Commutator characterization of pseudo-differential operators on compact manifolds

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Dissertação apresentada
ao
Instituto de Matemática e Estatística
da
Universidade de São Paulo
para
obtenção do título
de
Mestre em Ciências

Programa: Matemática
Orientador: Prof. Dr. Severino Toscano do Rêgo Melo

Durante o desenvolvimento deste trabalho o autor recebeu auxílio financeiro da CNPq

São Paulo, julho de 2020

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Esta é a versão original da dissertação elaborada pelo candidato Artur Henrique de Oliveira Andrade, tal como submetida à Comissão Julgadora.

Resumo

Andrade, A. H. O. **Caracterização via comutadores de operadores pseudo-diferenciais em variedades compactos**. 2020. 72 f. Dissertação (Mestrado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2020.

Nesse trabalho buscamos apresentar de forma original e detalhada a caracterização de operadores pseudo-diferenciais em variedades compactas via comutadores apresentada em M. Ruzhansky e V. Turunen [15]. A organização e apresentação também tem como objetivo ser autocontida e acessível a qualquer estudante com conhecimentos básicos em teoria da medida e integração como em G. Folland [5].

Palavras-chave: Operadores pseudo-diferenciais, comutadores, variedades compactas.

Abstract

Andrade, A. H. O. **Commutator characterization of pseudo-differential operators on compact manifolds**. 2020. 72 f. Thesis (Masters) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2020.

In this work we seek to present in an original and detailed way the commutator characterization of pseudo-differential operators on compact manifolds presented in M. Ruzhansky and V. Turunen [15]. The organization and presentation also aims to be self-contained and accessible to any student with basic knowledge in measure and integration theory such as in G. Folland [5].

Keywords: Pseudo-differential operators, commutators, compact manifolds.

Contents

Introduction

Roughly speaking, a pseudo-differential operator on $\mathbb{R}^n$ is an integral operator acting on a suitable space of functions and is of the form

$$Au(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, a(x,\xi)\, \widehat{u}(\xi)\, d\xi, \tag{1}$$

where $\widehat{u}$ is the Fourier transform of u. The function a in (1) is called the *symbol* of A.

It is easier to work with pseudo-differential operators when we impose some quantitative control over its symbol. We shall be concerned with the *symbol class* $S^m_{1,0}$ for some $m \in \mathbb{R}$. More specifically, $a \in S^m_{1,0}$ when $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is smooth on $\mathbb{R}^n \times \mathbb{R}^n$ and when for all α, β multiindices there exists a constant $A_{\alpha\beta} > 0$ such that

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leqslant A_{\alpha\beta}(1+|\xi|)^{m-|\alpha|} \tag{2}$$

holds for all and x, $\xi \in \mathbb{R}^n$. We then say that a pseudo-differential operator is of order (less or equal than) $m \in \mathbb{R}$ if its symbol belongs to $S^m_{1,0}$. We denote such class of pseudo-differential operators by $Op\, S^m$.

In addition to the symbol classes, there is another way of characterizing pseudo-differential operators on $\mathbb{R}^n$, namely, via commutators. In 1977 R. Beals [1] showed how one can characterize the pseudo-differential operators of a given class among the maps from $\mathcal{S}$ to $\mathcal{S}'$ in terms of boundedness of certain commutators.

In a closely related result, J. Dunau [4, Théorème 1] obtained a characterization of the topology in the space of pseudo-differential operators on compact manifolds in terms of the boundedness of their commutators with differential operators of order at most 1. Then in 1978 R. Coifman and Y. Meyer [2] stated and proved characterization of pseudo-differential operators of 0-order on compact manifolds via commutators with smooth vector fields. A more general statement for the characterization of pseudo-differential operators of any given order on compact manifolds was formulated by V. Turunen [20]. This later characterization is also presented in M. Ruzhansky and V. Turunen [15].

Although the work of M. Ruzhansky and V. Turunen [15] is a self-contained introduction to the theory of pseudo-differential operators, we believe that a more detailed presentation should be welcomed by those that had no previous exposition to the subject. Thus the main goal of this study is to provide an original and comprehensive presentation, starting from basic facts about Fourier transform and distribution theory, of the result presented by V. Turunen. More specifically, our presentation shall prove a slightly stronger version of the commutator characterization stated by V. Turunen.

The structure of this masters thesis is the following:

In chapter 1 we provide the necessary background on distribution theory, Fourier transform, Sobolev spaces and smooth manifolds. All the results are either accompanied by proof or a reference for the interested reader.

Chapter 2 contains the needed results from pseudo-differential operators on $\mathbb{R}^n$. We also provide a commutator characterization that is similar to that presented by R. Beals [1], but for a different class of pseudo-differential operators. All the results presented in this chapter are prerequisites for our commutator characterization on compact manifolds.

Chapter 3 starts with an introduction to distributions and Sobolev spaces over manifolds. We then introduce the definition and some basic properties of pseudo-differential operators on compact manifolds. Lastly, we state and prove our commutator characterization for pseudo-differential operators on compact manifolds.

Chapter 1

Background and prerequisites

The purpose of this first chapter is to present the necessary background that is needed for a complete understanding of the material.

In section 1.1 we recall some basic rules of calculus and set up notation that will be used through the rest of the thesis.

Section 1.2 gives some elements of the theory of topological vector spaces and distributions on open sets of the Euclidean space. The content and presentation are mostly based on G. Grubb [8].

In section 1.3 we discuss the Fourier transform and some of its properties. Detailed proofs and more properties may be found on L. Grafakos [6].

Section 1.4 covers Sobolev spaces on the Euclidean space and their local versions for open subsets. It contains results that may be found on G. Grubb [8] or on F. Trèves [19].

Finally, in section 1.5 we present some basic ingredients on the theory of smooth manifolds. Its presentation is based on J. M. Lee [11] and F. Warner [21].

1.1 Notation

We denote by $\mathbb{Z}$ the integers, by $\mathbb{N}$ the positive integers and by $\mathbb{N}_0$ the nonnegative integers. We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{C}$ the set of complex numbers. $\mathbb{R}^n$ is the n-dimensional Euclidean space, with points $x = (x_1, \ldots, x_n)$ and distance $\operatorname{dist}(x, y) = |x - y|$, where $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. When $x, y \in \mathbb{R}^n$, we write $x \cdot y = \sum_{1 \leqslant i \leqslant n} x_i y_i$.

For a, $b \in \mathbb{R}$ with $a < b$ we denote

$$\begin{aligned}
\{t \in \mathbb{R} \,:\, a < t < b\} &= (a, b),\\
\{t \in \mathbb{R} \,:\, a \leqslant t < b\} &= [a, b),\\
\{t \in \mathbb{R} \,:\, a < t \leqslant b\} &= (a, b],\\
\{t \in \mathbb{R} \,:\, a \leqslant t \leqslant b\} &= [a, b].
\end{aligned}$$

If $x, y \in \mathbb{R}^n$, then $[x, y]$ is used to denote the line segment connecting x and y.

Differentiation of functions on $\mathbb{R}$ is denoted by ∂ or by ∂_x in order to emphasize that the differentiation is with respect to the x-variable. Partial differentiation of functions on $\mathbb{R}^n$ is indicated by

$$\frac{\partial}{\partial x_j} = \partial_{x_j}.$$

We will make use of the multiindex notation. A multiindex is a vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. Given $\alpha \in \mathbb{N}_0^n$ we define its length by $|\alpha| = \sum_{i=1}^n \alpha_i$. For $x \in \mathbb{R}^n$ we define

$$\begin{aligned}
\partial_x^\alpha &= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}},\\
x^\alpha &= x_1^{\alpha_1} \ldots x_n^{\alpha_n}.
\end{aligned}$$

We will also use the conventions

$$
\begin{aligned}
&\beta \leqslant \alpha \text{ means } \beta_j \leqslant \alpha_j \text{ for all } 1 \leqslant j \leqslant n,\\
&\alpha! = \alpha_1! \cdots \alpha_n!,\\
&\alpha \pm \beta = (\alpha_1 \pm \beta_1, \ldots, \alpha_n \pm \beta_n),\\
&\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.
\end{aligned}
$$

Constants appearing in inequalities will be usually denoted by a capital letter with an index as e.g. $C_\alpha > 0$ denotes a positive constant that depends on α.

When $M \subset V$ for some set V, we define the indicator function of M over V by

$$\mathbf{1}_M(x) = \begin{cases} 1 & \text{for when } x \in M, \\ 0 & \text{for when } x \in V \setminus M. \end{cases}$$

If f is any real-valued or complex-valued function on a topological space X, we denote the *support* of f by

$$\operatorname{supp} f := \overline{\{x \in X \,:\, f(x) \neq 0\}}.$$

A function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be of class C^N if it has continuous derivatives up order N and is said to be *smooth* or C^∞ if $f \in C^N$ for all $N \in \mathbb{N}$. Recall Taylor's formula and Leibniz's rule:

Theorem 1.1 (Taylor's formula)**.** *Let $f \in C^N(\mathbb{R}^n)$. Then for any $x, y \in \mathbb{R}^n$ we have*

$$f(x+y) = \sum_{|\alpha|<N} \frac{y^\alpha}{\alpha!} \partial^\alpha f(x) + \sum_{|\alpha|=N} \frac{N}{\alpha!} y^\alpha \int_0^1 (1-\theta)^{N-1}\, \partial^\alpha f(x+\theta y)\, d\theta.$$

Proof. X. Saint Raymond [16, Theorem 1.1, p. 2] □

Theorem 1.2 (Leibniz's rule)**.** *Let $\alpha \in \mathbb{N}_0^n$ and $\Omega \subset \mathbb{R}^n$ an open set. Then for any $f, g \in C^{|\alpha|}(\Omega)$ we have*

$$\partial^\alpha (fg) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f \, \partial^\beta g.$$

Proof. X. Saint Raymond [16, Theorem 1.2, p. 3]. □

Recall the following result about smooth functions with compact support.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be open and let $K \subset \Omega$ be compact. There exists a smooth function $\phi : \Omega \to \mathbb{R}$ such that $0 \leqslant \phi \leqslant 1$, $\phi \equiv 1$ on a neighbourhood of K, and $\operatorname{supp} \phi \subset \Omega$.*

Proof. G. Grubb [8, Theorem 2.13, p. 22] and its corollary. □

Let (X, d) be a metric space. For $x \in X$ and $r > 0$, we denote the *closed ball* and the *open ball* of radius r by

$$
\begin{aligned}
\{z \in \mathbb{R}^n \,:\, d(x,z) \leqslant r\} &= \mathbb{B}_d(x,r),\\
\{z \in \mathbb{R}^n \,:\, d(x,z) < r\} &= \mathbb{U}_d(x,r),
\end{aligned}
$$

respectively. When $X = \mathbb{R}^n$ we shall omit the subscript d .

Let V be a vector space over a scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $\Sigma \subset \mathbb{K}$. When X, Y are subsets of V, we define

$$
\begin{aligned}
X \pm Y &:= \{x \pm y \,:\, x \in X, y \in Y\},\\
\Sigma X &:= \{\alpha x \,:\, \alpha \in \Sigma, x \in X\}.
\end{aligned}
$$

In particular, for $x \in V$ and $\alpha \in \mathbb{K}$ we write

$$\{x\} \pm Y = x \pm Y,$$
$$\{\alpha\}X = \alpha X.$$

When X is a normed vector space we denote the norm on X by $\|\cdot\|_X$. When X, Y are normed spaces we denote the space of bounded linear maps from X to Y by $\mathcal{B}(X, Y)$.

1.2 Topological vector spaces, Function spaces and Distributions

When X, Y are topological spaces we endow $X \times Y$ with the usual product topology unless otherwise stated.

Definition 1.4. A *topological vector space* (T.V.S.) over a scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a vector space X provided with a topology τ having the following properties:

1. A set consisting of one point $\{x\}$ is closed.
2. The bilinear maps

$$X \times X \ni (x, y) \mapsto x + y \in X,$$
$$\mathbb{K} \times X \ni (\lambda, x) \mapsto \lambda x \in X,$$

   are continuous.

We here follow the terminology of G. Grubb [8] and W. Rudin [14] where condition 1 is included in the definition of a topological vector space. In this way, a T.V.S. is in particular a topological Hausdorff space (check W. Rudin [14, Theorem 1.12, p. 11]).

Let X be a topological vector space. We associate to each $a \in X$ and to each scalar $\lambda \in \mathbb{K}\setminus\{0\}$ the *translation operator* T_a and the *multiplication operator* M_λ, by the formulas

$$T_a(x) := x + a, \quad \text{for all } x \in X,$$
$$M_\lambda(x) := \lambda x \quad \text{for all } x \in X.$$

It is not hard to see that their inverses are T_{-a} and $M_{1/\lambda}$, respectively, and that these four maps are continuous. Hence each of them is a homeomorphism of X onto X.

This implies that the topology of a T.V.S. X is *translation invariant*, i.e., $U \in \tau \iff x_0 + U \in \tau$ for all $x_0 \in X$. The topology is therefore determined from the system of neighbourhoods of 0.

Definition 1.5. A set $Y \subset X$ is said to be

1. convex, if for any $y_1, y_2 \in Y$ and $t \in (0, 1)$ we have that $ty_1 + (1-t)y_2 \in Y$,
2. balanced, when $y \in Y$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leqslant 1$ imply $\lambda y \in Y$,
3. bounded (with respect to τ), when for every neighbourhood $U \ni 0$ there exists $t > 0$ such that $Y \subset tU$.

X is said to be *locally convex*, when it has a local basis of neighbourhoods at 0 consisting of convex sets.

X is said to be *metrizable*, when there exists a metric $d : X \times X \to [0, \infty)$ such that the topology on X is identical with the topology defined by this metric. A metric is said to be *translation invariant* when

$$d(x, y) = d(x + z, y + z) \quad \text{for all } x, y, z \in X.$$

A *Cauchy sequence* in a T.V.S. X is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that for any neighbourhood $U \ni 0$ there exists $N_U \in \mathbb{N}$ so that $x_n - x_m \in U$ for all $m, n \geqslant N_U$.

In a metric space (M, d), a *Cauchy sequence* is a sequence such that $d(x_n, x_m) \to 0$ in $\mathbb{R}$ as $n, m \to \infty$. In particular, if the topology of a T.V.S. is given by a *translation invariant* metric d, then those concepts of Cauchy sequences coincide.

A metric space is called *complete* when every Cauchy sequence is convergent. More generally we call a T.V.S. *sequentially complete*, when every Cauchy sequence is convergent.

Banach spaces and Hilbert spaces are examples of complete metrizable topological vector spaces.

It is possible to define some topological vector spaces through the use of *seminorms*; we shall now take a closer look at this method.

Definition 1.6. Let X be a vector space. A *seminorm* is a map $p : X \to [0, \infty)$ such that

1. $p(x + y) \leqslant p(x) + p(y) \quad \forall x, y \in X$,
2. $p(\lambda x) = |\lambda| p(x) \quad \forall \lambda \in \mathbb{K},\ \forall x \in X$.

A family $\{p_i\}_{i \in I}$ is called *separating*, when $p_i(x) = 0$ for all $i \in I$ imply $x = 0$.

Theorem 1.7. *Let X be a vector space and let $\{p_i\}_{i \in I}$ be a separating family of seminorms on X. Define the topology on X by taking, as a local basis $\mathcal{B}$ for the system of neighbourhoods at 0, the sets*

$$V(p_i; \varepsilon) := \{x \in X : p_i(x) < \varepsilon\}, \quad \varepsilon > 0,$$

together with their finite intersections

$$W(p_{i_1}, \ldots, p_{i_N}; \varepsilon_{i_1}, \ldots, \varepsilon_{i_N}) = \bigcap_{k=1}^{N} V(p_{i_k}; \varepsilon_{i_k})\,; \tag{1.1}$$

and letting a local basis for the system of neighbourhoods at each $x \in X$ consist of the translated sets $x + W(p_{i_1}, \ldots, p_{i_N}; \varepsilon_{i_1}, \ldots, \varepsilon_{i_N})$.

With this topology, X is a topological vector space. The seminorms $\{p_i\}_{i \in I}$ are continuous maps from X into $\mathbb{R}$. Moreover, a set $E \subset X$ is bounded if and only if for every $i \in I$ there exists a constant $C_i > 0$ such that $p_i(E) \leqslant C_i$ for every $x \in E$.

Proof. G. Grubb [8, Theorem B.5, p. 434]. □

Note that the sets defined in (1.1) are open, convex and balanced sets, and thus any T.V.S. endowed with a topology such as in the above theorem is locally convex.

Additionally, this theorem implies that a sequence $\{x_k\}_{k \in \mathbb{N}}$ converges to x in X if and only if for every $i \in I$ we have $p_i(x_k - x) \to 0$ as $k \to \infty$. Furthermore, a sequence $\{x_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in X if and only if for every $i \in I$ we have $p_i(x_n - x_m) \to 0$ as $n, m \to \infty$.

We also have the following characterization for the continuity of linear maps

Lemma 1.8. *Let X, Y be topological vector spaces with topologies given as in Theorem 1.7 by separating families of seminorms $\{p_i\}_{i \in I}$ and $\{q_j\}_{j \in J}$, respectively. Then we have the following:*

1º A linear functional $\Lambda : X \to \mathbb{K}$ is continuous if and only if there exist a constant $C > 0$ and $i_1, \ldots, i_N \in I$ such that

$$|\Lambda(x)| \leqslant C \sum_{k=1}^{N} p_{i_k}(x) \quad \forall x \in X.$$

2º A linear map $T : X \to Y$ is continuous if and only if for every $j \in J$, there are $i_1, \ldots, i_{N_j} \in I$ and a constant $C_j > 0$ such that

$$q_j(Tx) \leqslant C_j \sum_{k=1}^{N_j} p_{i_k}(x) \quad \forall x \in X.$$

Proof. G. Grubb [8, Lemma B.7 and Remark B.8, p. 436]. □

In particular, the above lemma implies that a continuous linear map $T : X \to Y$ takes bounded sets of X into bounded sets of Y. Moreover, if Z is another T.V.S. with topology given as in Theorem 1.7 by a separating family of seminorms $\{r_\ell\}_{\ell\in L}$, then a bilinear map $B : X \times Y \to Z$ is continuous if and only if for every $\ell \in L$, there are $i_1, \dots, i_{N_\ell} \in I$, and $j_1, \dots, j_{M_\ell} \in J$, and a constant $C_\ell > 0$ such that

$$r_\ell(B(x,y)) \leqslant C_\ell \sum_{\substack{1\leqslant k\leqslant N_\ell \\ 1\leqslant n\leqslant M_\ell}} p_{i_k}(x)\, q_{j_n}(y) \quad \forall x \in X, \forall y \in Y.$$

Definition 1.9. A T.V.S. is called a *Fréchet space*, when X is metrizable with a translation invariant metric, is complete and is locally convex.

We have the following characterization of Fréchet spaces

Theorem 1.10. *When X is a T.V.S. where the topology is defined by a countable separating family of seminorms $\{p_n\}_{n\in\mathbb{N}}$, then X is locally convex and metrizable, and the topology on X is determined by the invariant metric*

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}.$$

Moreover, if X is complete in this metric, then X is a Fréchet space.

Proof. G. Grubb [8, Theorem B.9, p. 437]. □

We now present some important examples of Fréchet spaces. Let $\Omega \subset \mathbb{R}^n$ be an open set and for each $j \in \mathbb{N}$ let K_j be the compact subset of Ω given by

$$K_j := \{x \in \Omega \,:\, x \in \mathbb{B}(0,j),\, \mathrm{dist}(x,\Omega^c) \geqslant 1/j\}. \tag{1.2}$$

Note that $\{K_j\}_{j\in\mathbb{N}}$ is an increasing sequence of compact sets such that $\bigcup_{j\in\mathbb{N}} \mathrm{int}(K_j) = \Omega$.

Let $1 \leqslant p \leqslant \infty$. Denote by $L^p_{loc}(\Omega)$ the space of all complex-valued functions on Ω whose restrictions to compact subsets of Ω are in L^p

$$L^p_{loc}(\Omega) := \{f \text{ is measurable } : \mathbf{1}_K f \in L^p(K), \text{ for all compact } K \subset \Omega\}.$$

This is a Fréchet space when provided with the family of seminorms

$$q_j(f) := \|\mathbf{1}_{K_j} f\|_{L^p(\Omega)}, \text{ for } j \in \mathbb{N}.$$

Moreover, one has $L^p_{loc}(\Omega) \subset L^q_{loc}(\Omega)$ for all $p > q$, with continuous injection.

We also have spaces of differentiable functions over open subsets of $\mathbb{R}^n$. Define the family of seminorms

$$\rho_{k,j}(f) := \sup\{\, |\partial^\alpha f(x)| : |\alpha| \leqslant k,\, x \in K_j\}, \text{ for } k \in \mathbb{N}_0,\, j \in \mathbb{N}.$$

Lemma 1.11. *1º For each $k \in \mathbb{N}_0$, $C^k(\Omega)$ is a Fréchet space when provided with the family of seminorms $\{\rho_{k,j}\}_{j\in\mathbb{N}}$.*

2º The space $C^\infty(\Omega) = \bigcap_{k\in\mathbb{N}_0} C^k(\Omega)$ is a Fréchet space when provided with the family of seminorms $\{\rho_{k,j}\}_{k\in\mathbb{N}_0, j\in\mathbb{N}}$

Proof. G. Grubb [8, Lemma 2.4, p. 13]. □

Lemma 1.12. *1º Let $\alpha \in \mathbb{N}_0^n$ be a multiindex. The mapping $\partial^\alpha : \phi \mapsto \partial^\alpha\phi$ is a continuous linear operator on $C^\infty(\Omega)$.*

2º Let $f \in C^\infty(\Omega)$. The mapping $M_f : \phi \mapsto f\phi$ is a continuous linear operator on $C^\infty(\Omega)$.

Proof. Clearly both maps ∂^α and M_f are linear from $C^\infty(\Omega)$ to itself.

For the continuity of ∂^α we see that for each $k \in \mathbb{N}_0$, $j \in \mathbb{N}$ we have

$$\begin{aligned}\rho_{k,j}(\partial^\alpha \phi) &= \sup\{\,|\partial^\beta \partial^\alpha \phi(x)| : \,|\beta| \leqslant k,\, x \in K_j\} \\ &\leqslant \sup\{\,|\partial^\gamma \phi(x)| : \,|\gamma| \leqslant k + |\alpha|,\, x \in K_j\} \\ &= \rho_{k+|\alpha|,j}(\phi).\end{aligned}$$

For continuity of M_f we have by Leibniz's rule and for each $k \in \mathbb{N}_0$, $j \in \mathbb{N}$ that

$$\begin{aligned}\rho_{k,j}(f\phi) &= \sup\{\,|\partial^\alpha (f\phi)(x)| : \,|\alpha| \leqslant k,\, x \in K_j\} \\ &\leqslant \sup\Big\{ \sum_{\beta \leqslant \alpha} \Big| \binom{\alpha}{\beta} \partial^{\alpha-\beta} f(x) \partial^\beta \phi(x) \Big| : \,|\alpha| \leqslant k,\, x \in K_j \Big\} \\ &\leqslant C_k\, \rho_{k,j}(f)\, \rho_{k,j}(\phi),\end{aligned}$$

for a suitably large constant $C_k > 0$. Since $f \in C^\infty(\Omega)$, we have that $\rho_{k,j}(f) < \infty$ for all $k \in \mathbb{N}_0$, $j \in \mathbb{N}$ and this finishes the proof. □

Similarly, we may define the family of seminorms

$$\rho_k(f) := \sup\{\,|\partial^\alpha f(x)| \,:\, |\alpha| \leqslant k,\, x \in \mathbb{R}^n\}, \text{ for } k \in \mathbb{N}_0,$$

and the space of bounded differentiable functions of order k

$$C^k_B(\mathbb{R}^n) := \{f \in C^k(\mathbb{R}^n) \,:\, \rho_k(f) < \infty\}.$$

It is not hard to check that $C^k_B(\mathbb{R}^n)$ endowed with the seminorm ρ_k becomes a Banach space and that $C^\infty_B(\mathbb{R}^n) = \bigcap_{k \in \mathbb{N}_0} C^k_B(\mathbb{R}^n)$ endowed with the family $\{\rho_k\}_{k \in \mathbb{N}_0}$ becomes a Fréchet space.

Definition 1.13. The *Schwartz space* or *Rapidly decreasing function space* $\mathcal{S}(\mathbb{R}^n)$ is the set of infinitely differentiable complex-valued functions f on $\mathbb{R}^n$ such that for all multiindices α, β we have

$$p_{\alpha,\beta}(f) := \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha f(x)| < \infty. \tag{1.3}$$

Thus, the functions in $\mathcal{S}(\mathbb{R}^n)$ are those functions which together with their derivatives fall off more quickly than the inverse of any polynomial.

One can directly check that $p_{\alpha,\beta}$ as in (1.3) defines a seminorm on $\mathcal{S}(\mathbb{R}^n)$ and we shall call them the *Schwartz seminorms.*

Theorem 1.14. *The vector space $\mathcal{S}(\mathbb{R}^n)$ with the natural topology induced by the family of Schwartz seminorms $\{p_{\alpha,\beta} :\, \alpha, \beta \in \mathbb{N}^n_0\}$ is a Fréchet space.*

Proof. M. Reed and B. Simon [13, Theorem V.9, p. 133]. □

An important observation is that for all $1 \leqslant p \leqslant \infty$ we have that $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ with continuous injection. Indeed, for a each $1 \leqslant p < \infty$, take $M \in \mathbb{N}$ such that $n/2 < Mp$. Then for all $\phi \in \mathcal{S}$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\phi(x)|^p \, dx &= \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{Mp}} [(1+|x|^2)^M |\phi(x)|]^p \, dx \\ &\leqslant \Big(\sup_{x \in \mathbb{R}^n} \Big| (1+|x|^2)^M \phi(x) \Big| \Big)^p \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{Mp}} \, dx.\end{aligned}$$

Thus there exists a constant $C_M > 0$ such that $\|\phi\|_{L^p} \leqslant C_M \sup_{x \in \mathbb{R}^n} |(1+|x|^2)^M \phi(x)|$ and the right-hand side of this expression is bounded by a linear combination of Schwartz seminorms. The case $p = \infty$ is immediate.

Definition 1.15. The vector space $\mathcal{O}_M(\mathbb{R}^n)$ consists of complex-valued functions $p \in C^\infty(\mathbb{R}^n)$ which together with all of its derivatives has at most polynomial growth. More precisely, $p \in \mathcal{O}_M(\mathbb{R}^n)$ if for each $\alpha \in \mathbb{N}_0^n$ there exists $C_\alpha > 0$ and $N_\alpha \in \mathbb{N}$ such that

$$|\partial^\alpha p(x)| \leqslant C_\alpha (1+|x|^2)^{N_\alpha} \quad \forall x \in \mathbb{R}^n.$$

Lemma 1.16. *1º Let $\alpha \in \mathbb{N}_0^n$ be a multiindex. The mapping $\partial^\alpha : \phi \mapsto \partial^\alpha \phi$ is a continuous linear operator on $\mathcal{S}(\mathbb{R}^n)$.*

2º Let $p \in \mathcal{O}_M(\mathbb{R}^n)$. The mapping $M_p : \phi \mapsto p\phi$ is a continuous linear operator on $\mathcal{S}(\mathbb{R}^n)$.

3º Let $\chi \in C_c^\infty(\mathbb{R}^n)$. The mapping $M_\chi : \phi \mapsto \chi\phi$ is a continuous linear map from $C^\infty(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Proof. It is clear that those maps are linear, so let us check continuity.

The proof of 1º follows from the fact that for any $f \in \mathcal{S}$ and any $\gamma \in \mathbb{N}_0^n$ we have

$$p_{\gamma,\beta}(\partial^\alpha f) = \sup_{x\in\mathbb{R}^n} \left| x^\beta \partial^\gamma \partial^\alpha f(x) \right| = \sup_{x\in\mathbb{R}^n} \left| x^\beta \partial^{\gamma+\alpha} f(x) \right| = p_{\gamma+\alpha,\beta}(f)$$

The proof of 2º follows by Leibniz's rule, as for each $p \in \mathcal{O}_M$ there exists a constant $C > 0$ such that

$$\begin{aligned}
\left| x^\beta \partial^\alpha (p(x) f(x)) \right| &= |x^\beta| \sum_{\delta \leqslant \alpha} \binom{\alpha}{\delta} \left| \partial^\delta p(x)\, \partial^{\alpha-\delta} f(x) \right| \\
&\leqslant |x^\beta| \sum_{\delta \leqslant \alpha} \binom{\alpha}{\delta} |C_\delta (1+|x|^2)^{N_\delta}\, \partial^{\alpha-\delta} f(x)| \\
&\leqslant |x^\beta| \sum_{\substack{\delta \leqslant \alpha \\ |\gamma| \leqslant 2N_\delta}} C_{\delta\gamma\alpha}\, |x^{2\gamma}\, \partial^{\alpha-\delta} f(x)| \\
&\leqslant C \sum_{\substack{\delta \leqslant \alpha \\ \gamma \leqslant 2N_\delta}} |x^{\beta+2\gamma}\, \partial^{\alpha-\delta} f(x)| \\
&\leqslant C \sum_{\substack{\delta \leqslant \alpha \\ \gamma \leqslant 2N_\delta}} p_{\alpha-\delta,\beta+2\gamma}(f).
\end{aligned}$$

Taking the supremum over $x \in \mathbb{R}^n$ we conclude 2º.

For 3º we note that there exists $j_0 \in \mathbb{N}$ such that $\operatorname{supp} \partial^\alpha \chi \subset \operatorname{supp} \chi \subset \mathbb{B}(0, j_0)$. By Leibniz's rule we get

$$\begin{aligned}
\left| x^\beta \partial^\alpha (\chi(x) f(x)) \right| &= |x^\beta| \sum_{\delta \leqslant \alpha} \binom{\alpha}{\delta} \left| \partial^\delta \chi(x)\, \partial^{\alpha-\delta} f(x) \right| \\
&\leqslant \mathbf{1}_{\mathbb{B}(0,j_0)}(x)\, |x^\beta| \sum_{\delta \leqslant \alpha} \binom{\alpha}{\delta} \|\partial^\delta \chi\|_{L^\infty}\, |\partial^{\alpha-\delta} f(x)| \\
&\leqslant j_0^{|\beta|} \sum_{\delta \leqslant \alpha} \binom{\alpha}{\delta} \|\partial^\delta \chi\|_{L^\infty}\, \rho_{|\alpha|,j_0}(f).
\end{aligned}$$

Taking the supremum over $x \in \mathbb{R}^n$ we conclude our result. □

For any $j \in \mathbb{N}$ we define the space of smooth functions with compact support in $K_j \subset \Omega$ as

$$C_{K_j}^\infty(\Omega) := \{\phi \in C^\infty(\Omega) \,:\, \operatorname{supp} \phi \subset K_j\}.$$

This space provided with the inherited topology of $C^\infty(\Omega)$ is also a Fréchet space.

We now turn our attention to another space of differentiable functions which is a cornerstone in the theory of distributions and is not a Fréchet space. The space of infinitely differentiable functions with compact support in Ω

$$C_c^\infty(\Omega) := \{\phi \in C^\infty(\Omega) : \text{ supp}\,\phi \text{ is compact in } \Omega\}.$$

Clearly, $C_{K_j}^\infty(\Omega) \subset C_c^\infty(\Omega) \subset C^\infty(\Omega)$ for any $j \in \mathbb{N}$. Now, if we provide $C_c^\infty(\Omega)$ with the topology inherited from $C^\infty(\Omega)$, we get an incomplete metric space.

Therefore we prefer to provide $C_c^\infty(\Omega)$ with a stronger and somewhat more complicated topology that makes it locally convex and sequentially complete as a topological vector space. For the definition and several important properties of such topology on $C_c^\infty(\Omega)$ we recommend G. Grubb [8] or W. Rudin [14].

The topological properties of this space that we shall need are summed up in the following theorem.

Theorem 1.17. *The topology on $C_c^\infty(\Omega)$ has the following properties:*

1. *A sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$ converges to an element ϕ in $C_c^\infty(\Omega)$ if and only if there is a $j \in \mathbb{N}$ such that $\operatorname{supp}\phi_k \subset K_j$ for all $k \in \mathbb{N}$, $\operatorname{supp}\phi \subset K_j$, and for all $\alpha \in \mathbb{N}_0^n$ we have*

$$\sup_{x\in K_j} |\partial^\alpha \phi_k(x) - \partial^\alpha \phi(x)| \to 0 \quad \text{as } k \to \infty.$$

2. *A linear functional $\Lambda : C_c^\infty(\Omega) \to \mathbb{C}$ is continuous if and only if for each $j \in \mathbb{N}$ and all $\phi \in C_c^\infty(\Omega)$ with $\operatorname{supp}\phi \in K_j$, there is an $N_j \in \mathbb{N}_0$ and a constant $C_j > 0$ such that*

$$|\Lambda(\phi)| \leqslant C_j \sup\{|\partial^\alpha \phi(x)| : x \in K_j, |\alpha| \leqslant N_j\}.$$

3. *Let Y be a locally convex topological vector space. A linear map T is continuous from $C_c^\infty(\Omega)$ to Y if and only if for each $j \in \mathbb{N}$ the linear map $T : C_{K_j}^\infty(\Omega) \to Y$ is continuous.*

Differentiation and multiplication by smooth functions are examples of continuous linear operators on $C_c^\infty(\Omega)$.

Theorem 1.18. *1º Let $\alpha \in \mathbb{N}_0^n$ be a multiindex. The mapping $\partial^\alpha : \phi \mapsto \partial^\alpha\phi$ is a continuous linear operator on $C_c^\infty(\Omega)$.*

2º Let $f \in C^\infty(\Omega)$. The mapping $M_f : \phi \mapsto f\phi$ is a continuous linear operator on $C_c^\infty(\Omega)$.

Proof. G. Grubb [8, Theorem 2.6, p. 15]. □

Theorem 1.19. *1º $C_c^\infty(\Omega)$ is dense in $C^\infty(\Omega)$.*

2º $C_c^\infty(\Omega)$ is dense in $L^p_{loc}(\Omega)$ for all $1 \leqslant p < \infty$.

3º $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for all $1 \leqslant p < \infty$.

Proof. G. Grubb [8, Theorem 2.15, p. 22] □

Since $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, it follows from the above theorem that $\mathcal{S}(\mathbb{R}^n)$ is dense in $C^\infty(\mathbb{R}^n)$, and in $L^p_{loc}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ for all $1 \leqslant p < \infty$.

Lemma 1.20. *$C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ with continuous dense embedding.*

Proof. G. Grubb [8, Lemma 5.9, p. 103] □

Definition 1.21. A *distribution* on Ω is a continuous linear functional $\Lambda : C_c^\infty(\Omega) \to \mathbb{C}$. The vector space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$. When $\Lambda \in \mathcal{D}'(\Omega)$ we denote its value on $\phi \in C_c^\infty(\Omega)$ by either $\langle \Lambda, \phi \rangle$ or $\Lambda(\phi)$.

The space $\mathcal{D}'(\Omega)$ itself is provided with a weak topology defined by the family of seminorms

$$p_\phi(u) := |\langle u, \phi\rangle|, \text{ for } \phi \in C_c^\infty(\Omega).$$

We here use Theorem 1.7 noticing that the family $\{p_\phi\}_{\phi\in C_c^\infty(\Omega)}$ is separating since $u = 0$ in $\mathcal{D}'(\Omega)$ if and only if $\langle u, \phi\rangle = 0$ for all $\phi \in C_c^\infty(\Omega)$.

An important example of distribution is the following: When $f \in L^1_{loc}(\Omega)$ the linear map

$$\Lambda_f : \phi \mapsto \int_\Omega f(x)\phi(x)\,dx$$

from $C_c^\infty(\Omega)$ to $\mathbb{C}$ is a distribution. Indeed, for each $j \in \mathbb{N}$ and all $\phi \in C_c^\infty(\Omega)$ with $\operatorname{supp}\phi \in K_j$ we have

$$|\Lambda_f(\phi)| = \left|\int_\Omega f(x)\phi(x)\,dx\right| \leqslant \left(\int_{K_j} |f(x)|\,dx\right) \sup\{|\phi(x)| : x \in K_j\}. \tag{1.4}$$

Here one can in fact identify f with Λ_f in view of the following lemma:

Lemma 1.22. *When $f \in L^1_{loc}(\Omega)$ with*

$$\int_\Omega f(x)\phi(x)\,dx = 0 \quad \forall\phi \in C_c^\infty(\Omega),$$

then $f \equiv 0$ in $L^1_{loc}(\Omega)$.

Proof. G. Grubb [8, Lemma 3.2, p. 28]. □

Thus by (1.4) and the above Lemma we have that the linear map

$$L^1_{loc}(\Omega) \ni f \longmapsto \Lambda_f \in \mathcal{D}'(\Omega)$$

is continuous and injective.

When T is a continuous linear operator on $C_c^\infty(\Omega)$, and $\Lambda \in \mathcal{D}'(\Omega)$, we have that the composition map $\Lambda \circ T$ is a distribution. Define the *adjoint map* of operator T to be the map $T' : \Lambda \mapsto \Lambda \circ T$.

Theorem 1.23. *Let T be a continuous linear operator on $C_c^\infty(\Omega)$. The adjoint map T' defined by*

$$\langle T'\Lambda, \phi\rangle := \langle \Lambda, T\phi\rangle, \textit{ for } \Lambda \in \mathcal{D}'(\Omega),\ \phi \in C_c^\infty(\Omega)$$

is a continuous linear operator on $\mathcal{D}'(\Omega)$.

Proof. G. Grubb [8, Theorem 3.8, p. 35]. □

Definition 1.24. 1° For any $\alpha \in \mathbb{N}_0^n$, we define the differentiation operator $\partial^\alpha : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ by

$$\langle \partial^\alpha\Lambda, \phi\rangle := \langle \Lambda, (-1)^{|\alpha|}\partial^\alpha\phi\rangle, \text{ for } \phi \in C_c^\infty(\Omega).$$

2° When $f \in C^\infty(\Omega)$, we define the multiplication operator $M_f : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ by

$$\langle M_f\Lambda, \phi\rangle := \langle \Lambda, f\phi\rangle, \text{ for } \phi \in C_c^\infty(\Omega).$$

Both of these maps are continuous by Theorem 1.23. Moreover, it can be shown that both of these operators are *extensions* of the operators in Theorem 1.18 originally defined on $C_c^\infty(\Omega)$.

Theorem 1.25 (Convergence in $\mathcal{D}'(\Omega)$)**.** *A sequence of distributions $\{u_k\}_{k\in\mathbb{N}}$ is convergent in $\mathcal{D}'(\Omega)$ if and only if for every $\phi \in C_c^\infty(\Omega)$ the sequence $\langle u_k, \phi\rangle$ is convergent in $\mathbb{C}$. The limit of u_k in $\mathcal{D}'(\Omega)$ is the distribution u defined by*

$$\langle u, \phi\rangle = \lim_{k\to\infty}\langle u_k, \phi\rangle, \textit{ for } \phi \in C_c^\infty(\Omega).$$

Proof. G. Grubb [8, Theorem 3.9, p. 36]. □

When $u \in \mathcal{D}'(\Omega)$ and Ω' is an open subset of Ω, we define the *restriction* of u to Ω' as the element $u\restriction_{\Omega'} \in \mathcal{D}'(\Omega')$ defined by

$$\langle u\restriction_{\Omega'}, \varphi\rangle_{\Omega'} = \langle u, \varphi\rangle_{\Omega} \quad \text{for } \varphi \in C_c^\infty(\Omega').$$

For the sake of precision, we here indicate the duality between $\mathcal{D}'(\omega)$ and $C_c^\infty(\omega)$ by $\langle\, ,\, \rangle_\omega$, when ω is an open set.

