# Tightness in Banach Spaces 

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A Luz y a toda mujer, migrante, madre, negra.

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## Resumo

Cáceres-Rigo, A. C. Bases apertadas em espaços de Banach. 2022. Tese (Doutorado)

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No presente trabalho, estudamos espaços de Banach com bases apertadas e provamos dicotomias que envolvem noções diferentes de minimalidade e novos tipos de bases apertadas. Introduzimos a noção de sistema admissível de blocos para codificar diferentes tipos de mergulhos entre espaços de Banach com base de Schauder. Dados um espaço de Banach $E$ com base de Schauder normalizada $\left(e_{n}\right)_{n}$ e um sistema admissível de blocos $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$, definimos a noção de $\mathcal{A}_{E}$-mergulho e suas respetivas noções de $\mathcal{A}_{E}$-minimalidade e $\mathcal{A}_{E^{-}}$ aperto associadas. Estendendo os métodos usados por V. Ferenczi e C. Rosendal para provar a 'terceira dicotomia' no programa de classificação de espaços de Banach por subespaços, provamos que um espaço de Banach $E$ com um sistema admissível de blocos ( $\mathcal{D}_{E}, \mathcal{A}_{E}$ ) contém um subespaço infinito dimensional com base que é ou $\mathcal{A}_{E}$-apertada ou $\mathcal{A}_{E}$-minimal. Como corolário obtém-se a 'terceira dicotomia' de Ferenczi e Rosendal: todo espaço de Banach contém um subespaço que ou é minimal, ou possui uma base apertada. Também como corolário provamos que toda sequência básica normalizada $\left(e_{n}\right)_{n}$ tem uma subsequência que ou é uma base apertada por sequências, ou é uma base spreading. Outras dicotomias entre noções de minimalidade e de aperto são demonstradas. Estendemos a definição de base de Schauder apertada e de base de Schauder apertada com constantes para o caso de espaços de Banach com base transfinita. Damos caracterizações de tais noções neste contexto e estudamos suas propriedades.
Palavras-chave: Bases de Schauder apertadas, espaços minimais, dicotomias em espaços de Banach, bases spreading, bases transfinitas.


#### Abstract

Cáceres-Rigo, A. C. Tightness in Banach spaces. 2022. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022. In this work, we study Banach spaces with tight bases and we prove dichotomies involving different types of minimality and new types of tightness. We introduce the notion of admissible system of blocks to code various kinds of embeddings between Banach spaces with Schauder bases. Given a Banach space $E$ with normalized Schauder basis $\left(e_{n}\right)_{n}$ and an admissible system of blocks $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$, we define an $\mathcal{A}_{E}$-embedding and the respective notions of $\mathcal{A}_{E}$-minimality and $\mathcal{A}_{E}$-tightness associated to it. Extending the methods used by V. Ferenczi and C. Rosendal to prove the 'third dichotomy' in the program of classification of Banach spaces up to subspaces, we prove that a Banach space $E$ with an admissible system of blocks $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$, contains an infinite dimensional subspace with a basis which is either $\mathcal{A}_{E}$-tight or $\mathcal{A}_{E}$-minimal. As a corollary, we obtain the 'third dichotomy' of Ferenczi and Rosendal: every Banach space contains a subspace that is either minimal or it has a tight basis. Also as a corollary we prove that every normalized basic sequence $\left(e_{n}\right)_{n}$ has a subsequence which is either a tight-by-sequences basis or it is spreading. Other dichotomies between notions of minimality and tightness are demonstrated. We extend the definition of tight Schauder basis and tight-with-constants Schauder basis to the case of Banach spaces with transfinite basis. We give characterizations of these notions in this context and study their properties.


Keywords: Tight bases, minimal spaces, dichotomies on Banach spaces, spreading bases, transfinite bases.

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## Chapter 1

## Introduction

### 1.1 The motivation: Classification of Banach spaces up to subspaces

The notion of tightness in Banach spaces with Schauder basis has its beginnings in the work of V. Ferenczi and Ch. Rosendal [22] as a fundamental concept in the search for a dichotomy for minimality in the Classification Program of Banach spaces up to subspaces started by W. T. Gowers in 2002, see [29].

The program aims to classify subspaces of Banach spaces into "inevitable" classes of infinite dimensional Banach spaces. Conditions for a class to be considered of interest for the Program were given by Gowers:
a) The classes must be inevitable, that is, every Banach space must belong to a class.
b) A class must be hereditary for closed subspaces or, if the property that determines the class is defined for basic sequences, then the class must be hereditary for block subspaces.
c) Two different classes must be disjoint.
d) The property that determines the class must give additional information about the space of operators defined over the space or over its subspaces.

A Banach space $X$ is decomposable if it can be written as the direct sum of two closed infinite-dimensional subspaces, otherwise $X$ is said indecomposable. A Banach space is said hereditarily indecomposable (or HI ) if all its infinite-dimensional subspaces are indecomposable. The first example of an HI space (we shall refer to this space as GM) was given in 1994 by W. Gowers and B. Maurey [30] and its construction was the answer to the unconditional basis problem, providing the first example of a Banach space ( $G M$ ) without unconditional basic sequences.

A version $G_{u}$ of $G M$ with an unconditional basis given by Gowers in [26] was used to solve the Banach's hyperplane problem: Is every Banach space isomorphic to its closed hyperplanes? A closed subspace $Y \subseteq X$ is a closed hyperplane if its codimension is equal to 1 . $G_{u}$ was
the first example of a space which is not isomorphic to any of its hyperplanes (recall that all subspaces of codimension 1 in any Banach space are mutually isomorphic). In [30] T. Gowers and B. Maurey noticed that every HI space is not isomorphic to any proper subspace. So, $G M$ is, in fact, not isomorphic to proper subspaces.

A space is asymptotically unconditional if there is a constant $C$ such that any sequence of $n$ successive vectors whose supports belong to $[n, \infty)$ is $C$-unconditional. An asymptotically unconditional and HI version $G$ of $G M$ was also given by Gowers in [27].

### 1.2 Working on dichotomies

In 1996 Gowers showed a first dichotomy (see [28]) giving the first two examples of inevitable classes:

Theorem (First dichotomy). Every Banach space has a separable subspace that is either hereditary indecomposable, or has an unconditional basis.

The first dichotomy provided the answer to the Homogeneous Banach space problem, formulated by S . Banach: is $\ell_{2}$ the only infinite-dimensional Banach space (up to isomorphism) which is isomorphic to all its infinite-dimensional subspaces? A space with such property of being isomorphic to all of its infinite dimensional subspaces is called an homogeneous space.

Gowers in [28] combined his dichotomy result with a corollary of the results of R. Komorowski and N. Tomczak-Jaegermann (see [34] and [35]): If X is an homogeneous Banach space then either X is isomorphic to $\ell_{2}$ or fails to have an unconditional basis. Using this result and the first dichotomy, if $X$ has an HI subspace it fails to be homogeneous and if it has a subspace with unconditional basis then itself has an unconditional basis, so by the result of Komorowski and Tomczak-Jaegermann, $X$ is isomorphic to $\ell_{2}$. Therefore,

Theorem (Theorem 1 in [28]). A Banach space is homogeneous if and only it is isomorphic to $\ell_{2}$.

Later in 2002, Gowers showed a second dichotomy (see [29]):
Theorem (Second dichotomy). Any Banach space contains a subspace with a basis such that no pair of disjointly supported block subspaces are isomorphic, or any two block subspaces have isomorphic subspaces.

This last condition was named as quasi-minimality by Gowers. A Banach space is minimal if it can be isomorphically embedded in all its closed subspaces. Clearly, every minimal space is quasi-minimal. Then it is possible to subdivide such spaces in those which do not have any minimal subspace (which Gowers called strictly quasi-minimal spaces) and those which have a minimal subspace. This division does not give additional information about the properties of the spaces, therefore does not seem to be the type of result the program is looking for. At
that moment, a dichotomy involving minimality was missing, and the program was at the following state:

Theorem (Theorem 7.7 in [29]). Let $X$ be a Banach space. Then $X$ has a subspace $Y$ with one of the following properties, which are mutually exclusive and all possible.
(1) $Y$ is hereditarily indecomposable.
(2) $Y$ has an unconditional basis and no pair of disjointly supported block subspaces are isomorphic.
(3) $Y$ has an unconditional basis and is strictly quasi-minimal.
(4) $Y$ has an unconditional basis and is minimal.

The spaces $G M$ and $G$ are examples of the first type, $G_{u}$ is an example of the second type. Tsirelson's space $\mathbf{T}$ and its $p$-convexifications $\mathbf{T}^{(p)}$ with $1<p<\infty$ (see [25]) have an unconditional basis and no minimal subspaces, indeed they are strictly quasi-minimal (see [10] for this) so they are examples of the third type. The classical spaces $c_{0}, \ell_{p}$, the dual of Tsirelson space $\mathbf{T}^{*}$ and Schlumprecht space $\mathcal{S}$ (see [2]) are of the fourth type.

### 1.3 The answer for the dichotomy for minimality

If $Y$ and $X$ are Banach spaces, we write $Y \hookrightarrow X$ to say that $Y$ is isomorphically embeddable in $X$. In 2007, V. Ferenczi and Ch. Rosendal (see [22]) introduced the notion of tightness and responded with three new dichotomies to the gap involving minimality left by Gowers.

Suppose $X=\left[x_{n}\right]_{n}$ is a Banach space with Schauder basis. A Banach space $Y$ is tight in $X$ if there is a sequence $\left(I_{n}\right)_{n}$ of successive finite subsets of $\mathbb{N}$, such that for all $A$ infinite subset of $\mathbb{N}, Y$ does not isomorphically embed in $\left[x_{n}, n \notin \cup_{i \in A} I_{i}\right] .\left(x_{n}\right)_{n}$ is a tight basis for $X$ if any Banach space $Y$ is tight in $X$, and $X$ is tight if it has a tight basis.

A useful and beatiful characterization of tightness was given by V. Ferenczi and G. Godefroy in 2012 using Baire category, see [18]: $Y$ is tight in $X=\left[x_{n}\right]_{n}$ if, and only if,

$$
E_{Y}=\left\{\mathfrak{u} \in 2^{\omega}: Y \hookrightarrow\left[x_{n}: n \in \operatorname{supp}(\mathfrak{u})\right]\right\}
$$

is meager in $2^{\omega}$.
Tightness is, in a sense, an opposite notion to minimality. It is clear that a tight space is not minimal. The dichotomy associated to minimality and tightness that Ferenczi and Rosendal obtained is called the third dichotomy and it is the pivotal point of this work.

Theorem (Third dichotomy). Every Banach space contains a subspace with a basis which is either tight or minimal.

In [22], some special types of tightness were also presented and studied. Suppose $X=\left[x_{n}\right]_{n}$ is tight and for all $Y=\left[y_{n}\right]_{n}$ block subspace of $X$ the subsets $I_{i}$ witnessing such tightness satisfy
$\forall i \in \mathbb{N}\left(I_{i}=\operatorname{supp}\left(y_{i}\right)\right)$, then we say that $X$ is tight by support if $\forall i \in \mathbb{N}\left(I_{i}=\operatorname{ran}\left(y_{i}\right)\right)$, $X$ is tight by range. Recall that if $x \in X$, then $\operatorname{supp}(x):=\left\{n \in \mathbb{N}: x_{n}^{*}(x) \neq 0\right\}$ and $\operatorname{ran}(x):=[\min \operatorname{supp}(x), \max \operatorname{supp}(x)]$, where $\left(x_{n}^{*}\right)_{n}$ is the sequence of biorthogonal functionals of the basis $\left(x_{n}\right)_{n}$.

Two Banach spaces are comparable if one embeds into the other. "No two disjointly supported block subspaces of a Banach space $X$ with basis are isomorphic" is equivalent to "no two disjointly supported block subspaces of $X$ are comparable". Those conditions are equivalent to saying that $X$ is tight by support. So, the second dichotomy of Gowers can be written in terms of a stronger form of tightness and a weaker form of minimality as follows:

Theorem (Second dichotomy). Any Banach space contains a subspace with a basis either tight by support or quasi-minimal.

Ferenczi and Rosendal also gave a fourth and a fifth dichotomies, contrasting forms of tightness and minimality, and presented the relations between different types of tightness and HI, unconditional basis and quasi-minimality notions. A basis $\left(x_{n}\right)_{n}$ in a Banach space $X$ is subsequentially minimal if every subspace of $X$ contains an isomorphic copy of a subsequence of $\left(x_{n}\right)_{n}$. The basis $\left(x_{n}\right)_{n}$ is locally-minimal if $X$ is $K$-crudely finitely representable in any of its subspaces for some $K$. We shall enunciate the fourth and fifth dichotomies.

Theorem (4 $4^{\text {th }}$ dichotomy, [22]). Every Banach space contains a basic sequence that is either tight by range, or subsequentially minimal.

Theorem ( $5{ }^{\text {th }}$ dichotomy, [22]). Every Banach space contains a basic sequence that is either tight with constants or locally-minimal.

With the important contribution of Ferenczi and Rosendal to the program, the list of classes was refined and improved to six main classes (19 secondary classes, see [22] for a detailed exposition), as follows:

Theorem (Theorem 8.4 in [22]). Any infinite dimensional Banach space contains a subspace of one of the types listed in the following list:
(1) HI, tight by range.
(2) HI, tight, sequentially minimal.
(3) Tight by support.
(4) Unconditional basis, tight by range, quasi-minimal.
(5) Unconditional basis, tight, sequentially minimal.
(6) Unconditional basis, minimal.

In a companion paper [23], Ferenczi and Rosendal gave several examples of tight Banach spaces. So, they proved in [23] that $G$ and $G^{*}$ are of type (1); $G_{u}$ and $G_{u}^{*}$ are of type (3)
(besides those, several examples were given); $\mathbf{T}$ and $\mathbf{T}^{(p)}$ are of type (5). Later in 2011, V. Ferenczi and Th. Schlumprecht provided an example $\chi_{G M}$ of a space of type (2), constructing a version of $G M$ tight and sequentially minimal, see [24]. Finally, in 2014, the first example $\chi_{(4)}$ of a Banach space of type (4) was given by S. Argyros, A. Manoussakis and A. Pelczar [4].

### 1.4 Going into the proofs

The main tool used in the proof of the third dichotomy is the generalized asymptotic game which is a generalization of the infinite asymptotic game. Namely, the infinite asymptotic game on a Banach space $E$ with Schauder basis $\left(e_{n}\right)_{n}$ between two players $I$ and $I I$ is a game with infinite rounds where players are taking turns alternatively. The player I in the $k$-th round chooses a natural number $n_{k}$ and player II plays a finitely supported vector $x_{k}$ with $n_{k}<\operatorname{supp}\left(x_{k}\right)$. The outcome is the sequence $\left(x_{n}\right)_{n}$, and player $I I$ wins if the outcome belongs to a certain pre-fixed subset of $E^{\omega}$. Infinite asymptotic games were studied in [44] and previously in [40].

A modification of the infinite asymptotic game was used by Ferenczi in [17] to prove that a space saturated with subspaces with a Schauder basis, which embed into the closed linear span of any subsequence of their basis, must contain a minimal subspace. This result generalized the methods and the result of Pelczar in [42]: a Banach space saturated with subsymmetric basic sequences contains a minimal subspace.

In [22] were combined the techniques in [17] and in [44] to prove the third dichotomy. The generalized asymptotic game on a Banach space $E$ with Schauder basis, with parameters $C \geq 1$ and the basic sequence $\left(y_{n}\right)_{n}$, used in such proof is a game with infinite rounds between player $I$ and player $I I$ where in the $k$-th round, $I$ plays a natural number $n_{k}$ and player $I I$ responds with a natural number $m_{k}$ and a not necessarily normalized finitely supported vector $x_{k}$ such that $\operatorname{supp}\left(x_{k}\right) \subseteq \cup_{i=0}^{k}\left[n_{i}, m_{i}\right]$. The outcome of the game is the not necessarily block sequence $\left(x_{n}\right)_{n}$, and player $I I$ wins the game if the outcome is $C$-equivalent to the sequence $\left(y_{n}\right)_{n}$.

In order to prove the third dichotomy they proceed by contradiction and, in a very summarized way they proceed as follows. In the generalized asymptotic game take $C \geq 1$ and a prefixed basis $\left(y_{n}\right)_{n}$. Say that $I I$ wins the generalized asymptotic game, which depends on $C$ and on $\left(y_{n}\right)_{n}$, if $\left(x_{n}\right)_{n}$ is $C$-equivalent to $\left(y_{n}\right)_{n}$. Such game is equivalent to an open Gale-Stewart game, so it is determined. By contradiction, suppose that $E$ is a Banach space with basis $\left(e_{n}\right)_{n}$ having no tight block subspaces. It is proved that if $E$ is in some way "saturated" with block subspaces where the player $I$ always has a winning strategy playing the generalized asymptotic game in those subspaces, then $E$ contains a tight block subspace. Therefore, supposing there is no tight block subspaces implies that there are some block subspaces where player $I$ fails to have a winning strategy. By the determination of the game, in such subspaces player $I I$ has a winning strategy for the game and this is used to
construct a minimal block subspace.

### 1.5 The problem to be solved

Ferenczi and Rosendal also state the following result:
Theorem (Theorem 3.16, [22]). Every Banach space with a basis contains a block subspace $E=\left[e_{n}\right]_{n}$ satisfying one of the following properties:
(1) For any $\left[y_{n}\right]_{n}$ block subspace of $E$, there is a sequence $\left(I_{n}\right)_{n}$ of successive intervals in $\mathbb{N}$ such that for any $A \in[\mathbb{N}]^{\infty},\left[y_{n}\right]_{n}$ does not embed into $\left[e_{n}, n \notin \cup_{i \in A} I_{i}\right]$, as a sequence of disjointly supported vectors, respectively as a block sequence.
(2) For any $\left[y_{n}\right]_{n}$ block subspace of $E$, $\left(e_{n}\right)_{n}$ is equivalent to a sequence of disjointly supported vectors of $\left[y_{n}\right]_{n}$, respectively $\left(e_{n}\right)_{n}$ is equivalent to a block sequence of $\left[y_{n}\right]_{n}$.

They suggested that the proof of the last theorem follows by modifying the embedding notions in each case, but no proof was given. In the context of the third dichotomy the underlying embedding is the isomorphic embedding. That is, we say that $Y=\left[y_{n}\right]_{n}$ isomorphically embeds in $E=\left[e_{n}\right]_{n}$ if $\left(y_{n}\right)_{n}$ is equivalent to a (basic) sequence in $E$. After a standard perturbation argument, one can ask that such basic sequence is a sequence of finitely supported vectors of $E$. In the frame of Theorem 3.16 in [22], we can think of other forms of embeddings: on the first case $Y$ "embeds" in $E$ if $\left(y_{n}\right)_{n}$ is equivalent to a sequence of disjointly supported vectors of $E$; and on the second case $Y$ "embeds" in $E$ if $\left(y_{n}\right)_{n}$ is equivalent to a block sequence of $E$. Another example of a different type of embedding is to say that $Y$ "embeds" in $E$ if $\left(y_{n}\right)_{n}$ is equivalent to some subsequence of $\left(e_{n}\right)_{n}$.

Those ways of interpreting the embedding notion are coded in what we called an admissible system of blocks, which basically is a pair $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$ associated to a Banach space $E$ with a fixed basis $\left(e_{n}\right)_{n}$, where $\mathcal{D}_{E}$ is a set of blocks for $E$ (that is, a set containing the possible bases of subspaces we admit to consider) and $\mathcal{A}_{E}$ is an admissible set for $E$ (the set of infinite sequences of vectors which are the images of the respective embedding). For example, in the case of "being equivalent to a subsequence of $\left(e_{n}\right)_{n} ", \mathcal{D}_{E}$ would be the set which elements are the vectors of the basis and $\mathcal{A}_{E}$ is the set of all subsequences $\left(e_{n}\right)_{n}$. This coding for embedding through admissible sets $\mathcal{A}$ of vectors naturally leads us to define the notions of $\mathcal{A}$-minimality and $\mathcal{A}$-tightness, which depend on $\mathcal{A}$. We modify the methods in [22] to use this admissible systems and prove the following theorem.

Theorem (Theorem 6.5.1). Let E be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$ be an admissible system of blocks for $E$. Then $E$ contains a $\mathcal{D}_{E}$-block subspace $X$ which is either $\mathcal{A}_{E}$-tight or $\mathcal{A}_{E}$-minimal.

The third dichotomy is a particular case for certain $\mathcal{D}_{E}$ and $\mathcal{A}_{E}$. We also obtain a proof of Theorem 3.6 in [22] as a corollary of Theorem 6.5.1.

We define a basic sequence $\left(y_{n}\right)_{n}$ to be tight by sequences in a Banach space $E$ with Schauder
basis $\left(e_{n}\right)_{n}$, if the set

$$
E_{y_{n}}:=\left\{\mathfrak{u} \in 2^{\omega}:\left[y_{n}\right]_{n} \stackrel{s}{\hookrightarrow}\left[e_{n}: n \in \operatorname{supp}(\mathfrak{u})\right]\right\}
$$

is meager in $2^{\omega}$, where $\left[y_{n}\right]_{n} \stackrel{s}{\hookrightarrow}\left[x_{n}\right]_{n}$ if there is a subsequence $\left(z_{n}\right)_{n}$ of $\left(x_{n}\right)_{n}$ equivalent to $\left(y_{n}\right)_{n}$. If all basic sequences $\left(y_{n}\right)_{n}$ are tight by sequences in $E$, we say that $\left(e_{n}\right)_{n}$ is a tight-by-sequences basis of $E$. If $E$ has such a basis, then $E$ is a tight-by-sequences space. This definition follows the same spirit of the definition of tightness via the characterization given in [18], but using the notion of "embedding as a subsequence". As we already mentioned, this embedding corresponds to a particular choice of sets $\mathcal{D}_{E}$ and $\mathcal{A}_{E}$. So, as corollary of Theorem 6.5.1 and using the Galvin-Prikry Theorem, we obtain:

Theorem (Corollary 6.5.5). For any normalized basic sequence $\left(e_{n}\right)_{n}$ in a Banach space, there is a subsequence of $\left(e_{n}\right)_{n}$ which is either a tight-by-sequences basis or spreading.

Another focal point in our research was to obtain a generalization of tightness for a Banach space with transfinite bases. In this direction we prove a characterization for comeager subsets of $2^{\alpha}$, with $\alpha$ a limit ordinal (see [32] for usual Set Theory notation). We use this characterization to define tightness in this context as follows: Let $X$ and $Y$ be Banach spaces, $X$ with transfinite normalized basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$. We say that $Y$ is tight in $\left(x_{\gamma}\right)_{\gamma<\alpha}$ if $E_{Y}=\left\{\mathfrak{u} \in 2^{\alpha}: Y \hookrightarrow X_{\operatorname{supp}(\mathfrak{u})}\right\}$ is meager in $2^{\alpha}$. If any Banach space $Y$ is tight in $\left(x_{\gamma}\right)_{\gamma<\alpha}$ then $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is a tight transfinite basis for $X$. We say that $X$ is tight if it has a tight basis.

We show that the properties obtained from this definition extend the ones presented in Chapter Four for the Schauder basis case. Finally we list some open problems.

We have divided this thesis in seven chapters. With the objective of providing a selfexplanatory document, in Chapter Two we present some preliminary notions in Banach spaces and the majority of the notation. Chapter Two is divided in two sections: the first one is about basic result in Banach spaces with Schauder basis and the second one is about Banach spaces with transfinite bases. In Chapter Three we introduce some concepts of infinite games theory and other tools of descriptive set theory. In Chapter Four, we expose known facts about minimality and tightness, and prove in detail (giving some different proofs from the originals) basic results involving tight Banach spaces which appear in [22] and [18]. In Chapter Five, we define and study the basic properties of admissible families, sets and systems of vectors. Also, we define the concepts of $\mathcal{A}$-minimality and $\mathcal{A}$-tightness and we give some basic properties. In Chapter Six, we prove the $\mathcal{A}$-tight - $\mathcal{A}$-minimal dichotomy and provide its corollaries. Finally, in Chapter Seven, we provide a definition of tight Banach spaces with transfinite basis and study some of its properties and relations.

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## Chapter 2

## Preliminaries on Banach spaces

In this chapter we will present some well known results in Banach space theory which will appear repeatedly in the arguments and constructions developed along this document. We begin with some general considerations about notation in Section 1. In Section 2, results in Banach spaces with Schauder basis are presented. In Section 3 we introduce some initial definitions and results on Banach spaces with transfinite basis, which will be used in Chapter 7.

### 2.1 Notation

The notation that we shall use is standard from the literature on this subject. We shall consider separable or non-separable infinite dimensional Banach spaces. When we refer to a Banach space we are assuming it is infinite dimensional and a subspace of it will be also infinite dimensional and closed, unless stated otherwise. Unless a distinction is necessary, we shall denote the Banach space $(X,\|\cdot\|)$ only by $X$. For $X$ Banach, we denote as $X^{*}$ its topological dual space (the Banach space of bounded functionals over $X$ ) with its usual norm.
$\mathbb{S}_{X}, \mathbb{B}_{X}$ and $\overline{\mathbb{B}}_{X}$ denote the unit sphere, the open and closed ball of $X$, respectively. For $\varepsilon>0$ and $x \in X, \mathbb{B}_{X}(x, \varepsilon)$ and $\overline{\mathbb{B}}_{X}(x, \varepsilon)$ denote the open and closed ball in $X$ centered in $x$ with radius $\varepsilon$, respectively.

Let $A$ be a non-empty set. $|A|$ denotes the cardinality of $A, \mathbb{P}(A)$ denotes the power set of $A,[A]^{<\infty}$ and $[A]^{\infty}$ denote the set of finite subsets of $A$ and the set of infinite subsets of $A$, respectively. For an ordinal $\kappa \geq 1$, we write $[A]^{\kappa}$ to denote the set of subsets of $A$ which have $\kappa$ elements, and write $[A]^{<\kappa}$ to denote the set of subsets of $A$ which have less than $\kappa$ elements.

If $X$ is a Banach space, we write $\left(x_{n}\right)_{n}$ instead of $\left(x_{n}\right)_{n<\omega}$ to denote a sequence of vectors of $X$ and we write $\left(y_{n}\right)_{n} \preceq\left(x_{n}\right)_{n}$ to denote that $\left(y_{n}\right)_{n}$ is a subsequence of $\left(x_{n}\right)_{n}$. If $\left(x_{n}\right)_{n} \in X^{\omega}$, we denote as $\operatorname{span}\left(x_{n}\right)$ the linear span of $\left\{x_{n}: n \in \mathbb{N}\right\}$ and $\left[x_{n}\right]_{n}$ the closed linear span of $X$ generated by $\left\{x_{n}: n \in \mathbb{N}\right\}$. If $A \in \mathbb{P}(\mathbb{N})$, then $\left[x_{n}: n \in A\right]$ (or $\left[x_{n}\right]_{n \in A}$ ) denotes the closed linear subspace generated by $\left\{x_{n}: n \in A\right\}$.

Given $A, B \subset \mathbb{N}$ and assuming that $\max A$ and $\min B$ exist, we write $A<B$ to mean that
$\max A<\min B$. So, when we refer to a sequence $\left(I_{n}\right)_{n}$ of successive finite subsets of $\mathbb{N}$, we are saying that $I_{n}<I_{n+1}$ for every $n \in \mathbb{N}$. Also, when we refer to an interval $I$ of natural numbers, we are meaning that $I=[a, b] \cap \mathbb{N}$, for some $0 \leq a \leq b$.

### 2.2 Banach spaces with Schauder basis

In the following section we will consider separable Banach spaces, specifically Banach spaces with basis. Our main references for this sections are [1], [36] and [38].

### 2.2.1 Schauder Basis and basis constant

Definition 2.2.1. A sequence of elements $\left(x_{n}\right)_{n}$ of a Banach space $X$ is a Schauder basis (or a basis) for $X$ if, and only if, for each $x \in X$ there is a unique sequence of scalars $\left(\lambda_{n}\right)_{n}$ such that

$$
x=\sum_{n=0}^{\infty} \lambda_{n} x_{n} .
$$

If there is $M>0$ such that for every $n \in \mathbb{N},\left\|x_{n}\right\| \leq M$, then the basis is bounded. $\left(x_{n}\right)_{n}$ is seminormalized if it is bounded and $0<\inf \left\|x_{n}\right\|$, for $n \in \mathbb{N}$. If $\left\|x_{n}\right\|=1$ for every $n \in \mathbb{N}$, then the basis $\left(x_{n}\right)_{n}$ is normalized.

Those types of bases are named after J. Schauder who first introduced them in 1927. It is easy to prove from the definition that if $\left(x_{n}\right)_{n}$ is a basis for $X$, then $\left\{x_{n}: n \in \mathbb{N}\right\}$ is a set of linearly independent vectors. Also, if $X$ has a basis then $X$ is separable because the set of rational finite linear combinations of vectors of the basis is dense in $X$.

Additionally, if $\left(x_{n}\right)_{n}$ is a basis for $X$ and $\left(a_{n}\right)_{n}$ is a sequence of non-zero scalars, then every $x \in X$ has a representation $\sum_{n=0}^{\infty} \lambda_{n} x_{n}$ in terms of the basis, which can be written as $\sum_{n=0}^{\infty}\left(\lambda_{n} / a_{n}\right) a_{n} x_{n}$, which is a unique representation of $x$ in terms of the sequence $\left(a_{n} x_{n}\right)_{n}$. This means that $\left(a_{n} x_{n}\right)_{n}$ is also a basis for $X$. Thus, from a basis $\left(x_{n}\right)_{n}$ we can always obtain a normalized basis $\left(x_{n} /\left\|x_{n}\right\|\right)_{n}$ for $X$.

The following norm can be defined in $X$

$$
\begin{equation*}
\|x\|=\sup _{k \in \mathbb{N}}\left\|\sum_{n=0}^{k} \lambda_{n} x_{n}\right\|, \tag{2.1}
\end{equation*}
$$

where $\sum_{n=0}^{\infty} \lambda_{n} x_{n}$ is the representation of $x$ in the basis $\left(x_{n}\right)_{n}$. It is easy to see that $(X,\|\cdot\| \|)$ is a Banach space and by the Open Mapping Theorem it can be proved that $\|\cdot\| \|$ is a norm equivalent to $\|\cdot\|$. It is important to notice that the norm $\|\mid \cdot\| \|$ in the preceding definition depends not only on the basis $\left(x_{n}\right)_{n}$, but also on the original norm $\|\cdot\|$ of the space.

Notation 2.2.2. For $X$ a Banach space with basis $\left(x_{n}\right)_{n}$ and $A \in[\mathbb{N}]^{<\infty}$. We denote by $P_{A}$ the natural projection over $X$ that takes a vector $\sum_{n=0}^{\infty} \lambda_{n} x_{n}$ to the vector $\sum_{n \in A} \lambda_{n} x_{n}$. For a natural number $k \geq 1$, we denote by $P_{k}$ the projection $P_{[0, k-1]}$.

Remark 2.2.3. We are considering infinite linear combinations indexed on $\omega$ (later in some limit ordinal). For that reason we preferred to start the sum in 0 . Also, because of the usual set theory identification of the natural number $k \geq 1$ with the set of its predecessors $\{0,1, \ldots, k-1\}$, we choose to denote by $P_{k}$ the projection $P_{[0, k-1]}$ (instead of $P_{[0, k]}$ ).

Notice that for every $k \geq 0$ and $x=\sum_{n=0}^{\infty} \lambda_{n} x_{n}$ we have

$$
\left\|P_{k}(x)\right\| \leq \sup _{l \leq k}\left\|\sum_{n=0}^{l} \lambda_{n} x_{n}\right\| \leq \sup _{l \in \mathbb{N}}\left\|\sum_{n=0}^{l} \lambda_{n} x_{n}\right\|=\|x\| \|
$$

therefore, $P_{k}$ is bounded and $\left\|\left\|P_{k}\right\|\right\|=1$. Also, notice that for $k \geq 1$, the projection $P_{\{k\}}$ is equal to $P_{k+1}-P_{k}$, thus $P_{\{k\}}$ is bounded. Since $P_{A}=\sum_{i=1}^{|A|} P_{\left\{a_{i}\right\}}, P_{A}$ is bounded for any $A=\left\{a_{1}, \ldots, a_{|A|}\right\} \in[\mathbb{N}]^{<\infty}$.

Definition 2.2.4. If $\left(x_{n}\right)_{n}$ is a basis for the Banach space $X$, then the number

$$
C=\sup _{k \geq 1}\left\|P_{k}\right\|
$$

is called the basis constant of the basis $\left(x_{n}\right)_{n}$. If such number $C$ is equal to 1 we say that the basis is monotone.

Notice that for $k \geq 0$, if $x=\sum_{n=0}^{k-1} \lambda_{n} x_{n} \in X$ is normalized, then $\left\|P_{k}(x)\right\|=1$, therefore $\left\|P_{k}\right\| \geq 1$ for every $k \geq 1$. This implies that the basis constant of any basis is always greater than or equal to 1 , so the basis $\left(x_{n}\right)_{n}$ is monotone if every $P_{k}$ has norm exactly 1 .

Renorming the space $(X,\|\cdot\|)$ with basis $\left(x_{n}\right)_{n}$ with the equivalent norm $\|\cdot\| \|$ defined in Equation (2.1), it is clear that $\left(x_{n}\right)_{n}$ is a monotone basis of $(X,\|\cdot\| \|)$.

Let $X$ be a Banach space over the field $\mathbb{K}$ and $\left(x_{n}\right)_{n}$ be a basis for $X$. We can consider for each $k \in \mathbb{N}$, the function $x_{k}^{*}: X \rightarrow \mathbb{K}$, defined by $x=\sum_{n=0}^{\infty} \lambda_{n} x_{n} \mapsto \lambda_{k}$. Notice that each $x_{k}^{*}$ can be seen as the composition of the canonical isomorphism between the linear span of the vector $x_{k}$ and $\mathbb{K}$, with the bounded operator $P_{\{k\}}$. Therefore, $x_{k}^{*}$ is bounded for every $k \in \mathbb{N}$ as we summarize in the next definition.

Definition 2.2.5. If $\left(x_{n}\right)_{n}$ is a basis for the Banach space $X$, we define the coordinate functionals $\left(x_{k}^{*}\right)_{k}$ as the functionals that map $x_{k}^{*}: x=\sum_{n=0}^{\infty} \lambda_{n} x_{n} \mapsto \lambda_{k}$, for every $k \in \mathbb{N}$.

Notice that if $\left(x_{n}\right)_{n}$ is a basis for $X$, then any $x \in X$ can be represented as $x=\sum_{n=0}^{\infty} x_{n}^{*}(x) x_{n}$.

### 2.2.2 Basic sequences and block subspaces

After defining a Schauder basis for a Banach space an immediate question is whether every separable Banach space has a basis. This question was formulated by S. Banach in 1932 and was an open problem for many years. The relation between the basis existence problem and the Approximation Problem was independently remarked by S. Mazur (there are no written
records before 1950) and A. Grothendieck (approximately in 1953), see 5.7.4.14 in [43] for more information.

The Approximation Problem asked whether every Banach space has the approximation property. Recall that a Banach space $Y$ has the approximation property if every compact operator from any Banach space $X$ to $Y$ is the norm limit of finite-rank operators from $X$ to $Y$. Banach spaces with basis were known to satisfy this property, but an answer for general Banach spaces was not given until 1973 when P. Enflo negatively solved the problem. In [15], Enflo gave an example of a separable reflexive Banach space lacking the approximation property and, therefore, failing to have a Schauder basis.

Nevertheless, the fact that every Banach space contains a subspace with basis was already known by S. Banach (it can be found in his 1932 book without demonstration; the first known demonstration was given 26 years later using techniques previously developed by S . Mazur, see pp. 361 of [38] for more detailed information). This leads us to the following definition and theorem.

Definition 2.2.6. A sequence $\left(x_{n}\right)_{n}$ in a Banach space $X$ is a basic sequence if it is a Schauder basis for $\left[x_{n}\right]_{n}$.

Theorem 2.2.7. Every infinite dimensional Banach space contains a basic sequence. Additionally, given $\varepsilon>0$, such basic sequence can be chosen with basis constant less than or equal to $1+\varepsilon$.

Proof. See [1], Theorem 1.4.4.
Suppose that $\left(x_{n}\right)_{n}$ is a basis for the Banach space $X$. We define the support of $x \in X$ (in symbols $\operatorname{supp}(x))$ in the basis $\left(x_{n}\right)_{n}$ as the set $\left\{n \in \mathbb{N}: x_{n}^{*}(x) \neq 0\right\}$. Given a subset $A$ of $\mathbb{N}$, we define $A x=\sum_{n \in A} x_{n}^{*}(x) x_{n}$, when it exists. Notice that the set of finitely supported vectors (vectors with support in $[\mathbb{N}]^{<\infty}$ ) in the basis $\left(x_{n}\right)_{n}$ of $X$ coincide with $\operatorname{span}\left(x_{n}\right)$. The range of a vector $x$ is defined as $\operatorname{ran}(x)=[\min \operatorname{supp}(x), \max \operatorname{supp}(x)]$, if $x$ is a finitely supported vector and $\operatorname{ran}(x)=[\min \operatorname{supp}(x), \infty)$, if it is not finitely supported. The support and the range of the zero vector of $X$ is the empty set.

If $n \in \mathbb{N}$ and $x \in X$, we write $n<x$ if $n<\min \operatorname{supp}(x)$. Given two finitely supported vectors $x$ and $y$ of $X$, we write $x<y$ if $\operatorname{supp}(x)<\operatorname{supp}(y)$. A sequence of finitely supported vectors $\left(y_{n}\right)_{n}$ is called a block basis if $y_{n}<y_{n+1}$, for all $n \in \mathbb{N}$.

It is important to notice that a block basis is a basic sequence and its basis constant is, at most, the basis constant of the basis. If $X$ is a Banach space with basis $\left(x_{n}\right)_{n}$, then a subspace $Y$ is a block subspace of $X$ (in symbols $Y \leq X$ ) if there is a block basis $\left(y_{n}\right)_{n}$ of $\left(x_{n}\right)_{n}$ such that $Y=\left[y_{n}\right]_{n}$. If $Y$ is a block subspace and $u \in Y$ we denote as $\operatorname{supp}_{Y}(u)$ the support of $u$ with respect to the natural block basis of $Y$.

### 2.2.3 Embedding and equivalence of basic sequences

Notation 2.2.8. Let $X$ and $Y$ be Banach spaces, and $K \geq 1$.

- We say that $X$ is isomorphic to $Y$ with constant $K$ (denoted as $Y \simeq_{K} X$ ) if there exists a one-to-one bounded linear operator $T$ from $X$ onto $Y$ such that $T^{-1}$ is bounded and $K \geq\|T\| \cdot\left\|T^{-1}\right\|$.
- We say that $X$ contains a $K$-isomorphic copy of $Y$, or $Y$ is $K$-embeddable in $X$ (denoted as $Y \hookrightarrow_{K} X$ ), if $Y \simeq_{K} Z$ for some $Z$ subspace of $X$.
- We say that $Y$ is isomorphically embeddable in $X$, or just embeddable, (in symbols $Y \hookrightarrow X)$, if $Y \hookrightarrow_{K} X$, for some $K \geq 1$. In this case we say that $X$ contains an isomorphic copy, or just a copy, of $Y$.

It is easy to prove that if $\left(x_{n}\right)_{n}$ is a basic sequence in $X$ and $T: X \rightarrow Y$ is an isomorphism into a Banach space $Y$, then $\left(T\left(x_{n}\right)\right)_{n}$ is a basic sequence in $Y$. The next definition can be found in [14].

Definition 2.2.9. Let $K \geq 1$. Two basic sequences $\left(x_{n}\right)_{n}$ and $\left(y_{n}\right)_{n}$ are $K$-equivalent (in symbols $\left.\left(x_{n}\right)_{n} \sim_{K}\left(y_{n}\right)_{n}\right)$ if, and only if, for all $k \in \mathbb{N}$ and every finite sequence of scalars $\left(a_{i}\right)_{i=0}^{k}$ we have

$$
\frac{1}{K}\left\|\sum_{n=0}^{k} a_{n} x_{n}\right\| \leq\left\|\sum_{n=0}^{k} a_{n} y_{n}\right\| \leq K\left\|\sum_{n=0}^{k} a_{n} x_{n}\right\|
$$

Two basic sequences are equivalent if there is $K \geq 1$ satisfying that such sequences are $K$-equivalent.

In the literature (see [12], for example it) is also common to define $K$-equivalence of basic sequences as: $\left(x_{n}\right)_{n} \sim_{K}\left(y_{n}\right)_{n}$ if, and only if, there is an isomorphism $T:\left[x_{n}\right]_{n} \rightarrow\left[y_{n}\right]_{n}$ such that $T\left(x_{n}\right)=y_{n}$ for all $n \in \mathbb{N}$ and such that $\|T\|\left\|T^{-1}\right\| \leq K$; and two basic sequences are equivalent if they are $K$-equivalent for some $K \geq 1$. This definition of $K$-equivalence is not the same as the definition of $K$-equivalence given in Definition 2.2.9. However, basic sequences are equivalent in the sense of existence the isomorphism $T$ mentioned above if, and only if, they are equivalent in the sense of Definition 2.2.9.

An equivalent way to define the equivalence of two basic sequences $\left(x_{n}\right)_{n}$ and $\left(y_{n}\right)_{n}$ is the following: a series $\sum_{n=0}^{\infty} \lambda_{n} x_{n}$ converges if, and only if, the series $\sum_{n=0}^{\infty} \lambda_{n} y_{n}$ converges. It is easy to verify that "being equivalent to" is an equivalence relation over the set of all basic sequences in a Banach space $X$. In particular, if $\left(x_{n}\right)_{n} \sim_{C}\left(y_{n}\right)_{n}$ and $\left(y_{n}\right)_{n} \sim_{K}\left(z_{n}\right)_{n}$ then $\left(x_{n}\right)_{n} \sim_{C K}\left(z_{n}\right)_{n}$. Another useful and basic fact we will use frequently is the following theorem proved by C. Bessaga and A. Pełczynski in 1958.

Theorem 2.2.10. Let $X$ be a Banach space with basis $\left(x_{n}\right)_{n}, Y$ a subspace of $X$ and $\varepsilon>0$. Then $Y$ contains a basic sequence $\left(y_{n}\right)_{n}(1+\varepsilon)$-equivalent to a block basis of $\left(x_{n}\right)_{n}$. Such
sequence $\left(y_{n}\right)_{n}$ can be chosen normalized.

Proof. See [36], Theorem I.1.14.

The next fundamental theorem (proved in 1940 by M. Krein, D. Milman, and M. Rutman) establishes that sufficiently small perturbations of basic sequences are basic sequences equivalent to the original one.

Theorem 2.2.11 (Principle of small perturbations). Let $\left(x_{n}\right)_{n}$ be a basic sequence in a Banach space $X$ with basis constant $C$. If $\left(y_{n}\right)_{n}$ is a sequence in $X$ such that

$$
2 C \sum_{n=0}^{\infty} \frac{\left\|x_{n}-y_{n}\right\|}{\left\|x_{n}\right\|}=\theta<1
$$

then $\left(x_{n}\right)_{n} \sim\left(y_{n}\right)_{n}$. In particular,

- $\left(y_{n}\right)_{n}$ is a basic sequence (with basis constant at most $\left.C(1+\theta)(1-\theta)^{-1}\right)$,
- If $\left(x_{n}\right)_{n}$ is a basis of $X$, so is $\left(y_{n}\right)_{n}$.

Proof. See [1], Theorem 1.3.9.

### 2.2.4 Shrinking and boundedly complete bases

Given a basis $\left(x_{n}\right)_{n}$ for the Banach space $X$, we can consider the associated coordinate functionals $\left(x_{n}^{*}\right)_{n}$ which form a basic sequence in $X^{*}$, but not always a basis for $X^{*}$. The classical example to illustrate this is $\ell_{1}$ with the canonical basis $e_{n}=(0, \ldots, 0,1,0, \ldots)$ with 1 in the position $n$, for every $n \in \mathbb{N}$. The sequence $\left(e_{n}\right)_{n}$ is a basis for $\ell_{1}$ but the coordinate functionals fail to be a basis for $\ell_{1}^{*}$, which is not separable. This leads us to the following definition.

Definition 2.2.12. Let $\left(x_{n}\right)_{n}$ be a basis for the Banach space $X$. We say that $\left(x_{n}\right)_{n}$ is shrinking if, and only if, $\left(x_{n}^{*}\right)_{n}$ is a basis for $X^{*}$, i.e. $\left[x_{n}^{*}\right]_{n}=X^{*}$.

Proposition 2.2.13. Let $\left(x_{n}\right)_{n}$ be a basis for the Banach space $X .\left(x_{n}\right)_{n}$ is shrinking if, and only if, whenever $x^{*} \in X^{*}$,

$$
\lim _{N \rightarrow \infty}\left\|\left.x^{*}\right|_{\left[x_{n}\right]_{n>N}}\right\|=0
$$

where

$$
\left\|\left.x^{*}\right|_{\left[x_{n}\right]_{n>N}}\right\|=\sup \left\{\left|x^{*}(y): y \in \mathbb{S}_{\left[x_{n}\right]_{n>N}}\right|\right\}
$$

Proof. See [1], Proposition 3.2.6.

A notion in a sense dual to the shrinking condition is the notion of boundedly complete basis.

Definition 2.2.14. Let $X$ be a Banach space. A basic sequence $\left(x_{n}\right)_{n}$ in $X$ is boundedly complete if for any sequence of scalars $\left(\lambda_{n}\right)_{n}$ such that $\sup _{m \in \mathbb{N}}\left\|\sum_{n=0}^{m} \lambda_{n} x_{n}\right\|<\infty$, the series $\sum_{n=0}^{\infty} \lambda_{n} x_{n}$ converges.

The main relation between shrinking and boundedly complete bases is that given a basis $\left(x_{n}\right)_{n}$ for a Banach space, $\left(x_{n}\right)_{n}$ is boundedly complete if, and only if, the sequence of coordinate functionals $\left(x_{n}^{*}\right)_{n}$ is a shrinking basic sequence, see Theorem 4.4.14 in [38]. The following theorem characterizes the bases of a reflexive Banach space, when it has one. It was proved by R. James in 1951.

Theorem 2.2.15. Let $X$ be a Banach space with a basis. Then, $X$ is reflexive if, and only $i f$, all bases of $X$ are shrinking and boundedly complete, which happens if, and only if, some basis of $X$ is both shrinking and boundedly complete.

Proof. See [38], Theorem 4.4.15.

### 2.2.5 Unconditional, symmetric and subsymmetric bases

Another important type of bases are unconditional bases which first appear in a work of Karlin in 1948. Recall that a series $\sum_{n=0}^{\infty} z_{n}$ converges unconditionally if $\sum_{n=0}^{\infty} z_{\pi(n)}$ converges for every permutation $\pi: \mathbb{N} \rightarrow \mathbb{N}$.

Definition 2.2.16. A basis $\left(x_{n}\right)_{n}$ for $X$ is unconditional if for every $x \in X$ the series $\sum_{n=0}^{\infty} x_{n}^{*}(x) x_{n}$ converges unconditionally. A basis which is not unconditional is called conditional.

It is possible to characterize unconditionally convergent series on a Banach space as follows.
Theorem 2.2.17. Let $X$ be a Banach space. The series $\sum_{n=0}^{\infty} x_{n}$ in $X$ is unconditionally convergent if, and only if, $\sum_{n=0}^{\infty} \alpha_{n} x_{n}$ converges whenever $\left(\alpha_{n}\right)_{n}$ in $\ell_{\infty}$.

Proof. See Theorem 4.2.8 in [38].
For each $\left(\alpha_{n}\right)_{n}$ in $\ell_{\infty}$, define $T_{\left(\alpha_{n}\right)_{n}}: X \rightarrow X$ such that $T_{\left(\alpha_{n}\right)_{n}}(x)=\sum_{n=0}^{\infty} \alpha_{n} x_{n}^{*}(x) x_{n}$. As a consequence of Theorem 2.2.17 and the Uniform Boundedness principle, the following theorem is obtained.

Theorem 2.2.18. Suppose that $\left(x_{n}\right)_{n}$ is an unconditional basis for a Banach space $X$. Then for each $\left(\alpha_{n}\right)_{n}$ in $\ell_{\infty}$ the map $T_{\left(\alpha_{n}\right)_{n}}$ is a bounded linear operator, and

$$
\sup \left\{\left\|T_{\left(\alpha_{n}\right)_{n}}\right\|:\left(\alpha_{n}\right)_{n} \in \mathbf{S}_{\ell_{\infty}}\right\}<\infty
$$

Proof. See Theorem 4.2.25 in [38].
Notice that for each $\left(\epsilon_{i}\right)_{i}$ sequence of signs (that is each $\epsilon_{i}$ is equal to +1 ou -1 ), $T_{\left(\epsilon_{i}\right)_{i}}$ is bounded and $\left(\epsilon_{i}\right)_{i} \in \mathbf{S}_{\ell_{\infty}}$. This is a motivation for the next definition.

Definition 2.2.19. If $\left(x_{n}\right)_{n}$ is an unconditional basis for $X$, we define the unconditional constant $K$ of $\left(x_{n}\right)_{n}$ as follows

$$
K:=\sup \left\{\left\|T_{\left.\left(\epsilon_{n}\right)\right)_{n}}\right\|:\left(\epsilon_{n}\right)_{n} \text { sequence of signs }\right\} .
$$

For such $K$ we say that $\left(x_{n}\right)_{n}$ is $K$-unconditional.
When the space has an unconditional basis, one obtain the following characterization.
Theorem 2.2.20. Suppose that $X$ is a Banach space with unconditional basis $\left(x_{n}\right)_{n}$.

- The basis $\left(x_{n}\right)_{n}$ is shrinking if, and only if, $\ell_{1}$ is not embedded in $X$.
- The basis $\left(x_{n}\right)_{n}$ is boundedly complete if, and only if, $c_{0}$ is not embedded in $X$.

Proof. See Theorem 4.4.21 of [38].

We now present the following theorem due to R. James which follows directly from Theorems 2.2.15 and 2.2.20.

Theorem 2.2.21. Let $X$ be a Banach space with an unconditional basis. Then, $X$ is reflexive if, and only if, $X$ does not contain copies of $c_{0}$ or $\ell_{1}$.

For the special case of a space with unconditional basis, two other types of basis can be defined: symmetric and subsymmetric. Symmetric bases were independently introduced by I. Singer in 1961 and by M. Kadets and A. Pełczynski in 1962, see 5.6.3.24 in [43] for this reference. First, we write $\Pi$ to denote the set of all permutations over $\mathbb{N}$.

Definition 2.2.22. Let $X$ be a Banach space with unconditional basis $\left(x_{n}\right)_{n}$. We say that $\left(x_{n}\right)_{n}$ is symmetric if, and only if, $\left(x_{n}\right)_{n}$ is equivalent to $\left(x_{\sigma(n)}\right)_{n}$, for any $\sigma \in \Pi$.

As we mention for general Schauder basis and unconditional basis, for a symmetric basis $\left(x_{n}\right)_{n}$ of $X$ the following operators over $X$ are bounded:

$$
T_{\sigma,\left(\epsilon_{i}\right), k}(x):=\sum_{i=0}^{k} x_{i}^{*}(x) x_{\sigma(i)},
$$

where $\sigma \in \Pi, k \in \mathbb{N}$ and $\left(\epsilon_{i}\right)_{i=0}^{k}$ is a finite sequence of signs, see Theorem 22.1 and Proposition 22.1 in [46]. By the Uniform Boundedness principle, the following number is well defined

$$
\begin{equation*}
C:=\sup _{\sigma \in \Pi}^{\sigma \in \sup _{i}= \pm 1,} ⿻ \sup _{k \in \mathbb{N}}\left\|T_{\sigma,\left(\epsilon_{i}\right), k}\right\|<\infty . \tag{2.2}
\end{equation*}
$$

Definition 2.2.23. Let $X$ be a Banach space and $\left(x_{n}\right)_{n}$ a symmetric basis for $X$. We define the symmetric constant $C$, as the constant given in Equation (2.2). In this case we say that
$\left(x_{n}\right)_{n}$ is a $C$-symmetric basis.
Also, it can be proven that the symmetric constant $C$ is the smallest constant such that

$$
\begin{equation*}
\left\|\sum_{i=0}^{\infty} \epsilon_{i} \lambda_{i} x_{\sigma(i)}\right\| \leq C\left\|\sum_{i=0}^{\infty} \lambda_{i} x_{i}\right\| \tag{2.3}
\end{equation*}
$$

for any $\sigma \in \Pi,\left(\epsilon_{i}\right)_{i}$ sequence of signs and $\left(\lambda_{i}\right) \in c_{00}$, see Proposition 22.1 in [46]. In fact such condition characterizes the symmetric constant of a symmetric basis $\left(x_{n}\right)_{n}$, and in several references ([1], for example) is given as definition. Now, we will define the notion of subsymmetric basis for a Banach space.

Definition 2.2.24. Let $X$ be a Banach space with unconditional basis $\left(x_{n}\right)_{n}$. We say that $\left(x_{n}\right)_{n}$ is subsymmetric if, and only if, $\left(x_{n}\right)_{n}$ is equivalent to any of its subsequences.

Any symmetric basis is subsymmetric (see Proposition 22.2 in [46]) but the converse is false (see Remark 9.2.5 in [1]). Let $\Sigma$ be the set of infinite increasing sequences of natural numbers. For the case in which $\left(x_{n}\right)_{n}$ is subsymmetric, the associated bounded operators are the following (see page 569 in [46]):

$$
T_{\left(m_{i}\right),\left(n_{i}\right)}(x):=\sum_{i \in \mathbb{N}} x_{n_{i}}^{*}(x) x_{m_{i}},
$$

where $\left(\left(m_{i}\right),\left(n_{i}\right)\right) \in \Sigma \times \Sigma$ and they are uniformly bounded too (see Theorem 21.2 in [46]). By doing an analogous procedure to the one we did before with symmetric basis, we can define the subsymmetric constant as follows

Definition 2.2.25. Let $X$ be a Banach space with subsymmetric basis $\left(x_{n}\right)_{n}$. The subsymmetric constant is the smallest constant $C \geq 1$ such that given any scalars $\left(\lambda_{i}\right) \in c_{00}$, we have

$$
\left\|\sum_{i=0}^{\infty} \epsilon_{i} \lambda_{i} x_{n_{i}}\right\| \leq C\left\|\sum_{i=0}^{\infty} \lambda_{i} x_{i}\right\|,
$$

for all $\left(n_{i}\right)_{i} \in \Sigma$ and $\left(\epsilon_{i}\right)_{i}$ sequence of signs. In this case we say that $\left(x_{n}\right)_{n}$ is $C$-subsymmetric. Without assuming unconditionality, we have the following definition:

Definition 2.2.26. Let $C \geq 1$. Define a basis $\left(x_{n}\right)_{n}$ to be $C$-spreading if it is $C$-equivalent to all its subsequences. $A$ basis $\left(x_{n}\right)_{n}$ is spreading if it is $C$-spreading for some $C \geq 1$.

Notice that every $C$-subsymmetric basic sequence is $C$-spreading.

### 2.2.6 Some examples of Banach spaces with Schauder basis

## Classical spaces

In this subsection we want to point out some well known properties of classical Banach spaces $c_{0}$ and $\ell_{p}$, with $p \in[1, \infty)$, see [1] and [38]. Let us note that such spaces are equipped
with the canonical monotone basis $\left(e_{n}\right)_{n}$, formed by the coordinate vectors $e_{n}(i)=1$ if $n=i$ and 0 otherwise. The spaces $c_{0}$ and $\ell_{p}$ and their respective subspaces are mutually non-isomorphic, see Corollary 2.1.6 in [1].

The spaces $\ell_{p}$ are reflexive for $1<p<\infty$ : if $1 / p+1 / q=1$, then $\ell_{p}^{*}$ is isometric to $\ell_{q}$. The dual space of $c_{0}$ can be isometrically identified with $\ell_{1}$, and the dual of $\ell_{1}$ is isometric to $\ell_{\infty}$, which is not separable.

Any basis of $\ell_{p}$ with $1<p<\infty$ is shrinking and boundedly complete, whereas the canonical basis is 1 -unconditional and symmetric. The canonical basis of $\ell_{1}$ is unconditional, symmetric, boundedly complete but not shrinking (in fact, $\ell_{1}$ cannot have a shrinking basis because its dual is not separable).

In contrast with the canonical basis of $\ell_{1}$, the canonical basis of $c_{0}$ is shrinking but not boundedly complete. For each $n \in \mathbb{N}$ consider $s_{n}=\sum_{i \leq n} e_{i}$. The sequence $\left(s_{n}\right)_{n}$ is called the summing basis of $c_{0}$, and it is neither shrinking nor boundedly complete. The canonical basis of $c_{0}$ is unconditional and symmetric, but the summing basis is conditional, and therefore cannot be either symmetric or subsymmetric.

## Tsirelson's space

We give the definition of the Tsirelson's space $\mathbf{T}$ following the Figiel-Johnson construction, see [25]. Actually, T is the dual space of the original space constructed by B. S. Tsirelson in 1974 (see [49]) which was the first example of a Banach space not containing copies of $c_{0}$ or any $\ell_{p}$. Over $c_{00}$ it is defined a norm as the solution to the equation:

$$
\begin{equation*}
\|x\|_{T}=\max \left\{\|x\|_{\infty}, \sup \left\{\frac{1}{2} \sum_{i=1}^{n}\left\|E_{i} x\right\|_{T}: n \leq E_{1}<E_{2}<\ldots<E_{n}, E_{i} \subseteq \mathbb{N} \text { interval }\right\}\right\} \tag{2.4}
\end{equation*}
$$

Tsirelson's space $\mathbf{T}$ is obtained as the completion of $c_{00}$ under this norm. Notice that the norm is given implicitly in the last equation, and this represents the most important difference between Tsirelson's type spaces and classical ones (what "implicit" norm should formally mean still remains to be understood). Such way of defining norms has led to the construction of several Tsirelson's type spaces with exotic properties. Examples of spaces with properties like the ones we are studying (mainly notions of tightness) will be of this type.

