number $\mathrm{sl}(L)$ of L is the oriented intersection number $L_\epsilon \cdot \Sigma$ of L_ϵ and Σ. It is independent of the choices of Z and Σ. Proofs of these facts can be found in [Gei08]. If $P = (x, T)$ is a closed Reeb orbit, we define its self-linking number by $\mathrm{sl}(P) = \mathrm{sl}(x(\mathbb{R}))$.

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[1] A Seifert surface for L is an orientable, embedded, connected and compact surface Σ such that $L = \partial \Sigma$.

The orbit $P = (x, T)$ is called *unknotted* if $x(\mathbb{R})$ is an unknot [2] We say that a set of orbits $\cup_{i=1}^n P_i = (x_i, T_i)$ is an unlink if $\cup_{i=1} x_{iT_i}$ is an unlink.

We say that two orbits P and $\bar{P}$ are *linked* if the linking number [3] $\mathrm{lk}(x_T, \bar{x}_{\bar{T}})$ is nonzero. [4]

We call the orbit $P = (x, T)$ *nondegenerate* if 1 is not an eigenvalue of $d\varphi^T|_{\xi_{x(0)}}$. If every orbit $P \in \mathcal{P}(\lambda)$ is nondegenerate, then the contact form λ is called *nondegenerate*.

Periods of Reeb orbits   We call a number $T > 0$ a *period* if there exists a T-periodic orbit $P \in \mathcal{P}(\lambda)$.

Fix $C > 0$. If the contact form λ is nondegenerate, then there exists a finite number of orbits in $\mathcal{P}(\lambda)$ with period less that C. This is a consequence of the following lemma.

Lemma 1.4. *Let λ be a nondegenerate contact form on a closed manifold M. Fix $C > 0$ and let $(x, T) \in \mathcal{P}(\lambda)$ be a prime orbit satisfying $T \le C$. Then there exists a neighborhood $V \subset M$ of $x(\mathbb{R})$ such that if $(\bar{x}, \bar{T}) \in \mathcal{P}(\lambda) \setminus \{(x, T)\}$ satisfies $\bar{T} \le C$, then $\bar{x}(\mathbb{R}) \cap V = \emptyset$. In particular, there exists a finite number of orbits in $\mathcal{P}(\lambda)$ with period $\le C$.*

Proof. Consider a Poincaré section $\mathcal{D}$ at $x(0)$ such that $T_{x(0)}\mathcal{D} = \xi_{x(0)}$. Let $U \subset \mathcal{D}$ be a neighborhood of $x(0)$ such that the first return map

$$\phi : U \to \mathcal{D}, \quad \phi(x) = \varphi^{\tau^*(x)}(x)$$

is defined and $\phi(U) \subset U$, so that the iterate $\phi^k : U \to U$ is defined for every $k \ge 1$. Here $\tau^* : U \to \mathbb{R}^+$ is the first return map defined by $\tau^*(x) = \inf\{t > 0 | \varphi^t(x) \in \mathcal{D}\}$. Computing the derivative of ϕ at $x(0)$ and using $T = \tau^*(x(0))$, we obtain

$$d\phi_{x(0)} = d\varphi^T|_{\xi(x(0))}.$$

Since the orbit (x, T) is nondegenerate, then

$$d\phi_{x(0)} - id = d\varphi^T|_{\xi(x(0))} - id$$

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[2]A *knot* is a copy of S^1 embedded in S^3. More generally, a *link* with n components is a disjoint union of n copies of S^1 embedded in S^3. A knot K is called *unknot* or *trivial knot* if there exists a embedded disk $D \subset S^3$ such that $K = \partial D$. A link $L = \bigsqcup_{i=1}^n L_i$ is called *unlink* or *trivial link* if there exist n disjoint embedded disks $D_1, \ldots, D_n$ in S^3 such that $\partial D_i = L_i$.

[3]$\mathrm{lk}(x_T, \bar{x}_{\bar{T}}) = \mathrm{lk}(\bar{x}_{\bar{T}}, x_T)$ and $\mathrm{lk}(x_T, \bar{x}_{\bar{T}}) = 0$ if and only if the homology class of $t \in \mathbb{R}/\mathbb{Z} \mapsto x(Tt)$ in $H_1(S^3 \setminus \bar{x}(\mathbb{R}), \mathbb{Z})$ is zero.

[4]not linked $\not\Rightarrow$ unlink, but unlink $\Rightarrow$ not linked.

is an isomorphism. Using the inverse function theorem, we conclude that there exists a neighborhood $U_1 \subset U$ of $x(0)$ such that $x(0)$ is the only fixed point of ϕ in U. For any $k > 1$, we have $d\phi^k x(0) = d\varphi^{nT}|_{\xi(x(0))}$. Since the orbit (x, nT) is nondegenerate, then $d\phi^k x(0) - id = d\varphi^{nT}|_{\xi(x(0))} - id$ is an isomorphism, so that $x(0)$ is the only fixed point of ϕ^k in a neighborhood $U_k \subset U$ of $x(0)$. Fix $N \in \mathbb{N}$ such that $NT > C$ and define $\mathcal{U} = \bigcap_{n=1}^{2N} U_k$. Then $x(0)$ is the only periodic point of ϕ in $\mathcal{U}$ with period $\leq 2N$. Shrinking $\mathcal{U}$ if necessary, we can assume that $\phi(\mathcal{U}) \subset \mathcal{U}$ and for every $y \in \mathcal{U}$,

$$|\tau^*(y) - \tau^*(x(0))| < \frac{T}{2}.$$

Let $(\bar{x}, \bar{T}) \in \mathcal{P}(\lambda) \setminus \{(x, T)\}$ be a prime orbit such that $\bar{x}(0) \in \mathcal{U}$. Then $\bar{x}(0)$ is a periodic point of ϕ with period $M > 2N$. It follows that $|\tau^*(\phi^k(\bar{x}(0))) - \tau^*(x(0))| < \frac{T}{2}$, for $k = 0, \ldots, M-1$. Since $\sum_{k=0}^{M-1} \tau^*(\phi^k(\bar{x}(0))) = \bar{T}$, we have

$$|\bar{T} - MT| \leq \sum_{k=0}^{M-1} |\tau^*(\phi^k(\bar{x}(0))) - T| < M\frac{T}{2}.$$

It follows that
$$C < NT < M\frac{T}{2} = MT - M\frac{T}{2} \leq \bar{T}.$$

We have proved that if an orbit $(\bar{x}, \bar{T}) \in \mathcal{P}(\lambda) \setminus \{(x, T)\}$ satisfy $\bar{T} \leq C$, then $\bar{x}(\mathbb{R})$ does not intersect the section $\mathcal{U}$. Let $\mathcal{V}$ be a neighborhood of $x(\mathbb{R})$ in M such that any orbit intersecting the neighborhood $\mathcal{V}$ also intersects the section $\mathcal{U}$. Then any orbit $(\bar{x}, \bar{T}) \in \mathcal{P}(\lambda) \setminus \{(x, T)\}$ satisfying $\bar{T} \leq C$ satisfies $\bar{x}(\mathbb{R}) \cap \mathcal{V} = \emptyset$. □

Thus, if λ is nondegenerate, we can define $\sigma(C)$ as any real number satisfying

$$0 < \sigma(C) < \min\{T', |T' - T''| : T' \neq T'' \text{ periods }, T', T'' \leq C\} . \qquad (1.5)$$

The number $\sigma(C)$ will be important later in Chapters 2 and 4.

1.2.1 The Conley-Zehnder index

An axiomatic definition Let $Sp(n)$ denote the symplectic group in dimension $2n$. Consider the set $\Sigma^*(1)$ of paths $\varphi \in C^\infty([0,1], \text{Sp}(1))$ such that $\varphi(0) = I$ and $\det(\varphi(1) - I) \neq 0$.

The Conley-Zehnder index is a map $\mu : \Sigma^*(1) \to \mathbb{R}$ that can be axiomatically characterized as follows.

Theorem 1.5 ([HWZ03, Theorem 8.1]). *There is a unique surjective map* $\mu : \Sigma^*(1) \to \mathbb{Z}$, *called the Conley-Zehnder index, that satisfies the following axioms.*[5]

(1) The map $s \mapsto \mu(\varphi^s)$ *is constant if* $\{\varphi^s\}$ *is a homotopy of paths in* $\Sigma^*(1)$;

(2) If $\psi : [0,1] \to \mathrm{Sp}(1)$ *is a smooth loop based at* I, *then*

$$\mu(\psi\varphi) = 2\,\mathrm{Maslov}(\psi) + \mu(\varphi), \quad \forall \varphi \in \Sigma^*(1)$$

(3) If $\varphi \in \Sigma^*(1)$, *then* $\mu(\varphi^{-1}) = -\mu(\varphi)$

(4) If $\varphi(t) = \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix}$, *then* $\mu(\varphi) = 1$.

Fix a nondegenerate orbit $P = (x, T) \in \mathcal{P}(\lambda)$. The vector bundle $x_T^*\xi \to S^1$ becomes a symplectic bundle with the bilinear form $d\lambda$.

Let $\Psi : x_T^*\xi \to S^1 \times \mathbb{R}^2$ be a symplectic trivialization and consider the arc of symplectic matrices $\Phi \in \mathbb{C}^\infty([0,1], Sp(1))$ defined by

$$\Phi(t) = \Psi_t \circ d\varphi^{Tt}|_{\xi_{x(0)}} \circ \Psi_0^{-1} \; .$$

The arc Φ satisfies $\Phi(0) = I$ and, since P is nondegenerate, $\det(\Phi(1) - I) \neq 0$. Thus, $\Phi \in \Sigma^*(1)$.

Definition 1.6. We define the Conley-Zehnder index of the orbit P relative to the trivialization Ψ by

$$\mu(P, \Psi) = \mu(\Phi) \in \mathbb{Z} \; .$$

This index only depends on the homotopy class [6] of the trivialization Ψ and we denote $\mu(P, [\Psi]) = \mu(P, \Psi)$.

Global trivializations of the tight contact structure $\boldsymbol{\xi_0}$ on $\boldsymbol{S^3}$ Later on, we will only deal with the tight contact structure $\xi_0 = \ker \lambda_0|_{S^3}$ on S^3, that is a trivial symplectic bundle. Here λ_0 is the Liouville 1-form defined

---

[5]The Maslov index is a function that assigns an integer to every path of symplectic matrices $\psi : \frac{\mathbb{R}}{\mathbb{Z}} \to Sp(n)$. This function is homotopy invariant and induces an isomorphism $\pi_1(Sp(n)) \to \mathbb{Z}$. Definitions can be found in [MS17] or [HWZ95b].

[6]Let $E \xrightarrow{p} B$ be a vector bundle of rank k. Two trivializations $\Psi, \Phi : E \to B \times \mathbb{R}^k$ are homotopic if there exists a trivialization $F : E \times I \to (B \times I) \times \mathbb{R}^k \cong (B \times \mathbb{R}^k) \times I$ of the vector bundle $E \times I \xrightarrow{p \times id} B \times I$ such that, for every $e \in E$, $F(e, 0) = (\Psi(e))$ and $F(e, 1) = (\Phi(e), 1)$.

in 1.3. Now we show that the Conley-Zehnder index is independent of the choice of global symplectic trivialization of ξ_0.

Let $\Psi, \Psi' : \xi_0 \to S^3 \times \mathbb{R}^2$ be two symplectic trivializations of ξ_0 and let $P \in \mathcal{P}(\lambda)$ be a nondegenerate orbit, where λ is a contact form such that $\xi_0 = \ker \lambda$. Then the trivializations $\Psi : x_T^* \xi_0 \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2$ and $\Psi' : x_T^* \xi_0 \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2$ are homotopic. This is a consequence of the fact that the orbit P is contractible and of the following lemma.

Lemma 1.7. *Let $\phi : \mathbb{D} \to S^3$ be a continuous map. Then any two trivializations of the bundle $\phi^* \xi_0$ are homotopic.*

Proof. Let $\Psi_1, \Psi_2 : \phi^* \xi_0 \to \mathbb{D} \times \mathbb{R}^2$ be two symplectic trivializations of the bundle $\phi^* \xi_0 \xrightarrow{p} \mathbb{D}$. Consider the map $f : \mathbb{D} \to Sp(1)$ defined by $a \mapsto \Psi_1 \circ \Psi_2^{-1}|_{\{a\} \times \mathbb{R}^2}$. Since $\mathbb{D}$ is contractible, the map $f : \mathbb{D} \to Sp(1)$ is homotopic to a constant map. Since $Sp(1)$ is path connected, any constant map $x \in \mathbb{D} \mapsto c \in Sp(1)$ is homotopic to the constant map $x \in \mathbb{D} \mapsto id \in Sp(1)$. Thus, there exists a homotopy $H : \mathbb{D} \times I \to Sp(1)$ between $f : \mathbb{D} \to Sp(1)$ and the constant map $x \in \mathbb{D} \mapsto id \in Sp(1)$. Define $G : \phi^* \xi_0 \times I \to \mathbb{D} \times \mathbb{R}^2 \times I$ by

$$G(e,t) = (p(e), H(p(e),t) \cdot \Psi_2|_{p^{-1}(p(e))}(e), t), \text{ for } (e,t) \in \phi^* \xi_0 \times I.$$

Then G defines a homotopy between the trivializations Ψ_1 and Ψ_2. In fact, G is a bundle morphism with inverse given by

$$G^{-1}(x,v,t) = (\Psi_2^{-1}|_{\{x\} \times \mathbb{R}^2}(H(x,t)^{-1} \cdot v), t), \text{ for } (x,v,t) \in \mathbb{D} \times \mathbb{R}^2 \times I,$$

$$G(e,0) = (p(e), H(p(e),0) \cdot \Psi_2|_{p^{-1}(p(e))}(e), 0)$$
$$= (p(e), (\Psi_1 \circ \Psi_2^{-1})|_{\{p(e)\} \times \mathbb{R}^2} \cdot \Psi_2|_{p^{-1}(p(e))}(e), 0) = \Psi_1(e), \ \forall e \in \phi^* \xi_0$$

and
$$G(e,1) = (p(e), H(p(e),1) \cdot \Psi_2|_{p^{-1}(p(e))}(e), 1)$$
$$= (p(e), id \cdot \Psi_2|_{p^{-1}(p(e))}(e), 1) = \Psi_2(e), \ \forall e \in \phi^* \xi_0.$$

$\square$

A geometric description of the Conley-Zehnder index Following [HWZ03], we present a geometric construction for the index $\mu : \Sigma^*(1) \to \mathbb{Z}$.

Fix $\Phi \in \Sigma^*(1)$. Let $z \in \mathbb{C} \setminus \{0\}$ and let $\theta : [0,1] \to \mathbb{R}$ be a continuous argument for $z(t) := \Phi(t)z$, that is, $e^{2\pi\theta(t)} = \frac{z(t)}{|z(t)|}, \ \forall \ 0 \leq t \leq 1$.

Define the *winding number* of $z(t) = \Phi(t)z$ by

$$\Delta(z) = \theta(1) - \theta(0) \in \mathbb{R}$$

and the *winding interval* of the arc Φ by

$$I(\Phi) = \{\Delta(z) | z \in \mathbb{C} \setminus \{0\}\} \ .$$

It is proved in [HWZ03] that the length of this interval is strictly smaller than $\frac{1}{2}$. Thus, the winding interval either lies between two consecutive integers or contains precisely one integer. Define

$$\mu(\Phi) = \begin{cases} 2k+1, \text{ if } I(\Phi) \subset (k, k+1) \\ 2k, \text{ if } k \in I(\Phi) \end{cases}$$

It is proved in [HK99] that $\Phi \mapsto \mu(\Phi)$ satisfies the assumptions of Theorem 1.5. Thus, it agrees with the Conley-Zehnder index.

Recall that the spectrum of a matrix $A \in Sp(1)$ with $\det(A - I) \neq 0$ satisfies the following: either $\sigma(A) = \{\lambda, \bar{\lambda}\} \subset S^1 \setminus \{1\}$, $\sigma(A) = \{\lambda, \lambda^{-1}\} \subset (0, \infty) \setminus \{1\}$ or $\sigma(A) = \{\lambda, \lambda^{-1}\} \subset (-\infty, 0)$.

Definition 1.8. We say that an arc $\Phi \in \Sigma^*(1)$ is *hyperbolic positive* if $\sigma(\Phi(1)) \subset (0, \infty) \setminus \{1\}$, *hyperbolic negative* if $\sigma(\Phi(1)) \subset (-\infty, 0) \setminus \{-1\}$, or *elliptic* if the eigenvalues are in $S^1 \setminus \{1\}$.

Let $z \in \mathbb{C} \setminus \{0\}$. The winding number $\Delta(z)$ is an integer if and only if there exists $\lambda > 0$ such that $\Phi(1)z = \lambda z$. Thus, an arc is positive hyperbolic if and only if it has even index. For a negative hyperbolic arc, there exists $z \in \mathbb{C} \setminus \{0\}$ such that $\Phi(1)z = -\lambda z$, for some $\lambda > 0$, so that $\Delta(z) = k + \frac{1}{2}$, for some $k \in \mathbb{Z}$. Thus $\mu(\Phi) = 2k + 1$. Elliptic arcs also necessarily have odd indices.

Let $P = (x, T) \in \mathcal{P}(\lambda)$ be nondegenerate and assume that for all $n \in \mathbb{Z}$, the orbit $P^n = (x, nT)$ is nondegenerate. Fix a symplectic trivialization $\Psi : x_T^* \xi \to S^1 \times \mathbb{R}^2$. The trivialization Ψ induces a trivialization of $x_{nT}^* \xi$ that we also denote by Ψ. The following Lemma is a consequence of the properties of the Conley-Zehnder index proved in [HWZ03].

Lemma 1.9. *Assume λ is nondegenerate, let $P = (x, T)$ be a closed Reeb orbit and fix positive integers $1 \leq l \leq k$. The following assertions hold.*

(1) If $\mu(P^k, \Psi) = 1$, then $\mu(P^l, \Psi) = 1$;

(2) $\mu(P^k, \Psi) \leq 0 \iff \mu(P^l, \Psi) \leq 0$;

(3) If P is a hyperbolic orbit, then $\mu(P^l, \Psi) = l\mu(P, \Psi)$;

(4) If $\mu(P^k, \Psi) = 2$, then k, l and $\mu(P^l, \Psi)$ belong to $\{1, 2\}$ and P is hyperbolic. If $l = 1$ and $k = 2$, then $\mu(P, \Psi) = 1$.

1.3   The asymptotic operator

Let $E \xrightarrow{\pi} X$ be a smooth vector bundle. A complex structure J on E is a smooth fiberwise linear map satisfying $J_x^2 = -I$ on $\pi^{-1}(x)$, for all $x \in X$.

Definition 1.10. Let $(E \xrightarrow{\pi} X, \omega)$ be a symplectic vector bundle and let J be a smooth complex structure on E. J is said to be ω-*compatible* if

$$\omega_x(\cdot, J_x\cdot)$$

defines a positive definite inner product on E_x, for all $x \in X$. This is equivalent to requiring that for all $x \in X$, $\omega_x(J_x\cdot, J_x\cdot) = \omega_x$ and $\omega_x(v, J_x v) > 0$, for $0 \neq v \in \pi^{-1}(x)$.

The set $\mathcal{J}(E, \omega)$ of ω-compatible complex structures on (E, ω) is non empty and contractible. A proof of this fact can be found in [MS17].

Definition of the asymptotic operator   Let (M, λ) be a closed contact manifold of dimension 3 and let R be the associated Reeb vector field. Let $P = (x, T)$ be a closed Reeb orbit and let h be a vector field along $x_T :$ $S^1 = \mathbb{R}/\mathbb{Z} \to M$, that is, $h : S^1 \to TM$ is a smooth function satisfying $h(t) \in T_{x(Tt)}M, \forall t \in S^1$. We can define the Lie derivative $\mathcal{L}_R h$ of h by

$$\mathcal{L}_R h(t) = \frac{d}{ds}\bigg|_{s=0} d\varphi^{-s}(x(Tt+s))h\left(t + \frac{s}{T}\right) , \tag{1.6}$$

where φ^t is the flow of R. Note that it is the same as defining the Lie derivative of h as the Lie derivative of any extension of h to a neighborhood of $h(t)$ in M. Let ∇ be a symmetric (torsion-free) connection on TM. We can use $dx_T(t)\partial_t = TR(x_T(t))$ to write

$$T\mathcal{L}_R h = \mathcal{L}_{TR} h = [TR, \tilde{h}] = \nabla_{TR} h - \nabla_h TR = \nabla_t h - T\nabla_h R, \tag{1.7}$$

where $\tilde{h}$ is any extension of h and ∇_t is the covariant derivative along x_T. We conclude that the differential operator $\nabla_t \cdot - T\nabla. R$ maps sections of $x_T^*\xi$ to sections of $x_T^*\xi$ and is independent of the choice of symmetric connection.

Choosing some $J \in \mathcal{J}(\xi, d\lambda)$, we associate to the orbit (x, T) the unbounded differential operator [7]

$$A_{P,J} : \mathcal{D}(A_{P,J}) = W^{1,2}(S^1, x_T^*\xi) \subset L^2(S^1, x_T^*\xi) \to L^2(S^1, x_T^*\xi) \tag{1.8}$$

---

[7]We say that $Y \in W^{1,p}(x_P^*\xi)$ (or $Y \in L(x_T^*\xi)$) if in any trivialization $\Psi : x_T^*\xi \to S^1 \times \mathbb{R}^2$, where Y takes the form $Y = (y_1(t), y_2(t))$, the function $(y_1, y_2) : S^1 \to \mathbb{R}^2$ is in $W^{1,p}(S^1, \mathbb{R}^2)$ (or $L^2(S^1, \mathbb{R}^2)$).

defined by

$$A_{P,J}(h) = -J(\nabla_t h - T\nabla_h R) \ .$$

Definition 1.11 (Asymptotic operator)**.** The operator $A_{P,J}$ defined by (1.8) is called the *asymptotic operator* associated to the orbit P and the complex structure J.

The asymptotic operator in unitary trivializations

Definition 1.12. A *unitary trivialization* of the bundle $(x_T^* \xi, d\lambda, J)$ is a trivialization $\Psi : x_T^* \xi \to S^1 \times \mathbb{R}^2$ of $x_T^* \xi$ satisfying

$$\begin{cases} d\lambda = \Psi^* dx_1 \wedge dx_2 \\ J_0 \Psi = \Psi J \end{cases} , \tag{1.9}$$

where (x_1, x_2) are coordinates on $\mathbb{R}^2$ and $J_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Proposition 1.13. *In any unitary trivialization Ψ of $(x_T^* \xi, d\lambda, J)$, the operator $A_{P,J}$ takes the form*

$$L_S := -J_0 \frac{d}{dt} - S(t)$$

where $S(t)$ is a path of symmetric matrices given by $S(t) = -J_0 \dot{\phi}(t)\phi(t)^{-1}$, and $\phi(t)$ is the linearized flow restricted to ξ in the trivialization Ψ.

Proof. Consider the path of symplectic matrices

$$\phi(t) = \Psi_t \circ d\varphi^{Tt}|_{\xi_{x(0)}} \circ \Psi_0^{-1} \ ,$$

where φ^t is the Reeb flow. Note that for every $t \in \mathbb{R}$, the matrix $S(t) = -J_0 \dot{\phi}(t)\phi(t)$ is symmetric. Indeed, since $\phi(t)$ is symplectic, we have the identity

$$J_0 \dot{\phi}(t)^T J_0 = \frac{d}{dt}(J_0 \phi(t)^T J_0) = \frac{d}{dt}(-\phi(t)^{-1}) = \phi(t)^{-1}\dot{\phi}(t)\phi(t)^{-1} \tag{1.10}$$

Using (1.10) and the identity $(\phi(t)^{-1})^T = -J_0\phi(t)J_0$, we get

$$\begin{aligned} S(t)^T &= (-J_0\dot{\phi}(t)\phi(t)^{-1})^T = (\phi(t)^{-1})^T \dot{\phi}(t)^T J_0 \\ &= (\phi(t)^{-1})^T (-J_0)\phi^{-1}(t)\dot{\phi}(t)\phi^{-1}(t) = -J_0\dot{\phi}(t)\phi^{-1}(t) = S(t). \end{aligned}$$

Along $x(Tt)$, $T_{x(Tt)}M = Span\{TR, \Psi_t^{-1}e_1, \Psi_t^{-1}e_2\}$, where $e_1 = (1,0)$ and $e_2 = (0,1)$. In this coordinates for x_T^*TM, the linearized flow can be represented by a 3×3 matrix written in blocks

$$d\varphi_{Tt}(x(0)) = \begin{bmatrix} 1 & 0 \\ 0 & \phi(t) \end{bmatrix}.$$

Using the identity

$$\frac{d}{dt}d\varphi_{Tt}(x(0)) = T(dR \circ \varphi_{Tt})d\varphi_{Tt}(x(0)),$$

we conclude that $dR(x(Tt))$ can be written in the same coordinates above as

$$
\begin{aligned}
TdR(x(Tt)) &= \frac{d}{dt}d\varphi_{Tt}(x(0)) \circ d\varphi_{Tt}(x(0))^{-1} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & \dot{\phi}(t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \phi(t)^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \dot{\phi}(t)\phi^{-1}(t) \end{bmatrix}.
\end{aligned}
\tag{1.11}
$$

Now we compute $\Psi \circ A_{P,J} \circ \Psi^{-1}$. Let $n : S^1 \to \mathbb{R}^2$, $n(t) = (a_1(t), a_2(t))$ be a smooth function. Let (t, x_1, x_2) be coordinates in a neighborhood of $x(Tt)$ in M such that $x(Tt) = (t, 0, 0)$, $\frac{\partial}{\partial t}(t, 0, 0) = TR$, $\frac{\partial}{\partial x_1}(t, 0, 0) = \Psi^{-1}e_1$ and $\frac{\partial}{\partial x_2}(t, 0, 0) = \Psi^{-1}e_2$. Then

$$
\begin{aligned}
\Psi \circ A_{P,J} \circ \Psi^{-1}(n)(t) &= \Psi \left[-J \left(\nabla_t a_k(t)\frac{\partial}{\partial x_k} - \nabla_{a_k(t)\frac{\partial}{\partial x_k}}TR \right)(t, 0, 0) \right] \\
&= -J_0\Psi \left[\dot{a}_k(t)\frac{\partial}{\partial x_k} + a_k(t)\nabla_{TR}\frac{\partial}{\partial x_k} - a_k(t)\nabla_{\frac{\partial}{\partial x_k}}TR \right](t, 0, 0) \\
&= -J_0\dot{n}(t) - J_0\Psi \left[a_k(t) \left(\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_k}}TR \right)(t, 0, 0) \right]
\end{aligned}
$$

We write $TR(t, x_1, x_2)$ in coordinates near $x(Tt)$ as

$$TR(t, x_1, x_2) = r_0(t, x_1, x_2)\frac{\partial}{\partial t} + r_1(t, x_1, x_2)\frac{\partial}{\partial x_1} + r_2(t, x_1, x_2)\frac{\partial}{\partial x_2},$$

so that $TR(t, 0, 0) = \frac{\partial}{\partial t}$. Using these coordinates and equation (1.11), we obtain

$$\nabla_{\frac{\partial}{\partial x_k}}TR(t, 0, 0) = \frac{\partial r_1}{\partial x_k}\frac{\partial}{\partial x_1} + \frac{\partial r_2}{\partial x_k}\frac{\partial}{\partial x_2} + \nabla_{\frac{\partial}{\partial x_k}}\frac{\partial}{\partial t}.$$

Since ∇ is symmetric, we conclude that

$$
\begin{aligned}
\Psi \circ A_{P,J} \circ \Psi^{-1}(n)(t) &= -J_0\dot{n}(t) + J_0\Psi\left(a_k(t)\frac{\partial r_i}{\partial x_k}\frac{\partial}{\partial x_i}\right) \\
&= -J_0\dot{n}(t) + J_0(\pi_\xi dR\pi_\xi)(t,0,0)n(t) \\
&= -J_0\dot{n}(t) + J_0\dot{\phi}(t)\phi(t)^{-1}n(t)
\end{aligned}
$$

$\square$

Properties of the operator L_S   The unbounded operator

$$
L_S : \mathcal{D}(L_S) = W^{1,2}(S^1,\mathbb{R}^2) \subset L^2(S^1,\mathbb{R}^2) \to L^2(S^1,\mathbb{R}^2)
$$

is self-adjoint. Its spectrum consists of real eigenvalues which only accumulate at $+\infty$ and $-\infty$. [8]

If $n(t)$ is an eigenfunction of L_S with corresponding eigenvalue $\lambda \in \mathbb{R}$ and $n(t)$ is not identically zero in $L^2(S^1,\mathbb{R}^2)$, then $n(t)$ solves the linear first order differential equation

$$
-J_0\dot{n}(t) - S(t)n(t) = \lambda n(t), \ n(t+1) = n(t) \ .
$$

Hence $n(t) \neq 0$ for all $t \in S^1$. It follows that $n(t)$ has a well defined winding number given by

$$
\mathrm{wind}(n,\lambda) := \deg\left(t \mapsto \frac{n(t)}{\|n(t)\|}\right) \in \mathbb{Z} \ .
$$

The following properties about this winding number are proved in [HWZ95b].

Proposition 1.14. *[HWZ95b]*

- *Given nonzero eigenfunctions $x(t)$ and $y(t)$ associated to the same eigenvalue λ, we have*

$$
\mathrm{wind}(x,\lambda) = wind(y,\lambda)
$$

---

[8]It is easy to see that L_S is symmetric with respect to the usual inner product in $L^2(S^1,\mathbb{R}^2)$, that is, $\mathcal{D}(L_S)$ is dense and $\int_S^1 \langle L_S x(t), y(t)\rangle dt = \int_S^1 \langle x(t), L_S y(t)\rangle$, for all $x,y \in W^{1,2}(S^1,\mathbb{R}^2)$. Since L_S is symmetric and surjective, then L_S is self-adjoin (see Proposition 8.3 in [Tay11]). Also, L_S is a bijection, because $0 = \mathcal{R}(L_S)^\perp = \ker L_S^* = \ker L_S$. Since L_S is symmetric and bijective, L_S^{-1} is symmetric. By Hellinger-Toeplitz Theorem, L_S^{-1} is bounded. Since the embedding $W^{1,2}(S^1,\mathbb{R}^2) \to L^2(S^1,\mathbb{R}^2)$ is compact, it follows that L_S^{-1} is a compact self-adjoint operator. Applying the Spectral Theorem to L_S^{-1}, we conclude that the spectrum of L_S consists of real eigenvalues which only accumulate at $+\infty$ and $-\infty$.

so that we can define $\mathrm{wind}(\lambda) := \mathrm{wind}(x, \lambda)$, *for any eigenfunction associated to* λ.

- *If* $\lambda \neq \mu \in \sigma(L_S)$ *satisfy* $\mathrm{wind}(\lambda) = \mathrm{wind}(\mu)$ *and* $x(t), y(t)$ *are non-vanishing* λ, μ-*eigenfunctions, respectively, then* $x(t), y(t)$ *are pointwise linearly independent.*

- *Given* $k \in \mathbb{Z}$, *there exists precisely two eigenvalues* $\lambda, \mu \in \sigma(L_S)$, *counting multiplicities, such that* $\mathrm{wind}(\lambda) = \mathrm{wind}(\mu) = k$

- *If* $\lambda, \mu \in \sigma(L_S)$ *and* $\lambda \leq \mu$, *then* $\mathrm{wind}(\lambda) \leq \mathrm{wind}(\mu)$.

Properties of the asymptotic operator $A_{P,J}$ Given $\Psi : x_T^* \xi \to S^1 \times \mathbb{R}^2$ a trivialization and $t \mapsto Y(t)$ a section such that $Y(t) \neq 0$ for all $t \in S^1$, we define

$$\mathrm{wind}(Y, \Psi) := \deg\left(t \mapsto \frac{\Psi(Y(t))}{\|\Psi(Y(t))\|}\right). \tag{1.12}$$

This definition just depends on the homotopy class of the trivialization Ψ, and we denote $\mathrm{wind}(Y, [\Psi]) = \mathrm{wind}(Y, \Psi)$, for any Ψ in class $[\Psi]$.

The properties of the operator L_S discussed above imply similar properties for the asymptotic operator $A_{P,J}$, that we summarize below.

Proposition 1.15. *The unbounded operator* A_P *has discrete real spectrum accumulating only at* $\pm\infty$. *Fix* $[\Psi]$ *a homotopy class of unitary trivializations. Then*

- *Given nonzero eigensections* $\eta_1(t)$ *and* $\eta_2(t)$ *associated to the same eigenvalue* λ, *we have*

$$\mathrm{wind}(\eta_1, [\Psi]) = \mathrm{wind}(\eta_2, [\Psi])$$

so that we can define $\mathrm{wind}(\lambda, [\Psi]) := \mathrm{wind}(\eta, [\Psi])$, *for any eigensection* η *associated to* λ.

- *If* $\lambda \neq \mu \in \sigma(A_{P,J})$ *satisfy* $\mathrm{wind}(\lambda, [\Psi]) = \mathrm{wind}(\mu, [\Psi])$ *and* $\eta(t), \nu(t)$ *are non-vanishing* λ, μ-*eigensections, respectively, then* $\eta(t), \nu(t)$ *are pointwise linearly independent.*

- *Given* $k \in \mathbb{Z}$, *there exists precisely two eigenvalues* $\lambda, \mu \in \sigma(A_{P,J})$, *counting multiplicities, such that* $\mathrm{wind}(\lambda, [\Psi]) = \mathrm{wind}(\mu, [\Psi]) = k$

- *If* $\lambda, \mu \in \sigma(A_{P,J})$ *and* $\lambda \leq \mu$, *then* $\mathrm{wind}(\lambda, [\Psi]) \leq \mathrm{wind}(\mu, [\Psi])$.

- $0 \notin \sigma(A_{P,J})$ *if and only if the orbit* $P = (x, T)$ *is nondegenerate.*

The Conley-Zehnder index   Define

$$\nu_P^{neg} = \max\{\nu < 0 | \nu \text{ is an eigenvalue of } A_P\} \tag{1.13}$$

$$\nu_P^{pos} = \min\{\nu \geq 0 | \nu \text{ is an eigenvalue of } A_P\} \tag{1.14}$$

and fixing a trivialization Ψ of $x_T^*\xi$, define

$$p = \text{wind}(\nu_P^{pos}) - \text{wind}(\nu_P^{neg}) \ .$$

One can check that $p \in \{0, 1\}$.

Definition 1.16. We define the (generalized) Conley-Zehnder index of the orbit P relative to the unitary trivialization Ψ as

$$\tilde{\mu}(P, \Psi) = 2\,\text{wind}(\nu_P^{neq}, \Psi) + p. \tag{1.15}$$

It is proved in [HWZ95b, Theorem 3.10] that for any nondegenerate orbit $P \in \mathcal{P}(\lambda)$,

$$\tilde{\mu}(P, \Psi) = \mu(P, \Psi) \ ,$$

where $\mu(P, \Psi)$ is the Conley-Zehnder index defined in 1.2.1.

1.4   The action functional

Let (M, λ) be a closed contact manifold of dimension 3. We define the *action functional* $\mathcal{A} : C^\infty(S^1 = \mathbb{R}/\mathbb{Z}, M) \to \mathbb{R}$ by

$$\mathcal{A}(\gamma) = \int_\gamma \lambda := \int_{S^1} \gamma^*\lambda \ .$$

First variation of the action functional   Let $\gamma \in C^\infty(S^1, M)$ and let $h \in \Gamma(\gamma^*TM)$ be a vector field along γ. Consider a function $u : (-\epsilon, \epsilon) \times S^1 \to M, \ (s, t) \mapsto u(s, t)$ satisfying

$$\begin{cases} u(0, t) = \gamma(t) \\ \dfrac{\partial u}{\partial s}(0, t) = h(t) \end{cases}, \forall t \in S^1$$

Then

$$
\begin{aligned}
\frac{d}{ds}\Big|_{s=0} \mathcal{A}(u(s,\cdot)) &= \frac{d}{ds}\Big|_{s=0} \int_{S^1} u(s,\cdot)^*\lambda \\
&= \int_{S^1} \frac{d}{ds}\Big|_{s=0} (u^*\lambda)_{(s,t)}\left(\frac{\partial}{\partial t}\right) dt \\
&= \int_{S^1} \mathcal{L}_{\frac{\partial}{\partial s}} (u^*\lambda)_{(0,t)}\left(\frac{\partial}{\partial t}\right) dt \\
&= \int_{S^1} \left(i_{\frac{\partial}{\partial s}} d(u^*\lambda) + d i_{\frac{\partial}{\partial s}}(u^*\lambda)\right)_{(0,t)}\left(\frac{\partial}{\partial t}\right) dt \\
&= \int_{S^1} d\lambda(h(t), \dot{\gamma}(t)) dt.
\end{aligned}
$$

In the last equality we used the fact that $\int_{S^1} d i_{\frac{\partial}{\partial s}}(u^*\lambda)_{(0,t)}\left(\frac{\partial}{\partial t}\right) dt = 0$, since the integrand is the differential of a function on S^1.

It follows that γ is a critical point of $\mathcal{A}$, that is, $\dfrac{d}{ds}\Big|_{s=0}\mathcal{A}(\gamma_s) = 0$ for all smooth curve $s \mapsto \gamma_s$ satisfying $\gamma_0 = \gamma$, if and only if $\dot{\gamma}(t)$ points in the direction of the Reeb vector field $R(\gamma(t))$ for all t. Indeed, assume that γ is a critical point of $\mathcal{A}$ and that there exist $Y \in \Gamma(\gamma^*TM)$ and an open set U of S^1 such that $d\lambda(Y(\gamma(t)), \dot{\gamma}(t)) > 0$ on U. Consider $f \geq 0$ a smooth function with support in U and define the vector field $X = fY \in \Gamma(\gamma^*TM)$. It follows that $d\lambda(X, \dot{\gamma}) \geq 0$ and $\int_{S^1} d\lambda(X(t), \dot{\gamma}(t)) dt > 0$ on U, a contradiction.

It is clear that for any T-periodic solution $x : \mathbb{R} \to M$ of $\dot{x}(t) = R(x(t))$, the loop $x_T : S^1 \to M$; $x_T(t) = x(Tt)$ is a critical point of the action functional. On the other hand, every critical point satisfying $\dot{\gamma}(t) \neq 0, \forall t \in S^1$ and $\mathcal{A}(\gamma) = T$ is the reparametrization of a T-periodic Reeb orbit. Indeed, assume γ is a critical point satisfying $\dot{\gamma}(t) \neq 0, \forall t \in S^1$, i.e, $\dot{\gamma}(t) = f(t)R(t)$ and $f(t) \neq 0$. Let $T = \mathcal{A}(\gamma)$. Then there exists a reparametrization $\sigma(t) = \gamma(g(t))$ such that $\dot{\sigma}(t) = TR(\sigma(t))$. Indeed, consider $f : \mathbb{R} \to \mathbb{R}$ 1-periodic and $h(t) = \int_0^t f(\tau)d\tau$. Since $h(t+1) = T + \int_0^t f(\tau)d\tau$, $\frac{1}{T}h$ can be seen as a degree one map $S^1 \to S^1$. Since $f(t) \neq 0 \; \forall t$, then $\frac{1}{T}h$ is a diffeomorphism (bijective immersion). Define $g(t) = (\frac{1}{T}h)^{-1}(t)$ and $\sigma(t) = \gamma(g(t))$. Then $\dot{\sigma}(t) = TR(\sigma(t))$.

The asymptotic operator as the "hessian of $\mathcal{A}_\lambda$ in the direction ξ"
Let $\gamma(t) = x(Tt)$ and let $X, Y \in \Gamma(\gamma^*\xi)$ be vector fields along γ in the contact direction. Let $u : U \times S^1 \to M, (r, s, t) \mapsto u(r, s, t)$ be a smooth function

such that

$$u(0,0,t) = \gamma(t), \ \frac{\partial u}{\partial r}(0,0,t) = X(t) \ \text{ and } \ \frac{\partial u}{\partial s}(0,0,t) = Y(t) \ ,$$

where $U \ni 0$ is an open set in $\mathbb{R}^2$. Define $\tilde{u} : U \to \mathbb{C}^\infty(S^1, M)$ by $\tilde{u}(r,s) = u(r,s,\cdot)$. It follows that

$$\frac{\partial}{\partial r}(\mathcal{A} \circ \tilde{u})(s,r) = \int_{S^1} d\lambda_{u(r,s,t)} \left(\frac{\partial u}{\partial s}(r,s,t), \frac{\partial u}{\partial t}(r,s,t) \right) dt.$$

Let J be a $d\lambda$-compatible smooth complex structure on ξ and let g be the Riemannian metric on M defined by $g(u,v) = \lambda(u)\lambda(v) + d\lambda(\pi_\xi u, J\pi_\xi v)$. Let ∇ be the Levi-Civitta connection associated with g. Then

$$
\begin{aligned}
\frac{\partial^2}{\partial s \partial r}(\mathcal{A} \circ \tilde{u})\bigg|_{(0,0)} &= \frac{\partial}{\partial r}\bigg|_{(0,0)} \int_{S^1} d\lambda_{u(r,s,t)} \left(\frac{\partial u}{\partial s}(r,s,t), \frac{\partial u}{\partial t}(r,s,t) \right) dt \\
&= \partial_r|_{(0,0)} \int_{S^1} -d\lambda(J\pi_\xi \partial_t u, J\pi_\xi \partial_s u) dt \\
&= \int_{S^1} \partial_r|_{(0,0)} g(-J\pi_\xi \partial_t u, \pi_\xi \partial_s u) dt \\
&= \int_{S^1} g\left(\nabla_{X(t)}(-J\pi_\xi \partial_t u(r,0,t)|_{r=0}, Y(t)) \right) \\
&\quad + g(\underbrace{-J\pi_\xi \dot{\gamma}(t)}_{=0}, \nabla_{X(t)} \pi_\xi \partial_s u|_{(0,0,t)}) dt.
\end{aligned}
$$

In the last equality we have used $\nabla_X g = 0$. For the first term in the integral above, we compute

$$
\begin{aligned}
\nabla_X(-J\pi_\xi \partial_t u(r,0,t))|_{r=0} &= -J\nabla_X \left(\partial_t u(r,0,t) - \lambda(\partial_t u(r,0,t)) R(u(r,0,t)) \right)|_{r=0} \\
&= -J\{\nabla_X \partial_t u(r,0,t)|_{r=0} - \nabla_X \lambda(\partial_t u(r,0,t))|_{r=0} R(\gamma(t)) \\
&\quad - \lambda(\dot{\gamma}(t)) \nabla_X R|_{\gamma(t)}\},
\end{aligned}
$$

where in the first equality, we have used $\pi_\xi \partial_t u(0,0,t) = 0$.

Remark 1.17. $\nabla_X \lambda(\partial_t u(r,0,t))|_{r=0} = 0$. Ideed,

$$\nabla_X \lambda(\partial_t u(r,0,t))|_{r=0} = (u^* \nabla)_{\frac{\partial}{\partial r}} u^* \lambda \left(\frac{\partial}{\partial t}\right)\bigg|_{(0,0,t)}$$

$$= \frac{\partial}{\partial r} \cdot u^* \lambda \left(\frac{\partial}{\partial t}\right)\bigg|_{(0,0,t)}$$

$$= du^* \lambda \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial t}\right)\bigg|_{(0,0,t)} + \frac{\partial}{\partial t} \cdot u^* \lambda \left(\frac{\partial}{\partial r}\right)\bigg|_{(0,0,t)}$$

$$- u^* \lambda \left(\left[\frac{\partial}{\partial r}, \frac{\partial}{\partial t}\right]\right)$$

$$= 0 .$$

In the last equality we have used $d\lambda(X(t), \dot{\gamma}(t)) = 0$ and $\lambda(\partial_r u) = 0$ along $t \mapsto (0,0,t)$.[9]

Remark 1.18. $\nabla_{X(t)} \partial_t u(\gamma(t)) = \nabla_{\dot{\gamma}(t)} X(\gamma(t))$. Indeed, consider local coordinates $(x_1, \dots, x_{2n+1})$ on M near $\gamma(t)$ and write

$$\partial_t u(r,0,t) = \sum_k \partial_t u_k(r,0,t) \frac{\partial}{\partial x_k}(u(r,0,t)),$$

$$X(t) = \sum_i \partial_r u_i(0,0,t) \frac{\partial}{\partial x_i}(\gamma(t)), \quad \dot{\gamma}(t) = \sum_l \dot{\gamma}_l(t) \frac{\partial}{\partial x_l}(\gamma(t)).$$

Thus,

$$\nabla_{X(t)} \partial_t u(\gamma(t)) = \sum_k \partial_r \partial_t u_k(0,0,t) \frac{\partial}{\partial x_k}(\gamma(t)) + \sum_{i,k} \dot{\gamma}_k(t) \partial_r u_i(0,0,t) \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k},$$

$$\nabla_{\dot{\gamma}} X(\gamma(t)) = \sum_i \partial_t \partial_r u_i(0,0,t) \frac{\partial}{\partial x_i}(\gamma(t)) + \sum_{i,l} \partial_r u_i(0,0,t) \dot{\gamma}_l(t) \nabla_{\frac{\partial}{\partial x_l}} \frac{\partial}{\partial x_i}.$$

The equality follows from ∇ being symmetric.

It follows from the remarks above that

$$\nabla_X(-J\pi_\xi \partial_t u(r,0,t))|_{r=0} = -J\left(\nabla_{\dot{\gamma}} X - T\nabla_X R\right)(\gamma(t)).$$

---

[9]Also, we have used the formula $d\omega(X,Y) = X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X,Y])$.

We conclude that

$$\left. \frac{\partial^2}{\partial s \partial r}(\mathcal{A} \circ \tilde{u}) \right|_{(0,0)} = \int_{S^1} d\lambda \left(A_{P,J}(X), JY \right) dt$$
$$= T \int_{S^1} d\lambda(Y(t), \mathcal{L}_R X(t)) dt \ , \tag{1.16}$$

where $A_{P,J}$ is the asymptotic operator defined in (1.8).

Chapter 2

Finite energy surfaces in symplectizations

2.1 Almost complex structures in symplectizations

Let W is a smooth manifold. An *almost complex structure* on W is a complex structure on the tangent bundle TW. If J is an almost complex structure on W, the pair (W, J) is called an *almost complex manifold*.

Definition 2.1. Let (W, J) be an almost complex manifold and let (Σ, j) be a Riemann surface. A smooth map $u : \Sigma \to W$ is called a *J-holomorphic curve* if the differential du is complex-linear, that is, satisfies the equation

$$J(u) \circ du = du \circ j \ .$$

Let (W, ω) be a symplectic manifold. An ω-*compatible almost complex structure* on (W, ω) is an ω-compatible complex structure on TW (see definition 1.10). We denote the set of ω-compatible almost complex structures on (W, ω) by $\mathcal{J}(W, \omega)$.

