Countably compact group topologies on torsion-free Abelian groups

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All we have to decide is what to do with the time that is given to us.
— Gandalf, the Grey

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Resumo

Matheus Koveroff Bellini. **Topologias de grupo enumeravelmente compactas em grupos abelianos livres de torsão**. Tese (Doutorado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022.

Este trabalho apresenta avanços obtidos em resultados de consistência na área da álgebra topológica, em particular sobre topologias de grupo enumeravelmente compactas e se é possível que elas possuam sequências convergentes não-triviais. Com melhorias e avanços nos métodos e técnicas já consolidados nessa linha de pesquisa, obtivemos os seguintes resultados, os dois primeiros já publicados em periódicos internacionais com arbitragem por pares: primeiro, obter topologias de grupo p-compactas sobre grupo abelianos livres de torsão sem sequências convergentes não-triviais, em que p é um ultrafiltro seletivo; segundo, obter topologias de grupo sobre grupos abelianos livres arbitrariamente grandes sem sequências convergentes não-triviais cujas potências finitas são todas enumeravelmente compactas, assumindo $\mathfrak{c}$ ultrafiltros seletivos incomparáveis; terceiro, um modelo de forcing em que um grupo abeliano livre de torsão cuja cardinalidade é enumeravelmente cofinal admite uma topologia de grupo p-compacta, em que p é um ultrafiltro seletivo. Estes resultados são fortalecimentos da teoria já estabelecida e apresentam os primeiros exemplos consistentes no que diz respeito às propriedades de p-compacidade e grandeza arbitrária em seus respectivos contextos.

Palavras-chave: Topologia geral. Teoria dos conjuntos. Álgebra topológica. Topologia conjuntista. Compacidade enumerável. Grupos enumeravelmente compactos. Grupos livres de torsão. Sequências convergentes. Combinatória infinitária. Ultrafiltros seletivos. Forcing.

Abstract

Matheus Koveroff Bellini. **Countably compact group topologies on torsion-free Abelian groups**. Thesis (Doctorate). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2022.

This work presents advancements obtained in consistency results on the field of topological algebra, especially concerning countably compact group topologies and whether they may contain non-trivial convergent sequences. Furthering the methods and techniques already established in this line of research, we have obtained the following results, the first two of which already published in international journals with peer arbitration: first, obtain p-compact group topologies on arbitrarily large torsion-free Abelian groups without non-trivial convergent sequences, for p a selective ultrafilter; second, obtain group topologies on arbitrarily large free Abelian groups without non-trivial convergent sequences all of whose finite powers are countably compact, assuming $\mathfrak{c}$ incomparable selective ultrafilters; third, a forcing model in which a torsion-free Abelian group whose cardinality is countably cofinal admits a p-compact group topology for p a selective ultrafilter. These results improve upon previously established theory and showcase the first consistent examples regarding the properties of p-compactness and arbitrarily largeness in their respective settings.

Keywords: General topology. Set theory. Topological algebra. Set-theoretic topology. Countable compactness. Countably compact groups. Torsion-free groups. Convergent sequences. Infinitary combinatorics. Selective ultrafilters. Forcing.

Contents

Chapter 1

Introduction

The study of group topologies on torsion-free Abelian groups, and whether they can be countably compact or have nontrivial convergent sequences can be traced back to a result by Halmos (Halmos, 1944), stating that $\mathbb{R}$ can be endowed with a compact group topology. This topology, in particular, is countably compact and has nontrivial convergent sequences.

The study of countably compact groups without nontrivial convergent sequences has three main questions:

1. What groups admit such topologies?
2. Do they exist in ZFC?
3. How large can the examples be? How productive can countable compactness be?

A lot of work has been done regarding the first question. Dikranjan and Tkachenko Dikranjan and Tkachenko, 2003, using Martin's Axiom, classified the Abelian groups of cardinality up to $\mathfrak{c}$ that admit such topologies. This was later improved in M. K. Bellini *et al.*, 2021 under the use of $\mathfrak{c}$ selective ultrafilters.

Dikranjan and Shakhmatov Dikranjan and Shakhmatov, 2005 used forcing to classify all Abelian groups of cardinality at most $2^{\mathfrak{c}}$ that admit a countably compact group topology without non-trivial convergent sequences, and in M. Bellini *et al.*, 2021, we use forcing to classify all the ones that admit such a topology with convergent sequences.

Question 2 was the most sought after question in the subject. It has finally been answered by M. Hrušák, U. A. Ramos-García, J. van Mill and S. Shelah in Hrušák *et al.*, 2021, who use new techniques. These new ideas have two limitations: the construction depends on the use of a group of finite order and the example has cardinality $\mathfrak{c}$. It is not yet known whether the techniques can be adapted to produce such topologies on torsion-free groups or on a larger torsion group. In this work it was also proved that if p is a selective ultrafilter, then the Boolean group $\mathrm{Ult}_p([\omega]^{<\omega})$ is p-compact, where Ult_p stands for "ultrapower by p", and the operation of the boolean group $[\omega]^{<\omega}$ is symmetric difference. Their technique does not work for non-torsion groups, or for larger groups.

Still speaking of torsion groups, regarding Question 3, Castro-Pereira and Tomita

Castro-Pereira and A. H. Tomita, 2010 classified, using some cardinal arithmetic and the existence of a selective ultrafilter p, all the torsion groups that admit a p-compact group topology (without non-trivial convergent sequences). This in particular generated large countably compact groups without non-trivial convergent sequences in torsion groups. Before the result in Hrušák *et al.*, 2021, it could still be hoped that every countably compact group is p-compact for some ultrafilter. Now this possibility is gone, since the existence of a countably compact group without non-trivial convergent sequences in ZFC implies the existence of a countably compact group whose square is not countably compact in ZFC A. Tomita, 2005. (We recall that a topological space is p-compact for some ultrafilter p if and only if all of its powers are countably compact.)

Moving on to torsion-free gorups, also in Hrušák *et al.*, 2021, the authors expected to produce an example for an ultrafilter in ZFC, but for now, the quest has foremost shifted to selective ultrafilters. The use of selective ultrafilters arises from improving on seminal results that relied upon CH or Martin's Axiom (which implies the existence of $2^{\mathfrak{c}}$ incomparable selective ultrafilters). These assumptions are useful in both settings which are explored in this work: sums of $\mathbb{Q}$ and free Abelian groups. Related to this and the techniques used throughout this work is the following question:

Problem: Assume $p \in \omega^*$ is a selective ultrafilter. Does $(\mathrm{Ult}_p(\mathbb{Z}), \tau_{\overline{Bohr}})$ contain non-trivial convergent sequences? (Here we recall that $\tau_{\overline{Bohr}}$ is the *weak* topology generated by the family of all homomorphisms into the circle group $\mathbb{R}/\mathbb{Z}$.)

On the topic of free Abelian groups, it is well known that a nontrivial free Abelian group does not admit a compact Hausdorff group topology. Tomita (A. H. Tomita, 1998) showed that it does not even admit a group topology whose countable power is countably compact.

Tkachenko (Tkachenko, 1990) showed in 1990 that the free Abelian group on $\mathfrak{c}$ generators can be endowed with a countably compact Hausdorff group topology under CH. Tomita (A. H. Tomita, 1998), Koszmider, Tomita and Watson (Koszmider *et al.*, 2000), and Madariaga-García and Tomita(Madariaga-Garcia and A. H. Tomita, 2007) obtained such examples using weaker assumptions. Boero, Castro-Pereira and Tomita obtained such an example using a single selective ultrafilter (A. C. Boero, Castro-Pereira, *et al.*, 2019). Using $2^{\mathfrak{c}}$ selective ultrafilters, the example in Madariaga-Garcia and A. H. Tomita, 2007 showed the consistency of a countably compact topology on the free Abelian group of cardinality $2^{\mathfrak{c}}$. All forcing examples so far have their cardinalities bounded by $2^{\mathfrak{c}}$. E. K. van Douwen showed in Douwen, 1980b that the cardinality of a countably compact group cannot be a strong limit of countable cofinality.

Boero and Tomita (A. C. Boero and A. H. Tomita, 2011) showed from the existence of $\mathfrak{c}$ selective ultrafilters that the free Abelian group of cardinality $\mathfrak{c}$ admits a group topology whose square is countably compact. Tomita (A. H. Tomita, 2015) showed that there exists a group topology on the free Abelian group of cardinality $\mathfrak{c}$ that makes all of its finite powers countably compact.

With respect to infinitely divisible torsion-free Abelian groups, in particular direct algebraic sums of $\mathbb{Q}$ (one of which is $\mathbb{R}$, since it is as a $\mathbb{Q}$-vector space it has basis size $\mathfrak{c}$), Tkachenko and Yashenko TKACHENKO and YASCHENKO, 2002 first showed from Martin's Axiom that $\mathbb{R}$ can be endowed with a countably compact group topology without non-trivial convergent sequences. In A. C. BOERO and A. H. TOMITA, 2010, such a topology was constructed using $\mathfrak{c}$ selective ultrafilters. In A. BOERO *et al.*, 2015, it is shown that there is a group topology without non-trivial convergent sequences such that $\mathbb{R}^2$ is countably compact, a first step to make larger powers of $\mathbb{R}$ countably compact.

The proof that a free Abelian group F does not admit a group topology such that F^ω is countably compact relies on the fact that the only element of F that is infinitely divisible is 0. Since $\mathbb{Q}$ is a divisible group, it seemed to be a candidate for a torsion free group that admits a p-compact group topology. Another good reason to look at direct sums of $\mathbb{Q}$ was the argument that they are the test space for pseudocompactness of non-torsion groups (W. COMFORT and REMUS, 1993).

This thesis is divided thus:

In Chapter 2, we show how $\mathbb{Q}^{(\kappa)}$ can be endowed with a p-compact group topology without nontrivial convergent sequences, given p a selective ultrafilter and κ an infinite cardinal such that $\kappa = \kappa^\omega$.

In Chapter 3, we assume the existence of $\mathfrak{c}$ incomparable selective ultrafilters to show that, for every infinite cardinal κ such that $\kappa^\omega = \kappa$, the free Abelian group on κ generators can be endowed with a group topology without nontrivial convergent sequences such that all of its finite powers are countably compact.

In Chapter 4, we build a forcing poset in order to show that it is consistent that, assuming a selective ultrafilter $\mathcal{U}$, for a cardinal λ of countable cofinality, $\mathbb{Q}^{(\lambda)}$ admits a $\mathcal{U}$-compact group topology.

Chapter 2

On the p-compactness of arbitrarily large sums of $\mathbb{Q}$

2.1 Introduction

This chapter will lay out and detail the achievement of two goals obtained through the use of a selective ultrafilter: construct a p-compact group topology without non-trivial convergent sequences over a torsion-free Abelian group and construct arbitrarily large countably compact group topologies over some torsion-free group.

On the direct sum of $\mathbb{Q}$'s: some history, the setting and the aim

Halmos Halmos, 1944 proved that $\mathbb{R}$ can be endowed with a compact group topology, which in particular contains non-trivial convergent sequences. Recall that algebraically $\mathbb{R}$ is the direct sum of $\mathfrak{c}$ copies of $\mathbb{Q}$.

Tkachenko and Yashenko Tkachenko and Yaschenko, 2002 showed from Martin's Axiom that $\mathbb{R}$ can be endowed with a countably compact group topology without non-trivial convergent sequences. In A. C. Boero and A. H. Tomita, 2010, such a topology was constructed using $\mathfrak{c}$ selective ultrafilters. In A. Boero *et al.*, 2015, it is shown that there is a group topology without non-trivial convergent sequences such that $\mathbb{R}^2$ is countably compact, a first step to make larger powers of $\mathbb{R}$ countably compact.

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The first advantage we noticed is that an ultrapower of a direct sum of $\mathbb{Q}$ is a vector space, a useful fact for the construction of large countably compact groups without non-trivial convergent sequences in Castro-Pereira and A. H. Tomita, 2010. As for Abelian groups, their ultrapowers are never free Abelian groups.

Our aim for this chapter is: given a free ultrafilter p and a cardinal κ such that $\kappa = \kappa^\omega$, to show that $\sum_{\alpha<\kappa} \mathbb{Q}$ has a p-compact Hausdorff group topology without non-trivial convergent sequences.

2.2 Notation

Throughout this chapter we fix an infinite cardinal κ such that $\kappa = \kappa^\omega$.

Let $\mathbb{T}$ be the Abelian group $\mathbb{R}/\mathbb{Z}$.

Let G be the Abelian additive group $\mathbb{Q}^{(\kappa)} := \{g \in \mathbb{Q}^\kappa : |\operatorname{supp} g| < \omega\}$ and let H be the Abelian additive group $\mathbb{Z}^{(\kappa)} := \{g \in \mathbb{Z}^\kappa : |\operatorname{supp} g| < \omega\}$. If $C \subseteq \kappa$, let $\mathbb{Q}^{(C)} := \{g \in G : \operatorname{supp} g \subseteq C\}$.

Definition 2.2.1. Given $\mu \in \kappa$, we denote by χ_μ the element of G such that $\operatorname{supp} \chi_\mu = \{\mu\}$ and $\chi_\mu(\mu) = 1$.

Given $\mu \in \kappa$, we define $\vec{\mu}$ as the constant sequence whose value is μ.

If $A \subseteq \omega$ and $\zeta : A \to \kappa$, then we define $\chi_\zeta \in G^A$ by $\chi_\zeta(n) = \chi_{\zeta(n)}$ for each $n \in A$. □

Definition 2.2.2. Given $A \subseteq \omega$ and $s = (s_n : n \in A)$ a sequence of rational numbers, we denote by sf the function in G^A given by $(sf)(n) = s_n f(n)$, for each $n \in A$.

Given $\mathcal{A} \subseteq G^A$, we define $s\mathcal{A} := \{sf : f \in \mathcal{A}\}$. If $s : A \to \mathbb{Q}\setminus\{0\}$, we define $\frac{\mathcal{A}}{s} = \{\frac{1}{s} f : f \in A\}$. □

Definition 2.2.3. Given an ultrafilter q, we define an equivalence relation on G^ω by letting $f \simeq_q g$ if and only if $\{n \in \omega : f(n) = g(n)\} \in q$. We denote by $[f]_q$ the equivalence class to which f belongs, and by G^ω/q the quotient $G^\omega/\simeq_q$. Notice that this set has a natural Q-vector space structure. This group is known as the *q-ultrapower* of G, and is denoted by $\operatorname{Ult}_q(G)$. □

2.3 Homomorphisms, arc functions and arc equations

Our approach to construct the group topology is to consider the weak topology generated by an appropriate family of homomorphisms from G into $\mathbb{T}$. These homomorphisms will be constructed by considering successive approximations by arcs. Thus, the following definition is helpful.

Definition 2.3.1. Given $a \in \mathbb{R}$, let $a + \mathbb{Z} = \{a + n : n \in \mathbb{Z}\}$. Also, if $I \subseteq \mathbb{R}$, let $I + \mathbb{Z} = \{a + \mathbb{Z} : a \in I\}$, which is a subset of $\mathbb{T}$.

If $S \in \mathbb{Z}$ and $a \in \mathbb{T}$, where $a = b + \mathbb{Z}$ for some $b \in \mathbb{R}$, let $Sa := (Sb) + \mathbb{Z}$. Note that given a and S, this definition does not depend on b. Moreover, given $S \in \mathbb{Z}$ and $I \subseteq \mathbb{T}$, let $SI = \{Sa : a \in I\}$.

Given $I, J \subseteq \mathbb{T}$, let $I + J = \{a + b : a \in I, b \in J\}$. Note that this operation is associative and commutative.

Let $\mathbb{B} = \{I + \mathbb{Z} : \emptyset \neq I \subseteq \mathbb{R}$ is an open interval$\}$ be the collection of all the nonempty open arcs in $\mathbb{T}$, including $\mathbb{T}$ itself.

An *arc function* is a function $\phi : \kappa \to \mathbb{B}$ such that $\operatorname{supp}\phi := \{\xi \in \kappa : \phi(\xi) \neq \mathbb{T}\}$ is finite. This set is called the *support* of ϕ. Given a positive $\epsilon < \frac{1}{2}$, we say that an arc function ϕ is an *ϵ-arc fucntion* if for every $\xi \in \operatorname{supp}\phi$, $\phi(\xi)$ has length ϵ.

Given two arc functions ψ and ϕ, we will say that $\psi \leq \phi$ if $\psi(\xi) = \phi(\xi)$ or $\overline{\psi(\xi)} \subseteq \phi(\xi)$, for each $\xi \in \kappa$.

Given an arc function ϕ and a positive integer S, $S\phi$ is the arc function such that $(S\phi)(\mu) = S\phi(\mu)$ for every $\mu \in \kappa$. □

We can interpret an arc function as an approximation of a homomorphism defined from $\mathbb{Z}^{(\kappa)}$ into $\mathbb{T}$. Intuitively, ϕ tells us that the homomorphism we are approaching sends χ_ξ into a point in the closure of the arc $\phi(\xi)$ for every $\xi < \kappa$. Thus, given an $a \in \mathbb{Z}^{(\kappa)}$, the homomorphism we are guessing will send a in the closure of the arc $\sum_{\xi\in\operatorname{supp} a} a(\xi)\phi(\xi)$. Therefore the following definition becomes useful:

Definition 2.3.2. Let ϕ be an arc function. Given $a \in \mathbb{Z}^{(\kappa)}$, we define $\phi(a) := \sum_{\xi\in\operatorname{supp} a} a(\xi)\phi(\xi)$. □

The domain of an arc function was defined as κ. No confusion arises from the previous definition since κ and $\mathbb{Z}^{(\kappa)}$ are disjoint.

Now we define the concept of an arc equation. We begin with an informal discussion.

Imagine we are given an arc function ϕ, some elements $\mathcal{A} \subseteq \mathbb{Z}^{(\kappa)}$, a positive integer S and arcs U_a for each $a \in \mathcal{A}$. We search for an arc function ψ such that $S\psi \leq \phi$ and such that $\psi(a) \subseteq U_a$ for each $a \in \mathcal{A}$.

By iterating this process and selecting an appropriate sequence os S's, the first condition helps us to extend the final homomorphism to $\mathbb{Q}^{(\kappa)}$ instead of being defined only on $\mathbb{Z}^{(\kappa)}$. The second condition helps us to control whither some elements of $\mathbb{Z}^{(\kappa)}$ are going to be taken.

To study p-limits, it will be useful to consider sequences of elements of $\mathbb{Z}(\kappa)$ instead of simply elements. These sequences need not be defined in the whole of ω, just in a member of the ultrafilter p.

Definition 2.3.3. An arc equation is a quintuple $(\phi, A, \mathcal{A}, S, U)$ where ϕ is an arc function, $A \subseteq \omega$, $\mathcal{A} \subseteq (\mathbb{Z}^{(\kappa)})^A$, S is a positive integer and $U = (U_f : f \in \mathcal{A})$ is a family of elements of $\mathbb{B}$.

Given $n \in A$, an *n-solution* for the arc equation $(\phi, A, \mathcal{A}, S, U)$ is an arc function ψ such that $S\psi \leq \phi$ and $\psi(f(n)) \subseteq U_f$, for each $f \in \mathcal{A}$. □

We will need results that tell us that n-solutions exist for many n's. In order to achieve such results, the notion of rational stack, which will be introduced later, shall be useful.

The goal is to use this machinery to prove the following:

Lemma 2.3.4. (Main Lemma). Fix a selective ultrafilter p. Let $\mathcal{F} \subseteq G^\omega$ be a countable collection of distinct elements mod p such that $\{[f]_p : f \in \mathcal{F}\} \dot\cup \{[\chi_{\vec{\mu}}]_p : \mu \in \kappa\}$ is $\mathbb{Q}$-linearly independent in $\mathrm{Ult}_p(G)$.

Let $d, d_0, d_1 \in G \setminus \{0\}$ with $\mathrm{supp}\, d$, $\mathrm{supp}\, d_0$, $\mathrm{supp}\, d_1$ pairwise disjoint, and C be a countably infinite subset of κ such that $\omega \cup \mathrm{supp}\, d \cup \mathrm{supp}\, d_0 \cup \mathrm{supp}\, d_1 \cup \bigcup_{f \in \mathcal{F}, n \in \omega} \mathrm{supp}\, f(n) \subseteq C$. For each $f \in \mathcal{F}$, fix a $\xi_f \in C$.

<u>Then</u>: There exists a homomorphism $\phi : \mathbb{Q}^{(C)} \to \mathbb{T}$ such that

(a) $\phi(d) \neq 0$, $\phi(d_0) \neq \phi(d_1)$, and

(b) $p - \lim(\phi(\frac{1}{N} f)) = \phi(\frac{1}{N} \chi_{\xi_f})$, for each $f \in \mathcal{F}$ and $N \in \omega$.

Now we use this Lemma to prove the result stated at the end of section 2.1.

For the remaining of this section, let $\{f_\alpha : \omega \leq \alpha < \kappa\}$ be an enumeration of G^ω such that $\bigcup_{n \in \omega} \mathrm{supp}\, f_\xi(n) \subseteq \xi$, for each $\xi \in [\omega, \kappa)$.

By applying the Main Lemma, we get the following result:

Lemma 2.3.5. Fix a selective ultrafilter p. Let $I \subseteq [\omega, \kappa)$ be such that $\{[f_\xi]_p : \xi \in I\} \cup \{[\chi_{\vec{\mu}}]_p : \mu \in \kappa\}$ is a $\mathbb{Q}$-basis for $\mathrm{Ult}_p(G)$.

Let $d \in G \setminus \{0\}$, $r \in \mathbb{Q}^{(I)} \setminus \{0\}$ and $B \in p$. Let C be a countably infinite subset of κ such that $\omega \cup \mathrm{supp}\, r \cup \mathrm{supp}\, d \subseteq C$ and $\bigcup_{n \in \omega} \mathrm{supp}\, f_\xi(n) \subseteq C$ for every $\xi \in C \cap I$.

Then there exists a homomorphism $\phi : \mathbb{Q}^{(C)} \to \mathbb{T}$ such that

(a) $\phi(d) \neq 0$,

(b) $p - \lim(\phi(\frac{1}{N} f_\xi)) = \phi(\frac{1}{N} \chi_\xi)$, for each $\xi \in C \cap I$ and each $N \in \omega$, and

(c) $(\phi(\sum_{\mu \in \mathrm{supp}\, r} r(\mu) f_\mu(n)) : n \in B)$ does not converge.

Proof. Let $D = \mathrm{supp}\, r$. Let $B' \in p$ be a subset of B such that $(\sum_{\mu \in D} r(\mu) f_\mu(n) : n \in B')$ is a 1-1 sequence, which is possible since the f_μ's are linearly independent mod p with the constant sequences and by the selectiveness of p.

Let $\mathbb{A}$ be an almost disjoint family on B' of cardinality $\mathfrak{c}$ and $h_x : \omega \to \{\sum_{\mu \in D} r(\mu) f_\mu(n) : n \in x\}$ be a bijection for each $x \in \mathbb{A}$.

Claim: There exist $x_0, x_1 \in \mathbb{A}$ such that $\{[f_\xi]_p : \xi \in C \cap I\} \cup \{[\chi_{\vec{\mu}}]_p : \mu \in \kappa\} \cup \{[h_{x_0}]_p, [h_{x_1}]_p\}$ is a linearly independent subset.

Proof of the claim: Given $x_0, x_1 \in \mathbb{A}$, notice that $h_{x_0}(n) \neq h_{x_1}(n)$ for all but a finite amount of n's, so $[h_{x_0}]_p \neq [h_{x_1}]_p$. Since $\mathbb{Q}$ is countable, it follows that $\langle [h_x]_p : x \in \mathbb{A} \rangle$ has cardinality $\mathfrak{c}$, so there is a $J \subset \mathbb{A}$ such that $|J| = \mathfrak{c}$ and that $([h_x]_p : x \in J)$ is linearly independent. Now notice that $\langle [f_\xi]_p : \xi \in C \cap I \rangle \oplus \langle [\chi_{\vec{\mu}}]_p : \mu \in C \rangle$ is countable, so there exist $x_0, x_1 \in J$ such that $\{[f_\xi]_p : \xi \in C \cap I\} \cup \{[\chi_{\vec{\mu}}]_p : \mu \in C\} \cup \{[h_{x_0}]_p, [h_{x_1}]_p\}$ is linearly independent. Since all the supports of these elements are contained in C, it is straightforward to see that $\{[f_\xi]_p : \xi \in C \cap I\} \cup \{[\chi_{\vec{\mu}}]_p : \mu \in \kappa\} \cup \{[h_{x_0}]_p, [h_{x_1}]_p\}$ is linearly independent.

Let $\mathcal{F} = \{f_\xi : \xi \in C \cap I\} \cup \{h_{x_0}, h_{x_1}\}$. Set $\xi_f = \mu$ if $f = f_\mu$ for some $\mu \in C \cap I$ and $\xi_f = m_i$ if $f = h_{x_i}$ for some $i < 2$ where $m_0 \neq m_1$ and $m_0, m_1 \in \omega \setminus \operatorname{supp} d$. Let $d_0 = \chi_{m_0}$, $d_1 = \chi_{m_1}$ and ϕ be as in Lemma 2.3.4.

Clearly conditions a) and b) of Lemma 2.3.5 are satisfied.

Furthermore, $(\phi(h_{x_i}(k)) : k \in \omega)$ has $(\phi(\chi_{m_i})$ as an accumulation point, for $i < 2$. Since these sequences are reorderings some subsequence of $(\phi(\sum_{\mu \in D} r(\mu) f_\mu(n)) : n \in B)$ and $\phi(\chi_{m_0}) \neq \phi(\chi_{m_1})$, it follows that c) is satisfied. ☐

Finally, we may extend the above homomorphism to G:

Lemma 2.3.6. Fix a selective ultrafilter p. Let $I \subseteq [\omega, \kappa)$ be such that $\{[f_\xi]_p : \xi \in I\} \cup \{[\chi_{\vec{\mu}}]_p : \mu \in \kappa\}$ is a $\mathbb{Q}$-basis for $\mathrm{Ult}_p(G)$.

Let $d \in G \setminus \{0\}$, $r \in \mathbb{Q}^{(I)} \setminus \{0\}$, and $B \in p$.

Then there exists a homomorphism $\phi : G \to \mathbb{T}$ such that

(a) $\phi(d) \neq 0$,

(b) $p - \lim \phi(\frac{1}{N} f_\xi) = \phi(\frac{1}{N}\chi_\xi)$, for each $\xi \in I$ and $N \in \omega$, and

(c) $(\phi(\sum_{\mu \in \operatorname{supp} r} r(\mu) f_\mu(n)) : n \in B)$ does not converge.

Proof. Let $D = \operatorname{supp} r$. Let C be a countably infinite subset of κ such that $\omega \cup \operatorname{supp} d \subseteq C$ and $\bigcup_{n \in \omega} \operatorname{supp} f_\xi(n) \subseteq C$ for every $\xi \in C \cap I$. Such a C exists by standard closing off arguments. Let $(\xi_\alpha : \alpha < \kappa)$ be a strictly increasing enumeration of $\kappa \setminus C$. Let ϕ be as in Lemma 2.3.5.

For each $\alpha \leq \kappa$, let $C_\alpha := C \cup \{\xi_\beta : \beta < \alpha\}$ (so $C_0 = C$ and $C_\kappa = \kappa$). Note that for each α and $n \in \omega$, $\operatorname{supp} f_{\xi_\alpha}(n) \subseteq \xi_\alpha \subseteq C_\alpha$.

Recursively we define homomorphisms $\phi_\alpha : \mathbb{Q}^{(C_\alpha)} \to \mathbb{T}$ for $\alpha \leq \kappa$ satisfying:

(i) $\phi_0 = \phi$,

(ii) $\phi_\beta \subseteq \phi_\alpha$ whenever $\beta \leq \alpha \leq \kappa$, and

(iii) $p - \lim \phi_\alpha(\frac{1}{N} f_\xi) = \phi_\alpha(\frac{1}{N}\chi_\xi)$, for each $\xi \in C_\alpha \cap I$ and $N \in \omega$.