The space $\mathbf{T}$ is reflexive, the coordinate vectors $\left(e_{n}\right)_{n}$ constitute an 1-unconditional basis for it, and it has no subsymmetric basic sequences, see Theorem I. 8 in [11]. $\mathrm{T}^{*}$ has similar properties, that is: it is reflexive with no copies of $c_{0}$ or any $\ell_{p}$, the coordinate functionals $\left(e_{n}^{*}\right)_{n}$ form a 1-unconditional basis for it, which has no subsymmetric basic sequences.

## p-convexification of Tsirelson's space

The procedure of convexification, and conditions to apply it was first presented by T. Figiel and W. B. Johnson in [25]. As Tsirelson's space has a 1-unconditional basis $\left(e_{n}\right)_{n}$, it is possible to define its $p$-convexification for $1<p<\infty$ (also described in [25]). For $1<p<\infty$ consider

$$
\mathbf{T}^{(p)}=\left\{x=\left(x_{n}\right)_{n}: x^{p}:=\sum_{n=0}^{\infty} \operatorname{sgn}\left(x_{n}\right)\left|x_{n}\right|^{p} e_{n} \in \mathbf{T}\right\}
$$

equipped with the norm

$$
\|x\|_{(p)}=\left\|x^{p}\right\|_{T}^{1 / p} .
$$

That is,
$\|x\|_{(p)}=\max \left\{\|x\|_{\infty}, \sup \left\{\frac{1}{2^{1 / p}}\left(\sum_{i=1}^{n}\left\|E_{i} x\right\|_{T}^{p}\right)^{1 / p}: n \leq E_{1}<E_{2}<\ldots<E_{n}, E_{i}\right.\right.$ intervals $\left.\}\right\}$.

The $p$-convexification of Tsirelson's space $\mathbf{T}^{(p)}$ is the Banach space $\left(\mathbf{T}^{(p)},\|\cdot\|_{(p)}\right)$. The coordinate vectors in $\mathbf{T}^{(p)}$ forms a 1-unconditional basis for $\mathbf{T}^{(p)}$. Also, $\mathbf{T}^{(p)}$ has no copies of $c_{0}$ or any $\ell_{q}$, with $q \in[1, \infty)$.

## Symmetrization of the p-convexification of Tsirelson's space

Consider the following procedure to symmetrize a Banach space (see X.B. in [11]). Let $X$ be a Banach space with basis $\left(x_{n}\right)_{n}$. Define the following norm over $\operatorname{span}\left(x_{n}\right)$ :

$$
\|x\| \|=\sup \left\{\sum_{n}\left|\lambda_{n}\right| x_{\sigma(n)}: \sigma: \mathbb{N} \rightarrow \mathbb{N} \text { is a permutation }\right\}
$$

for $x=\sum_{n} \lambda_{n} x_{n} \in \operatorname{span}\left(x_{n}\right)$. The symmetrization $S(X)$ of $X$ is the completion of $X$ under that norm. It is immediate that $\left(x_{n}\right)_{n}$ is a symmetric basis for $S(X)$.

Symmetrizing Tsirelson's space results in a space isomorphic to $\ell_{1}$ (see [11], page 97). In this section we are interested in mentioning some properties of $S\left(\mathbf{T}^{(p)}\right), 1<p<+\infty$ : it is a reflexive Banach space whose canonical basis is symmetric, it is saturated with isomorphic copies of subspaces of $\mathbf{T}^{(p)}$ (see Section X.E. in [11]), for that reason $S\left(\mathbf{T}^{(p)}\right)$ has no copies of $c_{0}$ or any $\ell_{p}$.

## Schlumprecht's space

Constructed by T. Schlumprecht in 1991 (see [45]), it is the first example of an arbitrarily distortable space (see [39] for a nice presentation of this notion and its relations with other concepts on Banach spaces). Its norm definition follows the Tsirelson's scheme, in fact it is a mixed Tsirelson space, see [5] page 15 for more information. Schlumprecht's space $\mathcal{S}$ is the completion of $c_{00}$ equipped with the norm defined by the implicit equation

$$
\|x\|=\max \left\{\|x\|_{\infty}, \sup \left\{\frac{1}{f(n)} \sum_{i=1}^{n}\left\|E_{i} x\right\|:\left(E_{i}\right)_{i=1}^{n} \text { sequence of successive intervals, } n \in \mathbb{N}\right\}\right\}
$$

where $f(n)$, in general, must satisfy certain conditions specified in Lemma 1 of [45], but is usually taken as $f(n)=\log _{2}(n+1)$. The coordinate sequence $\left(e_{n}\right)_{n}$ is a 1-unconditional, subsymmetric basis for $\mathcal{S}$. It has no copies of $c_{0}$ and any $\ell_{p}$, with $1 \leq p<\infty$ (and therefore is reflexive and all bases are boundedly complete and shrinking).

### 2.2.7 Finite representability and some technical propositions

Definition 2.2.27. Let $X$ and $Y$ be Banach spaces. We say that $X$ is finitely representable in $Y$ if given any finite subspace $F$ of $X$ and $\varepsilon>0, F$ is $(1+\varepsilon)$-embeddable in $Y$.

Proposition 2.2.28. Every Banach space $X$ is finitely representable in $c_{0}$.
Proof. See [1], Example 11.1.2.
Also we have the following well known proposition which will be useful in the next chapters.
Proposition 2.2.29. Let $X$ be a Banach space with basis $\left(x_{n}\right)_{n}$ with basis constant $C$ and let $M \geq 1$. Then, there is a constant $c \geq 1$ which depends on $C$ and $M$, such that if $\left(z_{n}\right)_{n}$ and $\left(y_{n}\right)_{n}$ are normalized block bases of $\left(x_{n}\right)_{n}$ which differ only in $M$ terms, then $\left(y_{n}\right)_{n} \sim_{c}\left(z_{n}\right)_{n}$.

Proof. Let $X$ be a Banach space with basis $\left(x_{n}\right)_{n}$ with basis constant $C$. Let $M \geq 1$, $\left(y_{n}\right)_{n}$ and $\left(z_{n}\right)_{n}$ be two normalized block bases of $\left(x_{n}\right)_{n}$ such that $\forall n \in J\left(y_{n} \neq z_{n}\right)$ and $\forall n \notin J\left(y_{n}=z_{n}\right)$ for some $J \in[\mathbb{N}]^{M}$. Consider $\mathbb{N} \backslash J$ with its increasing order.

By the Proposition 2.2.28 applied to $\left[y_{n}\right]_{n}$, we know that for every finite sequence of scalars $\left(\alpha_{n}\right)_{n \in J}$, we have

$$
\begin{equation*}
\sup _{n \in J}\left|\alpha_{n}\right| \leq 2\left\|\sum_{n \in J} \alpha_{n} y_{n}\right\| . \tag{2.5}
\end{equation*}
$$

Using the relations we mentioned between projections and the basis constant of a block basis with the basis constant $C$ of $\left(x_{n}\right)_{n}$ it is seen that

$$
\begin{equation*}
\left\|\sum_{n \in J} \lambda_{n} y_{n}\right\| \leq 2 C M\left\|\sum_{n=0}^{\infty} \lambda_{n} y_{n}\right\| \tag{2.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\sum_{n \in \mathbb{N} \backslash J} \lambda_{n} y_{n}\right\| \leq\left\|\sum_{n=0}^{\infty} \lambda_{n} y_{n}\right\|+\left\|\sum_{n \in J} \lambda_{n} y_{n}\right\| \leq(2 C M+1)\left\|\sum_{n=0}^{\infty} \lambda_{n} y_{n}\right\| . \tag{2.7}
\end{equation*}
$$

Thus, using Equations (2.6) in (2.5) we obtain

$$
\begin{equation*}
M \sup _{n \in J}\left|\lambda_{n}\right| \leq 4 C M^{2}\left\|\sum_{n=0}^{\infty} \lambda_{n} y_{n}\right\| . \tag{2.8}
\end{equation*}
$$

Therefore, by Equations (2.7) and (2.8), and using that $\left(z_{n}\right)_{n}$ is normalized, we obtain

$$
\begin{aligned}
\left\|\sum_{n=0}^{\infty} \lambda_{n} z_{n}\right\| & =\left\|\sum_{n \in J} \lambda_{n} z_{n}+\sum_{n \in \mathbb{N} \backslash J} \lambda_{n} z_{n}\right\| \\
& =\left\|\sum_{n \in J} \lambda_{n} z_{n}+\sum_{n \in \mathbb{N} \backslash J} \lambda_{n} y_{n}\right\| \\
& \leq\left\|\sum_{n \in J} \lambda_{n} z_{n}\right\|+(2 C M+1)\left\|\sum_{n=0}^{\infty} \lambda_{n} y_{n}\right\| \\
& \leq M \sup _{n \in J}\left|\lambda_{n}\right|+(2 C M+1)\left\|\sum_{n=0}^{\infty} \lambda_{n} y_{n}\right\| \\
& \leq 4 C M^{2}\|y\|+(2 C M+1)\left\|\sum_{n=0}^{\infty} \lambda_{n} y_{n}\right\| \\
& =C(2 M+1)^{2}\left\|\sum_{n=0}^{\infty} \lambda_{n} y_{n}\right\| .
\end{aligned}
$$

Then, $\left\|\sum_{n=0}^{\infty} \lambda_{n} z_{n}\right\| \leq C(2 M+1)^{2}\left\|\sum_{n=0}^{\infty} \lambda_{n} y_{n}\right\|$. Notice that $C(2 M+1)^{2}$ only depends on $M$ and $C$. In a similar way, we can prove that $\left\|\sum_{n=0}^{\infty} \lambda_{n} y_{n}\right\| \leq C(2 M+1)^{2}\left\|\sum_{n=0}^{\infty} \lambda_{n} z_{n}\right\|$ So, if $c=C(2 M+1)^{2}$, for example, we have that $\left(y_{n}\right)_{n} \sim_{c}\left(z_{n}\right)_{n}$.

### 2.3 Banach spaces with transfinite basis

In this section we shall introduce the definition and some results involving transfinite basis (also known as long basis) of a Banach space. A transfinite basis is a generalization of Schauder basis which is indexed on an ordinal greater than $\omega$. For this section we are following the references [31] and [47].

The class of all the ordinals is denoted by Ord. In general we shall use Greek letters to denote ordinals. As usual, we denote by $\omega$ and $\omega_{1}$ the first infinite ordinal and the first uncountable ordinal, respectively. Given $\alpha$ and $\beta$ ordinal numbers, we denote as $\alpha+\beta, \alpha \cdot \beta$ and $\alpha^{\beta}$ the usual arithmetic operations with ordinal numbers (see Definitions 2.18, 2.19 and 2.20 in [32]).

We say that two well-ordered sets $(A,<)$ and $\left(B,<^{\prime}\right)$ are isomorphic (or have the same order-type) if, and only if, there is an order preserving bijection $f: A \rightarrow B$, that is, $\forall a \in A \forall b \in A\left(a<b \Rightarrow f(a)<^{\prime} f(b)\right)$. Since every well-ordered set is isomorphic to a unique ordinal number (see Theorem 2.12 in [32]), we say that the well-ordered set $(A,<)$
has order type $\alpha \in \operatorname{Ord}$ (in symbols $\operatorname{otp}(A)=\alpha$ ), if $A$ is isomorphic to the ordinal number $\alpha$.

We define $\alpha<\beta \Longleftrightarrow \alpha \in \beta$, for every $\alpha, \beta \in \operatorname{Ord}$. $<$ is a linear ordering of the class Ord. Each ordinal is identified with the set of its predecessors: $\alpha=\{\beta \in \operatorname{Ord}: \beta<\alpha\}$, and this set is denoted by the interval of ordinals $[0, \alpha)$. If $\alpha<\beta$ are ordinals the interval $[\alpha, \beta]$ denotes the set $\{\gamma \in$ Ord : $\alpha \leq \gamma \leq \beta\}$. In a similar way we define intervals of the type ( $\alpha, \beta$ ), $[\alpha, \beta)$ or ( $\alpha, \beta]$. Over any already defined interval of ordinals we consider the order topology (see 39 in Part II of [48]). Notice that $[\alpha, \beta]$ with such topology is compact (see 43 in Part II of [48]). Given $A, B \subseteq \gamma \in \operatorname{Ord}$, we write $A<B$ to mean that $\forall \alpha \in A \forall \beta \in B(\alpha<\beta)$.

### 2.3.1 Transfinite basic sequences

A transfinite sequence is a function whose domain is an ordinal and we shall denote it by $\left(a_{\beta}\right)_{\beta<\alpha}$ or $\left(a_{\beta}: \beta<\alpha\right)$, if it is convenient. We extend the natural definitions of normalized, seminormalized and bounded sequences to this context of transfinite sequences. In this work we shall distinguish between sequences (always over $\omega$ ) and transfinite sequences (over a previously fixed ordinal). A subsequence $\left(a_{\beta_{n}}\right)_{n}$ of a transfinite sequence $\left(a_{\beta}\right)_{\beta<\alpha}$ shall be indexed over $\omega$, meanwhile a transfinite subsequence $\left(a_{\beta_{\gamma}}\right)_{\gamma<\delta}$, can be indexed over any ordinal $\delta<\alpha$.

Definition 2.3.1. Let $X$ be a Banach space and $\alpha \in \operatorname{Ord}$. Consider $\left(y_{\beta}\right)_{\beta<\alpha}$ a transfinite sequence of vectors of $X$. We say that the transfinite series $\sum_{\beta<\alpha} y_{\beta}$ converges to $y \in X$ (in symbols $y=\sum_{\beta<\alpha} y_{\beta}$ ) if there is a unique continuous function $S:[1, \alpha] \rightarrow X$ such that

$$
S(1)=y_{0}, \quad S(\alpha)=y, \quad S(\beta+1)=S(\beta)+y_{\beta}, \quad \text { for } \beta<\alpha .
$$

Notice that in the particular case where $\alpha=\omega$, the last definition coincides with the usual definition of a sum of a series in a Banach space. If $y=\sum_{\beta<\alpha} y_{\beta}$, since $[1, \alpha]$ is compact and the function $S$ in Definition 2.3.1 is continuous, $S([1, \alpha])$ is a compact subset of $X$, and therefore separable. Also, for each $\varepsilon>0$ the set $\left\{\beta<\alpha:\left\|y_{\beta}\right\|>\varepsilon\right\}$ is finite and therefore $y_{\beta}=0$ for every except a countable number of $\beta<\alpha$. The following definition can be found in [31].

Definition 2.3.2. A transfinite sequence $\left(x_{\beta}\right)_{\beta<\alpha}$ of vectors in a Banach space $X$ is called a transfinite basis for $X$ if for every $x \in X$ there is a unique transfinite sequence of scalars $\left(a_{\beta}\right)_{\beta<\alpha}$ such that $x=\sum_{\beta<\alpha} a_{\beta} x_{\beta}$.

In the same way we observed for the case of Schauder basis, if $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is a transfinite basis for $X$, it is possible to define the canonical projections $P_{\beta}: X \rightarrow X$ for $0 \leq \beta<\alpha$ by $P_{\beta}\left(\sum_{\gamma<\alpha} a_{\gamma} x_{\gamma}\right)=\sum_{\gamma<\beta} a_{\gamma} x_{\gamma}$. Such projections are uniformly bounded (see Theorem 4.6 in [31]). This gives us the next definition.

Definition 2.3.3. Let $\left(x_{\beta}\right)_{\beta<\alpha}$ be a transfinite basis for the Banach space $X$. The basis
constant $C$ associated to $\left(x_{\beta}\right)_{\beta<\alpha}$ is defined as

$$
C:=\sup \left\{\left\|P_{\beta}\right\|: \beta<\alpha\right\} .
$$

If $C=1$, the transfinite basis is called monotone.
Again, it is possible to define the coordinate functionals associated to a transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$ of a Banach space $X$ as the functions $x_{\beta}^{*}$ defined over $X$ such that $x_{\beta}^{*}\left(\sum_{\gamma<\alpha} a_{\gamma} x_{\gamma}\right)=$ $a_{\beta}$, for each $\beta<\alpha$. Such functions are bounded and it is possible to represent each $x \in X$ as $x=\sum_{\beta<\alpha} x_{\beta}^{*}(x) x_{\beta}$.

Let $\left(x_{\beta}\right)_{\beta<\alpha}$ be a transfinite basis. We denote as $\operatorname{span}\left(x_{\beta}\right)_{\beta<\alpha}$ and $\left[x_{\beta}\right]_{\beta<\alpha}$ the linear span and the closed subspace of $X$ generated by $\left\{x_{\beta}: \beta<\alpha\right\}$, respectively. A transfinite sequence $\left(x_{\beta}\right)_{\beta<\alpha}$ of vectors in a Banach space $X$ is a transfinite basic sequence if $\left(x_{\beta}\right)_{\beta<\alpha}$ is a transfinite basis for $\left[x_{\beta}\right]_{\beta<\alpha}$. The support $\operatorname{supp}(x)$ of $x \in\left[x_{\beta}\right]_{\beta<\alpha}$ is the set $\left\{\beta<\alpha: x_{\beta}^{*}(x) \neq\right.$ $0\}$. For $x, y \in\left[x_{\beta}\right]_{\beta<\alpha}$ finitely supported vectors, we write $x<y$ if $\operatorname{supp}(x)<\operatorname{supp}(y)$.

Definition 2.3.4. Let $\left(x_{\gamma}\right)_{\gamma<\alpha}$ be a transfinite basic sequence. We say that a transfinite sequence $\left(y_{\xi}\right)_{\xi<\beta}$ is a transfinite block subsequence of $\left(x_{\gamma}\right)_{\gamma<\alpha}$ if, and only if, for all $\xi<\beta$, $y_{\xi}$ is a non-zero finitely supported vector and for all $\nu<\xi<\beta, y_{\nu}<y_{\xi}$. In the case that $\beta=\omega$, then $\left(y_{\xi}\right)_{\xi<\beta}$ is a block subsequence of $\left(x_{\gamma}\right)_{\gamma<\alpha}$.

Proposition 2.3.5. A transfinite block subsequence of a transfinite basis is a transfinite basic sequence.

As in the case of a Schauder basis, the constant basis of a transfinite block subsequence is controlled by the constant basis of the transfinite basic sequence.

Proposition 2.3.6. Let $\left(x_{\gamma}\right)_{\gamma<\alpha}$ be a transfinite basis for a space $X$. Let $\left(y_{n}\right)_{n}$ be a block subsequence of $\left(x_{\gamma}\right)_{\gamma<\alpha}$. Then, there is $\left(\gamma_{n}\right)_{n}$ an increasing sequence with elements in $\alpha$, such that $\left(y_{n}\right)_{n} \leq\left(x_{\gamma_{n}}\right)_{n}$.

Proof. Let $\left(x_{\gamma}\right)_{\gamma<\alpha}$ and $\left(y_{n}\right)_{n}$ be as in the hypothesis. Since $\left(\operatorname{supp}\left(y_{n}\right)\right)_{n}$ is a sequence of successive finite subsets of $\alpha$, $\operatorname{otp}\left(\cup_{n \in \omega} \operatorname{supp}\left(y_{n}\right)\right)=\omega$. Thus, it is possible to index $\cup_{n \in \omega} \operatorname{supp}\left(y_{n}\right)$ as $\left\{\gamma_{i}: i \in \omega\right\}$ with $x_{\gamma_{i}}<x_{\gamma_{i+1}}$, for every $i \in \omega$. Consequently, $\left(y_{n}\right)_{n}$ is a block basis of $\left(x_{\gamma_{i}}\right)_{i}$.

Notation 2.3.7. Let $X$ be a Banach space with transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$. For each $u \subseteq \alpha$, denote the closed subspace of $X$ spanned by the transfinite subsequence $\left(x_{\gamma}\right)_{\gamma \in u}$ by

$$
X_{u}=\left[x_{\gamma}: \gamma \in u\right] .
$$

As it is known, the structure of the subspaces of a Banach space with basis is strongly described by the block subspaces. The standard gliding hump argument, method used
to prove Theorem 2.2.10, among others, is not extendable to the case of transfinite basic sequences. The following theorem shows the conditions valid in this context:

Theorem 2.3.8. Let $\left(x_{\gamma}\right)_{\gamma<\alpha}$ be a transfinite basis of $X$ and $Y$ an infinite dimensional closed subspace of $X$. Then, there exists $\beta \leq \alpha$ and a closed subspace $Z=\left[z_{n}\right]_{n}$ of $Y$ such that
(i) $P_{\beta}: Z \rightarrow X_{\beta}$ is an embedding.
(ii) For every $\varepsilon>0$ there exists a semi-normalized block basis $\left(w_{n}\right)_{n}$ in $X_{\beta}$ and a normalized sequence $\left(z_{n}\right)_{n}$ in $Z$ such that $\sum_{n}\left\|P_{\beta} z_{n}-w_{n}\right\|<\varepsilon$.
(iii) There is a block sequence $\left(w_{n}\right)_{n}$ of $X$ which is equivalent to $\left(z_{n}\right)_{n}$.
(iv) If we additionally assume that $Y$ has a Schauder basis $\left(y_{n}\right)_{n}$, then the sequence $\left(z_{n}\right)_{n}$ in (ii) can be chosen to be a block basis of $\left(y_{n}\right)_{n}$.

Proof. See Proposition 1.3 in [3].
The next definitions can be found in [3]:
Definition 2.3.9. Let $X$ be a Banach space with basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$.
(i) $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is called unconditional if for all subsets $A$ of $\alpha$, the operators $P_{A}: \rightarrow X$, given by $P_{A}(x)=\sum_{\gamma \in A} x_{\gamma}^{*}(x) x_{\gamma}$, are uniformly bounded.
(ii) Let $C \geq 1 .\left(x_{\gamma}\right)_{\gamma<\alpha}$ is called $C$-spreading if, and only if, all its subsequences are $C$ spreading.
(iii) Let $C \geq 1 .\left(x_{\gamma}\right)_{\gamma<\alpha}$ is called $C$-subsymmetric if, and only if, all its subsequences are $C$-subsymmetric. $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is subsymmetric if it is $C$-subsymmetric for some $C \geq 1$.
(iv) $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is called shrinking if, and only if, every subsequence $\left(x_{\gamma_{n}}\right)_{n}$ of $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is shrinking.
(v) $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is called boundedly complete if, and only if, every subsequence $\left(x_{\gamma_{n}}\right)_{n}$ of $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is boundedly complete.

Theorem 2.3.10. Let $\left(x_{\gamma}\right)_{\gamma<\alpha}$ be a transfinite basis of a Banach space $X$. Then $X$ is reflexive if, and only if $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is both shrinking and boundedly complete.

Proof. See Proposition 1.6 in [3].

## Chapter 3

## Preliminaries of Descriptive Set Theory

In this chapter, we will introduce some basic concepts and notations of descriptive set theory. Particularly, those related with Polish spaces, trees and infinite games (fundamental tools involved in the proof of the dichotomy theorem in the fifth chapter). Our main references in this chapter are [33] for descriptive set theory and [16] for definitions and notations of general topology.

Let us recall some definitions from topology. Let $(X, \tau)$ be a topological space. There are three topological properties that a subset $A \subseteq X$ can satisfy which shall be recurrent in our work and it is worth to recall: $A$ being meager in $X, A$ being comeager in $X$ and $A$ satisfying the Baire property. We say that $A$ is meager in $X$ if there is a sequence of nowhere dense subsets $\left(F_{i}\right)_{i}$ (i.e. the interior of the closure of each $F_{i}$ is empty) such that $A=\cup_{i} F_{i}$. $A$ is comeager in $X$ if $X \backslash A$ is meager in $X$. $A$ has the Baire property (BP) if it is almost an open set of $X$, that is, if there is some meager set $B$ on $X$ and $O \in \tau$ such that $A=O \triangle B$.

### 3.1 Trees and sequence spaces

Let $A$ be a nonempty set. Define $A^{<\omega}:=\bigcup_{n \geq 1} A^{n}$, the set of all finite sequences from $A$. Let $n \geq 1$. For $s=\left(s_{0}, \ldots, s_{n-1}\right) \in A^{n}$, we say that the length of $a$ is $n$, in symbols length $(a)=n$. Define $A^{0}:=\{\emptyset\}$, where $\emptyset$ is the empty sequence and set length $(\emptyset)=0$. If a sequence $s$ is infinite we say that length $(s)=\infty$.

If $s$ is a sequence from $A$ and $m \in \mathbb{N}$ (if $s$ is finite we require $m \leq \operatorname{length(s)\text {),wedefine}}$ $s \upharpoonright m:=\left(s_{0}, \ldots, s_{m-1}\right)$ as the restriction of $s$ to $m$. Given $s$ and $t$ sequences of elements of $A$ with $s$ finite, we say that $s$ is an initial segment of $t$ and $t$ is an extension of $s$ (in symbols $s \sqsubseteq t)$ if there is some $m \leq \operatorname{length}(t)$ such that $s=t \upharpoonright m$. If $s \sqsubseteq t$ but $s \neq t$, then we say that $t$ is a proper extension of $s$. Also, $\emptyset \sqsubseteq s$ for every sequence $s$. For $s \in A^{n}$ and $t$ a sequence from $A$, we define the concatenation $s^{\wedge} t$ of $s$ with $t$ as the sequence such that $\left(s^{\wedge} t\right)_{i}=s_{i}$ for $0 \leq i<n$ and $\left(s^{\wedge} t\right)_{n+i}=t_{i}$, for $n \leq i<l e n g t h(t)$.

Definition 3.1.1. $A$ tree on a set $A$ is a subset $T \subseteq A^{<\omega}$ closed under taking initial segments: if $t \in T$ and $s \sqsubseteq t$, then $s \in T$ (in particular $\emptyset \in T$ ). The elements of $T$ are called nodes of $T$. An infinite branch of $T$ is a sequence $x \in A^{\omega}$ such that $\forall n \in \mathbb{N}(x \upharpoonright n \in T)$. The set of all infinite branches of $T$ is called the body of $T$ and is denoted as $[T]$. A tree $T$ is pruned if
all $s \in T$ have a proper extension $t \in T$.
Notation 3.1.2. Denote by $\operatorname{Tr}$ the set of trees over $\mathbb{N}$.
Consider $A$ with its discrete topology and $A^{\omega}$ with the product topology. Note that $A^{\omega}$ is a completely metrizable topological space. The standard basis for the topology over $A^{\omega}$ consists of the sets

$$
\mathcal{N}_{s}=\left\{t \in A^{\omega}: s \sqsubseteq t\right\},
$$

where $s$ is a finite sequence from $A$. It is easy to see that $s \sqsubseteq t$ if, and only if, $\mathcal{N}_{t} \subseteq \mathcal{N}_{s}$. Also, $\left(A^{\omega}\right)^{n}(n \geq 1)$, and $\left(A^{\omega}\right)^{\omega}$ are homeomorphic to $A^{\omega}$.

Remark 3.1.3. If $x_{n} \in A^{\omega}$, for every $n \in \mathbb{N}$ and $x \in A^{\omega}$, then $x_{n} \underset{n \rightarrow \infty}{ } x$ if, and only if, $\forall i \in \mathbb{N}\left(x_{n}(i)=x(i)\right.$, for all $n$ large enough $)$.

In the particular case of $2^{\omega}=\{0,1\}^{\omega}$ all the properties above are valid. The space $2^{\omega}$ with such topology is called the Cantor space. If $\mathfrak{s}=\left(s_{i}\right)_{i} \in 2^{\omega}$, define $\operatorname{supp}(\mathfrak{s})=\left\{i \in \mathbb{N}: s_{i}=1\right\}$. Notice that $\mathbb{P}(\mathbb{N})$ can be identified with $2^{\omega}$ using the characteristic functions: let $A \in \mathbb{P}(\mathbb{N})$, then the characteristic function $\chi_{A}$ belongs to $2^{\omega}$ and $A=\operatorname{supp}\left(\chi_{A}\right)$. Thus, families of subsets of $\mathbb{N}$ sometimes will be seen as families of sequences of $\mathbb{N}$. Therefore, any $\mathcal{F} \subseteq \mathbb{P}(\mathbb{N})$ can be seen as a topological subspace of $2^{\omega}$. For convenience, a basic open subset of $2^{\omega}$ determined by $\mathfrak{s} \in 2^{\omega}$ and $J \in[\mathbb{N}]^{<\infty}$ is given by

$$
\mathcal{N}_{\mathfrak{s}, J}:=\left\{\mathfrak{u}=\left(u_{n}\right)_{n} \in 2^{\omega}: \forall n \in J\left(u_{n}=s_{n}\right)\right\} .
$$

If $\mathfrak{s}=\left(s_{i}\right)_{i}$ and $\mathfrak{t}=\left(t_{i}\right)_{i}$ belong to $2^{\omega}$, then we define the sequence

$$
\mathfrak{s} \cup \mathfrak{t}=\chi_{\operatorname{supp}(s) \cup \operatorname{supp}(t)} \in 2^{\omega} .
$$

The following proposition presents a characterization of meager sets in $2^{\omega}$.
Proposition 3.1.4. Let $B$ be a subset of $2^{\omega}$. The following assertions are equivalent:
(i) $B$ is comeager,
(ii) there is a sequence $\left(I_{n}\right)_{n}$ of successive intervals of $\omega$ and $a_{n} \subseteq I_{n}$, such that for any $\mathfrak{u} \in 2^{\omega}$, if the set $\left\{n: \operatorname{supp}(\mathfrak{u}) \cap I_{n}=a_{n}\right\}$ is infinite, then $\mathfrak{u} \in B$.

Proof. See [21], Lemma 7.
Proposition 3.1.4 is still true assuming that each $\left(I_{n}\right)_{n}$ is a sequence of finite subsets not necessarily intervals with $I_{n}<I_{n+1}$. As a corollary of this proposition we have:

Corollary 3.1.5. Let $B$ be a subset of $2^{\omega}$ such that

$$
\forall \mathfrak{u} \in B, \forall \mathfrak{v} \in 2^{\omega}, \text { if } \operatorname{supp}(\mathfrak{u}) \subseteq \operatorname{supp}(\mathfrak{v}) \text { then } \mathfrak{v} \in B
$$

Then:
(i) $B$ is meager if, and only if, there exist a sequence $\left(I_{i}\right)_{i}$ of successive intervals in $\mathbb{N}$ such that

$$
\mathfrak{u} \in B \Rightarrow\left\{n \in \omega: \operatorname{supp}(\mathfrak{u}) \cap I_{n}=\emptyset\right\} \text { is finite }
$$

(ii) $B$ is comeager if, and only if, there exist a sequence $\left(I_{i}\right)_{i}$ of successive intervals in $\mathbb{N}$ such that

$$
\left\{n \in \omega: I_{n} \subseteq \operatorname{supp}(\mathfrak{u})\right\} \text { is infinite } \Rightarrow \mathfrak{u} \in B
$$

Proof. See [18], Corollary 2.4.
In Chapter 7, Proposition 3.1.4 and Corollary 3.1.5 are generalized by taking a limit ordinal $\alpha$ instead of $\omega$. We finish this section by observing that there is a correspondence between pruned trees on $A$ and closed subsets of $A^{\omega}$.

Theorem 3.1.6. Let $A$ be a nonempty set. We have:
(i) For every pruned tree $T$ on $A,[T]$ is closed in $A^{\omega}$.
(ii) The map $T \mapsto[T]$ is an isomorphism between pruned trees on $A$ and closed subsets of $A^{\omega}$. Its inverse is given by

$$
F \mapsto T_{F}:=\{x \upharpoonright n: x \in F, n \in \mathbb{N}\}
$$

Proof. See [33], Proposition 2.4.

### 3.2 Polish spaces and Borel sets

Let $(X, \tau)$ be a topological space. We say that $(X, \tau)$ is a completely metrizable space if there is a metric $d$ over $X$ compatible with $\tau$ (i.e. the topology induced by $d$ coincide with $\tau$ ) such that the metric space $(X, d)$ is complete. A separable completely metrizable topological space is called Polish. We say that $A \subseteq X$ is $F_{\sigma}$ in $X$ if it is a countable union of closed subsets in $X$ and $A$ is a $G_{\delta}$ in $X$ if it is a countable intersection of open sets in $X$.

As we already mentioned before, the space $\left(A^{\omega}, \tau\right)$, where $A \neq \emptyset$ and $\tau$ is the product topology obtained after endowing $A$ with the discrete topology, is a completely metrizable space, so it is Polish if $A$ is countable. In particular, the Cantor space $2^{\omega}$ is Polish.

Let $X \neq \emptyset$ and $A$ be a nonempty subset of $X$, we denote by $\sigma(A)$ to the smallest $\sigma$-algebra containing $A$. Let $\tau$ be a topology on $X$. The class of Borel sets $\mathcal{B}(X)$ of $(X, \tau)$ is the $\sigma$ algebra generated by $\tau$. It is clear that if $\mathcal{S}$ is a countable subbasis for the topology $\tau$, then $\sigma(\mathcal{S})=\mathcal{B}(X)$. In particular, if $X$ is a Polish space, then $\mathcal{B}(X)$ is countably generated, that is, since $X$ is separable and metrizable, there is countable basis $\mathcal{S}$ for $X$ and $\mathcal{B}(X)=\sigma(\mathcal{S})$.

Obviously $\mathcal{B}(X)$ contains all open, closed, $G_{\delta}$ and $F_{\sigma}$ sets in $X$. The measurable space $(X, \mathcal{B}(X))$ is called the Borel space of $X$.

We say that the function $f$ between topological spaces $X$ and $Y$ is Borel if the inverse image of a Borel set in $Y$ is a Borel set in $X$. Two measurable spaces $(X, \mathcal{S})$ and $(Y, \mathcal{R})$ are isomorphic if there is an invertible measurable function $f: X \rightarrow Y$ with measurable inverse.

Definition 3.2.1. A measurable space $(X, \mathcal{S})$ is a standard Borel space if it is isomorphic to $(Y, \mathcal{B}(Y))$ for some Polish space $Y$, or equivalently, if there is a Polish topology $\tau$ on $X$ with $\mathcal{S}=\mathcal{B}(\tau)$.

Another result we shall use in the next Chapters is known as the first topological 0-1 law:
Theorem 3.2.2 (First topological 0-1 law). Let $X$ be a Polish space, and $G$ be a group of homeomorphisms of $X$ with the following property: for any $U$ and $V$ non-empty open subsets of $X$, there is $g \in G$ such that $g(U) \cap V \neq \emptyset$. If $A \subseteq X$ has the Baire Property and is $G$-invariant (i.e. $g(A)=A$, for every $g \in G$ ), then $A$ is meager or comeager in $X$.

Proof. See Theorem 8.46 in [33].

Another theorem we shall use in the next chapter is the well known Galvin-Prikry Theorem.
Theorem 3.2.3 (Galvin-Prikry). Let $[\mathbb{N}]^{\infty}=P_{0} \cup \ldots \cup P_{k-1}$, where each $P_{i}$ is Borel and $k \in \mathbb{N}$. Then there is $H \in[\mathbb{N}]^{\infty}$ and $i<k$ with $[H]^{\infty} \subseteq P_{i}$.

Proof. See Theorem 19.11 in [33].

### 3.3 Infinite games

For this section, our main references are [33] and [6]. An infinite game with rules is a contest between two players, I and II, where three sets are involved, $A, X$ and $Y$, and where both players can always move according to previously established rules and knowing what was previously played (the game is of perfect information).

The nonempty sets $X$ and $Y$ contain the possible objects that players I and II can play, respectively. In his first move, player I (to which we shall refer as masculine) chooses some $u_{0} \in X$ according to the rules of the game. Immediately player $I I$ (to which we will refer as feminine) responds with her move $v_{0} \in Y$ in compliance with the rules, and completing the first round. I plays $u_{1} \in X$ as his second move. $I I$ plays $v_{1} \in Y$ completing second round, and so on. A diagrammatical way of representing such games is the following:

| I | $u_{0} \in X$ | $u_{1} \in X$ | $u_{2} \in X$ | $\ldots$ |
| :---: | :---: | :--- | :--- | :--- |
| II |  | $v_{0} \in Y$ |  | $v_{1} \in Y$ |

A position of the game is any finite stage of the game, that is any finite sequence of the type:

- $\left(u_{0}, v_{0}, \ldots, u_{n}, v_{n}\right)$ such that $u_{0}, \ldots, u_{n}$ have been played by $\mathrm{I}, v_{0}, \ldots, v_{n}$ have been played by II, and it corresponds to I to make his move, or
- $\left(u_{0}, v_{0}, \ldots, u_{n}\right)$ such that $u_{0}, \ldots, u_{n}$ have been played by I, $v_{0}, \ldots, v_{n-1}$ have been played by II, and it corresponds to $I I$ to make her move.

A run $\vec{p}=\left(u_{0}, v_{0}, u_{1}, v_{1}, \ldots\right)$ in the game is an infinite sequence of rounds where I and $I I$ took turns, i.e. $\vec{p}$ is an element of $(X \times Y)^{\omega}$ such that for each $n \in \mathbb{N}$, the projection of $\vec{p}$ over the first $n+1$ pairs of coordinates $p_{n}=\left(u_{0}, v_{0}, \ldots, u_{n}, v_{n}\right)$ is a position of the game.

The outcome of the game is an infinite sequence $\left(w_{i}\right)_{i}$ obtained from the plays of I and $I I$ after a run. In some cases, the outcome can be a specific subsequence of the run, for example the single sequence $\left(v_{i}\right)_{i}$ of plays of II. The nonempty set $A$ is called the payoff set of the game and it determines the winning condition for one specific player. A player (player I, for example) wins the game if the outcome $\left(w_{n}\right)_{n} \in A$ (equivalently, this condition determines a winning condition for the other player: $I I$ wins the game if $\left.\left(w_{n}\right)_{n} \notin A\right)$. Obviously, only one player can win the game.

A move of the players is legal if each moves according to the rules of the game. So, each player has a set of legal moves to make in each round. Those legal moves correspond to the nodes of a pruned tree. Knowing the rules of the game, we shall construct the tree of possible legal positions of the game as follows: formally, let $X, Y, A \neq \emptyset$ and $T \subseteq(X \times Y)^{<\infty}$ be a nonempty tree which is defined recursively in the length of the legal position:

1. $\emptyset \in T$.
2. Suppose $\left(u_{0}, v_{0}, \ldots, u_{n}, v_{n}\right)$ a legal position in the game. So, it is the turn for player I to make a legal move. For every legal move $u_{n+1} \in X$ for player I in his $n+2$-th round and for every legal move $v_{n+1} \in Y$ for player $I I$ in her $n+2$-th round, we ask that $\left(\emptyset, u_{0}, v_{0}, \ldots, u_{n}, v_{n}, u_{n+1}, v_{n+1}\right) \in T$.

Such tree will be pruned because we asked the game to be infinite (if I and $I I$ have played legally $n$ rounds, then the set of legal moves for the $(n+1)$-th round is nonempty). Of course, a nonempty pruned tree can be interpreted as the rules of a game or the tree of legal moves. For that reason, the tree $T$ is called the tree of rules or the tree of legal moves of the game. A run $\vec{p}=\left(u_{0}, v_{0}, u_{1}, v_{1}, \ldots\right)$ in the game is legal if I and $I I$ played legal moves along the complete run. We denote a game on sets $X$ and $Y$ with rules $T$ and payoff set $A$ by $G(X, Y ; A)$ (in the literature, the notation $G(T, X)$ is usual when $X=Y$ and $A \subseteq X$ ).

A strategy for I in the game $G(X, Y ; A)$ is a function $\sigma:(X \times Y)^{<\infty} \rightarrow X$ such that $\sigma(\emptyset)=u_{0}$ is a legal first move for I , and for every $n$, given the legal position $\left(u_{0}, v_{0}, \ldots, u_{n}, v_{n}\right)$ of the game, $\sigma\left(\left(u_{0}, v_{0}, \ldots, u_{n}, v_{n}\right)\right)=u_{n+1}$ is a legal move in the round $n+2$ for I. In a similar way we define a strategy for II. Notice that a strategy for I can be seen also as a tree such that:

1. $\sigma$ is nonempty and pruned.
2. If $\left(\emptyset, u_{0}, v_{0}, \ldots, u_{n}\right) \in \sigma$ then for all legal move $v_{n}$ for II, we have $\left(\emptyset, u_{0}, v_{0}, \ldots, u_{n}, v_{n}\right) \in \sigma$ (every legal move that $I I$ can make is considered for the next move of I).
3. If $\left(\emptyset, u_{0}, v_{0}, \ldots, u_{n}, v_{n}\right) \in \sigma$ then for a unique legal move $u_{n+1}$ for I , we have $\left(\emptyset, u_{0}, v_{0}, \ldots, u_{n}, v_{n}, u_{n+1}\right) \in \sigma$ (after $n+1$ complete rounds of the game, the strategy shows only one move for I to do).

We say that I follows a strategy $\sigma$ in the game $G(X, Y ; A)$ if in his first move he plays $\sigma(\emptyset)$, and in the $(n+1)$-th round I plays $\sigma\left(\left(u_{0}, v_{0}, \ldots, u_{n-1}, v_{n-1}\right)\right)$. In an analogous way we define that $I I$ plays according to a strategy $\rho:(X \times Y)^{<\infty} \rightarrow Y$. A strategy $\sigma$ for I (respectively for II) is a winning strategy for $G(X, Y ; A)$ if whenever I (respectively II) follows the strategy $\sigma$, then I (respectively II) wins the game. Both I and $I I$ cannot have winning strategies in the same run of the game $G(X, Y ; A)$.

We say that the game $G(X, Y ; A)$ is determined if one of the two players has a winning strategy. Two games $G\left(X_{1}, Y_{1} ; A_{1}\right)$ and $H\left(X_{2}, Y_{2} ; A_{2}\right)$ are equivalent if player I (respectively II) has a winning strategy for the game $G\left(X_{1}, Y_{1} ; A_{1}\right)$ if, and only if, player I (respectively II) has a winning strategy for the game $H\left(X_{2}, Y_{2} ; A_{2}\right)$.

### 3.4 Determinacy of open games

The next well known theorem allows us to find a winning strategy for player I in the game G(X,X;A), described below.

For this theorem, known as the determinacy theorem for closed or open games, consider $X=Y \neq \emptyset$ endowed with the discrete topology, and over $X^{\omega}$ consider the natural product topology. For $T$ a pruned tree over $X$ take the set of infinite branches $[T] \subseteq X^{\omega}$ endowed with the relative topology. Set $A \subseteq[T]$. Then a game $G(X, X ; A)$ is called a Gale-Stewart game.

Theorem 3.4.1 (Gale-Stewart). Let $T$ be a nonempty pruned tree on $X$. Let $A \subseteq[T]$ be closed or open in $[T]$. Then $G(X, X ; A)$ is determined.

Proof. See [33], Theorem (20.1).

Such a game $G(X, X ; A)$ where the payoff set $A \subseteq[T]$ establishes the winning condition for player I (respectively II) is open (respectively closed) in $[T]$ is called an open (respectively closed) game for player I (respectively for player II). For the uses of Theorem 3.4.1, we shall refer to such games as open games (since if the game is closed for one of the players, it is open for the other).

## Chapter 4

## Minimality and Tightness

In this chapter we shall present some properties and examples of minimal spaces. We also present the central notion of this work: tightness. We shall explain in details some basic properties of tight Banach spaces given in [22] and [18].

### 4.1 Minimal Banach spaces

According to P. Casazza and E. Odell, the notion of minimality was first introduced by H. Rosenthal. In this section we will present some known results involving this notion.

Definition 4.1.1. An infinite dimensional Banach space $X$ is minimal if, and only if, it isomorphically embeds in any closed infinite dimensional subspace of $X$.

Notice that minimal spaces have to be separable and that any subspace of a minimal space is also minimal. Banach already knew that $c_{0}$ is a minimal Banach space, and it was proved in 1960 by A. Pełczynski that the spaces $\ell_{p}$ are minimal for $1 \leq p<\infty$. Indeed, for those cases we have the following result.

Proposition 4.1.2. Let $X=c_{0}$ or $X=\ell_{p}(1 \leq p<\infty)$, and $\left(e_{n}\right)_{n}$ be the canonical basis for $X$. Then $\left(e_{n}\right)_{n}$ is equivalent to any of its seminormalized block bases.

Proof. See [1], Lemma 2.1.1.

Theorem 2.2.10 and Proposition 4.1.2 imply the minimality of $c_{0}$ and $\ell_{p}$ with $1 \leq p<\infty$.
The spaces $c_{0}$ and $\ell_{p}$ were the only examples of minimal spaces until P. Casazza, W. Johnson and L. Tzafriri proved in 1984 that $\mathbf{T}^{*}$, the dual of Tsirelson's space, is minimal (see Theorem 14 of [9]).

Tsirelson's space is not minimal. In fact, as proved by P. Casazza and E. Odell in [10], it satisfies a stronger condition:

Theorem 4.1.3. Tsirelson's space does not contain any minimal subspace.

Proof. See [11], Corollary VI.b.6.

On the other hand, the space $\mathcal{S}$ is minimal, as it was noticed by Schlumprecht right after he presented $\mathcal{S}$ in 1991 and published in [2], see Theorem 2.1.

Theorem 4.1.4. Let $1<p<\infty$. The p-convexification of Tsirelson's space $\mathbf{T}^{(p)}$ has no minimal subspaces.

Proof. In next section of this chapter a stronger result is proved, that $\mathbf{T}^{(p)}$ is a tight space and therefore that it has no minimal subspaces.

Theorem 4.1.5. The space $S\left(\mathbf{T}^{(p)}\right)$ has no minimal subspaces.
Proof. The space $S\left(\mathbf{T}^{(p)}\right)$ is saturated with isomorphic copies of subspaces of $\mathbf{T}^{(p)}$, i.e. every subspace of $S\left(\mathbf{T}^{(p)}\right)$ contains a subspace that is isomorphic to a subspace of $\mathbf{T}^{(p)}$, see X.E., Remark 7-a) in [11]. Since all subspaces of a minimal subspace are minimal, by Theorem 4.1.4, $S\left(\mathbf{T}^{(p)}\right)$ has no minimal subspaces.

### 4.2 Tight Banach spaces

The objective of this section is to review in detail some well known results involving the notion of tightness.
V. Ferenczi and Ch. Rosendal in [22] defined tight spaces as follows:

Definition 4.2.1. Let $X$ be a Banach space with Schauder basis $\left(x_{n}\right)_{n}$. We say that a Banach space $Y$ is tight in $X$ if there is a sequence $\left(I_{n}\right)_{n}$ of successive finite subsets of $\mathbb{N}$, such that for all $A \in[\mathbb{N}]^{\infty}$,

$$
Y \nLeftarrow\left[x_{n}, n \notin \cup_{i \in A} I_{i}\right] .
$$

We say that $\left(x_{n}\right)_{n}$ is a tight basis for $X$ if any Banach space $Y$ is tight in $X$. Finally, $X$ is tight if it has a tight basis.

It is important to notice that if $Y$ is tight in $X$ and $\left(J_{i}\right)_{i}$ is the sequence of successive finite subsets of $\mathbb{N}$ witnessing the definition, then

$$
Y \nLeftarrow\left[x_{n}, n \notin \cup_{i \in A} I_{i}\right],
$$

where $I_{i}=\left[\min J_{i}, \max J_{i}\right]$ for every $i$, and all $A \in[\mathbb{N}]^{\infty}$. Thus, without loss of generality, we can suppose that $\left(I_{i}\right)_{i}$ is a sequence of successive intervals in Definition 4.2.1.

Also, suppose $Y$ is tight in $X$ and $\left(I_{i}\right)_{i}$ is a sequence that witnesses the definition. If there is $B \in[\mathbb{N}]^{\infty}$ such that $Y$ embeds into $\left[x_{n}, n \in B\right]$, then $B$ intersects all but finitely many intervals $I_{i}$; indeed if $A:=\left\{n \in \mathbb{N}: B \cap I_{n}=\emptyset\right\}$ were infinite, then $Y \hookrightarrow\left[x_{n}: n \notin \cup_{j \in A} I_{j}\right]$, contradicting tightness.

Proposition 4.2.2. Let $X$ be a Banach space with normalized basis $\left(x_{n}\right)_{n}$. If for every $\left(y_{n}\right)_{n}$ normalized block basis of $\left(x_{n}\right)_{n}$ we have that $\left[y_{n}\right]_{n}$ is tight in $X$. Then every Banach space is tight in $X$.

Proof. Suppose $Z$ is a Banach space not tight in $X$ and $Z \hookrightarrow X$. Then a copy of $Z$ in $X$ contains a basic sequence equivalent to a block basis of $X$ (see Theorem 2.2.10). So, there is a block subspace $Y=\left[y_{n}\right]_{n}$ of $X$ with $\left(y_{n}\right)_{n}$ normalized such that $Y \hookrightarrow Z$. Since $Y$ is tight in $X$, there is $\left(I_{i}\right)_{i}$ such that for every $A \in[\mathbb{N}]^{<\infty}$

$$
\begin{equation*}
Y \nrightarrow\left[x_{n}: n \notin \cup_{i \in A} I_{i}\right] . \tag{4.1}
\end{equation*}
$$

Also, there is $B \in[\mathbb{N}]^{<\infty}$ such that

$$
Z \hookrightarrow\left[x_{n}: n \notin \cup_{i \in B} I_{i}\right] .
$$

Since $Y \hookrightarrow Z$, this last equation contradicts Equation (4.1).
Remark 4.2.3. To prove that a basis $\left(x_{n}\right)_{n}$ of $X$ is tight it is sufficient to show that any block subspace is tight in $X$.

In [18], V. Ferenczi and G. Godefroy gave a beautiful characterization of tightness using Baire category:

Theorem 4.2.4. Let $X$ be a Banach space with normalized basis $\left(x_{n}\right)_{n}$ and let $Y$ be $a$ Banach space. Then the following statements are equivalent:
(i) $Y$ is tight in $X$.
(ii) $E_{Y}:=\left\{\mathfrak{u} \in 2^{\omega}: Y \hookrightarrow\left[x_{n}: n \in \operatorname{supp}(\mathfrak{u})\right]\right\}$ is meager in $2^{\omega}$.

Proof:
$(\mathrm{i}) \Rightarrow($ ii $)$ Without loss of generality, we can suppose that $Y$ is a block subspace of $X$. If $Y$ is tight in $X$, there are intervals $I_{0}<I_{1}<\ldots$ such that, for any $A \in[\mathbb{N}]^{\infty}$,

$$
\begin{equation*}
Y \nLeftarrow\left[x_{n}: n \notin \cup_{i \in A} I_{i}\right] . \tag{4.2}
\end{equation*}
$$

Let $\mathfrak{u} \in E_{Y}\left(\right.$ clearly $\left.\operatorname{supp}(\mathfrak{u}) \in[\mathbb{N}]^{\infty}\right)$ and suppose by contradiction that

$$
A_{\mathfrak{u}}:=\left\{i \in \mathbb{N}: I_{i} \cap \operatorname{supp}(\mathfrak{u})=\emptyset\right\}
$$

is infinite. Then

$$
\operatorname{supp}(\mathfrak{u}) \subseteq \mathbb{N} \backslash \bigcup_{i \in A_{\mathfrak{u}}} I_{i},
$$

contradicting Equation (4.2). Thus, $A_{\mathfrak{u}}$ is finite and, by Corollary 3.1.5, $E_{Y}$ is meager in $2^{\omega}$.
(ii) $\Rightarrow$ (i) Again by using Corollary 3.1.5, there are subsets $I_{0}<I_{1}<\ldots$ such that if $\mathfrak{u} \in E_{Y}$, then $\left\{i \in \mathbb{N}: I_{i} \cap \operatorname{supp}(\mathfrak{u})=\emptyset\right\}$ is finite. If there is $A \in[\mathbb{N}]^{\infty}$ such that $Y \hookrightarrow\left[x_{n}\right.$ : $\left.n \notin \cup_{i \in A} I_{i}\right]$, then take $v:=\mathbb{N} \backslash \cup_{i \in A} I_{i}$. Clearly $\chi_{v} \in E_{Y}$ and $\left\{i \in \mathbb{N}: I_{i} \cap v=\emptyset\right\}$ is infinite, which contradicts that $E_{Y}$ is a meager subset of $2^{\omega}$.

Lemma 4.2.5. If $X$ is a Banach space with normalized basis $\left(x_{n}\right)_{n}$ and $Y$ is a Banach space, then $E_{Y}$ is meager or comeager in $2^{\omega}$.

Proof. See the proof of Theorem 3.2 in [18].
In Chapter 5 we shall generalize this result (see Proposition 5.6.6) using the scheme of the proof of Theorem 3.2 in [18] for different kinds of embbedings, including isomorphic embeddings.

As we already noticed, the notion of tightness depends not only on the space but also on the basis chosen for such space. Therefore it is expected to be hereditary by taking block subsequences of the considered tight basis, as it is established in the following proposition. To prove this proposition we have used the techniques given by Ferenczi and Godefroy in [18] instead of the arguments in [22].

Proposition 4.2.6. Let $X$ be a Banach space with normalized basis $\left(x_{n}\right)_{n}$ e $Y=\left[y_{n}\right]_{n} a$ block subspace of $X$. Let $Z$ be a Banach space. If $Z$ is tight in $X$, then $Z$ is tight in $Y$.

Proof. Let $X=\left[x_{n}\right]_{n}, Y=\left[y_{n}\right]_{n}$ and $Z$ be as in the hypothesis. Let us denote as

$$
E_{Z}^{X}:=\left\{\mathfrak{u} \in 2^{\omega}: Z \hookrightarrow\left[x_{n}: n \in \operatorname{supp}(\mathfrak{u})\right]\right\}
$$

and

$$
E_{Z}^{Y}:=\left\{\mathfrak{u} \in 2^{\omega}: Z \hookrightarrow\left[y_{n}: n \in \operatorname{supp}(\mathfrak{u})\right]\right\} .
$$

By hypothesis, we know that $E_{Z}^{X}$ is meager in $2^{\omega}$. Using Lemma 4.2.5, $E_{Z}^{Y}$ is meager or comeager in $2^{\omega}$. If it is meager, $Z$ is tight in $Y$ and the demonstration ends. Suppose that $E_{Z}^{Y}$ is comeager. By Corollary 3.1.5, there are sequences of successive intervals $\left(I_{i}\right)_{i}$ and $\left(J_{i}\right)_{i}$ such that

$$
\begin{equation*}
\mathfrak{u} \in E_{Z}^{X} \Rightarrow\left\{n \in \omega: \operatorname{supp}(\mathfrak{u}) \cap I_{n}=\emptyset\right\} \text { is finite, } \tag{4.3}
\end{equation*}
$$

and if $\mathfrak{v} \in 2^{\omega}$ satisfies

$$
\begin{equation*}
\left\{n \in \omega: J_{n} \subseteq \operatorname{supp}(\mathfrak{v})\right\} \text { is infinite, then } \mathfrak{v} \in E_{Z}^{Y} \tag{4.4}
\end{equation*}
$$

Let $A \in[\mathbb{N}]^{\infty}$ be such that

$$
\left\{k \in \mathbb{N}: \bigcup_{n \in A} \bigcup_{i \in J_{n}} \operatorname{supp}_{X}\left(y_{i}\right) \cap I_{k}=\emptyset\right\}
$$

is infinite. Such $A$ exists because each $I_{i}$ and $J_{i}$ are finite and each $y_{i}$ is finitely supported. Let $v=\cup_{n \in A} J_{n}$. By Equation (4.4), $\chi_{v} \in E_{Z}^{Y}$.

If $u=\cup_{k \in v} \operatorname{supp}_{X}\left(y_{k}\right)$, then

$$
Z \hookrightarrow\left[y_{n}: n \in v\right] \Rightarrow Z \hookrightarrow\left[x_{n}: n \in u\right] .
$$

Therefore, $\chi_{u}$ would be in $E_{Z}^{X}$ but it is disjoint of infinitely many intervals $I_{k}$, contradicting Equation (4.3).

Corollary 4.2.7. Let $X$ be a tight Banach space and $\left(x_{n}\right)_{n}$ a tight basis for $X$. Then any block basis $\left(y_{n}\right)_{n}$ of $\left(x_{n}\right)_{n}$ is tight.

Proof. If $Z$ is a block subspace of $Y=\left[y_{n}\right]_{n}$, then $Z$ is a block subspace of $X$. As $Z$ is tight in $X$, by Proposition 4.2.6, $Z$ is tight in $Y$.

Proposition 4.2.8. If $\left(x_{n}\right)_{n}$ is a spreading basic sequence in a Banach space $E$, then $\left(x_{n}\right)_{n}$ is not a tight basis.

Proof. Let $\left(I_{i}\right)_{i}$ be a sequence of successive intervals and for each $k \in \mathbb{N}$ set $n_{k} \in I_{2 k}$. Let $A=\left\{n_{k}: k \in \mathbb{N}\right\}$. It is clear that $\left\{n: A \cap I_{n}=\emptyset\right\}$ is infinite (the set of even numbers is contained in it) but

$$
\left[x_{n}\right]_{n} \hookrightarrow\left[x_{n}: n \in A\right] .
$$

Then, $\left(x_{n}\right)_{n}$ is not a tight basis for $\left[x_{n}\right]_{n}$.
In particular, subsymmetric basic sequences fails to be tight. Furthermore, a space with a tight basis fails to have spreading basic sequences, since it can be proved that a space with basis containing a spreading basic sequence has a spreading block sequence, see [42].

As we already mentioned, the notion of tightness is incompatible with minimality. The following proposition was proved in [22]. The proof we present is different from the original since it uses arguments of Baire Category introduced in [18].

