Relative Homological Dimensions and Controllable Extensions

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Let not future things disturb thee, for thou wilt come to them,
if it shall be necessary, having with thee the same
reason which now thou usest for present things
— Marcus Aurelius

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Resumo

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Em 1956, Hochschild desenvolveu uma teoria homológica para extensões de álgebras associativas. Sua teoria ficou dormente pelas próximas décadas, mas resultados recentes a relacionaram com a Conjectura da Dimensão Finitística: uma conjectura de 60 anos e central para teoria homológica de álgebras de dimensão finita.

Neste trabalho daremos uma visão panorâmica das relações entre Teoria Homológica Relativa e a Conjectura da Dimensão Finitística. Depois definimos e examinamos uma nova classe de extensões de álgebras, chamadas de extensões controláveis. Provamos que essa classe transporta muito do comportamento das dimensões homológicas clássicas para as dimensões homológicas relativas, resultando no cálculo da dimensão global relativa de algumas extensões. Nós também traduzimos alguns resultados da teoria clássica para o ambiente relativo, com destaque para uma generalização do comportamento homológico de álgebras de caminho.

Palavras-chave: álgebra homológica relativa, dimensões homológicas, extensões controláveis, Conjectura da Dimensão Finitística, álgebras de dimensão finita.

Abstract

Roger Ramirez Primolan. **Relative Homological Dimensions and Controllable Extensions**. Thesis (Master's). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2023.

In 1956, Hochschild developed an homological theory for extensions of associative algebras. His theory went dormant for the next decades, but recent results related it to the Finitistic Dimension Conjecture: a 60 years old central conjecture for homological theory of finite dimensional algebras.

In this work we will present a panoramic view on the relations between Relative Homological Algebra and the Finitistic Dimension Conjecture. We define and analyse a new class of extensions, the controllable extensions. We proved that this class preserves much of the properties of classical homological dimensions to the relative realm, in particular we are able to compute the relative global dimension of some extensions. We also translated some results of homological algebra to relative homological algebra, in particular we obtained a generalization of the homological behaviour of path algebras.

Keywords: relative homological algebra, homological dimensions, controllable extensions, Finitistic Dimension Conjecture, finite dimensional algebras.

Contents

Introduction

In 1956, Hochschild developed an homological theory to study extensions of algebras $B \subseteq A$. His theory went dormant until resent efforts related it to the Finitistic Dimension Conjecture. In [XX13], the autors where able to bound the finitistic dimension of B using that of A provided, among other things, that $B \subseteq A$ is an extension of finite relative global dimension. In [IM21], the authors explored relations among the algebras of an extension in order to prove that the finiteness of the finitistic dimension of any one of them implies that of the other. Again they needed extensions with certain finite conditions using relative homological dimensions. The extensions considered in [IM21] are a generalization of extensions defined in [CLMS22]. In both articles, the authors study how Han's Conjecture interacts with this extensions.

The aim of this work is to provide examples and ways to compute the relative global dimension of extensions, focusing on the finite dimensional cases. To do so, we proved some relative results analogous to classical theorems in Homological Algebra. For instance, we proved a generalization of the homological behaviour of path algebras. Then we defined a new class of extensions, proved some properties and constructed examples.

This work is organized as follows:

Part I consists of the theorical background needed.

Chapter 1 gives an overview of finite dimensional basic algebras with emphasis in the connection between Algebra and Combinatorics.

Chapter 2 introduces the reader to the relevant homological definitions and results, then we specialize it to finite dimensional basic algebras.

Chapter 3 discuss Relative Homological Algebra and its relations with the Finitistic Dimension Conjecture.

Part II consists of our results.

Chapter 4 proves relative homological results that generalizes the behaviour observed from classical homological theory or involves known extensions of algebras.

Chapter 5 proves our main results. We define a new class of extensions, discuss its properties and show, via examples, that this class is not trivial. In particular, we provide new examples of extensions with finite global dimension. We give a counterexample showing that this class does not encompass all possible extensions.

Chapter 6 is a small conclusion about our work and what we were able to achieve.

In order to make it easier to find the new results, definitions, and examples, they are organized in the following table.

Table 1: *Table with the new results together with a small description and the page where they are found.*

Novelty	**Description**	**Page**
Theorem 4.1.1	Equivalence for $\text{gldim}(T[B, N], B) = 1$	61
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Part I

Theory

Chapter 1

Finite Dimensional Algebras

In this chapter we present the theory of path algebras and representations of quivers. The main references are [ASS06] and [Kir16]. We assume the reader is familiar with module theory, tensor products, and category theory.

1.1 Historical Remarks

This section is heavily based on [Kle+07] and [Van13]. For an in depth analysis of History of Algebra we recommend the reader to look at [Kle+07] and [Van13], and references therein.

Non commutative algebra started its blooming by a new example that behaved differently from the objects studied at the time. That example was the *quaternion numbers*, discovered by William Rowan Hamilton (1805–1865) in a paper from 1843 [Van13, Chapter 10, Hamilton's Discovery of Quaternions]. Quaternions form a 4-dimensional real non commutative and unital algebra, with basis given by $1, i, j$ and k and multiplication is given by the rules:

- 1 is the identity.
- $i^2 = j^2 = k^2 = -1$.
- $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$.

Nowadays we call the quaternions a *division algebra*.

After Hamilton's discovery, the next decades saw an increase number of other examples of what we now call associative algebras. For instance, Arthur Cayley (1863-1895) introduced full matrix rings in a series of two papers between 1855 and 1858, remarking the non commutativity of this new object [Kle+07, Section 3.1.1.(iv)]. After a new theory gains a considerable amount of examples it is possible to start looking at the patterns that they, as a collective, have. This led to a first abstract definition of a finite dimensional algebra by Benjamin Peirce (1809-1880) in 1870 [Kle+07, Section 3.1.2.(i)].

In the 1890's, independently, Élie Cartan (1869-1951), Ferdinand Georg Frobenius (1849-1917) and Theodor Molien (1861-1941) showed that any finite dimensional algebra over $\mathbb{R}$

or $\mathbb{C}$ is the sum of a nilpotent ideal and a semisimple algebra $A = N \oplus B$ (a semisimple algebra B was an algebra without non trivial nilpotent bilateral ideals), every semisimple algebra is the sum of simple algebras (algebras without non trivial bilateral ideals), and that simple algebras are full matrix rings over a division algebra, see [Kle+07, Section 3.1.3.(i)].

The particularity of this classification is two fold: first it was only for specific fields, second it was not an structural proof. Joseph Wedderburn (1882-1948) was able to improve on both points in the main theorem of his paper "On Hypercomplex Numbers", from 1908. See [Van13, Chapter 11, Maclagan Wedderburn]. Later this result was generalized by Emil Artin (1898-1962), in 1927, for algebras satisfying the *descending chain condition* [Van13, Chapter 11, Emil Artin]. For a modern version of the Wedderburn-Artin theory we recommend [Lam91].

Theorem 1.1.1. *[Van13, Chapter 11, Maclagan Wedderburn, Theorem 13] (Wedderburn' Main Theorem) Let A be a finite dimensional algebra over a field. Then:*

1. $A = N \oplus \Sigma$, *where N is a maximal nilpotent ideal of A and Σ is semisimple.*
2. $\Sigma = \bigoplus_{i=1}^{n} S_i$, *where each S_i is a simple algebra.*
3. $S_i = \mathrm{Mat}_{n_i}(D_i)$, *where D_i is a division ring.*

An important message that we take from this theorem is that semisimple algebras are just sum of matrix rings.

To summarize, a finite dimensional algebra can be decompose as

$$A = \Sigma \oplus N, \tag{1.1}$$

where Σ is a semisimple algebra and N is a nilpotent ideal. Intuitively, this result says that any algebra is a sum of two things: a good part Σ and a complicated part N. This ideal nowadays is called *Jacobson radical* of A and is one of the objects we are going to study.

1.2 Algebras

In this section we define basic objects that will set the background of this work. Fix $\mathbb{K}$ a field.

1.2.1 Basic Definitions

The main object of study are algebras over a field.

Definition 1.2.1. By a $\mathbb{K}$-*algebra* or simply an *algebra (over* $\mathbb{K}$*)* A we mean a $\mathbb{K}$-vector space together with a bilinear map $\cdot : A \times A \longrightarrow A$ such that

1. *Associativity*: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, $\forall\, x, y, z \in A$.
2. *Identity*: there exists $1 \in A$ such that $1 \cdot x = x \cdot 1 = x$, for any $x \in A$.
3. *Compatibility with vector space structure*: For any $x, y \in A$ and $\lambda \in \mathbb{K}$, $\lambda(x \cdot y) = (\lambda x) \cdot y = x \cdot (\lambda y)$.

We say that an algebra is *finite dimensional* if it has finite dimension as a vector space. By a *homomorphism of* $\mathbb{K}-$*algebras* or *algebra homomorphism* between A and B we mean a linear transformation $\phi : A \longrightarrow B$ such that $\phi(1_A) = 1_B$ and $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in A$.

Example 1.2.2. Some simple examples of algebras are:

1. Every field $\mathbb{K}$ is an algebra over itself.
2. $\mathbb{C}$ is a $\mathbb{R}$-algebra.
3. The full matrix ring $\mathrm{Mat}_n(\mathbb{K})$ is a $\mathbb{K}$-algebra.
4. The upper triangular $n \times n$ matrices over $\mathbb{K}$, $UT_n(\mathbb{K})$, form a $\mathbb{K}$-algebra. The inclusion of vector spaces $\iota : UT_n(\mathbb{K}) \longrightarrow \mathrm{Mat}_n(\mathbb{K})$ is an algebra homomorphism.
5. Given an algebra A with multiplication $\cdot$ we can define its *opposite algebra* A^{op} as follows: as vector spaces $A^{op} = A$ and the multiplication $\star$ of A^{op} is $x \star y = y \cdot x$, for all $x, y \in A$.
6. Given an algebra A, we define its *enveloping algebra* as the tensor product of A with its opposite algebra, that is, $A^e \doteq A \otimes_{\mathbb{K}} A^{op}$.

Next we define modules. Morally, they are how an algebra interact with the world and, therefore, we study them as a “concrete” way to study the abstract object algebra. In this text, we will work mainly with left modules.

Definition 1.2.3. By a *left module* over a $\mathbb{K}$-algebra we mean a $\mathbb{K}$-vector space M together with an algebra homomorphism $\phi : A \longrightarrow \mathrm{End}_{\mathbb{K}}(M)$. The image of $x \in A$ via ϕ is called the *action of* x *on* M and denoted by $\phi(x)(m) = xm$, $\forall\, m \in M$. We say M is a *finite dimensional* module if it has finite dimension as a vector space. By an *homomorphism of* A*-modules* between M and N we mean a linear transformation $f : M \longrightarrow N$ such that

$f(xm) = xf(m)$, for all $x \in A$ and $m \in M$. We say that a homomorphism is an *epimorphism* if it is surjective and a *monomorphism* if it is injective.

We denote by A-**Mod** the category whose objects are all A-modules, the morphisms are homomorphisms of A-modules and the composition is the usual function composition. The full subcategory of A-**Mod** generated by all finite dimensional modules will be denoted A-**mod**.

By an $(A - B)$-*bimodule* we mean a left module over $A \otimes_{\mathbb{K}} B^{op}$.

We will use interchangeably the following results/notations:

- (Right modules) The category of (finite dimensional) right A-modules is isomorphic to the category of (finite dimensional) left A^{op}-modules, **Mod**-$A \cong A^{op}$-**Mod** (**mod**-$A \cong A^{op}$-**mod**).
- (Bimodules) The category of bimodules is $(A - B)$-**Bimod**$\cong A \otimes_{\mathbb{K}} B^{op}$-**Mod**. The finite dimensional version will be denoted by lower case letters.
- The symbols $\langle X \rangle$ will always mean structure generated by X. If X is a subset of an algebra $\langle X \rangle$ will be the subalgebra generated by X; if X is a subset of a module, $\langle X \rangle$ will mean a submodule; and so on. The context will make clear what type of creature $\langle X \rangle$ is.
- For an $(A - B)$-bimodule M we write ${}_AM_B$, for left modules, ${}_AM$, and for a right module, M_B.

Next we name different properties of modules.

Definition 1.2.4. Let $M \in A$-**Mod**, we say that

1. $M \neq 0$ is *simple* if it has no submodules apart from 0 and M.
2. M is *semisimple* if it is a direct sum of simple modules
3. M is *indecomposable* if it can not be written as a direct sum $M = N \oplus N'$ with $N, N' \notin \{0, M\}$.

Next we state a famous theorem about the category of finite dimensional modules.

Theorem 1.2.5. *[ASS06, I.4 Theorem 4.10](Krull–Schmidt theorem) If A is a finite dimensional $\mathbb{K}$-algebra and M is a finite dimensional A-module, then there exist a decomposition of M into indecomposable modules*

$$M \cong M_1 \oplus \cdots \oplus M_n \tag{1.2}$$

and if

$$M \cong N_1 \oplus \cdots \oplus N_r$$

is another decomposition of M into indecomposable modules, then $n = r$ and there exists a permutation $\sigma \in \mathrm{Sym}(n)$ such that $M_i \cong N_{\sigma(i)}$.

The above theorem says that if one is interested in studying $A - \mathbf{mod}$, then it suffices to study indecomposable modules and homomorphisms between them. Once this specific objects are well understood it is just a matter of taking the correct direct sum to obtain any particular module.

Let us use this theorem to understand a finite dimensional algebra A. By definition there exist $1 \in A$. This element has the following property: $1^2 = 1$. We say that 1 is *idempotent*. There are two possibilities: or we can write $1 = e_1 + e_2$, with $e_1 \neq 1$ and $e_2 \neq 1$ idempotents and orthogonal (meaning $e_1e_2 = e_2e_1 = 0$), or we can not. If we can decompose 1 as above, we then ask the same question about e_1 and e_2. Since non zero orthogonal idempotents are linearly independent, we eventually will obtain a maximal finite set of them $\{e_1, \dots, e_n\}$ with the added property that none e_i can be written as a sum of two non trivial orthogonal idempotents (excluding the sum $e_i = 0 + e_i$). We call such a set a *complete set of orthogonal primitive idempotents* or *copi* for short.

For each copi we can obtain a decomposition of A viewed as an A-module as follows

$$A = Ae_1 \oplus \cdots \oplus Ae_n, \tag{1.3}$$

and each submodule Ae_i is indecomposable, see [ASS06, I.4 Corollaries 4.7 and 4.8.(a)]. By (1.2.5) we obtain that two different copi's have the same number of elements and, up to reordering, they generate isomorphic modules $Ae_i \cong Af_i$. An algebra is said to be *basic* if there exists a copi such that $Ae_i \not\cong Ae_j$, for all $e_i \neq e_j$. By the (1.2.5), this definition does not depend on the copi.

Basic algebras are important because one can show that, for any finite dimensional algebra A, there exists a basic algebra A_{bas} such that their module categories are equivalent $A-\mathbf{mod} \cong A_{\mathrm{bas}}-\mathbf{mod}$. In other words, up to equivalence of categories, if module categories are the correct way to study algebras, then for finite dimensional algebras one need to worry only about the basic ones. See [ASS06, p. I.6] for details about this result.

1.2.2 Basic Constructions

Now we are going to construct some abstract algebras.

(a) Tensor algebra: Let A be any algebra and N any $(A - A)$-bimodule. Define

$$N^1 = N \text{ and } N^r = N^{r-1} \otimes_A N, \ r > 1,$$

where $\otimes_A$ denotes the tensor product over A. The *tensor algebra of A and N* is

- as vector spaces

$$T[A, N] \doteq A \oplus N \oplus N^2 \oplus \cdots \oplus N^r \oplus \cdots.$$

- the product of $a \in A$ with other element will be the actions on the left and right.
- the product of $n_r \in N^r$ and $n_s \in N^s$ will be the element $n_r \otimes_A n_s \in N^{r+s}$.

The tensor algebra is an associative algebra with unit $1_{T[A,N]} = 1_A$.

(b) Matrix Algebras: If A and B are two algebras and $M \in (B-A)$-**Bimod**. The *matrix algebra* of this data is defined as follows: as vector spaces it is

$$\Lambda \doteq \begin{pmatrix} A & 0 \\ M & B \end{pmatrix}.$$

The multiplication of Λ is defined as

$$\begin{pmatrix} a & 0 \\ m & b \end{pmatrix} \cdot \begin{pmatrix} a' & 0 \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' & 0 \\ ma' + bm' & bb'. \end{pmatrix}.$$

This structure makes Λ an associative algebra with unity given by

$$\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}.$$

(c) Trivial Extension: For an algebra A and an $(A-A)$-bimodule N, the *trivial extension* of A by N is the algebra whose underlying vector space is

$$A \ltimes N \doteq A \oplus N$$

and product given by

$$(a, n) \cdot (a', n') = (aa', na' + an').$$

It is possible to prove that trivial extensions are associative and that their unity is $(1_A, 0)$. If one takes

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \text{ and } N = \begin{pmatrix} 0 & 0 \\ {}_BM_A & 0 \end{pmatrix},$$

where A and B are algebras and M is a $(B-A)$-bimodule, then $C \ltimes N \cong \Lambda$, where Λ is the matrix algebra constructed using A, B, and M.

1.3 Jacobson Radical

This section is devoted to results regarding the Jacobson radical of an algebra and is based on [ASS06], but here we work with left modules. Fix a $\mathbb{K}$-algebra A.

Definition 1.3.1. [ASS06, I.1 Definition 1.2] The *Jacobson radical of* A or simply *radical of* A is defined as

$$J(A) \doteq \bigcap_{I \in \mathcal{I}} I, \tag{1.4}$$

where $\mathcal{I}$ is the set of all maximal left ideals of A.

Example 1.3.2. By (1.1.1) we know that $\mathbb{K}^2 = \mathrm{Mat}_1(\mathbb{K}) \oplus \mathrm{Mat}_1(\mathbb{K})$ is semisimple. Its maximum ideals are precisely all the subspaces of dimension 1. But the intersection of all the lines containing the origin (subspaces) in the plane ($\mathbb{K}^2$) is zero, therefore $J(\mathbb{K}^2) = 0$. This argument can be generalized to show that $J(\mathbb{K}^n) = 0$, for all $n \in \mathbb{N}$.

Here is a compilation of important results regarding the (Jacobson) radical, they allow us to carry several computations.

Proposition 1.3.3. *[ASS06, I.1 Corrolary 1.4] For an algebra A:*

1. *$J(A)$ is the intersection of all maximal right ideals of A.*
2. *$J(A)$ is a bilateral ideal.*
3. *If $I \subseteq A$ is a two sided bilateral ideal of A that is* nilpotent *(meaning $I^n = 0$ for some power $n \in \mathbb{N}$), then $I \subseteq J(A)$. If, in addition, $A/I \cong \bigoplus_{i=1}^{r} \mathbb{K}$, then $I = J(A)$.*

The Jacobson radical of a finite dimensional algebra is particularly well behaved

Proposition 1.3.4. *[ASS06, I.2 Corrolary 2.3] If A is finite dimensional, then $J(A)$ is nilpotent.*

And we have a characterization of basic algebras using their radical.

Proposition 1.3.5. *[ASS06, I.6 Proposition 6.2.(a)] A finite dimensional $\mathbb{K}$-algebra is basic if and only if the algebra $\frac{A}{J(A)}$ is isomorphic to a sum of copies of $\mathbb{K}$.*

Now we use the above propositions in order to compute the radical of some tensor algebras.

Corollary 1.3.6. *Let B be a basic finite dimensional $\mathbb{K}$-algebra and N a $B \otimes_{\mathbb{K}} B^{op}$-module. If $A = T[B, N]$ is finite dimensional, then*

$$J(T[B,N]) = J(B) \oplus N \oplus N^2 \oplus \cdots . \tag{1.5}$$

Proof. Denote $J = J(B) \oplus N \oplus N^2 \oplus \cdots$, it is clear that

$$\frac{A}{J} = \frac{B \oplus N \oplus N^2 \oplus \cdots}{J(B) \oplus N \oplus N^2 \oplus \cdots} \cong \frac{B}{J(B)} = \bigoplus_{i=1}^{m} \mathbb{K} . \tag{1.6}$$

Since $A = T[B, N]$ is finite dimensional, there must exist $r \in \mathbb{N}$ such that $N^r = 0$. By the above proposition, let $s \in \mathbb{N}$ be such that $J(B)^s = 0$.

Consider $s \cdot r$ elements in a matrix like array $x_{11}, \dots, x_{1s}, x_{21}, \dots, x_{rs} \in J$ and write them, uniquely, as $x_{ij} = \rho_{ij} + n_{ij}$, with $\rho_{ij} \in J(B)$ and n_{ij} in the bilateral ideal $\langle N \rangle$. Then $\prod_{j=1}^{s} x_{ij} = \prod_{j=1}^{s} \rho_{ij} + n_i = 0 + n_i$, for some $n_i \in \langle N \rangle$. Since $N^r = 0$ implies $\langle N \rangle^r = 0$, we obtain

$$\prod_{i=1}^{r} \prod_{j=1}^{s} x_{ij} = \prod_{i=1}^{r} n_i = 0. \tag{1.7}$$

This proves that $J^{r \cdot s} = 0$ and, therefore, J is nilpotent. Finally we conclude that $J(T[B, N]) = J$. □

We are also interested in the action of $J(A)$ on a left A-module. To understand what happens we need another definition.

Definition 1.3.7. For $M \in A$–**Mod**, the *radical of* M *with respect to* A or simply the *radical of* M is defined as

$$J_A(M) \doteq \bigcap_{N \in \mathcal{N}} I, \tag{1.8}$$

where $\mathcal{N}$ is the set of all maximal left submodules of M.

Remark 1.3.8. In the literature there is another notation $\operatorname{rad}_A M \doteq J_A(M)$. We opted to use $J_A(M)$ because of the next proposition.

It is easy to see that $J(A) = J_A(A) = J(A)A$. Next we have a result that generalizes this behaviour and will be useful for us.

Proposition 1.3.9. *[ASS06, I.3 Proposition 3.7.(d)] If $M \in A$-**mod**, then*

$$J_A(M) = J(A)M. \tag{1.9}$$

In terms of the radical, a finite dimensional module will be semisimple if and only if its Jacobson radical is zero, see [ASS06, I.3 Corrolary 3.9.(c)]. Finally there is one more useful thing we can do with radicals: to select the “semisimple” part of a module.

Definition 1.3.10. For $M \in A$-**mod** we define its *top with respect to A* or simply *top* as

$$\operatorname{top}_A(M) \doteq \frac{M}{J_A(M)}. \tag{1.10}$$

1.4 Path Algebras

This section is mainly based on [ASS06].

There are several ways to study an abstract class of objects, in our case this class takes the form of all (finite dimensional) algebras. The approach that we are going to take can be summarized as follows: we construct a specific subclass of objects in such a way that we can improve our intuition and computational power, then we study how to translate this newly gained knowledge to (almost) all objects. This will be the aim of this section: to construct path algebras and to convince one that this particular class of algebras is well behaved.

Path algebras are, inherently, of combinatorial nature and we will begin by defining the combinatorial foundation of path algebras: quivers.

Definition 1.4.1. [ASS06, II.1 Definition 1.1] A *(finite) quiver* is a 4-tuple $Q = (Q_0, Q_1, s, t)$ where

1. Q_0 is a set of finite elements called *vertices.*
2. Q_1 is a set of finite elements called *arrows.*
3. $s : Q_1 \longrightarrow Q_0$ is a function called *source.*
4. $t : Q_1 \longrightarrow Q_0$ is a function called *target.*

For simplicity we will denote a quiver by Q when all the necessary information is clear. One may note that quivers are the way algebraists refer to *oriented graphs.*

By a *subquiver* of a quiver $Q = (Q_0, Q_1, s, t)$ we mean a 4-tuple $R = (R_0, R_1, \sigma, \tau)$ such that

1. $R_0 \subseteq Q_0$.
2. $R_1 \subseteq Q_1$.
3. the restriction of s and t to R_1 coincides with σ and τ, respectively.

There is a pictorial way to represent quivers (or oriented graphs), see Figure (1.1). The vertices are represented by dots or numbers in a plane, the arrows as edges connecting vertices, and the functions s and t as orientations of the edges.

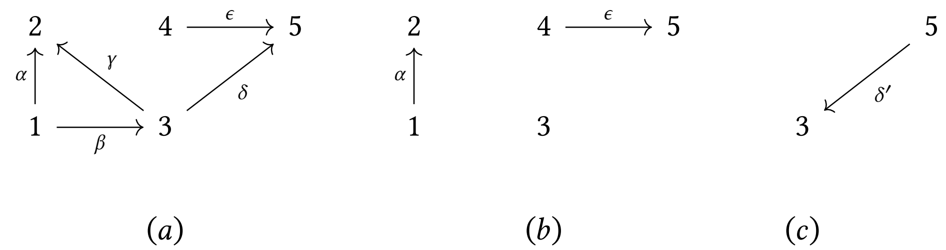


Figure 1.1: *Depiction of a quiver Q: Q_0 is the set of vertex, the edges are Q_1 and the orientation, s and t, can be inferred as follows: $s(\alpha) = 1$ and $t(\alpha) = 2$. In (a) we have a quiver Q, in (b) a subquiver R of Q, and in (c) we do not have a subquiver because the orientation of the edge δ changed.*

By a *path* of length $l \geqq 0$ from $e \in Q_0$ to $f \in Q_0$ in a quiver Q we mean a sequence of arrows $(\alpha_l, \dots, \alpha_1)$ such that $s(\alpha_1) = e$, $t(\alpha_l) = f$ and $s(\alpha_i) = t(\alpha_{i-1})$, for all $i \in \{2, \dots, l\}$, for simplicity we will denote a path by a product $\alpha_l \cdot \dots \cdot \alpha_1$. We identify all the paths of length one of $\mathbb{K}Q$ with Q_1. The elements of Q_0 are called *paths of length zero* or *trivial paths* and to make a distinction between the vertex $a \in Q_0$ and the path $a \in Q_0$ we denote the later by e_a. The idea of a path in a quiver is that you are walking in the graph preserving its orientation, with that interpretation a path of length zero corresponds to standing still on a vertex.

Definition 1.4.2. [Kir16, Definition 1.6] Let Q be a quiver. The *path algebra* of Q, denoted by $\mathbb{K}Q$ is the algebra such that

- Its underlying vector space has all the paths of Q as a basis.
- The multiplication of the basis' elements is

$$(\beta_m \cdot \dots \cdot \beta_0) \star (\alpha_l \cdot \dots \cdot \alpha_0) = \delta_{t(\alpha_l), s(\beta_0)} \beta_m \cdot \dots \cdot \beta_0 \cdot \alpha_l \cdot \dots \cdot \alpha_0, \tag{1.11}$$

  where δ is the Kronecker delta. The multiplication of a general element of $\mathbb{K}Q$ is extended by distributivity.

Continuing with our analogy, the multiplication of two paths in $\mathbb{K}Q$ is the path you get if you walk through the first path and then proceed to walk the second path if their end and beginning, respectively, coincide, and zero otherwise. For example, in quiver (a) of Figure (1.1) the multiplication $\delta \cdot \gamma = 0$ because the paths don't match, but we can multiply δ and β in this order to obtain the path $\delta \cdot \beta$ of length 2.