When $u_1 \in \mathcal{D}'(\Omega_1)$ and $u_2 \in \mathcal{D}'(\Omega_2)$, and $\omega \subset \Omega_1 \cap \Omega_2$ is open, we say that $u_1 = u_2$ on ω, when

$$u_1\restriction_\omega - u_2\restriction_\omega = 0 \quad \text{as an element of } \mathcal{D}'(\omega).$$

Theorem 1.26. *Let $\{\omega_i\}_{i\in I}$ be an arbitrary family of open sets in $\mathbb{R}^n$ and let $\Omega = \bigcup \omega_i$. Assume that there is given a family of distributions $u_i \in \mathcal{D}'(\omega_i)$ with the property that $u_i = u_j$ on $\omega_i \cap \omega_j$ for each pair of indices $i, j \in I$. Then there exists one and only one distribution $u \in \mathcal{D}'(\Omega)$ such that u_i is the restriction of u to ω_i for every $i \in I$.*

Proof. G. Grubb [8, Theorem 3.14, p. 38]. □

Theorem 1.27. *For any $u \in \mathcal{D}'(\Omega)$ the exists a sequence $\{u_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that $u_j \to u$ in $\mathcal{D}'(\Omega)$.*

Proof. G. Grubb [8, Theorem 3.18, p. 42]. □

In other words, the space $C_c^\infty(\Omega)$ is *sequentially dense* in $\mathcal{D}'(\Omega)$.

Let Ω, Ω' be open sets of $\mathbb{R}^n$, and let κ be a diffeomorphism of Ω onto Ω'. More precisely, κ is a bijective map

$$\Omega \ni x = (x_1, \dots, x_n) \longmapsto (\kappa_1(x), \dots, \kappa_n(x)) \in \Omega',$$

where each κ_j is a C^∞-function from Ω to $\mathbb{R}$ and the inverse map $\kappa^{-1} : \Omega' \to \Omega$ is likewise smooth. The *Jacobian matrix* of κ is given by

$$J_\kappa(x) = \begin{bmatrix} \frac{\partial\kappa_1}{\partial x_1} & \frac{\partial\kappa_1}{\partial x_2} & \cdots & \frac{\partial\kappa_1}{\partial x_n} \\ \frac{\partial\kappa_2}{\partial x_1} & \frac{\partial\kappa_2}{\partial x_2} & \cdots & \frac{\partial\kappa_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\kappa_n}{\partial x_1} & \frac{\partial\kappa_n}{\partial x_2} & \cdots & \frac{\partial\kappa_n}{\partial x_n} \end{bmatrix}.$$

Define $J(x) := |\det J_\kappa(x)|$ and note that $J(x) > 0$ for all $x \in \Omega$ and that both J and $1/J$ are C^∞-functions. For $\phi \in C_c^\infty(\Omega')$ we have that $\phi \circ \kappa \in C_c^\infty(\Omega)$. The following definition then *extends* C^∞-coordinate change from C_c^∞ to the space of distributions.

Definition 1.28. We define the coordinate change map T_κ by

$$\langle T_\kappa \Lambda, \phi\rangle_{\Omega'} := \langle \Lambda, J\,(\phi\circ\kappa)\rangle_\Omega, \text{ for } \Lambda \in \mathcal{D}'(\Omega),\ \phi \in C_c^\infty(\Omega').$$

It defines a continuous linear map from $\mathcal{D}'(\Omega)$ to $\mathcal{D}'(\Omega')$.

Definition 1.29. A *tempered distribution* is a continuous linear functional $\Lambda : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$. The vector space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$ or simply by $\mathcal{S}'$. When $\Lambda \in \mathcal{S}'$ we denote its value on $\phi \in \mathcal{S}$ by either $\langle \Lambda, \phi\rangle$ or $\Lambda(\phi)$.

In a similar fashion to what we have done for the space $\mathcal{D}'(\Omega)$, we provide $\mathcal{S}'(\mathbb{R}^n)$ with a weak topology defined by the family of seminorms

$$p_\phi(u) := |\langle u, \phi\rangle|, \text{ for } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Note that Lemma 1.20 implies that the restriction of $\Lambda \in \mathcal{S}'(\mathbb{R}^n)$ on $C_c^\infty(\mathbb{R}^n)$ defines an element in $\mathcal{D}'(\mathbb{R}^n)$. More precisely

Theorem 1.30. *The linear map $J : \Lambda \to \Lambda'$ from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$ defined by restriction of Λ on $C_c^\infty(\mathbb{R}^n)$,*

$$\langle \Lambda', \phi \rangle := \Lambda(\phi), \quad \forall \phi \in C_c^\infty(\mathbb{R}^n),$$

is injective, and hence allows an identification of $J[\mathcal{S}'(\mathbb{R}^n)]$ with a subspace of $\mathcal{D}'(\mathbb{R}^n)$, also called $\mathcal{S}'(\mathbb{R}^n)$.

Proof. G. Grubb [8, Theorem 5.10, p. 104]. □

Lemma 1.31. *For $1 \leqslant p \leqslant \infty$ and $f \in L^p(\mathbb{R}^n)$, the linear map*

$$\mathcal{S} \ni \phi \longmapsto \int_{\mathbb{R}^n} f(x)\phi(x)\, dx \in \mathbb{C}$$

defines a tempered distribution. In this way one has a continuous injection of $L^p(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ for each $1 \leqslant p \leqslant \infty$.

Proof. G. Grubb [8, Lemma 5.11, p. 105]. □

The above results combined with Theorem 1.27 give us that $C_c^\infty(\mathbb{R}^n)$, and thus $\mathcal{S}(\mathbb{R}^n)$, are sequentially dense in $\mathcal{S}'(\mathbb{R}^n)$.

The operators ∂^α and M_p for $p \in C^\infty(\Omega)$ in Definition 1.24 are continuous linear operators on $\mathcal{D}'(\Omega)$. Concerning their act on $\mathcal{S}'$ we have

Lemma 1.32. *1º For any $\alpha \in \mathbb{N}_0^n$, ∂^α maps $\mathcal{S}'$ continuously into $\mathcal{S}'$.*

2º When $p \in \mathcal{O}_M$, M_p maps $\mathcal{S}'$ continuously into $\mathcal{S}'$.

Proof. G. Grubb [8, Lemma 5.13, p. 106] □

Proposition 1.33. *Let X, Y be Banach spaces such that $X, Y \subset \mathcal{S}'$ with continuous injection and such that $\mathcal{S} \subset X$ with continuous dense embedding. Let $A\colon \mathcal{S}' \to \mathcal{S}'$ be a continuous linear operator such that $A(\mathcal{S}) \subset Y$ and such that there exists a constant $C > 0$ for which*

$$\|Af\|_Y \leqslant C\|f\|_X \quad \forall f \in \mathcal{S}. \tag{1.5}$$

Then A defines a continuous linear map from X to Y with operator norm at most equal to the constant $C > 0$.

Proof. Let $f \in X$ and let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ be such that $f_n \to f$ in X. Inequality (1.5) shows that $\{Af_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y, and therefore converges to some element $g \in Y$. On the other hand, if $f_n \to f$ in X, then $f_n \to f$ in $\mathcal{S}'$ by hypothesis. Since $A\colon \mathcal{S}' \to \mathcal{S}'$ is a continuous linear map we have that $Af_n \to Af$ in $\mathcal{S}'$. This implies that $Af = g$ in $\mathcal{S}'$ and thus also in Y, finishing the proof. □

1.3 Fourier Transformation on Distributions

Definition 1.34. Given f in $L^1(\mathbb{R}^n)$ we define the *Fourier transform* of f to be

$$\mathcal{F}(f)(\xi) \equiv \widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i x\cdot\xi}\, dx.$$

Theorem 1.35. *1º The mapping $\mathcal{F} : f \mapsto \mathcal{F}(f)$ is a bounded linear transformation from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. In fact $\|\widehat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}$.*

2º If $f \in L^1(\mathbb{R}^n)$ then $\widehat{f}$ is uniformly continuous

Theorem 1.36 (Riemann–Lebesgue)**.** *If $f \in L^1(\mathbb{R}^n)$ then $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$; thus, in view of the above result, we can conclude that $\widehat{f} \in C_0(\mathbb{R}^n)$, where*

$$C_0(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \,:\, f \text{ is continuous and } f(x) \to 0 \text{ as } |x| \to \infty\}.$$

A more suitable space to work with the Fourier transform is the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. In this space we have the following properties

Proposition 1.37. *Given $f \in \mathcal{S}(\mathbb{R}^n)$, $\alpha \in \mathbb{N}_0^n$ and $y \in \mathbb{R}^n$ we have*

1. $\mathcal{F}(\partial_x^\alpha f)(\xi) = (2\pi i\xi)^\alpha \widehat{f}(\xi)$,
2. $(\partial_\xi^\alpha \widehat{f})(\xi) = \mathcal{F}((-2\pi i x)^\alpha f(x))(\xi)$,
3. $\mathcal{F}(e^{2\pi x\cdot y} f(x))(\xi) = \widehat{f}(\xi - y)$,
4. $\mathcal{F}(f(x-y))(\xi) = e^{-2\pi\xi\cdot y}\, \widehat{f}(\xi)$,
5. $\widehat{f} \in \mathcal{S}(\mathbb{R}^n)$.

Proof. L. Grafakos [6, Proposition 2.2.11, p. 100] for the proof of these results and many others. □

Definition 1.38. Given $f \in \mathcal{S}(\mathbb{R}^n)$, we define the *inverse Fourier transform* as

$$\overline{\mathcal{F}}(f)(x) \equiv \check{f}(x) := \int_{\mathbb{R}^n} f(y)\, e^{2\pi i y\cdot x}\, dy.$$

Note that, apart from some minor sign changes, the inverse Fourier transform shares the same properties stated in Proposition 1.37 as the Fourier transform. The next theorem shows that the inverse Fourier transform is indeed the inverse operator of $\mathcal{F}$ in $\mathcal{S}(\mathbb{R}^n)$. This property is referred to as *Fourier inversion formula.*

Theorem 1.39. *Given $f, g \in \mathcal{S}(\mathbb{R}^n)$ we have*

1. $\displaystyle\int_{\mathbb{R}^n} \widehat{f}(x)\, g(x)\, dx = \int_{\mathbb{R}^n} f(x)\, \widehat{g}(x)\, dx$
2. $\overline{\mathcal{F}}(\widehat{f}) = f = \mathcal{F}(\check{f})$ *(Fourier Inversion)*
3. $\displaystyle\int_{\mathbb{R}^n} f(x)\, \overline{g(x)}\, dx = \int_{\mathbb{R}^n} \widehat{f}(x)\, \overline{\widehat{g}(x)}\, dx$ *(Parseval's identity)*
4. $\|f\|_{L^2} = \|\widehat{f}\|_{L^2} = \|\check{f}\|_{L^2}$ *(Plancherel's identity)*

Proof. L. Grafakos [6, Theorem 2.2.14, p. 102] □

Theorem 1.40. *The Fourier transform is a homeomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto itself.*

Proof. L. Grafakos [6, Corollary 2.2.15, p. 103] □

Definition 1.41. Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution. We define the Fourier transform of a tempered distribution u to be

$$\langle \widehat{u},\, f\rangle := \langle u,\, \widehat{f}\rangle$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 1.40 implies that the Fourier transform of a tempered distribution is a well-defined tempered distribution.

Additionally, Theorems 1.23, 1.30, and 1.40 assure that the linear map $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ given by

$$\langle \mathcal{F}u,\, \phi\rangle := \langle u,\, \mathcal{F}\phi\rangle, \text{ for } \phi \in \mathcal{S}(\mathbb{R}^n)$$

is a continuous linear homeomorphism from $\mathcal{S}'(\mathbb{R}^n)$ onto itself.

Moreover, if we also invoke 3 of Theorem 1.39, we can see that $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ *extends* the Fourier transform defined initially on $\mathcal{S}(\mathbb{R}^n)$.

Furthermore, properties *1-4* presented in Proposition 1.37 also hold in the space of tempered distributions.

1.4 Sobolev Spaces

Definition 1.42 (Sobolev spaces)**.** For each $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ is defined by

$$H^s(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) \mid (1+|\xi|^2)^{s/2}\,\widehat{f}(\xi) \in L^2(\mathbb{R}^n)\}.$$

It is a Hilbert space with inner product given by

$$\langle f,\, g\rangle_s = \int_{\mathbb{R}^n} (1+|\xi|^2)^s\, \widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}\, d\xi,$$

and norm given by

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^s\, |\widehat{f}(\xi)|^2\, d\xi.$$

It is not hard to check that $H^s(\mathbb{R}^n) \subset H^{s'}(\mathbb{R}^n)$ for $s' < s$ and that the inclusion is a continuous injection. Moreover, if $m \in \mathbb{N}_0$, then the Sobolev space $H^m(\mathbb{R}^n)$ may be defined by

$$H^m(\mathbb{R}^n) := \{\, u \in L^2(\mathbb{R}^n) \,:\, \partial^\alpha u \in L^2(\mathbb{R}^n) \text{ for } |\alpha| \leqslant m\},$$

where ∂^α is applied in the distribution sense. In this case $H^m(\mathbb{R}^n)$ is equipped with the scalar product and norm

$$\langle u,\, v\rangle_{H^m} := \sum_{|\alpha|\leqslant m} \langle \partial^\alpha u,\, \partial^\alpha v\rangle_{L^2},$$

$$\|u\|_{H^m} := \Big(\langle u,\, u\rangle_{H^m}\Big)^{1/2}.$$

Using the Fourier inversion formula it is not hard to obtain that both definitions induce equivalent norms on $H^m(\mathbb{R}^n)$ whenever $m \in \mathbb{N}_0$.

Lemma 1.43. *$C_c^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for every $s \in \mathbb{R}$.*

Proof. G. Grubb [8, Lemma 6.10, p. 129] □

In particular, $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for every $s \in \mathbb{R}$. Moreover, it can be shown that for each $s \in \mathbb{R}$ we have that $H^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ with continuous dense embedding.

Theorem 1.44 (Sobolev embedding theorem)**.** *Let $k \geqslant 0$ be an integer, and let $s > k + n/2$. Then*

$$H^s(\mathbb{R}^n) \subset C_B^k(\mathbb{R}^n),$$

with continuous injection. More precisely, there exists a constant $C > 0$ such that for any $u \in H^s(\mathbb{R}^n)$,

$$\sup\{\, |\partial^\alpha u(x)| \,:\, |\alpha| \leqslant k,\, x \in \mathbb{R}^n\} \leqslant C\|u\|_{H^s}.$$

Proof. G. Grubb [8, Lemma 6.11, p. 130] □

Lemma 1.45. *Let $s \in \mathbb{R}$.*

1^o For each $\alpha \in \mathbb{N}_0^n$, ∂^α is a continuous linear map from $H^s(\mathbb{R}^n)$ to $H^{s-|\alpha|}(\mathbb{R}^n)$.

2^o For each $f \in \mathcal{S}(\mathbb{R}^n)$, the multiplication by f defines a continuous linear operator on $H^s(\mathbb{R}^n)$.

Proof. G. Grubb [8, Lemma 6.17, p. 134]. □

Definition 1.46. Let $s \in \mathbb{R}$ and let $K \subset \mathbb{R}^n$ be a compact set. We denote

$$H^s(K) := \{u \in H^s(\mathbb{R}^n) : \langle u, \phi \rangle = 0 \text{ for all } \phi \in C_c^\infty(K^c)\ \}.$$

In other words, the set $H^s(K)$ is composed by those elements of $H^s(\mathbb{R}^n)$ which *vanish* outside of $K \subset \mathbb{R}^n$.

Note that $H^s(K)$ is a closed linear subspace of $H^s(\mathbb{R}^n)$, hence a Hilbert space for the inherited structure.

Now let κ be a diffeomorphism from Ω onto Ω' and suppose that $K \subset \Omega$. Let $K' := \kappa(K)$ be a compact subset of Ω' and consider the mapping $T_\kappa : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega')$ from Definition 1.28. It easily follows that for any $u \in H^s(K)$ we have $T_\kappa u \in H^s(K')$. Moreover, we have

Proposition 1.47. *Let $s \in \mathbb{R}$ and let K be an arbitrary compact subset of Ω. There is a constant $C_s(K) > 0$ such that*

$$\|T_\kappa u\|_{H^s} \leqslant C_s(K)\, \|u\|_{H^s} \quad \forall u \in H^s(K).$$

Proof. F. Trèves [19, Proposition 25.7, p. 230]. □

Definition 1.48. Let $s \in \mathbb{R}$ and let $\Omega \subset \mathbb{R}^n$ be an open set. The space $H^s_{loc}(\Omega)$ is defined as the set of distributions $u \in \mathcal{D}'(\Omega)$ for which $\phi u \in H^s(\mathbb{R}^n)$ for all $\phi \in C_c^\infty(\Omega)$ (where ϕu is understood to be extended by zero outside of Ω).

Consider $K_j \subset \Omega$ such as in (1.2) and consider a sequence of smooth functions $\{\phi_j\}_{j\in\mathbb{N}}$ satisfying $0 \leqslant \phi \leqslant 1$, $\phi \equiv 1$ in K_j and $\operatorname{supp} \phi_j \subset \operatorname{int}(K_{j+1})$. Then it can be shown that $H^s_{loc}(\Omega)$ is a Fréchet space with family of seminorms given by

$$p_j(u) := \|\phi_j u\|_{H^s}, \text{ for } j \in \mathbb{N}.$$

Lemma 1.49. *Let $s \in \mathbb{R}$. Let $f \in C^\infty(\Omega)$ and $\alpha \in \mathbb{N}_0^n$, then the operator $u \mapsto f\partial^\alpha u$ is a continuous linear mapping from $H^s_{loc}(\Omega)$ into $H^{s-|\alpha|}_{loc}(\Omega)$.*

Proof. G. Grubb [8, Lemma 6.27, p. 141]. □

Denoting the restriction of $T_\kappa : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega')$ on $H^s_{loc}(\Omega)$ by T_κ we have

Proposition 1.50. *Let κ be a diffeomorphism from Ω onto Ω'. Then the mapping T_κ is an isomorphism from $H^s_{loc}(\Omega)$ onto $H^s_{loc}(\Omega')$. Its inverse is the mapping $T_{\kappa^{-1}}$.*

Proof. F. Trèves [19, Proposition 25.8, p. 231]. □

1.5 Smooth Manifolds

Definition 1.51. An *n-dimensional manifold* is a Hausdorff space M with countable basis and for which each point has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$. If $\kappa : U \to \kappa(U)$ is a homeomorphism from an open set $U \subset M$ onto an open set $\kappa(U) \subset \mathbb{R}^n$, then κ is called a *coordinate map*, the pair (U, κ) is called *coordinate chart* or simply *chart*, and the open set $U \subset M$ is called *chart neighbourhood.*

In particular, since any open subset of $\mathbb{R}^n$ can be written as a union of countably many compact subsets, the same holds for any n-dimensional manifold.

Definition 1.52. A *smooth structure* $\mathcal{F}$ on an n-dimensional manifold M is a collection of charts $\{(U_\alpha, \kappa_\alpha)\}_{\alpha\in A}$ satisfying the following three properties:

1. $M \subset \bigcup_{\alpha\in A} U_\alpha$

2. $\kappa_\beta \circ \kappa_\alpha^{-1} : \kappa_\alpha(U_\alpha \cap U_\beta) \to \kappa_\beta(U_\alpha \cap U_\beta)$ is a diffeomorphism for all $\alpha, \beta \in A$.
3. The collection $\mathcal{F}$ is *maximal* with respect to 2; that is, if (U, κ) is a chart such that $\kappa \circ \kappa_\alpha^{-1}$ and $\kappa_\alpha \circ \kappa^{-1}$ are diffeomorphisms for all $\alpha \in A$, then $(U, \kappa) \in \mathcal{F}$.

Property 2 of the above definition is usually referred as *compatibility condition* and we say that two charts $(U_\alpha, \kappa_\alpha)$ and (U_β, κ_β) are *compatible* if

$$\kappa_\beta \circ \kappa_\alpha^{-1} : \kappa_\alpha(U_\alpha \cap U_\beta) \to \kappa_\beta(U_\alpha \cap U_\beta)$$

and

$$\kappa_\alpha \circ \kappa_\beta^{-1} : \kappa_\beta(U_\alpha \cap U_\alpha) \to \kappa_\alpha(U_\alpha \cap U_\beta)$$

are diffeomorphisms.

Proposition 1.53. *Let M be an n-dimensional manifold. If $\mathcal{F}_0$ is a family of charts satisfying conditions* 1 *and* 2 *of Definition 1.52, then there exists one and only one smooth structure $\mathcal{F}$ containing $\mathcal{F}_0$.*

Proof. Let $\mathcal{F}_0 := \{U_i, \kappa_i\}_{i \in I}$. Define the collection of charts

$$\mathcal{F} := \{\, (U, \kappa) \,:\, U \subset M \text{ is open, } \kappa \circ \kappa_i^{-1} \text{ and } \kappa_i \circ \kappa^{-1} \text{ are diffeomorphisms for all } \kappa_i \in \mathcal{F}_0\}.$$

Note that $\mathcal{F}_0 \subset \mathcal{F}$, thus $\mathcal{F}$ forms an open cover of M. Because every $(U, \kappa) \in \mathcal{F}$ satisfies the compatibility condition with the elements of $\mathcal{F}_0$, it follows that κ defines a homeomorphism from U to $\kappa(U) \subset \mathbb{R}^n$. Moreover, if $(U_\alpha, \kappa_\alpha)$ and (U_β, κ_β) are in $\mathcal{F}$, then for all (U_i, κ_i) in $\mathcal{F}_0$ we have

$$\kappa_\alpha \circ \kappa_\beta^{-1} = (\kappa_\alpha \circ \kappa_i^{-1}) \circ (\kappa_i \circ \kappa_\beta^{-1}) \quad \text{on } \kappa_\beta(U_\alpha \cap U_\beta \cap U_i),$$

and therefore on $\kappa_\beta(U_\alpha \cap U_\beta)$. Since the same holds if we exchange the roles of α and β, we conclude that the family $\mathcal{F}$ satisfies the compatibility condition. Now $\mathcal{F}$ is maximal by construction, and so $\mathcal{F}$ is a smooth structure containing $\mathcal{F}_0$. Uniqueness follows from the observation that any other family of charts on M that contains $\mathcal{F}_0$ and satisfies the compatibility condition is contained in $\mathcal{F}$. □

Definition 1.54. Let M be a n-dimensional manifold and let $\mathcal{A}$ denote a family of charts. Then $\mathcal{A}$ is called an *atlas* on M, when it satisfies conditions 1 and 2 of Definition 1.52. Two atlases are called *compatible* or *equivalent* if they define the same smooth structure on M.

Definition 1.55. A *smooth n-manifold* is an n-dimensional manifold endowed with a smooth structure $\mathcal{F}$. We shall usually denote a smooth n-manifold $(M, \mathcal{F})$ simply by M.

If M is a smooth n-manifold, then a *compatible atlas* on M is an atlas $\mathcal{A}$ that is compatible with the smooth structure on M.

Let M be a smooth n-manifold and let $\mathcal{F} := \{(U_\alpha, \kappa_\alpha)\}_{\alpha \in A}$ be its smooth structure. If U is an open subset of M, then U can be given a smooth structure

$$\mathcal{F}_U := \{(U \cap U_\alpha, \kappa_\alpha \restriction_U)\}_{\alpha \in A}.$$

With this smooth structure U is called an *open submanifold* of M. Open subsets of smooth manifolds will always be given this natural smooth structure.

Let us recall some concepts from general topology. Let X be a topological space and suppose that one-point sets are closed in X. Then X is said *regular* if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively. The space X is said to be *normal* if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively. X is said *locally compact*, when every point $x \in X$ has a compact neighbourhood, i.e., there exists an open set U and a compact set K, such that $x \in U \subset K$.

Recall Urysohn Metrization Theorem, for whose proof we refer the reader to J. Munkres [12, Theorem 34.1, p. 215].

Theorem 1.56 (Urysohn Metrization Theorem)**.** *Every regular space X with a countable basis is metrizable.*

It follows at once from the above definitions and Theorem 1.56 that a smooth n-manifold is a locally compact normal space, hence metrizable.

Definition 1.57. A *compact smooth n-manifold* is a smooth n-manifold that is compact as a topological space.

Note that if $\mathcal{F} := \{(U_\alpha, \kappa_\alpha)\}_{\alpha \in A}$ is the smooth structure of a compact smooth n-manifold M, then by compactness we may extract a finite subcover $\{U_i\}_{i=1}^N$ of M. By Proposition 1.53 the atlas $\{(U_i, \kappa_i)\}_{i=1}^N$ suffices to describe the smooth structure $\mathcal{F}$. Moreover, the compactness of M allow us to assume that the $\kappa_i(U_i)$ are bounded and mutually disjoint in $\mathbb{R}^n$.

Let M be a smooth n-manifold and let $f : M \to \mathbb{C}$ be any function. Given a chart (U, κ) on M we define

$$f_\kappa := f \circ \kappa^{-1} \quad \text{on } \kappa(U).$$

Definition 1.58. Let M be a smooth n-manifold and let $f : M \to \mathbb{C}$ be any function. We say that f is a *smooth function on M* if for every chart (U, κ) on M we have that $f_\kappa : \kappa(U) \subset \mathbb{R}^n \to \mathbb{C}$ is a C^∞-function on $\kappa(U)$. We denote the space of all smooth complex-valued functions on M by $C^\infty(M)$ and the space of all smooth complex-valued functions with compact support on M by $C_c^\infty(M)$.

Note if M is a smooth n-manifold and $\mathcal{A} := \{(U_i, \kappa_i)\}_{i \in I}$ is a compatible atlas on M, then the compatibility condition implies that a function $f : M \to \mathbb{C}$ is smooth on M if and only if for each $i \in I$ the function f_{κ_i} is in $C^\infty(\kappa_i(U_i))$. To see this, let (U, κ) be any compatible chart on M and write

$$f_\kappa = f_{\kappa_i} \circ (\kappa_i \circ \kappa^{-1}) \quad \text{on } \kappa(U \cap U_i).$$

Since this holds for all $i \in I$ we have that f_κ is a C^∞-function on $\kappa(U)$. Also note that if $v \in C_c^\infty(\kappa(U))$ for some chart (U, κ) and we set

$$u = v \circ \kappa \quad \text{on } U, \quad u = 0 \text{ elsewhere},$$

it follows that $u \in C^\infty(M)$.

Definition 1.59. Let M be a smooth n-manifold. A *smooth partition of unity* is an indexed family $\{\rho_i\}_{i \in I}$ of smooth functions $\rho_i : M \to \mathbb{C}$ with the following properties:

1. $\rho_i(x) \geqslant 0$ for all $i \in I$ and all $x \in M$,
2. The family of supports $\{\operatorname{supp} \rho_i\}_{i \in I}$ is *locally finite*; that is, for each $x \in M$ there exists a neighbourhood $U_x \ni x$ that intersect $\operatorname{supp} \rho_i$ for only finitely many $i \in I$,
3. $\displaystyle\sum_{i \in I} \rho_i(x) = 1$ for all $x \in M$ (the sum is finite in view of 2).

Let $\mathcal{U} := \{U_\alpha\}_{\alpha \in A}$ be an arbitrary open cover of a smooth n-manifold M. A smooth partition of unity $\{\rho_i\}_{i \in I}$ is said to be *subordinated* to $\mathcal{U}$ if for each $i \in I$ there exists $\alpha \in A$ such that $\operatorname{supp} \rho_i \subset U_\alpha$, and it is said to be *strictly subordinated* to $\mathcal{U}$ if we may take I to be the same index set as A and $\operatorname{supp} \rho_\alpha \subset U_\alpha$ for each $\alpha \in A$. We have the following theorem about the existence of partitions of unit, for whose proof we refer F. Warner [21, 1.11 Theorem, p. 10].

Theorem 1.60 (Existence of partitions of unity)**.** *Let M be a smooth n-manifold and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Then there exists a countable partition of unity $\{\rho_i : i \in \mathbb{N}\}$ subordinate to $\{U_\alpha\}_{\alpha \in A}$ with* $\operatorname{supp} \rho_i$ *compact for each $i \in \mathbb{N}$. If one does not require compact supports, then there is a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ strictly subordinate to $\{U_\alpha\}_{\alpha \in A}$ with at most countably many of the ρ_α not zero.*

Corollary 1.60.1. *Let G be open in M, and let A be closed in M, with $A \subset G$. Then there exists a C^∞-function $\varphi : M \to \mathbb{R}$ such that*

1. $0 \leqslant \varphi(x) \leqslant 1$ *for all* $x \in M$.
2. $\varphi(x) = 1$ *if* $x \in A$.
3. $\operatorname{supp} \varphi \subset G$.

Proof. F. Warner [21, Corollary, p. 11] □

Let us show that when M is a compact smooth n-manifold we can refine the above theorem to obtain a strictly subordinate partition of unity where each smooth function has compact support. We start by discussing a similar statement for compact subsets of $\mathbb{R}^n$.

The next two lemmas were based on G. Grubb [8].

Lemma 1.61. *Let K be a compact subset of $\mathbb{R}^n$ and let $\{V_j\}_{j=1}^N$ be a bounded open cover of K, i.e., each V_j is a bounded open subset of $\mathbb{R}^n$ and $K \subset \bigcup_{j=1}^N V_j$. Then there exists a family $\{K_j\}_{j=1}^N$ of compact subsets of $\mathbb{R}^n$ with $K_j \subset V_j$, and $K \subset \bigcup_{j=1}^N K_j$. Moreover, there exists a family of smooth functions $\{\rho_j\}_{j=1}^N$ such that*

1. $0 \leqslant \rho_j(x) \leqslant 1$ *for all* $x \in K$,
2. $\operatorname{supp} \rho_j \subset V_j$ *for all* $1 \leqslant j \leqslant N$,
3. $\sum_{1 \leqslant j \leqslant N} \rho_j(x) = 1$ *for all* $x \in K$.

Proof. For each $1 \leqslant j \leqslant N$ consider the increasing sequence of open sets of the form

$$V_{j,k} := \{x \in V_j \; : \; \operatorname{dist}(x, V_j^c) > 1/k\}.$$

Thus $V_j \subset \bigcup_{k \in \mathbb{N}} V_{j,k}$ and each $V_{j,k}$ is relatively compact in $\mathbb{R}^n$. Note that the family $\{V_{j,k}\}_{1 \leqslant j \leqslant N, k \in \mathbb{N}}$ forms an open cover of K. Since K is compact, there exists a finite subfamily that still covers K and since $V_{j,k} \subset V_{j,k+1}$ for each $k \in \mathbb{N}$, we can reduce this subfamily to one which contains at most one V_{j,k_j} for each $1 \leqslant j \leqslant N$. Now take $K_j = \overline{V_{j,k_j}}$ and for those values of $1 \leqslant j \leqslant N$ not in the reduced subfamily supplement it with $K_j = \overline{V_{j,1}}$.

Now for each j we take a smooth function $\zeta_j \in C_c^\infty(\mathbb{R}^n)$ such that $\zeta_j \equiv 1$ in K_j, $\operatorname{supp} \zeta_j \subset V_j$ and $0 \leqslant \zeta_j \leqslant 1$ and define

$$\Psi(x) = \sum_{1 \leqslant j \leqslant N} \zeta_j(x) \geqslant 1 \quad \text{for all } x \in \bigcup_{1 \leqslant j \leqslant N} K_j.$$

Since $K \subset \bigcup_{j=1}^N V_{j,k_j}$ we can find $\phi \in C_c^\infty(\mathbb{R}^n)$ taking values in $[0,1]$ and such that $\phi \equiv 1$ in K and $\operatorname{supp} \phi \subset \bigcup_{j=1}^N V_{j,k_j}$. Finally, we set

$$\rho_j(x) := \begin{cases} \zeta_j(x) \frac{\phi(x)}{\Psi(x)} & \text{for when } x \in \bigcup_{i=1}^N V_{i,k_i}, \\ 0 & \text{elsewhere.} \end{cases}$$

Since for each j we have $V_{j,k_j} \subset K_j$, it follows that $\{\rho_j\}_{j=1}^N$ is family of a well-defined smooth functions satisfying conditions 1 to 3 and this finishes the proof. □

Lemma 1.62. *Let M be a compact smooth n-manifold and let $\{U_j\}_{j=1}^N$ be a finite open cover of M by chart neighbourhoods. Then there exists a family $\{K_j\}_{j=1}^N$ of compact subsets of M with $K_j \subset U_j$ and $M \subset \bigcup_{j=1}^N K_j$. Moreover, there exists a family of smooth functions $\{\rho_j\}_{j=1}^N$ such that*

1. $0 \leqslant \rho_j(x) \leqslant 1$ *for all* $x \in M$,

2. $\operatorname{supp}\rho_j \subset U_j$ *for all* $1 \leqslant j \leqslant N$,

3. $\sum_{1\leqslant j\leqslant N} \rho_j(x) = 1$ *for all* $x \in M$.

Proof. Since $\kappa_j : U_j \to \kappa_j(U_j)$ are homeomorphisms and we may assume that the $\kappa_j(U_j)$ are bounded open sets, it follows from the construction on proof of Lemma 1.61 that for each $1 \leqslant j \leqslant N$ we can choose compact subsets K_j, K_j' of U_j such that $K_j \subset \operatorname{int}(K_j')$, $M \subset \bigcup K_j$, and we can find smooth functions ζ_j that are equal to 1 on K_j and have support in $\operatorname{int}(K_j')$. Set $\zeta_j(x) = 0$ for all $x \in M \setminus \operatorname{int}(K_j')$.

Now take

$$\rho_j(x) = \frac{\zeta_j(x)}{\sum_{1\leqslant k\leqslant N} \zeta_k(x)}$$

to conclude the proof. □

Definition 1.63. Let M be a smooth n-manifold and let p be a point of M. We say that a linear map $v : C^\infty(M) \to \mathbb{C}$ is a *derivation at* p if it satisfies

$$v(fg) = f(p)v(g) + g(p)v(f) \quad \text{for all } f, g \in C^\infty(M).$$

The set of all derivations of $C^\infty(M)$ at p, denoted by T_pM, is a vector space called the *tangent space to* M *at* p. An element of T_pM is called a *tangent vector at* p.

Let M be a smooth n-manifold and let $p \in M$ be a point. Suppose that $(U, \kappa = (x^1, \ldots, x^n))$ is a compatible chart on M such that $p \in U$. Denote by $(r_1, \ldots, r_n)$ the canonical coordinates on $\mathbb{R}^n$. Then for $1 \leqslant i \leqslant n$ and any $f \in C^\infty(U)$ we set

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \left.\frac{\partial(f \circ \kappa^{-1})}{\partial r_i}\right|_{\kappa(p)}.$$

Then $\partial/\partial x^i|_p$ is the derivation that takes the ith partial derivative of (the coordinate representation of) f at the (coordinate representation of) p. We call $\partial/\partial x^i|_p$ a *coordinate vector at* p. We also use the notation

$$\left.\frac{\partial f}{\partial x^i}\right|_p = \left.\frac{\partial}{\partial x^i}\right|_p f.$$

Note that if $(U, \kappa = (x^1, \ldots, x^n))$ and $(V, \mu = (y^1, \ldots, y^n))$ are compatible charts around $p \in M$, then the chain rule implies

$$\left.\frac{\partial}{\partial y^j}\right|_p = \sum_{i=1}^n \left.\frac{\partial x^i}{\partial y^j}\right|_p \left.\frac{\partial}{\partial x^i}\right|_p.$$

Proposition 1.64. *Let* M *be a smooth* n*-manifold and let* $p \in M$. *Then* T_pM *is an* n*-dimensional vector space, and for any compatible chart* $(U, \kappa = (x^1, \ldots, x^n))$ *containing* p*, the coordinate vectors* $\{\partial/\partial x^i|_p\}_{i=1}^n$ *form a basis for* T_pM.

Proof. J. M. Lee [11, Proposition 3.15, p. 61] □

Definition 1.65. Let M be a smooth n-manifold. We define the *tangent bundle of* M, denoted by TM, to be the disjoint union of the tangent spaces at all points of M:

$$TM := \dot{\bigcup_{p\in M}} T_pM.$$

Definition 1.66. A *vector field* on a smooth n-manifold M is a map that associates to each point $p \in M$ a tangent vector $X(p) := X_p \in T_pM$ on the tangent space of M at p. Equivalently, a vector field X is a map from a smooth n-manifold M into the tangent bundle TM.

Let X be a vector field on M. If $U \subset M$ is a chart neighbourhood and $f \in C^\infty(U)$, the *directional derivative* $X(f) : U \to \mathbb{C}$ is defined to be the function

$$X(f) : p \in U \mapsto X_p(f).$$

Further, if $\kappa = (x^1, \dots, x^n)$ is the coordinate map on U, we have already seen that $\{\partial/\partial x^i|_p\}_{i=1}^n$ form a basis for T_pM for any $p \in U$. It then follows that there are functions $a_i : U \to \mathbb{C}$ such that

$$X\big|_U = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}. \tag{1.6}$$

Definition 1.67. Let M be a smooth n-manifold. A vector field X on M is called a *smooth vector field* if for every compatible chart $(U, \kappa = (x^1, \dots, x^n))$ on M, the functions a_i defined by (1.6) are smooth.

It follows from the definition that a smooth vector field X on a smooth n-manifold M is a linear map $X : f \mapsto X(f)$ from $C^\infty(M)$ to itself. Furthermore, for any $f, g \in C^\infty(M)$ we have $X(fg) = X(f)g + fX(g)$. Note that for any smooth vector field X on M and any smooth function $g \in C^\infty(M)$ we have that the commutator $[X, g] = Xg - gX$ defines a linear map from $C^\infty(M)$ to itself. More explicitly, for any $f \in C^\infty(M)$ we have

$$[X, g]f = X(gf) - gX(f) = X(g)f + gX(f) - gX(f) = X(g)f,$$

i.e., the operator $[X, g] : C^\infty(M) \to C^\infty(M)$ is equivalent to the multiplication operator by the smooth function $X(g)$.

Now suppose $U \subset M$ is an open submanifold and Y is a smooth vector field on U. Then for any point $p \in Y$ there exists a neighbourhood V, $p \in V \subset U$ and a smooth vector field $\tilde{Y}$ on M such that $\tilde{Y}$ and Y induce the same smooth vector field on V. To see this, let $K \subset U$ be any compact set containing p. Set $V := \operatorname{int}(K)$. Then by Corollary 1.60.1 there exists a smooth function $\varphi : M \to \mathbb{R}$ such that $\varphi \equiv 1$ on K and $\operatorname{supp} \varphi \subset U$. For any $g \in C^\infty(M)$, let $g\restriction_U$ denote its restriction to U and define $\tilde{Y}$ to be the map

$$M \ni p : \tilde{Y}_p(g) \longmapsto \begin{cases} \varphi(p)\, Y_p(g\restriction_U) & \text{for } p \in U, \\ 0 \quad \text{elsewhere.} \end{cases}$$

Let M be a smooth n-manifold. A linear operator $D : C^\infty(M) \to C^\infty(M)$ is called a *differential operator of order at most* 1 on M if it is of the form $Df := X(f) + gf$, where X is a smooth vector field on M and g is a smooth function on M. The space of all differential operators of order at most 1 on M is denoted by $DO^1(M)$.

Chapter 2

Pseudo-differential Operators on $\mathbb{R}^n$

This chapter about pseudo-differential operators on $\mathbb{R}^n$ has the intention to cover the essential elements of the theory that will be useful through the rest of this thesis.

In section 2.1 we present the definition of pseudo-differential operators and then turn our attention to such operators whose symbols are in the Hörmander class S^m.

Section 2.2 concerns about an L^2 estimate for pseudo-differential operators with symbol in S^0.