We let $\phi_0 = \phi$. For limit steps, just take unions. For a successor step $\alpha + 1$ we proceed as follows:

Notice that $\mathbb{Q}^{(C_{\alpha+1})} = \mathbb{Q}^{(C_\alpha)} \oplus \{q\chi_{\xi_\alpha} : q \in \mathbb{Q}\}$.

First, we define $\widetilde{\phi}_\alpha : \{q\chi_{\xi_\alpha} : q \in \mathbb{Q}\} \to \mathbb{T}$ by letting $\widetilde{\phi}_\alpha(\frac{M}{N}\chi_{\xi_\alpha}) = M(p - \lim \phi_\alpha(\frac{1}{N} f_{\xi_\alpha}))$.

Since multiplying a group element by an integer is a continuous function and since ϕ_α is a homomorphism, it follows that $\widetilde{\phi}_\alpha$ is well-defined and a group homomorphism. Now let $\phi_{\alpha+1} = \phi_\alpha \oplus \widetilde{\phi}_\alpha$.

The required homomorphism is ϕ_κ. ☐

We apply this lemma to obtain the main result of this chapter:

Theorem 2.3.7. Assume that p is a selective ultrafilter and $\kappa = \kappa^\omega$ is an infinite cardinal. Then there exists a p-compact group topology on $G = \mathbb{Q}^{(\kappa)}$ without non-trivial convergent sequences.

Proof. Let I be as in the previous lemma. For each $d \in G\setminus\{0\}$, $r \in \mathbb{Q}^{(I)}\setminus\{0\}$ and $B \in p$, take $\phi_{d,r,B} : G \to \mathbb{T}$ as in the previous lemma.

The group topology induced by these homomorphisms is such that the $p-\lim(\frac{1}{N}f_\xi) = \frac{1}{N}\chi_\xi$, for each $\xi \in I$ and $\mathbb{N} \in \omega$.

If h is any element of G^ω, there exist families $(r_\xi : \xi \in I)$ and $(s_\mu : \mu \in \kappa)$ of rational numbers where all but a finite amount of them are 0 such that:

$[h]_p = \sum_{\xi\in I} r_\xi[f_\xi]_p + \sum_{\mu\in\kappa} s_\mu[\chi_{\vec{\mu}}]_p$.

It follows that $\sum_{\xi\in I} r_\xi\chi_\xi + \sum_{\mu\in\kappa} s_\mu\chi_\mu$ is the p-limit of h. Therefore, G is p-compact.

To check that there are no non-trivial convergent sequences, fix a one-to-one sequence g. Let $r \in \mathbb{Q}^{(I)}\setminus\{0\}$ and $s \in \mathbb{Q}^{(\kappa)}$ be such that:

$[g]_p = \sum_{\xi\in I} r_\xi[f_\xi]_p + \sum_{\mu\in\kappa} s_\mu[\chi_{\vec{\mu}}]_p$. Let $D = \operatorname{supp} r$. Then there is $B \in p$ such that: $g(n) = \sum_{\xi\in I} r_\xi f_\xi(n) + \sum_{\mu\in\kappa} s_\mu\chi_\mu$, for all $n \in B$. By Lemma 2.3.6(c), we have that $(\phi_{d,r,B}(\sum_{\xi\in D} r_\xi f_\xi(n)) : n \in B)$ does not converge in $\mathbb{T}$, and so $(\sum_{\xi\in D} r_\xi f_\xi(n) : n \in B)$ does not converge in G. Since $\sum_{\mu\in\kappa} s_\mu\chi_\mu$ is constant, it follows that $(g(n) : n \in B)$ does not converge, and so g does not converge. $\square$

Malykhin and Shapiro Malykhin and Shapiro, 1985 showed that under GCH there are no pseudocompact groups without non-trivial convergent sequences whose weight has countable cofinality. The second example in A. H. Tomita, 2003 showed that it is consistent that there exists a countably compact group without non-trivial convergent sequences whose weight is $\aleph_\omega < 2^{\mathfrak{c}}$. In Castro-Pereira and A. H. Tomita, 2010, the authors obtained consistent arbitrarily large examples of weight of countable cofinality, but the examples are finite torsion groups.

By applying an argument similar to the one in the proof of Theorem 4.1 of A. H. Tomita, 2003, it is possible to set the weight of the group to any cardinal between κ and 2^κ.

2.4 A preliminar discussion on rational stacks

We will start this section with an informal discussion about rational stacks.

A rational stack wil be defined as a nonuple $(\mathcal{B}, \nu, \zeta, K, A, k_0, k_1, l, T)$, where:

- $A \subseteq \omega$ is infinite,
- $T > 0$ is an integer,
- $K : A \to \omega\setminus 2$ is such that for every $n \in A$, $(n!T) \mid K_n$,

- $k_0 \leq k_1$ are natural numbers with $k_1 > 0$,
- $l : k_1 \to \omega$,
- $\nu : k_0 \to \kappa$,
- $\zeta : k_1 \to \kappa^\omega$
- $\mathcal{B} = (\mathcal{B}_{i,j} : i < k_1, j < l_i)$ is such that each $\mathcal{B}_{i,j} \subseteq H^\omega$ is finite.

In order to be a rational stack, this nonuple must satisfy additional properties. The full definition of rational stack will be given in Section 2.6. This definition was designed to solve arc equations when constructing the homomorphisms of Lemma 2.3.4.

Before we even define the stack, we will list the main results that motivated its definition.

The Lemma below associates each finite subset of functions to a stack that will be used to solve arc equations associated to this family.

Lemma 2.4.1. Let $B \in p$ and $\mathcal{G}$ be a finite subset of G^ω whose elements are distinct mod p and none of them are constant mod p such that $\{[f]_p : f \in \mathcal{G}\} \cup \{[\chi_{\vec{\nu}}]_p : \nu \in \kappa\}$ is linearly independent. Then there exists a rational stack $\mathcal{S} = (\mathcal{B}, \nu, K, A, k_0, k_1, l, T)$ such that, by defining $\mathcal{A} = \mathcal{G} \cup \{\chi_{\vec{\nu}_i} : i < k_0\}$ and $\mathcal{C} = \frac{\bigcup_{i<k_1, j<l_i} \mathcal{B}_{i,j}}{K}$, there exist $\mathcal{M} : \mathcal{A} \times \mathcal{C} \to \mathbb{Z}$, $\mathcal{N} : \mathcal{C} \times \mathcal{A} \to \mathbb{Z}$ satisfying:

(1) $\{[f]_p : f \in \mathcal{A}\}$ and $\{[h]_p : h \in \mathcal{C}\}$ generate the same subspace of $\mathrm{Ult}_p(G)$,

(2) $f(n) = \sum_{h\in \mathcal{C}} \mathcal{M}_{f,h} h(n)$, for each $n \in A$ and $f \in \mathcal{A}$,

(3) $h(n) = \frac{1}{T^2} \sum_{f\in\mathcal{A}} \mathcal{N}_{h,f} f(n)$, for each $n \in A$ and $h \in \mathcal{C}$,

(4) $K\mathcal{A} \subseteq H^\omega$,

(5) $K\mathcal{C} \subseteq H^\omega$, and

(6) $A \in p$ and $A \subseteq B$.

Proof. The proof is quite technical and will be presented in a later section. □

Notice that if we interpret $\mathcal{M}$ and $\mathcal{N}$ and matrices, the $\mathcal{M}$ and $\frac{1}{T^2}\mathcal{N}$ are inverse matrices.

Roughly speaking, to prove Lemma 2.3.4, we write the countable set $\mathcal{F}$ as a countable union of finite sets $\mathcal{F}_n$ and associate each of these finite sets of sequences to a stack. Then, working inductively, in each step we need to solve some arc equations. We transform the arc equations associated to these finite families to arc equations associated to a stack using 2.4.1, solve the arc equations using properties of the stack, then return to a solution of the original arc equations. We want to solve infinitely many equations; thus, this process is made back and forth. At each stage, the stack is different and there is no containment relation between them, even though we use a larger finite subfamily of sequences.

In each step of the recursive construction of the arcs used to define the homomorphism, we have arc equations related to a certain arc size. The following two Lemmas are used to solve these equations in a back and forth manner.

Lemma 2.4.2. Let $\mathcal{S}$, $\mathcal{A}$, $\mathcal{C}$, $\mathcal{M}$ and $\mathcal{N}$ be as in Lemma 2.4.1. Let ϵ be a positive real and D be a finite subset of κ. Then there exist $B \subseteq A$ cofinite in A and a family of positive real numbers $(\gamma_n : n \in B)$ such that:

For every $n \in B$, for every family $W = (W_h : h \in \mathcal{C})$ of open arcs of length ϵ, and for every arc function ψ of length ϵ such that $\operatorname{supp} \psi \subseteq D \backslash \{v_i : i < k_0\}$, there exists and n-solution of length γ_n for the arc equation $(\psi, B, K\mathcal{C}, K_n, W)$.

Proof. The proof is quite technical and will be presented in a later section. □

Lemma 2.4.3. Let $\mathcal{S}$, $\mathcal{A}$, $\mathcal{C}$, $\mathcal{M}$ and $\mathcal{N}$ be as in Lemma 2.4.1. Let δ be a positive real such that $\epsilon = \frac{\delta}{\sum_{f \in \mathcal{A}, h \in \mathcal{C}} |\mathcal{M}_{f,h}|} < \frac{1}{2}$.

Let $(U_f : f \in \mathcal{A})$ be a family of open arcs of length δ. Let ρ be an arc function of length δ such that $U_{\chi_{\vec{v_i}}} = \rho(v_i)$ for $i < k_0$. Furthermore, assume that $\{v_i : i < k_0\} \mathit{subseteq} \operatorname{supp} \rho$.

Then, there exist $W = (W_h : h \in \mathcal{C})$ a family of open arcs of length ϵ and ψ and ϵ-arc function with support $\operatorname{supp} \rho \backslash \{v_i : i < k_0\}$ such that for every $n \in A$, every n-solution for the arc equation $(\psi, A, K\mathcal{C}, K_n, W)$ is an n-solution for $(\rho, A, K\mathcal{A}, K_n, U)$.

Proof. Given $f \in \mathcal{A}$, let $y_f \in \mathbb{R}$ be such that $y_f + \mathbb{Z}$ is the center of the arc U_f.

For each $h \in \mathcal{C}$, let $z_h = \sum_{f \in \mathcal{A}} \mathcal{N}_{h,f} \frac{y_f}{T^2}$. Since $\mathcal{N}$ is an integer matrix, it follows that $z_h + \mathbb{Z} = \sum_{f \in \mathcal{A}} \mathcal{N}_{h,f}(\frac{y_f}{T^2} + \mathbb{Z})$. Let W_h be an arc centered on $z_h + \mathbb{Z}$ whose length is ϵ.

Let $\psi(\mu)$ be an arc with the same center as $\rho(\mu)$ of length ϵ for each $\mu \in \operatorname{supp} \rho \backslash \{v_i : i < k_0\}$ and $\psi(v_i) = \mathbb{T}$ for each $i < k_0$.

Suppose ϕ is an n-solution for $(\psi, A, K\mathcal{C}, K_n, W)$.

Then $\phi(K_n h(n)) \subseteq W_h$, for each $h \in \mathcal{C}$. Also, we have that, for each $\mu \in \kappa \backslash \{v_i : i < k_0\}$:

$$K_n \phi(\mu) \leq \psi(\mu) \leq \rho(\mu). \qquad (\#)$$

Let $f \in \mathcal{A}$.

Notice that for each μ, $\sum_{h \in C} \mathcal{M}_{f,h} K_n h(n)(\mu)\phi(\mu) = K_n f(n)(\mu)\phi(\mu)$. Therefore,

$$\sum_{h \in C} \mathcal{M}_{f,h} \sum_{\mu \in \operatorname{supp} h(n)} K_n h(n)(\mu)\phi(\mu) = \sum_{\mu \in \operatorname{supp} f(n)} K_n f(n)(\mu)\phi(\mu).$$

It follows that:

$$\phi(K_n f(n)) = \sum_{\mu \in \operatorname{supp} f(n)} K_n f(n)(\mu)\phi(\mu) \subseteq \sum_{h \in C} \mathcal{M}_{f,h} W_h.$$

The arc $\sum_{h\in C}\mathcal{M}_{f,h}W_h$ is centered on $\sum_{h\in C}\mathcal{M}_{f,h}(z_h+\mathbb{Z})=\sum_{h\in C}\mathcal{M}_{f,h}\sum_{g\in\mathcal{A}}\mathcal{N}_{h,g}(\frac{y_g}{T^2}+\mathbb{Z})=\sum_{g\in\mathcal{A}}\sum_{h\in C}\mathcal{M}_{f,h}\mathcal{N}_{h,g}(\frac{y_g}{T^2}+\mathbb{Z})=y_f+\mathbb{Z}$, and has length $\epsilon\cdot\sum_{h\in C}\mid\mathcal{M}_{f,h}\mid\leq\delta$. Therefore,

$$\sum_{h\in C}\mathcal{M}_{f,h}W_h\subseteq U_f. \qquad (*)$$

Thus, ϕ is an n-solution for $(\rho, A, K\mathcal{A}, K_n, U)$, as required, provided that we show $K_n\phi\leq\rho$.

From $(*)$, if $f=\chi_{\vec{v}_i}$, then $K_n\phi(v_i)=K_n\chi_{\vec{v}_i}(n)(v_i)\phi(v_i)\subseteq U_{\chi_{\vec{v}_i}}=\rho(v_i)$, hence $K_n\phi(v_i)\leq\rho(v_i)$ for each $i<k_0$. This and (#) imply that $K_n\phi\leq\rho$. □

2.5 Proof of the Main Lemma using the properties of the stacks

Our goal in this section is to prove Lemma 2.3.4 using the lemmas in the previous section. First we state the following lemma:

Lemma 2.5.1. Fix a selective ultrafilter p. Let $\mathcal{F}\subseteq G^\omega$ be a countable collection of distinct elements mod p such that $\{[f]_p : f\in\mathcal{F}\}\,\dot\cup\,\{[\chi_{\vec{\mu}}]_p : \mu\in\kappa\}$ is $\mathbb{Q}$-linearly independent in $\mathrm{Ult}_p(G)$.

Let $d, d_0, d_1\in G\setminus\{0\}$ with $\mathrm{supp}\, d$, $\mathrm{supp}\, d_0$, $\mathrm{supp}\, d_1$ pairwise disjoint, and C be a countably infinite subset of κ such that $\omega\cup\mathrm{supp}\, d\cup\mathrm{supp}\, d_0\cup\mathrm{supp}\, d_1\subseteq C$ and $\bigcup_{f\in\mathcal{F},n\in\omega}\mathrm{supp}\, f(n)\subseteq C$. For each $f\in\mathcal{F}$, choose $\xi_f\in C$. Let $(\mathcal{F}^n : n\in\omega)$ be an increasing sequence of finite sets whose union is $\mathcal{F}$.

Then there exist:

- stacks $\mathcal{S}^m=(\mathcal{B}^m, v^m, \zeta^m, K^m, A^m, k_0^m, k_1^m, l^m, T^m)$ and $\mathcal{A}^m, \mathcal{C}^m, \mathcal{M}^m, \mathcal{N}^m$ related to the stacks as in Lemma 2.4.1,
- $r:\omega\cup\{-1\}\to\omega$ such that $r[\omega]\in p$ and $r(-1)=0$,
- a sequence of arc functions $(\rho^{r(m)} : m\geq -1)$ with $C\subseteq\bigcup_{m\geq -1}\mathrm{supp}\,\rho^{r(m)}$, and
- a sequence of integers $(N_m : m\in\omega\cup\{-1\})$

satisfying, for every m, m' such that $-1\leq m'\leq m<\omega$, the following:

(a) $0\notin\overline{\rho^0(d)}$ and $\overline{\rho^0(d_0)}\cap\overline{\rho^0(d_1)}=\emptyset$,

(b) for every $\xi\in\mathrm{supp}\,\rho^{r(m+1)}$, we have $\frac{N_{m+1}}{N_{m'}}\rho^{r(m+1)}(\xi)\subseteq\rho^{r(m')}(\xi)$ and $\frac{N_{m+1}}{N_{m'}}\rho^{(m+1)}(\xi)$ has length at most $\frac{1}{2^{r(m)+1}}$,

(c) for every $f\in\mathcal{F}^{r(m)}$, we have $K^{r(m)}_{r(m+1)}\rho^{r(m+1)}(f(r(m+1)))\subseteq\rho^{r(m)}(\xi_f)$

(d) for every $f\in\mathcal{F}^{r(m)}$, we have $\frac{N_{m+1}}{N_{m'}}\rho^{r(m+1)}(f(r(m+1)))\subseteq\rho^{r(m')}(\xi_f)$,

(e) $\mathrm{supp}\,\rho^{r(m)}\subseteq\mathrm{supp}\,\rho^{r(m+1)}$, and

(f) $N_{-1}=1$ and $N_{m+1}=\prod_{i=-1}^{m}K^{r(i)}_{r(i+1)}$.

Proof. Let p, $\mathcal{F}$, d, d_0, d_1, C and $(\xi_f : f \in \mathcal{F})$ be given. Write C as an increasing sequence of finite sets $(C^n : n \in \omega)$ such that for each $n \in \omega$, $\bigcup\{\operatorname{supp} f(k) : f \in \mathcal{F}^n \text{ and } k \leq n\} \subseteq C^n$, and $\operatorname{supp} d \cup \operatorname{supp} d_0 \cup \operatorname{supp} d_1 \subseteq C^0$.

Apply Lemma 2.4.1 to $\mathcal{G} = \mathcal{F}^0$ and $B = \omega$ to obtain a rational stack $\mathcal{S}^0 = (\mathcal{B}^0, \nu^0, \zeta^0, K^0, A^0, k_0^0, k_1^0, l^0, T^0)$ and $\mathcal{A}^0$, $\mathcal{C}^0$, $\mathcal{M}^0$ and $\mathcal{N}^0$ satisfying (1)-(7) as in the Lemma.

Fix $\delta^0 \in \mathbb{R}$ such that $0 < \delta^0 < \frac{1}{2}$ and ρ^0 a δ^0-arc function such that $0 \notin \overline{\rho^0(d)}$ and $\overline{\rho^0(d_0)} \cap \overline{\rho^0(d_1)} = \emptyset$. We will also assume that $C^0 \cup \{\nu_i^0 : i < k_0^0\} \subseteq \operatorname{supp} \rho^0$.

Let $\epsilon^0 = \frac{\delta^0}{\sum_{f \in \mathcal{A}^0, h \in \mathcal{B}^0} |\mathcal{M}_{f,h}^0|}$. Notice that with this ϵ^0 we may apply Lemma 2.4.3.

Now we apply Lemma 2.4.2 with $D = C^0$ to obtain $B^0 \subseteq A^0 \setminus 1$ and $\gamma^0 = (\gamma_n^0 : n \in B^0)$ as in the Lemma.

Suppose the following are defined: $(B^t : t \leq m)$ a decreasing family of elements of p and γ_n^t for $t \leq m$ and $n \in B^t$.

Define $\delta^{m+1} = \frac{1}{2^{m+2}} \frac{1}{\prod_{i,n \leq m+1} K_n^i} \min(\{\gamma_n^t : t < n \leq m+2, n \in B^t\} \cup \{1\})$.

Apply Lemma 2.4.1 with $\mathcal{G} = \mathcal{F}^{m+1}$ and $B = B^m$ to obtain a stack $\mathcal{S}^{m+1} = (\mathcal{B}^{m+1}, \nu^{m+1}, \zeta^{m+1}, K^{m+1}, A^{m+1}, k_0^{m+1}, k_1^{m+1}, l^{m+1}, T^{m+1})$ and $\mathcal{A}^{m+1}$, $\mathcal{C}^{m+1}$, $\mathcal{M}^{m+1}$ and $\mathcal{N}^{m+1}$ related to the stack as in the lemma. Then $A^{m+1} \subseteq B^m$. Let $\epsilon^{m+1} = \frac{\delta^{m+1}}{\sum_{f \in \mathcal{A}^{m+1}, h \in \mathcal{B}^{m+1}} |\mathcal{M}_{f,h}^{m+1}|}$. Notice that with this ϵ^{m+1} we may apply Lemma 2.4.3.

Now we apply Lemma 2.4.2 with $D = C^{m+1}$ to obtain $B^{m+1} \subseteq A^{m+1} \setminus m+2$ and $\gamma^{m+1} = (\gamma_n^{m+1} : n \in B^{m+1})$ as in the Lemma.

We will use the *happiness* of the selective ultrafilter p: the sets constructed previously $B^0 \supseteq B^1 \supseteq \ldots$ are all elements of p, so there exists a function $r \in \omega^\omega$ such that $r[\omega] \in p$, $r(0) \in B^0$ and, for all $n \in \omega$, $r(n+1) \in B^{r(n)}$. (This follows from Proposition 11.6 of Halbeisen, 2012.)

Define $U^0 = (U_f^0 : f \in \mathcal{A}^0)$, where $U_f^0 = \rho^0(\xi_f)$ if $f \in \mathcal{F}^0$ or $U_f^0 = \rho(\nu_i^0)$ if $f = \chi_{\vec{\nu}_i^0}$. By Lemma 2.4.3 applied on stage 0, we obtain ψ and W as in the conclusion of Lemma 2.4.3. Now, according to the conclusion of Lemma 2.4.2 applied to stage 0 of the construction, since $r(0) \in B^0$, we obtain an $r(0)$-solution ϕ^0 of length $\gamma_{r(0)}^0$ to the arc equation $(\psi, B^0, K^0 C^0, K_{r(0)}^0, W)$.

Now, using the conclusion of Lemma 2.4.3, we have that ϕ^0 is an $r(0)$-solution to $(\rho^0, B^0, K^0 \mathcal{A}^0, K_{r(0)}^0, U^0)$. In particular, we have, for each $f \in \mathcal{F}^0$, $\phi^0(f(r(0))) \subseteq \rho^0(\xi_f)$.

Let $r(-1) = 0$ and $(N_m : m \geq -1)$ as in (f). We can recursively construct: $\phi^{r(m)}$, $\rho^{r(m)}$ a $\delta^{r(m)}$-arc function with $C^{r(m)} \subseteq \operatorname{supp} \rho^{r(m)}$, and $U^{r(m)}$ such that, for every m, m' such that $-1 \leq m' \leq m < \omega$:

(1) $\phi^{r(m)}$ is an $r(m+1)$-solution of length $\gamma_{r(m+1)}^{r(m)}$ for the arc equation

$$(\rho^{r(m)}, B^{r(m)}, K^{r(m)} \mathcal{A}^{r(m)}, , K_{r(m+1)}^{r(m)}, U^{r(m)}),$$

(2) $\rho^{r(m+1)} \leq \phi^{r(m)}$,

(3) for every $\xi \in \operatorname{supp} \rho^{r(m+1)}$, we have $\frac{N_{m+1}}{N_{m'}}\rho^{r(m+1)}(\xi) \subseteq \rho^{r(m')}(\xi)$ and $\frac{N_{m+1}}{N_{m'}}\rho^{r(m+1)}(\xi)$ has length at most $\frac{1}{2^{r(m)+1}}$,

(4) for every $f \in \mathcal{F}^{r(m)}$, we have $K^{r(m)}_{r(m+1)}\rho^{r(m+1)}(f(r(m+1))) \subseteq \rho^{r(m)}(\xi_f)$,

(5) for every $f \in \mathcal{F}^{r(m)}$, we have $\frac{N_{m+1}}{N_{m'}}\rho^{r(m+1)}(f(r(m+1))) \subseteq \rho^{r(m')}(\xi_f)$,

(6) $\operatorname{supp} \rho^{r(m)} \subseteq \rho^{r(m+1)}$, and

(7) $U^{r(m)} = (U^{r(m)}_f : f \in \mathcal{A}^{r(m)})$, where $U^{r(m)}_f = \rho^{r(m)}(\xi_f)$ if $f \in \mathcal{F}^{r(m)}$ or $U^{r(m)}_f = \rho^{r(m)}(v^{r(m)}_i)$ if $f = \chi_{\vec{v}^{r(m)}_i}$.

The base of the recursion is already done. Suppose the construction is done until step m and let us define $\phi^{r(m+1)}$, $\rho^{r(m+1)}$ and $U^{r(m+1)}$.

Let $\rho^{r(m+1)}$ be a $\delta^{r(m+1)}$-arc function such that $\operatorname{supp} \rho^{r(m)} \cup C^{r(m+1)} \cup \{v^{r(m+1)}_i : i < k^{r(m+1)}_0\} \subseteq \operatorname{supp} \rho^{r(m+1)}$ and $\rho^{r(m+1)} \leq \phi^{r(m)}$.

Now define $U^{r(m+1)} = (U^{r(m+1)}_f : f \in \mathcal{A}^{r(m+1)})$, where $U^{r(m+1)}_f = \rho^{r(m+1)}(\xi_f)$ if $f \in \mathcal{F}^{r(m+1)}$ or $U^{r(m+1)}_f = \rho^{r(m+1)}(v^{r(m+1)}_i)$ if $f = \chi_{\vec{v}^{r(m+1)}_i}$. By Lemma 2.4.3 applied on stage $m+1$, we obtain ψ and W as in the conclusion of Lemma 2.4.3. Now, according to the conclusion of Lemma 2.4.2 applied to stage $m+1$ of the construction, since $r(m+2) \in B^{r(m+1)}$, we obtain an $r(m+2)$-solution $\phi^{r(m+1)}$ of length $\gamma^{r(m+1)}_{r(m+2)}$ to the arc equation $(\psi, B^{r(m+1)}, K^{r(m+1)}C^{r(m+1)}, K^{r(m+1)}_{r(m+2)}, W)$.

Now, using the conclusion of Lemma 2.4.3, we have that $\phi^{r(m+1)}$ is an $r(m+2)$-solution to $(\rho^{r(m+1)}, B^{r(m+1)}, K^{r(m+1)}\mathcal{A}^{r(m+1)}, K^{r(m+1)}_{r(m+2)}, U^{r(m+1)})$.

Having $\rho^{r(m+1)}$ and $\phi^{r(m+1)}$ been thusly defined, items (1), (2), (6) and (7) of the recursion are immediately satisfied.

In order to verify item (3): the second statement follows from the definition of $\delta^{r(m+1)}$. As for the first statement, use items (1) and (2) and then use item (3) iteratively.

Item (4) follows from items (1) and (2) and the definition of $U^{r(m)}$.

Item (5) follows from multiplying the expression in (4) by $\frac{N_m}{N_{m'}}$ and then applying item (3) for m' and $m-1$.

Now that the recursion is complete, notice that items (a)-(e) of the statement of the Lemma are clearly satisfied. □

Now we are ready to prove Lemma 2.3.4.

Lemma (Main Lemma). Fix a selective ultrafilter p. Let $\mathcal{F} \subseteq G^\omega$ be a countable collection of distinct elements mod p such that $\{[f]_p : f \in \mathcal{F}\} \dot\cup \{[\chi_{\vec{\mu}}]_p > \mu \in \kappa\}$ is $\mathbb{Q}$ linearly independent in $\mathrm{Ult}_p(G)$.

Let $d, d_0, d_1 \in G \setminus \{0\}$ with $\operatorname{supp} d$, $\operatorname{supp} d_0$, $\operatorname{supp} d_1$ pairwise disjoint, and C be a countably infinite subest of κ such that $\operatorname{supp} d \cup \operatorname{supp} d_0 \cup \operatorname{supp} d_1 \cup \bigcup_{f \in \mathcal{F}, n \in \omega} \operatorname{supp} f(n) \subseteq C$. For each $f \in \mathcal{F}$, choose $\xi_f \in C$.

Then there exists a homomorphism $\phi : \mathbb{Q}^{(C)} \to \mathbb{T}$ such that

(a) $\phi(d) \neq 0$, $\phi(d_0) \neq \phi(d_1)$, and

(a) $p - \lim(\phi(\frac{1}{P} f)) = \phi(\frac{1}{P}\chi_{\xi_f})$, for each $f \in \mathcal{F}$ and $P \in \omega \setminus \{0\}$.