Let (M, λ) be a closed contact 3-manifold and let $\xi = \ker d\lambda$ be the associated contact structure. Recall that $(\xi, d\lambda|_\xi)$ is a symplectic vector bundle. Fixed a $d\lambda$-compatible complex structure $J \in \mathcal{J}(\xi, d\lambda)$, there is a canonical extension $\tilde{J}$ of the complex structure J to $T(\mathbb{R} \times M)$ defined by

$$\tilde{J}\partial_a = R_\lambda, \quad \tilde{J}|_\xi = J. \tag{2.1}$$

Here a is the coordinate on $\mathbb{R}$ and R_λ is the Reeb vector field associated with

λ. These equations uniquely define the extension $\tilde{J}$ because of the splitting

$$T(\mathbb{R} \times M) = \mathbb{R}\partial_a \oplus \mathbb{R}R_\lambda \oplus \xi \ .$$

Note that for all $(h, k) \in T(\mathbb{R} \times M)$,

$$\tilde{J}(h, k) = (-\lambda(k), J\pi_\xi k + hR_\lambda) \ .$$

Proposition 2.2. *The almost complex structure $\tilde{J}$ defined by (2.1) is $d\lambda_\phi$-compatible, where λ_ϕ is the 1-form defined by (1.2), for any $\phi \in C^\infty(\mathbb{R}, (0, +\infty))$ satisfying $\phi' > 0$.*

Proof. Let $(a, m) \in \mathbb{R} \times M$ and $(h, k), (h', k') \in T_{(a,m)}(\mathbb{R} \times M)$. First note that

$$d\lambda_\phi(a, m) = d\phi_{(a,m)} \wedge \lambda_m + \phi(a)d\lambda_m \ ,$$

where ϕ is seen as a function $(a, m) \mapsto \phi(a)$. Then

$$
\begin{aligned}
d\lambda_\phi(\tilde{J}(h, k), \tilde{J}(h', k')) &= d\lambda_\phi\left((-\lambda(k), J\pi k + hR_\lambda), (-\lambda(k'), J\pi k' + h'R_\lambda)\right) \\
&= \phi'(a)(-\lambda(k))h' - \phi'(a)(-\lambda(k'))h + \phi(a)d\lambda_m(J\pi k, J\pi k') \\
&= d\phi_a \wedge \lambda((h, k), (h', k')) + \phi(a)d\lambda(k, k') \\
&= d\lambda_\phi((h, k), (h', k')),
\end{aligned}
$$

proving that $\tilde{J}$ is a linear symplectomorphism. Also

$$
\begin{aligned}
d\lambda_\phi((h, k), \tilde{J}(h, k)) &= d\lambda_\phi((h, k), (-\lambda(k), J\pi k + hR_\lambda)) \\
&= \phi'(a)hh - \phi'(a)(-\lambda(k))\lambda(k) + \phi(a)d\lambda(k, J\pi k) \\
&= \phi'(a)(h^2 + \lambda(k)^2) + \phi(a)d\lambda(\pi k, J\pi k),
\end{aligned}
$$

so that $d\lambda_\phi((h, k), \tilde{J}(h, k)) > 0$, if $(h, k) \neq 0$. $\qquad\qquad\square$

2.2   Finite energy surfaces

Let (M, λ) be a closed contact manifold of dimension 3. Fixing an almost complex structure $J \in \mathcal{J}(\xi, d\lambda)$, we are interested in $\tilde{J}$-holomorphic curves

$$\tilde{u} : (a, u) : (\dot{S}, j) \to (\mathbb{R} \times M, \tilde{J})$$

where $\dot{S} = S \setminus \Gamma$, (S, j) is a closed Riemann surface and Γ is a finite set of points in S.

Define

$$\Sigma = \{\phi \in C^\infty(\mathbb{R}, [0, 1]) | \phi' \geq 0)\} \tag{2.2}$$

and for $\phi \in \Sigma$, let λ_ϕ be the 1-form on $\mathbb{R} \times M$ defined by

$$\lambda_\phi(a,x)(h,k) = \phi(a)\lambda(m)(k).$$

If $\tilde{u} = (a,u)$ is a $\tilde{J}$-holomorphic curve, we can compute in local holomorphic coordinates $s + it \in \mathbb{C}$

$$
\begin{aligned}
\tilde{u}^* d\lambda_\phi(\partial s, \partial t) &= d\lambda_\phi(\tilde{u}_s, \tilde{u}_t) \\
&= \frac{1}{2}\left(d\lambda_\phi(\tilde{u}_s, \tilde{J}\tilde{u}_s) + d\lambda_\phi(\tilde{u}_t, \tilde{J}\tilde{u}_t)\right) \\
&= \frac{1}{2}\left(\phi'(a)(a_s^2 + a_t^2 + \lambda(u_s)^2 + \lambda(u_t)^2) + \phi(a)(|\pi u_s|_J^2 + |\pi u_t|_J^2)\right) \\
&\geq 0,
\end{aligned}
\tag{2.3}
$$

where $|\cdot|_J^2 = d\lambda(\cdot, J\cdot)$. Therefore, if $\tilde{u}$ is $\tilde{J}$-holomorphic, then

$$0 \leq \int_{\dot{S}} \tilde{u}^* d\lambda_\phi \leq \infty,$$

for any $\phi \in \Sigma$, since (2.3) is non-negative. Also, $E(\tilde{u}) = 0$ if and only if $\tilde{u}$ is constant.

Definition 2.3 (Energy)**.** We define the *energy* of a $\tilde{J}$-holomorphic curve $\tilde{u} : \dot{S} \to \mathbb{R} \times M$ by

$$E(\tilde{u}) = \sup_{\phi \in \Sigma} \int_{S \backslash \Gamma} \tilde{u}^* d(\phi\lambda) \ ,$$

where Σ is defined by (2.2).

Definition 2.4 (Finite energy surface)**.** Let (S, j) be a closed Riemann surface and let $\Gamma \subset S$ be a finite set. A smooth map $\tilde{u} : S \setminus \Gamma \to \mathbb{R} \times M$ is called a *finite energy surface* if it is $\tilde{J}$-holomorphic and satisfies the energy condition

$$0 < E(\tilde{u}) < +\infty \ . \tag{2.4}$$

The elements of Γ are called *punctures*.

Write $\tilde{u} = (a,u)$. Let $z \in \Gamma$ be a puncture and take a holomorphic chart $\varphi : (U,0) \to (\varphi(U), z)$ centered at z. We call $(s,t) \simeq \varphi(e^{-2\pi(s+it)})$ *positive exponential coordinates* and $(s,t) \simeq \varphi(e^{2\pi(s+it)})$ *negative exponential coordinates* around z.

Set $\tilde{u}(s,t) = \tilde{u} \circ \varphi(e^{-2\pi(s+it)})$, for $s >> 1$. Using (2.3), Stokes Theorem

and the energy condition (2.4), one can prove that the limit

$$m(z) = \lim_{s \to +\infty} \int_{\{s\} \times S^1} u^* \lambda \qquad (2.5)$$

exists.

The puncture z is called *removable* if $m = 0$, *positive* if $m > 0$ and *negative* if $m < 0$. By an application of Gromov's removable singularity theorem [Gro85], one can prove that $\tilde{u}$ can be smoothly extended to a removable puncture. Thus, in the following we assume that all punctures are positive or negative and use the notation $\Gamma = \Gamma^+ \cup \Gamma^-$ to distinguish positive and negative punctures.

If $\tilde{u} : S \backslash \Gamma \to \mathbb{R} \times M$ is a finite energy surface, then $\Gamma \neq \emptyset$. Indeed, suppose $\Gamma = \emptyset$. Since $\partial S = \emptyset$, we conclude by Stokes Theorem that $\int_S \tilde{u}^* d\lambda_\phi = 0$ for any $\phi \in \Sigma$, which contradicts $E(\tilde{u}) > 0$. Again using Stokes Theorem and (2.3), one can show that $\Gamma^+ \neq \emptyset$.

Definition 2.5 ($d\lambda$-area)**.** The $d\lambda$-*area* of a $\tilde{J}$-holomorphic curve is given by the formula

$$A(\tilde{u}) = \int_{\dot{S}} \tilde{u}^* d\lambda = \int_{\dot{S}} u^* d\lambda \ .$$

By (2.3), $A(\tilde{u}) \geq 0$ and $A(\tilde{u}) = 0$ if and only if $\pi \cdot du \equiv 0$.

Theorem 2.6. *[HWZ95b, Theorem 6.11] Let $\tilde{u} = (a, u) : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times M$ be a finite-energy punctured sphere, where $\Gamma \subset \mathbb{C}$ is the finite set of negative punctures and ∞ is the unique positive puncture. If $\pi \cdot du \equiv 0$, where $\pi : TS^3 = \mathbb{R}R_\lambda \oplus \xi \to \xi$ is the projection, then there exists a nonconstant polynomial $p : \mathbb{C} \to \mathbb{C}$ and a periodic Reeb orbit $P = (x, T) \in \mathcal{P}(\lambda)$ such that*

$$p^{-1}(0) = \Gamma \quad and \quad \tilde{u} = F_P \circ p \ ,$$

where $F_P : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times M$ is defined by $F_P(z = e^{2\pi(s+it)}) = (Ts, x(Tt))$.

Remark 2.7. Note that in Theorem 2.6, if p has degree k, then the asymptotic limit of $\tilde{u}$ at ∞ is $P^k = (x, kT)$.

Corollary 2.8. *If $\tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times M$ is a finite-energy plane, then $\int_{\mathbb{C}} u^* d\lambda > 0$.*

2.2.1   Asymptotic behavior

The existence of a finite energy surface in the symplectization $\mathbb{R} \times M$ is equivalent to the existence of closed Reeb orbits in (M, λ). This is a consequence of the following theorem proved in [Hof93] and [HWZ96].

Theorem 2.9. *Let $\tilde{u} = (a, u) : S \setminus \Gamma \to \mathbb{R} \times M$ be a finite energy surface. Assume z is non removable and let $\epsilon = \pm 1$ be the sign of $m(z)$, defined in (2.5). Fix a sequence $s_n \to +\infty$. Then there exists a non constant trajectory of the Reeb flow $x : \mathbb{R} \to M$ with period $T > 0$ and a subsequence s_{n_k} such that*

$$\lim_{k \to +\infty} \{t \mapsto u(s_{n_k}, t)\} = \{t \mapsto x(\epsilon Tt)\}$$

in the $C^\infty(S^1, M)$ topology. If the orbit (x, T) is nondegenerate, then

$$\lim_{s \to +\infty} \{t \mapsto u(s, t)\} = \{t \mapsto x(\epsilon Tt)\} .$$

In the nondegenerate case, there is a unique periodic orbit (x, T) associated with the puncture z. It has period $T = |m|$ and is called the *asymptotic limit* of $\tilde{u}$ at z.

Definition 2.10. A *Martinet's tube* for a simply covered orbit $P = (x, T) \in \mathcal{P}(\lambda)$ is a pair (U, ψ), where U is a neighborhood of $x(\mathbb{R})$ in M and $\psi : U \to S^1 \times B$ is a diffeomorphism (here $B \subset \mathbb{R}^2$ is an open ball centered at the origin) satisfying

- There exists $f : S^1 \times B \to \mathbb{R}^+$ such that $f|_{S^1 \times \{0\}} \equiv T$, $df|_{S^1 \times \{0\}} \equiv 0$ and $\psi^*(f(d\theta + x_1 dx_2)) = \lambda$, where θ is the coordinate on S^1 and (x_1, x_2) are coordinates on $\mathbb{R}^2$;

- $\psi(x_T(t)) = (t, 0, 0)$.

The coordinates (θ, x_1, x_2) are referred to as *Martinet's coordinates*.

The existence of such Martinet's tubes is proved in [HWZ96] for any simply covered orbit $P \in \mathcal{P}(\lambda)$.

From now on we assume that λ is nondegenerate.

A more precise description of the asymptotic behavior of a finite energy surface is given by the following theorem of [HWZ96].

Theorem 2.11 ([HWZ96])**.** *Let λ be a nondegenerate contact form on M. Let $z_0 \in \Gamma$ be a positive puncture of the finite energy $\tilde{J}$-holomorphic sphere $\tilde{u} = (a, u) : S^2 \setminus \Gamma \to \mathbb{R} \times M$ and let (s, t) be positive exponential coordinates near z_0. Let $P = (x, T)$ be the asymptotic limit of $\tilde{u}$ at z_0 and let k be a positive integer such that $T = kT_{min}$, where T_{min} is the least positive period of x. Let (θ, x_1, x_2) be Martinet's coordinates in a neighborhood $U \subset M$ of $P_{min} = (x, T_{min})$ given by a Martinet's tube (U, ψ). Then the map $\psi \circ u(s, t) = (\theta(s, t), x_1(s, t), x_2(s, t))$ is defined for all large s and either*

$(x_1(s,t), x_2(s,t)) \equiv 0$ or there are constants $s_0, A_{ij}, r_0 \in \mathbb{R}^+$, $a_0 \in \mathbb{R}$, a function $R : \mathbb{R} \times S^1 \to \mathbb{R}^2$ and an eigensection $v(t)$ of the asymptotic operator $A_{P,J}$ (1.8), associated with a negative eigenvalue $\alpha \in \sigma(A_{P,J})$, such that

$$\left| \partial_s^i \partial_t^j (a(s,t) - (Ts + a_0)) \right| \le A_{ij} e^{-r_0 s}$$
$$\left| \partial_s^i \partial_t^j (\theta(s,t) - kt) \right| \le A_{ij} e^{-r_0 s}$$
$$(x_1(s,t), x_2(s,t)) = e^{\int_{s_0}^s \alpha(r) dr} (e(t) + R(s,t)) \tag{2.6}$$
$$\left| \partial_s^i \partial_t^j R(s,t) \right|, \left| \partial_s^i \partial_t^j (\alpha(s) - \alpha) \right| \le A_{ij} e^{-r_0 s}$$

for all large s and $i, j \in \mathbb{N}$. Here $\theta(s,t)$ is seen as a map on the universal cover $\mathbb{R}$ of S^1, $e : S^1 \to \mathbb{R}^2$ represents the eigensection $v(t)$ in the coordinates induced by ψ, and $\alpha : [s_0, \infty) \to \mathbb{R}$ is a smooth function such that $\alpha(s) \to \alpha$, as $s \to \infty$.

A similar statement holds if z_0 is a negative puncture. In this case, we use negative exponential coordinates near z_0, $e^{-r_0 s}$ is replaced by $e^{r_0 s}$ and the eigenvalue α of $A_{P,J}$ is positive.

The eigenvalue α and the eigensection $v(t)$, as in Theorem 2.11, will be refered to as the *asymptotic eigenvalue* and *asymptotic eigensection* of $\tilde{u}$ at the puncture z_0.

2.2.2   Somewhere injective curves

Definition 2.12. A $\tilde{J}$-holomorphic curve $\tilde{u} : S \setminus \Gamma \to \mathbb{R} \times M$ is called *somewhere injective* if there exists a point $z_0 \in S \setminus \Gamma$ satisfying

$$\tilde{u}^{-1}(\tilde{u}(z_0)) = \{z_0\} \quad \text{and} \quad d\tilde{u}(z_0) \ne 0 .$$

Every finite energy curve with nondegenerate asymptotic limits factors through a somewhere injective one. This is the content of Theorem 2.13 and Corollary 2.15 below, that are generalizations of the corresponding result about finite energy planes proved in Theorem 6.2 of [HWZ95b].

Theorem 2.13. *Let $\tilde{u} : (S \setminus \Gamma, j) \to (\mathbb{R} \times M, \tilde{J})$ be a nonconstant finite energy $\tilde{J}$-holomorphic curve asymptotic to nondegenerate Reeb orbits, where (S, j) is a closed Riemann surface and $\Gamma \subset S$ is a finite set. Then there exists a factorization*

$$\tilde{u} = \tilde{v} \circ \varphi ,$$

where

- *$\tilde{v} : (S' \setminus \Gamma', j') \to (\mathbb{R} \times M, \tilde{J})$ is a finite energy $\tilde{J}$-holomorphic curve that is embedded outside a finite set of critical points and self-intersections.*

- $\varphi : (S, j) \to (S', j')$ *is a holomorphic map of positive degree.*

A proof of theorem 2.13 can be found in [Nel15].

One of the ingredients of the proof, that will be useful to us later, is the lemma below, that follows from results in [Sie08].

Lemma 2.14. *Assume* $u : (S \backslash \Gamma, j) \to (\mathbb{R} \times M, \tilde{J})$ *is a finite energy curve and* $z_0 \in \Gamma$ *is a puncture whose asymptotic limit* $P = (x, T)$ *is nondegenerate. Then a punctured neighborhood* $\dot{U}$ *of* z_0 *in* $S \setminus \Gamma$ *can be biholomorphically identified with the punctured unit disk* $\dot{\mathbb{D}} = \mathbb{D} \setminus \{0\} \subset \mathbb{C}$ *and*

$$u(z) = v(z^k) \ , \forall z \in \dot{\mathbb{D}} \ ,$$

where $k \in \mathbb{Z}^+$ *divides the multiplicity of the orbit* P *and* $v : (\dot{\mathbb{D}}, i) \to (\mathbb{R} \times M, \tilde{J})$ *is an embedded finite energy map. If* $w : (S' \setminus \Gamma', j') \to (\mathbb{R} \times M, \tilde{J})$ *is another finite energy curve with puncture* $z_0' \in S'$ *then the images of* u *near* z_0 *and* w *near* z_0' *are either identical or disjoint.*

In particular, if $\tilde{u} : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times M$ is a finite energy sphere with a unique positive puncture at ∞, then we have the following corollary, which will be useful later.

Corollary 2.15. *Let* $\tilde{u} : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times M$ *be a finite energy sphere asymptotic to nondegenerate Reeb orbits and with a unique positive puncture at* ∞*. Then there exists a somewhere injective finite energy sphere* $\tilde{v} : \mathbb{C} \setminus \Gamma' \to \mathbb{R} \times M$ *and a polynomial* $p : \mathbb{C} \to \mathbb{C}$ *mapping* Γ *to* Γ' *such that*

$$\tilde{u} = \tilde{v} \circ p.$$

Proof. Using Theorem 2.13, the proof follows the same arguments found in the end of the proof of [HWZ95b, Theorem 6.2]. By Theorem 2.13, $\tilde{u}$ factors as $\tilde{v} \circ \varphi$, for a finite energy curve $\tilde{v} : (S', j') \to \mathbb{R} \times M$ and a holomorphic map $\varphi : (S^2, i) \to (S', j')$ with $\deg \varphi > 0$.

First we show that we can assume $(S', j') = (S^2, i)$. Using Poincaré duality, the formula $\varphi_*(\cdot) \cap \cdot = \varphi_*(\cdot \cap \varphi^*(\cdot))$ for the cap product and $\deg \varphi > 0$, we conclude that $\varphi^* : H^1(S') \to H^1(S^2)$ is injective, and consequently that $H_1(S') = 0$. This implies that S' is the topological 2-sphere. By the Uniformization theorem, (S', j') is biholomorphic to the Riemann sphere (S^2, i). Thus, we can assume $\tilde{u} = \tilde{v} \circ \varphi$, where $\varphi : (S^2, i) \to (S^2, i)$ is a holomorphic map with positive degree.

After a reparametrization, we can assume ∞ is a positive puncture of $\tilde{v}$. Note that $\tilde{v}$ must have only one positive puncture, $\varphi(\infty) = \infty$ and $\varphi(\mathbb{C}) = \mathbb{C}$. This follows from the surjectivity of φ and the fact that $\tilde{u}$ has

a unique positive puncture at ∞. The conclusion of the proof follows form the fact that any homolomorphic map $S^2 \to S^2$ mapping $\mathbb{C}$ to itself is a extension of a polynomial to the Riemman sphere.

$\square$

2.2.3   Algebraic invariants

Let λ be a nondegenerate contact form on M, fix $J \in \mathcal{J}(\xi, d\lambda)$ and let $\tilde{u} : S \setminus \Gamma \to \mathbb{R} \times M$ be a $\tilde{J}$-holomorphic finite energy surface.

Let $\pi : TM = \mathbb{R}R_\lambda \oplus \xi \to \xi$ be the projection along the Reeb vector field and assume that $\pi \cdot du$ is not identically zero. In [HWZ95b], it is proved that the set where $\pi \cdot du$ vanishes is finite, and it is defined a local degree associated to each zero of $\pi \cdot du$, that is always positive. The integer

$$\mathrm{wind}_\pi(\tilde{u}) \geq 0$$

is defined as the sum of such local degrees over all zeros of $\pi \cdot du$.

The bundle $(u^*\xi, d\lambda, J) \to \dot{S}$ has a unitary trivialization. Indeed, since $\Gamma \neq \emptyset$, $S \setminus \Gamma$ has the homotopy type of a wedge of circles. This implies that the first Chern class of the complex bundle $(u^*\xi, J)$ is zero and, thus, $(u^*\xi, J)$ is trivial. [1]

Consider a unitary trivialization $\Psi : (u^*\xi, d\lambda, J) \to \dot{S} \times \mathbb{R}^2$. For $z \in \Gamma$ fix positive cylindrical coordinates (s, t) at z and define

$$\mathrm{wind}_\infty(\tilde{u}, z, \Psi) = \lim_{s \to +\infty} \mathrm{wind}\left(t \mapsto \pi \cdot \partial_s u(s, \epsilon_z t), \Psi|_{u(s, \epsilon_z \cdot)^*\xi}\right) \in \mathbb{Z} \qquad (2.7)$$

where ϵ_z is the sign of the puncture z. The winding number on the right is defined as in (1.12), and is independent of the choice of J and of the holomorphic chart. This limit is well defined since $\pi \cdot \partial_s u(s, t)$ does not vanish for s sufficiently large. The asymptotic winding number of $\tilde{u}$ is defined by

$$\mathrm{wind}_\infty(\tilde{u}) = \sum_{z \in \Gamma^+} \mathrm{wind}_\infty(\tilde{u}, z, \Psi) - \sum_{z \in \Gamma^-} \mathrm{wind}_\infty(\tilde{u}, z, \Psi) .$$

It is proved in [HWZ95b] that this sum does not depend on the chosen trivialization Ψ.

Remark 2.16. Later on we will only deal with the tight contact structure ξ_0 on S^3, which is a trivial symplectic bundle. Consider a global symplectic

---

[1] $(u^*\xi, J)$ trivial complex bundle $\Rightarrow (u^*\xi, d\lambda, J) \to \dot{S}$ has a unitary trivialization. See [MS17], sections 2.6 and 2.7 for more details.

trivialization

$$\Psi : \xi_0 \to S^3 \times \mathbb{R}^2 \ .$$

Then $\text{wind}_\infty(\tilde{u}, z, \Psi)$ is the winding number of the asymptotic eigensection given by Theorem 2.11, with respect to the trivialization Ψ. Also, by Lemma 1.7, $\text{wind}_\infty(\tilde{u}, z, \Psi)$ does not depend on the chosen global symplectic trivialization.

Theorem 2.17. *[HWZ95b, Theorem 5.6] Let $\tilde{u} = (a, u)$ be a finite energy surface defined on $S \setminus \Gamma$, where (S, j) is a closed Riemann surface and $\Gamma \subset S$ is a finite set consisting of nonremovable punctures. Assume that $\pi \cdot du$ is not identically zero. Then*

$$\text{wind}_\pi(\tilde{u}) = \text{wind}_\infty(\tilde{u}) - \chi(S) + \#\Gamma, \tag{2.8}$$

where $\chi(S)$ is the Euler characteristic of S.

Definition 2.18. A finite energy plane $\tilde{u}$ is called *fast* if its asymptotic limit at ∞ is a simply covered Reeb orbit and $\text{wind}_\pi(\tilde{u}) = 0$.

2.2.4 Fredholm theory

Fix $J \in \mathcal{J}(\xi, d\lambda)$ and let $\tilde{u} : S \setminus \Gamma \to \mathbb{R} \times M$ be a $\tilde{J}$-holomorphic finite energy surface.

Let $\Psi : (u^*\xi, d\lambda) \to \dot{S} \times \mathbb{R}^2$ be a symplectic trivialization. Fix a puncture $z \in \Gamma$ and let $P = (x, T)$ be the asymptotic limit of $\tilde{u}$ at z. The trivialization Ψ induces a homotopy class of oriented trivializations $[\Psi_z]$ of $x_T^*\xi$. Define $\mu^\Psi(z) := \mu(P, \Psi_z)$, where $\mu(P, \Psi_z)$ is the Conley-Zehnder index of the asymptotic limit P of $\tilde{u}$ at z, with respect to the trivialization Ψ_z.

Definition 2.19. We define the Conley-Zehnder index of $\tilde{u}$ by

$$\mu(\tilde{u}) = \sum_{z \in \Gamma^+} \mu^\Psi(z) - \sum_{z \in \Gamma^-} \mu^\Psi(z) \ .$$

It is proved in [HWZ95b] that this sum does not depend on the chosen trivialization Ψ.

Let (S, j) and (S', j') be Riemman surfaces and let

$$\tilde{u} : (S \setminus \Gamma, j) \to (\mathbb{R} \times M, \tilde{J}), \ \ \tilde{v} : (S' \setminus \Gamma', j') \to (\mathbb{R} \times M, \tilde{J})$$

be finite energy $\tilde{J}$ holomorphic surfaces. We say that $(\tilde{u}, (S, j), \Gamma)$ and $(\tilde{v}, (S', j'), \Gamma')$ are equivalent if there exists a biholomorphic map $\phi : (S, j) \to$

(S', j') satisfying $\phi(\Gamma) = \Gamma'$ and $\tilde{u} = \tilde{v} \circ \phi$. We denote by $[(\tilde{u}, (S, j), \Gamma)]$ the equivalence class of $(\tilde{u}, (S, j), \Gamma)$.

In [Dra04], it is proved that the (equivalence classes of) finite energy spheres in the neighborhood of $[(\tilde{u}, (S, j), \Gamma)]$, where $\tilde{u}$ is a somewhere injective finite energy surface, are described by a nonlinear Fredholm equation having Fredholm index equal to

$$\mathrm{ind}(\tilde{u}) := \mu(\tilde{u}) - \chi(S) + \#\Gamma. \qquad (2.9)$$

Due to the $\mathbb{R}$-action, the kernel of the linearized Fredholm operator is at least one-dimensional unless the image of $\tilde{u}$ is a cylinder over a periodic orbit, in which case $\pi \circ du \equiv 0$.

The following result was proved for embedded finite energy surfaces in [HWZ99b] and generalized for somewhere injective finite energy surfaces in [Dra04].

Theorem 2.20. *There exists a residual set $\mathcal{J}_{reg} \subset \mathcal{J}(\xi, d\lambda)$ such that, if $\tilde{u} = (a, u) : S \setminus \Gamma \to \mathbb{R} \times M$ is a somewhere injective finite energy surface for $\tilde{J} \in \mathcal{J}_{reg}$, then*

$$0 \leq \mathrm{ind}(\tilde{u}) := \mu(\tilde{u}) - \chi(S) + \#\Gamma \ .$$

If $\pi \circ du$ is not identically zero, then

$$1 \leq \mathrm{ind}(\tilde{u}) := \mu(\tilde{u}) - \chi(S) + \#\Gamma \ .$$

2.3   Bubbling-off analysis

This section follows mainly the exposition of [HSa11] and [HLSa15].

In this section, λ is a nondegenerate tight contact form on S^3. We fix a Riemannian metric g on S^3 and consider the Riemannian metric g_0 on $\mathbb{R} \times S^3$ defined by

$$g_0 = da \otimes da + \pi_{S^3}^* g,$$

where $\pi_{S^3} : \mathbb{R} \times S^3 \to S^3$ and $a : \mathbb{R} \times S^3 \to \mathbb{R}$ are the projections onto the second and the first coordinates respectively. For any $(a, x) \in \mathbb{R} \times S^3$ and any linear map $L : \mathbb{C} \to T_{(a,x)}(\mathbb{R} \times S^3)$ we denote by $|L|$ the norm induced by the Euclidean inner product of $\mathbb{C}$ and the metric g_0.

2.3.1   Elliptic Regularity and compactness

Theorem 2.21 (Elliptic regularity). *Let (W, J) be an almost complex manifold and (S, j) be a Riemannian surface without boundary. If $u : S \to W$ is*

a C^1 map and satisfies $\bar{\partial}_J(u) = 0$, then u is C^∞.

Theorem 2.22. *Let (W, J) be an almost complex manifold and (S, j) a closed Riemann surface. Let $\{U_j\}_{j \geq 1}$ be an increasing sequence of open sets in S and $u_j : U_j \to W$ a sequence of C^∞ pseudoholomorphic maps. Defining $U = \cup_j U_j$, assume that for all compact set $A \subset U$, there exists a compact set $K \subset W$ and a constant $C > 0$ such that*

$$u_j(A) \subset K, \text{ for } j \text{ large and } \limsup_{j \to \infty} |du_j|_{L^\infty(A)} \leq C$$

Then there exists a C^∞ map $u : U \to W$ satisfying $\bar{\partial}_J(u) = 0$ and a subsequence u_{j_k} such that

$$u_{j_k} \to u \quad in \ C^\infty_{loc} \ ,$$

that is, $u_{j_k} \to u$ uniformly with all derivatives on compact subsets of U.

Proofs of theorems 2.21 and 2.22 can be found in [HSa09].

2.3.2 Bubbling

Lemma 2.23 (Ekeland-Hofer). *Let (X, d) be a complete metric space and $f : X \to [0, +\infty)$ a continuous function. For any $\epsilon_0 > 0$ and $x_0 \in X$, there exist $\epsilon_0' \in (0, \epsilon_0]$ and $x_0' \in \overline{B_{2\epsilon_0}(x_0)}$ such that*

$$\begin{cases} f(x_0')\epsilon_0' \geq f(x_0)\epsilon_0 \\ d(x, x_0') \leq \epsilon_0' \Rightarrow f(x) \leq 2f(x_0') \end{cases}$$

Proof. Suppose the claim is false, that is, there exist $x_0 \in X$ and $\epsilon_0 > 0$ such that, for every $\epsilon' \in (0, \epsilon_0]$ and $x' \in \overline{B_{2\epsilon_0}(x_0)}$, at least one of the following is true

- $f(x')\epsilon' < f(x_0)\epsilon_0$

- $\exists x \in \overline{B_{\epsilon'}(x')}$ such that $f(x) > 2f(x')$.

We will find a Cauchy sequence $\{x_N\}$ satisfying $f(x_N) \to \infty$, in contradiction to the fact that X is complete. Taking $x' = x_0$ and $\epsilon' = \epsilon_0$, we find $x_1 \in \overline{B_{\epsilon_0}(x_0)}$ satisfying $f(x_1) > 2f(x_0)$. Assume we have $\{x_0, \ldots, x_N\}$ satisfying $d(x_{n+1}, x_n) \leq \epsilon_0 2^{-n}$ and $f(x_{n+1}) > 2f(x_n)$ for all $n = 0, \ldots, N - 1$. Since $d(x_N, x_0) \leq \epsilon_0(2^{-(N-1)} + \ldots, +2^0) \leq 2\epsilon_0$, we have $x_N \in \overline{B_{2\epsilon_0}(x_0)}$. Consider $x' = x_N$ and $\epsilon' = \epsilon_0 2^{-N}$. Since, by assumption, $f(x_N) > 2^N f(x_0)$, we conclude there exists x_{N+1} satisfying $d(x_{N+1}, x_N) \leq \epsilon_0 2^{-N}$ and $f(x_{N+1}) > 2f(x_N)$. Thus $\{x_N\}$ is a Cauchy sequence and $f(x_N) \to \infty$, as $N \to \infty$. $\square$

Proposition 2.24. *Fix $J \in \mathcal{J}(\xi, d\lambda)$ and let $\tilde{J}$ be the induced almost complex structure on $\mathbb{R} \times S^3$ defined by formula (2.1). Let $\{U_n\}$ be an increasing sequence of open sets in $\mathbb{C}$ and let $\tilde{u}_n = (a_n, u_n) : U_n \to \mathbb{R} \times S^3$ be a sequence of smooth maps satisfying $\bar{\partial}_{\tilde{J}}(\tilde{u}_n) = 0$, $\forall n$ and $\sup_n E(\tilde{u}_n) = C < \infty$. Define $U = \cup_n U_n$ and assume there exits a sequence $z_n \in U_n$ such that $|d\tilde{u}_n(z_n)| \to +\infty$ and $z_n \to z_\infty \in U$. Then there exist subsequences $\{\tilde{u}_{n_k}\}$ and sequences $z_k' \to z_\infty$, $\delta_k, r_k \to 0^+$ with $\frac{\delta_k}{r_k} \to 0^+$ such that the family of maps*

$$\tilde{v}_k : B_{\frac{1}{r_k}}(0) \to \mathbb{R} \times S^3$$

$$z \mapsto (a_{n_k}(z_k' + \delta_k z) - a_{n_k}(z_k'), u_{n_k}(z_k' + \delta_k z))$$

converges in C^∞_{loc} to a $\tilde{J}$-holomorphic plane $\tilde{v} : \mathbb{C} \to \mathbb{R} \times M$ satisfying $0 < E(\tilde{v}) \leq C$ and $\sup_{\mathbb{C}} |d\tilde{v}| < \infty$.

Proof. The proof follows Proposition 3.67 of [HSa09]. Define $R_n = |d\tilde{u}_n(z_n)|$ and $\epsilon_n = R_n^{-\frac{1}{2}}$. By applying Lemma 2.23 to $|d\tilde{u}_n(z)|$, z_n and ϵ_n, we find $0 < \epsilon' \leq \epsilon_n$ and z_n' with $|z_n' - z_n| \leq 2\epsilon_n$ such that

$$|d\tilde{u}_n(z_n')|\epsilon_n' \geq |d\tilde{u}_n(z_n)|\epsilon_n = \epsilon_n R_n$$

and

$$|z - z_n'| \leq \epsilon_n' \Rightarrow |d\tilde{u}_n(z)| \leq 2|d\tilde{u}_n(z_n')| .$$

It follows that

$$\sup_{|z - z_n'| \leq \epsilon_n} |d\tilde{u}_n(z)| \leq 2|d\tilde{u}_n(z_n')| . \tag{2.10}$$

Define $R_n' = \|d\tilde{u}_n(z_n')\|$. Note that $\epsilon_n' R_n' \geq \epsilon_n R_n \to +\infty$ and define

$$\tilde{v}_n : B_{\epsilon_n R_n'}(0) \to \mathbb{R} \times S^3$$

$$z \mapsto \left(a_n\left(z_n' + \frac{z}{R_n'} \right) - a_n(z_n'), u_n\left(z_n' + \frac{z_n'}{R_n'} \right) \right)$$

If $z \in B_{\epsilon_n' R_n'}(0)$, using that the norm $|\cdot|$ is $\mathbb{R}$-invariant and (2.10) we get

$$|d\tilde{v}_n(z)| = \left| d\tilde{u}_n\left(z_n' + \frac{z}{R_n'} \right) \right| \frac{1}{R_n'} \leq 2, \; \forall n . \tag{2.11}$$

Fixed n_0, using the mean value inequality and (2.11), and noting that $\tilde{v}_n(0) \in \{0\} \times M$, we conclude that $\tilde{v}_n(B_{\epsilon_{n_0}' R_{n_0}'}(0))$ is contained in a compact set. By Theorem 2.22, there is a subsequence $\tilde{v}_{n_k}$ and a $\tilde{J}$-holomorphic curve

$\tilde{v} : \mathbb{C} \to \mathbb{R} \times M$, such that

$$\tilde{v}_{n_k} \to \tilde{v} \quad \text{in } C_{loc}^\infty \ .$$

Clearly $|d\tilde{v}|_{\mathbb{C}} < \infty$ and, by Fatou's Lemma, we get $E(\tilde{v}) \leq C$. Since $|d\tilde{v}_{n_k}(0)| = \lim_k |d\tilde{v}_{n_k}(0)| = 1$, it follows that $E(\tilde{v}) > 0$.

Defining $\delta_k = \frac{1}{R'_{n_k}}$ and $r_k = \frac{1}{\epsilon_{n_k} R'_{n_k}}$, we have $\delta_k, r_k \to 0^+$ and $\frac{\delta_k}{r_k} \to 0^+$. The family of maps $\tilde{v}_k := \tilde{v}_{n_k} : B_{\frac{1}{r_k}}(0) \to \mathbb{R} \times S^3$ satisfies the statement of Proposition 2.24. $\qquad \square$

2.3.3 Germinating sequences

Fix $C > 0$. Since λ is nondegenerate, there is just a finite number of orbits in $\mathcal{P}(\lambda)$ with period $\leq C$. Define $\sigma(C)$ as any real number satisfying

$$0 < \sigma(C) < \min\{T', |T' - T''| : T' \neq T'' \text{ periods }, T', T'' \leq C\} \ . \qquad (2.12)$$

Now we fix an arbitrary $J \in \mathcal{J}(\xi, d\lambda)$ and consider a sequence of $\tilde{J}$-holomorphic curves

$$\tilde{v}_n = (b_n, v_n) : B_{R_n}(0) \subset \mathbb{C} \to \mathbb{R} \times S^3$$

satisfying

$$R_n \to \infty, \quad R_n \in (0, +\infty] \qquad (2.13)$$

$$E(\tilde{v}_n) \leq C, \ \forall n \qquad (2.14)$$

$$\int_{B_{R_n}(0) \setminus \mathbb{D}} v_n^* d\lambda \leq \sigma(C), \ \forall n \qquad (2.15)$$

$$\{b_n(2)\} \text{ is uniformily bounded} \qquad (2.16)$$

Definition 2.25. Such a sequence $\tilde{v}_n$ of $\tilde{J}$-holomorphic curves satisfying (2.13)-(2.16) will be referred to as a *germinating sequence*.

Proposition 2.26. *Let $\tilde{v}_n$ be a germinating sequence. Then there exists a finite set $\Gamma \subset \mathbb{D}$, a $\tilde{J}$-holomorphic map $\tilde{v} = (b, v) : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times S^3$ and a subsequence of $\tilde{v}_n$, still denoted by $\tilde{v}_n$, such that*

$$\tilde{v}_n \to \tilde{v} \quad in \ C_{loc}^\infty(\mathbb{C} \setminus \Gamma, \mathbb{R} \times S^3).$$

Also, $E(\tilde{v}) \leq C$.

Proof. Let $\Gamma_0 \subset \mathbb{C}$ be the set of points $z \in \mathbb{C}$ such that there exists a

subsequence $\tilde{v}_{n_j}$ and a sequence $\zeta_j \in B_{R_{n_j}}(0)$ with $\zeta_j \to z$ and

$$|d\tilde{v}_{n_j}(\zeta_l)| \to \infty, \ j \to \infty \ .$$

If $\Gamma_0 = \emptyset$, then by Theorem 2.22, we find a $\tilde{J}$-holomorphic map $\tilde{v} : \mathbb{C} \to \mathbb{R} \times S^3$ such that, up to a subsequence, $\tilde{v}_n \to \tilde{v}$ in $C^\infty_{loc}(\mathbb{C}, \mathbb{R} \times S^3)$. In this case, $\Gamma = \emptyset$.

Now assume $\Gamma_0 \neq \emptyset$. If $z_0 \in \Gamma_0$, there exists a period $0 < T_0 \leq C$ and sequences $r_j^0 \to 0^+$, $n_j^0 \to \infty$ and $z_j^0 \to z_0$ such that

$$\lim_{j \to \infty} \int_{B_{r_j}(z_j^0)} v^*_{n_j^0} d\lambda \geq T_0 \ .$$

Indeed, by Proposition 2.24, there exists a subsequence $\{\tilde{v}_{n_k}\}$ and sequences $z'_k \to z_0, r_k, \delta_k \to 0^+$, with $\frac{\delta_k}{r_k} \to 0^+$, such that the sequence of maps

$$\tilde{u}_k : B_{1/r_k}(0) \to \mathbb{R} \times S^3$$
$$z \mapsto (b_{n_k}(z'_k + \delta_k z) - b_{n_k}(z'_k), v_{n_k}(z'_k + \delta_k z))$$

converges in C^∞_{loc} to a $\tilde{J}$-holomorphic plane $\tilde{u} : \mathbb{C} \to \mathbb{R} \times M$ satisfying $0 < E(\tilde{u}) \leq C$ and $\sup_{\mathbb{C}} |d\tilde{u}| < \infty$. Then there exists a period $T_0 \leq C$ such that $T_0 = \int_{\mathbb{C}} u^* d\lambda$. Using Fatou's Lemma, we obtain

$$T_0 = \int_{\mathbb{C}} u^* d\lambda \leq \liminf_{k \to \infty} \int_{B_{1/r_k}(0)} u_k^* d\lambda = \liminf_{k \to \infty} \int_{B_{\delta_k/r_k}(z'_k)} v^*_{n_k} d\lambda.$$

Consider $\tilde{v}_{n_j^0}$ as the new sequence $\tilde{v}_n$. Now let $\Gamma_1 \subset \mathbb{C} \setminus \{z_0\}$ be the set of points $z_1 \neq z_0$ such that there exists a subsequence $\tilde{v}_{n_j}$ and sequence $\zeta_j \in B_{R_{n_j}}(0)$ with $\zeta \to z_1$ and $|d\tilde{v}_{n_j}(\zeta_j)| \to \infty$.

As before, if $\Gamma_1 = \emptyset$, we have a $\tilde{J}$-holomorphic map $\tilde{v} : \mathbb{C} \setminus \{z_0\} \to \mathbb{R} \times S^3$ such that, up to subsequence, $\tilde{v}_n \to \tilde{v}$ in $C^\infty_{loc}(\mathbb{C} \setminus \{z_0\}, \mathbb{R} \times S^3)$. In this case, we define $\Gamma = \Gamma_0 = \{z_0\}$.

If $\Gamma_1 \neq \emptyset$ and $z_1 \in \Gamma_1$, there exist a period $0 < T_1 \leq C$ and sequences $r_j^1 \to 0$, $n_j^1 \to \infty$ and $z_j^1 \to z_1$ such that

$$\lim_{j \to \infty} \int_{B_{r_j}(z_j^1)} v^*_{n_j^1} d\lambda \geq T_1 \ .$$

Considering $\tilde{v}_{n_j^1}$ as the new sequence $\tilde{v}_n$, define $\Gamma_2 \subset \mathbb{C} \setminus \{z_0, z_1\}$ as before.

Repeating this argument, let $z_i \in \Gamma_i \subset \mathbb{C} \setminus \{z_0, \ldots, z_{i-1}\}$. Note that

$$C \geq \lim_{n \to \infty} E(\tilde{v}_n) \geq \lim_{n \to \infty} \int_{\mathbb{C}} v_n^* d\lambda$$

$$\geq \sum_{l=0}^{i} \lim_{j \to \infty} \int_{B_{r_j^l}(z_j^l)} v_{n_j^l}^* d\lambda \geq T_0 + \cdots + T_i$$

It follows that there exists i_0 such that $\Gamma_{i_0} \neq 0$ and $\Gamma_i = \emptyset$ for $i > i_0$.

We end up with a finite set $\Gamma = \{z_0, \ldots, z_{i_0}\}$ and a $\tilde{J}$-holomorphic map

$$\tilde{v} : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times S^3$$

such that up to a subsequence,

$$\tilde{v}_n \to \tilde{v} \text{ in } C_{loc}^\infty .$$

It follows from (2.15) that $\Gamma \subset \mathbb{D}$. The inequality $E(\tilde{v}) \leq C$ follows from (2.14) and Fatou's Lemma. $\qquad\square$

Definition 2.27. A $\tilde{J}$-holomorphic map $\tilde{v} : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times S^3$ as in Proposition 2.26 is called a limit of the germinating sequence.

If $\Gamma \neq \emptyset$, then $\tilde{v}$ is non-constant. In this case, all the punctures $z = z_i \in \Gamma$ are negative and ∞ is a positive puncture. To prove this, define for any $\epsilon > 0$

$$m_\epsilon(z) := \int_{\partial B_\epsilon(z)} v^* \lambda .$$

Here $\partial B_\epsilon(z)$ is oriented counterclockwise. This is equivalent to use negative exponential coordinates as defined in Section 2.2.

$$m_\epsilon(z) = \int_{\partial B_\epsilon(z)} v^* \lambda = \lim_{n \to \infty} \int_{\partial B_\epsilon(z)} v_n^* \lambda = \lim_{n \to \infty} \int_{B_\epsilon(z)} v_n^* d\lambda .$$

For j large, $B_{r_j^i}(z_j^i)$, defined as in the proof of Proposition 2.26, is contained in $B_\epsilon(z)$. It follows that

$$m_\epsilon(z) = \lim_{n \to \infty} \int_{B_\epsilon(z)} v_n^* d\lambda \geq \lim_{j \to \infty} \int_{B_{r_j}(z_j^i)} v_{n_j^i}^* d\lambda \geq T_i > 0 .$$

This implies that $\tilde{v}$ is non constant and the puncture z is negative. Also, as a consequence of $0 < E(\tilde{v}) < \infty$, we know that $\tilde{v}$ has at least one positive puncture. Thus, ∞ is a positive puncture.

2.3.4   Soft-rescaling near a negative puncture

Assume $\Gamma \neq \emptyset$ and let $\tilde{v} = (b, v) : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times M$ be a limit of a germinating sequence $\tilde{v}_n = (b_n, v_n)$. Let $z \in \Gamma$. We define the mass $m(z)$ of z by

$$m(z) = \lim_{\epsilon \to 0^+} m_\epsilon(z) = \lim_{\epsilon \to 0^+} \int_{\partial B_\epsilon(z)} v^* \lambda = T_z > \sigma(C) > 0 \ , \qquad (2.17)$$

where T_z is the period of the asymptotic limit of $\tilde{v}$ at z.

Since $m_\epsilon(z)$ is a non-decreasing function of ϵ, we can fix ϵ small enough so that
$$0 \leq m_\epsilon(z) - m(z) \leq \frac{\sigma(C)}{2}. \qquad (2.18)$$
Choose sequences $z_n \in \overline{B_\epsilon(z)}$ and $0 < \delta_n < \epsilon$, $\forall n$, so that

$$b_n(z_n) \leq b_n(\zeta), \ \forall \zeta \in B_\epsilon(z), \qquad (2.19)$$

$$\int_{B_\epsilon(z) \setminus B_{\delta_n}(z_n)} v_n^* d\lambda = \sigma(C). \qquad (2.20)$$

Since z is a negative puncture, (2.19) implies that $z_n \to z$. Hence the existence of δ_n as in (2.20) follows from (2.17). We claim that $\liminf \delta_n = 0$. Otherwise, we choose $0 < \epsilon' < \liminf \delta_n \leq \epsilon$. From (2.18), we get the contradiction
$$\frac{\sigma(C)}{2} \geq m_\epsilon(z) - m(z) \geq m_\epsilon(z) - m_{\epsilon'}(z)$$

$$= \lim_{n \to \infty} \int_{B_\epsilon(z) \setminus B_{\epsilon'}(z)} v_n^* d\lambda$$

$$\geq \lim_{n \to \infty} \int_{B_\epsilon(z) \setminus B_{\delta_n}(z_n)} v_n^* d\lambda = \sigma(C) \ .$$
Thus, we can assume $\delta_n \to 0$.

Now take any sequence $R_n \to +\infty$ satisfying

$$\delta_n R_n < \frac{\epsilon}{2}$$

and define the sequence of $\tilde{J}$-holomorphic maps $\tilde{w}_n = (c_n, w_n) : B_{R_n}(0) \to \mathbb{R} \times S^3$ by

$$\tilde{w}_n(\zeta) = (b_n(z_n + \delta_n \zeta) - b_n(z_n + 2\delta_n), v_n(z_n + \delta_n \zeta)) \ . \qquad (2.21)$$

It follows from (2.20) that

$$\int_{B_{R_n}(0)\setminus\mathbb{D}} w_n^* d\lambda \leq \sigma(C), \ \forall n \ .$$

Moreover, by the definition of $\tilde{w}_n$, $E(\tilde{w}_n) \leq E(\tilde{v}_n) \leq C$ and $\tilde{w}_n(2) \in \{0\} \times M$. Thus, $\tilde{w}_n$ is a germinating sequence.

Let $\tilde{w} = (c, w) : \mathbb{C} \setminus \Gamma' \to \mathbb{R} \times S^3$ be a limit of $\tilde{w}_n$, as in Proposition 2.26. If $\Gamma' \neq \emptyset$, then $\tilde{w}$ is not constant. If $\Gamma' = \emptyset$, then

$$
\begin{aligned}
\int_{\mathbb{D}} w^* d\lambda &= \lim_{n\to\infty} \int_{\mathbb{D}} w_n^* d\lambda = \lim_{n\to\infty} \int_{B_{\delta_n}(z_n)} v_n^* d\lambda \\
&= \lim_{n\to\infty} \left(\int_{B_\epsilon(z)} v_n^* d\lambda - \int_{B_\epsilon(z)\setminus B_{\delta_n}(z_n)} v_n^* d\lambda \right) \qquad (2.22) \\
&= m_\epsilon(z) - \sigma(C) \\
&\geq T_z - \sigma(C) > 0 \ .
\end{aligned}
$$

Thus $\tilde{w}$ is non-constant as well. From Fatou's Lemma we get $0 < E(\tilde{w}) \leq C$. This also implies that the period of the asymptotic limits of $\tilde{w}$ are bounded by C.

Proposition 2.28. *The asymptotic limit P_∞ of $\tilde{w}$ at ∞ coincides with the asymptotic limit P_z of $\tilde{v}$ at the negative puncture $z \in \Gamma$.*

To prove Proposition 2.28, we need the following Lemma from [HWZ03].