Proposition 4.2.9. A tight Banach space contains no minimal subspaces.
Proof. Suppose $\left(e_{n}\right)_{n}$ is a tight basis for the space $E$ and $X$ is a minimal subspace of $E$. We can assume without loss of generality that $X=\left[x_{n}\right]_{n}$ is a block subspace. Let $Y \leq X$. We know that $Y$ is tight in $E$. By Proposition 4.2.6, $Y$ is tight in $X$ but
$E_{Y}^{X}=\left\{\mathfrak{u} \in 2^{\omega}: Y \hookrightarrow\left[x_{n}: n \in \operatorname{supp}(\mathfrak{u})\right]\right\}$ is not meager in $2^{\omega}$. In fact, $E_{Y}^{X}$ is the subset of all the characteristic functions over infinite subsets of $\mathbb{N}$ (which is not meager): indeed, let $\mathfrak{v} \in 2^{\omega}$ be a characteristic function of an infinite subset of $\mathbb{N}$, then by the minimality of $X$

$$
X \hookrightarrow\left[x_{n}: n \in \operatorname{supp}(\mathfrak{v})\right],
$$

and then $Y \hookrightarrow\left[x_{n}: n \in \operatorname{supp}(\mathfrak{v})\right]$. Thus, $\mathfrak{v} \in E_{Y}^{X}$.
Remark 4.2.10. Notice that the last theorem also implies that minimal spaces fail to have tight subspaces since every subspace of a minimal space is also minimal. The classical spaces $c_{0}$ and $\ell_{p}(1 \leq p<\infty)$, the dual of Tsirelson's space $\mathbf{T}^{*}$ and Schlumprecht space $\mathcal{S}$ are minimal, therefore they are not tight and have no tight subspaces.

### 4.2.1 Shrinking bases and tightness

The results in this section are presented and proved in [22]. We study the proofs in detail.
Lemma 4.2.11. Let $Y$ and $X$ be Banach spaces, $X$ with normalized basis $\left(x_{n}\right)_{n}$, and $\left(I_{n}\right)_{n}$ a sequence of finite intervals such that $\min I_{n} \xrightarrow[n \rightarrow \infty]{ } \infty$ and for every $A \in[\mathbb{N}]^{\infty}$,

$$
Y \nrightarrow\left[x_{n}: n \notin \cup_{k \in A} I_{k}\right]
$$

Then, for any embedding $T: Y \rightarrow X$, we have $\liminf _{n \rightarrow \infty}\left\|P_{I_{n}} T\right\|>0$.
Proof. Let us proceed by contradiction. Let $Y, X=\left[x_{n}\right]_{n}$ and $\left(I_{n}\right)_{n}$ be as in the hypothesis. Let $T: Y \rightarrow X$ be an embedding with $\liminf _{n \rightarrow \infty}\left\|P_{I_{K}}\right\|=0$, so there exists $B \in[\mathbb{N}]^{\infty}$ such that $\lim _{n \rightarrow \infty, n \in B}\left\|P_{I_{K}}\right\|=0$. Then there exists an infinite subset $A$ of $B$ such that

$$
\begin{equation*}
\sum_{k \in A}\left\|P_{I_{k}} T\right\|<\frac{1}{2}\left\|T^{-1}\right\|^{-1} \tag{4.5}
\end{equation*}
$$

We can suppose that the intervals in $\left\{I_{k}: k \in A\right\}$ are disjoint. If they were not, as $\min I_{n} \xrightarrow[n \rightarrow \infty]{ } \infty$, by passing to an infinite subset of $A$, we would obtain disjoint intervals. Equation (4.5) says that the sequence of operators $\left(P_{I_{k}} T\right)_{k \in A}$ is absolutely summable, so consider the operator $\sum_{k \in A} P_{I_{k}} T: Y \rightarrow X$.

If $y \in Y$, we have

$$
\begin{aligned}
\left\|\sum_{k \in A} P_{I_{k}} T y\right\| & \leq\left\|\sum_{k \in A} P_{I_{k}} T\right\|\|y\| \\
& \leq \frac{1}{2\left\|T^{-1}\right\|}\|y\| \\
& \leq \frac{\|T(y)\|}{2}
\end{aligned}
$$

By this and the backwards triangular inequality, we obtain

$$
\begin{equation*}
\frac{\|T(y)\|}{2} \leq\left\|\left(T-\sum_{k \in A} P_{I_{k}} T\right) y\right\| \tag{4.6}
\end{equation*}
$$

As the difference of bounded operators is bounded, there is $K_{1}>0$ such that

$$
\begin{equation*}
\left\|\left(T-\sum_{k \in A} P_{I_{k}} T\right) y\right\| \leq K_{1}\|y\| . \tag{4.7}
\end{equation*}
$$

As $T$ is an embedding, there exists $K_{2}>0$ such that

$$
\frac{\|T(y)\|}{2} \geq K_{2}\|y\| .
$$

Combining all this equations we obtain

$$
K_{2}\|y\| \leq\left\|\left(T-\sum_{k \in A} P_{I_{k}} T\right) y\right\| \leq K_{1}\|y\| .
$$

So, $T-\sum_{k \in A} P_{I_{k}} T$ is an embedding of $Y$ in $\left[x_{n}: n \notin \cup_{i \in A} I_{i}\right]$, which contradicts the hypothesis.

Theorem 4.2.12. Any shrinking basic sequence of a tight space is a tight basis.
Proof. Let $X$ be a Banach space with a tight basis $\left(x_{n}\right)_{n}$, and $\left(y_{n}\right)_{n}$ be a shrinking basic sequence in $X$. Let $Y$ be an arbitrary Banach space. Using that $X$ is tight for $Y$ there exists a sequence $\left(I_{n}\right)_{n}$ of successive intervals that testifies $Y$ tight in $X$.

Let $m$ be a natural number and $\varepsilon>0$ arbitrary. For each $i \leq m$, there exists $N_{i} \in \mathbb{N}$, such that $\left\|y_{i}-P_{N_{i}}\left(y_{i}\right)\right\|<\varepsilon$. Let $N>\max \left\{N_{i}: i \leq m\right\}$ and $K \in \mathbb{N}$ be such that $N<I_{K}$. Then for any $k>K$ and $i \leq m$, we have that

$$
\left\|P_{I_{k}}\left(y_{i}\right)\right\| \leq\left\|y_{i}-P_{N}\left(y_{i}\right)\right\|<\varepsilon .
$$

Then,

$$
\begin{equation*}
\left\|\left.P_{I_{k}}\right|_{\left[y_{i}: i \leq m\right]}\right\| \xrightarrow[k \rightarrow \infty]{ } 0 \tag{4.8}
\end{equation*}
$$

Let $k \in \mathbb{N}$ and $m \in \mathbb{N}$. Denote by $F_{m}$ the closed subspace $\left[y_{n}: n>m\right.$ ]. Then we have for any $y \in \mathbb{S}_{F_{m}}$

$$
\begin{equation*}
\left\|P_{I_{K}}(y)\right\| \leq \sum_{i \in I_{k}}\left|x_{i}^{*}(y)\right| \tag{4.9}
\end{equation*}
$$

Then,

$$
\begin{aligned}
\left\|\left.P_{I_{k}}\right|_{F_{m}}\right\| & =\sup \left\{\left\|P_{I_{k}}(y)\right\|: y \in \mathbb{S}_{F_{m}}\right\} \\
& \leq \sup \left\{\sum_{i \in I_{k}}\left|x_{i}^{*}(y)\right|: y \in \mathbb{S}_{F_{m}}\right\} \\
& \leq \sum_{i \in I_{k}} \sup \left\{\left|x_{i}^{*}(y)\right|: y \in \mathbb{S}_{F_{m}}\right\} \\
& =\sum_{i \in I_{k}}\left\|\left.x_{i}^{*}\right|_{F_{m}}\right\| .
\end{aligned}
$$

As each $\left.x_{i}^{*}\right|_{F_{m}}$ is a functional of $F_{m}$ and the basis $\left(y_{n}\right)_{n}$ is shrinking, then, by Proposition 2.2.13, for each $i \in I_{k}$ we obtain

$$
\left.\lim _{m \rightarrow \infty} x_{i}^{*}\right|_{F_{m}}=0
$$

So, Equation (4.9) implies that

$$
\lim _{m \rightarrow \infty}\left\|\left.P_{I_{k}}\right|_{F_{m}}\right\|=0
$$

Now we are going to construct inductively the sequence of sets $\left(J_{n}\right)_{n}$ which will testify that $Y$ is tight. By Equation (4.8), for $\varepsilon=1$ there exists $k_{0} \in \mathbb{N}$ such that for all $k \geq k_{0}$

$$
\left\|\left.P_{I_{k}}\right|_{\left.y_{i}: i \leq m\right]}\right\|<1
$$

and, by Equation (4.2.1) there exists $M_{0}>0$ such that for all $m \geq M_{0}$

$$
\left\|P_{I_{k_{0}}}| |_{F_{m}}\right\|<1 .
$$

Let $J_{0}=\left[0, M_{0}\right]$. Then

$$
\left\|P_{I_{k_{0}}}\left|{ }_{\left.y_{i}: i \notin J_{0}\right]}\|=\| P_{I_{k_{0}}}\right|_{F_{M_{0}}}\right\|<1 .
$$

Suppose we had found $k_{0}<k_{1}<\ldots k_{n-1}$ natural numbers and $J_{0}<J_{1}<\ldots<J_{n-1}$ intervals such that for all $j \in\{0, \ldots, n-1\}$

$$
\begin{equation*}
\left\|\left.P_{I_{k_{j}}}\right|_{\left[y_{i}: i \notin J_{j}\right]}\right\| \leq \frac{2}{j+1} . \tag{4.10}
\end{equation*}
$$

Then, by Equation (4.8) there is $k_{n}>k_{n-1}$ such that

$$
\left\|\left.P_{I_{k_{n}}}\right|_{\left[y_{i}: i \leq \max J_{n-1}\right]}\right\|<\frac{1}{n+1} .
$$

Define $M_{n}>M_{n-1}$ such that for all $m \geq M_{n}$

$$
\left\|\left.P_{I_{k_{n}}}\right|_{F_{m}}\right\| \leq \frac{1}{n+1} .
$$

Let $J_{n}=\left\{M_{n-1}+1, M_{n}\right\}$. By using the last equations obtained

$$
\left.\begin{array}{rl}
\left\|\left.P_{I_{k_{n}}}\right|_{\left.y_{i}: i \notin J_{n}\right]}\right\| & \leq \| P_{I_{k_{n}}} \mid\left[y_{i}: i \leq M_{n-1}\right]
\end{array}\right]+\left\|\left.P_{I_{k_{n}}}\right|_{F_{M_{n}}}\right\|
$$

Let us suppose that there is $A \in[\mathbb{N}]^{\infty}$ such that $Y \hookrightarrow\left[y_{n}: n \notin \cup_{i \in A} J_{i}\right]$. Then $Y \hookrightarrow\left[x_{n}\right]_{n}$ and let $T$ be such embedding. As $Y$ is tight in $\left(x_{n}\right)_{n}$, the hypotheses of the lemma 4.2.11 are satisfied for the intervals $\left(I_{n}\right)_{n}$, obtaining on one hand

$$
\begin{equation*}
\underset{k}{\liminf }\left\|P_{I_{k}} T\right\|>0 \tag{4.11}
\end{equation*}
$$

and, on the other, that

$$
\lim _{n \in A}\left\|P_{I_{k_{n}}} T\right\|=0
$$

which contradicts Equation (4.11).
Corollary 4.2.13. If $X$ is tight and reflexive, then every basic sequence in $X$ is tight.
Proof. Let $\left(y_{n}\right)_{n}$ be a basic sequence in $X$. Then, $\left[y_{n}\right]_{n}$ is reflexive. By Theorem 2.2.15, $\left(y_{n}\right)_{n}$ is a shrinking basic sequence of a tight space. Theorem 4.2.12 implies that $\left[y_{n}\right]_{n}$ is tight.

Corollary 4.2.14. If $X$ is tight and has an unconditional basis, then every basic sequence in $X$ is tight.

Proof. Let us suppose $X$ satisfies the hypothesis. Then, by Proposition 4.2.9, $X$ cannot contain copies of minimal spaces $c_{0}$ and $\ell_{1}$. Consequently, by Theorem 2.2.21, $X$ is reflexive. It follows from Corollary 4.2.13 that all basic sequences of $X$ are tight.

The following example shows that a Banach space may not have minimal subspaces and not be tight. It can be found in [22], Example 3.6.

Example 4.2.15. The symmetrization $S\left(\mathbf{T}^{(p)}\right)$ of the $p$-convexification $T^{(p)}$ of Tsirelson's space, with $p \in(1, \infty)$, does not contain a minimal subspace, yet it is not tight.

Proof. As we remarked in Theorem 4.1.5, $S\left(\mathbf{T}^{(p)}\right)$ has no minimal subspaces. The canonical basis $\left(e_{n}\right)_{n}$ of $S\left(\mathbf{T}^{(p)}\right)$ is symmetric (thus, subsymmetric), then it is not a tight basis (see Proposition 4.2.8). Since $S\left(\mathbf{T}^{(p)}\right)$ is reflexive, if it were tight then by Corollary 4.2.13 all basic sequences, including $\left(e_{n}\right)_{n}$, would be tight, which is false.

### 4.2.2 Strongly asymptotic $\ell_{p}$ spaces and tightness

An example of a tight Banach space is Tsirelson's space. Following [22], it will be shown in this section that strongly asymptotic $\ell_{p}$ spaces without copies of $\ell_{p}$ are, in fact, tight if $1 \leq p<+\infty$. Definitions of asymptotic and strongly asymptotic $\ell_{p}$ spaces can be found
in [13]. It is important to mention that other slightly different definitions of asymptotic $\ell_{p}$ spaces exist in the literature, see [37].

Definition 4.2.16. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $1 \leq p \leq \infty$. We say that $\left(e_{n}\right)_{n}$ is an asymptotic $\ell_{p}$ basis, if there are a finite constant $C$ and an increasing function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that, for all natural n, every normalized block basis $\left(x_{i}\right)_{i=0}^{n}$ of $\left(e_{i}\right)_{i=f(n)}^{\infty}$ is $C$-equivalent to the unit basis of $\ell_{p}^{n}$. $E$ is an asymptotic- $\ell_{p}$ space if it has an asymptotic- $\ell_{p}$ basis.

Definition 4.2.17. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $1 \leq p \leq \infty$. We say that $\left(e_{n}\right)_{n}$ is a strongly asymptotic $\ell_{p}$ basis if there are a finite constant $C$ and an increasing function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that, for all $n$, every normalized sequence $\left(y_{i}\right)_{i=0}^{n}$ of disjointly supported vectors from $\left[x_{i}: i \geq f(n)\right]$ is $C$-equivalent to the unit basis of $\ell_{p}^{n}$. $E$ is a strongly asymptotic- $\ell_{p}$ space if it has a strongly asymptotic- $\ell_{p}$ basis.

Remark 4.2.18. Every strongly asymptotic $\ell_{p}$ basis is also asymptotic $\ell_{p}$.
Proposition 4.2.19. Tsirelson's space $\mathbf{T}$ is strongly asymptotic $\ell_{1}$ and for $1<p<\infty$, the $p$-convexification $\mathbf{T}^{(p)}$ of Tsirelson's space is strongly asymptotic $\ell_{p}$.

## Proof.

$\mathbf{T}$ : The so called modified Tsirelson's space $\mathbf{T}_{\mathbf{M}}$ is proved to be strongly asymptotic $\ell_{1}$ (see Proposition V. 8 in [11]). Also, the unit bases of $\mathbf{T}$ and $\mathbf{T}_{\mathbf{M}}$ are equivalent (see Theorem V. 3 in [11]). Then Tsirelson's space is strongly asymptotic $\ell_{1}$.
$\mathbf{T}^{(p)}$ : For proving this for the modified version $\mathbf{T}_{M}^{(p)}$ of $\mathbf{T}^{(p)}$ it is done in the same way as in the case of $\mathbf{T}$, see Proposition 7.3 in [7].

The next definition was given in [22].
Definition 4.2.20. Let $X$ be a Banach space with normalized basis $\left(x_{n}\right)_{n}$. We say that a Banach space $Y$ is tight in $X$ with constants if, and only if, there is a sequence $\left(I_{i}\right)_{i}$ of successive intervals such that for all integers $K$,

$$
Y \nrightarrow \sim_{K}\left[x_{n}: n \notin I_{K}\right] .
$$

We say that $\left(x_{n}\right)_{n}$ is a tight with constant basis for $X$ if $Y$ tight in $X$ with constants, for every Banach space $Y$. Finally, $X$ is tight with constants if it has a tight with constant basis.

In the same way that V. Ferenczi and G. Godefroy characterized tightness using Baire category, they showed a natural characterization of tightness with constant:

Proposition 4.2.21. A Banach space $Y$ is tight with constants in a Banach space $X$ with normalized basis $\left(x_{n}\right)_{n}$, if, and only if, for all $K$

$$
E_{Y}(K)=\left\{\mathfrak{u} \in 2^{\omega}: Y \hookrightarrow_{K}\left[x_{n}: n \in \operatorname{supp}(\mathfrak{u})\right]\right\}
$$

is nowhere dense in $2^{\omega}$.

Proof. See Proposition 3.5 in [18].
Tightness with constants is a stronger form of tightness, where extra information about how a space does not embed into the other is given. It is easy to see that if $Y$ is tight in $X$ with constants, then $Y$ does not embed uniformly into the tail subspaces of $X$, in fact this characterize tightness with constants, as was shown in [22]. We present the following proposition which we prove in a different way than in [22], using Proposition 4.2.21. First, recall that a basic open subset of $2^{\omega}$ determined by $\mathfrak{s} \in 2^{\omega}$ and $J \in[\mathbb{N}]^{<\infty}$ is given by

$$
\mathcal{N}_{\mathfrak{s}, J}:=\left\{\mathfrak{u}=\left(u_{n}\right)_{n} \in 2^{\omega}: \forall n \in J\left(u_{n}=s_{n}\right)\right\} .
$$

Proposition 4.2.22. The basis $\left(x_{n}\right)_{n}$ of the space $X$ is tight with constants if, and only if, for every $Y$ Banach space there is no $K$ such that $Y K$-embeds in all the tail subspaces of $X$.

Proof. For the direct implication, suppose by contradiction that there is a Banach space $Y$ which is uniformly embeddable in all the tail subspaces of $X$. Without loss of generality we can suppose $Y=\left[y_{n}\right]_{n}$ is a block subspace of $X$ and let $K$ be such that for every $m$,

$$
\begin{equation*}
Y \hookrightarrow_{K}\left[x_{n}: n \geq m\right] . \tag{4.12}
\end{equation*}
$$

This means that for every $m, \chi_{[m, \infty)} \in E_{Y}(K)$, so $\emptyset \in \overline{E_{Y}(K)}$. Since $E_{Y}(K)$ is closed under taking supersets, for every $u \in F I N$, we can find a sequence of elements of $E_{Y}(K)$ converging to $\chi_{u}$, so $\chi_{u} \in \overline{E_{Y}(K)}$. Since the copy of FIN in $2^{\omega}$ is dense in $2^{\omega}$, we conclude that $\overline{E_{Y}(K)}=2^{\omega}$ which is a contradiction.

For the converse implication we shall proceed by contradiction again. Recall that a basic open subset of $2^{\omega}$ determined by $\mathfrak{s} \in 2^{\omega}$ and $J \in[\mathbb{N}]^{<\infty}$ is given by

$$
\mathcal{N}_{\mathfrak{s}, J}=\left\{\mathfrak{u}=\left(u_{n}\right)_{n} \in 2^{\omega}: \forall n \in J\left(u_{n}=s_{n}\right)\right\} .
$$

Suppose the basis $\left(x_{n}\right)_{n}$ is not tight with constants, so there is a block subspace $Y$, some $K, \mathfrak{s} \in 2^{\omega}$ and $I \in[\mathbb{N}]^{<\infty}$ such that $\mathcal{N}_{\mathfrak{s}, I} \subseteq \overline{E_{Y}(K)}$. In particular, $\overline{E_{Y}(K)}$ "contains" finite subsets of $\mathbb{N}$. Let $\mathfrak{u} \in \mathcal{N}_{\mathfrak{s}, I}$ be such that $\operatorname{supp}(\mathfrak{u})$ is finite and non-empty. Let $\left(\mathfrak{u}^{i}\right)_{i}$ be a
sequence of elements of $E_{Y}(K) \cap \mathcal{N}_{\mathfrak{s}, I}$ converging to $\mathfrak{u}$ ．Let $c$ be the constant that depends on the basis constant of $\left(x_{n}\right)_{n}$ and $|\operatorname{supp}(\mathfrak{u})|$ ，which exists via Proposition 2．2．29．

Let $m \in \mathbb{N}$ ．Set $k>\max \{m, \operatorname{supp}(\mathfrak{u}), m+|\operatorname{supp}(\mathfrak{u})|\}$ ．Then，there is $N$ such that for every $i>N$

$$
[0, k] \cap \operatorname{supp}\left(\mathfrak{u}^{\mathfrak{i}}\right)=\operatorname{supp}(\mathfrak{u}) .
$$

We know that $Y \hookrightarrow_{K}\left[x_{n}: n \in \operatorname{supp}\left(\mathfrak{u}^{\mathfrak{i}}\right)\right]$ ，for every $i$ ．Set $j>N$

$$
v=\left(\operatorname{supp}\left(\mathfrak{u}^{j}\right) \backslash \operatorname{supp}(\mathfrak{u})\right) \cup[k-|\operatorname{supp}(\mathfrak{u})|, k) .
$$

Since for $i>N$ we have that $\operatorname{supp}\left(\mathfrak{u}^{\mathfrak{i}}\right)$ and $v$ differ only in $|\operatorname{supp}(\mathfrak{u})|$ elements，it follows that

$$
\left[x_{n}: n \in \operatorname{supp}\left(\mathfrak{u}^{\mathfrak{j}}\right)\right] \hookrightarrow_{c^{2}}\left[x_{n}: n \in v\right] \subseteq\left[x_{n}: n \geq m\right] .
$$

Therefore，$Y \hookrightarrow_{K c^{2}}\left[x_{n}: n \geq m\right]$ ．

Recall that a Banach space $X$ is $K$－crudely finitely representable in another Banach space $Y$ if，and only if，for any finite－dimensional subspace $F$ of $X$ there is a $K$－embedding $T: F \rightarrow Y$ ． Other useful characterizations are known，as can be seen in the following proposition．

Proposition 4．2．23．Let $X$ be a Banach space with normalized basis $\left(x_{n}\right)_{n}$ ．The following are equivalent：
（i）For any block basis $\left(y_{n}\right)_{n}$ there is a sequence $\left(I_{i}\right)_{i}$ of successive intervals such that for all $K$ ，

$$
\left[y_{n}: n \in I_{K}\right] \not_{\nmid}\left[x_{n}: n \notin I_{K}\right] .
$$

（ii）For any space $Y$ there is a sequence $\left(I_{i}\right)_{i}$ of successive intervals such that for all $K$ ，

$$
Y \nrightarrow ⿱ 丶 万 ⿱ ⿰ ㇒ 一 乂, ~\left[x_{n}: n \notin I_{K}\right] .
$$

（iii）No space embeds uniformly into the tail subspaces of $X$ ．
（iv）There is no $K$ and no subspace of $X$ which is $K$－crudely finitely representable in any tail subspace of $X$ ．

Proof．See［22］，Proposition 4．1．
Finally，we remark the result proved in［22］which implies that Tsirelson＇s space is tight．
Theorem 4．2．24．Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ strongly asymptotic $\ell_{p}$ and not containing a copy of $\ell_{p}$ ，for $1<p<\infty$ ．Then $\left(e_{n}\right)_{n}$ is tight with constants．

Proof．See［22］，Proposition 4．2．

Corollary 4.2.25. Tsirelson's space $\mathbf{T}$ and its p-convexifications $\mathbf{T}^{(p)}(1<p<\infty)$, are tight with constant.

Proof. It follows directly from Proposition 4.2.18 and Theorem 4.2.24.
Remark 4.2.26. The dual of Tsirelson's space $\mathbf{T}^{*}$ is strongly asymptotic $\ell_{\infty}$ and it has no copies of $\ell_{\infty}$ but it is minimal.

## Chapter 5

## New forms of tightness

Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. We want to code different types of embeddings to define the corresponding notion of minimality and tightness associated with such embeddings. Suppose that $X$ and $Y$ are two block subspaces of $E$, generated by the block bases $\left(x_{n}\right)_{n}$ and $\left(y_{n}\right)_{n}$, respectively. We are interested in studying the following types of embeddings:

- The subspace $X$ embeds in $Y$ as a block basis, that is, there is $\left(z_{n}\right)_{n}$ block basis of $\left(y_{n}\right)_{n}$ such that $\left(x_{n}\right)_{n} \sim\left(z_{n}\right)_{n}$.
- The subspace $X$ embeds in $Y$ as a sequence of disjointly supported vectors of $\left(y_{n}\right)_{n}$, that is, there is a sequence $\left(z_{n}\right)_{n}$ of finitely supported vectors with respect to $\left(y_{n}\right)_{n}$ such that $i \neq j \Rightarrow \operatorname{supp}_{Y}\left(z_{i}\right) \cap \operatorname{supp}_{Y}\left(z_{j}\right)=\emptyset$ and $\left(x_{n}\right)_{n} \sim\left(z_{n}\right)_{n}$.
- The subspace $X$ embeds in $Y$ as a subsequence of $\left(y_{n}\right)_{n}$, that is, there is $\left(y_{k_{n}}\right)_{n}$ subsequence of $\left(y_{n}\right)_{n}$ such that $\left(x_{n}\right)_{n} \sim\left(y_{k_{n}}\right)_{n}$.

In order to code such embeddings, we shall define a set of blocks $\mathcal{D}_{E}$ for $E$ and an admissible set $\mathcal{A}_{E}$ contained in $\left(\mathcal{D}_{E}\right)^{\omega}$. The set $\mathcal{D}_{E}$ will be a subset of a countable subset $\mathbb{D}_{E}$ of finitely supported vectors of $E$ with some additional property we shall define in this Chapter. Block sequences which elements belongs to ( $\mathcal{D}_{E}$ shall represent the basis of the subspaces we are interested in. The set $\mathcal{A}_{E}$ is an infinite set which contains infinite sequences of vectors in $\mathcal{D}_{E}$ and it shall be thought of as the images of the embeddings we are allowing to occur.

The objective of this chapter is to study the properties of admissible sets, and to use this notions to define $\mathcal{A}_{E}$-minimality and $\mathcal{A}_{E}$-tightness for a space $E$.

### 5.1 Admissible system of blocks

Along this section let us fix a Banach space $E$ with normalized Schauder basis $\left(e_{n}\right)_{n}$.
Proposition 5.1.1. There is $\mathbf{F}_{E}$ a countable subfield of $\mathbb{R}$ containing the rationals such that for all $\sum_{i=0}^{n} \lambda_{i} e_{i}$, with $n \in \mathbb{N}$ and $\left(\lambda_{i}\right)_{i=0}^{n} \in\left(\mathbf{F}_{E}\right)^{n+1}$, the norm $\left\|\sum_{i=0}^{n} \lambda_{i} e_{i}\right\| \in \mathbf{F}_{E}$.

Proof. Given $A$ a non-empty subset of $\mathbb{R}$ define the following sets:

$$
\operatorname{span}_{A}\left(e_{n}\right):=\left\{\sum_{i \in I}: \lambda_{i} e_{i}: \lambda_{i} \in A, \forall i \in I, I \in[\mathbb{N}]^{<\infty}\right\}
$$

and

$$
F_{A}:=\left\{\|x\|: x \in \operatorname{span}_{A}\left(e_{n}\right)\right\} .
$$

Now, take the set of rational numbers $\mathbb{Q}$ and the set $F_{\mathbb{Q}}$. Clearly $A_{0}:=\mathbb{Q} \cup F_{\mathbb{Q}}$ is countable. Take the countable subfield $A_{1}$ of $\mathbb{R}$ generated by $A_{0}$. Now, suppose we have found $A_{1} \subseteq A_{2} \subseteq \ldots \subseteq A_{n}$ countable subfields of $\mathbb{R}$, satisfying:

$$
A_{i} \cup F_{A_{i}} \subseteq A_{i+1}, \text { for every } i \in\{0, \ldots, n-1\}
$$

Take $A_{n+1}$ the subfield of $\mathbb{R}$ generated by $A_{n} \cup F_{A_{n}}$.
Let

$$
\mathbf{F}_{E}:=\bigcup_{i \in \mathbb{N}} A_{i} .
$$

$\mathbf{F}_{E}$ is a countable subfield of $\mathbb{R}$ because is the union of nested countable subfields of $\mathbb{R}$. Also, if $n \in \mathbb{N}$ and $\left(\lambda_{i}\right)_{i=0}^{n} \in\left(\mathbf{F}_{E}\right)^{n+1}$, there is some $k$ such that $\left(\lambda_{i}\right)_{i=0}^{n} \in\left(A_{k}\right)^{n+1}$, so the norm $\left\|\sum_{i=0}^{n} \lambda_{i} e_{i}\right\| \in A_{k+1} \subseteq \mathbf{F}_{E}$.

Notation 5.1.2. We denote by $\mathbb{D}_{E}$ the set of nonzero not necessarily normalized finite $\mathbf{F}_{E}$-linear combinations of $\left(e_{n}\right)_{n}$.

Remark 5.1.3. $\mathbb{D}_{E}$ is countable.
Definition 5.1.4. Let $\left(x_{n}\right)_{n}$ be a sequence of successive finitely supported vectors of $E$. For $X=\left[x_{n}\right]_{n}$, let us define the operation $*_{X}:\left(\mathbb{D}_{E} \cap X\right)^{\omega} \times\left(\mathbb{D}_{E}\right)^{\omega} \rightarrow\left(\mathbb{D}_{E}\right)^{\omega}$ as follows: if $v=\left(v_{n}\right)_{n}$ belongs to $\left(\mathbb{D}_{E}\right)^{\omega}$ and $u=\left(u_{n}\right)_{n} \in\left(\mathbb{D}_{E} \cap X\right)^{\omega}$ such that for each $n \in \mathbb{N}$

$$
u_{n}=\sum_{i \in s u p p_{X}\left(u_{n}\right)} \lambda_{i}^{n} x_{i},
$$

then $u *_{X} v$ is the sequence $\left(w_{n}\right)_{n}$, such that for each $n \in \mathbb{N}$

$$
w_{n}=\sum_{i \in{s u p p_{X}\left(u_{n}\right)} \lambda_{i}^{n} v_{i} . . . . . . . .}
$$

Remark 5.1.5. Notice that the set $\mathbb{D}_{E} \cap X$ could be empty. In our work we shall take subspaces generated by vectors on $\mathbb{D}_{E}$, so this will not occur.

Remark 5.1.6. Under the hypothesis of Definition 5.1.4 we have that each element of $u *_{X} v$ is a finite linear combination of the vectors of the sequence $v$, and for each $n \in \mathbb{N}$

$$
\begin{equation*}
\operatorname{supp}_{E}\left(w_{n}\right)=\bigcup_{i \in \operatorname{supp}_{X}\left(u_{n}\right)} \operatorname{supp}_{E}\left(v_{i}\right) \tag{5.1}
\end{equation*}
$$

Also, if $\left(v_{i}\right)_{i}$ is a basic sequence and $V=\left[v_{i}\right]_{i}$ we have for each $n$

$$
\begin{equation*}
\operatorname{supp}_{V}\left(w_{n}\right)=\operatorname{supp}_{X}\left(u_{n}\right) \tag{5.2}
\end{equation*}
$$

Definition 5.1.7. We define a set of blocks for the space $E$ to be a set $\mathcal{D}_{E}$ satisfying the following conditions
a) $\mathcal{D}_{E}$ is a subset of $\mathbb{D}_{E}$.
b) The set $\left\{e_{n}: n \in \mathbb{N}\right\}$ is contained in $\mathcal{D}_{E}$.
c) If $u \in \mathcal{D}_{E}$, then $\frac{u}{\|u\|} \in \mathcal{D}_{E}$.
d) For every $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$ and $\left(v_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$, we have $\left(u_{n}\right)_{n} *_{E}\left(v_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$.
e) Let $\left(x_{i}\right)_{i=0}^{n} \in\left(\mathcal{D}_{E}\right)^{n+1}$ with $x_{i}<x_{i+1}$ for every $0 \leq i \leq n$. If $u \in \mathcal{D}_{E}$ is such that

$$
u=\sum_{i=0}^{n} \lambda_{i} x_{i}
$$

then

$$
v=\sum_{i=0}^{n} \lambda_{i} e_{i} \in \mathcal{D}_{E}
$$

We say that a vector $u$ is a $\mathcal{D}_{E}$-block if $u$ is an element of the set $\mathcal{D}_{E}$.
Remark 5.1.8. Actually, the item d) in Definition 5.1.7 establishes a condition of closeness for the elements of $\mathcal{D}_{E}$ that we shall explain in the following lines. Define $\star: \mathbb{D}_{E} \times\left(\mathcal{D}_{E}\right)^{\omega} \rightarrow$ $\mathbb{D}_{E}$ as follows: If $\left(v_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$ is a block sequence and $u \in \mathcal{D}_{E}$ with

$$
u=\sum_{i \in s u p p_{E}(u)} \lambda_{i} e_{i}
$$

then, define

$$
u \star\left(v_{n}\right)_{n}=\sum_{i \in \operatorname{supp}_{E}(u)} \lambda_{i} v_{i} .
$$

Notice that in the construction of the vector $u \star\left(v_{n}\right)_{n}$ only a finite amount of coordinates of sequence $\left(v_{n}\right)_{n}$ are involved, specifically those $v_{n}$ with $n \in \operatorname{supp}_{E}(v)$.

Thus, condition d) in Definition 5.1.7 is equivalent to: For every $u \in \mathcal{D}_{E}$ and $\left(v_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$, we have

$$
u \star\left(v_{n}\right)_{n} \in \mathcal{D}_{E}
$$

Item e) in Definition 5.1.7, establishes another condition of "closeness": If some vector $u$ belongs to the set of blocks and it has some coordinates in terms of a block basis $\left(x_{n}\right)_{n}$ of $\left(e_{n}\right)_{n}$, then the vector with coordinates in the basis $\left(e_{n}\right)_{n}$ is also a block.

Notation 5.1.9. Let us denote by $\mathcal{B}_{E}$ the set whose elements are the vector of the normalized basis $\left\{e_{n}: n \in \mathbb{N}\right\}$ considered for $E$.

Example 5.1.10. The smallest set of blocks for $E$ is $\mathcal{B}_{E}$.

Proof. Clearly, condition $a), b$ ) and $c$ ) in Definition 5.1.7 are satisfied. Take $\left(u_{n}\right)_{n}$ and $\left(v_{n}\right)_{n}$ in $\left(\mathcal{B}_{E}\right)^{\omega}$. So, for each $n \in \mathbb{N}, u_{n}=e_{k_{n}}$ and $v_{n}=e_{l_{n}}$ for some $\left(k_{n}\right)_{n} \in \mathbb{N}^{\omega}$ and $\left(l_{n}\right)_{n} \in \mathbb{N}^{\omega}$. If $\left(w_{n}\right)_{n}:=\left(u_{n}\right)_{n} *_{E}\left(v_{n}\right)_{n}$, then for each $n$

$$
w_{n}=\sum_{i \in\left\{k_{n}\right\}} v_{i}=e_{l_{k_{n}}} \in \mathcal{B}_{E} .
$$

Thus, condition $d$ ) is satisfied. For condition $e)$ : Let $\left(x_{i}\right)_{i=0}^{n} \in\left(\mathcal{B}_{E}\right)^{n+1}$ with $x_{i}<x_{i+1}$ for every $0 \leq i \leq n$. Then, there is $\left(k_{i}\right)_{i=0}^{n}$ increasing sequence such that $x_{i}=e_{k_{i}}$. If $u \in \mathcal{B}_{E}$ is such that

$$
u=\sum_{i=0}^{n} \lambda_{i} e_{k_{i}}=e_{l},
$$

for some $l \in \mathbb{N}$. Then, due to the linear independence of $e_{i}$ 's, we have that for some $j \in\{0, \ldots, n\}$ such that $\lambda_{j}=1$ and $\lambda_{i}=0$, for all $i \in\{0, \ldots, n\} \backslash\{j\}$.
then,

$$
v=\sum_{i=0}^{n} \lambda_{i} e_{i}=e_{k_{j}} \in \mathcal{B}_{E} .
$$

Notation 5.1.11. Let us denote by $\mathcal{B}_{E}^{ \pm}$the set $\left\{e_{n}: n \in \mathbb{N}\right\} \cup\left\{-e_{n}: n \in \mathbb{N}\right\}$.
Example 5.1.12. The set $\mathcal{B}_{E}^{ \pm}$is a set of blocks for $E$.
Proof. This follows in the same way as the proof of Example 5.1.10. Notice that in this case we use that the product of signs is again a sign.

Example 5.1.13. The set $\mathbb{D}_{E}$ is obviously a set of blocks for $E$.
Remark 5.1.14. Clearly, a set of blocks $\mathcal{D}_{E}$ for $E$ is countable. Endow $\left(\mathcal{D}_{E}\right)^{\omega}$ with the product topology obtained by considering $\mathcal{D}_{E}$ with the discrete topology. As noticed in Section 3.2 of Chapter 3, $\left(\mathcal{D}_{E}\right)^{\omega}$ is a Polish space. Also, the set $\left(\mathbb{N} \times \mathbb{N} \times \mathcal{D}_{E}\right)^{\omega}$ with its natural product topology is Polish.

Definition 5.1.15. Let $D \subseteq \mathbb{D}_{E}$ be an infinite subset such that $D^{\omega}$ contains a block basis of $\left(e_{n}\right)_{n}$.
(i) We say that $\left(y_{n}\right)_{n} \in E^{\omega}$ is a D-block sequence if, and only if, $\left(y_{n}\right)_{n}$ is a block basis of $\left(e_{n}\right)_{n}$ and for each $n \in \mathbb{N}$ we have $y_{n} \in D$.
(ii) A normalized D-block sequence is a D-block sequence such that each element of the sequence is a normalized vector.
(iii) A D-block subspace is the closed subspace spanned by a D-block sequence.

Remark 5.1.16. Suppose $X$ is a $\mathcal{D}_{E}$-block subspace with $\mathcal{D}_{E}$-block basis. Without loss of generality, by condition c) in Definition 5.1.7, we can suppose that $\left(x_{n}\right)_{n}$ is a normalized $\mathcal{D}_{E}$-block sequence.

Notation 5.1.17. Let $D \subseteq \mathbb{D}$ be as in the hypothesis of Definition 5.1.15. In order to simplify the notation, we write that $X=\left[x_{n}\right]_{n}$ is a D-block subspace to say that the sequence $\left(x_{n}\right)_{n}$ is a D-block sequence and $X=\left[x_{n}\right]_{n}$. In the case that $D=\mathcal{D}_{E}$ we write $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace to say that the sequence $\left(x_{n}\right)_{n}$ is a normalized $\mathcal{D}_{E}$-block sequence and $X=\left[x_{n}\right]_{n}$.

Remark 5.1.18. Every $\mathcal{D}_{E}$-block subspace is a $\mathbb{D}_{E}$-block subspace.
Notation 5.1.19. If $X=\left[x_{n}\right]_{n}$ is a $\mathbb{D}_{E}$-block subspace, we denote by $\mathbb{D}_{X}$ the set $\mathbb{D}_{E} \cap X$.
Remark 5.1.20. The basis $\left(e_{n}\right)_{n}$ we have fixed for $E$ is a $\mathcal{D}_{E}$-block sequence, so $E$ is a $\mathcal{D}_{E}$-block subspace.

Notation 5.1.21. Let $\mathcal{D}_{E}$ be a set of blocks for $E$. Let $X$ be a $\mathcal{D}_{E}$-block subspace.
(i) We denote by $\mathcal{D}_{X}$ the set formed by the blocks which belong to $X$, that is

$$
\mathcal{D}_{X}:=\mathcal{D}_{E} \cap X
$$

(ii) We denote by $b b_{\mathcal{D}}(E)$ the set of normalized $\mathcal{D}_{E}$-block sequences of $E$, i.e.

$$
b b_{\mathcal{D}}(E):=\left\{\left(x_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}:\left(x_{n}\right)_{n} \text { is a } \mathcal{D}_{E} \text {-block sequence of } E \text { and } \forall n \in \mathbb{N}\left(\left\|x_{n}\right\|=1\right)\right\} .
$$

(iii) Let $X$ be an $\mathcal{D}_{E}$-block subspace of $E$, we denote by $b b_{\mathcal{D}}(X)$ the set of normalized $\mathcal{D}_{X}$-block sequences of $E$, i.e.
$b b_{\mathcal{D}}(X):=\left\{\left(y_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}:\left(y_{n}\right)_{n}\right.$ is a $\mathcal{D}_{X}$-block sequence of $E$ and $\left.\forall n \in \mathbb{N}\left(\left\|x_{n}\right\|=1\right)\right\}$.

Remark 5.1.22. The set $b b_{\mathcal{D}}(E)$ is a non-empty closed topological subspace of $\left(\mathcal{D}_{E}\right)^{\omega}$, therefore, it is Polish.

Remark 5.1.23. If $\mathcal{D}_{E}$ is a set of blocks for $E$ and $X$ is a $\mathcal{D}_{E}$-block subspace, then we sometimes identify an element $\left(y_{n}\right)_{n}$ of $b b_{\mathcal{D}}(X)$ with the $\mathcal{D}_{E}$-block subspace that it generates.

Definition 5.1.24. Let $\mathcal{D}_{E}$ be a set of blocks for $E$. We say that a set $\mathcal{A}_{E}$ is an admissible set for $E$ if, and only if, it satisfies the following conditions:
a) $\mathcal{A}_{E}$ is a closed subset of $\left(\mathcal{D}_{E}\right)^{\omega}$.
b) $\mathcal{A}_{E}$ is a subset of $\left(\mathcal{D}_{E}\right)^{\omega}$ which contains all the $\mathcal{D}_{E}$-block sequences.
c) For every $\left(y_{n}\right)_{n} \in \mathcal{A}_{E}$ and every $\mathcal{D}_{E}$-block subspace $X=\left[x_{n}\right]_{n}$ we have that if $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$, then

$$
\left(u_{n}\right)_{n} \in \mathcal{A}_{E} \Longleftrightarrow\left(u_{n}\right)_{n} *_{X}\left(y_{n}\right)_{n} \in \mathcal{A}_{E} .
$$

d) Let $\left(y_{n}\right)_{n}$ be a $\mathcal{D}_{E}$-block sequence and $Y=\left[y_{n}\right]_{n}$. For every $\left(u_{n}\right)_{n} \in \mathcal{A}_{E}$ and $k \in \mathbb{N}$, there is $\left(v_{n}\right)_{n} \in Y^{\omega}$ such that $\left(u_{0}, \ldots, u_{k}, v_{0}, v_{1}, \ldots\right) \in \mathcal{A}_{E}$.

Remark 5.1.25. Notice that an admissible set depends on the set of blocks that has been settled for $E$.

Notation 5.1.26. Let $\mathcal{D}_{E}$ be a set of blocks for $E, \mathcal{A}_{E}$ an admissible set for $E$ and $X$ be $a$ $\mathcal{D}_{E}$-block subspace.
(i) We denote by $\mathcal{A}_{X}$ the intersection $\mathcal{A}_{E} \cap X^{\omega}$.
(ii) We denote by $\left[\mathcal{A}_{X}\right]$ the set of initial parts of $\mathcal{A}_{X}$, that is:

$$
\left[\mathcal{A}_{X}\right]:=\bigcup_{n \in \mathbb{N}}\left\{\left(u_{0}, u_{1}, \ldots, u_{n}\right) \in\left(\mathcal{D}_{X}\right)^{n+1}: \exists\left(w_{i}\right)_{i} \in \mathcal{A}_{X} \text { s.t. } w_{i}=u_{i}, \text { for } 0 \leq i \leq n\right\}
$$

Remark 5.1.27. Since $\left(\mathcal{D}_{E}\right)^{i}$ is a discrete topological space, the set $\left[\mathcal{A}_{E}\right] \cap\left(\mathcal{D}_{E}\right)^{i}$ is a clopen subset of $\left(\mathcal{D}_{E}\right)^{i}$, for every $i \geq 1$.

Remark 5.1.28. If $X$ and $Y$ are $\mathcal{D}_{E}$-block subspaces such that $Y \subseteq X$, then $\mathcal{A}_{Y} \subseteq \mathcal{A}_{X}$.
Before we show examples of admissible sets and to finish giving the new definitions, we shall introduce a technical condition that relates the admissible set and the set of blocks. This condition is used only in Chapter 6, it is not necessary for the results obtained in this Chapter.

Definition 5.1.29. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ be an admissible set for $E$. We say that the pair $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$ is an admissible system of blocks for $E$ if, and only if, $\mathcal{D}_{E}$ and $\mathcal{A}_{E}$ satisfy the following relation: For all $\mathcal{D}_{E}$-block subspace $X$ of $E$, for all sequence $\left(\delta_{n}\right)_{n}$ with $0<\delta_{n}<1$, and $K \geq 1$, there is a collection $\left(A_{n}\right)_{n}$ of non-empty subsets of $\mathcal{D}_{X}$ such that
a) For each $n$ and for each $d \in[\mathbb{N}]^{<\infty}$ such that there is $w \in \mathcal{D}_{X}$ with $\operatorname{supp}_{X}(w)=d$, we have that there are finitely many vectors $u \in A_{n}$ such that $\operatorname{supp}_{X}(u)=d$.
b) For each sequence $\left(w_{i}\right)_{i} \in \mathcal{A}_{X}$ satisfying $1 / K \leq\left\|w_{i}\right\| \leq K$, for every $i$, there is $\left(u_{i}\right)_{i} \in \mathcal{A}_{X}$ such that for each $n$ we have
b.1) $u_{n} \in A_{n}$,
b.2) $\operatorname{supp}_{X}\left(u_{n}\right) \subseteq \operatorname{supp}_{X}\left(w_{n}\right)$,
b.3) $\left\|w_{n}-u_{n}\right\|<\delta_{n}$.

Remark 5.1.30. We know that closed balls of finite dimensional subspaces of a Banach space $E$ are totally bounded using balls centered on vectors of $E$. Roughly speaking, this condition allows us to choose the centers of such balls as $\mathcal{D}_{E}$-vectors.

### 5.2 Properties of admissible sets

In this section we shall describe how to pass from an admissible set for a Banach space $E$ to an admissible set for a $\mathcal{D}_{E}$-block subspace. Also, we present and detail some basic properties that a set of blocks and an admissible set satisfy.

Proposition 5.2.1. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of blocks and $\mathcal{A}_{E}$ be an admissible set for $E$, then the following are equivalent:
(i) For every $\left(y_{n}\right)_{n} \in \mathcal{A}_{E}$ and every $\mathcal{D}_{E}$-block subspace $X=\left[x_{n}\right]_{n}$ we have that if $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$, then

$$
\left(u_{n}\right)_{n} \in \mathcal{A}_{E} \Longleftrightarrow\left(u_{n}\right)_{n} *_{X}\left(y_{n}\right)_{n} \in \mathcal{A}_{E}
$$

(ii) For every $\left(y_{n}\right)_{n}$ and $\left(z_{n}\right)_{n}$ in $\mathcal{A}_{E}$ we have that if $\left(w_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$ then

$$
\left(w_{n}\right)_{n} *_{E}\left(y_{n}\right)_{n} \in \mathcal{A}_{E} \Longleftrightarrow\left(w_{n}\right)_{n} *_{E}\left(z_{n}\right)_{n} \in \mathcal{A}_{E} .
$$

(iii) For every $\left(y_{n}\right)_{n}$ and $\left(z_{n}\right)_{n}$ in $\mathcal{A}_{E}$ and every $\mathcal{D}_{E}$-block subspace $X=\left[x_{n}\right]_{n}$ we have that if $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$ then

$$
\left(u_{n}\right)_{n} *_{X}\left(y_{n}\right)_{n} \in \mathcal{A}_{E} \Longleftrightarrow\left(u_{n}\right)_{n} *_{X}\left(z_{n}\right)_{n} \in \mathcal{A}_{E}
$$

Proof. Suppose $E, \mathcal{D}_{E}$ and $\mathcal{A}_{E}$ as in the hypothesis. Notice that iii) implies $\left.i i\right)$.
ii) $\Rightarrow$ i) Let $\left(y_{n}\right)_{n}$ be in $\mathcal{A}_{E}$ and $X=\left[x_{n}\right]_{n}$ be a $\mathcal{D}_{E}$-block subspace. Take $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$. Suppose that for each $n \in \mathbb{N}$

$$
u_{n}=\sum_{i \in \operatorname{supp}_{X}\left(u_{n}\right)} \lambda_{i}^{n} x_{i} .
$$

For each $n \in \mathbb{N}$ set

$$
v_{n}=\sum_{i \in s u p p_{X}\left(u_{n}\right)} \lambda_{i}^{n} e_{i} .
$$

By condition $e$ ) of Definition 5.1.7, each $v_{n} \in \mathcal{D}_{E}$. Notice that $\left(u_{n}\right)_{n}=\left(v_{n}\right)_{n} *_{E}\left(x_{n}\right)_{n}$. Also, by the definition of the operation $*_{X}$ we have that $\left(u_{n}\right)_{n} *_{X}\left(y_{n}\right)_{n}=\left(v_{n}\right)_{n} *_{E}\left(y_{n}\right)_{n}$, and by condition $d$ ) in Definition 5.1.7 the sequence $\left(u_{n}\right)_{n} *_{X}\left(y_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$. So,

$$
\begin{aligned}
\left(u_{n}\right)_{n} *_{X}\left(y_{n}\right)_{n} \in \mathcal{A}_{E} & \Longleftrightarrow\left(v_{n}\right)_{n} *_{E}\left(y_{n}\right)_{n} \in \mathcal{A}_{E} \\
& \Longleftrightarrow\left(v_{n}\right)_{n} *_{E}\left(x_{n}\right)_{n} \in \mathcal{A}_{E} \\
& \Longleftrightarrow\left(u_{n}\right)_{n} \in \mathcal{A}_{E} .
\end{aligned}
$$

i) $\Rightarrow$ iii) Let $\left(y_{n}\right)_{n}$ and $\left(z_{n}\right)_{n}$ be in $\mathcal{A}_{E}$ and $X=\left[x_{n}\right]_{n}$ be a $\mathcal{D}_{E}$-block subspace. Take $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$, then

$$
\begin{aligned}
\left(u_{n}\right)_{n} *_{X}\left(y_{n}\right)_{n} \in \mathcal{A}_{E} & \Longleftrightarrow\left(u_{n}\right)_{n} \in \mathcal{A}_{E} \\
& \Longleftrightarrow\left(u_{n}\right)_{n} *_{X}\left(z_{n}\right)_{n} \in \mathcal{A}_{E}
\end{aligned}
$$

Remark 5.2.2. Items ii) or iii) in Proposition 5.2.1 are both equivalent to the Condition c) in Definition 5.1.24.

Proposition 5.2.3. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of blocks and $\mathcal{A}_{E}$ be an admissible set for $E$. If $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace, then we have
(i) $\mathcal{D}_{X} \subseteq \mathbb{D}_{X}$.
(ii) $\left\{x_{n}: n \in \mathbb{N}\right\} \subseteq \mathcal{D}_{X}$.
(iii) If $u \in \mathcal{D}_{X}$, then $\frac{u}{\|u\|} \in \mathcal{D}_{X}$.
(iv) For every $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$ and $\left(v_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$, we have $\left(u_{n}\right)_{n} *_{X}\left(v_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$. In particular, if $\left(v_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$, then $\left(u_{n}\right)_{n} *_{X}\left(v_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$.
(v) Let $\left(y_{i}\right)_{i=0}^{n} \in\left(\mathcal{D}_{X}\right)^{n+1}$ with $y_{i}<y_{i+1}$ for every $0 \leq i \leq n$. If $u \in \mathcal{D}_{X}$ is such that

$$
u=\sum_{i=0}^{n} \lambda_{i} y_{i},
$$

then

$$
v=\sum_{i=0}^{n} \lambda_{i} x_{i} \in \mathcal{D}_{X} .
$$

Proof. Let $X=\left[x_{n}\right]_{n}$ as in the hypothesis. For items (i), (ii) and (iii) we only use that $\mathcal{D}_{X}=\mathcal{D}_{E} \cap X$. Let us prove item (iv). Let $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$ and $\left(v_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$. If for each $n$
we have

$$
u_{n}=\sum_{i \in s u p p_{X}\left(u_{n}\right)} \lambda_{i}^{n} x_{i},
$$

then from condition $e$ ) in Definition 5.1.7, for each $n \in \mathbb{N}$ the vector

$$
w_{n}=\sum_{i \in \text { supp }_{X}\left(u_{n}\right)} \lambda_{i}^{n} e_{i}
$$

belongs to $\mathcal{D}_{E}$. By condition d) in Definition 5.1.7, $\left(w_{n}\right)_{n} *_{E}\left(v_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$. Since $\left(u_{n}\right)_{n} *_{X}\left(v_{n}\right)_{n}=\left(w_{n}\right)_{n} *_{E}\left(v_{n}\right)_{n}$, we have $\left(u_{n}\right)_{n} *_{X}\left(v_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$. If $\left(v_{n}\right)_{n} \in X^{\omega}$, then we also have $\left(u_{n}\right)_{n} *_{X}\left(v_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$

Now, let us prove $(v)$. Let $\left(y_{i}\right)_{i=0}^{n} \in\left(\mathcal{D}_{X}\right)^{n+1}$ with $y_{i}<y_{i+1}$ for every $0 \leq i \leq n$, and $Y=\left[y_{i}\right]_{i \leq n}$. Let

$$
u=\sum_{i=0}^{n} \lambda_{i} y_{i} \in \mathcal{D}_{X}=\mathcal{D}_{E} \cap X
$$

We want to prove that $v=\sum_{i=0}^{n} \lambda_{i} x_{i} \in \mathcal{D}_{X}$.
Take $w=\sum_{i=0}^{n} \lambda_{i} e_{i}$. For condition (e) in Definition 5.1.7, $w \in \mathcal{D}_{E}$. Consider the constant sequence $\left(w_{i}\right)_{i} \in\left(\mathcal{D}_{E}\right)^{\omega}$ where each $w_{i}=w$.

Using $d$ ) in Definition 5.1.7, we obtain

$$
\left(v_{i}\right)_{i}:=\left(w_{i}\right)_{i} *_{E}\left(x_{i}\right)_{i} \in\left(\mathcal{D}_{E}\right)^{\omega} .
$$

Notice that for each $i$ we have

$$
v_{i}=\sum_{j=0}^{n} \lambda_{j} x_{j}=v \in \mathcal{D}_{E} \cap X=\mathcal{D}_{X}
$$

Proposition 5.2.4. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of blocks and $\mathcal{A}_{E}$ be an admissible set for $E$. Let $X=\left[x_{n}\right]_{n}$ be a $\mathcal{D}_{E}$-block subspace. The set $\mathcal{A}_{X}$ satisfies the following properties:
(i) $\mathcal{A}_{X}$ is a closed subset of $\left(\mathcal{D}_{X}\right)^{\omega}$.
(ii) Any block basis $\left(y_{n}\right)_{n}$ in $\left(\mathcal{D}_{X}\right)^{\omega}$ belongs to $\mathcal{A}_{X}$.
(iii) For every $\left(v_{n}\right)_{n} \in \mathcal{A}_{X}$ and every $\mathcal{D}_{X}$-block subspace $Y=\left[y_{n}\right]_{n}$ we have that if $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{Y}\right)^{\omega}$, then

$$
\left(u_{n}\right)_{n} \in \mathcal{A}_{X} \Longleftrightarrow\left(u_{n}\right)_{n} *_{Y}\left(v_{n}\right)_{n} \in \mathcal{A}_{X} .
$$

(iv) Let $Y=\left[y_{n}\right]_{n}$ be a $\mathcal{D}_{X}$-block subspace. For every $\left(u_{n}\right)_{n} \in \mathcal{A}_{X}$ and $k \in \mathbb{N}$, there is $\left(v_{n}\right)_{n} \in Y^{\omega}$ such that $\left(u_{0}, \ldots, u_{k}, v_{0}, v_{1}, \ldots\right) \in \mathcal{A}_{X}$.