Before we continue, let us analyse some examples of path algebras that generate well known algebras.

Example 1.4.3. Consider the quiver Q that has only a vertex and an arrow

$$Q : \quad \bullet \circlearrowleft^{\alpha}$$

Then $\mathbb{K}Q$ has as basis $\{e_1, \alpha, \alpha^2, \alpha^3, \ldots\}$ and is easy to see that $\mathbb{K}[x] \cong \mathbb{K}Q$, via $x \mapsto \alpha$.

Example 1.4.4. Consider the quiver

$$R : \qquad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$$

For each $1 \leqq a \leqq b \leqq 4$ there is only one path that connects a to b. Denote this path ρ_{ba}, e.g. $\rho_{11} = e_1$ and $\rho_{42} = \gamma\beta$. It is easy to see that those are all the possible paths. Their multiplication table is given by $\rho_{ba}\rho_{dc} = \delta_{ad}\rho_{bc}$, which is precisely the multiplication table of the usual basis for the lower triangular 4×4 matrices $LT_4(\mathbb{K})$ and we have $\mathbb{K}R \cong LT_4(\mathbb{K})$, via $\rho_{ba} \mapsto e_{ba}$.

One of the main reasons to work with path algebras (and its quotients) is that we can infer data from the algebra using combinatorics of quivers. This process allows one to create a combinatoric-algebraic dictionary that helps comprehend the algebraic structure of path algebras and of its quotients. Now we begin the construction of this dictionary.

- The concatenation of paths in a quiver (for the last time, oriented graph) is an associative operation on a graph. This implies that the multiplication on $\mathbb{K}Q$ is associative, therefore path algebras are associative algebras.

- For any path ρ in $\mathbb{K}Q$, there exist a vertex $a \in Q_0$ such that ρ ends in a. In particular, if we concatenate ρ with the trivial path $e_a\rho = \rho$ we do nothing. Similarly for the starting point of ρ. The algebraic property that arises from this is the following: consider the element $\sum_{a \in Q_0} e_a \in \mathbb{K}Q$ (that is well defined because Q_0 is finite), then $\left(\sum_{a \in Q_0} e_a\right)\rho = \rho = \rho\left(\sum_{a \in Q_0} e_a\right)$. In other words, $\mathbb{K}Q$ is an unital algebra with unit given by $1 = \sum_{a \in Q_0} e_a$.

The next property of $\mathbb{K}Q$ that we are going to deduce is more complex. Essentially we are going to show that the paths of length zero $\{e_a \mid a \in Q_0\}$ form a copi for $\mathbb{K}Q$. We have seen that the sum of its elements equals $1 \in \mathbb{K}Q$, so we are done with completeness. If we take $\rho = e_a$ in the above discussion we get $e_a^2 = e_a$, this means that each e_a is idempotent. By the multiplication rule of $\mathbb{K}Q$, $e_a e_b = \delta_{a,b} e_a e_b = 0$ if $a \neq b$, the idempotents are orthogonal. The primitivity of this set of idempotents is a bit technical and we refer the reader to [ASS06, I.4 Corollary 4.7 and II.1 Corollary 1.5] for a proof.

The above discussion provides a way to algebraically manipulate the vertex data of our quiver. The other part of a quiver, its arrows, also encode a lot of combinatorics and now we are going to construct a bridge that takes this data, algebraically process it, and returns a very special ideal.

Definition 1.4.5. [ASS06, II.1 Definition 1.9] If Q is a connected quiver. The *ideal of arrows* of $\mathbb{K}Q$ is the bilateral ideal generated by $Q_1 \subseteq \mathbb{K}Q$ and denoted by

$$R_Q = \langle Q_1 \rangle \subseteq \mathbb{K}Q. \tag{1.12}$$

Let us analyse R_Q in order to understand how it translates combinatorial information to the realm of algebra.

- Since R_Q is generated by the arrows, when we square this ideal

$$R_Q^2 = \left\{ \sum_{i,j=1}^{n} r_i r_j \mid r_i, r_j \in Q_1 \right\} \tag{1.13}$$

  the resulting ideal is generated by all the paths of length 2. We can carry on this process as many times as we want and obtain that, if we are interested in looking at the paths of length n or greater, then we simply study R_Q^n.

- What if we want to work with all the paths of length *precisely* n? Fear no more, the space that we seek is

$$\mathbb{K}Q_n \doteq \frac{R_Q^n}{R_Q^{n+1}}. \tag{1.14}$$

  But we must know that we pay a price in order to consider just this paths: $\mathbb{K}Q_n$ is a vector space, not an ideal.

- We can specialize this idea even further. By the above discussion, the dimension of

$$\mathbb{K}Q_1 = \frac{R_Q}{R_Q^2} \tag{1.15}$$

  is the cardinality of Q_1 (the number of arrows of Q), but even if we know all the vertices of Q, in general, this information does not specify Q. To solve this problem, given $a, b \in Q_0$, it is easy to see that $e_b R_Q e_a$ is the vector space of all paths that begin at a and end at b. In particular, if we instead consider the vector space

$$(e_b \mathbb{K}Qe_a)_1 = e_b \frac{R_Q}{R_Q^2} e_a \tag{1.16}$$

  its dimension is *precisely* the number of arrows of Q that start at a and finish at b. Changing the idempotents, we can reconstruct Q from $\mathbb{K}Q$ (spoiler for the next section!).

- Ok... but what if we want to compute multiplications on $\mathbb{K}Q$ without having to worry about an infinite basis? Then we can use the fact that R_Q^n is a bilateral ideal and construct the algebra

$$\mathbb{K}\,Q_{<n} \doteq \frac{\mathbb{K}\,Q}{R_Q^n}, \tag{1.17}$$

whose natural basis, as vector space, is the paths of length up to n. Well, if you are given a finite number of bridges (arrows) that connect a finite number of islands (vertices) and you can only cross (multiply) up to a fixed number of them, then you have only a finite number of paths (elements of a basis) to take (express an arbitraty element).

1.5 Quotient of Path Algebras

This section is based on [ASS06]; it is necessary to make a disclaimer: we are working with contravariant path concatenation, meaning that our path algebras are the opposite algebras of the path algebras of [ASS06].

From now on we will start working with *connected quivers*, there will be only a few exceptions to this rule but they will be as examples. To illustrate what a connected quiver is it suffices to look at Figure (1.1): the quiver (a) is connected, the quiver (b) is not (it has three connected components, but we will not use this terminology anywhere else) and the quiver (c) is connected. The reason we can do this without compromising generality is that if $Q = R \cup S$ is not connected, then $\mathbb{K}\,Q = \mathbb{K}\,R \oplus \mathbb{K}\,S$, so if we study only connected quivers, when we finish our studies we can glue the results together in an algebraic way.

All of the efforts of the previous section were to construct a dictionary looking in the direction *Combinatorics* $\mapsto$ *Algebra*. In this section, we are going to reverse this direction and construct a "*codictionary*". This means that we want to obtain Q from $\mathbb{K}\,Q$ without knowing Q, or, in a less enigmatic way, given an algebra A, how can we construct a useful quiver for A? This question can be separated into three:

1. How to obtain Q_0?
2. How to obtain Q_1?
3. How to obtain s and t, the directions of Q_1?

For the first question we already have a partial answer: for finite dimensional algebras Q_0 is related to the existence of a copi set.

For the second question we already have a *hint*: look at R_Q. The problem is that, by assumption, we are working with an abstract algebra A, therefore this question need to be specialized to: How to recover R_Q in an algebraic way?

To do so, we change our approach back to path algebras. Given a quiver, consider the following

$$\mathbb{K}\,Q_0 \doteq \bigoplus_{i \in Q_0} \mathbb{K}\,e_i \text{ and } \mathbb{K}\,Q_1 \doteq \bigoplus_{\alpha \in Q_1} \mathbb{K}\,\alpha. \tag{1.18}$$

They are vector spaces, but we can add algebraic structures to them: $\mathbb{K}\,Q_0$ is a basic algebra with copi given by $\{e_i \mid i \in Q_0\}$ and Q_1 is a $\mathbb{K}\,Q_0 \otimes_{\mathbb{K}} \mathbb{K}\,{Q_0}^{op}$-module via $e_j \alpha e_i = \delta_{j,t(\alpha)}\delta_{s(\alpha),i}\alpha$.

With this we can construct the tensor algebra $T[\mathbb{K}Q_0, \mathbb{K}Q_1]$. It is easy to see that

$$\mathbb{K}Q \cong T[\mathbb{K}Q_0, \mathbb{K}Q_1] \tag{1.19}$$

and that $\mathbb{K}Q$ is finite if Q is a (finite) quiver without any *cycles* (non trivial paths starting and finishing in the same vertex). Therefore, if we have such a quiver, (1.3.6) says that

$$J(\mathbb{K}Q) = R_Q. \tag{1.20}$$

En passant, we showed that for a acyclic quiver, $\mathbb{K}Q$ is basic (Second part of [ASS06, II.1 Proposition 1.10]). But this result may not be true if Q has cycles. To see why, consider $\mathbb{K}Q$, where Q is the quiver  , then

$$\mathbb{K}Q \cong \mathbb{K}[x], \; J(\mathbb{K}Q) = 0 = J(\mathbb{K}[x]), \text{ and } R_Q \cong \langle x \rangle \neq 0.$$

The above discussion restricts in a natural way our scope: assume that A is finite dimensional and basic. Note that, upon the first assumption, the second one is not unreasonable, once that every finite dimensional algebra over an algebraically closed field has its module category equivalent to the module category of a basic finite dimensional algebra.

In order to not discard the data provided by cyclic quivers we consider a cleaver ideal

Definition 1.5.1. [ASS06, II.2 Definition 2.1] A bilateral ideal $I \subseteq \mathbb{K}Q$ is said to be *admissible* if there exists an integer $m > 1$ such that

$$R_Q^m \subseteq I \subseteq R_Q^2. \tag{1.21}$$

The idea of I is simple: it contain all lengthy enough paths without containing any arrow. Therefore, when we consider the quotient algebra $\mathbb{K}Q/I$ we have a finite dimensional algebra and the data of all the arrows of Q. Most of the above behaviour is carried to $\mathbb{K}Q/I$. For example, $R_Q/I \subseteq \mathbb{K}Q/I$ is a nilpotent bilateral ideal (since $R_Q^m \subseteq I$ for some m) such that

$$\frac{\mathbb{K}Q/I}{R_Q/I} \cong \frac{\mathbb{K}Q}{R_Q} \cong \mathbb{K}Q_0.$$

In other words, $\mathbb{K}Q/I$ is a basic finite dimensional algebra with $J(\mathbb{K}Q/I) = R_Q/I$, see [ASS06, II.2 Lemma 2.10]. We can also recover everything that we saw in the previous chapter: $\{\bar{e_a} = e_a + I \mid a \in Q_0\}$ is a copi [ASS06, II.2 Lemma 2.10] and the number of arrows from a to b is the dimension of

$$\bar{e_b}\frac{(R_Q/I)}{R_Q^2/I}\bar{e_a} \cong e_b\frac{(R_Q)}{R_Q^2}e_a$$

This allow us to answer the remaining questions and we can define a quiver for an arbitrary finite dimensional basic algebra.

Definition 1.5.2. [ASS06, III.3 Definition 3.1] Let A be a finite dimensional basic $\mathbb{K}$-algebra. Its *ordinary quiver* or *Gabriel quiver* Q_A is defined as

1. $(Q_A)_0$ is in bijection with a copi $\{e_1, \dots, e_n\}$ of A.

2. The arrows of $(Q_A)_1$ from i to j are in bijection with a basis of

$$e_j \frac{J(A)}{J^2(A)} e_i.$$

It can be shown that the Gabriel quiver of an algebra does not depend on the choice of a copi [ASS06, II.3 Lemma 3.2]. Note that if we start with $A = \mathbb{K}Q/I$, then we get that $Q_A = Q$, see [ASS06, III.3 Lemma 3.6], this is a way to see that we are able to shift all the data from Algebra to Combinatorics.

But we can go a step further if to each arrow $\alpha : i \mapsto j$ we associate a representative x_α of a class of $e_j J(A) e_i / J^2(A)$ in such a way that the set $\{x_\alpha + J^2(A) \mid \alpha : i \mapsto j \in (Q_A)_1\}$ is a basis for $e_j J(A) e_i / J^2(A)$, and to each $i \in Q_0$ associate the idempotent e_i in the chosen copi, then we obtain an well defined epimorphism of $\mathbb{K}$-algebras

$$\begin{aligned} \phi : \mathbb{K}Q_A &\longrightarrow A \\ i &\longmapsto e_i \\ \alpha &\longmapsto x_\alpha \end{aligned}$$

such that $\mathrm{Ker}(\phi) \subseteq \mathbb{K}Q_A$ is admissible, see [ASS06, III.3 Lemma 3.3 and Theorem 3.7]. Therefore we have the following result

Theorem 1.5.3. *[ASS06, III.3 Theorem 3.7] If A is a finite dimensional basic $\mathbb{K}$-algebra, then there exists an admissible ideal $I \subseteq \mathbb{K}Q_A$ such that*

$$A \cong \frac{\mathbb{K}Q_A}{I}. \tag{1.22}$$

This is a super useful result for computations and manipulation of algebras. To familiarize with this result we look some examples of basic algebras defined using the above isomorphism in mind.

Example 1.5.4. This examples that we are going to look will be useful down the line.

(a) Algebra of Dual Numbers: consider the quiver

$$Q \,:\quad \overset{\alpha}{\circlearrowleft} 1$$

The *algebra of dual numbers* is defined as $A \doteq \mathbb{K}\,Q/\langle \alpha^2 \rangle$ and has dimension 2.

(b) Oriented $\mathbb{A}_n$ with relations: Oriented $\mathbb{A}_n$ is defined as the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} n-1 \xrightarrow{\alpha_{n-1}} n$$

and we can consider the algebra

$$A \doteq \frac{\mathbb{K}\,\mathbb{A}_n}{J^2(\mathbb{K}\,\mathbb{A}_n)}.$$

That is we are considering the quotient of $\mathbb{K}\,\mathbb{A}_n$ by all the paths of length two $\alpha_{i+1} \cdot \alpha_i$, $1 \leqq i < n$.

1.6 Representations of Quivers

We have seen how useful quivers are to study algebra. Through the path algebra, we are able to study all finite dimensional basic algebras and, when we look at their modules, we are able to study, up to equivalence of categories, the module categories of any finite dimensional algebra over $\mathbb{K}$. The idea of this section is to use the combinatorics that was so fruitful in the algebra setting to study the module category. This section is based on [ASS06] and [Kir16].

As per usual, we start with exploratory computations. To facilitate matters, fix a quiver $Q = (Q_0, Q_1)$ and its path algebra $A = \mathbb{K}\,Q$. Let $M \in A$-**Mod** be any module and $\phi : A \longrightarrow \operatorname{End}_{\mathbb{K}}(M)$ the action of A on M. By the previous sections we can decompose the unity $1_A = \sum_{i \in Q_0} e_i$ as the trivial paths. If $m \in M$, then

$$m = 1m = \left(\sum_{i \in Q_0} e_i\right) m = \sum_{i \in Q_0} e_i m,$$

this shows that $M = \sum_{i \in Q_0} e_i M$, but this sum is actually direct. In fact, define $M_i = e_i M$ and suppose that $0 = \sum_{i \in Q_0} m_i$, with each $m_i \in M_i$. Then multiplying by e_i on the left we obtain

$$0 = e_i 0 = e_i \left(\sum_{i \in Q_0} m_i\right) = m_i, \text{ by orthogonality.} \tag{1.23}$$

Now consider an arrow $\alpha : i \mapsto j$, then for $m \in M_r$ we have $\alpha \cdot m = \alpha e_i e_r m = \delta_{i,r} \alpha \cdot m$. This means that the domain of the linear transformation $\phi(\alpha)$ can be restrict to M_i, since for any other vertex $r \neq i$, $M_r \subseteq \operatorname{Ker}(\phi(\alpha))$. On the codomain, if $m \in M$, then $\alpha \cdot m = e_j \alpha \cdot m \in M_j$, shows that we can restrict it to M_j. Our findings can be summarize as follows:

1. Each module has a direct decomposition $M = \bigoplus_{i \in Q_0} M_i$, as vector spaces.

2. For any $\alpha \in Q_1$, we can restrict both the domain and codomain of $\phi(\alpha) : M_{s(\alpha)} \longrightarrow M_{t(\alpha)}$, obtaining a linear transformation between this vector spaces.

How a homomorphism of A-modules $f : (M, \phi) \longrightarrow (N, \psi)$ behaves in this new approach? By definition we know that $f(\alpha m) = \alpha f(m)$, for any $m \in M$. But by our discussion we can assume that $m \in M_{s(\alpha)}$, $\alpha m \in M_{t(\alpha)}$ and $\alpha f(m) \in N_{t(\alpha)}$. If $m \in M_i$, then $f(m) = f(e_i m) = e_i f(m) \in M_i$, i.e, for each $i \in Q_0$ we can consider the linear transformation $f_i : M_i \longrightarrow N_i$. This says that the following diagram

$$\begin{array}{ccc} M_{s(\alpha)} & \xrightarrow{\phi(\alpha)} & M_{t(\alpha)} \\ \downarrow{\scriptstyle f_{s(\alpha)}} & & \downarrow{\scriptstyle f_{t(\alpha)}} \\ N_{s(\alpha)} & \xrightarrow{\psi(\alpha)} & N_{t(\alpha)} \end{array} \tag{1.24}$$

is commutative, for any $\alpha \in Q_1$. We arrive at the following definition.

Definition 1.6.1. [Kir16, Definition 1.2] Let Q be a quiver. The *category of representations of* Q, denoted by $\mathbf{Rep}(Q)$, is defined as follows

1. *(Objects)* The objects are a pair. The first entry is a tuple of vector spaces indexed by the vertices of Q $(M_i)_{i \in Q_0}$. The second entry is a tuple of linear transformations indexed by the arrows of Q $(T_\alpha : M_{s(\alpha)} \to M_{t(\alpha)})_{\alpha \in Q_1}$.

2. *(Morphisms)* We define a morphism $f : M = ((M_i)_{i \in Q_0}, (T_\alpha : M_{s(\alpha)} \to M_{t(\alpha)})_{\alpha \in Q_1}) \longrightarrow N = ((N_i)_{i \in Q_0}, (T_\alpha : N_{s(\alpha)} \to N_{t(\alpha)})_{\alpha \in Q_1})$ as a tuple of linear transformations indexed by Q_0 $f = (f_i : M_i \to N_i)_{i \in Q_0}$ such that

$$\begin{array}{ccc} M_{s(\alpha)} & \xrightarrow{\phi(\alpha)} & M_{t(\alpha)} \\ \downarrow{\scriptstyle f_{s(\alpha)}} & & \downarrow{\scriptstyle f_{t(\alpha)}} \\ N_{s(\alpha)} & \xrightarrow{\psi(\alpha)} & N_{t(\alpha)} \end{array}$$

   commutes for every $\alpha \in Q_1$

3. *(Composition)* Composition is done point-wise, i.e, $g \circ f = (g_i)_{i \in Q_0} \circ (f_i)_{i \in Q_0} = (g_i \circ f_i)_{i \in Q_0}$

The category $\mathbf{rep}(Q)$ is the full subcategory of $\mathbf{Rep}(Q)$ generated by all the objects such that $\dim M_i < \infty$, for all $i \in Q_0$. We call the objects of this subcategory *finite dimensional* *representations.*

The next theorem show us how to study modules over $\mathbb{K}Q$.

Theorem 1.6.2. *Let Q be a quiver, then there exist equivalences of categories*

$$\mathbb{K}Q - \mathbf{Mod} \equiv \mathbf{Rep}(Q) \text{ and } \mathbb{K}Q - \mathbf{mod} \equiv \mathbf{rep}(Q) \tag{1.25}$$

Proof. See [Kir16, Theorem 1.7]. □

Remark 1.6.3. The categories $\mathbf{Rep}(Q)$ and $\mathbf{rep}(Q)$ are abelian, see [ASS06, III.1 Lemma 1.3].

By a *relation* on $A = \mathbb{K}Q$, we mean a sum of paths $\rho = \sum_{i=1}^{r} \rho_i$ such that every ρ_i has the same beginning and end. Given an admissible ideal $I \subseteq A$, it is possible to show that it is generated, as an ideal, by a *finite* number of relations [ASS06, II.2 Corrolary 2.9]. For a path $\rho = \alpha_n \cdots \alpha_1$, we define the *evaluation* of a representation of Q on ρ as the linear transformation $T_\rho = T_{\alpha_n} \circ \cdots \circ T_{\alpha_1}$. The *evaluation* of a representation on a relation $\rho = \sum_{i=1}^{r} \rho_i$ is defined as $T_\rho = \sum_{i=1}^{r} T_{\rho_i}$.

Definition 1.6.4. [ASS06, III.1 Definition 1.4] A representation (M_i, T_α) is said to be *bound* by the relation ρ if $T_\rho = 0$. Given an admissible ideal I, we say that the representation is *bound by I* if it is bound by every relation on I.

The full subcategory of all representations of Q bound by I will be denoted by $\mathbf{Rep}(Q, I)$. The full subcategory of all finite dimensional representations of Q bound by I will be denoted $\mathbf{rep}(Q, I)$

Theorem 1.6.5. *There exist equivalences of categories*

$$\frac{\mathbb{K}Q}{I} - \mathbf{Mod} \equiv \mathbf{Rep}(Q, I), \text{ and } \frac{\mathbb{K}Q}{I} - \mathbf{mod} \equiv \mathbf{rep}(Q, I) \tag{1.26}$$

Proof. See [ASS06, III.1 Theorem.6] for an analogous proof for a similar result for right modules. □

To summarize this chapter, we formalize the dictionary that we have constructed.

Table 1.1: *dictionary between Combinatorics and Algebra*

Combinatorics	**Algebra**
Quivers	Path algebras
concatenation is associative	multiplication is associative
Q_0 finite	unital algebra and $\{e_a \mid a \in Q_0\}$ is copi
walks	R_Q
Q_1	R_Q/R_Q^2
arrows from a to b	$(e_b \, \mathbb{K} \, Q e_a)_1 = e_b \frac{R_Q}{R_Q^2} e_a$
paths of length up to n	$\mathbb{K} \, Q_{<n} = \frac{\mathbb{K} \, Q}{R_Q^n}$
paths of length n	$\mathbb{K} \, Q_n = \frac{R_Q^n}{R_Q^{n+1}}$
representations of quivers	modules over path algebras
Gabriel quivers	finite dimensional basic algebras
representations of quivers bound by relations	modules over finite dimensional basic algebras

In (1.1), the line *Gabriel quiver* ↔ *finite dimensional basic algebras* is not a bijection. To see this, consider the quiver

$$Q : \quad \bullet \circlearrowleft \quad .$$

Then, for any integer $n > 1$, Q is the Gabriel quiver of

$$\frac{\mathbb{K}[x]}{\langle x^n \rangle}.$$

Finally, notice that if one is interested in studying module categories of finite dimensional algebras up to categorical equivalence, then the algebraic side of the dictionary can be considered without the basic restriction.

Chapter 2

Homological Algebra

In this chapter we present Homological Theory of Associative Algebras. Then we apply it to finite dimensional algebras and classification problems using the technology developed in the first chapter. Our main references are [Rot09] and [Wei94].

2.1 Complexes and Homology

In this section we discuss homology theory for complexes of modules. This will be the background theory for what we aim to study in Chapters 3, 4, and 5. We mainly follow [Rot09] and [Wei94]. Fix A a $\mathbb{K}$-algebra and all modules will be left modules.

Definition 2.1.1. [Wei94, Definition 1.1.1] A *chain complex* of A modules is a sequence of A-modules $\{M_n \mid n \in \mathbb{Z}\}$ and homomorphisms of A-modules $\{d_n : M_n \to M_{n-1} \mid n \in \mathbb{Z}\}$

$$\cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots$$

such that $d_n \circ d_{n+1} = 0$, for all $n \in \mathbb{Z}$. Algebraically, this means that $\mathrm{Im}(d_{n+1}) \subseteq \mathrm{Ker}(d_n)$. We denote

$$(M_\bullet, d_\bullet) \doteq (\{M_n\}_{n\in\mathbb{Z}}, \{d_n\}_{n\in\mathbb{Z}})$$

or simply by $M_\bullet$. We call the maps $d_\bullet$ *differentials.*

As we do for most objects, we want a way for chain complexes to interact between them. By a *morphism of chain complexes* $f_\bullet : M_\bullet \longrightarrow N_\bullet$ we mean a sequence of homomorphisms of A-modules $\{f_n : M_n \to N_n \mid n \in \mathbb{Z}\}$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M_{n+1} & \xrightarrow{d_{n+1}} & M_n & \xrightarrow{d_n} & M_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & N_{n+1} & \xrightarrow{\delta_{n+1}} & N_n & \xrightarrow{\delta_n} & N_{n-1} & \longrightarrow & \cdots
\end{array}$$

such that each square commutes, i.e., $f_{n-1} \circ d_n = \delta_n \circ f_n, \ \forall\, n \in \mathbb{Z}$. We define the composition of morphisms of chain complexes as point-wise, that is, $(g \circ f)_n = g_n \circ f_n, \ \forall\, n \in \mathbb{Z}$. With this structure we obtain a category that we call *category of chain complexes of A-modules* and denoted by $\mathbf{Ch}(A - \mathbf{mod})$.

The above objects can be constructed in an analogous way for right A-modules and $(A-B)$-bimodules, as those are simply left A^{op}-modules and $A \otimes_{\mathbb{K}} B^{op}$-modules, respectively. Context will make clear what type of modules we are considering chain complexes of. The category of chain complexes over a module category is well behaved, in the sense that it has kernels, cokernels, zero object, finite sums, exact sequences of complexes, etc., see [Rot09, Proposition 5.100].

Example 2.1.2. For any exact sequence of modules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

we can add infinitely many zero modules to obtain a chain complex. For instance, if we define the position of N as index 0, we can consider

$$\cdots \xrightarrow{d_4=0} M_3 = 0 \xrightarrow{d_3=0} M_2 = L \xrightarrow{d_2=f} M_1 = M \xrightarrow{d_1=g} M_0 = N \xrightarrow{d_0=0} M_{-1} = 0 \xrightarrow{d_{-1}=0} \cdots$$

as an element of $\mathbf{Ch}(A - \mathbf{mod})$.

If for a chain complex $M_\bullet$ exists an integer $n \in \mathbb{Z}$ such that $M_k = 0$ for all $k > n+1$ or $k < n-1$ we will omit all zero modules and differentials. So in the case of the exact sequence, if we view one as a complex, we will keep writing it as an exact sequence.

One of the reasons to work with chain complexes is that we have a way to derive information from it. The n^{th} *homology* of a chain complex $(M_\bullet, d_\bullet)$ is defined as

$$H_n(M_\bullet, d_\bullet) \doteq \frac{\mathrm{Ker}\,(d_n)}{\mathrm{Im}\,(d_{n+1})} \subseteq \frac{M_n}{\mathrm{Im}\,(d_{n+1})}. \tag{2.1}$$

Essentially, the n^{th} homology of a chain complex measures how far from an equality $\mathrm{Im}\,(d_{n+1}) \subseteq \mathrm{Ker}\,(d_n)$ is, since if $H_n(M_\bullet) = 0$, then the equality holds. If all $H_n(M_\bullet) = 0$, we call $M_\bullet$ an *acyclic chain complex*. By *homology of a chain complex* we mean the collection $\{H_n(M_\bullet) \mid n \in \mathbb{Z}\}$. Next we discuss the interaction between homology and chain complex morphisms.