Lemma 2.29. *[HWZ03, Lemma 4.9] Consider constants $e > 0$ and $C > 0$ and let $\sigma(C)$ be as defined in (2.12). Identifying $S^1 = \mathbb{R}/\mathbb{Z}$, let $\mathcal{W} \subset C^\infty(S^1, S^3)$ be an open neighborhood of the set of periodic orbits $P = (x, T) \in \mathcal{P}(\lambda)$ with $T \leq C$, viewed as maps $x_T : S^1 \to S^3$, $x_T(t) = x(Tt)$. We assume that $\mathcal{W}$ is S^1-invariant, meaning that $y(\cdot + c) \in \mathcal{W} \Leftrightarrow y \in \mathcal{W}, \forall c \in S^1$, and that each of the connected components of $\mathcal{W}$ contains at most one periodic orbit modulo S^1-reparametrizations. Then there exists a constant $h > 0$ such that the following holds. If $\tilde{u} = (a, u) : [r, R] \times S^1 \to \mathbb{R} \times S^3$ is a $\tilde{J}$-holomorphic cylinder satisfying*

$$E(\tilde{u}) \leq C, \quad \int_{[r,R]\times S^1} u^* d\lambda \leq \sigma(C), \quad \int_{\{r\}\times S^1} u^*\lambda \geq e \quad \text{and } r+h \leq R-h,$$

then each loop $t \in S^1 \to u(s, t)$ is contained in $\mathcal{W}$ for all $s \in [r + h, R - h]$.

Proof of Proposition 2.28. Let $\mathcal{W}$ be an open neighborhood of the set of

periodic orbits $P = (x, T) \in \mathcal{P}(\lambda)$ with $T \leq C$ as in Lemma 2.29. Let $\mathcal{W}_\infty$ and $\mathcal{W}_z$ be connected components of $\mathcal{W}$ containing P_∞ and P_z respectively.

Since $\tilde{v}_n \to \tilde{v}$, we can choose $0 < \epsilon_0 < \epsilon$ small enough so that, if $0 < \rho \leq \epsilon_0$ is fixed, then the loop

$$t \in S^1 \mapsto v_n(z_n + \rho e^{i2\pi t})$$

belongs to $\mathcal{W}_z$ for n large. Since $\tilde{w}_n \to \tilde{w}$, we can choose $R_0 > 1$ large enough so that, if $R \geq R_0$ is fixed, then the loop

$$t \in S^1 \mapsto w_n(Re^{i2\pi t}) = v_n(z_n + \delta_n Re^{i2\pi t})$$

belongs to $\mathcal{W}_\infty$ for n large.

By (2.17) and (2.20), we can show that

$$e := \liminf \int_{\partial B_{\delta_n R_0}(z_n)} v_n^* \lambda > 0 \tag{2.23}$$

Consider, for each n, the $\tilde{J}$-holomorphic cylinder $\tilde{C}_n : \left[\frac{\ln R_o \delta_n}{2\pi}, \frac{\ln \epsilon_0}{2\pi}\right] \times S^1 \to \mathbb{R} \times S^3$, defined by $\tilde{C}_n(s, t) = \tilde{v}_n(z_n + e^{2\pi(s+it)})$. It follows from (2.20) that

$$\int_{\left[\frac{\ln R_o \delta_n}{2\pi}, \frac{\ln \epsilon_0}{2\pi}\right] \times S^1} C_n^* d\lambda \leq \sigma(C) \tag{2.24}$$

for n large. Using (2.23) and (2.24) and applying Lemma 2.29, we find $h > 0$ so that the loop

$$t \mapsto C_n(s, t)$$

is contained in $\mathcal{W}$ for all $s \in \left[\frac{\ln R_0 \delta_n}{2\pi} + h, \frac{\ln \epsilon_0}{2\pi} - h\right]$ and n large. But

$$C_n\left(\frac{\ln \epsilon_0}{2\pi} - h, t\right) = v_n(z_n + \epsilon_0 e^{-2\pi h} e^{2\pi it}) \in \mathcal{W}_z,$$

for all n large and

$$C_n\left(\frac{\ln R_0 \delta_n}{2\pi} + h, t\right) = v_n(z_n + R_0 \delta_n e^{2\pi h} e^{2\pi t}) \in \mathcal{W}_\infty,$$

for all n large. Thus $\mathcal{W}_\infty = \mathcal{W}_z$ and $P_\infty = P_z$.

$\square$

Proposition 2.30. *Either*

- $\int_{\mathbb{C} \backslash \Gamma'} w^* d\lambda > 0$ *or*

- $\int_{\mathbb{C}\setminus\Gamma'} w^*d\lambda = 0$ *and* $\#\Gamma' \geq 2$.

Proof. If $\Gamma' \neq \emptyset$, then $0 \in \Gamma'$. This fact follows from $c_n(0) = \inf c_n(B_{R_n})$ and the fact that the punctures in Γ' are negative. Arguing by contradiction, assume

$$\int_{\mathbb{C}\setminus\Gamma'} w^*d\lambda = 0 \text{ and } \#\Gamma' = 1 \ .$$

Thus, $\Gamma' = \{0\}$. By Theorem 2.6, there exists a polynomial $p : \mathbb{C} \to \mathbb{C}$ and an orbit $P = (x,T)$ such that $\Gamma' = p^{-1}(0)$ and $\tilde{v} = F_P \circ p$, where $F_P : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ is defined by $F_P(\zeta = e^{2\pi(s+it)}) = (Ts, x(Tt))$. Since 0 is the only root of p, we have $p(\zeta) = A\zeta^n$, for some $A \neq 0$ and $n \geq 1$. It follows that $P^n = P_z$, where $P_z = (x_z, T_z)$ is the asymptotic limit of $\tilde{v}$ at z, that coincides with the asymptotic limit of $\tilde{w}$ at ∞, and

$$\tilde{w}(\zeta = e^{2\pi(s+it)}) = F_P(Ae^{2\pi n(s+it)}) = \left(T_z \left(s + \frac{\log A}{2\pi n} \right), x_z(T_z t) \right).$$

Thus, we have the contradiction

$$m(z) = T_z = \int_{\partial\mathbb{D}} w^*\lambda = \lim_{n\to\infty} \int_{\partial\mathbb{D}} w_n^*\lambda = \lim_{n\to\infty} \int_{\partial B_\epsilon(z)} v_n^*\lambda - \sigma(C)$$

$$= \int_{\partial B_\epsilon(z)} v^*\lambda - \sigma(C) \leq m(z) - \frac{\sigma(C)}{2}.$$

Here we have used (2.18), (2.20) and (2.22). $\qquad\square$

2.3.5 Bubbling-off tree

Consider a finite rooted tree [2] $\mathcal{T} = (E, \{r\}, V)$, with edges oriented away from the root, and a finite set $\mathcal{U}$ of finite energy $\tilde{J}$-holomorphic spheres. The pair $\mathcal{B} = (\mathcal{T}, \mathcal{U})$ is called a bubbling-off tree if it satisfies the following properties

---

[2]A *tree* $\mathcal{T} = (E, V)$ is a connected graph with no cycles. A *rooted tree* is a tree $T = (V, E)$ with a distinguished vertex $r \in V$, called the root of $\mathcal{T}$. We denote a rooted tree $\mathcal{T} = (V, E)$ with root $r \in V$ by $\mathcal{T} = (V, \{r\}, E)$. The edges of a rooted tree have a natural orientation away from the root. Consider a rooted tree $\mathcal{T} = (V, \{r\}, E)$ with edges oriented away from the root. Each vertex v distinct from the root has a unique incoming edge $e = (u, v)$ and possibly many outgoing edges $\{f_i = (v, w_i)\}, i = 1, \ldots, N$. We call the vertex u the *parent* of v and the vertices $w_i, i = 1, \ldots, N$ the *children* of v. A *leaf* is a vertex with no children. There is a level structure on V defined by the following: The level of a vertex is the minimal number of edges necessary to reach the root plus one.

- There is a bijective correspondence between vertices $q \in V$ and finite-energy punctured spheres $\tilde{u}_q : \mathbb{C} \setminus \Gamma_q \to \mathbb{R} \times S^3 \in \mathcal{U}$;

- Each sphere $\tilde{u}_q$ has exactly one positive puncture at ∞ and $0 \leq \#\Gamma_q \leq \infty$ negative punctures, where Γ_q is the set of negative punctures of $\tilde{u}_q$;

- If the vertex q is not the root then q has an incoming edge e from a vertex q', and $\#\Gamma_q$ outgoing edges $f_1, \ldots, f_{\#\Gamma_q}$ to vertices $p_1, \ldots, p_{\#\Gamma_q}$ of $\mathcal{T}$, respectively. The edge e is associated to the positive puncture of $\tilde{u}_q$ and the edges $f_1, \ldots, f_{\#\Gamma_q}$ are associated to the negative punctures of $\tilde{u}_q$. The asymptotic limit of $\tilde{u}_q$ at its positive puncture coincides with the asymptotic limit of $\tilde{u}_{q'}$ at its negative puncture associated to e. In the same way, the asymptotic limit of $\tilde{u}_q$ at a negative puncture corresponding to f_i coincides with the asymptotic limit of $\tilde{u}_{p_i}$ at its unique positive puncture;

- If the $d\lambda$-area of $\tilde{u}_q$ vanishes then $\#\Gamma_q \geq 2$.

The following Theorem is a simpler case of the SFT Compactness theorem from [BEH+03].

Theorem 2.31. *Let $\{\tilde{u}_n = (a_n, u_n)\}$ be a germinating sequence with a non-constant limit $\tilde{u} : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times S^3$. Up to a subsequence of $\tilde{u}_n$, still denoted by $\tilde{u}_n$, there exists a bubbling-off tree $\mathcal{B} = (\mathcal{T}, \mathcal{U})$ with the following properties*

- $\tilde{u}_r = \tilde{u}$, *where r is the root of the tree $\mathcal{T}$.*

- *For every vertex q of $\mathcal{T}$ there exist sequences z_n^q, $\delta_n^q \in \mathbb{C}$ and $c_n^q \in \mathbb{R}$ such that*

$$\tilde{u}_n(z_n^q + \delta_n^q \cdot) + c_n^q \to \tilde{u}_q(\cdot) \text{ in } C^\infty_{loc}(\mathbb{C} \setminus \Gamma_q) \text{ as } n \to \infty. \qquad (2.25)$$

Here $\tilde{u} + c := (a + c, u)$, where $\tilde{u} = (a, u)$ and $c \in \mathbb{R}$.

Proof. After a selection of a subsequence, we can assume $\tilde{u}_n \to \tilde{u}$ in $C^\infty_{loc}(\mathbb{C} \setminus \Gamma, \mathbb{R} \times S^3)$. We start with a tree containing just the root r as a vertex and let $\tilde{u}_r = \tilde{u}$. If $\Gamma = \emptyset$, we define $\mathcal{T} = (\{r\}, r, \emptyset)$ and $\mathcal{B} = (\mathcal{T}, \{\tilde{u}_r\})$ and we have finished the construction of the bubbling-off tree.

Otherwise, let $z \in \Gamma$. As in Section 2.3.4, we define a germinating sequence $\{\tilde{w}_{z,n}\}$ by (2.21). Let $\tilde{w}_z : \mathbb{C} \setminus \Gamma_z \to \mathbb{R} \times M$ be a limit of the germinating sequence $\{\tilde{w}_{z,n}\}$. By Proposition 2.28, the asymptotic limit of $\tilde{u}$ at the negative puncture z coincides with the asymptotic limit of $\tilde{w}_z$ at ∞. By Proposition 2.30, if the $d\lambda$-area of $\tilde{w}_z$ vanishes, then $\#\Gamma_z \geq 2$. We add a vertex q_z to the tree, an edge f_z from r to q_z and let $\tilde{u}_{q_z} = \tilde{w}_z$. We do the

same for all $z \in \Gamma$. If $\Gamma_z = \emptyset$ for all $z \in \Gamma$ we have finished the construction and define $\mathcal{T} = (\{r\} \cup \{q_z\}_{z \in \Gamma_z}, r, \{f_z\}_{z \in \Gamma})$. Otherwise, we continue this process for every puncture $\zeta_z \in \Gamma_z$, for every $z \in \Gamma$.

By Proposition 2.30, the periods of the asymptotic limits strictly decreases when going down the tree. Since $T, |T - T'| > \sigma(C) > 0$, for T, T' periods and $T \neq T'$, this process has to finish after a finite number of steps. □

2.3.6 Estimating Conley-Zehnder indices

Remark 2.32. The tight contact structure ξ_0 on S^3 is a trivial symplectic bundle. We will fix a global symplectic trivialization $\Psi : \xi_0 \to S^3 \times \mathbb{R}^2$ and compute the index wind_∞ defined by (2.7) and the Conley-Zender index μ with respect to this trivialization. We will denote $\mathrm{wind}_\infty(\cdot, \cdot) = \mathrm{wind}_\infty(\cdot, \cdot, \Psi)$ and $\mu(\cdot) = \mu(\cdot, \Psi)$. Recall that on (S^3, ξ_0), the index wind_∞ and the Conley-Zehnder index do not depend on the chosen global trivialization of ξ_0.

Lemma 2.33. *Let* $\tilde{v} : (S \setminus \Gamma, j) \to \mathbb{R} \times S^3$ *be a finite energy curve and let* $z \in \Gamma$ *be a puncture with asymptotic limit* $P = (x, T)$. *Define* ν_P^{pos} *and* ν_P^{neg} *by (1.13) and (1.14) respectively. If* $\pi \cdot dv$ *does not vanish identically, then*

(1) $\mathrm{wind}_\infty(\tilde{v}, z) \leq \mathrm{wind}(\nu^{neg})$ *if* z *is a positive puncture.*

(2) $\mathrm{wind}_\infty(\tilde{v}, z) \geq \mathrm{wind}(\nu^{pos})$ *if* z *is a negative puncture.*

The proof follows immediately from the definition of wind_∞, Proposition 1.15 and Theorem 2.11.

The following Lemma will be useful later.

Lemma 2.34. *Let* $\tilde{u} = (a, u) : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times S^3$ *be a finite energy* $\tilde{J}$-*holomorphic curve such that every puncture in* Γ *is negative,* $\pi \circ du$ *does not vanish identically and for every asymptotic limit* P *of* $\tilde{u}$, $\mu(P) \in \{1, 2, 3\}$. *Then* $\mathrm{wind}_\pi(\tilde{u}) = 0$ *and for all* $z \in \Gamma \cup \{\infty\}$, $\mathrm{wind}_\infty(z) = 1$.

Proof. By the definition of the Conley-Zehnder index in (1.15), for every asymptotic limit P of $\tilde{u}$, one of the following options hold

$$
\begin{aligned}
\mu(P) = 1 &\Rightarrow \mathrm{wind}(\nu_P^{neg}) = 0, \ \ \mathrm{wind}(\nu_P^{pos}) = 1 \\
\mu(P) = 2 &\Rightarrow \mathrm{wind}(\nu_P^{neg}) = 1, \ \ \mathrm{wind}(\nu_P^{pos}) = 1 \\
\mu(P) = 3 &\Rightarrow \mathrm{wind}(\nu_P^{neg}) = 1, \ \ \mathrm{wind}(\nu_P^{pos}) = 2
\end{aligned} \tag{2.26}
$$

Then, by the definition of wind_∞, if P is the asymptotic limit of $\tilde{u}$ at z we have

$$
\begin{aligned}
\mathrm{wind}_\infty(z) &\leq \mathrm{wind}(\nu_P^{neg}) \leq 1, \text{ if } z = \infty, \\
\mathrm{wind}_\infty(z) &\geq \mathrm{wind}(\nu_P^{pos})) \geq 1, \text{ if } z \in \Gamma.
\end{aligned} \tag{2.27}
$$

It follows that $\text{wind}_\infty(\tilde{u}) \leq 1 - \#\Gamma$. Since $\text{wind}_\pi(\tilde{u}) \geq 0$, we have

$$0 \leq \text{wind}_\pi(\tilde{u}) = \text{wind}_\infty(\tilde{u}) - 1 + \#\Gamma \leq 0 \ .$$

Thus $\text{wind}_\pi = 0$ and $\text{wind}_\infty(\tilde{u}) = 1 - \#\Gamma$. Using (2.27), we conclude

$$0 \geq \text{wind}_\infty(\infty) - 1 = \sum_{z \in \Gamma} \text{wind}_\infty(z) - \#\Gamma = \sum_{z \in \Gamma} \text{wind}_\infty(z) - 1 \geq 0 \ .$$

Then $\text{wind}_\infty(z) = 1$, for all $z \in \Gamma \cup \{\infty\}$. $\square$

Now we present some results of [HSa11] that will be useful later.

Lemma 2.35. *[HSa11, Lemma 3.3] Let $\Gamma = \{z_1, \cdots, z_N\} \subset \mathbb{C}$ be non-empty and finite, and let $\tilde{u} = (a, u) : \dot{\mathbb{C}} = \mathbb{C} \setminus \Gamma \to \mathbb{R} \times S^3$ be a finite energy surface with exactly one positive puncture at ∞ and negative punctures at the points of Γ. If $\tilde{u}$ is asymptotic to P_∞ at ∞ and is asymptotic to P_j at z_j, for $j = 1, \ldots, N$, then the following assertions are true:*

i. $\int_{\dot{\mathbb{C}}} u^* d\lambda > 0$ and $\mu(P_\infty) \leq 1 \Rightarrow \exists \, j \in 1, \ldots, n$ such that $\mu(P_j) \leq 1$.

ii. $\int_{\dot{\mathbb{C}}} u^* d\lambda = 0$ and $\mu(P_\infty) = 1 \Rightarrow \mu(P_j) = 1, \ \forall j = 1, \ldots, N$.

iii. $\int_{\dot{\mathbb{C}}} u^* d\lambda = 0$ and $\mu(P_\infty) \leq 0 \Rightarrow \mu(P_j) \leq 0, \ \forall j = 1, \ldots, N$.

Proof. Let $\nu_j^{pos} > 0 > \nu_j^{neg}$ be the special eigenvalues of the operators A_{P_j}, for $j = 1, \ldots, N$, as defined in (1.14) and (1.13). In the same way, let $\nu_\infty^{pos} > 0 > \nu_\infty^{neg}$ be the special eigenvalues of the operator A_{P_∞}. Suppose, by contradiction, that $\mu(P_j) \geq 2, \ \forall j = 1, \ldots, N$, $\mu(P_\infty) \leq 1$ and $\pi \circ du$ does not vanish identically. It follows from (1.15) and Lemma 2.33 that

$$\text{wind}_\infty(\tilde{u}, \infty) \leq \text{wind}(\nu_\infty^{neg}) \leq 0,$$
$$\text{wind}_\infty(\tilde{u}, z_j) \geq \text{wind}(\nu_j^{pos}) \geq 1, \ \forall j = 1, \ldots, N.$$

Theorem 2.17 implies that

$$\begin{aligned}
1 - N &\leq \text{wind}_\pi(\tilde{u}) + 1 - N \\
&= \text{wind}_\pi(\tilde{u}) + \chi_{S^2} - (N+1) \\
&= \text{wind}_\infty(\tilde{u}) \\
&= \text{wind}_\infty(\tilde{u}, \infty) - \sum_{j=1}^{n} \text{wind}_\infty(\tilde{u}, z_j) \\
&\leq 0 - N,
\end{aligned}$$

which is a contradiction. Assertion *i.* is proved. Now assume $\pi \circ du$ vanishes identically. If $P_\infty = (x_\infty, T_\infty)$, denote by $\tau > 0$ the minimal positive period of the Reeb trajectory x_∞. Theorem 2.6 implies that each orbit P_j is of the form $P_j = (x_\infty, k_j\tau)$ for some integer $k_j \geq 1$ and $T_\infty = \sum_{j=1}^N k_j\tau$. It follows from Lemma 1.9 that

$$\mu(P_\infty) = 1 \Rightarrow \mu(P_j) = 1, \ \forall j = 1, \ldots, N$$
$$\mu(P_\infty) \leq 0 \Rightarrow \mu(P_j) \leq 0, \ \forall j = 1, \ldots, N$$

□

Lemma 2.36. *[HSa11, Lemma 3.4] Let $\{z_1, \ldots, z_N\}$, $\tilde{u} = (a, u)$, $P_\infty, P_1, \ldots P_N$ be as in the statement of Lemma 2.35. Assume that at least one of the following assertions is true*

i. $\mu(P_j) \geq 2, \ \forall j = 1, \ldots, N, \int_{\dot{\mathbb{C}}} u^*d\lambda > 0$ *and* $\mathrm{wind}_\infty(\tilde{u}, \infty) \leq 1$.

ii. $\mu(P_j) \geq 2, \ \forall j = 1, \ldots, N, \ $ *and* $\mu(P_\infty) \leq 2$.

Then $\mu(P_j) = 2, \forall j = 1, \ldots, N$.

Proof. Assume *i.* As in the proof of Lemma 2.35, it follows from the definition of wind_∞ and (1.15) that $\mathrm{wind}_\infty(\tilde{u}, z_j) \geq 1$, for $j = 1, \ldots, N$. Suppose, by contradiction, that there exists j_0 such that $\mu(P_{j_0}) \geq 3$. Then, again from the definition of wind_∞ and (1.15), we have $\mathrm{wind}_\infty(\tilde{u}, z_{j_0}) \geq 2$. Consequently,

$$\begin{aligned}
1 - N &= \chi(S^2) - (N+1) \\
&\leq \mathrm{wind}_\pi(\tilde{u}) + \chi(S^2) - (N+1) \\
&= \mathrm{wind}_\infty(\tilde{u}) \\
&= \mathrm{wind}_\infty(\tilde{u}, \infty - \sum_{j=1}^N \mathrm{wind}_\infty(\tilde{u}, z_j) \\
&\leq 1 - (N-1) - 2 \\
&= -N,
\end{aligned}$$

a contradiction.

Now assume *ii.* If $\int_{\dot{\mathbb{C}}} u^*d\lambda > 0$, by the definition of wind_∞ and equation (1.15), $\mu(P_\infty) \leq 2$ implies $\mathrm{wind}_\infty(\tilde{u}, \infty) \leq 1$. Then *i.* holds. If $\int_{\dot{\mathbb{C}}} u^*d\lambda = 0$, $P_\infty = (x_\infty, T_\infty)$ and τ is the minimal period of x_∞, then it follows from Theorem 2.6 that each orbit P_j is of the form $P_j = (x_\infty, k_j\tau)$ for integers $k_j \geq 1$ and $T_\infty = \sum_{j=1}^N k_j\tau$. Lemma 1.9 implies that $\mu(P_j) \in \{1, 2\}$ for $j = 1, \ldots, N$. Our assumption implies $\mu(P_j) = 2, \ \forall j = 1, \ldots, N$. □

Lemma 2.37. *[HSa11, Lemma 3.9] Let $\tilde{v} = (b, v)$ be a non-constant limit of a germinating sequence $\{\tilde{v}_n = (b_n, v_n)\}$. Then the asymptotic limit P_∞ at the (unique) positive puncture of $\tilde{v}$ satisfies $\mu(P_\infty) \geq 2$.*

Proof. Following Theorem 2.31, let $\mathcal{B} = (\mathcal{T}, \mathcal{U})$ be the bubbling-off tree associated with the germinating sequence $\tilde{v}_n$ and the limit $\tilde{v}$, so that if r is the root of $\mathcal{T}$, $\tilde{u}_r = \tilde{v}$. Suppose, by contradiction, that $\tilde{u}_r : \mathbb{C} \setminus \Gamma_r \to \mathbb{R} \times S^3$ is asymptotic to P_∞ at its positive puncture ∞ and $\mu(P_\infty) \leq 1$. If $\Gamma_r = \emptyset$, then, by the formula (1.15) and Lemma 2.33, $\tilde{u}_r$ is a finite energy plane satisfying

$$\text{wind}_\infty(\tilde{u}_r, \infty) \leq \text{wind}(\nu_{P_\infty}^{neg}) \leq 0 \ .$$

By Theorem 2.17, we have $\text{wind}_\pi(\tilde{u}_r) < 0$, a contradiction. If $\Gamma_r \neq \emptyset$, we can use Lemma 2.35 to find a negative puncture $z \in \Gamma_r$ and a closed Reeb orbit P_1 such that $\tilde{u}_r$ is asymptotic to P_1 at z and $\mu(P_1) \leq 1$. Thus, if q is the vertex immediately below r, corresponding to the puncture z, then the asymptotic limit of $\tilde{u}_q : \mathbb{C} \setminus \Gamma_q \to \mathbb{R} \times S^3$ at its unique positive puncture is P_1. If q is a leaf, we have a contradiction. If q is not a leaf, we can use Lemma 2.35 again to find a negative puncture $z \in \Gamma_q$ and a closed Reeb orbit P_2 such that $\tilde{u}_q$ is asymptotic to P_2 at z and $\mu(P_2) \leq 1$. Since the tree has a finite number of vertices, continuing this process we end up finding a leaf l so that $\tilde{u}_l : \mathbb{C} \to \mathbb{R} \times S^3$ is a finite energy plane with asymptotic limit P_N satisfying $\mu(P_N) \leq 1$, which implies $\text{wind}_\pi(\tilde{u}_l) < 0$, a contradiction. $\qquad \square$

Proposition 2.38. *[HSa11, Proposition 3.10] Let $\{\tilde{v}_n = (b_n, v_n)\}$ be a germinating sequence and $\tilde{v} = (b, v) : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times S^3$ a non-constant limit of $\tilde{v}_n$. Assume that at least one of the following holds:*

(i) $\Gamma \neq \emptyset$, $\int_{\mathbb{C} \setminus \Gamma} v^* d\lambda > 0$ and $\text{wind}_\infty(\tilde{v}, \infty) \leq 1$;

(ii) $\tilde{v}$ is asymptotic at ∞ to some Reeb orbit P_∞ satisfying $\mu(P_\infty) \leq 2$.

Then there exists a Reeb orbit $P_0 = (x_0, T_0)$ and a finite energy plane $\tilde{u}_0 = (a_0, u_0) : \mathbb{C} \to \mathbb{R} \times S^3$, satisfying:

(1) $u_0 : \mathbb{C} \to S^3$ is an immersion transversal to the Reeb vector field;

(2) $\mu(P_0) = 2$ and $\tilde{u}_0$ is asymptotic to P_0 at the puncture ∞;

(3) if some Reeb orbit $P = (x, T)$ satisfies $v_n(\mathbb{C}) \cap x(\mathbb{R}) = \emptyset \ \forall n$, then $u_0(\mathbb{C}) \cap x(\mathbb{R}) = \emptyset$. If in addition P is simply covered and satisfies $\mu(P) \geq 3$, then $\overline{u_0(\mathbb{C})} \cap x(\mathbb{R}) = \emptyset$.

Proof. Following Theorem 2.31, let $\mathcal{B} = (\mathcal{T}, \mathcal{U})$ be the bubbling-off tree associated with the germinating sequence $\tilde{v}_n$ and the limit $\tilde{v}$, so that if r is the root of $\mathcal{T}$, then $\tilde{u}_r = \tilde{v}$. To prove Proposition 2.38, we need the following lemma.

Lemma 2.39. *If a vertex q of $\mathcal{T}$ is not the root and $\tilde{u}_q$ is asymptotic to the closed Reeb orbit P at its (unique) positive puncture, then $\mu(P) = 2$.*

Proof. Let Γ be the set of negative punctures of $\tilde{v}$. Lemma 2.37 implies that $\mu(P_z) \geq 2, \forall z \in \Gamma$, where P_z is the asymptotic limit of $\tilde{v}$ at z. Using Lemma 2.36 we conclude that $\mu(P_z) = 2 \; \forall z \in \Gamma$. The orbits P_z are precisely the asymptotic limits at the positive punctures of the curves in the second level of the bubbling-off tree. We can apply Lemmas 2.37 and 2.36 to the curves in the second level of the tree to conclude that all negative asymptotic limits of these curves have Conley-Zenhder index equal to 2. We can repeat this argument inductively on each level of the tree to conclude the proof. $\qquad\square$

First assume that $\Gamma = \emptyset$. Then we are in case *(ii)*: $\tilde{v} : \mathbb{C} \to \mathbb{R} \times S^3$ is a finite energy plane and its asymptotic limit satisfies $\mu(P_\infty) \leq 2$. By Theorem 2.17, Lemma 2.33 and the formula (1.15), we have

$$0 \leq \text{wind}_\pi(\tilde{v}) = \text{wind}_\infty(\tilde{v}, \infty) - 1 \leq \frac{1}{2}\mu(P_\infty) - 1$$

so that $\mu(P_\infty) = 2$ and $\text{wind}_\pi(\tilde{v}) = 0$. We set $\tilde{u}_0 = \tilde{v}$ and $P_0 = P_\infty$. As a consequence of the definition of wind_π, we conclude that

$$\mathbb{R}R_{\lambda u_0(z)} \oplus du_0(z)(T_z\mathbb{C}) = T_{u_0(z)}S^3 \; , \; \forall z \in \mathbb{C} \qquad (2.28)$$

proving that u_0 is an immersion transverse to R_λ.

Now assume $\Gamma \neq 0$. The tree $\mathcal{T}$ has a leaf q distinct from the root r. We set $\tilde{u}_0 := \tilde{u}_q$. By Lemma 2.39, we know that

$$\tilde{u}_0 = (a_0, u_0) : \mathbb{C} \to \mathbb{R} \times S^3$$

is a finite energy plane asymptotic to a closed Reeb orbit satisfying $\mu(P_0) = 2$. Now we show that u_0 is an immersion transverse to the Reeb vector field. By Theorem 2.17, Lemma 2.33 and the formula (1.15), we have

$$0 \leq \text{wind}_\pi(\tilde{u}_0) = \text{wind}_\infty(\tilde{u}_0, \infty) - 1 \leq \frac{1}{2}\mu(P_0) - 1 = 0,$$

so that $\text{wind}_\pi(\tilde{u}_0) = 0$. As a consequence of the definition of wind_π, we conclude that $\tilde{u}_0$ satisfies (2.28), proving that u_0 is an immersion transverse

to R_λ.

Thus, in both cases, we found a finite energy plane $\tilde{u}_0 = (a_0, u_0) : \mathbb{C} \to \mathbb{R} \times S^3$ satisfying (1) and (2).

Now we prove that (3) holds for the plane $\tilde{u}_0$. Let $P = (x, T) \in \mathcal{P}(\lambda)$ satisfying $v_n(B_{R_n}(0)) \cap x(\mathbb{R}) = \emptyset$, $\forall n$. Consider the finite energy immersion

$$F : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3, \qquad \zeta \mapsto \left(\frac{T \log |\zeta|}{2\pi}, x \left(\frac{T \arg \zeta}{2\pi} \right) \right)$$

and define
$$A = \{ (z, \zeta) \in \mathbb{C} \times (\mathbb{C} \setminus 0) | \tilde{u}_0(z) = F(\zeta) \} \ .$$

If the set A is not empty, it consists of isolated points. Indeed, assume that (z^*, ζ^*) is not an isolated point of A. Since both $\tilde{u}_0$ and F are immersions, we can use Lemma 2.4.3 of [MS04] to find open neighborhoods $\mathcal{O}$ and $\mathcal{O}'$ of z^* and ζ^* respectively, and a holomorphic diffeomorphism $f : \mathcal{O} \to \mathcal{O}'$ such that $F \circ f = \tilde{u}_0$ on $\mathcal{O}$. Since u_0 is transverse to the Reeb flow, we get a contradiction. Now assume $A \neq \emptyset$ and choose $(z^*, \zeta^*) \in A$. The maps $\tilde{u}_0$ and F intersect transversally at the pair (z^*, ζ^*). By positivity and stability of intersections of pseudo-holomorphic immersions, (see Theorem 2.6.3 and exercise 2.6.7 of [MS04]) we find $z_j \to z^*$ and $n_j \to \infty$ such that $\tilde{v}_{n_j}(z_j) \in F(\mathbb{C} \setminus \{0\})$. This contradicts the fact that $v_n(\mathbb{C}) \cap x(\mathbb{R}) = \emptyset$, $\forall j$. So we have proved that $A = \emptyset$ and consequently $u_0(\mathbb{C} \setminus \{0\}) \cap P = \emptyset$.

If, in addition, $P = (x, T)$ is simply covered and $\mu(P) \geq 3$, then it follows from the fact that $\mu(P_0) = 2$ and item (4) in Lemma 1.9 that P and P_0 are geometrically distinct. This proves (3).

$\square$

2.4   Transverse foliations

To state the main theorem of this Thesis, which will be done in Chapter 3, we need the concept of *transverse foliation* adapted to a flow. This is a refined version of the concept of global systems of transverse sections introduced in [HWZ03].

Definition 2.40 (Transverse Foliation). [HSa18] Let M be a closed oriented 3-manifold and let ϕ^t be a flow on M. A *transverse foliation* for ϕ^t consists of

- A finite set $\mathcal{P}$ of simple periodic orbits of ϕ^t, called *binding orbits*;

- A smooth foliation of $M \setminus \cup_{P \in \mathcal{P}} P$ by properly embedded surfaces. Every leaf is transverse to ϕ^t and has an orientation induced by ϕ^t and M.

For every leaf $\dot{\Sigma}$ there exists a compact embedded surface $\Sigma \hookrightarrow M$ so that $\dot{\Sigma} = \Sigma \setminus \partial\Sigma$ and $\partial\Sigma$ is a union of connected components of $\cup_{P\in\mathcal{P}} P$. An end z of $\dot{\Sigma}$ is called a puncture. To each puncture z there is an associated component $P_z \in \mathcal{P}$ of $\partial\Sigma$, called the asymptotic limit of $\dot{\Sigma}$ at z. A puncture z of $\dot{\Sigma}$ is called positive if the orientation on P_z induced by Σ coincides with the orientation induced by ϕ^t. Otherwise z is called negative.

Transverse foliations adapted to Reeb flows on S^3 can be obtained as the projection onto S^3 of a stable finite energy foliation in the symplectization $\mathbb{R} \times S^3$. See Definition 2.41 below.

2.4.1 Finite energy foliations

Consider S^3 equipped with a nondegenerate tight contact form $\lambda = f\lambda_0|_{S^3}$.

Definition 2.41. [HWZ03] A *stable finite energy foliation* for (S^3, λ, J) is a smooth foliation $\mathcal{F}$ of $\mathbb{R} \times S^3$ with the following properties.

- There exists a uniform constant $C > 0$ such that for every leaf $F \in \mathcal{F}$, there exists an embedded finite energy $\tilde{J}$-holomorphic sphere $\tilde{u} : S^2 \setminus \Gamma \to \mathbb{R} \times S^3$ satisfying

$$F = \tilde{u}(S^2 \setminus \Gamma) \quad \text{and} \quad E(\tilde{u}) \leq C \ ;$$

- The translation

$$T_r(F) = r + F := \{(r + a, m) | (a, m) \in F\}, F \in \mathcal{F}, r \in \mathbb{R}$$

  defines an $\mathbb{R}$-action on $\mathcal{F}$;

- The asymptotic limits of a leaf $F = \tilde{u}(S^2 \setminus \Gamma) \in \mathcal{F}$ are defined to be the asymptotic limits of $\tilde{u}$ at its punctures. For every leaf F, the asymptotic limits are simply covered, their Conley-Zehnder indices are contained in $\{1, 2, 3\}$ and their self-linking numbers are equal to -1;

- If $F = \tilde{u}(S^2 \setminus \Gamma) \in \mathcal{F}$, then $\tilde{u}$ has precisely one positive puncture, but an arbitrary number of negative punctures. We define $\mathrm{ind}(F) := \mathrm{ind}(\tilde{u})$. The index does not depend on the choice of $\tilde{u}$. If F is not a fixed point of the $\mathbb{R}$-action, then $\mathrm{ind}(F) \in \{1, 2\}$.

Consider a leaf $F \in \mathcal{F}$ that is not a fixed point of the $\mathbb{R}$-action. Write $\Gamma = \Gamma^+ \cup \Gamma^- = \Gamma^+ \cup \Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3^-$, where Γ_j^- is the set of punctures having Conley-Zehnder index j. Note that by definition $\Gamma^+ = 1$. Denote by μ^+ the Conley-Zehnder index of the unique positive puncture. Then, by formula (2.9) for the Fredholm index,

$$\begin{aligned}
\operatorname{ind} F &= \mu^+ - 3\#\Gamma_3^- - 2\#\Gamma_2^- - \#\Gamma_1^- - 2 + 1 + \#\Gamma_1^- + \#\Gamma_2^- + \#\Gamma_3^- \\
&= \mu^+ - 1 - 2\#\Gamma_3^- - \#\Gamma_2^-
\end{aligned}$$

Since, by definition, $\operatorname{ind}(F) \geq 1$, we get

$$2\#\Gamma_3^- + \#\Gamma_2^- \leq \mu^+ - 2.$$

Using $\mu^+ \in \{2, 3\}$, we conclude

$$\mu^+ \in \{2, 3\}, \quad \#\Gamma_3^- = 0 \quad \text{and} \quad \#\Gamma_2^- \leq 1 .$$

Note that there is no restriction on $\#\Gamma_1^-$.

Therefore, a leaf $F \in \mathcal{F}$ which is not a fixed point of the $\mathbb{R}$-action satisfies one of the following alternatives.

- $\mu^+ = 3$, $\#\Gamma_2^- = 0$ and $\operatorname{ind} F = 2$.

- $\mu^+ = 3$, $\#\Gamma_2^- = 1$ and $\operatorname{ind}(F) = 1$.

- $\mu^+ = 2$, $\#\Gamma_2^- = 0$ and $\operatorname{ind}(F) = 1$.

In any case, the number of negative punctures with Conley-Zehnder index 1 is arbitrary.

By definition, the energies $E(\tilde{u})$ are uniformly bounded. Since the periods of the asymptotic limits are bounded by the energy and λ is nondegenerate, the number of all asymptotic limits appearing in $\mathcal{F}$ is finite.

The existence of stable finite energy foliations for generic almost complex structures $\tilde{J}$ was proved in [HWZ03].

Theorem 2.42 (Hofer-Wysocki-Zehnder). *Let λ be a nondegenerate tight contact form on S^3. There exists a residual subset of $\mathcal{J}(\xi, d\lambda)$ in the C^∞-topology such that there exists a stable finite energy foliation for (S^3, λ, J) containing a finite energy plane and satisfying the following properties:*

- *The fixed points of the $\mathbb{R}$-action on $\mathcal{F}$ are orbit cylinders;*

- *Let $p : \mathbb{R} \times S^3 \to S^3$ be the projection onto the second factor. If two leaves F and G do not belong to the same orbit of the action, then $p(F) \cap p(G) = \emptyset$;*

- *If $F \in \mathcal{F}$ is not a fixed point of the $\mathbb{R}$-action, then $p(F)$ is an embedded punctured sphere in S^3 which is transverse to the Reeb flow and coverges at the punctures to the asymptotic limits of F.*

- *Denote by $\mathcal{P}$ the set of asymptotic limits of $\mathcal{F}$. Then $p(\mathcal{F})$ is a singular foliation of S^3 having $\mathcal{P}$ as the singular set.*

It follows from the Fredholm theory of [HWZ99b] that a leaf satisfying $\mathrm{ind}(F) = 2$ belongs to a 2-parameter family of leaves, all having the same asymptotic limits, where one parameter is defined by the $\mathbb{R}$-action on $\mathcal{F}$. The image of the 2-parameter family under the projection $p : \mathbb{R} \times S^3 \to S^3$ is a 1-parameter family of embedded punctured spheres.

A leaf satisfying $\mathrm{ind}(F) = 1$ belongs to a 1-parameter family defined by the $\mathbb{R}$-action. The projection of this family onto S^3 is an isolated embedded punctured sphere, that is called a *rigid surface*.

The existence of a transverse foliation adapted to any nondegenerate Reeb flow on (S^3, ξ_0), where ξ_0 is the tight contact structure on S^3, is a corollary of Theorem 2.42. We state this corolary below.

Theorem 2.43 ([HWZ03]). *Let ϕ^t be a nondegenerate Reeb flow on (S^3, ξ_0). Then ϕ^t admits a transverse foliation. The binding orbits have self-linking number -1 and Conley-Zehnder indices in $\{1, 2, 3\}$. Every leaf $\dot{\Sigma}$ is a punctured sphere and has precisely one positive puncture. One of the following conditions holds:*

- *The asymptotic limit of $\dot{\Sigma}$ at its positive puncture has Conley-Zehnder index 3 and the asymptotic limit of $\dot{\Sigma}$ at any negative puncture has Conley-Zehnder index 1 or 2. There exists at most one negative puncture whose asymptotic limit has Conley-Zehnder index 2.*

- *The asymptotic limit of $\dot{\Sigma}$ at its positive puncture has Conley-Zehnder index 2 and the asymptotic limit of $\dot{\Sigma}$ at any negative puncture has Conley-Zehnder index 1.*

2.4.2 Open book decompositions

Assume that the foliation $\mathcal{F}$ given by Theorem 2.42 has precisely one fixed point of the $\mathbb{R}$-action, that is, there is precisely one asymptotic limit P in $\mathcal{P}$. Then $\mu(P) = 3$ and $\mathcal{F}$ consists of a cylinder over the orbit P and a family of pseudoholomorphic planes asymptotic to P. The projection of $\mathcal{F}$ onto S^3 is an open book decomposition [3] with pages of disk type, all having

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[3]An open book decomposition of a 3-manifold M is a pair (L, p) where $L \subset M$ is a link and $p : M \setminus L \to S^1$ is a fibration such that each fiber $p^{-1}(\theta)$ is the interior of a compact
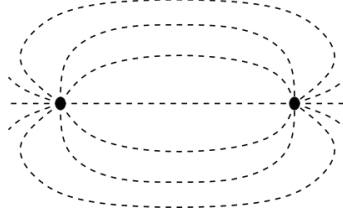
Figure 2.1: An open book decomposition cut by a plane. The dots represent a periodic orbit with Conley-Zehnder index 3. The dashed curves represent a family of planes with boundary P. The 3-sphere is viewed as $\mathbb{R}^3 \cup \{\infty\}$.

the orbit P as boundary. By the arguments in [HWZ98], one can show that the regular leaves of $p(\mathcal{F})$ are global surfaces of section for the Reeb flow associated with λ. Moreover, the first return map is conjugated to an area-preserving diffeomorphism of the open unit disk. Since the area of the disk is finite, Brouwer's translation theorem gives a periodic point of the first return map. It is a initial condition to a periodic solution of the Reeb vector field which is different from P. If the return map has a second periodic point, then by a Theorem of J. Franks [Fra92] on area-preserving homeomorphisms of the open annulus, the return map has infinitely many periodic points, so that X_λ has infinitely many periodic orbits. This is the case of dynamically convex contact forms.

Definition 2.44. A contact form λ on S^3 is called dynamically convex if $\mu(P) \geq 3$ for all periodic solution P of the associated Reeb vector field X_λ.

If the contact form λ is nondegenerate and dynamically convex, then the finite energy foliation given by Theorem 2.42 has precisely one fixed point of the $\mathbb{R}$-action, and we conclude the following corollary of Theorem 2.42.

Corollary 2.45. *The Reeb vector field X_λ associated with a nondegenerate and dynamically convex contact form $\lambda = f\lambda_0$ on S^3 possesses a global surface of section. Further, there are either 2 or infinitely many periodic orbits of X_λ.*

It is proved in [HWZ98] that the statement above holds true without the nondegeneracy condition on λ.

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embedded surface $S_\theta \hookrightarrow M$ satisfying $\partial S_\theta = L$. L is called the binding and the fibers are called pages. If the pages are disks we say that (L,p) has disk-like pages.

Chapter 3

3-2-1 Foliations and the main theorem

In this chapter, we define $3-2-1$ foliations, state the main theorem of this thesis and give a sketch of its proof.

3.1   Main Theorem

Let λ be a nondegenerate tight contact form on S^3. Let R_λ be the associated Reeb vector field and denote by φ^t the flow of R_λ.

Before defining $3-2-1$ foliations, we need the definition of *strong transverse sections*.

Definition 3.1 (Strong transverse section). Let $\Sigma \hookrightarrow S^3$ be a compact embedded surface such that $\dot{\Sigma} = \Sigma \backslash \partial \Sigma$ is transverse to φ^t and $\partial \Sigma$ consists of a finite number of simple orbits in $\mathcal{P}(\lambda)$. Σ is called a *strong transverse section* if every orbit $P_z = (x, T)$ associated to an end z of $\dot{\Sigma}$ has a neighborhood on Σ parametrized by $\psi : (r_0, 1] \times \mathbb{R}/\mathbb{Z} \to \Sigma$, such that $\psi(1, t) = x_T(t)$, $\forall t \in \mathbb{R} \times \mathbb{Z}$, and the section of $x_T^* \xi$ defined by

$$\eta(t) = \frac{\partial}{\partial r} \psi(r, t)\big|_{r=1}$$

satisfies

$$d\lambda(\eta(t), \mathcal{L}_R \eta(t)) \neq 0, \ \forall t \in \mathbb{R} \times \mathbb{Z},$$

where $\mathcal{L}_R$ is the Lie derivative in the direction of R_λ as defined in (1.6).

Remark 3.2. Any transverse section given by the projection of a $\tilde{J}$-holomorphic curve satisfies the definition above. Indeed, assume $\dot{\Sigma} = u(\dot{S})$, where $\tilde{u} =$

$(a, u) : \dot{S} \to \mathbb{R} \times S^3$ is a finite energy $\tilde{J}$-holomorphic curve. Then, by Theorem 2.11, for every puncture z with asymptotic limit P, the section $\pi_\xi \eta$, where η is defined as in Definition 3.1, is an eigensection of the asymptotic operator $A_{P,J}$ associated with a nonzero eigenvalue c and $\eta(t) \neq 0$, $\forall t \in S^1$. Thus, by the definition of $A_{P,J}$ in (1.8) and equation (1.7), we have

$$d\lambda(\eta(t), \mathcal{L}_R \eta(t)) = \frac{1}{T} d\lambda(\eta(t), J A_{P,J} \eta(t)) = \frac{1}{T} c \, d\lambda(\eta(t), J\eta(t)) \neq 0, \ \forall t.$$

Remark 3.3. In what follows we fix a global unitary trivialization

$$\Psi : \xi_0 \to S^3 \times \mathbb{R}^2 \tag{3.1}$$

of the tight contact structure $\xi_0 = \ker \lambda$ on S^3. All indices are computed with respect to this fixed trivialization. Recall that in (S^3, ξ_0) the Conley-Zehnder index and the index wind$_\infty$ do not depend on the chosen global trivialization.

Definition 3.4 (3-2-1 Foliation). A 3-2-1 foliation for the Reeb flow φ^t on $(S^3, \xi_0 = \ker \lambda)$ is a transverse foliation for φ^t satisfying the following properties: The set $\mathcal{P}$ of binding orbits consists of three prime orbits P_1, P_2 and P_3 with Conley-Zehnder indices respectively 1, 2 and 3. $P_1 \cup P_2 \cup P_3$ is a unlink. The foliation of $S^3 \setminus \cup_{P \in \mathcal{P}} P$ consists of

- A pair of cylinders V_1 and V_2, asymptotic to P_3 at their positive punctures and to P_2 at their negative punctures. $T := V_1 \cup V_2 \cup P_2 \cup P_3$ is homeomorphic to a torus and $T \setminus x_3(\mathbb{R})$ is C^1-embedded. T divides S^3 into two closed regions $\mathcal{R}_1$ and $\mathcal{R}_2$ with boundary T.

- A disk $D \subset \mathcal{R}_1$ asymptotic to P_2 at its positive puncture;

- A cylinder $U \subset \mathcal{R}_2$ asymptotic to P_2 at its positive puncture and to P_1 at its negative puncture;

- $D \cup U$ is a C^1-embedded disk with boundary P_1 and transverse to T;

- A 1-parameter family $F_\tau \subset \mathcal{R}_1$ of planes asymptotic to P_3 at its positive puncture;

- A 1-parameter family of cylinders $C_\tau \subset \mathcal{R}_2$ asymptotic to P_3 at its positive puncture and to P_1 at its negative puncture;

- Every regular leaf of the foliation is a strong transverse section.
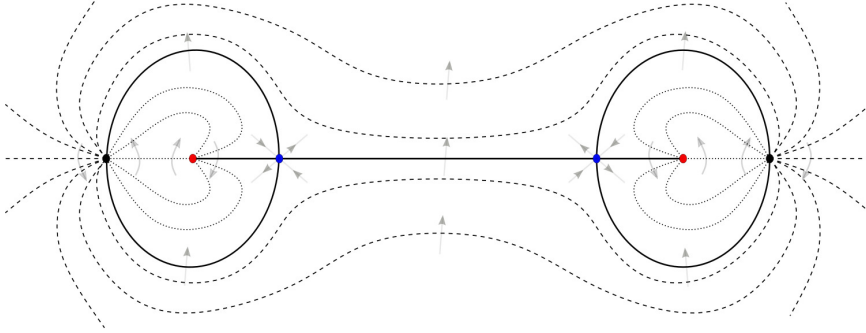
See figure 3.1.