Proof. Let $\mathcal{S}^m$, $\mathcal{A}^m$, $\mathcal{C}^m$, $\mathcal{M}^m$, $\mathcal{N}^m$, $\mathcal{F}^m$, $(N_m : m \geq -1)$, r, $(\rho^{r(m)} : m \in \omega)$, and ρ^0 be as in the previous lemma.

Given a positive integer m' and $\xi \in C \cap \operatorname{supp} \rho^{r(m')}$, define $\phi(\frac{1}{N_{m'}}\chi_\xi)$ as the unique element of $\bigcap_{m \geq m'} \frac{N_m}{N_{m'}} \rho^{r(m)}(\xi)$. Furthermore, if P divides $N_{m'}$ then define $\phi(\frac{1}{P}\chi_\xi) = \frac{N_{m'}}{P}\phi(\frac{1}{N_{m'}}\chi_\xi)$. Then, since $n! \mid K_n^m$ for every n and m, $\phi(\frac{1}{P}\chi_\xi)$ is well-defined and does not depend on $N_{m'}$, and thus ϕ can be extended to a homomorphism.

Notice that since $0 \notin \overline{\rho^0(d)}$ and $\overline{\rho^0(d_0)} \cap \overline{\rho^0(d_1)} = \emptyset$, it follows that $\phi(d) \neq 0$ and $\phi(d_0) \neq \phi(d_1)$.

Let $f \in \mathcal{F}$ and P be a positive integer. Let M be a positive integer such that $f \in \mathcal{F}^M$.

Claim: $(\phi(\frac{1}{P} f(r(m))) : m \in \omega)$ converges to $\phi(\frac{1}{P}\chi_{\xi_f})$.

Proof. Let $m \geq M$ be such that P divides N_{m-1} and $\xi_f \in C \cap \operatorname{supp} \rho^{r(m-1)}$. Then

$$\phi(\tfrac{1}{P} f(r(m))) =$$
$$\phi(\tfrac{1}{P} \textstyle\sum_{\mu \in \operatorname{supp} f(r(m))} f(r(m))(\mu)\chi_\mu) =$$
$$\phi(\textstyle\sum_{\mu \in \operatorname{supp} f(r(m))} \frac{f(r(m))(\mu)}{P}\chi_\mu) =$$
$$\textstyle\sum_{\mu \in \operatorname{supp} f(r(m))} \phi(\frac{f(r(m))(\mu)}{P}\chi_\mu) =$$
$$\textstyle\sum_{\mu \in \operatorname{supp} f(r(m))} \frac{f(r(m))(\mu)}{P} N_m \phi(\frac{1}{N_m}\chi_\mu) =$$
$$\tfrac{1}{P} N_{m-1} \textstyle\sum_{\mu \in \operatorname{supp} f(r(m))} K_{r(m)}^{r(m-1)} f(r(m))(\mu)\phi(\frac{1}{N_m}\chi_\mu) \in$$
$$\tfrac{1}{P} N_{m-1} \textstyle\sum_{\mu \in \operatorname{supp} f(r(m))} K_{r(m)}^{r(m-1)} f(r(m))(\mu)\rho_{r(m)}(\mu) \subseteq$$
$$\tfrac{1}{P} N_{m-1}\rho_{r(m-1)}(\xi_f).$$

This last set is a neighborhood of $\phi(\frac{1}{P}\chi_{\xi_f})$ and has length at most $\frac{1}{2^{r(m-2)+1}}$.

This proves the claim. $\square$

Since $r[\omega] \in p$, it follows that the p-limit of $(\phi(\frac{1}{P} f(n)) : n \in \omega)$ is $\phi(\frac{1}{P}\chi_{\xi_f})$. $\square$

2.6 Defining rational stacks

We define stacks as a tool to solve a system of arc equations. The definitions and the ideas of the construction of rational stacks are motivated by A. H. Tomita, 2015.

We need to solve arc equations related to representatives of a basis for $\operatorname{Ult}_p(G)$. Thus, we construct a stack, associate the original arc equations to arc equations for the stack, solve the arc equations for the stack and convert these solutions to solutions to the original system of arc equations.

Now we give the full definition of rational stack.

Definition 2.6.1. A rational stack is a nonuple $(\mathcal{B}, \nu, \zeta, K, A, k_0, k_1, l, T)$, where:

- $A \subseteq \omega$ is infinite,
- $k_0 \leq k_1$ are natural numbers with $k_1 > 0$,
- $l : k_1 \to \omega$,
- $\nu : k_0 \to \kappa$,
- $\zeta : k_1 \to \kappa^\omega$,
- $K : A \to \omega \setminus 2$ is such that for every $n \in A$, $n!T \mid K_n$,
- $\mathcal{B} = (\mathcal{B}_{i,j} : i < k_1, j < l_i)$ is such that each $\mathcal{B}_{i,j} \subseteq H^\omega$ is finite, and
- $T > 0$ is an integer,

satisfying the following requirements:

(i) $\zeta_i(n) = \nu_i$ for every $i < k_0$ and $n \in A$,

(ii) ν_i, for $i < k_0$, and $\zeta_j(n)$, for $k_0 \leq j < k_1$ and $n \in A$, are all pairwise distinct,

(iii) $\zeta_i(n) \in \operatorname{supp} h(n)$, for each $i < k_1$, $j < l_i$, $h \in \mathcal{B}_{i,j}$ and $n \in A$,

(iv) $\zeta_i(n) \notin \operatorname{supp} h(n)$, for each $i < i_* < k_1$, $j < l_{i_*}$, $h \in \mathcal{B}_{i_*,j}$ and $n \in A$,

(v) $\left(\frac{h(n)(\zeta_i(n))}{K_n}\right)_{n \in A}$ converges, monotonically, to $+\infty$, $-\infty$ or a real number, for each $i < k_1$, $j < l_i$ and $h \in \mathcal{B}_{i,j}$,

(vi) for every $i < k_1$ and $j < l_i$, there exists a $h_* \in \mathcal{B}_{i,j}$ such that for every $h \in \mathcal{B}_{i,j}$, $\left(\frac{h(n)(\zeta_i(n))}{h_*(n)(\zeta_i(n))}\right)_{n \in A}$ converges to a real number $\theta^h_{h_*}$ and $(\theta^h_{h_*} : h \in \mathcal{B}_{i,j})$ is linearly independent (as a family of elements of $\mathbb{R}$ considered as a $\mathbb{Q}$-vector space),

(vii) for each $i < k_1$, $j' < j < l_i$, $h' \in \mathcal{B}_{i,j'}$ and $h \in \mathcal{B}_{i,j}$, $\left(\frac{h(n)(\zeta_i(n))}{h'(n)(\zeta_i(n))}\right)_{n \in A}$ converges, monotonically, to 0,

(viii) for each $i < k_0$, there exists $j < l_i$ such that $\frac{K}{T}\chi_{\vec{\nu}_i} \in \mathcal{B}_{i,j}$,

(ix) $(\mid h(n)(\zeta_i(n)) \mid)_{n \in A}$ is strictly increasing, for each $i < k_1$, $j < l_i$ and $h \in \mathcal{B}_{i,j}$,

(x) for each $i < k_1$, $j < l_i$ and $h, h_* \in \mathcal{B}_{i,j}$, either
   - $\mid h(n)(\zeta_i(n)) \mid > \mid h_*(n)(\zeta_i(n)) \mid$, for each $n \in A$, or
   - $\mid h(n)(\zeta_i(n)) \mid = \mid h_*(n)(\zeta_i(n)) \mid$, for each $n \in A$, or
   - $\mid h(n)(\zeta_i(n)) \mid < \mid h_*(n)(\zeta_i(n)) \mid$, for each $n \in A$,

   and

(xi) for all $\mu \in \kappa$, given $g \in \bigcup_{j<l_i} \mathcal{B}_{i,j}$ with $k_0 \leq i < k_1$, if $\{n \in \omega : \mu \in \operatorname{supp} g(n)\} \in p$, then $\left(\frac{g(n)(\mu)}{K_n}\right)_{n \in A}$ is constant.

The family $\mathcal{B}_{i,j}$ is called the (i, j)-brick of the stack. □

Notice that (vi) implies that for any $i < k_1$, $j < l_i$ and $h_* \in \mathcal{B}_{i,j}$, we have that for every $h \in \mathcal{B}_{i,j}$, $\left(\frac{h(n)(\zeta_i(n))}{h_*(n)(\zeta_i(n))}\right)_{n \in A}$ converges to a real number $\theta^h_{h_*}$ and $(\theta^h_{h_*} : h \in \mathcal{B}_{i,j})$ is linearly independent. This is why the notation $\theta^h_{h_*}$ carries the h_*, even though in (vi) it seems we are singling out an $h_* \in \mathcal{B}_{i,j}$.

The idea to use the stacks to solve arc equations back and forth is based in A. C. Boero, Castro-Pereira, *et al.*, 2019.

2.7 Solving arc equations on a level of a rational stack

The main tool to solve the arc equations of rational stacks is the same used for integer stacks and we state the lemmas used in A. H. Tomita, 2015. However, there is a crucial difference in the way the stack was defined, so we separate when the denominator grows compared to the numerator, when the numerator and denominator are pretty much at even speed and when the numerator grows compared to the denominator.

In this section we prove Lemma 2.4.2, which tells us that there exist extensive families of solutions for arc equations related to stacks.

An application of Kronecker's theorem

Kronecker's theorem says that if $\{1, \theta_0, \dots, \theta_{k-1}\}$ is a linearly independent family of the $\mathbb{Q}$-vector space $\mathbb{R}$ then $\{(\theta_0 n + \mathbb{Z}, \dots, \theta_{k-1} n + \mathbb{Z}) : n \in \mathbb{Z}\}$ is a dense subset of $\mathbb{T}^k$ (see Bröcker and Dieck, 1985). From this theorem it is possible to prove the following lemma:

Lemma 2.7.1. (Lemma 4.3 of A. H. Tomita, 2015) If $(\theta_0, \dots, \theta_{r-1})$ is a linearly independent family of the $\mathbb{Q}$-vector space $\mathbb{R}$ and $\epsilon > 0$, then there exists a positive integer L such that $\{(\theta_0 x + \mathbb{Z}, \dots, \theta_{r-1} x + \mathbb{Z}) : x \in I\}$ is ϵ-dense in the usual Euclidian metric, for any interval I of length at least L.

Given $\epsilon > 0$ and $\theta = (\theta_i : i \in I)$ a finite linearly independent family of $\mathbb{R}$ as a $\mathbb{Q}$-vector space, fix an integer $L(\theta, \epsilon)$ satisfying the conditions in Lemma 2.7.1.

Lemma 2.7.2. (Lemma 4.4 of A. H. Tomita, 2015) Fix a positive real $\epsilon^* < \frac{1}{8}$. Let $\theta = (\theta_0, \dots, \theta_{r-1})$ be a linearly independent family of $\mathbb{R}$ as a $\mathbb{Q}$-vector space.

Set $L = L(\theta, \epsilon^*)$ and let $(a_0, \dots, a_{r-1})$ be a sequence of integers such that

(i) $| a_0 | > \dots > | a_{r-1} |$ and

(ii) $| \theta_k - \frac{a_k}{a_0} | < \frac{\epsilon^*}{\sqrt{r}L}$ for each $k < r$.

Then

(a) $\{(a_0 x, \dots, a_{r-1} x) : x \in J\}$ is $2\epsilon^*$-dense for any arc J of length at least $\frac{L}{|a_0|}$ and

(b) for any arc J of length at least $3\frac{L}{|a_0|}$ and $\mathcal{U}$ any open ball of radius $4\epsilon^*$ (in $\mathbb{T}^r$ with the Euclidean metric), there exists an arc K contained in J of length $\frac{4\epsilon^*}{\sqrt{r}|a_0|}$ such that

$\{(a_0 x, \ldots, a_{r-1} x) : x \in K\} \subseteq \mathcal{U}.$

Solving arc equations on a level of a rational stack

Now we are ready to prove Lemma 2.4.2.

Lemma (2.4.2) Let $\mathcal{S}$, $\mathcal{A}$, $\mathcal{C}$, $\mathcal{M}$ and $\mathcal{N}$ be as in Lemma 2.4.1. Let ϵ be a positive real and D be a finite subset of κ. Then there exist $B \subseteq A$ cofinite in A and a family of positive real numbers $(\gamma_n : n \in B)$ such that:

For every $n \in B$, for every family $(W_h : h \in \mathcal{C})$ of open arcs of length ϵ, and for every arc function ψ of length ϵ such that $\operatorname{supp}\psi \subseteq D \backslash \{v_i : i < k_0\}$, there exists an n-solution of length γ_n for the arc equation $(\psi, B, K\mathcal{C}, K_n, W)$.

Proof. For each (i, j) with $i < k_1$ and $j < l_i$, fix $u_{i,j}, v_{i,j} \in \mathcal{B}_{i,j}$ such that for every $h \in \mathcal{B}_{i,j}$ and $n \in A$, $\mid u_{i,j}(n)(\zeta_i(n)) \mid \leq \mid h(n)(\zeta_i(n)) \mid \leq \mid v_{i,j}(n)(\zeta_i(n)) \mid$. Fix an $\epsilon^* < \min\{\frac{1}{8}, \frac{1}{8}\epsilon\}$. For each $i < k_1$, $j < l_i$ and $h \in \mathcal{B}_{i,j}$, let θ_h be the limit of $\left(\frac{h(n)(\zeta_i(n))}{v_{i,j}(n)(\zeta_i(n))}\right)_{n \in A}$. Let $\theta_{i,j} = (\theta_h : h \in \mathcal{B}_{i,j})$ and let L be a fixed integer greater than $L(\theta_{i,j}, \epsilon^*)$ for any $i < k_1$ and $j < l_i$.

Let $B \subseteq A$ be the set of n's in A such that

(a) $\mid \theta_h - \frac{h(n)(\zeta_i(n))}{v_{i,j}(n)(\zeta_i(n))} \mid < \frac{\epsilon^*}{\sqrt{|\mathcal{A}|+1}L}$, for each $i < k_1$, $j < l_i$ and $h \in \mathcal{B}_{i,j}$,

(b) $\frac{3L}{|v_{i,l_i-1}(n)(\zeta_i(n))|} < \epsilon^*$, for each $i < k_1$,

(c) $\frac{3L}{|u_{i,j-1}(n)(\zeta_i(n))|} \leq \frac{4\epsilon^*}{\sqrt{|\mathcal{A}|+1}L}$, for each $i < k_1$ and $j < l_i$, and

(d) $\{\zeta_i(n) : k_0 \leq i < k_1\} \cap D = \emptyset$.

Notice that B is cofinite in A, and therefore is in p. Let $\gamma_n = \frac{\epsilon^*}{(|\mathcal{A}|+1)\max\{\|h(n)\| : h \in \mathcal{B}\}}$ for each $n \in B$, where $\|h(n)\| = \sum_{\mu \in \operatorname{supp} h(n)} \mid h(n)(\mu) \mid$. Now let $(W_h : h \in \mathcal{C})$ and ψ be given. Fix $n \in B$.

For each $h \in \mathcal{C}$, fix $V_h \subseteq W_h$ an arc of length $4\epsilon^*$.

Given an arbitrary ϵ-arc function ψ as required, fix ψ^* and ϵ^*-arc function such that $\operatorname{supp}\psi^* \subseteq D$, $\operatorname{supp}\psi^* \cap \{\zeta_i(n) : i < k_1\} = \emptyset$, $\psi^* \leq \psi$ and $\operatorname{supp} h(n) \backslash \{\zeta_i(n) : k_0 \leq i < k_1\} \subseteq \operatorname{supp}\psi^*$, for each $i < k_1$, $j < l_i$ and $h \in \mathcal{B}_{i,j}$.

For each $\mu \in \operatorname{supp}\psi^*$ choose $x_\mu \in \mathbb{T}$ such that $K_n x_\mu$ is the center of $\psi^*(\mu)$.

For each $i < k_1$, $j < l_i$, and $h \in \mathcal{B}_{i,j}$, notice that $\{\zeta_0(n), \ldots, \zeta_{k_1-1}(n)\} \cap \operatorname{supp} h(n) \subseteq \{\zeta_i(n), \ldots, \zeta_{k_1-1}(n)\}$.

We will define, by downward recursion, for $i < k_1$, an arc $Q_{i,0} \subseteq \mathbb{T}$.

Let $O_h = V_h - \sum_{\mu \in \operatorname{supp} h(n) \backslash \{\zeta_i(n)\}} h(n)(\mu) x_\mu$, for each $h \in \mathcal{B}_{i,j}$, $i < k_1$ and $j < l_i$.

For the first step $i_\star = k_1 - 1$, we will define $Q_{i_\star,j}$ for $j < l_{i_\star}$, also by downward recursion. So let $j_\star = l_{i_\star} - 1$ be the first step.

Fix an arbitrary arc J of length at least $\frac{3L}{|v_{i_\star,j_\star}(n)(\zeta_{i_\star}(n))|}$ and $\mathcal{U}_{i_\star,j_\star}$ the ball of radius $4\epsilon^*$ contained in the Cartesian product $\prod_{h\in\mathcal{B}_{i_\star,j_\star}} O_h$ (which is a subset of $\mathbb{T}^{\mathcal{B}_{i_\star,j_\star}}$). By Lemma 2.7.2, there exists an arc $Q_{i_\star,j_\star}$ contained in J of length $\frac{4\epsilon^*}{\sqrt{|\mathcal{A}|+1}|v_{i_\star,j_\star}(n)(\zeta_{i_\star}(n))|}$ such that $\{(h(n)(\zeta_{i_\star}(n))x : h\in\mathcal{B}_{i_\star,j_\star}) : x\in Q_{i_\star,j_\star}\}\subseteq\mathcal{U}_{i_\star,j_\star}$.

Now suppose $j'<l_{i_\star}$ and we have defined $Q_{i_\star,j}$ for all $j'\le j<l_{i_\star}$. If $j'=0$ then we are done for step $i_\star=k_1-1$.

If not: by (c), it follows that $\frac{3L}{|u_{i_\star,j'-1}(n)(\zeta_{i_\star}(n))|}\le\frac{4\epsilon^*}{\sqrt{|\mathcal{A}|+1}|v_{i_\star,j'}(n)(\zeta_{i_\star}(n))|}$ and $Q_{i_\star,j'}$ has length exactly the right side of the inequality above. Let $\mathcal{U}_{i_\star,j'-1}$ be a ball of length $4\epsilon^*$ contained in $\prod_{h\in\mathcal{B}_{i_\star,j'-1}} O_h$. Applying Lemma 2.7.2, there exists an arc $Q_{i_\star,j'-1}$ of length $\frac{4\epsilon^*}{\sqrt{|\mathcal{A}|+1}|v_{i_\star,j'-1}(n)(\zeta_{i_\star}(n))|}$ contained in $Q_{i_\star,j'}$ such that $\{(h(n)(\zeta_{i_\star}(n))x : h\in\mathcal{B}_{i_\star,j'-1}) : x\in Q_{i_\star,j'-1}\}\subseteq\mathcal{U}_{i_\star,j'-1}$.

We thus obtain, for $i_\star=k_1-1$, $Q_{i_\star,j}$ an arc of length $\frac{4\epsilon^*}{\sqrt{|\mathcal{A}|+1}|v_{i_\star,0}(n)(\zeta_{i_\star}(n))|}$ such that $\{(h(n)(\zeta_{i_\star}(n))x : h\in\mathcal{B}_{i_\star,j}) : x\in Q_{i_\star,j}\}\subseteq\prod_{h\in\mathcal{B}_{i_\star,j}} O_h$, for each $j<l_{i_\star}$. At the end, we will have defined $Q_{i_\star,0}$.

Let $x_{\zeta_{i_\star}(n)}$ be the center of $Q_{i_\star,0}$. By the definition of O_h, we have $O_h = V_h - \sum_{\mu\in\operatorname{supp}h(n)\setminus\{\zeta(n)\}} h(n)(\mu)x_\mu$, for each $j<l_{i_\star}$ and $h\in\mathcal{B}_{i_\star,j}$.

It follows then that $\sum_{\mu\in\operatorname{supp}h(n)} h(n)(\mu)x_\mu \in \sum_{\mu\in\operatorname{supp}h(n)\setminus\{\zeta_{i_\star}(n)\}} h(n)(\mu)x_\mu + O_h = V_h$, for each $j<l_{i_\star}$ and $h\in\mathcal{B}_{i_\star,j}$.

The first step of the recursion has been carried out. Suppose now $i'<k_1$ and $Q_{i,0}$ has been defined for all $i'\le i<k_1$.

If $i'=0$ then we are done. Otherwise if $i'>0$:

First it is important to notice that $\{\zeta_0(n),\dots,\zeta_{i'-1}(n)\}\cap\operatorname{supp}h(n)=\{\zeta_{i'-1}(n)\}$, for each $j<l_{i'-1}$ and $h\in\mathcal{B}_{i'-1,j}$.

We are in conditions for $i'-1$, analogous to the first step $i_\star$, that allow us to obtain $x_{\zeta_{i'-1}(n)}$ the center of $Q_{i'-1,0}$, the latter being an arc of length $\frac{4\epsilon^*}{\sqrt{|\mathcal{A}|+1}|v_{i'-1,0}(n)(\zeta_{i'-1}(n))|}$ such that $\sum_{\mu\in\operatorname{supp}h(n)} h(n)(\mu)x_\mu \in \sum_{\mu\in\operatorname{supp}h(n)\setminus\{\zeta_{i'-1}(n)\}} h(n)(\mu)x_\mu + O_h = V_h$, for each $j<l_{i'-1}$ and $h\in\mathcal{B}_{i'-1,j}$.

This ends the construction of x_μ for each $\mu\in\operatorname{supp}\psi^*\cup\{\zeta_i(n) : i<k_1-1\}$. Choose an arbitrary x_μ for $\mu\in D\setminus(\operatorname{supp}\psi^*\cup\{\zeta_i(n) : i<k_1-1\})$.

Let $\phi(\mu)$ be the arc of center x_μ and length γ_n. We show that ϕ is the solution for which we are looking.

By the choice of x_μ and since $\psi^*\le\psi$, it follows $K_n\phi\le\psi$.

Secondly, if $h\in\mathcal{B}_{i,j}$ then $\sum_{\mu\in\operatorname{supp}h(n)} h(n)(\mu)x_\mu\in V_h$. It follows that the center of $\sum_{\mu\in\operatorname{supp}h(n)} h(n)(\mu)\phi(\mu)$ is contained in V_h and this arc has length at most

$$\sum_{\mu\in\operatorname{supp}h(n)} \mid h(n)(\mu)\mid\gamma_n = \sum_{\mu\in\operatorname{supp}h(n)} \mid h(n)(\mu)\mid \frac{\epsilon^*}{(\mid\mathcal{A}\mid+1)\max\{\|h(n)\| : h\in\mathcal{B}\}}<\epsilon^*.$$

Therefore, any point of the arc $\phi(h(n))$ is at a distance smaller than $\epsilon^* + 2\epsilon^*$ from the center of V_h. Since W_h and V_h have the same center and $3\epsilon^* < \frac{\epsilon}{2}$ it follows that $\phi(h(n)) \subseteq W_h$. Thus, ϕ is an n-solution of length γ_n for the arc equation $(\psi, B, K\mathcal{C}, K_n, W)$, as was required. □

2.8 Constructing a sequence of rational stacks

All that remains to be done is to prove Lemma 2.4.1, which guarantees the existence of stacks associated to a linearly independent finite set of sequences.

Given a finite sequence of functions we start by finding an element of the ultrafilter p that makes the restricted functions closer to the properties we want for the stack.

Lemma 2.8.1. Suppose that $\mathcal{G}$ is a finite subset of G^ω, p is a selective ultrafilter and $C \in p$. Suppose $\zeta, \zeta_* \in \kappa^\omega$ are such that there exist $g_* \in \mathcal{G}$ such that $\{n \in C : \zeta(n) \in \operatorname{supp} g_*(n)\} \in p$ and $\{n \in C : \zeta(n) = \zeta_*(n)\} \notin p$.

Then there exist $B' \in p$ with $B' \subseteq C$ and $\mathcal{H} \subseteq \mathcal{G}$ such that:

($\star$1) $(\zeta(n))_{n\in B'}$ is either constant or one-to-one,

($\star$2) for each $g \in \mathcal{G}$, either $\zeta(n) \in \operatorname{supp} g(n)$ for all $n \in B'$ or $\zeta(n) \notin \operatorname{supp} g(n)$ for all $n \in B'$,

($\star$3) $\mathcal{H} = \{g \in \mathcal{G} : \forall\, n \in B', \zeta(n) \in \operatorname{supp} g(n)\}$ is nonempty,

($\star$4) $(g(n)(\zeta(n)))_{n\in B'}$ either converges strictly monotonically to $+\infty$, $-\infty$ or a real number, or is constant and equal to a rational number, for each $g \in \mathcal{H}$,

($\star$5) given $f, g \in \mathcal{H}$, either $|\, g(n)(\zeta(n)) \,|>|\, f(n)(\zeta(n)) \,|$ for all $n \in B'$, $|\, g(n)(\zeta(n)) \,|=|\, f(n)(\zeta(n)) \,|$ for all $n \in B'$, or $|\, g(n)(\zeta(n)) \,|<|\, f(n)(\zeta(n)) \,|$ for all $n \in B'$,

($\star$6) for each pair $g, h \in \mathcal{H}$, the sequence $\left(\frac{g(n)(\zeta(n))}{h(n)(\zeta(n))}\right)_{n\in B'}$ converges to $+\infty$, $-\infty$ or a real number, and

($\star$7) $\zeta(n) \neq \zeta_*(m)$ for all $n, m \in B'$.

Proof. Everything follows from the selectivity of p. For instance, to get B' for which ($\star$1) holds, let $\xi : [C]^2 \to 2$ be given by $\zeta(\{n, m\}) = 0$ iff $\zeta(n) = \zeta(m)$ and let $B' \in p$ be such that $B' \subseteq C$ and $\xi|_{[B']^2}$ is constant (which exists by the selectivity of p). We refine B' using similar straightforward techniques to obtain conditions ($\star$2) – ($\star$7), leaving the details to the reader. □

Notice that if $B \in p$ is such that $B \subseteq B'$, then ($\star$1) – ($\star$7) also hold for B.

Lemma 2.8.2. Suppose that $\mathcal{G}$, C, p, ζ, ζ_*, B' and $\mathcal{H}$ are as in Lemma 2.8.1.

Suppose $g_\# \in \mathcal{H}$ is such that for every $g \in \mathcal{H}$, $\left(\frac{g(n)(\zeta(n))}{g_\#(n)(\zeta(n))}\right)_{n\in B'}$ converges to a real number (or, equivalently, is bounded).

Then there exist $\mathcal{B} \subseteq \mathcal{H}$, $\sigma : \mathcal{H} \setminus \mathcal{B} \to G^\omega$ and a family of real numbers $(\theta_{g_\#,g} : g \in \mathcal{H})$ such that, for every $B \in p$ with $B \subseteq B'$:

($\star$8) $g_\# \in \mathcal{B}$,

($\star$9) $\left(\frac{g(n)(\zeta(n))}{g_\#(n)(\zeta(n))}\right)_{n\in B}$ converges to $\theta_{g_\#,g}$, for every $g \in \mathcal{H}$,

($\star$10) $(\theta_{g_\#,g} : g \in \mathcal{B})$ is a linearly independent set that generates the same $\mathbb{Q}$-vector space as $(\theta_{g_\#,g} : g \in \mathcal{H})$,

($\star$11) for each $g \in \mathcal{H}\setminus\mathcal{B}$, $\mathcal{B} \cup \{g\}$ and $\mathcal{B} \cup \{\sigma(g)\}$ generate the same $\mathbb{Q}$-vector subspace of G^ω,

($\star$12) for each $g \in \mathcal{H}\setminus\mathcal{B}$ and $h \in \mathcal{B}$, $\left(\frac{\sigma(g)(n)(\zeta(n))}{h(n)(\zeta(n))}\right)_{n\in B}$ converges to 0,

($\star$13) if $g \in \mathcal{H}\setminus\mathcal{B}$ and $\theta_{g_\#,g} = 0$, then $\sigma(g) = g$,

($\star$14) if ζ is constantly equal to ν, $\mathcal{B} = \{\chi_{\vec{\nu}}\}$ and $g^* \in \mathcal{H}$ is such that $(g^*(n)(\nu))_{n\in B}$ is not constant mod p, then $\{n \in \omega : \nu \in \operatorname{supp}\sigma(g^*)(n)\} \in p$.