Proposition 2.1.3. *[Wei94, Exercise 1.1.2] If $f_\bullet : (M_\bullet, d_\bullet) \longrightarrow (N_\bullet, \delta_\bullet)$ is a morphism of complexes, then*

$$\begin{aligned} \bar{f_n} : H_n(M_\bullet) &\longrightarrow H_n(N_\bullet) \\ \bar{x} &\longmapsto \bar{f_n(x)} \end{aligned}$$

is well defined. In particular, for each $n \in \mathbb{Z}$, $H_n : \mathbf{Ch}(A-\mathbf{mod}) \longrightarrow A-\mathbf{mod}$ is a covariant functor.

Proof. If $x \in \operatorname{Ker}(d_n)$, then $\delta_n f_n(x) = f_{n-1}d_n(x) = 0$ implies $f_n(x) \in \operatorname{Ker}(\delta_n)$ and it makes sense to consider $\bar{f_n(x)} \in H_n(N_\bullet)$. To show that it does not depend on the representative $x \in \operatorname{Ker}(d_n)$, suppose that $y = d_{n+1}(w) \in \operatorname{Im}(d_{n+1})$, then $f_n(y) = f_n d_{n+1}(w) = \delta_{n+1} f_{n+1}(w) \in \operatorname{Im}(\delta_{n+1})$. □

A natural question to ask is: is there a property of $\mathbf{Ch}(A - \mathbf{mod})$ that the homology does not distinguish. To answer it consider the following definition.

Definition 2.1.4. By an *homotopy* between two parallel morphisms of chain complexes $f_\bullet, g_\bullet : M_\bullet \longrightarrow N_\bullet$ we mean a sequence of homomorphisms of A-modules $\{h_n : M_n \to N_{n+1}\}_{n\in\mathbb{Z}}$ such that $\delta_{n+1}h_n + h_{n-1}d_n = f_n - g_n$. Pictorially,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M_{n+1} & \xrightarrow{d_{n+1}} & M_n & \xrightarrow{d_n} & M_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}-g_{n+1}} & \swarrow{\scriptstyle h_n} & \downarrow{\scriptstyle f_n-g_n} & \swarrow{\scriptstyle h_{n-1}} & \downarrow{\scriptstyle f_{n-1}-g_{n-1}} & & \\ \cdots & \longrightarrow & N_{n+1} & \xrightarrow[\delta_{n+1}]{} & N_n & \xrightarrow[\delta_n]{} & N_{n-1} & \longrightarrow & \cdots \end{array} \tag{2.2}$$

In this case we say that $f_\bullet$ and $g_\bullet$ are *homotopic.*

Proposition 2.1.5. *[Rot09, Theorem 6.14] If $f_\bullet$ and $g_\bullet$ are homotopic, then $H_n(f_\bullet) = H_n(g_\bullet)$, for all $n \in \mathbb{Z}$.*

Proof. If $x \in \operatorname{Ker}(d_n)$, then $(f_n - g_n)(x) = (\delta_{n+1}h_n + h_{n-1}d_n)(x) = \delta_{n+1}h_n(x) \in \operatorname{Im}(\delta_{n+1})$. □

Example 2.1.6. Suppose, for a complex $M_\bullet$, that $1_{M_\bullet}$ is homotopic to 0. If $x \in \operatorname{Ker}(d_n)$, then $x = 1x = (d_{n+1}h_n + h_{n-1}d_n)(x) = d_{n+1}h_n(x) \in \operatorname{Im}(d_{n+1})$. This means that $M_\bullet$ is an exact sequence.

We end with a technical but enlightening result about homotopy.

Theorem 2.1.7. *[Rot09, Theorem 6.10] If*

$$0 \longrightarrow M_\bullet \xrightarrow{f_\bullet} N_\bullet \xrightarrow{g_\bullet} L_\bullet \longrightarrow 0$$

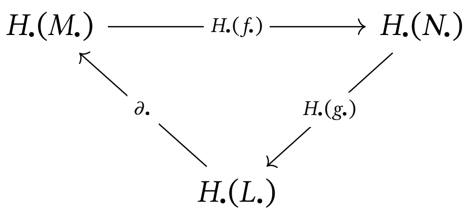
is an exact sequence of chain complexes, then there exists an exact sequence of A-modules

$$\cdots \longrightarrow H_{n+1}(L_\bullet) \xrightarrow{\partial_{n+1}} H_n(M_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(N_\bullet) \xrightarrow{H_n(g_\bullet)} H_n(L_\bullet) \xrightarrow{\partial_n} H_{n-1}(M_\bullet) \longrightarrow \cdots \tag{2.3}$$

The morphisms ∂_n are called connecting homomorphisms.

Proof. See, [Rot09] Proposition 6.9 and Theorem 6.10. □

Remark 2.1.8. There is a pictorial way to remember the above result



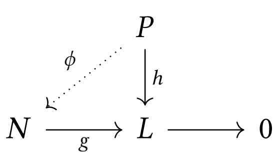
(2.4)

Dually one can define *cochain complexes*, *morphisms of cochain complexes*, and *cohomology*. Since they are analogous to the above definition, we omit them.

2.2 Projectives, Ext, and Tor

In this section we are going to discuss how to obtain (co)chain complexes from a module. The main references are the same, [Rot09] and [Wei94]. Before we are able to obtain a chain complex from a module, we will first derive a way to obtain an acyclic chain complex. To do so we need some definitions.

Definition 2.2.1. An A-module P is said to be *projective* if for every epimorphism $g : N \longrightarrow L$ and every homomorphism $h : P \longrightarrow L$ there exists a homomorphism $\phi : P \longrightarrow N$ such that $h = g \circ \phi$. In diagrams,



An *injective* module I has the dual property: for every monomorphism $f : M \longrightarrow N$ and every homomorphism $j : M \longrightarrow I$ there exists a homomorphism $\psi : N \longrightarrow I$ such that $j = \psi \circ f$. In diagrams,

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{f} & N \\ & & {\scriptstyle j}\downarrow & \swarrow_{\psi} & \\ & & I & & \end{array}$$

Example 2.2.2. The projective modules are precisely direct summands of free modules, see [Rot09, Theorem 3.5]. Therefore, if $\{e_1, \dots, e_n\}$ is a copi for a finite dimensional $\mathbb{K}$-algebra A, then each direct summand of A

$$P(i) \doteq Ae_1$$

is projective. If A is basic, then one can think of $P(i)$ as all the paths of A starting at the vertex i.

It is possible to prove that for each module M there exists a monomorphism $M \to I$ with I injective (we say that the category has *enough injectives*) and an epimorphism $P \to M$ with P projective (similarly, *enough projectives*). We can use this information to obtain chain complexes of modules.

Fix $M \in A - \mathbf{Mod}$ and consider an epimorphism $g : P_0 \to M$, think of it as a way to approximate M using "nice" modules. What is the error of this approximation? If M were to be projective, we could take g to be the identity and we would not commit any error, this intuition matches with the kernel of g being zero; therefore we can think of $\ker(g)$ as a measure of how good we can approximate M via g. What to do with this error? We could try to approximate it and we already have a way to proceed: let's discover how far from being projective it is! Consider an epimorphism $d_1 : P_1 \to \mathrm{Ker}(g)$ with P_1 projective. Recursively, we will get a long exact sequence that has a name.

Definition 2.2.3. By a *projective resolution of* M we mean an exact sequence

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{g} M \longrightarrow 0$$

such that each P_i is projective. An *injective resolution* is an exact sequence

$$0 \longrightarrow M \xrightarrow{f} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots$$

such that each I^j is injective.

The *deleted projective resolution* or simply *deleted resolution* of a projective resolution for M is, by definition, the chain complex

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0.$$

In a similar way, we define the *deleted injective resolution* or simply *deleted resolution* of an injective resolution as the chain complex

$$0 \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots$$

One can always recover the original resolution looking at the 0-(co)homology of the deleted resolution.

Fix $M, N \in A - \mathbf{mod}$ and consider a projective resolution for M in $A - \mathbf{mod}$:

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

Applying the contravariant functor $\mathrm{Hom}_A(-, N)$ to the deleted projective resolution above we obtain a cochain complex

$$\mathrm{Hom}_A(P_\bullet, N) : \; 0 \longrightarrow \mathrm{Hom}_A(P_0, N) \longrightarrow \mathrm{Hom}_A(P_1, N) \longrightarrow \mathrm{Hom}_A(P_2, N) \longrightarrow \cdots \tag{2.5}$$

We define the n^{th} Ext *group of* M *and* N by the cohomology of the above cochain complex, that is,

$$\mathrm{Ext}^n_A(M, N) \doteq H^n(\mathrm{Hom}_A(P_\bullet, N))$$

It is possible to show that the above groups (or $\mathbb{K}$-vector spaces in our framework) do not depend on the chosen projective resolution of M, as another resolution would produce an isomorphic group. We have the following easy proposition.

Proposition 2.2.4. *An A-module M is projective if, and only if,* $\mathrm{Ext}^1(M, N) = 0$, *for all A-modules N.*

With this we conclude one of the ways of studying modules using (co)chain complexes: fix a module, construct a projective resolution, construct a cochain complex applying $\mathrm{Hom}_A(-, N)$, for some $N \in A - \mathbf{mod}$, take its cohomology, use it to derive properties from the fixed module. The above proposition is one example of it: one can look at all the first Ext groups with M in the first coordinate in order to decide if M is projective or not. We also mention that it is possible to compute the Ext groups taking injective resolutions on the second module, for more details see [Rot09] and [Wei94].

Consider M and a projective resolution for it as above, but now suppose that N is an A^{op}-module. Then consider the following chain complex obtained applying $N \otimes_A -$ to the deleted projective resolution of M

$$N \otimes_A P_\bullet : \ \cdots \longrightarrow N \otimes_A P_2 \longrightarrow N \otimes_A P_1 \longrightarrow N \otimes_A P_0 \longrightarrow 0$$

We define the n^{th} Tor *group of* N *and* M by the homology of the above chain complex

$$\mathrm{Tor}_n^A(N, M) \doteq H_n(N \otimes_A P_\bullet).$$

Again, it is possible to show that the above groups (or $\mathbb{K}$-vector spaces in our framework) do not depend on the chosen projective resolution of M, as another resolution would produce an isomorphic group. It is also possible to compute Tor groups using projective resolutions of N as an A^{op}-module. The intuition for why to study Tor is the same as to study Ext: we want to derive properties of a module looking at its Tor groups. Finally, the long exact sequences for (co)homology groups yields.

Proposition 2.2.5. *If* $0 \to M \to N \to L \to 0$ *is an exact sequence of* A*-modules, then*

1. *for any* $X \in A - \mathbf{mod}$ *there is a long exact sequence for the* Ext *groups*

$$0 \longrightarrow \mathrm{Hom}_A(L, X) \longrightarrow \mathrm{Hom}_A(N, X) \longrightarrow \mathrm{Hom}_A(M, X) \longrightarrow \mathrm{Ext}_A^1(L, X) \longrightarrow \mathrm{Ext}_A^1(N, X) \longrightarrow \mathrm{Ext}_A^1(M, X) \longrightarrow \mathrm{Ext}_A^2(L, X) \longrightarrow \cdots$$

2. *for any* $Y \in A^{op} - \mathbf{mod}$ *there is a long exact sequence for the* Tor *groups*

$$\cdots \longrightarrow \mathrm{Tor}_2^A(Y, L) \longrightarrow \mathrm{Tor}_1^A(Y, M) \longrightarrow \mathrm{Tor}_1^A(Y, N) \longrightarrow \mathrm{Tor}_1^A(Y, L) \longrightarrow Y \otimes_A M \longrightarrow Y \otimes_A N \longrightarrow Y \otimes_A L \longrightarrow 0$$

There is also analogous long exact sequences for the functors $\mathrm{Ext}_A^*(X, -)$ *and* $\mathrm{Tor}_*^A(-, X)$.

In this work, the above groups will serve as a way to compute homological dimensions that we will define in what follows. For that reason we will not give any explicit example of said groups.

2.3 Homological Dimensions

In this section we will define some important homological invariants for our work. The references are [ASS06], [Kir16], and [Hui95].

As we discussed on the previous section, we may view a projective resolution by a succession of projective approximations for a module. Naturally, it would be interesting to find what is *the best* approximation among all the possible ones. This is the idea that we develop in what follows.

Definition 2.3.1. Let $M \in A - \mathbf{mod}$ and $P. \to M \to 0$ a projective resolution of M as an A-modules. We define the length of $P.$ as the least index n such that $P_n \neq 0$ and $P_{n+k} = 0$ whenever $k > 0$ and denote $\ell(P.) = n$. In case that there is no such index, we say that $\ell(P.) = \infty$. The *projective dimension* of M as an A-module is the non negative integer

$$\operatorname{pd}_A M \doteq \min\{\ell(P.) \mid P. \text{ is a projective resolution for } M \text{ as an } A\text{-module}\}.$$

Similarly one defines injective dimension of an A-module.

Example 2.3.2. If P is a projective A-module, then $1_P : P \to P$ is a projective resolution for P as A-module, therefore $\operatorname{pd}_A P = 0$.

A simple connection between projective dimensions and Ext-Tor groups is a consequence of the definition of these groups using projective resolutions

Lemma 2.3.3. *If* $\operatorname{Tor}_n^A(X, M) \neq 0$ *or* $\operatorname{Ext}_A^n(M, Y) \neq 0$*, for some appropriate* X *and* Y*, then* $n \leqq \operatorname{pd}_A M$*.*

We can use this point wise definition of dimension to derive an homological dimension for the category $A - \mathbf{mod}$ or, if you prefer, for A.

Definition 2.3.4. The *global dimension* of A is defined as

$$\operatorname{gldim}(A) = \sup\{\operatorname{pd}_A M \mid M \in A - \mathbf{mod}\}.$$

Example 2.3.5. An algebra A is said to be semisimple when every A-module is projective. Therefore $\operatorname{pd}_A M = 0$, for all $M \in A - \mathbf{mod}$, and $\operatorname{gldim}(A) = 0$. Simple computations show that this is an equivalence: the semisimple algebras are precisely the algebras with global dimension zero.

In a perfect world, the global dimensions would give us a measure of how complicated

$A - \mathbf{mod}$ is and, therefore, how complicated A is. But this is not the case. Consider B the algebra of dual numbers, that is,

$$B = \frac{\mathbb{K}[x]}{\langle x^2 \rangle}.$$

The category $B - \mathbf{mod}$ has only two indecomposable modules ${}_B\mathbb{K}$, the simple module with x acting as zero, and ${}_BB$, the regular module. Obviously, ${}_BB$ is B-projective, but ${}_B\mathbb{K}$ is not and the best projective resolution for it is

$$\cdots \longrightarrow P_n = B \longrightarrow \cdots \longrightarrow B \longrightarrow B \longrightarrow \mathbb{K} \longrightarrow 0$$

therefore $\operatorname{pd}_B \mathbb{K} = \infty$ and $\operatorname{gldim}(B) = \infty$. This is not intuitive, since the small number of indecomposable modules of $B - \mathbf{mod}$ should not be measured as infinitely complicated. In order to obtain a better invariant for this situation we define.

Definition 2.3.6. The *finitistic dimension* of A is defined as

$$\operatorname{findim}(A) = \sup\{\operatorname{pd}_A M \mid M \in A - \mathbf{mod} \text{ and } \operatorname{pd}_A M < \infty\}.$$

Example 2.3.7. If B is as above, then $\operatorname{findim}(B) = 0$.

Notice that, if A were simple enough in the previous measure of complexity, that is, $\operatorname{gldim}(A) = n < \infty$, then $\operatorname{findim}(A) = \operatorname{gldim}(A) = n$. In other words, the finitistic dimension is a measure of complexity that is equal to the global dimension, when the global dimension is finite, but can be finite in cases of infinite global dimensions. But in what cases the finitistic dimension is finite and the global dimension is infinite? Well, this is a conjecture...

Conjecture 2.3.8. (Little) Finitistic Dimension Conjecture If A is a finite dimensional $\mathbb{K}$-algebra, then

$$\operatorname{findim}(A) < \infty.$$

For an overview of the Finitistic Dimension Conjecture, we refer the reader to [Hui95] and references therein. The Finitistic Dimension Conjecture is central in representation theory of algebras. As one can see in [ARS95, Conjecture 11], it implies several other conjectures. For instance, the Nakayama Conjecture (see [ARS95, Conjectures 8]) holds if the Finitistic Dimension Conjecture holds.

2.4 How to Compute Classical Homological Dimensions

In this section we are going to discuss some results towards computing the classical homological dimensions. The main reference is [Aus55]. We restrict the results for finite dimensional $\mathbb{K}$-algebras.

When we introduced the Tor and Ext functors we alluded to the idea of deriving properties of modules using these groups. Then, on the next section, we associated an homological invariant to each module, namely $\operatorname{pd}_A M$. As a simple example, we enunciated that M is projective if, and only if, $\operatorname{Ext}^1_A(M, X) = 0$, for every $X \in A - \mathbf{mod}$, if, and only if, $\operatorname{pd}_A M = 0$. Now, suppose that $0 \to M \to N \to L \to 0$ is an exact sequence of A-modules, and that $\operatorname{Ext}^n_A(-, M) = \operatorname{Ext}^n_A(-, L) = 0$ are the zero functors, for all $n > k$, then the long exact sequence says that $\operatorname{Ext}^n_A(-, N) = 0$, for all $n > k$. If we inductively carry on this process using a composition series for N we obtain that the nullity of $\operatorname{Ext}^*_A(-, N)$ is controlled by the nullity of $\operatorname{Ext}^*_A(-, S)$ for any simple module that appears as a factor in the composition series. The above commentary, for the Ext groups, indicates that what matters for projective dimensions is to understand when simple modules have trivial Ext groups. This is formalized on the next result.

Proposition 2.4.1. *[Aus55, Proposition 7 and Corollary 9] Let A is a finite dimensional $\mathbb{K}$-algebra and $M \in A - \mathbf{mod}$. The following are equivalent:*

1. $\operatorname{pd}_A M < n$
2. $\operatorname{Tor}^A_n(S, M) = 0$*, for every simple A^{op}-module.*
3. $\operatorname{Ext}^n_A(M, S) = 0$ *for every simple A-module.*

But we can specialize the above result for $M = S$ a simple module, by doing so we obtain that is enough to compute Tor and Ext on pairs of simple modules is order to compute the projective dimension of simple modules. But if we apply the argument about the composition series on the first coordinate of Ext (and the second of Tor) we see that by computing it on all pairs of simple modules, we are actually computing $\operatorname{gldim}(A)$. Again, this is formalized on the next result.

Proposition 2.4.2. *[Aus55, Corollary 12] Let A be a finite dimensional basic algebra, then the following are equivalent*

1. $\operatorname{gldim}(A) < n$,
2. $\operatorname{Ext}^n_A(S_i, S_j) = 0$*, for all simple modules.*

3. $\operatorname{Tor}_A^n(S_{iA}, {}_AS_j) = 0$, *for all simple modules.*

Corollary 2.4.3. *If A is a finite dimensional $\mathbb{K}$-algebra, then*

$$\operatorname{gldim}(A) = \max\{\operatorname{pd}_A S \mid S \textit{ is a simple } A\textit{-module}\} = \operatorname{pd}_A \frac{A}{J(A)},$$

where $J(A)$ is the Jacobson radical of A.

Finally we end this section with results about how the global dimension change if we do some operations with algebras. First, taking the opposite algebra does not change the global dimension.

Proposition 2.4.4. *[Aus55, Corollary 5] If A is a finite dimensional basic algebra, then*

$$\operatorname{gldim}(A) = \operatorname{gldim}(A^{op}).$$

Secondly, if we take tensor products of algebras, then we need only to sum each global dimension.

Proposition 2.4.5. *[Aus55, Theorem 16] If A and B are finite dimensional basic $\mathbb{K}$-algebras, then*

$$\operatorname{gldim}(A \otimes B) = \operatorname{gldim}(A) + \operatorname{gldim}(B).$$

In particular, we get.

Corollary 2.4.6. *If A is a finite dimensional algebra, then*

$$\operatorname{gldim}(A^e) = 2\operatorname{gldim}(A).$$

Proof.

$$\operatorname{gldim}(A^e) = \operatorname{gldim}(A \otimes_{\mathbb{K}} A^{op}) = \operatorname{gldim}(A) + \operatorname{gldim}(A^{op}) = 2\operatorname{gldim}(A).$$

□

2.5 Applications to Bound Quiver Algebras

Recall that bound quiver algebras are algebras of the form

$$A = \frac{\mathbb{K}\, Q_A}{I},$$

where $I \lhd \mathbb{K}\, Q_A$ is an admissible ideal and Q_A is the Gabriel quiver of A. In this section we will discuss how to compute global dimensions of this type of algebras. The main references are [ASS06], [Kir16], and [Far07].

A first approach to bound quiver algebras is to consider a finite and acyclic quiver with $I = 0$. The next result show how to compute a projective resolution for an arbitrary module. Note that Q need not to be acyclic.

Proposition 2.5.1. *[Kir16, Theorem 1.19] Let Q be a quiver. For any $\mathbb{K}\, Q$-module V we have the following projective resolution*

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1} P(t(\alpha)) \otimes \mathbb{K}\, \alpha \otimes V_{s(\alpha)} \xrightarrow{d_1} \bigoplus_{i \in Q_0} P(i) \otimes V_i \xrightarrow{d_0} V \longrightarrow 0 \tag{2.6}$$

where $d_0(p \otimes v) = pv$ and $d_1(p \otimes \alpha \otimes v) = p\alpha \otimes v - p \otimes \alpha v$. In particular

$$\operatorname{gldim}(\mathbb{K}\, Q) \leqq 1.$$

It is easy to see that if $Q_1 = \emptyset$, then the above resolution has length 0 for any module. Therefore $\operatorname{gldim}(\mathbb{K}\, Q) = 0$. By the previous section, we need only to compute the projective dimension of simple modules in order to understand when $\operatorname{gldim}(\mathbb{K}\, Q) = 1$. Specializing the above resolution for the simple module concentrated on the vertex i we obtain

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1, s(\alpha) = i} P(t(\alpha)) \longrightarrow P(i) \longrightarrow S(i) \longrightarrow 0$$

In particular, if there is an arrow on Q, then there exists a vertex i such that the above resolution has length one. It is easy to prove that in this case, for the simple module concentrated on i, $\operatorname{pd}_{\mathbb{K}\, Q} S(i) = 1$ and, therefore, $\operatorname{gldim}(\mathbb{K}\, Q) = 1$.

What is not so obvious is that a basic finite dimensional $\mathbb{K}$-algebra with global dimension one is a path algebra of a finite acyclic quiver.

Theorem 2.5.2. *[ASS06, VII.1 Theorem 1.7.(b)] If A is a basic, connected, and* $\operatorname{gldim}(A) = 1$, *then $A \cong \mathbb{K}\, Q_A$, where Q_A is the Gabriel quiver of A and it is finite, connected, acyclic quiver with $Q_1 \neq \emptyset$.*

Remark 2.5.3. Algebras with global dimension at most one are called *hereditary algebras*. The above results are a classification of hereditary, finite dimensional and basic algebras.

Now we turn our attention to quotients by non trivial admissible ideals. What makes it easier to work with such algebras is that for finite dimensional algebras we can restrict the scope to good epimorphisms. The main notion behind it is that o a *superfluous* submodule: $L \subseteq M$ is superfluous if $L + X = M$, for X another submodule, implies $X = M$. We say that an epimorphism $f : M \to N$ is *minimal* if $\mathrm{Ker}(f)$ is superfluous. Interacting this notions with homological algebra we get the following definitions.

Definition 2.5.4. [ASS06, I.5 Definition 5.5.(b) and Definition 5.7.(b)] A *projective cover* is a minimal epimorphism $h : P \to M$ with P projective. A projective resolution $P_\bullet \to M \to 0$ for M is said to be *minimal* if each $d_i : P_i \to \mathrm{Im}(d_i)$ is a projective cover

Proposition 2.5.5. *[ASS06, I.5 Theorem 5.8.(a)] Let A be a finite dimensional $\mathbb{K}$-algebra and $\{e_1, \dots, e_n\}$ a copi. Then for any $M \in A - \mathbf{mod}$ there exists a projective cover*

$$P(M) \xrightarrow{\ h\ } M \longrightarrow 0,$$

where $P(M) \cong \bigoplus_{i \in Q_0} P(i)^{n_i}$, *with* $n_i \geqq 0$, *and* $\mathrm{top}\, P(M) \cong \mathrm{top}\, M$.

Remark 2.5.6. The projective modules $P(i) = Ae_i$ for a complete set of representatives of the isoclasses of indecomposable projective modules, see [ASS06, I.5. Corollary 5.17.(b)]. This means that the only relevant projective modules of $A - \mathbf{mod}$ are the $P(i)$'s.

The projective cover is unique up to an isomorphism (see [ASS06, I.5.8.(b)]). As a consequence of the above proposition, the category $A - \mathbf{mod}$ for A finite dimensional admits minimal projective resolutions. One can show that if $P_\bullet \to M \to 0$ is a minimal projective resolution for M, then its length is precisely $\mathrm{pd}_A M$, hence the name. The kernel of a projective cover for $M \in A - \mathbf{mod}$ is unique up to isomorphism. We will call it the *A-syzygy of M* and we denote it by

$$\Omega_A(M) \in A - \mathbf{mod}.$$

We recursively denote $\Omega_A^0(M) = M$ and $\Omega_A^n(M) = \Omega_A^{n-1}(\Omega_A(M))$, for an integer $n > 0$. We call $\Omega_A^n(M)$ the *n^{th}-syzygy of M (with respect to A).*

Lemma 2.5.7. *$M \in A - \mathbf{mod}$ is projective if, and only if, $\Omega_A(M) = 0$.*

Proof. If M is projective, then the identity $1_M : M \to M$ is a projective cover and $\Omega_A(M) = 0$. Conversely, if $\Omega_A(M) = 0$, then a projective cover for M is injective, and therefore an isomorphism. □

Now we turn to some computations using the notion of minimal resolutions.

Example 2.5.8. Let Q be the quiver

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4 \xrightarrow{d} 5$$

and $A = \frac{\mathbb{K}Q}{J^2(\mathbb{K}Q)}$. Then the indecomposable projectives are

$$P(1) : \mathbb{K} \xrightarrow{1} \mathbb{K} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

$$P(2) : 0 \longrightarrow \mathbb{K} \xrightarrow{1} \mathbb{K} \longrightarrow 0 \longrightarrow 0$$

$$P(3) : 0 \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{1} \mathbb{K} \longrightarrow 0$$

$$P(4) : 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{1} \mathbb{K}$$

$$P(5) \cong S(5) : 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{K}.$$

A minimal projective resolution for $S(1)$ is

$$0 \longrightarrow P(5) \cong S(5) \longrightarrow P(4) \longrightarrow P(3) \longrightarrow P(2) \longrightarrow P(1) \longrightarrow S(1) \longrightarrow 0$$

and $\operatorname{pd}_A S(1) = 4$. Computing the minimal projective dimension of the other simple modules, one sees that the projective dimension of $S(1)$ is the maximum of said dimensions. Therefore

$$\operatorname{gldim}(A) = 4.$$

Generalizing on the number of vertices, we obtain that

$$\operatorname{gldim}(A_n) = n - 1,$$

where $A_n \doteq \frac{\mathbb{K}\mathbb{A}_n}{J^2(\mathbb{K}\mathbb{A}_n)}$, where the orientation of $\mathbb{A}_n$ is all the arrows pointing to the same direction.