Figure 3.1: A $3 - 2 - 1$ foliation cut by a plane. The dots represent periodic orbits P_1, P_2 and P_3 with Conley-Zehnder indices respectively 1, 2 and 3. The bold curves represent two rigid cylinders connecting P_2 and P_3, a rigid cylinder connecting P_1 and P_2, and a rigid disk with boundary P_2. The dashed curves represent a family of planes with boundary P_3. The dotted curves represent a family of cylinders connecting P_1 and P_3. The arrows indicate the Reeb flow. The 3-sphere is viewed as $\mathbb{R}^3 \cup \{\infty\}$.

Now we state the main theorem of this thesis.

Theorem 3.5. *Let λ be a nondegenerate tight contact form on S^3. Let $P_1 = (x_1, T_1)$, $P_2 = (x_2, T_2)$, $P_3 = (x_3, T_3) \in \mathcal{P}(\lambda)$ be simply covered closed Reeb orbits with Conley-Zehnder indices respectively 1, 2 and 3. Assume that the orbits P_1, P_2 and P_3 are unknotted, P_i and P_j are not linked for $i \neq j$, $i, j \in \{1, 2, 3\}$ and the following conditions hold:*

(i) P_3 spans an embedded disk whose interior is transverse to the Reeb flow;

(ii) $T_1 < T_2 < T_3 < 2T_1$;

(iii) P_2 is the only Reeb orbit with Conley-Zehnder index 2 not linked to P_3 with period $< T_3$;

(iv) P_1 is the only Reeb orbit with Conley-Zehnder index 1 not linked to P_2 with period $< T_2$;

(v) There is no C^1-embedding $\Psi : S^2 \to S^3$ such that $x_2(T_2 \cdot) = \Psi|_{S^1 \times \{0\}}$ and each hemisphere is a strong transverse section.

Then P_1, P_2 and P_3 are the binding orbits of a $3 - 2 - 1$ foliation.

Condition *(v)* in Theorem 3.5 is also necessary. This is the content of Proposition 3.6 below.

Proposition 3.6. *Assume there exists a 3-2-1 foliation for the Reeb flow φ^t and let $P_2 = (x_2, T_2)$ be the binding orbit with Conley-Zehnder index 2, as in Definition 3.4. Then there is no C^1-embedding $\psi : S^2 \to S^3$ such that $x_2(T_2 \cdot) = \psi|_{S^1 \times \{0\}}$ and each hemisphere is a strong transverse section.*

3.1.1   Sketch of the proof of Theorem 3.5

Following [Hof93, HWZ95a, HWZ99a, HSa11], we consider a Bishop family of disks with boundary on a suitable disk $\mathcal{D}$ spanning P_3. By hypothesis there is an orbit with Conley-Zehnder index 2 not linked to P_3. So, we have to consider the possibility that the Bishop family breaks before it approaches $\partial \mathcal{D}$. We divide our analysis into two cases:

- Either the boundaries of the disks in the Bishop family tend to P_3, or

- The Bishop family breaks before it approaches $\partial \mathcal{D}$.

In the first case, the Bishop family produces either a plane asymptotic to P_3, so that we have a family of such planes, or a rigid cylinder asymptotic to P_3 and to P_2 and a rigid plane asymptotic to P_2, so that we can apply the gluing theorem to produce a family of planes asymptotic to P_3. Using our assumptions, we prove that the family of planes is noncompact. The closure of the family of projected planes is a solid torus $\mathcal{R}_1 \subset S^3$. The family of planes, together with a pair of rigid cylinders and a rigid plane to P_2 form part of the $3 - 2 - 1$ foliation. By the Fredholm theory of [HWZ99b], the plane asymptotic to P_2 lies in a one dimensional family of such planes in a symplectic cobordism. We show that the family is non compact, and produces a rigid cylinder asymptotic to P_2 and P_1. Applying the gluing theorem again, we produce a 1-parameter family of cylinders asymptotic to the orbits P_3 and P_1. This completes the $3 - 2 - 1$ foliation.

We do not give a complete proof to the second case in this thesis, however, we present the main ideas of the proof in Chapter 5. In the second case, we just obtain a rigid plane to P_2 from the Bishop family. As in the first case, using the rigid plane to P_2, we can prove the existence of a rigid cylinder connecting P_2 and P_1. Using gluing, we can obtain a family of pesudoholomorphic half-cylinders with boundary in $\mathcal{D}$ and a negative puncture asymptotic to P_1. We show that the boundaries of the half-cylinders in this family "go in the direction of the boundary of $\mathcal{D}$". If this family breaks before it approaches the boundary, we use gluing theorem again to produce a family of disks with boundary in $\mathcal{D}$. We repeat this process until we obtain a family of disks or half-cylinders whose boundaries converge to $\partial \mathcal{D}$. After this step we produce either a cylinder connecting P_3 and P_1 or a plane to P_3. From this point on, the analysis is basically the same as in the first case.

3.2 Proof of Proposition 3.6

In this section we use the notation of Definition 3.4.

Lemma 3.7. *If there exists a C^1-embedding $\psi : S^2 \to S^3$ such that $x_2(T_2\cdot) = \psi|_{S^1 \times \{0\}}$ and $\psi(S^2) \setminus x_2(\mathbb{R})$ is transverse to the Reeb flow, then we can assume that $\psi|_{S^2 \setminus S^1 \times \{0\}}$ is also transverse to $T \setminus (x_3(\mathbb{R}) \cup x_2(\mathbb{R})) = V_1 \cup V_2$.*

Proof. Define
$$F : \mathbb{R} \times S^2 \to S^3$$
$$(t, x) \mapsto \varphi^t \circ \psi(x).$$
Then F satisfies $F(t, \cdot)(S^1 \times \{0\}) = \psi(S^1 \times \{0\})$, $\forall t \in \mathbb{R}$, and

$$dF_{(t,x)}(a, v) = aR(\varphi^t \circ \psi(x)) + d\varphi^t_{\psi(x)} \cdot d\psi_x v \ ,$$

for every $(t, x) \in \mathbb{R} \times S^2$ and $(a, v) \in \mathbb{R} \times T_x S^2$. We claim that for every $x \in S^2 \setminus S^1 \times \{0\}$ and $t \in \mathbb{R}$, $dF(t, x)$ is surjective. Since ψ is transverse to the Reeb flow away from $S^1 \times \{0\}$, we know that $\pi_\xi \circ d\psi_x$ is surjective. Since $d\varphi^t$ preserves the splitting $TS^3 = \mathbb{R}R \oplus \xi$, it follows that $\pi_\xi \circ d\varphi^t_{\psi(x)} \circ d\psi_x$ is surjective. We conclude that $dF(t, x)$ is surjective for all $x \in S^2 \setminus S^1 \times \{0\}$ and $t \in \mathbb{R}$. In particular, F is transverse to $T \setminus x_3(\mathbb{R})$ on $\mathbb{R} \times (S^2 \setminus S^1 \times \{0\})$. This implies [1] that for almost all $t \in \mathbb{R}$, $F(t, \cdot)|_{S^2 \setminus S^1 \times \{0\}}$ is transverse to T.

Also, $F(t, \cdot)^* d\lambda = (\varphi^t \circ \psi)^* d\lambda = \psi^* d\lambda$, and it follows that $F(t, \cdot)$ is still transverse to the flow for all $t \in \mathbb{R}$. Thus, we can replace ψ with $F(t, \cdot)$ for some t such that $F(t, \cdot)|_{S^2 \setminus S^1 \times \{0\}} \pitchfork T$ to get the desired embedding. $\qquad\square$

Lemma 3.9. *Assume that $\psi : S^2 \to S^3$ is a C^1 embedding such that $x_2(T_2\cdot) = \psi|_{S^1 \times \{0\}}$ and such that each hemisphere is a strong transverse section. Then ψ is transverse to the torus T along P_2.*

The proof of Lemma 3.9 is given in Appendix A.

Proof of Proposition 3.6. Suppose, by contradiction, that $\psi : S^2 \to S^3$ is a C^1 embedding such that $x_2(T_2\cdot) = \psi|_{S^1 \times \{0\}}$ and such that each hemisphere is a strong transverse section.

Let $T = V_1 \cup V_2 \cup P_2 \cup P_3$ be the torus given by Definition 3.4. T divides S^3 into two connected regions $\mathcal{R}_1$ e $\mathcal{R}_2$. The region $\mathcal{R}_1$ contains an

---

[1] Applying the following theorem to F restricted to $S^2 \setminus (S^1 \times \{0\})$:

Theorem 3.8 ([GP10]). *Let X, S and Y be smooth manifolds, Z submanifold of Y and $F : X \times S \to Y$ smooth. If F is transverse to Z, then for almost all $s \in S$, $f_s := F(s, \cdot)$ is transverse to Z.*

embedded disk with boundary $x_2(\mathbb{R})$, so that P_2 is contractible in $\mathcal{R}_1$. One can also show, using Mayer-Vietoris sequence, that the holomology class of x_{T_2} generates $H_1(\mathcal{R}_2, \mathbb{Z})$.

Since, by Lemma 3.9, ψ is transverse to the torus T along P_2, we know that the image of any neighborhood of $S^1 \times \{0\} \subset S^2$ by ψ intersects both $\mathcal{R}_1$ and $\mathcal{R}_2$ away from $\psi(S^1 \times \{0\})$. This implies that the sphere $\psi(S^2)$ intersects the torus T away from $x_2(\mathbb{R})$. Indeed, one of the hemispheres of $\psi(S^2)$ intersects $\mathcal{R}_2$ and can not be contained in $\mathcal{R}_2$, because this would imply P_2 contractible in $\mathcal{R}_2$, which is a contradiction since the holomology class of x_{T_2} generates $H_1(\mathcal{R}_2, \mathbb{Z})$.

By Definition 3.4, the orbits P_3 and P_2 are not linked, that is, the linking number $\mathrm{lk}(P_2, P_3)$ is zero. Note that the image of each closed hemisphere of S^2 by ψ is a Seifert surface for P_2, which is transverse to the Reeb flow. Then $\mathrm{lk}(P_2, P_3)$ is the intersection number of P_3 and ψ restricted to one of the hemispheres of S^2. Since $\mathrm{lk}(P_2, P_3) = 0$ and $\psi|_{S^2 \setminus S^1 \times \{0\}}$ is transverse to P_3, we conclude that $\psi(S^2) \cap P_3 = \emptyset$. By Lemmas 3.7 and 3.9, we know that $\psi \pitchfork T \setminus P_3$. Thus, ψ intersects T transversely and the intersection $\psi(S^2) \cap T$ is contained in a closed subset of $T \setminus P_3$. We conclude that the preimage of the intersection $\psi(S^2) \cap T$ by ψ is a 1-dimensional submanifold of S^2 which is a closed subset of S^2. It follows that each connected component of $\psi^{-1}(\psi(S^2) \cap T)$ is diffeomorphic to S^1.

$S^1 \times \{0\}$ is one of the connected components of the boundary of a region $R \subset S^2$ such that $\psi(R) \subset \mathcal{R}_2$. Thus, one of the other connected components of the boundary of R, that we denote by S, is such that $\psi|_S$ is homologous to x_{2T_2} in $\mathcal{R}_2$. Denoting $H^1(T, \mathbb{Z}) = \mathbb{Z}[x_{2T_2}] \oplus \mathbb{Z}[m]$, the homology class of $\psi|_S$ in $H^1(T, \mathbb{Z})$ is $(1, l)$ for some $l \in \mathbb{Z}$. Since S does not intersect $x_2(\mathbb{R})$, l must be zero, that is, the class of $\psi|_S$ is $(1, 0)$. This implies that $\psi(S)$ and $x_2(\mathbb{R})$ divide T into two connected regions with boundary $x_2(\mathbb{R}) \cup \psi(S)$.

Now fix an orientation on $S^1 \times \{0\} \subset S^2$ in such a way that $\psi|_{S^1 \times \{0\}}$ preserves orientation. Consider the closed hemispheres of S^2, that we call H_+ e H_-, with the orientation induced by the orientation of $S^1 \times \{0\}$. It follows that

$$0 < T_2 = \int_{S^1} x_{2T_2}^* \lambda = \int_{H_\pm} \psi^* d\lambda \ .$$

Since ψ is transverse to the Reeb flow R_λ in $H_\pm \setminus S^1 \times \{0\}$, then $\psi^* d\lambda > 0$ in $H_\pm \setminus S^1 \times \{0\}$. Let B be the connected region of S^2 bounded by $S^1 \times \{0\}$ and S. Since B is contained in one of the hemispheres of S^2, then

$$0 < \int_B \psi^* d\lambda = \int_{S^1} x_{2T_2}^* \lambda - \int_S \psi^* \lambda = T_2 - \int_{\psi(S)} \lambda \ . \tag{3.2}$$

Since $\psi(S^2)$ does not intersect P_3, we know that one of the regions of T bounded by P_2 and $\psi(S)$, that we denote by A, satisfies either $A \subset V_1$ or $A \subset V_2$. Recall that V_1 and V_2 are oriented in such a way that $d\lambda|V_{1,2}$ is an area form, P_3 is a positive asymptotic limit and P_2 is a negative asymptotic limit. Then we have

$$0 < \int_A d\lambda = \int_{\psi(S)} \lambda - \int_{x_2(\mathbb{R})} \lambda = \int_{\psi(S)} \lambda - T_2 \ . \tag{3.3}$$

Leading to a contradiction.

$\square$

3.3 Proof of the main theorem

In this section we present the first steps in the proof of Theorem 3.5.

Consider S^3 equipped with a nondegenerate tight contact form λ. Let P_1, P_2 and P_3 be closed Reeb orbits satisfying the hypotheses of Theorem 3.5.

3.3.1 A special spanning disk for the orbit P_3

Let F be a closed disk embedded in S^3 such that ∂F is transverse to the contact structure $\xi = \ker \lambda$. The contact structure induces a singular distribution

$$(\xi \cap TF)^{d\lambda}$$

on F, called the *characteristic distribution* of F. Observe that

$$T_pF = \xi|_p \Rightarrow (\xi|_p \cap T_pF)^{d\lambda} = 0$$

$$T_pF \cap \xi|_p \subsetneq \xi|_p \Rightarrow (\xi|_p \cap T_pF)^{d\lambda} = \xi|_p \cap T_pF \ \ .$$

The characteristic foliation can be parametrized by a smooth vector field V on F in a canonical way, see [HWZ95b, Section 4] for details. In other words, $V_p = 0$ if and only if $T_pF = \xi_p$ and $\mathbb{R}V_p = T_pF \cap \xi_p$ if and only if $T_pF \cap \xi_p \subsetneq \xi_p$. Moreover, we can assume that V points outwards at the boundary ∂F.

Let $p \in F$ be a nondegenerate zero of V, that is, the linearization dV_p is an isomorphism of $\xi_p = T_pF$. If a and b are the eigenvalues of dV_p, then $ab \neq 0$. The point p is called *elliptic* if $ab > 0$ and *hyperbolic* if $ab < 0$. An elliptic point is called *nicely elliptic* if a and b are real numbers.

Denote by o the orientation of F so that the induced orientation on the boundary satisfies $\lambda|_{T\partial F} > 0$. We also have a fiberwise orientation o' of $\xi|_F$ induced by $d\lambda$. A nondegenerate zero p of V, where by definition $T_pF = \xi_p$,

is *positive* if o coincides with o' and *negative* otherwise. Negative elliptic points are sinks and positive elliptic points are sources for the vector field V, see [HWZ95b] for details.

The following theorem is a particular case of [HWZ95a, Theorem 3.2].

Theorem 3.10 ([HWZ95a]). *Let λ be a contact form on S^3 with associated contact structure ξ and Reeb vector field R_λ. Assume there exists a non-degenerate and simply covered orbit $P = (x, T)$ satisfying $\mu(P) = 3$, which spans an embedded disk F_0 whose interior is transverse to R_λ. Then there exists an embedded disk F spanning P whose interior is transverse to R and whose characteristic foliation is transverse to ∂F and possesses precisely one singular point e, which is positive and nicely elliptic.*

By hypothesis *(i)* of Theorem 3.5, we know that the orbit P_3 spans an embedded disk whose interior is transverse to R_λ. Applying Theorem 3.10 to the orbit $P = P_3$ we obtain an embedded disk $\mathcal{D}$ spanning P_3 with special properties. The vector field V that parametrizes the characteristic distribution $(\xi \cap T\mathcal{D})^{d\lambda}$ points outwards at $\partial\mathcal{D}$ and has precisely one zero e in the interior of $\mathcal{D}$, which is a positive nicely elliptic singularity and a source for the dynamics of V.

3.3.2   The Bishop family

In this section, we follow the exposition of [HSa11] and [HLSa15].

Let $\mathcal{D}$ be the disk obtained by applying Theorem 3.10 to the orbit P_3. Identifying S^3 with $\{0\} \times S^3 \subset \mathbb{R} \times S^3$ we identify $\mathcal{D}$ with $\{0\} \times \mathcal{D} \subset \mathbb{R} \times S^3$. Following ideas of [Hof93, HWZ95a, HWZ99a], we consider the boundary value problem

$$
\begin{cases}
\tilde{u} = (a, u) : \mathbb{D} \to \mathbb{R} \times S^3 \text{ is an embedding,} \\
d\tilde{u} \cdot i = \tilde{J}(\tilde{u}) \cdot d\tilde{u}, \\
a \equiv 0 \text{ on } \partial\mathbb{D}, \quad u(\partial\mathbb{D}) \subset \mathcal{D} \setminus \{e\}, \\
u(\partial\mathbb{D}) \text{ winds once positively around } e.
\end{cases}
\tag{3.4}
$$

Here i denotes the standard complex structure on $\mathbb{C}$ and J is any complex structure in $\mathcal{J}(\xi, d\lambda)$. The disk $\mathcal{D}$ is oriented so that $\lambda|_{\partial\mathcal{D}} > 0$ and the loop $u(\partial\mathbb{D})$ is oriented by orienting $\partial\mathbb{D}$ counterclockwise.

Define, for any $J \in \mathcal{J}(\xi, d\lambda)$,

$$\mathcal{M}(J) = \left\{ \tilde{u} = (a, u) : \mathbb{D} \to \mathbb{R} \times S^3 \text{ solution of (3.4)} \mid u(\partial\mathbb{D}) \cap \partial\mathcal{D} = \emptyset \right\}.$$

It is proved in [Hof93] that there exists $J \in \mathcal{J}(\xi, d\lambda)$ and a continuous map $\Psi : [0, \epsilon) \times \mathbb{D} \to \mathbb{R} \times S^3$ such that $\Psi : (0, \epsilon) \times \mathbb{D} \to \mathbb{R} \times S^3$ is a smooth

embedding, $\Psi(0, \mathbb{D}) = \{e\}$ and

$$\Psi(\tau, \cdot) : \mathbb{D} \to \mathbb{R} \times S^3 \text{ satisfies } (3.4), \text{ for } \tau \in (0, \epsilon).$$

Note that if $\tilde{u}$ is a solution of (3.4), then also $\tilde{u} \circ \varphi$ is a solution for every biholomorphism $\varphi : \mathbb{D} \to \mathbb{D}$. The group G of all such biholomorphisms has dimension 3. It is proved in [Hof93] that for any $J \in \mathcal{J}(\xi, d\lambda)$, the linearization of $\partial_{\tilde{J}}$ at any $\tilde{u} \in \mathcal{M}(J)$ is surjective and its Fredholm index is 4. An application of the implicit function theorem turns $\mathcal{M}(J)$ into a 4-dimensional smooth manifold. One can show that $(\tilde{u}, \varphi) \mapsto \tilde{u} \circ \varphi$ is a smooth, proper and free right action of G on $\mathcal{M}(J)$. This implies[2] that $\Pi : \mathcal{M}(J) \to \frac{\mathcal{M}(J)}{G}$ is a smooth principal G-bundle with one dimensional base space $\frac{\mathcal{M}(J)}{G}$, where Π is the quotient projection.

It is also proved in [Hof93] that if $\tilde{u}_n \in \mathcal{M}(J)$ satisfies $\tilde{u}_n \to \tilde{u}$ in $C^\infty_{loc}(\mathbb{D}, \mathbb{R} \times S^3)$ and $\tilde{u}$ is non-constant, then $\tilde{u}$ solves (3.4).

Next, we want to parametrize the one dimensional base space $\frac{\mathcal{M}(J)}{G}$. To do so, we need some additional facts.

Fix $\tilde{u}_0 \in \mathcal{M}(J)$ and let $t_0 = \Pi(\tilde{u}_0) \in \frac{\mathcal{M}(J)}{G}$. It is proved in [Hof93] that if s is a section of $\mathcal{M}(J) \to \frac{\mathcal{M}(J)}{G}$ defined around t_0 satisfying $s(t_0) = \tilde{u}_0$, then there exists a neighborhood U of t_0 in $\frac{\mathcal{M}(J)}{G}$ such that the map

$$\begin{aligned} \Phi : U \times \mathbb{D} &\to \mathbb{R} \times S^3 \\ (t, z) &\mapsto s(t)(z) \end{aligned} \tag{3.5}$$

is a smooth embedding onto its image.

Each leaf l of the characteristic foliation is a trajectory of the vector field V. Since there are no singularities other than e, $\partial \mathcal{D}$ is the image of a Reeb trajectory and V points outwards at $\partial \mathcal{D}$, the α-limit of l is the source e and l hits $\partial \mathcal{D}$ transversally in forward and finite time. Moreover, l has finite length since e is nicelly elliptic.

As proved in [Hof93, Lemma 19], the strong maximum principle[3] implies that if $\tilde{u} \in \mathcal{M}(J)$, then $u(\partial \mathbb{D})$ intersects the leaves transversally. Since $u(\partial \mathbb{D})$ winds once around e in $\mathcal{D}$, it hits every leaf exactly once.

Following [HWZ95a], choose a leaf l_1 of the characteristic foliation and denote its length by $\bar{\tau}$. We can define a smooth map $\tau : \frac{\mathcal{M}(J)}{G} \to (0, \bar{\tau})$ as follows.

Fix $\tilde{u} = (a, u) \in \mathcal{M}(J)$. For any $\varphi \in G$, $u \circ \varphi(\partial \mathbb{D})$ intersect l_1 at the

---

[2] By the caracterization of smooth principal bundles.

[3] a.k.a Hopf Lemma.

same point $p_*(\tilde{u})$. Set

$$\tau(\Pi(\tilde{u})) = \text{ length of the piece of } l_1 \text{ connecting } e \text{ to } p_*(\tilde{u}) \ .$$

It follows from the existence of local embeddings as in (3.5) that τ is a local diffeomorphism.

Intersections   The following result is a consequence of Theorem 4.4 of [HWZ95a].

Theorem 3.11 ([HWZ95a]). *Let $\mathcal{Y} \subset \frac{M}{G}$ be a connected component containing disks arbitrarily close to the constant $(0, e)$. If $\tilde{u} : \mathbb{D} \to \mathbb{R} \times M$ is a smooth map and $\tilde{u}_k \in \Pi^{-1}(\mathcal{Y})$ is a sequence satisfying $\tilde{u}_k \to \tilde{u}$ in C^∞, then $\tilde{u}(\mathbb{D}) \cap (\mathbb{R} \times x_3(\mathbb{R})) = \emptyset$.*

The following proposition is a consequence of the arguments in the proof of [HWZ95a, Theorem 4.4] and will be useful later.

Proposition 3.12. *Let $J \in \mathcal{J}(\xi, d\lambda)$. Consider a point $z \in \partial\mathbb{D}$, an open neighborhood U of z in $\mathbb{D}$ and a sequence $\tilde{u}_k = (a_k, u_k) : U \to \mathbb{R} \times S^3$ of $\tilde{J}$-holomorphic maps satisfying $u_k(U \cap \partial\mathbb{D}) \subset \mathcal{D} \setminus \partial\mathbb{D}$, $\forall k$. Assume $\tilde{u} = (a, u) : U \to \mathbb{R} \times S^3$ is a $\tilde{J}$-holomorphic map such that $\tilde{u}_k \to \tilde{u}$ in $C^\infty_{loc}(U)$ and $\int_U u^* d\lambda > 0$. If $u(z) \in x_3(\mathbb{R})$, then for k large enough, there exists $\delta_k \in U$ satisfying $u_k(\delta_k) \in x_3(\mathbb{R})$.*

A proof of Proposition 3.12 can be found in [HLSa15, Corollary 6.11].

Generic almost complex structures   Let $\Gamma \subset \mathring{\mathbb{D}}$ be a finite non empty set and consider the mixed boundary value problem

$$\begin{cases} \tilde{u} = (a, u) : \mathbb{D} \setminus \Gamma \to \mathbb{R} \times S^3 \text{ is an embedding} \\ d\tilde{u} \cdot i = \tilde{J}(\tilde{u}) \cdot d\tilde{u} \\ a \equiv 0 \text{ on } \partial\mathcal{D}, \ u(\partial\mathbb{D}) \subset \mathcal{D} \setminus \{e\} \\ u(\partial\mathbb{D}) \text{ winds once positively around } e \\ \displaystyle\int_{\mathbb{D}\setminus\Gamma} u^* d\lambda > 0, \ E(\tilde{u}) < \infty \\ \text{Every } z \in \Gamma \text{ is a negative puncture} \end{cases} \qquad (3.6)$$

Here we do not fix the complex structure on $\mathbb{D}$.

The following proposition is a consequence of results in [HWZ99b].

Proposition 3.13. *There exists a residual set $\mathcal{J}_{gen}^2 \subset \mathcal{J}(\xi, d\lambda)$ such that the following holds. Assume $J \in \mathcal{J}_{gen}^2$ and $\tilde{u}$ is a $\tilde{J}$-holomorphic solution of (3.6) asymptotic to a closed Reeb orbit P_z at each puncture $z \in \Gamma$. If $\mu(P_z) \geq 2$, $\forall z \in \Gamma$, then there exists at most one puncture in Γ and if there is a puncture z_0, then $\mu(P_{z_0}) = 2$.*

Let $\mathcal{J}_{gen}^1 \subset \mathcal{J}(\xi, d\lambda)$ be the residual set obtained by Theorem 2.20, and define
$$\mathcal{J}_{reg} := \mathcal{J}_{reg}^1 \cap \mathcal{J}_{reg}^2.$$

Limiting behavior of the Bishop family There exists $J \in \mathcal{J}(\xi, d\lambda)$ and a disk $\tilde{v}_* = (b, v) \in \mathcal{M}(J)$ arbitrarily close to the constant $(0, e)$ satisfying

$$\tilde{v}(\mathbb{D}) \cap (\mathbb{R} \times x_3(\mathbb{R})) = \emptyset.$$

Every such $\tilde{v}_*$ is automatically Fredholm regular for a Fredholm theory of pseudoholomorphic disks with boundary in the embedded surface $\{0\} \times \mathcal{D} \setminus \{e\}$, which is totally real in $(\mathbb{R} \times M, \tilde{J})$, as proved in [Hof93]. If $J' \in \mathcal{J}_{reg}$ is a C^∞ small perturbation of J, then we can find a disk $\tilde{v}_* \in \mathcal{M}(J')$ as a C^∞ small perturbation of $\tilde{v}_*$. Reverting the notation back to J and $\tilde{v}_*$, it follows that we could have assumed $J \in \mathcal{J}_{reg}$ from the beginning.

Let $\mathcal{Y}$ be the connected component of $\frac{\mathcal{M}(J)}{G}$ containing $\Pi(\tilde{v}_*)$. The set of solutions $\tilde{u}$ of (3.4) in $\Pi^{-1}(\mathcal{Y})$ satisfying $\tilde{u}(\mathbb{D}) \cap (\mathbb{R} \times x_3(\mathbb{R})) = \emptyset$ is closed in $\Pi^{-1}(\mathcal{Y})$. This follows from Theorem 3.11. Since it is also open, non-empty and $\Pi^{-1}(\mathcal{Y})$ is connected, every $\tilde{u} \in \Pi^{-1}(\mathcal{Y})$ satisfies $\tilde{u}(\mathbb{D}) \cap (\mathbb{R} \times x_3(\mathbb{R})) = \emptyset$.

Define
$$\tau^* := \sup_{\tilde{u} \in \Pi^{-1}(\mathcal{Y})} \tau(\tilde{u}) .$$

Fix two other leaves l_i and l_{-1} of the characteristic foliation of $\mathcal{D}$ distinct of l_1 in such a way that $\{l_1, l_i, l_{-1}\}$ is ordered according to the Reeb flow along the boundary.

Now we consider a sequence $\tau_n \to \tau^*$ and choose disks $\tilde{u}_n = (a_n, u_n) \in \pi^{-1}(\mathcal{Y})$ satisfying $\tau_n = \tau(\tilde{u}_n)$ and the normalization condition $u_n(1) \in l_1$, $u_n(i) \in l_i$ and $u_n(l_{-1}) \in l_{-1}$. Define

$$\Gamma_0 = \{z \in \mathbb{D} | \exists n_j \to \infty \text{ and } z_j \to z \text{ such that } |d\tilde{u}_{n_j}(z_j)| \to \infty\}.$$

We need the following Theorem of [HWZ95a].

Theorem 3.14 ([HWZ95a])**.** $\exists \; 0 < \rho < 1$ *such that for every sequence*

$\tilde{u}_n \in \mathcal{M}(J)$ *satisfying* $\tilde{u}_n(w) \in l_w$ *for* $w \in \{1, i, -1\}$, *we have*

$$\sup_n \sup_{\rho < |z| \leq 1} |d\tilde{u}_n(z)| < \infty \ .$$

By Theorem 3.14, $\Gamma_0 \subset \mathbb{D} \setminus \partial\mathbb{D}$.

Note that

$$0 < C := \sup_{\tilde{u} \in \mathcal{M}(J)} E(\tilde{u}) < T_3. \tag{3.7}$$

Indeed, let $\tilde{u} \in \mathcal{M}(J)$. Since $\tilde{u}$ is nonconstant, we have $E(\tilde{u}) > 0$. Denote by $\mathcal{D}_u$ the component of $\mathcal{D} \setminus u(\partial\mathbb{D})$ which contains the singular point e. Recall that $\mathcal{D}$ is transverse to the Reeb vector field and oriented in such a way that $d\lambda|_F > 0$. Using Stokes theorem and $a(\partial\mathbb{D}) = 0$, we can estimate for any $\phi \in C^\infty(\mathbb{R}, [0, 1])$

$$\int_{\mathbb{D}} \tilde{u}^* d\lambda_\phi = \int_{\partial\mathbb{D}} \phi(0) u^* \lambda = \int_{\mathcal{D}_u} \phi(0) d\lambda$$

$$\leq \int_{\mathcal{D}_u} d\lambda < \int_{\mathcal{D}} d\lambda = \int_{P_3} \lambda = T_3.$$

Passing to a subsequence, still denoted by $\tilde{u}_n$, we may assume Γ_0 is finite. This follows from (3.7) and from the same arguments used in the proof of Proposition 2.26. Thus, by Theorem 2.22, there exists a smooth $\tilde{J}$-holomorphic map

$$\tilde{u}_0 = (a_0, u_0) : \mathbb{D} \setminus \Gamma_0 \to \mathbb{R} \times S^3$$

and a subsequence, still denoted by $\tilde{u}_n$, such that

$$\tilde{u}_n \to \tilde{u}_0, \quad \text{in } C^\infty_{loc}(\mathbb{D} \setminus \Gamma_0) \ .$$

Arguing as in section 2.3.3, we conclude that Γ_0 consists of negative punctures of $\tilde{u}_0$.

Lemma 3.15. $\Gamma_0 \neq \emptyset$.

Proof. Suppose $\Gamma_0 = \emptyset$. Then $\tilde{u}_n \to \tilde{u}_0$ in $C^\infty_{loc}(\mathbb{D}, \mathbb{R} \times S^3)$ and $\tilde{u}_0$ is non constant. As remarked before, this implies that $\tilde{u}_0$ solves (3.4). By Theorem 3.11, $u_0(\partial\mathbb{D}) \cap x_3(\mathbb{R}) = \emptyset$. Thus, $\tilde{u}_0 \in \Pi^{-1}(\mathcal{Y})$ and $\tau(\tilde{u}_0) = \tau^*$. We take a local section s_0 of $\mathcal{M}$ defined on a neighborhood U_0 of $\Pi(\tilde{u}_0)$ in $\frac{\mathcal{M}}{G}$ and define $\Phi_0 : U_0 \times \mathbb{D} \to \mathbb{R} \times S^3$ by $\Phi_0(t, z) = s_0(t)(z)$. As explained before, Φ_0 is a smooth embedding into $\mathbb{R} \times S^3$. Thus, we can find elements $\tilde{u} \in \Pi^{-1}(\mathcal{Y})$ satisfying $\tau(\tilde{u}) > \tau(\tilde{u}_0) = \tau^*$, a contradiction. $\qquad\square$

Lemma 3.16. *$\tilde{u}_0$ is an embedding, $a_0 \equiv 0$ on $\partial\mathbb{D}$ and $u_0(\partial\mathbb{D}) \subset \mathcal{D} \setminus \{e\}$ winds once and positively around e.*

Proof. Since $\tilde{u}_n \to \tilde{u}_0$ in C_{loc}^∞, $a_n \equiv 0$ on $\partial\mathbb{D}$ and $u_n(\partial\mathbb{D}) \subset \mathcal{D} \setminus \{e\}$ winds once and positively around e for every n, we have $a_0 \equiv 0$ on $\partial\mathbb{D}$ and $u_0(\partial\mathbb{D}) \subset \mathcal{D} \setminus \{e\}$ winds once and positively around e. Using the fact that a_0 satisfies the strong maximum principle, one can show that $\tilde{u}_0$ is an embedding near $\partial\mathbb{D}$. It follows from results in [McD91] that self-intersections or critical points of $\tilde{u}_0$ would imply self-intersections or critical points of $\tilde{u}_n$ for n large, which contradicts the fact that $\tilde{u}_n$ is an embedding for every n. Thus, $\tilde{u}_0$ is an injective immersion. It follows from Lemma 2.14 that $\tilde{u}_0$ is an embedding. $\qquad\square$

Fix any $z_0 \in \Gamma_0$ and let $P_0 = (x_0, T_0)$ be the asymptotic limit of $\tilde{u}_0$ at z_0. As in Section 2.3.4, we define the mass of z_0 by

$$m(z_0) = \lim_{\epsilon \to 0^+} m_\epsilon(z_0),$$

where

$$m_\epsilon(z_0) = \lim_{n\to\infty} \int_{B_\epsilon(z_0)} u_n^* d\lambda \ .$$

Now we proceed as in the *soft rescaling* done in Section 2.3.4. Fix $\epsilon > 0$ such that

$$m_\epsilon(z_0) - m(z_0) \le \frac{\sigma(C)}{2} \ .$$

Choose z_n defined by $a_n(z_n) = \inf(a_n(B_\epsilon(z_0)))$ and let $0 < \delta_n < \epsilon$ be defined by

$$\int_{B_\epsilon(z_0) \setminus B_{\delta_n}(z_n)} u_n^* d\lambda = \sigma(C).$$

It follows that $z_n \to z_0$ and, up to a subsequence, $\delta_n \to 0$. Take $R_n \to +\infty$ such that $\delta_n R_n < \frac{\epsilon}{2}$ and define

$$\begin{aligned}
\tilde{v}_n = (b_n, v_n) : B_{R_n}(0) &\to \mathbb{R} \times S^3 \\
z &\mapsto (a_n(z_n + \delta_n z) - a_n(z_n + 2\delta_n), u_n(z_n + \delta_n z))
\end{aligned} \tag{3.8}$$

The sequence $\tilde{v}_n$ is a germinating sequence, according to Definition 2.25. Let

$$\Gamma_1 = \{z \in \mathbb{C} | \exists z_j \to z \text{ and subsequence } \tilde{u}_{n_j} \text{ s.t. } |d\tilde{u}_{n_j}(z_j)| \to \infty\}. \tag{3.9}$$

Passing to a subsequence, we can assume Γ_1 is finite. Let

$$\tilde{v} = (b, v) : \mathbb{C} \setminus \Gamma_1 \to \mathbb{R} \times S^3$$

be a limit of $\tilde{v}_n$, according to Definition 2.27. Then $\tilde{v}$ has a unique positive puncture at ∞ and $\tilde{v}$ is asymptotic to P_0 at ∞. Thus, using Lemma 2.37, we obtain the following.

Lemma 3.17. *If $z \in \Gamma_0$ and $\tilde{u}_0$ is asymptotic to P at z, then $\mu(P) \geq 2$.*

Now we divide our analysis into two cases:

1) $\pi \cdot du_0 \equiv 0$

2) $\int_{\mathbb{D} \setminus \Gamma_0} u_0^* d\lambda > 0$

3.3.3   The Case $\pi \cdot du_0 \equiv 0$

In this section, we use the same notation as in 3.3.2.

Proposition 3.18. *Assume $\pi \cdot du_0 \equiv 0$. After reparametrizing, we can asssume $\Gamma_0 = \{0\}$ and*

$$\tilde{u}_n \to F_{P_3}, \quad in \ C^\infty_{loc}(\mathbb{D} \setminus \{0\}, \mathbb{R} \times S^3) \ ,$$

where F_{P_3} denotes the map $e^{2\pi(s+it)} \mapsto (T_3 s, x_3(T_3 t))$ on $\mathbb{D} \setminus \{0\}$.

Proof. Since $\pi \cdot du_0 \equiv 0$, there exists a Reeb trajectory $\tilde{x}$ such that $u_0(\mathbb{D} \setminus \Gamma_0) \subset \tilde{x}(\mathbb{R})$. It follows from Lemma 3.16 that $\tilde{x}$ is periodic and $\tilde{x}(\mathbb{R}) = u_0(\partial \mathbb{D}) \subset \mathcal{D}$. By the properties of the disk $\mathcal{D}$ given by Proposition 3.10, we have $\tilde{x}(\mathbb{R}) = x_3(\mathbb{R})$. Consider the map $F : \mathbb{D} \setminus \{0\} \to \mathbb{R} \times S^3$ given by

$$z = e^{2\pi(s+it)} \mapsto (T_3 s, x_3(T_3 t)) \ .$$

One can prove using Carleman's Similarity Principle that $\exists k \in \mathbb{Z}^+$ and a holomorphic map $\varphi : \mathbb{D} \to \mathbb{D}$ satisfying $\varphi(\partial \mathbb{D}) = \partial \mathbb{D}$, $\Gamma_0 = \varphi^{-1}(0)$ and $\varphi|_{\partial \mathbb{D}} : \partial \mathbb{D} \to \partial \mathbb{D}$ has degree k, and such that

$$\tilde{u}_0 = F \circ \varphi \ .$$

Since T_3 is the minimal positive period of x_3, $u_0(\partial \mathbb{D})$ winds k times around e in $\mathcal{D}$. It follows from Lemma 3.16 that $k = 1$ and φ is a biholomorphism. $\square$

Let $\tilde{v}_n = (b_n, v_n)$ be the germinating sequence defined in (3.8) for $z_0 = 0$ and let Γ_1 be defined as in (3.9). Let $\tilde{v} := (b, v) : \mathbb{C} \setminus \Gamma_1 \to \mathbb{R} \times S^3$ be a limit of the sequence $\tilde{v}_n$. Then $\tilde{v}$ has a unique positive puncture at ∞ and the points in Γ_1 are negative punctures. By Proposition 3.18, we know that $\tilde{v}$ is asymptotic to P_3 at ∞. Also, if $\Gamma_1 \neq \emptyset$, then $0 \in \Gamma_1$. This is because $b_n(0) = \inf b_n(B_{R_n}(0))$ and the points in Γ_1 are negative punctures.
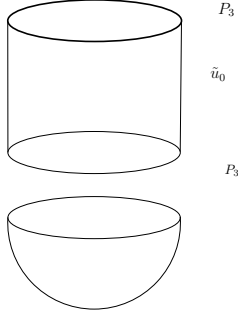
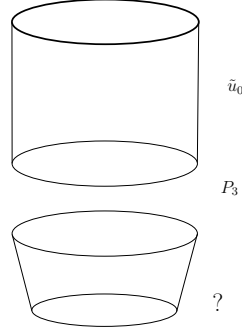Figure 3.2: Case $\pi \cdot du_0 \equiv 0$ and $\Gamma_1 = \emptyset$

Figure 3.3: Case $\pi \cdot du_0 \equiv 0$ and $\Gamma_1 \neq \emptyset$

Lemma 3.19. $\int_{\mathbb{C} \backslash \Gamma_1} v^* d\lambda > 0$.

Proof. Suppose that $\pi \circ dv$ vanishes identically. Then, by Theorem 2.6, there exists a polynomial $p : \mathbb{C} \to \mathbb{C}$ such that $\Gamma_1 = p^{-1}(0)$ and $\tilde{v} = F_{P_3} \circ p$. Since P_3 is prime, p must have degree 1. This implies $\Gamma_1 = \{0\}$, a contradiction with Proposition 2.30. $\qquad\square$

Corollary 2.15 together with the fact that P_3 is simply covered implies that $\tilde{v} = (b, v)$ is somewhere injective.

Thus, we can apply Theorem 2.20 to conclude

$$1 \le \mu(P_3) - \sum_{z \in \Gamma_1} \mu(P_z) - \chi_{S^2} + \#\Gamma_1 + 1 \le 2 - \sum_{z \in \Gamma_1} \mu(P_z) + \#\Gamma_1,$$

where P_z is the asymptotic limit of $\tilde{v}$ at z. The only possibilities for the set of punctures Γ_1 are

- Either $\Gamma_1 = \emptyset$ or

- $\Gamma_1 = \{0\}$ and $\mu(P_0) = 2$.

The bubbling-off tree in the case $\Gamma_1 = \{0\}$ Assume that $\Gamma_1 = \{0\}$ and $\mu(P_0) = 2$. Let $\tilde{w} : \mathbb{C} \setminus \Gamma_2 \to \mathbb{R} \times S^3$ be the unique vertex immediately below $\tilde{v}$ in the bubbling-off tree $\mathcal{B}$ associated with the germinating sequence $\tilde{v}_n$ and the limit $\tilde{v}$.

We can apply Lemma 2.39 to the curve $\tilde{w}$ to conclude that every asymptotic limit of $\tilde{w}$ has Conley-Zehnder index equal to 2. Using this fact we can prove the following lemma.

Lemma 3.20. $\tilde{w} : \mathbb{C} \setminus \Gamma_2 \to \mathbb{R} \times S^3$ *is somewhere injective.*

Proof. Arguing by contradiction, we assume that $\tilde{w}$ is not somewhere injective. Then there exist a polynomial $p : \mathbb{C} \to \mathbb{C}$ and a somewhere injective finite energy sphere $\tilde{u} : \mathbb{C} \setminus \Gamma' \to \mathbb{R} \times S^3$ such that $\deg p \geq 1$, $p^{-1}(\Gamma') = \Gamma_2$ and $\tilde{w}$ factors as

$$\tilde{w} = \tilde{u} \circ p .$$

Let P be the asymptotic limit of $\tilde{w}$ at ∞ and P_∞ be the asymptotic limit of $\tilde{u}$ at ∞. Then $P_\infty^{\deg p} = P$. Using Lemma 1.9 *(iii)* and $\mu(P) = 2$, we conclude that $\deg p = 2$ and $\mu(P_\infty) = 1$. By Theorem 2.20 applied to $\tilde{u}$, we have the estimate

$$0 \leq \mu(P_\infty) - \sum_{z' \in \Gamma'} \mu(P_{z'}) - \chi_{S^2} + \#\Gamma' + 1 = 1 - \sum_{z' \in \Gamma'} \mu(P_{z'}) - 2 + \#\Gamma' + 1 ,$$

where $P_{z'}$ is the asymptotic limit of $\tilde{u}$ at z'. We conclude $\sum_{z' \in \Gamma'} \mu(P_{z'}) \leq \#\Gamma'$ and consequently $\mu(P_{z'}) \leq 1, \forall z' \in \Gamma'$.

Suppose that $\zeta \in \Gamma$ is a regular point of p. Then p is a biholomorphism in a neighborhood of ζ and, if $z' = p(\zeta)$, then

$$P_\zeta = P_{z'} .$$

This leads to the contradiction $\mu(P_\zeta) = 2 > 1 \geq \mu(P_{z'})$. We conclude that if $\Gamma_2 \neq \emptyset$, it consists of critical points of p. Since $\deg p = 2$, we have $\# \operatorname{Crit} p = 1$. It follows that $\#\Gamma_2 \leq 1$ and consequently $\#\Gamma' \leq 1$.

By Proposition 2.30, we know that $\int_{\mathbb{C} \setminus \Gamma} w^* d\lambda > 0$. Consequently, we have $\int_{\mathbb{C} \setminus \Gamma'} u^* d\lambda > 0$. By Theorem 2.20 applied to $\tilde{u}$, we have the estimate

$$1 \leq \mu(P_\infty) - \sum_{z' \in \Gamma'} \mu(P_{z'}) - 2 + \#\Gamma' + 1 = - \sum_{z' \in \Gamma'} \mu(P_{z'}) + \#\Gamma' .$$

We conclude that $\Gamma' = \{z'\}$ and $\mu(P_{z'}) = 0$. It follows that $\Gamma_2 = \{\zeta\}$, $p(\zeta) = z'$, $p'(\zeta) = 0$ and
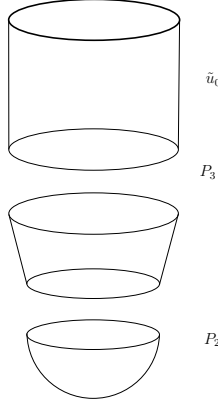
$$P_{z'}^2 = P_\zeta .$$

By lemma 1.9 this leads to a contradiction. We have proved that $\tilde{w}$ is somewhere injective. $\square$

By Theorem 2.20, we have the estimate

$$0 \leq \operatorname{ind}(\tilde{w}) \leq 2 - 2\#\Gamma_2 - 2 + \#\Gamma_2 + 1 = 1 - \#\Gamma_2.$$

Thus, we conclude that $\#\Gamma_2 \leq 1$ and consequently, by Lemma 2.30, we have

Figure 3.4: Case $\pi \cdot du_0 \equiv 0$ and $\Gamma_1 \neq \emptyset$

$\int_{\mathbb{C} \setminus \Gamma_2} w^* d\lambda > 0$. Again by Theorem 2.20 we obtain

$$1 \leq \mathrm{ind}(\tilde{w}) \leq 1 - \#\Gamma_2 .$$

Thus, $\#\Gamma_2 = 0$ and $\tilde{w}$ is the only leaf in the bubbling-off tree. By Proposition 2.38, the leaf is asymptotic to an orbit $\bar{P} = (\bar{x}, \bar{T})$ such that $\mu(\bar{P}) = 2$ and P_3 do not intersect the image of w. Then $\bar{P}$ is not linked to P_3. Since $\bar{T} < T_3$, it follows from the hypotheses of Theorem 3.5 that

$$\bar{P} = P_2.$$

So far, we have a $\tilde{J}$-holomorphic cylinder $\tilde{v} : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ asymptotic to P_3 at its positive puncture ∞ and to P_2 at its negative puncture 0, and a $\tilde{J}$-holomorphic plane asymptotic to P_2.

The bubbling off tree obtained in the case $\pi \cdot du_0 \equiv 0$ In conclusion, in the case $\pi \cdot du_0 \equiv 0$, we have two possibilities:

1. Either $\Gamma_1 = \emptyset$ and $\tilde{v} : \mathbb{C} \to \mathbb{R} \times S^3$, or

2. $\Gamma_1 = \{0\}$, $\tilde{v} : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ is asymptotic to P_3 at its positive puncture ∞ and to P_2 at its negative puncture $z = 0$. In this case we also obtain a $\tilde{J}$-holomorphic plane $\tilde{w} : \mathbb{C} \to \mathbb{R} \times S^3$ asymptotic to P_2.