Proof. Consider $\{\theta_{g_\#,g} : g \in \mathcal{H}\}$ as a subset of the $\mathbb{Q}$-vector space $\mathbb{R}$ and take $\mathcal{B} \subseteq \mathcal{H}$ containing $g_\#$ such that $(\theta_{g_\#,h} : h \in \mathcal{B})$ is a basis for the subspace generated by $\{\theta_{g_\#,g} : g \in \mathcal{H}\}$.

For the existence of σ, define $(r_{g,h} : g \in \mathcal{H}, h \in \mathcal{B})$ by the expressions $\theta_{g_\#,g} = \sum_{h\in\mathcal{B}} r_{g,h}\theta_{g_\#,h}$. Now define $\sigma(g) = g - \sum_{h\in\mathcal{B}} r_{g,h}h$ for each $g \in \mathcal{H}\setminus\mathcal{B}$.

Observe now that any $B \in p$ with $B \subseteq B'$ satisfies ($\star$8) – ($\star$13).

In case the conditions in ($\star$14) are met, first note that $g_\# = \chi_{\vec{\nu}}$, and thus $\theta_{g_\#,g^*} = r_{g^*,g_\#}\theta_{g_\#,g_\#}$, so that $\sigma(g^*) = g^* - r_{g^*,g_\#}g_\#$. Now, suppose $Z := \{n \in \omega : \nu \notin \operatorname{supp}\sigma(g^*)(n)\} \in p$; then, letting $Y = B \cap Z$, $\sigma(g^*)(n)(\nu) = 0$ for all $n \in Y$, which implies $g^*(n)(\nu) = r_{g^*,g_\#}g_\#(n)(\nu) = r_{g^*,g_\#}$ (since $g_\# = \chi_{\vec{\nu}}$) for all $n \in Y$; that means $(g^*(n)(\nu))_{n\in B}$ is constant mod p, contrary to the assumptions. Thus, item ($\star$14) is also satisfied. □

Lemma 2.8.3. Let $B \in p$. Suppose that $\zeta \in \kappa^\omega$, $m \in \omega$, $\zeta_i \in \kappa^\omega$ for $i < m$, are such that $\{n \in B : \forall\, i < m, \zeta(n) \neq \zeta_i(n)\} \in p$. Suppose $\mathcal{G}$ is a finite subset of G^ω whose elements are distinct mod p, none of them are constant mod p and such that $\{[f]_p : f \in \mathcal{G}\} \cup \{[\chi_{\vec{\mu}}]_p : \mu \in \kappa\}$ is a linearly independent subset of $\mathrm{Ult}_p(G)$ and there exists $g \in \mathcal{G}$ such that $\{n \in \omega : \zeta(n) \in \operatorname{supp} g(n)\} \in p$.

If ζ is constant, let ν be its value.

Then there exist finite $\mathcal{G}' \subseteq G^\omega$, $l \in \omega\setminus\{0\}$, finite nonempty $\mathcal{B}_j \subseteq G^\omega$ for each $j < l$, and $A \subseteq B$ such that:

(1) for every $g \in \mathcal{G}'$, $\{n \in \omega : \zeta(n) \in \operatorname{supp} g(n)\} \notin p$ and $\{n \in \omega : \zeta_i(n) \in \operatorname{supp} g(n)\} \notin p$ for each $i < m$,

(2) $\mathcal{B}_j \cap \mathcal{B}_{j'} = \emptyset$ for $j \neq j'$, $\mathcal{B}_j \cap \mathcal{G}' = \emptyset$ for each $j < l$ and $\{[f]_p : f \in \mathcal{G}' \cup \bigcup_{j<l}\mathcal{B}_j\}$ is a linearly independent subset of $\mathrm{Ult}_p(G)$; also, if $f, h \in \mathcal{G}' \cup \bigcup_{j<l}\mathcal{B}_j$ are distinct, then $[f]_p \neq [h]_p$,

(3) as vector subspaces of G^ω, $\langle \mathcal{G}' \cup \bigcup_{j<l} \mathcal{B}_j \rangle = \langle \mathcal{G} \cup \{\chi_{\vec{v}}\} \rangle$ if ζ is constant and $\langle \mathcal{G}' \cup \bigcup_{j<l} \mathcal{B}_j \rangle = \langle \mathcal{G} \rangle$ otherwise,

(4) $\zeta(n) \in \operatorname{supp} h(n)$, for each $j < l$, $h \in \mathcal{B}_j$ and $n \in A$,

(5) $\zeta_i(n) \notin \operatorname{supp} h(n)$, for each $i < m$, $j < l$, $h \in \mathcal{B}_j$ and $n \in A$,

(6) $(h(n)(\zeta(n)))_{n \in A}$ either converges strictly monotonically to $+\infty$, $-\infty$ or a real number, or is constant and equal to a rational number, for each $j < l$ and $h \in \mathcal{B}_j$,

(7) for every $j < l$, there exists $h_* \in \mathcal{B}_j$ such that for every $h \in \mathcal{B}_j$, $\left(\frac{h(n)(\zeta(n))}{h_*(n)(\zeta(n))}\right)_{n \in A}$ converges to a real number $\theta^h_{h_*}$ and $(\theta^h_{h_*} : h \in \mathcal{B}_j)$ is linearly independent (as a $\mathbb{Q}$-vector space),

(8) for each $j' < j < l$, $h \in \mathcal{B}_j$ and $h' \in \mathcal{B}_{j'}$, $\left(\frac{h(n)(\zeta(n))}{h'(n)(\zeta(n))}\right)_{n \in A}$ converges monotonically to 0,

(9) if ζ is constant, there exists $j < l$ such that $\chi_{\vec{v}} \in \mathcal{B}_j$,

(10) for each $j < l$ and distinct $h, h' \in \mathcal{B}_j$, either
   - $| h(n)(\zeta(n)) | > | h'(n)(\zeta(n)) |$ for all $n \in A$, or
   - $| h(n)(\zeta(n)) | = | h'(n)(\zeta(n)) |$ for all $n \in A$, or
   - $| h(n)(\zeta(n)) | < | h'(n)(\zeta(n)) |$ for all $n \in A$,

(11) no element of $\mathcal{G}'$ is constant mod p and $\{[f]_p : f \in \mathcal{G}'\} \cup \{[\chi_{\vec{\mu}}]_p : \mu \in \kappa\}$ is linearly independent,

(12) if $i < m$ and $n, n' \in A$ are distinct then $\zeta(n) \neq \zeta_i(n')$, and

(13) $| \mathcal{G}' | < | \mathcal{G} |$.

Proof. Since p is selective, we may suppose by shrinking B if necessary that, for all $i < m$ and $n, n' \in B$ distinct, $\zeta(n) \neq \zeta_i(n')$. Clearly, this property will hold for any subset of B.

If ζ is constant, let $\mathcal{G}_0 = \mathcal{G} \cup \{\chi_{\vec{v}}\}$. If not, let $\mathcal{G}_0 = \mathcal{G}$. We will construct by recursion on $j \in \omega$:

- $A_j \in p$ with $A_j \subseteq B$,
- $\mathcal{G}_j \subseteq G^\omega$,
- $\mathcal{H}_j \subseteq \mathcal{G}_j$,
- $\mathcal{B}_j \subseteq \mathcal{H}_j$, and
- $\sigma_j : \mathcal{H}_j \setminus \mathcal{B}_j \to G^\omega$,

satisfying:

(i) $A_j \subseteq A_{j-1}$ for every $j \in \omega$,

(ii) for each $j \in \omega$, $\{[f]_p : f \in \mathcal{G}_j \dot\cup \dot{\bigcup}_{k<j} \mathcal{B}_k\}$ is a linearly independent subset of $\mathrm{Ult}_p(G)$; also, if $f, h \in \mathcal{G}_j \dot\cup \dot{\bigcup}_{k<j} \mathcal{B}_k$ are distinct, then $[f]_p \neq [h]_p$,

(iii) for each $j \in \omega$, $h \in \mathcal{H}_j$, and $n \in A_j$, $\zeta(n) \in \operatorname{supp} h(n)$,

(iv) for each $j \in \omega$, $h \in \mathcal{G}_j \backslash \mathcal{H}_j$ and $n \in A_j$, then $\zeta(n) \notin \operatorname{supp} h(n)$,

(v) for each $j \in \omega$, $h \in \mathcal{H}_j$, $i < m$ and $n \in A_j$, then $\zeta_i(n) \notin \operatorname{supp} h(n)$,

(vi) for each $j \in \omega$ and $h \in \mathcal{H}_j$, $(h(n)(\zeta(n)))_{n \in A_j}$ either converges strictly monotonically to $+\infty$, $-\infty$ or a real number, or is constant and equal to a rational number,

(vii) for every $j \in \omega$, $\mathcal{B}_j \neq \emptyset$ iff there exists $g \in \mathcal{G}_j$ such that $\{n \in \omega : \zeta(n) \in \operatorname{supp} g(n)\} \in p$,

(viii) for every $j \in \omega$, if $\mathcal{B}_j \neq \emptyset$, then there exists $h_* \in \mathcal{B}_j$ such that for every $h \in \mathcal{B}_j$, $\left(\frac{h(n)(\zeta(n))}{h_*(n)(\zeta(n))}\right)_{n \in A_j}$ converges to a real number $\theta^h_{h_*}$ and $(\theta^h_{h_*} : h \in \mathcal{B}_j)$ is linearly independent (as a $\mathbb{Q}$-vector space),

(ix) for each $j < j'$, $h \in \mathcal{B}_j$ and $h' \in \mathcal{B}_{j'}$, then $\left(\frac{h(n)(\zeta(n))}{h'(n)(\zeta(n))}\right)_{n \in A_j}$ converges monotonically to 0,

(x) if ζ is constant, there exists $j \in \omega$ such that $\chi_{\vec{v}} \in \mathcal{B}_j$,

(xi) given $j \in \omega$ and $h, h' \in \mathcal{B}_j$, either

- $| h(n)(\zeta(n)) | > | h'(n)(\zeta(n)) |$ for all $n \in A_j$, or
- $| h(n)(\zeta(n)) | = | h'(n)(\zeta(n)) |$ for all $n \in A_j$, or
- $| h(n)(\zeta(n)) | < | h'(n)(\zeta(n)) |$ for all $n \in A_j$,

(xii) for each $j \in \omega$ and $g \in \mathcal{H}_j$, $(| g(n)(\zeta(n)) |)_{n \in A_j}$ is either constant or strictly increasing,

(xiii) for each $j \in \omega$ and $g \in \mathcal{H}_j \backslash \mathcal{B}_j$, $\mathcal{B}_j \cup \{g\}$ and $\mathcal{B}_j \{\sigma(g)\}$ generate the same $\mathbb{Q}$-vector subspace of G^ω,

(xiv) for each $j \in \omega$, $g \in \mathcal{G}_j$ and $i < m$, $\{n \in \omega : \zeta_i(n) \in \operatorname{supp} g(n) \notin p$,

(xv) for each $j, j' \in \omega$, $\mathcal{G}_j \cup \bigcup_{k<j} \mathcal{B}_k$ generates the same subspace of G^ω as $\mathcal{G}_{j'} \cup \bigcup_{k<j'} \mathcal{B}_k$,

(xvi) if ζ is constant, then for all $j \in \omega$, $\chi_{\vec{v}} \in \mathcal{B}_j \cup \mathcal{G}_j$,

(xvii) $\mathcal{G}_{j+1} = (\mathcal{G}_j \backslash \mathcal{H}_j) \cup \operatorname{ran} \sigma_j$,

(xviii) if ζ is constant, $j \in \omega$ and $\chi_{\vec{v}} \in \mathcal{H}_j \backslash \mathcal{B}_j$, then $\sigma_j(\chi_{\vec{v}}) = \chi_{\vec{v}}$, and

(xix) if ζ is constant and $\mathcal{B}_0 = \{\chi_{\vec{v}}\}$, then there exists $g \in \mathcal{H}_0 \backslash \mathcal{B}_0$ such that $\{n \in \omega : v \in \operatorname{supp} \sigma_0(g)\} \in p$.

Suppose we have carried on such a recursion. By (ii) and (xv), one of the BB_j's must be empty. Let l be the first j such that $BB_j = \emptyset$. By (vii), for all $g \in \mathcal{G}_l$, $\{n \in \omega : \zeta(n) \notin \operatorname{supp} g(n)\} \in p$. Since there exists $g \in \mathcal{G}$ such that $\{n \in \omega : \zeta(n) \in \operatorname{supp} g(n)\} \in p$, it follows that $l > 0$. Let $A = A_{l-1}$ and $\mathcal{G}' = \mathcal{G}_l$. Notice that every BB_j is nonempty for $j < l$.

(1) holds by the previous observation, (i), (v) and by the fact that $\mathcal{B}_j \subseteq \mathcal{H}_j$. (2) holds by (ii). (3) follows from (xvii) using $j = l$, $j' = 0$. (4)=(11) follow easily from (i), (iii)-(ix), (xi) and (xii). Suppose (9) doesn't hold. Then by (xvi), $\chi_{\vec{v}} \in \mathcal{G}_l$. But then, by (vii), $\mathcal{B}_l \neq \emptyset$, a contradiction.

(12) holds by (ii), because $\langle \mathcal{G}' \rangle \subseteq \langle \mathcal{G}_0 \rangle$ and because if ζ is constant then, by (xvi) and (9), $\mathcal{G}' \cup \{\chi_{\vec{v}}\}$ is linearly independent.

(13) holds: if ζ is not constant, it follows from (2) e (3). If it is constant, first, notice that, by (xix), (xvii) for $j = 0$, and (vii) for $j = 1$, it follows that $l > 1$ or $\mathcal{B}_0 \neq \{\chi_{\vec{v}}\}$. Either way, $\mathcal{B} = \bigcup_{i<l} \mathcal{B}_i \setminus \{\chi_{\vec{v}}\}$ is nonempty. By (2), (3) and (9), analayzing dimensions it follows that $1+ \mid \mathcal{B} \mid + \mid \mathcal{G}' \mid = 1+ \mid \mathcal{G} \mid$, and therefore $\mid \mathcal{G}' \mid < \mid \mathcal{G} \mid$.

Construction: For step 0, $\mathcal{G}_0$ is already defined. We apply Lemma 2.8.1 m times using $\zeta(n) = \zeta(n)$ and $\zeta_*(n) = \zeta_i(n)$ for every n. If $m = 0$ we apply it once using $\zeta_*(n) = \zeta(n)'$ for every n for some $\zeta(n)' \neq \zeta(n)$. We now have $\mathcal{H}_0$ and $A_0' \subseteq B$.

If it is the case that $(h(n)(\zeta(n)))_{n\in A_0'}$ converges to a real number for every $h \in \mathcal{H}_0$ and that ζ is constant, then we apply Lemma 2.8.2 with $g_\# = \chi_{\vec{v}}$, and obtain $A_0 \subseteq A_0'$, $\mathcal{B}_0 \subseteq \mathcal{H}_0$ and σ_0. If not, then we take any $g_\# \in \mathcal{H}_0$ that satisfies the hypothesis of Lemma 2.8.2 - one does exist because of $(\star 5)$, which also implies that for such a $g_\#$, $\left(\frac{1}{g_\#(n)(\zeta(n))}\right)_{n\in A_0}$ converges to 0, and thus in case ζ is constant, $\sigma_0(\chi_{\vec{v}}) = \chi_{\vec{v}}$. Either way, we obtain σ_0, $\mathcal{B}_0$ and A_0. It is straightforward to verify that (i)-(xix) hold for this step.

For the inductive step, we define $\mathcal{G}_{j+1}$ as in (xvii). If there is no $g \in \mathcal{G}_{j+1}$ such that $\{n \in \omega : \zeta(n) \in \operatorname{supp} g(n)\} \in p$, then we define $\mathcal{H}_{j+1} = \emptyset$, $A_{j+1} \subseteq A_j$ satisfying (v) with j swapped by $j+1$ and $\mathcal{B}_{j+1} = \sigma_{j+1} = \emptyset$. Otherwise, we proceed as in step 0: we first apply Lemma 2.8.1 to obtain $\mathcal{H}_{j+1}$ and $A_{j+1}' \subseteq A_j$ and then similarly apply Lemma 2.8.2 to obtain $\mathcal{B}_{j+1}$, A_{j+1} and σ_{j+1}. It is straightforward to verify that (i)-(xix) hold for this step. $\square$

Lemma 2.8.4. Suppose $\mathcal{G}$ is a finite subset of G^ω such that $([f]_p : f \in \mathcal{G}) \cup ([\chi_{\vec{\mu}}]_p : \mu \in \kappa)$ is a linearly indepedent family of elements of $\mathrm{Ult}_p(G)$. Then: either there exist $\mu \in \kappa$, $g^* \in \mathcal{G}$ and $A \in p$ such that $(g^*(n)(\mu))_{n\in A}$ is one-to-one, or there exists $A \in p$ such that for every $g \in \mathcal{G}$ there exists $\zeta_g \in \kappa^\omega$ satisfying $\zeta_g(n) \in \operatorname{supp} g(n)$ for all $n \in A$ and $\zeta_g|A$ is one-to-one.

Proof. Suppose that for all $\mu \in \kappa$, for all $g \in \mathcal{G}$ and for all $A \in p$, $(g(n)(\mu))_{n\in A}$. Then, by the selectivity of p, for all $\mu \in \kappa$ there exists $B_\mu \in p$ such that for all $g \in \mathcal{G}$, $(g(n)(\mu))_{n\in B_\mu}$ is constant.

Fix a $g \in \mathcal{G}$. By selectivity, there exists $B \in p$ such that either the sequence $(\mid \operatorname{supp} g(n) \mid)_{n\in B}$ is strictly increasing or it is constant. If it is strictly increasing, then we may pick recursively $\tilde{\zeta}_g(n) \in \operatorname{supp} g(n)$ for each $n \in B$ in a way such that $\tilde{\zeta}_g$ is one-to-one. Define $A_g = B$.

Otherwise if it is constant, let $k \in \omega$ be that constant. Since g is not 0 mod p, $k \geq 1$. For each $i < k$, let $w_i \in \kappa^\omega$ be such that $\operatorname{supp} g(n) = \{w_i(n) : i < k\}$ for each $n \in B$. Then there exists $C \subseteq B$, $C \in p$ such that for each $i < k$, $(w_i(n))_{n\in C}$ is one-to-one or constant.

We claim that there is a $j < k$ such that $(w_j(n))_{n\in C}$ is one-to-one. Suppose all of them are constant; take μ_j for each $j < k$ such that $w_j(n) = \mu_j$ for all $n \in C$. Then, since $(g(n)(\mu_j))_{n\in B_{\mu_j}}$ is constant for each $j < k$, let r_j be those constants. Let $D = C \cap \bigcap_{j<k} B_{\mu_j}$. We have that $D \in p$ and $g(n) = \left(\sum_{j<k} r_j \chi_{\vec{\mu}_j}\right)(n)$ for all $n \in D$, and so $[g]_p = \sum_{j<k} r_j [\chi_{\vec{\mu}_j}]_p$. This

contradicts the hypothesis that $([f]_p : f \in \mathcal{G}) \cup ([\chi_{\vec{\mu}}]_p : \mu \in \kappa)$ is a linearly indepedent family of elements of $\mathrm{Ult}_p(G)$.

Therefore, there exists a $j < k$ such that $(w_j(n))_{n\in C}$ is one-to-one, and so we define $\tilde{\zeta}_g = w_j$ and $A_g = C$.

Thus if we define $A = \bigcap_{g\in\mathcal{G}} A_g$ and $\zeta_g \in \kappa^\omega$ such that $\zeta_g|A = \tilde{\zeta}_g|A$, we have the desired result. □

Now we restate Lemma 2.4.1, which we are going to prove.

Lemma. Let $B \in p$ and $\mathcal{G}$ be a finite subset of G^ω such that $([f]_p : f \in \mathcal{G}) \cup ([\chi_{\vec{\mu}}]_p : \mu \in \kappa)$ is a linearly independent family of elements of $\mathrm{Ult}_p(G)$. Then there exist a rational stack $\mathcal{S} = \langle \mathcal{B}, \nu, \zeta, K, A, k_0, k_1, l, T\rangle$ such that, by defining $\mathcal{A} = \mathcal{G} \cup \{\chi_{\vec{\nu_i}} : i < k_0\}$ and $\mathcal{C} = \frac{\bigcup_{i<k_1, j<l_i} \mathcal{B}_{i,j}}{K}$, there exist $\mathcal{M} : \mathcal{A} \times \mathcal{C} \to \mathbb{Z}$, $\mathcal{N} : \mathcal{C} \times \mathcal{A} \to \mathbb{Z}$ satisfying:

(1) $\{[f]_p : f \in \mathcal{A}\}$ and $\{[h]_p : h \in \mathcal{C}\}$ generate the same subspace,

(2) $f(n) = \sum_{h\in\mathcal{C}} \mathcal{M}_{f,h} h(n)$, for each $n \in A$ and $f \in \mathcal{A}$,

(3) $h(n) = \frac{1}{T^2} \sum_{f\in\mathcal{A}} \mathcal{N}_{h,f} f(n)$, for each $n \in A$ and $h \in \mathcal{C}$,

(4) $K\mathcal{A} \subseteq H^\omega$,

(5) $K\mathcal{C} \subseteq H^\omega$, and

(6) $A \in p$ and $A \subseteq B$.

Proof. (of Lemma 2.4.1) We will start building a sequence that will almost be the stack $\mathcal{S}$ which we will associate with $\mathcal{G}$.

Claim: There exist:

- $A' \in p$ with $A' \subseteq B$,
- $k_0 \in \omega$,
- $l' : k_0 \to \omega\{0\}$,
- $\nu : k_o \to \kappa$,
- $\zeta' : k_0 \to \kappa^\omega$,
- $\mathcal{G}' \subseteq G^\omega$, and
- $(\hat{\mathcal{B}}_{i,j} : i < k_0, j < l_i)$ a family of nonempty subsets of G^ω,

satisfying:

(i) $\zeta'_i(n) = \nu_i$ for every $i < k_0$ and $n \in A'$,

(ii) the elements ν_i for $i < k_0$ are pairwise distinct,

(iii) $\nu_i \in \operatorname{supp} h(n)$, for each $i < k_0$, $j < l_i$, $h \in \hat{\mathcal{B}}_{i,j}$ and $n \in A'$,

(iv) $\nu_i \notin \operatorname{supp} h(n)$, for each $i < i_* < k_0$, $j < l_{i_*}$ and $h \in \hat{\mathcal{B}}_{i_*,j}$ and $n \in A'$,

(v) $(h(n)(\nu_i))_{n\in A'}$ converges, strictly monotonically, to $+\infty$, $-\infty$ or a real number, or is constantly equal to a rational number, for each $i < k_0$, $j < l_i$ and $h \in \hat{\mathcal{B}}_{i,j}$,

(vi) for every $i < k_0$ and $j < l_i$, there exists $h_* \in \hat{\mathcal{B}}_{i,j}$ such that for every $h \in \hat{\mathcal{B}}_{i,j}$, $\left(\frac{h(n)(\nu_i)}{h_*(n)(\nu_i)}\right)_{n\in A'}$ converges to a real number $\theta^h_{h_*}$ and $(\theta^h_{h_*} : h \in \hat{\mathcal{B}}_{i,j})$ is linearly independent (as a $\mathbb{Q}$-vector space),

(vii) for each $i < k_0$, $j' < j < l_i$, $h \in \hat{\mathcal{B}}_{i,j}$ and $h' \in \mathcal{B}_{i,j'}$, $\left(\frac{h(n)(\nu_i)}{h'(n)(\nu_i)}\right)_{n\in A'}$ converges, monotonically, to 0,

(viii) $(| h(n)(\nu_i) |)_{n\in A'}$ is constant or strictly increasing, for each $i < k_0$, $j < l_i$ and $h \in \hat{\mathcal{B}}_{i,j}$,

(ix) for each $i < k_0$ there exists a $j < l_i$ such that $\chi_{\vec{\nu}_i} \in \hat{\mathcal{B}}_{i,j}$,

(x) for each $i < k_0$, $j < l_i$ and $h, h_* \in \hat{\mathcal{B}}_{i,j}$, either

- $| h(n)(\nu_i) |>| h_*(n)(\nu_i) |$ for each $n \in A'$, or
- $| h(n)(\nu_i) |=| h_*(n)(\nu_i) |$ for each $n \in A'$, or
- $| h(n)(\nu_i) |<| h_*(n)(\nu_i) |$ for each $n \in A'$,

(xi) for all $\mu \in \kappa$, for every $g \in \mathcal{G}'$, if $\{n \in \omega : \mu \in \operatorname{supp} g(n)(\mu)\} \in p$ then $(g(n)(\mu))_{n\in A'}$ is constant,

(xii) for all $g \in \mathcal{G}'$ and all $i < k_0$, $\{n \in \omega : \nu_i \in \operatorname{supp} g(n)\} \notin p$,

(xiii) if $i, i' < k_0$, $j < l'_i$, $j' < l'_{i'}$ and $(i, j) \neq (i', j')$, then $\hat{\mathcal{B}}_{i,j} \cap \hat{\mathcal{B}}_{i',j'} = \emptyset$, $\hat{\mathcal{B}}_{i,j} \cap \mathcal{G}' = \emptyset$ and $([f]_p : f \in \mathcal{G}' \cup \bigcup_{i<k_0,j<l_i} \hat{\mathcal{B}}_{i,j})$ is a linearly independent family of elements of $\mathrm{Ult}_p(G)$,

(xiv) as vector subspaces of G^ω, $\langle \mathcal{G}' \cup \bigcup_{i<k_0,j<l_i} \hat{\mathcal{B}}_{i,j}\rangle = \langle \mathcal{G} \cup \{\chi_{\vec{\nu}_i} : i < k_0\}\rangle$, and

(xv) $([f]_p : f \in \mathcal{G}') \cup ([\chi_{\vec{\xi}}]_p : \xi \in \kappa)$ is a linearly independent family.

If (xi) already holds for $\mathcal{G}$ and $A = B$, we let $\mathcal{G}' = \mathcal{G}$, $A' = B$, $k_0 = 0$ and the other sequences be $\emptyset$.

If not, then we may take a $\nu_0 \in \kappa$ such that there exist $g \in \mathcal{G}$ and $B' \subseteq B$, $B' \in p$ such that $\{n \in \omega : \nu_0 \in \operatorname{supp} g(n)\} \in p$ and $(g(n)(\nu_0))_{n\in B'}$ is one-to-one.

We define $\mathcal{G}_0 = \mathcal{G}$ and apply Lemma 2.8.3 to B', $m = 0$, $\mathcal{G}_0$, $\zeta(n) = \nu_0$ for all $n \in B'$ and obtain $\mathcal{G}'_0$, A_0, l_0 and $\hat{\mathcal{B}}_{0,j}$ for $j < l_0$ satisfying everything but (xi) (possibly) by using $k_0 = 0$.

Suppose the recursion has been done up to $m \in \omega$ and we have, for $i < m$, ν_i, $\mathcal{G}'_i$, A_i, l_i and $\hat{\mathcal{B}}_{i,j}$ for $j < l_i$ satisfying everything but (x) (possibly) by using $k_0 = m$. If (x) holds for $\mathcal{G}'_{m-1}$, then we let $A' = A_{m-1}$, $k_0 = m$ and $\mathcal{G}' = \mathcal{G}'_{m-1}$ and the recursion is over. If not, we take $\nu_m \in \kappa$ such that there exist $g \in \mathcal{G}'_{m-1}$ and $B' \subseteq A_{m-1}$, $B' \in p$ such that $\{n \in \omega : \nu_m \in \operatorname{supp} g(n)\} \in p$ and $(g(n)(\nu_m))_{n\in B'}$ is one-to-one. Notice that item (xi) implies that $\nu_m \neq \nu_i$ for every $i < m$. We then apply Lemma 2.8.3 to B', m, $\mathcal{G}'_{m-1}$, $\zeta(n) = \nu_m$ for all $n \in B'$, $\zeta_i(n) = \nu_i$ for all $n \in B'$ and $i < m$, and obtain $\mathcal{G}'_m$, A_m, l_m and $\hat{\mathcal{B}}_{m,j}$ for $j < l_m$ satisfying everything but (xi) (possibly) by using $k_0 = m + 1$.