This is actually the upper bound of the following theorem.

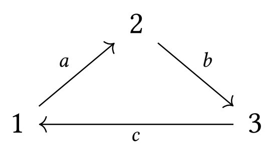
Theorem 2.5.9. *[Far07, Theorem] Let A be a bound quiver algebra of an acyclic quiver and Q_A its Gabriel quiver. Then*

$$\mathrm{gldim}(A) \leqq n - 1,$$

where n is the number of vertices of Q_A.

The next two examples show that no such result can be obtained for cyclic quivers.

Example 2.5.10. Let Q be the quiver



and consider $I = J^2(\mathbb{K}Q) = \langle ba, cb, ac\rangle$ and $L = \langle ba, cb\rangle$. It is obvious that I is admissible, and $R_Q^4 \subseteq L \subseteq R_Q^2$ is also admissible. But

$$\mathrm{gldim}(\frac{\mathbb{K}Q}{I}) = \infty \text{ and } \mathrm{gldim}(\frac{\mathbb{K}Q}{L}) = 3.$$

Chapter 3

Relative Homological Algebra

In this chapter we discuss the theory developed by Hochschild in [Hoc56]. Then we discuss the connections between of Hochschild's relative homological theory and classical homological problems made by [XX13], [Guo18], and [IM21].

3.1 Relative Homological Algebra

Definition 3.1.1. Let A be a $\mathbb{K}$-algebra. By a *subalgebra of* A we mean a subspace $B \subseteq A$ such that $1_A \in B$ and $x \cdot y \in B$ whenever $x, y \in B$, where $\cdot$ stands for the multiplication of A. We will call $B \subseteq A$ an *extension of algebras.*

Throughout this chapter fix $B \subseteq A$ an extension of algebras.

Example 3.1.2. Let A be any $\mathbb{K}$-algebra, then A has two trivial subalgebras:

1. $\mathbb{K} \equiv \mathbb{K}\,1_A \subseteq A$: the only one dimensional subalgebra of A
2. $A \subseteq A$: the only $\dim_{\mathbb{K}} A$-dimensional subalgebra of A when A is finite dimensional.

Example 3.1.3. Let $A = \mathrm{Mat}_2(\mathbb{K})$ denote the full 2×2 matrix ring over $\mathbb{K}$. Let $B = UT_2(\mathbb{K})$ the upper triangular 2×2 matrices, $C = \mathbb{K}\,e_{21}$ be the strictly upper triangular 2×2 matrices and $D = \mathbb{K}\,e_{11}$. Then $B \subseteq A$ is an extension of algebras. Even though both C and D are closed under the multiplication of A, they are not considered subalgebras since C is a non-unital ring and the identity of D is $1_D = e_{11} \neq e_{11} + e_{22} = 1_A$

Example 3.1.4. Let $A = kQ$, where Q is the following quiver

$$\overset{1}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet} \tag{3.1}$$

and consider B the subalgebra generated by the elements $1_A = e_1 + e_2$ and α. Then $B \subseteq A$ is an extension of algebras. In this case B is a bound quiver algebra and one can depict its Gabriel quiver as

$$\underset{1+2}{\bullet} \circlearrowright \alpha \tag{3.2}$$

and the relation ideal of B is generated by α^2. Notice that B is the algebra of dual numbers.

We want to study some homological theory that measures how complex an extension $B \subseteq A$ is. To do so, we first need to define a class of chain complexes to work with. In order to define them we need the notion of an *(A,B)-exact sequence.*

Definition 3.1.5. By an *(A,B)-exact sequence* we mean an exact sequence of A-homomorphisms and A-modules

$$\cdots \xrightarrow{f_{n+2}} M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \cdots \tag{3.3}$$

such that, for any index, $\operatorname{Ker}(f_n)$ is a direct summand of M_n as B-modules. In other words, there exists a B-module M'_n such that

$$_BM_n = {}_B\operatorname{Ker}(f_n) \oplus {}_BM'_n. \tag{3.4}$$

There is a useful equivalence for an (A, B)-exact sequence.

Lemma 3.1.6. *Let*

$$(M_\bullet, d_\bullet) : \cdots \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots \tag{3.5}$$

be an exact sequence of A-modules. Then the following are equivalent:

1. *$M_\bullet$ is (A, B)-exact;*
2. *there exists a family of B-homomorphisms $h_n : M_n \longrightarrow M_{n+1}$, $n \in \mathbb{Z}$, such that $d_{n+1} \circ h_n + h_{n-1} \circ d_n = 1_{M_n}$, and;*

3. *there exists a family of B-homomorphisms $h_n : M_n \longrightarrow M_{n+1}$, $n \in \mathbb{Z}$, such that $d_n = d_n \circ h_{n-1} \circ d_n$.*

Example 3.1.7. If $B \subseteq A$ is an extension of $\mathbb{K}$-algebras and B is a semisimple algebra, then every exact sequence of A-modules is (A, B)-exact. Indeed, every B-submodule of a B-module admits a B-complement. This type of extension always exist, just consider $\mathbb{K} \subseteq A$. For the other trivial extension, that is $A \subseteq A$, the exact sequences are precisely all the sequences of A-modules that split, see (3.1.6.2).

Algebraic homological theories are the study of complexes constructed using "special well behaved" objects. In our case, those are going to be called *relatively projectives* and *relatively injectives.* Our study will focus on relative projective modules.

Definition 3.1.8. An A-module P is *(A, B)-projective* or simply *relatively projective* when for any diagram

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & \overset{\psi}{\swarrow} & \downarrow \phi & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \underset{f}{\longrightarrow} & N & \longrightarrow & 0 \end{array} \tag{3.6}$$

where the horizontal complex is (A, B)-exact and $\phi : P \to N$ is a homomorphism of A-modules, there exists an A-homomorphism $\psi : P \longrightarrow M$ such that the triangle commutes, that is, $f \circ \psi = \phi$. *Relatively injectives* or (A, B)-*injectives* are defined dually.

Example 3.1.9. The (A, A)-projective modules are $A - \mathbf{mod}$. On the other side, the $(A, \mathbb{K})$-projectives are the usual projective modules of $A - \mathbf{mod}$.

In general, for a tower of extensions of $\mathbb{K}$-algebras $C \subseteq B \subseteq A$, there is a similar relation between the relative exact sequences and relative projectives that extends the discussion made about the trivial extensions. One can see this relation at Figure 3.1.

The next results guarantees that there is always a relative homological theory for any extension of algebras.

Lemma 3.1.10. *[Hoc56, Lemma 2] For any $Y \in B$-**mod**, $A \otimes_B Y$ is (A, B)-projective*

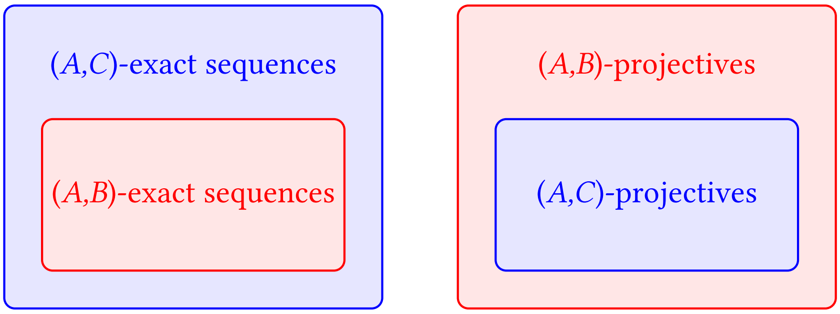


Figure 3.1: *Venn diagrams showing the relation between two extensions $B \subsetneq A$ (in red) and $C \subsetneq A$ (in blue) that form a tower $C \subsetneq B \subsetneq A$. Specializing $B = \mathbb{K}$ we obtain the relation between Relative Homological Theory for $B \subsetneq A$ and Classical Homological Theory.*

Example 3.1.11. Suppose that $P \in A - \mathbf{mod}$ is projective in the classical theory. In particular, P is $(A, \mathbb{K})$-projective. The above result states that P is a direct summand of $A \otimes_{\mathbb{K}} P \cong A^n$, where $n = \dim_{\mathbb{K}} P$. We have concluded, using relative homological theory, that projectives in $A - \mathbf{mod}$ are direct summand of free modules.

Corollary 3.1.12. *For any extension of algebras $B \subsetneq A$ there exists enough relative projective, that means, for any $M \in A$-$\mathbf{mod}$ there exists an epimorphism of A-modules that admits a section in B-$\mathbf{mod}$*

$$P \twoheadrightarrow M, \tag{3.7}$$

with P a relative projective module.

Proof. Simply take the multiplication map $\mu : A \otimes_B M \longrightarrow M$ and the B-section $\nu : M \longrightarrow A \otimes_B M$ is given by $\nu(m) = 1 \otimes_B m$, for all $m \in M$. □

There are analogous results for (A, B)-injectives, for those we refer the reader to [Hoc56]. Now we make the main definitions of this work: the relative homological dimensions we are going to study.

Definition 3.1.13. By an (A, B)-*projective resolution* or simply a *relative projective resolution* of $M \in A$-$\mathbf{mod}$ we mean an (A, B)-exact sequence

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0 \tag{3.8}$$

where all P_i's are (A, B)-projectives. Dually we define (A, B)-injective resolution.

Proposition 3.1.14. *For any $M \in A$-**mod** there exists an (A, B)-projective resolution.*

Proof. Via restriction consider $M \in B$-**mod**. Therefore, by (3.1.10), $A \otimes_B M$ is (A, B)-projective. We know that the kernel of the multiplication (A-)homomorphism $\mu_M : A \otimes_B M \longrightarrow M$ is a B-direct summand of $A \otimes_B M$ since μ_M admits a B-section. Denote $K_0 = \mathrm{Ker}(\mu_M)$.
Again the module $A \otimes_B K_0$ is (A, B)-projective, and $\mu_{K_0} : A \otimes_B K_0 \longrightarrow K_0$ is a B-split epimorphism, meaning that $K_1 = \mathrm{Ker}(\mu_{K_0})$ is a B-direct summand of $A \otimes_B K_0$. Continuing this process recursively one gets an exact sequence

$$\cdots \longrightarrow A \otimes_B K_r \xrightarrow{\mu_{K_r}} \cdots \xrightarrow{\mu_{K_1}} A \otimes_B K_0 \xrightarrow{\mu_{K_0}} A \otimes_B M \xrightarrow{\mu_M} M \longrightarrow 0 \tag{3.9}$$

that is (A, B)-exact by definition and, therefore, an (A, B)-projective resolution of M. □

Remark 3.1.15. The relative projective resolution constructed in (3.9) is called *standard relative (projective) resolution* or simply *standard (projective) resolution* if the data is clear from context.

Now we are able to define numbers that will measure how complicated our extension is and will be the main focus of this work.

Definition 3.1.16. [Guo18] For $M \in A$-**mod** we define its *(A, B)-relative projective dimension* (or *relative projective dimension*) as

$$\mathrm{pd}_{(A,B)} M \doteq \min\{\ell(P_\bullet) \mid P_\bullet \twoheadrightarrow M \text{ is a relative projective resolution}\}. \tag{3.10}$$

The *relative global dimension* of $B \subseteq A$ is defined as

$$\mathrm{gldim}(A, B) \doteq \sup\{\mathrm{pd}_{(A,B)} M \mid M \in A - \mathbf{mod}\}. \tag{3.11}$$

Remark 3.1.17. This definition is equivalent to the one in [XX13].

Example 3.1.18. Consider the extension $A \subseteq A$, then every module is relatively projective. This means that

$$\mathrm{pd}_{(A,A)} M = 0, \forall\, M \in A - \mathbf{mod} \text{ and } \mathrm{gldim}(A, A) = 0.$$

This makes sense, if we don't add anything to A, then we can not infer any *data* from the extension, so all the relative homological dimensions are zero.

Example 3.1.19. On the other hand, for the extension $\mathbb{K} \subseteq A$, the relative projective modules are, precisely, the A-projective modules. Therefore

$$\operatorname{pd}_{(A,\mathbb{K})} M = \operatorname{pd}_A M, \forall\, M \in A - \textbf{mod} \text{ and } \operatorname{gldim}(A,\mathbb{K}) = \operatorname{gldim}(A).$$

Again this makes sense, we are considering the data that constructs A from its smallest subalgebra (in terms of dimension as vector spaces), therefore its relative homological dimension must coincide with the classical homological dimensions.

Finally, we end this section discussing relative Ext and Tor groups. Let $M = M_A \in A^{op} - \textbf{mod}$ and $N = {}_AN \in A - \textbf{mod}$. Take $P_\bullet \to M \to 0$ an (A^{op}, B^{op})-projective resolution for M_A and $Q_\bullet \to N \to 0$ an (A,B)-projective resolution for ${}_AN$. Then we can consider the following complexes:

$$M \otimes_A Q_\bullet \text{ and } P_\bullet \otimes_A N.$$

We define the n^{th} *(A,B)* Tor *group* or simply n^{th} *relative* Tor *group* as

$$\operatorname{Tor}_n^{(A,B)}(M,N) = H_n(M \otimes_A Q_\bullet).$$

It is possible to show that $\operatorname{Tor}_n^{(A,B)}(M,N) = H_n(P_\bullet \otimes_A N)$ and that $\operatorname{Tor}_n^{(A,B)}(M,N)$ does not depend on $P_\bullet$ or $Q_\bullet$.

Similarly, if $M, N \in A - \textbf{mod}$, $P_\bullet \to M \to 0$ is an (A,B)-projective resolution, and $0 \to N \to I^\bullet$ is an (A,B)-injective resolution, then we define the n^{th} *(A,B)* Ext *group* or simply n^{th} *relative* Ext *group* as

$$\operatorname{Ext}^n_{(A,B)} = H^n(\operatorname{Hom}_A(P_\bullet, N)) = H^n(\operatorname{Hom}_A(M, I^\bullet)).$$

Again, the above groups do not depend on the resolutions.

We have the following useful consequence.

Proposition 3.1.20. *Let* $M \in A - \textbf{mod}$. *If* $\operatorname{pd}_{(A,B)} M = r$, *then* $\operatorname{Ext}^n_{(A,B)}(M,X) = 0$ *and* $\operatorname{Tor}_n^{(A,B)}(Y,M) = 0$, *for all* $n > r$, $X \in A - \textbf{mod}$, *and* $Y \in A^{op} - \textbf{mod}$.

Proof. Simply compute the Ext and Tor groups using a relative projective resolution for M of length r. □

In particular, whenever $\operatorname{Ext}^n_{(A,B)}(M,X) \neq 0$ or $\operatorname{Tor}_n^{(A,B)}(Y,M) \neq 0$, for some module, then $\operatorname{pd}_{(A,B)} M \geqq n$.

3.2 Relations with the Finitistic Dimension Conjecture

In this section we discuss the connection between Relative Homological Algebra and the Finitistic Dimension Conjecture made by the authors of [XX13]. In their paper, Xi and Xu worked with Artin algebras, but we are going to state the results in terms of finite dimensional $\mathbb{K}$-algebras. There is also other ways to relate the Finitistic Dimension Conjecture with extension of algebras, but they are not directely connected to Relative Homological Algebra. For the other ways we refer the reader to references [23], [24], and [26] of [XX13] and references therein. We also start introducing non trivial examples of extensions while computing their relative homological dimensions.

As we have seen on the previous chapter, Relative Homological Algebra is defined for all possible extensions of $\mathbb{K}$-algebras. This universality has positive and negative consequences. The positive is its potential for applications: if one is able to master Relative Homological Algebra and discover all that this theory has to offer, then that person is capable to prove several facts about finite dimensional algebras, for instance to decide if the Finitistic Dimension is true or not, as we are going to see in this section. On the negative side, even the specializations of Relative Homological Algebra, say for $B = \mathbb{K}$ in order to get the classical case, are too complicated to tackle with our current technologies. To work around this negative side, we are going to see some algebraic, combinatorial, homological, and categorical conditions that allow one to “control ” Relative Homological Algebra of some extensions. We begin by discussing the conditions imposed by the authors of [XX13].

First we need some notations. Denote by $\mathcal{P}(A, B)$ the full subcategory of $A - \mathbf{mod}$ generated by all (A, B)-projective A-modules.

Definition 3.2.1. An extension $B \subseteq A$ is said to be *n-hereditary* if, for any exact sequence of A-modules

$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0,$$

such that each X_i is (A, B)-projective for $0 \leqq i \leqq n-1$, X_n is also (A, B)-projective. An extension is said to be *relatively hereditary* if it is n-hereditary for some integer $n \geqq 0$.

Corollary 3.2.2. *If $B \subseteq A$ is n-hereditary, then* $\operatorname{gldim}(A, B) \leqq n$.

Remark 3.2.3. The condition for an extension to be n-hereditary is quite strong. If we take $C = \mathbb{K}$ in Figure 3.1, we see that it corresponds to a relation between the large squares. The above corollary is simply the observation that (A, B)-exact sequences (left red square)

are a particular case of (A, C)-exact sequences (left blue square).

Xi and Xu provide the following example of extension that is n-hereditary but is not $(n-1)$-hereditary. The details are available on Section 2 of [XX13].

Example 3.2.4. Let A be the algebra whose Gabriel quiver is

$$1 \xrightarrow{\alpha_1} 2 \longrightarrow \cdots \xrightarrow{\alpha_n} n+1 \xrightarrow{\alpha_{n+1}} n+2$$

and relations are all paths of length two. Let B be the subalgebra of A generated by all primitive idempotents and α_{n+1}. The extension $B \subseteq A$ is n-hereditary and is not $(n-1)$-hereditary.

The first categorical and homological condition required by Xi and Xu is to work with a category that interacts well with the homological theory for A.

Definition 3.2.5. [XX13] A subcategory C of $A - \mathbf{mod}$ is said to be *closed under syzygies* if, for any $M \in C$, its first syzygy $\Omega_A(X) \in C$ is also an object of C. In particular, $\Omega_A^n(X) \in C$, for any non negative integer n.

Lemma 3.2.6. *[XX13, Lemma 2.4.(2)] If $B \subseteq A$ is relatively hereditary, then $\mathcal{P}(A, B)$ is closed under kernels of surjective homomorphisms in $\mathcal{P}(A, B)$. In particular, it is closed under taking A-syzygies.*

The algebraic condition proposed in [XX13] is to impose that $J(B)$ is a left ideal in A. One of the major consequences of this fact is that if X is an A-module, then $J_B(X) = J(B)X$ is a left A-module. This structure then can be extended to show that $\mathrm{top}_B(X)$ and $\Omega_B(X)$ are A-modules. For a stronger relation, in [XX13, Lemma 2.2], the authors show that

$$\Omega_B^i({}_BY)$$

is an A-module provided $i \geqq 2$.

When one consider extensions that satisfies both conditions, $J(B)$ is a left ideal and $\mathcal{P}(A, B)$ is closed for syzygies, it is possible to prove several connections about A and B-modules. For instance, the radical condition assures that ${}_A\Omega_B(X) \cong {}_A\Omega_A(A \otimes_B X)$ for any $X \in A - \mathbf{mod}$ (see [XX13, Lemma 2.5]), but being close to syzygies and (3.1.10) says that ${}_A\Omega_A(A \otimes_B X) \in \mathcal{P}(A, B)$, this is [XX13, Corollary 2.7.(3)].

With the above discussion, if $X \in \mathcal{P}(A, B)$, then

$$ {}_A\Omega_B(X) \cong {}_A\Omega_A(A \otimes_B X) \cong {}_A\Omega_A(X) \oplus {}_A\Omega_A(X'), $$

where $X \oplus X' = A \otimes_B X$ as A-modules. Now, if we suppose that $\operatorname{pd}_B X < \infty$, then when we restrict the above isomorphism to $B - \mathbf{mod}$ we obtain that

$$ {}_B\Omega_B(X) \cong {}_B\Omega_A(X) \oplus {}_B\Omega_A(X'). $$

By hypothesis, $\Omega_B^n(X) = 0$, for some $n \geqq 1$, and the above isomorphism says that $\Omega_B^{n-1}(\Omega_A(X)) = 0$, therefore $\operatorname{pd}_B \Omega_A(X) < \infty$, see [XX13, Corollary 2.8].

Finally, before we present an idea of the proof of Xi and Xu's main theorem, we need the following technical lemma.

Lemma 3.2.7. *[XX13, Lemma 2.9.(1)] Suppose that $B \subsetneq A$ is an extension of finite dimensional $\mathbb{K}$-algebras such that $J(B)$ is a left ideal in A. If $X \in A - \mathbf{mod}$ and $n \in \mathbb{N}$ are such that ${}_A\Omega_B^i(X) \in \mathcal{P}(A, B)$ for $0 \leqq i \leqq n - 1$, then we have an isomorphism of A-modules*

$$ {}_A\Omega_B^j(X) \cong {}_A\Omega_A^j(X) \oplus \bigoplus_{i=1}^{j} {}_A\Omega_A^{j-i+1}(T_i) $$

where $T_i = \operatorname{Ker}(\mu_{\Omega_B^i(X)} : A \otimes_B \Omega_B^i(X) \to X)$, for $1 \leqq j \leqq n$.

Theorem 3.2.8. *[XX13, Theorem 1.1] Let A be a $\mathbb{K}$-algebra and $B \subsetneq A$ a subalgebra such that $J(B)$ is a left ideal in A.*

1. *Suppose that $\mathcal{P}(A, B)$ is closed under taking A-syzygies. Then*

$$ \operatorname{findim}(B) \leqq \operatorname{findim}(A) + \operatorname{findim}({}_BA) + 3, $$

where

$$ \operatorname{findim}({}_BA) = \sup\{\operatorname{pd}_B X \mid \text{there exists a decomposition } {}_BX \oplus_B \cong {}_BA \text{ and } \operatorname{pd}_B X < \infty\}. $$

2. *Suppose that $B \subsetneq A$ is n-hereditary for a non negative integer n, then*

$$ \operatorname{gldim}(A) \leqq \operatorname{gldim}(B) + n \leqq \operatorname{gldim}(A) + \operatorname{pd}_B A + n + 2. $$

Proof. We will only sketch the item (i). Let $Y \in B - \mathbf{mod}$ with finite projective dimension. If $\operatorname{pd}_B Y < 3$, the proposed upper bound is obviously true, so we can assume that $3 \leqq \operatorname{pd}_B Y$.

Then $Y' = \Omega_B^2(Y)$ is an A-module. By the above discussion, $\Omega_B^j(Y') = \Omega_B^{j+2}(Y)$ is (A, B)-projective for all $j \geqq 1$. Denote $M = \Omega_B(Y') = \Omega_B^3(Y)$, then all A-syzygies of M are relatively projective and, if $\operatorname{pd}_B Y = m$, then, by the above lemma,

$$0 = {}_A\Omega_B(Y) \cong {}_A\Omega_A^{m-2}(M) \oplus N,$$

where N is some A-module. Therefore $\operatorname{pd}_A M \leqq m - 3$. Let

$$0 \longrightarrow P_t \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a minimal projective resolution for M as an A-module. The finiteness of $\operatorname{pd}_B {}_BM$ implies the finiteness of $\operatorname{pd}_B {}_B\Omega_A^j(M)$, for any j a non negative integer, therefore all modules in the above sequence have finite projective dimension when viewed as B-modules. So $\operatorname{pd}_B M \leqq t + \max\{\operatorname{pd}_B P_i\} \leqq t + \operatorname{findim}({}_BA)$. Finally, we have $\operatorname{pd}_B X = \operatorname{pd}_B M + 3$, $t \leqq \operatorname{findim} A$ and

$$\operatorname{pd}_B X = \operatorname{pd}_B M + 3 \leqq \operatorname{findim} A + \operatorname{findim}({}_BA) + 3.$$

Since X is an arbitrary B-module of finite projective dimension, we obtain

$$\operatorname{findim} B \leqq \operatorname{findim} A + \operatorname{findim}({}_BA) + 3.$$

□

Remark 3.2.9. The above theorem states that, if $\operatorname{findim} A$ is finite, then $\operatorname{findim} B$ is finite provided the conditions assumed on the extension $B \subseteq A$.

We end [XX13] discussion with an equivalence that relates the Finitistic Dimension Conjecture to Relative Homological Algebra.

Proposition 3.2.10. *[XX13, Proposition 2.19] If $B \subseteq A$ be an extension of $\mathbb{K}$-algebras such that $J(B)$ is a left ideal in A and $J(A) = J(B)A$, then*

$$\operatorname{gldim}(A, B) \leqq 1.$$

The condition $J(A) = J(B)A$ is called radical full.

At the end of [XX13, Section 2] the authors provide the following construction.

Let B be a finite dimensional $\mathbb{K}$-algebra, with $\mathbb{K}$ a perfect field. Write $B = \Sigma \oplus J(B)$, where Σ is a maximal semisimple subalgebra of B, and define

$$\bar{B} \doteq \frac{B}{J^{n-1}(B)},$$

where $n = \min\{k \mid J^k(B) = 0\}$. Then we can construct the matrix algebra

$$A \doteq \begin{pmatrix} \Sigma & 0 \\ J(B) & \bar{B}. \end{pmatrix}$$

The $\mathbb{K}$-algebra homomorphism

$$b = s + j \longmapsto \begin{pmatrix} s & 0 \\ j & \bar{b} \end{pmatrix}$$

is injective and we can think about the extension $B \subseteq A$ via this homomorphism. The extension $B \subseteq A$ is radical full, $J(B)$ is a left ideal in A, and, therefore, $\mathrm{gldim}(A, B) \leqq 1$

Example 3.2.11. Consider $B = \frac{\mathbb{K}[x]}{\langle x^2 \rangle}$, the algebra of dual numbers. Then

$$A = \begin{pmatrix} \Sigma & 0 \\ J(B) & \bar{B} \end{pmatrix} = \begin{pmatrix} \mathbb{K} & 0 \\ \mathbb{K} & \mathbb{K} \end{pmatrix} \cong \mathbb{K}Q,$$

where $Q : \bullet \longrightarrow \bullet$. In particular $\mathrm{gldim}(A, B) \leqq 1$.

This bound is not sharp, to see this we will compute all (A, B)-projectives up to isomorphism of A-modules. Notice that the category of finite dimensional B-modules has only two indecomposable modules (up to isomorphism): ${}_BB$ and ${}_B\mathbb{K}$. Therefore all (A, B)-projective modules can be constructed using the A-modules ${}_AA \otimes_B B \cong {}_AA$ and ${}_AA \otimes_B \mathbb{K}$. For the first case we have ${}_AA \otimes_B B = {}_AA = M \otimes N$ with

$$M \cong \mathbb{K} \xrightarrow{1} \mathbb{K} \qquad \text{and} \qquad N \cong S(2).$$

For the second case, let $x \in {}_B\mathbb{K}$ be a generator (as B-modules), then $\alpha \otimes_B x = e_2 \otimes_B \alpha x = e_2 \otimes_B 0 = 0$, by the simplicity of ${}_B\mathbb{K}$. Therefore $A \otimes_B \mathbb{K}$ is generated by $e_1 \otimes_B x$ and $e_2 \otimes_B x$ with α acting trivially. In other words, $A \otimes_B \mathbb{K} \cong S(1) \oplus S(2)$.