Proof. Let $E, \mathcal{D}_{E}$ and $\mathcal{A}_{E}$ be as in the hypothesis. Let $X=\left[x_{n}\right]_{n}$ be a $\mathcal{D}_{E}$-block subspace of $E$.
(i) Recall that $\mathcal{A}_{X}=\mathcal{A}_{E} \cap X^{\omega}$ and $\left(\mathcal{D}_{X}\right)^{\omega}=\left(\mathcal{D}_{E}\right)^{\omega} \cap X^{\omega}$. Since $\mathcal{A}$ is admissible for $E$, we have that $\mathcal{A}_{E}$ is closed in $\left(\mathcal{D}_{E}\right)^{\omega}$, then $\mathcal{A}_{X}=\mathcal{A}_{E} \cap X^{\omega}$ is closed in $\mathcal{D}_{X}{ }^{\omega}$.
(ii) Clearly, any block basis in $\left(\mathcal{D}_{X}\right)^{\omega}$ is also a block basis in $\left(\mathcal{D}_{E}\right)^{\omega}$, so it belongs to $\mathcal{A}_{E}$ and also to $X^{\omega}$, therefore it belongs to $\mathcal{A}_{X}$.
(iii) Let $\left(v_{n}\right)_{n} \in \mathcal{A}_{X}$ and $Y=\left[y_{n}\right]_{n}$ be a $\mathcal{D}_{X}$-block subspace (so, it is also a $\mathcal{D}_{E}$-block subspace). Let $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{Y}\right)^{\omega}$, then using that $\mathcal{A}_{E}$ is an admissible set for $E$ we obtain

$$
\begin{aligned}
\left(u_{n}\right)_{n} \in \mathcal{A}_{X} & \Longleftrightarrow\left(u_{n}\right)_{n} \in \mathcal{A}_{E} \cap X^{\omega} \\
& \Longleftrightarrow\left(u_{n}\right)_{n} *_{Y}\left(v_{n}\right)_{n} \in \mathcal{A}_{E} \\
& \Longleftrightarrow\left(u_{n}\right)_{n} *_{Y}\left(v_{n}\right)_{n} \in \mathcal{A}_{X}
\end{aligned}
$$

(iv) Let $Y=\left[y_{n}\right]_{n}$ be a $\mathcal{D}_{X}$-block subspace. Let $\left(u_{n}\right)_{n} \in \mathcal{A}_{X}$ and $k \in \mathbb{N}$. Since $Y$ is also a $\mathcal{D}_{E}$-block subspace, there is $\left(v_{n}\right)_{n} \in Y^{\omega}$ such that $\left(u_{0}, \ldots, u_{k}, v_{0}, v_{1}, \ldots\right) \in \mathcal{A}_{E}$. Since $Y \subseteq X$, we have that $\left(u_{0}, \ldots, u_{k}, v_{0}, v_{1}, \ldots\right) \in \mathcal{A}_{X}$.

Remark 5.2.5. Notice that if $X$ is a $\mathcal{D}_{E}$-block subspace, then using ii) in Proposition 5.2.4 we conclude that $\left[\mathcal{A}_{X}\right]$ is infinite.

Remark 5.2.6. If $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$ is an admissible system of blocks for $E$ and $X$ is a $\mathcal{D}_{E}$-block subspace, then as a consequence of propositions 5.2 .3 and 5.2.4, the pair $\left(\mathcal{D}_{X}, \mathcal{A}_{X}\right)$ can be thought as an "admissible subsystem of blocks" relative to $X$. The "relativization" of the condition given in Definition 5.1.29 to $X$ is clearly true: For all $\mathcal{D}_{X}$-block subspace $Y$ of $X$, for all sequence $\left(\delta_{n}\right)_{n}$ with $0<\delta_{n}<1$, and $K \geq 1$, there is a collection $\left(A_{n}\right)_{n}$ of non-empty subsets of $\mathcal{D}_{Y}$ such that
a) For each $n$ and for each $d \in[\mathbb{N}]^{<\infty}$ such that there is $w \in \mathcal{D}_{Y}$ with $\operatorname{supp}_{Y}(w)=d$, we have that there are finitely many vectors $u \in A_{n}$ such that $\operatorname{supp}_{Y}(u)=d$.
b) For all sequence $\left(w_{i}\right)_{i} \in \mathcal{A}_{Y}$ satisfying $1 / K \leq \min _{i}\left\|w_{i}\right\| \leq \sup _{i}\left\|w_{i}\right\| \leq K$, there is $\left(u_{i}\right)_{i} \in \mathcal{A}_{Y}$ such that for each $n$ :
b.1) $u_{n} \in A_{n}$,
b.2) $\operatorname{supp}_{Y}\left(u_{n}\right) \subseteq \operatorname{supp}_{Y}\left(w_{n}\right)$,
b.3) $\left\|w_{n}-u_{n}\right\|<\delta_{n}$.

Proposition 5.2.7. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ be an admissible set for $E$. The following statements are true:
(i) Let $X$ be a $\mathcal{D}_{E}$-block subspace. If $\left(u_{i}\right)_{i} \in\left(\mathcal{D}_{X}\right)^{\omega}$ satisfies that for every $n \in \mathbb{N}$ the finite sequence $\left(u_{i}\right)_{i=0}^{n} \in\left[\mathcal{A}_{E}\right]$, then $\left(u_{i}\right)_{i} \in \mathcal{A}_{X}$.
(ii) If $X=\left[x_{n}\right]_{n}$ and $Y=\left[y_{n}\right]_{n}$ are $\mathcal{D}_{E}$-block subspaces such that $\left(x_{n}\right)_{n} \sim\left(y_{n}\right)_{n}$, and $T$ is the map such that $\forall n \in \mathbb{N}\left(T\left(x_{n}\right)=y_{n}\right)$, then

$$
T\left(\mathcal{A}_{X}\right)=\mathcal{A}_{Y}
$$

(iii) If $X$ is a $\mathcal{D}_{E}$-block subspace, then

$$
\left[\mathcal{A}_{X}\right]=\left[\mathcal{A}_{E}\right] \bigcap \bigcup_{i \geq 1} X^{i}
$$

Proof. (i) Let $X$ and $u=\left(u_{i}\right)_{i} \in\left(\mathcal{D}_{X}\right)^{\omega}$ be as in the hypothesis. For each $n \in \mathbb{N}$ let $v^{n}=\left(v_{i}^{n}\right)_{i} \in \mathcal{A}_{E}$ such that $u_{i}=v_{i}^{n}$ for every $0 \leq i \leq n$. Without loss of generality, we can suppose each $v^{n} \in \mathcal{A}_{X}$ (using $d$ ) in Definition 5.1.24 we can find a sequence in $\mathcal{A}_{X}$ which coincide with $v^{n}$ in the first $n$ coordinates). Thus, $v_{j}^{n}=u_{j}$, for every $n \geq j$. Which means that for each $j \in \mathbb{N}$ we have $\left(v_{j}^{n}\right) \xrightarrow[n \rightarrow \infty]{\longrightarrow} u_{j}$ in $\mathcal{D}_{X}$. Therefore, $v^{n} \xrightarrow[n \rightarrow \infty]{\longrightarrow} u$ in $\left(\mathcal{D}_{X}\right)^{\omega}$. Using $(i)$ of the Proposition 5.2.4, u $\underset{\in \mathcal{A}_{X}}{n \rightarrow \infty}$.
(ii) Let $X=\left[x_{n}\right]_{n}$ and $Y=\left[y_{n}\right]_{n}$ be $\mathcal{D}_{E}$-block subspaces of $E$, and let $T: X \rightarrow Y$ be as in the hypothesis. Notice that by $b$ ) in Definition 5.1.24, $\left(x_{n}\right)_{n},\left(y_{n}\right)_{n},\left(e_{n}\right)_{n} \in \mathcal{A}_{E}$.

Let $\left(u_{n}\right)_{n} \in \mathcal{A}_{X}$ with

$$
u_{n}=\sum_{i \in s u p p_{X}\left(u_{n}\right)} \lambda_{i}^{n} x_{i},
$$

for each $n \in \mathbb{N}$. We want to show that $\left(T\left(u_{n}\right)\right)_{n} \in \mathcal{A}_{Y}$. Notice that $\left(T\left(u_{n}\right)\right)_{n}=$ $\left(u_{n}\right)_{n} *_{X}\left(y_{n}\right)_{n}$, so by $c$ ) in Definition 5.1.24, $\left(T\left(u_{n}\right)\right)_{n} \in \mathcal{A}_{E} \cap Y^{\omega}=\mathcal{A}_{Y}$.

On the other hand, let $\left(v_{n}\right)_{n} \in \mathcal{A}_{Y}$ with

$$
v_{n}=\sum_{i \in s u p p_{Y}\left(v_{n}\right)} \alpha_{i}^{n} y_{i},
$$

for every $n \in \mathbb{N}$. Set for each $n \in \mathbb{N}$

$$
u_{n}=\sum_{i \in s u p p_{Y}\left(v_{n}\right)} \alpha_{i}^{n} x_{i} .
$$

Clearly, $T\left(u_{n}\right)=v_{n}$ for every $n$ and $\left(u_{n}\right)_{n}=\left(v_{n}\right)_{n} *_{Y}\left(x_{n}\right)_{n}$. By $\left.c\right)$ of Definition 5.1.24 $\left(u_{n}\right)_{n} \in \mathcal{A}_{E} \cap X^{\omega}=\mathcal{A}_{X}$.
(iii) Let $X=\left[x_{n}\right]_{n}$ be a $\mathcal{D}_{E}$-block subspace. Since $\mathcal{A}_{E} \cap X^{\omega}=\mathcal{A}_{X}$, it follows that

$$
\left[\mathcal{A}_{X}\right] \subseteq\left[\mathcal{A}_{E}\right] \bigcap \bigcup_{i \geq 1} X^{i}
$$

Suppose that $\left(u_{i}\right)_{i=0}^{n} \in\left[\mathcal{A}_{E}\right] \cap X^{n+1}$, for some $n \in \mathbb{N}$. Using $d$ ) in Definition 5.1.24, there is $\left(u_{i}\right)_{i=n+1}^{\infty} \in X^{\omega}$, such that $u=\left(u_{0}, \ldots, u_{n}, u_{n+1}, \ldots\right) \in \mathcal{A}_{X}$. Then, $\left(u_{i}\right)_{i=0}^{n} \in\left[\mathcal{A}_{X}\right]$.

### 5.3 Admissible families

In this section we shall study admissible sets which are determined by families of sequences of sets in FIN, which we have called admissible families, and give some examples.

Notation 5.3.1. (i) Let us denote the set of nonempty finite sets of $\mathbb{N}$ by FIN, that is FIN $:=[\mathbb{N}]^{<\infty} \backslash\{\emptyset\}$.
(ii) We denote by $\mathrm{FIN}^{\omega}$ the set of infinite sequences of non-empty finite subsets of $\mathbb{N}$.

Remark 5.3.2. We shall consider FIN ${ }^{\omega}$ as a topological subspace of $\left(2^{\omega}\right)^{\omega}$, where $\left(2^{\omega}\right)^{\omega}$ is endowed with the product topology which results from considering $2^{\omega}$ as the Cantor space with its topology (see Section 3.1 of Chapter 3).

Notation 5.3.3. (i) We denote by $b b(\mathbb{N})$ the set of sequences of successive non-empty finite subsets of $\mathbb{N}$, that is

$$
b b(\mathbb{N})=\left\{\left(U_{i}\right)_{i} \in \operatorname{FIN}^{\omega}: \forall i \in \mathbb{N}\left(U_{i}<U_{i+1}\right)\right\} .
$$

(ii) We denote by $d b(\mathbb{N})$ the set of sequences of non-empty finite subsets of $\mathbb{N}$ whose elements are mutually disjoint:

$$
d b(\mathbb{N})=\left\{\left(U_{i}\right)_{i} \in \operatorname{FIN}^{\omega}: \forall i \neq j\left(U_{i} \cap U_{j}=\emptyset\right)\right\}
$$

Definition 5.3.4. We define the operation $\circledast: \mathbb{P}(\omega)^{\omega} \times \mathbb{P}(\omega)^{\omega} \rightarrow \mathbb{P}(\omega)^{\omega}$, as follows: given $U=\left(U_{i}\right)_{i}$ and $V=\left(V_{i}\right)_{i}$ in $\mathbb{P}(\omega)^{\omega}$, we define $U \circledast V=\left(W_{i}\right)_{i}$ as $W_{i}=\cup_{j \in U_{i}} V_{j}$, for every $i \in \mathbb{N}$.

Remark 5.3.5. Observe that:
(i) $\mathrm{FIN}^{\omega}$ is closed under the operation $\circledast$, that is, if $U, V \in \mathrm{FIN}^{\omega}$, then $U \circledast V \in \mathrm{FIN}^{\omega}$.
(ii) If $U=\left(U_{i}\right)_{i}$ and $V=\left(V_{i}\right)_{i}$ in $\mathbb{P}(\omega)^{\omega}$ and $U \circledast V=\left(W_{i}\right)_{i}$, then

$$
\bigcup_{i \in \mathbb{N}} W_{i} \subseteq \bigcup_{i \in \mathbb{N}} V_{i} .
$$

(iii) The sets $b b(\mathbb{N})$ and $d b(\mathbb{N})$ are closed under the operation $\circledast$.
(iv) Note that $e:=(\{i\})_{i}$ is a neutral element for operation $\circledast$, that is, if $U \in \mathbb{P}(\omega)^{\omega}$, then $U \circledast e=e \circledast U=U$.

Proposition 5.3.6. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\mathcal{D}_{E}$ be a set of blocks for $E$. Let $X$ be a $\mathcal{D}_{E}$-block subspace of $E$. Suppose $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$ and $\left(v_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$. If

$$
\left(w_{n}\right)_{n}=\left(u_{n}\right)_{n} *_{X}\left(v_{n}\right)_{n}
$$

then

$$
\begin{equation*}
\left(\operatorname{supp}_{E}\left(w_{n}\right)\right)_{n}=\left(\operatorname{supp}_{X}\left(u_{n}\right)\right)_{n} \circledast\left(\operatorname{supp}_{E}\left(v_{n}\right)\right)_{n} . \tag{5.3}
\end{equation*}
$$

Also, if $\left(v_{n}\right)_{n}$ is a basic sequence, then for each $n$

$$
\begin{equation*}
\operatorname{supp}_{\left[v_{i}\right]_{i}}\left(w_{n}\right)=\operatorname{supp}_{X}\left(u_{n}\right) \tag{5.4}
\end{equation*}
$$

Proof. This follows directly from the definition of operations $*_{X}$ and $\circledast$.
Definition 5.3.7. We say that a non-empty subset $\mathfrak{B} \subseteq \mathrm{FIN}^{\omega}$ is an admissible family if, and only if, the following conditions are satisfied:
a) $\mathfrak{B}$ is a closed subset of $\mathrm{FIN}^{\omega}$.
b) $b b(\mathbb{N}) \subseteq \mathfrak{B}$.
c) For every $\left(U_{i}\right)_{i},\left(V_{i}\right)_{i} \in \mathfrak{B}$ and every $\left(W_{i}\right)_{i} \in \operatorname{FIN}^{\omega}$, we have

$$
\begin{equation*}
\left(W_{i}\right)_{i} \circledast\left(U_{i}\right)_{i} \in \mathfrak{B} \Longleftrightarrow\left(W_{i}\right)_{i} \circledast\left(V_{i}\right)_{i} \in \mathfrak{B} . \tag{5.5}
\end{equation*}
$$

d) For every $\left(U_{i}\right)_{i},\left(V_{i}\right)_{i} \in \mathfrak{B}$ and $n \in \mathbb{N}$, there is $\left(\left\{t_{i}\right\}\right)_{i}$ subsequence of $e$ such that

$$
\left(U_{0}, U_{1}, \ldots, U_{n}, W_{0}, W_{1}, \ldots\right) \in \mathfrak{B}
$$

where $\left(W_{i}\right)_{i}=\left(t_{i}\right)_{i} \circledast\left(V_{i}\right)_{i}$.
Remark 5.3.8. If $\mathfrak{B}$ is an admissible set, condition b) implies that the neutral element $e$ belongs to $\mathfrak{B}$.

Remark 5.3.9. It is easy to see that the condition c) in Definition 5.3.7 is equivalent to the following statement: For every $\left(V_{i}\right)_{i} \in \mathfrak{B}$ and every $\left(W_{i}\right)_{i} \in \mathrm{FIN}^{\omega}$, we have

$$
\begin{equation*}
\left(W_{i}\right)_{i} \in \mathfrak{B} \Longleftrightarrow\left(W_{i}\right)_{i} \circledast\left(V_{i}\right)_{i} \in \mathfrak{B} . \tag{5.6}
\end{equation*}
$$

Proposition 5.3.10. The sets $\mathrm{FIN}^{\omega}, b b(\mathbb{N})$ and $d b(\mathbb{N})$ are admissible families.

Proof. (i) It is clear that $\mathrm{FIN}^{\omega}$ is an admissible set.
(ii) Let $V=\left(V_{j}\right)_{j} \in \operatorname{FIN}^{\omega}$ be in the closure of $b b(\mathbb{N})$, and $U_{i}=\left(U_{j}^{i}\right)_{j} \in b b(\mathbb{N})$ such that $U_{i} \xrightarrow[i \rightarrow \infty]{ } V$. Then $U_{j}^{i} \xrightarrow[i \rightarrow \infty]{\longrightarrow} V_{j}$, for every $j \in \mathbb{N}$. We have to verify that for any $j$, $V_{j}<V_{j+1}$.

Suppose, on the contrary, that for some $j$ there exist $k \in V_{j+1}$ and $m \in V_{j}$ such that $k<m$. Then, there are $N>0$ and $M>0$ such that $\forall i \geq N\left(k \in U_{j+1}^{i}\right)$ and $\forall i \geq M\left(m \in U_{j}^{i}\right)$. Therefore, $U_{i}$ is not in $b b(\mathbb{N})$, for $i>\max (N, M)$. Thus, $b b(\mathbb{N})$ satisfies condition $a$ ) of Definition 5.3.7.

Condition b) in Definition 5.3.7 is immediate. Now, suppose that $\left(U_{i}\right)_{i},\left(V_{i}\right)_{i} \in b b(\mathbb{N})$ and consider $\left(W_{i}\right)_{i} \in$ FIN $^{\omega}$. We have

$$
\begin{aligned}
\left(W_{i}\right)_{i} \circledast\left(U_{i}\right)_{i} \in b b(\mathbb{N}) & \Longleftrightarrow\left(\bigcup_{i \in W_{n}} U_{i}\right)_{n} \in b b(\mathbb{N}) \\
& \Longleftrightarrow\left(W_{i}\right)_{i} \in b b(\mathbb{N}) \\
& \Longleftrightarrow\left(W_{i}\right)_{i} \circledast\left(V_{i}\right)_{i} \in b b(\mathbb{N}) .
\end{aligned}
$$

So, condition $c$ ) in Definition 5.3.7 is satisfied.
Let us verify condition $d)$. Let $\left(U_{i}\right)_{i}$ and $\left(V_{i}\right)_{i}$ be elements of $b b(\mathbb{N})$, and $n \in \mathbb{N}$. Then, there is $m \in \mathbb{N}$ such that

$$
\begin{equation*}
\forall i \geq m\left(\bigcup_{j=0}^{n} U_{j}<V_{i}\right) \tag{5.7}
\end{equation*}
$$

Let $\left(t_{i}\right)_{i}=(\{m+i\})_{i} \in b b(\mathbb{N})$ and $\left.\left(W_{i}\right)_{i}:=\left(t_{i}\right)\right)_{i} \circledast\left(V_{i}\right)_{i}=\left(V_{m+i}\right)_{i}$. By Equation (5.7), it is clear that

$$
\left(U_{0}, U_{1}, \ldots, U_{n}, W_{0}, W_{1}, \ldots\right) \in b b(\mathbb{N})
$$

(iii) Let $V=\left(V_{j}\right)_{j} \in \mathrm{FIN}^{\omega}$ be in the closure of $d b(\mathbb{N})$, and $U_{i}=\left(U_{j}^{i}\right)_{j} \in d b(\mathbb{N})$ such that $U_{i} \xrightarrow[i \rightarrow \infty]{ } V$. Then $U_{j}^{i} \xrightarrow[i \rightarrow \infty]{ } V_{j}$, for every $j \in \mathbb{N}$. We have to verify that $V_{i} \cap V_{j}$, for any $i \neq j$.

Suppose $k \in V_{m} \cap V_{l}$, for some $m \neq l$. Then, there are $N, M>0$ such that $k \in U_{m}^{i} \cap U_{l}^{i}$, for every $i>\max (N, M)$. Thus, $U_{i} \notin d b(\mathbb{N})$, for any $i>N, M$, which is a contradiction. Therefore, $d b(\mathbb{N})$ satisfies condition $a)$ of Definition 5.3.7.

For the case $d b(\mathbb{N})$ condition $b$ ) is immediate. Conditions $c$ ) and $d$ ) can be proven in exactly the same way that in the $b b(\mathbb{N})$ case.

Proposition 5.3.11. The set

$$
\operatorname{per}(\mathbb{N}):=\left\{\left(U_{i}\right)_{i} \in \operatorname{FIN}^{\omega}: \exists \pi \text { a permutation of } \mathbb{N} \text { s.t. } \forall i \in \mathbb{N}\left(U_{\pi(i)}<U_{\pi(i+1)}\right)\right\}
$$

is not an admissible family.

Proof. Consider the sequences

$$
\begin{aligned}
U & =(\{0,1\},\{2\},\{3\},\{4\}, \ldots) \in \operatorname{per}(\mathbb{N}) \\
V & =(\{0\},\{2\},\{1\},\{3\},\{4\}, \ldots) \in \operatorname{per}(\mathbb{N}) .
\end{aligned}
$$

Notice that $U=U \circledast e$ and $V$ belong to $\operatorname{per}(\mathbb{N})$, but $U \circledast V=(\{0,2\},\{1\},\{3\},\{4\}, \ldots)$ does not. Then, $\operatorname{per}(\mathbb{N})$ fails to satisfy condition $c$ ) in Definition 5.3.7.

Proposition 5.3.12. Let $\mathfrak{B}$ be an admissible family. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\mathcal{D}_{E}$ be a set of blocks for $E$. Define the set $\mathcal{A}$ as follows: $\left(u_{i}\right)_{i} \in\left(\mathcal{D}_{E}\right)^{\omega}$ belongs to $\mathcal{A}$ if, and only if, $\left(\operatorname{supp}_{E}\left(u_{i}\right)\right)_{i} \in \mathfrak{B}$. Then, $\mathcal{A}$ is an admissible set for $E$.

Proof. Suppose $\mathfrak{B}, E,\left(e_{n}\right)_{n}$ and $\mathcal{D}_{E}$ as in the hypothesis. Define

$$
\begin{equation*}
\mathcal{A}=\left\{\left(u_{i}\right)_{i} \in\left(\mathcal{D}_{E}\right)^{\omega}:\left(\operatorname{supp}_{E}\left(u_{i}\right)\right)_{i} \in \mathfrak{B}\right\} . \tag{5.8}
\end{equation*}
$$

Let us check each condition of Definition 5.1.24.
a) Suppose $v:=\left(v_{i}\right)_{i} \in \overline{\mathcal{A}} \subseteq\left(\mathcal{D}_{E}\right)^{\omega}$ and let $\left(u_{i}\right)_{i}$ be a sequence in $(\mathcal{A})^{\omega}$ which converges to $v$. Then, if for each $i, u_{i}=\left(u_{j}^{i}\right)_{j}$, then $u_{j}^{i} \xrightarrow[i \rightarrow \infty]{\longrightarrow} v_{j}$ in $\left(\mathcal{D}_{E}\right)^{\omega}$, for every $j \in \mathbb{N}$. Thus, for each $j \in \mathbb{N}$ there is $N_{j}>0$ such that $u_{j}^{i}=v_{j}\left(\right.$ in particular $\left.\operatorname{supp}_{E}\left(u_{j}^{i}\right)=\operatorname{supp}_{E}\left(v_{j}\right)\right)$, for every $i>N_{j}$. This means that for each $j \in \mathbb{N}$,

$$
\begin{equation*}
\operatorname{supp}_{E}\left(u_{j}^{i}\right) \underset{i \rightarrow \infty}{\longrightarrow} \operatorname{supp}_{E}\left(v_{j}\right) \quad \text { in FIN. } \tag{5.9}
\end{equation*}
$$

For each $i \in \mathbb{N}, u_{i} \in \mathcal{A} \Rightarrow U_{i}:=\left(\operatorname{supp}_{E}\left(u_{j}^{i}\right)\right)_{j} \in \mathfrak{B}$. Equation (5.9) shows that $\left(U_{i}\right)_{i}$ converges to $\left(\operatorname{supp}_{E}\left(v_{j}\right)\right)_{j} \in \operatorname{FIN}^{\omega}$. Since $\mathfrak{B}$ is closed in $\operatorname{FIN}^{\omega},\left(\operatorname{supp}_{E}\left(v_{j}\right)\right)_{j} \in \mathfrak{B}$. By the definition of $\mathcal{A}$, this means that $v \in \mathcal{A}$.
b) Let $\left(y_{n}\right)_{n}$ be a sequence of successive blocks, that is $\forall n \in \mathbb{N}\left(y_{n} \in \mathcal{D} \& y_{n}<y_{n+1}\right)$. Then, $\left(\operatorname{supp}_{E}\left(y_{i}\right)\right)_{i} \in b b(\mathbb{N})$. By item $\left.b\right)$ in Definition 5.3.7, $b b(\mathbb{N}) \subseteq \mathfrak{B}$, so $\left(y_{n}\right)_{n} \in \mathcal{A}$.
c) Let $\left(y_{n}\right)_{n} \in \mathcal{A}$ and $X=\left[x_{n}\right]_{n}$ be a $\mathcal{D}_{E}$-block subspace. Suppose $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$, where for each $n \in \mathbb{N}$,

$$
u_{n}=\sum_{i \in \text { supp }_{X}\left(u_{n}\right)} \lambda_{i}^{n} x_{i} .
$$

We want to see that

$$
\begin{equation*}
\left(u_{n}\right)_{n} \in \mathcal{A} \Longleftrightarrow\left(v_{n}\right)_{n}:=\left(u_{n}\right)_{n} *_{X}\left(y_{n}\right)_{n} \in \mathcal{A} \tag{5.10}
\end{equation*}
$$

Observe that $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$ and, due to (iv) in Proposition 5.2.3, we know that $\left(u_{n}\right)_{n} *_{X}\left(y_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$.

By Proposition 5.3.6, we know that

$$
\begin{equation*}
\left(\operatorname{supp}_{E}\left(v_{n}\right)\right)_{n}=\left(\operatorname{supp}_{X}\left(u_{n}\right)\right)_{n} \circledast\left(\operatorname{supp}_{E}\left(y_{n}\right)\right)_{n} . \tag{5.11}
\end{equation*}
$$

As a consequence of the last equation, the definition of $\mathcal{A}$ and condition $c$ ) of Definition 5.3.7, we obtain

$$
\begin{aligned}
\left(u_{n}\right)_{n} \in \mathcal{A} & \Longleftrightarrow\left(\operatorname{supp}_{E}\left(u_{n}\right)\right)_{n} \in \mathfrak{B} \\
& \Longleftrightarrow\left(\operatorname{supp}_{X}\left(u_{n}\right)\right)_{n} \circledast\left(\operatorname{supp}_{E}\left(x_{n}\right)\right)_{n} \in \mathfrak{B} \\
& \Longleftrightarrow\left(\operatorname{supp}_{x}\left(u_{n}\right)\right)_{n} \circledast\left(\operatorname{supp}_{E}\left(y_{n}\right)\right)_{n} \in \mathfrak{B} \\
& \Longleftrightarrow\left(\operatorname{supp}_{E}\left(v_{n}\right)\right)_{n} \in \mathfrak{B} \\
& \Longleftrightarrow\left(v_{n}\right)_{n} \in \mathcal{A}
\end{aligned}
$$

d) Let $\left(y_{n}\right)_{n}$ a $\mathcal{D}_{E}$-block sequence and $Y=\left[y_{n}\right]_{n}$. By using item $\left.b\right)$ we have $\left(\operatorname{supp}_{E}\left(y_{n}\right)\right)_{n} \in$ $\mathfrak{B}$. Let $\left(u_{i}\right)_{i} \in \mathcal{A}$, so $\left(\operatorname{supp}_{E}\left(u_{i}\right)\right)_{i} \in \mathfrak{B}$. By condition $d$ ) in Definition 5.3.7 there is $\left(\left\{a_{i}\right\}\right)_{i} \in b b(\mathbb{N})$ such that

$$
\begin{equation*}
\left(\operatorname{supp}_{E}\left(u_{0}\right), \operatorname{supp}_{E}\left(u_{1}\right), \ldots, \operatorname{supp}_{E}\left(u_{n}\right), B_{0}, B_{1}, \ldots\right) \in \mathfrak{B}, \tag{5.12}
\end{equation*}
$$

where $\left(B_{i}\right)_{i}=\left(\left\{a_{i}\right\}\right)_{i} \circledast\left(\operatorname{supp}_{E}\left(y_{i}\right)\right)_{i}$. For each $i \in \mathbb{N}$, let $z_{i}=y_{a_{i}}$. It is clear that $\left(z_{i}\right)_{i} \in \mathcal{D}_{Y}{ }^{\omega}$ and $\operatorname{supp}_{E}\left(z_{i}\right)=B_{i}$, for every $i \in \mathbb{N}$. Then, by Equation (5.12) we have

$$
\left(u_{0}, \ldots, u_{n}, z_{0}, z_{1}, \ldots\right) \in \mathcal{A} .
$$

Definition 5.3.13. Under the hypothesis of Proposition 5.3.12, we shall refer to the obtained set $\mathcal{A}$ as the admissible set for $E$ determined by the admissible family $\mathfrak{B}$.

Proposition 5.3.14. Let $\mathfrak{B}$ be an admissible family. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\mathcal{D}_{E}$ be a set of blocks for $E$. Let $X$ be a $\mathcal{D}_{E}$-block subspace of $E$. If $\mathcal{A}_{E}$ is the admissible set for $E$ determined by the admissible family $\mathfrak{B}$, then

$$
\begin{equation*}
\mathcal{A}_{X}=\left\{\left(u_{i}\right)_{i} \in\left(\mathcal{D}_{X}\right)^{\omega}:\left(\operatorname{supp}_{X}\left(u_{i}\right)\right)_{i} \in \mathfrak{B}\right\} . \tag{5.13}
\end{equation*}
$$

Proof. It is sufficient to prove that for every $\left(u_{n}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$,

$$
\begin{equation*}
\left(\operatorname{supp}_{E}\left(u_{n}\right)\right)_{n} \in \mathfrak{B} \Longleftrightarrow\left(\operatorname{supp}_{x}\left(u_{n}\right)\right)_{n} \in \mathfrak{B} . \tag{5.14}
\end{equation*}
$$

Recall that $\left(x_{n}\right)_{n} \in \mathcal{A}_{E}$. By Remark 5.3 .9 we know that

$$
\left(\operatorname{supp}_{X}\left(u_{n}\right)\right)_{n} \in \mathfrak{B} \Longleftrightarrow\left(\operatorname{supp}_{X}\left(u_{n}\right)\right)_{n} \circledast\left(\operatorname{supp}_{E}\left(x_{n}\right)\right)_{n} \in \mathfrak{B} .
$$

And by Proposition 5.3.6 we have

$$
\left(\operatorname{supp}_{E}\left(u_{n}\right)\right)_{n}=\left(\operatorname{supp}_{X}\left(u_{n}\right)\right)_{n} \circledast\left(\operatorname{supp}_{E}\left(x_{n}\right)\right)_{n}
$$

So, Equation (5.14) is true.

Notation 5.3.15. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\mathcal{D}_{E}$ a set of blocks for $E$. We denote by:
(i) $B S_{\mathcal{D}}(E)$ the set of not necessarily normalized $\mathcal{D}_{E}$-block sequences of $E$.
(ii) $D S_{\mathcal{D}}(E)$ the set of infinite sequences of pairwise disjointly supported $\mathcal{D}_{E}$-blocks.

Proposition 5.3.16. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\mathcal{D}_{E}$ a set of blocks for $E$. The following sets are admissible for $E$ :
(i) The set $\left(\mathcal{D}_{E}\right)^{\omega}$ of infinite sequences of $\mathcal{D}_{E}$-blocks.
(ii) The set $B S_{\mathcal{D}}(E)$.
(iii) The set $D S_{\mathcal{D}}(E)$.

Proof. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Consider $\mathcal{D}_{E}$ a set of blocks for $E$. The idea of this proof is to show that each of the three sets is admissible using Proposition 5.3.12, and they are determined by each of the admissible families given in Proposition 5.3.10.
(i) A sequence $\left(x_{n}\right)_{n}$ in $E^{\omega}$ is a sequence of $\mathcal{D}_{E}$-blocks if, and only if, $\left(x_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}$. So, for this case

$$
\begin{equation*}
\mathcal{A}_{E}=\left(\mathcal{D}_{E}\right)^{\omega} \tag{5.15}
\end{equation*}
$$

Also, notice that

$$
\left\{\left(u_{i}\right)_{i} \in\left(\mathcal{D}_{E}\right)^{\omega}:\left(\operatorname{supp}_{E}\left(u_{i}\right)\right)_{i} \in \operatorname{FIN}^{\omega}\right\}=\left(\mathcal{D}_{E}\right)^{\omega}=\mathcal{A}_{E}
$$

So, by Proposition 5.3.12, $\mathcal{A}$ is an admissible set for $E$.
(ii) For this case

$$
\begin{aligned}
\mathcal{A}_{E} & =\left\{\left(x_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}: \forall n\left(x_{n}<x_{n+1}\right)\right\} \\
& =\left\{\left(x_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}:\left(\operatorname{supp}_{E}\left(x_{n}\right)\right)_{n} \in b b(\mathbb{N})\right\} \\
& =B S_{\mathcal{D}}(E)
\end{aligned}
$$

So, by Proposition 5.3.12, $\mathcal{A}_{E}$ is an admissible set for $E$ determined by the admissible family $b b(\mathbb{N})$.
(iii) For this case we have

$$
\begin{aligned}
\mathcal{A}_{E} & =\left\{\left(x_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}: \operatorname{supp}_{E}\left(x_{i}\right) \cap \operatorname{supp}_{E}\left(x_{j}\right)=\emptyset, \text { whenever } i \neq j\right\} \\
& =\left\{\left(x_{n}\right)_{n} \in\left(\mathcal{D}_{E}\right)^{\omega}:\left(\operatorname{supp}_{E}\left(x_{n}\right)\right)_{n} \in d b(\mathbb{N})\right\} \\
& =D S_{\mathcal{D}}(E) .
\end{aligned}
$$

By Proposition 5.3.12, $\mathcal{A}$ is an admissible set for $E$ determined by the admissible family $d b(\mathbb{N})$.

### 5.3.1 Interpretations for the set of blocks

Depending on the set of blocks $\mathcal{D}_{E} \subseteq \mathbb{D}_{E}$ we have chosen for the Banach space $E$, it is possible to give different interpretations for the admissible set considered. In this section we shall explore various sets of blocks and analyze the admissible sets obtained in Proposition 5.3.16 in each context. In this section, let us fix $E$ being a Banach space with normalized basis $\left(e_{n}\right)_{n}$.

## Blocks as the non-zero F-linear combinations

We shall start this exposition with the biggest set of blocks possible. Consider the set of blocks $\mathbb{D}_{E}$, that is, the set which elements are all non-zero finitely supported $\mathbf{F}_{E}$-linear combinations of the basis $\left(e_{n}\right)_{n}$. This was the set considered by Pelczar in [42] and also by Ferenczi and Rosendal in [22].

In this context, a $\mathbb{D}_{E}$-block sequence is a block basis which elements are non-zero finitely supported $\mathbf{F}_{E}$-linear combinations and

$$
b b_{\mathbb{D}}(E)=\left\{\left(x_{n}\right)_{n} \in\left(\mathbb{D}_{E}\right)^{\omega}: \forall n \in \mathbb{N}\left(x_{n}<x_{n+1} \&\left\|x_{n}\right\|=1\right)\right\} .
$$

Remark 5.3.17. Without loss of generality we can suppose that any normalized finitely supported basic sequence $\left(y_{n}\right)_{n}$ in $E=\left[e_{n}\right]_{n}$, is equivalent to $\left(z_{n}\right)_{n} \in\left(\mathbb{D}_{E}\right)^{\omega}$ with supp ${ }_{E}\left(z_{n}\right)=$ $\operatorname{supp}_{E}\left(y_{n}\right)$, for every $n$. This is a consequence of the density of $\mathbb{D}_{E}$ in $E$ and the principle of small perturbations (presented as Theorem 2.2.11).

Proposition 5.3.18. Suppose that we are considering the set of blocks for $E$ as $\mathbb{D}_{E}$ and that $\mathcal{A}_{E}$ is an admissible set for $E$ determined by an admissible family. Then, the pair $\left(\mathbb{D}_{E}, \mathcal{A}_{E}\right)$ is an admissible system of blocks for $E$.

Proof. Let $X=\left[x_{n}\right]_{n}$ be a $\mathbb{D}_{E}$-block subspace, $\left(\delta_{n}\right)_{n}$ with $0<\delta_{n}<1$ and $K \geq 1$. We are going to construct for each $n \in \mathbb{N}$ sets $D_{n}$ of not necessarily normalized $\mathbb{D}_{X}$-blocks with the following properties:

1. For each $d \in[\mathbb{N}]^{<\infty}$, there are a finite number of vectors $u \in D_{n}$ such that $\operatorname{supp}_{X}(u)=d$.
2. If $w$ is a $\mathbb{D}_{X}$-block vector with norm in $\left[\frac{1}{K}, K\right]$, then there is some $u \in D_{n}$, with the same support in $X$ of $w$ such that $\|w-u\|<\delta_{n}$.

Before the proof of the existence of such sets $D_{n}$, let us show why this is sufficient: Let $\left(v_{i}\right)_{i} \in \mathcal{A}_{X}$ satisfying $\frac{1}{K} \leq\left\|v_{i}\right\| \leq K$, for every $i \in \mathbb{N}$. Since $\left(v_{i}\right)_{i} \in \mathcal{A}_{X}$ and $\mathcal{A}_{E}$ is the admissible set for $E$ determined by an admissible family $\mathfrak{B}$, it follows that

$$
\begin{equation*}
\left(\operatorname{supp}_{X}\left(v_{i}\right)\right)_{i} \in \mathfrak{B} . \tag{5.16}
\end{equation*}
$$

Using 2), for each $i$ there is $w_{i} \in D_{i}$ with $\left\|w_{i}-v_{i}\right\|<\delta_{i}$ and $\operatorname{supp}_{X}\left(w_{i}\right)=\operatorname{supp}_{X}\left(v_{i}\right)$, so by Equation (5.16) $\left(\operatorname{supp}_{X}\left(w_{i}\right)\right)_{i} \in \mathfrak{B}$, what means that $\left(w_{i}\right)_{i} \in \mathcal{A}_{X}$. Therefore, $\left(\mathbb{D}_{E}, \mathcal{A}_{E}\right)$ is an admissible system of blocks for $E$.

Let us prove that such sets $D_{n}$ exist: Set $n \in \mathbb{N}$. We proceed by induction: If $d \in[\mathbb{N}]^{1}$, then, since the closed K-ball of $\left[x_{i}\right]_{i \in d}$ is totally bounded and $\mathbb{D}_{E}$ is dense in $E$, it is possible to find a finite $U_{d}=\left\{u_{1}^{d}, \ldots u_{m(d)}^{d}\right\} \subset \overline{\mathbb{B}}_{K}\left(\left[x_{i}\right]_{i \in d}\right) \cap \mathbb{D}_{E}$ such that if $w \in\left[x_{i}\right]_{i \in d}$ and $\frac{1}{K} \leq\|w\| \leq K$, then there is some $j \leq m(d)$ with $\left\|w-u_{j}^{d}\right\|<\delta_{n}$.

Suppose we have found for every $d \in[\mathbb{N}]^{<m}$ such vectors $U_{d}=\left\{u_{1}^{d}, \ldots u_{m(d)}^{d}\right\} \subset \overline{\mathbb{B}}_{K}\left(\left[x_{i}\right]_{i \in d}\right) \cap$ $\mathbb{D}_{E}$ with the desired property. Let $d \in[\mathbb{N}]^{m}$, then as the closed K-ball of

$$
\left[x_{i}\right]_{i \in d} \backslash \bigcup_{d^{\prime} \subset d}\left[x_{i}\right]_{i \in d^{\prime}}
$$

is again totally bounded and $\mathbb{D}_{E}$ is dense in $E$, there is $U_{d}=\left\{u_{1}^{d}, \ldots u_{m(d)}^{d}\right\} \subset \overline{\mathbb{B}}\left(\left[x_{i}\right]_{i \in d}\right) \cap \mathbb{D}_{E}$ such that if $w \in\left[x_{i}\right]_{i \in d}, \frac{1}{K} \leq\|w\| \leq K$ and $\operatorname{supp}(w)=d$, then there is some $j \leq m(d)$ such that $\left\|w-u_{j}^{d}\right\|<\delta_{n}$. Finally, set

$$
D_{n}=\bigcup_{d \in[\mathbb{N}]<\infty} U_{d}
$$

Corollary 5.3.19. The pairs $\left(\mathbb{D}_{E},\left(\mathbb{D}_{E}\right)^{\omega}\right)$, $\left(\mathbb{D}_{E}, B S_{\mathbb{D}}(E)\right)$ and $\left(\mathbb{D}_{E}, D S_{\mathbb{D}}(E)\right)$ are admissible systems of blocks for $E$.

Proof. The result follows directly from Proposition 5.3.16 and Proposition 5.3.18.
As it was proved in Proposition 5.3.11, the family $\operatorname{per}(\mathbb{N})$ is not admissible for FIN ${ }^{\omega}$. So, Proposition 5.3.12 can not be used to determined whether is an admissible set for the Banach space $E$. In the next proposition we prove that the set of infinite sequences of blocks that are a permutation of a block basis of sequences is not admissible for $E$.

Proposition 5.3.20. The set

$$
\mathcal{P}_{E}:=\left\{\left(x_{n}\right)_{n} \in\left(\mathbb{D}_{E}\right)^{\omega}:\left(\operatorname{supp}_{E}\left(x_{n}\right)\right)_{n} \in \operatorname{per}(\mathbb{N})\right\}
$$

is not admissible for $E$.
Proof. Let

$$
z_{0}=e_{0}, z_{1}=e_{2}, z_{2}=e_{1}, \text { and } z_{i}=e_{i}, \text { for all } i \geq 3
$$

The sequences $\left(e_{n}\right)_{n}$ and $\left(z_{n}\right)_{n}$ belong to $\mathcal{P}_{E}$. Set now

$$
w_{0}=e_{0}+e_{1}, \text { and } w_{i}=e_{i+1}, \text { for all } i \geq 1
$$

Notice that $\left(w_{n}\right)_{n} *_{E}\left(e_{n}\right)_{n}=\left(w_{n}\right)_{n}$ so it is a permutation of a $\mathbb{D}_{E}$-block sequence (it is itself a $\mathbb{D}_{E}$-block sequence), but $\left(w_{n}\right)_{n} *_{E}\left(z_{n}\right)_{n}=\left(e_{0}+e_{2}, e_{1}, e_{3}, \ldots\right)$ is not. So, condition $\left.c\right)$ in Definition 5.1.24 is not satisfied.

## Blocks as the set of vectors of the basis

The smallest set of blocks we can consider is that where the blocks are exclusively the vectors of the basis:

$$
\mathcal{B}_{E}=\left\{e_{n}: n \in \mathbb{N}\right\}
$$

Notice that in this case all blocks are normalized. An element of $\left(\mathcal{B}_{E}\right)^{\omega}$ can be represented as $\left(e_{f(i)}\right)_{i}$, where $f: \mathbb{N} \rightarrow \mathbb{N}$ is a function.

In this context a $\mathcal{B}_{E}$-block sequence is a subsequence of the basis, and a sequence of disjointly supported blocks is a sequence of different elements of the the basis (not necessarily in increasing order). So, we can fix the following notation:

Notation 5.3.21. Let us denote by $b b_{\mathcal{B}}(E)$ the set of $\mathcal{B}_{E}$-block sequences

$$
\begin{aligned}
b b_{\mathcal{B}}(E) & :=\left\{\left(e_{n_{i}}\right)_{i}:\left(n_{i}\right)_{i} \text { is increasing }\right\} \\
& =\left\{\left(e_{f(n)}\right)_{i}: f \in \mathbb{N}^{\mathbb{N}} \text { is increasing }\right\} .
\end{aligned}
$$

We denote

$$
d b_{\mathcal{B}}(E):=\left\{\left(e_{f(n)}\right)_{i}: f \in \mathbb{N}^{\mathbb{N}} \text { is injective }\right\} .
$$

Proposition 5.3.22. Let $\mathfrak{B}$ be an admissible family. Let $\mathcal{A}_{E}$ be the admissible set for $E$ determined by $\mathfrak{B}$. We have that $\left(\mathcal{B}_{E}, \mathcal{A}_{E}\right)$ is an admissible system of blocks for $E$.

Proof. It follows directly from the fact that for each $n \in \mathbb{N}$ only one $\mathcal{B}_{E}$-block has support $\{n\}$. In this case, the conditions asked in Definition 5.1.29 are trivial. What we are saying is that for the case of embedding, minimality or tightness by sequences, it is not necessary to perturb the vectors along the proofs.

Remark 5.3.23. Let $\mathcal{D}_{E}$ be a set of blocks for the Banach space $E$ and $\mathcal{A}_{E}$ be an admissible set determined by an admissible family. Notice that the Proposition 5.3.22 is true in the case where for each $d \in[\mathbb{N}]^{<\infty}$ such that there is $w \in \mathcal{D}_{E}$ with $\operatorname{supp}_{E}(w)=d$, we have that the set $\left\{u \in \mathcal{D}_{E}: d=\operatorname{supp}_{E}(u)\right\}$ is finite. Under this hypothesis a pair $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$ is an admissible system of blocks for $E$.

Proposition 5.3.24. The following sets are admissible for $E$ :
(i) The set $b b_{\mathcal{B}}(E)$ of subsequences of $\left(e_{n}\right)_{n}$.
(ii) The set $d b_{\mathcal{B}}(E)$ of sequences of pairwise distinct elements of the basis $\left(e_{n}\right)_{n}$.

Proof. In each case, the considered sets are determined by an admissible family.
(i) The set of subsequences of $\left(e_{n}\right)_{n}$ is given by

$$
\begin{aligned}
b b_{\mathcal{B}}(E) & =\left\{\left(e_{n_{i}}\right)_{i}:\left(n_{i}\right)_{i} \text { is increasing }\right\} \\
& =\left\{\left(x_{n}\right)_{n} \in\left(\mathcal{B}_{E}\right)^{\omega}:\left(\operatorname{supp}_{E}\left(x_{n}\right)\right)_{n} \in b b(\mathbb{N})\right\} .
\end{aligned}
$$

By $i i)$ in Proposition 5.3.16, $b b_{\mathcal{B}}(E)$ is an admissible set for $E$.
(ii) The set of sequences of pairwise distinct elements of the basis $\left(e_{n}\right)_{n}$ is given by

$$
\begin{aligned}
d b_{\mathcal{B}}(E) & =\left\{\left(e_{f(n)}\right)_{n}: f \in \mathbb{N}^{\mathbb{N}} \text { is injective }\right\} \\
& =\left\{\left(x_{n}\right)_{n} \in\left(\mathcal{B}_{E}\right)^{\omega}:\left(\operatorname{supp}_{E}\left(x_{n}\right)\right)_{n} \in d b(\mathbb{N})\right\} .
\end{aligned}
$$

By iii) $^{\text {) in Proposition 5.3.16 }}, d b_{\mathcal{B}}(E)$ is an admissible set for $E$.

Remark 5.3.25. Notice that the set of sequences of elements of the basis $\left(e_{n}\right)_{n}$ is immediately an admissible set. In this case, the set of sequences of elements of the basis $\left(e_{n}\right)_{n}$ is given by

$$
\begin{aligned}
\mathcal{A}_{E} & :=\left\{\left(e_{f(n)}\right)_{n}: f \in \mathbb{N}^{\mathbb{N}}\right\} \\
& =\left\{\left(x_{n}\right)_{n} \in\left(\mathcal{B}_{E}\right)^{\omega}:\left(\operatorname{supp}_{E}\left(x_{n}\right)\right)_{n} \in \operatorname{FIN}^{\omega}\right\}
\end{aligned}
$$

By i) in Proposition 5.3.16, $\mathcal{A}_{E}$ is an admissible set for $E$.
Remark 5.3.26. Notice that a sequence $\left(x_{n}\right)_{n} \in \mathcal{D}_{E}^{\omega}$ with $x_{m} \neq x_{n}$ whenever $n \neq m$ is a permutation of a $\mathcal{D}_{E}$-block sequence and vice-versa. Thus, despite the set per $(\mathbb{N})$ not being admissible, it "determines" an admissible set for this set of blocks. This is in contrast to the result we have obtained in Proposition 5.3.20 for $\mathcal{D}_{E}=\mathbb{D}_{E}$.

Corollary 5.3.27. The pair $\left(\mathcal{B}_{E}, \mathcal{A}_{E}\right)$ is an admissible system of blocks for $E$ for $\mathcal{A}_{E}$ in any of the three cases given in Proposition 5.3.24.

Proof. This corollary is a consequence of propositions 5.3.24 and 5.3.22.

## Blocks as signed elements of the basis

We already saw that $\mathcal{B}_{E}^{ \pm}$is a set of blocks for $E$. An element $x \in \mathcal{B}_{E}^{ \pm}$is a vector $x=\varepsilon e_{k}$, for some $k \in \mathbb{N}$ and $\varepsilon \in\{-1,1\}$ ( $\varepsilon$ is a sign). In this context a $\mathcal{B}_{E}^{ \pm}$-block sequence is a "sign subsequence of the basis" (as we define in the following lines), i.e.

$$
b b_{\mathcal{B}^{ \pm}}(E):=\left\{\left(\varepsilon_{i} e_{n_{i}}\right)_{i}:\left(n_{i}\right)_{i} \in \mathbb{N}^{\omega} \text { is increasing and }\left(\varepsilon_{i}\right)_{i} \in\{-1,1\}^{\omega}\right\} .
$$

Definition 5.3.28. We say that $\left(x_{n}\right)_{n}$ is a signed subsequence of $\left(e_{n}\right)_{n}$ if and only if, $\left(x_{n}\right)_{n} \in b b_{\mathcal{B}^{ \pm}}(E)$.

We can also define the set of permutations of signed subsequences of the basis $\left(e_{n}\right)_{n}$ as follows

$$
d b_{\mathcal{B}^{ \pm}}(E):=\left\{\left(\varepsilon_{n} e_{f(n)}\right)_{n}: f \in \mathbb{N}^{\mathbb{N}} \text { is injective and }\left(\varepsilon_{i}\right)_{i} \in\{-1,1\}^{\omega}\right\} .
$$

Proposition 5.3.29. Let $\mathfrak{B}$ be an admissible family. Let $\mathcal{A}_{E}$ be the admissible set for $E$ determined by $\mathfrak{B}$. We have that $\left(\mathcal{B}_{E}^{ \pm}, \mathcal{A}_{E}\right)$ is an admissible system of blocks for $E$.

Proof. It follows directly from the fact that for each $n \in \mathbb{N}$ only two vectors $e_{n}$ and $-e_{n}$ in $\mathcal{B}_{E}$ have as support $\{n\}$. In this case, the conditions asked in Definition 5.1.29 are trivial. See also Remark 5.3.23.

Corollary 5.3.30. Consider the set of blocks $\mathcal{B}_{E}^{ \pm}$for $E$. We have that the following sets are admissible for $E$ :
(i) The set $b b_{\mathcal{B}^{ \pm}}(E)$ of signed subsequences of $\left(e_{n}\right)_{n}$.
(ii) The set $d b_{\mathcal{B}^{ \pm}}(E)$ of permutations of signed subsequences of $\left(e_{n}\right)_{n}$.

So, the pairs $\left(\mathcal{B}_{E}^{ \pm}, b b_{\mathcal{B}^{ \pm}}(E)\right)$ and $\left(\mathcal{B}_{E}^{ \pm}, d b_{\mathcal{B}^{ \pm}}(E)\right)$ are admissible systems of blocks for $E$.

Proof. (i) The set of signed subsequences of $\left(e_{n}\right)_{n}$ is given by $b b_{\mathcal{B}^{ \pm}}(E)$ and

$$
\begin{aligned}
b b_{\mathcal{B}^{ \pm}}(E) & =\left\{\left(\varepsilon_{n} e_{f(n)}\right)_{n}: f \in \mathbb{N}^{\mathbb{N}} \text { is increasing and }\left(\varepsilon_{i}\right)_{i} \in\{-1,1\}^{\omega}\right\} \\
& =\left\{\left(x_{n}\right)_{n} \in\left(\mathcal{B}_{E}^{ \pm}\right)^{\omega}:\left(\operatorname{supp}_{E}\left(x_{n}\right)\right)_{n} \in b b(\mathbb{N})\right\} .
\end{aligned}
$$

By Proposition 5.3.12, $b b_{\mathcal{B}^{ \pm}}(E)$ is an admissible set for $E$ determined by the admissible family $b b(\mathbb{N})$. By Proposition 5.3.29, $\left(\mathcal{B}_{E}^{ \pm}, b b_{\mathcal{B}^{ \pm}}(E)\right)$ is and admissible system of blocks for $E$.
(ii) The set of sequences of pairwise distinct elements of the basis $\left(e_{n}\right)_{n}$ is given by

$$
\begin{aligned}
d b_{\mathcal{B}^{ \pm}}(E) & =\left\{\left(\varepsilon_{n} e_{f(n)}\right)_{n}: f \in \mathbb{N}^{\mathbb{N}} \text { is injective } \&\left(\varepsilon_{i}\right)_{i} \in\{-1,1\}^{\omega}\right\} \\
& =\left\{\left(x_{n}\right)_{n} \in\left(\mathcal{B}_{E}^{ \pm}\right)^{\omega}:\left(\operatorname{supp}_{E}\left(x_{n}\right)\right)_{n} \in d b(\mathbb{N})\right\} .
\end{aligned}
$$

By Proposition 5.3.12, $d b_{\mathcal{B}^{ \pm}}(E)$ is an admissible set for $E$ determined by the admissible family $d b(\mathbb{N})$. By Proposition 5.3.29, $\left(\mathcal{B}_{E}^{ \pm}, d b_{\mathcal{B}^{ \pm}}(E)\right)$ is and admissible system of blocks for $E$.

## Summary of the possible sets of blocks

The interpretations for the elements of each admissible set determined by admissible families in the cases of the three sets of the blocks we have mentioned above can be summarized in the following table.

| $\underset{\mathfrak{B}}{ } \mathcal{D}_{E}$ | $\mathcal{B}_{E}$ | $\mathcal{B}_{E}^{ \pm}$ | $\mathbb{D}_{E}$ |
| :---: | :---: | :---: | :---: |
| FIN ${ }^{\omega}$ | Sequences of $e_{n}{ }^{\text {'s }}$ | Sequences of $\pm e_{n}$ 's | Sequences of finitely supported $\mathbf{F}_{E}$-linear combinations |
| $b b(\mathbb{N})$ | Subsequences of $\left(e_{n}\right)_{n}$ | Signed subsequences of $\left(e_{n}\right)_{n}$ | Block sequences of finitely supported $\mathbf{F}_{E}$-linear combinations |
| $d b(\mathbb{N})$ | Sequences of distinct elements of $\left\{e_{n}: n \in \mathbb{N}\right\}$ | Permutations of signed subsequences $\left(e_{n}\right)_{n}$ | Sequences of disjointly finitely supported $\mathbf{F}_{E}$-linear combinations |

Table 5.1: Elements of the admissible sets determined by the respective admissible family and for the three sets of blocks we have considered.

## $5.4 \mathcal{A}$-embeddings

In this section we shall introduce some notation we use in the following chapters and give some interpretation for the $\mathcal{A}$-embeddings depending on the set of blocks we defined in the previous sections.

Definition 5.4.1. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose that $X$ is a $\mathcal{D}_{E}$-block subspace. Let $Y$ be a Banach space with normalized basis $\left(y_{n}\right)_{n}$ and suppose $K \geq 1$.
(i) We shall say that $Y \mathcal{A}_{X}$-embeds in $X$ with constant $K$ (in symbols $Y \stackrel{\mathcal{A}}{\hookrightarrow}_{K} X$ ) if, and only if, there is some sequence $\left(u_{n}\right)_{n} \in \mathcal{A}_{X}$ of blocks such that $\left(u_{n}\right)_{n} \sim_{K}\left(y_{n}\right)_{n}$.
(ii) We say that $Y \mathcal{A}_{X}$-embeds by $\mathcal{D}_{X}$-blocks in $X$ (in symbols $Y \stackrel{\mathcal{A}}{\hookrightarrow} X$ ), if $Y \stackrel{\mathcal{A}}{\hookrightarrow}{ }_{K} X$ for some constant $K \geq 1$.

Proposition 5.4.2. Let $E$ and $Z$ be Banach spaces with normalized bases $\left(e_{n}\right)_{n}$ and $\left(z_{n}\right)_{n}$, respectively. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose that $X$ is a $\mathcal{D}_{E}$-block subspace.
(i) If $K^{\prime} \geq K$, then

$$
Z \stackrel{\mathcal{A}}{\hookrightarrow}_{K} X \Rightarrow Z{\stackrel{\mathcal{A}}{K^{\prime}}} X .
$$

(ii) If $Y$ is a $\mathcal{D}_{X}$-block subspace of $X$ and $Z \stackrel{\mathcal{A}}{\hookrightarrow} Y$, then $Z \stackrel{\mathcal{A}}{\hookrightarrow} X$.

Proof. (i) It follows directly from the fact that if $K^{\prime} \geq K$ and $\left(z_{n}\right)_{n} \sim_{K}\left(w_{n}\right)_{n}$, then $\left(z_{n}\right)_{n} \sim_{K^{\prime}}\left(w_{n}\right)_{n}$.
(ii) If $Y$ is a $\mathcal{D}_{X}$-block subspace of $X$ and $Z \stackrel{\mathcal{A}}{\hookrightarrow} Y$, then there is $\left(w_{n}\right)_{n} \in \mathcal{A}_{Y}$ such that $\left(w_{n}\right)_{n} \sim\left(z_{n}\right)_{n}$. Since $\mathcal{A}_{Y}=\mathcal{A}_{E} \cap Y^{\omega} \subseteq \mathcal{A}_{X},\left(w_{n}\right)_{n} \in \mathcal{A}_{X}$, so $Z \stackrel{\mathcal{A}}{\hookrightarrow} X$.