The recursion must eventually stop due to items (xii), (xiii) and the fact that the $\hat{\mathcal{B}}_{i,j}$'s are nonempty. We now have $A', k_0, l', v, \zeta', \mathcal{G}'$ and $(\hat{\mathcal{B}}_{i,j} : i < k_0, j < l_i)$ as in the Claim above.

Claim: There exist:

- $A'' \in p$ with $A'' \subseteq A'$,
- $k_1 \in \omega \setminus \{0\}$,
- $l : k_1 \to \omega \setminus \{0\}$ extending l',
- $\zeta : k_1 \to \kappa^\omega$ extending ζ',
- $(\hat{\mathcal{B}}_{i,j} : i < k_1, j < l_i)$ a family of nonempty subsets of G^ω,

satisfying:

(I) $\zeta_i(n) = v_i$ for every $i < k_0$ and $n \in A''$,

(II) the elements v_i, for $i < k_0$, and $\zeta_j(n)$, for $k_0 \leq j < k_1$ and $n \in A''$, are all pairwise distinct,

(III) $\zeta_i(n) \in \operatorname{supp} h(n)$, for each $i < k_1$, $j < l_i$, $h \in \hat{\mathcal{B}}_{i,j}$ and $n \in A''$,

(IV) $\zeta_i(n) \notin \operatorname{supp} h(n)$, for each $i < i_* < k_1$, $j < l_{i_*}$ and $h \in \hat{\mathcal{B}}_{i_*,j}$ and $n \in A''$,

(V) $(h(n)(\zeta_i(n)))_{n \in A''}$ either converges, strictly monotonically, to $+\infty$, $-\infty$ or a real number, or is constantly equal to a rational number, for each $i < k_1$, $j < l_i$ and $h \in \hat{\mathcal{B}}_{i,j}$,

(VI) for every $i < k_1$ and $j < l_i$, there exists $h_* \in \hat{\mathcal{B}}_{i,j}$ such that for every $h \in \hat{\mathcal{B}}_{i,j}$, $\left(\frac{h(n)(\zeta_i(n))}{h_*(n)(\zeta_i(n))}\right)_{n \in A''}$ converges to a real number $\theta^h_{h_*}$ and $(\theta^h_{h_*} : h \in \hat{\mathcal{B}}_{i,j})$ is linearly independent (as a $\mathbb{Q}$-vector space),

(VII) for each $i < k_1$, $j' < j < l_i$, $h \in \hat{\mathcal{B}}_{i,j}$ and $h' \in \hat{\mathcal{B}}_{i,j'}$, $\left(\frac{h(n)(\zeta_i(n))}{h'(n)(\zeta_i(n))}\right)_{n \in A''}$ converges, monotonically, to 0,

(VIII) $(| h(n)(\zeta_i(n)) |)_{n \in A''}$ is strictly increasing, for each $i < k_1$, $j < l_i$ and $h \in \hat{\mathcal{B}}_{i,j}$,

(IX) for each $i < k_0$ there exists $j < l_i$ such that $\chi_{\vec{v}_i} \in \hat{\mathcal{B}}_{i,j}$,

(X) for each $i < k_1$, $j < l_i$ and $h, h_* \in \hat{\mathcal{B}}_{i,j}$, either

- $| h(n)(\zeta_i(n)) | > | h_*(n)(\zeta_i(n)) |$ for each $n \in A''$, or
- $| h(n)(\zeta_i(n)) | = | h_*(n)(\zeta_i(n)) |$ for each $n \in A''$, or
- $| h(n)(\zeta_i(n)) | < | h_*(n)(\zeta_i(n)) |$ for each $n \in A''$,

(XI) for all $\mu \in \kappa$, for every $i \geq k_0$, $j < l_i$ and $g \in \hat{\mathcal{B}}_{i,j}$, if $\{n \in \omega : \mu \in \operatorname{supp} g(n)(\mu)\} \in p$ then $(g(n)(\mu))_{n \in A''}$ is constant,

(XII) as vector subspaces of G^ω, $\langle \bigcup_{i<k_1, j<l_i} \hat{\mathcal{B}}_{i,j} \rangle = \langle \mathcal{G} \cup \{\chi_{\vec{v}_i} : i < k_0\} \rangle$,

(XIII) if $i, i' < k_1$, $j < l'_i$, $j' < l'_i$, and $(i,j) \neq (i',j')$, then $\hat{\mathcal{B}}_{i,j} \cap \hat{\mathcal{B}}_{i',j'} = \emptyset$, and $([f]_p : f \in \bigcup_{i<k_1, j<l_i} \hat{\mathcal{B}}_{i,j})$ is a linearly independent family of elements of $\operatorname{Ult}_p(G)$.

□

For the initial step k_0 of the recursion, we first notice that, by item (xi) of the previous Claim and Lemma 2.8.4 there exists $A''_{k_0} \in p$ such that for every $g \in \mathcal{G}'$ there exists $\zeta_g \in \kappa^\omega$ such that $\zeta_g(n) \in \operatorname{supp} g(n)$ for all $n \in A''_{k_0}$ and $\zeta_g|A''_{k_0}$ is one-to-one. Define $\mathcal{G}_{k_0} = \mathcal{G}'$, and apply Lemma 2.8.3 to $B = A''_{k_0}$, $m = k_0$, $\mathcal{G}_{k_0}$, $\zeta = \zeta_g$ for an arbitrary $g \in \mathcal{G}'$, and obtain $\mathcal{G}'_{k_0}$, A''_{k_0+1}, l_{k_0+1} and $\hat{\mathcal{B}}_{k_0+1,j}$ for $j < l_{k_0+1}$. We then repeat this step until for some $k_1 \geq k_0$, $\mathcal{G}'_{k_1} = \emptyset$. Such a k_1 exists since, by Lemma 2.8.3, $|\ \mathcal{G}'_m\ |<|\ \mathcal{G}'_{m-1}\ |$ for each $m > k_0$.

It follow from items (XII) and (XIII) that $\bigcup_{i<k_1, j<l_i} \hat{\mathcal{B}}_{i,j}$ and $\mathcal{G} \cup \{\chi_{\vec{v}_i} : i < k_0\}$, mod p, are bases for the same subspace of $\mathrm{Ult}_p(G)$. Let $\mathcal{A} = \mathcal{G} \cup \{\chi_{\vec{v}_i} : i < k_0\}$ and $\mathcal{B}' = \bigcup_{i<k_1, j<l_i} \hat{\mathcal{B}}_{i,j}$.

Fix families of integers $\hat{\mathcal{M}} = (\hat{\mathcal{M}}_{f,h} : f \in \mathcal{A}, g \in \mathcal{B}')$ and $\hat{\mathcal{N}} = (\hat{\mathcal{N}}_{h,f} : h \in \mathcal{B}', f \in \mathcal{A})$ and a positive integer T such that:

(3′) $[f]_p = \frac{1}{T} \sum_{h \in \mathcal{B}'} \hat{\mathcal{M}}_{f,h}[h]_p$, for each $f \in \mathcal{A}$ and

(4′) $[h]_p = \frac{1}{T} \sum_{f \in \mathcal{A}} \hat{\mathcal{N}}_{h,f}[f]_p$, for each $h \in \mathcal{B}'$.

Let $C = \{\frac{h}{T} : h \in \mathcal{B}'\}$ and $\mathcal{M}$ and $\mathcal{N}$ be such that $\mathcal{M}_{f,\frac{h}{T}} = \hat{\mathcal{M}}_{f,h}$ and $\mathcal{N}_{\frac{h}{T},f} = \hat{\mathcal{N}}_{h,f}$, for each $f \in \mathcal{A}$ and $h \in \mathcal{B}'$. Then we have:

(3″) $[f]_p = \sum_{h \in C} \mathcal{M}_{f,h}[h]_p$, for each $f \in \mathcal{A}$ and

(4″) $[h]_p = \frac{1}{T^2} \sum_{f \in \mathcal{A}} \mathcal{N}_{h,f}[f]_p$, for each $h \in C$.

Let $A \subseteq A''$, $A \in p$ be such that for every $n \in A$, $f \in \mathcal{A}$ and $h \in C$, $f(n) = \sum_{h \in C} \mathcal{M}_{f,h} h(n)$ and $h(n) = \frac{1}{T^2} \sum_{f \in \mathcal{A}} \mathcal{N}_{h,f} f(n)$.

Now let K be a strictly increasing sequence of positive integers such that $K_0 > 1$, $n!T \mid K_n$ for all $n \in \omega$, and $KC \subseteq H^\omega$. We now have that by defining $\mathcal{B}_{i,j} = K\frac{K}{T}\hat{\mathcal{B}}_{i,j}$, we have the desired rational stack.

2.9 A note on free Abelian groups

With such results, we can now improve the example from A. C. Boero, Castro-Pereira, *et al.*, 2019 and A. H. Tomita, 2015:

Example 2.9.1. Assume the existence of a selective ultrafilter. Then for each $\alpha \leq \omega$ there exists a group topology on the free Abelian group F of cardinality $\mathfrak{c}$ such that F^n is countably compact for each $n < \alpha$ and F^α is not countably compact.

Proof. Let p be a selective ultrafilter. Then the direct sum of $\mathfrak{c}$ copies of $\mathbb{Q}$ has a p-compact group topology without nontrivial convergente sequences. In particular, all of its powers are countably compact.

Now, Tomita (A. H. Tomita, 2019) showed that if a torsion-free group H without nontrivial convergent sequences admits a topology such that H^n is countably compact for each $n < \alpha$ then the free Abelian group F of cardinality $\mathfrak{c}$ admits a group topology without nontrivial convergent sequences such that F^n is countably compact for each $n < \alpha$ and F^α is not countably compact. $\square$

2.10 Questions

Some natural questions that remain open are:

Question 2.10.1. Is it consistent that there are p-compact groups on $\mathbb{Q}^{(\lambda)}$ for some λ of countable cofinality? In addition, with weight of countable cofinality and without nontrivial convergent sequences?

We recall that van Douwen (Douwen, 1980b) showed that there is no pseudocompact group whose cardinality has countable cofinality under GCH.

Boolean example have been obtained in A. H. Tomita, 2005b: A consistent countably compact group without nontrivial convergent sequences of weight and cardinality $\aleph_\omega$.

The following is still an open question related to van Douwen's question:

Question 2.10.2. Is there a cardinal $\lambda > 2^{\mathfrak{c}}$ of countable cofinality for which there exists a countably compact group of cardinality λ?

Some questions from Hrušák *et al.*, 2021 related to this chapter remain open:

Question 2.10.3. Is there in ZFC a Hausdorff (infinite) p-compact topological group without nontrivial convergent sequences?

Question 2.10.4. Is it consistent with ZFC that for some ultrafilter $p \in \omega^*$ there is a Hausdorff (infinite) p-compact topological group without nontrivial convergent sequences of weight $< \mathfrak{c}$?

Chapter 3

On the consistency of arbitrarily large, countably compact, free Abelian groups

3.1 Introduction

3.1.1 Some history

As was discussed in the 1, Tomita (A. H. Tomita, 1998) showed that a nontrivial free Abelian group does not admit a group topology such that its countably infinite power is coutanbly compact.

However, it was later shown (A. H. Tomita, 2015) that there exists a group topology on the free Abelian group of cardinality $\mathfrak{c}$ that makes all of its finite powers countably compact, when assuming $\mathfrak{c}$ selective ultrafilters.

In this chapter, assuming the existence of $\mathfrak{c}$ *incomparable* selective ultrafilters, we prove that there is a group topology of the free Abelian group of cardinality κ without nontrivial convergent sequences and such that all finite powers are countably compact, for any κ cardinal such that $\kappa^\omega = \kappa$.

With such a result, we obtain the following:

Theorem 3.1.1. Assume GCH. Then a free Abelian group of infinite cardinality κ can be endowed with a countably compact group topology (without nontrivial convergent sequences) if and only if $\kappa = \kappa^\omega$.

The result above answers a question of Dikranjan and Shakhmatov that was posed in the survey by Comfort, Hofmann and Remus (W. W. Comfort *et al.*, 1992).

Because of the way our examples are constructed we can raise their weights in the same way as in the papers A. H. Tomita, 2003 or Castro-Pereira and A. H. Tomita, 2010 and obtain the following result – the examples in these references are Boolean but the trick is similar.

Theorem 3.1.2. It is consistent that there is a proper class of cardinals of countable cofinality that can occur as the weight of a countably compact free Abelian group.

3.1.2 Basic results, notations and terminology

We recall that a topological space is *countably compact* if, and only if, every countable open cover of it has a finite subcover.

Definition 3.1.3. Let $\mathcal{U}$ be a filter on ω and $(x_n : n \in \omega)$ be a sequencee in a topological space X. We say that $x \in X$ is a *$\mathcal{U}$-limit point* of $(x_n : n \in \omega)$ if, for every neighborhood U of x, the set $\{n \in \omega : x_n \in U\} \in \mathcal{U}$.

If X is Hausdorff, every sequence has at most one $\mathcal{U}$-limit and in that case we denote $x = \mathcal{U} - \lim(x_n : n \in \omega)$. □

The set of all free ultrafilters on ω is denoted by ω^*. The following proposition is a well known result on ultrafilter limits.

Proposition 3.1.4. A topological space is countably compact if and only if each sequece in it has a $\mathcal{U}$-limit point for some $\mathcal{U} \in \omega^*$.

The concept of almost disjoint families will be useful in our construction.

Definition 3.1.5. An almost disjoint family is an infinite family $\mathcal{A}$ of infinite subsets of ω such that if $A, B \in \mathcal{A}$ are distinct, then $| A \cap B |< \omega$. □

It is well known that there exists an almost disjoint family of size continuum (see Kunen, 1983).

Definition 3.1.6. The unit circle group $\mathbb{T}$ will be the metric group $(\mathbb{R}/\mathbb{Z}, \delta)$ where the metric δ is given by $\delta(x + \mathbb{Z}, y + \mathbb{Z}) = \min\{| x - y + a | : a \in \mathbb{Z}\}$ for every $x, y \in \mathbb{R}$.

Given an open interval (a, b) of $\mathbb{R}$ with $a < b$, we let $\delta((a, b)) = b - a$.

An arc of $\mathbb{T}$ is a set of the form $I + \mathbb{Z} = \{a + \mathbb{Z} : a \in I\}$, where I is an open interval of $\mathbb{R}$. An arc is said to be proper if it is distinct from $\mathbb{T}$.

If U is a proper arc and $U = \{a + \mathbb{Z} : a \in I\} = \{b + \mathbb{Z} : b \in J\}$, where I and J are open intervals of $\mathbb{R}$, then $\delta(I) = \delta(J)$, and so we define the length of U as $\delta(U) = \delta(I)$. We also define $\delta(\mathbb{T}) = 1$. □

Given an arc U such that $\delta(U) \leq \frac{1}{2}$, it follows that $\operatorname{diam}_\delta U = \delta(U)$.

Our free Abelian group will all be represented as direct sums of copies of the group of integers $\mathbb{Z}$; we fix some notation. The additive group of rationals will also be used, so in the following definition one should read $\mathbb{Z}$ or $\mathbb{Q}$ for G.

Definition 3.1.7. If f is a map from a set X to a group G, then the *support* of f, denoted by $\operatorname{supp} f$, is the set $\{x \in X : f(x) \neq 0\}$.

We define $G^{(X)} = \{f \in G^X :| \operatorname{supp} f |< \omega\}$.

If Y is a subset of X, then, as an abuse of notation, we often write $G^{(Y)} = \{x \in G^{(X)} : \operatorname{supp} x \subseteq Y\}$.

Given $x \in X$, we denote by $\chi_x \in G^{(X)}$ the characteristic function of $\{x\}$, whose support is $\{x\}$ and whose value is $\chi_x(x) = 1$.

For a sequence $\zeta : \omega \to X$, we define $\chi_\zeta : \omega \to G^{(X)}$ by $\chi_\zeta(n) = \chi_{\zeta(n)}$.

Finally, for $x \in X$, we let $\vec{x} : \omega \to X$ be constantly equal to x. □

Note that thus $\chi_{\vec{x}}$ is also constant, with value χ_x.

Definition 3.1.8. Let $\mathcal{U}$ be a filter on ω and X be a set. We say that the sequences $f, g \in X^\omega$ are *$\mathcal{U}$-equivalent* and write $f \equiv_{\mathcal{U}} g$ if and only if $\{n \in \omega : f(n) = g(n)\} \in \mathcal{U}$. □

It is easy to verify that $\equiv_{\mathcal{U}}$ is an equivalence relation. We denote the equivalence class of and $f \in X^\omega$ by $[f]_{\mathcal{U}}$ and the quotient $X^\omega/\mathcal{U}$.

If R is a ring and X is an R-module, then $X^\omega/\mathcal{U}$ has a natural R-module structure given by addition, identity element, opposite and scalar multiplication "representativewise" (that is, $[f]_{\mathcal{U}} + [g]_{\mathcal{U}} = [f+g]_{\mathcal{U}}$, $0_{X^\omega/\mathcal{U}} = [\vec{0}]_{\mathcal{U}}$, $-[f]_{\mathcal{U}} = [-f]_{\mathcal{U}}$ and $r \cdot [f]_{\mathcal{U}} = [r \cdot f]_{\mathcal{U}}$).

If p is a free ultrafilter, then the *ultrapower* of the R-module X by p, is the R-module X^ω/p.

For the remainder of this chapter we will fix a cardinal number κ that satisfies $\kappa = \kappa^\omega$.

Throughout this chaper, we will work inside ultrapowers of $\mathbb{Q}^{(\kappa)}$. These ultrapowers contain copies of ultrapowers of $\mathbb{Z}^{(\kappa)}$, which will be useful for the construction. So we lay down some notation.

Definition 3.1.9. Let p be a free ultrafilter on ω. We define $\mathrm{Ult}_p(\mathbb{Q})$ as the $\mathbb{Q}$-vector space $(\mathbb{Q}^{(\kappa)})^\omega/p$ and $\mathrm{Ult}_p(\mathbb{Z}) = \{[g]_p : g \in \mathbb{Z}^\omega\}$ with the subgroup structure. □

Notice that each $[g]_p$ in $\mathrm{Ult}_p(\mathbb{Z})$ is formally an element of $(\mathbb{Q}^{(\kappa)})^\omega/p$, not of $(\mathbb{Z}^{(\kappa)})^\omega/p$. Nevertheless, it is clear that $(\mathbb{Z}^{(\kappa)})^\omega/p$ is isomorphic to $\mathrm{Ult}_p(\mathbb{Z})$ via the obvious isomorphism that carries the equivalence class of a sequence $g \in (\mathbb{Z}^{(\kappa)})^\omega$ in $(\mathbb{Z}^{(\kappa)})^\omega/p$ to its class in $(\mathbb{Q}^{(\kappa)})^\omega/p$.

3.2 Selective ultrafilters

In this section we review some basic facts about selective ultrafilters, the Rudin-Keisler order and some lemmas we will use in the next sections.

Definition 3.2.1. A selective ultrafilter (on ω), also called Ramsey ultrafilter, is a free ultrafilter p on ω with the property that for every partition $(A_n : n \in \omega)$ of ω, either there exists n such that $A_n \in p$, or there exists $B \in p$ such that $|B \cap A_n| = 1$ for every $n \in \omega$. □

The following proposition is well known. We provide Jech, 2003 as a reference.

Proposition 3.2.2. Let p be a free ultrafilter on ω. The the following are equivalent:

(a) p is a selective ultrafilter;

(b) for every $f \in \omega^\omega$, there exists $A \in p$ such that f is either constant or one-to-one on A;

(c) for every function $f : [\omega]^2 \to 2$ there exists $A \in p$ such that f is constant on $[A]^2$.

The Rudin-Keisler order is defined as follows:

Definition 3.2.3. Let $\mathcal{U}$ be a filter on ω and $f : \omega \to \omega$. We define $f_*(\mathcal{U}) = \{A \subseteq \omega : f^{-1}[A] \in \mathcal{U}\}$. □

It is easy to verify that $f_*(\mathcal{U})$ is a filter; if $\mathcal{U}$ is an ultrafilter, then so is $f_*(\mathcal{U})$; if $f, g : \omega \to \omega$, then $(f \circ g)_* = f_* \circ g_*$; and $(\mathrm{id}_\omega)_*$ is the identity over the set of all filters. This implies that if f is bijective, then $(f^{-1})_* = (f_*)^{-1}$.

Definition 3.2.4. Let $\mathcal{U}$ and $\mathcal{V}$ be filters. We say that $\mathcal{U} \leq \mathcal{V}$ (or $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{V}$, if we need to be clear) if and only if there exists $f \in \omega$ such that $f_*(\mathcal{U}) = \mathcal{V}$.

The *Rudin-Keisler order* is the set of all free ultrafilters on ω ordered by $\leq_{\mathrm{RK}}$. Notice that this is technically a *preorder*, and thus naturally we say that two ultrafilters are equivalent if and only if $p \leq q$ and $q \leq p$. □

It is easy to verify that $\leq$ is a preorder and so the equivalence defined above is indeed an equivalence relation. Moreover, the equivalence class of a fixed ultrafilter is the set of all fixed ultrafilters, so the relation restricts to ω^* without modifying the equivalence classes. We refer to Jech, 2003 for the following proposition:

Proposition 3.2.5. The following hold:

(1) If p and q are ultrafilters, then $p \leq q$ and $q \leq p$ if and only if there exists a bijection $f : \omega \to \omega$ such that $f_*(p) = q$.

(2) The selective ultrafilters are exactly the minimal elements of the Rudin-Keisler order.

This implies that if $f : \omega \to \omega$ and p is a selective ultrafilter, then $f_*(p)$ is either a fixed ultrafilter or a selective ultrafilter. If $f_*(p)$ is the ultrafilter generated by n, then $f^{-1}[\{n\}] \in p$, so, in particular, if f is finite-to-one and p is selective, then $f_*(p)$ is a selective ultrafilter equivalent to p.

The existence of selective ultrafilters is independent of ZFC. Martin's Axiom for countable orders implies the existence of $2^{\mathfrak{c}}$ pairwise incomparable selective ultrafilters in the Rudin-Keisler order.

The Lemma below appears in A. H. Tomita, 2005a.

Lemma 3.2.6. Let $(p_k : k \in \omega)$ be a family of pairwise incomparable selective ultrafilters. For each $k \in \omega$ let $(a_{k,i} : i \in \omega)$ be a strictly increasing sequence in ω such that $\{a_{k,i} : i \in \omega\} \in p_k$ and $i < a_{k,i}$ for all $i \in \omega$. Then there exists $\{I_k : k \in \omega\}$ such that:

(a) $\{a_{k,i} : i \in I_k\} \in p_k$, for each $k \in \omega$,

(b) $I_i \cap I_j = \emptyset$ for distinct $i, j \in \omega$, and

(c) $\{[i, a_{k,i}] : k \in \omega \text{ and } i \in I_k\}$ is a pairwise disjoint family.

In the course of the construction we will often use families of ultrafilters indexed by ω and finite sequences of infinite subsets of ω. It is thus convenient to establish the following notation:

Definition 3.2.7. A *finite tower* in ω is a finite sequence $(A_0, \dots, A_k)$ of infinite subsets of ω such that $A_{t+1} \subseteq A_t$ for every $t < k$. The set of all finite towers in ω is called $\mathcal{T}$. If $T = (A_0, \dots, A_k)$ then $l(T) = A_k$, the last term of the sequence T. For the empty sequence we write $l(\emptyset) = \omega$. □

Lemma 3.2.8. Assume there are $\mathfrak{c}$ incomparable selective ultrafilters. Then there is a family of incomparable ultrafilters $(p_{T,n} : T \in \mathcal{T}, n \in \omega)$ such that $l(T) \in p_{T,n}$ for all $T \in \mathcal{T}$ and $n \in \omega$.

Proof. Index the $\mathfrak{c}$ incomparable ultrafilters faithfully as $\{q_{T,n} : T \in \mathcal{T}, n \in \omega\}$. Fo each T, let $f_T : \omega \to l(T)$ be a bijection and define $p_{T,n} = f_{T*}(q_{T,n})$. Since f is one-to-one, it follows that $p_{T,n}$ is a selective ultrafilters equivalent to $q_{T,n}$. Therefore, the family $(p_{T,n} : T \in \mathcal{T}, n \in \omega)$ is as required. □

3.3 Main ideas

From now on we fix a family $(p_{T,n} : T \in \mathcal{T}, n \in \omega)$ of selective ultrafilters as provided by 3.2.8.

The idea is to use these ultrafilters to assign p-limits to enough injective sequences in $\mathbb{Z}^{(\kappa)}$ to ensure countable compactness of the resulting topology. We take some inspiration from A. C. Boero, Castro-Pereira, *et al.*, 2019 where a large independent family was used such that, up to a permutation, every injective sequence in $\mathbb{Z}^{(\mathfrak{c})}$ was part of this family. Since this group has cardinality $\mathfrak{c}$, there were indeed enough permutations to accomplish this. For an arbitrarily large group, we shall consider large linearly independent pieces to make sure every sequence has an accumulation point.

The following definition will be used to construct a witness for linearly independence in an ultraproduct that does not depende on the free ultrafilter.

Definition 3.3.1. Let $\mathcal{F}$ be a subset of $(\mathbb{Z}^{(\kappa)})^\omega$ and $A \in [\omega]^\omega$. We shall call $\mathcal{F}$ *linearly independent* mod A^* if and only if for every free ultrafilter p with $A \in p$ the family $([f]_p : f \in \mathcal{F}) \mathbin{\dot\cup} ([\chi_{\vec{\xi}}]_p : \xi < \kappa)$ is linearly independent in the $\mathbb{Q}$-vector space $\mathrm{Ult}_p(\mathbb{Q})$. □

Notice that it is implicit in our definition that $\{[f]_p : f \in \mathcal{F}\}$ and $\{[\chi_{\vec{\xi}}]_p : \xi < \kappa\}$ are disjoint. We will abbreviate "linearly independent mod A^*" to l.i. mod A^*.

An application of Zorn's Lemma will establish the following fact.

Lemma 3.3.2. Every set of sequences that is l.i. mod A^* can be extended to a maximal linearly independent set mod A^*.

It should be clear that if $A \subseteq B \subseteq \omega$ and A and B are infinite, then a set that is l.i. mod B^* is also l.i. mod A^*. Through the use of recursion, this easily implies the following Corollary:

Corollary 3.3.3. There exists a family $(\mathcal{E}_T : T \in \mathcal{T})$ such that:

(1) for every $T \in \mathcal{T}$ the set $\mathcal{E}_T$ is maximal l.i. mod $l(T)^*$, and

(2) for every $T \in \mathcal{T}$, if $n \leq | T |$ then $\mathcal{E}_{T|n} \subseteq \mathcal{E}_T$.

We note explicitly that even though $\mathcal{E}_T$ is only required to be maximal l.i. mod $l(T)^*$, it will, by virtue of item (2), depend on all of T, not just on $l(T)$.

Lemma 3.3.4. Let g be an element of $(\mathbb{Z}^{(\kappa)})^\omega$ and let $\mathcal{E} \subseteq (\mathbb{Z}^{(\kappa)})^\omega$ be maximal l.i. mod B^*. Then there exist an infinite subset A of B, a finite subset E of $\mathcal{E}$, a finite subset D of κ, and sets $\{r_f : f \in E\}$ and $\{s_\nu : \nu \in D\}$ of rational numbers such that $g|_A = \sum_{f \in E} r_f f|_A + \sum_{\nu \in D} s_\nu \chi_{\vec{\nu}}|_A$.

Proof. If $g \in \mathcal{E}$ of $g = \chi_{\vec{\nu}}$ for some $\nu < \kappa$, then we are done. Otherwise, by the maximality of $\mathcal{E}$, there exists a free ultrafilter p with $B \in p$ such that the set $\{[g]_p\} \cup \{[h]_p : h \in \mathcal{E}\} \cup \{[\chi_{\vec{\xi}}]_p : \xi < \kappa\}$ is not linearly independent.