We have shown that, up to isomorphism, all the indecomposable A-modules are (A, B)-projectives, that is, $S(1)$, $S(2)$ and M are (A, B)-projectives. Since every A-module is a sum of indecomposable modules, we conclude that the (A, B)-projectives are, precisely, $A - \textbf{mod}$. Therefore $\mathrm{gldim}(A, B) = 0$.

We end this example by saying that one could prove that $\mathrm{gldim}(A, B) = 0$ using [XX13, Lemma 2.12].

The above theory and construction can be used to prove the following equivalence.

Corollary 3.2.12. *[XX13] The following statements are equivalent:*

1. *Every finite dimensional algebra over a perfect field has finite finitistic dimension.*
2. *Let $B \subseteq A$ is a finite dimensional extension of algebras over a perfect field $\mathbb{K}$ with $J(B)$ a left ideal in A and $\operatorname{findim}(A) < \infty$. If $\operatorname{gldim}(A, B) \leqq 1$, then $\operatorname{findim}(B) < \infty$.*

This equivalence obtained by Xi and Xu indicates that one can prove the Finitistic Dimension Conjecture using relative homological data of extensions with finite relative global dimension.

Xi and Xu's work is not the only one that tackles the Finitistic Dimension Conjecture using relative homology of extensions of algebras. In [IM21] the authors study *strongly proj-bounded extensions* $B \subseteq A$ in order to understand when the finiteness of the finitistic dimension of B implies that $\operatorname{findim} A < \infty$ and vice-versa. One of the conditions for an extension to be strongly proj-bounded is $\operatorname{gldim}(A^e, B^e) < \infty$. Their main theorem is.

Theorem 3.2.13. *[IM21, Theorem 6.11] Let $B \subseteq A$ be a strongly proj-bounded extension. Then $\operatorname{findim}(B)$ is finite if, and only if, $\operatorname{findim}(A)$ is finite.*

For an in depth discussion of [IM21] see 3.3.

In both cases, [XX13] and [IM21] indicates that it is interesting to obtain results regarding extensions of finite relative global dimension. For the remainder of this text we will take a look at some efforts to produce examples of such extensions.

Guo, in [Guo18], was able to construct examples of extensions of algebras while obtaining bounds for their relative homological dimensions. We will not get into the details, for that the reader is refered to [Guo18]. To understand this examples, we first need a definition.

Definition 3.2.14. [Guo18, Definition 5.3] Let A be a $\mathbb{K}$-algebra and $B, C, S \subseteq A$ three subalgebras. We say that A decomposes as a twisted tensor product of B and C over S if the following conditions hold:

1. S is a semisimple subalgebra of A such that $A = S \oplus J(A)$ as S^e-modules.
2. $B \cap C = S$.
3. the multiplication map $\mu : C \otimes_S B \to A$ is an isomorphism of $(C - B)$-bimodules.
4. $J(B)J(C) \subseteq J(C)J(B)$.

We can understand twisted tensor products using quivers. Suppose that $C = \mathbb{K} R_1/I$, I generated by the relations σ_i, and $B = \mathbb{K} R_2/J$, J generated by τ_j, and that $(R_1)_0 = (R_2)_0$

holds. Consider the quiver Q such that $Q_0 = (R_1)_0 = (R_2)_0$ and Q_1 is the disjoint union of $(R_1)_1$ and $(R_2)_1$. If $L \lhd \mathbb{K}\, Q$ is the ideal generated by the relations σ_i, τ_j and $\beta\alpha$, for $\alpha \in (R_1)_1$ and $\beta \in (R_2)_1$, then

$$A = \frac{\mathbb{K}\, Q}{L}$$

is the twisted tensor product of B and C over $\Sigma = \sum_{i \in Q_0} \mathbb{K}\, e_i$.

Corollary 3.2.15. *[Guo18, Corollary 1.2] Let A be a $\mathbb{K}$-algebra and $B, C, S \subseteq A$ three subalgebras. If A decomposes as a twisted tensor product of B and C over S, then* $\operatorname{gldim}(A, B) \geqq \operatorname{gldim}(C)$.

Example 3.2.16. Let C be the algebra whose Gabriel quiver is

$$\mathbb{A}_{n+1} \,:\, 1 \longrightarrow \cdots \longrightarrow n \longrightarrow n+1$$

and relations are $J^2(\mathbb{K}\, \mathbb{A}_{n+1})$. Suppose that B is any bound quiver algebra over the same vertices of C. If A is the twisted tensor product of B and C over $\Sigma = \sum_{i \in (\mathbb{A}_{n+1})_0} \mathbb{K}\, e_i$, then

$$\operatorname{gldim}(A, B) \geqq \operatorname{gldim}(C) = n.$$

3.3 Combinatorial Approach to Relative Global Dimensions

In this section we discuss the theory of [IM21] related to relative homological dimensions. In their paper, Iusenko and MacQuarrie prove an upper bound for the relative global dimension using combinatorics. This particular result will be very useful and important for the next section, as it is the primary upper bound used to provide examples for the extension that we will introduce.

As is the case for some combinatorial results, the proof of [IM21] main theorem for relative homological algebra is technical and we will not enter in the details. We highly recommend the reader to see the details by themselves in the original paper, since often such combinatorial arguments can add a lot of intuition for what is happening.

Their theorem is

Theorem 3.3.1. *[IM21, Theorem 3.2] Let Q be a quiver together with a partition $V_1, \ldots, V_n$ of the vertices set of Q, satisfying the properties:*

1. There are no arrows from a vertex in V_i to a vertex in V_j if $i < j$;

2. There are no arrows between distinct vertices in each V_i.

Let $A = \mathbb{K}Q/I$, with I admissible, and let B be a subalgebra of A generated as a subalgebra by the elements of a quiver R which satisfies the following properties:

i. Each vertex of R is a sum of vertices of Q;

ii. The arrows β of R are linear combinations of paths in Q and have the property that for any vertex e of Q there are vertices h, g of Q such that $\beta e = h\beta e$ and $e\beta = e\beta g$;

iii. for each vertex e of Q and loop γ of Q at e, there is an element β of B such that $\gamma = e\beta$.

Then the relative global dimension $\operatorname{gldim}(A, B)$ *is at most $n - 1$.*

Before we discuss what each condition in the theorem is saying, it is interesting to discuss a particular case. Let A be any finite dimensional $\mathbb{K}$-algebra whose Gabriel quiver is acyclic and let B be the semisimple subalgebra generated by all the vertices of A. Then the above theorem says that

$$\operatorname{gldim}(A) = \operatorname{gldim}(A, B) \leqq n - 1,$$

where n is the dimension of B or number of vertices of Q_A. This recovers the theorem [Far07, Theorem].

And that is the idea of the proof: the several combinatorial conditions of the theorem are needed so that we can treat the extension $B \subseteq A$ as an "acyclic quiver ", in the sense that we only add data to B in order to obtain A that does not become a loop or cycle. This is formalized in condition (iii), (1) and (2).

The other conditions are necessary so that the authors can use the partition V_i to get the finiteness of the relative projective dimension of any module. By studying the induced modules $A \otimes_B M$ under this combinatorial constrain, they are able to obtain a direct sum ${}_A A \otimes_B M = {}_A A_S \oplus {}_A A_D$ such that:

- the restriction of the multiplication $A \otimes_B M$ to A_S is surjective, denote K the kernel of the restriction.
- If M is supported on vertices in $V_1 \cup \cdots \cup V_m$, in the sense that the vector spaces M_i are trivial for all $i \notin V_1 \cup \cdots \cup V_m$, then the support of K is $V_1 \cup \cdots \cup V_{m-1}$.

With the two facts above, a simple induction shows that

$$\operatorname{gldim}(A, B) \leqq n - 1.$$

Corollary 3.3.2. *[IM21, Corollary 3.3] Let Q be a quiver with vertices $1, \ldots, n$ such that there are no arrows $i \to j$ when i is strictly smaller than j, and $A = \mathbb{K}Q/I$, with $I \lhd \mathbb{K}Q$ admissible.*

If B is a subalgebra of A generated as a subalgebra by elements of a quiver R satisfying the following properties

1. *the vertices of R are sums of vertices of Q.*
2. *the arrows β of R are such that $\beta = f\beta e$, for some e, f vertices of Q.*
3. *R contains every loop of Q.*

Then $\operatorname{gldim}(A, B) \leqq n - 1$.

The reason why Iusenko and MacQuarrie where interested in finding extensions of finite relative global dimension is because they where interested in strongly proj-bounded extensions.

Definition 3.3.3. [IM21, Definition 4.1] An extension of $\mathbb{K}$-algebras $B \subseteq A$ is *proj-bounded* if it satisfies the following three conditions:

1. A/B has finite projective dimension as a B^e-module.
2. A/B is projective as either a left or a right B-module.
3. There exist a integer $p \geqq 1$ such that $(A/B)^{\otimes_B n}$ is projective as a B^e-module for any $n \geqq p$. The least such p is called *index of projectivity.*

Moreover, we say that $B \subseteq A$ is *strongly proj-bounded* if it is proj-bounded and satisfies $\operatorname{pd}_{(A^e, B^e)} A < \infty$.

In particular, $\operatorname{pd}_{(A^e, B^e)} A < \infty$ holds provided $\operatorname{gldim}(A^e, B^e) < \infty$. To apply the above theorem for $B^e \subseteq A^e$ we need to understand tensor product of two algebras in terms of quivers.

Definition 3.3.4. If Q and R are two quivers, the *product* of Q and R, denoted by $Q \times R$ is the quiver whose vertices are

$$(Q \times R)_0 \doteq Q_0 \times R_0,$$

that is the usual product of sets of the original vertices, and whose arrows are

$$(Q \times R)_1 = (Q_0 \times R_1) \cup (Q_1 \times R_0).$$

The source of the arrow (α, i), with $\alpha \in Q_1$ and $i \in R_0$, is $(s(\alpha), i) \in Q_0 \times R_0$ and its target is $(t(\alpha), i)$. The definition for an arrow (j, β), $j \in Q_0$ and $\beta \in R_1$, is analogous.

If $I \lhd \mathbb{K}Q$ and $J \lhd \mathbb{K}R$ are admissible, we define their product $I \times J$ to be the admissible ideal of $\mathbb{K}Q \times R$ generated by the following relations

$$\begin{cases} (\alpha, t(\beta))(s(\alpha), \beta) - (t(\alpha), \beta)(\alpha, s(\beta)), \alpha \in Q_1, \beta \in R_1, \\ Q_0 \times J \\ I \times R_0 \end{cases}$$

The above definition is a bit abstract, therefore an example should help elucidate what is happening. If the reader is interested in other examples and more detailed computations, we suggest [Ska11].

Example 3.3.5. Consider

$$Q \,:\, 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \qquad \text{and} \qquad R \,:\, 1' \xrightarrow{\psi} 2' \xrightarrow{\omega} 3'.$$

Then $Q \times R$ is given by

$$\begin{array}{ccccc} (1,1') & \xrightarrow{(\alpha,1')} & (2,1') & \xrightarrow{(\beta,1')} & (3,1') \\ {\scriptstyle (1,\psi)}\downarrow & & \downarrow{\scriptstyle (2,\psi)} & & \downarrow{\scriptstyle (3,\psi)} \\ (1,2') & \xrightarrow{(\alpha,2')} & (2,2') & \xrightarrow{(\beta,2')} & (3,2') \\ {\scriptstyle (1,\omega)}\downarrow & & \downarrow{\scriptstyle (2,\omega)} & & \downarrow{\scriptstyle (3,\omega)} \\ (1,3') & \xrightarrow[(\alpha,3')]{} & (2,3') & \xrightarrow[(\beta,3')]{} & (3,3'). \end{array}$$

If $I = J = 0$, then $I \times J \neq 0$ and it is comprised of the diagonal relations on all the squares above. If one of them is not zero, say $\beta\alpha \in I$, then we simply make all the possible copies of this relation (in the same way that we made copies of the arrows)

$$(\beta, 1')(\alpha, 1'),\ (\beta, 2')(\alpha, 2'),\ \text{and } (\beta, 3')(\alpha, 3')$$

Lemma 3.3.6. *[Les94, Lemma 1.3] For any quivers Q and R, and any admissible ideals $I \lhd \mathbb{K}Q$ and $J \lhd \mathbb{K}R$, we have*

$$\frac{\mathbb{K}Q}{I} \otimes \frac{\mathbb{K}R}{J} \cong \frac{\mathbb{K}Q \times R}{I \times J}.$$

Corollary 3.3.7. *[IM21, Corollary 3.6] Let A, B be as in (3.3.2), then*

$$\mathrm{gldim}(A^e, B^e) \leqq 2n - 2.$$

Proof. By the above discussion, the vertices of A^e are pairs (i, j), with i and j vertices of A. Simply consider the partition $V_s = \{(i, j) \mid j - i = s\}$. It is easy to see that all the conditions are meet and that the above partition has $2n - 1$ elements, therefore

$$\mathrm{gldim}(A^e, B^e) \leqq 2n - 1 - 1 = 2n - 2.$$

□

The above result was used by the authors to construct a class of strongly proj-bounded extensions, see [IM21, Proposition 4.7]. Finally, they show that for $B \subseteq A$ a strongly proj-bounded extension of finite dimensional $\mathbb{K}$-algebras, it suffices to know the finiteness of the finitistic dimension of only one of the algebras. This adds another connection between Relative Homological Algebra and the Finitistic Dimension Conjecture.

Theorem 3.3.8. *[IM21, Theorem 6.11] Let $B \subseteq A$ as above. Then* $\mathrm{findim}(B)$ *is finite if, and only if,* $\mathrm{findim}(A)$ *is finite.*

Remark 3.3.9. The strongly proj-bounded extensions also preserve another homological conjecture, namely Han's Conjecture. See [IM21, Corollary 6.17].

Part II

Results

Chapter 4

Classical-Relative Parallel

In this chapter we discuss some advances made in the computation of relative homological dimensions of extensions of known constructions. Most of the results in this chapter are based on results of the classical theory and use cleaver ways to transfer classical resolutions to relative resolutions.

4.1 Tensor Algebra Extensions

Let B be a $\mathbb{K}$-algebra, N a $(B-B)$-bimodule. By the *tensor algebra extension* we mean $B \subseteq T[B,N]$. In this section we deduce some results about this extension that generalize results about path algebras of finite acyclic quivers.

The results presented in this section are true if one consider gldim$(T[B,N],B)$ to be the supremum of the relative projective dimension of any T-module, instead of the finite dimensional modules like our definition. Therefore one could consider infinite dimensional T, B, and N. But for the sake of clarity and compatibility with the rest of the text, we will keep working with finite dimensional modules and algebras. For that reason, let B be a finite dimensional $\mathbb{K}$-module, and N a finite dimensional $(B-B)$-bimodule such that $N^{\otimes_B r} = 0$ for some integer $r > 0$. The main result of this section is.

Theorem 4.1.1. *Let B be a finite dimensional $\mathbb{K}$-algebra and N a finite dimensional $(B-B)$-bimodule such that $T[B,N]$ is finite dimensional. Then*

$$\mathrm{gldim}(T,B) = 1 \iff N \neq 0 \tag{4.1}$$

We construct our proves based on the following result.

Theorem 4.1.2. *[CLMS20, Theorem 2.5] There exists a (T^e, B^e)-projective resolution for T of* *the form*

$$0 \longrightarrow T \otimes_B N \otimes_B T \xrightarrow{g} T \otimes_B T \xrightarrow{f} T \longrightarrow 0 \tag{4.2}$$

where $f(x \otimes y) = xy$ *and* $g(x \otimes n \otimes y) = xn \otimes y - x \otimes ny$.

Proposition 4.1.3. *Let* B, N, *and* T *as above. Then*

$$\operatorname{gldim}(T, B) \leqq 1. \tag{4.3}$$

Remark 4.1.4. The above proposition generalizes (2.5.1) in the following sense: if one consider $B = \mathbb{K}Q_0$ and $N = \mathbb{K}Q_1$, then $T[B, N] = \mathbb{K}Q$ is the path algebra of Q and $\operatorname{gldim}(\mathbb{K}Q) \leqq 1$.

Proof. Let M be a finite dimensional T-module. Apply the functor $- \otimes_T M$ to (4.2) in order to obtain a chain complex of T-modules

$$0 \longrightarrow (T \otimes_B N \otimes_B T) \otimes_T M \xrightarrow{g \otimes_T 1_M} (T \otimes_B T) \otimes_T M \xrightarrow{f \otimes_T 1_M} T \otimes_T M \longrightarrow 0. \tag{4.4}$$

Use the well known isomorphisms

$$\begin{cases} (T \otimes_B N \otimes_B T) \otimes_T M \cong T \otimes_B N \otimes_B M \\ (T \otimes_B T) \otimes_T M \cong T \otimes_B M \\ T \otimes_T M \cong M, \end{cases}$$

to write it as

$$0 \longrightarrow T \otimes_B N \otimes_B M \xrightarrow{\psi} T \otimes_B M \xrightarrow{\phi} M \longrightarrow 0, \tag{4.5}$$

where $\psi(t \otimes n \otimes m) = tn \otimes m - t \otimes nm$, with nm being the restriction to N of the action of T on M, and $\phi(t \otimes m) = tm$.

To show that (4.5) is a (T, B)-projective resolution for M, which would imply that $\operatorname{gldim}(T, B) \leqq 1$, it suffices to show that it admits a B-homotopy, see (3.5). The construction of said B-homotopy is fully based on the proof of [CLMS20, Theorem 2.5].

We want homomorphisms of B-modules

$$s : M \longrightarrow T \otimes_B M \quad \text{and} \quad r : T \otimes_B M \longrightarrow T \otimes_B \otimes_B N \otimes_B M,$$

such that

$$\phi \circ s = 1_M \tag{4.6}$$
$$r \circ \psi = 1_{T \otimes_B N \otimes_B M} \tag{4.7}$$
$$\psi \circ r + s \circ \phi = 1_{T \otimes_B M}. \tag{4.8}$$

Define

$$\begin{aligned} s \,:\, M &\longrightarrow T \otimes_B M \\ m &\longmapsto 1 \otimes_B m. \end{aligned}$$

It is clear that s, as above, is a homomorphism of B-modules and it satisfies (4.6).

We know that T is $\mathbb{Z}$-graded, its negative homogeneous components are zero and $T_i = N^{\otimes_B i}$ for non-negative integers. Using that tensor products commute with direct sums, we can consider $T \otimes_B M$ as a $\mathbb{Z}$-graded module with homogeneous components given by

$$(T \otimes_B M)_i = \begin{cases} 0, \text{ if } i < 0 \\ N^{\otimes_B i} \otimes_B M, \text{ if } i \geqq 0. \end{cases}$$

For each $i \in \mathbb{N}$ we will define a linear map $r_i \,:\, (T \otimes_B M)_i \longrightarrow T \otimes_B N \otimes_B M$. If $i = 0$, define $r_0 \doteq 0$. If $i > 0$, let r_i be the unique homomorphism of groups induced by the following B-multi-additive map

$$\begin{aligned} R_i \,:\, N^i \times M &\longrightarrow T \otimes_B N \otimes_B M \\ (n_1, \dots, n_i; m) &\longmapsto \sum_{j=1}^{i} (n_1 \otimes \cdots \otimes n_{j-1}) \otimes n_j \otimes [(n_{j+1} \otimes \cdots \otimes n_i)m], \end{aligned}$$

with $n_0 = n_{i+1} \doteq 1_B \in B$. One way to understand R_i, and therefore r_i, is that it adds a summand for each index $1 \leqq j \leqq i$ with n_j viewed as an element on the second "coordinate" of $T \otimes_B N \otimes_B M$. Define

$$r = \bigoplus_{i=0}^{\infty} r_i \,:\, T \otimes_B M \longrightarrow T \otimes_B N \otimes_B M,$$

then r is a homomorphism of B-modules. In fact,

$$\begin{aligned} r(b \cdot (n_1 \otimes \cdots \otimes n_i \otimes m)) &= r_i(bn_1 \otimes \cdots \otimes n_i \otimes m) \\ &= 1 \otimes bn_1 \otimes [(n_2 \otimes \cdots \otimes n_i)m] + \sum_{j=2}^{i} (bn_1 \otimes \cdots \otimes n_{j-1}) \otimes n_j \otimes [(n_{j+1} \otimes \cdots \otimes n_i)m] \\ &= b \cdot (\sum_{j=1}^{i} (n_1 \otimes \cdots \otimes n_{j-1}) \otimes n_j \otimes [(n_{j+1} \otimes \cdots \otimes n_i)m]) \\ &= b \cdot r(n_1 \otimes \cdots \otimes n_i \otimes m). \end{aligned}$$

We now proceed to show that r satisfies (4.7) and (4.8).

For (4.7) we have

$$\begin{aligned}
r\psi((n_1 \otimes \cdots \otimes n_i) \otimes n_{i+1} \otimes m) = \\
&= r((n_1 \otimes \cdots \otimes n_{i+1}) \otimes m - (n_1 \otimes \cdots \otimes n_i) \otimes [n_{i+1}m]) \\
&= \sum_{j=1}^{i+1}(n_1 \otimes \cdots \otimes n_{j-1}) \otimes n_j \otimes [(n_{j+1} \otimes \cdots \otimes n_{i+1})m] \\
&- \sum_{j=1}^{i}(n_1 \otimes \cdots \otimes n_{j-1}) \otimes n_j \otimes [(n_{j+1} \otimes \cdots \otimes n_i)n_{i+1}m] \\
&= (n_1 \otimes \cdots \otimes n_i) \otimes n_{i+1} \otimes [1 \cdot m] \\
&= 1_{T\otimes_B N \otimes_B M}((n_1 \otimes \cdots \otimes n_i) \otimes n_{i+1} \otimes m).
\end{aligned}$$

For (4.8) we compute

$$\begin{aligned}
(\psi \circ r + s \circ \phi)((n_1 \otimes \cdots \otimes n_i) \otimes m) = \\
&= \psi\left(\sum_{j=1}^{i}(n_1 \otimes \cdots \otimes n_{j-1}) \otimes n_j \otimes [(n_{j+1} \otimes \cdots \otimes n_i)m]\right) + 1 \otimes [n_1 \otimes \cdots \otimes n_i]m \\
&= \sum_{j=1}^{i}(n_1 \otimes \cdots \otimes n_{j-1} \otimes n_j) \otimes [(n_{j+1} \otimes \cdots \otimes n_i)m] \\
&- \sum_{j=1}^{i}(n_1 \otimes \cdots \otimes n_{j-1}) \otimes [(n_j \otimes n_{j+1} \otimes \cdots \otimes n_i)m] \\
&+ 1 \otimes [n_1 \otimes \cdots \otimes n_i]m \\
&= (n_1 \otimes \cdots \otimes n_i) \otimes m - 1 \otimes [(n_1 \otimes \cdots \otimes n_i)m] + 1 \otimes [(n_1 \otimes \cdots \otimes n_i)m] \\
&= 1_{T\otimes_B M}((n_1 \otimes \cdots \otimes n_i) \otimes m).
\end{aligned}$$

Therefore by (3.5) we proved that (4.5) is a $(T[B,N],B)$-projective resolution for any $M \in T[B,N] - \mathbf{mod}$ and $\operatorname{pd}_{(T[B,N],B)} M \leqq 1$. In particular

$$\operatorname{gldim}(T[B,N],B) \leqq 1. \tag{4.9}$$

□

Remark 4.1.5. Before we continue to prove this section's main theorem, let us understand how to derive 4.1.2 from 4.1.3. Begin by specializing 4.5 with $M = T$, then we recover the complex 4.2, that is, all the modules are the same, $\psi = g$ and $\phi = f$. It is easy to verify that the specialized complex is a complex of T^e-modules.

Now we turn our attention to the B-homotopy obtained in 4.1.3. This homotopy shows that the specialized complex is, actually, a short exact sequence, since the proposition proves that it is a (T,B)-exact sequence. Therefore, to completely recover what was obtained in [CLMS20, Theorem 2.5] it suffices to show that this B-homotopy is a is a B^e-homotopy, since the equalities 4.6, 4.7, and 4.8 follows from the left module structures and equalities of functions are not disturbed by new added structure. Therefore, we compute

$$
\begin{aligned}
r((n_1 \otimes \cdots \otimes n_i \otimes t) \cdot b) &= r_i(n_1 \otimes \cdots \otimes n_i \otimes (tb)) \\
&= 1 \otimes n_1 \otimes [(n_2 \otimes \cdots \otimes n_i)tb] + \sum_{j=2}^{i} (n_1 \otimes \cdots \otimes n_{j-1}) \otimes n_j \otimes [(n_{j+1} \otimes \cdots \otimes n_i)tb] \\
&= (\sum_{j=1}^{i} (n_1 \otimes \cdots \otimes n_{j-1}) \otimes n_j \otimes [(n_{j+1} \otimes \cdots \otimes n_i)t]) \cdot b \\
&= r(n_1 \otimes \cdots \otimes n_i \otimes t) \cdot b, \ \ \forall\, b \in B, \ \ \forall\, t \in T.
\end{aligned}
$$

and

$$s(t \cdot b) = s(tb) = 1_T \otimes_B (tb) = (1_T \otimes t) \cdot b = s(t) \cdot b, \ \ \forall\, b \in B, \ \ \forall\, t \in T.$$

In conclusion, by specializing 4.5 with $M = T$, one obtain the (T^e, B^e)-exact sequence of [CLMS20, Theorem 2.5].

Now we are able to prove this section's main theorem.

(Proof of Theorem 4.1.1) It's clear that if $N = 0$, then

$$\mathrm{gldim}(T, B) = \mathrm{gldim}(B, B) = 0.$$

Conversely, if $N \neq 0$, then compute the group $\mathrm{Tor}_1^{(T,B)}(B, B)$ using the (T, B)-projective resolution

$$0 \longrightarrow T \otimes_B N \xrightarrow{g} T \xrightarrow{f} B \longrightarrow 0 \tag{4.10}$$

obtained from (4.5) specifying $M = B$. The differentials become

$$\begin{cases} f(b + n_1 + n_2 \otimes n_3 + \cdots) = b \\ g(t \otimes n) = tn. \end{cases} \tag{4.11}$$

Applying $B \otimes_T -$ to the deleted sequence we obtain a complex

$$0 \longrightarrow B \otimes_T T \otimes_B N \xrightarrow{1_B \otimes_T g} B \otimes_T T \longrightarrow 0, \tag{4.12}$$

with

$$(1_B \otimes_T g)(b \otimes_T t \otimes_B n) = b \otimes tn = b \cdot (tn) \otimes 1_T = 0 \otimes 1_T = 0, \tag{4.13}$$

since T acts on B on the right by projection, that is,

$$B \cong \frac{T[B, N]}{\langle N \rangle} \tag{4.14}$$

Therefore $1_B \otimes_T g = 0$ and

$$\mathrm{Tor}_1^{(T,B)}(B, B) = B \otimes_T T \otimes_B N \cong B \otimes_B N \cong N \neq 0.$$

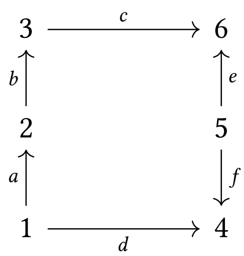
In conclusion,

$$\mathrm{pd}_{(T,B)} B = 1 \text{ and } \mathrm{gldim}(T, B) = 1. \tag{4.15}$$

□

There are a myriad of tensor algebras extensions. For instance, one could take $B = \mathbb{K}Q$ and add to it a finite number of arrows between elements of Q_0. In that case, N would have as basis all the added arrows, and the paths that they create. In order to see non obvious constructions, the next example show a way in which one could add relations while constructing a tensor extension.