In section 2.3 we discuss the symbolic calculus that emerges when composing two pseudo-differential operators. Here we show that there is an asymptotic formula for the composition of two such operators, whose main term is the pointwise product of their symbols.

In section 2.4 we use previous results developed in this chapter to show that if $a \in S^m$, then the pseudo-differential operator T_a extends to a bounded linear map from $H^s(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$ for any $s \in \mathbb{R}$.

Section 2.5 shows how one can characterize pseudo-differential operators using commutators. Such characterization dates back to R. Beals [1].

Lastly, section 2.6 is a preparation for our next chapter, which deals with pseudo-differential operators on manifolds. The main result of this section shows how certain pseudo-differential operators behave under a smooth change of variables.

The presentation of this chapter follows closely A. Grigis and J. Sjöstrand [7], G. Grubb [8], L. Hörmander [10], M. Ruzhansky and V. Turunen [15], and E. M. Stein [18]. For more general results dealing with other classes of pseudo-differential operators we recommend H. O. Cordes [3].

2.1 Definition and First Results

Pseudo-differential operators generalize linear variable-coefficient differential operators. To motivate the definition of pseudo-differential operators, consider a function $f \in \mathcal{S}(\mathbb{R}^n)$ and a differential operator $p(X, D) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\partial_x^\alpha$. By applying the Fourier inversion formula to $\partial_x^\alpha f$ we obtain

$$\partial_x^\alpha f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\,(2\pi i\xi)^\alpha\,\widehat{f}(\xi)\,d\xi.$$

Thus we can see that

$$p(X, D)f(x) = \sum_{|\alpha|\leqslant m} a_\alpha(x)\partial_x^\alpha f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\,p(x,\xi)\,\widehat{f}(\xi)\,d\xi,$$

where $p(x,\xi) = \sum_{|\alpha|\leqslant m} a_\alpha(x)(2\pi i\xi)^\alpha$.

We are now in position to define pseudo-differential operators more precisely. Given a smooth function $a\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ with at most polynomial growth[1] on the ξ-variable, we could define T_a

[1]For a more precise definition using very mild conditions over the function a we recommend [3].

to be the *pseudo-differential operator* on Schwartz functions $f \in \mathcal{S}(\mathbb{R}^n)$ by the formula

$$T_a f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, a(x,\xi)\, \widehat{f}(\xi)\, d\xi. \tag{2.1}$$

The function $a(x,\xi)$ is called *symbol* of the operator T_a.

Through the rest of this thesis we will restrict ourselves to a more suitable class of symbols denoted by $S^m_{1,0}$ or simply by S^m.

Definition 2.1. We say that $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ if it is a smooth complex-valued function on $\mathbb{R}^n \times \mathbb{R}^n$ and if for all α, β multiindices there exists a constant $A_{\alpha\beta} > 0$ such that

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leqslant A_{\alpha\beta}(1+|\xi|)^{m-|\alpha|} \tag{2.2}$$

holds for all and x, $\xi \in \mathbb{R}^n$.

An important example of an element in S^m is the function $(1+|\xi|^2)^{m/2}$. Indeed, let us show that for each $m \in \mathbb{R}$ and for every $\beta \in \mathbb{N}_0^n$ there exists a constant $C_{m\beta} > 0$ such that

$$\partial_\xi^\beta (1+|\xi|^2)^{m/2} \leqslant C_{m\beta}(1+|\xi|)^{m-|\beta|}. \tag{2.3}$$

The case $|\beta| = 0$ follows from the inequalities

$$1+|\xi|^2 \leqslant (1+|\xi|)^2 \leqslant 3(1+|\xi|^2).$$

We now argue by induction. Suppose that (2.3) holds for all $m \in \mathbb{R}$ and for all multiindices with length less or equal than k for some $k \in \mathbb{N}_0$. Let $\beta \in \mathbb{N}_0^n$ be a multiindex such that $\beta = \alpha + \gamma$, with $|\alpha| = k$ and $|\gamma| = 1$. Therefore

$$\partial_\xi^\beta (1+|\xi|^2)^{m/2} = \partial_\xi^\alpha \partial_\xi^\gamma (1+|\xi|^2)^{m/2} = \partial_\xi^\alpha \partial_{\xi_j} (1+|\xi|^2)^{m/2},$$

for some $j \in \{1,\dots,n\}$. Since $\partial_{\xi_j}(1+|\xi|^2)^{m/2} = m\,\xi_j(1+|\xi|^2)^{(m-2)/2}$, we have by Leibniz's rule and the induction hypothesis

$$\begin{aligned}
|\partial_\xi^\beta (1+|\xi|^2)^{m/2}| &= |\partial_\xi^\alpha \partial_{\xi_j} (1+|\xi|^2)^{m/2}| \\
&= |\partial_\xi^\alpha m\,\xi_j (1+|\xi|^2)^{(m-2)/2}| \\
&\leqslant m \sum_{\delta\leqslant\alpha} \binom{\alpha}{\delta} |\partial_\xi^\delta \xi_j|\, |\partial_\xi^{\alpha-\delta}(1+|\xi|^2)^{(m-2)/2}| \\
&\leqslant m \sum_{\delta\leqslant\alpha} C_{m\alpha\delta}(1+|\xi|)^{1-|\delta|}\,(1+|\xi|)^{m-2-|\alpha-\delta|} \\
&\leqslant m \sum_{\delta\leqslant\alpha} C_{m\alpha\delta}(1+|\xi|)^{m-1-|\alpha|} \\
&\leqslant C_{m\beta}(1+|\xi|)^{m-|\beta|},
\end{aligned}$$

and this finishes the proof.

For each $a \in S^m$ the function $T_a f$ given by formula (2.1) is well-defined for any $f \in \mathcal{S}(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$ since

$$\begin{aligned}
|T_a f(x)| &\leqslant \int_{\mathbb{R}^n} |a(x,\xi)||\widehat{f}(\xi)|\, d\xi \\
&\leqslant A_{00} \int_{\mathbb{R}^n} (1+|\xi|)^m\, |\widehat{f}(\xi)|\, d\xi \\
&\leqslant A_{00} \sup_{\xi\in\mathbb{R}^n} \Big|(1+|\xi|)^{m+n+1}\, |\widehat{f}(\xi)|\Big| \int_{\mathbb{R}^n} \frac{1}{(1+|\xi|)^{n+1}}\, d\xi < \infty.
\end{aligned}$$

In fact, we will show on Theorem 2.5 that for any $a \in S^m$, T_a is a continuous operator on $\mathcal{S}(\mathbb{R}^n)$ and we shall call then *pseudo-differential operators*.

A pseudo-differential operator is said to be of order m if its symbol is in S^m. The class of pseudo-differential operators with symbol in $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ are denoted by $\Psi^m(\mathbb{R}^n \times \mathbb{R}^n)$ or $Op\, S^m(\mathbb{R}^n \times \mathbb{R}^n)$.

According to our previous discussion, all partial differential operators $p(X, D) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\partial_x^\alpha$ are in $Op\, S^m$ if each $a_\alpha(x)$ and all of its derivatives are smooth and bounded.

Let us explore some properties of the symbol class S^m. First, it is clear that S^m is a vector space. Now, for all α, β multiindices and for all $a \in S^m$ we have by (2.2) that

$$P^{(m)}_{\alpha,\beta}(a) := \sup_{(x,\xi)\in\mathbb{R}^{2n}} \left| \frac{\partial_x^\beta \partial_\xi^\alpha a(x,\xi)}{(1+|\xi|)^{m-|\alpha|}} \right| < \infty.$$

It is not hard to check that for each $\alpha, \beta \in \mathbb{N}_0^n$ we have that $P^{(m)}_{\alpha,\beta}$ is a seminorm on S^m. Moreover, we have the following theorem.

Theorem 2.2. *The vector space $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ with the natural topology induced by the family of seminorms $\{P^{(m)}_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}_0^n\}$ is a Fréchet space.*

Proof. It is clear that the family of seminorms $\{P^{(m)}_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}_0^n\}$ separates points of S^m. Since there are countably many of then, it follows that S^m is metrizable. It remains to show that S^m is complete with respect to this topology.

Let $\{a_k\}_{k\in\mathbb{N}} \subset S^m$ be a Cauchy sequence. Note that for any $\alpha, \beta \in \mathbb{N}_0^n$, any $a \in S^m$ and any compact $K \subset \mathbb{R}^{2n}$ we have

$$\begin{aligned} \sup_{(x,\xi)\in K} \left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| &= \sup_{(x,\xi)\in K} \left| \frac{\partial_x^\beta \partial_\xi^\alpha a(x,\xi)}{(1+|\xi|)^{m-|\alpha|}} (1+|\xi|)^{m-|\alpha|} \right| \\ &\leqslant \sup_{\xi\in K} \left|(1+|\xi|)^{m-|\alpha|}\right| \sup_{(x,\xi)\in\mathbb{R}^{2n}} \left| \frac{\partial_x^\beta \partial_\xi^\alpha a(x,\xi)}{(1+|\xi|)^{m-|\alpha|}} \right| \\ &= C_{K,\alpha}\, P^{(m)}_{\alpha,\beta}(a), \end{aligned}$$

where $C_{K,\alpha} = \sup_{\xi\in K} \left|(1+|\xi|)^{m-|\alpha|}\right| < \infty$. This inequality shows that any Cauchy sequence in S^m is also a Cauchy sequence with respect to the natural topology of $C^\infty(\mathbb{R}^{2n})$ which makes it a Fréchet space (check Lemma 1.11). Therefore there exists $a \in C^\infty(\mathbb{R}^{2n})$ such that $a_k \to a$ in $C^\infty(\mathbb{R}^{2n})$. In particular for all α, β multiindices we have $\partial_x^\beta \partial_\xi^\alpha a_k(x,\xi) \to \partial_x^\beta \partial_\xi^\alpha a(x,\xi)$ for all $x, \xi \in \mathbb{R}^n$.

To check that $a \in S^m$, we note that since $\{a_k\}_{k\in\mathbb{N}}$ is Cauchy in S^m, it is bounded, i.e., for all α, β multiindices there exists constants $A_{\alpha\beta} > 0$ which does not depend on $k \in \mathbb{N}$ and such that

$$|\partial_x^\beta \partial_\xi^\alpha a_k(x,\xi)| \leqslant A_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}$$

for all x, $\xi \in \mathbb{R}^n$ and all $k \in \mathbb{N}$. Taking the limit as $k \to \infty$ we conclude our result. □

Proposition 2.3. *1º If $m \leqslant m'$, then S^m is continuously injected in $S^{m'}$.*

2º Let $\gamma, \delta \in \mathbb{N}_0^n$. The mapping $\partial_x^\gamma \partial_\xi^\delta : a \mapsto \partial_x^\gamma \partial_\xi^\delta a$ is a continuous linear map from S^m to $S^{m-|\delta|}$.

3º Let $a \in S^{m_1}$ and $b \in S^{m_2}$ and let $M : (a, b) \mapsto a \cdot b$ be the pointwise multiplication of symbols. Then $M : S^{m_1} \times S^{m_2} \to S^{m_1+m_2}$ is a continuous bilinear map.

Proof. The proof of 1° follows from the fact that for all $\alpha, \beta \in \mathbb{N}_0^n$ and for all $m' \geqslant m$ we have

$$\begin{aligned} P_{\alpha,\beta}^{(m')}(a) = \sup_{(x,\xi)\in\mathbb{R}^{2n}} \left| \frac{\partial_x^\beta \partial_\xi^\alpha a(x,\xi)}{(1+|\xi|)^{m'-|\alpha|}} \right| &= \sup_{(x,\xi)\in\mathbb{R}^{2n}} \left| \frac{\partial_x^\beta \partial_\xi^\alpha a(x,\xi)}{(1+|\xi|)^{m-|\alpha|}} (1+|\xi|)^{m-m'} \right| \\ &\leqslant \sup_{(x,\xi)\in\mathbb{R}^{2n}} \left| \frac{\partial_x^\beta \partial_\xi^\alpha a(x,\xi)}{(1+|\xi|)^{m-|\alpha|}} \right| \sup_{\xi\in\mathbb{R}^n} \left|(1+|\xi|)^{m-m'}\right| \\ &= P_{\alpha,\beta}^{(m)}(a). \end{aligned}$$

To prove 2° we note that the mapping $\partial_x^\gamma \partial_\xi^\delta$ is linear and that

$$\begin{aligned} P_{\alpha,\beta}^{(m-|\delta|)}(\partial_x^\gamma \partial_\xi^\delta a) &= \sup_{(x,\xi)\in\mathbb{R}^{2n}} \left| \frac{\partial_x^\beta \partial_\xi^\alpha [\partial_x^\gamma \partial_\xi^\delta a](x,\xi)}{(1+|\xi|)^{m-|\delta|-|\alpha|}} \right| \\ &= \sup_{(x,\xi)\in\mathbb{R}^{2n}} \left| \frac{\partial_x^{\beta+\gamma} \partial_\xi^{\alpha+\delta} a(x,\xi)}{(1+|\xi|)^{m-|\delta+\alpha|}} \right| \\ &= P_{\alpha+\delta,\beta+\gamma}^{(m)}(a). \end{aligned}$$

For 3° we note that the mapping $M(a,b) \mapsto a \cdot b$ is clearly bilinear. Now, for every $a \in S^{m_1}$, $b \in S^{m_2}$ and for all $\alpha, \beta \in \mathbb{N}_0^n$, it follows by Leibniz's rule that

$$\begin{aligned} \left|\partial_x^\beta \partial_\xi^\alpha (a \cdot b)\right| &= \left| \sum_{\substack{\alpha' \leqslant \alpha \\ \beta' \leqslant \beta}} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} (\partial_x^{\beta'} \partial_\xi^{\alpha'} a)\, (\partial_x^{\beta-\beta'} \partial_\xi^{\alpha-\alpha'} b) \right| \\ &\leqslant \sum_{\substack{\alpha' \leqslant \alpha \\ \beta' \leqslant \beta}} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \left|(\partial_x^{\beta'} \partial_\xi^{\alpha'} a)\right| \left|(\partial_x^{\beta-\beta'} \partial_\xi^{\alpha-\alpha'} b)\right|. \end{aligned}$$

Since for all $\alpha' \leqslant \alpha \in \mathbb{N}_0^n$ we have $|\alpha| = |\alpha'| + |\alpha - \alpha'|$, it follows that

$$\frac{\left|\partial_x^\beta \partial_\xi^\alpha (a \cdot b)\right|}{(1+|\xi|)^{m_1+m_2-|\alpha|}} \leqslant \sum_{\substack{\alpha' \leqslant \alpha \\ \beta' \leqslant \beta}} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \frac{\left|(\partial_x^{\beta'} \partial_\xi^{\alpha'} a)\right|}{(1+|\xi|)^{m_1-|\alpha'|}} \frac{\left|(\partial_x^{\beta-\beta'} \partial_\xi^{\alpha-\alpha'} b)\right|}{(1+|\xi|)^{m_2-|\alpha-\alpha'|}},$$

which then implies

$$P_{\alpha,\beta}^{(m_1+m_2)}(a \cdot b) \leqslant \sum_{\substack{\alpha' \leqslant \alpha \\ \beta' \leqslant \beta}} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} P_{\alpha',\beta'}^{(m_1)}(a)\, P_{\alpha-\alpha',\beta-\beta'}^{(m_2)}(b),$$

finishing the proof. □

Proposition 2.4. *Let $\{a_k\}_{k\in\mathbb{N}}$ be a bounded sequence in S^m such that for all $\alpha, \beta \in \mathbb{N}_0^n$ we have that $\partial_x^\beta \partial_\xi^\alpha a_k(x,\xi)$ converges at each point $(x,\xi) \in \mathbb{R}^{2n}$. Then there exists $a \in S^m$ such that $a_k \to a$ in $C^\infty(\mathbb{R}^{2n})$. Moreover, for every $m' > m$ we have that $a_k \to a$ in $S^{m'}$.*

Proof. Let us show that under these hypotheses we have that $\{a_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $C^\infty(\mathbb{R}^{2n})$. More precisely, given $\varepsilon > 0$, $j \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}_0^n$, we will show that there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{(x,\xi)\in\mathbb{B}(0,j)} \left|\partial_x^\beta \partial_\xi^\alpha a_k(x,\xi) - \partial_x^\beta \partial_\xi^\alpha a_\ell(x,\xi)\right| < \varepsilon \quad \text{for all } k, \ell > n_0. \tag{2.4}$$

Since $\{a_k\}_{k\in\mathbb{N}} \subset S^m$ is bounded we have that

$$\begin{aligned}
\sup_{(x,\xi)\in\mathbb{B}(0,j)}\left|\partial_x^\beta\partial_\xi^\alpha a_k(x,\xi)\right| &= \sup_{(x,\xi)\in\mathbb{B}(0,j)}\left|\frac{\partial_x^\beta\partial_\xi^\alpha a_k(x,\xi)}{(1+|\xi|)^{m-|\alpha|}}(1+|\xi|)^{m-|\alpha|}\right| \\
&\leqslant \sup_{\xi\in\mathbb{B}(0,j)}\left|(1+|\xi|)^{m-|\alpha|}\right| \sup_{(x,\xi)\in\mathbb{R}^{2n}}\left|\frac{\partial_x^\beta\partial_\xi^\alpha a_k(x,\xi)}{(1+|\xi|)^{m-|\alpha|}}\right| \\
&= A^{(j)}_{\alpha\beta},
\end{aligned}$$

where $A^{(j)}_{\alpha\beta} > 0$ is a constant which depends on $j \in \mathbb{N}$ and $\alpha,\beta \in \mathbb{N}_0^n$, but not on $k \in \mathbb{N}$.

Now, given points $(x,\xi),(y,\eta) \in \mathbb{B}(0,j)$, we have by Taylor's formula

$$\partial_x^\beta\partial_\xi^\alpha a_k(x,\xi) = \partial_x^\beta\partial_\xi^\alpha a_k(y,\eta) + \sum_{|\gamma|=1}[(x,\xi)-(y,\eta)]^\gamma\int_0^1 \partial^\gamma(\partial_x^\beta\partial_\xi^\alpha a_k)(\theta y+(1-\theta)x,\theta\eta+(1-\theta)\xi)\,d\theta.$$

Taking $C^{(j)}_{\alpha\beta} = 2n\cdot\max\{A^{(j)}_{\alpha'\beta'} : |\alpha'| \leqslant |\alpha|+1, |\beta'| \leqslant |\beta|+1\}$, we conclude that $\forall\,(x,\xi),(y,\eta) \in \mathbb{B}(0,j)$

$$\left|\partial_x^\beta\partial_\xi^\alpha a_k(x,\xi) - \partial_x^\beta\partial_\xi^\alpha a_k(y,\eta)\right| \leqslant |(x,\xi)-(y,\eta)|\,C^{(j)}_{\alpha\beta}.$$

Thus given $\varepsilon > 0$ we have that

$$\left|\partial_x^\beta\partial_\xi^\alpha a_k(x,\xi) - \partial_x^\beta\partial_\xi^\alpha a_k(y,\eta)\right| \leqslant \frac{\varepsilon}{3},$$

for all $|(x,\xi)-(y,\eta)| \leqslant \frac{\varepsilon}{3C^{(j)}_{\alpha\beta}}$ and all $k \in \mathbb{N}$.

Now fix $r_\varepsilon = \frac{\varepsilon}{3C^{(j)}_{\alpha\beta}}$ and consider the open covering $\mathbb{B}(0,j) \subset \bigcup_{(x,\xi)\in\mathbb{B}(0,j)}\mathbb{U}((x,\xi),r_\varepsilon)$. Since $\mathbb{B}(0,j)$ is compact, there exists a finite subcover satisfying $\mathbb{B}(0,j) \subset \bigcup_{i=1}^N \mathbb{U}((x_i,\xi_i),r_\varepsilon)$. This implies that for every $(x,\xi) \in \mathbb{B}(0,j)$ there exists $1 \leqslant i \leqslant N$ such that $(x,\xi) \in \mathbb{U}((x_i,\xi_i),r_\varepsilon)$. Therefore for every $(x,\xi) \in \mathbb{B}(0,j)$ and for every $k,\ell \in \mathbb{N}$ we have

$$\begin{aligned}
\left|\partial_x^\beta\partial_\xi^\alpha a_k(x,\xi) - \partial_x^\beta\partial_\xi^\alpha a_\ell(x,\xi)\right| &\leqslant \left|\partial_x^\beta\partial_\xi^\alpha a_k(x,\xi) - \partial_x^\beta\partial_\xi^\alpha a_k(x_i,\xi_i)\right| \\
&\quad + \left|\partial_x^\beta\partial_\xi^\alpha a_k(x_i,\xi_i) - \partial_x^\beta\partial_\xi^\alpha a_\ell(x_i,\xi_i)\right| + \left|\partial_x^\beta\partial_\xi^\alpha a_\ell(x_i,\xi_i) - \partial_x^\beta\partial_\xi^\alpha a_\ell(x,\xi)\right|,
\end{aligned}$$

which provides, for every $(x,\xi) \in \mathbb{B}(0,j)$, the following inequality:

$$\left|\partial_x^\beta\partial_\xi^\alpha a_k(x,\xi) - \partial_x^\beta\partial_\xi^\alpha a_\ell(x,\xi)\right| \leqslant \frac{\varepsilon}{3} + \max_{1\leqslant i\leqslant N}|\partial_x^\beta\partial_\xi^\alpha a_k(x_i,\xi_i) - \partial_x^\beta\partial_\xi^\alpha a_\ell(x_i,\xi_i)| + \frac{\varepsilon}{3}.$$

Since for every $1 \leqslant i \leqslant N$ the sequence $\{\partial_x^\beta\partial_\xi^\alpha a_k(x_i,\xi_i)\}_{k\in\mathbb{N}}$ converges, there exists $n_i \in \mathbb{N}$ such that $|\partial_x^\beta\partial_\xi^\alpha a_k(x_i,\xi_i) - \partial_x^\beta\partial_\xi^\alpha a_\ell(x_i,\xi_i)| \leqslant \varepsilon/3$ for all $k,\ell > n_i$. Taking $n_0 := \max\{n_i : 1 \leqslant i \leqslant N\}$ it follows that for all $(x,\xi) \in \mathbb{B}(0,j)$ and all $k,\ell > n_0$

$$\left|\partial_x^\beta\partial_\xi^\alpha a_k(x,\xi) - \partial_x^\beta\partial_\xi^\alpha a_\ell(x,\xi)\right| < \varepsilon.$$

Since the estimate does not depend on $(x,\xi) \in \mathbb{B}(0,j)$, we conclude (2.4) by taking the supremum over $(x,\xi) \in \mathbb{B}(0,j)$.

Now, since $\{a_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $C^\infty(\mathbb{R}^{2n})$, there exists $a \in C^\infty(\mathbb{R}^{2n})$ such that $a_k \to a$ in $C^\infty(\mathbb{R}^{2n})$ and therefore for all α,β multiindices we have $\partial_x^\beta\partial_\xi^\alpha a_k(x,\xi) \to \partial_x^\beta\partial_\xi^\alpha a(x,\xi)$ for all $x,\xi \in \mathbb{R}^n$.

To check that $a \in S^m$, we note that since $\{a_k\}_{k\in\mathbb{N}}$ is bounded on S^m, we have that for all α, β

multiindices there exists a constant $A_{\alpha\beta} > 0$ which does not depend on $k \in \mathbb{N}$ and such that

$$|\partial_x^\beta \partial_\xi^\alpha a_k(x,\xi)| \leqslant A_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}$$

for all $x, \xi \in \mathbb{R}^n$ and all $k \in \mathbb{N}$. Taking the limit as $k \to \infty$ we conclude that $a \in S^m$.

In order to prove the convergence in $S^{m'}$ for $m' > m$, we let $\alpha, \beta \in \mathbb{N}_0^n$ and note that for all $j \in \mathbb{N}$ we have

$$\begin{aligned} P_{\alpha,\beta}^{(m')}(a-a_k) &= \sup_{(x,\xi)\in\mathbb{R}^{2n}} \left| \frac{\partial_x^\beta \partial_\xi^\alpha a(x,\xi) - \partial_x^\beta \partial_\xi^\alpha a_k(x,\xi)}{(1+|\xi|)^{m'-|\alpha|}} \right| \\ &\leqslant \sup_{(x,\xi)\in\mathbb{B}(0,j)} \left| \frac{\partial_x^\beta \partial_\xi^\alpha a(x,\xi) - \partial_x^\beta \partial_\xi^\alpha a_k(x,\xi)}{(1+|\xi|)^{m'-|\alpha|}} \right| \\ &\quad + \sup_{(x,\xi)\in\mathbb{B}(0,j)^c} \left| \frac{\partial_x^\beta \partial_\xi^\alpha a(x,\xi) - \partial_x^\beta \partial_\xi^\alpha a_k(x,\xi)}{(1+|\xi|)^{m-|\alpha|}} (1+|\xi|)^{m-m'} \right|. \end{aligned}$$

Since $\{a_k\}_{k\in\mathbb{N}} \subset S^m$ is bounded and $a \in S^m$, we have that there exists a constant $B_{\alpha\beta} > 0$, which does not depend on $k \in \mathbb{N}$ nor $j \in \mathbb{N}$, and such that

$$\sup_{(x,\xi)\in\mathbb{B}(0,j)^c} \left| \frac{\partial_x^\beta \partial_\xi^\alpha a(x,\xi) - \partial_x^\beta \partial_\xi^\alpha a_k(x,\xi)}{(1+|\xi|)^{m-|\alpha|}} (1+|\xi|)^{m-m'} \right| \leqslant B_{\alpha\beta}(1+j)^{m-m'}. \tag{2.5}$$

Therefore given $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that (2.5) is less than $\varepsilon/2$.

For such $j_0 \in \mathbb{N}$ we have

$$\sup_{(x,\xi)\in\mathbb{B}(0,j_0)} \left| \frac{\partial_x^\beta \partial_\xi^\alpha a(x,\xi) - \partial_x^\beta \partial_\xi^\alpha a_k(x,\xi)}{(1+|\xi|)^{m'-|\alpha|}} \right| \leqslant \sup_{(x,\xi)\in\mathbb{B}(0,j_0)} \left| \partial_x^\beta \partial_\xi^\alpha a(x,\xi) - \partial_x^\beta \partial_\xi^\alpha a_k(x,\xi) \right| \sup_{(x,\xi)\in\mathbb{B}(0,j_0)} (1+|\xi|)^{|\alpha|-m'}.$$

Hence, because $a_k \to a$ in $C^\infty(\mathbb{R}^{2n})$, we conclude that there exists $k_0 \in \mathbb{N}$ such that the above expression is less than $\varepsilon/2$ for all $k > k_0$, and thus $P_{\alpha,\beta}^{(m')}(a-a_k) < \varepsilon$ for all $k > k_0$, finishing the proof. ☐

We now discuss the behaviour of pseudo-differential operators of order m on Schwartz functions.

Theorem 2.5 (Pseudo-differential Operators on $\mathcal{S}$)**.** *If $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, we have that $T_a f \in \mathcal{S}(\mathbb{R}^n)$. More generally, the bilinear map*

$$S^m(\mathbb{R}^n \times \mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (a,f) \longmapsto T_a f \in \mathcal{S}(\mathbb{R}^n)$$

is continuous.

Proof. We start by showing that for any $f \in \mathcal{S}(\mathbb{R}^n)$ we have $T_a f \in C^\infty(\mathbb{R}^n)$ whenever $a \in S^m$.

Given any $\alpha \in \mathbb{N}_0^n$, if we differentiate under the integral sign on $T_a f$ we get

$$\partial_x^\alpha T_a f(x) = \int_{\mathbb{R}^n} \partial_x^\alpha [e^{2\pi i x\cdot\xi} a(x,\xi)]\, \widehat{f}(\xi)\, d\xi. \tag{2.6}$$

Let us now show that the above integral is uniformly bounded on $x \in \mathbb{R}^n$.

By Leibniz's rule we have

$$\begin{aligned} \partial_x^\alpha [e^{2\pi i x\cdot\xi}\, a(x,\xi)] &= \sum_{\gamma\leqslant\alpha} \binom{\alpha}{\gamma} \partial_x^\gamma e^{2\pi i x\cdot\xi}\, \partial_x^{\alpha-\gamma} a(x,\xi) \\ &= e^{2\pi i x\cdot\xi} \sum_{\gamma\leqslant\alpha} \binom{\alpha}{\gamma} (2\pi i \xi)^\gamma\, \partial_x^{\alpha-\gamma} a(x,\xi). \end{aligned} \tag{2.7}$$

Since $a \in S^m$ we have the upper bound

$$\left|\partial_x^\alpha[e^{2\pi i x\cdot\xi}\, a(x,\xi)]\right| \leqslant \sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}|2\pi\xi|^{|\gamma|}\, A_{\alpha\gamma}\,(1+|\xi|)^m \leqslant B_\alpha(1+|\xi|)^{m+|\alpha|} \tag{2.8}$$

for some $B_\alpha > 0$. Inequality (2.8) allow us to justify the differentiation under the integral sign and to show that (2.6) is well defined since

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\partial_x^\alpha[e^{2\pi i x\cdot\xi}a(x,\xi)]\,\widehat{f}(\xi)\,d\xi\right| &\leqslant \int_{\mathbb{R}^n}|\partial_x^\alpha[e^{2\pi i x\cdot\xi}a(x,\xi)]|\,|\widehat{f}(\xi)|\,d\xi\\ &\leqslant B_\alpha\int_{\mathbb{R}^n}(1+|\xi|)^{m+|\alpha|}\,|\widehat{f}(\xi)|\,d\xi\\ &\leqslant B_\alpha\sup_{\xi\in\mathbb{R}^n}\left|(1+|\xi|)^{m+|\alpha|+n+1}\,|\widehat{f}(\xi)|\right|\int_{\mathbb{R}^n}\frac{1}{(1+|\xi|)^{n+1}}\,d\xi,\end{aligned}$$

and the right-hand side is finite because $f \in \mathcal{S}(\mathbb{R}^n)$. Since this is true for any multiindex α we conclude that $T_a f \in C^\infty(\mathbb{R}^n)$.

Next we simultaneously show that $T_a f \in \mathcal{S}$ and that the (obviously bilinear) map $(a, f) \mapsto T_a f$ is continuous. By (2.7) we have

$$x^\beta\partial_x^\alpha T_a f(x) = x^\beta\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\int_{\mathbb{R}^n}e^{2\pi i x\cdot\xi}\,(2\pi i\xi)^\gamma\,\partial_x^{\alpha-\gamma}a(x,\xi)\,\widehat{f}(\xi)\,d\xi. \tag{2.9}$$

Let $\Delta_\xi = \sum_{1\leqslant i\leqslant n}\partial_{\xi_i}^2$ be the Laplacian operator on ξ. If we set

$$L_\xi = (1+4\pi^2|x|^2)^{-1}(1-\Delta_\xi), \tag{2.10}$$

then $L_\xi(e^{2\pi i x\cdot\xi}) = e^{2\pi i x\cdot\xi}$. Inserting this operator N times in (2.9) and using integration by parts $2N$ times we have

$$\begin{aligned}x^\beta\partial_x^\alpha T_a f(x) &= x^\beta\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\int_{\mathbb{R}^n}(L_\xi)^N(e^{2\pi i x\cdot\xi})\,(2\pi i\xi)^\gamma\,\partial_x^{\alpha-\gamma}a(x,\xi)\,\widehat{f}(\xi)\,d\xi\\ &= \frac{x^\beta}{(1+4\pi^2|x|^2)^N}\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\int_{\mathbb{R}^n}e^{2\pi i x\cdot\xi}\,(1-\Delta_\xi)^N[(2\pi i\xi)^\gamma\,\partial_x^{\alpha-\gamma}a(x,\xi)\,\widehat{f}(\xi)]\,d\xi.\end{aligned}$$

Taking $2N > |\beta|$, we have for every $x \in \mathbb{R}^n$

$$\begin{aligned}\left|x^\beta\partial_x^\alpha T_a f(x)\right| &= \left|\frac{x^\beta}{(1+4\pi^2|x|^2)^N}\right|\left|\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\int_{\mathbb{R}^n}e^{2i\pi x\cdot\xi}(1-\Delta_\xi)^N[(2\pi\xi)^\gamma\,\partial_x^{\alpha-\gamma}a(x,\xi)\,\widehat{f}(\xi)]\,d\xi\right|\\ &\leqslant \frac{|x|^{|\beta|}}{(1+4\pi^2|x|^2)^N}\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\int_{\mathbb{R}^n}\left|(1-\Delta_\xi)^N[(2\pi\xi)^\gamma\,\partial_x^{\alpha-\gamma}a(x,\xi)\,\widehat{f}(\xi)]\right|d\xi\\ &\leqslant \sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\int_{\mathbb{R}^n}\left|(1-\Delta_\xi)^N[(2\pi\xi)^\gamma\,\partial_x^{\alpha-\gamma}a(x,\xi)\,\widehat{f}(\xi)]\right|d\xi.\end{aligned} \tag{2.11}$$

For a given multiindex $\gamma \leqslant \alpha$ we can check that

$$(1-\Delta_\xi)^N[(2\pi i\xi)^\gamma\,\partial_x^{\alpha-\gamma}a(x,\xi)\,\widehat{f}(\xi)] \subset \operatorname{span}\{\partial_\xi^\delta\partial_x^{\alpha-\gamma}a(x,\xi)\,\partial_\xi^\theta[(2\pi\xi)^\gamma\,\widehat{f}(\xi)],\ \text{ where } |\delta+\theta| \leqslant 2N\}. \tag{2.12}$$

Since we have

$$|\partial_\xi^\delta\partial_x^{\alpha-\gamma}a(x,\xi)| \leqslant P_{\delta,\alpha-\gamma}^{(m)}(a)\,(1+|\xi|)^{m-|\delta|}\,,\ \text{for all } x,\xi\in\mathbb{R}^n, \tag{2.13}$$

and

$$\partial_\xi^\theta[(2\pi\xi)^\gamma\,\widehat{f}(\xi)] = (-i)^{|\gamma|}\partial_\xi^\theta\,\widehat{(\partial_x^\gamma f)}(\xi), \tag{2.14}$$

we conclude that for each $\gamma \leqslant \alpha$ there exist constants $C_{\alpha\gamma\theta\delta} > 0$ such that

$$\left|(1-\Delta_\xi)^N[(2\pi i\xi)^\gamma\,\partial_x^{\alpha-\gamma}a(x,\xi)\,\widehat{f}(\xi)]\right| \leqslant \sum_{|\delta+\theta|\leqslant 2N} C_{\alpha\gamma\theta\delta}\,P^{(m)}_{\delta,\alpha-\gamma}(a)\,(1+|\xi|)^{m-|\delta|}|\partial_\xi^\theta\,\widehat{(\partial_x^\gamma f)}(\xi)|. \tag{2.15}$$

By applying the above inequality to the right-hand side of (2.11) we get

$$\begin{aligned}\left|x^\beta\partial_x^\alpha T_a f(x)\right| &\leqslant \sum_{\substack{\gamma\leqslant\alpha\\|\delta+\theta|\leqslant 2N}}\binom{\alpha}{\gamma}C_{\alpha\gamma\theta\delta}\,P^{(m)}_{\delta,\alpha-\gamma}(a)\int_{\mathbb{R}^n}(1+|\xi|)^{m-|\delta|}|\partial_\xi^\theta\,\widehat{(\partial_x^\gamma f)}(\xi)|\,d\xi\\ &\leqslant \sum_{\substack{\gamma\leqslant\alpha\\|\delta+\theta|\leqslant 2N}}\binom{\alpha}{\gamma}C_{\alpha\gamma\theta\delta}\,P^{(m)}_{\delta,\alpha-\gamma}(a)\sup_{\xi\in\mathbb{R}^n}\left|(1+|\xi|)^{m-|\delta|+n+1}\,|\partial_\xi^\theta\,\widehat{(\partial_x^\gamma f)}(\xi)|\right|\int_{\mathbb{R}^n}\frac{1}{(1+|\xi|)^{n+1}}\,d\xi.\end{aligned} \tag{2.16}$$

Note that the right-hand side of (2.16) does not depend on $x \in \mathbb{R}^n$. Taking the supremum over $x \in \mathbb{R}^n$ and using the fact that differentiation and Fourier transformation are continuous operators on $\mathcal{S}(\mathbb{R}^n)$ we conclude our result. □

Proposition 2.6. *Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ in $\mathbb{B}(0,1)$ and* $\operatorname{supp}\psi \subset \mathbb{B}(0,2)$. *Given $a \in S^m$ and $0 < \varepsilon \leqslant 1$, define $a_\varepsilon(x,\xi) = a(x,\xi)\psi(\varepsilon x)\psi(\varepsilon\xi)$. Then $\{a_\varepsilon\}_{0<\varepsilon\leqslant 1} \subset S^m$ is bounded and $T_{a_\varepsilon}f \to T_a f$ in $\mathcal{S}$ for any $f \in \mathcal{S}$.*

Proof. Let us first show that $\psi(\varepsilon x)\psi(\varepsilon\xi) \in S^0$ uniformly in $0 < \varepsilon \leqslant 1$.

Fix $\varepsilon \in (0,1]$. For any $\alpha,\beta \in \mathbb{N}_0^n$ we have $\partial_x^\beta\partial_\xi^\alpha\psi(\varepsilon x)\psi(\varepsilon\xi) = \partial_x^\beta\psi(\varepsilon x)\partial_\xi^\alpha\,\psi(\varepsilon\xi)$. For the derivatives in the x-variable we use the rough estimate

$$\left|\partial_x^\beta\psi(\varepsilon x)\right| = \varepsilon^{|\beta|}|(\partial_x^\beta\psi)(\varepsilon x)| \leqslant \|\partial_x^\beta\psi\|_{L^\infty}.$$

For the derivatives in the ξ-variable we write

$$|\partial_\xi^\alpha\psi(\varepsilon\xi)| = \varepsilon^{|\alpha|}|(\partial_\xi^\alpha\psi)(\varepsilon\xi)| \tag{2.17}$$

and we note that $(\partial_\xi^\alpha\psi)(\varepsilon\xi) \neq 0$ only if $|\xi| \leqslant 2/\varepsilon$. This allows us to conclude that

$$0 < \varepsilon \leqslant \min\left\{1, \frac{2}{|\xi|}\right\} \leqslant \frac{3}{1+|\xi|} \quad \forall\,\xi \in \mathbb{B}(0,2/\varepsilon).$$

Using this in conjunction with (2.17) we obtain

$$|\partial_\xi^\alpha\psi(\varepsilon\xi)| = \varepsilon^{|\alpha|}|(\partial_\xi^\alpha\psi)(\varepsilon\xi)| \leqslant \frac{3^{|\alpha|}}{(1+|\xi|)^{|\alpha|}}\,\|\partial_\xi^\alpha\psi\|_{L^\infty},$$

and hence

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}}\left|\partial_x^\beta\partial_\xi^\alpha\psi(\varepsilon x)\psi(\varepsilon\xi)\right| \leqslant 3^{|\alpha|}(1+|\xi|)^{-|\alpha|}\,\|\partial_x^\beta\psi\|_{L^\infty}\,\|\partial_\xi^\alpha\psi\|_{L^\infty}.$$

Since the right-hand side does not depend on ε we conclude that $\psi(\varepsilon x)\psi(\varepsilon\xi) \in S^0$ uniformly in $0 < \varepsilon \leqslant 1$.

By 3° in Proposition 2.3 we have that for every $0 < \varepsilon \leqslant 1$

$$a(x,\xi)\psi(\varepsilon x)\psi(\varepsilon\xi) = a_\varepsilon(x,\xi) \in S^m.$$

Moreover, by the continuity of the pointwise multiplication of symbols we get that $\{a_\varepsilon\}_{0<\varepsilon\leqslant 1} \subset S^m$

is bounded.