This means that we can find finite subsets E and D of $\mathcal{E}$ and κ respectively and finite sets $\{r_f : f \in E\}$ and $\{s_\nu : \nu \in D\}$ of rational numbers such that $[g]_p = \sum_{f \in E} r_f [f]_p + \sum_{\nu \in D} s_\nu [\chi_{\vec{\nu}}]_p$.

Now choose $A \in p$ with $A \subseteq B$ that witnesses this equality. □

Corollary 3.3.5. If $\mathcal{E} \subseteq (\mathbb{Z}^{(\kappa)})^\omega$ is maximal l.i. mod B^*, then $| \mathcal{E} | = \kappa$.

Proof. First notice that $| \mathcal{E} | \leq | (\mathbb{Z}^{(\kappa)})^\omega | = \kappa^\omega = \kappa$. Assume $| \mathcal{E} | < \kappa$. Then the set $C = \bigcup \{\operatorname{supp} f(n) : n \in \omega, f \in \mathcal{E}\}$ has cardinality less than κ.

Take some injective sequence $(\xi_n : n \in \omega)$ in $\kappa \setminus C$ and define $g : \omega \to \mathbb{Z}^{(\kappa)}$ by $g(n) = \chi_{\vec{\xi_n}}$ for each $n \in \omega$. Clearly then $\bigcup \{\operatorname{supp} g(n) : n \in \omega\}$ is disjoint from C.

Apply Lemma 3.3.4 to obtain sets A, E, D, $\{r_f : f \in E\}$ and $\{s_\nu : \nu \in D\}$ such that $g|_A = \sum_{f \in E} r_f f|_A + \sum_{\nu \in D} s_\nu \chi_{\vec{\nu}}|_A$.

Since A is infinite and D is finite, there is a $k \in A$ such that $\xi_k \notin D$. Now $f(k)(\xi_k) = 0$ when $f \in E$ because $\xi_k \notin C$, and $\chi_{\vec{\nu}}(k)(\xi_k) = 0$ when $\nu \in D$ because $\xi_k \notin D$. Since $g(k)(\xi_k) = 1$, we have a contradiction. □

Henceforth we fix a family $(\mathcal{E}_T : T \in \mathcal{T})$ as in Corollary 3.3.3 and enumerate each $\mathcal{E}_T$ faithfully as $\mathcal{E}_T = \{f^T_\xi : \kappa \leq \xi \leq \kappa + \kappa\}$.

Definition 3.3.6. For each $T \in \mathcal{T}$ and $n \in \omega$, we denote by $G_{T,n}$ the intersection of $\mathrm{Ult}_{p_{T,n}}(\mathbb{Z})$ and the free Abelian group generated by $\{\frac{1}{n!}[f^T_\xi]_{p_{T,n}} : \kappa \leq \xi \leq \kappa + \kappa\} \cup \{\frac{1}{n!}[\chi_{\vec{\xi}}]_{p_{T,n}} : \xi < \kappa\}$. □

For the next lemma, we are going to use the following proposition:

Proposition 3.3.7. If G is an Abelian group and H is a subgroup of G such that G/H is an infinite cyclic group, then there exists $a \in G$ such that $G = H \oplus \langle a \rangle$.

A proof may be found in Fuchs, 1970, p. 14.4. This is not the statement of the theorem but it is exactly what is proved by the author.

The main idea of the proof of the following lemma is to mimic the well known proof of the fact that every subgroup of a free Abelian group is free.

Lemma 3.3.8. The group $G_{T,n}$ has a basis of the form $\{[\chi_{\vec{\xi}}]_{p_{T,n}} : \xi < \kappa\} \dot\cup \{[f]_{p_{T,n}} : f \in \mathcal{F}_{T,n}\}$ for some $\mathcal{F}_{T,n}(\mathbb{Z}^{(\kappa)})^\omega$.

Proof. Let H_μ be the group generated by $\{\frac{1}{n!}[\chi_{\vec{\xi}}]_{p_{T,n}} : \xi < \mu\}$ if $\mu \leq \kappa$ and by the union of $\{\frac{1}{n!}[\chi_{\vec{\xi}}]_{p_{T,n}} : \xi < \kappa\}$ and $\{\frac{1}{n!}[f^T_\xi]_{p_{T,n}} : \kappa \leq \xi < \mu\}$ when $\kappa < \mu \leq \kappa + \kappa$.

Let $G_\mu = H_\mu \cap \mathrm{Ult}_{p_{T,n}}(\mathbb{Z})$ for all $\mu \leq \kappa + \kappa$.

For every $\mu < \kappa + \kappa$ we shall determine an h_μ such that $G_{\mu+1} = G_\mu \oplus \langle\{[h_\mu]_{p_{T,n}}\}\rangle$ as follows:

For $\mu < \kappa$ the group G_μ is generated by $\{[\chi_{\vec{\xi}}]_{p_{T,n}} : \xi < \mu\}$, so $G_{\mu+1} = G_\mu \oplus \langle\{[\chi_{\vec{\mu}}]_{p_{T,n}}\}\rangle$ and we set $h_\mu = \chi_{\vec{\mu}}$.

For $\mu \geq \kappa$ observe that $\mathcal{G}_{\mu+1} \cap H_\mu = G_\mu$, so:

$$\frac{G_{\mu+1}}{G_\mu} = \frac{G_{\mu+1}}{G_{\mu+1} \cap H_\mu} \cong \frac{G_{\mu+1} + H_\mu}{H_\mu} \leq \frac{H_{\mu+1}}{H_\mu}.$$

The group $\frac{H_{\mu+1}}{H_\mu}$ is infinite and cyclic, so either $\frac{G_{\mu+1}}{G_\mu}$ is also infinite and cyclic, or $G_{\mu+1} = G_\mu$. By Proposition 3.3.7 there exists $a_\mu \in G_{\mu+1}$ such that $G_{\mu+1} = G_\mu \oplus \langle\{a_\mu\}\rangle$ (and $a_\mu = 0$ in the case $G_{\mu+1} = G_\mu$). Take h_μ such that $[h_\mu]_{p_{T,n}} = a_\mu$.

For every $\mu < \kappa + \kappa$, it follows that $G_{\mu+1} = G_\mu \oplus \langle\{[h_\mu]_{p_{T,n}}\}\rangle$. Since $G_{T,n} = \bigcup_{\mu<\kappa+\kappa} G_\mu$, it follows that $G_{T,n} = \bigoplus_{\mu<\kappa+\kappa}\langle\{[h_\mu]_{p_{T,n}}\}\rangle$.

Thus, the set $\mathcal{F}_{T,n} = \{h_\mu : \kappa \leq \mu < \kappa + \kappa, [h_\mu]_{p_{T,n}} \neq 0\}$ is as required. □

For the remainder of this chapter we fix such a set $\mathcal{F}_{T,n}$ as above for each $(T, n) \in \mathcal{T} \times \omega$.

The next lemma makes good on the promise from the beginning of this section as it shows how to make our topology countably compact.

Lemma 3.3.9. Assume that for every $(T, n) \in \mathcal{T} \times \omega$, every sequence $f \in \mathcal{F}_{T,n}$ has a $p_{T,n}$-limit in $\mathbb{Z}^{(\kappa)}$. Then every finite power of $\mathbb{Z}^{(\kappa)}$ is countably compact.

Proof. A sequence in some finite power of $\mathbb{Z}^{(\kappa)}$ is represented by finitely many members of $(\mathbb{Z}^{(\kappa)})^\omega$, say $g_0, \ldots, g_m$. We show that there is one ultrafilter p such that $p - \lim g_i$ exists for all $i \leq m$, namely $p_{T,n}$ for a suitable T and n.

Recursively, we define a tower $T = (A_0, \ldots, A_m)$ and for $i \leq m$ finite subsets E_i and D_i of $\mathcal{E}_{T|i}$ and κ respectively together with finite sets $(r^i_f : f \in E_i)$ and $(s^i_\nu : \nu \in D_i)$ of rational numbers such that

$$(*) \qquad g_i|_{A_i} = \sum_{f \in E_i} r^i_f f|_{A_i} + \sum_{\nu \in D_i} s^i_\nu \chi_{\vec{\nu}}|_{A_i}.$$

For $i = 0$, use Lemma 3.3.4 applied to $\mathcal{E}_\varnothing$ to obtain A_0, E_0, D_0, $(r^0_f : f \in E_0)$ and $(s^0_\nu : \nu \in D_0)$ such that $(*)$ holds with $i = 0$.

To go from i to $i+1$, apply Lemma 3.3.4 to $\mathcal{E}_{(A_0,\ldots,A_i)}$ to obtain $A_{i+1}, E_{i+1}, D_{i+1}, (r_f^{i+1} : f \in E_{i+1})$ and $(s_\nu^{i+1} : \nu \in D_{i+1})$ so that $(*)$ holds for $i+1$.

Let $A := A_m$ and let n be sufficiently large so that $n!r_f^i$ and $n!s_\nu^i$ are integers, for all $i \leq m$, $f \in E_i$, and $\nu \in D_i$. Then $g_i|_A = \sum_{f\in E_i} n!r_f^i(\frac{1}{n!}f)|_A + \sum_{\nu\in D} n!s_\nu^i(\frac{1}{n!}\chi_{\vec{\nu}})|_A$ for all $i \leq m$.

Since $l(T) = A \in p_{T,n}$ and each $E_i \subseteq \mathcal{E}_T$, it follows that $[g_i]_{p_{T,n}} \in G_{T,n}$. Therefore, each $[g_i]_{p_{T,n}}$ is an integer combination of $\{[f]_{p_{T,n}} : f \in \mathcal{F}_{T,n}\} \cup \{[\chi_{\vec{\xi}}]_{p_{T,n}} : \xi < \kappa\}$. Then, by hypothesis, it follows that each g_i has a $p_{T,n}$-limit. This completes the proof. □

3.4 Constructing homomorphisms

Through this section, we let $G = \mathbb{Z}^{(\kappa)}$ $\{h_\xi : \omega \leq \xi < \kappa\}$ be an enumeration of G^ω such that $\operatorname{supp} h_\ni(n) \subseteq \xi$ for every $n \in \omega$ and $\omega \leq \xi < \kappa$, and such that each element of G^ω appears at least $\mathfrak{c}$ many times.

Lemma 3.4.1. There exists a family $(J_{T,n} : T \in \mathcal{T}, n \in \omega)$ of pairwise disjoint subsets of κ such that $\{h_\xi : \xi \in J_{T,n}\} = \mathcal{F}_{T,n}$.

Proof. For each $f \in G^\omega$ there is a one-to-one map $\phi_f : \mathcal{T} \times \omega \to \{\xi \in \kappa : f = h_\xi\}$. Let $J_{T,n} = \{\phi_f(T,n) : f \in \mathcal{F}_{T,n}\}$ and we are done. □

For the rest of this section, we fix a family $(J_{T,n} : T \in \mathcal{T}, n \in \omega)$ as above.

The following lemma is the key to the main result.

Lemma 3.4.2. Assume we have $d \in G\backslash\{0\}$, $r \in G^\omega$ one-to-one, and $D \in [\kappa]^\omega$ such that

(i) $\omega \cup \operatorname{supp} d \cup \bigcup_{n\in\omega} \operatorname{supp} r(n) \subseteq D$,

(ii) $D \cap J_{T,n} \neq \emptyset$ for infinitely many (T,n))'s, and

(iii) $\operatorname{supp} h_\xi(n) \subseteq D$ for all $n \in \omega$ and $\xi \in D\backslash\omega$.

Then there exists a homomorphism $\phi : \mathbb{Z}^{(D)} \to \mathbb{T}$ such that:

(1) $\phi(d) \neq 0$,

(2) $p_{T,n} - \lim_{k\in\omega} \phi(h_\xi(k)) = \phi(\chi_\xi)$, whenever $T \in \mathcal{T}$, $n \in \omega$, and $\xi \in D \cap J_{T,n}$, and

(3) $\phi \circ r$ does not converge.

Before proving this lemma, we show how to use it to prove the main result. First, we use it to prove another lemma:

Lemma 3.4.3. Assume $d \in G\backslash\{0\}$ and $r \in G^\omega$ is one-to-one. Then there exists a homomorphism $\phi : \mathbb{Z}^{(\kappa)} \to \mathbb{T}$ such that

(1) $\phi(d) \neq 0$,

(2) $p_{T,n} - \lim_{k\in\omega} \phi(h_\xi(k)) = \phi(\chi_\xi)$, whenever $T \in \mathcal{T}$, $n \in \omega$ and $\xi \in J_{T,n}$, and

(3) $\phi \circ r$ does not converge.

Proof. Using a closing-off argument, construct a $D \in [\kappa]^\omega$ that intersects infinitely many sets $J_{T,n}$, and that contains ω, $\operatorname{supp} d$, $\bigcup_{n\in\omega} \operatorname{supp} r(n)$ and $\bigcup_{\xi\in D\setminus\omega, n\in\omega} \operatorname{supp} h_\xi(n)$.

By the previous Lemma, there exists a homomorphism $\phi_0 : \mathbb{Z}^{(D)} \to \mathbb{T}$ such that $\phi_0(d) \neq 0$, $\phi \circ r$ does not converge, and $p_{T,n} - \lim_{k\in\omega} \phi_0(h_\xi(k)) = \phi_0(\chi_\xi)$ whenever $T \in \mathcal{T}$, $n \in \omega$ and $\xi \in D \cap J_{T,n}$.

We let $(\alpha_\delta : \delta < \kappa)$ be the monotone enumeration of $\kappa\setminus D$. For $\gamma \leq \kappa$, let $D_\gamma := D \cup \{\alpha_\delta : \delta < \gamma\}$. Thus, $D_0 = D$ and $D_\kappa = \kappa$.

Recursively, we construct, for $\gamma \leq \kappa$, an increasing sequence of homomorphisms $\phi_\gamma : \mathbb{Z}^{(D_\gamma)} \to \mathbb{T}$ such that $p_{T,n} - \lim_{k\in\omega} \phi_\gamma(h_\xi(k)) = \phi_\gamma(\chi_\xi)$ whenever $T \in \mathcal{T}$, $n \in \omega$ and $\xi \in D_\gamma \cap J_{T,n}$. The desired homomorphism ϕ will be ϕ_κ. The basis step 0 is already done, and for limit steps, we simply take the union of all previous homomorphisms.

To define $\phi_{\gamma+1} given \phi_\gamma$, it suffices to specify the value $\phi_{\gamma+1}(\chi_{\alpha_\gamma})$.

If $\alpha_\gamma \in J_{T,n}$ for some $T \in \mathcal{T}$ and $n \in \omega$, then we ascribe $\phi_{\gamma+1}(\chi_{\alpha_\gamma}) =$ $p_{T,n} \lim_{k\in\omega} \phi_\gamma(h_\gamma(k))$. This is well-defined because $\operatorname{supp} h_\gamma(n) \subseteq \gamma \subseteq D_\gamma$ for all $n \in \omega$ and because $\mathbb{T}$ is compact. Otherwise, just let $\phi_{\gamma+1}(\chi_{\alpha_\gamma}) = 0$. □

We can now prove our main result.

Theorem 3.4.4. Assume the existence of $\mathfrak{c}$ pairwise incompatible selective ultrafilters and that κ is an infinite cardinal such that $\kappa = \kappa^\omega$. Then the free Abelian group of cardinality κ has a Hausdorff group topology without nontrivial convergent sequences such that all of its finite powers are countably compact.

Proof. Following the notation ofn the rest of the chapter, given $d \in G\setminus\{0\}$, and a one-to-one $r \in G^\omega$, Lemma 3.4.3 provides a homomorphism $\phi_{d,r} : G \to \mathbb{T}$ such that $\phi_{d,r}(d) \neq 0$, $\phi_{d,r} \circ r$ does not converge, and $p_{T,n} - \lim_{k\in\omega} \phi_{d,r}(h_\xi(k)) = \phi_{d,r}(\chi_\xi)$ whenever $T \in \mathcal{T}$, $n \in \omega$, and $\xi \in J_{T,n}$. We give G the initial topology generated by the collection of homomorphisms $\{\phi_{d,r} : d \in G\setminus\{0\}, r \in G^\omega$ is one-to-one$\}$ thus obtained and the usual topology of $\mathbb{T}$.

Since the initial topology generated by any collection of group homomorphisms is a group topology we do indeed obtain a group topology. Since $\mathbb{T}$ is Hausdorff and for every $d \neq 0$ there are many $\phi_{d,r}$ with $\phi_{d,r} \neq 0$ it follows that our topology is Hausdorff.

To see that every finite power of G is countably compact, we now use Lemma 3.3.9. Given $T \in \mathcal{T}$, $n \in \omega$ and $f \in \mathcal{F}_{T,n}$, there exists $\xi \in J_{T,n}$ such that $h_\xi = f$. For every $d \in G\setminus\{0\}$ and one-to-one $r \in G^\omega$, we have $p_{T,n} - \lim_{k\in\omega} \phi_{d,r}(h_\xi(k)) = \phi_{d,r}(\chi_\xi)$. So $p_{T,n} - \lim f(n) = \chi_\xi$ and we are done.

Since for a given one-to-one sequence r and any $d \neq 0$ the sequence $\phi_{d,r} \circ r$ does not converge and $\phi_{d,r}$ is continuous, it follows that r does not converge. So G has no nontrivial convergent sequences. □

Towards the proof of Lemma 3.4.2 we formulate a definition and a very technical lemma.

Definition 3.4.5. Let $\epsilon > 0$. An *ϵ-arc function* is a function $\psi : \kappa \to \mathbb{B}$ such that for all $\alpha < \kappa$ either $\psi(\alpha) = \mathbb{T}$ or the length of $\psi(\alpha)$ is ϵ, and the set $\{\alpha \in \kappa : \psi(\alpha) \neq \mathbb{T}\}$ is finite. We will call this finite set the support of ψ and denote it by $\operatorname{supp} \psi$.

Given two arc functions ψ and ρ we write $\psi \leq \rho$ if $\overline{\psi(\alpha)} \subseteq \rho(\alpha)$ or $\psi(\alpha) = \rho(\alpha)$ for each $\alpha \in \kappa$. □

We shall obtain our homomorphisms using limits of such arc functions. The following lemmas are instrumental in its construction.

The following result follows from an argument implicit in the construction of A. C. Boero, Castro-Pereira, *et al.*, 2019, but it may be difficult to extracit it from that paper. We postpone its rather technical proof to the next section.

Lemma 3.4.6. Let p be a selective ultrafilter and $\mathcal{F}$ be a finite subset of G^ω such that the family $([f]_p : f \in \mathcal{F}) \cup ([\chi_{\vec{\alpha}}]_p : \alpha < \kappa)$ is linearly independent.

Then given $\epsilon > 0$ and a finite $E \subseteq \kappa$, there exist $A \in p$ and a sequence $(\delta_n : n \in A)$ of positive real numbers such that

($\star$) whenever $(U_f : f \in \mathcal{F})$ is a family of arcs of length ϵ and ρ is an arc function of length at least ϵ with $\operatorname{supp} \rho \subseteq E$, there exist, for each $n \in A$, a δ_n-arc function $\psi_n \leq \rho$ such that $\operatorname{supp} \psi_n = \bigcup_{f \in \mathcal{F}} \operatorname{supp} f(n) \cup E$ and $\sum_{\mu \in \operatorname{supp} f} f(n)(\mu)\psi_n(\mu) \subseteq U_f$ for each $f \in \mathcal{F}$.

Now we proceed to prove Lemma 3.4.2. We will use the following lemma:

Lemma 3.4.7. Let $(\mathcal{F}^k : k \in \omega)$ be a sequence of countable subsets of G^ω and let $(p_k : k \in \omega)$ be a sequence of pairwise incompatible selective ultrafilters such that for each $k \in \omega$ the family $([f]_{p_k} : f \in \mathcal{F}^k) \cup ([\chi_{\vec{\xi}}]_{p_k} : \xi \in \kappa)$ is linearly independent. Also let for every $k \in \omega$ and $f \in \mathcal{F}^k$ an ordinal $\xi_{f,k} < \kappa$ be given. In addition let $d, d' \in G \setminus \{0\}$ such that $\operatorname{supp} d \cap \operatorname{supp} d' = \emptyset$. Finally, let $D \in [\kappa]^\omega$ that contains ω, $\operatorname{supp} d$, $\operatorname{supp} d'$, and $\operatorname{supp} f(n)$ for every $k \in \omega$, $f \in \mathcal{F}^k$, and $n \in \omega$.

Then: there exists a homomorphism $\phi : \mathbb{Z}^{(D)} \to \mathbb{T}$ such that $\phi(d) \neq 0$, $\phi(d') \neq 0$ and $p_k - \lim_{n \in \omega} \phi(f(n)) = \phi(\chi_{\xi_{f,k}})$, for all $k \in \omega$ and $f \in \mathcal{F}^k$.

Proof. Write D as the union of an increasing sequence $(D_n : n \in \omega)$ of finite nonempty susbets, and likewise take $(\mathcal{F}^k_n : n \in \omega)$ for each $\mathcal{F}^k$.

Take a sufficiently small positive number ϵ_0 and an ϵ_0-arc function ρ_* such that $\operatorname{supp} d \cup \operatorname{supp} d' \subseteq \operatorname{supp} \rho_*$ and $0 \notin \overline{\sum_{\mu \in \operatorname{supp} d} d(\mu)\rho_*(\mu)} \cup \overline{\sum_{\mu \in \operatorname{supp} d'} d'(\mu)\rho_*(\mu)}$.

Let $E_0 = \operatorname{supp} \rho_* \cup D_0$ and $B^k_0 = \omega$ for each $k \in \omega$.

We will define, by recursion, for $m \in \omega$: finite sequences $(B^k_m : m \in \omega)$, finite sets $E_m \subseteq \kappa$, and real numbers $\epsilon_m > 0$ satisfying:

(1) for all $k, m \in \omega$, $B^k_m \in p_k$,

(2) for each $m \geq 1$ and $k \leq m$, we have a sequence $(\delta^k_{m,n} : n \in \omega)$ of positive real numbers such that: if $(U_f : f \in \mathcal{F}^k_m)$ is a family of arcs of length ϵ_{m-1} and ρ is an ϵ_{m-1}-arc function with $\operatorname{supp} \rho \subseteq E_{m-1}$, then for each $n \in \omega$ there exists a $\delta^k_{m,n}$-arc function ψ

with $\psi \leq \rho$, $\operatorname{supp} \psi = \bigcup_{f \in \mathcal{F}^k_m} \operatorname{supp} f(n) \cup E_{m-1}$, and $\sum_{\mu \in \operatorname{supp} f(n)} f(n)(\mu)\psi(\mu) \subseteq U_f$ for each $f \in \mathcal{F}^k_m$,

(3) for all $k, m \in \omega$ we have $B^k_{m+1} \subseteq B^k_m$, and

(4) $\epsilon_{m+1} = \frac{1}{2} \min(\{\delta^k_{l,n} : k \leq l \leq m+1 \text{ and } n \in (m+2) \cap B^k_l\} \cup \{\epsilon_m\})$.

Suppose we have defined B^k_l for all $k \in \omega$ as well as E_l and ϵ_l for all $l \leq m$. As will be clear from the step below the set B^k_m is only nontrivial when $k \leq m$. Therefore we let $B^k_{m+1} = B^k_m = \omega$ for $k > m+1$ and we concentrate on the case $k \leq m+1$.

Let $k \leq m+1$. By Lemma 3.4.6, there exist $B^k_{m+1} \in p_k$ and $(\delta^k_{m+1,n} : n \in \omega)$ that satisfy (2) for $m+1$. Without loss of generality we can assume $B^k_{m+1} \subseteq B^k_m$.

Condition (4) now specifies ϵ_{m+1}.

Setting $E_{m+1} = E_m \cup \bigcup \{\operatorname{supp} f(k) : k \leq m, f \in \bigcup_{k \leq m+1} \mathcal{F}^k_{m+1}\} \cup D_{m+1}$ completes the recursion.

For each $k \in \omega$, apply the selectivity of p_k to choose an increasing sequence $(a_{k,i} : i \in \omega)$ with $\{a_{k,i} : i \in \omega\} \in p_k$ and such that $a_{k,i} \in B^k_i$ and $a_{k,i} > i$ for all $i \in \omega$.

Next apply Lemma 3.2.6 and let $(I_k : k \in \omega)$ be a sequence of pairwise disjoint subsets of ω such that $\{a_{k,i} : i \in I_k\} \in p_k$ and the family of intervals $\{[i, a_{k,i}] : k \in \omega, i \in I_k\}$ is pairwise disjoint. Without loss of generality we can assume that $k < \min I_k$.

Enumerate $\bigcup_{k \in \omega} I_k$ in increasing order as $(i_t : t \in \omega)$. For each $t \in \omega$, let k_t be such that $i_t \in I_{k_t}$. Notice that for each $t \in \omega$ we have $i_i \in I_{k_t}$, and hence $i_t \geq \min I_{k_t} > k_t$ and $a_{k_t,i_t} > i_t$.

By recursion we define a sequence of arc functions, $(\rho_{i_t} : t \in \omega)$, such that $\rho_{i_0} \leq \rho_*$ and $\rho_{i_{t+1}} \leq \rho_{i_t}$.

We start with $t = 0$. In this case we have $k_0 < i_0 < a_{k_0,i_0}$, $a_{k_0,i_0} \in B^{k_0}_{i_0}$, and $\epsilon_{i_0-1} \leq \epsilon_0$.

Since ρ_* has length at least ϵ_{i_0-1}, there exists an arc function ρ_{i_0} of length $\delta^{k_0}_{i_0,a_{k_0,i_0}}$ such that $\sum_{\mu \in \operatorname{supp} f(a_{k_0,i_0})}(\mu)\rho_{i_0}(\mu) \subseteq \rho_*(\xi_{f,k_0})$, for each $f \in \mathcal{F}^{k_0}_{i_0}$. We have by the definition that $\delta^{k_0}_{i_0,a_{k_0,i_0}} > \epsilon_{i_1-1}$.

Suppose $t > 0$ and that $\rho_{i_{t-1}}$ has been defined with length at least $\epsilon_{i_{t-1}}$.

Apply item (2) to the arc function $\rho_{i_{t-1}}$, the finite set $\mathcal{F} = \mathcal{F}^{k_t}_{i_t}$, the number $\epsilon_{i_{t-1}}$, the finite set $E_{i_{t-1}}$, the arcs $U_f = \rho_{i_{t-1}}(\xi_{f,k_t})$ for $f \in \mathcal{F}^{k_t}_{i_t}$, and $n = a_{k_t,i_t} \in B^{k_t}_{i_t}$ to obtain an arc function $\rho_{i_t} \leq \rho_{i_{t-1}}$ such that $\sum_{\mu \in \operatorname{supp} f(a_{k_t,i_t})} f(a_{k_t,i_t})(\mu)\rho_{i_{t-1}}(\xi_{f,k_t})$ for all $f \in \mathcal{F}^{k_t}_{i_t}$, and $\rho_{i_{t-1}}$ has length $\delta^{k_t}_{i_t,a_{k_t,i_t}}$.

Since $k_t < i_t < a_{k_t,i_t} \leq i_{t+1} - 1$ and $a_{k_t,i_t} \in B^{k_t}_{i_t}$, we have that $\delta^{k_t}_{i_t,a_{k_t,i_t}} > \epsilon_{i_{t+1}-1}$.

If $\xi \in D_{i_t}$, then $\xi \in \operatorname{supp} \rho_{i_t}$ and the length of $\rho_{i_t}(\xi)$ is not greater than ϵ_{i_t-1}, and $\epsilon_{i_t-1} \leq \frac{1}{2^{i_t-1}} \leq \frac{1}{2^t}$.

It follows that for all $\xi \in D$ the intersection $\bigcap_{t \in \omega} \rho_{i_t}(\xi)$ consists of a unique element; we define $\phi(\chi_\xi)$ to be that element and extend ϕ to a group homomorphism.