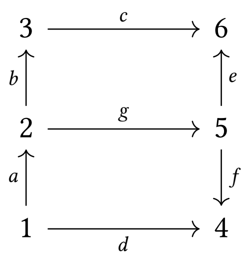
Example 4.1.6. Let B be the algebra given by



with relations cb and ba. Then we can take $N = \mathbb{K}g \oplus \mathbb{K}\alpha \oplus \mathbb{K}\beta \oplus \mathbb{K}\gamma$, with bimodule structure given by

$$g = e_5 g e_2, \ \alpha = g \cdot a, \ \beta = e \cdot g, \ \gamma = e \cdot \alpha = \beta \cdot a = e \cdot g \cdot a, \text{ and } f \cdot g = 0.$$

Then $T[B, N]$ is the algebra whose Gabriel quiver is



and relations are generated by cb, ba, and fg. In particular, $\mathrm{gldim}(T[B, N], B) = 1$.

4.2 Inequality: Tensor Product of Two Algebras

In this section we present a lower bound for gldim$(A\otimes B, B)$ using classical homological theory. Then we adapt the only combinatorial upper bound found in the literature to this type of extension. By the extension $B \subseteq A\otimes B$ we mean the identification $B \equiv \mathbb{K}\otimes B \subseteq A\otimes B$. In this section $\otimes$ means $\otimes_{\mathbb{K}}$.

We begin by proving some technical results.

Lemma 4.2.1. *Let A and B be any $\mathbb{K}$-algebras. If*

$$M_\bullet : \cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots$$

is an exact sequence of A-modules (not necessarily finite dimensional), then

$$M_\bullet \otimes B : \cdots \longrightarrow M_{n+1}\otimes B \xrightarrow{d_{n+1}\otimes 1_B} M_n \otimes B \xrightarrow{d_n\otimes 1_B} M_{n-1}\otimes B \longrightarrow \cdots$$

is an $(A\otimes B, B)$-exact sequence.

Proof. It is trivial that $M_\bullet \otimes B$ is a complex of $A\otimes B$-modules. By (3.1.6) there exists a family of linear transformations $T_n : M_n \to M_{n+1}$ such that $T_{n-1}d_n + d_{n+1}T_n = 1_{M_n}$. Apply $-\otimes B$ to this equation and use that it is a covariant functor to obtain

$$\begin{aligned} 1_{M_n\otimes B} &= 1_{M_n}\otimes 1_B = (T_{n-1}d_n + d_{n+1}T_n)\otimes 1_B \\ &= (T_{n-1}\otimes 1_B)\circ(d_n\otimes 1_B) + (d_{n+1}\otimes 1_B)\circ(T_n\otimes 1_B). \end{aligned}$$

This means that $M_\bullet\otimes B$ admits an $\mathbb{K}\otimes B \equiv B$-homotopy and, therefore, is an $(A\otimes B, B)$-exact sequence. □

Lemma 4.2.2. *Suppose that $M \in A-\textbf{mod}$ is projective, then $M\otimes B \in A\otimes B-\textbf{mod}$ is $A\otimes B$-projective.*

Proof. Since M is projective, there exists $N \in A-\textbf{mod}$ and an integer $n > 0$ such that $M\oplus N \cong A^n$. Applying $-\otimes B$ we obtain

$$(M\oplus N)\otimes B \cong (M\otimes B)\oplus(N\otimes B) \cong (A^n)\otimes B \cong (A\otimes B)^n.$$

□

Corollary 4.2.3. *If $P_\bullet \to M$ is a projective resolution of $M \in A-\textbf{mod}$, then $P_\bullet\otimes B \to M\otimes B$ is an $(A\otimes B, B)$-projective resolution for $M\otimes B$.*

Now that we are able to shift classical homological resolutions to the relative context, we can obtain a lower bound.

Proposition 4.2.4. *Let A and B be finite dimensional $\mathbb{K}$-algebras. Consider B as a subalgebra of $A \otimes B$, then*

$$\operatorname{gldim}(A \otimes B, B) \geqq \operatorname{gldim} A. \tag{4.16}$$

Proof. Let $M \in A - \mathbf{mod}$ and consider a minimal projective resolution for M

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0. \tag{4.17}$$

By (4.2.3),

$$\cdots \longrightarrow P_n \otimes B \xrightarrow{d_n \otimes 1_B} P_{n-1} \otimes B \longrightarrow \cdots \longrightarrow P_0 \otimes B \xrightarrow{d_0 \otimes 1_B} M \otimes B \longrightarrow 0 \tag{4.18}$$

is an $(A \otimes B, B)$-projective resolution for $M \otimes B$.

Since (4.17) is a minimal resolution of M as A-module, each induced A-epimorphism $d_n : P_n \longrightarrow \operatorname{Im}(d_n)$ is a projective cover. Without losing any generality, by (2.5.5), assume that each d_n induces an isomorphism $\bar{d}_n : \operatorname{top} P_n \longrightarrow \operatorname{top} \operatorname{Im}(d_n)$. This yields a commutative diagram

$$\begin{array}{ccc} P_n & \xrightarrow{d_n} & \operatorname{Im} d_n \\ \downarrow & & \downarrow \\ \frac{P_n}{J_A(P_n)} & \xrightarrow[\bar{d}_n]{\cong} & \frac{\operatorname{Im} d_n}{J_A(\operatorname{Im} d_n)} \end{array}$$

where the vertical maps are the usual projections. In particular,

$$\operatorname{Ker} d_n \subseteq J_A(P_n) = J(A)P_n.$$

and

$$\operatorname{Ker}(d_n \otimes 1_B) = \operatorname{Ker}(d_n) \otimes B \subseteq J_A(P_n) \otimes B.$$

Computing the extension groups $\operatorname{Ext}^*_{(A\otimes B,B)}(M \otimes B, S \otimes B)$, where $S \in A - \mathbf{mod}$ is any simple module, using (4.18) one obtains the cochain complex

$$0 \longrightarrow \operatorname{Hom}_{A\otimes B}(P_0 \otimes B, S \otimes B) \xrightarrow{(d_1 \otimes 1_B)^*} \operatorname{Hom}_{A\otimes B}(P_1 \otimes B, S \otimes B) \xrightarrow{(d_2 \otimes 1_B)^*} \cdots. \tag{4.19}$$

For any $\phi \in \operatorname{Hom}_{A\otimes B}(P_n \otimes B, S \otimes B)$,

$$\begin{aligned} (d_{n+1} \otimes 1_B)^*(\phi)(P_{n+1} \otimes B) &= \phi(d_{n+1} \otimes 1_B(P_{n+1} \otimes B)) \\ &\subseteq \phi(\operatorname{Im} d_{n+1} \otimes B) = \phi(\operatorname{Ker} d_n \otimes B) \\ &\subseteq \phi(J_A(P_n) \otimes B) = \phi(J(A)P_n \otimes B). \end{aligned}$$

Since ϕ is $(A \otimes B)$-linear, given $a \in J(A)$ and $p \otimes b \in P_n \otimes B$ we have the following computation

$$\phi(ap \otimes b) = \phi((a \otimes 1_B) \cdot (p \otimes b)) = (a \otimes 1_B)\phi(p \otimes b) = 0$$

because S is a simple A-module and $a \in J(A)$. This means that $\phi(J(A)P_n \otimes B) = 0$. In particular, the differentials of (4.19) are all zero, and its cohomology becomes

$$\operatorname{Ext}^n_{(A\otimes B,B)}(M \otimes B, S \otimes B) = \operatorname{Hom}_{A\otimes B}(P_n \otimes B, S \otimes B), \text{ for } n \geqq 1.$$

If there exists $f \in \operatorname{Hom}_A(P_n, S)$ such that $f \neq 0$, then $f \otimes 1_B \in \operatorname{Hom}_{A\otimes B}(P_n \otimes B, S \otimes B)$ is a non zero homomorphism. Again by the minimality of (4.17) this shows that

$$\operatorname{Ext}^n_A(M, S) = \operatorname{Hom}_A(P_n, S) \neq 0 \implies \operatorname{Ext}^n_{(A\otimes B,B)}(M \otimes B, S \otimes B) \neq 0.$$

By (2.4.1), we conclude that

$$\operatorname{pd}_{(A\otimes B,B)} M \otimes B \geqq \operatorname{pd}_A M, \ \forall \ M \in A - \mathbf{mod}. \tag{4.20}$$

In particular, there exists $S \in A - \mathbf{mod}$ simple such that

$$\operatorname{gldim}(A \otimes B, B) \geqq \operatorname{pd}_{(A\otimes B,B)} S \otimes B \geqq \operatorname{pd}_A S = \operatorname{gldim} A. \tag{4.21}$$

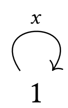
□

The above result states an inequality, but we are interested in computing $\operatorname{gldim}(A, B)$. We can do that by considering algebras whose global dimension in not finite.

Corollary 4.2.5. *If A and B are finite dimensional $\mathbb{K}$-algebras and* $\operatorname{gldim}(A) = \infty$*, then*

$$\operatorname{gldim}(A \otimes B, B) = \infty.$$

Example 4.2.6. Consider B any basic $\mathbb{K}$-algebra and write it as $B \cong \mathbb{K}Q/I$, where I is an admissible ideal, and consider A as the algebra whose Gabriel quiver is

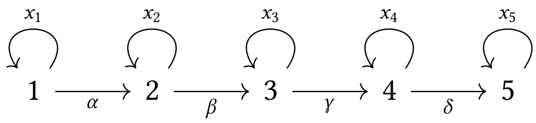


and the relations are generated by x^n, with $n > 1$. Then $\operatorname{gldim}(A) = \infty$ and the above corollary says that $\operatorname{gldim}(A \otimes B, B) = \infty$.

For a more concrete example, consider A as above and B as $\mathbb{A}_5$ with relations. That is, its Gabriel quiver is

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \xrightarrow{\delta} 5$$

and all the paths of length 2 generate the relations. By (3.3.6), $A \otimes B \cong \mathbb{K}Q/J$, where Q is the quiver



and J is generated by all the paths of length two using arrows of B, x_i^n, and $x_2\alpha - \alpha x_1$, $x_3\beta - \beta x_2$, etc. Then

$$\mathrm{gldim}\Big(\frac{\mathbb{K}Q}{J}, B\Big) = \infty.$$

It would be nice to find an upper bound for gldim$(A \otimes B, B)$. To do that for a reasonable class of $\mathbb{K}$-algebras we are going to use the combinatorial upper bound for relative global dimensions, and their proofs, obtained by the authors of [IM21]. One can read their results in Section (3.3).

We are interested in applying the theorem (3.3.1) to $B \subseteq A \otimes B$. The first thing we need to do is to understand how the theorem's conditions can be satisfied by this particular type of extension. But this is easier than it sounds, because a good thing about (3.3.1) is that the necessary conditions make it easy to see what A and B can *not* be. For example, conditions 1 and 2 together say that both A and B can not have oriented cycles besides loops.

Suppose that $A = \mathbb{K}Q/I$ and $B = \mathbb{K}R/J$, with I and J admissible, such that Q and R don't have oriented cycles besides, possibly, loops. Then, by (3.3.6), $A \otimes B \cong \mathbb{K}(Q \times R)/L$, for some L admissible. The image of an arrow $\beta \in R_1$ via the above isomorphism is

$$\sum_{i \in Q_0} (i, \beta)$$

and that of a vertex $e \in R_0$ is

$$\sum_{i \in Q_0} (i, e)$$

With the equations above, it is easy to see that the conditions *i* and *ii* are always satisfied. For the third condition to hold, any loop of $Q \times R$ must come from R, which implies that Q must have no oriented cycles, while B can have only loops. If B has a loop, then condition *iii* is clearly satisfied.

But what can we do about conditions 1 and 2? Suppose that $V_1, \dots, V_n$ and $U_1, \dots, U_m$ are partitions of the vertices of Q and of R, respectively, such that both satisfy conditions 1 and 2 of (3.3.1). Consider

$$W_r = \bigcup_{i+j=r} V_i \times U_j,$$

then the family $\{W_2, \dots, W_{n+m}\}$ is a partition for $(Q \times R)_0$.

Suppose that there is an arrow $\xi \in (Q \times R)_1$ from a vertex of W_r to a vertex of W_s. Then we have two options

$$\begin{cases} \xi : (a, c) \longrightarrow (b, c), \\ \xi : (a, c) \longrightarrow (a, d), \end{cases}$$

where $a, b \in Q_0$ and $c, d \in R_0$.

If $\xi : (a, c) \longrightarrow (b, c)$, then $a \neq b$ because Q has no oriented cycles, even loops. Therefore, for some $j > i$, $a \in V_j$ and $b \in V_i$, once $V_1, \dots, V_n$ satisfy condition 1. Say that $c \in U_l$, then $r = j + l > i + l = s$.

If $\xi : (a, c) \longrightarrow (a, d)$ and $c, d \in U_l$, then by condition 2 applied to the partition $U_1, \dots, U_m$ we must have $c = d$ and ξ is an arrow of from a vertex of W_r to itself, in particular also holds that $r = s$. If $c \in U_h$ and $d \in U_l$, then by condition 1, we have $h > l$. Say that $a \in V_i$, then $r = i + h > i + l = s$.

This discussion shows that the partition $W_2, \dots, W_{n+m}$ of the vertices of $Q \times R$ satisfy conditions 1 and 2. Now we can apply (3.3.1) to the extension $B \subseteq A \otimes B$ to obtain

$$\text{gldim}(A) \leqq \text{gldim}(A \otimes B, B) \leqq n + m - 2 + 1 - 1 = n + m - 2 < \infty. \tag{4.22}$$

The above discussion can be summarized in the following proposition

Proposition 4.2.7. *Let Q and R be quivers such that*

1. *Q has no oriented cycles of any length;*
2. *There exists a partition $V_1, \dots, V_n$ of Q_0 satisfying conditions i and ii of (3.3.1);*
3. *R has only loops as oriented cycles;*
4. *There exists a partition $U_1, \dots, U_m$ of R_0 satisfying conditions i and ii of (3.3.1).*

If $A = \mathbb{K}Q/I$ and $B = \mathbb{K}R/J$, with I and J admissible, then

$$\text{gldim}(A \otimes B, B) \leqq n + m - 2. \tag{4.23}$$

Example 4.2.8. Let us build on the previous example: where B's quiver is $\mathbb{A}_5$, A's is a loop, and we keep the same relations. We have seen that

$$\text{gldim}(A \otimes B, B) = \text{gldim}(A) = \infty.$$

But if we change their places, we get

$$\mathrm{gldim}(B \otimes A, A) \geqq \mathrm{gldim}(B) = 4.$$

We can apply the above proposition to this pair of algebras $A \subseteq B \otimes A$, since all the loops are in A. The partition of B's vertices is simply the unitary sets $\{i\}$, $i \in \mathbb{A}_5$, and the partition of A is just its vertex. Therefore $n = 5$, $m = 1$, and

$$4 = \mathrm{gldim}(B) \leqq \mathrm{gldim}(B \otimes A, A) \leqq 5 + 1 - 2 = 4 \implies \mathrm{gldim}(B \otimes A, A) = 4.$$

This example can be generalized for $\mathbb{A}_{n+1}$ with relations to obtain that there exist a pair of algebras A and B such that

$$\mathrm{gldim}(A \otimes B, B) = \infty \text{ and } \mathrm{gldim}(B \otimes A, A) = n.$$

4.3 Inequality: Trivial Extensions

Let B be a finite dimensional $\mathbb{K}$-algebra and N a finite dimensional $(B - B)$-bimodule. By a trivial extension we mean the extension $B \subseteq E = B \ltimes N$. In this section we obtain a lower bound for $\mathrm{gldim}(E, B)$ based on the structure of N.

Lemma 4.3.1. *Let $E = B \oplus N$ be the trivial extension of B by N. Then the kernel of the multiplication map $\mu_n : E \otimes_B N^{\otimes_B n} \longrightarrow N^{\otimes_B n}$ is $N^{\otimes_B (n+1)}$, for all $n \in \mathbb{N}$.*

Proof. We have $E_B = B_B \oplus N_B$ and we can write

$$E \otimes_B N^{\otimes_B n} = B \otimes_B N^{\otimes_B n} \oplus N \otimes_B N^{\otimes_B n}.$$

The multiplication map sends $B \otimes_B N^{\otimes_B n}$ bijectively to $N^{\otimes_B n}$ and $N \otimes_B N^{\otimes_B n}$ to zero, therefore

$$\mathrm{Ker}\,(\mu) = N \otimes_B N^{\otimes_B n} = N^{\otimes_B (n+1)}.$$

□

Proposition 4.3.2. *One of the two holds:*

1. *if there exist $m \in \mathbb{N}$ such that $N^{\otimes_B m} = 0$ and r is the maximal integer such that $N^{\otimes_B r} \neq 0$, then $\mathrm{pd}_{(E,B)} B = r$;*

2. *if $N^{\otimes_B m} \neq 0$ for all $m \in \mathbb{N}$, then $\mathrm{pd}_{(E,B)} B = \infty$.*

In particular, $\mathrm{gldim}(E,B) \geqq r$ *or* $\mathrm{gldim}(E,B) = \infty$*, respectively.*

Proof. We will prove this result computing the standard resolution for this extension, to recall what is a standard resolution see (3.9). To do so, we begin by looking at the kernel of the multiplication homomorphism

$$\begin{aligned} \mu_B : E \otimes_B B &\to B \\ e \otimes_B b &\mapsto e \cdot b. \end{aligned}$$

By (4.3.1) for $n = 0$ we obtain that $\mathrm{Ker}(\mu_B) = N$. The next step is to compute the kernel of the multiplication map $\mu_N : E \otimes_B N \to B$. Again we evoke (4.3.1) for $n = 1$ to get $\mathrm{Ker}(\mu_N) = N^{\otimes_B 2}$. Now one sees that recursive applications of (4.3.1) shows that

$$\cdots \xrightarrow{\mu_3} E \otimes_B N^{\otimes_B 2} \xrightarrow{\mu_2} E \otimes_B N \xrightarrow{\mu_1} E \otimes_B B \xrightarrow{\mu_0} B \xrightarrow{0} 0 \tag{4.24}$$

is the standard (E,B)-projective resolution of B.

Consider the deleted version of (4.24) and apply the functor $B \otimes_E -$ to get the chain complex

$$\cdots \xrightarrow{1_B \otimes_E \mu_3} B \otimes_E E \otimes_B N^{\otimes_B 2} \xrightarrow{1_B \otimes_E \mu_2} B \otimes_E E \otimes_B N \xrightarrow{1_B \otimes_E \mu_1} B \otimes_E E \otimes_B B \longrightarrow 0 \tag{4.25}$$

And we have the following computations regarding the differentials

$$\begin{aligned} &1_B \otimes_E \mu_{s+1} : B \otimes_E E \otimes_B N^{\otimes_B (s+1)} \longrightarrow B \otimes_E E \otimes_B N^{\otimes_B s} \\ &\quad 1_B \otimes_E \mu_{s+1}(b \otimes_E (\beta + \eta) \otimes (n_1 \otimes \ldots \otimes n_{s+1})) = \\ &\qquad = (b \otimes_E \beta n_1 \otimes (n_2 \otimes \ldots \otimes n_{s+1})) + (b \otimes_E \eta n_1 \otimes (n_2 \otimes \ldots \otimes n_{s+1})) \\ &\qquad = b\beta n_1 \otimes_E 1_E \otimes (n_2 \otimes \ldots \otimes n_{s+1}) \\ &\qquad = 0. \end{aligned}$$

Since all the differentials of (4.25) are zero, we have

$$\mathrm{Tor}_s^{(E,B)}(B,B) = B \otimes_E E \otimes_B N^{\otimes_B s} \cong N^{\otimes_B s}. \tag{4.26}$$

If $N^{\otimes_B s} \neq 0$, for all $s \in \mathbb{N}$, then $\mathrm{Tor}_s^{(E,B)}(B,B) \neq 0$ for any s and $\mathrm{pd}_{(E,B)} B = \infty$.

If some tensor power of N eventually vanishes and $r = \max\{N^{\otimes_B s} \neq 0\}$, then (4.24) has length r, this implies $\mathrm{pd}_{(E,B)} B \leqq r$. On the other hand,

$$\mathrm{Tor}_r^{(E,B)}(B,B) \cong N^{\otimes_B r} \neq 0, \tag{4.27}$$

implying $\mathrm{pd}_{(E,B)} B \geqq r$. In conclusion

$$\mathrm{pd}_{(E,B)} B = r.$$

□

Example 4.3.3. Consider B any finite dimensional $\mathbb{K}$-algebra. Write

$$B = \frac{\mathbb{K}Q}{I}.$$

Let L be a quiver of loops in the vertices of B, that is, $L_0 = Q_0$ and for each $x \in L_1$ $s(x) = t(x)$ holds. Think of $N = \mathbb{K}L/J^2(\mathbb{K}L)$ as a $(B-B)$-bimodule in the following way: each vertex acts via the natural identification, since $L_0 = Q_0$, and each arrow of Q_1 acts as zero. Consider

$$A = \frac{\mathbb{K}Q'}{J}$$

where $Q'_0 = Q_0$, $Q'_1 = Q_1 \cup L_1$, and J is generated by I, x^2 for each $x \in L_1$, and $\alpha x, x\alpha$ for each $x \in L_1$ and $\alpha \in Q_1$. Then

$$A \cong B \ltimes N.$$

In this case $N^{\otimes_B r} \neq 0$, for all $r \in \mathbb{N}$. Therefore

$$\text{gldim}(A, B) = \infty.$$

If we take $B = \mathbb{K}$ and L_1 with only one arrow, then $A = B \ltimes N = \frac{\mathbb{K}[x]}{\langle x^2 \rangle}$ and we recover

$$\text{gldim}\left(\frac{\mathbb{K}[x]}{\langle x^2 \rangle}\right) = \infty.$$

Chapter 5

Controllable Extensions

In this chapter we will define a new class of extensions that generalize much of the behaviour of classical global dimensions to the relative realm. Then we state and prove a theorem that, together with the literature, allows one to construct several classes of extensions that are controllable and that sheds light on Jacobson radical compatibility's. Finally we give an example of extension that is not controllable.

5.1 Definition and Examples

In this section we define our main object: *controllable extensions*. As we have seen in the previous part, relative homological algebra is built using trivial exact sequences of B-modules, in the sense that every kernel is a direct summand as a B-module. Morally, we are discarding any algebraic complexity related to B. If we apply this idea with the decomposition $B = \Sigma \oplus J(B)$ in mind, where Σ is a maximal semisimple subalgebra and $J(B)$ is the Jacobson radical of B, then we are discarding the data provided by $J(B)$ (recall that Σ is homologically trivial). The definition below is a way to formalize this intuition.

Definition 5.1.1. An extension $B \subseteq A$ is said to be *controllable* if

$$\operatorname{gldim}(A, B) = \operatorname{gldim}\left(\frac{A}{AJ(B)A}\right). \tag{5.1}$$

Denote the bilateral ideal $\downarrow B \doteq (AJ(B)A) \lhd A$ and the quotient algebra $\mathrm{A}_{\downarrow \mathrm{B}} \doteq A/(AJ(B)A)$. The controllable property becomes

$$\operatorname{gldim}(A, B) = \operatorname{gldim}(\mathrm{A}_{\downarrow \mathrm{B}}).$$

Before we study the implications of controllable extensions, we present a few examples.

Example 5.1.2. If B is semisimple and A a finite dimensional $\mathbb{K}$-algebra, then $\frac{A}{AJ(B)A} = \frac{A}{0} = A$ and (5.1) simply becomes

$$\operatorname{gldim}(A, B) = \operatorname{gldim}(A) \tag{5.2}$$

which is *always* true.

Example 5.1.3. If $B = A$ and A is a finite dimensional $\mathbb{K}$-algebra, then $\frac{A}{J(A)}$ is semisimple and

$$\operatorname{gldim}(A, A) = 0 = \operatorname{gldim}\left(\frac{A}{J(A)}\right) = \operatorname{gldim}\left(\frac{A}{AJ(A)A}\right) \tag{5.3}$$

We now use the theory obtained in the last chapter to get non trivial controllable extensions. Basically the next results proves the following: if we take a quiver R with relations, add as many arrows as we want without adding any vertex or relations using the new arrows, then we obtain a controllable extension.

Proposition 5.1.4. *Let Q be a finite quiver, with or without loops, R be a subquiver of Q such that*

- $R_0 = Q_0$, *and;*
- $\emptyset \neq R_1 \neq Q_1$.

If $I \lhd \mathbb{K}R$ is admissible, $J = \langle I \rangle_{\mathbb{K}Q}$ is the ideal generated by all the relations of I in $\mathbb{K}Q$, $A = \mathbb{K}Q/J$ and $B = \mathbb{K}R/I$, then $B \subsetneq A$ is an extension of algebras and

$$\operatorname{gldim}(A, B) = 1 = \operatorname{gldim}\left(\frac{A}{AJ(B)A}\right). \tag{5.4}$$

Proof. In this proof, $[-]_I$ denote the class of an element in B and $[-]_J$ that of an element in A. It is easy to see that there exists an injection $B \to A$ via $B \ni [b]_I \mapsto [b]_J \in A$, therefore we have an extension $B \subsetneq A$.

Define $0 \neq N = \operatorname{span}_{B^e}\{Q_1 \setminus R_1\} \subseteq A$ and consider $T = T[B, N]$, the tensor algebra of B and N. The inclusion $\iota : N \longrightarrow A$ is a (B, B)-bimodule homomorphism and it induces a unique $\mathbb{K}$-algebra homomorphism

$$\Phi : T = T[B, N] \longrightarrow A \tag{5.5}$$

such that $\Phi(b) = b,\ \forall\, b \in B$, and $\Phi(n) = \iota(n) = n \in A,\ \forall\, n \in N$. Notice that $\Phi(B) = B1_A = B$, that is, Φ *preserves* B.