We continue the analysis of the case $\pi \circ du_0 \equiv 0$ in Chapter 4. In the case $\Gamma_1 = \emptyset$, we show that $\tilde{v}$ belongs to a 1-parameter family of $\tilde{J}$-holomorphic planes asymptotic to P_3, whose projection foliates a region in S^3. This is the first step to obtain the $3 - 2 - 1$ foliation. In the case $\Gamma_1 = \{0\}$, we apply the Gluing theorem 5.2 to the cylinder $\tilde{v} : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ and the plane

$\tilde{w} : \mathbb{C} \to \mathbb{R} \times S^3$ to obtain a 1-parameter family of $\tilde{J}$-holomorphic planes asymptotic to P_3.

3.3.4   The case $\int_{\mathbb{D} \backslash \Gamma_0} u_0^* d\lambda > 0$

We continue using the notation of 3.3.2. In this case, $(\tilde{u}_0, \tilde{J})$ is a solution of (3.6). Remember we have assumed $J \in \mathcal{J}_{reg}^2$, where $\mathcal{J}_{reg}^2$ is given by Proposition 3.13. By Lemma 3.15, we know that $\Gamma_0 \neq \emptyset$ and, if $z \in \Gamma_0$ and $\tilde{u}_0$ is asymptotic to P at z, then $\mu(P) \geq 2$. Thus, Proposition 3.13 implies that Γ_0 consists of a single point $z = 0$ and $\mu(P_0) = 2$, where P_0 is the asymptotic limit of $\tilde{u}_0$ at 0.

Proposition 3.21. $u_0(\partial \mathbb{D}) \cap x_3(\mathbb{R}) = \emptyset$.

Proof. Assume there exists $z \in \partial \mathbb{D}$ such that $u_0(z) \in x_3(\mathbb{R})$. By Lemma 3.12 applied to the sequence $\tilde{u}_n$, we obtain a sequence $\{\delta_k\}$ in $\mathbb{D} \setminus \{0\}$ satisfying $u_n(\delta_n) \in x_3(\mathbb{R})$, for n large enough. It is a contradiction, since $u_n(\mathbb{D}) \cap x_3(\mathbb{R}) = \emptyset$, $\forall n$. □

Consider the germinating sequence $\tilde{v}_n = (b_n, v_n) : \mathbb{C} \setminus \Gamma_1 \to \mathbb{R} \times S^3$ as defined in (3.8), and let $\tilde{v} = (b, v)$ be a limit of the sequence $\tilde{v}_n$. Then $\tilde{v}$ satisfies the hypotheses of Lemma 2.39 and it follows that for all $z \in \Gamma_1 \cup \{\infty\}$,

$$\mu(P_z) = 2 \ ,$$

where P_z is the asymptotic limit of $\tilde{v}$ at z. Using Lemma 3.20, we conclude that $\tilde{v}$ is somewhere injective.

By Theorem 2.20, we have the estimate

$$\begin{aligned}
0 \leq \mathrm{ind}(\tilde{v}) &\leq \mu(P_2) - \mu^-(\tilde{v}) - \chi_{S^2} + \#\Gamma_1 + 1 \\
&= 2 - 2\#\Gamma_1 - 2 + \#\Gamma_1 + 1 \\
&= 1 - 2\#\Gamma_1 \ .
\end{aligned}$$

We conclude that $\#\Gamma_1 \leq 1$ and consequently, by Proposition 2.30, $\int_{\mathbb{C} \backslash \Gamma_1} v^* d\lambda > 0$. By Theorem 2.20 again, we have the estimate

$$1 \leq \mathrm{ind}(\tilde{v}) \leq 1 - 2\#\Gamma_1 \ .$$

Hence, $\#\Gamma_1 = 0$.

Let $P_0 = (x_0, T_0)$ be the asymptotic limit of $\tilde{v}$ at ∞. It follows from Proposition 2.38 that P_0 is not linked to P_3. Also, $T_0 < T_3$ and $\mu(P_0) = 2$. By the hypotheses of Theorem 3.5, P_2 is the only orbit satisfying these properties.
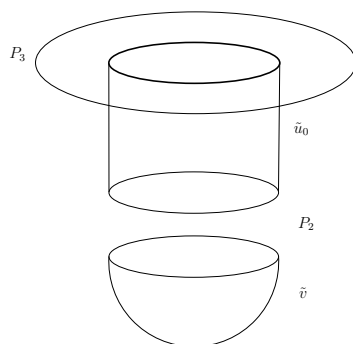
Figure 3.5: Case $\int_{\mathbb{D}\backslash\Gamma_0} u_0^* d\lambda > 0$

Thus,

$$P_0 = P_2 \ .$$

We continue the analysis of the case $\int_{\mathbb{D}\backslash\Gamma_0} u_0^* d\lambda > 0$ in Section 5.2.

Chapter 4

Proof of Theorem 3.5 in the case $\pi \cdot du_0 \equiv 0$

Following Section 3.3.2, we consider a sequence $\tilde{u}_n = (a_n, u_n) \in \mathcal{M}(J)$ and a $\tilde{J}$-holomorphic map

$$\tilde{u}_0 = (a_0, u_0) : \mathbb{D} \setminus \Gamma_0 \to \mathbb{R} \times S^3$$

such that, passing to a subsequence, $\tilde{u}_n \to \tilde{u}_0$ in $C^\infty_{loc}(\mathbb{D} \setminus \Gamma_0)$. In this chapter we assume

$$\pi \circ du_0 \equiv 0 .$$

With this assumption, by Proposition 3.18, we have

$$\tilde{u}_0 = F_{P_3} ,$$

where F_{P_3} denotes the cylinder $e^{2\pi(s+it)} \mapsto (T_3 s, x_3(T_3 t))$ in $\mathbb{D} \setminus \{0\}$.

Fix $\epsilon > 0$ such that $m_\epsilon(0) - m(0) \leq \frac{\sigma(C)}{2}$ and choose z_n defined by $a_n(z_n) = \inf(a_n(B_\epsilon(0)))$. We also choose $0 < \delta_n < \epsilon$ satisfying

$$\int_{B_\epsilon(0) \setminus B_{\delta_n}(z_n)} u_n^* d\lambda = \sigma(C).$$

It follows that $z_n \to 0$ and, up to a subsequence, $\delta_n \to 0$. Take $R_n \to \infty$ such that $\delta_n R_n < \frac{\epsilon}{2}$ and define

$$\tilde{v}_n = (b_n, v_n) : B_{R_n}(0) \to \mathbb{R} \times S^3$$
$$z \mapsto (a_n(z_n + \delta_n z) - a_n(z_n + 2\delta_n), u_n(z_n + \delta_n z)) \tag{4.1}$$

The sequence $\tilde{v}_n$ is a germinating sequence as in Definition 2.25. Let

$$\Gamma_1 = \{z \in \mathbb{C} | \exists z_j \to z \text{ and subsequence } \tilde{u}_{n_j} \text{ s.t. } |d\tilde{u}_{n_j}(z_j)| \to \infty\}. \qquad (4.2)$$

Passing to a subsequence, we can assume $\#\Gamma_1 < \infty$. Let

$$\tilde{v} = (b, v) : \mathbb{C} \setminus \Gamma_1 \to \mathbb{R} \times S^3$$

be a limit of the germinating sequence $\tilde{v}_n$ as in Definition 2.27. Then $\tilde{v}$ has a unique positive puncture at ∞ and the points of Γ_1 are negative punctures. Also, $\tilde{v}$ is asymptotic to P_3 at ∞. As proved in Section 3.3.3, we have two possibilities for Γ_1:

- Either $\Gamma_1 = \emptyset$ and $\tilde{v}$ is a plane asymptotic to P_3

- or $\Gamma_1 = \{0\}$ and $\tilde{v}$ is a cylinder asymptotic to P_3 at its positive puncture and to P_2 at its negative puncture.

4.1   Foliating a solid torus in the case $\Gamma_1 = \emptyset$

In this case $\tilde{v} : \mathbb{C} \to \mathbb{R} \times S^3$ is a $\tilde{J}$-holomorphic finite energy plane asymptotic to the orbit P_3. It follows from Lemma 2.34 that

$$\text{wind}_\pi(\tilde{v}) = 0 .$$

As a consequence of the definition of wind_π, we conclude

$$\mathbb{R}X_{v(z)} \oplus dv(z)(T_z\mathbb{C}) = T_{v(z)}S^3, \ \forall z \in \mathbb{C} ,$$

proving that v is an immersion transverse to the Reeb flow. In particular, this implies that $\tilde{v}$ is an immersion. Now we prove that $\tilde{v}$ is an embedding. Since $\tilde{v}$ is an immersion, it is enough to prove that $\tilde{v}$ is injective to conclude that it is an embedding in any closed ball $\bar{B}_{R'} \subset \mathbb{C}$, $R' > 0$. Let $\Delta \subset \mathbb{C} \times \mathbb{C}$ be the diagonal and consider the set

$$D := \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \setminus \Delta | \tilde{v}(z_1) = \tilde{v}(z_2)\} .$$

D consists only of isolated points in $\mathbb{C} \times \mathbb{C} \setminus \Delta$. Indeed, if D has a limit point in $\mathbb{C} \times \mathbb{C} \setminus \Delta$, then we find, using Carleman's similarity principle as in [HWZ95b], a polynomial p of degree at least 2 and a $\tilde{J}$-holomorphic map $f : \mathbb{C} \to \mathbb{R} \times S^3$ such that $\tilde{v} = f \circ p$. This forces zeros of $d\tilde{v}$, contradicting the fact that $\tilde{v}$ is an immersion. If $D \neq \emptyset$, using positivity and stability of intersections of pseudo-holomorphic immersions, we obtain self-intersections
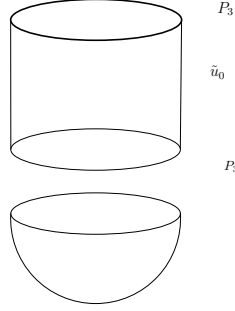
Figure 4.1: Case $\pi \cdot du_0 \equiv 0$ and $\Gamma_1 = \emptyset$

of the maps $\tilde{v}_n$. However, we know that each $\tilde{u}_n$, and consequently each $\tilde{v}_n$ is an embedding. It follows, form Theorem 2.11, Lemma 2.14 and the fact that the orbit P_3 is prime, that $\tilde{v}$ is an embedding near the boundary, that is, there exists $R > 0$ such that $\tilde{v}^{-1}(\tilde{v}(B_R)) = B_R$ and $\tilde{v}|_{\mathbb{C} \backslash B_R}$ is an embedding. This shows that $\tilde{v}$ is an embedding.

A family of planes asymptotic to P_3   The following Theorem is proved in [HWZ99b].

Theorem 4.1 ([HWZ99b]). *Assume that the $\tilde{J}$-holomorphic finite energy plane $\tilde{w} = (d, w) : \mathbb{C} \to \mathbb{R} \times S^3$ is an embedding asymptotic to a nondegenerate simply covered orbit $P = (x, T)$ satisfying $\mu(P) = 3$. Then there exists $\epsilon > 0$ and an embedding*

$$\tilde{\Phi} = (a, \Phi) : (-\epsilon, \epsilon) \times \mathbb{C} \to \mathbb{R} \times S^3$$

so that

(i) $\tilde{\Phi}(0, \cdot) = \tilde{w}$;

(ii) For any $\tau \in (-\epsilon, \epsilon)$, the map $\tilde{\Phi}_\tau := \tilde{\Phi}(\tau, \cdot) : \mathbb{C} \to \mathbb{R} \times S^3$ is an embedded finite energy $\tilde{J}$-holomorphic plane asymptotic to P;

(iii) $\Phi(\tau, \mathbb{C}) \cap P = \emptyset$, $\forall \tau$ and the map $\Phi : (-\epsilon, \epsilon) \times \mathbb{C} \to S^3 \setminus P$ is an embedding;

(iv) Let $\tilde{w}_n$ be a sequence of embedded fast finite energy $\tilde{J}$-holomorphic planes all asymptotic to P satisfying $\tilde{w}_n \to \tilde{w}$ in $C^\infty_{loc}(\mathbb{C})$ as $n \to +\infty$. Then there exist sequences A_n, B_n in $\mathbb{C}$ with $A_n \to 1$, $B_n \to 0$, c_n in $\mathbb{R}$ with $c_n \to 0$ and τ_n in $(-\epsilon, \epsilon)$ with $\tau_n \to 0$ such that

$$\tilde{w}_n(z) = (a_{\tau_n}(A_n z + B_n) + c_n, \Phi_{\tau_n}(A_n z + B_n))$$

for sufficiently large n.

By applying the above theorem to the plane $\tilde{v}$ we obtain a maximal 1-parameter family of finite energy planes asymptotic to the orbit P_3

$$\tilde{v}_\tau = (b_\tau, v_\tau), \quad \tau \in (\tau_-, \tau_+) \ .$$

The family satisfies $v_{\tau_1}(\mathbb{C}) \cap v_{\tau_2}(\mathbb{C}) = \emptyset, \ \forall \tau_1 \neq \tau_2$. Indeed, if there are $\tau_1 \neq \tau_2$ such that $v_{\tau_1}(\mathbb{C}) \cap v_{\tau_2}(\mathbb{C}) \neq \emptyset$, then $v_{\tau_1}(\mathbb{C}) = v_{\tau_2}(\mathbb{C})$ and we would obtain an S^1-family of embedded planes that provides an open book decomposition [1] with disk-like pages to S^3, with binding orbit P_3. Since each plane in the family is fast and consequently transverse to the Reeb flow, it clearly contradicts the fact that P_2 and P_1 are not linked to P_3.

Now we describe how the family $\{v_\tau\}$ breaks as $\tau \to \tau_\pm$. We assume that $\tau_- = 0$ and $\tau_+ = 1$ and τ strictly increases in the direction of R_λ.

Limiting behavior

Proposition 4.2. *Consider a sequence $\tilde{v}_n := \tilde{v}_{\tau_n}$ in the family $\{\tilde{v}_\tau\}$ satisfying $\tau_n \to 0^+$. Then there exists a $\tilde{J}$-holomorphic finite energy cylinder $\tilde{u}_r : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ which is asymptotic to P_3 at the positive puncture ∞ and to P_2 at its negative puncture and a finite energy $\tilde{J}$-holomorphic plane $\tilde{u}_q : \mathbb{C} \to \mathbb{R} \times S^3$ asymptotic to P_2 at ∞, such that, after suitable reparametrizations and $\mathbb{R}$-translations, the following hold*

(i) Up to subsequence, $\tilde{v}_n \to \tilde{u}_r$ in $C^\infty_{loc}(\mathbb{C} \setminus \{0\})$ as $n \to \infty$.

(ii) There exist sequences $\delta_n^+ \to 0^+$, $z_n \in \mathbb{C}$ and $c_n \in \mathbb{R}$ such that, up to subsequence, $\tilde{v}_n(\delta_n \cdot) + c_n \to \tilde{u}_q$ in $C^\infty_{loc}(\mathbb{C})$ as $n \to \infty$.

Proof. After a holomorphic reparametrization of $\tilde{v}_n$ and $\mathbb{R}$ translation, we can assume $\{\tilde{v}_n\}$ is a germinating sequence as defined in (2.13)-(2.16). Indeed, we can take sequences $z_n \in \mathbb{C}$ and $0 < \delta_n$, such that $b_n(z_n) = \inf b_n(\mathbb{C})$ and $\int_{\mathbb{C} \setminus B_{\delta_n}(z_n)} v_n^* d\lambda = \sigma(T_3)$, and define

$$\tilde{u}_n(z) = (a_n(z), u_n(z)) = (b_n(z_n + \delta_n z) - b_n(z_n + 2\delta_n), v_n(z_n + \delta_n z)) \ .$$

Then $a_n(2) = 0, \forall n$ and

$$\int_{\mathbb{C} \setminus \mathbb{D}} u_n^* d\lambda = \int_{\mathbb{C} \setminus B_{\delta_n}(z_n)} v_n^* d\lambda = \sigma(T_3). \tag{4.3}$$

---

[1]An open book decomposition of a 3-manifold M is a pair (L, p), where $L \subset M$ is a link and $p : M \setminus L \to S^1$ is a fibration such that each fiber $p^{-1}(\theta)$ is the interior of a compact embedded surface $S_\theta \hookrightarrow M$ satisfying $\partial S_\theta = L$. L is called the binding and the fibers are called pages. If the pages are disks we say that (L, p) has disk-like pages.

Hence, changing the notation back to $\tilde{v}_n$, we can assume $\{\tilde{v}_n\}$ is a germinating sequence.

Let $\tilde{u}_r : \mathbb{C} \setminus \Gamma_r \to \mathbb{R} \times S^3$ be a limit of $\tilde{v}_n$ and let $\mathcal{B} = (\mathcal{T}, \mathcal{U})$ be the bubbling-off tree obtained as in Theorem 2.31.

We claim that P_3 is the asymptotic limit of $\tilde{u}_r$ at ∞. Indeed, let $\mathcal{W} \subset C^\infty(S^1, S^3)$ be an S^1-invariant neighborhood of the set of periodic orbits $P = (x, T) \in \mathcal{P}(\lambda)$ with $T \leq T_3$, viewed as maps $x_T : S^1 \to S^3$, $x_T(t) = x(Tt)$, such that each connected component of $\mathcal{W}$ contains at most one periodic orbit modulo S^1-reparametrizations. Using the normalization condition (4.3), we can apply Lemma 2.29 and find $R_0 > 1$ such that for $R \geq R_0$, the loops $t \mapsto u_r(Re^{i2\pi t})$ and $\{t \mapsto v_n(Re^{i2\pi t})\}, n \in \mathbb{N}$ belong to $\mathcal{W}$. For R fixed, the sequence of loops $t \mapsto v_n(Re^{i2\pi t})$ converges to $t \mapsto u_r(Re^{i2\pi t})$ in $C^\infty(S^1, S^3)$, so that for n large and $R \geq R_0$, $t \mapsto u_r(Re^{i2\pi t})$ and $t \mapsto v_n(Re^{i2\pi t})$ belong to the same connected component of $\mathcal{W}$. This implies that P_3 is the asymptotic limit of $\tilde{u}_r$ at ∞.

We first note that $\#\Gamma_r \neq 0$. Indeed, if $\Gamma_r = \emptyset$, then $\tilde{u}_r : \mathbb{C} \to \mathbb{R} \times S^3$ would satisfy the hypotheses of Theorem 4.1, which contradicts the fact that the family $\{\tilde{v}_\tau\}$ is maximal.

Since P_3 is simply covered, we know that $\tilde{u}_r$ is somewhere injective. By Theorem 2.20, we have the estimate

$$0 \leq \mathrm{ind}(\tilde{u}_r) = 3 - \sum_{z \in \Gamma_r} \mu(P_z) - 2 + \#\Gamma_r + 1.$$

By Lemma 2.37, $\mu(P_z) \geq 2$, $\forall z \in \Gamma_r$. Hence,

$$2\#\Gamma_r \leq \sum_{z \in \Gamma_r} \mu(P_z) \leq 2 + \#\Gamma_r \ ,$$

which implies that $\#\Gamma_r \leq 2$.

Now we prove that $\#\Gamma_r = 1$. Suppose, by contradiction, that $\#\Gamma_r = 2$. Since $\tilde{u}_r$ is somewhere injective, we can use Theorem 2.20 to obtain the estimate

$$0 \leq \mathrm{ind}(\tilde{u}_r) = \mu(P_3) - \sum_{z \in \Gamma_r} \mu(P_z) - 2 + \#\Gamma_r + 1 = 4 - \sum_{z \in \Gamma_r} \mu(P_z) \ .$$

Since $\mu(P_z) \geq 2$ for all $z \in \Gamma_r$, the only possibility is $\mu(P_z) = 2$ for all $z \in \Gamma_r$ and $\mathrm{ind}(\tilde{u}_r) = 0$. By Theorem 2.20, this implies that $\int_{\mathbb{C} \setminus \Gamma_r} u_r^* d\lambda = 0$. By Theorem 2.6, there exists a non-constant polynomial $p : \mathbb{C} \to \mathbb{C}$ and a Reeb orbit $P = (x, T)$ such that $p^{-1}(0) = \Gamma_r$ and $\tilde{u}_r = F_P \circ p$, where $F_P : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times M$ is defined by $F_P(z = e^{2\pi(s+it)}) = (Ts, x(Tt))$, and

the asymptotic limit of $\tilde{u}_r$ at ∞ is $P^{\deg p} = (x, \deg p \cdot T)$. Since P_3 is simply covered, $\deg p = 1$. This implies that $\# p^{-1}(0) = 1$, a contradiction. Thus we have proved that $\#\Gamma_r = 1$. Since $b_n(0) = \inf b_n(\mathbb{C})$ and the puncture in Γ_r is negative, we have $\Gamma_r = \{0\}$.

Now we show that $\int_{\mathbb{C} \backslash \{0\}} u_r^* d\lambda > 0$. Suppose $\pi \cdot du_r \equiv 0$. Then, by Theorem 2.6, there exists a polynomial $p : \mathbb{C} \to \mathbb{C}$ and an orbit $P = (x, T)$ such that $\Gamma' = p^{-1}(0)$ and $\tilde{v} = F_P \circ p$, where $F_P : \mathbb{C} \backslash \{0\} \to \mathbb{R} \times S^3$ is defined by $F_P(\zeta = e^{2\pi(s+it)}) = (Ts, x(Tt))$. Since 0 is the only root of p, we have $p(\zeta) = A\zeta^n$, for some $A \neq 0$ and $n \geq 1$. Since P_3 is prime, we have $n = 1$ and $P = P_3$. Thus,

$$\tilde{u}_r(\zeta = e^{2\pi(s+it)}) = F_{P_3}(Ae^{2\pi(s+it)}) = \left(T_3 \left(s + \frac{\log A}{2\pi} \right), x_3(T_3 t) \right).$$

and we have the contradiction

$$T_3 = \int_{\partial \mathbb{D}} u_r^* \lambda = \lim_{n \to \infty} \int_{\mathbb{D}} v_n^* d\lambda = T_3 - \sigma(C).$$

Here we have used (4.3). We conclude that $\int_{\mathbb{C} \backslash \{0\}} u_r^* d\lambda > 0$.

By Theorem 2.20, we have

$$1 \leq \text{ind}(\tilde{u}_r) = 3 - \mu(P_0) - 2 + \#\Gamma_r + 1 = 3 - \mu(P_0) \,,$$

where P_0 is the asymptotic limit of $\tilde{u}_r$ at 0. Hence $\mu(P_0) = 2$.

Following the arguments of Section 2.3.4, we find sequences $\delta_n \to 0^+$ and $z_n \to z_0$ such that the sequence

$$\tilde{w}_n := (b(z_n + \delta_n \cdot) - b_n(z_n + 2\delta_n), v_n(z_n + 2\delta_n \cdot))$$

is a germinating sequence. Let $\tilde{u}_q = (a_q, u_q) : \mathbb{C} \backslash \Gamma_q \to \mathbb{R} \times S^3$ be a limit of the sequence $\tilde{w}_n$. By Lemma 2.39, all asymptotic limits associated to the punctures of $\tilde{u}_q$ have Conley-Zehnder index equal to 2. By Lemma 3.20 this implies that $\tilde{u}_q$ is somewhere injective. Then, we can apply Theorem 2.20 to $\tilde{u}_q$ and obtain

$$0 \leq \text{ind}(\tilde{u}_q) = 2 - 2\#\Gamma_q - 2 + \#\Gamma_q + 1 \,.$$

Hence $\#\Gamma_q \leq 1$. By Proposition 2.30, we have $\int_{\mathbb{C} \backslash \Gamma_q} u_q^* d\lambda > 0$. Again by Theorem 2.20, we have $\text{ind}(\tilde{u}_q) \geq 1$ and consequently $\#\Gamma_q = 0$. By Proposition 2.38, we conclude that the asymptotic limit of $\tilde{u}_q$ at ∞ is an orbit P_0 satisfying $\mu(P_0) = 2$ not linked to P_3. Since the period of P_0 is less

than T_3, by the hypotheses of Theorem 3.5, we have

$$P_0 = P_2$$

and $\tilde{u}_q$ is a plane asymptotic to P_2. $\square$

By Lemma 2.34, $\text{wind}_\pi(\tilde{u}_r) = 0$ and $\text{wind}_\pi(\tilde{u}_q) = 0$. It follows that the projections $u_r : \mathbb{C} \setminus \{0\} \to S^3$ and $u_q : \mathbb{C} \to S^3$ are immersions transverse to the Reeb flow.

Proposition 4.2 is still valid if we consider a sequence $\{v_{\tau_n}\}$, $\tau_n \to 1^-$. In this case, denote by $\tilde{u}'_r : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ and $\tilde{u}'_q : \mathbb{C} \to \mathbb{R} \times S^3$ respectively the finite energy $\tilde{J}$-homolorphic cylinder and the finite energy $\tilde{J}$-homolorphic plane obtained as the SFT-limit of a subsequence of $\{v_{\tau_n}\}$.

Proposition 4.3.   *(i) For any S^1-invariant neighborhood $\mathcal{W}_3$ of the loop $t \mapsto x_3(T_3 t)$ in $C^\infty(\mathbb{R}/\mathbb{Z}, S^3)$, there exists $R_0 >> 1$ such that for $R \geq R_0$, the loop $t \mapsto v_n(Re^{2\pi it})$ belongs to $\mathcal{W}_3$.*

(ii) For any S^1-invariant neighborhood $\mathcal{W}_2$ of the loop $t \mapsto x_2(T_2 t)$ in $C^\infty(\mathbb{R}/\mathbb{Z}, S^3)$, there exist $\epsilon_1 > 0$ and $R_1 >> 0$ such that the loop $t \mapsto v_n(z_n + Re^{2\pi it})$ belongs to $\mathcal{W}$ for $R_1 \delta_n \leq R \leq \epsilon_1$.

(iii) Given any neighborhood $\mathcal{V}$ of $u_r(\mathbb{C} \setminus \{0\}) \cup u_q(\mathbb{C}) \cup P_2 \cup P_3$, we have $v_{\tau_n}(\mathbb{C}) \subset \mathcal{V}$ for n large. A similar statement works for any sequence $\tau_n \to 1^-$ with u_q and u_r replaced by u'_q and u'_r respectively.

Proof. The proof of *(i)* and *(ii)* is an application of Lemma 2.29. Let $\mathcal{W}$ be an S^1-invariant neighborhood of the set of periodic orbits $P = (x, T) \in \mathcal{P}(\lambda)$ with $T \leq T_3$, viewed as maps $x_T : S^1 \to S^3$, $x_T(t) = x(Tt)$, such that each of the connected components of $\mathcal{W}$ contains at most one periodic orbit modulo S^1-reparametrizations and such that $\mathcal{W}_2, \mathcal{W}_3 \subset \mathcal{W}$. Using the normalization condition (4.3) we can apply Lemma 2.29 and find $R_0 >> 1$ such that for $R \geq R_0$, the loops $t \mapsto u_r(Re^{i2\pi t})$ and $\{t \mapsto v_n(Re^{i2\pi t})\}, n \in \mathbb{N}$ belong to $\mathcal{W}$. By the asymptotic behavior of the planes $\tilde{v}_n$, we know that for each n, the loop $t \mapsto v_n(Re^{s+it})$ belongs to $\mathcal{W}_3$ for R large enough. We conclude that for any $R \geq R_0$ and n large, the loop $t \mapsto v_n(Re^{2\pi it})$ belongs to $\mathcal{W}_3$.

Let $z_n \to 0$ and $\delta_n \to 0$ be the sequences obtained by soft rescaling near $z = 0$ as in the proof of Proposition 4.2, such that passing to a subsequence, we have

$$\tilde{v}_n(z_n + \delta_n \cdot) - b_n(z_n + 2\delta_n) \to \tilde{u}_q$$

in $C^\infty_{loc}(\mathbb{C})$, as $n \to +\infty$. Applying Lemma 2.29 as in the proof of Proposition 2.28, we find $\epsilon_1 > 0$ small and $R_1 >> 1$ such that for every R satisfying $\delta_n R_1 \leq R \leq \epsilon_1$ and n large, the loop $t \mapsto v_n(z_n + Re^{2\pi it})$ belongs to $\mathcal{W}_2$.

The proof of *(iii)* is a consequence of *(i)*, *(ii)* and the fact that v_n converges uniformly to u_r on compact subsets of $\mathbb{C} \setminus \{0\}$ and $w_n = v_n(z_n + \delta_n \cdot)$ converges uniformly to u_q on compact subsets of $\mathbb{C}$.

$\square$

The projected curves are embeddings

Proposition 4.4. *The curves $u_r(\mathbb{C} \setminus \{0\})$, $u'_r(\mathbb{C} \setminus \{0\})$, $u_q(\mathbb{C})$ and $u'_q(\mathbb{C})$ do not intersect $P_2 \cup P_3$.*

Proof. We first prove that $u_r(\mathbb{C} \setminus \{0\}) \cap P_3 = \emptyset$. Consider $F : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ defined by $F(e^{2\pi(s+it)}) = (Ts, x_3(Tt))$. Note that F is a finite energy $\tilde{J}$-holomorphic immersion. Define

$$A = \{(z, \zeta) \in \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\} \mid \tilde{u}_r(z) = F(\zeta)\} .$$

If the set A is not empty, it consists of isolated points. Indeed, assume that (z^*, ζ^*) is not an isolated point of A. Since both $\tilde{u}_r$ and F are immersions, we can use Lemma 2.4.3 of [MS04] to find open neighborhoods $\mathcal{O}$ and $\mathcal{O}'$ of z^* and ζ^* respectively, and a holomorphic diffeomorphism $f : \mathcal{O} \to \mathcal{O}'$ such that $F \circ f = \tilde{u}_r$ on $\mathcal{O}$. Since u_r is transversal to the Reeb flow, we get a contradiction. Now assume $A \neq \emptyset$ and choose $(z^*, \zeta^*) \in A$. The maps $\tilde{u}_r$ and F intersect transversally at the pair (z^*, ζ^*). By positivity and stability of intersections of pseudo-holomorphic immersions, we find $z_j \to z^*$ and $n_j \to \infty$ such that $\tilde{v}_{n_j}(z_j) \in F(\mathbb{C} \setminus \{0\})$. This contradicts the fact that $v_{n_j}(\mathbb{C}) \cap x_3(\mathbb{R}) = \emptyset$, $\forall j$. So we have proved that $A = \emptyset$ and consequently $u_r(\mathbb{C} \setminus \{0\}) \cap P_3 = \emptyset$.

To prove that $u_r(\mathbb{C} \setminus \{0\}) \cap P_2 = \emptyset$, we proceed in the same way, noting that $v_n(\mathbb{C}) \cap x_2(\mathbb{R}) = \emptyset$, $\forall n$. Indeed, by assumption, P_2 and P_3 are not linked. Then the linking number $\mathrm{lk}(x_2(\mathbb{R}), x_3(\mathbb{R}))$ is zero. Since each $v_n(\mathbb{C})$ is a Seifert surface for $x_3(\mathbb{R})$, $\mathrm{lk}(x_2(\mathbb{R}), x_3(\mathbb{R}))$ is the intersection number of $x_2(\mathbb{R})$ and $v_n(\mathbb{C})$. If $P_2 \cap v_n(\mathbb{C}) \neq \emptyset$, there are positive and negative intersections. But this contradicts the fact that v_n is transverse to R_λ.

By the same arguments above, we prove that $u'_r(\mathbb{C} \setminus \{0\})$, $u_q(\mathbb{C})$ and $u'_q(\mathbb{C})$ do not intersect $P_2 \cup P_3$. $\square$

Using Proposition 4.4 and Theorem B.5, we conclude the following.

Proposition 4.5. *The projected curves $u_r : \mathbb{C} \setminus \{0\} \to S^3$, $u'_r : \mathbb{C} \setminus \{0\} \to S^3$, $u_q : \mathbb{C} \to S^3$ and $u'_q : \mathbb{C} \to S^3$ are embeddings which do not intersect any of their asymptotic limits.*

Uniqueness

Definition 4.6 ([Sie11])**.** *Let $\tilde{u} : S \setminus \Gamma_u \to \mathbb{R} \times S^3$ and $\tilde{v} : S \setminus \Gamma_v \to \mathbb{R} \times S^3$ be finite energy $\tilde{J}$-holomorphic curves asymptotic to the same nondegenerate periodic orbit $P \in \mathcal{P}(\lambda)$ at certain punctures $z_u \in \Gamma_u$ and $z_v \in \Gamma_v$. We say that $\tilde{u}$ and $\tilde{v}$ approach P in the same direction at these punctures if $\eta_u = c\eta_v$ for $c > 0$, where η_u and η_v are the asymptotic eigensections of $\tilde{u}$ at z_u and of $\tilde{v}$ at z_v respectively, defined in Theorem 2.11. In case $\eta_u = c\eta_v$, with $c < 0$, we say that $\tilde{u}$ and $\tilde{v}$ approach P in opposite directions.*

Proposition 4.7. *Up to reparametrization and $\mathbb{R}$-translation, $\tilde{u}_q = \tilde{u}'_q$, and $\tilde{u}_q$ is the unique finite energy $\tilde{J}$-holomorphic plane asymptotic to P_2. Also, if $u_r(\mathbb{C} \setminus \{0\}) \neq u'_r(\mathbb{C} \setminus \{0\})$, then (up to reparametrization and $\mathbb{R}$-translation) $\tilde{u}_r : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ and $\tilde{u}'_r : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ are the unique finite energy $\tilde{J}$-holomorphic cylinders asymptotic to P_3 at its positive puncture and to P_2 at its negative puncture that do not intersect $P_2 \cup P_3$. Moreover, $\tilde{u}_r$ and $\tilde{u}'_r$ approach P_2 in opposite directions.*

Proof. The proof follows the ideas of [dPSa18, Proposition C.1].

First we prove that $\tilde{u}_q$ and $\tilde{u}'_q$ coincide up to reparametrization and $\mathbb{R}$-translation. Suppose that $\tilde{u}_q$ and $\tilde{u}'_q$ do not coincide. This is equivalent to $u_q(\mathbb{C}) \neq u'_q(\mathbb{C})$. Since P_2 is unknotted, $\mu(P_2) = 2$ and $\pi_2(S^3) = 0$, we have from Theorem 1.3 in [HWZ95b] that

$$u_q(\mathbb{C}) \cap P_2 = \emptyset, \quad u'_q(\mathbb{C}) \cap P_2 = \emptyset$$

and u_q, u'_q are embeddings. By Theorem 1.4 in [HWZ95b], $u_q(\mathbb{C}) \neq u'_q(\mathbb{C})$ and $u_q(\mathbb{C}), u'_q(\mathbb{C}) \cap P_2 = \emptyset$ imply

$$u_q(\mathbb{C}) \cap u'_q(\mathbb{C}) = \emptyset .$$

Let σ be the asymptotic eigensection of $\tilde{u}_q$ at ∞ and σ' the asymptotic eigensection of $\tilde{u}'_q$ at ∞. Since by Lemma 2.34 $\operatorname{wind} \sigma = \operatorname{wind} \sigma' = 1$ and $\nu_{P_2}^{neg}$ is the only negative eigenvalue of A_{P_2} with winding number equal to 1, it follows that σ, σ' are $\nu_{P_2}^{neg}$-eigensections. Since the eigenspace of $\nu_{P_2}^{neg}$-eigensections is one dimensional, we find a constant $c \neq 0$ such that

$$\sigma = c\sigma' .$$

Assume $c > 0$. Since $u_q(\mathbb{C}) \cap u'_q(\mathbb{C}) = \emptyset$, by Theorem B.4 we conclude that $[\tilde{u}_q] * [\tilde{u}'_q] > 0$. Since $u_q(\mathbb{C})$, $u'_q(\mathbb{C})$ does not intersect P_2, the orbit P_2 is even and

$$d_0(\tilde{u}_q, \infty) := \operatorname{wind}(\nu_{P_2}^{neg}) - \operatorname{wind}(\sigma) = 0, \quad d_0(\tilde{u}'_q, \infty) := \operatorname{wind}(\nu_{P_2}^{neg}) - \operatorname{wind}(\sigma') = 0 ,$$

then we obtain, using Theorem B.1,

$$[\tilde{u}_q] * [\tilde{u}_q'] = 0 \ .$$

This contradiction shows that the only possibility is $c < 0$. This implies that $u_q(\mathbb{C}) \cup u_q'(\mathbb{C}) \cup x_2(\mathbb{R})$ form a C^1-embedded sphere, where each hemisphere is a strong transverse section, a contradiction with the hypotheses of Theorem 3.5. We have proved that $\tilde{u}_q$ and $\tilde{u}_q'$ coincide up to reparametrization and $\mathbb{R}$-translation. By the same arguments, we prove that any finite energy $\tilde{J}$-holomorphic plane asymptotic to P_2 coincides with $\tilde{u}_q$ up to reparametrization and $\mathbb{R}$-translation.

Now we prove the assertions about the cylinders $\tilde{u}_r$ and $\tilde{u}_r'$. Assume $u_r(\mathbb{C} \setminus \{0\}) \neq u_r'(\mathbb{C} \setminus \{0\})$. Following Theorem 2.11, let η_+ be the asymptotic eigensection of $\tilde{u}_r$ at $+\infty$ and let η_- be the asymptotic eigensection of $\tilde{u}_r$ at 0. Similarly, define η_+' and η_-' for $\tilde{u}_r'$. Denote by $\mathrm{wind}(\eta_\pm)$, $\mathrm{wind}(\eta_\pm')$ the winding numbers of $\eta_\pm$, $\eta_\pm'$ computed with respect to a global trivialization of ξ. By Lemma 2.34, $\mathrm{wind}(\eta_-) = \mathrm{wind}(\eta_-') = 1$. Also, η_- and η_-' are associated to the eigenvalue $\nu_{P_2}^{pos}$ defined in (1.13), since there is just one positive eigenvalue of the asymptotic operator A_{P_2} with winding number equal to 1. Since the eigenspace of $\nu_{P_2}^{pos}$ is one dimensional, we conclude that there exists a nonzero constant c so that $\eta_-' = c\eta_-$.

Assume $c > 0$, that is, $\tilde{u}_r$ and $\tilde{u}_r'$ approach P_2 in the same direction. Since we assume $u_r(\mathbb{C} \setminus \{0\}) \neq u_r'(\mathbb{C} \setminus \{0\})$, it follows, by Carleman's similarity principle, that the images of u_r and u_r' do not coincide in any neighborhood of 0. By Theorem B.4, this implies that $[\tilde{u}_r] * [\tilde{u}_r'] > 0$. Since $u_r(\mathbb{C} \setminus \{0\})$ and $u_r'(\mathbb{C} \setminus \{0\})$ do not intersect $P_2 \cup P_3$, the orbit P_2 is even,

$$d_0(\tilde{u}_r, 0) := \mathrm{wind}(\nu_{P_3}^{pos}) - \mathrm{wind}(\eta_+), \ \ d_0(\tilde{u}_r, \infty) := \mathrm{wind}(\nu_{P_2}^{neg}) - \mathrm{wind}(\eta_-) = 0$$

and similarly, $d_0(\tilde{u}_r', 0) = d_0(\tilde{u}_r', \infty) = 0$, then, by Theorem B.1, we obtain $[\tilde{u}_r] * [\tilde{u}_r'] = 0$. This contradiction shows that $c < 0$, that is, $\tilde{u}_r$ and $\tilde{u}_r'$ approach P_2 in opposite directions.

By the same arguments, we conclude that $\tilde{u}_r$ and $\tilde{u}_r'$ are the unique cylinders with the properties given in the statement.

$\square$

The following proposition can be proved by Proposition 4.4 and Theorem B.2.

Proposition 4.8. *The images of the projected curves* u_r, u_q *and* $\{v_\tau\}, \tau \in (0, 1)$ *are mutually disjoint. Assume* $u_r(\mathbb{C} \setminus \{0\}) \neq u_r'(\mathbb{C} \setminus \{0\})$. *Then the images of* u_r, u_r', u_q *and* $\{v_\tau\}, \tau \in (0, 1)$ *are mutually disjoint.*

The foliation

Proposition 4.9. *The cylinders u_r and u'_r satisfy $u_r(\mathbb{C}\setminus\{0\})\cap u'_r(\mathbb{C}\setminus\{0\}) = \emptyset$. The union of the image of the family $\{v_\tau : \mathbb{C} \to S^3\}$, $\tau \in (0,1)$ with the images of u_q, u_r, u'_r, x_2 and x_3 determine singular foliation of a closed region $\mathcal{R}_1$ such that $\partial\mathcal{R}_1 = T$, where $T = x_2(\mathbb{R})\cup x_3(\mathbb{R})\cup u_r(\mathbb{C}\setminus\{0\})\cup u'_r(\mathbb{C}\setminus\{0\})$.*

Proof. First we prove that $u_r(\mathbb{C}\setminus\{0\}) \cap u'_r(\mathbb{C}\setminus\{0\}) = \emptyset$. Suppose by contradiction that $u_r(\mathbb{C}\setminus\{0\}) = u_r(\mathbb{C}\setminus\{0\})$. We will show that every point

$$p \in S^3 \setminus (u_r(\mathbb{C}\setminus\{0\}) \cup u_q(\mathbb{C}) \cup x_2(\mathbb{R}) \cup x_3(\mathbb{R}))$$

lies in the image of the family $\{v_\tau\}$. Let

$$p \in S^3 \setminus (u_r(\mathbb{C}\setminus\{0\}) \cup u_q(\mathbb{C}) \cup x_2(\mathbb{R}) \cup x_3(\mathbb{R}))$$

and consider a neighborhood $\mathcal{V}$ of $u_r(\mathbb{C}\setminus\{0\}) \cup u_q(\mathbb{C}) \cup x_2(\mathbb{R}) \cup x_3(\mathbb{R})$ such that $p \notin \mathcal{V}$. By Proposition 4.3, we have $v_{\tau_0}(\mathbb{C}), v_{\sigma_0}(\mathbb{C}) \subset \mathcal{V}$ for some τ_0 sufficiently close to 0 and σ_0 sufficiently close to 1. The surface $S = v_{\tau_0}(\mathbb{C}) \cup v_{\sigma_0}(\mathbb{C}) \cup x_3(\mathbb{R})$ is a piecewise smooth embedded sphere. By Jordan-Brouwer separation theorem, S divides S^3 into two regions A_1 and A_2 with boundary S and disjoint interior. One of these regions, say A_1 contains p and the other contains $u_r(\mathbb{C}\setminus\{0\}) \cup u_q(\mathbb{C}) \cup x_2(\mathbb{R}) \cup x_3(\mathbb{R})$. The intersection of the image of the family $\{v_\tau\}_{\tau\in(0,1)}$ with A_1 is non empty, open and closed. Thus p is in the image of the family $\{v_\tau\}_{\tau\in(0,1)}$.

It follows that $S^3 = u_r(\mathbb{C}\setminus\{0\}) \cup u_q(\mathbb{C}) \cup x_2(\mathbb{R}) \cup x_3(\mathbb{R}) \cup \{v_\tau(\mathbb{C})\}_{\tau\in(0,1)}$. But this contradicts the fact that P_1 is not linked to P_3 and the curves u_r, u_q and v_τ are transverse to the Reeb flow. We conclude that

$$u_r(\mathbb{C}\setminus\{0\})\cap u'_r(\mathbb{C}\setminus\{0\}) = \emptyset.$$

The surface $T = x_2(\mathbb{R}) \cup x_3(\mathbb{R}) \cup u_r(\mathbb{C}\setminus\{0\}) \cup u'_r(\mathbb{C}\setminus\{0\})$ is a piecewise smooth embedded torus. By Jordan-Brouwer separation theorem, T divides S^3 into two closed regions $\mathcal{R}_1$ and $\mathcal{R}_2$. Only one of the regions, say $\mathcal{R}_1$ contains the image of the family $\{v_\tau\}$ and the plane $u_q(\mathbb{C})$. Now we show that the images of u_q, u_r, u'_r, x_2, x_3 and $\{v_\tau\}$ foliate the closed region $\mathcal{R}_1$. Let

$$p \in \mathcal{R}_1 \setminus (x_2(\mathbb{R}) \cup x_3(\mathbb{R}) \cup u_r(\mathbb{C}\setminus\{0\}) \cup u'_r(\mathbb{C}\setminus\{0\}) \cup u_q(\mathbb{C}))$$

and let $\mathcal{V}$ be a neighborhood of $x_2(\mathbb{R})\cup x_3(\mathbb{R})\cup u_r(\mathbb{C}\setminus\{0\})\cup u'_r(\mathbb{C}\setminus\{0\})\cup u_q(\mathbb{C})$ such that $p \notin \mathcal{V}$. By Proposition 4.3, we have $v_{\tau_0}(\mathbb{C}), v_{\sigma_0}(\mathbb{C}) \subset \mathcal{V}$ for some τ_0 sufficiently close to 0 and σ_0 sufficiently close to 1. The surface $S =$
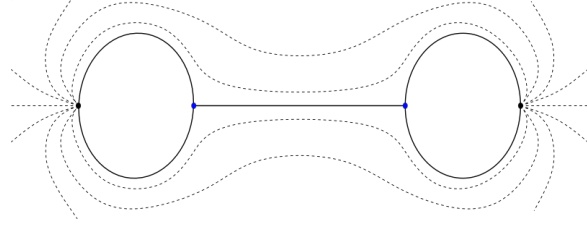
Figure 4.2: The foliation obtained so far

$v_{\tau_0}(\mathbb{C}) \cup v_{\sigma_0}(\mathbb{C}) \cup x_3(\mathbb{R})$ is a piecewise smooth embedded sphere. By Jordan-Brouwer separation theorem, S divides S^3 into two regions A_1 and A_2 with boundary S and disjoint interiors. One of these regions, say A_1 contains p and is contained in $\mathcal{R}_1$. Thus the intersection of the image of the family $\{v_\tau\}$ with A_1 is open, closed and non empty. This implies that p is in the image of v_τ, for some $\tau \in (0,1)$. $\qquad\square$

Proposition 4.10. *The closure of the image of the family $\{v_\tau : \mathbb{C} \to S^3\}$, that is, the union of the image of the family with the images of u_q, u_r, u_r', x_2 and x_3 is homeomorphic to a solid torus.*

Proof. We follow the proof of the Solid torus theorem in [Rol03]. Consider the (piecewise smooth) embedded torus $T = x_2(\mathbb{R}) \cup x_3(\mathbb{R}) \cup u_r(\mathbb{C} \setminus \{0\}) \cup u_r'(\mathbb{C}\setminus\{0\})$. Let $\mathcal{R}_1$ be the union of the image of the family of planes $\{v_\tau\}$ with the image of u_q and T, so that $\partial \mathcal{R}_1 = T$. The curve $x_2(\mathbb{R})$ is the boundary of an embedded disk $\mathcal{D}$ contained in $\mathcal{R}_1$ such that $\mathring{\mathcal{D}} \subset \mathring{\mathcal{R}}_1$. Let N be a bicollar neighborhood of $\mathcal{D}$ in $\mathcal{R}_1$, so that $N \cap T$ is an annular neighborhood of $x_2(\mathbb{R})$ in T. The boundary of $\mathcal{R}_1 \setminus N$ is the union of two disks on ∂N and the set $T \setminus N$, which is an annulus. Thus $\mathcal{R}_1 \setminus N$ is bounded by a piecewise smooth 2-sphere in S^3. By the generalized Schönflies theorem, $\mathcal{R}_1 \setminus N$ is homeomorphic to a closed 3-ball. It follows that $\mathcal{R}_1$ is homeomorphic to a 3-ball with a $D^2 \times [0,1]$ attached (by mapping $D^2 \times \{0\}$ and $D^2 \times \{1\}$ onto disjoint disks on the boundary of the 3-ball). Since $\mathcal{R}_1$ is orientable, $\mathcal{R}_1$ is homeomorphic to $D^2 \times S^1$. $\qquad\square$

4.2   Foliating a solid torus in the case $\mathbf{\Gamma_1 = \{0\}}$

Let
$$\tilde{v} = (b, v) : \mathbb{C} \setminus \Gamma_1 \to \mathbb{R} \times S^3$$
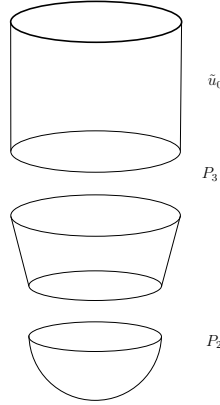
Figure 4.3: Case $\pi \cdot du_0 \equiv 0$ and $\Gamma_1 \neq \emptyset$
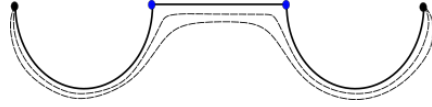
Figure 4.4: Gluing theorem $\Rightarrow$ family of planes

be a limit of the germinating sequence $\tilde{v}_n$, as defined in (3.8). In this section, we assume $\Gamma_1 = \{0\}$. In this case,

$$\tilde{v} : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$$

is a finite energy cylinder asymptotic to P_3 at its positive puncture and to P_2 at its negative puncture. The unique vertex immediately below $\tilde{v}$ in the bubbling-off tree $\mathcal{B}$ associated with the germinating sequence $\tilde{v}_n$ and the limit $\tilde{v}$ is a plane

$$\tilde{w} : \mathbb{C} \to \mathbb{R} \times S^3$$

asymptotic to P_2.