For the specific case when we consider $\left(\mathbb{D}_{E},\left(\mathbb{D}_{E}\right)^{\omega}\right)$ as the set of blocks we have the following:
Proposition 5.4.3. Let $E$ and $Y$ be Banach spaces with normalized bases $\left(e_{n}\right)_{n}$ and $\left(y_{n}\right)_{n}$, respectively. Consider $\mathbb{D}_{E}$ as the set of blocks and $\mathcal{A}_{E}=\left(\mathbb{D}_{E}\right)^{\omega}$. Suppose that $X=\left[x_{n}\right]_{n}$ is $a \mathbb{D}_{E}$-block subspace. Then,

$$
Y \stackrel{\mathcal{A}}{\hookrightarrow} X \Longleftrightarrow Y \hookrightarrow X .
$$

Furthermore, if $Y \hookrightarrow_{K} X$ for some $K \geq 1$, then for any $\varepsilon>0$ we have $Y \stackrel{\mathcal{A}}{\hookrightarrow}_{K+\varepsilon} X$.

Proof. We can suppose $\left(y_{n}\right)_{n}$ normalized with basis constant $C$. If $Y \mathcal{A}_{X}$-embeds in $X$, then $Y$ embeds in $X$. Conversely, suppose that $Y \hookrightarrow X$, then there is a basic sequence $\left(v_{n}\right)_{n} \in X^{\omega}$ such that $\left(y_{n}\right)_{n} \sim_{K}\left(v_{n}\right)_{n}$, for some $K \geq 1$. Since $\left(y_{n}\right)_{n}$ is normalized, $1 / K \leq\left\|v_{n}\right\| \leq K$, for all $n$.

Let $0<\varepsilon<1$. We will show that is possible to perturb $\left(v_{n}\right)_{n}$ and obtain $\left(w_{n}\right)_{n} \in \mathcal{A}_{X}$ such that $\left(v_{n}\right)_{n} \sim_{1+\varepsilon}\left(w_{n}\right)_{n}$. Let $\theta \leq \frac{\varepsilon}{K}\left(2+\frac{\varepsilon}{K}\right)^{-1}$ be a nonzero real number. Notice that $\theta<1$ and

$$
\begin{equation*}
(1+\theta)(1-\theta)^{-1}<1+\frac{\varepsilon}{K} . \tag{5.17}
\end{equation*}
$$

For each $n$, let $w_{n}$ be a nonzero block in $\mathbb{D}_{E}$ such that $\left\|w_{n}-v_{n}\right\|<\varepsilon_{n}$ (recall that $\mathbb{D}_{E}$ is dense in $E$, see the Subsection 5.3.1) and

$$
2 K^{2} C \sum_{n=0}^{\infty} \varepsilon_{n}<\theta
$$

Then,

$$
2(K C) \sum_{n=0}^{\infty} \frac{\left\|w_{n}-v_{n}\right\|}{\left\|v_{n}\right\|} \leq 2 K^{2} C \sum_{n=0}^{\infty} \varepsilon_{n}<\theta<1 .
$$

By Theorem 2.2.11, $\left(v_{n}\right)_{n} \sim_{(1+\theta)(1-\theta)^{-1}}\left(w_{n}\right)_{n}$ and by Equation (5.17), $\left(v_{n}\right)_{n} \sim_{1+\varepsilon / K}\left(w_{n}\right)_{n}$. Therefore, $\left(y_{n}\right)_{n} \sim_{K+\varepsilon}\left(w_{n}\right)_{n}$ and $\left(w_{n}\right)_{n} \in \mathcal{A}_{X}$, so $Y \stackrel{\mathcal{A}}{\hookrightarrow} X$.

Notation 5.4.4. Let $E=\left[e_{n}\right]_{n}$ and $Y=\left[y_{n}\right]_{n}$ be two Banach spaces with their respective normalized bases. We write $Y \stackrel{s}{\hookrightarrow} E$ to denote that $\left(y_{n}\right)_{n}$ is equivalent to a subsequence of $\left(e_{n}\right)_{n}$.

Remark 5.4.5. Suppose that $E=\left[e_{n}\right]_{n}$ and $Y=\left[y_{n}\right]_{n}$ are two Banach spaces with their respective normalized bases. Then $Y \stackrel{s}{\hookrightarrow} E$ if, and only if, $Y \stackrel{\mathcal{A}}{\hookrightarrow} E$ where $\mathcal{B}_{E}$ is the set of blocks and $\mathcal{A}_{E}=b b_{\mathcal{B}}(E)$ is the admissible set for $E$.

Notation 5.4.6. Let $E=\left[e_{n}\right]_{n}$ and $Y=\left[y_{n}\right]_{n}$ be two Banach spaces with their respective normalized bases. We write $Y \stackrel{ \pm s}{\hookrightarrow} E$ to denote that $\left(y_{n}\right)_{n}$ is equivalent to a signed subsequence of $\left(e_{n}\right)_{n}$.

Remark 5.4.7. Suppose that $E=\left[e_{n}\right]_{n}$ and $Y=\left[y_{n}\right]_{n}$ are two Banach spaces with their
respective normalized bases. We have that $Y \stackrel{ \pm s}{\hookrightarrow} E$ if, and only if, $Y \stackrel{\mathcal{A}}{\rightarrow} E$ where $\mathcal{B}_{E}^{ \pm}$is the set of blocks and $\mathcal{A}_{E}=b b_{\mathcal{B}_{E}^{ \pm}}(E)$ is the admissible set for $E$.

As we did in the previous section, we can summarize the interpretation of each embedding as follows: Set $E=\left[e_{n}\right]_{n}$ and $Y=\left[y_{n}\right]_{n}$ two Banach spaces with their respective normalized bases. Suppose that we are considering the set of blocks $\mathcal{D}_{E}$ being $\mathcal{B}_{E}, \mathcal{B}_{E}^{ \pm}$or $\mathbb{D}_{E}$ (depending of the case) and $\mathcal{A}_{E}$ the admissible set determined for any of the admissible families FIN ${ }^{\omega}$, $b b(\mathbb{N})$ or $d b(\mathbb{N})$. To say that $Y \stackrel{\mathcal{A}}{\hookrightarrow} E$ means in each case that there is $\left(x_{n}\right)_{n}$ in $E^{\omega}$ such that it satisfies the respective condition we have represented in the following table.

| $\overline{\mathcal{B}} \quad \mathcal{D}_{E}$ | $\mathcal{B}_{E}$ | $\mathcal{B}_{E}^{ \pm}$ | $\mathbb{D}_{E}$ |
| :---: | :---: | :---: | :---: |
| FIN ${ }^{\omega}$ | $\mathcal{A}_{E}=\left(\mathcal{B}_{E}\right)^{\omega}$ <br> $\left(x_{n}\right)_{n}=\left(e_{f(n)}\right)_{n}$ for some $f \in \mathbb{N}^{\mathbb{N}}$ injective | $\begin{gathered} \mathcal{A}_{E}=\left(\mathcal{B}_{E}^{ \pm}\right)^{\omega} \\ \left(x_{n}\right)_{n}=\left(\varepsilon_{n} e_{f(n)}\right)_{n} \text { for some } \\ f \in \mathbb{N}^{\mathbf{N}} \text { injective and } \\ \left(\varepsilon_{n}\right)_{n} \in\{-1,1\}^{\omega} \end{gathered}$ | $\mathcal{A}_{E}=\left(\mathbb{D}_{E}\right)^{\omega}$ <br> $\left(x_{n}\right)_{n}$ is a sequence of finitely supported vectors |
| $b b(\mathbb{N})$ | $\mathcal{A}_{E}=b b_{\mathcal{B}}(E)$ <br> $\left(x_{n}\right)_{n}$ is a subsequence of $\left(e_{n}\right)_{n}$ | $\mathcal{A}_{E}=b b_{\mathcal{B}^{ \pm}}(E)$ <br> $\left(x_{n}\right)_{n}$ is a signed subsequence of $\left(e_{n}\right)_{n}$ | $\mathcal{A}_{E}=B S_{\mathbb{D}}(E)$ <br> $\left(x_{n}\right)_{n}$ is a $\mathbb{D}_{E}$-block sequence |
| $d b(\mathbb{N})$ | $\mathcal{A}_{E}=d b_{\mathcal{B}}(E)$ <br> $\left(x_{n}\right)_{n}=\left(e_{f(n)}\right)_{n}$ for some $f \in \mathbb{N}^{\mathbb{N}}$ injective | $\mathcal{A}_{E}=d b_{\mathcal{B}^{ \pm}}(E)$ <br> $\left(x_{n}\right)_{n}=\left(\varepsilon_{n} e_{f(n)}\right)_{n}$ for some $f \in \mathbb{N}^{\mathbb{N}}$ injective and $\left(\varepsilon_{n}\right)_{n} \in\{-1,1\}^{\omega}$ | $\mathcal{A}_{E}=D S_{\mathbb{D}}(E)$ <br> $\left(x_{n}\right)_{n}$ is a sequence of disjointly finitely supported vectors of $\mathbb{D}_{E}$ |

Table 5.2: $\mathcal{A}$-embeddings for $\mathcal{A}$ an admissible set determined by an admissible family

Notice that since $\left(y_{n}\right)_{n}$ is a basic sequence, in the trivial cases when the admissible family is FIN ${ }^{\omega}$, then, necessarily, $\left(x_{n}\right)_{n}$ must be also basic, so at least $x_{n} \neq x_{m}$ for $n \neq m$. For that reason, the first and third rows of the $\mathcal{B}_{E}$ and $\mathcal{B}_{E}^{ \pm}$columns are the same.

### 5.5 Results on $\mathcal{A}$-minimality

Definition 5.5.1. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose that $X$ is a $\mathcal{D}_{E}$-block subspace. We say that $X$ is $\mathcal{A}_{E}$-minimal if, and only if, for all $\mathcal{D}_{X}$-block subspace $Y$ we have that $X \stackrel{\mathcal{A}}{\hookrightarrow} Y$.

The following proposition establishes that the property of being $\mathcal{A}_{E}$-minimal is hereditary by taking $\mathcal{D}_{E^{-}}$-subspaces.

Proposition 5.5.2. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set
of blocks for $E$ and $\mathcal{A}_{E}$ be an admissible set for $E$. Suppose that $X$ is a $\mathcal{D}_{E}$-block subspace which is $\mathcal{A}_{E}$-minimal. If $Y$ is a $\mathcal{D}_{X}$-block subspace of $X$, then $Y$ is $\mathcal{A}_{E}$-minimal.

Proof. Let $E, \mathcal{D}_{E}, \mathcal{A}_{E}$ and $X$ be as in the hypothesis. Let $Y=\left[y_{n}\right]_{n}$ be a $\mathcal{D}_{X}$-block subspace of $X$. Let $Z=\left[z_{n}\right]_{n}$ be a $\mathcal{D}_{Y}$-block subspace of $Y$ (so it is also a $\mathcal{D}_{X}$-block subspace of $\left.\left(x_{n}\right)_{n}\right)$. We want to see that $Y \stackrel{\mathcal{A}}{\hookrightarrow} Z$.
By the $\mathcal{A}_{E}$-minimality of $X$, we have $X \stackrel{\mathcal{A}}{\hookrightarrow} Z$, thus there is $\left(u_{n}\right)_{n} \in \mathcal{A}_{Z} \subseteq \mathcal{A}_{X}$ such that $\left(x_{n}\right)_{n} \sim\left(u_{n}\right)_{n}$. By (iii) in Proposition 5.2.4 we have

$$
\left(y_{n}\right)_{n} \in \mathcal{A}_{X} \Rightarrow\left(w_{n}\right)_{n}:=\left(y_{n}\right)_{n} *_{X}\left(u_{n}\right)_{n} \in \mathcal{A}_{X} \cap Z=\mathcal{A}_{Z}
$$

Then, $\left(w_{n}\right)_{n}$ is a block basis of the basic sequence $\left(u_{n}\right)_{n}$ of $\mathcal{D}_{Z}$-blocks (it is not necessarily a block sequence of $X$ because $\left(u_{n}\right)_{n}$ need not be a block sequence). Since $\left(u_{n}\right)_{n} \sim\left(x_{n}\right)_{n}$ and each $w_{n}$ has the same scalars in its expansion than $y_{n}$, we have that $\left(y_{n}\right)_{n} \sim\left(w_{n}\right)_{n}$. So, $Y \stackrel{\mathcal{A}}{\hookrightarrow} Z$.

It is clear that for every entry in Table 5.2 we can associate a different type of minimality. In the following we shall explore some of these notions. Let us begin with the classic case: when the set of blocks is $\mathbb{D}_{E}$.

The following definition was introduced in [20]:
Definition 5.5.3. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. The basis $\left(e_{n}\right)_{n}$ is said to be equivalence block-minimal if, and only if, any block sequence has a further block sequence equivalent to $\left(e_{n}\right)_{n}$.

As a direct consequence of the results of Subsection 5.3 .1 we obtain the following proposition.
Proposition 5.5.4. Let E be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Consider the set of blocks $\mathbb{D}_{E}$ for $E$. Let $X=\left[x_{n}\right]_{n}$ be a $\mathbb{D}_{E}$-block subspace. We have:
(i) $X$ is minimal if, and only if, it is $\mathcal{A}_{E}$-minimal, for $\mathcal{A}_{E}=\left(\mathbb{D}_{E}\right)^{\omega}$.
(ii) $\left(x_{n}\right)_{n}$ is a equivalence block-minimal basis if, and only if, $X$ is $\mathcal{A}_{E}$-minimal, for $\mathcal{A}_{E}=B S_{\mathbb{D}}(E)$.
(iii) For every $\mathbb{D}_{X}$-block sequence $\left(y_{n}\right)_{n}$ of $\left(x_{n}\right)_{n}$ we have that $\left(x_{n}\right)_{n}$ is equivalent to a sequence of disjointly supported blocks of $Y=\left[y_{n}\right]_{n}$ if, and only if, $X$ is $\mathcal{A}_{E}$-minimal, for $\mathcal{A}_{E}=D S_{\mathbb{D}}(E)$.

Let us discuss a little about the three types of minimality characterized in (i), (ii) and (iii) of Proposition 5.5.4.

It is clear that if $X=\left[x_{n}\right]_{n}$ is a $\mathbb{D}_{E}$-block subspace with $\left(x_{n}\right)_{n}$ a equivalence block-minimal basis (case (ii) in Proposition 5.5.4), then $X$ is $\mathcal{A}_{E}$-minimal, for $\mathcal{A}_{E}=D S_{\mathbb{D}}(E)$ (case (iii)
in Proposition 5.5.4). In addition, if $X$ satisfies the minimality condition given in (iii) of Proposition 5.5.4, then $X$ is minimal ( $(i)$ in Proposition 5.5.4).

The canonical basis of $c_{0}$ and $\ell_{p}$, with $1 \leq p<\infty$, is, in each case, equivalence block-minimal. In [2] was proved that the canonical basis of the Schlumprecht space $\mathcal{S}$ is equivalence blockminimal. So, these are examples of the three cases $(i)$, (ii) and (iii). In [17] it was observed that $\mathbf{T}^{*}$ has no block minimal block subspaces. In [20] it was defined a block minimal space $Z=\left[z_{n}\right]_{n}$ as a space such that for every block subspace there is a further block subspace isomorphic to $Z$. Clearly, being a equivalence block- minimal space implies being a block minimal space. In particular, $\mathbf{T}^{*}$ is not an equivalence block-minimal space and it has no equivalence block-minimal block subspaces. Therefore, it satisfies (i) and does not satisfy the minimality conditions of (iii) (nor (ii)) in Proposition 5.5.4. We do not know of spaces satisfying condition (iii) of minimality but not being equivalence block-minimal (not satisfying condition (ii)).

Now, we shall consider the set of blocks for $E$ as $\mathcal{B}_{E}$. In [19] was observed that if a basis $\left(x_{n}\right)_{n}$ satisfies that for every $\left(y_{n}\right)_{n} \preceq\left(x_{n}\right)_{n},\left(x_{n}\right)_{n}$ is equivalent to a subsequence of $\left(y_{n}\right)_{n}$, then $\left(x_{n}\right)_{n}$ is spreading. Let us prove this in the next proposition.

Lemma 5.5.5. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. If $\left(e_{n}\right)_{n}$ satisfies that every subsequence has a further subsequence equivalent to the basis, then $\left(e_{n}\right)_{n}$ is spreading.

Proof. Let $\mathcal{C} \subseteq[\mathbb{N}]^{\infty}$ be such that

$$
\left\{n_{k}: k \in \mathbb{N}\right\} \in \mathcal{C} \Longleftrightarrow\left(e_{n_{k}}\right)_{k} \sim\left(e_{k}\right)_{k}
$$

Notice that we are identifying an infinite subset $M=\left\{m_{k}: k \in \mathbb{N}\right\}$ with the increasing sequence determined by its elements. $\mathcal{C}$ can be written as follows:

$$
\mathcal{C}=\bigcup_{c=1}^{\infty} \bigcap_{k=0}^{\infty}\left\{\left\{n_{m}: m \in \mathbb{N}\right\} \in[\mathbb{N}]^{\infty}:\left(e_{n_{i}}\right)_{i}^{k} \sim_{c}\left(e_{i}\right)_{i}^{k}\right\}
$$

Since $\left\{\left\{n_{m}: m \in \mathbb{N}\right\} \in[\mathbb{N}]^{\infty}:\left(e_{n_{i}}\right)_{i}^{k} \sim_{c}\left(e_{i}\right)_{i}^{k}\right\}$ is open in $[\mathbb{N}]^{\infty}$, for $k$ and $c$ fixed, the set $\mathcal{C}$ is a Borel subset of $[\mathbb{N}]^{\infty}$ and so it is its complement. By the Galvin-Prikry Theorem (see Theorem 3.2.3) there is some $H \in[\mathbb{N}]^{\infty}$ such that either $[H]^{\infty} \subseteq \mathcal{C}$ or $[H]^{\infty} \subseteq[\mathbb{N}]^{\infty} \backslash \mathcal{C}$.

If $[H]^{\infty} \subseteq[\mathbb{N}]^{\infty} \backslash \mathcal{C}$ then the sequence $\left(e_{n}\right)_{n \in H}$ satisfies that all its subsequences are not equivalent to $\left(e_{n}\right)_{n}$, which is a contradiction. On the contrary, if $[H]^{\infty} \subseteq \mathcal{C}$, then $\left(e_{n}\right)_{n \in H}$ is spreading, and so it is $\left(e_{n}\right)_{n}$ because it is equivalent to $\left(e_{n}\right)_{n \in H}$.

So, for the case when $\mathcal{A}_{E}=b b_{\mathcal{B}}(E)$, results that $E$ is $\mathcal{A}_{E}$-minimal if and only if, for all subsequence $\left(x_{n}\right)_{n}$ of $\left(e_{n}\right)_{n}$, there is a further subsequence $\left(y_{n}\right)_{n}$ of $\left(x_{n}\right)_{n}$ equivalent to $\left(e_{n}\right)_{n}$.

Therefore, by Lemma $5.5 .5,\left(e_{n}\right)_{n}$ is spreading. Clearly an spreading sequence is $b b_{\mathcal{B}}(E)$ minimal. We can summarize this in the following proposition.

Proposition 5.5.6. Consider the admissible set $b b_{\mathcal{B}}(E)$ for $E$. We have that $E$ is $b b_{\mathcal{B}}(E)$ minimal if and only if, $\left(e_{n}\right)_{n}$ is spreading.

Let us summarize in the following proposition the notions of $\mathcal{A}_{E}$-minimality which follows from each non-trivial $\mathcal{A}_{E}$-embedding notion given in the Table 5.2. That is, the entries corresponding $(1,3),(2,1),(2,2),(2,3),(3,1),(3,2)$ and $(3,3)$ of the Table 5.2.

Proposition 5.5.7. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. We have,

- Consider the set of blocks $\mathcal{B}_{E}$ for $E$, and $X=\left[x_{n}\right]_{n}$ a $\mathcal{B}_{E}$-block subspace of $E$. We have
(i) $X$ is $b b_{\mathcal{B}}(E)$-minimal if, and only if, $\left(x_{n}\right)_{n}$ is spreading.
(ii) $X$ is $d b_{\mathcal{B}}(E)$-minimal if, and only if, for every $\left(y_{n}\right)_{n}$ subsequence of $\left(x_{n}\right)_{n}$ there is a function $f \in \mathbb{N}^{\mathbb{N}}$ injective such that $\left(y_{f(n)}\right)_{n} \sim\left(x_{n}\right)_{n}$.
- Consider the set of blocks $\mathcal{B}_{E}^{ \pm}$for $E$, and $X=\left[x_{n}\right]_{n}$ a $\mathcal{B}_{E}^{ \pm}$-block subspace of $E$, that is $\left(x_{n}\right)_{n}$ is a signed subsequence of $\left(e_{n}\right)_{n}$. We have
(iii) $X$ is $b b_{\mathcal{B}}^{ \pm}(E)$-minimal if, and only if, for every $\left(y_{n}\right)_{n}$ subsequence of $\left(x_{n}\right)_{n}$ there is a further signed subsequence $\left(z_{n}\right)_{n}$ of $\left(y_{n}\right)_{n}$ such that $\left(z_{n}\right)_{n} \sim\left(x_{n}\right)_{n}$.
(iv) $X$ is db $b_{\mathcal{B}}^{ \pm}(E)$-minimal if, and only if, for every $\left(y_{n}\right)_{n}$ signed subsequence of $\left(x_{n}\right)_{n}$ there is a function $f \in \mathbb{N}^{\mathbb{N}}$ injective and some sequence of signs $\left(\varepsilon_{n}\right)_{n}$ such that $\left(\varepsilon_{n} y_{f(n)}\right)_{n} \sim\left(x_{n}\right)_{n}$.
- Consider the set of blocks $\mathbb{D}_{E}$ for $E$, and $X=\left[x_{n}\right]_{n}$ a $\mathbb{D}_{E}$-block subspace of $E$. We have
(v) $X$ is $\left(\mathbb{D}_{E}\right)^{\omega}$-minimal if, and only if, $X$ is minimal.
(vi) $X$ is $B S_{\mathbb{D}}(E)$-minimal if, and only if, $\left(x_{n}\right)_{n}$ is an equivalence block- minimal basis.
(vii) $X$ is $D S_{\mathbb{D}}(E)$-minimal if, and only if, for every $\mathbb{D}_{X}$-block sequence $\left(y_{n}\right)_{n}$ of $\left(x_{n}\right)_{n}$ there is a sequence of disjointly supported blocks $\left(z_{n}\right)_{n}$ of $Y=\left[y_{n}\right]_{n}$ such that $\left(z_{n}\right)_{n} \sim\left(x_{n}\right)_{n}$.

Proof. Item ( $i$ ) follows from Proposition 5.5.6. Items (ii) and (iv) follow directly from the definition. Item (iii) follows from the definition and the following easily proved fact.

For every $\left(y_{n}\right)_{n}$ signed subsequence of $\left(x_{n}\right)_{n}$ there is a further signed subsequence $\left(z_{n}\right)_{n}$ of $\left(y_{n}\right)_{n}$ such that $\left(z_{n}\right)_{n} \sim\left(x_{n}\right)_{n}$, if and only if, for every $\left(y_{n}\right)_{n}$ subsequence of $\left(x_{n}\right)_{n}$ there is signed subsequence $\left(z_{n}\right)_{n}$ of $\left(y_{n}\right)_{n}$ such that $\left(z_{n}\right)_{n} \sim\left(x_{n}\right)_{n}$.

Items $(v),(v i)$ and (vii) follow from Proposition 5.5.4.

### 5.6 Results on $\mathcal{A}$-tightness

Definition 5.6.1. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose that $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace. We say that a Banach space $Y$ with Schauder basis is $\mathcal{A}_{E}$-tight in the basis $\left(x_{n}\right)_{n}$ if, and only if, there is a sequence of successive intervals $\left(I_{i}\right)_{i}$ such that for every $A \in[\mathbb{N}]^{\infty}$

$$
\begin{equation*}
Y \stackrel{A}{\hookrightarrow}\left[x_{n}: n \notin \cup_{i \in A} I_{i}\right] . \tag{5.18}
\end{equation*}
$$

Definition 5.6.2. Let $E$ be a Banach space with a normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose that $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace. The basis $\left(x_{n}\right)_{n}$ is $\mathcal{A}_{E}$-tight if, and only if, every $\mathcal{D}_{X}$-block subspace $Y$ of $X$ is $\mathcal{A}_{E}$-tight in the basis $\left(x_{n}\right)_{n}$. The $\mathcal{D}_{E}$-block subspace $X$ is $\mathcal{A}_{E}$-tight if, and only if, $\left(x_{n}\right)_{n}$ is an $\mathcal{A}_{E}$-tight basis.

Remark 5.6.3. Let $E$ be a Banach space with a normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. From Definition 5.6.2, $E$ is $\mathcal{A}_{E}$-tight if, and only if, every $\mathcal{D}_{E}$-block subspace $X$ is $\mathcal{A}_{E}$-tight in $\left(e_{n}\right)_{n}$.

Remark 5.6.4. It follows from the definition of an $\mathcal{A}_{E}$-tight space that $X=\left[x_{n}\right]_{n}$ is $\mathcal{A}_{E}$ tight if, and only if, for every $\mathcal{D}_{X}$-block subspace $Y$ of $X$ there are intervals $I_{0}<I_{1}<\ldots$, such that if $Y \mathcal{A}_{E}$-embeds into $\left[x_{n}: n \in B\right]$, then $B$ intersects all but finitely many intervals $I_{i}^{\prime} s$.

Proposition 5.6.5. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose that $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace and $Y$ is a $\mathcal{D}_{X}$-block subspace of $X$. Then, $Y$ is $\mathcal{A}_{X}$-tight in $\left(x_{n}\right)_{n}$ if, and only if, the set

$$
\begin{equation*}
E_{Y, X}^{\mathcal{A}}:=\left\{\mathfrak{u} \in 2^{\omega}: Y \stackrel{\mathcal{A}}{\longrightarrow}\left[x_{n}: n \in \operatorname{supp}(u)\right]\right\} \tag{5.19}
\end{equation*}
$$

is meager in $2^{\omega}$.
Proof. Let $E, \mathcal{D}_{E}$ and $\mathcal{A}_{E}$ be as in the hypothesis. Suppose that $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace and let $Y$ be a $\mathcal{D}_{X}$-block subspace of $X$. If $Y$ is $\mathcal{A}_{E}$-tight in $\left(x_{n}\right)_{n}$, then there are intervals $I_{0}<I_{1}<\ldots$ such that for any $A \in[\mathbb{N}]^{\infty}$,

$$
\begin{equation*}
Y \stackrel{A}{\hookrightarrow}\left[x_{n}: n \notin \cup_{i \in A} I_{i}\right] . \tag{5.20}
\end{equation*}
$$

Let $\mathfrak{u} \in E_{Y, X}^{\mathcal{A}}\left(\right.$ clearly $\left.\operatorname{supp}(\mathfrak{u}) \in[\mathbb{N}]^{\infty}\right)$ and suppose by contradiction that $A_{\mathfrak{u}}=\{i \in \mathbb{N}$ : $\left.I_{i} \cap \operatorname{supp}(\mathfrak{u})=\emptyset\right\}$ is infinite. We have

$$
\operatorname{supp}(\mathfrak{u}) \subseteq \mathbb{N} \backslash \bigcup_{i \in A_{u}} I_{i} .
$$

By the Remark 5.1.28, we obtain

$$
Y \stackrel{\mathcal{A}}{\hookrightarrow}\left[x_{n}: n \in \operatorname{supp}(\mathfrak{u})\right] \Rightarrow Y \stackrel{\mathcal{A}}{\rightarrow}\left[x_{n}: n \notin \cup_{i \in A_{u}} I_{i}\right],
$$

contradicting Equation (5.20). Therefore $A_{u}$ is finite and, by Corollary 3.1.5, $E_{Y}^{\mathcal{A}}$ is meager in $2^{\omega}$.

For the opposite implication, suppose that $E_{Y}^{\mathcal{A}}$ is meager in $2^{\omega}$. Then, by using Corollary 3.1.5, there are subsets $I_{0}<I_{1}<\ldots$ such that if $\mathfrak{u} \in E_{Y}^{\mathcal{A}}$, then $\left\{i \in \mathbb{N}: I_{i} \cap \operatorname{supp}(\mathfrak{u})=\emptyset\right\}$ is finite. If there is $A \in[\mathbb{N}]^{\infty}$ such that $Y \stackrel{\mathcal{A}}{\rightarrow}\left[x_{n}: n \notin \cup_{i \in A} I_{i}\right]$, then take $v=\mathbb{N} \backslash \cup_{i \in A} I_{i}$. Clearly $\chi_{v} \in E_{Y}^{\mathcal{A}}$ and $\left\{i \in \mathbb{N}: I_{i} \cap v=\emptyset\right\}$ is infinite, which contradicts that $E_{Y}^{\mathcal{A}}$ is a meager subset of $2^{\omega}$.

Lemma 5.6.6. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose that $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace and $Y$ is a $\mathcal{D}_{X}$-block subspace of $X$. Then $E_{Y, X}^{\mathcal{A}}$ defined in Equation (5.19) is either meager or comeager in $2^{\omega}$.

Proof. The proof follows the same scheme used by V. Ferenczi and G. Godefroy in [18] to prove that the set $E_{Y}=\left\{u \subseteq \omega: Y \hookrightarrow\left[x_{n}: n \in u\right]\right\}$ is meager or comeager. As it is affirmed in Example 2.2 in [18], the relation $E_{0}^{\prime}$ defined on $\mathbb{P}(\omega)$ as follows

$$
u E_{0}^{\prime} v \Longleftrightarrow \exists n \geq 0((u \cap[n, \infty)=v \cap[n, \infty)) \&(|u \cap[0, n-1]|=|v \cap[0, n-1]|))
$$

is an equivalence relation and its equivalence classes are the orbits of the group $G_{0}^{\prime}$ of permutation of $\mathbb{N}$ with finite support. Once we see $\mathbb{P}(\omega)$ as the Cantor space, it is Polish and clearly $G_{0}^{\prime}$ satisfies that for any $U$ and $V$ non-empty open subsets of $\mathbb{P}(\omega)$, there is $g \in G$ such that $g(U) \cap V \neq \emptyset$. We want to use the first topological 0-1 law (Theorem 3.2.2) to conclude that $E_{Y, X}^{\mathcal{A}}$ is meager or comeager in $2^{\omega}$, or more specifically we have to prove:
(i) $E_{Y, X}^{\mathcal{A}}$ has the Baire Property.
(ii) $E_{Y, X}^{\mathcal{A}}$ is $G_{0^{\prime}}^{\prime}$-invariant.

To prove ( $i$ ) we shall see that $E_{Y, X}^{\mathcal{A}}$ is an analytic subset of $2^{\omega}$ (See Theorem 21.6 in [33]). Notice that we can write the set $E_{Y, X}^{\mathcal{A}}$ as the projection on the first coordinate of the set $B:=\cup_{k \in \omega} B_{k}$, where for each $k \in \omega$

$$
B_{k}:=\left\{\left(u,\left(w_{n}\right)_{n}\right) \in 2^{\omega} \times\left(\mathcal{D}_{X}\right)^{\omega}:\left(y_{n}\right)_{n} \sim_{k}\left(w_{n}\right)_{n} \&\left(w_{n}\right)_{n} \in\left[x_{i}: i \in u\right] \&\left(w_{n}\right)_{n} \in \mathcal{A}_{X}\right\} .
$$

Each $B_{k}$ is Borel in $2^{\omega} \times\left(\mathcal{D}_{X}\right)^{\omega}$ since the relation of two sequences being equivalent is closed and $\mathcal{A}_{X}$ is a closed subset of $\left(\mathcal{D}_{X}\right)^{\omega}$.

In order to prove (ii) we show that $E_{Y, X}^{\mathcal{A}}$ is $E_{0}^{\prime}$-saturated (this is sufficient because the orbits
of the group $G_{0}^{\prime}$ coincide with the equivalence classes of the relation $E_{0}^{\prime}$ ), that is:

$$
E_{Y, X}^{\mathcal{A}}=E_{Y, X}^{\mathcal{A}}{ }^{E_{0}^{\prime}}:=\left\{v \subseteq \omega: \exists u \in E_{Y, X}^{\mathcal{A}}\left(u E_{0}^{\prime} v\right)\right\}
$$

Clearly, $E_{Y, X}^{\mathcal{A}} \subseteq E_{Y, X}^{\mathcal{A}}{ }^{E_{0}^{\prime}}$. Take $v \in E_{Y, X}^{\mathcal{A}}{ }^{E_{0}^{\prime}}$ and let $u \in E_{Y, X}^{\mathcal{A}}$ be such that $u E_{0}^{\prime} v$. Notice that there is $M$ such that $u$ and $v$ only differ on $M$ elements and $\left(x_{n}\right)_{n \in u}$ and $\left(x_{n}\right)_{n \in v}$ are $\mathcal{D}_{X^{-}}$ block sequences. So, by the Proposition 2.2.29, there is $K \geq 1$ such that $\left(x_{n}\right)_{n \in u} \sim_{K}\left(x_{n}\right)_{n \in v}$. Let $T$ be such $K$-isomorphism from $X_{u}:=\left[x_{n}\right]_{n \in u}$ to $X_{v}:=\left[x_{n}\right]_{n \in v}$. By Proposition 5.2.7, part (ii) we have $T\left[\mathcal{A}_{X_{u}}\right]=\mathcal{A}_{X_{v}}$. Therefore,

$$
\begin{aligned}
u \in E_{Y, X}^{\mathcal{A}} & \Rightarrow \exists\left(z_{n}\right)_{n} \in \mathcal{A}_{X_{u}}\left(\left(y_{n}\right)_{n} \sim\left(z_{n}\right)_{n}\right) \\
& \Rightarrow\left(y_{n}\right)_{n} \sim\left(T\left(z_{n}\right)\right)_{n} \text { and }\left(T\left(z_{n}\right)\right)_{n} \in \mathcal{A}_{X_{v}} \\
& \Rightarrow v \in E_{Y, X}^{\mathcal{A}} .
\end{aligned}
$$

Proposition 5.6.7. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose that $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace, $Y$ is a $\mathcal{D}_{X}$-block subspace of $X$ and $Z$ is a $\mathcal{D}_{Y}$-block subspace. If $Z$ is $\mathcal{A}_{E}$-tight in $X$, then $Z$ is $\mathcal{A}_{E}$-tight in $Y$.

Proof. Let $E, \mathcal{D}_{E}$ and $\mathcal{A}_{E}$ be as in the hypothesis. Suppose that $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace, $Y$ is a $\mathcal{D}_{X}$-block subspace of $X$ and $Z$ is a $\mathcal{D}_{Y}$-block subspace. Let us denote as

$$
E_{Z, X}^{\mathcal{A}}:=\left\{u \subseteq \omega: Z \stackrel{\mathcal{A}}{\hookrightarrow}\left[x_{n}: n \in u\right]\right\}
$$

and

$$
E_{Z, Y}^{\mathcal{A}}:=\left\{u \subseteq \omega: Z \stackrel{\mathcal{A}}{\hookrightarrow}\left[y_{n}: n \in u\right]\right\} .
$$

By hypothesis, we know that $E_{Z, X}^{\mathcal{A}}$ is meager in $\mathbb{P}(\omega)$ after the identification of $\mathbb{P}(\omega)$ with $2^{\omega}$. Using Lemma 5.6.6, $E_{Z, Y}^{\mathcal{A}}$ is meager or comeager. If it were meager, by the Proposition 5.6.5 the demonstration ends. Suppose that $E_{Z, Y}^{\mathcal{A}}$ is comeager in $\mathbb{P}(\omega)$. By Corollary 3.1.5, there are sequences of successive intervals $\left(I_{i}\right)_{i}$ and $\left(J_{i}\right)_{i}$ such that

$$
\begin{equation*}
u \in E_{Z, X}^{\mathcal{A}} \Rightarrow\left\{n \in \omega: u \cap I_{n}=\emptyset\right\} \text { is finite } \tag{5.21}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\{n \in \omega: J_{n} \subseteq v\right\} \text { is infinite } \Rightarrow v \in E_{Z, Y}^{\mathcal{A}} . \tag{5.22}
\end{equation*}
$$

Let $A \in[\mathbb{N}]^{\infty}$ be such that

$$
\left\{k \in \mathbb{N}:\left(\bigcup_{n \in A} \bigcup_{i \in J_{n}} \operatorname{supp}_{x}\left(y_{i}\right)\right) \cap I_{k}=\emptyset\right\}
$$

is infinite. Such $A$ exists because each $I_{i}$ and $J_{i}$ are finite and each $y_{i}$ is finitely supported. Let $v=\bigcup_{n \in A} J_{n}$, then by Equation (5.22), we have $v \in E_{Z, Y}^{\mathcal{A}}$. If $u=\bigcup_{k \in v} \operatorname{supp}_{X}\left(y_{k}\right)$, then

$$
Z \stackrel{\mathcal{A}}{\hookrightarrow}\left[y_{n}: n \in v\right] \Rightarrow Z \stackrel{\mathcal{A}}{\hookrightarrow}\left[x_{n}: n \in u\right] .
$$

The last implication follows from the Proposition 5.4.2. Therefore, $u \in E_{Z, X}^{\mathcal{A}}$ but it is disjoint of infinitely many intervals $I_{k}$ 's, contradicting Equation (5.21).

Corollary 5.6.8. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose that $X=\left[x_{n}\right]_{n}$ is a $\mathcal{A}_{E}$-tight $\mathcal{D}_{E}$-block subspace. Then, any $\mathcal{D}_{E}$-block sequence $\left(y_{n}\right)_{n}$ of $\left(x_{n}\right)_{n}$ is an $\mathcal{A}$-tight basis.

Proof. Let $Z$ be a $\mathcal{D}_{Y}$-block subspace of $Y$. Since $Z$ is a $\mathcal{D}_{X}$-block subspace of $X$ and $Z$ is $\mathcal{A}_{E}$-tight in $X$, by Proposition 5.6.7, $Z$ is $\mathcal{A}_{E}$-tight in $Y$.

Theorem 5.6.9. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose that $X=\left[x_{n}\right]_{n}$ is an $\mathcal{A}_{E}$-tight $\mathcal{D}_{E}$-block subspace. If $X$ is $\mathcal{A}_{E}$-tight then it contains no $\mathcal{A}_{E}$-minimal $\mathcal{D}_{X}$-block subspaces.

Proof. Let $E, \mathcal{D}_{E}, \mathcal{A}_{E}$ and $X$ be as in the hypothesis. By contradiction, suppose $Y=\left[y_{n}\right]_{n}$ is an $\mathcal{A}_{E}$-minimal $\mathcal{D}_{X}$-block subspace of $X$. Let $Z=\left[z_{n}\right]_{n}$ be a $\mathcal{D}_{Y}$-block subspace of $Y$, so $Z$ is $\mathcal{A}_{E}$-tight in $X$. By Proposition 5.6.7, $Z$ is $\mathcal{A}_{E}$-tight in $Y$, then

$$
E_{Z, Y}^{\mathcal{A}}=\left\{u \subseteq \omega: Z \stackrel{\mathcal{A}}{\hookrightarrow}\left[y_{n}: n \in u\right]\right\}
$$

must be meager in $\mathbb{P}(\omega)$.
Notice that the set $E_{Z, Y}^{\mathcal{A}}$ coincide with the subset of all the characteristic functions over infinite subsets of $\mathbb{N}$, which is comeager (for a more general proof of this fact, see Example 7.1.8) what leads us to a contradiction. Let us prove this: suppose $v \subseteq \omega$ infinite, then by the $\mathcal{A}_{E}$-minimality of $Y$

$$
Y \stackrel{\mathcal{A}}{\longrightarrow}\left[y_{n}: n \in v\right],
$$

so, there is $\left(u_{n}\right)_{n} \in \mathcal{A}_{E} \cap\left[y_{n}: n \in v\right]$ such that $\left(y_{n}\right)_{n} \sim\left(u_{n}\right)_{n}$.
We know that $\left(y_{n}\right)_{n},\left(u_{n}\right)_{n},\left(z_{n}\right)_{n} \in \mathcal{A}_{Y}$ and by $\left.i i i\right)$ in Proposition 5.2 .1 we have

$$
\left(z_{n}\right)_{n} *_{Y}\left(y_{n}\right)_{n}=\left(z_{n}\right)_{n} \in \mathcal{A}_{Y} \Rightarrow\left(w_{n}\right)_{n}:=\left(z_{n}\right)_{n} *_{Y}\left(u_{n}\right)_{n} \in \mathcal{A}_{Y} .
$$

Then, $\left(w_{n}\right)_{n}$ is a $\mathcal{D}_{Y^{-}}$-block sequence of the basic sequence $\left(u_{n}\right)_{n}$ (not necessarily is a block sequence of $X$ because $\left(u_{n}\right)_{n}$ need not to be a block sequence). Also, each $w_{n}$ has the same scalars in its expansion than $z_{n}$. Since $\left(u_{n}\right)_{n} \sim\left(y_{n}\right)_{n}$, we have that $\left(z_{n}\right)_{n} \sim\left(w_{n}\right)_{n}$ and also
we already know that $\left(w_{n}\right)_{n} \in \mathcal{A}_{E} \cap\left[y_{n}: n \in v\right]$. So, $Z \xrightarrow{\mathcal{A}}\left[y_{n}: n \in v\right]$, which means that $v \in E_{Z, Y}^{\mathcal{A}}$. We just proved that $[\mathbb{N}]^{\infty}$ is contained in $E_{Z, Y}^{\mathcal{A}}$, thus they are the same set.

Proposition 5.6.10. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Consider $\mathbb{D}_{E}$ as the set of blocks for $E$ and set $\mathcal{A}_{E}=\left(\mathbb{D}_{E}\right)^{\omega}$. A $\mathbb{D}_{E}$-block sequence $\left(x_{n}\right)_{n}$ is a $\mathcal{A}_{X}$-tight basis if, and only if, $\left(x_{n}\right)_{n}$ is a tight basis.

Proof. This follows directly from Proposition 5.4.3 and Definition 5.6.1.

The following Corollary is the Proposition 3.3. in [22]
Corollary 5.6.11. If $E$ is a Banach space with normalized tight basis $\left(e_{n}\right)_{n}$, then $E$ has no minimal subspaces.

Proof. Take $\mathbb{D}_{E}$ as the set of blocks for $E$ and set $\mathcal{A}_{E}=\left(\mathbb{D}_{E}\right)^{\omega}$. This result is obtained as a consequence of the Proposition 5.6.9 and the Proposition 5.6.10.

### 5.6.1 Tightness by sequences

Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. In this section we shall define a specific case of tightness called "tight by sequences". This is a specific case when in Definition 5.6.2 we consider $\mathcal{D}_{E}=\mathcal{B}_{E}, \mathcal{A}_{E}$ is the set of subsequences of the basis, and the embedding as a subsequence $\stackrel{\mathrm{s}}{\hookrightarrow}$.

Definition 5.6.12 (Tight by sequences). Let E be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\left(y_{n}\right)_{n}$ a basic sequence in a Banach space. We say that $\left(y_{n}\right)_{n}$ is tight by sequences in $E$, if the set

$$
E_{y_{n}}:=\left\{u \in 2^{\omega}:\left[y_{n}\right]_{n} \stackrel{s}{\hookrightarrow}\left[e_{n}: n \in u\right]\right\}
$$

is meager in $2^{\omega}$. If any basic sequence $\left(y_{n}\right)_{n}$ is tight by sequences in $E$, we say that $\left(e_{n}\right)_{n}$ is a tight-by-sequences basis of $E$.

Remark 5.6.13. The basic sequence $\left(e_{n}\right)_{n}$ is tight by sequences if, and only if, every $\left(e_{n_{k}}\right)_{k}$ subsequence of $\left(e_{n}\right)_{n}$ is tight by sequences in $E$.

Corollary 5.6.14. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ tight by sequences, then for every $\left(x_{n}\right)_{n}$ subsequence of $\left(e_{n}\right)_{n}$, there exists a subsequence $\left(y_{n}\right)_{n}$ of $\left(x_{n}\right)_{n}$, such that $\left(z_{n}\right)_{n} \nsim\left(x_{n}\right)_{n}$, for every subsequence $\left(z_{n}\right)_{n}$ of $\left(y_{n}\right)_{n}$.

Proof. It follows directly from Theorem 5.6.5.

### 5.6.2 Summary of types of tightness

Let us summarize in the following proposition the notions of $\mathcal{A}_{E}$-tightness which follows from each non-trivial $\mathcal{A}_{E}$-embedding notion given in the Table 5.2. That is, the entries corresponding to $(1,3),(2,1),(2,2),(2,3),(3,1),(3,2)$ and $(3,3)$ of the Table 5.2.

Proposition 5.6.15. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$.

- Consider the set of blocks $\mathcal{B}_{E}$ for $E$, and $X=\left[x_{n}\right]_{n}$ a $\mathcal{B}_{E}$-block subspace of $E$. We have
(i) $\left(x_{n}\right)_{n}$ is a bb $b_{\mathcal{B}}(E)$-tight basis if, and only if, $\left(x_{n}\right)_{n}$ is tight by sequences.
(ii) $\left(x_{n}\right)_{n}$ is a db $b_{\mathcal{B}}(E)$-tight basis if, and only if, for every $\left(y_{n}\right)_{n}$ subsequence of $\left(x_{n}\right)_{n}$ there is a sequence of successive intervals $\left(I_{n}\right)_{n}$ such that for every $A \in[\mathbb{N}]^{\infty}$ and for every injection $f: \mathbb{N} \rightarrow \mathbb{N} \backslash \cup_{i \in A} I_{i}$, we have $\left(y_{n}\right)_{n} \nsim\left(x_{f(n)}\right)_{n}$.
- Consider the set of blocks $\mathcal{B}_{E}^{ \pm}$for $E$, and $X=\left[x_{n}\right]_{n}$ a $\mathcal{B}_{E}^{ \pm}$-block subspace of $E$, that is $\left(x_{n}\right)_{n}$ is a signed subsequence of $\left(e_{n}\right)_{n}$. We have
(iii) $\left(x_{n}\right)_{n}$ is a bb $b_{\mathcal{B}}^{ \pm}(E)$-tight basis if, and only if, for every $\left(y_{n}\right)_{n}$ subsequence of $\left(x_{n}\right)_{n}$ there is a sequence of successive intervals $\left(I_{n}\right)_{n}$ such that for every $A \in[\mathbb{N}]^{\infty}$ we have that $\left(y_{n}\right)_{n}$ is not equivalent to any signed subsequence of $\left(x_{n}: n \in \mathbb{N} \backslash \cup_{i \in A} I_{i}\right)$.
(iv) $\left(x_{n}\right)_{n}$ is a db $b_{\mathcal{B}}^{ \pm}(E)$-tight basis if, and only if, for every $\left(y_{n}\right)_{n}$ signed subsequence of $\left(x_{n}\right)_{n}$ there is a sequence of successive intervals $\left(I_{n}\right)_{n}$ such that for every $A \in[\mathbb{N}]^{\infty}$, we have that $\left(y_{n}\right)_{n}$ is not equivalent to any sequence of the form $\left(\delta_{n} x_{f(n)}\right)_{n}$, where $\left(\delta_{n}\right)_{n}$ is a sequence of signs and $f: \mathbb{N} \rightarrow \mathbb{N} \backslash \cup_{i \in A} I_{i}$ is an injective function.
- Consider the set of blocks $\mathbb{D}_{E}$ for $E$, and $X=\left[x_{n}\right]_{n}$ a $\mathbb{D}_{E}$-block subspace of $E$. We have
(v) $\left(x_{n}\right)_{n}$ is a $\left(\mathbb{D}_{E}\right)^{\omega}$-tight if, and only if, $\left(x_{n}\right)_{n}$ is tight.
(vi) $\left(x_{n}\right)_{n}$ is a $B S_{\mathbb{D}}(E)$-tight basis if, and only if, every $\mathbb{D}_{X}$-block sequence $\left(y_{n}\right)_{n}$ of $\left(x_{n}\right)_{n}$, there is a sequence $\left(I_{n}\right)_{n}$ of successive intervals in $\mathbb{N}$ such that for any $A \in[\mathbb{N}]^{\infty},\left[y_{n}\right]_{n}$ does not embed into $\left[x_{n}, n \notin \cup_{i \in A} I_{i}\right]$ as a block sequence.
(vii) $\left(x_{n}\right)_{n}$ is a $D S_{\mathbb{D}}(E)$-tight basis if, and only if, for every $\mathbb{D}_{X}$-block sequence $\left(y_{n}\right)_{n}$ of $\left(x_{n}\right)_{n}$ there is a sequence $\left(I_{n}\right)_{n}$ of successive intervals in $\mathbb{N}$ such that for any $A \in[\mathbb{N}]^{\infty},\left[y_{n}\right]_{n}$ does not embed into $\left[x_{n}, n \notin \cup_{i \in A} I_{i}\right]$, as a sequence of disjointly supported vectors.

Proof. Item (i) follows from Corollary 5.6.14. Items (ii), (iv), (vi) and (vii) follow directly from the definition. Item (iii) follows from the definition and the fact that if $\left(w_{n}\right)_{n} \sim\left(z_{n}\right)_{n}$, then $\left(\varepsilon_{n} w_{n}\right)_{n} \sim\left(\varepsilon_{n} z_{n}\right)_{n}$, for any sequence of signs $\left(\varepsilon_{n}\right)_{n}$.

Item $(v)$ is a consequence of Proposition 5.6.10.

Remark 5.6.16. Notice that $\left(x_{n}\right)_{n}$ is a db $b_{\mathcal{B}}^{ \pm}(E)$-tight basis if, and only if, for every $\left(y_{n}\right)_{n}$ signed subsequence of $\left(x_{n}\right)_{n}$ there is a sequence of successive intervals $\left(I_{n}\right)_{n}$ such that for every $A \in[\mathbb{N}]^{\infty}$ we have that $\left(y_{n}\right)_{n}$ is not equivalent to any sequence of the form $\left(\varepsilon_{n} z_{n}\right)_{n}$, where $\left(\varepsilon_{n}\right)_{n}$ is a sequence of signs and $\left(z_{n}\right)_{n}$ is an infinite sequence of distinct elements of the set $\left\{x_{n}: n \in \mathbb{N} \backslash \cup_{i \in A} I_{i}\right\}$.

### 5.7 Final considerations about support and range

We have seen so far extensions of the tightness and minimality notions for different kinds of embeddings. It is natural, also, to extend the notions of tightness by range and by support in this context. We shall present the definitions without entering into the study of their properties.

Definition 5.7.1. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose that $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace. We say that a Banach space $Y$ with Schauder basis is $\mathcal{A}_{E}$-tight by support in the basis $\left(x_{n}\right)_{n}$ if, and only if, for every $A \in[\mathbb{N}]^{\infty}$

$$
\begin{equation*}
Y \stackrel{A}{\hookrightarrow}\left[x_{n}: n \notin \cup_{i \in A} \operatorname{supp}_{X}\left(y_{i}\right)\right] . \tag{5.23}
\end{equation*}
$$

Definition 5.7.2. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose that $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace. We say that a Banach space $Y$ with Schauder basis is $\mathcal{A}_{E}$-tight by range in the basis $\left(x_{n}\right)_{n}$ if, and only if, for every $A \in[\mathbb{N}]^{\infty}$

$$
\begin{equation*}
Y \stackrel{A}{\hookrightarrow}\left[x_{n}: n \notin \cup_{i \in A} \operatorname{ran}_{X}\left(y_{i}\right)\right] . \tag{5.24}
\end{equation*}
$$

Definition 5.7.3. Let $E$ be a Banach space with a normalized basis $\left(e_{n}\right)_{n}$. Let $\mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose that $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace. The basis $\left(x_{n}\right)_{n}$ is $\mathcal{A}_{E}$-tight by support (respectively $\mathcal{A}_{E}$-tight by range) if, and only if, every $\mathcal{D}_{X}$-block subspace $Y$ of $X$ is $\mathcal{A}_{E}$-tight by support (respectively $\mathcal{A}_{E}$ - tight by range) in the basis $\left(x_{n}\right)_{n}$. The $\mathcal{D}_{E}$-block subspace $X$ is $\mathcal{A}_{E}$-tight by support (respectively $\mathcal{A}_{E}$-tight by range) if, and only if, $\left(x_{n}\right)_{n}$ is an $\mathcal{A}_{E}$-tight by support (respectively $\mathcal{A}_{E}$-tight by range) basis.

Notice that it is natural to consider the extension of tightness by range and support using the embedding "as a subsequence".

## Chapter 6

## Tight - minimal dichotomy

In [22] it is defined that a space $E=\left[e_{n}\right]_{n}$ is continuously tight if, and only if, there is a continuous function $f: b b_{\mathbb{D}}\left(e_{n}\right) \rightarrow[\mathbb{N}]^{<\infty}$ such that for all normalized block bases $X$, if $I_{j}=\left[f(X)_{2 j}, f(X)_{2 j+1}\right]$, then

$$
X \nsim\left(E, I_{j}\right) .
$$

Clearly, every continuously tight space is also tight. The third dichotomy, as it is stated and proved in [22] is the following:

Theorem (Third dichotomy, [22]). Let E be a Banach space with a basis $\left(e_{n}\right)_{n}$. Then either E contains a minimal block subspace or a continuously tight block subspace.

The main theorem of this chapter (Theorem 6.5.1) is an $\mathcal{A}$-tight - $\mathcal{A}$-minimal dichotomy, for an admissible set $\mathcal{A}$. The result is the following:

Theorem (Theorem 6.5.1). Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$ be an admissible system of blocks for $E$. Then $E$ contains a $\mathcal{D}_{E}$-block subspace $X$ which is either $\mathcal{A}_{E}$-tight or $\mathcal{A}_{E}$-minimal.

It is possible to obtain the third dichotomy as a particular case of Theorem 6.5.1 by taking the admissible system of blocks being $\left(\mathbb{D}_{E},\left(\mathbb{D}_{E}\right)^{\omega}\right)$, as is proved in Corollary 6.5.2.

We also show a proof of the next theorem which was stated by Ferenczi and Rosendal:
Theorem (Theorem 3.16, [22]). Every Banach space with a basis contains a block subspace $E=\left[e_{n}\right]_{n}$ satisfying one of the following properties:
(1) For any $\left[y_{n}\right]_{n} \leq E$, there is a sequence $\left(I_{n}\right)_{n}$ of successive intervals in $\mathbb{N}$ such that for any $A \in[\mathbb{N}]^{\infty},\left[y_{n}\right]_{n}$ does not embed into $\left[e_{n}, n \notin \cup_{i \in A} I_{i}\right]$, as a sequence of disjointly supported vectors, respectively as a block sequence.
(2) For any $\left[y_{n}\right]_{n} \leq E,\left(e_{n}\right)_{n}$ is equivalent to a sequence of disjointly supported vectors of $\left[y_{n}\right]_{n}$, respectively $\left(e_{n}\right)_{n}$ is equivalent to a block sequence of $\left[y_{n}\right]_{n}$.

This theorem is proved at the end of this chapter in Corollary 6.5.3 (for the embedding as a sequence of disjointly supported vectors) and Corollary 6.5.4 (for the embedding as a block sequence).

The authors of [22] stated that modifying the notion of embedding in the definition of tight basis, and consequently modifying the games involved in the proof of the Third Dichotomy, the following statement is true:

Every Banach space with a basis contains a block subspace $E=\left[e_{n}\right]_{n}$ satisfying that either for any $\left[y_{n}\right]_{n} \leq E$, there is a sequence $\left(I_{n}\right)_{n}$ of successive intervals in $\mathbb{N}$ such that for any $A \in[\mathbb{N}]^{\infty},\left[y_{n}\right]_{n}$ does not embed into $\left[e_{n}, n \notin \cup_{i \in A} I_{i}\right]$ as a permutation of a block sequence; or for any $\left[y_{n}\right]_{n} \leq E,\left(e_{n}\right)_{n}$ is permutatively equivalent to a block sequence of $\left[y_{n}\right]_{n}$.

But, as we saw in Proposition 5.3.20 a basic sequence $\left(y_{n}\right)_{n}$ being embedded in $\left[e_{n}\right]_{n}=E$ as a permutation of $\left(e_{n}\right)_{n}$, is not an $\mathcal{A}_{E}$-embbedding obtained from an admissible set for $E$, and this is fundamental for the proofs in this chapter to work. We have no evidence that in this case the statement is true, but it can not be obtained just by modifying the embedding notion in the proof of the third dichotomy.

We also obtain as a Corollary of Theorem 6.5.1 a dichotomy involving tight-by-sequences and spreading bases:

Theorem (Corollary 6.5.5). For any normalized basic sequence $\left(e_{n}\right)_{n}$ in a Banach space, there is $\left(x_{n}\right)_{n} \preceq\left(e_{n}\right)_{n}$ such that either $\left(x_{n}\right)_{n}$ is a tight-by-sequences basis, or spreading.

In order to prove Theorem 6.5.1, we follow the demonstration of the third dichotomy adapting the arguments for the context of $\mathcal{A}$-minimality and $\mathcal{A}$-tightness, using the results obtained in Chapter 5 for admissible systems of blocks. In the first place, we shall adapt for $\mathcal{D}_{E}$-block subspaces two technical Lemmas, whose original versions for block subspaces were proved in [22] and in [41], respectively.

Then, we shall define the asymptotic game $H_{Y, X}^{\mathcal{A}}$ for $\mathcal{A}$-tightness depending on a an admissible set for a Banach space $E . H_{Y, X}^{\mathcal{A}}$ differs from $H_{Y, X}$ in [22] in two natural aspects: the types of blocks that player $I I$ can choose and the winning condition. In the same way that was observed in [22] for the game $H_{Y, X}$ with constant $C$, the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$ is open for player $I$ and then, by the determinacy of open Gale-Stewart games, it is determined. This is proved in subsection 6.2.1.