Conversely, consider the following functions $\psi_i : Q_i \longrightarrow T[B,N]$, $i = 0,1$ where $\psi_0(r) = [e_r]_I \in B \subseteq T$ and

$$\psi(\alpha) = \begin{cases} [\alpha]_I \in B \subseteq T, \text{ if } \alpha \in R_1 \\ [\alpha]_J \in N \subseteq T, \text{ if } \alpha \in Q_1 \setminus R_1. \end{cases}$$

One can easily verify that $\psi_0(r)^2 = \psi_0(r)$, $\psi_0(s)\psi_0(r) = 0$, if $s \neq r$, and $\psi_1(\alpha) = \psi_0(t(\alpha))\psi_1(\alpha)\psi_0(s(\alpha))$. By the universal property of path algebras, see [ASS06, II.1 Theorem 1.8], there is a unique homomorphism of $\mathbb{K}$-algebras

$$\Psi : \mathbb{K}Q \longrightarrow T = T[B,N] \tag{5.6}$$

such that $\psi(e_i) = \psi_0(i)$, for any $i \in Q_0$, and $\psi(\alpha) = \psi_1(\alpha)$, for any $\alpha \in Q_1$. It is also easy to see that if $\rho \in I$ is a relation, then by our hypothesis ρ can be viewed as an element of $\mathbb{K}Q$ and $\psi(\rho) = [\rho]_I = 0$. In particular, since $J = \langle I \rangle_{\mathbb{K}Q}$, $J \subseteq \mathrm{Ker}(\psi)$. By the universal property of quotient algebras, there is a unique homomorphism of $\mathbb{K}$-algebras

$$\begin{aligned} \Psi : A &\longrightarrow T = T[B,N] \\ [x]_J &\longmapsto \psi(x). \end{aligned}$$

Using the universal properties above and their uniqueness one verifies that

$$\Psi \circ \Phi = 1_T \quad \text{and} \quad \Phi \circ \Psi = 1_A. \tag{5.7}$$

In particular,

$$\mathrm{gldim}(A,B) = \mathrm{gldim}(T,B) = 1 \tag{5.8}$$

where the second equality follows from $R_1 \neq \emptyset$ and (4.1.1).

Finally, by the hypothesis on R, the Gabriel quiver of $\frac{A}{AJ(B)A}$ is

$$Q \setminus R = \begin{cases} (Q \setminus R)_0 = Q_0 \\ (Q \setminus R)_1 = Q_1 \setminus R_1 \neq \emptyset \end{cases}$$

Since I is admissible we know that $I \subseteq J(B)$, which implies $J = AIA \subseteq AJ(B)A$. In particular,

$$\mathrm{gldim}\left(\frac{A}{AJ(B)A}\right) = 1. \tag{5.9}$$

□

Before we look at some examples using the above proposition, we state a corollary.

Corollary 5.1.5. *Let Q be a finite quiver, with or without loops, R be a subquiver of Q such that*

- $R_0 = Q_0$, *and;*
- $\emptyset \neq R_1 \neq Q_1$.

If $A = \mathbb{K}Q$ *and* $B = \mathbb{K}R$, *then*

$$\operatorname{gldim}(A, B) = 1 = \operatorname{gldim}\left(\frac{A}{AJ(B)A}\right). \tag{5.10}$$

Example 5.1.6. Consider $B = \mathbb{K}Q_B$ and $A = \mathbb{K}Q_A$, where

$$Q_B : \begin{array}{ccc} & & 3 \\ & \nearrow & \\ 1 & \longrightarrow & 2 \end{array} \qquad Q_A : \begin{array}{ccc} & & 3 \\ & \nearrow & \uparrow \\ 1 & \longrightarrow & 2 \end{array}$$

In this case, both A and B are hereditary algebras, meaning that their classical global dimension is equal to 1. The same is true about $\mathrm{A}_{\downarrow\mathrm{B}}$, since this algebra is the path algebra of

$$\begin{array}{ccc} & & 3 \\ & & \uparrow \\ 1 & & 2 \end{array}$$

By the above corollary, $\operatorname{gldim}(A, B) = 1 = \operatorname{gldim}(\mathrm{A}_{\downarrow\mathrm{B}})$

Example 5.1.7. We can also construct some examples that are more exotic. Let B be the algebra whose Gabriel quiver is the $\mathbb{A}_5$ with an extra vertex

$$\begin{array}{ccccccccc} & & & & 6 & & & & \\ 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & 4 & \longrightarrow & 5 \end{array}$$

and the relations of B are generated by all the paths of length 2. Then $\operatorname{gldim}(B) = 4$. If we take A to be the algebra given by

$$\begin{array}{ccccccccc} & & & & 6 & & & & \\ & & & \nearrow & & & & & \\ 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & 4 & \longrightarrow & 5 \end{array}$$

with the same relations of B, then it is also true that $\operatorname{gldim}(A) = 4$. Yet the above proposition says that the extension $B \subseteq A$ is controllable and

$$\text{gldim}(A, B) = 1.$$

Generalizing this example we obtain a class of controllable extensions of finite dimensional algebras $B \subseteq A$ such that $\text{gldim}(B) = \text{gldim}(A)$ is an arbitrary (positive) integer and $\text{gldim}(A, B) = 1$.

5.2 Properties

A definition is not justified only using abstract motivations/ideas and examples, we also need some theory to backup its importance. The next proposition shows that controllable extensions generalize some equalities that appear in classical global dimensions. In this section $\otimes$ means $\otimes_{\mathbb{K}}$. Recall that $A^e \doteq A \otimes_{\mathbb{K}} A^{op}$.

Proposition 5.2.1. *Let $B \subseteq A$ and $D \subseteq C$ be controllable extensions. We have the following:*

1. *if $B^{op} \subseteq A^{op}$ is controllable, then* $\text{gldim}(A^{op}, B^{op}) = \text{gldim}(A, B)$.
2. *if $B \otimes D \subseteq A \otimes C$ is controllable, then* $\text{gldim}(A \otimes C, B \otimes D) = \text{gldim}(A, B) + \text{gldim}(C, D)$.
3. *if $B \subseteq A \otimes B$ is controllable, then* $\text{gldim}(A \otimes B, B) = \text{gldim}(A)$.
4. *if $B^{op} \subseteq A^{op}$ and $B^e \subseteq A^e$ are controllable, then* $\text{gldim}(A^e, B^e) = 2\,\text{gldim}(A, B)$.

Proof. It is easy to see that the kernel of the homomorphism of algebras

$$\begin{aligned} \tau \,:\, A^{op} &\longrightarrow (\mathrm{A}_{\downarrow \mathrm{B}})^{op} \\ a &\longmapsto [a] \end{aligned}$$

is $A^{op} J(B)^{op} A^{op} = A^{op} J(B^{op}) A^{op}$. Therefore $A^{op}{}_{\downarrow B^{op}} \cong (\mathrm{A}_{\downarrow \mathrm{B}})^{op}$ as algebras and, by (2.4.4), we get

$$\text{gldim}(A^{op}, B^{op}) = \text{gldim}(A^{op}{}_{\downarrow B^{op}}) = \text{gldim}((\mathrm{A}_{\downarrow \mathrm{B}})^{op}) = \text{gldim}(\mathrm{A}_{\downarrow \mathrm{B}}) = \text{gldim}(A, B),$$

proving *1*.

For *2*, we have that

$$J(B \otimes D) = J(B) \otimes D + B \otimes J(D),$$

for finite dimensional algebras. In particular,

$$(A \otimes C)(J(B \otimes D))(A \otimes C) = AJ(B)A \otimes C + A \otimes CJ(D)C = (\downarrow B) \otimes C + A \otimes (\downarrow D).$$

We claim that

$$(A \otimes C)_{\downarrow (B \otimes D)} \cong \mathrm{A}_{\downarrow \mathrm{B}} \otimes C_{\downarrow D}.$$

In fact, let $\pi_A : A \longrightarrow \mathrm{A}_{\downarrow\mathrm{B}}$ and $\pi_C : C \longrightarrow C_{\downarrow D}$ be the canonical algebra projections. Then $\mathrm{Ker}(\pi_A \otimes \pi_C) = (\downarrow B) \otimes C + A \otimes (\downarrow D) = (\downarrow B \otimes D)$ and we have an induced isomorphism of algebras

$$\begin{aligned} \phi : (A \otimes C)_{\downarrow(B\otimes D)} &\longrightarrow \mathrm{A}_{\downarrow\mathrm{B}} \otimes C_{\downarrow D} \\ [a \otimes c] &\longmapsto [a] \otimes [c]. \end{aligned}$$

All the above yields

$$\begin{aligned} \mathrm{gldim}(A \otimes C, B \otimes D) &= \mathrm{gldim}((A \otimes C)_{\downarrow(B\otimes D)}), \text{ by hypothesis} \\ &= \mathrm{gldim}(\mathrm{A}_{\downarrow\mathrm{B}} \otimes C_{\downarrow D}), \text{ by the isomorphism} \\ &= \mathrm{gldim}(\mathrm{A}_{\downarrow\mathrm{B}}) + \mathrm{gldim}(C_{\downarrow D}), \text{ by (2.4.5)} \\ &= \mathrm{gldim}(A, B) + \mathrm{gldim}(C, D), \text{ by hypothesis.} \end{aligned}$$

To prove *3*, it is just the above argument for $\mathbb{K} \subseteq A$ and $B \subseteq B$ since both of this extensions are always controllable for finite dimensional basic algebras.

For *4*, the argument for *2* says that $\mathrm{gldim}(A^e, B^e) = \mathrm{gldim}(\mathrm{A}_{\downarrow\mathrm{B}}) + \mathrm{gldim}({A^{op}}_{\downarrow B^{op}})$. We know that ${A^{op}}_{\downarrow B^{op}} = (\mathrm{A}_{\downarrow\mathrm{B}})^{op}$, and $\mathrm{gldim}(\mathrm{A}_{\downarrow\mathrm{B}}) = \mathrm{gldim}({A^{op}}_{\downarrow B^{op}})$, see (2.4.4). Therefore,

$$\mathrm{gldim}(A^e, B^e) = \mathrm{gldim}(\mathrm{A}_{\downarrow\mathrm{B}}) + \mathrm{gldim}({A^{op}}_{\downarrow B^{op}}) = 2\,\mathrm{gldim}(\mathrm{A}_{\downarrow\mathrm{B}}).$$

□

We have seen, in (4.2.4), that for any finite dimensional $\mathbb{K}$-algebras, $\mathrm{gldim}(A \otimes B, B) \geqq \mathrm{gldim}(A)$ holds. Moreover, in the example (4.2.8), we have seen a family of extensions $\{B_n \subseteq A_n \mid n \in \mathbb{N}\}$ such that

$$\mathrm{gldim}(A_n, B_n) = \mathrm{gldim}(A_n) = n.$$

With the introduction of controllable extensions and (5.2.1.*3*), we now know that this family is composed by controllable extensions.

5.3 Lower Bound for the Controllable Extension Equality

In this section we will explore a way to obtain the inequality $\mathrm{gldim}(A, B) \geqq \mathrm{gldim}(\mathrm{A}_{\downarrow\mathrm{B}})$ theoretically. In the next section we will apply this result to compute the relative global dimension of some extensions.

Fix A a finite dimensional $\mathbb{K}$-algebra, $I \lhd A$ a bilateral ideal, and $\pi : A \longrightarrow A/I$ the canonical projection of algebras. Before we begin with relative dimensions, we explore a way to study $A/I - \mathbf{mod}$ in $A - \mathbf{mod}$ without losing any information.

Consider the functor $\Pi : (A/I) - \mathbf{mod} \longrightarrow A - \mathbf{mod}$ induced by $\pi : A \longrightarrow (A/I)$. On the objects, it takes an (A/I)-module (M, ϕ_M), where $\phi_M : (A/I) \longrightarrow \mathrm{End}_{\mathbb{K}}(M)$ is a homomorphism of $\mathbb{K}$-algebras, to the A-module (M, ψ_M) defined by precomposition

$$\psi : A \xrightarrow{\pi} (A/I) \xrightarrow{\phi} \mathrm{End}_{\mathbb{K}}(M) \tag{5.11}$$

and on the (homo)morphisms Π is the identity. To check that it is well defined we just need to compute how the action of an arbitrary element $a \in A$ affects a morphism $f : M \longrightarrow N$ of (A/I)-modules:

$$f(a \cdot m) = f(\phi \circ \pi(a)[m]) = \phi \circ \pi(a) f(m) = a \cdot f(m), \ \forall \ m \in M, \tag{5.12}$$

in other words, every (A/I)-homomorphism is an A-homomorphism.

Denote by $K(I) \subseteq A$–**mod** the full subcategory of all A-modules (M, ψ_M) such that $I \subseteq \mathrm{Ker}(\psi)$. If $(M, \psi_M) \in K(I)$, then by the universal property of the quotient algebra there exists a unique algebra homomorphism $\phi_M : (A/I) \longrightarrow \mathrm{End}_{\mathbb{K}}(M)$ such that

$$\begin{array}{ccc} A & \xrightarrow{\psi_M} & \mathrm{End}_{\mathbb{K}}(M) \\ {\scriptstyle\pi}\big\downarrow & \nearrow_{\phi_M} & \\ (A/I) & & \end{array} \tag{5.13}$$

commutes, i.e., $\psi_M = \phi_M \circ \pi$.

Let $f : M \longrightarrow N$ be a homomorphism of A-modules such that $M, N \in K(I)$ and $a \in A$, then

$$f((a + I) \cdot m) = f(am) = af(m) = (a + I) \cdot f(m), \ \forall \ m \in M, \tag{5.14}$$

that is f is also a homomorphism of (A/I)-modules. This can be summarized in another functor $\mathcal{I} : K(I) \longrightarrow (A/I)$–**mod** such that $\mathcal{I}(M, \psi_M) = (M, \psi_M)$ and is the identity on the morphisms.

The idea that we can view the category of (A/I)-modules inside A-**mod** without losing information is formalized on the next lemma

Lemma 5.3.1. *It holds $\mathcal{I} \circ \Pi = 1_{(A/I)-\mathbf{mod}}$ and $\Pi \circ \mathcal{I} = 1_{K(I)}$, where 1 denotes the identity functor of the respective category.*

Proof. Since both functors are the identity in the morphisms of their respective domains, it suffices to compute only what is happening to the objects.

Let $(M, \phi_M) \in (A/I)$–**mod**, then $\mathcal{I} \circ \Pi(M, \phi_M) = \mathcal{I}(M, \phi_M \circ \pi) = (M, \phi_M) = 1_{(A/I)-\mathbf{mod}}(M, \phi_M)$, by the uniqueness of the universal property. On the other hand, $\Pi \circ \mathcal{I}(M, \psi_M) = \Pi(M, \phi_M) = (M, \psi_M) = 1_{K(I)}(M, \psi_M)$, by the commutativity and uniqueness of the universal property. □

This allows one to identify the categories (A/I)–**mod**$\equiv K(I) \subseteq A$–**mod**. Since $K(I)$ is a full subcategory, we get the following result

Corollary 5.3.2. *(*Hom*-functor Equality) Let A be a finite dimensional $\mathbb{K}$-algebra, $I \lhd A$ a bilateral ideal, and $\pi : A \longrightarrow \frac{A}{I}$ the canonical projection. Then*

$$\mathrm{Hom}_{A/I}(M,N) = \mathrm{Hom}_A(M,N), \tag{5.15}$$

*for all $M,N \in (A/I)$–**mod**. In particular, if $M,N,L \in (A/I)$–**mod** and $f : M \longrightarrow N$ is an (A/I)-morphism, then $f^*_{(A/I)} = \mathrm{Hom}_{A/I}(f,L) = \mathrm{Hom}_A(f,L) = f^*_A$.*

There is an underlying interesting property of the Hom-functor Equality. By (2.4.2), the classical homological dimensions are completely determined by homomorphisms. Therefore, we have a way to study $(A/I) -$ **mod** in $A -$ **mod** that preserves the part of homological theory that interest us, and it becomes natural to investigates what is missing for us to connect relative homological theory to this results. The next lemma is a link between the theories.

Fix an extension $B \subseteq A$ and a decomposition $B = \Sigma \oplus J(B)$. In the next results we want to apply the Hom-functor Equality for the ideal $I = \downarrow B$.

Lemma 5.3.3. *If $f : M \longrightarrow N$ is an epimorphism in* $\mathrm{A}_{\downarrow\mathrm{B}}$-**mod**, *then $f : M \longrightarrow N$ as an epimorphism of A-modules admits a B-section*

Proof. Since $\Sigma \subseteq B$ is semisimple and $\mathrm{A}_{\downarrow\mathrm{B}}$-modules are Σ-modules via restriction, f admits a Σ-section

$$M \underset{h}{\overset{f}{\rightleftarrows}} N \longrightarrow 0 \tag{5.16}$$

that is, a homomorphism of Σ-modules that satisfy $f \circ h = 1_N$.

The action of $a \in A$ in any $\mathrm{A}_{\downarrow\mathrm{B}}$-module is to compute $\pi(a)$ and then consider its action on the $\mathrm{A}_{\downarrow\mathrm{B}}$-module. Remember that $B = \Sigma \oplus J(B)$ as vector spaces, so if $b \in B$, then there is a unique way to write $b = \sigma + \beta$, with $\sigma \in \Sigma$ and $\beta \in J(B)$. Compute

$$h(b \cdot m) = h((\sigma + \beta) \cdot m) = h(\sigma \cdot m) + h(\beta \cdot m) \tag{5.17}$$

$$= \sigma \cdot h(m) + h(0) = \sigma \cdot h(m) + \beta \cdot h(m) = b \cdot h(m), \ \forall \ m \in M. \tag{5.18}$$

This means that $h \in B -$ **mod** and the identity $f \circ h = 1_N$ still holds. □

Now we are ready to compute relative Ext-functors

Theorem 5.3.4. *Let $B \subseteq A$ be an extension of $\mathbb{K}$-algebras such that all indecomposable* *projectives of* $\mathrm{A}_{\downarrow\mathrm{B}}$ $-\mathbf{mod}$ *are* (A, B)*-projective* A*-modules. Then*

$$\mathrm{Ext}^*_{(A,B)}(M, N) = \mathrm{Ext}^*_{\mathrm{A}_{\downarrow\mathrm{B}}}(M, N), \tag{5.19}$$

for all $M, N \in \mathrm{A}_{\downarrow\mathrm{B}}$ $-\mathbf{mod}$.

Proof. Let $M \in \mathrm{A}_{\downarrow\mathrm{B}}$ $-\mathbf{mod}$ and

$$P_{\bullet} : \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0 \tag{5.20}$$

a projective resolution for M as an $\mathrm{A}_{\downarrow\mathrm{B}}$-module. Applying the functor $\mathrm{Hom}_{\mathrm{A}_{\downarrow\mathrm{B}}}(-, N)$ to the deleted version of $P_{\bullet}$ we obtain a cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{A}_{\downarrow\mathrm{B}}}(P_0, N) \xrightarrow{d^*_{1(\mathrm{A}_{\downarrow\mathrm{B}})}} \mathrm{Hom}_{\mathrm{A}_{\downarrow\mathrm{B}}}(P_1, N) \xrightarrow{d^*_{2(\mathrm{A}_{\downarrow\mathrm{B}})}} \mathrm{Hom}_{\mathrm{A}_{\downarrow\mathrm{B}}}(P_2, N) \longrightarrow \cdots \tag{5.21}$$

whose cohomology groups are $\mathrm{Ext}^*_{\mathrm{A}_{\downarrow\mathrm{B}}}(M, N)$.

By our hypothesis on the indecomposable projectives and the above lemma, the exact sequence $P_{\bullet}$ is also an (A, B)-projective resolution for M as an A module. This allow us to use relative homology to study it. So applying the functor $\mathrm{Hom}_A(-, N)$ to the deleted version of $P_{\bullet}$ we obtain the following cochain complex

$$0 \longrightarrow \mathrm{Hom}_A(P_0, N) \xrightarrow{d^*_{1A}} \mathrm{Hom}_A(P_1, N) \xrightarrow{d^*_{2A}} \mathrm{Hom}_A(P_2, N) \longrightarrow \cdots, \tag{5.22}$$

whose cohomology groups are $\mathrm{Ext}^*_{(A,B)}(M, N)$.

But by the Hom-functor Equality we have an equality of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathrm{A}_{\downarrow\mathrm{B}}}(P_0, N) & \xrightarrow{d^*_{1\,\mathrm{A}_{\downarrow\mathrm{B}}}} & \mathrm{Hom}_{\mathrm{A}_{\downarrow\mathrm{B}}}(P_1, N) & \xrightarrow{d^*_{2\,\mathrm{A}_{\downarrow\mathrm{B}}}} & \mathrm{Hom}_{\mathrm{A}_{\downarrow\mathrm{B}}}(P_2, N) & \longrightarrow & \cdots \\ \| & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathrm{Hom}_A(P_0, N) & \xrightarrow{d^*_{1A}} & \mathrm{Hom}_A(P_1, N) & \xrightarrow{d^*_{2A}} & \mathrm{Hom}_A(P_2, N) & \longrightarrow & \cdots \end{array} \tag{5.23}$$

therefore they have the same cohomology

$$\mathrm{Ext}^*_{\mathrm{A}_{\downarrow\mathrm{B}}}(M, N) = \mathrm{Ext}^*_{(A,B)}(M, N). \tag{5.24}$$

□

This way of computing the classical and relative Ext-functors has some nice consequences regarding dimensions.

Corollary 5.3.5. *Let* $B \subseteq A$ *be an extension of* $\mathbb{K}$*-algebras of finite dimension. If all the indecomposable projective* $\mathrm{A}_{\downarrow\mathrm{B}}$*-modules are* (A, B)*-projective, then*

1. $\mathrm{pd}_{A_{\downarrow B}} M = \mathrm{pd}_{(A,B)} M$, *for any* $A_{\downarrow B}$*-module* M. *In particular it holds for simple modules.*

2. $\mathrm{gldim}(A_{\downarrow B}) \leqq \mathrm{gldim}(A, B)$.

3. *If* $\mathrm{gldim}(A_{\downarrow B})$ *is infinite, then* $B \subseteq A$ *is controllable.*

Proof. Suppose that $\mathrm{pd}_{A_{\downarrow B}} M < \infty$. Then a minimal $A_{\downarrow B}$-projective resolution for M is also an (A, B)-projective resolution and $\mathrm{pd}_{A_{\downarrow B}} M \geqq \mathrm{pd}_{(A,B)} M$. On the other hand, since $A_{\downarrow B}$ is a finite dimensional algebra, there exists a simple $S \in A_{\downarrow B}$-**mod** such that $0 \neq \mathrm{Ext}_{A_{\downarrow B}}^{\mathrm{pd}_{A_{\downarrow B}} M}(M, S) = \mathrm{Ext}_{(A,B)}^{\mathrm{pd}_{A_{\downarrow B}} M}(M, S)$ and $\mathrm{pd}_{A_{\downarrow B}} M \leqq \mathrm{pd}_{(A,B)} M$, see (2.4.1). The infinite dimensional case is treated similarly. This proves the first item and the rest follows trivially.

□

Remark 5.3.6. (Applications to Relative Dimensions) The above corollary states that if one is interested in computing relative projective and relative global dimensions, then a way to do that is by looking for the indecomposable projectives $A_{\downarrow B}$-modules among the (A, B)-projectives. For finite dimensional basic algebras in the category of finite dimensional modules, the main category studied in this work, there are only a finite number of (isoclasses of) indecomposable projectives, so the above theorem says that you can only look at the direct summands of $A \otimes_B P_{A_{\downarrow B}}(i)$, for $i \in (Q_{A_{\downarrow B}})_0$ in order to (possibly) have a lower bound for the relative global dimension.

5.4 Application: Jacobson Radical Compatibility

For this discussion we assume that $B \subseteq A$ is an extension of finite dimensional basic $\mathbb{K}$-algebras such that $AJ(B)$ is a right ideal in A, where $J(B)$ denotes the *Jacobson radical* of B. Remember that, for any finite dimensional B-module X, $J_B(X)$ denotes the radical of X as a B-module and $\mathrm{top}_B(X) \doteq \frac{X}{J_B(X)}$.

With our assumptions we can consider an exact sequence of $A \otimes_k B^{op}$-modules induced by $J(B)$ as follows

$$0 \longrightarrow AJ(B) \longrightarrow A \longrightarrow \frac{A}{AJ(B)} \longrightarrow 0, \tag{5.25}$$

Apply the funtor $- \otimes_B X$ to (5.25), in order to obtain the following long exact sequence of vector spaces

$$\cdots \longrightarrow \mathrm{Tor}_1^B(A_{\downarrow B}, X) \longrightarrow AJ(B) \otimes_B X \longrightarrow A \otimes_B X \longrightarrow A_{\downarrow B} \otimes_B X \longrightarrow 0. \tag{5.26}$$

Analysing the modules that appear above, one obtains that

1. $AJ(B) \otimes_B X = A \otimes_B J(B)X \cong A \otimes_B J_B(X)$.

2. $A \otimes_B X$ is an (A, B)-projective module and it appears with degree zero at X's standard (A, B)-projective resolution.

3. For the last module we claim that $\mathrm{A}_{\downarrow\mathrm{B}} \otimes_B X \cong \mathrm{A}_{\downarrow\mathrm{B}} \otimes_\Sigma \operatorname{top}_B(X)$, where $\Sigma \subsetneq B$ is a semisimple subalgebra such that $B = \Sigma \oplus J(B)$ as vector spaces. This is proved in the next lemma.

Lemma 5.4.1. $\mathrm{A}_{\downarrow\mathrm{B}} \otimes_B X \cong \mathrm{A}_{\downarrow\mathrm{B}} \otimes_\Sigma \operatorname{top}_B(X)$*, as A-modules (or* $\mathrm{A}_{\downarrow\mathrm{B}}$*-modules).*

Proof. Since Σ is semisimple, and all B-modules are Σ-modules by restriction, we get a (B, Σ)-exact sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J_B(X) & \xrightarrow{\iota} & X & \xrightarrow{\pi} & \operatorname{top}_B(X) & \longrightarrow & 0 \\
 & & \| & \swarrow_{h_1} & \| & \swarrow_{h_0} & \| & & \\
0 & \longrightarrow & J_B(X) & \xrightarrow[\iota]{} & X & \xrightarrow[\pi]{} & \operatorname{top}_B(X) & \longrightarrow & 0
\end{array}
$$

where h_0 and h_1 are the Σ-homotopy, that is, they satisfy $\pi \circ h_0 = 1_{\operatorname{top}_B(X)}$ and $\iota \circ h_1 + h_0 \circ \pi = 1_X$.