To conclude that for any $f \in \mathcal{S}(\mathbb{R}^n)$ we have that $T_{a_\varepsilon} f \to T_a f$ in $\mathcal{S}$ we fix $m' > m$ and note that for all $\alpha, \beta \in \mathbb{N}_0^n$ we have that $\partial_x^\beta \partial_\xi^\alpha a_\varepsilon(x,\xi) \to \partial_x^\beta \partial_\xi^\alpha a(x,\xi)$ for all $x, \xi \in \mathbb{R}^n$ as $\varepsilon \to 0^+$. Thus the boundedness of $\{a_\varepsilon\}_{0<\varepsilon\leqslant 1} \subset S^m$ and Proposition 2.4 let us conclude that $a_\varepsilon \to a$ in $S^{m'}$.

By Theorem 2.5 we have that the bilinear map

$$S^{m'}(\mathbb{R}^n \times \mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (a, f) \longmapsto T_a f \in \mathcal{S}(\mathbb{R}^n)$$

is continuous and this finishes the proof. □

It is worth noting that if $\psi \in C_c^\infty(\mathbb{R}^n)$ is as in Proposition 2.6 and $a \in S^m$, then by defining $a^\varepsilon(x,\xi) = a(x,\xi)\psi(\varepsilon\xi)$, we also have that $\{a^\varepsilon\}_{0<\varepsilon\leqslant 1} \subset S^m$ uniformly on $0 < \varepsilon \leqslant 1$ and that $T_{a^\varepsilon} f \to T_a f$ in $\mathcal{S}$ for all $f \in \mathcal{S}$.

Let us discuss the *Adjoint map* of a pseudo-differential operator. This notion will help us extend the pseudo-differential operator T_a, initially defined on $\mathcal{S}$, to the space of tempered distributions $\mathcal{S}'$.

For each $a \in S^m$ and $0 < \varepsilon \leqslant 1$, let $a_\varepsilon \in S^m$ be as above. Interpreting $T_a f$ as an element of $\mathcal{S}'$ and using that $T_{a_\varepsilon} f \to T_a f$ in $\mathcal{S}$ for any $f \in \mathcal{S}$, we have for every $g \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
\langle T_a f, g\rangle = \lim_{\varepsilon\to 0^+} \langle T_{a_\varepsilon} f, g\rangle &= \lim_{\varepsilon\to 0^+} \int (T_{a_\varepsilon} f)(x)\, g(x)\, dx \\
&= \lim_{\varepsilon\to 0^+} \int \left[\int e^{2\pi i x\cdot\xi}\, a_\varepsilon(x,\xi)\, \widehat{f}(\xi) d\xi\right] g(x)\, dx \\
&= \lim_{\varepsilon\to 0^+} \int \left[\int\int e^{2\pi i (x-y)\cdot\xi}\, a_\varepsilon(x,\xi)\, f(y)\, dy\, d\xi\right] g(x)\, dx \\
&= \lim_{\varepsilon\to 0^+} \int f(y) \left[\int\int e^{2\pi i (x-y)\cdot\xi}\, a_\varepsilon(x,\xi)\, g(x)\, dx\, d\xi\right] dy.
\end{aligned} \tag{2.18}$$

For each $g \in \mathcal{S}$, let us define

$$T_a' g(y) := \lim_{\varepsilon\to 0^+} \int\int e^{2\pi i (x-y)\cdot\xi}\, a_\varepsilon(x,\xi)\, g(x)\, dx\, d\xi. \tag{2.19}$$

We claim that this limit exists and that $y \mapsto T_a' g(y)$ is a well-defined function.

Indeed, let us set

$$L_x = (1 + 4\pi^2|\xi|^2)^{-1}(1 - \Delta_x),$$

so that $L_x(e^{2\pi i (x-y)\cdot\xi}) = e^{2\pi i (x-y)\cdot\xi}$. Inserting this operator N times on the right-hand side of (2.19) and using integration by parts $2N$ times we have

$$\begin{aligned}
T_a' g(y) &= \lim_{\varepsilon\to 0^+} \int\int (L_x)^N (e^{2\pi i (x-y)\cdot\xi})\, a_\varepsilon(x,\xi)\, g(x)\, dx\, d\xi \\
&= \lim_{\varepsilon\to 0^+} \int\int \frac{e^{2\pi i (x-y)\cdot\xi}}{(1 + 4\pi^2|\xi|^2)^N} (1 - \Delta_x)^N [a_\varepsilon(x,\xi)\, g(x)]\, dx\, d\xi.
\end{aligned} \tag{2.20}$$

Now,

$$(1 - \Delta_x)^N [a_\varepsilon(x,\xi)\, g(x)] \subset \operatorname{span}\{\partial_x^\delta a_\varepsilon(x,\xi)\, \partial_x^\theta g(x), \text{ where } |\delta + \theta| \leqslant 2N\}. \tag{2.21}$$

Since $\{a_\varepsilon\}_{0<\varepsilon\leqslant 1} \subset S^m$ is bounded, we have that there exist constants $A_\delta > 0$, which do not depend on ε, and such that

$$|\partial_x^\delta a_\varepsilon(x,\xi)| \leqslant A_\delta (1 + |\xi|)^m \quad \forall (x,\xi) \in \mathbb{R}^{2n}. \tag{2.22}$$

Thus there exist constants $C_{\delta\theta} > 0$ such that

$$\int\int\left|\frac{e^{2\pi i(x-y)\cdot\xi}}{(1+4\pi^2|\xi|^2)^N}(1-\Delta_x)^N[a_\varepsilon(x,\xi)\,g(x)]\right|dx\,d\xi \leqslant \int\int\frac{1}{(1+4\pi^2|\xi|^2)^N}\left|(1-\Delta_x)^N[a_\varepsilon(x,\xi)\,g(x)]\right|dx\,d\xi$$
$$\leqslant \int\int\sum_{|\delta+\theta|\leqslant 2N} C_{\delta\theta}\frac{(1+|\xi|)^m}{(1+4\pi^2|\xi|^2)^N}\,|\partial_x^\theta g(x)|\,dx\,d\xi.$$

Choosing $2N > m+n$, we have that the integrand of the right-hand side of the above expression is an L^1 function which does not depend on $0 < \varepsilon \leqslant 1$. Therefore by (2.20) and the Dominated Convergence Theorem we get

$$T'_a g(y) = \int\int\frac{e^{2\pi i(x-y)\cdot\xi}}{(1+4\pi^2|\xi|^2)^N}(1-\Delta_x)^N[a(x,\xi)\,g(x)]\,dx\,d\xi, \tag{2.23}$$

for all $N \in \mathbb{N}$ provided $2N > m+n$.

More generally, we have the following theorem.

Theorem 2.7. *If $a \in S^m(\mathbb{R}^n\times\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$, we have that $T'_a g \in \mathcal{S}(\mathbb{R}^n)$. Moreover, the linear map*

$$\mathcal{S}(\mathbb{R}^n) \ni g \longmapsto T'_a g \in \mathcal{S}(\mathbb{R}^n)$$

is continuous.

Proof. First, we claim that $T'_a g \in C^\infty(\mathbb{R}^n)$. Indeed, let $\alpha \in \mathbb{N}_0^n$ and take $2N > m+n+|\alpha|$ on (2.23). Differentiation under the integral sign gives

$$\partial_y^\alpha T'_a g(y) = \int\int e^{2\pi i(x-y)\cdot\xi}\frac{(-2\pi i\xi)^\alpha}{(1+4\pi^2|\xi|^2)^N}(1-\Delta_x)^N[a(x,\xi)\,g(x)]\,dx\,d\xi. \tag{2.24}$$

By equations (2.21)–(2.22) we obtain that there exist constants $C_{\delta\theta\alpha} > 0$ such that

$$\left|\partial_y^\alpha T'_a g(y)\right| = \left|\int\int e^{2\pi i(x-y)\cdot\xi}\frac{(-2\pi i\xi)^\alpha}{(1+4\pi^2|\xi|^2)^N}(1-\Delta_x)^N[a(x,\xi)\,g(x)]\,dx\,d\xi\right|$$
$$\leqslant \int\int\frac{|2\pi\xi|^{|\alpha|}}{(1+4\pi^2|\xi|^2)^N}\left|(1-\Delta_x)^N[a(x,\xi)\,g(x)]\right|dx\,d\xi$$
$$\leqslant \int\int\sum_{|\delta+\theta|\leqslant 2N} C_{\delta\theta\alpha}\frac{(1+|\xi|)^{m+|\alpha|}}{(1+4\pi^2|\xi|^2)^N}\,|\partial_x^\theta g(x)|\,dx\,d\xi.$$

Since this is true for any multiindex α we conclude that $T'_a g \in C^\infty(\mathbb{R}^n)$.

Second, we simultaneously show that $T'_a g \in \mathcal{S}$ and that the map $g \mapsto T'_a g$ is linear and continuous.

The linearity of T'_a is clear from (2.23). As for continuity we proceed as follows: For any $\alpha,\beta \in \mathbb{N}_0^n$, we take $2M \geqslant |\beta|$ and $2N > m+n+|\alpha|$. By (2.24) we have that

$$y^\beta\partial_y^\alpha T'_a g(y) = y^\beta\int\int e^{2\pi i(x-y)\cdot\xi}\frac{(-2\pi i\xi)^\alpha}{(1+4\pi^2|\xi|^2)^N}(1-\Delta_x)^N[a(x,\xi)\,g(x)]\,dx\,d\xi. \tag{2.25}$$

If we set

$$L_\xi = (1+4\pi^2|x-y|^2)^{-1}(1-\Delta_\xi),$$

then $L_\xi(e^{2\pi i(x-y)\cdot\xi}) = e^{2\pi i(x-y)\cdot\xi}$. Inserting this operator M times on the right-hand side of (2.25) and using integration by parts $2M$ times we have

$$\begin{aligned} y^\beta \partial_y^\alpha T_a' g(y) &= y^\beta \int\int (L_\xi)^M (e^{2\pi i(x-y)\cdot\xi}) \frac{(-2\pi i\xi)^\alpha}{(1+4\pi^2|\xi|^2)^N} (1-\Delta_x)^N [a(x,\xi)\, g(x)]\, dx\, d\xi \\ &= y^\beta \int\int \frac{e^{2\pi i(x-y)\cdot\xi}}{(1+4\pi^2|x-y|^2)^M} (1-\Delta_\xi)^M \left[\frac{(-2\pi i\xi)^\alpha}{(1+4\pi^2|\xi|^2)^N} (1-\Delta_x)^N [a(x,\xi)\, g(x)] \right] dx\, d\xi. \end{aligned}$$

Now, since for any $x, y \in \mathbb{R}^n$ we have

$$\begin{aligned} 1 + 2\pi^2|y|^2 &\leqslant 1 + 2\pi^2 \left(|x-y| + |x|\right)^2 \\ &\leqslant 1 + 2\pi^2 \left(2|x-y|^2 + 2|x|^2\right) \\ &\leqslant (1+4\pi^2|x-y|^2)(1+4\pi^2|x|^2), \end{aligned}$$

which implies

$$\frac{1}{1+4\pi^2|x-y|^2} \leqslant \frac{1+4\pi^2|x|^2}{1+2\pi^2|y|^2},$$

it follows that

$$\begin{aligned} \left|y^\beta \partial_y^\alpha T_a' g(y)\right| &\leqslant \left| y^\beta \int\int \frac{e^{2\pi i(x-y)\cdot\xi}}{(1+4\pi^2|x-y|^2)^M} (1-\Delta_\xi)^M \left[\frac{(-2\pi i\xi)^\alpha}{(1+4\pi^2|\xi|^2)^N} (1-\Delta_x)^N [a(x,\xi)\, g(x)] \right] dx\, d\xi \right| \\ &\leqslant \int\int \frac{|y|^{|\beta|}\,(1+4\pi^2|x|^2)^M}{(1+2\pi^2|y|^2)^M} \cdot \left| (1-\Delta_\xi)^M \left[\frac{(2\pi\xi)^\alpha}{(1+4\pi^2|\xi|^2)^N} (1-\Delta_x)^N [a(x,\xi)\, g(x)] \right] \right| dx\, d\xi \\ &\leqslant \int\int (1+4\pi^2|x|^2)^M \left| (1-\Delta_\xi)^M \left[\frac{(2\pi\xi)^\alpha}{(1+4\pi^2|\xi|^2)^N} (1-\Delta_x)^N [a(x,\xi)\, g(x)] \right] \right| dx\, d\xi. \end{aligned} \tag{2.26}$$

By (2.21) we know that there exist $B_{\delta\theta} \in \mathbb{R}$ such that

$$(1-\Delta_x)^N [a(x,\xi)\, g(x)] = \sum_{|\delta+\theta|\leqslant 2N} B_{\delta\theta}\, \partial_x^\delta a(x,\xi)\, \partial_x^\theta g(x).$$

By a similar argument we may obtain $D_{\delta\theta\sigma\tau} \in \mathbb{R}$ such that

$$(1-\Delta_\xi)^M \left[\frac{(2\pi\xi)^\alpha}{(1+4\pi^2|\xi|^2)^N} (1-\Delta_x)^N [a(x,\xi)\, g(x)] \right] = \sum_{\substack{|\delta+\theta|\leqslant 2N \\ |\sigma+\tau|\leqslant 2M}} D_{\delta\theta\sigma\tau}\, \partial_\xi^\sigma \frac{(2\pi\xi)^\alpha}{(1+4\pi^2|\xi|^2)^N}\, \partial_\xi^\tau \partial_x^\delta a(x,\xi) \partial_x^\theta g(x).$$

Since $a \in S^m$ we have that for all multiindices τ, δ there exists a constant $A_{\tau\delta} > 0$ such that

$$\left|\partial_\xi^\tau \partial_x^\delta a(x,\xi)\right| \leqslant A_{\tau\delta} (1+|\xi|)^{m-|\tau|} \leqslant A_{\tau\delta}(1+|\xi|)^m \quad \forall (x,\xi) \in \mathbb{R}^{2n}.$$

By (2.3) we have that for all $\sigma \in \mathbb{N}_0^n$ there exists a constant $C_{\sigma\alpha} > 0$ such that

$$\left| \partial_\xi^\sigma \frac{(2\pi\xi)^\alpha}{(1+4\pi^2|\xi|^2)^N} \right| \leqslant C_{\sigma\alpha} (1+|\xi|)^{|\alpha|-2N}.$$

Thus there exist $C_{\delta\theta\sigma\tau\alpha} > 0$ such that

$$\left| (1-\Delta_\xi)^M \left[\frac{(2\pi\xi)^\alpha}{(1+4\pi^2|\xi|^2)^N} (1-\Delta_x)^N [a(x,\xi)\, g(x)] \right] \right| \leqslant (1+|\xi|)^{m+|\alpha|-2N} \sum_{\substack{|\delta+\theta|\leqslant 2N \\ |\sigma+\tau|\leqslant 2M}} C_{\delta\theta\sigma\tau\alpha}\, |\partial_x^\theta g(x)|.$$

Inserting this inequality into the right-hand side of (2.26) and taking $C = \max\{\, C_{\delta\theta\sigma\tau\alpha} : \; |\delta+\theta| \leqslant$

$2N,\ |\sigma+\tau| \leqslant 2M\}$ we get

$$\left|y^\beta \partial_y^\alpha T_a' g(y)\right| \leqslant \sum_{\substack{|\delta+\theta|\leqslant 2N\\ |\sigma+\tau|\leqslant 2M}} C_{\delta\theta\sigma\tau\alpha} \int\int (1+|\xi|)^{m+|\alpha|-2N} (1+4\pi^2|x|^2)^M |\partial_x^\theta g(x)|\, dx\, d\xi$$

$$\leqslant \sum_{\substack{|\delta+\theta|\leqslant 2N\\ |\sigma+\tau|\leqslant 2M}} C \left(\int (1+|\xi|)^{m+|\alpha|-2N} d\xi\right)\left(\int \frac{1}{(1+4\pi^2|x|^2)^n}\, dx\right) \sup_{x\in\mathbb{R}^n} \left|(1+4\pi^2|x|^2)^{M+n} |\partial_x^\theta g(x)|\right|.$$

Since the right-hand side is independent of $y \in \mathbb{R}^n$ and is bounded by a continuous seminorm on $\mathcal{S}(\mathbb{R}^n)$ we conclude our result. □

Proposition 2.8 (Pseudo-differential operators on $\mathcal{S}'$). *Let $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$. Given $u \in \mathcal{S}'$, define $\widetilde{T}_a u$ as*

$$\langle \widetilde{T}_a u,\, g\rangle := \langle u,\, T_a' g\rangle \quad \text{for all } g \in \mathcal{S}.$$

Then $\widetilde{T}_a u \in \mathcal{S}'$. Moreover, the linear map $\widetilde{T}_a\colon \mathcal{S}' \to \mathcal{S}'$ is continuous and it extends the pseudo-differential operator $T_a : \mathcal{S} \to \mathcal{S}$ to the space of tempered distributions.

Proof. Let us recall that $u \in \mathcal{S}'$ means that there exist $k, m \in \mathbb{N}_0$ and a constant $C > 0$ such that

$$|\langle u,\, g\rangle| \leqslant C \sum_{\substack{|\alpha|\leqslant k\\ |\beta|\leqslant m}} \sup_{x\in\mathbb{R}^n} \left|x^\beta \partial_x^\alpha g(x)\right| \quad \text{for all } g \in \mathcal{S}.$$

Therefore by definition we have

$$|\langle \widetilde{T}_a u,\, g\rangle| = \left|\langle u,\, T_a' g\rangle\right| \leqslant C \sum_{\substack{|\alpha|\leqslant k\\ |\beta|\leqslant m}} \sup_{x\in\mathbb{R}^n} \left|x^\beta \partial_x^\alpha T_a' g(x)\right| \quad \text{for all } g \in \mathcal{S}.$$

By Theorem 2.7 we have that $T_a' : \mathcal{S} \to \mathcal{S}$ is linear and continuous, thus implying that $\widetilde{T}_a u \in \mathcal{S}'$.

It is also clear by the definition that $\widetilde{T}_a\colon \mathcal{S}' \to \mathcal{S}'$ is linear. For the continuity we note that if $\{u_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{S}'$ such that $u_n \to u$ in $\mathcal{S}'$ for some $u \in \mathcal{S}'$, then for all $g \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\langle \widetilde{T}_a u_n,\, g\rangle = \langle u_n,\, T_a' g\rangle \xrightarrow{n\to\infty} \langle u,\, T_a' g\rangle = \langle \widetilde{T}_a u,\, g\rangle,$$

showing that $(\widetilde{T}_a u_n) \to (\widetilde{T}_a u)$ in $\mathcal{S}'$.

To check that $\widetilde{T}_a$ is the extension of T_a to the set of tempered distribuitions, we note that $\mathcal{S}$ is a dense subset of $\mathcal{S}'$ and thus T_a admits at most one continuous extension to $\mathcal{S}'$. We also note that if $f \in \mathcal{S}(\mathbb{R}^n)$, we have by formula (2.18) that

$$\langle \widetilde{T}_a f,\, g\rangle = \langle f,\, T_a' g\rangle = \langle T_a f,\, g\rangle \quad \forall g \in \mathcal{S},$$

thus $\widetilde{T}_a\restriction_{\mathcal{S}} = T_a$. Since $\widetilde{T}_a\colon \mathcal{S}' \to \mathcal{S}'$ is continuous we conclude our result. □

From now on we shall make no distinction between the pseudo-differential operator T_a initially defined on $\mathcal{S}$ and its extension to $\mathcal{S}'$.

The next result is about how one can determine if a given operator is a pseudo-differential operator simply by testing it against a plane wave of the form $e^{2\pi i x\cdot\xi}$.

Theorem 2.9. *A continuous linear operator $T\colon \mathcal{S}' \to \mathcal{S}'$ is a pseudo-differential operator with symbol $a \in S^m$ if and only if*

$$a(x,\xi) = e^{-2\pi i x\cdot\xi}\, T(e^{2\pi i x\cdot\xi}) \in S^m. \tag{2.27}$$

In particular, a pseudo-differential operator $T \in \Psi^m$ defines its symbol $a \in S^m$ uniquely by formula (2.27), so that $T = T_a$.

Before we jump into the proof of Theorem 2.9, let us give a formal reasoning for one of the implications. If $u \in \mathcal{S}(\mathbb{R}^n)$, then the Fourier inversion formula gives us

$$u(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\,\widehat{u}(\xi)\,d\xi.$$

Therefore if $T(e^{2\pi i x\cdot\xi}) = e^{2\pi i x\cdot\xi}\,a(x,\xi)$ for some $a \in S^m$ we would have

$$\begin{aligned} T(u)(x) &= T\left(\int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\,\widehat{u}(\xi)\,d\xi\right) \\ &= \int_{\mathbb{R}^n} T(e^{2\pi i x\cdot\xi})\,\widehat{u}(\xi)\,d\xi \\ &= \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\,a(x,\xi)\,\widehat{u}(\xi)\,d\xi \end{aligned}$$

and we would conclude that $T = T_a$. To justify the above argument rigorously, we will need a convergence lemma for the Riemann sums defining the Fourier transform.

Let us denote the cube of center $c \in \mathbb{R}^n$ and length size $h > 0$ by $Q(c,h)$, i.e, $Q(c,h) = \{x \in \mathbb{R}^n\colon\ |x_i - c_i| \leqslant h/2 \text{ for every } 1 \leqslant i \leqslant n\}$.

Lemma 2.10. *Let $u \in \mathcal{S}$. For each $0 < h \leqslant 1$, define $u_h \in C^\infty(\mathbb{R}^n)$ by*

$$u_h(x) = \sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})}} e^{2\pi i x\cdot hk}\,\widehat{u}(hk)\,h^n.$$

Then $u_h \to u$ in $C^\infty(\mathbb{R}^n)$ as $h \to 0^+$. Moreover, $u_h \to u$ in $\mathcal{S}'$.

Proof. We will show that given $\varepsilon > 0$, $j \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$, there exists $h_0 \in (0,1]$ such that

$$\sup_{x\in\mathbb{B}(0,j)} |\partial_x^\alpha u(x) - \partial_x^\alpha u_h(x)| < \varepsilon \quad \text{for all } 0 < h \leqslant h_0. \tag{2.28}$$

Thus let $\varepsilon > 0$ and fix $0 < h \leqslant 1$. Note that $\mathbb{R}^n = \bigcup_{k\in\mathbb{Z}^n} Q(kh,h)$ and that those cubes have disjoint interior. Since $u \in \mathcal{S}$, we have by the Fourier inversion formula that

$$\partial_x^\alpha u(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\,(2\pi i\xi)^\alpha\,\widehat{u}(\xi)\,d\xi = \sum_{k\in\mathbb{Z}^n}\int_{Q(kh,h)} e^{2\pi i x\cdot\xi}\,(2\pi i\xi)^\alpha\,\widehat{u}(\xi)\,d\xi.$$

Since $m(Q(kh,h)) = h^n$, we have that $\partial_x^\alpha u_h(x)$ can be expressed as

$$\partial_x^\alpha u_h(x) = \sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})}} e^{2\pi i x\cdot hk}\,(2\pi i hk)^\alpha\,\widehat{u}(hk)\,h^n = \sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})}} \int_{Q(kh,h)} e^{2\pi i x\cdot hk}\,(2\pi i hk)^\alpha\,\widehat{u}(hk)\,d\xi.$$

Denoting $g_\alpha(\xi) := (2\pi i\xi)^\alpha\,\widehat{u}(\xi)$, we have for each $x \in \mathbb{R}^n$

$$\begin{aligned} |\partial_x^\alpha u(x) - \partial_x^\alpha u_h(x)| \leqslant{}& \sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})}} \int_{Q(kh,h)} \left|e^{2\pi i x\cdot\xi}\,g_\alpha(\xi) - e^{2\pi i x\cdot hk}\,g_\alpha(hk)\right| d\xi \\ &+ \sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})^c}} \int_{Q(kh,h)} |g_\alpha(\xi)|\,d\xi. \end{aligned} \tag{2.29}$$

Let us analyse the second term on the right-hand side of (2.29). First, we note that this term does not depend on the point $x \in \mathbb{R}^n$, implying that our estimate will hold uniformly on $x \in \mathbb{R}^n$. Now, $k \in \mathbb{B}(0,h^{-2})^c$ implies that $h|k| \geqslant 1/h$, and since for every $k \in \mathbb{Z}^n$ and $h > 0$ we have $Q(hk,h) \subset \mathbb{B}(hk, \frac{\sqrt{n}h}{2})$, it follows that

$$\bigcup_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})^c}} Q(hk,h) \subset \bigcup_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})^c}} \mathbb{B}(hk, \frac{\sqrt{n}h}{2}) \subset \mathbb{B}\left(0, \frac{1}{h} - \frac{\sqrt{n}h}{2}\right)^c \quad \text{for all } 0 < h \leqslant \min\left\{1, \frac{\sqrt{2}}{n^{1/4}}\right\}.$$

Because $g_\alpha \in \mathcal{S}$ we conclude by the Dominated Convergence Theorem that

$$\sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})^c}} \int_{Q(kh,h)} |g_\alpha(\xi)|\, d\xi \leqslant \int_{\mathbb{B}\left(0,\frac{1}{h}-\frac{\sqrt{n}h}{2}\right)^c} |g_\alpha(\xi)|\, d\xi \to 0 \quad \text{as } h \to 0^+.$$

In particular, there exists $h_1 \in (0,1]$ such that

$$\sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})^c}} \int_{Q(kh,h)} |g_\alpha(\xi)|\, d\xi \leqslant \frac{\varepsilon}{2} \quad \text{for all } 0 < h \leqslant h_1.$$

Now we analyse the first term on the right-hand side of (2.29). Consider $f(\xi) = e^{2\pi i x\cdot\xi} g_\alpha(\xi) \in \mathcal{S}$. For each $k \in \mathbb{Z}^n$ and for any $\eta \in \mathbb{R}^n$ we may use Taylor's formula to obtain

$$f(hk+\eta) = f(hk) + \sum_{|\gamma|=1} \eta^\gamma \int_0^1 \partial^\gamma f(hk+t\eta)\, dt.$$

Hence we have the inequality

$$\begin{aligned} |f(hk+\eta) - f(hk)| &\leqslant \sum_{|\gamma|=1} |\eta| \max_{t\in[0,1]} |\partial^\gamma f(hk+t\eta)| \\ &\leqslant n\,|\eta| \max_{\substack{t\in[0,1]\\ |\gamma|=1}} |\partial^\gamma f(hk+t\eta)|. \end{aligned} \tag{2.30}$$

Thus the change of variable $\xi \mapsto hk+\eta$ and equation (2.30) and gives us

$$\begin{aligned} \int_{Q(kh,h)} \left| e^{2\pi i x\cdot\xi} g_\alpha(\xi) - e^{2\pi i x\cdot hk} g_\alpha(hk)\right| d\xi &= \int_{Q(kh,h)} |f(\xi) - f(hk)|\, d\xi \\ &= \int_{Q(0,h)} |f(hk+\eta) - f(hk)|\, d\eta \\ &\leqslant n \int_{Q(0,h)} |\eta| \max_{\substack{t\in[0,1]\\ |\gamma|=1}} |\partial^\gamma f(hk+t\eta)|\, d\eta \\ &\leqslant n \frac{\sqrt{n}\, h^{n+1}}{2} \max_{\substack{y\in Q(hk,h)\\ |\gamma|=1}} |\partial^\gamma f(y)| \end{aligned}$$

for every $k \in \mathbb{Z}^n$. Consequently, we have the following inequality for the first term on the right-hand side of (2.29)

$$\sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})}} \int_{Q(kh,h)} \left| e^{2\pi i x\cdot\xi} g_\alpha(\xi) - e^{2\pi i x\cdot hk} g_\alpha(hk)\right| d\xi \leqslant \sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})}} \frac{n^{3/2}}{2} h^{n+1} \max_{\substack{y\in Q(hk,h)\\ |\gamma|=1}} |\partial^\gamma f(y)|. \tag{2.31}$$

Now, for all $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = 1$ we have $\partial^\gamma f(y) = e^{2\pi i x\cdot y}\, \partial^\gamma g_\alpha(y) + (2\pi i x)^\gamma\, e^{2\pi i x\cdot y}\, g_\alpha(y)$ and thus

$$|\partial^\gamma f(y)| \leqslant |\partial^\gamma g_\alpha(y)| + 2\pi|x|\,|g_\alpha(y)|.$$

Because $g_\alpha \in \mathcal{S}$ we have that for every $M \in \mathbb{N}$ there exists a constant $C_M > 0$ such that

$$\max_{|\gamma|=1} |\partial^\gamma f(y)| \leqslant C_M \frac{1 + 2\pi|x|}{(1+|y|)^M} \quad \text{for all } y \in \mathbb{R}^n.$$

If $y \in Q(hk, h) \subset \mathbb{B}(hk, \frac{\sqrt{n}h}{2})$, then $|y| \geqslant h|k| - h\sqrt{n}/2 \geqslant h|k| - 1/2$ for all $0 < h \leqslant 1/\sqrt{n}$. Thus,

$$\max_{\substack{y\in Q(hk,h)\\ |\gamma|=1}} |\partial^\gamma f(y)| \leqslant C_M\, 2^M \frac{1+2\pi|x|}{(1+2h|k|)^M} \quad \text{for all } 0 < h \leqslant 1/\sqrt{n}.$$

Taking $M > n$ and inserting this inequality into (2.31) we obtain

$$\begin{aligned}
\sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})}} \int_{Q(kh,h)} \left| e^{2\pi i x\cdot\xi}\, g_\alpha(\xi) - e^{2\pi i x\cdot hk}\, g_\alpha(hk)\right| d\xi &\leqslant \sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})}} \frac{n^{3/2}}{2}\, h^{n+1} \max_{\substack{y\in Q(hk,h)\\ |\gamma|=1}} |\partial^\gamma f(y)| \\
&\leqslant h\, \frac{n^{3/2}}{2}\, 2^M\, C_M \sum_{k\in\mathbb{Z}^n} \frac{(1+2\pi|x|)}{(1+2h|k|)^M}\, h^n \\
&\leqslant h\, \frac{n^{3/2}}{2}\, 2^M\, C_M'(1+2\pi|x|) \int_{\mathbb{R}^n} \frac{1}{(1+2|y|)^M}\, dy,
\end{aligned} \tag{2.32}$$

For some $C_M' > 0$. Since $\sup_{x\in\mathbb{B}(0,j)}(1+2\pi|x|) = 1 + 2\pi j$ is finite, we conclude that the right-hand side of (2.32) goes to zero as $h \to 0^+$, uniformly for $x \in \mathbb{B}(0, j)$. Hence, for the given $\varepsilon > 0$, there exists $h_2 > 0$ such that

$$\sup_{x\in\mathbb{B}(0,j)} \left| \sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})}} \int_{Q(kh,h)} \left| e^{2\pi i x\cdot\xi}\, g_\alpha(\xi) - e^{2\pi i x\cdot hk}\, g_\alpha(hk)\right| d\xi \right| \leqslant \frac{\varepsilon}{2} \quad \text{for all } 0 < h \leqslant h_2.$$

Taking $h_0 = \min\{h_1, h_2\}$ we conclude (2.28) and consequently that $u_h \to u$ in $C^\infty(\mathbb{R}^n)$.

To conclude that $u_h \to u$ in $\mathcal{S}'$, note that $u_h(x)$ is a smooth bounded function that converges pointwise to $u \in \mathcal{S}$ and that $\{u_h\}_{0<h\leqslant 1} \subset L^\infty(\mathbb{R}^n)$ uniformly on $0 < h \leqslant 1$. Hence by the Dominated Convergence Theorem

$$\lim_{h\to 0^+} |\langle u - u_h,\, f\rangle| \leqslant \lim_{h\to 0^+} \int_{\mathbb{R}^n} |u(x) - u_h(x)||f(x)|\, dx = 0 \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n),$$

finishing the proof. □

Proof of theorem 2.9. Suppose that $T\colon \mathcal{S}' \to \mathcal{S}'$ is a continuous operator that satisfies $T(e^{2\pi i x\cdot\xi}) = e^{2\pi i x\cdot\xi}\, a(x,\xi)$ for some $a \in S^m$. Then for any $u \in \mathcal{S}$ we use Lemma 2.10 to conclude that

$$
\begin{aligned}
T(u)(x) = \lim_{h\to 0^+} T(u_h)(x) &= \lim_{h\to 0^+} \sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})}} T(e^{2\pi i x\cdot hk})\,\widehat{u}(hk)\,h^n \\
&= \lim_{h\to 0^+} \sum_{\substack{k\in\mathbb{Z}^n\\ k\in\mathbb{B}(0,h^{-2})}} e^{2\pi i x\cdot hk} a(x,hk)\,\widehat{u}(hk)\,h^n \\
&= \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\,a(x,\xi)\,\widehat{u}(\xi)\,d\xi.
\end{aligned}
$$

For the other implication we must show that if $a \in S^m$, then $T_a(e^{2\pi i x\cdot\xi}) = e^{2\pi i x\cdot\xi}\,a(x,\xi)$. Indeed, for each fixed $\eta \in \mathbb{R}^n$ and each $0 < \varepsilon \leqslant 1$, consider $f_\varepsilon(x) = e^{2\pi i x\cdot\eta}\,\psi(\varepsilon x)$, where the function ψ is the same as in Proposition 2.6. It is clear that $f_\varepsilon \in \mathcal{S}$ and that $f_\varepsilon(x) \to e^{2\pi i x\cdot\eta}$ pointwise and in $\mathcal{S}'$ as $\varepsilon \to 0^+$. Now,

$$
\widehat{f}_\varepsilon(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\,e^{2\pi i x\cdot\eta}\,\psi(\varepsilon x)\,dx \overset{x\mapsto x'/\varepsilon}{=} \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} e^{2\pi i x'\cdot\frac{(\xi-\eta)}{\varepsilon}}\,\psi(x')\,dx' = \frac{1}{\varepsilon^n}\,\widehat{\psi}\left(\frac{\xi-\eta}{\varepsilon}\right).
$$

It follows from Proposition 2.8 that T_a is continuous from $\mathcal{S}'$ to itself, thus

$$
\begin{aligned}
T_a(e^{2\pi i x\cdot\eta}) = \lim_{\varepsilon\to 0^+} T_a(f_\varepsilon) &= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\,a(x,\xi)\,\widehat{f}_\varepsilon(\xi)\,d\xi \\
&= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\,a(x,\xi)\,\widehat{\psi}\left(\frac{\xi-\eta}{\varepsilon}\right)\frac{1}{\varepsilon^n}\,d\xi \qquad (2.33)\\
&= \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n} e^{2\pi i x\cdot(\eta+\varepsilon z)}\,a(x,\eta+\varepsilon z)\,\widehat{\psi}(z)\,dz.
\end{aligned}
$$

Note that this equality holds both pointwise and in $\mathcal{S}'$. By hypothesis $a \in S^m$, so that there exists $A > 0$ such that

$$
|a(x,\eta+\varepsilon z)| \leqslant A\,(1+|\eta+\varepsilon z|)^m.
$$

If $m \leqslant 0$, then $|a(x,\eta+\varepsilon z)| \leqslant A$. If $m > 0$, then $(1+|\eta+\varepsilon z|)^m \leqslant (1+|\eta|+|z|)^m$ and since $\eta \in \mathbb{R}^n$ is fixed, we have that $a(x,\eta+\varepsilon z)$ has at most polynomial growth on the z-variable (independent of ε). This implies that the absolute value of the integrand on the right-hand side of (2.33) is uniformly bounded by an L^1 function and by the Dominated Convergence Theorem we have

$$
T_a(e^{2\pi i x\cdot\eta}) = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n} e^{2\pi i x\cdot(\eta+\varepsilon z)}\,a(x,\eta+\varepsilon z)\,\widehat{\psi}(z)\,dz = e^{2\pi i x\cdot\eta}\,a(x,\eta)\int_{\mathbb{R}^n}\widehat{\psi}(z)\,dz.
$$

Since $\int \widehat{\psi}(z)\,dz = \psi(0) = 1$ we have our result. □

2.2 L^2-Boundedness

In the previous section we have shown that if $a \in S^m$ then the pseudo-differential operator associated with a, and initially defined from $\mathcal{S}$ into itself, can be extended as a continuous linear mapping from $\mathcal{S}'$ to itself. Now we want to understand how the pseudo-differential operator T_a behaves when acting on $L^2(\mathbb{R}^n)$. More precisely, we shall show that if $a \in S^0$, then the pseudo-differential operator T_a defines a bounded linear operator on L^2.

Our proof is based on E. M. Stein [18], but it is given in a more detailed and elementary fashion. The main difference is that we do not evoke any results about the Schwartz Kernel of operator T_a. For a discussion over the Schwartz Kernel of a pseudo-differential operator we recommend L. Hörmander [10, p. 69].

Now, to prove the L^2-boundedness of a pseudo-differential operator of order 0, we first observe that given any $a \in S^m$, we can define $a^\varepsilon(x,\xi) = a(x,\xi)\,\psi(\varepsilon\xi)$ where ψ is such as in Proposition 2.6. Then $T_{a^\varepsilon} f \to T_a f$ in $\mathcal{S}$ for all $f \in \mathcal{S}$ and we can write

$$\begin{aligned} T_a f(x) = \lim_{\varepsilon\to 0^+} T_{a^\varepsilon} f(x) &= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i\xi\cdot(x-y)}\, a^\varepsilon(x,\xi)\, f(y)\, dy\, d\xi \\ &= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} f(y) \left[\int_{\mathbb{R}^n} e^{2\pi i\xi\cdot(x-y)}\, a^\varepsilon(x,\xi)\, d\xi\right] dy \\ &= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} f(y)\, K_\varepsilon(x,y)\, dy, \end{aligned}$$

with $K_\varepsilon(x,y) = \displaystyle\int e^{2\pi i\xi\cdot(x-y)}\, a^\varepsilon(x,\xi)\, d\xi$.

If $x \neq y$ and we set $\mathcal{L}_\xi = (-4\pi^2|x-y|^2)^{-1}\Delta_\xi$, we have $(\mathcal{L}_\xi)(e^{2\pi i\xi\cdot(x-y)}) = e^{2\pi i\xi\cdot(x-y)}$. Inserting this operator N times on the expression of K_ε and integrating by parts $2N$ times we get

$$|K_\varepsilon(x,y)| \leqslant \frac{1}{(4\pi^2|x-y|^2)^N}\int_{\mathbb{R}^n}\left|(\Delta_\xi)^N[a^\varepsilon(x,\xi)]\right| d\xi.$$

Since $\{a^\varepsilon\}_{0<\varepsilon\leqslant 1} \subset S^m$ is bounded, we can find a constant $A_N > 0$ such that

$$\left|(\Delta_\xi)^N[a^\varepsilon(x,\xi)]\right| \leqslant A_N(1+|\xi|)^{m-2N} \quad \text{for all } (x,\xi)\in\mathbb{R}^{2n}.$$

In particular, if we take $2N > m+n$ there exists a constant B_N such that

$$|K_\varepsilon(x,y)| \leqslant \frac{1}{(4\pi^2|x-y|^2)^N}\int_{\mathbb{R}^n}\left|(\Delta_\xi)^N[a^\varepsilon(x,\xi)]\right| d\xi \leqslant \frac{B_N}{|x-y|^{2N}}.$$

Thus if $x \notin \operatorname{supp} f$ we have

$$|T_a f(x)| \leqslant B_N \int_{\operatorname{supp} f} \frac{|f(y)|}{|x-y|^{2N}}\, dy. \tag{2.34}$$

We can now move to the main result of this section.