By construction, $\phi(f(a_{k_t,i_t}))$ is in $\sum_{\mu\in\operatorname{supp} f(a_{k_t,i_t})} f(a_{k_t,i_t})(\mu)\rho_{i_t}(\mu)$ which is a subset of $\rho_{i_{t-1}}(\xi_{f,k_t})$, for all $f \in \mathcal{F}^{k_t}_{i_t}$. Therefore, the sequence $(\phi(f(a_{k,i})))_{i\in I_k}$ converges to $\phi(\chi_{\xi_{f,k_t}})$, for each $k \in \omega$ and $f \in \mathcal{F}^k$.

Furthermore, $\phi(d) \in \sum_{\mu\in\operatorname{supp} d} d(\mu)\rho_*(\mu)$, so it follows that $\phi(d) \neq 0$; and likewise $\phi(d') \neq 0$.

It is clear that this implies the conclusion of Lemma 3.4.7. □

We now give the proof of Lemma 3.4.2.

Proof of Lemma 3.4.2:

Proof. There are only countably many pairs $(T, n) \in \mathcal{T} \times \omega$ such that $J_{T,n} \cap D \neq 0$. We enumerate them faithfully as $((T_m, n_m) : m \geq 2)$.

For $m \geq 2$ let $\mathcal{F}^m = \{h_\xi : \xi \in D \cap J_{T_m,n_m}\}$ and $p_m = p_{T_m,n_m}$. Let p_0 and p_1 be two ultrafilters such that $(p_m : m \geq 0)$ is a family of pairwise incompatible selective ultrafilters, and let $\mathcal{F}^0 = \mathcal{F}^1 = \{r\}$. For each $m \geq 2$ and $\xi \in J_{T_m,n_m} \cap D$, let $\xi_{h_\xi,m} = \xi$. Let $\xi_{r,0} = k$ and $\xi_{r,1} = k'$ with $k, k' \in \omega \setminus \operatorname{supp} d$. Then, by applying Lemma 3.4.7 with $d' = \chi_k - \chi_{k'}$, there exist $\phi : \mathbb{Z}^{(D)} \to \mathbb{T}$ satisfying (1) and (2). In order to see (3) is also satisfied, notice that $p_0 - \lim \phi \circ r \neq p_1 - \lim \phi \circ r$. □

3.5 Proof of Lemma 3.4.6

In this section we present a proof of Lemma 3.4.6. We will need the notion of integer stack, which was defined in A. H. Tomita, 2015.

The integer stacks are collections of sequences in $\mathbb{Z}^{(\mathfrak{c})}$ that are usually associated to a selective ultrafilter. Given a finite set of sequences $\mathcal{F}$ it is possible to associate it to an integer stack which generates the same $\mathbb{Q}$-vector space as $\mathcal{F}$. The sequences in the stack have some nice properties that help us construct well behaved arcs when constructing homomrphisms, and the linear relations between $\mathcal{F}$ and the sequences of the stack help us transform these arcs into arcs that work for the functions of $\mathcal{F}$. Below, we give the definition of integer stack.

Definition 3.5.1. An *integer stack $\mathcal{S}$ on A* consists of

(i) $A \in [\omega]^\omega$,

(ii) natural numbers s, t, and M, positive integers r_i for $0 \leq i < s$ and positive integers $r_{i,j}$ for $0 \leq i < s$ and $0 \leq j < r_i$,

(iii) functions $f_{i,j,k} \in (\mathbb{Z}^{(\mathfrak{c})})^A$ for $0 \leq i < s$, $0 \leq j < r_i$ and $0 \leq k < r_{i,j}$, and $g_l \in (\mathbb{Z}^{(\mathfrak{c})})^A$ for $0 \leq< t$,

(iv) sequences $\xi_i \in \mathfrak{c}^A$ for $0 \leq i < s$ and $\mu_l \in \mathfrak{c}^A$ for $0 \leq l < t$, and

(v) real numbers $\theta_{i,j,k}$ for $0 \leq i < s$, $0 \leq j < r_i$ and $0 \leq k < r_{i,j}$.

These are required to satisfy the following conditions:

(1) $\mu_l(n) \in \operatorname{supp} g_l(n)$ for each $n \in A$ and $l < t$,

(2) $\mu_{l^*}(n) \notin \operatorname{supp} g_l(n)$ for each $n \in A$ and $l^* < l < t$,

(3) the elements of $\{\mu_l(n) : 0 \leq l < t \text{ and } n \in A\}$ are pairwise distinct,

(4) $| g_l(n) | \leq M$ for each $n \in A$ and $l < t$,

(5) for each $i < s$ and $j < r_i$, $(\theta_{i,j,k} : k < r_{i,j})$ is a linearly independent family of elements of $\mathbb{R}$ viewed as a $\mathbb{Q}$-vector space,

(6) $\lim_{n \in A} \frac{f_{i,j,k}(n)(\xi_i(n))}{f_{i,j,0}(n)(\xi_i(n))} = \theta_{i,j,k}$ for each $i < s$, $j < r_i$ and $k < r_{i,j}$,

(7) the sequence $(| f_{i,j,k}(n)(\xi_i(n)) |)_{n \in A}$ diverges monotonically to $+\infty$, for each $i < s$, $j < r_i$ and $k < r_{i,j}$,

(8) $| f_{i,j,k}(n)(\xi_i(n)) | > | f_{i,j,k^*}(n)(\xi_i(n)) |$ for each $n \in A$, $i < s$, $j < r_i$ and $k < k^* < r_{i,j}$,

(9) $\left(\frac{| f_{i,j,k}(n)(\xi_i(n)) |}{| f_{i,j^*,k^*}(n)(\xi_i(n)) |} \right)_{n \in A}$ converges monotonically to 0 for each $i < s$, $j^* < j < r_i$ $k < r_{i,j}$ and $k^* < r_{i,j^*}$, and

(10) $\{f_{i,j,k}(n)(\xi_{i^*}(n)) : n \in A\} \subseteq [-M, M]$ for each $i^* < i < s$, $j < r_i$ and $k < r_{i,j}$.

□

It follows from the definition the sequences that comprise the stack are linearly independent. Moreover, if p is a free ultrafilter, $\mathcal{S}$ is a stack over A, and $A \in p$, then it also follows that that $([g_l]_p : l < t) \cup ([f_{i,j,k}]_p : i < s, j < r_i, k < r_{i,j})$ is linearly independent in the $\mathbb{Q}$-vector space $\mathbb{Q}^{(\mathfrak{c})}/p$.

Definition 3.5.2. Given an integer stack $\mathcal{S}$ and a positive integer N, the *Nth root of* $\mathcal{S}$, written $\frac{1}{N}\mathcal{S}$, is obtained by keeping all the structure in $\mathcal{S}$ with the exception of the functions; these are divided by N. Thus a function $f_{i,j,k}$ in $\mathcal{S}$ is replaced by $\frac{1}{N} f_{i,j,k}$ in $\frac{1}{N}\mathcal{S}$ for each $i < s$, $j < r_i$ and $k < r_{i,j}$, and a function g_l in $\mathcal{S}$ is replaced by $\frac{1}{N} g_l$ in $\frac{1}{N}\mathcal{S}$ for each $l < t$.

A *stack* (unspecified) is then defined to be the Nth root of an integer stack for some positive integer N. □

The lemma below gives the relation between a finite set of sequences in $\mathbb{Z}^{(\mathfrak{c})}$ and a stack $\mathcal{S}$ that is associated to it. The first part of this lemma is proved in A. H. Tomita, 2015. The second part was stated in A. C. Boero, Castro-Pereira, *et al.*, 2019 with no proof presented there, since it follows directly from statements of several lemmas and constructions from A. H. Tomita, 2015. Since the construction there is long and complex, we sketch here, for the sake of completeness, a proof for the second part indicating which statements and proofs from A. H. Tomita, 2015 are used, without repeating the arguments.

Lemma 3.5.3. Let $h_i \in (\mathbb{Z}^{(\mathfrak{c})})^\omega$, for $i < m$, and $\mathcal{U} \in \omega^*$ be a selective ultrafilter. Then there exists $A \in \mathcal{U}$ and a stack $\frac{1}{N}\mathcal{S}$ on A such that: if the elements of the stack have a $\mathcal{U}$-limit in $\mathbb{Z}(\mathfrak{c})$ then h_i has a $\mathcal{U}$-limit in $\mathbb{Z}^{(\mathfrak{c})}$ for each $i < m$.

We will say in this case that the finite set $\{h_i : i < m\}$ is associated to $(\frac{1}{N}\mathcal{S}, A, \mathcal{U})$.

(#) If $([h_i]_{\mathcal{U}} : i < m)$ is a $\mathbb{Q}$-linearly independent family and the group generated by it does not contain nonzero constant classes, then each restriction $h_i|_A$ is an integer combination of the stack $\frac{1}{N}\mathcal{S}$ on A. Also each element of the integer stack $\mathcal{S}$ is an integer combination of $(h_i|_A : i < m)$.

Proof. We prove (#). All numbered references in this proof are from the paper A. H. Tomita, 2015.

First, notice that if $([h_i]_{\mathcal{U}} : i < m)$ is a $\mathbb{Q}$-linearly independent family and the group generated by it does not contain nonzero constant classes, the it satisfies the conclusion of Lemma 4.1. Then, following the proof of Lemma 7.1, using the f's as the h's themselves, we see that the functions h_i for $i < m$ are integer combinations of the stack $\frac{1}{N}\mathcal{S}$ that was constructed.

It remains to be seen that the functions of $\mathcal{S}$ are integer combinations of the functions h_i restricted to A. First, notice that in the statement of Lemma 5.4, by x), xi), xii) and xiv), the functions $f_q^{i,j}$ and g_q^0 are integer combinations of the h_i. This Lemma is used in the proof of Lemma 5.5, where the functions $f_q^{i,j}$ become the functions $f_{i,j,k}$, so they are integer commbinations of the h_i's.

Now notice that in Lemma 6.1, by g), c) and finite induction, the functions g_j^i are integer combinations of the h_i, and some of these become the g_i's in the proof of Lemma 6.2. Since in the proof of 7.1 the stack is constructed by applying Lemma 5.5, or Lemma 6.2, or Lemma 5.5 followed by Lemma 6.2 (depending on the case), it followd that the stack constructed consists of functions that are linear combinations of the functions h_i (when restricted to A). □

Now we define some integers related to Kronecker's Theorem that will be useful in our proof. The existence of these integers follows from Lemma 4.3 of A. H. Tomita, 2015. These integers were also defined and used in that paper.

Definition 3.5.4. If $(\theta_0, \dots, \theta_{r-1})$ is a linearly independent family of elements of the $\mathbb{Q}$-vector space $\mathbb{R}$ and $\epsilon > 0$, then $L(\theta_0, \dots, \theta_{r-1}, \epsilon)$ denotes a positive integer L such that $\{(\theta_0 x + \mathbb{Z}, \dots, \theta_{r-1} x + \mathbb{Z}) : x \in I\}$ is ϵ-dense in $\mathbb{T}^r$ in the usual Euclidean metric product topology, for any interval I of length at least L. □

The last lemma we are going to need is Lemma 8.3 from A. H. Tomita, 2015, stated below.

Lemma 3.5.5. Let ϵ, γ and α be positive reals, N be a positive integer and ψ be an arc function. Let $\mathcal{S}$ be an integer stack on an $A \in [\omega]^\omega$ and s, t, r_i, $r_{i,j}$, M, $f_{i,j,k}$, g_l, ξ_i, μ_j and $\theta_{i,j,k}$ be as in Definition 3.5.1.

Let L be an integer greater than or equal to $\max\{L(\theta_{i,j,0}, \dots, \theta_{i,j,r_{i,j}-1}, \frac{\epsilon}{24}) : i < s \text{ and } j < r_i\}$ and let $r := \max\{r_{i,j} : i < s \text{ and } j < r_i\}$.

Suppose that $n \in A$ is such that

(a) $\{V_{i,j,k} : i < s, j < r_i \text{ and } k < r_{i,j}\} \cup \{W_l : l < t\}$ is a collection of open arcs of length ϵ,

(b) $\delta(\psi(\beta)) \geq \epsilon$ for each $\beta \in \operatorname{supp}\psi$,

(c) $\epsilon > 3N\alpha \max(\{\|g_l(n)\| : l < t\} \cup \{\|f_{i,j,k}(n)\| : i < s, j < r_i, k < r_{i,j}\})$,

(d) $3MNs\gamma < \epsilon$,

(e) $|f_{i,r_i-1,0}(n)(\xi_i(n))|\gamma > 3L$ for each $i < s$,

(f) $|f_{i,j-1,0}(n)(\xi_i(n))|\frac{\epsilon}{6\sqrt{r_{i,j}}|f_{i,j,0}(n)(\xi_i(n))|} > 3L$ for each $i < s$ and $j < r_i$,

(g) $\left|\theta_{i,j,k} - \frac{f_{i,j,k}(n)(\xi_i(n))}{f_{i,j,0}(n)(\xi_i(n))}\right| < \frac{\epsilon}{24\sqrt{r}L}$ for each $i < s$, $j < r_i$ and $k < r_{i,j}$, and

(h) $\operatorname{supp}\psi \cap \{\mu_l(n) : l < t\} = \emptyset$.

Then there exists an arc function ϕ such that

(A) $N\overline{\phi(\beta)} \subseteq \psi(\beta)$ for each $\beta \in \operatorname{supp}\psi$,

(B) $\sum_{\beta\in\operatorname{supp} g_l(n)} g_l(n)(\beta)\phi(\beta) \subseteq W_t$ for each $l < t$,

(C) $\sum_{\beta\in\operatorname{supp} f_{i,j,k}(n)} f_{i,j,k}(n)(\beta)\phi(\beta) \subseteq V_{i,j,k}$ for each $i < s< j <_{r_i}$ and $k < r_{i,j}$,

(D) $\delta(\psi(\beta) = \alpha$ for each $\beta \in \operatorname{supp}\phi$, and

(E) $\operatorname{supp}\phi$ can be chosen to be any finite set containing $\operatorname{supp}\psi$, $\operatorname{supp} f_{i,j,k}(n)$ for $i < s$, $j < r_i$ and $k < r_{i,j}$, and $\operatorname{supp} g_l(n)$ for $l < t$.

□

Now we are ready to prove Lemma 3.4.6.

Proof of Lemma 3.4.6: Write $\mathcal{F} = \{u_0, \dots, u_{q-1}\}$ without repetition. Let $\mathcal{S}$ be an integer stack on and $A' \in p$ and let N be a positive integer such that $(\frac{1}{N}\mathcal{S}, A', p)$ is associated to $\mathcal{F}$.

As in Definition 3.5.1, the components of $\mathcal{S}$ will be denoted s, t, M, $(r_i : i < s)$, $(r_{i,j} : i < s, j < r_i)$, $(f_{i,j,k} : i < s, j < r_i, k < r_{i,j})$, $(g_l : l < t)$, $(\xi_i : i < s)$, $(\mu_l : l < t)$ and $(\theta_{i,j,k} : i < s, j < r_i, k < r_{i,j})$.

We write $\{f_{i,j,k} : i < s, j < r_i, k < r_{i,j}\} \cup \{g_l : l < t\}$ as $\{v_0, \dots, v_{q-1}\}$.

Let $\mathcal{M}$ be the $q \times q$ matrix of integer numbers such that $Nu_i(n) = \sum_{j<q} \mathcal{M}_{i,j}v_j(n)$ for all $n \in A$ and $i < q$.

By (#) in Lemma 3.5.3, each v_j is an integer combination of the u_i's, therefore the inverse matrix of $\frac{1}{N}\mathcal{M}$, which we denote by $\mathcal{N}$, has integer entries.

Let $\epsilon' := \epsilon(\sum_{i,j<q} |\mathcal{M}_{i,j}|)^{-1]}$ and $\gamma < \frac{\epsilon'}{3MNs}$. Let L be greater than or equal to $\max\{L(\theta_{i,j,0}, \dots \theta_{i,j,r_{i,j}-1}, \frac{\epsilon'}{24}) : i < s, j < r_i\}$.

For each $n \in A'$, let $\delta_n < \frac{1}{2}$ be such that:

$$\epsilon' > 3N \max(\{\|g_l(n)\| : l < t\} \cup \{\|f_{i,j,k}(n)\| : i < s, j < r_i, k < r_{i,j}) \frac{\delta_n}{N}.$$

We note that both N's above cancel each other out, but we write the expression this way as we will use $\frac{\delta_n}{N}$ in the role of α in item (c) of Lemma 3.5.5.

Let $r := \max\{r_{i,j} : i < s, j < r_i\}$. Let A be the set of n's in A' such that:

- $|f_{i,r_i-1,0}(n)(\xi_i(n))|\gamma > 3L$ for each $i < s$,
- $|f_{i,j-1,0}(n)(\xi_i(n))|\frac{\epsilon'}{6\sqrt{r_{i,j}}|f_{i,j,0}(n)|} > 3L$ for each $i < s$ and $j < r_i$,
- $\left|\theta_{i,j,k} - \frac{f_{i,j,k}(n)(\xi_i(n))}{f_{i,j,0}(n)(\xi_i(n))}\right| < \frac{\epsilon'}{24\sqrt{r}L}$ for each $i < s$, $j < r_i$ and $k < r_{i,j}$, and
- $E \cap \{\mu_l(n) : l < t\} = \emptyset$.

Notice that A is cofinitr in A', and so $A \in p$.

We claim this A and this sequence $(\delta_n : n \in A)$ work.

Fix $n \in A$.

Let $(U_f : f \in \mathcal{F})$ be a family of arcs of length ϵ and let ρ be an arc function of length at least ϵ such that $\operatorname{supp}\rho \subseteq E$. We reindex the family of arcs as $(U_i : i < q)$ by means of $U_i := U_{f_i}$ for each $i < q$. Given $i < q$, let y_i be a real numbers such that $y_i + \mathbb{Z}$ is the center of U_i. Given $j < q$, let $z_j = \sum_{i<q} \mathcal{N}_{j,i}\frac{y_i}{N}$ and R_j be the arc of center z_j and length ϵ'. Since $\mathcal{N}$ is an integer matrix, we have that $z_j + \mathbb{Z} = \sum_{i<q} \mathcal{N}_{j,i}(\frac{y_i}{N} + \mathbb{Z})$. Then the arc $\sum_{j<q} \mathcal{M}_{i,j}R_j$ is a subset of U_i for each $i < q$.

Now we aim to apply Lemma 3.5.5. Set $\psi = \rho$, $\alpha = \frac{\delta_n}{N}$ and ϵ' in the place of ϵ. For $i < s$, $j < r_i$, $k < r_{i,j}$, we put $V_{i,j,k} = R_x$ if $f_{i,j,k} = v_x$ for some $x < q$, and for $l < t$ we put $W_j = R_x|$ if $g_l = v_x$ for some $x < q$.

Then there exists an arc function $\widetilde{\psi}_n$ such that

(A) $N\overline{\widetilde{\psi}_n(\beta)} \subseteq \rho(\beta)$ for each $\beta \in \operatorname{supp}\rho$,

(B) $\sum_{\beta\in\operatorname{supp} g_l(n)} g_l(n)(\beta)\widetilde{\psi}_n(\beta) \subseteq W_l$ for each $l < t$,

(C) $\sum_{\beta\in\operatorname{supp} f_{i,j,k}(n)} f_{i,j,k}(n)(\beta)\widetilde{\psi}_n(\beta) \subseteq V_{i,j,k}$ for each $i < s$, $j < r_i$ and $k < r_{i,j}$,

(D) $\delta(\widetilde{\psi}_n(\beta)) = \frac{\delta_n}{N}$ for each $\beta \in \operatorname{supp}\widetilde{\psi}_n$, and

(E) $\operatorname{supp}\widetilde{\psi}_n$ is equal to

$$\bigcup_{i<s,j<r_i,k<r_{i,j}} \operatorname{supp} f_{i,j,k}(n) \cup \bigcup_{l<t} \operatorname{supp} g_l(n) \cup E = \bigcup_{f\in\mathcal{F}} \operatorname{supp} f(n) \cup E.$$

Let $\psi_n := N\widetilde{\psi}_n$. By (A), $\psi_n \leq \rho$. By (E) and (D), $\operatorname{supp}\psi_n = \bigcup_{f\in\mathcal{F}} \operatorname{supp} f(n) \cup E$ and for each $\beta \in \operatorname{supp}\psi_n$, we have $\delta(\psi_n(\beta)) = \delta_n$. Let $S := \operatorname{supp}\psi_n$. Now notice that given $u_i \in \mathcal{F}$, we have:

$$\begin{aligned}\sum_{\mu\in\operatorname{supp} u_i} u_i(n)(\mu)\psi_n(\mu) &= \sum_{\mu\in S} u_i(n)(\mu)N\widetilde{\psi}_n(\mu)\\ &= \sum_{\mu\in S}\left(\sum_{j<q}\mathcal{M}_{i,j}v_j(n)(\mu)\right)\widetilde{\psi}_n(\mu)\\ &= \sum_{j<q}\mathcal{M}_{i,j}\left(\sum_{\mu\in S}v_j(n)(\mu)\widetilde{\psi}_n(\mu)\right).\end{aligned}$$

Then, by (B), (C) and the definitions of the W_l's and the $V_{i,j,k}$'s:

$$\sum_{\mu\in\operatorname{supp} u_i} u_i(n)(\mu)\psi_n(\mu) = \sum_{\mu\in S} u_i(n)(\mu)N\widetilde{\psi}_n(\mu) \subseteq \sum_{j<q}\mathcal{M}_{i,j}R_j \subseteq U_i.$$

As intended.

3.6 Final comments

This method of obtaining countably compact free Abelian groups came from the technique developed to construct countably compact groups without nontrivial convergent sequences. It is not known whether there is an easier method to produce countably compact group topologies on free Abelian groups if the resulting topology is allowed to have nontrivial convergent sequences.

In fact, even in the construction of a countably compact group topology *with* nontrivial convergent sequences in nontorsion groups, a modification of the technique to produce countably compact groups without nontrivial convergent sequences; see Matheus Koveroff Bellini *et al.*, 2019 and A. C. Boero, Castro-Pereira, *et al.*, 2019.

The first examples of countably compact groups without nontrivial convergent sequences were obtained by Hajnal and Juhász (Hajnal and Juhász, 1976) under CH. E. van Douwen (Douwen, 1980a) obtained an example from MA and asked for a ZFC example. Other examples were obtained using $\mathrm{MA}_{\mathrm{countable}}$ (Koszmider *et al.*, 2000), a selective ultrafilter (Garcia-Ferreira *et al.*, 2005) and in the random reals model (Szeptycki and A. H. Tomita, 2009). Only recently, Hrušák, van Mill, Shelah and Ramos obtained an example in ZFC (Hrušák *et al.*, 2021).

This motivates the following questions in ZFC:

Question 3.6.1. Are there large countably compact group without nontrivial convergent sequences in ZFC? Is there an example of cardinality $2^{\mathfrak{c}}$?

The example of Hrušák et. al. has size $\mathfrak{c}$ and it is not clear whether their construction could yield larger examples.

Question 3.6.2. Is there a countably compact free Abelian group in ZFC? A countably compact free Abelian group without nontrivial convergent sequences in ZFC?

It is still open whether there exists a torsion-free group in ZFC that admits a countably compact group topology without nontrivial convergent sequences. If such example exists then there is a countably compact group topology without nontrivial convergent sequences in the free Abelian group of cardinality $\mathfrak{c}$ (see A. Tomita, 2005 or A. H. Tomita, 2019).

Question 3.6.3. Is there a both-sided cancellative semigroup that is not a group that admits a countably compact semigroup topology (a *Wallace semigroup*) in ZFC?

The known examples were obtained in Robbie and Svetlichny, 1996 under CH, in A. H. Tomita, 1996 under $\mathrm{MA}_{\mathrm{countable}}$, in Madariaga-Garcia and A. H. Tomita, 2007 from $\mathfrak{c}$ incomparable selective ultrafilters and in A. C. Boero, Castro-Pereira, *et al.*, 2019 from one selective ultrafilter. The last two use the known fact that a free Abelian group without nontrivial convergent sequences contains a Wallace semigroup, which was used in Robbie and Svetlichny, 1996. The example in A. H. Tomita, 1996 was a modification of Hart and Mill, 1991.

Chapter 4

On a $\mathcal{U}$-compact topology for a torsion-free group whose cardinality has countable cofinality

This chapter gives a partial answer to a question posed in Chapter 2, using forcing to obtain a model where λ is a cardinal whose cofinality is ω and such that $\mathbb{Q}^{(\lambda)}$ has a $\mathcal{U}$-compact Hausdorff group topology without non-trivial convergent sequences, where $\mathcal{U}$ is a given selective ultrafilter.

4.1 Notation

We shall fix throughout this article a cardinal λ and a selective ultrafilter $\mathcal{U}$.

As usual, given $a \in \mathbb{Q}^\lambda$, its *support* is $\operatorname{supp} a = \{\xi \in \lambda : a_\xi \neq 0\}$.

Let G be the Abelian group $\mathbb{Q}^{(\lambda)} = \{a \in \mathbb{Q}^\lambda : \operatorname{supp} a \text{ is finite}\}$ (considering coordinate-wise addition as its operation).

If $E \subseteq \lambda$, we do a standard abuse of notation and consider $\mathbb{Q}^{(E)} = \{a \in G : \operatorname{supp} a \subseteq E\}$.

Given $\mu \in \lambda$, we define $\chi_\mu \in G$ by $\chi_\mu(\mu) = 1$ and $\chi_\mu(\beta) = 0$ for all $\beta \in \lambda$, $\beta \neq \mu$. Now, given $\zeta : \omega \to \lambda$, we define $\chi_\zeta : \omega \to G$ by $\chi_\zeta(n) = \chi_{\zeta(n)}$ for each $n \in \omega$. And given a $\mu \in \lambda$, we define $\vec{\mu}$ as the constant sequence of value μ. Thus, $\chi_{\vec{\mu}}$ is the constant sequence of value χ_μ (this will appear often in our work).

Since $\mathcal{U}$ is an ultrafilter on ω, the *ultrapower* of G by $\mathcal{U}$, denoted $\operatorname{Ult}_{\mathcal{U}}(G)$, is the quotient of the set G^ω by the following equivalence relation: $g \sim h$ if and only if $\{n \in \omega : g(n) = h(n)\} \in \mathcal{U}$. We will make frequent use of the fact that $\operatorname{Ult}_{\mathcal{U}}(G)$ is a $\mathbb{Q}$-vector space with all operations defined naturally (that is via representatives). If $g \in G^\omega$, we will denote its class in $\operatorname{Ult}_{\mathcal{U}}(G)$ by $[g]_{\mathcal{U}}$.

We now fix $\mathcal{H} \subseteq G^\omega$ such that $([g]_\mathcal{U} : g \in \mathcal{H}) \cup ([\chi_{\vec{\mu}}]_\mathcal{U} : \mu < \lambda)$ is a $\mathbb{Q}$-basis for $\mathrm{Ult}_\mathcal{U}(G)$.

4.2 Forcing poset

Definition 4.2.1. We define $\mathcal{P}$ as the set of the tuples of the form $(E, \alpha, \mathcal{G}, \xi, \phi)$ such that:

- E is a countable subset of λ containing ω,
- $\alpha < \mathfrak{c}$,
- $\mathcal{G}$ is a countable subset of $\mathcal{H}$,
- $\xi = (\xi_g : g \in \mathcal{G})$ is a family of elements of $\mathfrak{c} \cap E$,
- $\phi : \mathbb{Q}^{(E)} \to \mathbb{T}^\alpha$ is a homomorphism,
- $\mathcal{U} - \lim(\phi \circ g) = \phi(\chi_{\xi_g})$ for each $g \in \mathcal{G}$,

We define $(E, \alpha, \mathcal{G}, \xi, \phi) \leq (E', \alpha', \mathcal{G}', \xi', \phi')$ if:

1. $E \supseteq E'$,
2. $\alpha \geq \alpha'$,
3. $\mathcal{G} \supseteq \mathcal{G}'$,
4. $\xi_g = \xi'_g$ for each $g \in \mathcal{G}'$, and
5. for every $\xi < \alpha'$ and $a \in \mathbb{Q}^{(E')}$, $\phi(a)(\xi) = \phi'(a)(\xi)$.