A family of planes asymptotic to P_3   By the Gluing Theorem 5.2 and Theorem 4.1, we obtain a maximal 1-parameter family of planes asymptotic to the orbit P_3

$$\tilde{v}_\tau = (b_\tau, v_\tau), \quad \tau \in (\tau_-, \tau_+)$$

such that $\tilde{v}_\tau$ converges, in the SFT sense, to the broken curve $(\tilde{v}, \tilde{w})$ as $\tau \to \tau_+$. Also, both the $\tilde{J}$-holomorphic planes $\tilde{v}_\tau$ and the projections $v_\tau : \mathbb{C} \to S^3$ are embeddings. This is a consequence of the following Theorem of [HWZ95b].

Theorem 4.11 ([HWZ95b, Theorem 1.3]). *Consider S^3 equipped with a tight*

contact form λ. Assume $\tilde{u} = (a, u) : \mathbb{C} \to \mathbb{R} \times S^3$ is a finite energy plane asymptotic to a simply covered and unknotted orbit P. If $\mu(P) \leq 3$, then $u(\mathbb{C}) \cap P = \emptyset$ and $u : \mathbb{C} \to S^3 \setminus P$ is an embedding.

By the same arguments used in Section 4.1, we conclude that $\tilde{v}_\tau$ converges, as $\tau \to \tau_-$, to a broken curve $(\tilde{v}', \tilde{w})$, where $\tilde{v}' : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ is a cylinder asymptotic to P_3 at its positive puncture ∞ and to P_2 at its negative puncture 0 such that $v'(\mathbb{C} \setminus \{0\}) \cap v(\mathbb{C} \setminus \{0\}) = \emptyset$. By Proposition 4.10, the union of the image of the family $\{v_\tau\}$ with the images of v, v', w, x_2 and x_3 is a solid torus.

4.3   A cylinder asymptotic to P_2 and P_1

In this section, we use a symplectic cobordism to find a 1-parameter family of generalized pseudoholomorphic planes asymptotic to P_2, which 'breaks' and produces a pseudoholomorphic cylinder asymptotic to P_2 at its positive puncture and to P_1 at its negative puncture.

Symplectic cobordism   Following [HWZ98] we define a symplectic cobordism between $(S^3, \lambda = f\lambda_0)$ and (S^3, λ_E), where λ_E is a dynamically convex contact form on S^3. Given $0 < r_1 < r_2$, with $\frac{r_1^2}{r_2^2}$ irrational, let $\lambda_E = f_E \lambda_0$ be the contact form associated to the ellipsoid

$$E = \left\{ (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \Big| \frac{x_1^2 + y_1^2}{r_1^2} + \frac{x_2^2 + y_2^2}{r_2^2} = 1 \right\} ,$$

that is, $f_E(x, y) = \left(\frac{x_1^2 + y_1^2}{r_1^2} + \frac{x_2^2 + y_2^2}{r_2^2} \right)^{-1}$.

The Reeb vector field X_E defined by λ_E has precisely two simply covered periodic orbits $\bar{P}_0$ and $\bar{P}_1$. Both periodic orbits and its iterates are nondegenerate. Their Conley-Zehnder indices are $\mu(\bar{P}_0) = 3$ and $\mu(\bar{P}_1) = 2k + 1 \geq 5$ where $k \geq 2$ is determined by $k < 1 + \left(\frac{r_1^2}{r_2^2} \right) < k + 1$. See [HWZ95a, Lemma 1.6] for a proof of these facts.

We choose $0 < r_1 < r_2$ small enough so that

$$f_E < f \text{ pointwise on } S^3$$

and a smooth function $h : \mathbb{R} \times S^3 \to \mathbb{R}^+$ satisfying

$$\begin{aligned} h(a, \cdot) &= f_E, \text{ if } a \leq -2, \\ h(a, \cdot) &= f, \text{ if } a \geq 2, \end{aligned} \tag{4.4}$$
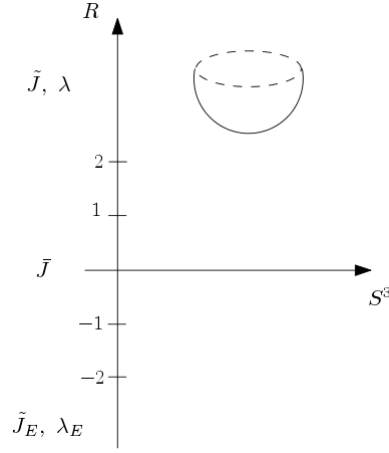
Figure 4.5: The almost complex structure $\bar{J}$

$$\frac{\partial h}{\partial a} \geq 0 \text{ on } \mathbb{R} \times S^3 \text{ and } \frac{\partial h}{\partial a} > \sigma > 0 \text{ on } [-1, 1] \times S^3. \tag{4.5}$$

In view of (4.5), the 2-form $d(h\lambda_0)$ restricted to $[-1, 1]$ is a symplectic form.

We consider the family of contact forms $\{\lambda_a = h(a, \cdot)\lambda_0, a \in \mathbb{R}\}$. The contact structure $\xi = \ker \lambda_a$ does not depend on a. Choose $J_E \in \mathcal{J}(\xi, d\lambda_E)$ and let $\{J_a \in \mathcal{J}(\xi, d\lambda_a), a \in \mathbb{R}\}$ be a smooth family of $d\lambda_a$-compatible complex structures on ξ so that $J_a = J$ if $a \geq 2$ and $J_a = J_E$ if $a \leq -2$. We consider smooth almost complex structures $\bar{J}$ on the symplectization $\mathbb{R} \times S^3$ with the following properties. On $(\mathbb{R} \setminus [-1, 1]) \times S^3$, we consider

$$\bar{J}|_\xi = J_a \text{ and } \bar{J}\partial_a = X_{\lambda_a} .$$

On $[-1, 1] \times S^3$ we only require $\bar{J}$ to be compatible with the symplectic form $d(h\lambda_0)$. The space of such almost complex structures on $\mathbb{R} \times S^3$ is non-empty and contractible in the C^∞-topology and will be denoted by $\mathcal{J}(\lambda, J, \lambda_E, J_E)$.

Generalized finite energy surfaces

Definition 4.12. Let (S, j) be a closed Riemann surface and let $\Gamma \subset S$ be a non empty finite set. A smooth map $\tilde{u} : S \setminus \Gamma \to \mathbb{R} \times S^3$ is called a *generalized finite energy surface* if it is $\bar{J}$-holomorphic, that is, satisfies $d\tilde{u} \circ j = \bar{J}(\tilde{u}) \circ d\tilde{u}$, for some $\bar{J} \in \mathcal{J}(\lambda, J, \lambda_E, J_E)$, as well as the energy condition

$$0 < E(\tilde{u}) < +\infty ,$$

where the energy is defined as follows. Let Σ be the collection of smooth

functions $\phi : \mathbb{R} \to [0,1]$ satisfying $\phi' \geq 0$ and $\phi = \frac{1}{2}$ on $[-1,1]$. Then

$$E(\tilde{u}) = \sup_{\phi \in \Sigma} \int_{S \setminus \Gamma} \tilde{u}^* d(\phi \lambda). \qquad (4.6)$$

Theorem 2.20 is still valid for almost complex structures in $\mathcal{J}(\lambda, J, \lambda_E, J_E)$.

Theorem 4.13 ([HWZ99b])**.** *There exists a residual set $\mathcal{J}_{reg}^E \subset \mathcal{J}(\lambda, J, \lambda_E, J_E)$ such that if $\tilde{u} = (a,u) : S^2 \setminus \Gamma \to \mathbb{R} \times S^3$ is a somewhere injective generalized finite energy sphere for $\bar{J} \in \mathcal{J}_{reg}^E$, then*

$$0 \leq \mathrm{ind}(\tilde{u}) = \mu(\tilde{u}) - \chi_{S^2} + \#\Gamma \ .$$

We are interested in the space of all finite energy generalized $\bar{J}$-holomorphic planes asymptotic to the orbit P_2, for fixed $\bar{J} \in \mathcal{J}_{reg}^E$.

The following theorem is a consequence of results in [HWZ99b].

Theorem 4.14 ([HWZ99b])**.** *Let $\tilde{u}_0 : \mathbb{C} \to \mathbb{R} \times S^3$ be an embedded finite energy $\bar{J}$-holomorphic plane, asymptotic to a non degenerate, simply covered orbit $P = (x,T)$ satisfying $\mu(P) = 2$. Then there exists a smooth embedding*

$$\tilde{\Phi} : \mathbb{C} \times (-\epsilon, \epsilon) \to \mathbb{R} \times S^3$$

with the following properties:

- $\tilde{\Phi}(\cdot, 0) = \tilde{u}_0$;

- *For every $\tau \in (-\epsilon, \epsilon)$, the map $z \mapsto \tilde{\Phi}(z, \tau)$ is a generalized finite energy $\bar{J}$-holomorphic plane asymptotic to P;*

- *If $\tilde{u}_n$ is a sequence of finite energy $\bar{J}$-holomorphic planes asymptotic to P satisfying $\tilde{u}_n \to \tilde{u}_0$ in $C_{loc}^\infty(\mathbb{C})$ as $n \to +\infty$, then there exist sequences A_n, B_n in $\mathbb{C}$ with $A_n \to 1$, $B_n \to 0$ and τ_n in $(-\epsilon, \epsilon)$ with $\tau_n \to 0$ such that*

$$\tilde{u}_n(z) = \tilde{\Phi}(A_n z + B_n, \tau_n)$$

  for sufficiently large n.

A family of $\bar{J}$-holomorphic planes asymptotic to P_2   From now on we fix $\bar{J} \in \mathcal{J}_{reg}^E$, where $\mathcal{J}_{reg}^E$ is given by Theorem 4.13.

Let Θ be the space of generalized finite energy $\bar{J}$-holomorphic planes asymptotic to P_2, modulo holomorphic reparametrizations. By Theorem 4.14, Θ is a smooth 1-dimensional manifold.

Lemma 4.15. *The space Θ is non-empty.*

Proof. Let $\tilde{u}_q = (a_q, u_q) : \mathbb{C} \to \mathbb{R} \times S^3$ be the $\tilde{J}$-holomorphic plane asymptotic to P_2 given by Theorem 4.2. After a $\mathbb{R}$-translation we can assume $\min_{z \in \mathbb{C}} a_q(z) > 2$, so that $\tilde{u}_q = (a_q, u_q)$ can be viewed as a generalized finite energy $\bar{J}$-holomorphic plane asymptotic to P_2. $\square$

By the proof of Lemma 4.15, we have $\tilde{u}_q \in \Theta$. Let Θ' be the connected component of Θ containing $\tilde{u}_q$.

4.3.1 Limiting behavior

Definition 4.16 (Bubbling-off tree). Consider a finite, rooted and oriented (away from the root) tree $\mathcal{T}$, and a finite set $\mathcal{U}$ of (generalized) finite energy pseudoholomorphic spheres. The pair $\mathcal{B} = (\mathcal{T}, \mathcal{U})$ is called a bubbling-off tree if satisfy the following properties

- There is a bijective correspondence between vertices $q \in \mathcal{T}$ and finite-energy punctured spheres $\tilde{u}_q : \mathbb{C} \setminus \Gamma_q \to \mathbb{R} \times S^3 \in \mathcal{U}$. Each $\tilde{u}_q : \mathbb{C} \setminus \Gamma_q \to \mathbb{R} \times S^3$ is pseudoholomorphic with respect to either $\tilde{J}$, $\tilde{J}_E$ or $\bar{J}$. Moreover, each ordered path $(q_1, \ldots, q_N)$ from the root $q_1 = r$ to a leaf q_N, where q_{k+1} is a direct descendant of q_k, contains at most one vertex q_i such that $\tilde{u}_{q_i}$ is $\bar{J}$-holomorphic, in which case $\tilde{u}_{q_j}$ is $\tilde{J}$-holomorphic $\forall 1 \leq j < i$, and $\tilde{u}_{q_j}$ is $\tilde{J}_E$-holomorphic $\forall i < j \leq N$.

- Each sphere $\tilde{u}_q$ has exactly one positive puncture at ∞ and $0 \leq \#\Gamma_q < \infty$ negative punctures, where Γ_q is the set of negative punctures of $\tilde{u}_q$.

- If the vertex q is not the root then q has an incoming edge e from a vertex q', and $\#\Gamma_q$ outgoing edges $f_1, \ldots, f_{\#\Gamma_q}$ to vertices $p_1, \ldots, p_{\#\Gamma_q}$ of $\mathcal{T}$, respectively. The edge e is associated to the positive puncture of $\tilde{u}_q$ and the edges $f_1, \ldots, f_{\#\Gamma_q}$ are associated to the negative punctures of $\tilde{u}_q$. The asymptotic limit of $\tilde{u}_q$ at its positive puncture coincides with the asymptotic limit of $\tilde{u}_{q'}$ at its negative puncture associated to e. In the same way, the asymptotic limit of $\tilde{u}_q$ at a negative puncture corresponding to f_i coincides with the asymptotic limit of $\tilde{u}_{p_i}$ at its unique positive puncture. If $\tilde{u}_q$ is $\tilde{J}$-holomorphic, then $\tilde{u}_{p_i}$ is either $\tilde{J}$ or $\bar{J}$-holomorphic. If $\tilde{u}_q$ is either $\bar{J}$ or $\tilde{J}_E$ holomorphic, then $\tilde{u}_{p_i}$ is necessarily $\tilde{J}_E$-holomorphic, $\forall i = 1, \ldots, \#\Gamma_q$.

- If the contact area of $\tilde{u}_q$ vanishes and $\tilde{u}_q$ is $\tilde{J}$ or $\tilde{J}_E$-holomorphic, then $\#\Gamma_q \geq 2$.

Consider a sequence $\tilde{v}_n = (a_n, v_n)$ of generalized finite energy planes representing elements of Θ'. The energy $E(\tilde{v}_n)$ is uniformly bounded by T_2.

The following statement is a corollary of the SFT compactness theorem of [BEH$^+$03].

Theorem 4.17 ([HS16],Theorem 3.11). *Up to a subsequence of $\tilde{v}_n$, still denoted by $\tilde{v}_n$, there exists a bubbling-off tree $\mathcal{B} = (\mathcal{T},\mathcal{U})$ with the following properties*

- *For every vertex q of $\mathcal{T}$ there exist sequences z_n^q, $\delta_n^q \in \mathbb{C}$ and $c_n^q \in \mathbb{R}$ such that*

$$\tilde{v}_n(z_n^q + \delta_n^q \cdot) + c_n^q \to \tilde{v}_q(\cdot) \text{ in } C_{loc}^\infty(\mathbb{C} \setminus \Gamma_q) \text{ as } n \to \infty \qquad (4.7)$$

  Here $\tilde{v} + c := (a + c, v)$, where $\tilde{v} = (a, v)$ and $c \in \mathbb{R}$

- *The curve $\tilde{v}_r$ is asymptotic to P_2 at ∞ and the asymptotic limits of all curves $\tilde{v}_q$ are closed orbits with periods $\leq T_2$ of the Reeb flow of either λ or λ_E.*

Lemma 4.18 ([HS16], Lemma 3.12). *Let z_n^q, δ_n^q, c_n^q be sequences such that (4.7) holds for all vertices q of $\mathcal{T}$. Then we can assume, up to a selection of a subsequence still denoted by $\tilde{u}_n$, that one of the three mutually excluding possibilities holds for each vertex q.*

(I) *c_n^q is bounded, $a_n(z_n + \delta_n^q \cdot)$ is $C_{loc}^0(\mathbb{C} \setminus \Gamma_q)$-bounded and $\tilde{v}_q$ is a $\bar{J}$-holomorphic curve;*

(II) *$c_n^q \to -\infty$, $a_n(z_n^q + \delta_n^q \cdot) \to +\infty$ in $C_{loc}^0(\mathbb{C} \setminus \Gamma_q)$ as $n \to \infty$ and $\tilde{v}_q$ is a $\bar{J}$-holomorphic curve;*

(III) *$c_n^q \to +\infty$, $a_n(z_n^q + \delta_n^q \cdot) \to -\infty$ in $C_{loc}^0(\mathbb{C} \setminus \Gamma_q)$ as $n \to \infty$ and $\tilde{v}_q$ is a $\bar{J}_E$-holomorphic curve.*

Moreover, if q is a vertex for which (III) holds, then $\tilde{u}_q$ is asymptotic at its positive puncture to a closed Reeb orbit having period strictly less than T_2. In particular, (III) does not hold for the root r.

The following lemma is proved using the maximum principle combined with estimates for cylinders with small area (Lemma 2.29).

Lemma 4.19 ([HS16, Lemma 3.13]). *Let z_n^r, δ_n^q, c_n^q be sequences such that (4.7) holds for the root r. Then, by Theorem 4.17, $P_2 = (x_2, T_2)$ is the asymptotic limit of $\tilde{u}_r$ at the positive puncture ∞. For every $\mathbb{R}/\mathbb{Z}$-invariant neighborhood $\mathcal{W}$ of $t \mapsto x_2(T_2 t)$ in $C^\infty(\mathbb{R}/\mathbb{Z}, S^3)$ and for every number $M > 0$, there exist $R_0 > 0$ and n_0 such that if $R > R_0$ and $n > n_0$, then the loop $t \mapsto v_n(z_n^r + \delta_n^r R e^{i2\pi t})$ belongs to $\mathcal{W}$ and $a_n^r(z_n^r + \delta_n^r e^{i2\pi t}) + c_n^r > M$.*

Let Θ' be the connected component of Θ containing $\tilde{u}_q$. Since Θ' is a connected 1-dimensional manifold without boundary, it is diffeomorphic either to S^1 or to an open interval. It can not be diffeomorphic to S^1, since it contains the family $\{[(a_q + t, u_q)]\}_{t>0}$ of equivalence classes of translations of $\tilde{u}_q$. Thus, the family is diffeomorphic to an interval and we can assume $\Theta' = \{[\tilde{v}_\tau = (b_\tau, v_\tau)]\}_{\tau \in (\tau_-,+\infty)}$, where for $\tau \geq 0$, $\tilde{v}_\tau = (a_q + \tau, u_q)$. Consider a sequence $\tilde{v}_n := \tilde{v}_{\tau_n}$ satisfying $\tau_n \to \tau_-$.

Proposition 4.20. *The bubbling-off tree obtained as an SFT-limit of the sequence $[\tilde{v}_n]$, as in Theorem 4.17, is as follows. The tree has vertices r, q, where r is the root and q is a leaf and direct descendant of r. The root r corresponds to a $\tilde{J}$-holomorphic cylinder $\tilde{v}_r : \mathbb{C} \setminus \{z_0\} \to \mathbb{R} \times S^3$ asymptotic to P_2 at its positive puncture ∞ and to P_1 at its negative puncture. The leaf q corresponds to a $\bar{J}$-holomorphic plane asymptotic to P_1.*

Proof. Let $\mathcal{B} = (\mathcal{T}, \mathcal{U})$ be the bubbling-off three given by Theorem 4.17 and let $\tilde{v}_r : \mathbb{C} \setminus \Gamma_r \to \mathbb{R} \times S^3$ be the finite energy sphere associated to the root of $\mathcal{T}$. By Lemma 4.18, $\tilde{v}_r$ is not $\tilde{J}_E$-holomorphic. Now we show that $\tilde{v}_r$ is $\tilde{J}$-holomorphic.

Suppose, by contradiction, that $\tilde{v}_r$ is $\bar{J}$-holomorphic. By Theorem 4.17, $\tilde{v}_r$ is asymptotic to P_2 at ∞. Since P_2 is simply covered, $\tilde{v}_r$ is somewhere injective. If $\Gamma_r = \emptyset$, then $\tilde{v}_r \in \Theta$, contradicting the fact that the interval Θ' is maximal. Assume $\Gamma_r \neq \emptyset$. By Theorem 4.13, it follows that

$$0 \leq \mathrm{ind}(\tilde{v}_r) = 2 - \sum_{z \in \Gamma_r} \mu(P_z) - 2 + \#\Gamma_r + 1 \ . \tag{4.8}$$

Then

$$\sum_{z \in \Gamma_r} \mu(P_z) - \#\Gamma_r \leq 1 \ . \tag{4.9}$$

Contradicting the fact that for all $z \in \Gamma_r$, the asymptotic limit P_z of $\tilde{v}_r$ at z is a closed Reeb orbit of λ_E, which is a dynamically convex contact from, that is, $\mu(P_z) \geq 3$, for all $z \in \Gamma_r$. Thus, $\tilde{v}_r$ is $\tilde{J}$-holomorphic.

Let $m := \min a_q(\mathbb{C})$. We claim that

$$\limsup_n \left(\min b_n(\mathbb{C})\right) \leq m \ .$$

To prove the claim, suppose by contradiction that $\limsup_n \left(\min b_n(\mathbb{C})\right) > m > 2$. Then there exists a subsequence $\tilde{v}_{n_k} = (b_{n_k}, v_{n_k})$ satisfying $\min b_{n_k}(\mathbb{C}) > m$. By the definiton of $\bar{J}$, the planes $\tilde{v}_{n_k}$ are $\tilde{J}$-holomorphic. By the hypotheses of Theorem 3.5, $\tilde{v}_q$ is the only $\tilde{J}$-holomorphic plane asymptotic to P_2, up

to reparametrization and $\mathbb{R}$-translation. This implies $[\tilde{v}_{n_k}] = [a_q + \tau_k, u_q]$ for a sequence $\tau_k > 0$. This contradicts the fact that Θ' is an interval.

Now we show that $\Gamma_r \neq \emptyset$. Suppose, by contradiction, that $\tilde{v}_r$ is a $\tilde{J}$-holomorphic plane. Let z_n, δ_n and c_n be the sequences given by Theorem 4.17, such that

$$\tilde{v}_n(z_n + \delta_n \cdot) + c_n \to \tilde{v}_r(\cdot) \text{ in } C^\infty_{loc}(\mathbb{C}) \text{ as } n \to \infty.$$

The limit $\tilde{v}_r$ satisfies 4.18(II), so that $c_n \to -\infty$ and $b_n(z_n + \delta_n \cdot) \to +\infty$ in $C^0_{loc}(\mathbb{C})$ as $n \to \infty$. By Lemma 4.19, there exists $R_0 > 0$ such that $b_n(z_n + \delta_n z) > m - c_n > m$ for $|z| > R_0$ and n large enough. For $|z| \leq R_0$, by Lemma 4.18(II), we have $b_n(z_n + \delta_n z) > m$, for n large enough. Thus, $\inf b_n(\mathbb{C}) > m$ for n large enough. This contradicts $\limsup_n \inf b_n(\mathbb{C}) \leq m$, and concludes the proof of $\Gamma_r \neq \emptyset$.

So far, we know that $\tilde{v}_r : \mathbb{C} \setminus \Gamma_r \to \mathbb{R} \times M$ is a $\tilde{J}$-holomorphic sphere and $\Gamma_r \neq \emptyset$. The next step is to prove that every negative asymptotic limit of $\tilde{v}_r$ has Conley-Zehnder index equal to 1.

I) If q is not the root and P_∞ is the asymptotic limit of $\tilde{v}_q$ at ∞, then $\mu(P_\infty) \geq 1$.

Suppose, by contradiction, that $\mu(P_\infty) \leq 0$. The curve $\tilde{v}_q : \mathbb{C} \setminus \Gamma_q \to \mathbb{R} \times S^3$ factors as

$$\tilde{v}_q = \tilde{u} \circ p ,$$

where $\tilde{u} : \mathbb{C} \setminus \Gamma' \to \mathbb{R} \times S^3$ is a somewhere injective finite energy sphere and p is a polynomial. If P is the asymptotic limit of $\tilde{u}$ at ∞, then $P^{\deg p} = P_\infty$. By Lemma 1.9, $\mu(P_\infty) \leq 0$ implies $\mu(P) \leq 0$.

By Theorem 4.13,

$$0 \leq \operatorname{ind} \tilde{u} := \mu(P) - \sum_{z' \in \Gamma'} \mu(P_{z'}) + \#\Gamma_q - 1,$$

where $P_{z'}$ is the asymptotic limit of $\tilde{u}$ at z'.

If $\Gamma' = \emptyset$, we already have a contradiction. Otherwise, there exists $z'_0 \in \Gamma'$ such that $\mu(P_{z'_0}) \leq 0$. Let $z_0 \in \Gamma_q$ be such that $p(z_0) = z'_0$. Then $P_{z_0} = P^k_{z'_0}$, for some $k \leq \deg p$, where P_{z_0} is the asymptotic limit of $\tilde{v}_q$ at z_0. By Lemma 1.9, we have $\mu(P_{z_0}) \leq 0$.

Since the tree has a finite number of vertices, by induction we find a leaf l of the tree such that the finite energy plane $\tilde{u}_l : \mathbb{C} \to \mathbb{R} \times S^3$ is asymptotic to an orbit P with $\mu(P) \leq 0$, a contradiction.

II) For every $z \in \Gamma_r$, we have $\mu(P_z) = 1$, where P_z is the asymptotic limit of $\tilde{v}_r$ at z.

By Theorem 2.20, we have

$$0 \leq \text{ind } \tilde{v}_r = 2 - \sum_{z \in \Gamma_r} \mu(P_z) + \#\Gamma_r - 1 \ .$$

It follows that

$$\sum_{z \in \Gamma_r} \mu(P_z) \leq \#\Gamma_r + 1 \ .$$

Since, by Claim I), we have $1 \leq \mu(P_z)$, for every $z \in \Gamma_r$, then there exists at most one puncture $z_0 \in \Gamma_r$ such that $\mu(P_{z_0}) \geq 2$. If there exists such puncture, we have $\text{ind } \tilde{v}_r = 0$. By Theorem 2.20, this implies $\pi \circ dv_r \equiv 0$. By the definition of bubbling-off tree (Definition 4.16), this implies $\#\Gamma_r \geq 2$. By Theorem 2.6, there exists a periodic orbit P and a polynomial $p : \mathbb{C} \to \mathbb{C}$ such that $p^{-1}(0) = \Gamma_r$ and $\tilde{v}_r = F_P \circ p$, where F_P is the cylinder over the orbit P. This contradicts the fact that P_2 is simply covered.

We conclude that $\mu(P_z) = 1$, for all $z \in \Gamma_r$. We also proved that $\pi \circ dv_r \neq 0$.

III) $\tilde{v}_r$ is a cylinder asymptotic to P_1 at its negative puncture.

By Lemma 2.34, we conclude that $\text{wind}_\infty(\tilde{v}_r, z) = 1$, $\forall z \in \Gamma_r \cup \{\infty\}$. Recall that

$$u_q(\mathbb{C}) \cap x_2(\mathbb{R}) = \emptyset \ .$$

Thus, the curves $\tilde{u}_q$ and $\tilde{v}_r$ satisfy condition (2) of Theorem B.2. Here $\tilde{u}_q$ is the plane asymptotic to P_2 obtained by Theorem 4.2. It follows from Theorem B.2 that the projected curve u_q does not intersect any of the negative asymptotic limits of $\tilde{v}_r$. This implies that P_2 is contractible in $S^3 \setminus P_z$, for every P_z asymptotic limit of $\tilde{v}_r$ at $z \in \Gamma_r$. Consequently,

$$\text{lk}(P_2, P_z) = \text{lk}(P_z, P_2) = 0, \ \forall z \in \Gamma_r.$$

Since, by hypothesis, the orbit P_1 is the only orbit with Conley-Zehnder index 1 and period less than T_2 that is not linked to P_2, it follows that

$$P_z = P_1, \ \ \forall z \in \Gamma_r \ .$$

Moreover, the hypothesis $T_2 < 2T_1$ implies that $\#\Gamma_r = 1$.

IV) Now we prove that the next (and last) level of the bubbling-off tree consists of a $\bar{J}$-holomorphic plane.

Let $\tilde{v}_q : \mathbb{C} \setminus \Gamma_q \to \mathbb{R} \times S^3$ be the finite energy sphere associated to the only vertex that is a direct descendant of the root r. The asymptotic limit of $\tilde{v}_q$ at ∞ is P_1, that is a simply covered orbit. It follows that $\tilde{v}_q$ is somewhere injective. By the definition of bubbling-off tree, $\tilde{v}_q$ is either $\tilde{J}$-holomorphic or $\bar{J}$-holomorphic. By Theorem 4.13, we have

$$0 \leq \operatorname{ind} \tilde{v}_q = 1 - \sum_{z \in \Gamma_q} \mu(P_z) + \#\Gamma_q - 1 = \#\Gamma_q - \sum_{z \in \Gamma_q} \mu(P_z) \ ,$$

where P_z is the asymptotic limit of $\tilde{v}_q$ at $z \in \Gamma_q$. Since by Claim I), $\mu(P_z) \geq 1$ for all $z \in \Gamma_q$, then $\operatorname{ind} \tilde{v}_q = 0$ and $\mu(P_z) = 1$, for all $z \in \Gamma_q$.

Suppose, by contradiction, that $\tilde{v}_q$ is $\tilde{J}$-holomorphic. By Theorem 2.20, we have $\pi \circ dv_q \equiv 0$. By the definition of bubbling-off tree 4.16, this implies $\#\Gamma_q \geq 2$. Theorem 2.6 and the fact that P_1 is simply covered lead to a contradiction. We have proved that $\tilde{v}_q$ is $\bar{J}$-holomorphic.

Suppose that $\Gamma_q \neq \emptyset$. By the definition of bubbling-off tree, if l is a vertex of the tree that is a direct descendant of q, then $\tilde{v}_l$ is necessarily $\tilde{J}_E$-holomorphic. The asymptotic limit of $\tilde{v}_l$ at ∞ is equal to P_z for some $z \in \Gamma_q$. But $\mu(P_z) = 1$ for all $z \in \Gamma_q$, contradicting the fact that all closed orbits of X_E have Conley-Zehnder index ≥ 3.

We have proved that $\Gamma_q = \emptyset$. This finishes the proof of Proposition 4.20.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proposition 4.21. *The curve* $\tilde{v}_r = (b_r, v_r) : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ *and the projection* $v_r : \mathbb{C} \setminus \{0\} \to S^3$ *are embeddings. The projection* v_r *does not intersect any of its asymptotic limits.*

To prove Proposition 4.21, we need the following.

Proposition 4.22. *Let* $\tilde{u} : \mathbb{R} \times S^1 \to \mathbb{R} \times S^3$ *be a finite energy cylinder asymptotic to prime orbits* $P = (x, T)$ *at* $+\infty$ *and* $\bar{P} = (\bar{x}, \bar{T})$ *at* $-\infty$, *and satisfying*

- $P \neq \bar{P}$,

- P *and* $\bar{P}$ *form an unlink,*

- $\mu(P),\ \mu(\bar{P}) \in \{1, 2, 3\}$.

Then $u(\mathbb{R} \times S^1) \cap P = \emptyset$ *and* $u(\mathbb{R} \times S^1) \cap \bar{P} = \emptyset$.

Remark 4.23. In Proposition 4.22, the fact that the contact form λ in S^3 is tight is crucial.

Proof. The proof follows the arguments of the proof of Theorem 1.3 in [HWZ95b], so we just sketch the proof here and refer to the results in [HWZ95b] when necessary.

A finite energy surface $\tilde{u} : \mathbb{C} \backslash \Gamma \to \mathbb{R} \times S^3$ such that $\pi \cdot du$ is not identically zero can intersect its asymptotic limits in at most finitely many points. This is proved in [HWZ96, Theorem 5.2]. This allows the definition of an algebraic intersection index as follows. Take a small embedded 2-disk $\mathcal{D}$ transversal to the periodic orbit at a point $x(t)$ of P and tangent to $\xi = \ker \lambda$ at $x(t)$, that is, $T_{x(t)}\mathcal{D} = \xi_{x(t)}$. We orient the disk in such a way that $T_{x(t)}\mathcal{D}$ and $\xi_{x(t)}$ have the same orientation. Let $M \in \mathbb{R}$ be such that all the intersection points of u with P are contained in $u((-\infty, M) \times S^1)$. Let $\varphi : \mathbb{D} \to S^3$ be a disk map to $\bar{P}$ such that $\varphi(\mathbb{D}) \cap P = \emptyset$. Such disk exists because $P \cup \bar{P}$ is the trivial link. Glue the disk $\mathbb{D}$ to $[-\infty, M] \times S^1$ along $-\infty \times S^1$ to form a new disk D. Let $\bar{u} : \bar{\mathbb{R}} \times S^1 \to S^3$ be the map obtained by defining $\bar{u}(-\infty, t) = \bar{x}(\bar{T}t)$ and $\bar{u}(+\infty, t) = x(Tt)$. Define $U : D \to S^3$ by

$$U|_{[-\infty, M] \times S^1} = \bar{u}, \quad U|_{\mathbb{D}} = \varphi .$$

Consider $U_* : H_1(\partial D, \mathbb{Z}) \to H_1(S^3 \backslash P, \mathbb{Z})$. If α is the generator of $H_1(\partial D)$, then since $U(\partial D) \cup P = \emptyset$, there exists an integer, that we call $int(u)$, such that

$$U_*(\alpha) = int(u)[\partial \mathcal{D}] \in H_1(S^3 \backslash P, \mathbb{Z}) .$$

Here we used the fact that $H_1(S^3 \backslash P)$ is generated by $[\partial \mathcal{D}]$.

It follows from the proof of Theorem 4.6 in [HWZ95b] that $int(u)$ is the oriented intersection number of U with P. Also, all the intersections are in the image of the map u, since there is no intersections of P with $\varphi(\mathbb{D})$. Following the proof of Theorem 4.6 in [HWZ95b], one can show that $int(u) \geq 0$, so that $int(u) = 0$ if and only if $u(\mathbb{R} \times S^1) \cup P = \emptyset$. Thus, to show that $u(\mathbb{R} \times S^1) \cap P = \emptyset$, it is sufficient to show that $int(u) = 0$.

Let $\Phi : \mathcal{U} \to S^3$ be an embedding of an open neighborhood $\mathcal{U}$ of the zero section of $\xi|_P$. We require that $\Phi(0_p) = p$ and the fiberwise derivative of Φ at 0_p is the inclusion of ξ into $T_p S^3$.

Consider a non vanishing section $v(t)$ of ξ along P which is contained in $\mathcal{U}$. Define the loop $\beta(v)$ by $\beta(v)(t) = \Phi \circ v(t)$ for $0 \leq t \leq T$. It is contained in $S^3 \backslash P$ and we denote by $[\beta(v)] \in H_1(S^3 \backslash P)$ the homology class generated by this loop. Fix the global trivialization $\Psi : \xi \to S^3 \times \mathbb{R}^2$. Let wind$(v, \Psi)$ be the winding number of the small section $v(t)$ with respect to the trivialization Ψ. It is proved in [HWZ95b] that

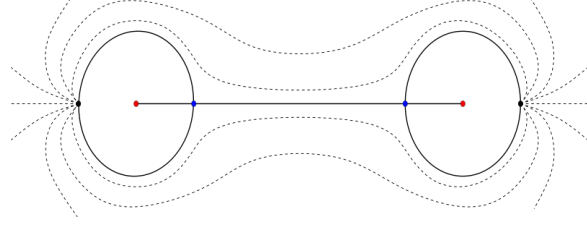$$\text{wind}(v, \Psi)[\partial \mathcal{D}] - [\beta(v)] \in H_1(S^3 \backslash P)$$

Figure 4.6: The foliation of the region $\mathcal{R}_1$ and the cylinder v_r connecting the orbits P_2 and P_1.

is independent of the section v as described above. We define a constant $c(u)$ by

$$(\text{wind}(v, \Psi) - c(u))[\partial \mathcal{D}] = [\beta(v)] \ .$$

If we choose, for example, the special section $v(t)$ such that $\Phi \circ v(t) = u(s^*, t)$ for some large $s^* \in \mathbb{R}$, then by definition

$$[\beta(v)] = int(u)[\partial \mathcal{D}]$$

and taking the limit as $s^* \to \infty$, we have

$$\text{wind}_\infty(\tilde{u}, \infty) - c(u) = int(u) \ .$$

It is proved in [HWZ95b] that there exists an embedded disc $F = \varphi(\mathbb{D})$ with $\varphi(\partial \mathbb{D}) = P$ whose characteristic distribution has $e^+ \geq 1$ positive elliptic points, and that

$$c(u) = 2e^+ - 1 \ \ .$$

By Lemma 2.34, we have $\text{wind}_\infty(\tilde{u}, \infty) = 1$, so that

$$int(u) = 2 - 2e^+ \ .$$

Since $int(u) \geq 0$ and $e^+ \geq 1$, then $int(u) = 0$ and $e^+ = 1$. This shows that $u(\mathbb{R} \times S^1) \cap P = \emptyset$. We can repeat the arguments replacing P by $\bar{P}$ and show that $u(\mathbb{R} \times S^1) \cap \bar{P} = \emptyset$ and conclude the proof. $\qquad \square$

Proof of Proposition 4.21. From Proposition 4.22, we conclude that $\tilde{v}_r$ does not intersect its asymptotic limits. Thus $\tilde{v}_r$ satisfies condition (3) of Theorem B.5. Applying Theorem B.5, we conclude the proof. $\qquad \square$

Proposition 4.24. *Up to reparametrization, $\tilde{v}_r$ is the unique $\tilde{J}$-holomorphic cylinder asymptotic to P_2 at ∞ and to P_1 at 0 that do not intersect the orbits P_1 and P_2. Moreover, the cylinder $\tilde{v}_r$ and the plane $\tilde{u}_q$ approach P_2 in opposite directions, according to Definition 4.6.*
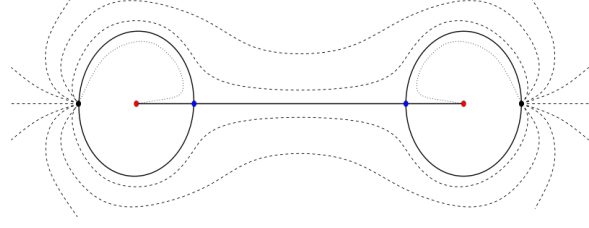
Figure 4.7: Gluing theorem $\Rightarrow$ family of cylinders asymptotic to P_3 and P_1

Proof. The proof follows the ideas of [dPSa18, Proposition C.1]. Following Theorem 2.11, let η_+ be the asymptotic eigensection of $\tilde{u}_q$ at ∞ and let η'_+ be the asymptotic eigensection of $\tilde{v}_r$ at ∞. By Lemma 2.34, $\text{wind}_\infty(\tilde{u}_q, \infty) = \text{wind}(\tilde{v}_r, \infty) = 1$. Using $\mu(P_2) = 2$, formula (1.15) and Proposition 1.15, we conclude that $\nu_{P_2}^{neg}$ is the unique negative eigenvalue of A_{P_2} with winding number 1. By the properties of the asymptotic operator A_{P_2} given in Section 1.3, we know that the eigenspace of $\nu_{P_2}^{neg}$ is one dimensional. Thus, there exists $c \neq 0$ such that $\eta'_+ = c\eta_+$. Suppose $c > 0$, that is, $\tilde{v}_r$ and $\tilde{u}_q$ approach P_2 in the same direction. By Carleman's similarity principle, the images of $\tilde{v}_r$ and $\tilde{u}_q$ do not coincide in any neighborhood of ∞. By Theorem B.4, we have $[\tilde{v}_r] * [\tilde{u}_q] > 0$. On the other hand, we can apply Theorem B.1 to the curves $\tilde{v}_r$ and $\tilde{u}_q$ to get $[\tilde{v}_r] * [\tilde{u}_q] = 0$. With this contradiction, we prove that $\tilde{v}_r$ and $\tilde{u}_q$ approach P_2 in opposite directions, that is, $c < 0$.

Using the same arguments, one can show that any cylinder with the properties given in the statement must have the same image as $\tilde{v}_r$. $\qquad\square$

4.4   A family of cylinders asymptotic to P_1 and P_3

So far we have foliated a region $\mathcal{R}_1 \subset S^3$ homeomorphic to a solid torus with boundary equal to the torus $T = P_3 \cup u_r(\mathbb{C} \setminus \{0\}) \cup u'_r(\mathbb{C} \setminus \{0\}) \cup P_2$. The complement of $\mathring{\mathcal{R}}_1$ in S^3 is a closed region that we denote by $\mathcal{R}_2$. In the previous section we found an embedded pseudoholomorphic cylinder $\tilde{v}_r : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$. The projected curve v_r is also embedded and its image is contained in the interior of $\mathcal{R}_2$.

Applying the Gluing Theorem 5.2 to the broken cylinder $(\tilde{u}_r, \tilde{v}_r)$, we obtain a family of pseudoholomorphic cylinders $\{\tilde{w}_\tau : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3\}, \tau \in [R, +\infty)$, asymptotic to the orbits P_3 and P_1, converging to the broken cylinder $(\tilde{u}_r, \tilde{v}_r)$ in the sense of the SFT compactness theorem as $\tau \to +\infty$.

Proposition 4.25. *For every $\tau \in [R, +\infty)$, $\tilde{w}_\tau$ is an embedding, the projec-*

tion $w_\tau : \mathbb{C} \backslash \{0\} \to S^3$ is an embedding which does not intersect its asymptotic limits and $w_\tau(\mathbb{C} \setminus \{0\}) \subset \mathcal{R}_2$.

Proof. By Proposition 4.22 and Theorem B.5, we conclude that $\tilde{w}_\tau$ and w_τ are embeddings and w_τ does not intersect its asymptotic limits. Applying Theorem B.2 to $\tilde{w}_\tau$ and any $\tilde{J}$-holomorphic plane asymptotic to P_3 whose projection is in $\mathcal{R}_1$, we conclude that $w_\tau(\mathbb{C} \setminus \{0\})) \subset \mathcal{R}_2$. $\qquad \square$

The following is Theorem 4.5.44 of [Wen05].

Theorem 4.26 ([Wen05]). *Let $\tilde{u} = (a, u) : S^2 \backslash \Gamma \to \mathbb{R} \times M$ be an embedded $\tilde{J}$-holomorphic finite energy sphere with $\mathrm{ind}(\tilde{u}) = 2$, such that every asymptotic limit is simply covered and has odd Conley-Zehnder index. Then there exists a number $\delta > 0$ and an embedding*

$$\tilde{F} : \mathbb{R} \times (-\delta, \delta) \times S^2 \setminus \Gamma \to \mathbb{R} \times M$$
$$(\sigma, \tau, z) \mapsto (a_\tau(z) + \sigma, u_\tau(z))$$

such that

- *For $\sigma \in \mathbb{R}$ and $\tau \in (-\delta, \delta)$, the maps $\tilde{u}_{(\sigma,\tau)} = \tilde{F}(\sigma, \tau, \cdot)$ are (up to parametrization) embedded $\tilde{J}$-holomorphic finite energy spheres and $\tilde{u}_{(0,0)} = \tilde{u}$.*

- *The map $F(\tau, z) = u_\tau(z)$ is an embedding $(-\delta, \delta) \times S^2 \setminus \Gamma \to M$ and its image never intersects the asymptotic limits. In particular, the maps $u_\tau : S^2 \setminus \Gamma \to M$ are embedded for each $\tau \in (-\delta, \delta)$ with mutually disjoint images which do not intersect their asymptotic limits.*

- *For any sequence $\tilde{v}_k : S^2 \setminus \Gamma \to \mathbb{R} \times M$ such that for each puncture in Γ, $\tilde{v}_k$ has the same asymptotic limit as $\tilde{u}$, with the same sign, and $\tilde{v}_k \to \tilde{u}$ in $C^\infty_{loc}(S^2 \backslash \Gamma)$, there is a sequence $(\sigma_k, \tau_k) \to (0, 0) \in \mathbb{R} \times (-\delta, \delta)$ such that $\tilde{v}_k = \tilde{u}_{(\sigma_k, \tau_k)} \circ \varphi_k$ for some sequence of diffeomorphisms $\varphi_k : S^2 \setminus \Gamma \to S^2 \setminus \Gamma$ and k sufficiently large.*

Remark 4.27. In Theorem 4.26, no genericity assumption is required for $\tilde{J}$.

Applying the theorem above to the maps $\tilde{w}_\tau$, we obtain a maximal smooth one parameter family of maps, that we denote again by $\tilde{w}_\tau$, $\tau \in (\tau_-, \tau_+)$. We assume the normalization $\tau_- = 0$, $\tau_+ = 1$ and $\tilde{w}_\tau \to (\tilde{u}_r, \tilde{v}_r)$, as $\tau \to 0^+$.

Proposition 4.28. *Consider a sequence $\tilde{w}_n = (c_n, w_n) : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$, where $\tilde{w}_n = \tilde{w}_{\sigma_n}$ and $\sigma_n \to 0^+$. Then after suitable reparametrizations and $\mathbb{R}$-translations, we have*

(i) *Up to subsequence, $\tilde{w}_n \to \tilde{u}_r$ in $C^\infty_{loc}(\mathbb{C} \setminus \{0\})$ as $n \to \infty$.*

(ii) *There exist sequences $\delta_n^+ \to 0^+$ and $d_n \in \mathbb{R}$ such that, up to subsequence, $\tilde{w}_n(\delta_n \cdot) + d_n \to \tilde{v}_r$ in $C^\infty_{loc}(\mathbb{C} \setminus \{0\})$ as $n \to \infty$.*

A similar statement holds for any sequence $\tau_n \to 1^-$ with $\tilde{u}_r$ replaced with $\tilde{u}_r'$.

The Proof of Proposition 4.28 is the subject of Sections 4.4.1 and 4.4.2.

4.4.1   Bubbling-off analysis for the family of cylinders

In this section, we follow the ideas of [HWZ03, Section 6.2].

Consider a sequence $\tilde{w}_n = (c_n, w_n) : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$, where $\tilde{w}_n = \tilde{w}_{\sigma_n}$ and $\sigma_n \to 0^+$.

Note that since all cylinders $\tilde{w}_n$ are asymptotic to P_3 and P_1, we have $0 < E(\tilde{w}_n) \le T_3$.

We reparametrize the sequence so that

$$\int_{\mathbb{C} \setminus \mathbb{D}} w_n^* d\lambda = \frac{\sigma(T_3)}{2} \; . \tag{4.10}$$

Define

$$\Theta = \{z \in \mathbb{C} \setminus \{0\} | \exists \text{ subsequence } \tilde{w}_{n_j} \text{ and } z_j \to z \text{ s.t. } |\nabla \tilde{w}_{n_j}(z_j)| \to \infty\}$$

By the same arguments used in the proof of Proposition 2.26, we can assume that Θ is finite and $\Theta \subset \mathbb{D} \setminus \{0\}$. Also, there exists a $\tilde{J}$-holomorphic map

$$\tilde{w} : \mathbb{C} \setminus (\{0\} \cup \Theta) \to \mathbb{R} \times S^3$$

such that, up to a subsequence, still denoted by $\tilde{w}_n$,

$$\tilde{w}_n \to \tilde{w} \text{ in } C^\infty_{loc}\left(\mathbb{C} \setminus (\{0\} \cup \Theta), \mathbb{R} \times S^3\right)$$

and $E(\tilde{w}) \le T_3$.