Theorem 2.11. *Suppose $a \in S^0$. Then the operator T_a, initially defined on $\mathcal{S}(\mathbb{R}^n)$, extends to a bounded operator on $L^2(\mathbb{R}^n)$.*

Proof. Since $\mathcal{S} \subset L^2 \subset \mathcal{S}'$ with continuous and dense embeddings, we have by Proposition 1.33 that it is enough to show that there exists a constant $C > 0$ such that

$$\|T_a f\|_{L^2} \leqslant C\,\|f\|_{L^2} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n). \tag{2.35}$$

So let us prove that (2.35) holds. We proceed in 3 steps.

Step 1: Firstly, we prove that the inequality holds under the assumption that the symbol a has compact support in x and that its support is independent of ξ, i.e., there exists a compact set $K \subset \mathbb{R}^n$ such that for all $\xi \in \mathbb{R}^n$ we have $\operatorname{supp}_x a(\cdot,\xi) \subset K$.

For fixed $\xi \in \mathbb{R}^n$ we have that $a(\cdot,\xi) \in C_c^\infty(\mathbb{R}^n)$. Taking the Fourier transform on the first variable, we conclude that $\widehat{a}(y,\xi) \in \mathcal{S}$, where

$$\widehat{a}(y,\xi) = \int_{\mathbb{R}^n} e^{-2\pi i y\cdot x}\, a(x,\xi)\, dx.$$

The Fourier inversion formula gives us

$$a(x,\xi) = \int_{\mathbb{R}^n} e^{2\pi i y\cdot x}\, \widehat{a}(y,\xi)\, dy.$$

Now we note that for all $\alpha \in \mathbb{N}_0^n$ we have $\operatorname{supp}_x \partial_x^\alpha a(\cdot,\xi) \subset \operatorname{supp}_x a(\cdot,\xi) \subset K$ and that for all

$y \in \mathbb{R}^n$

$$\begin{aligned}|(2\pi i y)^\alpha \, \widehat{a}(y,\xi)| &= \left| \int_{\mathbb{R}^n} e^{-2\pi i x\cdot y} \, \partial_x^\alpha a(x,\xi)\, dx \right| \\ &\leqslant \int_K |\partial_x^\alpha a(x,\xi)|\, dx.\end{aligned}$$

By hypothesis $a \in S^0$, thus there exists a constant $A_\alpha > 0$ such that $|\partial_x^\alpha a(x,\xi)| \leqslant A_\alpha$ for all $(x,\xi) \in \mathbb{R}^{2n}$. Therefore

$$|(2\pi i y)^\alpha \, \widehat{a}(y,\xi)| \leqslant A_\alpha \, m(K) \quad \text{for all } (y,\xi) \in \mathbb{R}^{2n}.$$

As a result, for all $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that

$$\sup_{\xi \in \mathbb{R}^n} |\widehat{a}(y,\xi)| \leqslant C_N (1+|y|)^{-N} \quad \text{for all } y \in \mathbb{R}^n. \tag{2.36}$$

Now,

$$\begin{aligned}(T_a f)(x) &= \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, a(x,\xi)\, \widehat{f}(\xi)\, d\xi \\ &= \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, \widehat{f}(\xi) \left[\int_{\mathbb{R}^n} e^{2\pi i y\cdot x}\, \widehat{a}(y,\xi)\, dy \right] d\xi \\ &= \int_{\mathbb{R}^n} e^{2\pi i y\cdot x} \left[\int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, \widehat{a}(y,\xi)\, \widehat{f}(\xi)\, d\xi \right] dy \\ &= \int_{\mathbb{R}^n} e^{2\pi i y\cdot x}\, (A_y f)(x)\, dy.\end{aligned}$$

where $(A_y f)(x) = \displaystyle\int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, \widehat{a}(y,\xi)\, \widehat{f}(\xi)\, d\xi.$

Since for each fixed $y \in \mathbb{R}^n$ we have $\widehat{(A_y f)}(\xi) = \widehat{a}(y,\xi)\, \widehat{f}(\xi)$, it follows by Plancherel's identity and (2.36) that

$$\|A_y f\|_{L^2} \leqslant \sup_\xi |\widehat{a}(y,\xi)| \cdot \|\widehat{f}\|_{L^2} = \sup_\xi |\widehat{a}(y,\xi)| \cdot \|f\|_{L^2} \leqslant \frac{C_N}{(1+|y|)^N} \|f\|_{L^2}. \tag{2.37}$$

By Minkowski's Inequality for Integrals (G. Folland [5, p. 194]) and (2.37) we conclude that

$$\|T_a f\|_{L^2} \leqslant \int_{\mathbb{R}^n} \|A_y f\|_{L^2}\, dy \leqslant C_N \|f\|_{L^2} \int_{\mathbb{R}^n} (1+|y|)^{-N}\, dy.$$

Fixing an $N_0 > n$ and taking $C = C_{N_0} \int (1+|y|)^{-N_0}\, dy$ we obtain (2.35) for the case where symbol a is compactly supported on x-variable.

Step 2: For the general case where the symbol $a \in S^0$ does not need to have compact support, we shall show that for each $x_0 \in \mathbb{R}^n$ there exists a constant $C_N > 0$, which is independent of x_0, and such that

$$\int_{|x-x_0|\leqslant 1} |T_a f(x)|^2\, dx \leqslant C_N \int_{\mathbb{R}^n} \frac{|f(x)|^2}{(1+|x-x_0|)^N}\, dx, \tag{2.38}$$

for all $N > n$. To see that the above inequality implies the Theorem, let $\chi_{|x-x_0|\leqslant 1}$ be the characteristic function of the ball of radius 1 centered at x_0. Fix $N > n$ and integrate (2.38) with respect to $x_0 \in \mathbb{R}^n$ to get

$$\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \chi_{|x-x_0|\leqslant 1} |T_a f(x)|^2 \, dx \right] dx_0 \leqslant C_N \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \frac{|f(x)|^2}{(1+|x-x_0|)^N} \, dx \right] dx_0.$$

Applying Fubini-Tonelli's theorem to exchange the integration order we get

$$m(\mathbb{B}(0,1)) \int_{\mathbb{R}^n} |T_a f(x)|^2 \, dx \leqslant C_N \int_{\mathbb{R}^n} |f(x)|^2 \left[\int_{\mathbb{R}^n} \frac{1}{(1+|x-x_0|)^N} \, dx_0 \right] dx,$$

which gives the desired result.

Step 3: Let us now prove (2.38). Let $x_0 \in \mathbb{R}^n$ be an arbitrary but fixed point. Take $\phi \in C_c^\infty(\mathbb{R}^n)$ to be such that $0 \leqslant \phi \leqslant 1$, $\operatorname{supp} \phi \subset \mathbb{B}(x_0, 3)$ and $\phi \equiv 1$ on $\mathbb{B}(x_0, 2)$. Then for every $f \in \mathcal{S}$ we can write $f = \phi f + (1-\phi) f := f_1 + f_2$. Since T_a is a linear operator, we have $T_a f = T_a f_1 + T_a f_2$, and thus the pointwise inequality $|T_a f|^2 \leqslant 2(|T_a f_1|^2 + |T_a f_2|^2)$ holds, so it suffices to analyse each of those terms separately. We do the calculations for f_1 first.

Consider $\eta \in C_c^\infty(\mathbb{R}^n)$ to be such that $0 \leqslant \eta \leqslant 1$, $\operatorname{supp} \eta \subset \mathbb{B}(x_0, 2)$ and $\eta \equiv 1$ on $\mathbb{B}(x_0, 1)$. Then $\eta(x) T_a f_1 = T_{\eta a} f_1$, which is a pseudo-differential operator of order m whose symbol has compact support on the x-variable, so that the result in **STEP 1** would apply. Therefore,

$$\begin{aligned}
\int_{|x-x_0|\leqslant 1} |T_a f_1(x)|^2 \, dx &\leqslant \int_{\mathbb{R}^n} |\eta(x)|^2 |T_a f_1(x)|^2 \, dx \\
&= \int_{\mathbb{R}^n} |T_{\eta a} f_1(x)|^2 \, dx \\
&\leqslant C \int_{\mathbb{R}^n} |f_1(x)|^2 \, dx \\
&= C \int_{\mathbb{R}^n} |\phi(x) f(x)|^2 \, dx \\
&\leqslant C \int_{\mathbb{B}(x_0,3)} \left(\frac{4}{1+|x-x_0|} \right)^N |f(x)|^2 \, dx \\
&\leqslant 4^N C \int_{\mathbb{R}^n} \frac{|f(x)|^2}{(1+|x-x_0|)^N} \, dx,
\end{aligned}$$

which satisfies (2.38). For the estimate on f_2, we first note that $\operatorname{supp} f_2 \subset \mathbb{B}(x_0, 2)^c$, thus for all $x \in \mathbb{B}(x_0, 1)$ we have that formula (2.34) holds for $2N > n$. Moreover, for all $y \in \mathbb{B}(x_0, 2)^c$ and $x \in \mathbb{B}(x_0, 1)$ we have $|y - x_0| \leqslant |y - x| + |x - x_0| \leqslant 2|y - x|$, and thus

$$\begin{aligned}
|T_a f_2(x)| &\leqslant A_N \int_{\mathbb{B}(x_0,2)^c} \frac{|f_2(y)|}{|x-y|^{2N}} \, dy \\
&\leqslant 4^N A_N \int_{\mathbb{B}(x_0,2)^c} \frac{|f_2(y)|}{|y-x_0|^{2N}} \, dy \\
&\leqslant B_N \int_{\mathbb{R}^n} \frac{|f(y)|}{(1+|y-x_0|)^{N/2}} \frac{1}{(1+|y-x_0|)^{3N/2}} \, dy \\
&\leqslant B_N \left(\int_{\mathbb{R}^n} \frac{|f(y)|^2}{(1+|y-x_0|)^N} \, dy \right)^{1/2} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|y-x_0|)^{3N}} \, dy \right)^{1/2},
\end{aligned}$$

where we used Cauchy-Schwarz to get the last inequality. Squaring and integrating the above inequality on $\mathbb{B}(x_0, 1)$ gives us inequality (2.38) for f_2. Finally, since both constants found on the estimates for $T_a f_1$ and $T_a f_2$ do not depend on the choosen point $x_0 \in \mathbb{R}^n$ we conclude our proof. □

2.3 Symbolic Calculus: Composition Formula

Next we shall prove that the composition of two pseudo-differential operators is a pseudo-differential operator and we shall derive an asymptotic expression relating their symbols.

Again, our exposition is based on E. M. Stein [18], but our calculations shall be done very explicitly.

The main result about the composition of pseudo-differential operators can be stated as follows.

Theorem 2.12 (Composition formula for pseudo-differential operators)**.** *Suppose $a \in S^{m_1}$ and $b \in S^{m_2}$. Then there exists a symbol $c \in S^{m_1+m_2}$ such that*

$$T_c = T_a \circ T_b.$$

Moreover,

$$c \;\sim\; \sum_{\alpha} \frac{(2\pi i)^{-|\alpha|}}{\alpha!}(\partial_\xi^\alpha a)\cdot(\partial_x^\alpha b)$$

Where the expression above has to be interpreted in the sense that

$$c - \sum_{|\alpha|\leqslant N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!}(\partial_\xi^\alpha a)\cdot(\partial_x^\alpha b) \;\in\; S^{m_1+m_2-N-1}\;, \tag{2.39}$$

for all $N \geqslant 0$.

Note that formula (2.39) of the above Theorem implies that if $a \in S^{m_1}$ and $b \in S^{m_2}$, then the *commutator* between operators T_a and T_b, given explicitly by

$$[T_a, T_b] = T_a \circ T_b - T_b \circ T_a,$$

defines a pseudo-differential operator of order $m_1 + m_2 - 1$.

Proof. We shall prove this Theorem in several steps, dealing first with the case where one of the symbols has compact support and then moving to the general case.

STEP 1: Let us start by proving the result for any $a \in S^{m_1}$ and under the assumption that the symbol $b \in S^{m_2}$ has compact support in x and that its support is independent of ξ, i.e., there exists a compact set $K \subset \mathbb{R}^n$ such that for all $\xi \in \mathbb{R}^n$ we have $\operatorname{supp}_x b(\cdot,\xi) \subset K$.

Let $b^\varepsilon(x,\xi) = b(x,\xi)\,\psi(\varepsilon\xi)$ where ψ is such as in Proposition 2.6. Then for any $f \in \mathcal{S}$ we can write

$$T_{b^\varepsilon} f(y) = \int e^{2\pi i\xi\cdot(y-z)}\, b^\varepsilon(y,\xi)\, f(z)\, dz\, d\xi.$$

Similarly, we can take $a_\varepsilon(x,\eta) = a(x,\eta)\psi(\varepsilon x)\psi(\varepsilon\eta)$. Applying T_{a_ε} in the above expression we get

$$(T_{a_\varepsilon} T_{b^\varepsilon} f)(x) = \int\left(\int e^{2\pi i\xi\cdot(y-z)}\, b^\varepsilon(y,\xi)\, f(z)\, dz\, d\xi\right) e^{2\pi i\eta\cdot(x-y)}\, a_\varepsilon(x,\eta)\, dy\, d\eta. \tag{2.40}$$

By Fubini's theorem and the fact that $e^{2\pi i\eta\cdot(x-y)}\, e^{2\pi i\xi\cdot(y-z)} = e^{2\pi i(x-y)\cdot(\eta-\xi)}\, e^{2\pi i(x-z)\cdot\xi}$ we have

$$(T_{a_\varepsilon} T_{b^\varepsilon} f)(x) = \int c_\varepsilon(x,\xi)\, e^{2\pi i(x-z)\cdot\xi}\, f(z)\, dz\, d\xi, \tag{2.41}$$

with $c_\varepsilon(x,\xi)$ given explicitly by

$$\begin{aligned} c_\varepsilon(x,\xi) &= \int e^{2\pi i(x-y)\cdot(\eta-\xi)}\, a_\varepsilon(x,\eta)\, b^\varepsilon(y,\xi)\, dy\, d\eta \\ &= \int e^{2\pi i x\cdot(\eta-\xi)}\, a_\varepsilon(x,\eta)\, \widehat{b}^\varepsilon(\eta-\xi,\xi)\, d\eta \\ &= \int e^{2\pi i x\cdot\eta}\, a_\varepsilon(x,\xi+\eta)\, \widehat{b}^\varepsilon(\eta,\xi)\, d\eta, \end{aligned} \tag{2.42}$$

where $\widehat{b}^\varepsilon(\cdot,\xi)$ is the Fourier transform of $b^\varepsilon(\cdot,\xi)$ as a function of $y \in \mathbb{R}^n$. Note that, for each fixed $0 < \varepsilon \leqslant 1$, this leads us to the following equality

$$(T_{a_\varepsilon} T_{b^\varepsilon} f)(x) = T_{c_\varepsilon} f(x) \quad \forall f \in \mathcal{S}. \tag{2.43}$$

STEP 1.1 - Asymptotic Formula: By hypothesis $b \in S^{m_2}$ has compact support in x, and thus for each fixed $\xi \in \mathbb{R}^n$, we have that $b^\varepsilon(\cdot,\xi) \in C_c^\infty(\mathbb{R}^n)$ which implies that $\widehat{b}^\varepsilon(\cdot,\xi) \in \mathcal{S}(\mathbb{R}^n)$. More explicitly, for each multiindex α and for each $\xi \in \mathbb{R}^n$ we have

$$(2\pi i\eta)^\alpha \widehat{b}^\varepsilon(\eta,\xi) = \widehat{(\partial_x^\alpha b^\varepsilon)}(\eta,\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\eta}\, \partial_x^\alpha b^\varepsilon(x,\xi)\, dx.$$

As $\operatorname{supp}_x \partial_x^\alpha b^\varepsilon(x,\xi) \subset \operatorname{supp}_x b^\varepsilon(x,\xi) \subset K$, we conclude that

$$\begin{aligned} \left|(2\pi i\eta)^\alpha \widehat{b}^\varepsilon(\eta,\xi)\right| &= \left|\int_{\mathbb{R}^n} e^{-2\pi i x\cdot\eta}\, \partial_x^\alpha b^\varepsilon(x,\xi)\, dx\right| \\ &\leqslant \int_K |\partial_x^\alpha b^\varepsilon(x,\xi)|\, dx \\ &\leqslant A_\alpha (1+|\xi|)^{m_2}\, m(K). \end{aligned}$$

This implies, in particular, that for every $M \in \mathbb{N}$ there exists a constant A_M such that

$$|\widehat{b}^\varepsilon(\eta,\xi)| \leqslant A_M (1+|\xi|)^{m_2} (1+|\eta|)^{-M}. \tag{2.44}$$

We now proceed to make a careful analysis of the right-hand side of (2.42). Using Taylor's formula we write

$$a_\varepsilon(x,\xi+\eta) = \sum_{|\alpha|\leqslant N} \frac{1}{\alpha!} \partial_\xi^\alpha a_\varepsilon(x,\xi)\eta^\alpha + R_N^\varepsilon(x,\xi,\eta), \tag{2.45}$$

where

$$R_N^\varepsilon(x,\xi,\eta) = \sum_{|\alpha|=N+1} \frac{\eta^\alpha}{\alpha!} \int_0^1 (1-\theta)^N \partial_\xi^\alpha a_\varepsilon(x,\xi+\theta\eta)\, d\theta.$$

Inserting (2.45) into (2.42) we get

$$\begin{aligned} c_\varepsilon(x,\xi) &= \int e^{2\pi i x\cdot\eta}\, a_\varepsilon(x,\xi+\eta)\, \widehat{b}^\varepsilon(\eta,\xi)\, d\eta \\ &= \sum_{|\alpha|\leqslant N} \frac{1}{\alpha!} \int e^{2\pi i x\cdot\eta}\, \partial_\xi^\alpha a_\varepsilon(x,\xi)\eta^\alpha \widehat{b}^\varepsilon(\eta,\xi)\, d\eta + \int e^{2\pi i x\cdot\eta}\, R_N^\varepsilon(x,\xi,\eta)\, \widehat{b}^\varepsilon(\eta,\xi)\, d\eta. \end{aligned} \tag{2.46}$$

Now, each term in the first sum of the above equality contributes with

$$\frac{1}{\alpha!} \int \partial_\xi^\alpha a_\varepsilon(x,\xi)\eta^\alpha\, \widehat{b}^\varepsilon(\eta,\xi)\, e^{2\pi i x\cdot\eta}\, d\eta = \frac{(2\pi i)^{-|\alpha|}}{\alpha!}\, \partial_\xi^\alpha a_\varepsilon(x,\xi) \int (2\pi i)^{|\alpha|}\eta^\alpha\, \widehat{b}^\varepsilon(\eta,\xi)\, e^{2\pi i x\cdot\eta}\, d\eta,$$

which upon using the Fourier inversion formula on $(2\pi i)^{|\alpha|}\eta^\alpha\,\widehat{b}^\varepsilon(\eta,\xi)$ gives us

$$\frac{(2\pi i)^{-|\alpha|}}{\alpha!}\,\partial_\xi^\alpha a_\varepsilon(x,\xi)\cdot\partial_x^\alpha b^\varepsilon(x,\xi),$$

and this is the term corresponding to each α in formula (2.39), except it carries $0<\varepsilon\leqslant 1$.

STEP 1.2 - Remainder Term: We will now show that the second term on the right-hand side of (2.46) belongs to $S^{m_1+m_2-N-1}$ uniformly in $0<\varepsilon\leqslant 1$. More precisely, we will show that given any multiindices γ,β there exists a constant $A_{\gamma\beta}>0$ such that

$$\left|\partial_x^\gamma\partial_\xi^\beta\int e^{2\pi i x\cdot\eta}\,R_N^\varepsilon(x,\xi,\eta)\,\widehat{b}^\varepsilon(\eta,\xi)\,d\eta\right|\leqslant A_{\gamma\beta}(1+|\xi|)^{m_1+m_2-N-1-|\beta|}\quad\forall x,\xi\in\mathbb{R}^n,\tag{2.47}$$

uniformly in $0<\varepsilon\leqslant 1$.

Note that differentiation under the integral sign gives us

$$\int\partial_x^\gamma\partial_\xi^\beta[e^{2\pi i x\cdot\eta}\,R_N^\varepsilon(x,\xi,\eta)\,\widehat{b}^\varepsilon(\eta,\xi)]\,d\eta.$$

By Leibniz's rule we get that

$$\partial_x^\gamma\partial_\xi^\beta[e^{2\pi i x\cdot\eta}\,R_N^\varepsilon(x,\xi,\eta)\,\widehat{b}^\varepsilon(\eta,\xi)]\subset\operatorname{span}\{e^{2\pi i x\cdot\eta}\,\eta^\delta\,\partial_\xi^\sigma\widehat{b}^\varepsilon(\eta,\xi)\,\partial_x^{\gamma-\delta}\partial_\xi^{\beta-\sigma}R_N^\varepsilon(x,\xi,\eta):\ \delta\leqslant\gamma,\sigma\leqslant\beta\}.$$

Since

$$\left|\int e^{2\pi i x\cdot\eta}\,\eta^\delta\,\partial_\xi^\sigma\widehat{b}^\varepsilon(\eta,\xi)\,\partial_x^{\gamma-\delta}\partial_\xi^{\beta-\sigma}R_N^\varepsilon(x,\xi,\eta)\,d\eta\right|\leqslant\int|\eta|^{|\delta|}\,|\partial_\xi^\sigma\widehat{b}^\varepsilon(\eta,\xi)|\,|\partial_x^{\gamma-\delta}\partial_\xi^{\beta-\sigma}R_N^\varepsilon(x,\xi,\eta)|\,d\eta,$$

we may obtain (2.47) by showing that there exist constants $C_{\gamma\beta\delta\sigma}>0$ such that

$$\int|\eta|^{|\delta|}\,|\partial_\xi^\sigma\widehat{b}^\varepsilon(\eta,\xi)|\,|\partial_x^{\gamma-\delta}\partial_\xi^{\beta-\sigma}R_N^\varepsilon(x,\xi,\eta)|\,d\eta\leqslant C_{\gamma\beta\delta\sigma}(1+|\xi|)^{m_1+m_2-N-1-|\beta|}\tag{2.48}$$

for all $x,\xi\in\mathbb{R}^n$ and uniformly in $0<\varepsilon\leqslant 1$. Let us prove the above inequality.

First, we observe that for any multiindices γ',β' we have

$$\partial_x^{\gamma'}\partial_\xi^{\beta'}R_N^\varepsilon(x,\xi,\eta)=\sum_{|\alpha|=N+1}\frac{\eta^\alpha}{\alpha!}\int_0^1(1-\theta)^N\partial_x^{\gamma'}\partial_\xi^{\alpha+\beta'}a_\varepsilon(x,\xi+\theta\eta)\,d\theta.$$

Thus we can find a constant $C_{N\gamma'\beta'}>0$, which does not depend on $0<\varepsilon\leqslant 1$, and such that

$$\begin{aligned}|\partial_x^{\gamma'}\partial_\xi^{\beta'}R_N^\varepsilon(x,\xi,\eta)|&\leqslant\sum_{|\alpha|=N+1}\frac{|\eta|^{N+1}}{\alpha!}\max\{\,|\partial_x^{\gamma'}\partial_\xi^{\alpha+\beta'}a_\varepsilon(x,\zeta)|:\ \xi\leqslant\zeta\leqslant\xi+\eta\}\\&\leqslant C_{N\gamma'\beta'}\,|\eta|^{N+1}\,\max\{\,(1+|\zeta|)^{m_1-N-1-|\beta'|}:\ \xi\leqslant\zeta\leqslant\xi+\eta\},\end{aligned}$$

for all $x,\xi,\eta\in\mathbb{R}^n$. We now consider the following different cases for our estimates.

Case 1: When $|\eta|\leqslant|\xi|/2$, we have $|\xi|/2\leqslant|\zeta|\leqslant 3|\xi|/2$, and therefore there exists a constant $A_{N\gamma'\beta'}>0$ such that

$$|\partial_x^{\gamma'}\partial_\xi^{\beta'}R_N^\varepsilon(x,\xi,\eta)|\leqslant A_{N\gamma'\beta'}|\eta|^{N+1}\,(1+|\xi|)^{m_1-N-1-|\beta'|}\quad\forall x\in\mathbb{R}^n.$$

Case 2.1: When $|\eta|>|\xi|/2$ and $m_1-N-1-|\beta'|>0$, we use that $|\zeta|\leqslant|\xi|+|\eta|\leqslant 3|\eta|$ to obtain a constant $A_{N\gamma'\beta'}>0$ such that

$$|\partial_x^{\gamma'}\partial_\xi^{\beta'}R_N^\varepsilon(x,\xi,\eta)|\leqslant A_{N\gamma'\beta'}|\eta|^{N+1}\,(1+|\eta|)^{m_1-N-1-|\beta'|}\quad\forall x\in\mathbb{R}^n.$$

Case 2.2: When $|\eta| > |\xi|/2$ and $m_1 - N - 1 - |\beta'| \leqslant 0$, we obtain a constant $A_{N\gamma'\beta'} > 0$ such that

$$|\partial_x^{\gamma'} \partial_\xi^{\beta'} R_N^\varepsilon(x, \xi, \eta)| \leqslant A_{N\gamma'\beta'} |\eta|^{N+1} \quad \forall x \in \mathbb{R}^n.$$

Now, in a similar fashion as in (2.44), we have that for every $M \in \mathbb{N}$ and $\sigma \in \mathbb{N}_0^n$ there exists constants $A_{M\sigma} > 0$ such that

$$|\partial_\xi^\sigma \widehat{b}^\varepsilon(\eta, \xi)| \leqslant A_{M\sigma}(1 + |\xi|)^{m_2 - |\sigma|}(1 + |\eta|)^{-M}. \tag{2.49}$$

Breaking the integral on the left-hand side of (2.48) according to cases 1 and 2 above we get

$$\int_{|\eta| \leqslant |\xi|/2} |\eta|^{|\delta|} \, |\partial_\xi^\sigma \widehat{b}^\varepsilon(\eta, \xi)| \, |\partial_x^{\gamma-\delta} \partial_\xi^{\beta-\sigma} R_N^\varepsilon(x, \xi, \eta)| \, d\eta \\ + \int_{|\eta| > |\xi|/2} |\eta|^{|\delta|} \, |\partial_\xi^\sigma \widehat{b}^\varepsilon(\eta, \xi)| \, |\partial_x^{\gamma-\delta} \partial_\xi^{\beta-\sigma} R_N^\varepsilon(x, \xi, \eta)| \, d\eta \tag{2.50}$$

For the first integral we use **Case 1** together with (2.49) to obtain a constant $A > 0$, which depends on $\gamma, \beta, \delta, \sigma, M$ and N, but does not depend on $0 < \varepsilon \leqslant 1$, and such that

$$\int_{|\eta| \leqslant |\xi|/2} |\eta|^{|\delta|} \, |\partial_\xi^\sigma \widehat{b}^\varepsilon(\eta, \xi)| \, |\partial_x^{\gamma-\delta} \partial_\xi^{\beta-\sigma} R_N^\varepsilon(x, \xi, \eta)| \, d\eta \leqslant A \, (1+|\xi|)^{m_1+m_2-N-1-|\beta|} \int_{\mathbb{R}^n} |\eta|^{N+1+|\delta|}(1+|\eta|)^{-M} \, d\eta.$$

It is enough to take $M > n + N + 1 + |\delta|$ to obtain our desired inequality.

We shall do only **Case 2.2** for the estimate of the second integral in (2.50) since **Case 2.1** follows by a similar calculation. Now, if $m_1 - N - 1 - |\beta'| \leqslant 0$, it follows from **Case 2.2** and (2.49) that we may obtain a constant $B > 0$, which depends on $\gamma, \beta, \delta, \sigma, M$ and N, but does not depend on $0 < \varepsilon \leqslant 1$, and such that

$$\begin{aligned} \int_{|\eta| > |\xi|/2} |\eta|^{|\delta|} \, |\partial_\xi^\sigma \widehat{b}^\varepsilon(\eta, \xi)| \, |\partial_x^{\gamma-\delta} \partial_\xi^{\beta-\sigma} R_N^\varepsilon(x, \xi, \eta)| \, d\eta &\leqslant B(1 + |\xi|)^{m_2 - |\sigma|} \int_{|\eta| > |\xi|/2} |\eta|^{N+1+|\delta|} \, (1 + |\eta|)^{-M} \, d\eta \\ &\leqslant B(1 + |\xi|)^{m_2 - |\sigma|} \int_{|\eta| > |\xi|/2} (1 + |\eta|)^{N+1+|\delta|-M} \, d\eta. \end{aligned}$$

Note that if $M > N + 1 + |\delta| + n$, then the last integral is absolutely convergent. Moreover, since the integrand is a radial function, we may use spherical coordinates to get

$$\int_{|\eta| > |\xi|/2} (1 + |\eta|)^{N+1+|\delta|-M} \, d\eta = \omega_{n-1} \int_{|\xi|/2}^\infty (1 + |r|)^{N+|\delta|+n-M} \, dr,$$

where ω_{n-1} denotes the surface area of the $(n-1)$-sphere of radius 1. Setting $M > N + |\delta| + n + k + 1$, where $k \in \mathbb{N}$ is still to be determined, we have that

$$\omega_{n-1} \int_{|\xi|/2}^\infty (1 + |r|)^{N+|\delta|+n-M} \, dr \leqslant \omega_{n-1} \int_{|\xi|/2}^\infty (1 + |r|)^{-k-1} \, dr = \frac{\omega_{n-1}}{k}(1 + |\xi|/2)^{-k}.$$

Hence, there exists a constant $C > 0$, which does not depend on $0 < \varepsilon \leqslant 1$ and such that

$$\int_{|\eta| > |\xi|/2} |\eta|^{|\delta|} \, |\partial_\xi^\sigma \widehat{b}^\varepsilon(\eta, \xi)| \, |\partial_x^{\gamma-\delta} \partial_\xi^{\beta-\sigma} R_N^\varepsilon(x, \xi, \eta)| \, d\eta \leqslant C(1 + |\xi|)^{m_2-|\sigma|-k}.$$

As $k \in \mathbb{N}$ can be choosen arbitrarily large, we may take it so that the above expression is bounded by a constant times $(1 + |\xi|)^{m_1+m_2-N-1-|\beta|}$.

Now that we have checked that (2.47) holds, we may proceed to the last step in the proof of the theorem for the scenario where $b \in S^{m_2}$ has x-compact support.

STEP 1.3 - Symbol $c \in S^{m_1+m_2}$: Combining the results proved so far, we conclude that $c_\varepsilon \in S^{m_1+m_2}$ uniformly in $0 < \varepsilon \leqslant 1$ and that for every $N \in \mathbb{N}$ we may write c_ε explicitly as

$$\begin{aligned} c_\varepsilon(x,\xi) &= \int e^{2\pi i x\cdot\eta}\, a_\varepsilon(x,\xi+\eta)\, \widehat{b^\varepsilon}(\eta,\xi)\, d\eta \\ &= \sum_{|\alpha|\leqslant N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha a_\varepsilon(x,\xi)\cdot \partial_x^\alpha b^\varepsilon(x,\xi) + \int e^{2\pi i x\cdot\eta}\, R_N^\varepsilon(x,\xi,\eta)\, \widehat{b^\varepsilon}(\eta,\xi)\, d\eta. \end{aligned}$$

By the Dominated Convergence Theorem and Proposition 2.6 we conclude that there exists $c \in S^{m_1+m_2}$ such that $c_\varepsilon \to c$ pointwise and in $S^{m'}$ for all $m' > m_1+m_2$. Moreover, we can write $c(x,\xi)$ explicitly as

$$\begin{aligned} c(x,\xi) &= \int e^{2\pi i x\cdot\eta}\, a(x,\xi+\eta)\, \widehat{b}(\eta,\xi)\, d\eta \\ &= \sum_{|\alpha|\leqslant N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x,\xi)\cdot \partial_x^\alpha b(x,\xi) + \int e^{2\pi i x\cdot\eta}\, R_N(x,\xi,\eta)\, \widehat{b}(\eta,\xi)\, d\eta. \end{aligned}$$

Finally, taking the limit as $\varepsilon \to 0^+$ in (2.43) and using Theorem 2.5 leads us to the following equality

$$(T_aT_bf)(x) = \lim_{\varepsilon\to 0^+} (T_{a_\varepsilon}T_{b^\varepsilon}f)(x) = \lim_{\varepsilon\to 0^+} T_{c_\varepsilon}f(x) = T_cf(x) \quad \forall f \in \mathcal{S},$$

and this finishes the proof for the case when $b \in S^{m_2}$ has compact x-support.

STEP 2 - General case: To consider the case when we do not assume that $b \in S^{m_2}$ has x-compact support we do as follows: Start by considering an arbitrary but fixed point $x_0 \in \mathbb{R}^n$. Take $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leqslant \phi \leqslant 1$, $\operatorname{supp}\phi \subset \mathbb{B}(x_0,2)$ and $\phi \equiv 1$ on $\mathbb{B}(x_0,1)$ and write $b = \phi b + (1-\phi)b := b' + b''$. It is clear that $b', b'' \in S^{m_2}$ and that

$$T_aT_bf(x) = T_aT_{b'}f(x) + T_aT_{b''}f(x) \quad \forall f \in \mathcal{S}.$$

Since $\operatorname{supp} b' \subset \mathbb{B}(x_0,2)$, it follows from **STEP 1** that there exists $c' \in S^{m_1+m_2}$ such that $T_aT_{b'} = T_{c'}$ and

$$c' - \sum_{|\alpha|\leqslant N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} (\partial_\xi^\alpha a)\cdot(\partial_x^\alpha b') \in S^{m_1+m_2-N-1} \tag{2.51}$$

for all $N \geqslant 0$.

We now show that there exists a symbol c'' such that $T_aT_{b''} = T_{c''}$ and such that for all $x \in \mathbb{B}(x_0,1/2)$ we have: For all γ,β multiindices there exists a constant $C_{N\gamma\beta}$ such that

$$|\partial_x^\gamma \partial_\xi^\beta c''(x,\xi)| \leqslant C_{N\gamma\beta}(1+|\xi|)^{m_1+m_2-N-1-|\beta|} \quad \text{for all } N \geqslant 0. \tag{2.52}$$

To see this, fix $0 < \varepsilon \leqslant 1$ and consider the symbols $a_\varepsilon, b''_\varepsilon$. By (2.40)–(2.42) we have $T_{a_\varepsilon}T_{b''_\varepsilon} = T_{c''_\varepsilon}$ with

$$c''_\varepsilon(x,\xi) = \int_{\mathbb{R}^n}\int_{\mathbb{B}(x_0,1)^c} e^{2\pi i(x-y)\cdot(\eta-\xi)}\, a_\varepsilon(x,\eta)\, b''_\varepsilon(y,\xi)\, dy\, d\eta.$$

In the above integral we note that $|x-y| \geqslant |x_0-y| - 1/2 > 0$ for all $x \in \mathbb{B}(x_0,1/2)$. We set $\mathcal{L}_\eta = (-4\pi^2|x-y|^2)^{-1}\Delta_\eta$ and thus

$$(\mathcal{L}_\eta)(e^{2\pi i(x-y)\cdot(\eta-\xi)}) = e^{2\pi i(x-y)\cdot(\eta-\xi)}.$$

Inserting this operator N_1 times on the above integral and integrating by parts $2N_1$ times in the

η-variable we get

$$c''_\varepsilon(x,\xi) = \int_{\mathbb{R}^n}\int_{\mathbb{B}(x_0,1)^c} e^{2\pi i(x-y)\cdot(\eta-\xi)}\,\frac{\Delta_\eta^{N_1} a_\varepsilon(x,\eta)}{(-4\pi^2|x-y|^2)^{N_1}}\, b''_\varepsilon(y,\xi)\,dy\,d\eta.$$

Next, we set $L_y = (1+4\pi^2|\xi-\eta|^2)^{-1}(1-\Delta_y)$, insert this operator N_2 times on the above integral, and integrate by parts $2N_2$ times with respect to the y-variable using the identity

$$L_y(e^{2\pi i(x-y)\cdot(\eta-\xi)}) = e^{2\pi i(x-y)\cdot(\eta-\xi)}.$$

This calculation gives us the equality

$$c''_\varepsilon(x,\xi) = \int_{\mathbb{R}^n}\int_{\mathbb{B}(x_0,1)^c} e^{2\pi i(x-y)\cdot(\eta-\xi)}\,\frac{\Delta_\eta^{N_1} a_\varepsilon(x,\eta)}{(1+4\pi^2|\xi-\eta|^2)^{N_2}}\,(1-\Delta_y)^{N_2}\left[\frac{b''_\varepsilon(y,\xi)}{(-4\pi^2|x-y|^2)^{N_1}}\right] dy\,d\eta. \tag{2.53}$$

Since

$$\begin{aligned} \left|\Delta_\eta^{N_1} a_\varepsilon(x,\eta)\right| &\leqslant A_{N_1}(1+|\eta|)^{m_1-2N_1}, \\ \left|(1-\Delta_y)^{N_2}\left[\frac{b''_\varepsilon(y,\xi)}{(-4\pi^2|x-y|^2)^{N_1}}\right]\right| &\leqslant A_{N_1N_2}\frac{(1+|\xi|)^{m_2}}{|x-y|^{2N_1}}, \end{aligned} \tag{2.54}$$

and

$$\frac{1}{1+4\pi^2|\xi-\eta|^2} \leqslant \frac{1+4\pi^2|\eta|^2}{1+2\pi^2|\xi|^2},$$

it follows that there exists a constant $B_{N_1N_2} > 0$ such that

$$\begin{aligned} \left|c''_\varepsilon(x,\xi)\right| &\leqslant \int_{\mathbb{R}^n}\int_{\mathbb{B}(x_0,1)^c} \frac{\left|\Delta_\eta^{N_1} a_\varepsilon(x,\eta)\right|}{(1+4\pi^2|\xi-\eta|^2)^{N_2}}\left|(1-\Delta_y)^{N_2}\left[\frac{b''_\varepsilon(y,\xi)}{(-4\pi^2|x-y|^2)^{N_1}}\right]\right| dy\,d\eta \\ &\leqslant B_{N_1N_2}\int_{\mathbb{R}^n}\int_{\mathbb{B}(x_0,1)^c}(1+|\eta|)^{m_1-2N_1}\,\frac{(1+4\pi^2|\eta|^2)^{N_2}}{(1+2\pi^2|\xi|^2)^{N_2}}\,\frac{(1+|\xi|)^{m_2}}{|x-y|^{2N_1}}\,dy\,d\eta. \end{aligned}$$

As $1+|\xi|^2 \leqslant (1+|\xi|)^2 \leqslant 3(1+|\xi|^2)$ for every $\xi \in \mathbb{R}^n$ and $|x-y| \geqslant |x_0-y|-1/2$ for all $x \in \mathbb{B}(x_0,1/2)$, we have that there exists $C_{N_1N_2} > 0$ such that

$$\left|c''_\varepsilon(x,\xi)\right| \leqslant C_{N_1N_2}\,(1+|\xi|)^{m_2-2N_2}\left(\int_{\mathbb{R}^n}(1+|\eta|)^{m_1+2N_2-2N_1}\,d\eta\right)\left(\int_{\mathbb{B}(0,1)^c}\frac{1}{(|z|-1/2)^{2N_1}}\,dz\right).$$

From the above inequality it is clear that if we take N_2 sufficiently large and $2N_1 > \max\{n, m_1 + n + 2N_2\}$ we get (2.52) for $c''_\varepsilon(x,\xi)$ when $\gamma=\beta=0$. The inequality for the general case involving derivatives follows from differentiation under the integral sign in (2.53), application of Leibniz's rule, and similar inequalities as in (2.54). Then (2.52) follows from taking the limit of $c''_\varepsilon(x,\xi)$ as $\varepsilon \to 0^+$ under the light of Theorem 2.5 and Proposition 2.6.