Given $p \in \mathcal{P}$, we may denote its components by E^p, α^p, $\mathcal{G}^p$, ξ^p and ϕ^p.

If H is a generic filter over $\mathcal{P}$ then the generic homomorphism defined by H is the mapping Φ of domain $\bigcup\{\mathrm{dom}(\phi^p) : p \in H\}$ into $\mathbb{T}^\mathfrak{c}$ defined by $\Phi(\cdot)(\xi) = \bigcup\{\phi^p(\cdot)(\xi) : p \in H, \xi < \alpha^p\}$. In other words, if $p \in H$, $a \in \mathbb{Q}^{(E^p)}$ and $\xi < \alpha_p$, then $\Phi(a)(\xi) = \phi^p(a)(\xi)$. □

Naturally, we must assure that such generic homomorphisms are well-defined and into $\mathbb{T}^\mathfrak{c}$. We will do so by showing that, assuming CH in the ground model, this forcing notion is ω_1-closed and has the ω_2-chain condition, and thus preserves cardinals and $\mathfrak{c}$.

Proposition 4.2.2. Let $e \in G\backslash\{0\}$. Then the set $C_e = \{p \in \mathcal{P} : e \in \mathbb{Q}^{(E^p)}$ and $\phi^p(e) \neq 0\}$ is open and dense in $\mathcal{P}$.

Proof. Openness: suppose $p \in C_e$ and $q \leq p$. Then since $\phi^p(e) \neq 0$, for some $\beta < \alpha^p$, $\phi^p(e)(\beta) \neq 0$. By (2), $\beta < \alpha^p \leq \alpha^q$, and by (1) $E^p \subseteq E^q$. It follows that $e \in \mathbb{Q}^{(E^q)}$ and, by (6), $\phi^q(e)(\beta) = \phi^p(e)(\beta) \neq 0$. Thus, $q \in C_e$ as well.

Denseness: now let $p \in \mathcal{P}$ be arbitrary. We shall produce a $q \leq p$ such that $q \in C_e$. First, take any $d_0, d_1 \in G\backslash\{0\}$ such that $\mathrm{supp}\, e$, $\mathrm{supp}\, d_0$ and $\mathrm{supp}\, d_1$ are pariwise disjoint. Now take C a countable subset of λ such that $\omega \cup \mathrm{supp}\, e \cup \mathrm{supp}\, d_0 \cup \mathrm{supp}\, d_1 \cup \bigcup_{g \in \mathcal{G}, n \in \omega} \mathrm{supp}\, g(n) \subseteq C$.

Lemma 2.3.4 tells us that there exists a homomorphism $\rho : \mathbb{Q}^{(C)} \to \mathbb{T}$ such that:

- $\rho(e) \neq 0$ and $\rho(d_0) \neq \rho(d_1)$;

- $\mathcal{U}-\lim(\rho\circ(\frac{1}{N}g)) = \rho(\frac{1}{N}\chi_{\xi_g})$, for every $g\in\mathcal{G}$ and $N\in\omega$.

Define now $E^q := E^p\cup C$ and extend $\phi^p : \mathbb{Q}^{(E^p)}\to\mathbb{T}^{\alpha^p}$ to a $\varphi:\mathbb{Q}^{(E^q)}\to\mathbb{T}^{\alpha^p}$ using the divisibility of the codomain. Also extend ρ to a $\psi:\mathbb{Q}^{(E^q)}\to\mathbb{T}$. Define then $\alpha^q=\alpha^p+1$, $\mathcal{G}^q=\mathcal{G}^p$, $\xi^q=\xi^p$, and $\phi^q=\varphi^\frown\psi$.

It follows that $q\in C_e$ and $q\leq p$. $\square$

Proposition 4.2.3. Let $\alpha<\mathfrak{c}$. Then the set $\mathcal{A}_\alpha=\{p\in\mathcal{P} : \alpha^p>\alpha\}$ is open and dense in $\mathcal{P}$.

Proof. Openness: suppose $p\in\mathcal{A}_\alpha$ and $q\leq p$. We have that $\alpha^p>\alpha$ and $\alpha^q\geq\alpha^p$, and so $q\in\mathcal{A}_\alpha$.

Denseness: let $p\in\mathcal{P}$. If $\alpha^p>\alpha$, then $p\in\mathcal{A}_\alpha$. Suppose now that $\alpha^p\leq\alpha$. We define q as follows: $E^q=E^p$, $\alpha^q=\alpha+1$, $\mathcal{G}^q=\mathcal{G}^p$, $\xi^q=\xi^p$, and $\phi^q=\phi^{p\frown}\psi$, where $\psi:\mathbb{Q}^{(E^q)}\to\mathbb{T}^{[\alpha^p,\alpha]}$ is the zero-homomorphism.

It follows that $q\in\mathcal{A}_\alpha$ and $q\leq p$. $\square$

Proposition 4.2.4. Let $g\in\mathcal{H}$. Then the set $\mathcal{S}_g:=\{p\in\mathcal{P} : g\in\mathcal{G}^p\}$ is open and dense in $\mathcal{P}$.

Proof. Openness: suppose $p\in\mathcal{S}_g$ and $q\leq p$. Since $g\in\mathcal{G}^p$ and $\mathcal{G}^p\subseteq\mathcal{G}^q$, it follows that $g\in\mathcal{G}^q$ and thus $q\in\mathcal{S}_g$.

Denseness: Let $p\in\mathcal{P}$ be given. First, take E a countable subset of λ such that $\omega\cup E^p\subseteq E$ and $\bigcup_{k\in\omega}\operatorname{supp} g(k)\subseteq E$. Take any $\mu\in\mathfrak{c}\setminus E$.

Define now: $E^q=E\cup\{\mu\}$, $\alpha^q=\alpha^p$, $\mathcal{G}^q=\mathcal{G}^p\cup\{g\}$, and $\xi^q=\xi^p\cup\{(g,\mu)\}$.

Extend $\phi^p : \mathbb{Q}^{(E^p)}\to\mathbb{T}^{\alpha^p}$ to a $\psi:\mathbb{Q}^{(E)}\to\mathbb{T}^{\alpha^p}$ using divisibility. Now, since $\mathbb{T}^{\alpha^p}$ is a compact space, let $z=\mathcal{U}-\lim(\psi\circ g)$. Define then $\phi^q:\mathbb{Q}^{(E^q)}\to\mathbb{T}^{\alpha^q}$ as an extension of ψ, declaring that $\phi^q(\chi_\mu)=z$.

It follows that $q=(E^q,\alpha^q,\mathcal{G}^q,\xi^q,\phi^q)\in\mathcal{S}_g$ and $q\leq p$. $\square$

Proposition 4.2.5. $\mathcal{P}$ is ω_1-closed.

Proof. Let $(p_t : t\in\omega)$ be a decreasing sequence in $\mathcal{P}$. We shall produce an $r\in\mathcal{P}$ such that $r\leq p_t$ for all $t\in\omega$.

Denote now $p_t=(E^t,\alpha^t,\mathcal{G}^t,\xi^t,\phi^t)$.

Define $E^r=\bigcup_{t\in\omega}E^t$, $\alpha^r=\sup_{t\in\omega}\alpha^t$, $\mathcal{G}^r=\bigcup_{t\in\omega}\mathcal{G}^t$ and given $g\in\mathcal{G}^r$, $\xi^r_g=\xi^t_g$ for any t such that $g\in\mathcal{G}^t$ (this does not depend on such t's).

Lastly, given $a\in\mathbb{Q}^{(E^r)}=\bigcup_{t\in\omega}\mathbb{Q}^{(E^t)}$ and $\xi<\alpha^r$, define $\phi^r(a)(\xi)=\phi^t(a)(\xi)$ for any t such that $\xi<\alpha^t$ (again, this assignment does not depend on such t's). $\square$

We will now need a technical Lemma in order to guarantee the divergence of non-trivial sequences.

Lemma 4.2.6. Let $\mathcal{G} \subseteq \mathcal{H}$ be countable and $B \in \mathcal{U}$. Let $\mathcal{H}'$ be a finite subset of $\mathcal{G}$ and $(r_g : g \in \mathcal{H}')$ a family of rational numbers. Let $E \subseteq \lambda$ countably infinite such that $\omega \bigcup \cup_{g\in\mathcal{G},n\in\omega} \operatorname{supp} g(n) \subseteq E$. Let $(\xi_g : g \in \mathcal{G})$ be a family in $\mathfrak{c} \cap E$.

Then there exists a homomorphism $\phi : \mathbb{Q}^{(E)} \longrightarrow \mathbb{T}$ such that

a) $\mathcal{U}$-$\lim(\phi(\frac{1}{N}g)) = \phi(\frac{1}{N}\chi_{\xi_g})$, for each $g \in \mathcal{G}$ and $N \in \omega$, and

b) $(\phi(\sum_{g\in\mathcal{H}'} r_g g(n)) : n \in B)$ does not converge.

Proof. Let $B' \in \mathcal{U}$ be a subset of B such that $(\sum_{g\in\mathcal{H}'} r_g g(n) : n \in B')$ is a one-to-one sequence, which is possible since the g's are linearly independent mod $\mathcal{U}$ with the constant sequences and by the selectiveness of $\mathcal{U}$.

Let $\mathbb{A}$ be an almost disjoint family on B' of cardinality $\mathfrak{c}$ and $h_x : \omega \longrightarrow \{\sum_{g\in\mathcal{H}'} r_g g(n) : n \in x\}$ be a bijection for each $x \in \mathbb{A}$.

Claim: There exist $x_0, x_1 \in \mathbb{A}$ such that $\{[g]_\mathcal{U} : g \in \mathcal{G}\} \cup \{[\chi_{\vec{\mu}}]_\mathcal{U} : \mu \in \lambda\} \cup \{[h_{x_0}]_\mathcal{U}, [h_{x_1}]_\mathcal{U}\}$ is a linearly independent subset.

Proof of the claim: Given $x_0, x_1 \in \mathbb{A}$, notice that $h_{x_0}(n) \neq h_{x_1}(n)$ for all but a finite numbers of n's, so $[h_{x_0}]_\mathcal{U} \neq [h_{x_1}]_\mathcal{U}$. Since $\mathbb{Q}$ is countable, it follows that $\langle [h_x]_\mathcal{U} : x \in \mathbb{A}\rangle$ has cardinality $\mathfrak{c}$, so there is $J \subseteq \mathbb{A}$ such that $|J| = \mathfrak{c}$ and that $([h_x]_\mathcal{U} : x \in J)$ is linearly independent. Now notice that $\langle \mathcal{G}\rangle \oplus \langle \chi_{\vec{\xi}} : \xi \in E\rangle$ is countable, so there exist $x_0, x_1 \in J$ such that $\{[g]_\mathcal{U} : g \in \mathcal{G}\} \cup \{[\chi_{\vec{\mu}}]_\mathcal{U} : \mu \in E\} \cup \{[h_{x_0}]_\mathcal{U}, [h_{x_1}]_\mathcal{U}\}$ is linearly independent. Since all the supports of these elements are contained in E, it is straightforward to see that $\{[g]_\mathcal{U} : g \in \mathcal{G}\} \cup \{[\chi_{\vec{\mu}}]_\mathcal{U} : \mu \in \lambda\} \cup \{[h_{x_0}]_\mathcal{U}, [h_{x_1}]_\mathcal{U}\}$ is linearly independent. □

We will now apply Lemma 2.3.4 with $\kappa = \lambda$, $p = \mathcal{U}$, $\mathcal{F} = \{g : g \in \mathcal{G}\} \cup \{h_{x_0}, h_{x_1}\}$, $\xi_g = \xi_g$ for $g \in \mathcal{G}$, $\xi_{h_{x_0}} = 0$, $\xi_{h_{x_1}} = 1$, $d_0 = \chi_0$, $d_1 = \chi_1$, $d = \chi_2$ and $C = E$. Take the homomorphism from $\mathbb{Q}^{(E)}$ to $\mathbb{T}$ given by the conclusion of the Lemma and call it ψ.

Clearly condition a) of this Lemma is satisfied.

Furthermore, $(\psi(h_{x_i}(k)) : k \in \omega)$ has $\psi(\chi_i)$ as an accumulation point for $i < 2$. Since these sequences are reorderings of a subsequence of $(\psi(\sum_{g\in\mathcal{H}'} r_g g(n)) : n \in B)$ and $\phi(\chi_0) \neq \phi(\chi_1)$, it follows that b) is satisfied. □

Proposition 4.2.7. Let $h \in G^\omega$ be a one-to-one sequence. Then the set $\mathcal{E}_h := \{p \in \mathcal{P} :$ there is $\beta < \alpha^p$ such that $(\phi^p(h(n))(\beta) : n \in \omega)$ does not converge$\}$

Proof. Openness: Suppose $p \in \mathcal{E}_h$ and $q \leq p$. Then for some $\beta < \alpha^p$, $(\phi^p(h(n))(\beta) : n \in \omega)$ does not converge. By (2), $\beta < \alpha^p \leq \alpha^q$, and by (1) $E^p \subseteq E^q$. It follows that $h(n) \in \mathbb{Q}^{(E^q)}$ for all $n \in \omega$ and, by (6), $\phi^q(h(n))(\beta) = \phi^p(h(n))(\beta)$ for all $n \in \omega$. Since $(\phi^p(h(n))(\beta) : n \in \omega)$ does not converge, then $(\phi^q(h(n))(\beta) : n \in \omega)$ does not converge. Thus, $q \in \mathcal{E}_h$ as well.

Denseness: Let $p \in \mathcal{P}$ be given. Take $\mathcal{H}' \subseteq \mathcal{H}$ finite, $\mu_0, \dots, \mu_{m-1} \in \lambda$ and $(r_g : g \in \mathcal{H}')$ and $(s_{\mu_j} : j < m)$ families of rational numbers such that $[h]_{\mathcal{U}} = \sum_{g \in \mathcal{H}'} r_g [g]_{\mathcal{U}} + \sum_{j<m} s_{\mu_j} [\chi_{\vec{\mu_j}}]_{\mathcal{U}}$.

We will define a decreasing sequence of conditions $(p_i : i \in \omega)$. First, we obtain $p_0 \leq p$ by applying Proposition 4.2.4 $|\mathcal{H}'|$ times in order to guarantee that $\mathcal{H}' \subseteq \mathcal{G}^{p_0}$.

Take an enumeration $(\gamma_k : k \geq 1)$ of $\cup_{g \in \mathcal{H}', n \in \omega} \operatorname{supp} g(n) \bigcup \cup_{n \in \omega} \operatorname{supp} h(n) \bigcup \{\mu_j : j < m\}$.

Apply Proposition 4.2.2 using $e = \chi_{\gamma_1}$ and obtain $p_1 \leq p_0$ such that $\chi_{\gamma_1} \in \mathbb{Q}^{(E^{p_1})}$, which implies $\gamma_1 \in E^{p_1}$.

Now recursively apply Proposition 4.2.2 for each $i > 1$ using $e = \chi_{\gamma_i}$ and obtain $p_i \leq p_{i-1}$ such that $\chi_{\gamma_i} \in \mathbb{Q}^{(E^{p_i})}$, which implies $\gamma_i \in E^{p_i}$.

Applying Proposition 4.2.5, we obtain a $p_\omega \leq p_i$ for all $i \in \omega$.

We will now use Lemma 4.2.6 with $\mathcal{G} = \mathcal{G}^{p_\omega}$, $\mathcal{H}' = \mathcal{H}'$, $r_g = r_g$ for $g \in \mathcal{H}'$, $E = E^{p_\omega}$, $\xi_g = \xi_g^{p_\omega}$ for $g \in \mathcal{G}^{p_\omega}$ and $B \in \mathcal{U}$ such that $h(n) = \sum_{g \in \mathcal{H}'} r_g g(n) + \sum_{j<m} s_{\mu_j} \chi_{\mu_j}$ for all $n \in B$. We thus obtain a $\psi : \mathbb{Q}^{(E^{p_\omega})} \to \mathbb{T}$ such that

a) $\mathcal{U}\text{-}\lim(\psi(\frac{1}{N} g)) = \psi(\frac{1}{N} \chi_{\xi_g})$, for each $g \in \mathcal{G}$ and $N \in \omega$, and

b) $(\psi(\sum_{g \in \mathcal{H}'} r_g g(n)) : n \in B)$ does not converge.

We define now a $q \leq p_\omega$. Define $E^q = E^{p_\omega}$, $\alpha^q = \alpha^{p_\omega} + 1$, $\mathcal{G}^q = \mathcal{G}^{p_\omega}$, $\xi^q = \xi^{p_\omega}$ and $\phi^q = \phi^{p_\omega} {}^\frown \psi$.

Since $(\psi(\sum_{g \in \mathcal{H}'} r_g g(n)) : n \in B)$ does not converge and $\sum_{j<m} s_{\mu_j} \chi_{\mu_j}$ is constant, it follows that $(\psi(h(n)) : n \in B)$ does not converge, and so $(\psi(h(n)) : n \in \omega)$ does not converge.

Let $\beta = \alpha^p$. The definition $\phi^q = \phi^{p_\omega} {}^\frown \psi$ means that for all $x \in \mathbb{Q}^{(E^q)}$, $\phi^q(x)(\beta) = \psi(x)$. Thus, $(\psi(h(n)) : n \in \omega)$ does not converge means that $(\phi^q(h(n))(\beta) : n \in \omega)$ does not converge, proving that $q \in \mathcal{E}_h$.

□

Proposition 4.2.8. Assume CH. Then the partial order $\mathcal{P}$ has the ω_2-chain condition.

Proof. Since under CH, $\mathfrak{c}^+ = \omega_2$, we will show that $\mathcal{P}$ has the $\mathfrak{c}^+$-c.c.. So let $\mathcal{Q} \subseteq \mathcal{P}$ of cardinality $\mathfrak{c}^+$. We will show that $\mathcal{Q}$ has a subset consisting of $\mathfrak{c}^+$ pairwise compatible elements.

First, take a $\mathcal{Q}_0 \subseteq \mathcal{Q}$ of cardinality $\mathfrak{c}^+$ and an $\alpha < \mathfrak{c}$ such that $\alpha_q = \alpha$ for each $q \in \mathcal{Q}_0$.

Using the $\triangle$-system Lemma, take a $\mathcal{Q}_1 \subseteq \mathcal{Q}_0$ of cardinality $\mathfrak{c}^+$ such that $\{E^q : q \in \mathcal{Q}_1\}$ is a $\triangle$-system of root $\tilde{E}$. Now, using CH, we have that $\left|(\mathbb{T}^\alpha)^{\mathbb{Q}^{(\tilde{E})}}\right| = \mathfrak{c}$, and thus we may take a $\mathcal{Q}_2 \subseteq \mathcal{Q}_1$ of cardinality $\mathfrak{c}^+$ such that $\phi^q|_{\mathbb{Q}^{(\tilde{E})}} = \phi^p|_{\mathbb{Q}^{(\tilde{E})}}$ for all $q, p \in \mathcal{Q}_2$.

Using the $\triangle$-system Lemma again, take $\mathcal{Q}_3 \subseteq \mathcal{Q}_2$ of cardinality $\mathfrak{c}^+$ such that $\{\mathcal{G}^q : q \in \mathcal{Q}_3\}$ is a $\triangle$-system of root $\tilde{\mathcal{G}}$. Now since for each $q \in \mathcal{Q}_3$, $(\xi^q_g : g \in \tilde{\mathcal{G}}) \in \mathfrak{c}^{\tilde{\mathcal{G}}}$ and $|\mathfrak{c}^{\tilde{\mathcal{G}}}| = \mathfrak{c}$, take $\mathcal{Q}_4 \subseteq \mathcal{Q}_3$ of cardinality $\mathfrak{c}^+$ such that for all $q, p \in \mathcal{Q}_4$ and each $g \in \tilde{\mathcal{G}}$, $\xi^q_g = \xi^p_g$.

A common extension to $q, p \in \mathcal{Q}_4$ is an r defined as follows: $E^r = E^q \cup E^p$; $\alpha^r = \alpha^q = \alpha^p$; $\mathcal{G}^r = \mathcal{G}^q \cup \mathcal{G}^p$; $\xi^r = \xi^q \cup \xi^p$.

To define ϕ^r, notice that $\mathbb{Q}^{(E^r)} = \mathbb{Q}^{(E^q \setminus \tilde{E})} \oplus \mathbb{Q}^{(\tilde{E})} \oplus \mathbb{Q}^{(E^p \setminus \tilde{E})}$. Then, let $\pi_0 : \mathbb{Q}^{(E^r)} \to \mathbb{Q}^{(E^q \setminus \tilde{E})}$, $\pi_1 : \mathbb{Q}^{(E^r)} \to \mathbb{Q}^{(\tilde{E})}$ and $\pi_2 : \mathbb{Q}^{(E^r)} \to \mathbb{Q}^{(E^p \setminus \tilde{E})}$ be the projections. Define $\phi^r = \phi^q \circ \pi_0 + \phi^q \circ \pi_1 + \phi^p \circ \pi_2 = \phi^q \circ \pi_0 + \phi^p \circ \pi_1 + \phi^p \circ \pi_2$ and we are done. □

Theorem 4.2.9. Assume CH. Then the forcing notion $\mathcal{P}$ preserves cofinalities and cardinals, preserves $\mathfrak{c}$ and does not add reals. Given H a $\mathcal{P}$-generic filter, the associated Φ (as in 4.2.1) is a well-defined monomorphism from G to $\mathbb{T}^{\mathfrak{c}}$. Also, given any $g \in \mathcal{H}$, there exists a $\xi \in \mathfrak{c}$ such that $\mathcal{U} - \lim(\Phi \circ g) = \Phi(\chi_\xi)$. Furthermore, given any one-to-one sequence $h \in G^\omega$, $\Phi \circ h$ does not converge.

Proof. By Propositions 4.2.5 and 4.2.8, $\mathcal{P}$ preserves cofinalities (and therefore cardinals), does not add reals and preserves $\mathfrak{c}$. Note that since being a basis for $\mathrm{Ult}_{\mathcal{U}}(G)$ is absolute for transitive models of ZFC, $\mathcal{H}$ is still a basis for $\mathrm{Ult}_{\mathcal{U}}(G)$ in the extension.

Let H be a $\mathcal{P}$-generic filter and let Φ be its associated homomorphism.

First, let us see that $\Phi : G \to \mathbb{T}^{\mathfrak{c}}$ is well-defined. Let $q, p \in H$ and suppose $\xi < \min\{\alpha^q, \alpha^p\}$ and $e \in \mathbb{Q}^{(E^q \cap E^q)}$. We must see that $\phi^q(e)(\xi) = \phi^p(e)(\xi)$. Take $r \in H$ such that $r \leq q, p$. Then $\xi < \alpha^r$ and $e \in \mathbb{Q}^{(E^r)}$, and by item (5) of Definition 4.2.1, $\phi^q(e)(\xi) = \phi^r(e)(\xi) = \phi^p(e)(\xi)$.

Now let $\alpha < \mathfrak{c}$ and $e \in \mathbb{Q}^{(\lambda)}$ such that $e \neq 0$. Since C_e and $\mathcal{A}_\alpha$ are open and dense, let $p \in H$ such that $e \in \mathbb{Q}^{(E^p)}$, $\phi^p(e) \neq 0$ and $\alpha^p > \alpha$. Since $\phi^p(e) \neq 0$, there is a $\xi \leq \alpha^p$ such that $\phi^p(e)(\xi) \neq 0$, and therefore $\Phi(e)(\xi) \neq 0$, so that $\Phi(e) \neq 0$. And since $\alpha \subseteq \alpha^p \subseteq \operatorname{dom} \Phi(e) \subseteq \mathfrak{c}$, and α was arbitrary, it follows that $\operatorname{dom} \Phi(e) = \mathfrak{c}$.

We have thus seen that the domain of Φ is $\mathbb{Q}^{(\lambda)}$, the codomain is $\mathbb{T}^{\mathfrak{c}}$ and that Φ is injective.

Now we see that Φ is a homomorphism. Let $e, e' \in \mathbb{Q}^{(\lambda)}$. Since C_e, $C_{e'}$ and $C_{e+e'}$ are dense and open, take $p \in H$ such that $e, e', e + e' \in \mathbb{Q}^{(E^p)}$. We know that ϕ^p is a homomorphism, and so $\Phi(e + e') = \phi^p(e + e') = \phi^p(e) + \phi^p(e') = \Phi(e) + \Phi(e')$.

Let now $g \in \mathcal{H}$. Since $\mathcal{S}_g$ is open and dense, let $p \in H$ such that $g \in \mathcal{G}^p$. We have then that $\mathcal{U} - \lim(\phi^p \circ g) = \phi^p(\xi_g)$. Let us see that $\mathcal{U} - \lim(\Phi \circ g) = \Phi(\xi_g)$. Let F be a finite subset of $\mathfrak{c}$ and let $\alpha < \mathfrak{c}$ such that $F \subseteq \alpha$. Since $\mathcal{A}_\alpha$ is open and dense, and $p \in H$, let $q \in H$ such that $q \leq p$ and $\alpha^q > \alpha$. We have then that $\mathcal{U} - \lim(\pi_F \circ \phi^q \circ g) = (\pi_F \circ \phi^q)(\xi_g)$. Since $\pi_F \circ \phi^q = \pi_F \circ \Phi$ (due to $\alpha^q \supseteq F$), it follows that $\mathcal{U} - \lim(\pi_F \circ \Phi \circ g) = (\pi_F \circ \Phi)(\xi_g)$, as we sought for.

Finally, let $h \in G^\omega$ be a one-to-one sequence. Since $\mathcal{E}_h$ is open and dense, let $p \in H \cap \mathcal{E}_h$. Take then a $\beta < \alpha^p$ such that $(\phi^p(h(n))(\beta) : n \in \omega)$ does not converge. Since $\Phi(h(n))(\beta) = \phi^p(h(n))(\beta)$ for each $n \in \omega$, it follows that $(\Phi(h(n))(\beta) : n \in \omega)$ does not converge, which in turn implies that $(\Phi(h(n)) : n \in \omega) = \Phi \circ h$ does not converge. □

Theorem 4.2.10. It is consistent with ZFC that given λ a countably cofinal cardinal and $\mathcal{U}$ a selective ultrafilter, G can be endowed with a $\mathcal{U}$-compact Hausdorff group topology without non-trivial convergent sequences.

Proof. Consider the forcing model obtained via forcing with $\mathcal{P}$. We fix a generic monomorphism Φ as in 4.2.9. Since we have that Φ is a monomorphism from G to $\mathbb{T}^{\mathfrak{c}}$, then Φ induces a Hausdorff group topology in G such that for any $g \in \mathcal{H}$, there exists a $\xi \in \mathfrak{c}$ such that $\mathcal{U} - \lim g = \chi_\xi$. Fix one such ξ_g for each $g \in \mathcal{H}$.

Now let us see that such topology is indeed $\mathcal{U}$-compact. Let $f \in G^\omega$. Since $([g]_\mathcal{U} : g \in \mathcal{H}) \cup ([\chi_{\vec{\mu}}]_\mathcal{U} : \mu < \lambda)$ is a $\mathbb{Q}$-basis for $\mathrm{Ult}_\mathcal{U}(G)$, there exist families $(r_g : g \in \mathcal{H})$ and $(s_\mu : \mu < \lambda)$ of rational numbers, all but finitely many of which are 0, such that $[f]_\mathcal{U} = \sum_{g\in\mathcal{H}} r_g \cdot [g]_\mathcal{U} + \sum_{\mu<\lambda} s_\mu \cdot [\chi_{\vec{\mu}}]_\mathcal{U}$. It follows then that $\mathcal{U} - \lim f = \sum_{g\in\mathcal{H}} r_g \cdot (\mathcal{U} - \lim g) + \sum_{\mu<\lambda} s_\mu \cdot (\mathcal{U} - \lim \chi_{\vec{\mu}}) = \sum_{g\in\mathcal{H}} r_g \cdot \chi_{\xi_g} + \sum_{\mu<\lambda} s_\mu \cdot \chi_\mu$ and f has a $\mathcal{U}$-limit.

Finally, there are no non-trivial convergent sequences since for each $h \in G^\omega$ one-to-one, $\Phi \circ h$ does not converge, which means that in the induced topology on G, h does not converge. $\square$

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