In Section 6.3 we shall prove various lemmas following the ideas of Ferenczi and Rosendal, adapted for the concepts of $\mathcal{A}$-minimality and $\mathcal{A}$-tightness. We show that if $E$ is in some way saturated by $\mathcal{D}_{E}$-block subspaces $X$ and $Y$ such that player $I$ has a winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$, then $E$ has an $\mathcal{A}_{E}$-tight subspace. This result is fundamental to prove Theorem 6.5.1.

Before the proof of our main theorem it is necessary to introduce two games for $\mathcal{A}$-minimality: the game $G_{Y, X}^{\mathcal{A}}$ with constant $C$ and a version of that game assuming that finitely many moves have been made in $G_{Y, X}^{\mathcal{A}}$. This shall be done in Section 6.4. The main result in that section relates the existence of a winning strategy for player $I I$ in the game $H_{Y, X}^{\mathcal{A}}$ with the existence
of a winning strategy for player $I I$ in the game $G_{Y, X}^{\mathcal{A}}$. Finally, after the proof of Theorem 6.5.1 which will be showed in Section 6.5, some corollaries are shown.

### 6.1 Preliminary lemmas

Notation 6.1.1. Let $\left(x_{n}\right)_{n}$ and $\left(y_{n}\right)_{n}$ be two sequences of successive and finitely supported vectors of $E$. Let $Y=\left[y_{n}\right]_{n}$ and $X=\left[x_{n}\right]_{n}$. We write $Y \leq^{*} X$ if there is some $N \geq 1$ such that $y_{n} \in X$, for every $n \geq N$.

The following proposition was proved by Ferenczi and Rosendal (see Lemma 2.2 in [22]) in the context of block subspaces of a Banach space with basis. We follow the ideas of the demonstration given by them, adapting the arguments for $\mathcal{D}_{E}$-block subspaces.

Proposition 6.1.2. Let $E$ be a Banach space and $\mathcal{D}_{E}$ be a set of blocks for $E$. Suppose that $X=\left[x_{n}^{0}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace and $\left[x_{n}^{1}\right]_{n} \geq\left[x_{n}^{2}\right]_{n} \geq \ldots$ is a decreasing sequence of $\mathcal{D}_{X}$-block subspaces. Then, there exists a $\mathcal{D}_{X}$-block sequence $\left(y_{n}\right)_{n}$ such that $\left(y_{n}\right)_{n}$ is $\sqrt{K}$-equivalent with a $\mathcal{D}_{X}$-block sequence of $\left[x_{n}^{K}\right]_{n}$, for every $K \geq 1$.

Proof. Let $X=\left[x_{n}^{0}\right]_{n} \geq\left[x_{n}^{1}\right]_{n} \geq \ldots$ be a decreasing sequence of $\mathcal{D}_{X}$-block subspaces as in the hypothesis and $C$ be the basis constant of $\left(x_{n}^{0}\right)_{n}$. Recall that for each $k,\left(x_{n}^{k}\right)_{n} \in\left(\mathcal{D}_{X}\right)^{\omega}$. For $M>0$, consider $c(M, C)$ the constant that exists by Proposition 2.2.29 applied to $X$.

For each $K \geq 1$, let $M_{K}$ be the greatest non-negative integer such that

$$
\begin{equation*}
c\left(M_{K}, C\right) \leq \sqrt{K} \tag{6.1}
\end{equation*}
$$

Notice that $M_{K} \leq M_{K+1}$. Let $\left(l_{i}\right)_{i \geq 1}$ be the strictly increasing sequence of natural numbers that shows the subscripts in which $\left(M_{K}\right)_{K \geq 1}$ increases its value, so $M_{l_{i}}<M_{l_{i}+1}$, for every $i \geq 1$. For convenience, set $l_{0}=0$. Since $\left(l_{i}\right)_{i}$ is strictly increasing, we have that $i \leq l_{i}$. Also, since $\left(l_{i}\right)_{i}$ registers the changes in $M_{K}$, we obtain that $i \leq M_{l_{i}}$, for all $i \geq 1$. This means that if $i$ and $K$ are such that $l_{i}<K$, then

$$
\begin{equation*}
i \leq M_{l_{i}} \leq M_{K} \tag{6.2}
\end{equation*}
$$

Now, let us inductively construct the $\mathcal{D}_{X}$-block sequence $\left(y_{n}\right)_{n}$ we are seeking for. Set $n_{0}=0$ and let

$$
y_{0} \in\left[x_{n}^{l_{1}}: n_{0} \leq n \leq l_{1}\right] \cap \mathcal{D} .
$$

Let $n_{1}>l_{1}$ be such that $x_{n}^{l_{2}} \in\left[x_{i}^{l_{1}}: i>l_{1}\right]$, for all $n \geq n_{1}$, and

$$
y_{1} \in\left[x_{n}^{l_{2}}: n_{1} \leq n \leq n_{1}+l_{2}\right] \cap \mathcal{D}_{X} .
$$

Notice that $\left[x_{n}^{l_{2}}: n_{1} \leq n \leq n_{1}+l_{2}\right] \leq\left[x_{n}^{l_{1}}: n>l_{1}\right]$.

Suppose now that we have already defined finite sequences $\left(n_{0}, \ldots, n_{m}\right)$ and $\left(y_{0}, \ldots, y_{m}\right)$ such that $n_{i}>n_{i-1}+l_{i}$ with

$$
\left[x_{n}^{l_{i+1}}: n_{i} \leq n \leq n_{i}+l_{i+1}\right] \leq\left[x_{n}^{l_{i}}: n>l_{i+1}\right]
$$

and $y_{i} \in\left[x_{n}^{l_{i+1}}: n_{i} \leq n \leq n_{i}+l_{i+1}\right]$, for every $i \in\{0, \ldots, m\}$. Notice that $n_{i}>l_{i} \geq i$.
Let $n_{m+1}>n_{m}+l_{m+1}$ be such that $x_{n}^{l_{m+2}} \in\left[x_{i}^{l_{m+1}}: i>n_{m}+l_{m+1}\right]$, for all $n \geq n_{m+1}$. Let

$$
y_{m+1} \in\left[x_{n}^{l_{m+2}}: n_{m+1} \leq n \leq n_{m+1}+l_{m+2}\right] \cap \mathcal{D}_{X} .
$$

Consider the sequence $\left(y_{n}\right)_{n}$ obtained with this process. First notice that by construction $\left(y_{n}\right)_{n}$ is in fact a $\mathcal{D}_{X}$-block sequence.
Let $K \geq 1$, and set $i$ such that $l_{i}<K \leq l_{i+1}$, then we have $y_{i} \in\left[x_{n}^{l_{i+1}}: n_{i} \leq n \leq n_{i}+l_{i+1}\right]$ and

$$
\begin{aligned}
{\left[x_{n}^{l_{i+1}}: n_{i} \leq n \leq n_{i}+l_{i+1}\right] } & \subseteq\left[x_{n}^{l_{i+1}}: n \geq n_{i}\right] & & \\
& \leq\left[x_{n}^{K}: n \geq n_{i}\right] & & \text { since }\left(K \leq l_{i+1}\right) \\
& \leq\left[x_{n}^{K}: n \geq i\right] & & \text { since }\left(i \leq n_{i}\right)
\end{aligned}
$$

So, $x_{i-1}^{K}<y_{i}$ and $\left(y_{m}\right)_{m \geq i}$ is a $\mathcal{D}_{X}$-block sequence of $\left[x_{n}^{K}: n \geq i\right]$. Therefore, $\left(y_{n}\right)_{n}$ differs in $i-1$ terms from the block sequence $\left(x_{0}^{K}, x_{1}^{K}, \ldots, x_{i-1}^{K}, y_{i}, y_{i+1}, \ldots\right)$. By Equation (6.2) we have $i-1<M_{K}$, then such sequences are $c\left(M_{K}, C\right)$-equivalent, by Equation (6.1) they are $\sqrt{K}$-equivalent.

The following lemma is obtained after a small modification of Lemma 2.1 in [41]. For our purposes, we need to work with $\mathcal{D}_{E}$-block subspaces instead of classical block subspaces. Recall that we can identify an element $\left(y_{n}\right)_{n}$ of $b b_{\mathcal{D}}(X)$ with the $\mathcal{D}_{E}$-block subspace that it generates, as was mentioned in Remark 5.1.23.

Lemma 6.1.3. Let $E$ be a Banach space and $\mathcal{D}_{E}$ a set of blocks for $E$. Suppose that $X$ is $a \mathcal{D}_{E}$-block subspace. Let $N$ be a countable set and let $\mu: b b_{\mathcal{D}}(X) \rightarrow \mathbb{P}(N)$ satisfying either of the following monotonic conditions:

$$
V \leq^{*} W \Rightarrow \mu(V) \subseteq \mu(W)
$$

or

$$
V \leq^{*} W \Rightarrow \mu(V) \supseteq \mu(W)
$$

Then, there exists a "stabilizing" $\mathcal{D}_{X}$-block subspace $V_{0} \leq E$, i.e. a $\mathcal{D}_{X}$-block subspace such that $\mu(V)=\mu\left(V_{0}\right)$, for all $V \leq^{*} V_{0}$.

Proof. Suppose $E, \mathcal{D}_{E}, X, N$ and $\mu$ as in the hypothesis with $\mu$ satisfying

$$
V \leq^{*} W \Rightarrow \mu(V) \subseteq \mu(W)
$$

By contradiction, suppose that for all $W \mathcal{D}_{X}$-block subspace there is $V \leq^{*} W$ such that $\mu(V) \varsubsetneqq \mu(W)$.
Let us construct a transfinite sequence $\left(W_{\gamma}\right)_{\gamma<\omega_{1}}$ of $\mathcal{D}_{X}$-block subspaces such that if $\gamma<\eta<\omega_{1}$, then $W_{\eta} \leq^{*} W_{\gamma}$ and $\mu\left(W_{\eta}\right) \varsubsetneqq \mu\left(W_{\gamma}\right)$.

First, set $W_{0}:=X$. Suppose that for some $\beta<\omega_{1}$, we have defined a sequence $\left(W_{\gamma}\right)_{\gamma<\beta}$ such that for $\gamma<\eta<\beta$ :

- $W_{\eta} \leq{ }^{*} W_{\gamma}$,
- $\mu\left(W_{\eta}\right) \varsubsetneqq \mu\left(W_{\gamma}\right)$.

If $\beta$ is a successor ordinal $\eta+1$, by hypothesis there is $V \leq^{*} W_{\eta}$ such that $\mu(V) \varsubsetneqq \mu\left(W_{\eta}\right)$, then set $W_{\beta}:=V$.

If $\beta$ is a limit ordinal, since $\beta<\omega_{1}$, there is $\left(\gamma_{n}\right)_{n}$ an increasing sequence of ordinals converging to $\beta$. Notice that for each $n \in \mathbb{N}$ :

- $W_{\gamma_{n+1}} \leq^{*} W_{\gamma_{n}}$,
- $\mu\left(W_{\gamma_{n+1}}\right) \varsubsetneqq \mu\left(W_{\gamma_{n}}\right)$,
- $\bigcap_{i \leq n} W_{\gamma_{i}}$ is infinite dimensional.

For each $n \in \mathbb{N}$, consider $\left(w_{k}^{n}\right)_{k}$ a normalized $\mathcal{D}_{X}$-block basis of $W_{\gamma_{n}}$. Take $v_{0}=w_{0}^{0}$. Since $W_{\gamma_{n+1}} \leq{ }^{*} W_{\gamma_{n}}$, for each $n \in \mathbb{N}$ we can pick $v_{n}$ such that:

- $v_{n} \in\left\{w_{k}^{n}: k \in \mathbb{N}\right\}$, so $v_{n}$ is a $\mathcal{D}_{X}$-block vector;
- $v_{n} \in \bigcap_{i \leq n} W_{\gamma_{i}}$,
- $v_{k}<v_{k+1}$. for $0 \leq k<n$.

Thus, $\left(v_{n}\right)_{n}$ is a $\mathcal{D}_{X}$-block sequence. Let $W_{\eta}:=\left[v_{n}\right]_{n}$. For all $n \in \mathbb{N}$ we have that $W_{\eta} \leq^{*} W_{\gamma_{n}}$ and $\mu\left(W_{\eta}\right) \varsubsetneqq \mu\left(W_{\gamma_{n}}\right)$, what ends the construction. The sequence $\left(\mu\left(W_{\eta}\right)\right)_{\eta<\omega_{1}}$ is an uncountable strongly decreasing chain (with respect to the inclusion) of subsets of $N$, which contradicts that $N$ is a countable set.

The case when $\mu$ satisfies the condition:

$$
V \leq^{*} W \Rightarrow \mu(V) \supseteq \mu(W)
$$

is analogous.

### 6.2 Games for tightness

In this section we shall define and study the game $H_{Y, X}^{\mathcal{A}}$ with constant $C \geq 1$. This asymptotic game is a modification of the game $H_{Y, X}$ given in [22]: In $H_{Y, X}^{\mathcal{A}}$ player $I I$ choses a $\mathcal{D}_{E}$-block instead of a block vector and we ask in the winning condition not only that the outcome sequence is equivalent to the previously fixed basis of $Y$, but also that the outcome belongs to the admissible set $\mathcal{A}_{E}$. The proofs are strongly based on the fact that the winning condition remains closed in this case.

After defining such game $H_{Y, X}^{\mathcal{A}}$ with constant $C$, we shall prove that its is determined by formulating it as an open Gale-Stewart game. Then, we shall use Theorem 3.4.1 to conclude.

Notation 6.2.1. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\mathcal{D}_{E}$ be a set of blocks for $E$. Let $X=\left[x_{n}\right]_{n}$ be a $\mathcal{D}_{E}$-block subspace. For $k \leq m$ natural numbers, we denote the set $\left[x_{n}: k \leq n \leq m\right] \cap \mathcal{D}_{X}$ by $X[k, m]$.

Definition 6.2.2. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ be an admissible set for $E$. Let $X=\left[x_{n}\right]_{n}$ be a $\mathcal{D}_{E}$-block subspace, and let $Y$ be a Banach space with normalized basis $\left(y_{n}\right)_{n}$. Suppose $C \geq 1$. We define the asymptotic game $H_{Y, X}^{\mathcal{A}}$ with constant $C$ between players $I$ and $I I$ taking turns as follows: I plays a natural number $n_{i}$, and II plays a natural number $m_{i}$ and a not necessarily normalized $\mathcal{D}_{X}$-block vector $u_{i} \in X\left[n_{0}, m_{0}\right]+\ldots+X\left[n_{i}, m_{i}\right]$. Diagramatically,

| I | $n_{0}$ | $n_{1}$ |  |  |
| :---: | :---: | :---: | :---: | :---: |
| II |  | $m_{0}, u_{0}$ |  | $m_{1}, u_{1}$ |$\quad \ldots$

The sequence $\left(u_{n}\right)_{n}$ is the outcome of the game and we say that II wins the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$, if $\left(u_{n}\right)_{n} \sim_{C}\left(y_{n}\right)_{n}$ and $\left(u_{n}\right)_{n} \in \mathcal{A}_{X}$.

The next subsection is dedicated to prove that the game $H_{Y, X}^{\mathcal{A}}$ is determined.

### 6.2.1 Determinacy of the games for tightness

To simplify the notation along this subsection, let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of blocks for $E$ and $\mathcal{A}_{E}$ an admissible set for $E$. Also, take $X=\left[x_{n}\right]_{n}$ a $\mathcal{D}_{E}$-block subspace and $Y$ a Banach space with normalized basis $\left(y_{n}\right)_{n}$. Let $C \geq 1$ be a constant. In this section we will show that the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$ is open for player $I$ and, therefore, it is determined.

Since we have fixed an arbitrary $\mathcal{D}_{E^{-}}$block subspace $X$ on which we shall play, let $\mathcal{D}:=\mathcal{D}_{X}$ and $\mathbb{X}:=\mathbb{N} \times \mathbb{N} \times \mathcal{D}$ (to simplify the notation). Endow $\mathbb{N}$ and $\mathcal{D}$ with their discrete topology. So, $\mathbb{X}$ is endowed with its discrete topology.

Remark 6.2.3. For each $k \in \mathbb{N}$, the set

$$
E Q_{k}:=\left\{\left(z_{i}\right)_{i=0}^{k} \in \mathcal{D}^{k+1}:\left(z_{i}\right)_{i=0}^{k} \sim_{C}\left(y_{i}\right)_{i=0}^{k}\right\}
$$

is a clopen subset of $\mathcal{D}^{k+1}$.
Theorem 6.2.4. The following sets are open in $\mathbb{X}^{\omega}$ :
(i) The set $N E$ of sequences $\left(\left(n_{i}, m_{i}, u_{i}\right)\right)_{i}$ in $\mathbb{X}^{\omega}$ such that the sequence of vectors $\left(u_{i}\right)_{i}$ are not $C$-equivalent to $\left(y_{i}\right)_{i}$.
(ii) The set $N A$ of sequences $\left(\left(n_{i}, m_{i}, u_{i}\right)\right)_{i}$ in $\mathbb{X}^{\omega}$ such that the sequence of vectors $\left(u_{i}\right)_{i}$ are not in $\mathcal{A}_{X}$.
(iii) The set $N L$ of non legal runs in the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$.

Proof. Consider the following continuous projections for $i \in \mathbb{N}$

$$
\begin{aligned}
& \mathbb{X}^{\omega} \xrightarrow{\Pi_{i}} \mathbb{X} \\
&\left(p_{i}\right)_{i} \mapsto \\
&\left(n_{i}, m_{i}, u_{i}\right),
\end{aligned}
$$

and the natural projections

$$
\begin{array}{lll}
(n, m, u) \in \mathbb{X} & \xrightarrow[\pi_{0}]{\longleftrightarrow} & n \in \mathbb{N}, \\
(n, m, u) \in \mathbb{X} & \xrightarrow[\pi_{1}]{\longmapsto} & m \in \mathbb{N}, \\
(n, m, u) \in \mathbb{X} & \xrightarrow[\pi_{2}]{\longmapsto} & u \in \mathcal{D} .
\end{array}
$$

Consider for each $k \in \mathbb{N}$, the continuous functions $f_{i}: \mathbb{X}^{\omega} \rightarrow \mathcal{D}^{k+1}$ (which are finite products of projections $\pi_{j} \circ \Pi_{i}$ ), given by

$$
\left(p_{i}\right)_{i} \stackrel{f_{k}}{\longrightarrow}\left(u_{0}, u_{1}, \ldots, u_{k}\right)
$$

(i) Since two sequences are $C$-equivalent if, and only if, each initial sequence of them are $C$-equivalent, we can conclude that the set $N E$ is open in $\mathbb{X}^{\omega}$. This is true because $N E$ may be written as follows:

$$
N E:=\bigcup_{k \in \mathbb{N}} f_{k}^{-1}\left[\mathcal{D}^{k+1} \backslash E Q_{k}\right]
$$

Recall that $E Q_{k}$ is clopen (see Remark 6.2.3).
(ii) Recall that an infinite sequence of blocks belongs to $\mathcal{A}_{X}$ if, and only if, its respective finite initial sequences are in $\left[\mathcal{A}_{X}\right]\left(\mathcal{A}_{X}\right.$ is closed in $\left(\mathcal{D}_{X}\right)^{\omega}$, see the Proposition 5.2.4), So, the set $N A$ is open in $\mathbb{X}^{\omega}$. Actually, $N A$ can be written as follows:

$$
N A:=\bigcup_{k \in \mathbb{N}} f_{k}^{-1}\left[\mathcal{D}^{k+1} \backslash\left[\mathcal{A}_{X}\right]\right] .
$$

Recall that $\mathcal{D}^{k+1} \backslash\left(\left[\mathcal{A}_{X}\right] \cap \mathcal{D}^{k+1}\right)$ is clopen (see Remark 5.1.27).
(iii) A run is not legal if there is some $k$ where player $I I$, in her $k+1$-th round, chooses $m_{k}<n_{k}$ or the block $u_{k}$ does no belong to $X\left[n_{0}, m_{0}\right]+X\left[n_{1}, m_{1}\right]+\ldots+X\left[n_{k}, m_{k}\right]$. We
shall show that $N L$ is open by writing it as the union of two open sets: the set $N L_{1}$ of runs for which some $m_{k}<n_{k}$, and the set $N L_{2}$ of runs such that some $u_{k}$ (chosen by $I I)$ does no belong to $X\left[n_{0}, m_{0}\right]+X\left[n_{1}, m_{1}\right]+\ldots+X\left[n_{k}, m_{k}\right]$.

Notice that the natural embedding $i: \mathbb{N} \rightarrow \mathbb{R}$, where $\mathbb{R}$ is endowed with the usual topology, is continuous. So, the functions $g_{k}: \mathbb{X}^{\omega} \rightarrow \mathbb{R}$ such that $\left(p_{i}\right)_{i} \stackrel{g_{k}}{\longrightarrow}$ $\left(i \circ \pi_{1} \circ \Pi_{k}-i \circ \pi_{0} \circ \Pi_{k}\right)\left(\left(p_{i}\right)_{i}\right)=n_{k}-m_{k}$ are continuous for all $k \in \mathbb{N}$. So, the set

$$
\begin{aligned}
N L_{1} & =\bigcup_{k \in \mathbb{N}}\left\{\left(p_{i}\right)_{i}=\left(\left(n_{i}, m_{i}, u_{i}\right)\right)_{i} \in \mathbb{X}^{\omega}: m_{k}<n_{k}\right\} \\
& =\bigcup_{k \in \mathbb{N}} g_{k}^{-1}[(0,+\infty)]
\end{aligned}
$$

Therefore, $N L_{1}$ is open.
Fix $k \in \mathbb{N}$. Notice that $u_{k} \in X\left[n_{0}, m_{0}\right]+X\left[n_{1}, m_{1}\right]+\ldots+X\left[n_{k}, m_{k}\right]$ if, and only if, $P_{\left[0, n_{0}\right)}\left(u_{k}\right)=0_{X}, \forall j \in\{0, \ldots, k-1\}\left(P_{\left(m_{j}, n_{j+1}\right)}\left(u_{k}\right)=0_{X}\right)$ and $P_{\left(m_{k},+\infty\right)}\left(u_{k}\right)=0_{X}$. Notice that the inclusion $l: \mathcal{D} \hookrightarrow X$ is continuous (after endowing $X$ with its norm topology), so the functions $h_{j}: \mathbb{X}^{\omega} \rightarrow X$ defined as

$$
\begin{aligned}
& \left(p_{i}\right)_{i}=\left(\left(n_{i}, m_{i}, u_{i}\right)\right)_{i} \quad \stackrel{h_{0}}{\longmapsto} P_{\left[0, n_{0}\right)} \circ l\left(u_{k}\right) \\
& \left(p_{i}\right)_{i}=\left(\left(n_{i}, m_{i}, u_{i}\right)\right)_{i} \\
& \xrightarrow{h_{j+1}} P_{\left(m_{j}, n_{j+1}\right)} \circ l\left(u_{k}\right) \quad \text { for } 0 \leq j \leq k-1 \\
& \left(p_{i}\right)_{i}=\left(\left(n_{i}, m_{i}, u_{i}\right)\right)_{i} \quad \stackrel{h_{k}}{\longmapsto} P_{\left(m_{k},+\infty\right)} \circ l\left(u_{k}\right)
\end{aligned}
$$

are continuous and

$$
\begin{aligned}
N L_{2}= & \bigcup_{k \in \mathbb{N}}\left\{\left(p_{i}\right)_{i}=\left(\left(n_{i}, m_{i}, u_{i}\right)\right)_{i} \in \mathbb{X}^{\omega}: P_{\left[0, n_{0}\right)}\left(u_{k}\right) \neq 0_{X}\right. \text { or } \\
& \left.\exists j \in\{0, \ldots, k-1\}\left(P_{\left(m_{j}, n_{j+1}\right)}\left(u_{k}\right) \neq 0_{X}\right) \text { or } P_{\left(m_{k},+\infty\right)}\left(u_{k}\right) \neq 0_{X}\right\} \\
= & \bigcup_{k \in \mathbb{N}} \bigcup_{j=0}^{k} h_{j}^{-1}\left[X \backslash\left\{0_{X}\right\}\right] .
\end{aligned}
$$

Therefore $N L_{2}$ is open and the set of non-legal runs $N L$ is open.

Theorem 6.2.5. The game $H_{Y, X}^{\mathcal{A}}$ with constant $C$ is determined.
To prove this proposition, we shall define an equivalent auxiliary Gale-Stewart game which will be open for player I, and then, use the determinacy theorem for open games (Theorem 3.4.1) to conclude that it is determined. Let us define such an auxiliary game.

## An auxiliary game

Let $x^{\prime} \in \mathcal{D}$ be an arbitrary vector. We define the asymptotic game $A_{Y, X}^{\mathcal{A}}\left(\mathbb{X} ; \mathbb{A}^{\prime}\right)$ with constant $C$ between players $I$ and $I I$ taking turns as follows: $I$ plays a triplet $\left(n_{i}, 0, x^{\prime}\right) \in \mathbb{X}$, and $I I$
plays $\left(0, m_{i}, u_{i}\right) \in \mathbb{X}$, with $u_{i} \in X\left[n_{0}, m_{0}\right]+\ldots+X\left[n_{i}, m_{i}\right]$. Diagramatically,
I $\quad\left(n_{0}, 0, x^{\prime}\right)$
$\left(n_{1}, 0, x^{\prime}\right)$
II $\quad\left(0, m_{0}, u_{0}\right)$ $\left(0, m_{1}, u_{1}\right) \quad \ldots$

Before giving the criterion of victory, we introduce some notation. Let $T \subset[\mathbb{X}]^{<\infty}$ be the pruned tree of legal moves of the game $A_{Y, X}^{\mathcal{A}}\left(\mathbb{X} ; \mathbb{A}^{\prime}\right)$ with constant $C$. We denote a legal run in such game as $\vec{p}=\left(p_{n}\right)_{n} \in[T]$ where

- if $n=2 k: p_{n}=\left(n_{k}, 0, x^{\prime}\right)$, which corresponds with the play of player $I$ in the $(k+1)$-th round,
- if $n=2 k+1: p_{n}=\left(0, m_{k}, u_{k}\right)$, which corresponds with the play of player $I I$ in the $(k+1)$-th - round,
for $k \in \mathbb{N}$.
With this notation in mind, consider the following subset of $\mathbb{X}^{\omega}$ :

$$
\begin{aligned}
\mathbb{A}^{\prime}:= & \left\{\left(p_{n}\right)_{n} \in[T]: \exists i \in \mathbb{N} \text { s.t. }\left(u_{k}\right)_{k=0}^{i} \in \mathcal{D}^{i+1} \backslash\left[\mathcal{A}_{X}\right]\right. \text { or } \\
& \left.\exists i \in \mathbb{N} \text { s.t. }\left(u_{k}\right)_{k=0}^{i} \not \chi_{C}\left(y_{k}\right)_{k=0}^{i}\right\} .
\end{aligned}
$$

We say that player $I$ wins the game $A_{Y, X}^{\mathcal{A}}\left(\mathbb{X} ; \mathbb{A}^{\prime}\right)$ with constant $C$ if the run $\left(p_{n}\right)_{n}$ belongs to $\mathbb{A}^{\prime}$.

The game $A_{Y, X}^{\mathcal{A}}\left(\mathbb{X} ; \mathbb{A}^{\prime}\right)$ is a version of the game $H_{Y, X}^{\mathcal{A}}$ where players $I$ and $I I$ make their movements in the same set $\mathbb{X}$. This is necessary to use the Gale-Stewart theorem.

Proposition 6.2.6. The game $A_{Y, X}^{\mathcal{A}}\left(\mathbb{X} ; \mathbb{A}^{\prime}\right)$ with constant $C$ is determined.
Proof. We shall prove that the pay-off set $\mathbb{A}^{\prime}$ of the game $A_{Y, X}^{\mathcal{A}}\left(\mathbb{X} ; \mathbb{A}^{\prime}\right)$ with constant $C$ is open as subset of $[T]$, then the conclusion follows from the determinacy theorem for open games, Theorem 3.4.1 of Chapter 3.

First notice that the pay-off set can be rewritten as $\mathbb{A}^{\prime}=A_{1} \cup A_{2}$, where

$$
\begin{align*}
A_{1} & :=\bigcup_{i \in \mathbb{N}}\left\{\left(q_{n}\right)_{n} \in[T]:\left(u_{k}\right)_{k=0}^{i} \in \mathcal{D}^{i+1} \backslash\left[\mathcal{A}_{X}\right]\right\},  \tag{6.3}\\
A_{2} & :=\bigcup_{i \in \mathbb{N}}\left\{\left(q_{n}\right)_{n} \in[T]:\left(u_{k}\right)_{k=0}^{i} \not \chi_{C}\left(y_{k}\right)_{k=0}^{i}\right\} . \tag{6.4}
\end{align*}
$$

Notice that the sets $A_{1}$ and $A_{2}$ are homeomorphic to the sets $N A \cap\left(\mathbb{X}^{\omega} \backslash N L\right)$ and $N E \cap\left(\mathbb{X}^{\omega} \backslash N L\right.$ ), where $N A, N E$ and $N L$ are open sets of $\mathbb{X}^{\omega}$ (see Theorem 6.2.4). So, they are open as subsets of the set of legal runs $\mathbb{X}^{\omega} \backslash N L$ in the game $H_{Y, X}^{\mathcal{A}}$ (endowed with the relative topology), which is homeomorphic to $[T]$. Then, $A_{1}, A_{2}$ and, therefore, $\mathbb{A}^{\prime}$ are open subsets of $[T]$ and the game $A_{Y, X}^{\mathcal{A}}$ with constant $C$ is determined. .

Proposition 6.2.7. The games $A_{Y, X}^{\mathcal{A}}\left(\mathbb{X} ; \mathbb{A}^{\prime}\right)$ with constant $C$ and $H_{Y, X}^{\mathcal{A}}$ with constant $C$ are equivalent.

Proof. Suppose that player $I$ has a winning strategy $\sigma: \mathbb{X}^{<\infty} \rightarrow \mathbb{X}$ for the game $A_{Y, X}^{\mathcal{A}}\left(\mathbb{X} ; \mathbb{A}^{\prime}\right)$ with constant $C$. Let us define inductively the strategy $\rho$ for $I$ to play in the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$.

Let $\sigma(\emptyset)=\left(n_{0}, 0, x^{\prime}\right)$, then define $\rho(\emptyset)=n_{0}$. Suppose $\left(n_{0}, m_{0}, u_{0}, \ldots, n_{k}, m_{k}, u_{k}\right)$ is a legal position in the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$ where $I$ has played following $\rho$. then $\left(n_{0}, 0, x^{\prime}\right),\left(0, m_{0}, u_{0}\right), \ldots,\left(n_{k}, 0, x^{\prime}\right),\left(0, m_{k}, u_{k}\right)$ is a legal position in the game $A_{Y, X}^{\mathcal{A}}$ with constant $C$. Let $\sigma\left(\left(n_{0}, 0, x^{\prime}\right),\left(0, m_{0}, u_{0}\right), \ldots,\left(n_{k}, 0, x^{\prime}\right),\left(0, m_{k}, u_{k}\right)\right)=\left(n_{k+1}, 0, x^{\prime}\right)$ and set $\rho\left(n_{0}, m_{0}, u_{0}, \ldots, n_{k}, m_{k}, u_{k}\right)=n_{k+1}$.

Since $\sigma$ is a winning strategy, the sequence $\left(u_{i}\right)_{i}$ is not $C$-equivalent to $\left(y_{i}\right)_{i}$ or $\left(u_{i}\right)_{i}$ does not belong to $\mathcal{A}_{X}$. Then, $\rho$ is a winning strategy for $I$ in the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$.
Analogously, we can obtain from a winning strategy for $I$ in the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$ a winning strategy for $I$ in the game $A_{Y, X}^{\mathcal{A}}$ with constant $C$. The proof follows with the same arguments if we consider strategies for player $I I$.

Proof of Theorem 6.2.5. consider the game $A_{Y, X}^{\mathcal{A}}\left(\mathbb{X} ; \mathbb{A}^{\prime}\right)$ with constant $C$, which is determined for Proposition 6.2.6. If player $I$ (respectively player $I I$ ) has a winning strategy for the game $A_{Y, X}^{\mathcal{A}}\left(\mathbb{X} ; \mathbb{A}^{\prime}\right)$, then by Proposition 6.2 .7 , player $I$ (respectively player $I I$ ) has a winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$. Thus, the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$ is determined.

Remark 6.2.8. We proved that the game $A_{Y, X}^{\mathcal{A}}\left(\mathbb{X} ; \mathbb{A}^{\prime}\right)$ with constant $C$ is open for the player $I$ and that it is equivalent to the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$. For those reasons we shall say that the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$ is open for player $I$.

### 6.3 Relation of games and existence of an $\mathcal{A}$-tight subspace

In this section we follow the ideas of Ferenczi and Rosendal to prove three lemmas. As we already mentioned, the main result of this section, Lemma 6.3.8, says that if $E$ is in some way saturated of $\mathcal{D}_{E}$-block subspaces $X$ and $Y$ such that player $I$ has a winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$, then $E$ has an $\mathcal{A}_{E}$-tight subspace. Intuitively, player $I$ in $H_{Y, X}^{\mathcal{A}}$ having a winning strategy for $H_{Y, X}^{\mathcal{A}}$, under certain hypothesis, is related to the existence of an $\mathcal{A}$-tight subspace.

Notation 6.3.1. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of blocks and $\mathcal{A}_{E}$ an admissible set for $E$. Let $X=\left[x_{n}\right]_{n}$ be a $\mathcal{D}_{E}$-block subspace, $Y$ be a Banach space and $\left(I_{i}\right)_{i}$ be a sequence of successive non-empty intervals of natural numbers.
(i) Let $K$ be a positive constant. Denote

$$
Y \stackrel{\mathcal{A}}{\hookrightarrow}_{K}\left(X, I_{i}\right)
$$

if there is $A \in[\mathbb{N}]^{\infty}$ containing 0 , such that $Y \stackrel{\mathcal{A}}{\hookrightarrow}_{K}\left[x_{n}, n \notin \cup_{i \in A} I_{i}\right]$.
(ii) Denote

$$
Y \stackrel{\mathcal{A}}{\hookrightarrow}\left(X, I_{i}\right)
$$

if there is $A \in[\mathbb{N}]^{\infty}$ such that $Y \stackrel{\mathcal{A}}{\longrightarrow}\left[x_{n}, n \notin \cup_{i \in A} I_{i}\right]$.
Remark 6.3.2. Notice that under the hypothesis of the Notation 6.3.1, if there is some $A \in[\mathbb{N}]^{\infty}$ such that $Y \xrightarrow{\mathcal{A}}\left[x_{n}, n \notin \cup_{i \in A} I_{i}\right]$ and $0 \notin A$, then there is some $B \in[\mathbb{N}]^{\infty}$ containing 0 such that $Y \stackrel{\mathcal{A}}{\hookrightarrow}\left[x_{n}, n \notin \cup_{i \in B} I_{i}\right]$.

Proof. Without loss of generality we can assume that $0 \in I_{0}$. Suppose that there is some $A \in[\mathbb{N}]^{\infty}$ such that $Y \stackrel{\mathcal{A}}{\hookrightarrow}\left[x_{n}, n \notin \cup_{i \in A} I_{i}\right]$ and $0 \notin A$. Since $Y$ is infinite dimensional, $\mathbb{N} \backslash \cup_{i \in A} I_{i}$ must be infinite. Let $k \in A$ be the least integer such that

$$
\left|\bigcup_{\substack{i \in A \\ i \leq k}} I_{i}\right| \geq\left|I_{0}\right|
$$

Let $l=\max I_{k}$. Consider $M:=[0, l] \backslash \bigcup_{\substack{i \in A \\ i \leq k}} I_{i}$ and $m$ its cardinal. Notice that $I_{0} \subseteq M$ and $I_{0}<l-m$. Let $M^{\prime}:=[l+1, \infty) \backslash \bigcup_{\substack{i \neq A \\ i>k}} I_{i}^{-}$and

$$
\sigma: \mathbb{N} \backslash \cup_{i \in A} I_{i}=M \cup M^{\prime} \rightarrow(l-m, l] \cup M^{\prime}
$$

being the order preserving bijection between those two sets. By the Proposition 2.2.29, there is an isomorphism $T$ between $\left[x_{n}: n \in \mathbb{N} \backslash \cup_{i \in A} I_{i}\right]$ and $\left[x_{f(n)}: n \in \mathbb{N} \backslash \cup_{i \in A} I_{i}\right]$ such that $T\left(x_{n}\right)=x_{f(n)}$. Thus, by the Proposition 5.2.7, if $Y \stackrel{\mathcal{A}}{\hookrightarrow}\left[x_{n}, n \notin \cup_{i \in A} I_{i}\right]$ then $Y \stackrel{\mathcal{A}}{\hookrightarrow}\left[x_{f(n)}: n \in \mathbb{N} \backslash \cup_{i \in A} I_{i}\right]$. Set $B:=\{0\} \cup\{n \in A: n>k\}$. Since

$$
\left[x_{f(n)}: n \in \mathbb{N} \backslash \cup_{i \in A} I_{i}\right] \subseteq\left[x_{n}: n \in \mathbb{N} \backslash \cup_{i \in B} I_{i}\right]
$$

we can conclude that

$$
Y \stackrel{\mathcal{A}}{\hookrightarrow}\left[x_{n}: n \in \mathbb{N} \backslash \cup_{i \in B} I_{i}\right] .
$$

Proposition 6.3.3. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of
blocks and $\mathcal{A}_{E}$ an admissible set for $E$. Let $X=\left[x_{n}\right]_{n}$ be a $\mathcal{D}_{E}$-block subspace, $Y$ be Banach space with normalized basis $\left(y_{n}\right)_{n}$ and suppose $K \geq 1$. If II has a winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ with constant $K$, then for any sequence $\left(I_{i}\right)_{i}$ of successive intervals we have

$$
Y \stackrel{\mathcal{A}}{\hookrightarrow}_{K}\left(X, I_{i}\right) .
$$

Proof. Suppose $X=\left[x_{n}\right]_{n}$ and $Y=\left[y_{n}\right]_{n}$ as in the hypothesis. Suppose that $I I$ has a winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ with constant $K$. Set $\left(I_{i}\right)_{i}$ a sequence of successive intervals of natural numbers.

Consider the following plays in the game $H_{Y, X}^{\mathcal{A}}$. In his first run, $I$ plays $n_{0} \in \mathbb{N}$ such that $n_{0}>I_{0}$. Player II move the pair $\left(m_{0}, u_{0}\right)$ following her winning strategy. Notice that $u_{0} \in\left[x_{n}: n \notin I_{0}\right]$ and set $a_{0}=0$. In the ( $i+1$ )-th round, player $I$ chooses an integer $n_{i}$ such that there is some $a_{i} \in \mathbb{N}$ with $\left.I_{a_{i}} \subset\right] m_{i-1}, n_{i}\left[\right.$, and $I I$ plays the pair ( $m_{i}, u_{i}$ ) according to her winning strategy. Notice that $u_{i} \in X\left[n_{0}, m_{0}\right]+\ldots+X\left[n_{i}, m_{i}\right] \subseteq\left[x_{n}: n \notin \cup_{j \in A_{i}} I_{j}\right]$, where $A_{i}=\left\{a_{j}: j \leq i\right\}$.

If we continue with this playing, $I I$ will produce a sequence $\left(u_{n}\right)_{n} \in \mathcal{A}_{X}$ which is $K$ equivalent to $\left(y_{n}\right)_{n}$ and, by construction, for each $i \in \mathbb{N}, u_{i} \in\left[x_{n}: n \notin \cup_{j \in A} I_{j}\right]$, where $A=\cup_{i \in \mathbb{N}} A_{i} \in[\mathbb{N}]^{\infty}$ and $0 \in A$. Thus,

$$
Y \stackrel{\mathcal{A}}{\hookrightarrow}\left[x_{n}: n \notin \bigcup_{j \in A} I_{j}\right],
$$

which ends the proof.

Remark 6.3.4. Proposition 6.3 .3 tells us that if II has a winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ with constant $K$ then $Y$ is not $\mathcal{A}_{E}$-tight in $X$.

In addition to what is said in Proposition 6.3.3, the next proposition shows that if player $I$ has a winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ with constant $K+\varepsilon$, for every $\mathcal{D}_{X}$-block subspace $Y$ of $X$, then for every such $Y$ there are some successive intervals $I_{0}<\ldots<I_{n}<\ldots$ such that $Y \underset{\overbrace{h}}{\mathcal{A}}\left(X, I_{j}\right)$.

Lemma 6.3.5. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$ be an admissible system of blocks for $E$. Suppose that $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace and that $K$ and $\varepsilon$ are positive constants such that for all $\mathcal{D}_{X}$-block subspace $Y$ of $X$ there is a winning strategy for player $I$ in the game $H_{Y, X}^{\mathcal{A}}$ with constant $K+\varepsilon$. Then, there is a function $f: b b_{\mathcal{D}}(X) \rightarrow[\mathbb{N}]^{\infty}$ such that for all $\mathcal{D}_{X}$-block subspace $Y$, if $I_{j}=\left[f(Y)_{2 j}, f(Y)_{2 j+1}\right]$, then $Y \underset{\overbrace{K}}{\mathcal{A}}\left(X, I_{j}\right)$.

Proof. Suppose $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace, $K$ and $\varepsilon$ as in the hypothesis. We will divide this proof in six steps:

1. By hypothesis, for each $\mathcal{D}_{X}$-block subspace $Y$ of $X$ there is a winning strategy for player $I$ in the game $H_{Y, X}^{\mathcal{A}}$ with constant $K+\varepsilon$. Let $\sigma$ be the function that maps each $Y$ to such winning strategy $\sigma_{Y}$.
2. Let $C \geq 1$ be the basis constant of $\left(x_{n}\right)_{n}$. Let $\rho=1+\frac{\varepsilon}{K}$. Now, let $0<\theta<1$ be such that $(1+\theta)(1-\theta)^{-1}=\rho$. Take $\Delta=\left(\delta_{n}\right)_{n}$ a sequence of positive numbers such that $2 C K^{2} \sum_{n \in \mathbb{N}} \delta_{n}=\theta$.
Let $\left(w_{n}\right)_{n}$ be a $K C$-basic sequence of not necessarily normalized blocks with $\frac{1}{K} \leq$ $\left\|w_{i}\right\| \leq K$, for any $i \in \mathbb{N}$. If $\left(u_{n}\right)_{n}$ is such that $\forall i \in \mathbb{N}\left(\left\|w_{i}-u_{i}\right\|<\delta_{i}\right)$, then

$$
2 K C \sum_{n \in \mathbb{N}} \frac{\left\|w_{n}-u_{n}\right\|}{\left\|w_{n}\right\|}=2 C K^{2} \sum_{n \in \mathbb{N}} \delta_{n}=\theta<1 .
$$

Thus, by Theorem 2.2.11, $\left(u_{n}\right)_{n} \sim_{\rho}\left(w_{n}\right)_{n}$.
3. We shall obtain some collection of sets of vectors $\left\{D_{n}: n \in \mathbb{N}\right\}$ which will be used in step 4 to assist in the construction of a strategy for player I. Since $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$ is an admissible system of blocks for $E$, we have that for $X$, the sequence $\left(\delta_{n}\right)_{n}$ and $K$, there is a collection $\left(D_{n}\right)_{n}$ of non-empty sets of vectors of $\mathcal{D}_{X}$ such that

C-1 For each $n$ and for each $d \in[\mathbb{N}]^{<\infty}$ such that there is $w \in \mathcal{A}_{X}$ with $\operatorname{supp}_{X}(w)=d$, we have that there are a finite number of vectors $u \in D_{n}$ such that $\operatorname{supp}_{x}(u)=d$.

C-2 For all sequence $\left(w_{i}\right)_{i} \in \mathcal{A}_{X}$ satisfying $1 / K \leq \min _{i}\left\|w_{i}\right\| \leq \sup _{i}\left\|w_{i}\right\| \leq K$, for all $n$ there is $u_{n} \in D_{n}$, such that

$$
\begin{aligned}
& \mathrm{C}-2.1 \sup _{X}\left(u_{n}\right) \subseteq \operatorname{supp}_{X}\left(w_{n}\right) . \\
& \mathrm{C}-2.2\left\|w_{n}-u_{n}\right\|<\delta_{n} . \\
& \mathrm{C}-2.3\left(u_{i}\right)_{i} \in \mathcal{A}_{X} .
\end{aligned}
$$

4. Suppose now that $Y$ is a $\mathcal{D}_{X}$-block subspace with normalized $\mathcal{D}_{X}$-block basis $\left(y_{n}\right)_{n}$. Suppose that $p=\left(n_{0}, u_{0}, m_{0}, \ldots, n_{i}, u_{i}, m_{i}\right)$, with $u_{j} \in D_{j}$ for $j \leq i$ is a legal position in the game $H_{Y, X}^{\mathcal{A}}$ in which $I$ has played according to $\sigma_{Y}$.

\[

\]

We write $p<k$ if $n_{j}, u_{j}, m_{j}<k$ for all $j \leq i$. Since $I I$ is playing in $\prod_{j \leq i} D_{j}$, using the condition C-1, for every $k$ there is only a finite number of such legal positions $p$ which satisfies $p<k$. So, for every $k \in \mathbb{N}$ the following maximum exists:

$$
\begin{equation*}
\alpha(k):=\max \left\{k, \max \left\{\sigma_{Y}(p): p<k\right\}\right\} . \tag{6.5}
\end{equation*}
$$

We set $I_{k}=[k, \alpha(k)]$. The intervals in $\left(I_{k}\right)_{k}$ are not necessarily disjoint, but it is
possible to extract a subsequence of successive intervals of it, with $I_{0}$ as first element.
5. To prove that $Y \stackrel{\mathcal{A}}{K}^{( }\left(X, I_{j}\right)$ we shall show that for every $A \in[\mathbb{N}]^{\infty}$, containing 0 , $Y \stackrel{A}{\nrightarrow}_{K}\left[x_{n}: n \notin \cup_{k \in A} I_{k}\right]$.

By contradiction, suppose there is $A \in[\mathbb{N}]^{\infty}$ containing 0 and a sequence of blocks $\left(w_{n}\right)_{n} \in \mathcal{A}_{X} \cap\left[x_{n}: n \notin \cup_{k \in A} I_{k}\right]$ such that

$$
\begin{equation*}
\left(y_{n}\right)_{n} \sim_{K}\left(w_{n}\right)_{n} . \tag{6.6}
\end{equation*}
$$

Recall that, since $\left(y_{n}\right)_{n}$ is normalized (recall the Remark 5.1.16), $\frac{1}{K} \leq\left\|w_{n}\right\| \leq K$, for all $n \in \mathbb{N}$.

By the step 3, condition C-2, we can find for each $i$ a block $u_{i} \in D_{i}$ such that $\left\|w_{i}-u_{i}\right\|<\delta_{i}, \operatorname{supp}_{x}\left(u_{i}\right) \subseteq \operatorname{supp}_{x}\left(w_{i}\right),\left(u_{n}\right)_{n} \in \mathcal{A}_{X}$, and

$$
\begin{equation*}
\left(u_{n}\right)_{n} \sim_{\rho}\left(w_{n}\right)_{n} \tag{6.7}
\end{equation*}
$$

By Equation (6.7), $\left(u_{n}\right)_{n} \sim_{K \rho}\left(y_{n}\right)_{n}$. Considering that $\rho=1+\frac{\varepsilon}{K}$, we can conclude that $\left(u_{n}\right)_{n} \sim_{K+\varepsilon}\left(y_{n}\right)_{n}$.
6. Finally, we will construct a playing $\vec{p}$ in the game $H_{Y, X}^{\mathcal{A}}$ with constant $K+\varepsilon$, where player $I$ will follow his winning strategy and the outcome will be the sequence $\left(u_{n}\right)_{n}$. Which means that $I$ wins the game, leading us to a contradiction. In order to do that define $n_{i}, m_{i}$ natural numbers and $a_{i} \in A$ as follows:

Let $a_{0}=0$ and $n_{0}=\sigma_{Y}(\emptyset)=\alpha(0)$, then, by definition of $I_{k}, I_{0}=[0, \alpha(0)]=\left[0, n_{0}\right]$. Find $a_{1} \in A$, such that $n_{0}, u_{0}, a_{0}<a_{1}$ and set $m_{0}=a_{1}-1$. Then $p_{0}=\left(n_{0}, m_{0}, u_{0}\right)$ is a legal position in $H_{Y, X}^{\mathcal{A}}$ in which $I$ has played according to his winning strategy $\sigma_{Y}$. Since $w_{0} \in X\left[n_{0}, m_{0}\right]$ and $\operatorname{supp}_{X}\left(u_{0}\right) \subseteq \operatorname{supp}_{X}\left(w_{0}\right)$, we have $u_{0} \in X\left[n_{0}, m_{0}\right]$.

Now, as $p_{0}<a_{1}$, by the definition of the function $\alpha$, if $n_{1}=\sigma_{Y}\left(n_{0}, m_{0}, u_{0}\right)$, we obtain $n_{1} \leq \alpha\left(a_{1}\right)$. Therefore, $] m_{0}, n_{1}\left[=\left[m_{0}+1, n_{1}-1\right]=\left[a_{1}, n_{1}-1\right] \subseteq\left[a_{1}, \alpha\left(a_{1}\right)\right]=I_{a_{1}}\right.$.

Suppose by induction that $n_{0}, \ldots, n_{i}, m_{0}, \ldots, m_{i}$ and $a_{0}, \ldots, a_{i} \in A$ have been defined. Since $\left[0, n_{0}\left[\subseteq I_{0}\right.\right.$ and $] m_{j}, n_{j}+1\left[\subseteq I_{a_{j+1}}\right.$, for all $j<i$, we have

$$
u_{i} \in X\left[n_{0}, m_{0}\right]+X\left[n_{1}, m_{1}\right]+\ldots+X\left[n_{i}, \infty[.\right.
$$

Find some $a_{i+1} \in A$ greater than $n_{0}, \ldots, n_{i}, u_{0}, \ldots, u_{i}$ and $a_{0}, \ldots, a_{i}$ and let $m_{i}=a_{i+1}-1$, then

$$
u_{i} \in X\left[n_{0}, m_{0}\right]+X\left[n_{1}, m_{1}\right]+\ldots+X\left[n_{i}, m_{i}\right] .
$$

Therefore $p_{i}=\left(n_{0}, m_{0}, u_{0}, \ldots, n_{i}, m_{i}, u_{i}\right)$ is a legal position of the game $H_{Y, X}^{\mathcal{A}}$ with
constant $K+\varepsilon$ in which $I$ has played according to $\sigma_{Y}$. Since $p_{i}<a_{i+1}$, we have

$$
n_{i+1}=\sigma_{Y}\left(n_{0}, m_{0}, u_{0}, \ldots, n_{i}, m_{i}, u_{i}\right) \leq \alpha\left(a_{i+1}\right)
$$

and

$$
] m_{i}, n_{i+1}\left[=\left[m_{i}+1, n_{i+1}-1\right]=\left[a_{i+1}, n_{i+1}-1\right] \subseteq\left[a_{i+1}, \alpha\left(a_{i+1}\right)\right]=I_{a_{i+1}} .\right.
$$

Set $\vec{p}$ the legal run such that each $p_{i}$ is a legal position for the game. Such $\vec{p}$ is the run we were looking for to produce a contradiction.

Remark 6.3.6. Notice that we are not dealing with "continuously $\mathcal{A}$-tight" spaces, so we do not need to use the strategic uniformization theorem (see Theorem (35.32), [33]) in the first step to ensure that such intervals are obtained in a Borel way, as was done in [22].

The following lemma gives us a criterion for passing from the existence of intervals dependent on $K$ for which $Y$ is not $\mathcal{A}$-embedded in $\left(I_{j}^{(K)}\right)$, to the existence of intervals $\left(J_{j}\right)_{j}$ for which $Y$ is not embedded for any constant $K$.

Lemma 6.3.7. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$ be an admissible system of blocks for $E$. Suppose that $X=\left[x_{n}\right]_{n}$ is a $\mathcal{D}_{E}$-block subspace and $Y$ is a Banach space with normalized basis $\left(y_{n}\right)_{n}$. If for every constant $K$ there are successive intervals of natural numbers $\left(I_{n}^{(K)}\right)$ such that $Y \stackrel{\mathcal{A}_{K}}{K}\left(X, I_{j}^{(K)}\right)$, then there is a sequence of successive intervals $\left(J_{j}\right)_{j}$ such that $Y \stackrel{\text { A }}{\stackrel{\leftrightarrow}{\rightarrow}}\left(X, J_{j}\right)$.

Proof. Let $E$, the pair $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right), X=\left[x_{n}\right]_{n}$ and $Y=\left[y_{n}\right]_{n}$ be as in the hypothesis. We will construct the intervals $\left(J_{j}\right)_{j}$ inductively. The idea is to find such a sequence satisfying:
(i) For each $n \geq 1, J_{n}$ contains one interval of each $\left(I_{i}^{(n)}\right)_{i}$.
(ii) If $M=\min J_{n}-1$ and $K=\lceil n \cdot c(M)\rceil$ (where $c(M)$ is the constant which existence is guaranteed by Proposition 2.2.29 for $\left.\left(x_{n}\right)_{n}\right)$, then $\max J_{n}>\max I_{0}^{(K)}+M$.

This can be done as follows: Take $J_{0}=I_{0}^{(1)}$ (it could be taken to be another interval or some bigger interval, what we want to prove does not depend on the initial intervals). Now, let $a=\min I_{1}^{(1)}, M=a-1$ and $K=\lceil c(M)\rceil$. Let $b=\max \left\{\max I_{0}^{(K)}+M, \max I_{1}^{(1)}\right\}+1$ and define $J_{1}=[a, b]$. Then, $J_{0}<J_{1}, I_{0}^{(1)} \subseteq J_{1}$ and $\max J_{1}>\max I_{0}^{(K)}+M$.

Now suppose that we have defined $J_{0}, \ldots, J_{n}$ satisfying $(i)$ and (ii). Let $a$ be a natural number greater than max $J_{n}$, put $M=a-1$ and $K=\lceil(n+1) \cdot c(M)\rceil$. Take $b>\max I_{0}^{(K)}+M$ and such that there exists $j_{i} \in \mathbb{N}$ with $I_{j(i)}^{(i)} \subseteq[a, b]$, for all $i \in\{1, \ldots, n+1\}$ (this can be done because the intervals are finite and we are looking at just the first $n+1$ sequences). Let $J_{n+1}:=[a, b]$. By construction, such $J_{n+1}$ satisfies the conditions $(i)$ and (ii).

By contradiction, suppose that $A \in[\mathbb{N}]^{\infty}$ and that for some integer $N$, we have

$$
Y \stackrel{\mathcal{A}}{\hookrightarrow} N\left[x_{n}: n \notin \cup_{i \in A} J_{i}\right] .
$$

This implies that there is a sequence $\left(w_{n}\right)_{n}$ of $\mathcal{D}_{X}$-blocks in $\mathcal{A}_{X} \cap\left[x_{n}: n \notin \cup_{i \in A} J_{i}\right]$ such that $\left(y_{n}\right)_{n} \sim_{N}\left(w_{n}\right)_{n}$.

Pick $a \in A$ such that $a \geq N$ and set $M=\min J_{a}-1$ and $K=\lceil a \cdot c(M)\rceil$. Let us define an isomorphic embedding $T$ from

$$
\left[x_{n}: n \notin \cup_{i \in A} J_{i}\right]
$$

into

$$
\left[x_{n}: \max I_{0}^{(K)}<n \leq \max J_{a}\right]+\left[x_{n}: n \notin \cup_{i \in A} J_{i} \& n>\max J_{a}\right]
$$

by setting

$$
T\left(x_{n}\right)= \begin{cases}x_{n}, & \text { if } n>\max J_{a}  \tag{6.8}\\ x_{\max I_{0}^{(K)}+n+1}, & \text { if } n \leq M\end{cases}
$$

Notice that $T$ is an isomorphism between those two $\mathcal{D}_{X}$-block subspaces. So, by ii) in Proposition 5.2.7, we have $\left(T\left(w_{n}\right)\right)_{n} \in \mathcal{A}_{X}$.

Since $T$ only changes at most $M$ vectors from $\left(x_{n}\right)_{n}$, it is a $C(M)$-embedding, then,

$$
\left(y_{n}\right)_{n} \sim_{N}\left(w_{n}\right)_{n} \sim_{C(M)}\left(T\left(w_{n}\right)\right)_{n}
$$

and because $N \cdot c(M) \leq a \cdot c(M) \leq K$, we obtain

$$
\begin{equation*}
Y \stackrel{\mathcal{A}}{\hookrightarrow}_{K}\left[x_{n}: n \notin \cup_{i \in A} J_{i} \& n>\max J_{a}\right]+\left[x_{n}: \max I_{0}^{(K)}<n \leq \max J_{a}\right] \tag{6.9}
\end{equation*}
$$

Now, since for each $n \geq 1, J_{n}$ contains one interval of each $\left(I_{i}^{(n)}\right)_{i}$, for any $l \in A$ such that $l \geq K$ there is $b(l) \in \mathbb{N}$ such that $I_{b(l)}^{(K)} \subseteq J_{l}$. Let $B=\{0\} \cup\{b(l): l \in A, l \geq K\}$. Then, $i d:\left[x_{n}: n \notin \cup_{i \in A} J_{i} \& n>\max J_{a}\right]+\left[x_{n}: \max I_{0}^{(K)}<n \leq \max J_{a}\right] \longrightarrow\left[x_{n}: n \notin \cup_{i \in B} I_{i}^{(K)}\right]$ is an isomorphism onto its image and by ii) in Proposition 5.2.7 and Equation (6.9) we have:

$$
Y \stackrel{\mathcal{A}}{\hookrightarrow}_{K}\left[x_{n}: n \notin \cup_{i \in B} I_{i}^{(K)}\right],
$$

which contradicts our initial hypothesis.