Finally, take $c_{x_0} = c' + c''$. Then $T_{c_{x_0}} = T_aT_b$ and, since $b' = b$ for all $x \in \mathbb{B}(x_0,1/2)$, it follows from (2.51) and (2.52) that

$$c_{x_0} - \sum_{|\alpha|\leqslant N}\frac{(2\pi i)^{-|\alpha|}}{\alpha!}(\partial_\xi^\alpha a)\cdot(\partial_x^\alpha b) \tag{2.55}$$

satisfies similar inequalities as of an element in $S^{m_1+m_2-N-1}$ for all $N \geqslant 0$, but uniformly only on $x \in \mathbb{B}(x_0,1/2)$. Since $x_0 \in \mathbb{R}^n$ is arbitrary and Theorem 2.9 guarantees that a pseudo-differential operator defines its symbol uniquely, it follows that there exists a unique symbol $c \in S^{m_1+m_2}$ such that $T_aT_b = T_c$ and such that (2.39) holds for all $N \geqslant 0$. □

2.4 Boundedness on Sobolev Spaces

We start this section by claiming that for any multiindex α, it follows that $\partial^\alpha\colon \mathcal{S} \to \mathcal{S}$ extends to a bounded linear map from H^s to $H^{s-|\alpha|}$. Indeed, since $|\xi^\alpha| \leqslant (1+|\xi|^2)^{|\alpha|/2}$ for any $\xi \in \mathbb{R}^n$ and any $\alpha \in \mathbb{N}_0^n$, we have for any $f \in \mathcal{S}$

$$\begin{aligned}\|\partial^\alpha f\|^2_{H^{s-|\alpha|}} &= \int_{\mathbb{R}^n} (1+|\xi|^2)^{s-|\alpha|}\, |(2\pi i\xi)^\alpha\, \widehat{f}(\xi)|^2\, d\xi \\ &\leqslant (2\pi)^{2|\alpha|} \int_{\mathbb{R}^n} (1+|\xi|^2)^s\, |\widehat{f}(\xi)|^2\, d\xi \\ &= (2\pi)^{2|\alpha|}\, \|f\|^2_{H^s}.\end{aligned}$$

Because $\mathcal{S} \subset H^s \subset \mathcal{S}'$ with continuous dense embedding for every $s \in \mathbb{R}$, and $\partial^\alpha\colon \mathcal{S}' \to \mathcal{S}'$ is continuous the claim holds by Proposition 1.33.

Inspired by the above result for derivatives, one has the following theorem for pseudo-differential operators.

Theorem 2.13 (Boundedness on Sobolev Spaces)**.** *Suppose $a \in S^m$. Then for any given $s \in \mathbb{R}$ the operator T_a, initially defined on $\mathcal{S}(\mathbb{R}^n)$, extends to a bounded linear map from $H^s(\mathbb{R}^n)$ into $H^{s-m}(\mathbb{R}^n)$.*

Proof. By Proposition 1.33 it suffices to show that there exists $C > 0$ such that

$$\|T_a f\|_{H^{s-m}} \leqslant C\, \|f\|_{H^s} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n). \tag{2.56}$$

From (2.3) we know that $(1+|\xi|^2)^{t/2} \in S^t$ for every $t \in \mathbb{R}$, thus we may define a pseudo-differential operator $\Lambda_t\colon \mathcal{S} \to \mathcal{S}$ by

$$\Lambda_t f(x) := \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, (1+|\xi|^2)^{t/2}\, \widehat{f}(\xi)\, d\xi.$$

Since $(1+|\xi|^2)^{t/2}\widehat{f}(\xi) \in \mathcal{S}$, we have that $\widehat{(\Lambda_t f)}(\xi) := (1+|\xi|^2)^{t/2}\widehat{f}(\xi)$ and that $\Lambda_t \circ \Lambda_{-t} = I$. Additionally, by Plancherel's theorem we have $\|f\|_{H^s} = \|\Lambda_s f\|_{L^2}$ for all $f \in \mathcal{S}(\mathbb{R}^n)$ and every $s \in \mathbb{R}$.

By writing $A_{s,m} = \Lambda_{s-m} \circ T_a \circ \Lambda_{-s}$, we have by Theorem 2.12 that $A_{s,m}$ is a pseudo-differential operator of order 0. Thus by Theorem 2.11 there exists $C > 0$ such that

$$\|A_{s,m} f\|_{L^2} \leqslant C\, \|f\|_{L^2} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n).$$

This implies that for any $f \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}\|T_a f\|_{H^{s-m}} = \|\Lambda_{s-m}\, T_a f\|_{L^2} &= \|\Lambda_{s-m}\, T_a\, \Lambda_{-s}\Lambda_s f\|_{L^2} \\ &= \|A_{s,m}\, \Lambda_s f\|_{L^2} \\ &\leqslant C\, \|\Lambda_s f\|_{L^2} \\ &= C\, \|f\|_{H^s}\end{aligned}$$

and this finishes the proof. ☐

2.5 Commutator characterization of Pseudo-differential Operators on Euclidean Spaces

In this section we will discuss the commutator characterization of local pseudo-differential operators on $\mathbb{R}^n$. We start by proving the following auxiliary lemmas.

Lemma 2.14 (Peetre's Inequality)**.** *For all $s \in \mathbb{R}$ and every x, $y \in \mathbb{R}^n$, we have*

$$(1+|x+y|)^s \leqslant (1+|x|)^s\, (1+|y|)^{|s|}$$

Proof. First note that

$$(1 + |x + y|) \leqslant 1 + (|x| + |y|) \leqslant (1 + |x|)(1 + |y|), \tag{2.57}$$

which clearly implies the result if $s \geqslant 0$.

If $s < 0$, then we exchange $x \mapsto x + y$ and $y \mapsto -y$ in (2.57) to get

$$(1 + |x|)^{-s} \leqslant (1 + |x + y|)^{-s}(1 + |-y|)^{-s}$$

which implies that

$$(1 + |x + y|)^s \leqslant (1 + |x|)^s(1 + |y|)^{|s|}$$

as we desired. □

Lemma 2.15. *Let $\phi(x, y) \in C^\infty(\Omega \times U)$, with $\Omega \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ open sets. Suppose that there is a compact set $K \subset \Omega$ such that* $\operatorname{supp}_x \phi(\cdot, y) \subset K$ *for all $y \in U$. Then for all $u \in \mathcal{D}'(\Omega)$ the mapping*

$$y \mapsto u(\phi(\cdot, y))$$

is a C^∞-function of y and

$$\partial_y^\alpha u(\phi(\cdot, y)) = u(\partial_y^\alpha \phi(\cdot, y)).$$

Proof. Fix $y_0 \in U$. For each $h \in \mathbb{R}^n$ such that $[y_0, y_0 + h] \subset U$ we have by Taylor's formula that

$$\phi(x, y_0 + h) = \phi(x, y_0) + \sum_{j=1}^m h_j \frac{\partial \phi(x, y_0)}{\partial y_j} + \sum_{|\beta|=2} \frac{2}{\beta!} h^\beta \int_0^1 (1 - t)\, \partial_y^\beta \phi(x, y_0 + th)\, dt.$$

Denoting the last term on the right-hand side by $\psi(x, y_0, h)$, we note that it defines a smooth function on the x-variable and that it has compact support on K.

Additionally, if we take $\delta > 0$ such that $\mathbb{B}(y_0, \delta) \subset U$, it follows that

$$\sup_{x \in K} |\partial_x^\gamma \psi(x, y_0, h)| \leqslant |h|^2 \sum_{|\beta|=2} \frac{2}{\beta!} \sup_{\substack{x \in K \\ y \in \mathbb{B}(y_0, \delta)}} \left| \partial_y^\beta \partial_x^\gamma \phi(x, y) \right|,$$

for all $h \in \mathbb{R}^n$ such that $|h| \leqslant \delta$. Thus there exists a constant $C > 0$ such that for all $h \in \mathbb{R}^n$ sufficiently small we have $|u(\psi)| \leqslant C|h|^2$.

By the linearity of u we conclude that

$$\begin{aligned}
\frac{\partial}{\partial y_k} u(\phi(\cdot, y_0)) &= \lim_{h_k \to 0} \frac{u(\phi(\cdot, y_0 + h_k)) - u(\phi(\cdot, y_0))}{h_k} \\
&= \lim_{h_k \to 0} u\left(\frac{\phi(\cdot, y_0 + h_k) - \phi(\cdot, y_0)}{h_k} \right) \\
&= \lim_{h_k \to 0} u\left(\frac{\partial \phi(\cdot, y_0)}{\partial y_k} + \frac{\psi(\cdot, y_0, h_k)}{h_k} \right) \\
&= u\left(\frac{\partial}{\partial y_k} \phi(\cdot, y_0) \right).
\end{aligned}$$

Iteration of this result proves the Lemma. □

Definition 2.16. A continuous linear map

$$A \colon C_c^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$$

is called a *local pseudo-differential operator of order m* if for all ϕ, $\psi \in C_c^\infty(\mathbb{R}^n)$ we have $\phi A \psi \in$

$Op\, S^m$. Here we have

$$(\phi A\psi)u := \phi\, A(\psi u) \quad \text{in } \mathcal{D}'(\mathbb{R}^n),$$

and when A is a local pseudo-differential operator the equality holds pointwise. We denote the space of all local pseudo-differential operators of order m by $\Psi^m_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$.

Note that if $A \in \Psi^m_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$ then for any $\phi, \psi \in C^\infty_c(\mathbb{R}^n)$ we have that $\phi A\psi$ maps $C^\infty(\mathbb{R}^n)$ continuously into $C^\infty_{\operatorname{supp}\phi}(\mathbb{R}^n)$.

Recall that if T is a pseudo-differential operator, then by Theorem 2.9 there exists a unique $a \in S^m$ such that $T = T_a$. Richard Beals [1] proved another characterization of pseudo-differential operators on $\mathbb{R}^n$, namely, via commutators. We provide a related result, Theorem 2.17, for the characterization of local pseudo-differential operators via commutators. Our result is a slightly stronger version of M. Ruzhansky and V. Turunen [15, Theorem 5.14, p. 414].

Let us introduce some notation before we prove our result.

Let T be a linear map from $C^\infty_c(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$. Let us define the commutators

$$L_j(T) := [\partial_{x_j}, T] \text{ and } R_k(T) := [T, M_{x_k}],$$

where M_{x_k} is the linear operator defined by $M_{x_k}(u) := 2\pi i x_k u$.

Note that for all $j \neq k$ we have $\partial_{x_j} M_{x_k} = M_{x_k}\partial_{x_j}$ and also $R_k R_j = R_j R_k$, $L_k L_j = L_j L_k$ and $R_k L_j = L_j R_k$. Furthermore, we note that $\partial_{x_k} M_{x_k} = 2\pi i\, I + M_{x_k}\partial_{x_k}$, and thus

$$\begin{aligned} R_k L_k(T) &= R_k([\partial_{x_k}, T]) \\ &= R_k(\partial_{x_k} T - T\partial_{x_k}) \\ &= \partial_{x_k} T M_{x_k} - T\partial_{x_k} M_{x_k} - M_{x_k}\partial_{x_k} T + M_{x_k} T\partial_{x_k} \\ &= \partial_{x_k} T M_{x_k} - T M_{x_k}\partial_{x_k} - \partial_{x_k} M_{x_k} T + M_{x_k} T\partial_{x_k} \\ &= [\partial_{x_k}, T M_{x_k}] - [\partial_{x_k}, M_{x_k} T] \\ &= L_k([T, M_{x_k}]) \\ &= L_k R_k(T). \end{aligned}$$

For all multiindices α, $\beta \in \mathbb{N}_0^n$ we set $R^\alpha = R_1^{\alpha_1}\cdots R_n^{\alpha_n}$ and $L^\beta = L_1^{\beta_1}\cdots L_n^{\beta_n}$, with the convention that $R_k^0 = L_j^0 = I$.

Theorem 2.17 (Commutator chracterization on $\mathbb{R}^n$)**.** *Let $m \in \mathbb{R}$ and let A be a linear map from $C^\infty_c(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$. Then the following conditions are equivalent:*

(i) $A \in \Psi^m_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$.

(ii) *For any $\phi, \psi \in C^\infty_c(\mathbb{R}^n)$ and for every α, $\beta \in \mathbb{N}_0^n$, the operator $R^\alpha L^\beta(\phi A\psi)$ has a continuous linear extension mapping $H^{m-|\alpha|}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.*

Proof. $(i) \Longrightarrow (ii)$: Let $A \in \Psi^m_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$ and fix ψ, $\phi \in C^\infty_c(\mathbb{R}^n)$. By definition we have $\phi A\psi \in Op\, S^m$, hence by Theorem 2.5 and Proposition 2.8 we have that $\phi A\psi$ is a continuous linear operator on $\mathcal{S}$ and on $\mathcal{S}'$. By Theorem 2.9 we have that there exists a unique symbol $a_{\phi,\psi} \in S^m$ such that

$$a_{\phi,\psi}(x,\xi) = e^{-2\pi i x\cdot\xi}(\phi A\psi)(e^{2\pi i x\cdot\xi})(x).$$

Let us denote $e^{2\pi i x\cdot\xi} = e_\xi(x)$. Now

$$\begin{aligned} \partial_{x_j} a_{\phi,\psi}(x,\xi) &= (-2\pi i\xi_j)e_{-\xi}(x)(\phi A\psi)(e_\xi)(x) + e_{-\xi}(x)\partial_{x_j}(\phi A\psi)(e_\xi)(x) \\ &= -e_{-\xi}(x)(\phi A\psi)(\partial_{x_j} e_\xi)(x) + e_{-\xi}(x)\partial_{x_j}(\phi A\psi)(e_\xi)(x) \\ &= e_{-\xi}(x) L_j(\phi A\psi)(e_\xi)(x). \end{aligned}$$

Let $\chi_\psi \in C^\infty_c(\mathbb{R}^n)$ be such that $\chi_\psi \equiv 1$ on a neighbourhood of the compact set $\operatorname{supp}\psi$ and write

$(\phi A\psi)(e_\xi) = (\phi A\psi)(\chi_\psi e_\xi)$. Since $A\psi \in \mathcal{D}'(\mathbb{R}^n)$ by definition, it follows from Lemma 2.15 that

$$\partial_{\xi_k}(\phi A\psi)(e_\xi)(x) = \partial_{\xi_k}(\phi A\psi)(\chi_\psi e_\xi)(x) = (\phi A\psi)(\partial_{\xi_k}\chi_\psi e_\xi)(x) = (\phi A\psi)(M_{x_k}e_\xi)(x).$$

Therefore

$$\begin{aligned}\partial_{\xi_k} a_{\phi,\psi}(x,\xi) &= (-2\pi i x_k)e_{-\xi}(x)(\phi A\psi)(e_\xi)(x) + e_{-\xi}(x)\partial_{\xi_k}(\phi A\psi)(e_\xi)(x)\\ &= -e_{-\xi}(x)M_{x_k}(\phi A\psi)(e_\xi)(x) + e_{-\xi}(x)(\phi A\psi)(M_{x_k}e_\xi)(x)\\ &= e_{-\xi}(x)R_k(\phi A\psi)(e_\xi)(x).\end{aligned}$$

Iteration of the above results implies that for any multiindices $\alpha, \beta \in \mathbb{N}_0^n$ we have

$$\partial_\xi^\alpha \partial_x^\beta a_{\phi,\psi}(x,\xi) = e_{-\xi}(x)R^\alpha L^\beta(\phi A\psi)(e_\xi)(x) \in S^{m-|\alpha|}. \tag{2.58}$$

Finally, by Theorem 2.13 we have that for any $s \in \mathbb{R}$ the linear map $R^\alpha L^\beta(\phi A\psi)$ is continuous from $H^s(\mathbb{R}^n)$ to $H^{s-(m-|\alpha|)}(\mathbb{R}^n)$, thus it is in particular continuous from $H^{m-|\alpha|}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$

$(ii) \Longrightarrow (i)$: Assume (ii) and fix ψ, $\phi \in C_c^\infty(\mathbb{R}^n)$. We now have to prove that $\phi A\psi \in Op\, S^m$.

Let $\chi_\psi \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leqslant \chi_\psi \leqslant 1$ and $\chi_\psi \equiv 1$ on a neighbourhood of the compact set $\operatorname{supp}\psi$. Note that for each $\xi \in \mathbb{R}^n$ we have $\chi_\psi e_\xi \in C_c^\infty(\mathbb{R}^n)$ and

$$(\phi A\psi)(\chi_\psi e_\xi) = (\phi A\psi)(e_\xi) \quad \text{in } L^2(\mathbb{R}^n)$$

since by hypothesis we have that for every $\alpha, \beta \in \mathbb{N}_0^n$ the linear map $R^\alpha L^\beta(\phi A\psi)$ is bounded from $H^{m-|\alpha|}$ to L^2, so that $R^\alpha L^\beta(\phi A\psi)(\chi_\psi e_\xi) \in L^2(\mathbb{R}^n)$.

Now define $a_{\phi,\psi}(x,\xi) = e_{-\xi}(x)(\phi A\psi)(e_\xi)(x)$ as an L^2-function of $x \in \mathbb{R}^n$ depending on a parameter $\xi \in \mathbb{R}^n$. We shall show that $a_{\phi,\psi}(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ and that

$$\partial_\xi^\alpha \partial_x^\beta a_{\phi,\psi}(x,\xi) = e_{-\xi}(x)R^\alpha L^\beta(\phi A\psi)(e_\xi)(x). \tag{2.59}$$

First note that by hypothesis there exists $C_0 > 0$ such that, for all $\xi \in \mathbb{R}^n$, we have

$$\|a_{\phi,\psi}(\cdot,\xi)\|_{L^2} \leqslant \|(\phi A\psi)(\chi_\psi e_\xi)(\cdot)\|_{L^2} \leqslant C_0\|\chi_\psi e_\xi\|_{H^m}.$$

Now, since $\chi_\psi e_\xi(\cdot)$ is an infinitely differentiable function of ξ with values in $C_c^\infty(\mathbb{R}^n)$ and for every $\alpha, \beta \in \mathbb{N}_0^n$ the linear map $R^\alpha L^\beta(\phi A\psi)$ is continuous from $H^{m-|\alpha|}$ to L^2, we obtain that $a_{\phi,\psi}(\cdot,\xi)$ is an infinitely differentiable function of ξ taking values in $L^2(\mathbb{R}^n)$. Moreover, we have that for each $1 \leqslant k \leqslant n$

$$\chi_\psi(x)\frac{e_{\xi+h_k}(x) - e_\xi(x)}{h_k} \to \chi_\psi(x)(2\pi i x_k)e_\xi(x) \quad \text{in } \mathcal{S} \text{ as a function in the } x\text{-variable.}$$

Therefore, in the L^2-sense

$$\begin{aligned}\partial_{\xi_k} R^\alpha L^\beta(\phi A\psi)(e_\xi) &= \lim_{h_k\to 0}\frac{R^\alpha L^\beta(\phi A\psi)(e_{\xi+h_k}) - R^\alpha L^\beta(\phi A\psi)(e_\xi)}{h_k}\\ &= \lim_{h_k\to 0} R^\alpha L^\beta(\phi A\psi)\left(\frac{e_{\xi+h_k} - e_\xi}{h_k}\right)\\ &= \lim_{h_k\to 0} R^\alpha L^\beta(\phi A\psi)\left(\chi_\psi\frac{e_{\xi+h_k} - e_\xi}{h_k}\right)\\ &= R^\alpha L^\beta(\phi A\psi)(\chi_\psi 2\pi i x_k e_\xi)\\ &= R^\alpha L^\beta(\phi A\psi)(M_{x_k}e_\xi).\end{aligned}$$

In particular, the above calculation implies that for each $\xi \in \mathbb{R}^n$ and almost every $x \in \mathbb{R}^n$

$$\begin{aligned}\partial_{\xi_k} a_{\phi,\psi}(x,\xi) &= \partial_{\xi_k}[e_{-\xi}(x)(\phi A\psi)(e_\xi)(x)]\\ &= (-2\pi i x_k)e_{-\xi}(x)(\phi A\psi)(e_\xi)(x) + e_{-\xi}(x)\partial_{\xi_k}(\phi A\psi)(e_\xi)(x)\\ &= -e_{-\xi}(x)M_{x_k}(\phi A\psi)(e_\xi)(x) + e_{-\xi}(x)(\phi A\psi)(M_{x_k}e_\xi)(x)\\ &= e_{-\xi}(x)R_k(\phi A\psi)(e_\xi)(x).\end{aligned}$$

Iteration of the above results gives us that for any multiindex $\alpha \in \mathbb{N}_0^n$ we have

$$\partial_\xi^\alpha a_{\phi,\psi}(x,\xi) = e_{-\xi}(x)R^\alpha(\phi A\psi)(e_\xi)(x).$$

It follows from the hypothesis on $R^\alpha(\phi A\psi)$ that there exists a constant $C_\alpha > 0$ such that

$$\left\|\partial_\xi^\alpha a_{\phi,\psi}(\cdot,\xi)\right\|_{L^2} \leqslant \|R^\alpha(\phi A\psi)(\chi_\psi e_\xi)(\cdot)\|_{L^2} \leqslant C_\alpha\|\chi_\psi e_\xi\|_{H^{m-|\alpha|}}.$$

Now let $\alpha \in \mathbb{N}_0^n$ be a multiindex. For each $\xi \in \mathbb{R}^n$, we calculate the distributional x-derivatives of $\partial_\xi^\alpha a_{\phi,\psi}(x,\xi)$ noting that for any $1 \leqslant j \leqslant n$ and almost every $x \in \mathbb{R}^n$ we have

$$\begin{aligned}\partial_{x_j}\partial_\xi^\alpha a_{\phi,\psi}(x,\xi) &= \partial_{x_j}[e_{-\xi}(x)R^\alpha(\phi A\psi)(e_\xi)(x)]\\ &= (-i2\pi\xi_j)e_{-\xi}(x)R^\alpha(\phi A\psi)(e_\xi)(x) + e_{-\xi}(x)\partial_{x_j}R^\alpha(\phi A\psi)(e_\xi)(x)\\ &= -e_{-\xi}(x)R^\alpha(\phi A\psi)(\partial_{x_j}e_\xi)(x) + e_{-\xi}(x)\partial_{x_j}R^\alpha(\phi A\psi)(e_\xi)(x)\\ &= e_{-\xi}(x)R^\alpha L_j(\phi A\psi)(\chi_\psi e_\xi)(x).\end{aligned} \tag{2.60}$$

By the hypothesis on $R^\alpha L_j(\phi A\psi)$ we obtain a constant $C_{j\alpha} > 0$ such that for any $\xi \in \mathbb{R}^n$

$$\begin{aligned}\left\|\partial_{x_j}\partial_\xi^\alpha a_{\phi,\psi}(\cdot,\xi)\right\|_{L^2} &\leqslant \|R^\alpha L_j(\phi A\psi)(\chi_\psi e_\xi)(\cdot)\|_{L^2}\\ &\leqslant \|R^\alpha L_j(\phi A\psi)(\chi_\psi e_\xi)(\cdot)\|_{L^2}\\ &\leqslant C_{j\alpha}\|\chi_\psi e_\xi\|_{H^{m-|\alpha|}}.\end{aligned} \tag{2.61}$$

By induction and equations (2.60)–(2.61) we may obtain that for each $\xi \in \mathbb{R}^n$, for any $\alpha, \beta \in \mathbb{N}_0^n$, and for almost every $x \in \mathbb{R}^n$,

$$\partial_x^\beta\partial_\xi^\alpha a_{\phi,\psi}(x,\xi) = e_{-\xi}(x)R^\alpha L^\beta(\phi A\psi)(e_\xi)(x),$$

and that there exists a constant $C_{\alpha\beta} > 0$ such that

$$\|\partial_x^\beta\partial_\xi^\alpha a_{\phi,\psi}(\cdot,\xi)\|_{L^2} \leqslant C_{\alpha\beta}\|\chi_\psi e_\xi\|_{H^{m-|\alpha|}}. \tag{2.62}$$

To evaluate $\|\chi_\psi e_\xi\|_{H^{m-|\alpha|}}$ we use that $1 + |\eta|^2 \leqslant (1 + |\eta|)^2 \leqslant 3(1 + |\eta|^2)$ for all $\eta \in \mathbb{R}^n$, together with Peetre's inequality in Lemma 2.14 to obtain a constant $C' > 0$ such that

$$\begin{aligned}
\|\chi_\psi e_\xi\|_{H^{m-|\alpha|}} &= \left(\int_{\mathbb{R}^n} (1+|\eta|^2)^{(m-|\alpha|)}\, |\widehat{\chi_\psi e_\xi}(\eta)|^2\, d\eta\right)^{1/2} \\
&\leqslant C' \left(\int_{\mathbb{R}^n} (1+|\eta|)^{2(m-|\alpha|)}\, |\widehat{\chi_\psi}(\eta-\xi)|^2\, d\eta\right)^{1/2} \\
&= C' \left(\int_{\mathbb{R}^n} (1+|\eta+\xi|)^{2(m-|\alpha|)}\, |\widehat{\chi_\psi}(\eta)|^2\, d\eta\right)^{1/2} \\
&\leqslant C' \left(\int_{\mathbb{R}^n} (1+|\xi|)^{2(m-|\alpha|)}\, (1+|\eta|)^{2(|m-|\alpha||)}\, |\widehat{\chi_\psi}(\eta)|^2\, d\eta\right)^{1/2} \\
&\leqslant C' \,(1+|\xi|)^{m-|\alpha|} \left(\int_{\mathbb{R}^n} (3^{|m-|\alpha||}(1+|\eta|^2)^{(|m-|\alpha||)}\, |\widehat{\chi_\psi}(\eta)|^2\, d\eta\right)^{1/2} \\
&= C' \,(1+|\xi|)^{m-|\alpha|}\, 3^{\frac{|m-|\alpha||}{2}}\, \|\chi_\psi\|_{H^{|m-|\alpha||}}.
\end{aligned}$$

Plugging the above inequality into the right-hand side of (2.62) we obtain $C'_{\alpha\beta} > 0$ such that

$$\|\partial_x^\beta \partial_\xi^\alpha a_{\phi,\psi}(\cdot,\xi)\|_{L^2} \leqslant C'_{\alpha\beta}\,(1+|\xi|)^{m-|\alpha|} \quad \text{for all } \xi \in \mathbb{R}^n. \tag{2.63}$$

To conclude that $a_{\phi,\psi} \in S^m$ we argue as follows: For each multiindex $\alpha \in \mathbb{N}_0^n$ and for each $\xi \in \mathbb{R}^n$, equation (2.63) implies that the distributional x-derivatives of any given order of $\partial_\xi^\alpha a_{\phi,\psi}(x,\xi)$ are bounded in L^2, which implies that $\partial_\xi^\alpha a_{\phi,\psi}(x,\xi) \in H^N(\mathbb{R}^n)$ for all $N \geqslant 0$. Hence by the Sobolev Embedding Theorem 1.44 we obtain that $\partial_\xi^\alpha a_{\phi,\psi}(\cdot,\xi) \in C_B^\infty(\mathbb{R}^n)$ and thus the mapping $\xi \mapsto a_{\phi,\psi}(\cdot,\xi)$ is smooth with values in $C_B^\infty(\mathbb{R}^n)$, so that $a_{\phi,\psi} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Moreover, for each $\beta \in \mathbb{N}_0^n$ we may take an integer $N > |\beta| + n/2$ and apply the Sobolev Embedding Theorem together with equation (2.63) to obtain a constant $C''_{\alpha\beta} > 0$ such that

$$\begin{aligned}
\left|\partial_x^\beta \partial_\xi^\alpha a_{\phi,\psi}(x,\xi)\right| &\leqslant C\, \|\partial_\xi^\alpha a_{\phi,\psi}(\cdot,\xi)\|_{H^N} \\
&\leqslant C' \sum_{|\gamma|\leqslant N} \|\partial_x^\gamma \partial_\xi^\alpha a_{\phi,\psi}(\cdot,\xi)\|_{L^2} \\
&\leqslant C''_{\alpha\beta}\,(1+|\xi|)^{m-|\alpha|} \quad \text{for all } x,\xi \in \mathbb{R}^n.
\end{aligned}$$

This implies that $a_{\phi,\psi} \in S^m$.

Next, it remains to check that the pseudo-differential operator with symbol $a_{\phi,\psi}$ defines the same operator on $\mathcal{S}$ as the linear map $\phi A \psi$.

Let $u \in \mathcal{S}$ and write u_h such as in Lemma 2.10. By that same Lemma, we have that $u_h \to u$ in $C^\infty(\mathbb{R}^n)$. Hence, since multiplication by a C_c^∞-function defines a continuous linear map from C^∞ to $\mathcal{S}$, it follows that $\chi_\psi u_h \to \chi_\psi u$ in $\mathcal{S}(\mathbb{R}^n)$. This implies that in the L^2-sense

$$\begin{aligned}
(\phi A \psi)(u) = \lim_{h\to 0^+} (\phi A \psi)(\chi_\psi u_h) &= \lim_{h\to 0^+} \sum_{\substack{k\in\mathbb{Z}^n \\ k\in\mathbb{B}(0,h^{-2})}} (\phi A \psi)(\chi_\psi e_{hk})\, \widehat{u}(hk)\, h^n \\
&= \lim_{h\to 0^+} \sum_{\substack{k\in\mathbb{Z}^n \\ k\in\mathbb{B}(0,h^{-2})}} e_{-hk}(\cdot) a_{\phi,\psi}(\cdot, hk)\, \widehat{u}(hk)\, h^n
\end{aligned} \tag{2.64}$$

Since $a_{\phi,\psi} \in S^m$ we have the pointwise convergence

$$\sum_{\substack{k\in\mathbb{Z}^n \\ k\in\mathbb{B}(0,h^{-2})}} e_{-hk}(x) a_{\phi,\psi}(x,hk)\,\widehat{u}(hk)\,h^n \longrightarrow \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, a_{\phi,\psi}(x,\xi)\,\widehat{u}(\xi)\,d\xi = (T_{a_{\phi,\psi}}u)(x) \tag{2.65}$$

as $h \to 0^+$. Moreover, if we let $\chi_\phi \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leqslant \chi_\phi \leqslant 1$ and $\chi_\phi \equiv 1$ on a neighbourhood of the compact set $\operatorname{supp}\phi$, then for each $h > 0$ and for almost every $x \in \mathbb{R}^n$

$$\begin{aligned}
|(\phi A\psi)(\chi_\psi u_h)(x)| &= |\chi_\phi(x)|\,|(\phi A\psi)(\chi_\psi u_h)(x)| \\
&\leqslant |\chi_\phi(x)| \sum_{\substack{k\in\mathbb{Z}^n \\ k\in\mathbb{B}(0,h^{-2})}} |a_{\phi,\psi}(x,hk)\,\widehat{u}(hk)\,h^n| \\
&\leqslant C\,|\chi_\phi(x)| \sum_{\substack{k\in\mathbb{Z}^n \\ k\in\mathbb{B}(0,h^{-2})}} |(1+|hk|)^m\,\widehat{u}(hk)\,h^n| \\
&\leqslant C'\,|\chi_\phi(x)| \int_{\mathbb{R}^n} |(1+|\xi|)^m\,\widehat{u}(\xi)|\,d\xi,
\end{aligned}$$

for some $C' > 0$. Therefore there exists $g \in L^2$ such that $|(\phi A\psi)(\chi_\psi u_h)| \leqslant g$ almost everywhere and by the Dominated Convergence Theorem the convergence in (2.65) holds in L^2. By (2.64) this means $\phi A\psi = T_{a_{\phi,\psi}}$ on $\mathcal{S}$. □

We now present a reformulation of Theorem 2.17 that shall be useful for our commutator characterization of pseudo-differential operators on smooth manifolds.

Let D be a partial differential operator with smooth coefficients. If $m \in \mathbb{N}_0$ is the order of the highest derivative that occurs in D, then we say that D has *order m*. We denote the order of D by $ord(D)$.

Theorem 2.18. *Let $m \in \mathbb{R}$ and let A be a linear map from $C_c^\infty(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$. Then the following conditions are equivalent:*

(i) $A \in \Psi^m_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$.

(ii) *For any $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ and for any sequence of partial differential operators with smooth coefficients $\mathcal{D} = \{D_j\}_{j\in\mathbb{N}_0} \subset \Psi^1_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$ we have*

$$\begin{cases} B_0 = \phi A\psi \in \mathcal{B}(H^m(\mathbb{R}^n), L^2(\mathbb{R}^n)), \\ B_{k+1} = [B_k, D_k] \in \mathcal{B}(H^{m-d_{\mathcal{D},k}}(\mathbb{R}^n), L^2(\mathbb{R}^n)), \end{cases}$$

where $d_{\mathcal{D},k} = \sum_{j=0}^k (1 - ord(D_j))$.

Proof. $(i) \Longrightarrow (ii)$: Let $A \in \Psi^m_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$ and fix ψ, $\phi \in C_c^\infty(\mathbb{R}^n)$. By definition we have $\phi A\psi \in Op\,S^m$, hence by Theorem 2.13 we have that the linear map $B_0 = \phi A\psi$ is bounded from H^m to L^2.

Now let $D_0 \in \Psi^1_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$ be a partial differential operator with smooth coefficients and consider $\chi_{\phi,\psi} \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leqslant \chi_{\phi,\psi} \leqslant 1$ and $\chi_{\phi,\psi} \equiv 1$ on a neighbourhood of the compact set $\operatorname{supp}\phi \bigcup \operatorname{supp}\psi$. Note that $\chi_{\phi,\psi} D_0 \in Op\,S^{ord(D_0)}$ and that

$$B_1 = [B_0, D_0] = [\phi A\psi, D_0] = [\phi A\psi, \chi_{\phi,\psi} D_0].$$

It follows from Theorem 2.12 that $B_1 \in Op\,S^{m+ord(D_0)-1} = Op\,S^{m-d_{\mathcal{D},0}}$ and therefore by Theorem 2.13 we have that the linear map B_1 is bounded from $H^{m-d_{\mathcal{D},0}}$ to L^2. The general statement follows from iteration of the above result.

$(ii) \Longrightarrow (i)$: This implication is an immediate consequence of Theorem 2.17 since the operators $M_{x_k}, \partial_{x_j} \in \Psi^1_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$, and $ord(M_{x_k}) = 0$, $ord(\partial_{x_j}) = 1$ for all $1 \leqslant j, k \leqslant n$. □

2.6 Amplitude representation and Change of Variables

Definition 2.19 (Amplitudes)**.** We say that a complex-valued function $c = c(x, y, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ is a *amplitude symbol* of order m if for all α, β, γ multiindices there exists a constant $C_{\alpha\beta\gamma} > 0$ such that

$$\left|\partial_y^\gamma \partial_x^\beta \partial_\xi^\alpha c(x, y, \xi)\right| \leqslant C_{\alpha\beta\gamma}(1 + |\xi|)^{m-|\alpha|} \tag{2.66}$$

for all x, y, $\xi \in \mathbb{R}^n$. We denote the space of amplitude symbols by $\mathcal{A}^m(\mathbb{R}^{3n})$.

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be any function such that $\varphi(0) = 1$. Then for any $\xi \in \mathbb{R}^n$ we have $\varphi(\varepsilon\xi) \to 1$ pointwise as $\varepsilon \to 0^+$. Inspired by previous formulas such as (2.19) we assign, to each $c \in \mathcal{A}^m$, a linear map $T_{[c]}$ defined by

$$(T_{[c]}f)(x) := \lim_{\varepsilon\to 0^+} \int\int e^{2\pi i\xi\cdot(x-y)}\, c(x, y, \xi)\, \varphi(\varepsilon\xi)\, f(y)\, dy\, d\xi \quad \text{for all } f \in \mathcal{S}. \tag{2.67}$$

By letting

$$L_y = (1 + 4\pi^2|\xi|^2)^{-1}(1 - \Delta_y)$$

and using that $L_y(e^{2\pi i(x-y)\cdot\xi}) = e^{2\pi i(x-y)\cdot\xi}$, we can insert this operator N times in (2.67) and, for any $N \in \mathbb{N}$ such that $2N > m+n$, we have by the Dominated Convergence Theorem and integration by parts

$$(T_{[c]}f)(x) = \int\int e^{2\pi i\xi\cdot(x-y)}\, (L_y)^N[c(x, y, \xi)f(y)]\, dy\, d\xi. \tag{2.68}$$

Since N can be made arbitrarily large, we may apply a similar procedure as the one used in the proof of Theorem 2.7 to conclude that $T_{[c]}$ is a continuous operator on $\mathcal{S}$.

The following Proposition shows that the space S^m in Definition 2.1 and the space $\mathcal{A}^m$ are, in some sense, related to each other.

Proposition 2.20. *Suppose $c \in \mathcal{A}^m$ is an amplitude symbol. Then there exists a unique symbol $a \in S^m$ so that $T_a = T_{[c]}$ for all Schwartz functions $f \in \mathcal{S}$.*
Moreover, the asymptotic expansion for $a \in S^m$ is given by

$$a(x, \xi) \;-\; \sum_{|\alpha|<N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_y^\alpha c(x, y, \xi)\Big|_{y=x} \;\in\; S^{m-N}(\mathbb{R}^n \times \mathbb{R}^n), \tag{2.69}$$

for all $N \geqslant 0$.

Sketch. The proof is essentially a reprise of the composition formula proof as in Theorem 2.12. For more details check E. M. Stein [18, p. 258]. ☐

One of the advantages of working with amplitude symbols is the following: Let $K \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and define

$$Tf(x) := \int_{\mathbb{R}^n} K(x, y)f(y)\, dy, \quad \text{for } f \in \mathcal{S}(\mathbb{R}^n).$$

We claim that there exists an amplitude symbol $c(x, y, \xi)$ such that $c \in \mathcal{A}^m$ for any $m \in \mathbb{R}$ and such that $T = T_{[c]}$ for all $f \in \mathcal{S}$. Indeed, consider $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\int \eta(z)\, dz \neq 0$ and set

$$c(x, y, \xi) = \left(\int \eta(z)\, dz\right)^{-1} K(x, y)\, \eta(\xi)\, e^{-2\pi i(x-y)\cdot\xi}.$$

Note that $c \in C_c^\infty(\mathbb{R}^{3n})$ and therefore is in $\mathcal{A}^m$ for any $m \in \mathbb{R}$. It then follows from Fubini's theorem that for any $f \in \mathcal{S}(\mathbb{R}^n)$ we have

$$T_{[c]}f(x) = \int\int e^{2\pi i(x-y)\cdot\xi}\, c(x, y, \xi)\, f(y)\, dy\, d\xi = \int K(x, y)f(y)\, dy = Tf(x),$$

and our claim holds.

Let us now discuss how certain pseudo-differential operators behave under smooth coordinate changes. The result that we present here is weaker than L. Hörmander [10, Theorem 18.1.17, p. 81], as we do not use results concerning the Schwartz Kernel of pseudo-differential operators.

Let U, V be open sets of $\mathbb{R}^n$, and let κ be a diffeomorphism of U onto V. Denote by J_κ the *Jacobian matrix* of κ.

For any $u \in C_c^\infty(V)$ we define its *pullback* by κ by

$$(\kappa^* u)(x) = u(\kappa(x)),\ x \in U.$$

It easily follows that $\kappa^* u \in C_c^\infty(U)$.

Now let $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ be such that there exists $K \subset V$ compact with $\operatorname{supp}_x a(\cdot, \xi) \subset K$ for all $\xi \in \mathbb{R}^n$ and denote the pseudo-differential operator with symbol $a \in S^m$ by A. Then for any $f \in \mathcal{S}(\mathbb{R}^n)$, the smooth function Af has compact support in V and thus can be identified as an element of $C_c^\infty(V)$.

