Minimally immersed surfaces in the unit tangent bundle of the 2-sphere arising from area-minimizing unit vector fields on $\mathbb{S}^2 \backslash \{N, S\}$

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Para minha mãe.

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Resumo

CONRADO, J. **Superfícies minimamente imersas no fibrado tangente unitário da esfera Euclidiana que surgem de campos vetoriais unitários minimizantes de área na** $\mathbb{S}^2 \backslash \{N, S\}$. 2022. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022.

Este trabalho tem dois objetivos. Primeiramente, para todo campo vetorial unitário sobre $\mathbb{S}^2 \backslash \{N, S\}$ com índice de Poincaré par diferente de zero e dois, provamos que o fecho topológico de sua imagem coincide com a imagem de uma garrafa de Klein minimamente imersa em $T^1\mathbb{S}^2$. Em segundo lugar, estabelecemos uma relação entre o Toro de Clifford e os campos vetoriais unitários Norte-Sul e Sul-Norte. Mais especificamente, provamos que o fecho topológico da união das imagens dos campos vetoriais Norte-Sul e Sul-Norte em $T^1\mathbb{S}^2$ é um Toro de Clifford mergulhado.

Palavras-chave: Garrafa de Klein, toro de Clifford, volume de campos vetoriais unitários, superfícies mínimas, espaço projetivo redondo, fibrado tangente unitário.

Abstract

CONRADO, J. **Minimally immersed surfaces in the unit tangent bundle of the 2-sphere arising from area-minimizing unit vector fields on** $\mathbb{S}^2\backslash\{N, S\}$. 2022. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022.

The aim of this work is twofold. Firstly, for the unit vector fields on $\mathbb{S}^2\backslash\{N, S\}$ with even Poincaré indexes other than zero or two, we prove that the topological closure of their image coincides with the image of minimally immersed Klein bottles in $T^1\mathbb{S}^2$. Secondly, we establish a relationship between the Clifford Torus and the North-South and South-North unit vector field. More specifically, we prove that the topological closure of the union of the images of the North-South and the South-North vector fields in $T^1\mathbb{S}^2$ is an embedded Clifford Torus.

Keywords: Klein bottle, Clifford Torus, volume of a unit vector field, minimal surface, round projective space, unit tangent bundle.

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Chapter 1

Introduction

The theory of minimal surfaces has its roots in variational calculus and emerged when the *Plateau Problem* was studied by Lagrange in 1760, [Lag62]. Throughout the years, many renowned mathematicians have contributed to the flowering and ramification of this theory. Such contributions provided us with several ways to find minimal surfaces in ambient spaces other than the classical - three-dimensional Euclidean space $\mathbb{R}^3$ - such as the round 3-sphere $\mathbb{S}^3$ and the unit tangent bundle of the Euclidean sphere $T^1\mathbb{S}^2$ or equivalently the round real projective space, $\mathbb{R}\mathrm{P}^3 = \mathbb{S}^3(2)/\mathbb{Z}_2$, which are the main ambient spaces of this thesis.

Among the equivalent definitions of minimal surfaces, we highlight the one from the point of view of variational calculus and differential geometry, respectively.

i) A hypersurface $\Sigma \subset M$ is minimal if and only if Σ is a critical point of the functional area.

ii) Let Σ be a submanifold of a Riemannian manifold (M, g). If the mean curvature H of Σ vanishes, then Σ is said to be a minimal submanifold.

Recently, old conjectures have been resolved and new and unexpected applications to other parts of mathematics and physics have been achieved using this theory. In [MN14], Fernando Codá Marques and André Neves using the min-max theory for minimal surfaces, proved the Willmore conjecture [Wil65]

"Every compact surface Σ of genus one in $\mathbb{R}^3$ must satisfy $\int_\Sigma H^2 dA \geq 2\pi^2$, where dA stands for the area form of Σ."

The equality is achieved for the torus of revolution whose generating circle has radius 1 and the distance from its center to the axis of revolution is $\sqrt{2}$. This torus can also be obtained as the stereographic projection of the Clifford Torus.

In [Bre13], Simon Brendle gave an affirmative answer to the Lawson conjecture, [Law70].

"The Clifford Torus is the only compact embedded minimal surface in $\mathbb{S}^3$ of genus 1".

Parallel to the growth of minimal surface theory, Herman Gluck and Wolfgang Ziller defined the volume of a unit vector field V over an n-dimensional manifold M as the measure of the volume of $V(M)$ in T^1M with respect to the Sasaki metric, [GZ86]. A unit vector field of minimum volume, if it exists, is to be found among the critical points of the volume functional restricted to $\mathcal{X}^1(M)$, the set of unitary vector fields over M. Also, if a vector field defines a minimal immersion, it should be a solution of the variational problem restricted to $\mathcal{X}^1(M)$. Intuitively speaking, one hopes that the *visually best organized* unit vector fields on M are rewarded with minimum possible volume.