The punctures in $\{0\} \cup \Theta$ are non-removable and negative, and the puncture $z = \infty$ is positive. Indeed for any ϵ sufficiently large or small, we have

$$\int_{\partial B_\epsilon(0)} w^* \lambda = \lim_{n \to \infty} \int_{\partial B_\epsilon(0)} w_n^* \lambda \in [T_1, T_3],$$

where $\partial B_\epsilon(z)$ is oriented counterclockwise. It follows that ∞ is a positive puncture and 0 is a negative puncture. If $z \in \Theta$, then for any sufficiently

small ϵ, we have

$$\int_{\partial B_\epsilon(z)} w^*\lambda = \lim_{n\to\infty} \int_{\partial B_\epsilon(z)} w_n^*\lambda = \lim_{n\to\infty} \int_{B_\epsilon(z)} w_n^*d\lambda \geq T > 0,$$

where $T \leq T_3$ is a period. This follows from the same arguments used in Section 2.3.3 . It follows that z is a negative puncture.

By the same arguments used in the proof of Proposition 4.2 we conclude that the asymptotic limit of $\tilde{w}$ at ∞ is P_3.

Lemma 4.29.

$$\int_{\mathbb{C}\backslash(\{0\}\cup\Theta)} w^*d\lambda > 0 \ .$$

Proof. If $\Theta = \emptyset$, then it follows from (4.10) that $\int_{\mathbb{C}\backslash\{0\}} w^*d\lambda \geq \frac{\sigma(T_3)}{2}$. In the case $\Theta \neq \emptyset$, suppose $\int_{\mathbb{C}\backslash(\{0\}\cup\Theta)} w^*d\lambda = 0$. By Theorem 2.6, there exists a polynomial $p : \mathbb{C} \to \mathbb{C}$ and a periodic orbit $P \in \mathcal{P}(\lambda)$ such that $p^{-1}(0) = \{0\}\cup\Theta$ and $\tilde{w} = F_P \circ p$, where F_P is the cylinder over the orbit P. But this implies $\deg p \geq 2$, contradicting the fact that the asymptotic limit of $\tilde{w}$ at ∞ is P_3, that is a prime orbit. $\square$

Soft rescaling near $z_0 \in \Theta$. Assume $\Theta \neq \emptyset$ and take a puncture $z_0 \in \Theta$. Let $P_{z_0} = (x_{z_0}, T_{z_0})$ be the asymptotic limit of $\tilde{w}$ at z_0. As in Section 2.3.4, we define the mass of z_0 by

$$m(z_0) = \lim_{\epsilon\to 0^+} m_\epsilon(z_0) \qquad m_\epsilon(z_0) = \lim_{n\to\infty} \int_{B_\epsilon(z_0)} w_n^*d\lambda \ ,$$

where $\partial B_\epsilon(z_0)$ is oriented counterclockwise. Now we proceed as in the *soft rescaling* done in Section 2.3.4. Fix $\epsilon > 0$ such that

$$m_\epsilon(z_0) - m(z_0) \leq \frac{\sigma(T_3)}{2} \ .$$

Choose z_n defined by $c_n(z_n) = \inf(c_n(B_\epsilon(z_0)))$ and $0 < \delta_n < \epsilon$ by

$$\int_{B_\epsilon(z_0)\backslash B_{\delta_n}(z_n)} w_n^*d\lambda = \sigma(T_3).$$

It follows that $z_n \to z_0$ and, up to a subsequence, $\delta_n \to 0$. Take $R_n \to \infty$ such that $\delta_n R_n < \frac{\epsilon}{2}$ and define

$$\begin{aligned}
\tilde{v}_n = (b_n, v_n) : B_{R_n}(0) &\to \mathbb{R} \times S^3 \\
z &\mapsto (c_n(z_n + \delta_n z) - c_n(z_n + 2\delta_n), w_n(z_n + \delta_n z))
\end{aligned} \tag{4.11}$$

The sequence $\tilde{v}_n$ is a germinating sequence according to Definition 2.25. Let

$$\Theta_1 = \{z \in \mathbb{C} | \exists z_j \to z \text{ and subsequence } \tilde{w}_{n_j} \text{ s.t. } |d\tilde{w}_{n_j}(z_j)| \to \infty\} . \quad (4.12)$$

Passing to a subsequence, we can assume Θ_1 is finite. Let

$$\tilde{v} = (b, v) : \mathbb{C} \setminus \Theta_1 \to \mathbb{R} \times S^3$$

be a limit of $\tilde{v}_n$ as defined in 2.27. Then $\tilde{v}$ has a unique positive puncture at ∞ and $\tilde{v}$ is asymptotic to P_0 at ∞. Thus, using Lemma 2.37, we conclude the following.

Lemma 4.30. *If $z \in \Theta$ and $\tilde{w}$ is asymptotic to P at z, then $\mu(P) \geq 2$.*

Since P_3 is prime, it follows that $\tilde{w}$ is somewhere injective. By Theorem 2.20, we have

$$1 \leq \text{ind}(\tilde{w}) = 3 - \sum_{z \in \{0\} \cup \Theta} \mu(P_z) + \#\Theta$$

and consequently

$$\sum_{z \in \{0\} \cup \Theta} \mu(P_z) \leq 2 + \#\Theta . \quad (4.13)$$

This proves the following lemma.

Lemma 4.31. *Assume $\Theta \neq \emptyset$. Then $\mu(P_0) \leq 1$. If $\mu(P_0) = 1$, then $\#\Theta = 1$ and $\mu(P_z) = 2$, where P_z is the asymptotic limit of $\tilde{w}$ at the unique point $z \in \Theta$.*

Soft-rescaling near $z = 0$. For any $\epsilon > 0$, define

$$m_\epsilon(0) = \lim_{n \to \infty} \int_{B_\epsilon(0) \setminus \{0\}} w_n^* d\lambda,$$

and define the *mass of the puncture $z = 0$* by

$$m(0) = \lim_{\epsilon \searrow 0} m_\epsilon(0) . \quad (4.14)$$

Note that, for n large and ϵ small, we have

$$\int_{B_\epsilon(0) \setminus \{0\}} w_n^* d\lambda = \int_{\partial B_\epsilon(0)} w_n^* \lambda - \lim_{\delta \to 0} \int_{\partial B_\delta(0)} w_n^* \lambda = \int_{\partial B_\epsilon(0)} w_n^* \lambda - T_1. \quad (4.15)$$

It follows that

$$m(0) = \lim_{\epsilon \searrow 0} \int_{\partial B_\epsilon(0)} w^*\lambda - T_1 = T_0 - T_1,$$

where T_0 is the period of the asymptotic limit $P_0 = (x_0, T_0)$ of $\tilde{w}$ at the puncture $z = 0$. We have two cases:

- either $m(0) = 0$ or

- $m(0) > 0 \Rightarrow m(0) > \sigma(T_3)$.

I)   First assume that $m(0) > \sigma(T_3) > 0$. We claim that there is a sequence $\delta_n \to 0$ satisfying

$$\int_{D_{\delta_n(0)}\backslash\{0\}} w_n^* d\lambda = m(0) - \frac{\sigma(C)}{2} \tag{4.16}$$

Indeed, there exists a sequence δ_n satisfying the equation above, since using (4.10) and $\int_{\mathbb{C}\backslash\{0\}} w_n^* d\lambda = T_3 - T_1 \geq m(0) > \sigma(C)$, we conclude

$$\int_{\mathbb{D}\backslash\{0\}} w_n^* d\lambda \geq m(0) - \frac{\sigma(C)}{2} > 0 \ .$$

Now we show that $\liminf \delta_n = 0$, so that, passing to a subsequence, still denoted by δ_n, the claim is true. Suppose that there exists $0 < \epsilon' < \liminf \delta_n$. Then we have the contradiction

$$m(0) - \frac{\sigma(C)}{2} = \lim_{j \to \infty} \int_{B_{\delta_n(0)}\backslash\{0\}} w_n^* d\lambda \geq \lim_{j \to \infty} \int_{B_{\epsilon'}(0)\backslash\{0\}} w_n^* d\lambda \geq m(0) \ ,$$

and the claim is proved.

Let $\epsilon_0 > 0$ be small enough so that the disks $B_{\epsilon_0}(\gamma), \ \gamma \in \Theta \cup \{0\}$ are disjoint. Define

$$\tilde{v}_n(z) = (b_n(z), v_n(z)) = (c_n(\delta_n z) - c_n(2\delta_n), w_n(\delta_n z)) \tag{4.17}$$

for $z \in B_{\frac{\epsilon_0}{\delta_n}}(0) \backslash \{0\}$. Note that $\tilde{v}_n$ satisfies the normalization

$$\int_{\mathbb{D}\backslash\{0\}} \tilde{v}_n^* d\lambda = \int_{B_{\delta_n}(0)\backslash\{0\}} w_n^* d\lambda = m(0) - \frac{\sigma(C)}{2} \ .$$

It follows that, for j large and ϵ_0 small we have the estimate

$$
\begin{aligned}
\int_{B_{\frac{\epsilon_0}{\delta_n}} \setminus \mathbb{D}} v_n^* d\lambda &= \int_{B_{\frac{\epsilon_0}{\delta_n}} \setminus \{0\}} v_n^* d\lambda - \int_{\mathbb{D} \setminus \{0\}} v_n^* d\lambda \\
&= \int_{B_{\epsilon_0}(0) \setminus \{0\}} w_n^* d\lambda - \left(m(0) - \frac{\sigma(C)}{2} \right) \qquad (4.18) \\
&\le m(0) + \frac{\sigma(C)}{2} - \left(m(0) - \frac{\sigma(C)}{2} \right) = \sigma(C)
\end{aligned}
$$

Define Θ_0 as the set of *bubbling* points of the sequence $\tilde{v}_n$. Using the proof of Proposition 2.26 we obtain, passing to a subsequence, Θ_0 finite and $\Theta_0 \subset \mathbb{D} \setminus \{0\}$. Also, there exists a $\tilde{J}$-holomorphic map $\tilde{v}_0 : \mathbb{C} \setminus \{0\} \cup \Theta_0 \to \mathbb{R} \times S^3$ such that, passing to a subsequence

$$
\tilde{v}_n \to \tilde{v}_0 \text{ in } C^\infty_{loc}(\mathbb{C} \setminus \{0\} \cup \Theta_0) \ .
$$

The map $\tilde{v}_0$ is non-constant, the punctures in $\{0\} \cup \Theta_0$ are non-removable and negative, and the puncture $z = \infty$ is positive.

Lemma 4.32. *The asymptotic limit of $\tilde{v}_0$ at its unique positive puncture $z = \infty$ is equal to P_0, the asymptotic limit of $\tilde{w}$ at $\{0\}$.*

Proof. Let $\mathcal{W}$ be an open neighborhood of the set of loops

$$
\{ t \in S^1 \mapsto x(Tt + c) | P = (x, T) \in \mathcal{P}(\lambda), \ c \in \mathbb{R} \}
$$

such that each connected component of $\mathcal{W}$ contains at most one periodic orbit modulo S^1-reparametrization. Let P_∞ be the asymptotic limit of $\tilde{v}$ at ∞. Let $\mathcal{W}_\infty$ and $\mathcal{W}_0$ be connected components of $\mathcal{W}$ containing P_∞ and P_0 respectively.

Since $\tilde{w}_n \to \tilde{w}$ in C^∞_{loc}, we can choose $0 < \epsilon_0' < \epsilon_0$ small enough so that, if $0 < \rho \le \epsilon_0'$ is fixed, then the loop

$$
t \in S^1 \mapsto w_n(\rho e^{i2\pi t})
$$

belongs to $\mathcal{W}_0$ for n large. Since $\tilde{v}_n \to \tilde{v}$ in C^∞_{loc}, we can choose $R_0 > 1$ large enough so that, if $R \ge R_0$ is fixed, then the loop

$$
t \in S^1 \mapsto v_n(R e^{i2\pi t}) = w_n(\delta_n R e^{i2\pi t})
$$

belongs to $\mathcal{W}_\infty$ for n large.

By (4.16), we can show that

$$e := \liminf \int_{\partial B_{\delta_n R_0}(0)} w_n^* \lambda > 0 \tag{4.19}$$

Consider, for each n, the $\tilde{J}$-holomorphic cylinder $\tilde{C}_n : \left[\frac{\ln R_o \delta_n}{2\pi}, \frac{\ln \epsilon_0'}{2\pi}\right] \times S^1 \to \mathbb{R} \times S^3$, defined by $\tilde{C}_n(s,t) = \tilde{w}_n(e^{2\pi(s+it)})$. It follows from (4.18) that

$$\int_{\left[\frac{\ln R_o \delta_n}{2\pi}, \frac{\ln \epsilon_0'}{2\pi}\right] \times S^1} C_n^* d\lambda \leq \sigma(C) \tag{4.20}$$

for n large. Using (4.19) and (4.20) and applying Lemma 2.29, we find $h > 0$ so that the loop

$$t \mapsto C_n(s,t)$$

belongs to $\mathcal{W}$ for every $s \in \left[\frac{\ln R_0 \delta_n}{2\pi} + h, \frac{\ln \epsilon_0}{2\pi} - h\right]$ and n large. Since $h > 0$, we have

$$C_n\left(\frac{\ln \epsilon_0}{2\pi} - h, t\right) = w_n(\epsilon_0 e^{-2\pi h} e^{2\pi it}) \in \mathcal{W}_0$$

for all n large and

$$C_n\left(\frac{\ln R_0 \delta_n}{2\pi} + h, t\right) = w_n(R_0 \delta_n e^{2\pi h} e^{2\pi t}) \in \mathcal{W}_\infty$$

for all n large. Thus $\mathcal{W}_\infty = \mathcal{W}_0$ and $P_\infty = P_0$.   □

Lemma 4.33. *Either*

- $\int_{\mathbb{C}\backslash\{0\}\cup\Theta_0} v_0^* d\lambda > 0$ *or*

- $\int_{\mathbb{C}\backslash\{0\}\cup\Theta_0} v_0^* d\lambda = 0$ *and* $\#\Theta_0 \geq 1$.

Proof. Indeed, suppose, by contradiction, that $\int_{\mathbb{C}\backslash\{0\}\cup\Theta_0} v^* d\lambda = 0$ and $\Theta_0 = \emptyset$. Then

$$\begin{aligned}
T_0 = \int_{\partial\mathbb{D}} v^* \lambda &= \lim_{n\to\infty} \int_{\partial B_{\delta_n}(0)} w_n^* \lambda \\
&= \int_{B_{\delta_n}(0)\backslash\{0\}} w_n^* \lambda + \lim_{\epsilon\to 0} \int_{\partial B_\epsilon(0)} w_n^* \lambda \\
&= m(0) - \frac{\sigma(T_3)}{2} + T_1 \\
&= T_0 - \frac{\sigma(T_3)}{2},
\end{aligned}$$

a contradiction.   □

II)   Now assume $m(0) = 0$. Let $\epsilon > 0$ be small enough so that $m_\epsilon(0) \leq \frac{\sigma(T_3)}{2}$.
Define

$$\tilde{v}_n(z) = \tilde{w}_n(\delta_n z), \quad z \in \mathbb{C} \setminus \{0\}$$

for any sequence $\delta_n \to 0$. We have

$$\int_{B_{\frac{\epsilon}{\delta_n}} \setminus B_{\delta_n}} v_n^* d\lambda = \int_{B_\epsilon \setminus B_{\delta_n^2}} w_n^* d\lambda \leq \int_{B_\epsilon \setminus \{0\}} w_n^* d\lambda.$$

Thus,

$$\lim_{n \to \infty} \int_{B_{\frac{\epsilon}{\delta_n}} \setminus B_{\delta_n}} v_n^* d\lambda \leq m_\epsilon(0) \leq \frac{\sigma(C)}{2} . \tag{4.21}$$

It follows that $\{\tilde{v}_n\}$ has no *bubbling* points. Ideed, if there is a bubbling-off point, arguing as in the proof of Proposition 2.26, we conclude that $\lim_{n \to \infty} \int_{B_{\frac{\epsilon}{\delta_n}} \setminus B_{\delta_n}} v_n^* d\lambda \geq T$, for some period T.

Thus, passing to a subsequence, there exists a $\tilde{J}$-holomorphic map $\tilde{v}_0 = (b_0, v_0) : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ such that

$$\tilde{v}_n \to \tilde{v}_0 \text{ in } C^\infty_{loc}(\mathbb{C} \setminus \{0\}) .$$

It follows from (4.21) that

$$\int_{\mathbb{C} \setminus \{0\}} v_0^* d\lambda = 0 .$$

Indeed, if $\int_{\mathbb{C} \setminus \{0\}} v_0^* d\lambda > 0$, then $\int_{\mathbb{C} \setminus \{0\}} v_0^* d\lambda > \sigma(T_3)$, which contradicts (4.21).
   We claim that $\tilde{v}_0$ is non-constant. Indeed,

$$\int_{\partial \mathbb{D}} v_n^* \lambda = \int_{\partial B_{\delta_n}} w_n^* \lambda = \int_{\partial B_\epsilon} w_n^* \lambda - \int_{B_\epsilon \setminus B_{\delta_n}} w_n^* \lambda .$$

Thus,

$$\int_{\partial \mathbb{D}} v_0^* \lambda = \lim_{n \to \infty} \int_{\partial \mathbb{D}} v_n^* \lambda = \lim_{n \to \infty} \left(\int_{\partial B_\epsilon} w_n^* \lambda - \int_{B_\epsilon \setminus B_{\delta_n}} w_n^* \lambda \right) = \int_{\partial B_\epsilon} w^* \lambda .$$

Taking the limit as $\epsilon \to 0$, we get

$$\int_{\partial \mathbb{D}} v_0^* \lambda = T_0 = T_1 .$$

It follows that $\tilde{v}_0$ is a cylinder over a T_0-periodic Reeb orbit $P = (x, T_0)$.

Lemma 4.34. $P_0 = P_1$.

Proof. Choose now an S^1-invariant neighborhood $\mathcal{W}$ in $C^\infty(S^1, S^3)$ separating the loops associated with periodic solutions of period $\leq T_3$ from each other. For fixed n, we know that

$$(t \mapsto w_n(\epsilon e^{2\pi i t})) \to x_1(Tt) \text{ as } \epsilon \to 0 .$$

We choose a sequence $\delta_n \to 0$ such that

$$(t \mapsto w_n(\delta_n e^{2\pi i t})) \in \mathcal{W}, \quad \forall n .$$

Recalling that

$$(t \mapsto w(\epsilon e^{2\pi i t})) \to x_0(Tt) \text{ as } \epsilon \to 0,$$

We conclude, from estimate (4.21) and Lemma 2.29, arguing as in Section 2.3.4, that $P = P_1 = P_0$. $\qquad\square$

We conclude the analysis of the case $m(0) = 0$.

Going back to the case $m(0) > \sigma(T_3) > 0$, if the mass of the puncture $z = 0$ of $\tilde{v}_0$ is positive or $\Theta_0 \neq 0$, we repeat the process. It necessarily stops after finitely many iterations, when we reach punctures with zero mass or run out of bubbling-off points. This follows from Lemmas 2.30 and 4.33. We obtain a tree of finite energy spheres having a unique positive puncture whose asymptotic limit agrees with the asymptotic limit of the corresponding negative puncture belonging to the previous level. The leaves of the tree correspond to finite energy planes originating from the bubbling-off points and a cylinder over the orbit P_1, originating from the puncture $z = 0$.

We are ready to proof Proposition 4.28.

4.4.2   Proof of Proposition 4.28

First we show that the mass of the puncture $z = 0$, as defined in (4.14), is positive. Suppose $m(0) = 0$. Then, by the analysis done in Section 4.4.1, either

1) $\Theta = \emptyset$ and $\tilde{w} : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ is asymptotic to P_3 at $z = \infty$ and asymptotic to P_1 at $z = 0$, or

2) $\Theta = \{z_1, \ldots, z_k\}$ and $\tilde{w} : \mathbb{C} \setminus \{0\} \cup \Theta : \mathbb{R} \times S^3$ is asymptotic to P_3 at $z = \infty$, to P_1 at $z = 0$ and to orbits $\tilde{P}_i$ satisfying $\mu(\tilde{P}_i) \geq 2$ at each z_i, $i \in \{1, \ldots, k\}$.
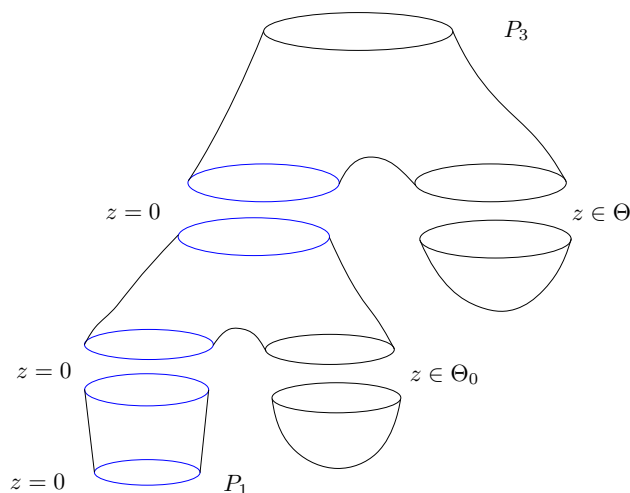
Figure 4.8: An (a priori) possible bubbling-off tree

Option 1) can not occur, since it would contradict the fact that the family of cylinders $\{\tilde{w}_\tau\}$ is maximal.

Now we show that option 2) also leads to a contradiction. Suppose 2) holds. By Lemma 4.31, we have $k = 1$ and $\mu(\tilde{P}) = 2$, where $\tilde{P} = (\tilde{x}, \tilde{T})$ is the asymptotic limit of $\tilde{w}$ at the unique puncture in Θ. The orbit $\tilde{P}$ is not linked to P_3. This follows from positivity and stability of intersections and the fact that $\tilde{P}$ is the limit of contractible links contained in the image of the embedded cylinders w_n, which do not intersect P_3. Since $\tilde{T} < T_3$, by the hypotheses of Theorem 3.5 we conclude that $\tilde{P} = P_2$. But $\tilde{T}$ also satisfies $T_3 > T_1 + \tilde{T} = T_1 + T_2$, contradicting the hypothesis $T_3 < 2T_1$ of Theorem 3.5. This contradiction shows that $m(0) > 0$.

As explained before, one of the leaves of the bubbling-off tree obtained from the sequence $\tilde{w}_n$ is a cylinder over the orbit P_1 originated from the puncture $z = 0$. By the same arguments used in the proof of theorem 4.20, using the fact that $\mu(P_1) = 1$, we conclude that $\mu(P_0) \geq 1$.

Now we show that $\Theta = \emptyset$. Suppose $\Theta \neq \emptyset$. Then by Lemma 4.31, we have $\mu(P_0) = 1$, $\#\Theta = 1$ and $\mu(\tilde{P}) = 2$, where $\tilde{P}$ is the asymptotic limit at the unique puncture in Θ. The orbit $\tilde{P}$ is not linked to P_3. This is because $\tilde{P}$ is the limit of contractible links contained in the image of the embedded cylinders w_n. Since $\tilde{T} \leq T_3$, by our hypotheses we conclude $\tilde{P} = P_2$. But $\tilde{T}$ also satisfies $T_3 > T_0 + \tilde{T} > T_1 + \tilde{T}$, contradicting the assumption $T_3 < 2T_1$. This contradiction proves $\Theta = \emptyset$.

So far, we know that $\tilde{w} : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ is a $\tilde{J}$-holomorphic cylinder asymptotic to P_3 at ∞ and to the orbit P_0 at 0. By (4.13), we have $\mu(P_0) \leq 2$,

so that

$$1 \leq \mu(P_0) \leq 2.$$

The second level of the bubbling-off tree obtained from the sequence $\tilde{w}_n$ consists of a unique vertex associated with a finite energy sphere $\tilde{v}_0 : \mathbb{C} \setminus \{0\} \cup \Theta_0 \to \mathbb{R} \times S^3$, asymptotic to P_0 at its positive puncture ∞.

Claim: $\mu(P_0) = 2$. To prove the claim, suppose by contradiction that $\mu(P_0) = 1$. If $\tilde{v}_0$ is somewhere injective, using Theorem 2.20, we have

$$0 \leq 1 - \mu(P_0^v) - \sum_{z \in \Theta_0} \mu(P_z^v) + \#\Theta_0,$$

where P_z^v is the asymptotic limit of $\tilde{v}_0$ at the puncture z. By the same arguments used in the proof of Theorem 4.20, using the fact that $\mu(P_1) = 1$, we conclude that $\mu(P_0^v) \geq 1$. By Lemma 2.37, we have $\mu(P_z^v) \geq 2$. We conclude that $\#\Theta_0 = 0$ and $\pi \cdot dv_0 \equiv 0$, which contradicts Lemma 4.33.

If $\tilde{v}_0$ is not somewhere injective, there exists a somewhere injective $\tilde{J}$-holomorphic curve $\tilde{u}_0 : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times S^3$ and a polynomial $p : \mathbb{C} \to \mathbb{C}$ such that $\tilde{v}_0 = \tilde{u}_0 \circ p$, $p(\Theta_0 \cup \{0\}) = \Gamma$ and $p^{-1}(\Gamma) = \Theta_0 \cup \{0\}$. This implies that $P_0 = P_\infty^{\deg p}$, where P_∞ is the asymptotic limit of $\tilde{u}_0$ at ∞. Using Lemma 1.9 and the assumption $\mu(P_0) = 1$, we conclude that $\mu(P_\infty) = 1$. For every $z \in \Theta_0 \cup \{0\}$, we have $P_z^v = (P_w^u)^k$, where $p(z) = w$, $k \mid \deg p$ and P_w^u is the asymptotic limit of $\tilde{u}_0$ at the puncture w. Since $\mu(P_z^v) \geq 1$, $\forall z \in \Theta_0 \cup \{0\}$, using Lemma 1.9 we conclude that $\mu(P_z^u) \geq 1$, $\forall z \in \Gamma$. Applying Theorem 2.20 to the curve $\tilde{u}_0$ we have

$$0 \leq 1 - \sum_{z \in \Gamma} \mu(P_z^u) + \#\Gamma - 1,$$

where P_z^u is the asymptotic limit of $\tilde{u}_0$ at the puncture $z \in \Gamma$. It follows that $\mu(P_z^u) = 1$, $\forall z \in \Gamma$. By Theorem 2.20, we have $\pi \cdot du_0 \equiv 0$, which implies $\pi \cdot dv_0 \equiv 0$. It follows from Theorem 2.6 that $\tilde{v}_0 = F_P \circ p$, where F_P is a cylinder over an orbit $P \in \mathcal{P}(\lambda)$, $p : \mathbb{C} \to \mathbb{C}$ is a polynomial and $P_0 = P^{\deg p}$. Since $\mu(P_0) = 1$, we conclude, using Lemma 1.9, that $\mu(P) = 1$. By Lemma 4.33, we have $\Theta_0 \neq \emptyset$. Let $z \in \Theta_0$. Then $\mu(P_z^v) \geq 2$ and $P_z^v = P^k$, where $k \mid \deg p$. This implies that $2 \leq \mu(P^k) \leq \mu(P^{\deg p}) = 1$, a contradiction. We have proved the claim $\mu(P_0) = 2$.

Now we prove that the mass $m(0)$ of the puncture $z = 0$ of $\tilde{v}_0$ is zero, according to definition (4.14). This implies that $\tilde{v}_0$ is asymptotic to P_1 at its unique negative puncture $z = 0$.

Claim: $m(0) = 0$.   Suppose, by contradiction, that $m(0) > 0$.

If $\tilde{v}_0$ is not somewhere injective, there exists a somewhere injective $\tilde{J}$-holomorphic curve $\tilde{u}_0 : \mathbb{C} \setminus \Gamma \to \mathbb{R} \times S^3$ and a polynomial $p : \mathbb{C} \to \mathbb{C}$ such that $\tilde{v}_0 = \tilde{u}_0 \circ p$, $p(\Theta_0 \cup \{0\}) = \Gamma$ and $p^{-1}(\Gamma) = \Theta_0 \cup \{0\}$. This implies that $P_0 = P_\infty^{\deg p}$, where P_∞ is the asymptotic limit of $\tilde{u}_0$ at ∞. Using Lemma 1.9, we conclude that $\mu(P_\infty) = 1$ and $\deg p = 2$. For every $z \in \Theta_0 \cup \{0\}$, we have $P_z^v = (P_w^u)^k$, where $p(z) = w$, $k \mid \deg p$ and P_w^u is the asymptotic limit of $\tilde{u}_0$ at the puncture w. Since $\mu(P_z^v) \geq 1$, $\forall z \in \Theta_0 \cup \{0\}$, using Lemma 1.9, we conclude that $\mu(P_z^u) \geq 1$, $\forall z \in \Gamma$. Applying Theorem 2.20 to the curve $\tilde{u}_0$ we have

$$0 \leq 1 - \sum_{z \in \Gamma} \mu(P_z^u) + \#\Gamma - 1,$$

where P_z^u is the asymptotic limit of $\tilde{u}_0$ at the puncture $z \in \Gamma$. It follows that $\mu(P_z^u) = 1$, $\forall z \in \Gamma$ and $\pi \cdot du_0 \equiv 0$. This implies that $\pi \cdot dv_0 \equiv 0$. It follows from Theorem 2.6 that $\tilde{v}_0 = F_P \circ p$, where F_P is a cylinder over an prime orbit $P \in \mathcal{P}(\lambda)$, $p : \mathbb{C} \to \mathbb{C}$ is a polynomial and $P_0 = P^{\deg p}$. By Lemma 1.9, we have $\deg p, \mu(P) \in \{1, 2\}$. Since, by assumption, $\tilde{v}_0$ is not somewhere injective, then $\mu(P) = 1$ and $\deg p = 2$. By Lemma 4.33, we have $\Theta_0 \neq \emptyset$. Let $z \in \Theta_0$. Then $P_z^v = P^k$, where $k \mid 2 = \deg p$. Since we know that $\mu(P^k) = \mu(P_z^v) \geq 2$ and $\mu(P^2) = 2$, by Lemma 1.9, we have $\mu(P_z^v) = 2$. The orbit $\tilde{P}_z^v$ is not linked to P_3. This follows from positivity and stability of intersections and the fact that P_z^v is the limit of contractible links contained in the image of the embedded cylinders w_n, which do not intersect P_3. Also, the period of P_z^v is $< T_3$. By the hypotheses of Theorem 3.5, we have $P_z^v = P_2$. This implies $T_3 \geq T_0^v + T_2 > T_1 + T_2$, where T_0^v is the period of P_0^v, a contradiction with the hypothesis $T_3 < 2T_1$.

If $\tilde{v}_0$ is somewhere injective, then we have

$$0 \leq \mathrm{ind}(\tilde{v}_0) = 2 - \mu(P_0^v) - \sum_{z \in \Theta_0} \mu(P_z^v) + \#\Theta_0.$$

If $\mathrm{ind}(\tilde{v}_0) = 0$, then $\pi \cdot dv_0 \equiv 0$ and by Lemma 4.33, we have $\Theta_0 \neq \emptyset$. Using $\mu(P_0^v) \geq 1$ and $\mu(P_z^v) \geq 2$, $\forall z \in \Theta_0$, we have

$$1 \leq \mu(P_0^v) = 2 - \sum_{z \in \Theta_0} \mu(P_z^v) + \#\Theta_0 \leq 2 - \#\Theta_0.$$

It follows that $\Theta_0 = \{z\}$ and $\mu(P_z^v) = 2$. The orbit $\tilde{P}_z^v$ is not linked to P_3 and has period $< T_3$. By the hypotheses of Theorem 3.5, we have $P_z^v = P_2$. This implies $T_3 \geq T_0^v + T_2 > T_1 + T_2$, where T_0^v is the period of P_0^v, a contradiction

with the hypothesis $T_3 < 2T_1$. If $\operatorname{ind}(\tilde{v}_0) \geq 1$, we have

$$1 \leq \mu(P_0^v) \leq 1 - \sum_{z \in \Theta_0} \mu(P_z^v) + \#\Theta_0.$$

Thus $\#\Theta_0 = \emptyset$ and $\mu(P_0^v) = 1$. Following the same arguments used to prove the claim $\mu(P_0) = 2$, we get a contradiction with $m(0) > 0$. We have proved the claim $m(0) = 0$. Consequently we have

$$P_0^v = P_1.$$

We claim that $\Theta_0 = \emptyset$. To prove this claim, first note that, since $\tilde{v}_0$ is asymptotic to P_1, which is a prime orbit, we know that $\tilde{v}_0$ is somewhere injective. Then we have

$$0 \leq \operatorname{ind}(\tilde{v}_0) = 2 - \mu(P_1) - \sum_{z \in \Theta_0} \mu(P_z^v) + \#\Theta_0 = 1 - \sum_{z \in \Theta_0} \mu(P_z^v) + \#\Theta_0.$$

It follows that $\Theta_0 = \{z\}$ and $\mu(P_z^v) = 2$. The orbit $\tilde{P}_z^v$ is not linked to P_3 and has period $< T_3$. By the hypotheses of Theorem 3.5, we have $P_z^v = P_2$. This implies $T_3 \geq T_1 + T_2$, which contradicts the hypothesis $T_3 < 2T_1$.

Now we show that $P_0 = P_2$. By positivity and stability of intersections and the fact that the images of the cylinders $\tilde{w}_n$ do not intersect P_3, we conclude that $v_0(\mathbb{C} \setminus \{0\})$ does not intersect P_3. Since P_1 is not linked to P_3, we conclude that P_0 is not linked to P_3. Since the period of P_0 is $< T_3$ and P_0 is not linked to P_3, by the hypotheses of Theorem 3.5, we conclude that

$$P_0 = P_2.$$

So far, we have proved the following.

Proposition 4.35. *Consider a sequence* $\tilde{w}_n = (c_n, w_n) : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$, *where* $\tilde{w}_n = \tilde{w}_{\sigma_n}$ *and* $\sigma_n \to 0^+$. *There exists a cylinder* $\tilde{w} : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ *asymptotic to* P_3 *at its positive puncture* ∞ *and to* P_2 *at its negative puncture* 0 *and a cylinder* $\tilde{v}_0 : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ *asymptotic to* P_2 *at its positive puncture* ∞ *and to* P_1 *at its negative puncture* 0, *such that, after suitable reparametrizations and* $\mathbb{R}$-*translations, we have*

(i) *Up to subsequence,* $\tilde{w}_n \to \tilde{w}$ *in* $C^\infty_{loc}(\mathbb{C} \setminus \{0\})$ *as* $n \to \infty$.

(ii) *There exist sequences* $\delta_n^+ \to 0^+$ *and* $d_n \in \mathbb{R}$ *such that, up to subsequence,* $\tilde{w}_n(\delta_n \cdot) + d_n \to \tilde{v}_0$ *in* $C^\infty_{loc}(\mathbb{C} \setminus \{0\})$ *as* $n \to \infty$.

A similar statement holds for any sequence $\tau_n \to 1^-$ with $\tilde{w}$ replaced with a cylinder $\tilde{w}'$ with the same asymptotics as $\tilde{w}$ and $\tilde{v}_0$ replaced with a cylinder $\tilde{v}'_0$ with the same asymptotics as $\tilde{v}_0$.

It follows from the uniqueness of the cylinder $\tilde{v}_r$ (Proposition 4.24) that $\tilde{v}_0 = \tilde{v}'_0 = \tilde{v}_r$, up to reparametrization and $\mathbb{R}$-translation. By the surjectivity of the gluing map, we conclude that $\tilde{w} = \tilde{u}_r$. It follows from the uniqueness of the cylinders $\tilde{u}_r$ and $\tilde{u}'_r$ (Proposition 4.7) that $\tilde{w}'$ is either $\tilde{u}_r$ or $\tilde{u}'_r$, up to reparametrization and $\mathbb{R}$-translation.

Proposition 4.36. *Consider a sequence $\tilde{w}_n = \tilde{w}_{\sigma_n}$ in the family $\{\tilde{w}_\tau\}_{\tau \in (0,1)}$ such that $\sigma_n \to 0^+$. Then*

(i) *For any S^1-invariant neighborhood $\mathcal{W}_3$ of the loop $t \mapsto x_3(T_3 t)$ in $C^\infty(\mathbb{R}/\mathbb{Z}, S^3)$, there exists $R_3 \gg 1$ such that for $R \geq R_3$, the loop $t \mapsto w_n(Re^{2\pi i t})$ belongs to $\mathcal{W}_3$.*

(ii) *For any S^1-invariant neighborhood $\mathcal{W}_2$ of the loop $t \mapsto x_2(T_2 t)$ in $C^\infty(\mathbb{R}/\mathbb{Z}, S^3)$, there exist $\epsilon_2 > 0$ and $R_2 > 1$ such that the loop $t \mapsto w_n(Re^{2\pi i t})$ belongs to $\mathcal{W}_2$ for $R_2 \delta_n \leq R \leq \epsilon_2$.*

(iii) *For any S^1-invariant neighborhood $\mathcal{W}_1$ of the loop $t \mapsto x_1(T_1 t)$ in $C^\infty(\mathbb{R}/\mathbb{Z}, S^3)$, there exist $\epsilon_1 > 0$ such that the loop $t \mapsto w_n(\rho e^{2\pi i t})$ belongs to $\mathcal{W}_1$ for $\rho \leq \epsilon_1 \delta_n$.*

(iv) *Given any neighborhood $\mathcal{V} \subset \mathcal{R}_2$ of $w(\mathbb{C} \setminus \{0\}) \cup v_r(\mathbb{C} \setminus 0) \cup P_1 \cup P_2 \cup P_3$, we have $w_n(\mathbb{C} \setminus \{0\}) \subset \mathcal{V}$, for n large.*

A similar statement holds for any sequence $\tilde{w}_{\tau_n}$ such that $\tau_n \to 1^-$, with w replaced by w'.

Proof. We can assume that $\mathcal{W}_i, i = 1, 2, 3$ contains only the periodic orbit $t \mapsto x_i(T_i \cdot)$ modulo S^1-reparametrizations. Let $\mathcal{W}$ be an S^1-invariant neighborhood of the set of periodic orbits $P = (x, T) \in \mathcal{P}(\lambda)$ with $T \leq T_3$, viewed as maps $x_T : S^1 \to S^3$, $x_T(t) = x(Tt)$, such that each of the connected components of $\mathcal{W}$ contains at most one periodic orbit modulo S^1-reparametrizations and such that $\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3 \subset \mathcal{W}$.

Using the normalization condition (4.10) we can apply Lemma 2.29 and find $R_3 \gg 1$ such that for $R \geq R_3$, the loops $\{t \mapsto w_n(Re^{i2\pi t})\}, n \in \mathbb{N}$ belong to $\mathcal{W}$. By the asymptotic behavior of the cylinders $\tilde{w}_n$, we know that for each n, the loop $t \mapsto w_n(Re^{2\pi i t})$ belongs to $\mathcal{W}_3$ for R large enough. We conclude that for any $R \geq R_3$ and n large, the loop $t \mapsto w_n(Re^{2\pi i t})$ belongs to $\mathcal{W}_3$. Assertion *(i)* is proved.

Now we prove *(ii)*. Recall that the asymptotic limit of $\tilde{w}$ at $z = 0$ is the orbit P_2. We can apply 2.29 as in the proof of Proposition 4.32 to find $\epsilon_2 > 0$ small and $R_2 >> 1$ such that for every R satisfying $\delta_n R_2 \leq R \leq \epsilon_2$ and n large, the loop $t \mapsto w_n(Re^{2\pi it})$ belongs to $\mathcal{W}_2$.

To prove *(iii)*, first recall that the mass of the puncture $z = 0$ of the sequence $\tilde{v}_n$ is zero. Applying 2.29 as in the proof of Lemma 4.34, we find $\epsilon_1 > 0$ small and a sequence $\delta'_n \to 0$ such that $t \mapsto w_n(\delta_n \rho' e^{2\pi it}) = v_n(\rho' e^{2\pi it}) \in \mathcal{W}_1$ for $\delta'_n \leq \rho' \leq \epsilon_1$. The sequence $v_n(\delta'_n \cdot)$ converges in C^∞_{loc} to the cylinder over P_1, so that applying Lemma 2.29 again, we conclude that for n large, the loop $t \mapsto v_r(\delta'_n r e^{2\pi it})$ belongs to $\mathcal{W}_1$ for every $r \leq 1$. We conclude that for any $\rho \leq \delta_n \epsilon_1$ and n large enough, the loop $t \mapsto w(\rho e^{2\pi it})$ belongs to $\mathcal{W}_1$.

The proof of *(iv)* is a consequence of *(i)*, *(ii)*, *(iii)*, the convergence of $\tilde{w}_n$ to $\tilde{w}$ in $C^\infty_{loc}(\mathbb{C} \setminus \{0\})$ and the convergence of $\tilde{w}_n(\delta_n \cdot) + d_n$ to $\tilde{v}_r$ in $C^\infty_{loc}(\mathbb{C} \setminus \{0\})$. $\qquad\square$

The only step left to conclude the proof of Proposition 4.28 is to prove that $\tilde{w}' = \tilde{u}'_r$.

Proposition 4.37. $\tilde{w}' = \tilde{u}'_r$.

Proof. Fix $\tau_0 \in (0, 1)$. The surface

$$S := w_{\tau_0}(\mathbb{C} \setminus \{0\}) \cup w(\mathbb{C} \setminus \{0\}) \cup v_r(\mathbb{C} \setminus \{0\}) \cup P_1 \cup P_2 \cup P_3$$

bounds two open connected regions in S^3. One of these open connected regions, that we call A, is contained in $\mathcal{R}_2$, because S does not intersect any of the curves foliating the interior of the region $\mathcal{R}_1$.

We will show that A is foliated by cylinders in the family $\{w_\tau\}$. Let $p \in A$ and let $\mathcal{V} \subset \mathcal{R}_2$ be a small neighborhood of S in $\mathcal{R}_2$ separating p and S. By Proposition 4.36, for $\tau'_0 < \tau_0$ sufficiently small, we have $w_{\tau'_0}(\mathbb{C} \setminus \{0\}) \subset \mathcal{V}$. Also, $w_{\tau'_0}(\mathbb{C} \setminus \{0\}) \subset \mathcal{V} \cap A$, since there are points in the image of the family w_τ converging to points in a compact subset of $\tilde{u}_r(\mathbb{C} \setminus \{0\})$, as $\tau \to 0^+$. Let $B \subset A$ be the region limited by $w_{\tau_0}(\mathbb{C} \setminus \{0\}) \cup P_3 \cup P_1 \cup w_{\tau'_0}(\mathbb{C} \setminus \{0\})$. Thus, the image of the family w_τ, $\tau \in [\tau'_0, \tau_0]$ is open, closed and non empty in $\bar{B}$. It follows that p is in the image of w_τ for some $\tau \in (\tau'_0, \tau_0)$. We conclude that A is foliated by the images of the cylinders $\{w_\tau\}$, $\tau \in (0, \tau_0)$.

Since every neighborhood of a compact set of $u_r(\mathbb{C} \setminus \{0\})$ in $\mathcal{R}_2$ is contained in $\bar{A}$, the family w_τ, $\tau \in (0, \tau_0)$, foliates A and the cylinders in the family w_τ do not intersect each other, we conclude that $\tilde{w}' = \tilde{u}'_r$. $\qquad\square$

The proof of Proposition 4.28 is complete.

4.4.3   The foliation

Proposition 4.38. *The images of $\{w_\tau\}, \tau \in (0,1)$, u_r, u'_r, v_r, P_1, P_2 and P_3 form a singular foliation of the region $\mathcal{R}_2$.*

Proof. From the arguments in the proof of Proposition 4.37, we have foliated a region A in $\mathcal{R}_2$ bounded by

$$S := w_{\tau_0}(\mathbb{C} \setminus \{0\}) \cup w(\mathbb{C} \setminus \{0\}) \cup v_r(\mathbb{C} \setminus 0) \cup P_1 \cup P_2 \cup P_3,$$

where $\tau_0 \in (0,1)$ is fixed. Repeating the same arguments in the proof of Proposition 4.37, with u_r replaced by u'_r, we foliate the complement of A in $\mathcal{R}_2$. $\square$

The proof of Theorem 3.5 is complete.

Chapter 5

Idea of the proof of Theorem 3.5 in case $\int_{\mathbb{D}\setminus\Gamma_0} u_0^* d\lambda > 0$

5.1 Gluing

The gluing process allows us to approximate a broken curve, consisting of two rigid pseudoholomorphic curves sharing a common asymptotic limit, with a family of pseudoholomorphic curves. In this section, we follow the exposition of [Nel13], [Wen16] and [AD14] to state the gluing theorem.

Let $\tilde{J}$ be a regular almost complex structure on the symplectization $\mathbb{R} \times S^3$ and let P_+ and P_- be two closed Reeb orbits. We denote by

$$\mathcal{S}(P_+, P_-)$$

the moduli space of somewhere injective $\tilde{J}$-holomorphic cylinders converging to P_+ at its positive puncture and to P_- at its negative puncture, and $\mathcal{S}(P_+, \emptyset)$ the moduli space of $\tilde{J}$-holomorphic planes asymptotic to P_+.

Remark 5.1. Unless otherwise stated, in what follows the notation $\mathcal{S}(P_+, P_-)$ can also denote the case "$P_- = \emptyset$", so that our statements can refer both to cylinders and planes.

We define the space of *unparametrized trajectories* by

$$\mathcal{M}(P_+, P_-) := \mathcal{S}(P_+, P_-)/(S^1 \times \mathbb{R})$$

(or $\mathcal{M}(P_+, \emptyset) := \mathcal{S}(P_+, \emptyset)/Aut(\mathbb{C})$) and denote by $\pi : \mathcal{S}(P_+, P_-) \to \mathcal{M}(P_+, P_-)$ the quotient map. Consider the $\mathbb{R}$-action on $\mathcal{M}(P_+, P_-)$ by translations $\tilde{u} = (a, u) \mapsto (a + c, u)$, $c \in \mathbb{R}$. We denote the quotient space by

$$\hat{\mathcal{M}}(P_+, P_-) = \mathcal{M}(P_+, P_-)/\mathbb{R}.$$

Theorem 5.2 (Gluing). *Let $P_x = (x, T_x)$, $P_y = (y, T_y)$ and $P_z = (z, T_z)$ (where possibly $P_z = \emptyset$) be closed Reeb orbits. Let $\tilde{u} \in \mathcal{S}(P_x, P_y)$ and $\tilde{v} \in \mathcal{S}(P_y, P_z)$ be parametrized solutions with Fredholm index 1, representing equivalence classes $(\mathcal{C}_u, \mathcal{C}_v) \in \hat{\mathcal{M}}(P_x, P_y) \times \hat{\mathcal{M}}(P_y, P_z)$. Then there exists $R_0 \in \mathbb{R}$ and a differentiable map*

$$\Psi : [R_0, \infty) \to \mathcal{S}(P_x, P_z)$$

such that $\hat{\Psi} = \pi \circ \Psi$ is an embedding

$$\hat{\Psi} : [R_0, \infty) \to \hat{\mathcal{M}}(P_x, P_z)$$

satisfying

$$\lim_{R \to +\infty} \hat{\Psi}(R) = (\mathcal{C}_u, \mathcal{C}_v) \ .$$

Theorem 5.3 (Surjectivity of the gluing map). *With the same notation of Theorem 5.2, assume that the orbit P_y is simply covered. If there exists a sequence $\tilde{w}_n$ in $\hat{\mathcal{M}}(P_x, P_z)$ converging to $(\mathcal{C}_u, \mathcal{C}_v)$, then $\tilde{w}_n$ belongs to the image of $\hat{\Psi}$, for n sufficiently large.*

5.1.1   Pregluing

Given $[\tilde{u}] \in \mathcal{M}(P_+^u, P_-^u)$ and $[\tilde{v}] \in \mathcal{M}(P_+^v, P_-^v)$ such that $P_-^u = P_+^v$, we will construct a one parameter family of curves

$$\bar{w}_R = \tilde{u} \#_R \tilde{v} \ ,$$

that are *not* $\tilde{J}$-holomorphic, but are asymptotic to P_+^u and P_-^v. For each R fixed, the curve $\bar{w}_R$ is called *preglued map*.