Restricting the action of B when necessary, we can consider the following homomorphisms of A-modules

$$
\begin{cases}
\Phi = 1_{\mathrm{A}_{\downarrow\mathrm{B}}} \otimes_\Sigma \pi \;:\; \mathrm{A}_{\downarrow\mathrm{B}} \otimes_\Sigma X \longrightarrow \mathrm{A}_{\downarrow\mathrm{B}} \otimes_\Sigma \operatorname{top}_B(X) \\
\Psi = 1_{\mathrm{A}_{\downarrow\mathrm{B}}} \otimes_\Sigma h_0 \;:\; \mathrm{A}_{\downarrow\mathrm{B}} \otimes_\Sigma \operatorname{top}_B(X) \longrightarrow \mathrm{A}_{\downarrow\mathrm{B}} \otimes_\Sigma X.
\end{cases}
$$

Let $T_B = \langle [a]b \otimes_\Sigma x - [a] \otimes_\Sigma bx \mid [a] \in \mathrm{A}_{\downarrow\mathrm{B}}, b \in B, x \in X \rangle = \langle [a] \otimes_\Sigma bx \mid [a] \in \mathrm{A}_{\downarrow\mathrm{B}}, b \in J(B), x \in X \rangle$ be the A-submodule of $A \otimes_\Sigma X$ such that

$$
\frac{\mathrm{A}_{\downarrow\mathrm{B}} \otimes_\Sigma X}{T_B} \cong \mathrm{A}_{\downarrow\mathrm{B}} \otimes_B X
$$

and compute

$$
\Phi([a] \otimes_\Sigma bx) = [a] \otimes_\Sigma bx = [a] \otimes_\Sigma 0 = 0,
$$

for any $a \in A$, $b \in J(B)$ and $x \in X$. Therefore $T_B \subseteq \operatorname{Ker}(\Phi)$ and there is a unique homomorphism of A-modules

$$
\phi \;:\; \mathrm{A}_{\downarrow\mathrm{B}} \otimes_B X \longrightarrow \mathrm{A}_{\downarrow\mathrm{B}} \otimes_\Sigma \operatorname{top}_B(X)
$$

such that $\Phi = \phi \circ \pi_{T_B}$, that is $\phi([a] \otimes_B x) = [a] \otimes_\Sigma [x]$, for any $a \in A$ and $x \in X$. Here $\pi_{T_B} : A \otimes_\Sigma X \longrightarrow A \otimes_B X$ is the canonical projection. Composing Ψ with π_{T_B} we get a homomorphism of A-modules

$$
\psi = \pi_{T_B} \circ \Psi \;:\; \mathrm{A}_{\downarrow\mathrm{B}} \otimes_\Sigma \operatorname{top}_B(X) \longrightarrow \mathrm{A}_{\downarrow\mathrm{B}} \otimes_B X
$$

that satisfies $\psi([a] \otimes_\Sigma [x]) = [a] \otimes_B h_0([x])$, for any $a \in A$ and $x \in X$.

Calculate

$$\begin{aligned}\phi \circ \psi([a] \otimes_\Sigma [x]) &= \phi([a] \otimes_B h_0([x])) \\ &= [a] \otimes_\Sigma \pi \circ h_0([x]) = [a] \otimes_\Sigma [x], \text{ by the homotopy}\end{aligned}$$

and

$$\begin{aligned}\psi \circ \phi([a] \otimes_B x) &= \psi \circ \phi([a] \otimes_B (\iota \circ h_1(x) + h_0 \circ \pi(x))) \\ &= \phi([a] \otimes_\Sigma (\pi \circ \iota \circ h_1(x) + \pi \circ h_0 \circ \pi(x))) \\ &= \phi([a] \otimes_\Sigma \bar{x}) = a \otimes_B h_0 \circ \pi(x) \\ &= [a] \otimes_B h_0 \circ \pi(x) + 0 \\ &= [a] \otimes_B h_0 \circ \pi(x) + [a] \otimes_B \iota \circ h_1(x) = [a] \otimes_B x,\end{aligned}$$

for any $a \in A$ and $x \in X$, we obtain that $\mathrm{A}_{\downarrow\mathrm{B}} \otimes_B X \cong \mathrm{A}_{\downarrow\mathrm{B}} \otimes_\Sigma \operatorname{top}_B(X)$ as modules over A (or $\mathrm{A}_{\downarrow\mathrm{B}}$). □

With the above remarks we can study the following short exact sequence of A-modules obtained from (5.26)

$$0 \longrightarrow \operatorname{Im}(A \otimes_B J_B(X)) \longrightarrow A \otimes_B X \longrightarrow \mathrm{A}_{\downarrow\mathrm{B}} \otimes_\Sigma \operatorname{top}_B(X) \longrightarrow 0 \tag{5.27}$$

Suppose that X is a semisimple B-module, that means that $J_B(X) = 0$. Then the above sequence yields

$$A \otimes_B X \cong \mathrm{A}_{\downarrow\mathrm{B}} \otimes_\Sigma \operatorname{top}_B(X), \tag{5.28}$$

as modules over A or $\mathrm{A}_{\downarrow\mathrm{B}}$.

Supose that $A = \frac{\mathbb{K}Q}{I}$, with $I \lhd \mathbb{K}Q$ is admissible, $B = \frac{\mathbb{K}R}{J}$, where $J \lhd \mathbb{K}R$ is admissible, and $Q_0 = R_0$ is a finite quiver and the arrows of R are linear combinations of non trivial paths of Q. The later condition implies that $J(B) \subseteq J(A)$ and that $\mathrm{A}_{\downarrow\mathrm{B}} \doteq A/(AJ(B)A)$ has the same vertices of A, since $AJ(B)A \subseteq AJ(A)A = J(A)$ and $J(\mathrm{A}_{\downarrow\mathrm{B}}) = \frac{\mathrm{A}_{\downarrow\mathrm{B}}}{J(A)/(AJ(B)A)}$.

Fix $\Sigma = \bigoplus_{i \in R_0} \mathbb{K} e_i$. If $i \in R_0 = Q_0$, then we can specify $X = S_B(i) = S_A(i)$ in (5.28), here $S_B(i)$ denotes the simple B (or A) module concentrated at vertex i. Doing so, we obtain that

$$A \otimes_B S_B(i) \cong \mathrm{A}_{\downarrow\mathrm{B}} \otimes_\Sigma S_B(i) = \mathrm{A}_{\downarrow\mathrm{B}}\, e_i \otimes_\Sigma S_B(i) \cong \mathrm{A}_{\downarrow\mathrm{B}}\, e_i = P_{\mathrm{A}_{\downarrow\mathrm{B}}}(i). \tag{5.29}$$

where $P_{\mathrm{A}_{\downarrow\mathrm{B}}}(i) = \mathrm{A}_{\downarrow\mathrm{B}}\, e_i$ are representatives of the isoclasses of indecomposable projectives of $\mathrm{A}_{\downarrow\mathrm{B}}$ $-$**mod**.

We proved the following statement.

Corollary 5.4.2. *Let $B \subseteq A$ be an extension of basic finite dimensional $\mathbb{K}$-algebras. Write $A = \mathbb{K}Q/I$, with $I \lhd \mathbb{K}Q$ admissible, and $B = \mathbb{K}R/J$, with $J \lhd \mathbb{K}R$. If*

- *the vertices of R and Q are the same.*
- *the arrows of R are linear combinations of non trivial paths of Q.*
- *$AJ(B)$ is a right ideal of A.*

Then

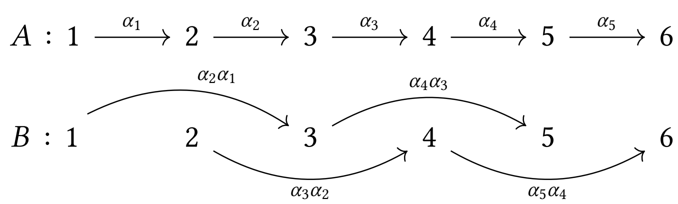
$$\operatorname{gldim}(\mathrm{A}_{\downarrow\mathrm{B}}) \leqq \operatorname{gldim}(A, B). \tag{5.30}$$

In particular, if $\operatorname{gldim}(\mathrm{A}_{\downarrow\mathrm{B}}) = \infty$, *then $B \subseteq A$ is controllable.*

Proof. The above discussion proved that every indecomposable $\mathrm{A}_{\downarrow\mathrm{B}}$-projective module is (A, B)-projective, therefore the conditions of (5.3.5) are satisfied and the result follows. □

Now we are able to construct examples of controllable extensions without doing any homological computations, simply relying on the literature and looking at ideals!

Example 5.4.3. Consider $A = \mathbb{K}\,\mathbb{A}_n$, denote the arrows of $\mathbb{A}_n$ by $\alpha_i : i \to i+1$. Let B be the path algebra of the quiver whose vertices are the same of $\mathbb{A}_n$ and whose arrows are $\alpha_{i+1}\alpha_i$, for $i = 1, \ldots, n-1$. Pictorially, we are considering cases like this one



In this case we have that $J(B)$ consist of all paths of $\mathbb{A}_n$ with even order and $AJ(B) = J^2(A) = J^2(\mathbb{K}\,\mathbb{A}_n)$ is a bilateral ideal. Therefore

$$\mathrm{A}_{\downarrow\mathrm{B}} = \frac{\mathbb{K}\,\mathbb{A}_n}{J^2(\mathbb{K}\,\mathbb{A}_n)},$$

and

$$\operatorname{gldim}(\mathrm{A}_{\downarrow\mathrm{B}}) = n - 1 \leqq \operatorname{gldim}(A, B).$$

But we can apply the theorem (3.3.1) to obtain that $\operatorname{gldim}(A, B) = n - 1$.

We can go a step further, $J(B)A$ is also a bilateral ideal since $AJ(B) = J(B)A$. This says that, if we work with the extension $B^{op} \subseteq A^{op}$, then an analogous result holds:

$$\operatorname{gldim}(A^{op}, B^{op}) = n - 1 = \operatorname{gldim}(A, B).$$

We can state our findings as

Corollary 5.4.4. *Fix $n \in \mathbb{N}$. There exist an extension $B_n \subseteq A_n$ of $\mathbb{K}$-algebras such that* $\operatorname{gldim}(B) = \operatorname{gldim}(A) = 1$*,* $B \subseteq A$ *is controllable and*

$$\operatorname{gldim}(A, B) = n.$$

Finally we end this section with a result that allows one to look at $A - \mathbf{mod}$ in order to bound $\operatorname{gldim}(A^{op}, B^{op})$. First we need a lemma that generalizes [ASS06, Theorem I.5.13.(a) and (b)].

Lemma 5.4.5. *Let A be a finite dimensional algebra and $D(-) = Hom_{\mathbb{K}}(-, \mathbb{K}) : A - \mathbf{mod} \longrightarrow A^{op} - \mathbf{mod}$ be the dual functor, then*

1. *the image of an (A, B)-exact sequence by $D(-)$ is an (A^{op}, B^{op})-exact sequence.*
2. *if $P \in A - \mathbf{mod}$ is (A, B)-projective, then $D(P)$ is (A^{op}, B^{op})-injective in $A^{op} - \mathbf{mod}$*
3. *if $E \in A - \mathbf{mod}$ is (A, B)-injective, then $D(E)$ is (A^{op}, B^{op})-projective in A^{op}*

Proof. Consider an (A, B)-exact sequence

$$0 \longrightarrow M \underset{r}{\overset{f}{\rightleftarrows}} N \underset{s}{\overset{g}{\rightleftarrows}} L \longrightarrow 0. \tag{5.31}$$

By definition r and s are homomorphisms of B-modules that satisfy

$$\begin{cases} gs = 1_L \\ fr + sg = 1_N \\ rf = 1_M. \end{cases}$$

Apply $D(-)$ to the above exact sequence to obtain the following structure

$$0 \longrightarrow D(L) \underset{D(s)}{\overset{D(g)}{\rightleftarrows}} D(N) \underset{D(r)}{\overset{D(f)}{\rightleftarrows}} D(M) \longrightarrow 0, \tag{5.32}$$

where $D(g)$ and $D(f)$ are homomorphisms of A^{op}-modules, and $D(r)$ and $D(s)$ are homomorphisms of B^{op}-modules, see [ASS06, I.2 Standard dualities 2.9].

Compute

$$\begin{cases} D(s)D(g) = D(gs) = D(1_L) = 1_{D(L)} \\ D(r)D(f) + D(g)D(s) = D(fr+sg) = D(1_N) = 1_{D(N)} \\ D(f)D(r) = D(rf) = D(1_M) = 1_{D(M)}, \end{cases}$$

which proves statement **i**.

Assume that $P \in A$-**mod** is (A, B)-projective and consider the following structure in the category of finite dimensional A^{op}-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \overset{f}{\underset{r}{\rightleftarrows}} & N & \overset{g}{\underset{s}{\rightleftarrows}} & L & \longrightarrow & 0, \\ & & \big\downarrow{\scriptstyle \phi} & & & & & & \\ & & D(P) & & & & & & \end{array} \tag{5.33}$$

where the horizontal line is (A^{op}, B^{op})-exact. Applying $D(-)$, using $D^2 \cong 1$ (see [ASS06, I.2 Standard dualities 2.9]), and statement **i** we get

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & & \big\downarrow{\scriptstyle D(\phi)} & & \\ 0 & \longrightarrow & D(L) & \xrightarrow{D(g)} & D(N) & \xrightarrow{D(f)} & D(M) & \longrightarrow & 0, \end{array} \tag{5.34}$$

with the horizontal line being (A, B)-exact. By hypothesis on P there exists a homomorphism of A-modules $\psi : P \longrightarrow D(N)$ such that $D(\phi) = D(f)\psi$. We compute

$$\phi = D^2(\phi) = D(D(f)\psi) = D(\psi)D^2(f) = D(\psi)f$$

and $D(\psi) : N \longrightarrow D(P)$. This proves statement **ii**.

A proof of statement **iii** is similar to that of statement **ii** and will be omitted. □

Corollary 5.4.6. *If every indecomposable* $\mathrm{A}_{\downarrow\mathrm{B}}$*-module is* (A, B)*-injective, then every indecomposable* $(\mathrm{A}_{\downarrow\mathrm{B}})^{op}$*-projective module is* (A^{op}, B^{op})*-projective. In particular every result from (5.3.5) holds for* $B^{op} \subseteq A^{op}$*.*

Remark 5.4.7. The finite dimensional condition is necessary because $D(-)$ is not algebraically well behaved for infinite dimensional modules.

If we assume that $B \subseteq A$ is an extension of algebras such that B and A have the same vertices, the arrows of B are linear combinations of non trivial paths of A, and $J(B)A$ is a bilateral ideal in A, then a similar (dual) argument involving the short exact sequence

$$0 \longrightarrow J(B)A \xrightarrow{\iota} A \xrightarrow{\pi} \frac{A}{J(B)A} \doteq \mathrm{A}_{\downarrow \mathrm{B}} \longrightarrow 0$$

proves that every indecomposable $\mathrm{A}_{\downarrow \mathrm{B}}$-injective module is (A, B)-injective. When a monomorphism $f : M \to N$ is lifted from $\mathrm{A}_{\downarrow \mathrm{B}} -\mathbf{mod}$ to $A - \mathbf{mod}$ it is possible to prove it admits a B-retraction, the dual result that we computed for epimorphisms. This culminates in the following result.

Corollary 5.4.8. *Let $B \subseteq A$ be an extension of basic finite dimensional $\mathbb{K}$-algebras. Write $A = \mathbb{K}Q/I$, with $I \lhd \mathbb{K}Q$ admissible, and $B = \mathbb{K}R/J$, with $J \lhd \mathbb{K}R$. If*

- *the vertices of R and Q are the same.*
- *the arrows of R are linear combinations of non trivial paths of Q.*
- *$J(B)A$ is a left ideal of A.*

Then

$$\mathrm{gldim}({\mathrm{A}_{\downarrow \mathrm{B}}}^{op}) \leqq \mathrm{gldim}(A^{op}, B^{op}). \tag{5.35}$$

In particular, if $\mathrm{gldim}({\mathrm{A}_{\downarrow \mathrm{B}}}^{op}) = \infty$, *then $B^{op} \subseteq A^{op}$ is controllable.*

We end this section with an example that uses the above theory to compute $\mathrm{gldim}(A^e, B^e)$.

Example 5.4.9. Again consider $A = \mathbb{K}\mathbb{A}_n$, denote the arrows of $\mathbb{A}_n$ by $\alpha_i : i \to i+1$. Let B be the path algebra of the quiver whose vertices are the same of $\mathbb{A}_n$ and whose arrows are $\alpha_{i+1}\alpha_i$, for $i = 1, \dots, n-1$.

As we have seen before we can use the results from this section and (3.3.1) to compute

$$\mathrm{gldim}(A, B) = n - 1 = \mathrm{gldim}(A^{op}, B^{op}).$$

Consider the extension $B^e \subseteq A^e$. Then we have $J(B^e) = J(B) \otimes B^{op} + B \otimes J(B^{op})$. Take an element $b \otimes j$, with $b \in B$ and $j \in J(B^{op})$. Then for any vectors $a_i \in A$, $i = 1, \dots, 4$, we have

$$(a_1 \otimes a_2)(b \otimes j)(a_3 \otimes a_4) = a_1 b a_3 \otimes a_2 j a_3 \in A \otimes A^{op} J(B^{op}),$$

since $A^{op}J(B^{op})A^{op} \subseteq A^{op}J(B^{op})$. Similarly, $A \otimes A^{op} \cdot J(B) \otimes B^{op} \cdot A \otimes A^{op} \subseteq AJ(B) \otimes A^{op}$.

This says that

$$A^e J(B^e) A^e \subseteq A^e J(B^e).$$

Therefore

$$\operatorname{gldim}(A^e, B^e) \geqq \operatorname{gldim}\left(\frac{A^e}{A^e J(B^e) A^e}\right) = \operatorname{gldim}((\mathrm{A}_{\downarrow \mathrm{B}})^e) = 2 \operatorname{gldim}(\mathrm{A}_{\downarrow \mathrm{B}}) = 2n - 2.$$

And (3.3.7) yields $\operatorname{gldim}(A^e, B^e) \leqq 2n - 2$. For the sake of completeness

$$\operatorname{gldim}(A^e, B^e) = 2n - 2 = 2 \operatorname{gldim}(A, B)$$

and all the involved extensions are controllable.

5.5 Counterexamples

We end this chapter with a section dedicated to counterexamples.

Example 5.5.1. A natural question would be if every controllable extension has the compatibility "$AJ(B)$ is a bilateral ideal in A". That is not the case. For example, consider the following extension

$$A: \quad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$$

$$B: \quad 1 \xrightarrow{\alpha} 2 \qquad 3 \xrightarrow{\beta} 4.$$

Then $A = T[B, N]$, where $N \doteq \langle \beta, \beta\alpha, \gamma\beta \rangle$. By 4.1.1 we have $\operatorname{gldim}(A, B) = 1$. Notice that

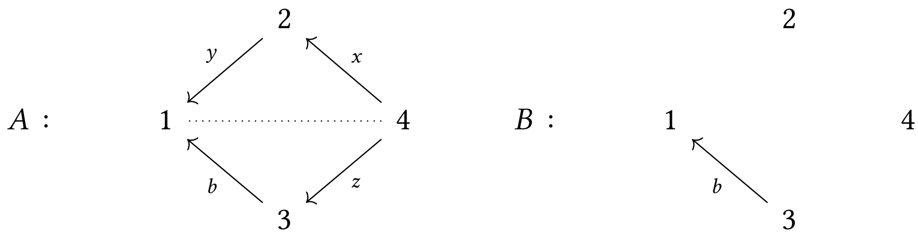
$$\mathrm{A}_{\downarrow \mathrm{B}}: \quad 1 \qquad 2 \xrightarrow{\beta} 3 \qquad 4,$$

by simple computations or using 5.1.4. Therefore

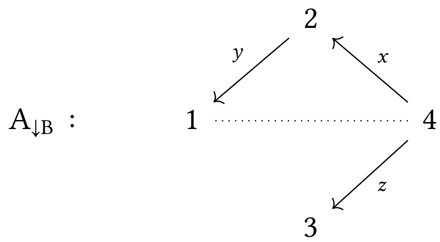
$$\operatorname{gldim}(A, B) = 1 = \operatorname{gldim}(\mathrm{A}_{\downarrow \mathrm{B}}),$$

but the Jacobson radical compatibilities explored in the results do not hold.

Example 5.5.2. This is another example of a controllable extension without the Jacobson radical compatibility. Consider

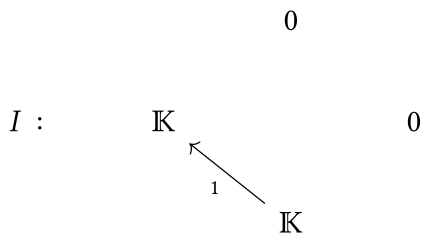


where the dotted line means that we are consider the relation $bz - yx$ in A. Then



with relation yx. It is easy to show that $\mathrm{gldim}(\mathrm{A}_{\downarrow \mathrm{B}}) = 2$.

Notice that $AJ(B) = \langle b \rangle$ is not a bilateral ideal since $bz = yx \notin AJ(B)$. To compute $\mathrm{gldim}(A, B)$ we will use that B has only 5 isomorphism classes of indecomposable modules. They are represented by $S(i)$, for $i = 1, 2, 3, 4$, and



Simple computations show that

$$A \otimes_B S(1) \cong P_A(1),\ A \otimes_B S(2) \cong P_A(2),\ A \otimes_B S(3) \cong S(3),\ A \otimes_B \cong P_A(4),\ \textit{and } A \otimes_B I \cong {}_A I.$$

Since we computed the induced modules [1] using indecomposable B-modules and obtained only indecomposable modules, we have that every (A, B)-projective module can be written as a direct sum of the above A-modules. Using this we obtain the following minimal

[1] Induced modules are modules of the form $A \otimes_B M$ for some B-module M.

(A, B)-projective resolution for $S(4)$

$$0 \longrightarrow P(1) \longrightarrow P(2) \oplus I \longrightarrow P(4) \longrightarrow S(4) \longrightarrow 0.$$

In particular, $\operatorname{gldim}(A, B) \geqq 2$. From (3.3.1) we obtain $2 \leqq \operatorname{gldim}(A, B)$. Therefore

$$\operatorname{gldim}(A, B) = 2 = \operatorname{gldim}(\mathrm{A}_{\downarrow \mathrm{B}}).$$

Example 5.5.3. The most natural question to ask is is every finite dimensional extension is controllable. Sadly, this is not true.

To see why, consider $A = \mathbb{K}Q$ and $B = \mathbb{K}R$ where

$$Q : \begin{array}{ccc} & & 3 \\ & \overset{b}{\nearrow} & \\ 1 & \xrightarrow[a]{} & 2 \end{array} \qquad R : 1 \xrightarrow{a+b} 2+3$$

that is, B is the subalgebra generated by $e_1, e_2 + e_3$, and $a + b$. The category $A - \mathbf{mod}$ has only 6 non isomorphic indecomposable modules, they are the simple modules $S(1)$, $S(2)$, and $S(3)$, and

$$M : \begin{array}{ccc} & & 0 \\ & \overset{0}{\nearrow} & \\ \mathbb{K} & \xrightarrow[1]{} & \mathbb{K} \end{array} \qquad N : \begin{array}{ccc} & & \mathbb{K} \\ & \overset{1}{\nearrow} & \\ \mathbb{K} & \longrightarrow & 0 \end{array} \qquad L : \begin{array}{ccc} & & \mathbb{K} \\ & \overset{1}{\nearrow} & \\ \mathbb{K} & \xrightarrow{1} & \mathbb{K} \end{array}$$

The indecomposable $S(2) = P(2)$, $S(3) = P(3)$, and $L = P(1)$ are (A, B)-projectives because they are A-projectives. To see that $S(1) = \langle x \rangle$ is (A, B)-projective we compute

$$\begin{cases} e_2 \otimes_B x = e_2(e_2 + e_3) \otimes_B x = e_2 \otimes_B (e_2 + e_3)x = 0 \\ e_3 \otimes_B x = 0 \end{cases}$$

And $(a + b) \otimes_B x = (e_2 + e_3) \otimes_B (a + b)x = 0$ by simplicity, implying $a \otimes_B x = -b \otimes_B x \in (A \otimes_B S(1))_2 \cap (A \otimes_B S(1))_3 = 0$. Therefore $A \otimes_B S(1) = \langle e_1 \otimes_B x \rangle \cong S(1)$.

Computing in a similar fashion $A \otimes_B M$, denote $M_1 = \mathbb{K}\,x$ and $M_2 = \mathbb{K}\,y$, then

$$\begin{cases} e_2 \otimes_B x = 0,\; e_3 \otimes_B x = 0 \\ a \otimes_B x = e_2(a+b) \otimes_B x = e_2 \otimes_B (a+b)x = e_2 \otimes_B y \\ b \otimes_B x = e_3(a+b) \otimes_B x = e_3 \otimes_B (a+b)x = e_3 \otimes_B y \\ e_1 \otimes_B y = 0 \\ a \otimes_B y = 0,\; b \otimes_B y = 0. \end{cases}$$

In terms of representations we get

$$\begin{array}{ccc} \begin{array}{ccc} & & 0 \\ & \overset{0}{\nearrow} & \\ M : \langle x \rangle & \xrightarrow[1]{} & \langle y \rangle \end{array} & \Longrightarrow & \begin{array}{ccc} & & \langle e_3 \otimes_B y \rangle \\ & \overset{1}{\nearrow} & \\ A \otimes_B M : \langle e_1 \otimes_B x \rangle & \xrightarrow{1} & \langle e_2 \otimes_B y \rangle \end{array} \end{array}$$

and $A \otimes_B M \cong L = P(1)$ is indecomposable, so M is *not* (A,B)-projective. Clearly the kernel of $A \otimes_B M \to M$ is $S(3)$ and is (A,B)-projective. Therefore $\mathrm{pd}_{(A,B)}\, M = 1$. Due to the symmetry of the algebras we also get $\mathrm{pd}_{(A,B)}\, N = 1$.
But $J(B) = \langle a+b \rangle \implies AJ(B)A = \langle a, b \rangle = J(A)$, so $\mathrm{A}_{\downarrow \mathrm{B}}$ is semisimple and we obtain

$$\mathrm{gldim}(A,B) = 1 > 0 = \mathrm{gldim}(\mathrm{A}_{\downarrow \mathrm{B}}).$$

The Examples (5.5.2) and (5.5.3) were presented to me by Prof. Kostiantyn Iusenko and Prof. John MacQuarrie during private meetings.

Chapter 6

Conclusions

Our work - first motivated by the results of [XX13] that connected Relative Homological Algebra to the Finitistic Dimension Conjecture, [Guo18] examples, and later by [IM21] combinatorial bound - faced a dormant theory that had not being updated with recent representation theoretical techniques. More precisely, there were little examples of extensions of algebras using Gabriel's approach to finite dimensional algebras: the use of a combinatorial data - quivers - to understand them.

We could summarize our findings as an effort to interact representation theory of quivers with Relative Homological Algebra, trying to use the computational ease of quivers to obtain relative dimensions.

In that sense, we defined a new class of extensions. *Controllable extensions* (5.1) are extensions of algebras that preserve much of the computation properties of classical global dimensions regarding algebraic operations (5.2.1).

Due to the universality of Hochschild's relative homological theory, when one studies generalizations of classical homological results, it is necessary to show that they encompass non trivial examples. With that in mind, we obtained the result computing the relative global dimension of tensor extensions of algebras in (4.1.1), and found lower bounds for other classically motivated extensions in (4.2.4) and (4.3.2).

These classically motivated results were used to compute relative homological dimensions. The tensor extension result, being an equality, was sufficient to obtain a class of non trivial controllable extensions of algebras with relative homological dimension equal to one (5.1.4). For the other results we were able to obtain controllable extensions with infinite relative global dimensions (4.3.3), or used the combinatorial upper bound of [IM21], see (3.3.1), to compute controllable extensions of algebras with any finite relative global dimension (4.2.8).

Finally, we obtained a sufficient condition for the global dimension an extension to be bounded bellow by the controllable condition, that is, the inequality

$$\mathrm{gldim}(\mathrm{A}_{\downarrow\mathrm{B}}) \leqq \mathrm{gldim}(A, B).$$

This sufficient condition is algebraic in the sense that it imposes compatibilities between the Jacobson radicals of A and B, see (5.4.2). This result was combined with (3.3.1) to get the relative finite dimension examples (5.4.3). Then we showed some examples that lie outside the scope of our results in 5.5.

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