Given any $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset U$ we set $M_\psi : \mathcal{S}(\mathbb{R}^n) \to C_c^\infty(U)$ by $M_\psi(f) := \psi f$. Then we may define a linear map with the help of the commutative diagram

$$\begin{array}{ccc} C_c^\infty(V) & \xrightarrow{\quad A \quad} & C_c^\infty(V) \\ {\scriptstyle (\kappa^{-1})^* \circ M_\psi}\Big\uparrow & & \Big\downarrow{\scriptstyle \kappa^*} \\ \mathcal{S}(\mathbb{R}^n) & \xrightarrow{\ A_{\kappa^{-1}}\psi\ } & C_c^\infty(U) \end{array}$$

More precisely,

$$[(A_{\kappa^{-1}}\psi)f](x) := A(\psi f \circ \kappa^{-1}) \circ \kappa(x), \quad \text{for } f \in \mathcal{S}(\mathbb{R}^n),\ \forall x \in U. \tag{2.70}$$

Note that $(A_{\kappa^{-1}}\psi)f \in C_c^\infty(U)$ for any $f \in \mathcal{S}(\mathbb{R}^n)$, and thus $(A_{\kappa^{-1}}\psi)f \in \mathcal{S}(\mathbb{R}^n)$ after extension by zero on the complement of U.

We shall show that $(A_{\kappa^{-1}}\psi)$ is a pseudo-differential operator of order $m \in \mathbb{R}$ and that its symbols is related to symbol $a \in S^m$.

Theorem 2.21. *Let U, V be open subsets of $\mathbb{R}^n$, $\kappa : U \to V$ be a diffeomorphism. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}\psi \subset U$ and suppose $a \in S^m$ is such that there exists a compact set $K \subset V$ for which $\operatorname{supp}_x a(\cdot, \xi) \subset K$ for all $\xi \in \mathbb{R}^n$. Denote the pseudo-differential operator with symbol $a \in S^m$ by A.*

If for any $f \in \mathcal{S}(\mathbb{R}^n)$ we define

$$[(A_{\kappa^{-1}}\psi)f](x) := A(\psi f \circ \kappa^{-1})(\kappa(x)), \quad \forall x \in U, \tag{2.71}$$

and $[(A_{\kappa^{-1}}\psi)f] \equiv 0$ on U^c, then $(A_{\kappa^{-1}}\psi) \in Op\, S^m$.

Moreover, if $a_{\psi,\kappa^{-1}} \in S^m$ is the symbol of $(A_{\kappa^{-1}}\psi)$, then $a_{\psi,\kappa^{-1}}(x, \xi) = a(\kappa(x), \left[\frac{\partial\kappa}{\partial x}\right]'\xi)\, \psi(x)$ modulo S^{m-1}, where $\left[\frac{\partial\kappa}{\partial x}\right]' = \left[J_\kappa(x)^T\right]^{-1}$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$. The right-hand side of (2.71) gives us that

$$\begin{aligned} [(A_{\kappa^{-1}}\psi)f](x) &= \lim_{\varepsilon \to 0^+} \int\!\!\int e^{2\pi i(\kappa(x) - y)\cdot\xi}\, a(\kappa(x), \xi)\, \varphi(\varepsilon\xi)\, (\psi f \circ \kappa^{-1})(y)\, dy\, d\xi \\ &= \lim_{\varepsilon \to 0^+} \int\!\!\int e^{2\pi i(\kappa(x) - \kappa(y))\cdot\xi}\, a(\kappa(x), \xi)\, \psi(y)\, |\det J_\kappa(y)|\, \varphi(\varepsilon\xi)\, f(y)\, dy\, d\xi, \end{aligned} \tag{2.72}$$

where we made the change of variables $y \mapsto \kappa(y)$ from the first to the second line. Note that this formula holds for all $x \in \mathbb{R}^n$ since $a \in S^m$ has compact support in V. We now reason that the main contribution of the above expression comes from when x is close to y.

In order to do that properly, we begin by considering the following: By applying Taylor's formula to each κ_j we then may obtain the following equality

$$\kappa(x) - \kappa(y) = M(x, y) \cdot (x - y), \tag{2.73}$$

with $M(x, y)$ given by

$$M(x, y) = \int_0^1 J_\kappa(x + t(y - x))\, dt. \tag{2.74}$$

Note that $M(x, y)$ is well defined on the open set $\{(x, y) \in U \times U : [x, y] \in U\}$. In particular, this set is a neighbourhood of the diagonal of $U \times U$ and $M(x, y)$ is a smooth map on x and y on that domain. We now claim that $M(x, y)$ is invertible on a small neighbourhood of the diagonal.

Indeed, note that for each $x \in U$, $M(x, x) = J_\kappa(x)$, hence is invertible. Therefore there exists an open set $A_x \subset U \times U$ such that $(x, x) \in A_x$ and $M(z, y)$ is invertible for all $(z, y) \in A_x$. Taking $\Omega = \bigcup_{x \in U} A_x$, we get that $(x, x) \in \Omega$ for all $x \in U$ and that M is invertible on this set.

Now consider $K' = \kappa^{-1}(K)$, where $K \subset V$ is the compact set containing the x-support of symbol a. We then have that $\Delta_{K' \times K'} := \{(z, z) \in U \times U : z \in K'\}$ is compact and is a subset of Ω. Hence there exists $\chi \in C_c^\infty(U \times U)$ such that $\chi \equiv 1$ on a neighbourhood of $K' \times K'$, $\operatorname{supp} \chi \subset \Omega$, and $0 \leqslant \chi \leqslant 1$. Writing $1 = (1 - \chi(x, y)) + \chi(x, y)$ and inserting this relation into (2.72) we now have to evaluate two different expressions.

Main contribution: Let us first estimate the term which contains the points where x is close to y. More precisely, let us analyse (2.72) after inserting $\chi(x, y)$ by writing

$$\lim_{\varepsilon \to 0^+} \int\!\!\int e^{2\pi i(\kappa(x) - \kappa(y)) \cdot \xi}\, \chi(x, y)\, a(\kappa(x), \xi)\, \psi(y)\, |\det J_\kappa(y)|\, \varphi(\varepsilon\xi)\, f(y)\, dy\, d\xi.$$

By (2.73) we have

$$\lim_{\varepsilon \to 0^+} \int\!\!\int e^{2\pi i M(x,y)(x-y) \cdot \xi}\, \chi(x, y)\, a(\kappa(x), \xi)\, \psi(y)\, |\det J_\kappa(y)|\, \varphi(\varepsilon\xi)\, f(y)\, dy\, d\xi.$$

Making the change of variables $M(x, y)^T \xi \mapsto \xi$ and letting $M(x, y)' = \left[M(x, y)^T\right]^{-1}$ we conclude that

$$\lim_{\varepsilon \to 0^+} \int\!\!\int e^{2\pi i(x-y) \cdot \xi}\, \chi(x, y)\, a(\kappa(x), M(x, y)'\xi)\, \psi(y)\, |\det J_\kappa(y)|\, |\det M(x, y)^{-1}|\, \varphi(\varepsilon M(x, y)'\xi)\, f(y)\, dy\, d\xi. \tag{2.75}$$

If we set

$$c_1(x, y, \xi) = \chi(x, y)\, a(\kappa(x), M(x, y)'\xi)\, \psi(y)\, |\det J_\kappa(y)|\, |\det M(x, y)^{-1}|,$$

then $c_1 \in \mathcal{A}^m$ is an amplitude symbol such that $T_{[c_1]} f$ coincides with (2.75).

By Proposition 2.20 we get that there exists a symbol $a_1 \in S^m$ such that $T_{a_1} = T_{[c_1]}$ and for which the first term of the asymptotic expansion is given by

$$\begin{aligned} c_1(x, y, \xi)\Big|_{y=x} &= \chi(x, x)\, a(\kappa(x), M(x, x)'\xi)\, \psi(x)\, |\det J_\kappa(x)|\, |\det M(x, x)^{-1}| \\ &= a(\kappa(x), \left[\frac{\partial \kappa}{\partial x}\right]' \xi)\, \psi(x), \end{aligned}$$

and all other terms from the asymptotic expansion have order less than or equal to $m - 1$.

Off-diagonal: We now analyse (2.72) after inserting $1 - \chi(x, y)$, i.e.,

$$\lim_{\varepsilon \to 0^+} \int\!\!\int e^{2\pi i(\kappa(x) - \kappa(y)) \cdot \xi}\, (1 - \chi(x, y))\, a(\kappa(x), \xi)\, \psi(y)\, |\det J_\kappa(y)|\, \varphi(\varepsilon\xi)\, f(y)\, dy\, d\xi. \tag{2.76}$$

Let us set $\widetilde{\mathcal{L}}_\xi = (4\pi^2|\kappa(x)-\kappa(y)|^2)^{-1}(\Delta_\xi)$. Since for all $x, y \in U$ with $x \neq y$ we have

$$\widetilde{\mathcal{L}}_\xi \left(e^{2\pi i(\kappa(x)-\kappa(y))\cdot\xi}\right) = e^{2\pi i(\kappa(x)-\kappa(y))\cdot\xi},$$

we can insert this operator N times in (2.76) and, after integration by parts $2N$ times on the ξ-variable we get

$$\lim_{\varepsilon\to 0^+} \int\int e^{2\pi i(\kappa(x)-\kappa(y))\cdot\xi}\,(1-\chi(x,y))\,\psi(y)\,|\det J_\kappa(y)|\,(\widetilde{\mathcal{L}}_\xi)^N[a(\kappa(x),\xi)\varphi(\varepsilon\xi)]\,f(y)\,dy\,d\xi.$$

Now fix an $N \in \mathbb{N}$ such that $2N > m + n$ to obtain that the above expression becomes

$$\int\int e^{2\pi i(\kappa(x)-\kappa(y))\cdot\xi}\,(1-\chi(x,y))\,\psi(y)\,|\det J_\kappa(y)|\,(\widetilde{\mathcal{L}}_\xi)^N[a(\kappa(x),\xi)]\,f(y)\,dy\,d\xi.$$

Define

$$K(x,y) := \int e^{2\pi i(\kappa(x)-\kappa(y))\cdot\xi}\,(1-\chi(x,y))\,\psi(y)\,|\det J_\kappa(y)|\,(\widetilde{\mathcal{L}}_\xi)^N[a(\kappa(x),\xi)]\,d\xi.$$

Note that $K \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and that (2.76) is the same as

$$\int K(x,y)\,f(y)\,dy.$$

Therefore there exists an amplitude symbol $c_2(x,y,\xi)$ such that $c_2 \in \mathcal{A}^m$ for every $m \in \mathbb{R}$ and such that the above expression coincides with $T_{[c_2]}f$. By Proposition 2.20 there exists a symbol a_2 such that $a_2 \in S^m$ for every $m \in \mathbb{R}$ and such that $T_{a_2} = T_{[c_2]}$.

Finally, define $a_{\psi,\kappa^{-1}} = a_1 + a_2$. It follows at once that $a_{\psi,\kappa^{-1}}(x,\xi) = a(\kappa(x), [\frac{\partial\kappa}{\partial x}]'\xi)\,\psi(x)$ modulo S^{m-1} and that $(A_{\kappa^{-1}}\psi) = T_{a_{\psi,\kappa^{-1}}}$, hence $(A_{\kappa^{-1}}\psi) \in Op\,S^m$. ☐

If $A \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{C}$, then $f\restriction_A$ denotes the restriction of f to the subset A.

Corollary 2.21.1. *Let U, V be open subsets of $\mathbb{R}^n$ and let $\kappa : U \to V$ be a diffeomorphism. Suppose we are given $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ both supported on V and let $A : C_c^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ be a linear map such that $\phi A\psi \in Op\,S^m$. If for any $f \in \mathcal{S}(\mathbb{R}^n)$ we set*

$$(\phi A\psi)_{\kappa^{-1}} f(x) := (\phi A\psi)(f\restriction_U \circ \kappa^{-1})(\kappa(x)), \quad \forall x \in U,$$

with extension by zero on U^c, it follows that $(\phi A\psi)_{\kappa^{-1}} \in Op\,S^m$.

Proof. It is clear that the symbol of $\phi A\psi$ has compact x-support contained in $\operatorname{supp}\phi \subset V$.

Now let $\widetilde{\psi} \in C_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}\widetilde{\psi} \subset U$ and $\widetilde{\psi} \equiv 1$ on the neighbourhood of $\kappa^{-1}(\operatorname{supp}\psi)$. Then for any function $f : \mathbb{R}^n \to \mathbb{C}$ we have

$$\psi[f\restriction_U \circ \kappa^{-1}] = \psi[(\widetilde{\psi} f) \circ \kappa^{-1}].$$

Thus for any $f \in \mathcal{S}(\mathbb{R}^n)$ and any $x \in U$ we have

$$\begin{aligned}(\phi A\psi)_{\kappa^{-1}} f(x) &= (\phi A\psi)(f\restriction_U \circ \kappa^{-1})(\kappa(x)) \\ &= (\phi A\psi)(\widetilde{\psi} f \circ \kappa^{-1})(\kappa(x)),\end{aligned} \tag{2.77}$$

and the result follows from Theorem 2.21. ☐

Chapter 3

Pseudo-differential Operators on Manifolds

On a compact smooth manifold the pseudo-differential operators can be characterized by taking commutators with smooth functions and vector fields. Such characterization was first stated by R. Coifman and Y. Meyer [2] in the case of 0-order operators. In 2000 V. Turunen [20] provided a result for pseudo-differential operators of any given order. In a closely related result, J. Dunau [4, Théorème 1] obtained a characterization of the topology in the space of pseudo-differential operators on compact manifolds in terms of the boundedness of their commutators with differential operators of order at most 1.

The goal of this chapter is to provide the necessary concepts of pseudo-differential operators on manifolds before we prove the main result about commutator characterization of pseudo-differential operators on compact smooth manifolds. Sections are organized as follows:

In section 3.1 we introduce distributions and Sobolev spaces over smooth manifolds. We also show that Sobolev spaces over compact smooth manifolds can be endowed with a Hilbert space structure.

Section 3.2 shows how to define pseudo-differential operators on compact smooth manifolds and some of their properties.

Finally, section 3.3 is entirely devoted to our main result about commutator characterization of pseudo-differential operators on compact smooth manifolds.

The references used for the exposition of this chapter were G. Grubb [8], L. Hörmander [9] and [10], M. Ruzhansky and V. Turunen [15], M. Shubin [17], and F. Trèves [19].

3.1 Distributions on Manifolds

Before we talk about pseudo-differential operators on compact smooth manifolds we need to discuss some vector spaces which are suitable for the development of the theory.

Let M be a smooth n-manifold. Recall that a function $f : M \to \mathbb{C}$ is in $C^\infty(M)$ if for any given compatible chart (U, κ) on M we have

$$f_\kappa := f \circ \kappa^{-1} \quad \text{is in } C^\infty(\kappa(U)).$$

Note that if $(U_\alpha, \kappa_\alpha)$ and (U_β, κ_β) are two compatible charts on M such that $U_\alpha \cap U_\beta \neq \varnothing$, then for all $f \in C^\infty(M)$ we have

$$f_{\kappa_\alpha} = f_{\kappa_\beta} \circ (\kappa_\beta \circ \kappa_\alpha^{-1}) \quad \text{in } \kappa_\alpha(U_\alpha \cap U_\beta). \tag{3.1}$$

We now introduce the definition of distributions on a smooth n-manifold. To motivate our definition, we use the compatibility condition (3.1) to describe smooth functions on manifolds.

Indeed, let M be a smooth n-manifold with smooth structure $\mathcal{F} = \{(U_\alpha, \kappa_\alpha)\}_{\alpha \in A}$. Let $\{f_\alpha\}_{\alpha \in A}$

be a family of smooth functions such that $f_\alpha \in C^\infty(\kappa_\alpha(U_\alpha))$ and such that for every $\alpha, \beta \in A$ we have

$$f_\alpha = f_\beta \circ (\kappa_\beta \circ \kappa_\alpha^{-1}) \quad \text{in } \kappa_\alpha(U_\alpha \cap U_\beta).$$

Then there exists one and only one $f \in C^\infty(M)$ such that $f_\alpha = f \circ \kappa_\alpha$ for every $\alpha \in A$.

Conversely, if $f \in C^\infty(M)$, then by definition the family $\{f_{\kappa_\alpha}\}_{\alpha \in A}$ satisfies the compatibility condition (3.1).

In analogy to this description we define distributions on a smooth n-manifold as follows:

Definition 3.1. Let M be a smooth n-manifold with smooth structure $\mathcal{F} = \{(U_\alpha, \kappa_\alpha)\}_{\alpha \in A}$. Let $T_{(\kappa_\beta \circ \kappa_\alpha^{-1})} : \mathcal{D}'(\kappa_\alpha(U_\alpha \cap U_\beta)) \to \mathcal{D}'(\kappa_\beta(U_\alpha \cap U_\beta))$ be as in Definition 1.28.

If for every chart $(U_\alpha, \kappa_\alpha)$ we are given a distribution $u_\alpha \in \mathcal{D}'(\kappa_\alpha(U_\alpha))$ and the family $\{u_\alpha\}_{\alpha \in A}$ is such that

$$T_{(\kappa_\beta \circ \kappa_\alpha^{-1})} u_\alpha = u_\beta \ \text{ in } \kappa_\beta(U_\alpha \cap U_\beta), \text{ for every } \alpha, \beta \in A, \tag{3.2}$$

then we call the family $\{u_\alpha\}_{\alpha \in A}$ a distribution u in M. The set of all distributions in M is denoted by $\mathcal{D}'(M)$.

With this definition we have that $\mathcal{D}'(M)$ appears as a natural extension of $C^\infty(M)$ when we identify a function $f \in C^\infty(M)$ with the family of smooth functions $\{f_\alpha\}_{\alpha \in A}$.

Let us show that this definition of distribution in M is consistent with the definition for open subsets of $\mathbb{R}^n$. First, we show that distributions on a smooth n-manifold can be identified by using only an atlas. For that we need the following lemma.

Lemma 3.2. *Let $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \Omega_3$ be diffeomorphisms of open subsets of $\mathbb{R}^n$. If $u \in \mathcal{D}'(\Omega_1)$, then*

$$T_{(g \circ f)} u = T_g(T_f u).$$

Proof. It follows from the chain rule that the Jacobian matrix of $g \circ f$ is given by

$$J_{(g \circ f)} = (J_g \circ f) J_f.$$

Now, for any $\varphi \in C_c^\infty(\Omega_3)$ we have

$$\begin{aligned}
\langle T_{(g \circ f)} u, \varphi \rangle_{\Omega_3} &= \langle u, \left|\det J_{(g \circ f)}\right| (\varphi \circ (g \circ f)) \rangle_{\Omega_1} \\
&= \langle u, |\det J_g \circ f||\det J_f| \left((\varphi \circ g) \circ f\right) \rangle_{\Omega_1} \\
&= \langle T_f u, |\det J_g| (\varphi \circ g) \rangle_{\Omega_2} \\
&= \langle T_g(T_f u), \varphi \rangle_{\Omega_3},
\end{aligned}$$

which proves the lemma. □

Theorem 3.3. *Let M be a smooth n-manifold and let $\mathcal{A} := \{(U_i, \kappa_i)\}_{i \in I}$ be a compatible atlas on M. If for every chart $(U_i, \kappa_i) \in \mathcal{A}$ we have a distribution $u_i \in \mathcal{D}'(\kappa_i(U_i))$ and the family $\{u_i\}_{i \in I}$ is such that (3.2) holds for all charts in $\mathcal{A}$, then $\{u_i\}_{i \in I}$ defines a unique distribution $u \in \mathcal{D}'(M)$.*

Proof. Let $\mathcal{F} = \{(U_\alpha, \kappa_\alpha)\}_{\alpha \in A}$ denote the smooth structure on M. For any chart $(U_\alpha, \kappa_\alpha)$ we have that

$$\kappa_\alpha(U_\alpha) = \bigcup_{i \in I} \kappa_\alpha(U_\alpha \cap U_i).$$

Since $\mathcal{A}$ is a compatible atlas on M we have that for each $i \in I$ the map $\kappa_\alpha \circ \kappa_i^{-1} : \kappa_i(U_\alpha \cap U_i) \to \kappa_\alpha(U_\alpha \cap U_i)$ is a diffeomorphism and thus

$$T_{(\kappa_\alpha \circ \kappa_i^{-1})} u_i \in \mathcal{D}'(\kappa_\alpha(U_\alpha \cap U_i)).$$

Moreover, it follows from the hypothesis and Lemma 3.2 that for any $\alpha \in A$, and any $i, j \in I$

$$\begin{aligned} T_{(\kappa_\alpha \circ \kappa_i^{-1})} u_i &= T_{(\kappa_\alpha \circ \kappa_i^{-1})} T_{(\kappa_i \circ \kappa_j^{-1})} u_j \\ &= T_{(\kappa_\alpha \circ \kappa_j^{-1})} u_j \end{aligned}$$

on $\kappa_\alpha(U_\alpha \cap U_i \cap U_j) = \kappa_\alpha(U_\alpha \cap U_i) \cap \kappa_\alpha(U_\alpha \cap U_j)$. By Theorem 1.26 we conclude that there exists a unique distribution $u_\alpha \in \mathcal{D}'(\kappa_\alpha(U_\alpha))$ such that $u_\alpha = T_{(\kappa_\alpha \circ \kappa_i^{-1})} u_i$ on $\kappa_\alpha(U_\alpha \cap U_i)$ for every $i \in I$.

To conclude that the family $\{u_\alpha\}_{\alpha \in A}$ defines a distribution in M we use Lemma 3.2 and note that for every $i \in I$ and for any $\alpha, \beta \in A$ we have

$$\begin{aligned} T_{(\kappa_\beta \circ \kappa_\alpha^{-1})} u_\alpha &= T_{(\kappa_\beta \circ \kappa_\alpha^{-1})} T_{(\kappa_\alpha \circ \kappa_i^{-1})} u_i \\ &= T_{(\kappa_\beta \circ \kappa_i^{-1})} u_i \\ &= u_\beta, \end{aligned}$$

on $\kappa_\beta(U_\beta \cap U_\alpha \cap U_i)$. Now given any $\varphi \in C_c^\infty(\kappa_\beta(U_\alpha \cap U_\beta))$ we can choose a finite family of compact sets $K_1, \ldots, K_N$ with $K_j \subset \kappa_\beta(U_\alpha \cap U_\beta \cap U_j)$ and so that $\operatorname{supp} \varphi \subset \bigcup_{j=1}^N K_j$. By Lemma 1.61 there exist smooth functions $\{\rho_j\}_{j=1}^N$ such that $\operatorname{supp} \rho_j \subset \kappa_\beta(U_\alpha \cap U_\beta \cap U_j)$ and $\sum_{j \leqslant N} \rho_j \equiv 1$ on $\operatorname{supp} \varphi$. This implies that

$$\begin{aligned} \langle T_{(\kappa_\beta \circ \kappa_\alpha^{-1})} u_\alpha,\, \varphi \rangle &= \sum_{j=1}^N \langle T_{(\kappa_\beta \circ \kappa_\alpha^{-1})} u_\alpha,\, \rho_j\, \varphi \rangle \\ &= \sum_{j=1}^N \langle u_\beta,\, \rho_j\, \varphi \rangle \\ &= \langle u_\beta,\, \varphi \rangle \end{aligned}$$

which is the desired equality. □

Corollary 3.3.1. *If Ω is an open subset of $\mathbb{R}^n$, then Definition 3.1 coincides with the usual definition of a distribution on Ω.*

Proof. It follows at once by considering the trivial atlas (Ω, I), where $I : \Omega \to \Omega$ is the identity map, and then applying Theorem 3.3. □

Note that if $\psi \in C^\infty(M)$ and $u \in \mathcal{D}'(M)$, then $\psi u \in \mathcal{D}'(M)$. Indeed, just note that in local coordinates we get

$$(\psi u)_i = (\psi \circ \kappa_i^{-1})\, u_i,$$

which is a well-defined distribution in $\mathcal{D}'(\kappa_i(U_i))$. It is not hard to check that the family $\{(\psi u)_i\}_{i \in I}$ satisfies the compatibility condition (3.2), and thus defines an element in $\mathcal{D}'(M)$.

Now recall that if $\Omega \subset \mathbb{R}^n$ is open we have $H^s_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$ for any $s \in \mathbb{R}$. Moreover, Proposition 1.50 shows that if $\kappa : \Omega \to \Omega'$ is a diffeomorphism between open sets of $\mathbb{R}^n$, then the induced map $T_\kappa : H^s_{loc}(\Omega) \to H^s_{loc}(\Omega')$ is a isomorphism. This allow us to introduce the following definition for Sobolev spaces $H^s_{loc}(M)$ when M is a smooth n-manifold.

Definition 3.4. Let $s \in \mathbb{R}$. Let M be a smooth n-manifold and let $\mathcal{F} = \{(U_\alpha, \kappa_\alpha)\}_{\alpha \in A}$ be its smooth structure. We denote by $H^s_{loc}(M)$ the space of distributions $u = \{u_\alpha\}_{\alpha \in A}$ in M such that $u_\alpha \in H^s_{loc}(\kappa_\alpha(U_\alpha))$ for every $\alpha \in A$.

By the above observations we have that $H^s_{loc}(M)$ is a well-defined vector space.

Suppose that M is a compact smooth n-manifold. In that case we adopt $H^s(M) := H^s_{loc}(M)$ and $L^2(M) := H^0_{loc}(M)$. The compactness of M will enable us to equip $H^s(M)$ with a Hilbert space structure. Namely, let $\mathcal{A} := \{(U_i, \kappa_i)\}_{i=1}^N$ be a finite compatible atlas on M; we may assume that the $\kappa_i(U_i)$ are mutually disjoint. It follows from Theorem 3.3 that a family of distributions

$u_i \in \mathcal{D}'(\kappa_i(U_i))$ that satisfies the compatibility condition (3.2) for every $1 \leqslant i \leqslant N$ suffices to describe $u \in \mathcal{D}'(M)$, and hence suffices to describe $u \in H^s(M)$ for any $s \in \mathbb{R}$. By Lemma 1.62 there exists a partition of unity $\{\rho_i\}_{i=1}^N$ strictly subordinate to $\{U_i\}_{i=1}^N$. Now define

$$\langle u, v\rangle_{H^s(M)} := \sum_{i=1}^{N} \big\langle (\rho_i \circ \kappa_i^{-1})\, u_i,\, (\rho_i \circ \kappa_i^{-1})\, v_i \big\rangle_{H^s(\mathbb{R}^n)}. \tag{3.3}$$

Proposition 3.5. *Let $s \in \mathbb{R}$. If M is a compact smooth n-manifold and we endow $H^s(M)$ with the above structure, then (3.3) defines an inner product on $H^s(M)$ that makes it a Hilbert space.*

Proof. It is clear from the definition that (3.3) is a sesquilinear form on $H^s(M) \times H^s(M)$ and that $\langle u, u\rangle_{H^s(M)} \geqslant 0$ for all $u \in H^s(M)$. To conclude that $\langle\, ,\, \rangle_{H^s(M)}$ defines an inner product on $H^s(M)$ we need to show that

$$\langle u, u\rangle_{H^s(M)} = 0 \iff u = 0 \quad \text{in } H^s(M).$$

Clearly $u = 0$ in $H^s(M)$ implies $\langle u, u\rangle_{H^s(M)} = 0$. For the converse we note that

$$\langle u, u\rangle_{H^s(M)} = 0 \implies (\rho_j \circ \kappa_j^{-1})u_j = 0 \quad \text{in } \mathcal{D}'(\kappa_j(U_j)) \text{ for every } 1 \leqslant j \leqslant N. \tag{3.4}$$

Now for each $1 \leqslant i \leqslant N$ and every $\varphi \in C_c^\infty(\kappa_i(U_i))$ we use that $\{\rho_j\}_{j=1}^N$ is a partition of unity and the compatibility condition $u_i = T_{(\kappa_i \circ \kappa_j^{-1})} u_j$ in $\kappa_i(U_i \cap U_j)$ to write

$$\begin{aligned} \langle u_i, \varphi\rangle &= \sum_{j=1}^{N} \langle u_i, (\rho_j \circ \kappa_i^{-1})\varphi\rangle \\ &= \sum_{j=1}^{N} \langle T_{(\kappa_i \circ \kappa_j^{-1})} u_j, (\rho_j \circ \kappa_i^{-1})\varphi\rangle. \end{aligned} \tag{3.5}$$

Let $\tilde{\rho}_j \in C_c^\infty(U_j)$ be such that $\tilde{\rho}_j \equiv 1$ on a neighbourhood of $\operatorname{supp} \rho_j$ to obtain

$$\begin{aligned} \langle u_i, \varphi\rangle &= \sum_{j=1}^{N} \langle T_{(\kappa_i \circ \kappa_j^{-1})} u_j, (\rho_j \circ \kappa_i^{-1})\varphi\rangle \\ &= \sum_{j=1}^{N} \langle (\rho_j \circ \kappa_i^{-1}) T_{(\kappa_i \circ \kappa_j^{-1})} u_j, (\tilde{\rho}_j \circ \kappa_i^{-1})\varphi\rangle \\ &= \sum_{j=1}^{N} \langle T_{(\kappa_i \circ \kappa_j^{-1})} [(\rho_j \circ \kappa_j^{-1}) u_j], (\tilde{\rho}_j \circ \kappa_i^{-1})\varphi\rangle \end{aligned} \tag{3.6}$$

It follows from (3.4) that each term on the sum in the right-hand side of (3.6) is zero. This implies $u_i = 0$ in $\mathcal{D}'(\kappa_i(U_i))$ for each i and therefore $u = 0$ in $H^s(M)$. This shows that (3.3) is an inner product on $H^s(M)$.

It now remains to verify the completeness of $H^s(M)$ with respect to the norm

$$\|u\|_{H^s(M)} = \left(\sum_{i=1}^{N} \big\| (\rho_i \circ \kappa_i^{-1})\, u_i \big\|^2_{H^s(\mathbb{R}^n)} \right)^{1/2}.$$

Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to $\|\cdot\|_{H^s(M)}$. Then for each $1 \leqslant i \leqslant N$ the sequence $\{(\rho_i \circ \kappa_i^{-1}) u_{i,n}\}_{n\in\mathbb{N}}$ is Cauchy in $H^s(\mathbb{R}^n)$. By completeness there exists $v_i \in H^s(\mathbb{R}^n)$ such that $(\rho_i \circ \kappa_i^{-1}) u_{i,n} \to v_i$ in $H^s(\mathbb{R}^n)$. Note that if we let $V_i \subset \kappa_i(U_i)$ be an open neighbourhood of $\operatorname{supp} \rho_i$, then $v_i \equiv 0$ on $\mathbb{R}^n \setminus \overline{V_i}$ because each $(\rho_i \circ \kappa_i^{-1}) u_{i,n}$ is zero on $\mathbb{R}^n \setminus \overline{V_i}$. In particular, we may identify v_i as an element of $\mathcal{D}'(\kappa_i(U_i))$.

Now for each $1 \leqslant i \leqslant N$, every $n \in \mathbb{N}$ and for any $\varphi \in C_c^\infty(\kappa_i(U_i))$ we use (3.5) and (3.6) to write

$$\langle u_{i,n}, \varphi \rangle = \sum_{j=1}^{N} \langle T_{(\kappa_i \circ \kappa_j^{-1})}[(\rho_j \circ \kappa_j^{-1}) u_{j,n}], (\tilde{\rho}_j \circ \kappa_i^{-1})\varphi \rangle.$$

Taking the limit as $n \to \infty$ and using the continuity of $T_{(\kappa_i \circ \kappa_j^{-1})} : \mathcal{D}'(\kappa_j(U_i \cap U_j)) \to \mathcal{D}'(\kappa_i(U_i \cap U_j))$ we get

$$\lim_{n\to\infty} \langle u_{i,n}, \varphi \rangle = \sum_{j=1}^{N} \langle T_{(\kappa_i \circ \kappa_j^{-1})} v_j, (\tilde{\rho}_j \circ \kappa_i^{-1})\varphi \rangle.$$

Hence by Theorem 1.25 there exists a distribution $\tilde{u}_i \in \mathcal{D}'(\kappa_i(U_i))$ such that $\tilde{u}_i = \lim_{n\to\infty} u_{i,n}$ and

$$\langle \tilde{u}_i, \varphi \rangle = \sum_{j=1}^{N} \langle T_{(\kappa_i \circ \kappa_j^{-1})} v_j, (\tilde{\rho}_j \circ \kappa_i^{-1})\varphi \rangle \quad \text{for } \varphi \in C_c^\infty(\kappa_i(U_i)).$$

We shall denote $\tilde{u}_i := \sum_{j=1}^{N} (\tilde{\rho}_j \circ \kappa_i^{-1})\, T_{(\kappa_i \circ \kappa_j^{-1})} v_j$.

By construction we have $(\rho_i \circ \kappa_i^{-1})\, \tilde{u}_i = v_i$ in $\mathcal{D}'(\kappa_i(U_i))$ and since both have compact support in $\kappa_i(U_i)$ we have the same equality in $H^s(\mathbb{R}^n)$ for every $1 \leqslant i \leqslant N$. Moreover, $\tilde{u}_i \in H^s_{loc}(\kappa_i(U_i))$ and for any $1 \leqslant i, l \leqslant N$ we have by Lemma 3.2 that

$$\begin{aligned} T_{(\kappa_l \circ \kappa_i^{-1})} \tilde{u}_i &= T_{(\kappa_l \circ \kappa_i^{-1})} \left(\sum_{j=1}^{N} (\tilde{\rho}_j \circ \kappa_i^{-1})\, T_{(\kappa_i \circ \kappa_j^{-1})} v_j \right) \\ &= \sum_{j=1}^{N} (\tilde{\rho}_j \circ \kappa_l^{-1})\, T_{(\kappa_l \circ \kappa_i^{-1})}\, T_{(\kappa_i \circ \kappa_j^{-1})} v_j \\ &= \sum_{j=1}^{N} (\tilde{\rho}_j \circ \kappa_l^{-1})\, T_{(\kappa_l \circ \kappa_j^{-1})} v_j \\ &= \tilde{u}_l \quad \text{on } \kappa_l(U_i \cap U_l). \end{aligned}$$

Therefore the family $\{\tilde{u}_i\}_{i=1}^N$ defines an element $u \in H^s(M)$. We claim that $u_n \to u$ in $H^s(M)$.

Indeed,

$$\begin{aligned} \|u - u_n\|^2_{H^s(M)} &= \sum_{i=1}^{N} \left\| (\rho_i \circ \kappa_i^{-1})\, \tilde{u}_i - (\rho_i \circ \kappa_i^{-1})\, u_{i,n} \right\|^2_{H^s(\mathbb{R}^n)} \\ &= \sum_{i=1}^{N} \left\| v_i - (\rho_i \circ \kappa_i^{-1})\, u_{i,n} \right\|^2_{H^s(\mathbb{R}^n)}, \end{aligned}$$

which goes to zero as $n \to \infty$. □

Let us show that $C^\infty(M)$ is dense in $H^s(M)$ for every $s \in \mathbb{R}$. Indeed, let $s \in \mathbb{R}$ and consider $u \in H^s(M)$. It follows from Lemma 1.43 that for each $1 \leqslant i \leqslant N$ there exists a sequence $\{v_{i,n}\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that $v_{i,n} \to (\rho_i \circ \kappa_i^{-1})\, u_i$ in $H^s(\mathbb{R}^n)$ as $n \to \infty$. Observe that since $(\rho_i \circ \kappa_i^{-1})\, u_i \equiv 0$ on the complement of any open set containing $\operatorname{supp} \rho_i$, we may take the $v_{i,n}$'s to be compactly supported on $\kappa_i(U_i)$.

Now set

$$\tilde{u}_{i,n} := \sum_{j=1}^{N} [\tilde{\rho}_j(\, v_{j,n} \circ \kappa_j)] \circ \kappa_i^{-1} = \sum_{j=1}^{N} T_{(\kappa_i \circ \kappa_j^{-1})}[(\tilde{\rho}_j \circ \kappa_j^{-1})\, v_{j,n}].$$

Note that for each $1 \leqslant i \leqslant N$ and for every $n \in \mathbb{N}$ we have $\tilde{u}_{i,n} \in C^\infty(\kappa_i(U_i))$. Furthermore, for each $n \in \mathbb{N}$ we have that the family $\{\tilde{u}_{i,n}\}_{i=1}^N$ satisfies the compatibility condition and thus defines an element $\tilde{u}_n \in C^\infty(M)$. Finally, we claim that $\tilde{u}_n \to u$ in $H^s(M)$.

To see this, note that for every $n \in \mathbb{N}$ we have

$$\begin{aligned}(\rho_i \circ \kappa_i^{-1})\, \tilde{u}_{i,n} &= \sum_{j=1}^{N} (\rho_i \circ \kappa_i^{-1})\, T_{(\kappa_i \circ \kappa_j^{-1})}[(\tilde{\rho}_j \circ \kappa_j^{-1})\, v_{j,n}] \\ &= \sum_{j=1}^{N} T_{(\kappa_i \circ \kappa_j^{-1})}[(\tilde{\rho}_j \rho_i \circ \kappa_j^{-1})\, v_{j,n}].\end{aligned}$$

Now for each i, j we let $K_{i,j} \subset U_i \cap U_j$ be any compact set such that $\operatorname{supp} \rho_i \cap \operatorname{supp} \tilde{\rho}_j \subset \operatorname{int}(K_{i,j})$. By proposition 1.47 we have that $T_{(\kappa_i \circ \kappa_j^{-1})}$ maps $H^s(\kappa_j(K_{i,j}))$ continuously into $H^s(\kappa_i(K_{i,j}))$, and thus

$$\begin{aligned}\lim_{n\to\infty} (\rho_i \circ \kappa_i^{-1})\, \tilde{u}_{i,n} &= \lim_{n\to\infty} \sum_{j=1}^{N} T_{(\kappa_i \circ \kappa_j^{-1})}[(\tilde{\rho}_j \rho_i \circ \kappa_j^{-1})\, v_{j,n}] \\ &= \sum_{j=1}^{N} T_{(\kappa_i \circ \kappa_j^{-1})}[(\rho_j \rho_i \circ \kappa_j^{-1})\, u_j] \\ &= (\rho_i \circ \kappa_i^{-1})\, u_i \quad \text{in } H^s(\mathbb{R}^n).\end{aligned}$$

Since $\|u - \tilde{u}_n\|_{H^s(M)}^2 = \sum_{i=1}^{N} \left\|(\rho_i \circ \kappa_i^{-1})\, u_i - (\rho_i \circ \kappa_i^{-1})\, \tilde{u}_{i,n}\right\|_{H^s(\mathbb{R}^n)}^2$ we conclude our result.

Remark: Note that formula (3.3) depends on many choices and is not "canonical" in general. However, it follows from the compatibility conditions and Proposition 1.47 that if we change our choice of finite compatible atlas on M or that of the partition of unity, we replace the inner product in (3.3) by another one that induces an equivalent norm on $H^s(M)$.

3.2 Pseudo-differential Operators on Compact Manifolds

Now consider M to be a compact smooth n-manifold and let $A : C^\infty(M) \to C^\infty(M)$ be a linear map. If $\phi, \psi \in C^\infty(M)$, we define the linear map $\phi A \psi : C^\infty(M) \to C^\infty(M)$ by

$$(\phi A \psi) u(x) := \phi(x) A(\psi u)(x) \quad \text{for all } u \in C^\infty(M).$$

For any function $f : M \to \mathbb{C}$ and any chart (U, κ), we have defined $f_\kappa = f \circ \kappa^{-1}$ on $\kappa(U)$. Now if $P : C^\infty(U) \to C^\infty(U)$ is a linear map, we define the operator $P_\kappa : C^\infty(\kappa(U)) \to C^\infty(\kappa(U))$ by

$$P_\kappa u := P(u \circ \kappa) \circ \kappa^{-1}.$$

Notice that if P and Q are linear maps from $C^\infty(U)$ to itself, then

$$(PQ)_\kappa u = (PQ)(u \circ \kappa) \circ \kappa^{-1} = P(Q_\kappa u \circ \kappa) \circ \kappa^{-1} = P_\kappa Q_\kappa u,$$

for any $u \in C^\infty(\kappa(U))$. Similarly, we have that $[P, Q]_\kappa = [P_\kappa, Q_\kappa]$ and that $(\phi P \psi\, u)_\kappa = (\phi P \psi)_\kappa (u_\kappa)$.