Let $P_-^u = P_+^v = (y, T_y)$, $P_+^u = (x, T_x)$ and $P_-^v = (z, T_z)$. We are assuming that, after fixing $p = y(0)$, cylindrical coordinates (s, t) near $-\infty$ for $\tilde{u}$ and (s', t') near $+\infty$ for $\tilde{v}$, we have

$$\lim_{s \to -\infty} u(s, t) = \lim_{s' \to +\infty} v(s', t) = y(Tt) \ .$$

Take a Riemannian metric g on M, that can be defined by $g = \lambda \cdot \lambda + d\lambda(\pi\cdot, J\pi\cdot)$. Writing in coordinates

$$\tilde{u}(s, t) = (a(s, t), u(s, t)), \qquad \tilde{v}(s', t') = (b(s', t'), v(s', t')),$$

we have

$$
\begin{aligned}
a(s,t) &= Ts + \eta(s,t) \\
u(s,t) &= exp_{y(Tt)} U(s,t), \quad \text{for } s \ll 0 \\
b(s,t) &= Ts' + \eta'(s',t') \\
v(s',t') &= exp_{y(Tt)} V(s',t'), \quad \text{for } s \gg 0
\end{aligned}
\tag{5.1}
$$

where U and η decay exponentially as $s \to -\infty$ and V, η' decay exponentially as $s' \to +\infty$. This follows from the asymptotic behavior of $\tilde{u}$ and $\tilde{v}$ near the punctures. See Theorem 2.11.

To define the preglued map, we need two cutoff functions β_R^+ and β_R^- in $C^\infty(\mathbb{R}, [0,1])$ satisfying for $0 < \epsilon < \frac{R}{2}$ fixed

$$
\beta_R^-(s) = \begin{cases} 1, & s \le -R \\ 0, & s \ge -\epsilon \end{cases} \qquad \beta_R^+(S) = \begin{cases} 1, & s \ge R \\ 0, & s \le \epsilon \end{cases}
\tag{5.2}
$$

Fix parametrizations of $\tilde{u}$ and $\tilde{v}$ and parametrization of the orbit (y,T) such that (5.1) holds.

Define the preglued map $\bar{w}_R(s,t) = (c_R(s,t), w_R(s,t))$ by

$$
w_R(s,t) = \begin{cases} v(s+2R,t), & s \le -R \\ exp_{y(Tt)}\left(\beta_R^-(s)V(s+2R,t) + \beta_R^+(s)U(s-2R,t)\right), & s \in [-R,R] \\ u(s-2R,t), & s \ge R \end{cases}
$$

$$
c_R(s,t) = \begin{cases} b(s+2R,t) - 2RT, & s \le -R \\ Ts + \beta_R^-(s)\eta'(s+2R,t) + \beta_R^+(s)\eta(s-2R,t), & s \in [-R,R] \\ a(s-2R,t) + 2RT, & s \ge R \end{cases}
$$

The map $\bar{w}_R$ is defined for R sufficiently large, such that for $s \in [-R, R]$, $u(s-2R,t) = exp_{y(Tt)} U(s-2R,t)$ and $v(s+2R,t) = exp_{y(Tt)} V(s+2R,t)$.

If $\tilde{v}$ is a plane, we have not defined $\bar{w}_R(0)$ yet. Recall that we write $\tilde{v}(s,t) = v(e^{2\pi(s+it)})$. We define for $z \in \overline{B_{e^{-2\pi R}}(0)}$,

$$
\bar{w}_R(z) = \tau_{-2RT} \circ \tilde{v}(e^{4\pi R}z),
$$

where the map $\tau_c : \mathbb{R} \times S^3 \to \mathbb{R} \times S^3$ is defined by $\tau_c(a,x) = (a+c, x)$, for $c \in \mathbb{R}$.

Properties of the preglued map

- $\bar{w}_R$ is asymptotic to P_+^u at $z = \infty$ and asymptotic to P_-^v at $z = 0$.

- for $s \in [-\epsilon, \epsilon]$, $\bar{w}_R(s,t) = (Ts, y(Tt))$.

- If $s \leq R$, $\bar{w}_R(s - 2R, t) = (b(s,t) - 2RT, v(s,t))$. It follows that $\tau_{2RT} \circ \bar{w}_R(s - 2R, t) \to \tilde{v}(s,t)$ in C_{loc}^∞ as $R \to \infty$.

- If $s \geq -R$, $\bar{w}_R(s + 2R, t) = (a(s,t) + 2RT, u(s,t))$. It follows that $\tau_{-2RT} \circ \bar{w}_R(s + 2R, t) \to \tilde{u}(s,t)$ in C_{loc}^∞ as $R \to \infty$.

- $\bar{w}_R$ converges to $(Ts, y(Tt))$ as $R \to \infty$ in C_{loc}^∞.

- $\bar{\partial}\bar{w}_R \to 0$ in C_{loc}^∞.

Construction of the gluing map   Using a version of the implicit function theorem, one can show that there exists a distinguished $\tilde{J}$-holomorphic curve $\tilde{w}_R = \Psi(R) \in \mathcal{S}(P_+^u, P_-^v)$ close to the preglued map $\bar{w}_R$ for R sufficiently large, which has the form $\Psi(R) = \exp_{\bar{w}_R}(\eta(R))$, where $\eta(R) \in W_\delta^{1,p}(\bar{w}_R^* TW)$. This can be proved by the same arguments found in [AD14] and the Fredholm theory of [Dra04].

5.2   Idea of the proof of Theorem 3.5 in the case $\int_{\mathbb{D}\backslash\Gamma_0} u_0^* d\lambda > 0$

In Section 3.3.4, we obtained a $\tilde{J}$-holomorphic curve $\tilde{u}_0 : \mathbb{D}\backslash\{0\} \to \mathbb{R} \times S^3$ satisfying

$$
\begin{cases}
\tilde{u}_0 = (a_0, u_0) : \mathbb{D} \setminus \{0\} \to \mathbb{R} \times S^3 \text{ is an embedding },\\
a_0 \equiv 0 \text{ on } \partial\mathbb{D}, \ u_0(\partial\mathbb{D}) \subset \mathcal{D} \setminus \{e\},\\
u_0(\partial\mathbb{D}) \text{ winds once positively around } e,\\
\tilde{u}_0 \text{ is asymptotic to } P_2 \text{ at } z = 0,
\end{cases}
\tag{5.3}
$$

where $\mathcal{D}$ is the disk spanning the orbit P_3 obtained by Theorem 3.10. We also obtained a $\tilde{J}$-holomorphic plane $\tilde{v} : \mathbb{C} \to \mathbb{R} \times S^3$ asymptotic to P_2.

By Proposition 4.20, we obtain a $\tilde{J}$-holomorphic cylinder $\tilde{v}_r : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ asymptotic to P_2 at its positive puncture $z = \infty$ and to P_1 at its negative puncture $z = 0$. By Proposition 4.24, we know that $\tilde{v}$ and $\tilde{v}_r$ approach the orbit P_2 in opposite directions, according to definition 4.6.

Now we can apply the Gluing theorem 5.2 to the curves $\tilde{u}_0$ and $\tilde{v}_r$ to obtain a 1-parameter family of $\tilde{J}$-holomorphic punctured disks

$$
\tilde{u}_R = (a_R, u_R) : \mathbb{D} \setminus \{0\} \to \mathbb{R} \times S^3, \ R \in [R_0, +\infty)
\tag{5.4}
$$

satisfying

$$\begin{cases} a_R \equiv 0 \text{ on } \partial\mathbb{D}, \ u_R(\partial\mathbb{D}) \subset \mathcal{D} \setminus \{e\}, \\ u_R(\partial\mathbb{D}) \text{ winds once and positively around } e, \\ \tilde{u}_R(\mathbb{D} \setminus \{0\}) \cap (\mathbb{R} \times x_3(\mathbb{R})) = \emptyset, \\ \tilde{u}_R \text{ is asymptotic to } P_1 \text{ at } z = 0. \end{cases} \tag{5.5}$$

We need the following proposition, whose proof is not given in this thesis.

Proposition 5.4. *Let $\tilde{u} = (a, u) : \mathbb{D} \setminus \{0\} \to \mathbb{R} \times S^3$ be a $\tilde{J}$-holomorphic curve satisfying*

$$\begin{cases} a \equiv 0 \text{ on } \partial\mathbb{D}, \\ u(\partial\mathbb{D}) \subset \mathcal{D} \setminus \{e\}, \\ \tilde{u} \text{ is asymptotic to } P_2 \text{ at } z = 0, \end{cases} \tag{5.6}$$

Consider two $\tilde{J}$-holomorphic curves $\tilde{v} : \mathbb{C} \setminus \Gamma_v \to \mathbb{R} \times S^3$ and $\tilde{v}' : \mathbb{C} \setminus \Gamma_{v'} \to \mathbb{R} \times S^3$, asymptotic to P_2 at its unique positive puncture $z = \infty$, with $\#\Gamma_v, \#\Gamma_{v'} \leq 1$ and such that $\tilde{v}$ and $\tilde{v}'$ approach P_2 in opposite directions. Let $\tilde{w}_R : \mathbb{D} \setminus \Gamma_v \to \mathbb{R} \times S^3$, $R \geq R_0$ be the family of $\tilde{J}$-holomorphic curves given by Theorem 5.2 applied to the curves $\tilde{u}$ and $\tilde{v}$, and let $\tilde{w}_R' : \mathbb{D} \setminus \Gamma_v \to \mathbb{R} \times S^3$, $R \geq R_0'$ be the family of $\tilde{J}$-holomorphic curves given by Theorem 5.2 applied to the curves $\tilde{u}$ and $\tilde{v}'$. Then for R and s sufficiently large, the following holds: If $z = e^{2\pi(s+it)}$ and $\tilde{w}_R(z) = \exp_{\tilde{u}\#_R\tilde{v}} V(z)$, where V is a section on $\tilde{u}\#_R\tilde{v}^* T(\mathbb{R} \times S^3)$, then $\tilde{w}_R'(z) = \exp_{\tilde{u}\#_R\tilde{v}'}(-V(z))$.

By surjectivity of gluing (Theorem 5.3), we know that the family of disks given by Theorem 5.2 applied to the curves $\tilde{u}_0$ and $\tilde{v}$ coincide with the family of disks obtained in Section 3.3.2.

Recall that in Section 3.3.2, we fixed a leaf l_1 of the characteristic foliation of $\mathcal{D}$ and denoted its length by $\bar{\tau}$. We defined for each disk $\tilde{u} \in \mathcal{M}(J)$,

$$\tau([u]) = \text{ length of the piece of } l_1 \text{ connecting } e \text{ to } p_*(\tilde{u})$$

where $p_*(\tilde{u})$ is the intersection point of l_1 and $u(\partial\mathbb{D})$. $\tau(u)$ is well defined because $u(\partial\mathbb{D})$ hits every leaf exactly once. In the same way we can define define $\tau^* = \tau(\tilde{u}_0) = \text{ length of the piece of } l_1 \text{ connecting } e \text{ to } p_*(\tilde{u}_0)$, where $p_*(\tilde{u}_0)$ is the intersection point of l_1 and $u_0(\partial\mathbb{D})$.

Now we apply Proposition 5.4 to the curves $\tilde{u}_0$, $\tilde{v}$ and $\tilde{v}_r$. One can show that for R large, any curve $\tilde{u}_R$ in the family (5.4) satisfies $\tau([\tilde{u}_R]) > \tau^*$, where we also define $\tau([\tilde{u}_R]) = \text{ length of the piece of } l_1 \text{ connecting } e \text{ to } p_*(\tilde{u}_R)$ and $p_*(\tilde{u}_R)$ is the intersection point of l_1 and $u_R(\partial\mathbb{D})$.

The family of cylinders $\tilde{u}_R$, $R \in [R_0, +\infty)$ extends to a maximal family of $\tilde{J}$-holomorphic curves satisfying (5.5), that we also denote by $\tilde{u}_R$, $R \in (R_-, +\infty)$.

We need the following facts about the family $\tilde{u}_R$, that are left without proof.

1. $\tilde{u}_R$ is an embedding near $\partial\mathbb{D}$;

2. $\tilde{u}_R$ is an immersion for all $R \in (R_-, +\infty)$;

3. for $R \neq R'$, $\tilde{u}_R(\partial\mathbb{D}) \cap \tilde{u}_{R'}(\partial\mathbb{D}) = \emptyset$.

Consider a sequence $\tilde{u}_n = \tilde{u}_{R_n}$, such that $R_n \to R_-^+$. After an analysis similar to the one done in Sections 4.4.1 and 4.4.2, we can show that, after suitable reparametrizations and $\mathbb{R}$-translations, we have the following

(i) There exists a $\tilde{J}$-holomorphic curve $\tilde{u}_1 : \mathbb{D} \setminus \{0\} \to \mathbb{R} \times S^3$ such that, up to subsequence, $\tilde{u}_n \to \tilde{u}_1$ in $C_{loc}^\infty(\mathbb{D} \setminus \{0\})$ as $n \to \infty$.

(ii) There exist sequences $\delta_n \to 0^+$, $R_n \to +\infty$ and $d_n \in \mathbb{R}$, and a $\tilde{J}$-holomorphic curve $\tilde{v}_1 : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$ such that, up to subsequence, defining $\tilde{w}_n : B_{R_n}(0) \setminus \{0\} \to \mathbb{R} \times S^3$ by $\tilde{w}_n = \tilde{u}_n(\delta_n \cdot) + d_n$, we have $\tilde{w}_n \to \tilde{v}_1$ in $C_{loc}^\infty(\mathbb{C} \setminus \{0\})$ as $n \to \infty$.

Note that, since each curve $\tilde{u}_R$ satisfies $\tau([\tilde{u}_R]) > \tau^*$ and $\tau([\tilde{u}_R])$ is a decreasing function of R, we have $\tau([\tilde{u}_1]) > \tau^*$.

If $\pi \circ du_1 \equiv 0$, then, by the same arguments of Proposition 3.18, we know that $\tilde{u}_1$ is a reparametrization of the $\tilde{J}$-holomorphic curve $F : \mathbb{D} \setminus \{0\} \to \mathbb{R} \times S^3$, $F(z = e^{2\pi(s+it)}) = (T_3 s, x_3(T_3 t))$, and $\tilde{v}_1$ is a cylinder asymptotic to P_3 at its unique positive puncture $z = \infty$ and asymptotic to either P_2 or P_1 at its unique negative puncture $z = 0$.

If $\int_{\mathbb{D}\setminus\{0\}} u_1^* d\lambda > 0$, then $\tilde{u}_1$ is asymptotic to P_2 at its unique negative puncture $z = 0$ and we can apply the Gluing theorem 5.2 to the curves $\tilde{u}_1$ and $\tilde{v}$ to obtain a family of disks

$$\tilde{u}_R^1 : \mathbb{D} \to \mathbb{R} \times S^3, \ R \in [R_1, +\infty),$$

such that for each R, $\tau([\tilde{u}_R^1]) > \tau([\tilde{u}_1]) > \tau^*$.

We claim that there exists a family of $\tilde{J}$-holomorphic disks $\tilde{u}_n = (a_n, u_n) : \mathbb{D} \to \mathbb{R} \times S^3$ satisfying

$$\begin{cases} a_n \equiv 0 \ \text{ on } \ \partial\mathbb{D}, \ \ u_n(\partial\mathbb{D}) \subset \mathcal{D} \setminus \{e\}, \\ u_n(\partial\mathbb{D}) \ \text{ winds once positively around } \ e, \\ \tilde{u}_n(\mathbb{D} \setminus \{0\}) \cap (\mathbb{R} \times x_3(\mathbb{R})) = \emptyset. \end{cases} \qquad (5.7)$$

such that $\tau([\tilde{u}_n]) \to \bar{\tau}$ as $n \to \infty$, or a family of $\tilde{J}$-holomorphic punctured disks $\tilde{w}_n = (c_n, w_n) : \mathbb{D} \backslash \{0\} \to \mathbb{R} \times S^3$ satisfying

$$\begin{cases} c_n \equiv 0 \text{ on } \partial\mathbb{D}, \ w_n(\partial\mathbb{D}) \subset \mathcal{D} \backslash \{e\}, \\ w_n(\partial\mathbb{D}) \text{ winds once and positively around } e, \\ \tilde{w}_n \text{ is asymptotic to } P_1 \text{ at } z = 0. \end{cases} \tag{5.8}$$

such that $\tau([\tilde{w}_n]) \to \bar{\tau}$ as $n \to \infty$. The argument to proof the claim is as follows.

Suppose, by contradiction, that no such families of $\tilde{J}$-holomorphic curves exist. Let τ_∞ be the supremum of $\tau([\tilde{u}])$ over all $\tilde{J}$-holomorphic disks $\tilde{u} : \mathbb{D} \to \mathbb{R} \times S^3$ satisfying (5.7) and let $\tilde{u}_n$ be a sequence of disks satisfying (5.7) and such that $\tau([\tilde{u}_n]) \to \tau_\infty$ as $n \to \infty$. Repeating the analysis done in Section 3.3.2, we find a $\tilde{J}$-holomorphic punctured disk

$$\tilde{u}_\infty : \mathbb{D} \backslash \{0\} \to \mathbb{R} \times S^3$$

asymptotic to P_2 at $z = 0$ such that $\tilde{u}_n \to \tilde{u}_\infty$ in $C^\infty_{loc}(\mathbb{D} \backslash \{0\})$ as $n \to \infty$ and $\tau(\tilde{u}_\infty) = \tau_\infty$. Also, there exist sequences $\delta_n \to 0^+$ and $d_n \in \mathbb{R}$ such that, up to subsequence, we have $\tilde{u}_n(\delta_n \cdot) + d_n \to \tilde{v}$ in $C^\infty_{loc}(\mathbb{C})$ as $n \to \infty$. Applying the Gluing theorem 5.2 to the punctured disk $\tilde{u}_\infty$ and the cylinder $\tilde{v}_r$, we obtain a family of $\tilde{J}$-holomorphic punctured disks

$$\tilde{u}_R^\infty = (a_R^\infty, u_R^\infty) : \mathbb{D} \backslash \{0\} \to \mathbb{R} \times S^3 \, , R \in [R^\infty, +\infty)$$

satisfying

$$\begin{cases} a_R^\infty \equiv 0 \text{ on } \partial\mathbb{D}, \ u_R^\infty(\partial\mathbb{D}) \subset \mathcal{D} \backslash \{e\}, \\ u_R^\infty(\partial\mathbb{D}) \text{ winds once and positively around } e, \\ \tilde{u}_R(\mathbb{D} \backslash \{0\}) \cap (\mathbb{R} \times x_3(\mathbb{R})) = \emptyset, \\ u_R^\infty \text{ is asymptotic to } P_1 \text{ at } z = 0. \end{cases} \tag{5.9}$$

Applying Proposition 5.4 to the curves $\tilde{u}_\infty$, $\tilde{v}$ and $\tilde{v}_r$, one can show that for R large, we have $\bar{\tau} > c > \tau([\tilde{u}_R^\infty]) > \tau_\infty$.

The family $\tilde{u}_R^\infty$ extends to a maximal family of $\tilde{J}$-holomorphic punctured disks satisfying (5.9), that we also denote by $\tilde{u}_R^\infty$, $R \in (R_-, +\infty)$. Taking the limit as $R \to R_-$, we obtain a $\tilde{J}$-holomorphic punctured disk $\tilde{w}_\infty : \mathbb{D} \backslash \{0\} \to \mathbb{R} \times S^3$ asymptotic to P_2 at its unique negative puncture and satisfying

$$\bar{\tau} > \tau([\tilde{w}_\infty]) > \tau_\infty.$$

Gluing $\tilde{w}_\infty$ with the plane $\tilde{v}$ and applying Proposition 5.4, one can show that there exists a family of disks $\tilde{w}_R : \mathbb{D} \to \mathbb{R} \times S^3$ satisfying (5.7) and such that $\tau([\tilde{w}_R]) > \tau_\infty$, a contradiction.

We conclude that there exists either

1. a family of $\tilde{J}$-holomorphic disks $\tilde{u}_n = (a_n, u_n) : \mathbb{D} \to \mathbb{R} \times S^3$ satisfying (5.7) and such that $\tau([\tilde{u}_n]) \to \bar{\tau}$ as $n \to \infty$. Following the same arguments used in Section 3.3.2, after suitable reparametrizations and $\mathbb{R}$-translations, we have the following:

   (i) Passing to subsequence, $\tilde{u}_n \to F$ in $C_{loc}^\infty(\mathbb{D}\backslash\{0\})$ as $n \to \infty$, where $F(z = e^{2\pi(s+it)}) = (T_3 s, x_3(T_3 t))$;

   (ii) There exist sequences $\delta_n \to 0^+$, $R_n \to +\infty$ and $d_n \in \mathbb{R}$, and a $\tilde{J}$-holomorphic curve $\tilde{v}_\infty : \mathbb{C} \backslash \Gamma_\infty \to \mathbb{R} \times S^3$ such that, up to subsequence, defining $\tilde{w}_n : B_{R_n}(0) \to \mathbb{R} \times S^3$ by $\tilde{w}_n = \tilde{u}_n(\delta_n \cdot) + d_n$, we have $\tilde{w}_n \to \tilde{v}_\infty$ in $C_{loc}^\infty(\mathbb{C} \backslash \Gamma_\infty)$ as $n \to \infty$. The curve $\tilde{v}_\infty$ is either a plane asymptotic to P_3 or $\Gamma_\infty = \{0\}$ and $\tilde{v}_\infty$ is a cylinder asymptotic to P_3 at its positive puncture ∞ and to P_2 at its negative puncture $z = 0$;

   (iii) If $\Gamma_\infty = \{0\}$, there exist sequences $\delta_n' \to 0^+$ and $d_n' \in \mathbb{R}$ such that, up to subsequence, we have $\tilde{u}_n(\delta_n' \cdot) \to \tilde{v}$ in $C_{loc}^\infty(\mathbb{C})$ as $n \to \infty$.

2. or a family of $\tilde{J}$-holomorphic punctured disks $\tilde{u}_n = (a_n, u_n) : \mathbb{D} \backslash \{0\} \to \mathbb{R} \times S^3$ satisfying (5.8) and such that $\tau([\tilde{u}_n]) \to \bar{\tau}$ as $n \to \infty$. Following the same arguments used in Sections 3.3.2 and 4.4.1, after suitable reparametrizations and $\mathbb{R}$-translations, we have the following

   (i) Passing to subsequence, $\tilde{u}_n \to F$ in $C_{loc}^\infty(\mathbb{D}\backslash\{0\})$ as $n \to \infty$, where $F(z = e^{2\pi(s+it)}) = (T_3 s, x_3(T_3 t))$;

   (ii) There exist sequences $\delta_n \to 0^+$, $R_n \to +\infty$ and $d_n \in \mathbb{R}$, and a $\tilde{J}$-holomorphic curve $\tilde{v}_\infty : \mathbb{C} \backslash \{0\} \to \mathbb{R} \times S^3$ such that, up to subsequence, we have $\tilde{u}_n(\delta_n \cdot) \to \tilde{v}_\infty$ in $C_{loc}^\infty(\mathbb{C} \backslash \Gamma_\infty)$ as $n \to \infty$. The curve $\tilde{v}_\infty$ is either a cylinder asymptotic to P_3 at its positive puncture $z = \infty$ and to P_1 at its negative puncture $z = 0$ or a cylinder asymptotic to P_3 at its positive puncture ∞ and to P_2 at its negative puncture $z = 0$;

   (iii) If $\tilde{v}$ is asymptotic to P_2 at $z = 0$, there exist sequences $\delta_n' \to 0^+$ and $d_n' \in \mathbb{R}$ such that, up to subsequence, we have $\tilde{u}_n(\delta_n' \cdot) \to \tilde{v}_r$ in $C_{loc}^\infty(\mathbb{C} \backslash \{0\})$ as $n \to \infty$.

In both cases, we can prove Theorem 3.5 by the same arguments of Chapter 4.

Appendix A

Proof of Lemma 3.9

In this Appendix, we prove Lemma 3.9, which is restated below.

Lemma A.1. *Assume that $\psi : S^2 \rightarrow S^3$ is a C^1 embedding such that $x_2(T_2 \cdot) = \psi|_{S^1 \times \{0\}}$ and such that each hemisphere is a strong transverse section. Then ψ is transverse to the torus T along P_2.*

Let $P = (x, T) \in \mathcal{P}(\lambda)$ be a prime orbit such that $\mu(P) = 2$. The orbit P is hyperbolic and lies in the intersection of its stable manifold $W^+(P)$ and its unstable manifold $W^-(P)$. The tangent spaces of $W^\pm(P)$ along the periodic solution $t \mapsto x(t)$ are spanned by the Reeb vector field $R(x(t))$ and vector fields $v^\pm(t) \in \xi_{x(t)}$.

Fix a symplectic trivialization $\Psi : x_T^* \xi \rightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2$. Let

$$\Phi(t) = \Psi_{t/T} \circ d\varphi^t|_{\xi(x(0))} \circ \Psi_0^{-1}.$$

We know that $\Phi(T)$ has two eigenvalues β, β^{-1}, where $\beta > 1$.

For each t, $v^-(t)$ is an eigenvector of $d\varphi^T|\xi(x(t))$ associated with the eigenvalue β. In the same way, $v^+(t)$ is an eigenvector of $d\varphi^T|\xi(x(t))$ associated with the eigenvalue β^{-1}. Changing $v^\pm(t)$ with $-v^\pm(t)$ if necessary, we can assume $\{v^-(t), v^+(t)\}$ is a positive basis for $\xi_{x(t)}$ for each t. The basis $\{v^-(t), v^+(t)\}$ determines four open quadrants in $\xi_{x(t)}$. Let (I) and (III) be the open quadrants between $\mathbb{R}v^-(t)$ and $\mathbb{R}v^+(t)$ following the positive orientation and (II) and (IV) the open quadrants between $\mathbb{R}v^+(t)$ and $\mathbb{R}v^-(t)$.

Proposition A.2. *Let $t \mapsto v(t) \in \xi_{x(Tt)}$ be a section such that*

$$\mathrm{wind}(v, \Psi) = 1 .$$

If $d\lambda(v(t), \mathcal{L}_R v(t)) > 0, \ \forall t \in S^1$, then $v(t)$ belongs to regions (I) or (III) for every $t \in S^1$. If $d\lambda(v(t), \mathcal{L}_R v(t)) < 0, \ \forall t \in S^1$, then $v(t)$ belongs to regions (II) or (IV) for every $t \in S^1$.

Proof. Let $S(t) = -J_0\dot{\Phi}(t)\Phi^{-1}(t)$. Recall that $t \mapsto S(t)$ is T-periodic, and for each t, $S(t)$ is real and symmetric. Define $A(t) = J_0 S(t)$.

The sections $v^{\pm}(t)$ are solutions of

$$\dot{x}(t) = A(t)x(t) \tag{A.1}$$

satisfying

$$v^+(T) = \frac{1}{\beta}v^+(0), \quad v^-(T) = \beta v^-(0) \ .$$

We will use Floquet theory to change coordinates so that the equation (A.1) changes to

$$\dot{y}(t) = By(t),$$

with B constant.

In the basis $\{v^-(0), v^+(0)\}$, $\Phi(T)$ has the form

$$\Phi(T) = \begin{bmatrix} \beta & 0 \\ 0 & \frac{1}{\beta} \end{bmatrix}$$

We want to find a matrix B satisfying $e^{TB} = \Phi(T)$ and a T-periodic map $t \mapsto P(t)$ such that $\Phi(t) = P(t)e^{tB}$, $\forall t \in \mathbb{R}$, so that with the change of variables $y = P^{-1}(t)x$, the equation (A.1) becomes $\dot{y}(t) = By(t)$.

Define

$$B = \frac{1}{T}\begin{bmatrix} \ln\beta & 0 \\ 0 & -\ln\beta \end{bmatrix},$$

and $t \mapsto P(t)$ by

$$\Phi(t) = P(t)e^{tB}.$$

The map $t \mapsto P(t)$ is T-periodic. In fact,

$$\Phi(t)e^{TB} = \Phi(t)\Phi(T) = \Phi(t+T) = P(t+T)e^{tB}e^{TB} \Rightarrow \Phi(t) = P(t+T)e^{tB}.$$

If $x(t)$ is a solution of $\dot{x} = A(t)x(t)$, then $y(t) = P(t)^{-1}x(t)$ satisfy

$$\begin{aligned}
\dot{y}(t) &= (\dot{P(t)^{-1}})x(t) + P(t)^{-1}\dot{x}(t) \\
&= -P(t)^{-1}\dot{P}(t) + P(t)^{-1}(P(t)y(t)) + P(t)^{-1}A(t)(P(t)y(t)) \\
&= (-P(t)^{-1}\dot{P}(t) + P(t)^{-1}A(t)P(t))y(t).
\end{aligned}$$

Using the identities $\dot{\Phi}(t) = A(t)\Phi(t)$ and $\Phi(t) = P(t)e^{tB}$, we get

$$A(t)P(t)e^{tB} = \dot{\Phi}(t) = P(t)e^{tB}B + \dot{P}(t)e^{tB}$$
$$\Rightarrow P(t)B + \dot{P}(t) = A(t)P(t)$$
$$\Rightarrow B = -P(t)^{-1}\dot{P}(t) + P(t)^{-1}A(t)P(t).$$

Then $\dot{y}(t) = By(t)$. In the same way, if $y(t)$ solves $\dot{y}(t) = By(t)$, then $x(t) = P(t)y(t)$ solves (A.1).

In the basis $\{P^{-1}(t)v^-(0), P^{-1}(t)v^+(0)\}$, the equation $\dot{y}(t) = By(t)$ becomes

$$\begin{cases} \dot{y}_1(t) = \lambda y_1(t) \\ \dot{y}_2(t) = -\lambda y_2(t), \end{cases}$$

where $\lambda = \frac{1}{T}\ln\beta$.

Writing

$$\begin{cases} y_1(t) = \rho(t)\cos\eta(t) \\ y_2(t) = \rho(t)\sin\eta(t), \end{cases}$$

We get

$$\rho^2(t)\dot{\zeta}(t) = \rho^2(t)\left(-2\lambda\sin\zeta(t)\cos\zeta(t)\right). \tag{A.2}$$

Let $t \mapsto v(t) \in \xi_{x(t)}$ be a section on $x^*\xi$. We want to compute

$$d\lambda(v(t), \mathcal{L}_R v(t)).$$

In the symplectic trivialization Ψ, $\mathcal{L}_R v(t)$ takes the form

$$\mathcal{L}_R v(t) = \frac{d}{dt}v(t) - J_0 S(t)v(t).$$

So that

$$d\lambda(v(t), \mathcal{L}_R v(t)) = d\lambda_0(v(t), \dot{v}(t) = J_0 S(t)v(t))$$
$$= d\lambda_0(v(t), \dot{v}(t)) - d\lambda_0(v(t), A(t)v(t)).$$

Writing $v(t)$ in the basis $\{v^-(0), v^+(0)\}$ as

$$v(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t)),$$

we have

$$d\lambda_0(v(t), \dot{v}(t)) = Cr(t)^2\dot{\theta}(t),$$

where C is a positive constant.

For t fixed, let $s \mapsto x_t(s)$ be the solution of (A.1) such that $x_t(t) = v(t)$.

Writing $x_t(s) = x_1(s)v^-(0) + x_2(s)v^+(0)$, we have

$$P^{-1}(t)v(t) = x_1(t)(P(t)^{-1}v^-(0)) + x_2(P(t)^{-1}v^+(0)),$$

so that the functions x_1 and x_2 satisfy

$$\begin{cases} \dot{x}_1(t) = \lambda x_1(t) \\ \dot{x}_2(t) = -\lambda x_2(t), \end{cases}$$

Then
$$\begin{aligned} A(t)v(t) = A(t)x_t(t) &= \dot{x}_t(t) \\ &= \dot{x}_1(t)v^-(0) + \dot{x}_2(t)v^+(0) \\ &= \lambda x_1(t)v^-(0) - \lambda x_2(t)v^+(0). \end{aligned}$$

It follows that

$$\begin{aligned} d\lambda_0(v(t), A(t)v(t)) &= C\left(-\lambda v_1(t)v_2(t) - \lambda v_1(t)v_2(t)\right) \\ &= C\left(-2\lambda r^2(t)\cos\theta(t)\sin\theta(t)\right) \\ &= C\left(-2\lambda\rho^2(t)\cos\eta(t)\sin\eta(t)\right) \\ &= C\rho^2(t)\dot{\eta}(t), \end{aligned}$$

where $x_1(s) = \rho(s)\cos\eta(s)$, $x_2 = \rho(s)\sin\eta(s)$. The last equality follows from (A.2).

We conclude that

$$d\lambda(v(t), \mathcal{L}_R(t)) = Cr^2(t)(\dot{\theta}(t) - \dot{\eta}(t)). \tag{A.3}$$

Assume that $t \mapsto v(t) \in \xi_{x(t)}$ satisfy $\text{wind}(v, \Phi) = 1$ and

$$d\lambda(v(t), \mathcal{L}_R v(t)) > 0, \ \forall t \in \mathbb{R}\ .$$

By (A.3), we have
$$\dot{\theta}(t) > \dot{\eta}(t), \ \forall t \in \mathbb{R}.$$

Note that $\text{wind}(v^\pm(t), \Psi) = 1$. This follows from $\mu(P) = 2$ and the geometric description of the Conley-Zehnder index given in 1.2.1. This implies that

$$\text{wind}(v(t), P(t)^{-1}v^\pm(0)) = \text{wind}(v(t), v^\pm(t)) = 0\ .$$

Now we show that for all $t \in \mathbb{R}$, $v(t)$ lies in regions (I) or (II). In the regions (II) and (IV), we have, by (A.2), $\dot{\theta}(t) > \dot{\eta}(t) > 0$. So that the existence of t such that $v(t) \in$ (II) or (IV) would force $\text{wind}(v(t), P(t)^{-1}v^\pm(0)) \geq 1$. Suppose that, for some t_0, $v(t_0)$ is in the direction of $P^{-1}(t_0)v^+(0)$, then, using

(A.2), we have $\dot{\theta}(t) > \dot{\eta}(t_0)$. But this would force $v(t)$ to go to (II) or (IV), which is impossible. Suppose that $v(t_0)$ is in the direction of $P^{-1}(t_0)v^-(0)$. Then $\dot{\theta}(t_0) > \dot{\eta}(t_0) = 0$. This implies that $\theta(t)$ is growing at t_0. Since wind$(v(t), P(t)^{-1}v^\pm(0)) = 0$, this would force the existence of $t_1 > t_0$ such that $\theta(t_1) = \theta(t_0)$ and $\dot{\theta}(t_1) < 0$, a contradiction with $\dot{\theta}(t_1) > \dot{\eta}(t_1) = 0$. We conclude that $v(t)$ lies in regions (I) or (II), for every $t \in \mathbb{R}$.

By a similar argument, we conclude that if $t \mapsto v(t) \in \xi_{x(t)}$ satisfies wind$(v, \Phi) = 1$ and $d\lambda(v(t), \mathcal{L}_R v(t)) > 0$, $\forall t \in \mathbb{R}$, then for every $t \in \mathbb{R}$, we have $v(t) \in$ (II) or $v(t) \in$ (IV). $\qquad \square$

Lemma A.3. *Let* $t \mapsto \eta(t) \in \xi$ *be a vector field along* $x_2(T_2 \cdot)$ *such that* $\{\eta(t), R(x_2(T_2 t))\}$ *generates* $d\Psi(TS^2)$ *along* $x_2(T_2 \cdot)$ *and let* $t \mapsto \eta'(t) \in \xi$ *be a section along* $x_2(T_2 \cdot)$ *such that* $\{\eta'(t), R(x_2(T_2 t))\}$ *generates the tangent space of* T *along* $x_2(T_2 \cdot)$. *Then* wind$(\eta', \Psi) = $ wind$(\eta, \Psi) = 1$.

Proof. Using the fact that ψ restricted to each hemisphere of S^2 is transverse to the Reeb flow, we can follow the arguments of [HSa11, Section 6] to conclude that

$$\text{wind}(\eta, \psi) = 1.$$

In the same way, if ν is a section such that $\{\nu(t), R(x_2(t))\}$ generates the tangent space of the disk D along x_2, where D is the disk given by the definition of 3-2-1 foliation 3.4, we have

$$\text{wind}(\nu, \Psi) = 1.$$

Since T is transverse to D along x_2, we obtain

$$\text{wind}(\eta', \Psi) = 1 \ .$$

$\qquad \square$

Proof of Lemma 3.9. Let $t \mapsto \eta(t) \in \xi$ be a section along $x_2(T_2 \cdot)$ such that $\{\eta(t), R(x_2(T_2 t))\}$ generates $d\Psi(TS^2)$ along $x_2(T_2 \cdot)$.

Let $u : (-\epsilon, \epsilon) \times S^1 \to \Psi(S^2)$; $(s, t) \mapsto u(s, t)$ be a smooth function such that

$$\begin{cases} u(0, \cdot) = x_{2T_2} \\ \frac{\partial}{\partial s} u(s, t) \big|_{s=0} = \eta(t) \end{cases}$$

Fix an orientation on $S^1 \times \{0\} \subset S^2$ in such a way that $\psi|_{S^1 \times \{0\}}$ preserves orientation. Consider the closed hemispheres of S^2, that we call H_+ e H_-, with the orientation induced by the orientation of $S^1 \times \{0\}$. It follows that

$$0 < T_2 = \int_{S^1} x_{2T_2}^* \lambda = \int_{H_\pm} \psi^* d\lambda \ .$$

Since ψ is transverse to the Reeb flow R_λ in $H_\pm \setminus S^1 \times \{0\}$, then $\psi^* d\lambda > 0$ in $H_\pm \setminus S^1 \times \{0\}$. This implies that 0 is a local maximum of the function $s \mapsto \mathcal{A}(u(s, \cdot))$.

It follows from (1.7) and (1.16) that

$$\frac{d^2}{ds^2}(\mathcal{A}(u(s, \cdot)))\bigg|_{s=0} = T_2 \int_{S^1} d\lambda(\eta(t), \mathcal{L}_R \eta(t)) dt < 0 . \qquad (A.4)$$

Since $\psi|_{H_\pm}$ is a strong transverse section, we have

$$d\lambda(\eta(t), \mathcal{L}_R \eta(t)) dt < 0, \ \forall t \in S^1.$$

Let $t \mapsto \eta'(t) \in \xi$ be a section along $x_2(T_2 \cdot)$ such that $\{\eta'(t), R(x_2(T_2 t))\}$ generates the tangent space of T along $x_2(T_2 \cdot)$.

Let $v : (-\epsilon, \epsilon) \times S^1 \to T$, $(s, t) \mapsto v(s, t)$ be a smooth function such that

$$\begin{cases} v(0, \cdot) = x_{2T_2} \\ \frac{\partial}{\partial s} v(s, t)\big|_{s=0} = \eta'(t). \end{cases}$$

Recall that V_1 and V_2 are oriented in such a way that $d\lambda|_{V_{1,2}}$ is an area form, P_3 is a positive asymptotic limit and P_2 is a negative asymptotic limit. This implies that 0 is a local minimum of the function $s \mapsto \mathcal{A}(v(s, \cdot))$.

It follows from (1.7) and (1.16) that

$$\frac{d^2}{ds^2}(\mathcal{A}(v(s, \cdot)))\bigg|_{s=0} = T_2 \int_{S^1} d\lambda(\eta'(t), \mathcal{L}_R \eta'(t)) dt > 0 . \qquad (A.5)$$

Since V_1 and V_2 are strong transverse sections, we have

$$d\lambda(\eta'(t), \mathcal{L}_R \eta'(t)) dt > 0, \ \forall t \in S^1.$$

Now we can apply Proposition A.2 to the sections η and η' to conclude the proof. $\qquad \square$

Appendix B

Intersection theory of punctured pseudoholomorphic curves

In this appendix we state some results of [Sie11] that were used in the thesis. Some of the theorems stated here are simpler versions of the results of [Sie11] cited, that are enough for our purposes.

Let (M, λ) be a contact manifold, fix $J \in \mathcal{J}(\xi, d\lambda)$ and consider the almost complex structure $\tilde{J}$ in $\mathbb{R} \times M$ defined by (2.1).

In [Sie11], it is defined a generalized intersection number of pseudoholomorphic curves, denoted by $[\cdot] * [\cdot]$, which counts the algebraic intersection number plus an asymptotic intersection number at punctures.

Theorem B.1. *[Sie11, Corollary 5.9] Let* $\tilde{u} : (\Sigma \setminus \Gamma, j) \to \mathbb{R} \times M$ *and* $\tilde{v} : (\Sigma' \setminus \Gamma', j') \to \mathbb{R} \times M$ *finite energy* $\tilde{J}$-*holomorphic curves. Assume that no component of* $\tilde{u}$ *or* $\tilde{v}$ *lies in an orbit cylinder. Then the following are equivalent:*

(1) $[\tilde{u}] * [\tilde{v}] = 0$.

(2) All of the following hold:

- *The map u does not intersect any of the positive asymptotic limits of v;*

- *The map v does not intersect any of the negative asymptotic limits of u;*

- *If P is a periodic orbit so that at* $z \in \Gamma$, $\tilde{u}$ *is asymptotic* P^{m_z} *and at* $w \in \Gamma'$, $\tilde{v}$ *is asymptotic to* P^{m_w} *then: If z and w are both positive, then* $d_0(\tilde{u}, z) = 0$ *and* $\frac{\operatorname{wind}(\nu_{P^{m_z}}^{neg}, [\Phi])}{m_z} \geq \frac{\operatorname{wind}(\nu_{P^{m_w}}^{neg}, [\Phi])}{m_w}$. *If z and w are both negative, then* $d_0(\tilde{v}, w) = 0$ *and* $\frac{-\operatorname{wind}(\nu_{P^{m_w}}^{pos}, [\Phi])}{m_w} =$

$\frac{-\,\mathrm{wind}(\nu_{Pm_z}^{pos},[\Phi]}{m_z}$. *If z is a negative puncture and w is a positive punc-
ture, then $d_0(\tilde{u},z) = d_0(\tilde{v},w) = 0$ and P^{m_z} and P^{m_w} are both even
orbits, or equivalently*

$$\frac{\mathrm{wind}_\infty^\Phi(\tilde{u},z)}{m_z} = \frac{\mathrm{wind}_\infty^\Phi(\tilde{u},w)}{m_z w}$$

(3) All of the following hold:

- *The map u does not intersect any of the asymptotic limits of v.*

- *The map v does not intersect any of the asymptotic limits of u.*

- *If P is a periodic orbit so that at $z \in \Gamma$, $\tilde{u}$ is asymptotic to P^{m_z} and
  at $w \in \Gamma'$, $\tilde{v}$ is asymptotic to P^{m_w}, then $d_0(\tilde{u},z) = d_0(\tilde{v},w) = 0$.
  Further, if P is elliptic, then either m_z and m_w are both positive
  and $\frac{\mathrm{wind}(\nu_{Pm_z}^{neg},[\Phi])}{m_z} = \frac{\mathrm{wind}(\nu_{Pm_w}^{neg},[\Phi])}{m_w}$, or m_z and m_w are both negative
  and $\frac{\mathrm{wind}(\nu_{Pm_z}^{pos},[\Phi])}{m_z} = \frac{\mathrm{wind}(\nu_{Pm_w}^{pos},[\Phi])}{m_w}$. If P is odd hyperbolic then either
  m_z and m_w are both even or $m_z = m_w$.*

Theorem B.2. *[Sie11, Theorem 2.4] Let $\tilde{u} : (\Sigma \setminus \Gamma, j) \to \mathbb{R} \times M$ and
$\tilde{v} : (\Sigma' \setminus \Gamma', j') \to \mathbb{R} \times M$ asymptotically cylindrical $\tilde{J}$-holomorphic curves.
Assume that no component of $\tilde{u}$ or $\tilde{v}$ lies in an orbit cylinder and that the
projected curves u and v do not have identical image on any component of
their domains. Then the following are equivalent:*

(1) The projected curves u and v do not intersect;

(2) All of the following hold:

- *The map u does not intersect any of the positive asymptotic limits
  of v;*

- *The map v does not intersect any of the negative asymptotic limits
  of u;*

- *If P is a periodic orbit so that at $z \in \Gamma$, $\tilde{u}$ is asymptotic P^{m_z}
  and at $w \in \Gamma'$, $\tilde{v}$ is asymptotic to P^{m_w} then: If z and w are both
  positive or both negative punctures, then $\frac{\mathrm{wind}_\infty^\Phi(\tilde{u},z)}{m_z} \geq \frac{\mathrm{wind}_\infty^\Phi(\tilde{v},w)}{m_w}$.
  If z is a negative puncture and w is a positive puncture, then*

$$\frac{\mathrm{wind}_\infty^\Phi(\tilde{u},z)}{m_z} = \frac{\lfloor \mu^\Phi(P^{m_z})/2 \rfloor}{m_z} = \frac{\lfloor \mu^\Phi(P^{m_w}) \rfloor}{m_w} = \frac{\mathrm{wind}_\infty^\Phi(\tilde{v},w)}{m_w}$$

(3) All of the following hold:

- *The map u does not intersect any of the asymptotic limits of v.*

- *The map v does not intersect any of the asymptotic limits of u.*

- *If P is a periodic orbit so that at $z \in \Gamma$, $\tilde{u}$ is asymptotic to P^{m_z} and at $w \in \Gamma'$, $\tilde{v}$ is asymptotic to P^{m_w}, then $\frac{\text{wind}_\infty^\Phi(\tilde{u},z)}{m_z} = \frac{\text{wind}_\infty^\Phi(\tilde{v},w)}{m_w}$.*

Definition B.3 ([Sie11])**.** Let $\tilde{u} : \Sigma \setminus \Gamma' \to \mathbb{R} \times M$ and $\tilde{v} : \Sigma' \setminus \Gamma' \to \mathbb{R} \times M$ be finite energy $\tilde{J}$-holomorphic curves asymptotic to the same nondegenerate periodic orbit $P \in \mathcal{P}(\lambda)$ at certain punctures $z \in \Gamma$ and $w \in \Gamma'$. We say that $\tilde{u}$ and $\tilde{v}$ approach P in the *same direction* at these punctures if $\eta_u = c\eta_v$ for $c > 0$, where η_u and η_v are the asymptotic eigensections of $\tilde{u}$ at z and of $\tilde{v}$ at w respectively, defined in Theorem 2.11. In case $\eta_u = c\eta_v$ with $c < 0$, we say that $\tilde{u}$ and $\tilde{v}$ approach P in *opposite directions*.

Theorem B.4. *[Sie11, Theorem 2.5] Let P be a simple even orbit and let $\tilde{u} : \Sigma \setminus \Gamma \to \mathbb{R} \times M$ and $\tilde{v} : \Sigma \setminus \Gamma' \to \mathbb{R} \times M$ be connected finite energy $\tilde{J}$-holomorphic curves. Assume that at punctures $z \in \Gamma$ and $w \in \Gamma'$, $\tilde{u}$ and $\tilde{v}$ approach P in the same direction, and that there do not exist neighborhoods U of z and V of w so that $u(U \setminus \{z\}) = v(V \setminus \{w\})$. Then*

$$[\tilde{u}] * [\tilde{v}] > 0 .$$

Theorem B.5. *[Sie11, Theorem 2.6] Let $\tilde{u} : \Sigma \setminus \Gamma \to \mathbb{R} \times M$ be a connected simple finite energy $\tilde{J}$-holomorphic curve. Assume that $\tilde{u}$ does not have image contained in an orbit cylinder. Then the following are equivalent:*

(1) *The projected map $u : S \setminus \Gamma \to M$ is an embedding.*

(2) *The algebraic intersection number $int(\tilde{u}, \tilde{u}_c)$ between $\tilde{u}$ and an $\mathbb{R}$-translate $\tilde{u}_c = (a + c, u)$ is zero for all $c \in \mathbb{R} \setminus \{0\}$.*

(3) *All of the following hold:*

- *u does not intersect any of its asymptotic limits.*

- *If P is a periodic orbit so that u is asymptotic at $z \in \Gamma$ to P^{m_z} and u is asymptotic at $w \in \Gamma$ to P^{m_w}, then $\frac{\text{wind}_\infty(\tilde{u},z)}{m_z} = \frac{\text{wind}_\infty(\tilde{u},w)}{m_w}$.*

If (1), (2) or (3) holds, then

- *The map $\tilde{u}$ is an embedding.*

- *The projected map u is an immersion which is everywhere transverse to the Reeb vector field.*

- *For each $z \in \Gamma$, we have $\gcd(m_z, \text{wind}_\infty(\tilde{u}, z)) = 1$.*

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