The next lemma uses a "diagonalization" argument to prove that if a space $E$ is saturated with $\mathcal{D}_{E}$-block subspaces $X$ such that for all $Y \leq X, I$ has a winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ for any constant $K$, then there is a $\mathcal{A}_{E}$-tight $\mathcal{D}_{E}$-block subspace $X$.

Lemma 6.3.8. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$ be an admissible system of blocks for $E$. Suppose that for every $\mathcal{D}_{E}$-block subspace $Z$ and constant $K$ there is a $\mathcal{D}_{Z}$-block subspace $X$ such that for all $\mathcal{D}_{X}$-block subspace $Y$, I has a winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ with constant $K$. Then, there is a $\mathcal{D}_{E}$-block subspace $X$ which is $\mathcal{A}_{E}$-tight.

Proof. Let $E$ and $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$ be as in the hypothesis. The idea of the proof is to construct inductively a sequence $X_{0} \geq X_{1} \geq X_{2} \geq \ldots$ of $\mathcal{D}_{E}$-block subspaces and corresponding functions $f_{K}: b b_{\mathcal{D}}\left(X_{K}\right) \rightarrow[\mathbb{N}]^{\infty}$ such that for all $V \leq X_{K}$, if $I_{j}=\left[f_{K}(V)_{2 j}, f_{K}(V)_{2 j+1}\right]$, then $V \stackrel{A}{\rightarrow}_{K^{2}}\left(X_{K}, I_{j}\right)$. Once constructed, we will use the Proposition 6.1.2 to obtain the desired $\mathcal{D}_{E}$-block subspace.

Consider $X_{0}=E$ and let $\varepsilon>0$. Using the hypothesis for $K=1+\varepsilon$, there is a $\mathcal{D}_{X_{0}}$ block subspace $X_{1} \leq X_{0}$ such that for all $\mathcal{D}_{X_{1}}$-block subspace $Y \leq X_{1}, I$ has a winning strategy for the game $H_{Y, X_{1}}^{\mathcal{A}}$ with constant $1+\varepsilon$. By Lemma 6.3.5 there is a function $f_{1}$ : $b b_{\mathcal{D}}\left(X_{1}\right) \rightarrow[\mathbb{N}]^{\infty}$ such that for all $\mathcal{D}_{X_{1}}$-block subspace $V \leq X_{1}$, if $I_{j}=\left[f_{1}(V)_{2 j}, f_{1}(V)_{2 j+1}\right]$, then $V \stackrel{\text { A }}{\boldsymbol{A}_{1}}\left(X_{1}, I_{j}\right)$.

Suppose we have defined a finite sequence $X_{0} \geq X_{1} \geq \ldots \geq X_{n}$ of $\mathcal{D}_{E}$-block subspaces and functions $f_{K}: b b_{\mathcal{D}}\left(X_{K}\right) \rightarrow[\mathbb{N}]^{\infty}$ such that for all $\mathcal{D}_{X_{K}}$-block subspace $V \leq X_{K}$, if $I_{j}=\left[f_{K}(V)_{2 j}, f_{K}(V)_{2 j+1}\right]$, then $V \stackrel{A}{\nrightarrow}_{K^{2}}\left(X, I_{j}\right)$, for all $K \leq n$.

Applying the hypothesis to $X_{n}$, there is a $\mathcal{D}_{X_{n}}$-block subspace $X_{n+1} \leq X_{n}$ such that for all $\mathcal{D}$-block subspace $Y \leq X_{n+1}$ and for all $\varepsilon>0, I$ has a winning strategy for the game $H_{Y, X_{n+1}}^{\mathcal{A}}$ with constant $(n+1)^{2}+\varepsilon$. By Lemma 6.3.5, there is a function $f_{n+1}: b b_{\mathcal{D}}\left(X_{n+1}\right) \rightarrow[\mathbb{N}]^{\infty}$ such that for all $\mathcal{D}_{X_{n+1}}$-block subspace $V \leq X_{n+1}$, if $I_{j}=\left[f_{n+1}(V)_{2 j}, f_{n+1}(V)_{2 j+1}\right]$, then $V \stackrel{A}{\hookrightarrow}_{(n+1)^{2}}\left(X_{n+1}, I_{j}\right)$.

Suppose that we continue this procedure and we obtain a sequence of subspaces

$$
X_{0} \geq \ldots \geq X_{K} \geq \ldots
$$

of $\mathcal{D}_{E}$-block subspaces and functions $f_{K}: b b_{\mathcal{D}}\left(X_{K}\right) \rightarrow[\mathbb{N}]^{\infty}$ such that for all $\mathcal{D}_{X_{K}}$-block subspaces $V$, if $I_{j}^{K}=\left[f_{K}(V)_{2 j}, f_{K}(V)_{2 j+1}\right]$, then $V \stackrel{\mathcal{A}}{\nmid}_{K^{2}}\left(X_{K}, I_{j}^{K}\right)$, for all $K \geq 1$.

Applying Lemma 6.1.2 to that sequence, we find a $\mathcal{D}_{E}$-block subspace $X_{\infty}=\left[x_{n}^{\infty}\right]_{n} \leq X_{0}=$ $E$, such that for each $K \geq 1$ there is a $\mathcal{D}_{X_{K}}$-block sequence $\left(z_{n}^{K}\right)_{n}$ with $Z_{K}=\left[z_{n}^{K}\right]_{n} \leq X_{K}$ such that

$$
\begin{equation*}
\left(x_{n}^{\infty}\right)_{n} \sim_{\sqrt{K}}\left(z_{n}^{K}\right)_{n} \tag{6.10}
\end{equation*}
$$

Let $Y=\left[y_{n}\right]_{n} \leq X_{\infty}$ be a $\mathcal{D}_{E^{-}}$-block subspace of $X_{\infty}$. For each $K \geq 1$ there exists a $\mathcal{D}_{Z_{K}}{ }^{-}$ block subspace $V_{K}=\left[v_{n}^{K}\right]_{n}$ (using the form of the isomorphism given in Equation (6.10) and
(ii) in Proposition 5.2.3) such that

$$
\begin{equation*}
\left(y_{n}\right)_{n} \sim_{\sqrt{K}}\left(v_{n}^{K}\right)_{n} \tag{6.11}
\end{equation*}
$$

and for such $V_{K}$ we have that if $I_{j}^{K}=\left[f_{K}\left(V_{K}\right)_{2 j}, f_{K}\left(V_{K}\right)_{2 j+1}\right]$, then

$$
\begin{equation*}
V_{K} \stackrel{\wedge}{\nrightarrow}_{K^{2}}\left(X_{K}, I_{j}^{K}\right) . \tag{6.12}
\end{equation*}
$$

Claim: There are successive intervals $\left(J_{j}^{K}\right)_{j}$ such that

$$
\begin{equation*}
V_{K} \stackrel{A}{\nrightarrow}_{K^{2}}\left(Z_{K}, J_{j}^{K}\right) . \tag{6.13}
\end{equation*}
$$

Proof. (of the claim) Let $\left(n_{j}\right)_{j}$ and $\left(m_{j}\right)_{j}$ be increasing sequences from $\mathbb{N}$, such that for each $j \in \mathbb{N}$ we have

- $n_{j}<m_{j}<n_{j+1}$,
- there is $k_{j}>0$ with

$$
\operatorname{supp}_{X_{K}}\left(z_{n_{j}}^{K}\right)<I_{k_{j}}^{K}<\operatorname{supp}_{X_{K}}\left(z_{m_{j}}^{K}\right) .
$$

Let $J_{j}^{K}=\left[n_{j}, m_{j}\right]$, for each $j \in \mathbb{N}$. Such sequences $\left(n_{j}\right)_{j}$ and $\left(m_{j}\right)_{j}$ exist because each $I_{j}^{K}$ and $\operatorname{supp}_{X_{K}}\left(z_{j}^{K}\right)$ are finite subsets. Notice that for each $A \in[\mathbb{N}]^{\infty}$ we have

$$
\begin{equation*}
\left[z_{n}^{K}: n \notin \bigcup_{j \in A} J_{j}^{K}\right] \subseteq\left[x_{n}^{K}: n \notin \bigcup_{j \in A} I_{k_{j}}^{K}\right] \tag{6.14}
\end{equation*}
$$

Now, suppose that there is $B \in[\mathbb{N}]^{\infty}$ such that

$$
V_{K}{\stackrel{\mathcal{A}}{K^{2}}}_{\hookrightarrow}\left[z_{n}^{K}: n \notin \bigcup_{i \in B} J_{i}^{K}\right] .
$$

Then, there is $\left(w_{n}\right)_{n} \in \mathcal{A}_{E} \cap\left[z_{n}^{K}: n \notin \bigcup_{i \in B} J_{i}^{K}\right]$ such that $\left(v_{n}^{K}\right)_{n} \sim_{K^{2}}\left(w_{n}\right)_{n}$. By Equation (6.14), $\left(w_{n}\right)_{n} \in \mathcal{A}_{E} \cap\left[x_{n}^{K}: n \notin \bigcup_{j \in B} I_{k_{j}}^{K}\right]$, then

$$
V_{K} \stackrel{\mathcal{A}}{\hookrightarrow}_{K^{2}}\left[x_{n}^{K}: n \notin \cup_{j \in A} I_{j}^{K}\right],
$$

where $A=\left\{k_{j}: j \in B\right\}$, which contradicts Equation (6.12).

Now, we will show that $\left.Y \stackrel{{\underset{\leftrightarrow}{\leftrightarrows}}_{K}}{( } X_{\infty}, J_{j}^{K}\right)$ : Suppose, on the contrary, that $Y \stackrel{\mathcal{A}}{\hookrightarrow}_{K}\left(X_{\infty}, J_{j}^{K}\right)$, then, there is $A \in[\mathbb{N}]^{\infty}$ with $0 \in A$ and a sequence $\left(w_{n}\right)_{n} \in \mathcal{A}_{E} \cap\left[x_{n}^{\infty}: n \notin \cup_{i \in A} J_{j}^{K}\right]$ such that

$$
\begin{equation*}
\left(y_{n}\right)_{n} \sim_{K}\left(w_{n}\right)_{n} \tag{6.15}
\end{equation*}
$$

Recall that $Y$ and each $Z_{K}$ are $\mathcal{D}_{E}$-block subspaces. By the isomorphism given in Equation (6.10) and using ii) in Proposition 5.2.7, we can find $\left(u_{n}^{K}\right)_{n} \in Z_{K}^{\omega}$ (image of $\left(w_{n}\right)_{n}$ by such isomorphism), such that $\left(u_{n}^{K}\right)_{n} \in \mathcal{A}_{E} \cap\left[z_{n}^{K}: n \notin \cup_{i \in A} J_{j}^{K}\right]$ and

$$
\begin{equation*}
\left(u_{n}^{K}\right)_{n} \sim_{\sqrt{K}}\left(w_{n}\right)_{n} . \tag{6.16}
\end{equation*}
$$

Then, using the Equations (6.11), (6.15) and (6.16), we obtain

$$
\begin{equation*}
\left(v_{n}^{K}\right)_{n} \sim_{\sqrt{K}}\left(y_{n}\right)_{n} \sim_{K}\left(w_{n}\right)_{n} \sim_{\sqrt{K}}\left(u_{n}^{K}\right)_{n} . \tag{6.17}
\end{equation*}
$$

Thus, $\left(v_{n}^{K}\right)_{n} \sim_{K^{2}}\left(u_{n}^{K}\right)_{n}$, which means that

$$
\left[v_{n}^{K}\right]_{n}=V_{K}{\stackrel{\mathcal{A}}{K^{2}}}^{K^{2}}\left[z_{n}^{K}: n \notin \bigcup_{i \in A} J_{j}^{K}\right] .
$$

This contradicts Equation (6.13).
We have proved that, for every $Y \leq X_{\infty}$ and for every $K \geq 1$, there is a sequence of successive intervals $\left(J_{j}^{K}\right)_{j}$, such that $Y \stackrel{\mathcal{A}}{\boldsymbol{\sim}}_{K}\left(X_{\infty}, J_{j}^{K}\right)$. Using Lemma 6.3.7 there exists a sequence of successive intervals $\left(L_{i}^{Y}\right)_{i}$ such that

$$
Y \stackrel{A}{\nrightarrow}\left(X_{\infty}, L_{j}^{Y}\right),
$$

which finishes our proof.

### 6.4 Games for minimality

In the following we shall define a game $G_{Y, X}^{\mathcal{A}}$ associated with an $\mathcal{A}$-minimality notion. The games $G_{Y, X}^{\mathcal{A}}$ and $G_{X, Y}$ both with constant $C$, are slightly different. In $G_{Y, X}^{\mathcal{A}}$, players $I$ and II must choose $\mathcal{A}_{E}$-block subspaces and vectors in $\left[\mathcal{A}_{E}\right]$, instead of block subspaces and any block vectors as in $G_{X, Y}$ from [22]. The main result in this section relates the existence of a winning strategy for the player $I I$ in the game $H_{Y, X}^{\mathcal{A}}$ with the existence of a winning strategy for player $I I$ in the game $G_{Y, X}^{\mathcal{A}}$. We can think that in a way the existence of a winning strategy for player $I I$ in $H_{Y, X}^{\mathcal{A}}$ "implies" the existence of an $\mathcal{A}$-minimal subspace. Let us define such a game.

Definition 6.4.1. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of blocks and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose $L$ and $M$ are $\mathcal{D}_{E}$-block subspaces of a Banach space $E$ and $C \geq 1$ a constant. We define the asymptotic game $G_{L, M}^{\mathcal{A}}$ with constant $C$ between players $I$ and II taking turns as follows. In the $(i+1)$-th round, I chooses a subspace $E_{i} \subseteq L$, spanned by a finite $\mathcal{D}_{L}$-block sequence, a not necessarily normalized $\mathcal{D}_{L^{-}}$ block $u_{i} \in E_{0}+\ldots+E_{i}$, and a natural number $m_{i}$. On the other hand, II plays in the first time an integer $n_{0}$, and in all successive rounds II plays a subspace $F_{i}$ spanned by a finite
$\mathcal{D}_{M}$-block sequence, a not necessarily normalized $\mathcal{D}_{M}$-block vector $v_{i} \in F_{0}+\ldots+F_{i}$ and an integer $n_{i+1}$.

For a move to be legal we demand that $n_{i} \leq E_{i}, m_{i} \leq F_{i}$ and that for each play in the game, the chosen vectors $u_{i}$ and $v_{i}$ satisfy $\left(u_{0}, \ldots, u_{i}\right) \in\left[\mathcal{A}_{E}\right]$ and $\left(v_{0}, \ldots, v_{i}\right) \in\left[\mathcal{A}_{E}\right]$. We present the following diagram:

$$
\text { I } \begin{array}{cc}
n_{0} \leq E_{0} \subseteq L & n_{1} \leq E_{1} \subseteq L \\
& u_{1} \in E_{0}+E_{1}, m_{1} \\
& \left(u_{0}, u_{1}\right) \in\left[\mathcal{A}_{E}\right]
\end{array}
$$

$$
\text { II } \begin{array}{cc}
n_{0} & m_{0} \leq F_{0} \subseteq M \\
& m_{1} \leq F_{1} \subseteq M \\
v_{0} \in F_{0}, n_{1} & v_{1} \in F_{0}+F_{1}, n_{2} \\
& \left(v_{0}, v_{1}\right) \in\left[\mathcal{A}_{E}\right]
\end{array}
$$

The sequences $\left(u_{i}\right)_{i}$ and $\left(v_{i}\right)_{i}$ are the outcome of the games and we say that II wins the game $G_{L, M}^{A}$ with constant $C$, if $\left(u_{i}\right)_{i} \sim_{C}\left(v_{i}\right)_{i}$.

Notice that in the game $G_{L, M}^{\mathcal{A}}$ stated in Definition 6.4.1 the outcome $\left(u_{i}\right)_{i}$ and $\left(v_{i}\right)_{i}$ belong to $\mathcal{A}_{E}$, since for each $n \in \mathbb{N}$, we have $\left(u_{i}\right)_{i \leq n},\left(v_{i}\right)_{i \leq n} \in\left[\mathcal{A}_{E}\right]$ and $\mathcal{A}_{E}$ is closed in $\left(\mathcal{D}_{E}\right)^{\omega}$.

Proposition 6.4.2. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of blocks and $\mathcal{A}_{E}$ an admissible set for $E$. Suppose $X$ and $Y$ are $\mathcal{D}_{E}$-block subspaces of a Banach space $E$ with basis and $C \geq 1$ is a constant. Suppose that $\vec{p}$ is a legal run in the game such that every finite stage of $\vec{p}$ is a finite stage of a run where II wins the game $G_{Y, X}^{\mathcal{A}}$ with constant $C$. Then $\vec{p}$ is a run where II wins the game $G_{Y, X}^{\mathcal{A}}$ with constant $C$.

Proof. Let $\vec{p}$ be a legal run in the game $G_{Y, X}^{\mathcal{A}}$ with constant $C$, where each legal position after $(i+1)$ complete rounds is given by $p_{i}=\left(n_{0}, E_{0}, u_{0}, m_{0}, F_{0}, v_{0}, n_{1}, \ldots, E_{i}, u_{i}, m_{i}, F_{i}, v_{i}, n_{i+1}\right)$. By hypothesis, for each $i \in \mathbb{N}, p_{i}$ is a legal position of a legal run where $I I$ wins, so $\left(u_{j}\right)_{j=0}^{i} \sim_{C}\left(v_{j}\right)_{j=0}^{i}$. Therefore $\vec{p}$ is a legal run where $I I$ wins the game.

The next lemma relates the games $H_{Y, X}^{\mathcal{A}}$ and $G_{Y, X}^{\mathcal{A}}$ with same constant.
Lemma 6.4.3. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of blocks and $\mathcal{A}_{E}$ an admissible set for $E$. If $X$ and $Y$ are $\mathcal{D}_{E}$-block subspaces of $E$ such that player II has a winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$, then II has a winning strategy for the game $G_{Y, X}^{\mathcal{A}}$ with constant $C$.

Proof. Let $E, \mathcal{D}_{E}, \mathcal{A}_{E}$ and $C \geq 1$ be as in the hypothesis. Suppose that $X=\left[x_{n}\right]_{n}$ and $Y=\left[y_{n}\right]_{n}$ are $\mathcal{D}_{E}$-block subspaces. We shall exhibit the move that player $I I$ has to do after $i$ rounds in the game $G_{Y, X}^{\mathcal{A}}$ with constant $C$, and we will prove that such moves determine a winning strategy for $I I$ in the game $G_{Y, X}^{\mathcal{A}}$ with constant $C$. For each $i$ (even $i=0$ ), suppose player $I$ has played $i$ times, and we have the following stage in the game $G_{Y, X}^{\mathcal{A}}$ :

I

$$
\begin{aligned}
& 0 \leq E_{0} \subseteq Y \\
& u_{0} \in E_{0}, m_{0}
\end{aligned}
$$

$$
\begin{gathered}
0 \leq E_{i} \subseteq Y \\
u_{i} \in E_{0}+\ldots+E_{i}, m_{i} \\
\left(u_{0}, \ldots, u_{i}\right) \in\left[\mathcal{A}_{E}\right]
\end{gathered}
$$

II 0

$$
\begin{array}{ccc}
m_{0} \leq F_{0} \subseteq X & \ldots & m_{i-1} \leq F_{i-1} \subseteq X \\
v_{0} \in F_{0}, 0 & & v_{i-1} \in F_{0}+\ldots+F_{i-1}, 0 \\
& & \left(v_{0}, \ldots, v_{i-1}\right) \in\left[\mathcal{A}_{E}\right]
\end{array}
$$

Notice that we are asking to player $I I$ to play $n_{j}=0$ for all $j$, so player $I$ has more possibilities to play and makes the game more difficult for $I I$. Let us write each block vector $u_{j}$ as $\sum_{k=0}^{k_{j}} \lambda_{k}^{j} y_{k}$, for all $j \leq i$. We can assume that $k_{j-1}<k_{j}$, for all $j \leq i$.

Consider the following run in the game $H_{Y, X}^{\mathcal{A}}$ :

where $I$ consecutively plays $m_{0}$ the first $\left(k_{0}+1\right)$-times, then consecutively plays $m_{j}$ for $\left(k_{j}-k_{j-1}\right)$-times, for any $j \in\{1, \ldots, i\}$, and then he plays $m_{i}$ constantly. Meanwhile, II moves according to her winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$, which means that

$$
w^{\prime}:=\left(w_{0}, \ldots, w_{k_{i}}, w^{\prime}{ }_{0}, w^{\prime}{ }_{1}, \ldots\right) \in \mathcal{A}_{X}=\mathcal{A}_{E} \cap X^{\omega} .
$$

Since $\left(u_{0}, \ldots, u_{i}\right) \in\left[\mathcal{A}_{E}\right]$, by condition $\left.d\right)$ in Definition 5.1.24, we have that there is $\left(t_{n}\right)_{n} \in Y^{\omega}$ such that $u^{\prime}=\left(u_{0}, \ldots, u_{i}, t_{0}, t_{1}, \ldots\right) \in \mathcal{A}_{Y}=\mathcal{A}_{E} \cap Y^{\omega}$. Notice that $u^{\prime} *_{Y}\left(y_{n}\right)_{n}=u^{\prime} \in \mathcal{A}_{E}$ and $\left(y_{n}\right)_{n} \in b b_{\mathcal{D}}(E) \subseteq \mathcal{A}_{E}$, thus, using condition $c$ ) of Definition 5.1.24, we have

$$
v^{\prime}:=u^{\prime} *_{Y} w^{\prime} \in \mathcal{A}_{E} \cap X^{\omega}=\mathcal{A}_{X}
$$

If $v^{\prime}=\left(v_{j}^{\prime}\right)_{j}$, then it follows from the inductive construction that:

- $v^{\prime}{ }_{j}=v_{j}$, for $j<i$,
- $v_{i}^{\prime}=\sum_{k=0}^{k_{i}} \lambda_{k}^{i} w_{k}$,
- $\left(v_{0}^{\prime}, \ldots, v_{i}^{\prime}\right) \in\left[\mathcal{A}_{X}\right]$.

Set $v_{i}:=v_{i}^{\prime}$ and

$$
F_{i}=X\left[m_{i}, \max \left\{p_{k_{i-1}+1}, \ldots, p_{k_{i}}\right\}\right] .
$$

Therefore, $\left(v_{0}, \ldots, v_{i}\right) \in\left[\mathcal{A}_{X}\right], v_{i} \in F_{0}+\ldots+F_{i}$, with $m_{i} \leq F_{i} \subseteq X$. This means that $\left(F_{i}, v_{i}, 0\right)$ is a legal position for $I I$ to play in the game $G_{Y, X}^{\mathcal{A}}$ with constant $C$ in its $(i+1)$-th round.

Suppose that we have continued with the game, where $I I$ have played by using the previously procedure in every round, and we have obtained the outcome: $\left(u_{i}\right)_{i}$ (which $I$ played) and $\left(v_{i}\right)_{i}$ (which $I I$ played).

Using i) of the Proposition 5.2.7, $\left(u_{i}\right)_{i}$ and $\left(v_{i}\right)_{i}$ are in $\mathcal{A}_{E}$ (each initial part is in $\left[\mathcal{A}_{E}\right]$ ). Since $\left(u_{i}\right)_{i}$ and $\left(v_{i}\right)_{i}$ are defined with the same coefficients over $\left(y_{i}\right)_{i}$ and $\left(w_{i}\right)_{i}$, respectively, we have that $\left(u_{i}\right)_{i} \sim_{C}\left(v_{i}\right)_{i}$. Hence, we have showed the moves that II can do in each round to win the game. Consequently, $I I$ has a winning strategy for the game $G_{Y, X}^{\mathcal{A}}$ with constant $C$.

Now, we give some definitions which shall be used in the proof of the main theorem of this chapter.

Notation 6.4.4. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\mathcal{D}_{E}$ be a set of blocks for $E$. We denote by $\mathcal{F}_{E}$ to the set of subspaces of $E$ generated by finite $\mathcal{D}_{E}$-block sequences.

Definition 6.4.5. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\mathcal{D}_{E}$ be a set of blocks for $E$. A state $s$ is a pair $(a, b)$ with $a, b \in\left(\mathcal{D}_{E} \times \mathcal{F}_{E}\right)^{<\omega}$, such that if $a=\left(a_{0}, A_{0}, \ldots, a_{i}, A_{i}\right)$ and $b=\left(b_{0}, B_{0}, \ldots, b_{j}, B_{j}\right)$, then $j=i$ or $j=i-1$. Let us denote by $\mathbf{S}_{E}$ the set of states.

Remark 6.4.6. In the hypothesis of Definition 6.4.5, $\mathbf{S}_{E}$ is countable because $\mathcal{D}_{E}$ and $\mathcal{F}_{E}$ are countable (see Remark 5.1.14).

Remark 6.4.7. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of blocks and $\mathcal{A}_{E}$ an admissible set for $E$. Take $M$ and $L$ two $\mathcal{D}_{E}$-block subspaces and $C \geq 1$. Consider the game $G_{L, M}^{A}$ with constant $C$. If we forget the integers $m_{i}^{\prime}$ 's played by I and $n_{i}$ 's played by II in such game, then the set $\mathbf{S}_{E}$ contains the set of possible positions after a finite number of runs were played.

Definition 6.4.8. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of blocks and $\mathcal{A}_{E}$ an admissible set for $E$. Let $M$ and $L$ be two $\mathcal{D}_{E}$-block subspaces and $C \geq 1$. We say that the state $s=\left(\left(a_{0}, A_{0}, \ldots, a_{i}, A_{i}\right),\left(b_{0}, B_{0}, \ldots b_{j}, B_{j}\right)\right) \in \mathbf{S}_{E}$ is valid for the game $G_{L, M}^{\mathcal{A}}$ with constant $C$ if, and only if, the finite sequences $\left(a_{0}, \ldots, a_{i}\right),\left(b_{0}, \ldots, b_{j}\right) \in\left[\mathcal{A}_{E}\right]$.

Definition 6.4.9. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}, \mathcal{D}_{E}$ be a set of blocks and $\mathcal{A}_{E}$ an admissible set for $E$. Let $M$ and $L$ be two $\mathcal{D}_{E}$-block subspaces and $C \geq 1$. Consider $s \in \mathbf{S}_{E}$ a valid state for the game $G_{L, M}^{A}$ with constant $C$. We define the game $G_{L, M}^{\mathcal{A}}(s)$ as the game $G_{L, M}^{\mathcal{A}}$ with constant $C$ in which the vectors and finite subspaces in the state $s$ have been played in the initial rounds. That is: if $s=(a, b)$ with $a=\left(a_{0}, A_{0}, \ldots, a_{i}, A_{i}\right)$ and $b=\left(b_{0}, B_{0}, \ldots, b_{i}, B_{i}\right)$ then the game $G_{L, M}^{\mathcal{A}}(s)$ goes like as follows:

I

$$
\begin{gathered}
n_{i+1} \leq E_{i+1} \subseteq L \\
u_{i+1} \in A_{0}+\ldots+A_{i}+E_{i+1}, m_{i+1} \\
\left(\left(a_{0}, \ldots, a_{i}, u_{i+1}\right) \in\left[\mathcal{A}_{E}\right]\right)
\end{gathered}
$$

II $n_{i+1}$

$$
\begin{aligned}
& m_{i+1} \leq F_{i+1} \subseteq M \\
& v_{i+1} \in B_{0}+\ldots+B_{i}+F_{i+1}, n_{i+1} \\
&\left(\left(b_{0}, \ldots, b_{i}, v_{i+1}\right)\right.\left.\in\left[\mathcal{A}_{E}\right]\right)
\end{aligned}
$$

The outcome of the game will be the pair of infinite sequences $\left(a_{0}, \ldots, a_{i}, u_{i+1}, \ldots\right)$ and $\left(b_{0}, \ldots, b_{i}, v_{i+1}, \ldots\right)$.

If $s=(a, b)$ with $a=\left(a_{0}, A_{0}, \ldots, a_{i}, A_{i}\right)$ and $b=\left(b_{0}, B_{0}, \ldots, b_{i-1}, B_{i-1}\right)$ then the game $G_{L, M}^{\mathcal{A}}(s)$ goes like as follows:
I $\quad m_{i}$

$$
\begin{gathered}
n_{i} \leq E_{i+1} \subseteq L \\
u_{i+1} \in A_{0}+\ldots+A_{i}+E_{i+1}, m_{i+1} \\
\left(\left(a_{0}, \ldots, a_{i}, u_{i+1}\right) \in\left[\mathcal{A}_{E}\right]\right)
\end{gathered}
$$

II

$$
\begin{gathered}
m_{i} \leq F_{i} \subseteq M \\
v_{i} \in B_{0}+\ldots+B_{i}+F_{i}, n_{i} \\
\left(\left(b_{0}, \ldots, b_{i-1}, v_{i}\right) \in\left[\mathcal{A}_{E}\right]\right)
\end{gathered}
$$

The outcome of the game will be the pair of infinite sequences $\left(a_{0}, \ldots, a_{i}, u_{i+1}, \ldots\right)$ and $\left(b_{0}, \ldots, b_{i}, v_{i+1}, \ldots\right)$. We say that player II wins the game $G_{L, M}^{A}(s)$ with constant $C$ if $\left(a_{0}, \ldots, a_{i}, u_{i+1}, \ldots\right) \sim_{C}\left(b_{0}, \ldots, b_{i}, v_{i+1}, \ldots\right)$.

### 6.5 Tight-minimal dichotomies

In this subsection, we shall proceed with the proof of our dichotomy theorem.
Theorem 6.5.1. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$ and $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$ be an admissible system of blocks for $E$. Then $E$ contains a $\mathcal{D}_{E}$-block subspace $X$ which is either $\mathcal{A}_{E}$-tight or $\mathcal{A}_{E}$-minimal.

Proof. Let $E,\left(e_{n}\right)_{n}$ and $\left(\mathcal{D}_{E}, \mathcal{A}_{E}\right)$ be as in the hypothesis. We shall prove that if no $\mathcal{D}_{E}$-block subspace is $\mathcal{A}_{E}$-tight, then there is a $\mathcal{D}_{E}$-block subspace which is $\mathcal{A}_{E}$-minimal.

If $E$ fails to have an $\mathcal{A}_{E}$-tight subspace then by Lemma 6.3 .8 there are a $\mathcal{D}_{E}$-block subspace $Z$ of $E$ and a constant $C \geq 1$ such that for every $\mathcal{D}_{Z}$-block subspace $X$ of $Z$ there is a further $\mathcal{D}_{X}$-block subspace $Y$ of $X$ such that $I$ has no winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$. If we prove that $Z$ has a $\mathcal{A}_{E}$-minimal $\mathcal{D}_{E}$-block subspace the proof will be completed. So, without loss of generality we can suppose that $Z=E$.

Summing up, we are supposing that for every $\mathcal{D}_{E^{-}}$-block subspace $X$ there is a further $\mathcal{D}_{X^{-}}$ block subspace $Y \leq X$ such that $I$ has no winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$. Since the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$ is determined (see Proposition 6.2.5), we can
conclude that for any $\mathcal{D}_{E}$-block subspace $X$, there is a $\mathcal{D}_{X}$-block subspace $Y$ such that $I I$ has a winning strategy for the game $H_{Y, X}^{\mathcal{A}}$ with constant $C$.

Let $\tau: b b_{\mathcal{D}}(E) \rightarrow \mathbb{P}(\mathbf{S})$ be defined by

$$
\begin{aligned}
s \in \tau(M) \Leftrightarrow & \exists L \mathcal{D}_{M} \text {-block subspace such that player } I I \text { has a winning strategy for } \\
& G_{L, M}^{\mathcal{A}}(s) \text { with constant } C .
\end{aligned}
$$

First observe that the elements of $\tau(M)$ are valid states for the game $G_{L, M}^{\mathcal{A}}$ and that $\tau(M)$ is non-empty for each $M \leq E$ a $\mathcal{D}_{E}$-block subspace: we already saw that there is $L \leq M$ a $\mathcal{D}_{M}$-block subspace such that $I I$ has a winning strategy for the game $H_{L, M}^{\mathcal{A}}$ with constant $C$, and by Lemma 6.4.3, II has a winning strategy for the game $G_{L, M}^{\mathcal{A}}$ with constant $C$. Then, it is possible to define a valid state $s=(a, b)$, with $b$ being chosen following the winning strategy for $I I$, such that player II has a winning strategy for $G_{L, M}^{\mathcal{A}}(s)$ and $s \in \tau(M)$.

Consider now $M^{\prime} \leq^{*} M$ a $\mathcal{D}_{E^{-}}$-block subspace and $s \in \tau\left(M^{\prime}\right)$. Therefore, there is $L^{\prime} \leq M^{\prime}$ a $\mathcal{D}_{M^{\prime}}$-block subspace such that $I I$ has a winning strategy for the game $G_{L^{\prime}, M^{\prime}}^{\mathcal{A}}(s)$ with constant $C$. Since player $I$ can always choose finite subspaces $E_{i}$ 's in $L^{\prime}$ inside of $M$ and choose integers $n_{i}$ 's large enough to force player $I I$ to play in $M^{\prime}$ and inside of $M$ (the game $G_{L^{\prime}, M^{\prime}}^{\mathcal{~}}$ is asymptotic, in the sense that it does not depend on the first coordinates), it follows that it is possible to find a $\mathcal{D}_{M}$-block subspace $L \leq M$ such that $I I$ has a winning strategy for the game $G_{L, M}^{\mathcal{A}}(s)$ with constant $C$. Therefore, $s \in \tau(M)$, and we conclude that $\tau\left(M^{\prime}\right) \subseteq \tau(M)$.

By Lemma 6.1.3 there is a $\mathcal{D}_{E}$-block subspace $M_{0} \leq E$ which is stabilizing for $\tau$, i.e. $\tau\left(M_{0}\right)=\tau\left(M^{\prime}\right)$, for all $M^{\prime} \leq^{*} M \mathcal{D}_{M}$-block subspace.

Define $\rho: b b_{\mathcal{D}}\left(M_{0}\right) \rightarrow \mathbb{P}(\mathbf{S})$ by setting

$$
s \in \rho(L) \Leftrightarrow \text { player } I I \text { has a winning strategy for } G_{L, M_{0}}^{\mathcal{A}}(s) \text { with constant } C \text {. }
$$

Notice that there is a $\mathcal{D}_{M_{0}}$-block subspace $L \leq M_{0}$ such that $\rho(L) \neq \emptyset$ (the same justification was used to show that $\tau(M) \neq \emptyset$, for all $\mathcal{D}_{E}$-block subspace $\left.M \leq E\right)$, so $\rho$ is a non-trivial function. As we did before, set $L^{\prime} \leq^{*} L$ a $\mathcal{D}_{M_{0}}$-block subspace and $s \in \rho(L)$. If player II has a winning strategy for $G_{L, M_{0}}^{\mathcal{A}}(s)$ then, by the asymptoticity of the game (same previous argument for $\tau), I I$ has a winning strategy for $G_{L^{\prime}, M_{0}}^{\mathcal{A}}(s)$, so $s \in \rho\left(L_{0}^{\prime}\right)$. Thus $\rho$ is decreasing. We can apply Lemma 6.1 .3 to $\rho$, to find a stabilizing $\mathcal{D}_{M_{0}}$-block subspace $L_{0}$ of $M_{0}$ for $\rho$. Additionally, we obtain that

$$
\begin{equation*}
\rho\left(L_{0}\right)=\tau\left(L_{0}\right)=\tau\left(M_{0}\right) . \tag{6.18}
\end{equation*}
$$

Let us prove Equation (6.18): Since $L_{0} \leq M_{0}$ and $M_{0}$ stabilizes $\tau, \tau\left(M_{0}\right)=\tau\left(L_{0}\right)$. If $s \in \rho\left(L_{0}\right)$, then player $I I$ has a winning strategy for $G_{L_{0}, M_{0}}^{\mathcal{A}}(s)$, which means that $s \in \tau\left(M_{0}\right)$,
so $\rho\left(L_{0}\right) \subseteq \tau\left(M_{0}\right)$. If $s \in \tau\left(M_{0}\right)=\tau\left(L_{0}\right)$, then there is some $L^{\prime} \leq L_{0}$ a $\mathcal{D}_{L_{0}}$-block subspace such that $I I$ has a winning strategy for $G_{L^{\prime}, L_{0}}^{\mathcal{A}}(s)$. Since $L_{0} \leq M_{0}$, in particular $I I$ has a winning strategy for the game $G_{L^{\prime}, M_{0}}^{\mathcal{A}}(s)$ with constant $C$. Thus, $s \in \rho\left(L^{\prime}\right)=\rho\left(L_{0}\right)$ because $L_{0}$ is stabilizing for $\rho$.

Claim. For every $\mathcal{D}_{L_{0}}$-block subspace $M$, II has a winning strategy for the game $G_{L_{0}, M}^{\mathcal{A}}$ with constant $C$.

Proof of the claim. Fix $M$ a $\mathcal{D}_{L_{0}}$-block subspace. The idea of the proof of this claim is to show inductively that for each valid state $s$ from which player $I I$ has a winning strategy for the game $G_{L_{0}, M}^{\mathcal{A}}(s)$ with constant $C$, there is another state $s^{\prime}$ which "extends" it in such a way that player $I I$ has a winning strategy for the game $G_{L_{0}, M}^{\mathcal{A}}\left(s^{\prime}\right)$. Then, we use the fact that the winning condition is closed for player $I I$ (see Proposition 6.4.2) to justify that $I I$ has a winning strategy for the game. This method was used by A. Pelczar in [42] and we are using it in the same way that V. Ferenczi and Ch. Rosendal did in [22].

First, let us prove that $(\emptyset, \emptyset) \in \tau\left(L_{0}\right)$. We know that there is a $\mathcal{D}_{L_{0}}$-block subspace $Y$ such that $I I$ has a winning strategy for the game $H_{Y, L_{0}}^{\mathcal{A}}$ with constant $C$. From Lemma 6.4.3 it follows that $I I$ has a winning strategy for the game $G_{Y, L_{0}}^{\mathcal{A}}$ with constant $C$, and, by definition of $\tau$, this means that $(\emptyset, \emptyset) \in \tau\left(L_{0}\right)$. Now, we will show that:
(i) For all valid states for the game $G_{L_{0}, M}^{\mathcal{A}}(s)$

$$
s=\left(\left(u_{0}, E_{0}, \ldots, u_{i}, E_{i}\right),\left(v_{0} \cdot F_{0}, \ldots, v_{i}, F_{i}\right)\right) \in \tau\left(L_{0}\right),
$$

there is an $n$ (which player $I I$ can play), such that for any subspace $E$ spanned by a finite $\mathcal{D}_{L_{0}}$-block sequence of $L_{0}$ with support greater than $n$, and any $u \in E_{0}+\ldots+E_{i}+E$ such that $\left(u_{0}, \ldots, u_{i}, u\right) \in\left[\mathcal{A}_{E}\right]$ (that is, any move that player $I$ could do in his $(i+1)$-th round in the game $G_{L_{0}, M}^{A}(s)$, disregarding the integer $\left.m_{i+1}\right)$, we have

$$
\left(\left(u_{0}, E_{0}, \ldots, u_{i}, E_{i}, u, E\right),\left(v_{0} . F_{0}, \ldots, v_{i}, F_{i}\right)\right) \in \tau\left(L_{0}\right) .
$$

(ii) For any $\left(\left(u_{0}, E_{0}, \ldots, u_{i+1}, E_{i+1}\right),\left(v_{0} . F_{0}, \ldots, v_{i}, F_{i}\right)\right) \in \tau\left(L_{0}\right)$, and for all $m$, there are $m \leq F$ a subspace spanned by a finite $\mathcal{D}_{M}$-block sequence and $v \in F_{0}+\ldots+F_{i}+F$ with $\left(v_{0}, \ldots, v_{i}, v\right) \in\left[\mathcal{A}_{E}\right]$ (which is a legal move that II can play), such that

$$
\left(\left(u_{0}, E_{0}, \ldots, u_{i+1}, E_{i+1}\right),\left(v_{0} \cdot F_{0}, \ldots, v_{i}, F_{i}, v, F\right)\right) \in \tau\left(L_{0}\right) .
$$

This will be the case in which both players has played $(i+1)$-rounds and player $I$ has played in his (i+1)th-move $\left(E_{i+1}, u_{i+1}, m\right)$, and it corresponds to player $I I$ making a legal move.

Let us prove statement $i$. Suppose that

$$
s=\left(\left(u_{0}, E_{0}, \ldots, u_{i}, E_{i}\right),\left(v_{0} . F_{0}, \ldots, v_{i}, F_{i}\right)\right) \in \tau\left(L_{0}\right)
$$

By Equation (6.18), II has a winning strategy for $G_{L_{0}, M_{0}}^{\mathcal{A}}(s)$, which means that there is $n$ such that for all subspace $n \leq E \subseteq L_{0}$ spanned by a finite $\mathcal{D}_{L_{0}}$-block sequence and $u \in E_{0}+\ldots+E_{i}+E$, II has a winning strategy for the game $G_{L_{0}, M_{0}}^{\mathcal{A}}\left(s^{\prime}\right)$, where

$$
s^{\prime}=\left(\left(u_{0}, E_{0}, \ldots, u_{i}, E_{i}, u, E\right),\left(v_{0} \cdot F_{0}, \ldots, v_{i}, F_{i}\right)\right)
$$

So, $s^{\prime} \in \rho\left(L_{0}\right)=\tau\left(L_{0}\right)$.
To prove $i i$ ), suppose

$$
\left(\left(u_{0}, E_{0}, \ldots, u_{i+1}, E_{i+1}\right),\left(v_{0} . F_{0}, \ldots, v_{i}, F_{i}\right)\right) \in \tau\left(L_{0}\right)
$$

and $m$ is given. Then, as $M \leq L_{0} \leq M_{0}$ and $\tau(M)=\tau\left(L_{0}\right), I I$ has a winning strategy for $G_{L, M}^{\mathcal{A}}(s)$, for some $\mathcal{D}_{M}$-block subspace $L \leq M$. Thus, there are $F \leq M$ with $m \leq F$ and $v \in F_{0}+\ldots+F_{i}+F$ such that $I I$ has a winning strategy for $G_{L, M}^{\mathcal{A}}\left(s^{\prime}\right)$, where

$$
s^{\prime}=\left(\left(u_{0}, E_{0}, \ldots, u_{i+1}, E_{i+1}\right),\left(v_{0} \cdot F_{0}, \ldots, v_{i}, F_{i}, v, F\right)\right)
$$

So, $s^{\prime} \in \tau(M)=\tau\left(L_{0}\right)$.
Starting at state $(\emptyset, \emptyset) \in \tau\left(L_{0}\right)$ and following inductively those two steps, we can obtain a sequence of states $\left(s_{i}\right)_{i}$ such that each $s_{i} \in \tau\left(L_{0}\right)$ is the initial part of the following one $s_{i+1} \in \tau\left(L_{0}\right)$. We can define a strategy for the player $I I$ as follows:

Since $(\emptyset, \emptyset) \in \tau\left(L_{0}\right)$, using $i$ ) there is $n_{0}$ such that whenever $m_{0}, E_{0} \leq L_{0}$ and $u_{0} \in E_{0}$ such that $n_{0} \leq E_{0}$, played by $I$, we have

$$
\left(\left(u_{0}, E_{0}\right), \emptyset\right) \in \tau\left(L_{0}\right)
$$

Let $\sigma((\emptyset, \emptyset))=\left(n_{0}\right)$. Using $\left.i i\right)$, there is $F_{0} \leq M$ and $v_{0} \in F_{0}$ such that

$$
\left(\left(u_{0}, E_{0}\right),\left(v_{0}, F_{0}\right)\right) \in \tau\left(L_{0}\right) .
$$

Again using $i$ ), there is $n_{1}$ such that whenever $m_{1}, E_{1} \leq L_{0}$ and $u_{1} \in E_{0}+E_{1}$ such that $n_{1} \leq E_{1}$, played by I, we have

$$
\left(\left(u_{0}, E_{0}, u_{1}, E_{1}\right),\left(v_{0}, F_{0}\right)\right) \in \tau\left(L_{0}\right) .
$$

Let $\sigma\left(\left(E_{0}, u_{0}, m_{0}\right)\right)=\left(F_{0}, v_{0}, n_{1}\right)$. Following this process inductively, supposing that player $I$
in the $(k+1)$-th round has played $\left(E_{k}, u_{k}, m_{k}\right)$, using $\left.i i\right)$ there is $F_{k} \leq M$ and $v_{k} \in F_{0}+\ldots+F_{k}$ such that $m_{k} \leq F_{n}$ and

$$
\left(\left(u_{0}, E_{0}, \ldots, u_{k}, E_{k}\right),\left(v_{0} . F_{0}, \ldots, v_{k}, F_{k}\right)\right) \in \tau\left(L_{0}\right)
$$

Using $i$ ) there is $n_{k+1}$ such that whatever $m_{k+1}, E_{k+1} \leq L_{0}$ and $u_{k+1} \in E_{0}+\ldots+E_{k+1}$ such that $n_{k+1} \leq E_{k+1}$, played by I, we have

$$
\left(\left(u_{0}, E_{0}, \ldots, u_{k+1}, E_{k+1}\right),\left(v_{0} \cdot F_{0}, \ldots, v_{k}, F_{k}\right)\right) \in \tau\left(L_{0}\right)
$$

Let $\sigma\left(\left(E_{0}, u_{0}, m_{0}, \ldots, E_{k}, u_{k}, m_{k}\right)\right)=\left(F_{k}, v_{k}, n_{k+1}\right) . \sigma$ is a strategy for $I I$ to play in the game $G_{L_{0}, M}^{\mathcal{A}}$ with constant $C$.

Let $\vec{p}=\left(n_{0}, E_{0}, u_{0}, m_{0}, F_{0}, v_{0}, n_{1}, \ldots\right)$ be a legal run of the game $G_{L_{0}, M}^{\mathcal{A}}$ where $I I$ follows the strategy $\sigma$. So, every finite stage ( $n_{0}, E_{0}, u_{0}, m_{0}, F_{0}, v_{0}, n_{1}, \ldots, E_{i}, u_{i}, m_{i}, F_{i}, v_{i}, n_{i+1}$ ) of $\vec{p}$ determines the state $s_{i}=\left(\left(u_{0}, E_{0}, \ldots, u_{i}, E_{i}\right),\left(v_{0} . F_{0}, \ldots, v_{i}, F_{i}\right)\right) \in \tau\left(L_{0}\right)=\rho\left(L_{0}\right)$, which satisfy that player $I I$ has a winning strategy for the game $G_{L_{0}, M_{0}}^{\mathcal{A}}\left(s_{i}\right)$. By construction of $\sigma$, $I I$ actually plays in $M \leq L_{0} \leq M_{0}$, so for every $i \in \mathbb{N} I I$ has a winning strategy for the game $G_{L_{0}, M}^{\mathcal{A}}\left(s_{i}\right)$.

Therefore, for every $i \in \mathbb{N}, p_{i}$ is a finite stage of a legal run in the game $G_{L_{0}, M}^{\mathcal{A}}$ with constant $C$ where $I I$ wins. By Proposition 6.4.2, $\vec{p}$ is a run in the game $G_{L_{0}, M}^{\mathcal{A}}$ with constant $C$ where $I I$ wins. Thus, $\sigma$ is a winning strategy for $I I$.

Returning to the proof of the theorem: For $L_{0}$ there is a $\mathcal{D}_{L_{0}}$-block subspace $Y=\left[y_{n}\right]_{n}$ such that $I I$ has a winning strategy for the game $H_{Y, L_{0}}^{\mathcal{A}}$ with constant $C$. We shall show that for every $\mathcal{D}_{L_{0}}$-block subspace $M \leq L_{0}, Y \stackrel{\mathcal{A}}{\mathcal{C}^{2}}$.

Since $I I$ has a winning strategy for $H_{Y, L_{0}}^{\mathcal{A}}$ with constant $C$, player $I$ can produce in the game $G_{L_{0}, M}^{\mathcal{A}}$ a sequence $\left(u_{i}\right)_{i} \in \mathcal{A}_{L_{0}}$ such that $\left(u_{i}\right)_{i} \sim_{C}\left(x_{i}\right)_{i}$. That is, in each round of the game $G_{L_{0}, M}^{\mathcal{A}}$, player $I$ can choose the pair $\left(0, u_{i}\right)$, where each $u_{i}$ is obtained by the moves of $I I$ in $H_{Y, L_{0}}^{\mathcal{A}}$. By the Claim, $I I$ has a winning strategy for the game $G_{L_{0}, M}^{\mathcal{A}}$ for producing $\left(v_{i}\right)_{i} \in \mathcal{A}_{M}$, such that $\left(u_{i}\right)_{i} \sim_{C}\left(v_{i}\right)_{i}$. By transitivity $\left(x_{i}\right)_{i} \sim_{C^{2}}\left(v_{i}\right)_{i}$, therefore $Y \stackrel{\mathcal{A}}{\rightarrow} C^{2} M$, which ends the proof.

### 6.5.1 Corollaries from the $\mathcal{A}$ - tight-minimal dichotomy

As a corollary of Theorem 6.5.1 we obtain the third dichotomy:
Corollary 6.5.2 (Third Dichotomy, [22]). Let E be a Banach space with normalized basis $\left(e_{n}\right)_{n}$, then $E$ contains a tight block subspace or a minimal block subspace.

Proof. In Theorem 6.5.1 consider the admissible system of blocks $\left(\mathbb{D}_{E},\left(\mathbb{D}_{E}\right)^{\omega}\right)$. As we already observed in Remark 5.6.4 and the Proposition 5.5.4 for this admissible set we obtain exactly our thesis.

Corollary 6.5.3. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$, then $E$ contains a block subspace $X=\left[x_{n}\right]_{n}$ satisfying one of the following properties:
(1) For any $\left[y_{n}\right]_{n} \leq X$, there is a sequence $\left(I_{n}\right)_{n}$ of successive intervals in $\mathbb{N}$ such that for any $A \in[\mathbb{N}]^{\infty},\left[y_{n}\right]_{n}$ does not embed into $\left[x_{n}, n \notin \cup_{i \in A} I_{i}\right]$ as a block sequence.
(2) $\left(x_{n}\right)_{n}$ is a equivalence block-minimal basis.

Proof. In Theorem 6.5.1 consider the admissible system of blocks $\left(\mathbb{D}_{E}, B S_{\mathbb{D}}(E)\right)$ and item (vi) of Propositions 5.5.7 and 5.6.15.
V. Ferenczi and Ch. Rosendal also remarked in [22] that the case of block sequences in this theorem implies the theorem of A. Pelczar in [42] and an extension of it due to Ferenczi [17].

Corollary 6.5.4. Let $E$ be a Banach space with normalized basis $\left(e_{n}\right)_{n}$, then $E$ contains a block subspace $X=\left[x_{n}\right]_{n}$ satisfying one of the following properties:
(1) For any $\left[y_{n}\right]_{n}$ block basis of $X$, there is a sequence $\left(I_{n}\right)_{n}$ of successive intervals in $\mathbb{N}$ such that for any $A \in[\mathbb{N}]^{\infty},\left[y_{n}\right]_{n}$ does not embed into $\left[x_{n}, n \notin \cup_{i \in A} I_{i}\right]$, as a sequence of disjointly supported vectors.
(2) For any $\left[y_{n}\right]_{n}$ block basis of $X,\left(x_{n}\right)_{n}$ is equivalent to a sequence of disjointly supported vectors of $\left[y_{n}\right]_{n}$.

Proof. In Theorem 6.5.1 consider the admissible system of blocks $\left(\mathbb{D}_{E}, D S_{\mathcal{D}}(E)\right)$ and item (vii) of Propositions 5.5.7 and 5.6.15.

Notice that both properties (1) and (2) in Corollary 6.5.3 and in Corollary 6.5.4 are incompatible (see Theorem 5.6.9). Corollaries 6.5.3 and 6.5.4 are stated as Theorem 3.16 in [22]. In its statement it is also considered the embedding as a permutation of a block sequence, specifically:

Every Banach space with a basis contains a block subspace $E=\left[e_{n}\right]_{n}$ satisfying that either for any $\left[y_{n}\right]_{n} \leq E$, there is a sequence $\left(I_{n}\right)_{n}$ of successive intervals in $\mathbb{N}$ such that for any $A \in[\mathbb{N}]^{\infty},\left[y_{n}\right]_{n}$ does not embed into $\left[e_{n}, n \notin \cup_{i \in A} I_{i}\right]$ as a permutation of a block sequence; or for any $\left[y_{n}\right]_{n} \leq E,\left(e_{n}\right)_{n}$ is permutatively equivalent to a block sequence of $\left[y_{n}\right]_{n}$.

Nevertheless, as we have already seen in the proofs of this chapter, such kind of embedding corresponds to a non-admissible set (see Proposition 5.3.20). So, the proofs we have presented do not work for the case of the embedding as permutation of a block sequence, and we see no reason to think that last statement is true.

Corollary 6.5.5. For any normalized basic sequence $\left(e_{n}\right)_{n}$ in a Banach space, there is $\left(x_{n}\right)_{n} \preceq\left(e_{n}\right)_{n}$ which is either a tight-by-sequences basis or spreading.

Proof. In Theorem 6.5.1 consider the admissible system of blocks $\left(\mathcal{B}_{E}, b b_{\mathcal{B}}(E)\right.$ ). The result follows from item (i) in Propositions 5.5.7 and 5.6.15.

Corollary 6.5.6. For any basic sequence $\left(e_{n}\right)_{n}$ in a Banach space, there is $\left(x_{n}\right)_{n} \preceq\left(e_{n}\right)_{n}$ satisfying one of the following properties:
(i) For any $\left(y_{n}\right)_{n} \preceq\left(x_{n}\right)_{n}$ there is a sequence of successive intervals $\left(I_{n}\right)_{n}$ such that for every $A \in[\mathbb{N}]^{\infty}$ and for every injection $f: \mathbb{N} \rightarrow \mathbb{N} \backslash \cup_{i \in A} I_{i}$, we have $\left(y_{n}\right)_{n} \nsim\left(x_{f(n)}\right)_{n}$.
(ii) For any $\left(y_{n}\right)_{n} \preceq\left(x_{n}\right)_{n}$ we have $\left(x_{n}\right)_{n} \sim\left(y_{f(n)}\right)_{n}$, for some $f \in \mathbb{N}^{\mathbb{N}}$ injective.

Proof. In Theorem 6.5.1 consider the admissible system of blocks $\left(\mathcal{B}_{E}, d b_{\mathcal{B}}(E)\right)$. The result follows from item (ii) in Propositions 5.5.7 and 5.6.15.

Corollary 6.5.7. For any basic sequence $\left(e_{n}\right)_{n}$ in a Banach space, there is a signed subsequence $\left(x_{n}\right)_{n}=\left(\epsilon_{n} e_{k_{n}}\right)_{n}$, which satisfies one of the following properties:
(i) For any $\left(y_{n}\right)_{n} \preceq\left(x_{n}\right)_{n}$ there is a sequence of successive intervals $\left(I_{n}\right)_{n}$ such that for every $A \in[\mathbb{N}]^{\infty}$ we have that $\left(y_{n}\right)_{n}$ is not equivalent to any signed subsequence of $\left(x_{n}: n \in \mathbb{N} \backslash \cup_{i \in A} I_{i}\right)$.
(ii) For any $\left(y_{n}\right)_{n} \preceq\left(x_{n}\right)_{n}$ there is $\left(z_{n}\right)_{n}$ a signed subsequence of $\left(y_{n}\right)_{n}$ equivalent to $\left(x_{n}\right)_{n}$.

Proof. In Theorem 6.5.1 consider the admissible system of blocks $\left(\mathcal{B}_{E}^{ \pm}, b b_{\mathcal{B}}^{ \pm}(E)\right)$. The result follows from item (iii) in Propositions 5.5.7 and 5.6.15.

Corollary 6.5.8. For any basic sequence $\left(e_{n}\right)_{n}$ in a Banach space, there is a signed subsequence $\left(x_{n}\right)_{n}=\left(\epsilon_{n} e_{k_{n}}\right)_{n}$ of $\left(e_{n}\right)_{n}$, which satisfies one of the following properties:
(i) For any signed subsequence $\left(y_{n}\right)_{n}$ of $\left(x_{n}\right)_{n}$ there is a sequence of successive intervals $\left(I_{n}\right)_{n}$ such that for every $A \in[\mathbb{N}]^{\infty}$, we have that $\left(y_{n}\right)_{n}$ is not equivalent to any sequence of the form $\left(\delta_{n} x_{f(n)}\right)_{n}$, where $\left(\delta_{n}\right)_{n}$ is a sequence of signs and $f: \mathbb{N} \rightarrow \mathbb{N} \backslash \cup_{i \in A} I_{i}$ is an injective function.
(ii) For any signed subsequence $\left(y_{n}\right)_{n}$ of $\left(x_{n}\right)_{n}$, there is an injective function $f \in \mathbb{N}^{\mathbb{N}}$ and a sequence of signs $\left(\delta_{n}\right)_{n}$ such that $\left(x_{n}\right)_{n} \sim\left(\delta_{n} y_{f(n)}\right)_{n}$.

Proof. In Theorem 6.5.1 consider the admissible system of blocks $\left(\mathcal{B}_{E}^{ \pm}, d b_{\mathcal{B}}^{ \pm}(E)\right)$. The result follows from item (iv) in Propositions 5.5.7 and 5.6.15.

## Chapter 7

## Tightness in general Banach spaces

One of the objectives of this research was to find a new definition of tightness which could be used for general separable and non-separable Banach spaces, and which would coincide with the original definition for Banach spaces with Schauder basis. The extended definition should satisfy the following items:
a) It must be hereditary. For example, if it is defined for spaces with transfinite basis it is desirable that it should be hereditary for subsequences or transfinite subsequences of the basis.
b) If a non-separable Banach space is tight in the new definition, then it should not contain certain "types" of minimal subspaces. One possible non-minimality condition could be that a tight space (in the new sense) does not contain minimal subspaces.
c) In the case of transfinite basis, a relation between the space being tight and a certain subset of $2^{\alpha}$ being meager is expected.