Definition 3.6. Let M be a compact smooth n-manifold. A *pseudo-differential operator* of order $m \in \mathbb{R}$ on M is a linear map $A : C^\infty(M) \to C^\infty(M)$ such that for every compatible chart (U, κ) and for any $\phi, \psi \in C^\infty(M)$ supported in U we have that $(\phi A \psi)_\kappa \in Op\, S^m$ (here an extension by zero in $\mathbb{R}^n \setminus \kappa(U)$ is understood). We denote the set of all such operators by $\Psi^m(M)$.

The next proposition shows that pseudo-differential operators on M satisfies similar compatibility conditions to that for smooth functions and distributions on M.

Proposition 3.7. *Let M be a compact smooth n-manifold and suppose $(U_\alpha, \kappa_\alpha)$ and (U_β, κ_β) are two compatible charts with $U_\alpha \cap U_\beta \neq \varnothing$. If $A : C^\infty(M) \to C^\infty(M)$ is a linear map and*

$\phi, \psi \in C^\infty(M)$ are supported in $U_\alpha \cap U_\beta$, then

$$(\phi A\psi)_{\kappa_\alpha} \in Op\, S^m \iff (\phi A\psi)_{\kappa_\beta} \in Op\, S^m.$$

Proof. Let $u \in C^\infty(M)$. A linear map $A : C^\infty(M) \to C^\infty(M)$ must satisfy

$$(\phi A\psi\, u)_{\kappa_\alpha} = (\phi A\psi\, u)_{\kappa_\beta} \circ (\kappa_\beta \circ \kappa_\alpha^{-1}).$$

Now let $\tilde{\psi} \in C_c^\infty(M)$ be supported in $U_\alpha \cap U_\beta$ and such that $\tilde{\psi} \equiv 1$ on a neighbourhood of $\operatorname{supp}\psi$. Note that

$$(\phi A\psi\, u)_{\kappa_\alpha} = (\phi A\psi)_{\kappa_\alpha}(\tilde{\psi}\, u)_{\kappa_\alpha},$$

and that

$$\begin{aligned}(\phi A\psi)_{\kappa_\alpha}(\tilde{\psi}\, u)_{\kappa_\alpha} &= (\phi A\psi\, u)_{\kappa_\beta} \circ (\kappa_\beta \circ \kappa_\alpha^{-1}) \\ &= [(\phi A\psi)_{\kappa_\beta}(\tilde{\psi} u)_{\kappa_\beta}] \circ (\kappa_\beta \circ \kappa_\alpha^{-1}) \\ &= (\phi A\psi)_{\kappa_\beta}[(\tilde{\psi} u)_{\kappa_\alpha} \circ (\kappa_\alpha \circ \kappa_\beta^{-1})] \circ (\kappa_\beta \circ \kappa_\alpha^{-1}).\end{aligned}$$

Since we may exchange the roles of $\alpha, \beta \in A$, it follows from Corollary 2.21.1 that

$$(\phi A\psi)_{\kappa_\alpha} \in Op\, S^m \iff (\phi A\psi)_{\kappa_\beta} \in Op\, S^m.$$

□

When M is a compact smooth n-manifold, it is possible to find a finite compatible atlas on M for which there exists a subordinated partition of unity such that any four of its elements are compactly supported in the same chart neighbourhood of the atlas. Such property is useful when dealing with pseudo-differential operators and we shall now turn our attention to prove it.

So let (X, d) be a metric space and let $S \subset X$ be a bounded subset. Recall that the *diameter* of S is defined by

$$\operatorname{diam}(S) := \sup\{d(x, y) \,:\, x, y \in S\}.$$

We then have the following result for compact metric spaces.

Lemma 3.8 (The Lebesgue number Lemma)**.** *Let $\mathcal{U}$ be an open cover of a metric space (X, d). If X is compact, there exists a number $\lambda > 0$ such that every subset of X having diameter less than λ is contained in some element of $\mathcal{U}$.*

A number $\lambda > 0$ satisfying this condition is called a Lebesgue number for the covering $\mathcal{U}$.

Proof. J. Munkres [12, Lemma 27.5, p. 175]. □

Lemma 3.9. *Let M be a compact smooth n-manifold. Then there exists a finite compatible atlas $\{(U_i, \kappa_i)\}_{i=1}^{I_1}$ for which there is a subordinated partition of unity $\{\rho_j\}_{j=1}^{J_0}$ such that any four of the functions $\rho_j, \rho_k, \rho_l, \rho_m$ are supported in some open set U_i.*

Proof. Recall that a smooth n-manifold is a locally compact normal space, and thus by Urysohn Metrization Theorem 1.56 it is metrizable. Let $d : M \times M \to [0, \infty)$ be a metric that induces the same topology as the one already defined on M. Let $\mathcal{A} := \{(U_i, \kappa_i)\}_{i=1}^{I_0}$ be a finite compatible atlas on M; we assume that the $\kappa_i(U_i)$ are mutually disjoint. Note that the sets U_i do not need to be connected sets. Since the U_i's form an open cover of (M, d), it follows from the compactness of M and Lemma 3.8 that there is a number $\lambda > 0$ such that any subset of M with diameter less than λ is contained in one of the U_i's.

Now for each $x \in M$ we consider the open ball

$$\mathbb{U}_d(x, \lambda/8) := \{\, y \in M \,:\, d(x, y) < \lambda/8\}.$$

It is clear that the diameter of each such set is less than or equal to $\lambda/4$. By compactness there exists $J_0 \in \mathbb{N}$ such that

$$M \subset \bigcup_{j=1}^{J_0} \mathbb{U}_d(x_j, \lambda/8).$$

Set $B_j = \mathbb{U}_d(x_j, \lambda/8)$ for each $1 \leqslant j \leqslant J_0$. We claim that the family $\{B_j\}_{1\leqslant j\leqslant J_0}$ has the following property: Any four sets B_{j_1}, B_{j_2}, B_{j_3}, B_{j_4} can be grouped in clusters that are mutually disjoint and where each cluster lies in one of the sets U_i.

This property is seen as follows: Let $1 \leqslant j_1, j_2, j_3, j_4 \leqslant J_0$ be given. First, adjoin to B_{j_1} those of the B_{j_k}, $k = 2, 3, 4$, that it intersects with. Second, adjoin to this union those of the remaining sets that it intersects with. Finally, do it once more. Those steps give us the first cluster. If any of the four sets are not contained in the first cluster, repeat the procedure with these (at most three) remaining sets. This gives us the second cluster. Now the procedure is repeated with the remaining sets, and so on. Note that this leads us to at most four clusters which, by construction, are mutually disjoint and each cluster has a diameter of less than λ, hence lies in a set U_i for some $1 \leqslant i \leqslant I_0$.

Now we construct a new finite atlas by considering the following new coordinate mappings: Assume that B_{j_1}, B_{j_2}, B_{j_3}, B_{j_4} gave rise to the disjoint clusters U', U'', ..., where $U' \subset U_{i'}$, $U'' \subset U_{i''}$, ... Then use $\kappa_{i'}$ on U', $\kappa_{i''}$ on U'', ... (if necessary, followed by a linear transformation Φ'', ... to separate the images) to define the mapping $\kappa : U' \cup U'' \cup \cdots \longrightarrow \kappa_{i'}(U') \cup \Phi''\kappa_{i''}(U'') \cup \cdots$. This gives a new coordinate mapping, for which $B_{j_1} \cup B_{j_2} \cup B_{j_3} \cup B_{j_4}$ equals the initial set $U' \cup U'' \cup \cdots$. In this way, finitely many new coordinate mappings, say $\{(U_i, \kappa_i)\}_{i=I_0+1}^{I_1}$, are used to form a new compatible atlas for M, and we have established a mapping $(j_1, j_2, j_3, j_4) \mapsto i = i(j_1, j_2, j_3, j_4)$ for $i > I_0$ and for which

$$B_{j_1} \cup B_{j_2} \cup B_{j_3} \cup B_{j_4} \subset U_{i(j_1,j_2,j_3,j_4)}.$$

It follows from Lemma 1.62 that there exists a partition of unity $\{\rho_j\}_{j=1}^{J_0}$ strictly subordinated to the open cover $\{B_j\}_{j=1}^{J_0}$. By the above construction we have that this partition of unity has the desired property with respect to the atlas $\{(U_i, \kappa_i)\}_{i=I_0+1}^{I_1}$. □

Let us show how convenient is the above lemma when we consider composition of pseudo-differential operators.

Proposition 3.10. *Let M be a compact smooth n-manifold. If $A \in \Psi^{m_1}(M)$ and $B \in \Psi^{m_2}(M)$, then the composition $AB \in \Psi^{m_1+m_2}(M)$.*

Proof. It is clear from the definitions that AB defines a linear map from $C^\infty(M)$ to itself.

Let (U, κ) be any compatible chart on M and let $\phi, \psi \in C^\infty(M)$ with support in U. By Lemma 3.9 there exists a compatible finite atlas $\mathcal{A} = \{(U_i, \kappa_i)\}_{i=1}^{I_1}$ for which there is a subordinated partition of unity $\{\rho_j\}_{j=1}^{J_0}$ such that any four of the functions $\rho_j, \rho_k, \rho_l, \rho_m$ are supported in some open set U_i. Write

$$\phi AB\psi = \sum_{j,k,l,m} \phi\rho_j A\rho_k\rho_l B\rho_m\psi.$$

Now each term $(\phi\rho_j A\rho_k\rho_l B\rho_m\psi)$ has support in a set U_i and we can write

$$(\phi\rho_j A\rho_k\rho_l B\rho_m\psi)_{\kappa_i} = (\phi\rho_j A\rho_k)_{\kappa_i}(\rho_l B\rho_m\psi)_{\kappa_i}.$$

By hypothesis $(\phi\rho_j A\rho_k)_{\kappa_i} \in Op\, S^{m_1}$ and $(\rho_l B\rho_m\psi)_{\kappa_i} \in Op\, S^{m_2}$, so Theorem 2.12 implies that

$$(\phi\rho_j A\rho_k\rho_l B\rho_m\psi)_{\kappa_i} \in Op\, S^{m_1+m_2}.$$

Because $\phi\rho_j, \rho_m\psi \in C_c^\infty(M)$ are both supported in $U \cap U_i$, it follows from Proposition 3.7 that

$$(\phi\rho_j A\rho_k\rho_l B\rho_m\psi)_\kappa \in Op\, S^{m_1+m_2}.$$

Since this holds for all $1 \leqslant j, k, l, m \leqslant J_0$ we conclude that $(\phi AB\psi)_\kappa \in Op\, S^{m_1+m_2}$, hence $AB \in \Psi^{m_1+m_2}(M)$. □

We can also use partitions of unity to extend a pseudo-differential operator $A \in \Psi^m(M)$, initially defined as a linear operator on $C^\infty(M)$, to a linear operator on the space of distributions $\mathcal{D}'(M)$. Indeed, let M be a compact smooth n-manifold and suppose $A \in \Psi^m(M)$. By Lemma 3.9 there exists a compatible finite atlas $\mathcal{A} = \{(U_i, \kappa_i)\}_{i=1}^{I_1}$ for which there is a subordinated partition of unity $\{\rho_k\}_{k=1}^{K_0}$ such that any two of the functions ρ_k, ρ_l are supported in some open set U_i. Write

$$A = \sum_{k,l} \rho_k A \rho_l.$$

Then one can extend each $\rho_k A \rho_l$ to $\mathcal{D}'(M)$ is the following way: Fix $1 \leqslant i \leqslant I_1$ such that ρ_k, ρ_l are both supported in U_i. Let $\tilde{\rho}_k, \tilde{\rho}_l \in C_c^\infty(M)$ be both supported on U_i and such that $\tilde{\rho}_k \equiv 1$ on a neighbourhood of $\operatorname{supp} \rho_k$, and $\tilde{\rho}_l \equiv 1$ on a neighbourhood of $\operatorname{supp} \rho_l$. For each $u \in C^\infty(M)$ we have that $\rho_k A \rho_l\, u \in C^\infty(M)$ so it follows from the compatibility condition that for any $1 \leqslant j \leqslant I_1$

$$\begin{aligned}(\rho_k A \rho_l\, u)_{\kappa_j} &= (\rho_k A \rho_l\, u)_{\kappa_i} \circ (\kappa_i \circ \kappa_j^{-1}) \\ &= (\tilde{\rho}_k \circ \kappa_j^{-1})\, T_{(\kappa_j \circ \kappa_i^{-1})}[(\rho_k A \rho_l)_{\kappa_i} (\tilde{\rho}_l u)_{\kappa_i}].\end{aligned}$$

Thus for $u \in \mathcal{D}'(M)$ we set

$$(\rho_k A \rho_l\, u)_j := (\tilde{\rho}_k \circ \kappa_j^{-1})\, T_{(\kappa_j \circ \kappa_i^{-1})}[(\rho_k A \rho_l)_{\kappa_i} (\tilde{\rho}_l u)_{\kappa_i}] \quad \text{for } 1 \leqslant j \leqslant I_1. \tag{3.7}$$

To show that the above induces a well-defined element in $\mathcal{D}'(M)$ we need to check that for each j we have $(\rho_k A \rho_l\, u)_j \in \mathcal{D}'(\kappa_j(U_j))$ and that the family $\{(\rho_k A \rho_l\, u)_j\}_{j=1}^{I_1}$ satisfies the compatibility condition for distributions. The later follows immediately from Lemma 3.2, for if $(\rho_k A \rho_l\, u)_j \in \mathcal{D}'(\kappa_j(U_j))$ for any $1 \leqslant j \leqslant I_1$, then for any j, j' we have

$$\begin{aligned}T_{(\kappa_{j'} \circ \kappa_j^{-1})}(\rho_k A \rho_l\, u)_j &= T_{(\kappa_{j'} \circ \kappa_j^{-1})}\, (\tilde{\rho}_k \circ \kappa_j^{-1})\, T_{(\kappa_j \circ \kappa_i^{-1})}[(\rho_k A \rho_l)_{\kappa_i} (\tilde{\rho}_l u)_{\kappa_i}] \\ &= (\tilde{\rho}_k \circ \kappa_{j'}^{-1})\, T_{(\kappa_{j'} \circ \kappa_i^{-1})}[(\rho_k A \rho_l)_{\kappa_i} (\tilde{\rho}_l u)_{\kappa_i}] \\ &= (\rho_k A \rho_l\, u)_{j'} \quad \text{in } \kappa_{j'}(U_j \cap U_{j'}).\end{aligned}$$

To see that $(\rho_k A \rho_l\, u)_j \in \mathcal{D}'(\kappa_j(U_j))$ we consider the following chain of maps

$$u_{\kappa_i} \xrightarrow{(1)} (\tilde{\rho}_l \circ \kappa_i^{-1})\, u_{\kappa_i} = (\tilde{\rho}_l u)_{\kappa_i} \xrightarrow{(2)} (\rho_k A \rho_l)_{\kappa_i} (\tilde{\rho}_l u)_{\kappa_i} \xrightarrow{(3)} T_{(\kappa_j \circ \kappa_i^{-1})}[(\rho_k A \rho_l)_{\kappa_i} (\tilde{\rho}_l u)_{\kappa_i}]$$

and

$$T_{(\kappa_j \circ \kappa_i^{-1})}[(\rho_k A \rho_l)_{\kappa_i} (\tilde{\rho}_l u)_{\kappa_i}] \xrightarrow{(4)} (\tilde{\rho}_k \circ \kappa_j^{-1})\, T_{(\kappa_j \circ \kappa_i^{-1})}[(\rho_k A \rho_l)_{\kappa_i} (\tilde{\rho}_l u)_{\kappa_i}].$$

Now, because $(\tilde{\rho}_l \circ \kappa_i^{-1}) \in C_c^\infty(\kappa_i(U_i))$ we get that (1) maps $u_{\kappa_i} \in \mathcal{D}'(\kappa_i(U_i))$ into $(\tilde{\rho}_l u)_{\kappa_i} \in \mathcal{S}'(\mathbb{R}^n)$. Next, we use the hypothesis $(\rho_k A \rho_l)_{\kappa_i} \in Op\, S^m$, so that (2) maps $\mathcal{S}'(\mathbb{R}^n)$ continuously into $\mathcal{S}'(\mathbb{R}^n)$. Moreover, if $K \subset \kappa_i(U_i)$ is a compact subset such that $\operatorname{supp}(\rho_k \circ \kappa_i^{-1}) \subset \operatorname{int}(K)$, then $(\rho_k A \rho_l)_{\kappa_i} (\tilde{\rho}_l u)_{\kappa_i} \equiv 0$ on $\mathbb{R}^n \setminus K$ and it follows that we may identify $(\rho_k A \rho_l)_{\kappa_i} (\tilde{\rho}_l u)_{\kappa_i}$ as an element in $\mathcal{D}'(\kappa_i(U_i))$.

Taking the restriction of $(\rho_k A \rho_l)_{\kappa_i} (\tilde{\rho}_l u)_{\kappa_i} \in \mathcal{D}'(\kappa_i(U_i))$ to the open subset $\kappa_i(U_i \cap U_j)$ and using that $T_{(\kappa_j \circ \kappa_i^{-1})} : \mathcal{D}'(\kappa_i(U_i \cap U_j)) \to \mathcal{D}'(\kappa_j(U_i \cap U_j))$ is a isomorphism, it follows that (3) maps $\mathcal{D}'(\kappa_i(U_i))$ into $\mathcal{D}'(\kappa_j(U_i \cap U_j))$.

Finally, since $(\tilde{\rho}_k \circ \kappa_j^{-1}) \in C^\infty(\kappa_j(U_j))$ is such that $(\tilde{\rho}_k \circ \kappa_j^{-1}) \equiv 0$ in a neighbourhood of $[\partial(\kappa_j(U_i \cap U_j))] \cap \kappa_j(U_j)$, it follows that (4) maps $\mathcal{D}'(\kappa_j(U_i \cap U_j))$ into $\mathcal{D}'(\kappa_j(U_j))$.

This implies that $(\rho_k A \rho_l\, u)_j \in \mathcal{D}'(\kappa_j(U_j))$ for every $1 \leqslant j \leqslant I_1$. In this way we have that the linear map $\rho_k A \rho_l : \mathcal{D}'(M) \to \mathcal{D}'(M)$ is well-defined, and so is the linear operator $A := \sum_{k,l} \rho_k A \rho_l$.

It is clear from the above construction that $A : \mathcal{D}'(M) \to \mathcal{D}'(M)$ is an extension of $A \in \Psi^m(M)$ to the space of distributions.

When we consider Sobolev spaces over M we have the following result

Theorem 3.11. *Let M be a compact smooth n-manifold and let $A \in \Psi^m(M)$. Then for all $s \in \mathbb{R}$ we have that the operator A, initially defined on $C^\infty(M)$, extends to a continuous linear mapping from $H^s(M)$ to $H^{s-m}(M)$.*

Proof. Let $s \in \mathbb{R}$ be an arbitrary but fixed real number. Let $\mathcal{A} = \{(U_i, \kappa_i)\}_{i=1}^{I_1}$ and $\{\rho_k\}_{k=1}^{K_0}$ be a compatible finite atlas on M and a partition of unity given by Lemma 3.9, respectively. Write

$$A = \sum_{k,l \leqslant K_0} \rho_k A \rho_l,$$

where for each k, l there is an $1 \leqslant i \leqslant I_1$ such that both ρ_k, ρ_l have support in U_i. Consider the extension $\rho_k A \rho_l : \mathcal{D}'(M) \to \mathcal{D}'(M)$ as above. To conclude the theorem, it suffices to show that for each k, l we have that $\rho_k A \rho_l$ maps $H^s(M)$ continuously into $H^{s-m}(M)$. We start by showing that if $u \in H^s(M)$, then $(\rho_k A \rho_l\, u) \in H^{s-m}(M)$.

Fix $1 \leqslant i \leqslant I_1$ such that ρ_k, ρ_l are both supported in U_i. Then for $u \in H^s(M)$ we use (3.7) to define $(\rho_k A \rho_l\, u) \in \mathcal{D}'(M)$ by the system

$$(\rho_k A \rho_l\, u)_j := (\tilde{\rho}_k \circ \kappa_j^{-1})\, T_{(\kappa_j \circ \kappa_i^{-1})}[(\rho_k A \rho_l)_{\kappa_i}(\tilde{\rho}_l u)_i] \quad \text{for } 1 \leqslant j \leqslant I_1.$$

We claim that for each j we have $(\rho_k A \rho_l\, u)_j \in H^{s-m}_{loc}(\kappa_j(U_j))$ and thus $(\rho_k A \rho_l\, u) \in H^{s-m}(M)$. To see this we consider the chain of maps

$$u_i \xrightarrow{(1)} (\tilde{\rho}_l \circ \kappa_i^{-1})\, u_i = (\tilde{\rho}_l u)_i \xrightarrow{(2)} (\rho_k A \rho_l)_{\kappa_i}(\tilde{\rho}_l u)_i \xrightarrow{(3)} T_{(\kappa_j \circ \kappa_i^{-1})}[(\rho_k A \rho_l)_{\kappa_i}(\tilde{\rho}_l u)_i]$$

and

$$T_{(\kappa_j \circ \kappa_i^{-1})}[(\rho_k A \rho_l)_{\kappa_i}(\tilde{\rho}_l u)_i] \xrightarrow{(4)} (\tilde{\rho}_k \circ \kappa_j^{-1})\, T_{(\kappa_j \circ \kappa_i^{-1})}[(\rho_k A \rho_l)_{\kappa_i}(\tilde{\rho}_l u)_i].$$

Now, (1) maps $H^s_{loc}(\kappa_i(U_i))$ continuously into $H^s(\mathbb{R}^n)$, so that $(\tilde{\rho}_l u)_i \in H^s(\mathbb{R}^n)$. Next, from Theorem 2.13 we obtain that $(\rho_k A \rho_l)_{\kappa_i} \in Op\, S^m$ maps $H^s(\mathbb{R}^n)$ continuously into $H^{s-m}(\mathbb{R}^n)$. Moreover, if $K \subset \kappa_i(U_i)$ is a compact subset such that $\operatorname{supp}(\rho_k \circ \kappa_i^{-1}) \subset \operatorname{int}(K)$, then $(\rho_k A \rho_l)_{\kappa_i}(\tilde{\rho}_l u)_i \equiv 0$ on $\mathbb{R}^n \setminus K$ as a distribution, and so (2) is a continuous map from $H^s(\mathbb{R}^n)$ to $H^{s-m}(K)$. In particular, (2) is continuous from $H^s(\mathbb{R}^n)$ to $H^{s-m}_{loc}(\kappa_i(U_i))$.

By Proposition 1.50 we have that $T_{(\kappa_j \circ \kappa_i^{-1})}$ is a isomorphism from $H^{s-m}_{loc}(\kappa_i(U_i \cap U_j))$ to $H^{s-m}_{loc}(\kappa_j(U_i \cap U_j))$, and hence (3) maps $H^{s-m}_{loc}(\kappa_i(U_i))$ into $H^{s-m}_{loc}(\kappa_j(U_i \cap U_j))$

Finally, (4) maps $H^{s-m}_{loc}(\kappa_j(U_i \cap U_j))$ into $H^{s-m}_{loc}(\kappa_j(U_j))$ since $(\tilde{\rho}_k \circ \kappa_j^{-1}) \in C^\infty(\kappa_j(U_j))$ is such that $(\tilde{\rho}_k \circ \kappa_j^{-1}) \equiv 0$ in a neighbourhood of $[\partial(\kappa_j(U_i \cap U_j))] \cap \kappa_j(U_j)$. This implies that $\rho_k A \rho_l\, u \in H^{s-m}(M)$ whenever $u \in H^s(M)$.

Let us now show that $\rho_k A \rho_j : H^s(M) \to H^{s-m}(M)$ is a continuous linear map. By Lemma 1.62 there exists a partition of unity $\{\psi_j\}_{j=1}^{I_1}$ strictly subordinated to the open cover $\{U_j\}_{j=1}^{I_1}$ of M. By equation (3.3) and Proposition 3.5 we may set

$$\|\rho_k A \rho_l\, u\|_{H^{s-m}(M)} := \left(\sum_{j=1}^{I_1} \left\| (\psi_j \circ \kappa_j^{-1})\, (\rho_k A \rho_l\, u)_j \right\|^2_{H^{s-m}(\mathbb{R}^n)} \right)^{1/2} \quad \text{for } u \in H^s(M).$$

Note that for any $u \in H^s(M)$ and for each $1 \leqslant j \leqslant I_1$ we have

$$\begin{aligned} (\psi_j \circ \kappa_j^{-1})\, (\rho_k A \rho_l\, u)_j &= (\psi_j\, \tilde{\rho}_k \circ \kappa_j^{-1})\, T_{(\kappa_j \circ \kappa_i^{-1})}[(\rho_k A \rho_l)_{\kappa_i}(\tilde{\rho}_l u)_i] \\ &= T_{(\kappa_j \circ \kappa_i^{-1})}[(\psi_j \tilde{\rho}_k \circ \kappa_i^{-1})\, (\rho_k A \rho_l)_{\kappa_i}(\tilde{\rho}_l u)_i], \end{aligned}$$

thus for any compact set $K_{i,j} \subset \kappa_i(U_j \cap U_i)$ such that $\operatorname{supp}(\psi_j\, \tilde{\rho}_k \circ \kappa_i^{-1}) \subset \operatorname{int}(K_{i,j})$ we have that $(\psi_j\tilde{\rho}_k \circ \kappa_i^{-1})\,(\rho_k A\rho_l)_{\kappa_i}(\tilde{\rho}_l u)_i \in H^{s-m}(K_{i,j})$. Setting $K'_{i,j} = (\kappa_j \circ \kappa_i^{-1})(K_{i,j})$, it follows from Proposition 1.47 that $T_{(\kappa_j\circ\kappa_i^{-1})}$ maps $H^{s-m}(K_{i,j})$ continuously into $H^{s-m}(K'_{i,j})$. Hence for each $1 \leqslant j \leqslant I_1$ there exists a constant $A_j > 0$ such that

$$\begin{aligned}\|(\psi_j \circ \kappa_j^{-1})\,(\rho_k A\rho_l\, u)_j\|_{H^{s-m}(\mathbb{R}^n)} &= \|T_{(\kappa_j\circ\kappa_i^{-1})}[(\psi_j\tilde{\rho}_k \circ \kappa_i^{-1})\,(\rho_k A\rho_l)_{\kappa_i}(\tilde{\rho}_l u)_i]\|_{H^{s-m}(\mathbb{R}^n)}\\ &\leqslant A_j\,\left\|(\psi_j \circ \kappa_i^{-1})\,(\rho_k A\rho_l)_{\kappa_i}(\tilde{\rho}_l u)_i\right\|_{H^{s-m}(\mathbb{R}^n)}.\end{aligned}$$

Since $0 \leqslant \psi_j \leqslant 1$ and $(\rho_k A\rho_l)_{\kappa_i}$ maps $H^s(\mathbb{R}^n)$ continuously into $H^{s-m}(\mathbb{R}^n)$ we have that there exists a constant $B_j > 0$ such that

$$\|(\psi_j \circ \kappa_j^{-1})\,(\rho_k A\rho_l\, u)_j\|_{H^{s-m}(\mathbb{R}^n)} \leqslant B_j\,\left\|(\tilde{\rho}_l \circ \kappa_i^{-1})\,u_i\right\|_{H^s(\mathbb{R}^n)}.$$

Taking $C = I_1 \cdot \max\{\, B_j \,:\, 1 \leqslant j \leqslant I_1 \,\}$ we conclude that

$$\|\rho_k A\rho_l\, u\|_{H^{s-m}(M)} \leqslant \sum_{j=1}^{I_1}\|(\psi_j \circ \kappa_j^{-1})\,(\rho_k A\rho_l\, u)_j\|_{H^{s-m}(\mathbb{R}^n)} \leqslant C\,\left\|(\tilde{\rho}_l \circ \kappa_i^{-1})\,u_i\right\|_{H^s(\mathbb{R}^n)}.$$

Since different choices of the partition of unity on M induces equivalent norms in $H^s(M)$, we conclude that $\rho_k A\rho_l$ maps $H^s(M)$ continuously into $H^{s-m}(M)$.

Lastly, since $C^\infty(M)$ is a dense subspace of $H^s(M)$, it follows that the above extension of $\rho_k A\rho_l$ is the unique extension that makes it continuous. □

3.3 Commutator characterization of Pseudo-differential Operators on Compact Manifolds

This section is dedicated to the commutator characterization of pseudo-differential operators on compact smooth n-manifolds. A characterization of pseudo-differential operators on compact manifolds via commutators with smooth vector fields was stated and proved by R. Coifman and Y. Meyer [2] for the case of 0-order operators. By considering commutators with smooth vector fields and smooth functions, V. Turunen [20] presented a characterization that holds for pseudo-differential operators of any given order $m \in \mathbb{R}$. Here we provide a slightly stronger version of that characterization.

Let us recall some concepts and results. Recall Theorem 2.18, whose proof lies at the end of section 2.5.

Theorem 2.18. *Let $m \in \mathbb{R}$ and let A be a linear map from $C_c^\infty(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$. Then the following conditions are equivalent:*

(i) $A \in \Psi^m_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$.

(ii) For any $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ and for any sequence of partial differential operators with smooth coefficients $\mathcal{D} = \{D_j\}_{j\in\mathbb{N}_0} \subset \Psi^1_{loc}(\mathbb{R}^n \times \mathbb{R}^n)$ we have

$$\begin{cases} B_0 = \phi A\psi \in \mathcal{B}(H^m(\mathbb{R}^n), L^2(\mathbb{R}^n)),\\ B_{k+1} = [B_k, D_k] \in \mathcal{B}(H^{m-d_{\mathcal{D},k}}(\mathbb{R}^n), L^2(\mathbb{R}^n)),\end{cases}$$

where $d_{\mathcal{D},k} = \sum_{j=0}^k (1 - ord(D_j))$.

An immediate consequence of Theorem 2.18 is that if we are given $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ and a linear map A from $C_c^\infty(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$ such that $\phi A\psi$ satisfies condition (ii), then $\phi A\psi \in Op\, S^m$.

Recall that if M is a smooth n-manifold, then $DO^1(M)$ is the space of all linear operators $D : C^\infty(M) \to C^\infty(M)$ of the form $Df := X(f) + gf$, where X is a smooth vector field on M and g is a smooth function on M.

For $D \in DO^1(M)$, define

$$ord(D) := \begin{cases} 0 & \text{when } X \equiv 0, \\ 1 & \text{otherwise.} \end{cases}$$

If we further assume that M is compact, then for any $g \in C^\infty(M)$ the multiplication map $f \mapsto gf$ is an element of $\Psi^0(M)$, and for any smooth vector field X on M, we have $X \in \Psi^1(M)$. It is also clear that the commutator $[X, g] \in \Psi^0(M)$. It then follows from Theorem 3.11 that $D \in DO^1(M)$ maps $H^s(M)$ continuously into $H^{s-ord(D)}(M)$ for any $s \in \mathbb{R}$.

Finally, we obtain the following commutator characterization for pseudo-differential operators on compact manifolds:

Theorem 3.12. *Let M be a compact smooth n-manifold and let $m \in \mathbb{R}$. Suppose A is a linear operator on $C^\infty(M)$. Then the following conditions are equivalent:*

(i) $A \in \Psi^m(M)$.

(ii) For any sequence $\mathcal{D} = \{D_j\}_{j\in\mathbb{N}_0} \subset DO^1(M)$ we have

$$\begin{cases} A_0 = A \in \mathcal{B}(H^m(M), L^2(M)), \\ A_{k+1} = [A_k, D_k] \in \mathcal{B}(H^{m-d_{\mathcal{D},k}}(M), L^2(M)), \end{cases}$$

where $d_{\mathcal{D},k} = \sum_{j=0}^{k}(1 - ord(D_j))$.

Proof. $(i) \implies (ii)$: Let $A \in \Psi^m(M)$. It follows from Theorem 3.11 that the linear map $A_0 := A$ is continuous from $H^m(M)$ to $L^2(M)$.

We shall show that for any $D \in DO^1(M)$ we have that $[A, D] \in \Psi^{m-(1-ord(D))}(M)$, and thus statement (ii) follows by iteration and Theorem 3.11.

So let $D \in DO^1(M)$ and let (U, κ) be an arbitrary but fixed compatible chart on M. Note that for any $\phi, \psi \in C_c^\infty(M)$ supported in U we have

$$\phi A[\psi, D] = \phi A\psi D - \phi AD\psi,$$
$$[\phi, D]A\psi = \phi DA\psi - D\phi A\psi,$$

and

$$[\phi A\psi, D] = \phi A\psi D - D\phi A\psi.$$

Therefore we have the identity

$$\begin{aligned} [\phi A\psi, D] - \phi A[\psi, D] - [\phi, D]A\psi &= (\phi A\psi D - D\phi A\psi) - (\phi A\psi D - \phi AD\psi) - (\phi DA\psi - D\phi A\psi) \\ &= \phi AD\psi - \phi DA\psi \\ &= \phi[A, D]\psi. \end{aligned}$$

The above equality then implies (see remarks at the beginning of section 3.2)

$$(\phi[A, D]\psi)_\kappa = [(\phi A\psi)_\kappa, D_\kappa] - (\phi A[\psi, D])_\kappa - ([\phi, D]A\psi)_\kappa. \tag{3.8}$$

Let $\chi_{\phi,\psi} \in C_c^\infty(M)$ be supported in U and such that $0 \leqslant \chi_{\phi,\psi} \leqslant 1$ and $\chi_{\phi,\psi} \equiv 1$ on the compact set $\operatorname{supp}\phi \bigcup \operatorname{supp}\psi$.

Now, if $ord(D) = 0$ then $[\psi, D] = [\phi, D] = 0$. This implies $(\phi[A, D]\psi)_\kappa = [(\phi A\psi)_\kappa, D_\kappa] = [(\phi A\psi)_\kappa, (\chi_{\phi,\psi}D)_\kappa]$. It then follows from the composition formula for pseudo-differential operators on $\mathbb{R}^n$ given in Theorem 2.12 that $(\phi[A, D]\psi)_\kappa \in Op\, S^{m-1}$.

If $ord(D) = 1$, then the composition formula for pseudos on $\mathbb{R}^n$ gives us $[(\phi A\psi)_\kappa, (\chi_{\phi,\psi}D)_\kappa] \in Op\, S^m$. Moreover, we would have that $[\psi, D]$ and $[\phi, D]$ define smooth functions on M with compact support in U and thus $(\phi A[\psi, D])_\kappa, ([\phi, D]A\psi)_\kappa \in Op\, S^m$ by hypothesis. Therefore by (3.8) we have $(\phi[A, D]\psi)_\kappa \in Op\, S^m$.

The conclusion is that for any $D \in DO^1(M)$ we have $[A, D] \in \Psi^{m-(1-ord(D))}(M)$ as desired.

$(ii) \implies (i)$: Suppose that the linear map $A : C^\infty(M) \to C^\infty(M)$ satisfies condition (ii). To conclude that $A \in \Psi^m(M)$ we need to show that for any compatible chart (U, κ) and for any $\phi, \psi \in C_c^\infty(M)$ supported in U we have that $(\phi A \psi)_\kappa \in Op\, S^m$. We shall do this by proving that if C is any partial differential operator of order at most 1 on $\mathbb{R}^n$ with smooth coefficients, then $[(\phi A\psi)_\kappa, C]$ maps $H^{m-(1-ord(C))}(\mathbb{R}^n)$ continuously into $L^2(\mathbb{R}^n)$. The conclusion that $(\phi A\psi)_\kappa \in Op\, S^m$ then follows from Theorem 2.18.

So fix a compatible chart (U, κ) on M and fix $\phi, \psi \in C_c^\infty(M)$ both supported in U. Consider $\chi_{\phi,\psi} \in C_c^\infty(\mathbb{R}^n)$ supported in $\kappa(U)$ and such that $0 \leqslant \chi_{\phi,\psi} \leqslant 1$ and $\chi_{\phi,\psi} \equiv 1$ on the compact set $\operatorname{supp} \phi_\kappa \bigcup \operatorname{supp} \psi_\kappa$. Thus if C is a partial differential operator of order at most 1 on $\mathbb{R}^n$ with smooth coefficients, then so is $\chi_{\phi,\psi} C$. Moreover, since $\chi_{\phi,\psi}$ has compact support in $\kappa(U)$ we get that there exists $D \in DO^1(M)$ such that $D_\kappa = \chi_{\phi,\psi} C$. Note that by construction $ord(D) \leqslant ord(C)$.

It follows from equation (3.8) that

$$[(\phi A\psi)_\kappa, D_\kappa] = (\phi A[\psi, D])_\kappa + ([\phi, D]A\psi)_\kappa + (\phi[A, D]\psi)_\kappa.$$

By hypothesis we have that $[A, D]$ maps $H^{m-(1-ord(D))}(M)$ continuously into $L^2(M)$, therefore $(\phi[A, D]\psi)_\kappa$ maps $H^{m-(1-ord(D))}(\mathbb{R}^n)$ continuously into $L^2(\mathbb{R}^n)$, and thus it maps $H^{m-(1-ord(C))}(\mathbb{R}^n)$ continuously into $L^2(\mathbb{R}^n)$.

If $ord(D) = 0$ then $[\psi, D] = [\phi, D] = 0$ and trivially we have that $\phi A[\psi, D]$ and $[\phi, D]A\psi$ are continuous from $H^{m-(1-ord(C))}(M)$ to $L^2(M)$.

If $ord(D) = 1$, then $ord(C) = 1$. Furthermore, $[\psi, D]$ and $[\phi, D]$ define smooth functions on M with compact support in U, so by hypothesis the maps $\phi A[\psi, D]$ and $[\phi, D]A\psi$ are continuous from $H^m(M)$ to $L^2(M)$. In particular, it follows that $(\phi A[\psi, D])_\kappa$ and $([\phi, D]A\psi)_\kappa$ are continuous linear maps from $H^m(\mathbb{R}^n) = H^{m-(1-ord(C))}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

The conclusion is that $[(\phi A\psi)_\kappa, D_\kappa] = [(\phi A\psi)_\kappa, \chi_{\phi,\psi} C] = [(\phi A\psi)_\kappa, C]$ is a continuous linear map from $H^{m-(1-ord(C))}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and this finishes the proof. □

We are now in position to discuss in precise terms why the above theorem is considered a slightly stronger version than the commutator characterization V. Turunen [20, Theorem 3.1], which we state here for the reader's convenience

Theorem (V. Turunen [20]). *Let $m \in \mathbb{R}$ and let $A : C^\infty(M) \to C^\infty(M)$ be a linear map. Then the following conditions are equivalent:*

(i) $A \in \Psi^m(M)$.

(ii) For any $s \in \mathbb{R}$ and for any sequence $\mathcal{D} = \{D_j\}_{j \in \mathbb{N}_0} \subset DO^1(M)$,

$$\begin{cases} A_0 = A \in \mathcal{B}(H^s(M), H^{s-m}(M)), \\ A_{k+1} = [A_k, D_k] \in \mathcal{B}(H^{s+m-d_{\mathcal{D},k}}(M), H^s(M)), \end{cases}$$

where $d_{\mathcal{D},k} = \sum_{j=0}^k (1 - ord(D_j))$.

On one hand, both theorems characterize pseudo-differential operators in terms of the boundedness of commutators. On the other hand, a direct comparison shows that the hypotheses on the above theorem are more stringent than that on Theorem 3.12.

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