In order to accomplish such objective, three possible definitions were considered: I-tight, $I I$-tight and tight, which will be studied in the following section.

Let us begin the exposition with the definition of a $I$-tight space.
Definition 7.0.1. Let $X$ be a separable or non-separable Banach space. $X$ is said to be I-tight if, and only if, for any closed subspace $Y$ of $X$ with Schauder basis, $Y$ has a tight basis (in the original sense).

The following proposition sums up some properties of $I$-tight spaces that follow immediately from the definition

Proposition 7.0.2. Suppose $X$ is a I-tight Banach space, then
(i) If $X$ has a Schauder basis, then $X$ is tight.
(ii) Any subspace of $X$ with Schauder basis is tight in the original sense.
(iii) If $Y \subseteq X$ is a subspace of $X$ (not necessarily with Schauder basis), then $Y$ is I-tight.

Proposition 7.0.3. If $X$ is a reflexive Banach space with Schauder basis, then $X$ is tight if, and only if, $X$ is I-tight.

Proof. It follows from Proposition 4.2.13 and item (i) of Proposition 7.0.2.
Proposition 7.0.4. Let $X$ be I-tight; then, $X$ has no minimal subspaces.
Proof. Suppose that there exists a minimal subspace $Y$ of $X$. Let $Z$ be a closed subspace of $Y$ with Schauder basis. By hypothesis, $Z$ has a tight basis, so $Z$ is tight. Due to the minimality of $Y, Y \hookrightarrow Z$, which result in $Z$ having a minimal subspace, contradicting the third dichotomy.

Recall that a minimal space has to be separable. In contrast to the original notion of tightness, the definition of a $I$-tight space does not depend on any of the bases of the space. Also, being $I$-tight does not extend the notion of tightness for the case when the space has a Schauder basis. As can be observed in Proposition 7.0.3, both notions coincide when the space is reflexive. This indicates that the definition of I-tight might not be adequate to our conditions. In the following we shall restrict the work considering Banach spaces with transfinite basis.

### 7.1 Banach spaces with transfinite basis

The next definitions are conceived for Banach spaces with a transfinite basis of any density. Let $\alpha$ be a limit ordinal. We shall follow the considerations and notations established in Section 2.3 of Chapter 2.

The correspondence between $\mathbb{P}(\alpha)$ and $2^{\alpha}$, allows us to identify families of subsets of $\alpha$ with topological subspaces of $2^{\alpha}$, where $2^{\alpha}$ is endowed with the natural product topology.

Definition 7.1.1. We say that a transfinite basic sequence $\left(x_{\beta}\right)_{\beta<\alpha}$ is a II-tight basis for its closed linear span if, and only if, for all $A \subseteq \alpha$ with order-type $\omega,\left(x_{\gamma}\right)_{\gamma \in A}$ is a tight basic sequence (in the original sense). A Banach space is II-tight if it admits a transfinite II-tight basis.

So, if $\left(x_{n}\right)_{n}$ is a Schauder basis, then $\left(x_{n}\right)_{n}$ is a $I I$-tight basis if, and only if, all its subsequences are tight. Notice that a basic sequence $\left(y_{n}\right)_{n}$ is tight if, and only if, for all $k$ the sequence $\left(y_{n}\right)_{n \geq k}$ is tight. So, $\left(x_{n}\right)_{n}$ is a tight Schauder basis if, and only if, $\left(x_{n}\right)_{n}$ is a $I I$-tight basis.

We have found no results relating a space $Y$ being $I I$-tight in a space with a transfinite basic sequence and a certain subset of $2^{\alpha}$ depending on $Y$ being meager, comeager, or having another Baire category type property, without passing to a subsequence. A positive result (which we will prove later) is that a $I I$-tight Banach space fails to have minimal subspaces.

Clearly, if $X$ is a $I$-tight space then it is a $I I$-tight space. We do not know whether the reserve implication is true.

Considering the above definitions and the properties that a tight space should satisfy in this context, we consider that the following definition of tightness is the correct in the context of Banach spaces with transfinite bases.

Definition 7.1.2. Let $X$ be a Banach space with transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$. We say that a Banach space $Y$ is tight in $X$ if, and only if,

$$
E_{Y}:=\left\{\mathfrak{u} \in 2^{\alpha}: Y \hookrightarrow X_{\operatorname{supp}(\mathfrak{u})}\right\}
$$

is meager in $2^{\alpha}$. The basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is a tight transfinite basis for $X$ if, and only if, any Banach space $Y$ is tight in $X . X$ is tight if it admits a tight transfinite basis.

As we may see in detail later, the definition of tight space given in Definition 7.1.2, satisfies the properties we were looking for. It is hereditary not only by considering the subsequences of the basis, but also by taking transfinite block subsequences of the basis. For $\alpha=\omega$, both notions coincide: the classical definition of tightness and the one given in Definition 7.1.2. Obviously we have a relation between being tight and a Baire Category notion. Also, since we shall show an extension of the Proposition 3.1 .4 for the case of $2^{\alpha}$, we will present a characterization of this notion of tightness in terms of the non-existence of an isomorphic embedding avoiding infinitely many block subspaces of the basis, as in the original definition.

We shall prove that if a transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is tight (in this new version), then it is $I I$-tight, so it has no minimal subspaces.

### 7.1.1 Characterizations of meager and comeager sets

In this section, we shall give some characterizations of a subset of $2^{\alpha}$ being meager or comeager and some examples of such sets. Consequently, we will show the properties mentioned above about tightness in this context.

Notation 7.1.3. Let $\mathfrak{s}=\left(s_{\gamma}\right)_{\gamma<\alpha} \in 2^{\alpha}$ and $J \subset \alpha$ finite, then we write $\left(s_{i}\right)_{i \in J}$ to say that $s_{i}=0$ for all $j \notin J$.

Generalizing the case $\alpha=\omega$, if $\mathfrak{s}=\left(s_{\gamma}\right)_{\gamma<\alpha} \in 2^{\alpha}$ and $\mathfrak{t}=\left(t_{\gamma}\right)_{\gamma<\alpha} \in 2^{\alpha}$, we define $\mathfrak{s} \cup \mathfrak{t}=\left(u_{\gamma}\right)_{\gamma<\alpha} \in 2^{\alpha}$ as the sequence which elements are $u_{\gamma}=\max \left\{s_{\gamma}, t_{\gamma}\right\}$, for every $\gamma<\alpha$. That is, $\mathfrak{s} \cup \mathfrak{t}$ is the characteristic function of $\chi_{\text {supp }(\mathfrak{s}) \cup \operatorname{supp}(\mathfrak{t})}$,

We obtain the following generalization of the Proposition 3.1.4.
Proposition 7.1.4. Let $A \subseteq 2^{\alpha}$. The following assertions are equivalent:
(i) $A$ is comeager in $2^{\alpha}$,
(ii) there are a sequence $\left(I_{n}\right)_{n<\omega}$ of non-empty finite pairwise disjoint subsets of $\alpha$, and
subsets $a_{n} \subseteq I_{n}$, such that for any $\mathfrak{u} \in 2^{\alpha}$, if $\left|\left\{n: I_{n} \cap \operatorname{supp}(\mathfrak{u})=a_{n}\right\}\right|=\aleph_{0}$, then $\mathfrak{u} \in A$.

Proof. To prove $(i i) \Rightarrow(i)$, let us define for each $n \in \omega$ the following sets

$$
O_{n}=\left\{\mathfrak{u} \in 2^{\alpha}: \exists k \geq n\left(\operatorname{supp}(\mathfrak{u}) \cap I_{k}=a_{k}\right)\right\} .
$$

Claim: For each $n, O_{n}$ is a dense open set in $2^{\alpha}$.
Set $n \in \omega$. To prove the density, let $\mathfrak{s}=\left(s_{\gamma}\right)_{\gamma<\alpha} \in 2^{\alpha}$ and let $J \subset \alpha$ be a finite non-empty set. Let $\mathcal{N}_{\mathfrak{s}, J}$ be the basic open set determined by $\mathfrak{s}$ and $J$, that is

$$
\mathcal{N}_{\mathfrak{s}, J}:=\left\{\mathfrak{u}=\left(u_{\gamma}\right)_{\gamma<\alpha} \in 2^{\alpha}: \forall \gamma \in J\left(u_{\gamma}=s_{\gamma}\right)\right\} .
$$

Let $D=\bigcup_{m \in \omega} I_{m} \cap J$. Then, for each $k \in D$, there exists a unique $n_{k}$ natural number such that $k \in I_{n_{k}}$ (because $\left(I_{n}\right)_{n}$ is a pairwise disjoint sequence). Since $J$ is finite, $n^{\prime}:=\max \left\{n, \max \left\{n_{k}: k \in D\right\}\right\}+1$ is finite. Then, for each $m \geq n^{\prime}$ we have that $I_{m} \cap J=\emptyset$. It is clear that there exists $\mathfrak{u}=\left(u_{\gamma}\right)_{\gamma<\alpha}$, such that

- $u_{\gamma}=s_{\gamma}$ if $\gamma \in J ;$
- $u_{\gamma}=1$ if $\gamma \in a_{n^{\prime}}$;
- $u_{\gamma}=0$ if $\gamma \in I_{n^{\prime}} \backslash a_{n^{\prime}}$.

In consequence, $\mathfrak{u} \in \mathcal{N}_{\mathfrak{s}, J} \cap O_{n} \neq \emptyset$.
To see that $O_{n}$ is an open set, let $\mathfrak{u} \in O_{n}$; then, there is $k \geq n$, such that $\operatorname{supp}(\mathfrak{u}) \cap I_{k}=a_{k}$. Let $\mathfrak{s}=\left(s_{\gamma}\right)_{\gamma<\alpha}$ satisfying

- $s_{\gamma}=1$, if $\gamma \in a_{k}$;
- $s_{\gamma}=0$, if $\gamma \in 2^{\alpha} \backslash a_{k}$.

Then. we obtain that $\mathfrak{u} \in \mathcal{N}_{\mathfrak{s}, I_{k}} \subset O_{n}$.
Now that the claim is proved, and since $2^{\alpha}$ is a Hausdorff and compact topological space, it is Baire. For that reason, $\bigcap_{n \in \omega} O_{n}$ is dense in $2^{\alpha}$. Finally, if $\mathfrak{u} \in \bigcap_{n \in \omega} O_{n}$, then $\left|\left\{n: I_{n} \cap \operatorname{supp}(\mathfrak{u})=a_{n}\right\}\right|=\aleph_{0}$, and by hypothesis, $\mathfrak{u} \in A$. This means that $\bigcap_{n \in \omega} O_{n} \subseteq A$, and $2^{\alpha} \backslash A \subseteq \bigcup_{n \in \omega}\left(2^{\alpha} \backslash O_{n}\right)$ which is meager, because $2^{\alpha} \backslash O_{n}$ is nowhere dense.
$(i) \Rightarrow(i i)$ :
Suppose that $A$ is comeager, then for each $n \in \omega$ there is a closed set $F_{n} \subset 2^{\alpha}$ with $\operatorname{int}\left(F_{n}\right)=\emptyset$, such that

$$
\begin{equation*}
2^{\alpha} \backslash A \subseteq \bigcup_{n \in \omega} F_{n} \tag{7.1}
\end{equation*}
$$

In the following, we are going to construct by induction the sets $\left(I_{n}\right)_{n}$ and $\left(a_{n}\right)_{n}$ as we need. Let $\mathfrak{u} \in 2^{\alpha} \backslash F_{0}$. Because $2^{\alpha} \backslash F_{0}$ is a dense open set, it is possible to find $\mathfrak{t}^{0}=\left(t_{\gamma}^{0}\right)_{\gamma<\alpha}$ and $I_{0} \subset \alpha$ finite subset, such that $\mathfrak{u} \in \mathcal{N}_{\mathfrak{t}^{0}, I_{0}} \subset 2^{\alpha} \backslash F_{0}$. Define

$$
\begin{equation*}
a_{0}=\left\{\gamma \in I_{0}: t_{\gamma}^{0}=1\right\} . \tag{7.2}
\end{equation*}
$$

By inductive hypothesis, suppose that we found a finite sequence $\left(I_{i}\right)_{i=0}^{m-1}$ of non-empty finite pairwise disjoint subsets of $\alpha$ and $a_{i} \subseteq I_{i}$, such that, for every $i \in\{1, \ldots, m-1\}$ we have

$$
\mathfrak{u} \in 2^{\alpha} \text { and } \operatorname{supp}(\mathfrak{u}) \cap I_{i}=a_{i} \Rightarrow \mathfrak{u} \notin \cup_{j=0}^{i} F_{j} .
$$

Let us consider $L=\cup_{i=0}^{m-1} I_{i}, l=2^{|L|}-1$, and $\left\{b^{0}, b^{1}, \ldots, b^{l}\right\}$ a enumeration of all the subsets of $L$.

Since $2^{\alpha} \backslash\left(\bigcup_{i=0}^{m} F_{i}\right)$ is a dense open set, $\left(2^{\alpha} \backslash\left(\bigcup_{i=0}^{m} F_{i}\right)\right) \cap \mathcal{N}_{\mathfrak{b} \mathfrak{o}, L}$ is a non-empty open set. Consequently, there are $J_{0} \subset \alpha$ finite and disjoint from $L$, and $\mathfrak{s}_{0}=\left(s_{i}^{0}\right)_{i \in J_{0}}$, such that

$$
\mathcal{N}_{\mathfrak{b} \circ \cup \mathfrak{s}_{\mathfrak{o}}, L \cup J_{0}} \subset \mathcal{N}_{\mathfrak{b} \mathfrak{o}, L} \bigcap\left(2^{\alpha} \backslash\left(\bigcup_{i=0}^{m} F_{i}\right)\right) \subseteq 2^{\alpha} \backslash\left(\bigcup_{i=0}^{m} F_{i}\right) .
$$

Using the density of $2^{\alpha} \backslash\left(\bigcup_{i=0}^{m} F_{i}\right)$, we have $\mathcal{N}_{\mathfrak{b}^{1} \cup \mathfrak{s}_{\mathfrak{o}}, L \cup J_{0}} \cap\left(2^{\alpha} \backslash\left(\bigcup_{i=0}^{m} F_{i}\right)\right)$ is a non-empty open set. So, we can find $J_{1} \subset \alpha$ finite disjoint from $J_{0} \cup L$, a sequence $\mathfrak{s}_{1}=\left(s_{i}^{1}\right)_{i \in J_{1}}$ e such that

$$
\mathcal{N}_{\mathfrak{b}^{1} \cup \mathfrak{s}_{0} \cup \mathfrak{s}_{1}, L \cup J_{0} \cup J_{1}} \subset \mathcal{N}_{\mathfrak{b}^{1} \cup \mathfrak{s}_{\mathfrak{o}}, L \cup J_{0}} \bigcap\left(2^{\alpha} \backslash\left(\bigcup_{i=0}^{m} F_{i}\right)\right) \subseteq 2^{\alpha} \backslash\left(\bigcup_{i=0}^{m} F_{i}\right)
$$

If we continue with this construction, we can find $J_{l} \subset \alpha$ finite set disjoint from with $\cup_{i=0}^{l-1} J_{i} \cup L$, and $\mathfrak{s}_{\mathfrak{l}}=\left(s_{i}^{l}\right)_{i \in J_{l}}$, such that if $\mathfrak{t}^{\mathfrak{m}}:=\mathfrak{s}_{0} \cup \ldots \cup \mathfrak{s}_{\mathfrak{l}}$, and $I_{m}:=\bigcup_{j \leq l} J_{j}$, then

$$
\begin{equation*}
\mathcal{N}_{\mathfrak{b}^{\mathfrak{l}} \cup \mathfrak{t} \boldsymbol{m}, L \cup I_{m}} \subset \mathcal{N}_{\mathfrak{b}^{\prime}, J^{\prime}} \bigcap\left(2^{\alpha} \backslash\left(\bigcup_{i=0}^{m} F_{i}\right)\right) \subseteq 2^{\alpha} \backslash\left(\bigcup_{i=0}^{m} F_{i}\right) \tag{7.3}
\end{equation*}
$$

where $\mathfrak{b}^{\prime}=\cup_{i=0}^{l-1} \mathfrak{s}_{\mathfrak{i}} \cup \mathfrak{b}^{\mathfrak{l}}$ and $J^{\prime}=\cup_{i=0}^{l-1} J_{i} \cup L$.
Let

$$
\begin{equation*}
a_{m}=\bigcup_{i=0}^{l} \operatorname{supp}\left(s_{i}\right) . \tag{7.4}
\end{equation*}
$$

Let us note that if $\mathfrak{u} \in 2^{\alpha}$ and $\operatorname{supp}(\mathfrak{u}) \cap I_{m}=a_{m}$, then $\mathfrak{u} \in \mathcal{N}_{\mathfrak{t}^{\mathrm{m}}, I_{m}}$; therefore, there is
$i \in\{0, \ldots, l\}$ such that $\mathfrak{u} \in \mathcal{N}_{\mathfrak{b}^{i}, L}$, thus using Equation (7.3)

$$
\begin{equation*}
\mathfrak{u} \in \mathcal{N}_{\mathfrak{b}^{\mathfrak{i}} \cup \mathfrak{t}^{\mathrm{m}}, L \cup I_{m}} \subseteq 2^{\alpha} \backslash\left(\bigcup_{i=0}^{m} F_{i}\right) \tag{7.5}
\end{equation*}
$$

Consequently, $\mathfrak{u} \notin \bigcup_{i=0}^{m} F_{i}$.
Finally, let $\mathfrak{u} \in 2^{\alpha}$ and $B=\left\{n: I_{n} \cap \operatorname{supp}(\mathfrak{u})=a_{n}\right\}$. If $|B|=\aleph_{0}$, then

$$
\forall m \in B\left(\mathfrak{u} \in \mathcal{N}_{\mathfrak{t}^{\mathrm{m}}, I_{m}}\right) .
$$

It follows from Equation (7.5) that for each $m \in B, \mathfrak{u} \notin \bigcup_{i=0}^{m} F_{i}$; then $\mathfrak{u} \notin \bigcup_{i \in \omega} F_{i}$. Due to Equation (7.1), we have $\mathfrak{u} \in A$, which ends the proof.

Remark 7.1.5. For the $2^{\omega}$ case the sequence $\left(I_{i}\right)_{i}$ can be chosen as a sequence of successive finite intervals, and even a partition of $\omega$ (see Proposition 3.1.4). But in the general case this can not be done preserving the finiteness of the sets.

We obtain the following corollary to the Proposition 7.1.4.
Corollary 7.1.6. Let $A$ be a subset of $2^{\alpha}$, such that for all $\mathfrak{u} \in A$ and $\mathfrak{v} \in 2^{\alpha}$ if $\operatorname{supp}(\mathfrak{u}) \subseteq \operatorname{supp}(\mathfrak{v})$, then $\mathfrak{v} \in A$. Then we have:
(i) $A$ is comeager in $2^{\alpha}$ if, and only if, there is a sequence $\left(I_{n}\right)_{n<\omega}$ of finite pairwise disjoint subsets of $\alpha$, such that if the support of $\mathfrak{u} \in 2^{\alpha}$ contains infinitely many subsets $I_{n}$, then $\mathfrak{u} \in A$.
(ii) $A$ is meager in $2^{\alpha}$ if, and only if, there is a sequence $\left(I_{n}\right)_{n<\omega}$ of finite pairwise disjoint subsets of $\alpha$, such that if $\mathfrak{u} \in A$, then $\left\{n: \operatorname{supp}(\mathfrak{u}) \cap I_{n}=\emptyset\right\}$ is finite.

Proof. (i) Suppose that $A$ is comeager in $2^{\alpha}$, then by Proposition 7.1.4, there are sequences $\left(I_{i}\right)_{i}$ and $\left(a_{i}\right)_{i}$ of finite pairwise disjoint subsets of $\alpha$ with $a_{n} \subseteq I_{n}$, such that for any $\mathfrak{u} \in 2^{\alpha}$ with $\left\{n: I_{n} \cap \operatorname{supp}(\mathfrak{u})=a_{n}\right\}$ infinite, we have $\mathfrak{u} \in A$.

Now, let $\mathfrak{u} \in 2^{\alpha}$ such that $B:=\left\{n: I_{n} \subset \operatorname{supp}(\mathfrak{u})\right\}$ is infinite and consider $v=\cup_{n \in B} a_{n}$. It is clear that $v \cap I_{n}=a_{n}$ for all $n \in B$, and $v$ is contained in $u$. So, by Proposition 7.1.4, $\chi_{v} \in A$, and by hypothesis, $\mathfrak{u} \in A$.

Now, suppose that there exist a sequence $\left(I_{i}\right)_{i}$ of finite subsets of $\alpha$ satisfying the hypothesis. Set $a_{i}:=I_{i}$, for every $i \in \mathbb{N}$. By Proposition 7.1.4 we obtain the result.
(ii) If $A$ is meager, then applying Proposition 7.1.4 to $2^{\alpha} \backslash A$, there are sequences $\left(I_{i}\right)_{i}$ and $\left(a_{i}\right)_{i}$ of finite pairwise disjoint subsets of $\alpha$ with $a_{n} \subseteq I_{n}$, such that for any $\mathfrak{u} \in 2^{\alpha}$ with $\left\{n: I_{n} \cap \operatorname{supp}(\mathfrak{u})=a_{n}\right\}$ infinite, we have $\mathfrak{u} \in 2^{\alpha} \backslash A$.

Let $\mathfrak{u} \in A$ and suppose that $B=\left\{n: \operatorname{supp}(\mathfrak{u}) \cap I_{n}=\emptyset\right\}$ is infinite. Let
$v=\operatorname{supp}(\mathfrak{u}) \cup\left(\cup_{n \in B} a_{n}\right)$. Since $\operatorname{supp}(\mathfrak{u}) \subseteq v, \chi_{v} \in A$, but $B \subseteq\left\{n: v \cap I_{n}=a_{n}\right\}$, so $\chi_{v} \in 2^{\alpha} \backslash A$, which is a contradiction.

For the reverse implication, let $\left(I_{i}\right)_{i}$ as the hypothesis and for each $n \in \omega$, consider the set

$$
F_{n}=\left\{\mathfrak{u} \in 2^{\alpha}: \forall k \geq n\left(\operatorname{supp}(\mathfrak{u}) \cap I_{k} \neq \emptyset\right)\right\}
$$

Claim: For each $n<\omega, F_{n}$ is nowhere dense.
Let $n<\omega$ be fixed. Let $\mathfrak{u}=\left(u_{\gamma}\right)_{\gamma<\alpha} \in 2^{\alpha} \backslash F_{n}$ and $k \geq n$ such that $\operatorname{supp}(\mathfrak{u}) \cap I_{k}=\emptyset$. Consider $\mathcal{N}_{\mathfrak{u}, I_{k}}$ the open set in $2^{\alpha}$ given by

$$
\mathcal{N}_{\mathfrak{u}, I_{k}}=\left\{\mathfrak{t}=\left(t_{\gamma}\right)_{\gamma<\alpha} \in 2^{\alpha}: \forall \gamma \in I_{k}\left(u_{\gamma}=t_{\gamma}\right)\right\} .
$$

Then, $\mathcal{N}_{\mathfrak{u}, I_{k}} \subseteq 2^{\alpha} \backslash F_{n}$, which implies that each $F_{n}$ is closed.
Now, let $J \subset \alpha$ finite, $\mathfrak{s}=\left(s_{\gamma}\right)_{\gamma<\alpha} \in 2^{\alpha}$ and consider the open set $\mathcal{N}_{\mathfrak{s}, J}$ determined by them. Let $\mathfrak{t}=\left(t_{\gamma}\right)_{\gamma<\alpha} \in \mathcal{N}_{\mathfrak{s}, J}$ such that $t_{\gamma}=0$, for all $\gamma \notin J$. Since $J$ is finite, there are infinitely many $k \geq n$, such that $\operatorname{supp}(\mathfrak{t}) \cap I_{k}=\emptyset$. Then $\mathfrak{t} \notin F_{n}$. Since $\mathfrak{s}$ and $J$ are arbitrary, the interior of $F_{n}$ is empty and we had proved the claim.

Let $\mathfrak{u} \in A$. By hypothesis $\left\{n: \operatorname{supp}(\mathfrak{u}) \cap I_{n}=\emptyset\right\}$ is finite, then we can consider $m$ any natural number greater than its maximum. So, $\mathfrak{u} \in F_{m}$. Therefore, $A \subseteq \cup_{n \in \omega} F_{n}$.

The Corollary 7.1.6 provide an easy way to construct meager and comeager subsets of $2^{\alpha}$.
Example 7.1.7. Let $\left(I_{i}\right)_{i}$ a sequence of finite pairwise disjoint subsets of $\alpha$. Then, the set

$$
\mathcal{U}:=\left\{\mathfrak{u} \in 2^{\alpha}: I_{n} \subseteq \operatorname{supp}(\mathfrak{u}) \text { for infinitely many } n\right\}
$$

is closed under taking supersets and by Corollary 7.1.6 is comeager in $2^{\alpha}$.
Example 7.1.8. The set $[\alpha]^{\infty}$ of infinite subsets of $\alpha$ is comeager in $2^{\alpha}$.
Proof. Notice that $[\alpha]^{\infty}$ is closed by taking supersets. Let $\left(\gamma_{n}\right)_{n}$ be a sequence of pairwise different ordinals in $\alpha$. For each $n \in \mathbb{N}$, let $I_{n}=\left\{\gamma_{n}\right\}$. Clearly, if $u \in 2^{\alpha}$ contains infinite $I_{i}$ 's, then $u$ itself is infinite. So, by Corollary 7.1.6, $[\alpha]^{\infty}$ is comeager.

Example 7.1.9. The set $2^{\alpha} \backslash[\alpha]^{\infty}$ of finite subsets of an infinite ordinal $\alpha$ is meager in $2^{\alpha}$.
Now, we are going to consider the set of countable subsets of an uncountable ordinal. Notice that $[\alpha]^{\Lambda_{0}}$ is not closed under taking supersets, so we can not use Corollary 7.1.6 to prove such set is meager or comeager.

Example 7.1.10. Suppose $\alpha$ is an uncountable ordinal, then, $[\alpha]^{\alpha_{0}}$ is neither meager or comeager in $2^{\alpha}$.

Proof. Suppose that $\alpha$ is an uncountable ordinal. We are identifying $[\alpha]^{\aleph_{0}}$ with the subset of $2^{\alpha}$ of sequences with countable support. Suppose that there are sequences $\left(I_{i}\right)_{i}$ and $\left(a_{i}\right)_{i}$ of finite pairwise disjoint subsets of $\alpha$ with $a_{i} \subseteq I_{i}$, for every $i$, such that

$$
\text { If } \mathfrak{u} \in 2^{\alpha} \text { and }\left|\left\{n: I_{n} \cap \operatorname{supp}(\mathfrak{u})=a_{n}\right\}\right|=\aleph_{0} \text {, then } \mathfrak{u} \in[\alpha]^{\aleph_{0}} .
$$

Let $t \subseteq \alpha$ uncountable. Let

$$
v:=\left(t \backslash \cup_{i \in \mathbb{N}} I_{i}\right) \cup \cup_{i \in \mathbb{N}} a_{i} .
$$

Clearly, $v$ is uncountable and $\left|\left\{n: I_{n} \cap v=a_{n}\right\}\right|=\aleph_{0}$, so $\chi_{v} \in[\alpha]^{\aleph_{0}}$, which is a contradiction. Therefore, $[\alpha]^{\alpha_{0}}$ is not comeager in $2^{\alpha}$.

By contradiction, suppose that $[\alpha]^{\aleph_{0}}$ is meager in $2^{\alpha}$, and $\left(I_{i}\right)_{i}$ and $\left(a_{i}\right)_{i}$ are the sequences of finite subsets of $\alpha$ which testify that $2^{\alpha} \backslash[\alpha]^{N_{0}}$ is comeager in $2^{\alpha}$. If $\left\{n: a_{n} \neq \emptyset\right\}$ is infinite then the set $u^{\prime}:=\cup_{i \in \mathbb{N}} a_{i}$ has cardinality $\aleph_{0}$, so $\left|\left\{n: I_{n} \cap u^{\prime}=a_{n}\right\}\right|=\aleph_{0}$ but $\chi_{u^{\prime}} \notin 2^{\alpha} \backslash[\alpha]^{\aleph_{0}}$, which is a contradiction.

If $\left\{n: a_{n} \neq \emptyset\right\}$ is finite, let $u^{\prime} \subseteq \alpha \backslash \cup_{n \in \mathbb{N}} I_{n}$ such that $u^{\prime}$ is countable. $u^{\prime}$ exists since $\alpha \backslash \cup_{n \in \mathbb{N}} I_{n}$ is uncountable. Then, $I_{n} \cap u^{\prime}=\emptyset=a_{n}$ for $\aleph_{0}$-many $n$ 's, but $\chi_{u^{\prime}} \notin 2^{\alpha} \backslash[\alpha]^{\aleph_{0}}$, which is a contradiction. Thus, $[\alpha]^{\aleph_{0}}$ is not meager in $2^{\alpha}$.

Remark 7.1.11. For $\alpha$ an uncountable ordinal, the set $[\alpha]^{\aleph_{0}}$ is dense in $2^{\alpha}$ with dense complement in $2^{\alpha}$ and do not have the Baire property.

Proof. Suppose that $\alpha$ is an uncountable ordinal. It is easy to see that $[\alpha]^{N_{0}}$ is dense with a dense complement in $2^{\alpha}$. Let us suppose that $[\alpha]^{\alpha_{0}}$ has the Baire property, so there are $O$ open set of $2^{\alpha}$ and $G$ meager in $2^{\alpha}$ such that

$$
\begin{equation*}
[\alpha]^{\aleph_{0}} \Delta O=G . \tag{7.6}
\end{equation*}
$$

After some calculations we obtain that

$$
\begin{equation*}
2^{\alpha} \backslash G=\left(\left(2^{\alpha} \backslash[\alpha]^{\aleph_{0}}\right) \cap 2^{\alpha} \backslash O\right) \cup\left([\alpha]^{\aleph_{0}} \cap O\right) \tag{7.7}
\end{equation*}
$$

Since $G$ is meager, by the Proposition 7.1.4, there are $\left(I_{n}\right)_{n}$ and $\left(a_{n}\right)_{n}$ such that

- $I_{i} \cap I_{j}=\emptyset$, if $i \neq j$.
- $a_{n} \subseteq I_{n}$, for every $n$.
- $A:=\left\{\mathfrak{u} \in 2^{\alpha}:\left|\left\{n: I_{n} \cap \operatorname{supp}(\mathfrak{u})=a_{n}\right\}\right|=\aleph_{0}\right\} \subseteq 2^{\alpha} \backslash G$.

Notice that Equation (7.7) implies that:
(i) if $\mathfrak{u} \in A$ is infinite and countable, then $\mathfrak{u} \in O$;
(ii) if $\mathfrak{u} \in A$ is finite or uncountable, then $\mathfrak{u} \in 2^{\alpha} \backslash O$.

Take $v:=\cup_{n \in \omega} a_{n}$ and notice that $\chi_{v} \in A \subseteq 2^{\alpha} \backslash G$.
If $v$ is infinite then, by $(i), \chi_{v} \in O$. Let $\mathfrak{s} \in 2^{\alpha}$ and $J \in[\alpha]^{<\infty}$ such that $\mathfrak{v} \in \mathcal{N}_{\mathfrak{s}, J} \subseteq O$. Let

$$
J_{1}:=\operatorname{supp}(\mathfrak{s}) \cap J \text { and } J_{0}:=J \backslash \operatorname{supp}(\mathfrak{s}) .
$$

Since $\chi_{v} \in \mathcal{N}_{\mathfrak{s}, J}, J_{1} \subseteq \cup_{n \in \omega} a_{n}$. Take $I \subseteq \alpha \backslash\left(J \cup \cup_{n \in \omega} I_{n}\right)$ uncountable. Consider the uncountable set $w:=v \cup I . \chi_{w} \in A$ because

$$
\left|\left\{n: I_{n} \cap w=a_{n}\right\}\right|=\left|\left\{n: I_{n} \cap v=a_{n}\right\}\right|=\aleph_{0} .
$$

So, by (ii), we have $\chi_{w} \notin O$, but

- if $\gamma \in J_{1}$, then $\chi_{w}(\gamma)=\chi_{v}(\gamma)=\mathfrak{s}(\gamma)=1$;
- if $\gamma \in J_{0}$, then $\chi_{w}(\gamma)=0=\mathfrak{s}(\gamma)$.

Then, $\chi_{w} \in \mathcal{N}_{\mathfrak{s}, J} \subseteq O$ which is a contradiction.
If $v$ is finite, then we have that $[\alpha]^{<\infty} \subseteq 2^{\alpha} \backslash G$, thus, using (ii), we have $[\alpha]^{<\infty} \subseteq 2^{\alpha} \backslash O$. But this means that $O=\emptyset$, since every non-empty open set in $2^{\alpha}$ contains infinitely many functions $\mathfrak{s}$ such that $s$ is finite. Therefore, $[\alpha]^{\aleph_{0}}$ is meager in $2^{\alpha}$ which contradicts the Example 7.1.10.

Example 7.1.12. For $\alpha$ a countable ordinal, $[\alpha]^{\infty}=[\alpha]^{\aleph_{0}}$, therefore $[\alpha]^{\aleph_{0}}$ is comeager in $2^{\alpha}$.

### 7.1.2 Definition of tight transfinite bases

Let us recall Definition 7.1.2: Let $X$ and $Y$ be Banach spaces, $X$ with transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$. We say that $Y$ is tight in $X$ if

$$
E_{Y}=\left\{\mathfrak{u} \in 2^{\alpha}: Y \hookrightarrow X_{\text {supp }(\mathfrak{u})}\right\}
$$

is meager in $2^{\alpha}$.
If $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is a transfinite basis for $X$ such that any $Y$ Banach space, $Y$ is tight in $X$, then we say that $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is a tight transfinite basis for $X$ and $X$ is tight.

Consider $X$ a Banach space with tight Schauder basis $\left(x_{n}\right)_{n}$, then we know that if $Y$ is an arbitrary Banach space, the set $E_{Y}=\left\{\mathfrak{u} \in 2^{\omega}: Y \hookrightarrow\left[x_{n}: n \in \operatorname{supp}(\mathfrak{u})\right]\right\}$ is either meager
or comeager in $2^{\omega}$ (see Lemma 4.2.5). This is consequence of the first topological 0-1 law applied to the set $E_{Y}$ which has the Baire Property (for being analytic in the Polish space $2^{\omega}$ ). For the case of a tight transfinite basis, $2^{\alpha}$ is not separable and therefore is not Polish. We do not know if $E_{Y}$ is always meager or comeager in $2^{\alpha}$.

Proposition 7.1.13. Let $X$ be a Banach space with transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$. Then $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is a tight transfinite basis for $X$ if, and only if, $Y$ is tight in $\left(x_{\gamma}\right)_{\gamma<\alpha}$, for every separable Banach space $Y$.

Proof. The proof follows easily from the following fact: if $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is not tight and $Z$ is a nonseparable Banach space which is not tight in $X$, then there is $Y \subseteq Z$ a separable subspace of $Z$ such that $Y$ is not tight in $X$.

Proposition 7.1.14. Let $X$ be a Banach space with transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$. Then $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is a tight transfinite basis for $X$ if, and only if, $\left[y_{n}\right]_{n}$ is tight in $\left(x_{\gamma}\right)_{\gamma<\alpha}$, for every block subsequence $\left(y_{n}\right)_{n}$ of $\left(x_{\gamma}\right)_{\gamma<\alpha}$.

Proof. Since we can always find a block subspace in every closed subspace of $X$ (see Theorem 2.3.8), the proof follows from Proposition 7.1.13.

Using Corollary 7.1.6 we obtain the following result.
Proposition 7.1.15. Let $X$ be a Banach space with transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$. Let $Y$ a Banach space. Then the following statements are equivalent:
(i) $Y$ is tight in $X$;
(ii) there exists a sequence $\left(I_{i}\right)_{i}$ of finite pairwise disjoint subsets of $\alpha$, such that for any $A \subset \omega$ infinite,

$$
Y \nrightarrow\left[x_{\gamma}: \gamma \notin \cup_{i \in A} I_{i}\right] .
$$

Proof. By definition, $Y$ is tight in $X$ if, and only if, $E_{Y}$ is meager in $2^{\alpha}$. Also, the set $E_{Y}$ is always closed under taking supersets. By Corollary 7.1.6, $E_{Y}$ meager is equivalent to the existence of a sequence $\left(I_{i}\right)_{i}$ of finite pairwise disjoint subsets of $\alpha$, such that if $\mathfrak{u} \in E_{Y}$, then $\operatorname{supp}(\mathfrak{u})$ intersects all but finitely many subsets $I_{i}$. That means that $Y$ cannot be embedded in $\left[x_{\gamma}: \gamma \in \operatorname{supp}(\mathfrak{u})\right]$, with $\operatorname{supp}(\mathfrak{u})$ avoiding infinitely many $I_{i}$. Therefore, $E_{Y}$ is meager in $2^{\alpha}$ if, and only if, there is a sequence $\left(I_{i}\right)_{i}$ of finite pairwise disjoint subsets of $\alpha$, such that for all $A \in[\mathbb{N}]^{\infty}$,

$$
Y \nrightarrow\left[x_{\gamma}: \gamma \notin \cup_{i \in A} I_{i}\right] .
$$

Proposition 7.1.16. Let $X$ and $Z$ be Banach spaces, $X$ with a transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$, and $\left(y_{\xi}\right)_{\xi<\beta}$ be a transfinite block subsequence of $\left(x_{\gamma}\right)_{\gamma<\alpha}$. If $Z$ is tight in $\left(x_{\gamma}\right)_{\gamma<\alpha}$, then $Z$ is tight in $\left(y_{\xi}\right)_{\xi<\beta}$.

Proof. Let $\left(y_{\xi}\right)_{\xi<\beta}$ be a transfinite block subsequence of $\left(x_{\gamma}\right)_{\gamma<\alpha}$. Let $Y=\left[y_{\xi}\right]_{\xi<\beta}$ and $Z$ be an arbitrary space such that $Z \hookrightarrow Y$. Since $Z$ is tight in $X$, find $\left(I_{n}\right)_{n}$ the sequence of pairwise disjoint finite subsets of $\alpha$ associated to $Z$, i.e.

$$
\begin{equation*}
\text { if } u \subseteq \alpha \text { is such that } Z \hookrightarrow X_{u} \text { then }\left\{n: u \cap I_{n}=\emptyset\right\} \text { is finite. } \tag{7.8}
\end{equation*}
$$

Define for each $n \in \mathbb{N}$ the set

$$
K_{n}:=\left\{\xi<\beta: \operatorname{supp}\left(y_{\xi}\right) \cap I_{n} \neq \emptyset\right\} .
$$

The set $\left\{n: K_{n}=\emptyset\right\}$ is finite: for contradiction suppose that $B:=\left\{n: K_{n}=\emptyset\right\}$ is infinite and notice that $B=\left\{n: \cup_{\xi<\beta} \operatorname{supp}\left(y_{\xi}\right) \cap I_{n}=\emptyset\right\}$. Since $Z \hookrightarrow\left[x_{\gamma}: \gamma \in \cup_{\xi<\beta} \operatorname{supp}\left(y_{\xi}\right)\right]$ and by Equation (7.8), $\cup_{\xi<\beta} \operatorname{supp}\left(y_{\xi}\right)$ intersects all but finitely many $I_{n}$ which contradicts $B$ infinite.

Each $K_{n}$ is finite because each $I_{n}$ is finite and the collection $\left\{\operatorname{supp}\left(y_{\xi}\right): \xi<\beta\right\}$ is pairwise disjoint.

The sequence $\left(K_{n}\right)_{n}$ is not necessarily pairwise disjoint but it is possible to extract a subsequence $\left(J_{n}\right)_{n}$ of $\left(K_{n}\right)_{n}$ which is. Indeed, let $J_{0}=K_{m_{0}}$, where $K_{m_{0}} \neq \emptyset$. Suppose we find an increasing finite sequence $\left(m_{n}\right)_{n<k}$ of natural numbers and $\left(J_{n}\right)_{n<k}$ such that $J_{i}=K_{m_{i}} \neq \emptyset$, for $0 \leq i<k$ and $J_{i} \cap J_{n}=\emptyset$ if $n \neq i$. If for every $i>m_{k-1}$ we have that $\cup_{n<k} J_{n} \cap K_{i}$ is non-empty, there is $\xi \in \cup_{n<k} J_{n}$ and $A \in[\mathbb{N}]^{\infty}$ such that $\forall n \in A\left(\operatorname{supp}\left(y_{\xi}\right) \cap I_{n} \neq \emptyset\right)$. Since $\operatorname{supp}\left(y_{\xi}\right)$ is finite, there is $\gamma \in \operatorname{supp}\left(y_{\xi}\right)$ which belongs to infinitely many $I_{n}$ 's, contradicting $\left(I_{n}\right)_{n}$ is pairwise disjoint. So, there must exist $m_{k}>m_{k-1}$ such that $K_{m_{k}} \cap \cup_{n<k} J_{n}=\emptyset$. Let $J_{k}=K_{m_{k}}$. The sequence $\left(J_{n}\right)_{n}$ is the one we sought for.

Let

$$
E_{Z}^{Y}=\left\{\mathfrak{u} \in 2^{\beta}: Z \hookrightarrow Y_{\text {supp }(\mathfrak{u})}\right\}
$$

We will show that if $\mathfrak{u} \in E_{Z}^{Y}$, then the set $B:=\left\{n: \operatorname{supp}(\mathfrak{u}) \cap J_{n}=\emptyset\right\}$ is finite to conclude by Corollary 7.1.6 that $E_{Z}^{Y}$ is meager. Let $u:=\operatorname{supp}(\mathfrak{u})$. By contradiction, suppose $B$ is infinite, then $\forall n \in B\left(\cup_{\xi \in u} \operatorname{supp}\left(y_{\xi}\right) \cap I_{m_{n}}=\emptyset\right)$.

Then $\cup_{\xi \in u} \operatorname{supp}\left(y_{\xi}\right)$ avoids infinitely many $I_{n}$ 's but

$$
Z \hookrightarrow Y_{u} \hookrightarrow\left[x_{\gamma}: \gamma \in \cup_{\xi \in u} \operatorname{supp}\left(y_{\xi}\right)\right]
$$

which contradicts Equation (7.8).

Proposition 7.1.17. Let $X$ be a Banach space with a tight transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$ and $\left(y_{\xi}\right)_{\xi<\beta}$ a transfinite block subsequence of $\left(x_{\gamma}\right)_{\gamma<\alpha}$, then $\left(y_{\xi}\right)_{\xi<\beta}$ is a tight transfinite basis.

Proof. Let $Z$ a separable Banach space and $\left(y_{\xi}\right)_{\xi<\beta}$ a transfinite block subsequence of $\left(x_{\gamma}\right)_{\gamma<\alpha}$. Since $X$ is tight, $Z$ is tight in $X$. By Proposition 7.1.16 $Z$ is tight in $\left[y_{\xi}\right]_{\xi<\beta}$. Since $Z$ is arbitrary, $\left(y_{\xi}\right)_{\xi<\beta}$ is a tight transfinite basis.

Corollary 7.1.18. Let $X$ be a Banach space with a tight transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$.
(i) If $\left(y_{n}\right)_{n}$ is a block subsequence of $\left(x_{\gamma}\right)_{\gamma<\alpha}$, then $\left(y_{n}\right)_{n}$ is a tight basis.
(ii) If $\beta<\alpha$, then $\left(x_{\gamma}\right)_{\gamma<\beta}$ is a tight transfinite basis.

Proof. It follows directly from Proposition 7.1.18.
Corollary 7.1.19. If $X$ is a Banach space with a tight transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$, then $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is a II-tight basis for $X$.

Proof. It follows directly from (i) of Corollary 7.1.18.

Proposition 7.1.20. If $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is a tight shrinking transfinite basic sequence, and $\left(\gamma_{n}\right)_{n}$ is an increasing sequence of ordinals in $\alpha$, then every basic sequence in $\left[x_{\gamma_{n}}\right]_{n}$ is tight.

Proof. It follows directly from the definition of transfinite shrinking basis, Corollary 7.1.18 and Theorem 4.2.12.

Proposition 7.1.21. If $X$ is a reflexive Banach space with tight transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$ and $\left(\gamma_{n}\right)_{n}$ is an increasing sequence of ordinals in $\alpha$, then every basic sequence in $\left[x_{\gamma_{n}}\right]_{n}$ is tight.

Proof. The proof follows as a direct consequence of Theorem 2.3.10 and 7.1.20.

Proposition 7.1.22. Let $X$ be a Banach space with an II-tight transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$, then $X$ does not contain minimal subspaces.

Proof. Suppose that there is $Y^{\prime}$ aa minimal subspace. Let $Y=\left[y_{n}\right]_{n}$ be a subspace of $Y^{\prime}$ with Schauder basis. As consequence of Theorem 2.3.8, there are $\left(z_{n}\right)_{n} \leq\left(y_{n}\right)_{n}$ and $\left(w_{n}\right)_{n}$ a block subsequence of $\left(x_{\gamma}\right)_{\gamma<\alpha}$ such that $\left[z_{n}\right]_{n} \hookrightarrow\left[w_{n}\right]_{n}$. Notice that $\left[z_{n}\right]_{n}$ is minimal.

Using the Proposition 2.3.6, there is $\left(\gamma_{n}\right)_{n}$ an increasing sequence with elements in $\alpha$, such that $\left(w_{n}\right)_{n} \leq\left(x_{\gamma_{n}}\right)_{n}$. By hypothesis, $\left(x_{\gamma_{n}}\right)_{n}$ is a tight Schauder basis but $\left[z_{n}\right]_{n} \hookrightarrow\left[x_{\gamma_{n}}\right]_{n}$ and $\left[z_{n}\right]_{n}$ is minimal, which contradicts the Proposition 4.2.9.

Theorem 7.1.23. Let $X$ be a Banach space with a tight transfinite basis, then $X$ does not have any minimal subspace.

Proof. If $X=\left[x_{\gamma}\right]_{\gamma<\alpha}$ is tight. By Proposition 7.1.18, $X$ is also $I I$-tight. Then, $X$ has no minimal subspaces.

### 7.2 Tight-with-constants transfinite bases

Based on Definition 7.1.2 and its implications, a generalization of Proposition 4.2.23 in the context of transfinite basis is expected. Also, there is a natural definition of a tight-bysupport transfinite basis. On the other hand, the extension of the notion of tightness by range is not natural, since given a block subsequence $\left(y_{n}\right)_{n}$ of a transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$, it could be the case that $\operatorname{ran}\left(y_{i}\right)=\left[\min \operatorname{supp}\left(y_{i}\right), \max \operatorname{supp}\left(y_{i}\right)\right]$ is not finite, for some $i \in \mathbb{N}$.

Notation 7.2.1. Let $X$ be a Banach space with transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$. Let $Y$ be a Banach space. For each $j \in \mathbb{N}$ we denote

$$
\begin{equation*}
E_{Y}(j):=\left\{\mathfrak{u} \in 2^{\alpha}: Y \hookrightarrow_{j} X_{\operatorname{supp}(\mathfrak{u})}\right\} . \tag{7.9}
\end{equation*}
$$

Definition 7.2.2. Let $X$ be a Banach space with transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$. Let $Y$ be a Banach space. We say that $Y$ is tight in $\left(x_{\gamma}\right)_{\gamma<\alpha}$ with constants if, and only if, the set $E_{Y}(j)$ is nowhere dense in $2^{\alpha}$ for all $j \geq 1$.

Proposition 7.2.3. Let $X$ be a Banach space with transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$. Let $Y$ be a Banach space. If $Y$ is tight in $\left(x_{\gamma}\right)_{\gamma<\alpha}$ with constants, then $Y$ is tight in $\left(x_{\gamma}\right)_{\gamma<\alpha}$.

Proof. It follows directly from Definition 7.2.2 and from the fact that $E_{Y}=\cup_{j \geq 1} E_{Y}(j)$.

Lemma 7.2.4. Let $O$ be an open set of $2^{\alpha}$ and let $B$ be a closed nowhere dense set of $2^{\alpha}$. Then, there are $\mathfrak{s} \in 2^{\alpha}, I \neq \emptyset$ a finite subset of $\alpha$ such that $I \cap \operatorname{supp}(\mathfrak{s}) \neq \emptyset$

$$
\mathcal{N}_{\mathfrak{s}, I} \cap O \cap B=\emptyset .
$$

Proof. Let $O$ and $B$ be as in the hypothesis. Set

$$
A:=O \cap\left(2^{\alpha} \backslash B\right) .
$$

Notice that $A$ is an nonempty open set. Take $\mathfrak{s}^{\prime} \in 2^{\alpha}$ and $I^{\prime} \in[\alpha]^{<\infty}$ such that $\mathcal{N}_{\mathfrak{s}^{\prime}, I^{\prime}} \subseteq A$. We can refine $\mathcal{N}_{\mathfrak{s}^{\prime}, I^{\prime}}$ and obtain $\mathcal{N}_{\mathfrak{s}, I} \subseteq \mathcal{N}_{\mathfrak{s}^{\prime}, I^{\prime}}$ such that $I \cap \operatorname{supp}(\mathfrak{s}) \neq \emptyset$.

Notation 7.2.5. Denote by $\mathbb{1}$ to the element of $2^{\alpha}$ such that $\mathbb{1}(\gamma)=1$ for all $\gamma \in \alpha$.
Proposition 7.2.6. Let $X$ be a Banach space with transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$. Let $Y$ be $a$ Banach space. Then, the following statements are equivalent:
(i) $Y$ is tight in $X$ with constants;
(ii) there exists a sequence $\left(I_{i}\right)_{i \geq 1}$ of finite pairwise disjoint subsets of $\alpha$, such that for each $n \geq 1$

$$
Y \not \oiiint_{n}\left[x_{\gamma}: \gamma \notin I_{n}\right] .
$$

Proof. $(i) \Rightarrow($ ii $)$ Let us construct inductively such finite subsets $\left(I_{i}\right)_{i \geq 1}$. From Lemma 7.2.4, there are $\mathfrak{s}_{1} \in 2^{\alpha}$ and $I_{1} \in[\alpha]^{<\infty}$ such that $K_{1}:=\operatorname{supp}\left(\mathfrak{s}_{1}\right) \cap I_{1}$ is not empty and

$$
\begin{equation*}
\mathcal{N}_{\mathfrak{s}_{1}, I_{1}} \cap \overline{E_{Y}(1)}=\emptyset \tag{7.10}
\end{equation*}
$$

Suppose that we have found $\mathfrak{s}_{1}, \ldots, \mathfrak{s}_{n-1}$ in $2^{\alpha}$ and $I_{1}, \ldots, I_{n-1}$ mutually disjoint sets in $[\alpha]^{<\infty}$, such that for each $i=1, \ldots, n-1$ we have

- $K_{i}:=\operatorname{supp}\left(\mathfrak{s}_{\mathfrak{i}}\right) \cap I_{i} \neq \emptyset$,
- $\bigcap_{j=1}^{i-1} \mathcal{N}_{1, I_{j}} \bigcap \mathcal{N}_{\mathfrak{s}_{i}, I_{i}} \bigcap \overline{E_{Y}(i)}=\emptyset$.

Using Lemma 7.2.4, there are $\mathfrak{s}_{\mathfrak{n}} \in 2^{\alpha}, I_{n} \in[\alpha]^{<\infty}$ disjoint from $\cup_{j=1}^{n-1} I_{j}$ such that $K_{n}:=\operatorname{supp}\left(\mathfrak{s}_{\mathfrak{n}}\right) \cap I_{n} \neq \emptyset$ and

$$
\begin{equation*}
\cap_{j=1}^{n-1} \mathcal{N}_{1, I_{j}} \cap \mathcal{N}_{\mathfrak{s}_{\mathfrak{n}}, I_{n}} \cap \overline{E_{Y}(n)}=\emptyset . \tag{7.11}
\end{equation*}
$$

Suppose $n \geq 1$ and $Y \hookrightarrow_{n}\left[x_{\gamma}: \gamma \notin I_{n}\right]$, that is, for $u:=\alpha \backslash I_{n}, \chi_{u} \in E_{Y}(n)$. Take $v=u \cup K_{n}$, then $\chi_{v} \in E_{Y}(n)$ and $\chi_{v} \in \cap_{j=1}^{n-1} \mathcal{N}_{1, I_{j}} \cap \mathcal{N}_{\mathfrak{s}_{\mathfrak{n}}, I_{n}}$, contradicting Equation (7.11).
$(i i) \Rightarrow(i)$ Suppose $\left(I_{i}\right)_{i \geq 1}$ a sequence of finite mutually disjoint subsets of $\alpha$ such that for each $n \geq 1$

$$
Y \not \oiiint_{n}\left[x_{\gamma}: \gamma \notin I_{n}\right] .
$$

So, if $\mathfrak{u} \in E_{Y}(j)$ for some $j \geq 1$, then $\operatorname{supp}(\mathfrak{u}) \cap I_{j} \neq \emptyset$. Since for $j \leq k$ we have that $E_{Y}(j) \subseteq E_{Y}(k)$, we obtain

$$
\begin{equation*}
E_{Y}(j) \subseteq \bigcap_{k \geq j} E_{Y}(k) \subseteq \bigcap_{k \geq j}\left\{\mathfrak{u} \in 2^{\alpha}: \operatorname{supp}(\mathfrak{u}) \cap I_{k} \neq \emptyset\right\} \tag{7.12}
\end{equation*}
$$

Let us prove that the set on the right in Equation (7.12), is closed with empty interior on $2^{\alpha}$. Since for each $k \geq j$ is true that

$$
\left\{\mathfrak{u} \in 2^{\alpha}: \operatorname{supp}(\mathfrak{u}) \cap I_{k} \neq \emptyset\right\}=\bigcup_{\gamma \in I_{k}}\left\{\mathfrak{u} \in 2^{\alpha}: \mathfrak{u}(\gamma)=1\right\}
$$

and $I_{k}$ is finite, it follows that each $\left\{\mathfrak{u} \in 2^{\alpha}: \operatorname{supp}(\mathfrak{u}) \cap I_{k} \neq \emptyset\right\}$ is closed, then so it is

$$
\bigcap_{k \geq j}\left\{\mathfrak{u} \in 2^{\alpha}: \operatorname{supp}(\mathfrak{u}) \cap I_{k} \neq \emptyset\right\} .
$$

Now suppose $\mathfrak{v} \in \bigcap_{k \geq j}\left\{\mathfrak{u} \in 2^{\alpha}: \operatorname{supp}(\mathfrak{u}) \cap I_{k} \neq \emptyset\right\}$ and let $\mathfrak{s} \in 2^{\alpha}$ and $J \subseteq \alpha$ finite, such that $\mathfrak{v} \in \mathcal{N}_{\mathfrak{s}, J}$. Let us find $\mathfrak{w} \in \mathcal{N}_{\mathfrak{s}, J}$, but not in $\bigcap_{k \geq j}\left\{\mathfrak{u} \in 2^{\alpha}: \operatorname{supp}(\mathfrak{u}) \cap I_{k} \neq \emptyset\right\}$.

Consider $J^{\prime}:=J \cap\left(\cup_{k \geq j} I_{k}\right)$. If $J^{\prime}=\emptyset$, then take $\mathfrak{s}^{\prime} \in 2^{\alpha}$ such that $\mathfrak{s}^{\prime}(\gamma)=\mathfrak{v}(\gamma)=\mathfrak{s}(\gamma)$ for $\gamma \in J$ and $\mathfrak{s}^{\prime}(\gamma)=0$ if $\gamma \in I_{j}$. So, $\mathfrak{s}^{\prime} \in \mathcal{N}_{\mathfrak{s}, J}$ and $\mathfrak{s}^{\prime} \notin\left\{\mathfrak{u} \in 2^{\alpha}: \operatorname{supp}(\mathfrak{u}) \cap I_{j} \neq \emptyset\right\}$.

If $J^{\prime} \neq \emptyset$, there is $l \geq j$ such that $J \cap\left(\cup_{k \geq l} I_{k}\right)=\emptyset$. Take $\mathfrak{s}^{\prime}(\gamma)=\mathfrak{v}(\gamma)$ for $\gamma \in J$ and $\mathfrak{s}^{\prime}(\gamma)=0$, for $\gamma \in I_{l}$. Thus, $\mathfrak{s}^{\prime} \in \mathcal{N}_{\mathfrak{s}, J}$ but $\mathfrak{s}^{\prime} \notin\left\{\mathfrak{u} \in 2^{\alpha}: \operatorname{supp}(\mathfrak{u}) \cap I_{l} \neq \emptyset\right\}$.

Now we shall define a tight with constants space in the natural way.
Definition 7.2.7. Let $X$ be a Banach space with transfinite basis $\left(x_{\gamma}\right)_{\gamma<\alpha}$. We say that $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is a tight with constants basis if, and only if, every Banach space $Y$ is tight in $\left(x_{\gamma}\right)_{\gamma<\alpha}$ with constants. If $\left(x_{\gamma}\right)_{\gamma<\alpha}$ is a tight with constants basis, then $X$ is a tight with constants Banach space.

Finding examples of spaces with tight transfinite bases is not an easy task. Recall that for the $\omega$ case, a tight space fail to have spreading basic sequences. Thus, a candidate for a tight space with transfinite basis should not contain spreading basic sequences. The only examples of spaces satisfying such property are presented by Ch. Brech, J. López-Abad and S. Todorcevic in [8]. During this research we proved the spaces in [8] fails to have minimal subspaces. The following question is open.

Question. The spaces defined in [8] are tight (or tight with constant)?

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