A new way of obtaining minimal surfaces in the tangent bundle of a Riemannian manifold M was presented by Olga Gil-Medrano and Elise Fuster-Llinhares in [GMLF02]: An element $V \in \mathcal{X}^1(M)$ is a critical point of the volume functional restricted to $\mathcal{X}^1(M)$ if and only if $V : M \to (T^1M, g^{Sas})$ is a minimal immersion, where g^{Sas} is the Sasaki metric. Therefore, one can find a volume-minimizing vector field $V : M \to T^1M$, that produces a minimal surface $V(M)$ in the unit tangent bundle T^1M.

In regards to the volume-minimizing unit vector fields, the first closed Riemannian manifold that was successfully studied was the unit round 3-sphere. In that case, the infimum is attained by the unit Hopf vector fields, i.e., those tangent to the fibers of a Hopf fibration.

In dimension 2, if $V(M)$ is a critical point of the volume functional, then we say that V is area-minimizing. In 2010, Olga Gil-Medrano and Vincent Borrelli prove that the Pontryagin vector field (unit vector field with one singularity) is an area-minimizing unit vector field on the round 2-sphere, and its image is homeomorphic to the projective plane, [BGM10].

If $M = \mathbb{S}^{2k+1}$, the volume of the North-South vector field $\overrightarrow{NS}$ on M – also known as the radial field – provides a lower bound for the volume of the unit vector field defined on odd-dimensional spheres, see [BCN04],

$$\operatorname{vol}\left(\overrightarrow{NS}\right) = 4^k \begin{pmatrix} 2k \\ k \end{pmatrix}^{-1} \operatorname{vol}\left(\mathbb{S}^{2k+1}\right).$$

Hence, for $k = 1$ we obtain

$$\operatorname{vol}\left(\overrightarrow{NS}\right) = 2\operatorname{vol}(\mathbb{S}^3) = 4\pi^2. \tag{1.1}$$

In [BCJ08], Fabiano Brito, Pablo Chacón and David Johnson established an explicit relationship between the volume of a unit vector field V on $\mathbb{S}^n\backslash\{N, S\}$, ($n = 2$ or 3) and the absolute values of the Poincaré index around its isolated singularities $I_V(N)$ and $I_V(S)$, see the Appendix, Section 5.1, for these definitions. Specifically,

i) for $n = 2$, $\operatorname{vol}(V) \geq \frac{1}{2}(\pi + |I_V(N)| + |I_V(S)| - 2)\operatorname{vol}(\mathbb{S}^2)$,

ii) for $n = 3$, $\operatorname{vol}(V) \geq (|I_V(N)| + |I_V(S)|)\operatorname{vol}(\mathbb{S}^3)$.

In this way, they computed

$$\operatorname{vol}\left(\overrightarrow{NS}\right) = \frac{1}{2}\pi\operatorname{vol}(\mathbb{S}^2) = 2\pi^2. \tag{1.2}$$

If $\pi : T^1\mathbb{S}^2 \to \mathbb{S}^2$ is the projection, one has that

$$\partial\left(\overrightarrow{NS}\right) = \pi^{-1}(p) \cup \pi^{-1}(-p); \tag{1.3}$$

therefore, $\overrightarrow{NS}$ is a cylinder whose boundary consists of two antipodal fibers of $T^1\mathbb{S}^2$.

Recently, we proved the most recent result about area-minimizing unit vector fields on the antipodally punctured round 2-sphere, [BCGN21]. We concluded that the volume of a unit vector field V is bounded below by the length of an ellipse naturally associated to it (see Theorem 2.10). In addition, we exhibit minimizing vector fields $V_{k,2-k}$ within each index class and show that they are the only ones whose volume attains the minimum. We provide a more precise definition of the unit vector fields $V_{k,2-k}$ in Subsection 2.1.3. For each integer k other than 0 or 2, the image of $V_{k,2-k}$ is a minimal surface of $T^1\mathbb{S}^2$.

In [BGM10], the authors presented one way to see that the image of the Pontryagin vector field $\mathcal{P}$ is the projective plane $\mathbb{RP}^2$. Let D be a disk of $\mathbb{S}^2(1)$ containing the singularity of $\mathcal{P}$. Since the Poincaré index of $\mathcal{P}$ at the singularity is 2, the image $\mathcal{P}(\partial D)$ is a closed curve that surrounds twice the fiber sitting above the singularity. If $\pi : T^1\mathbb{S}^2 \to \mathbb{S}^2$ is the natural projection, we have that

$$\overline{\mathcal{P}} \cap \pi^{-1}(D) \cong \mathcal{M}, \tag{1.4}$$

where $\overline{\mathcal{P}}$ is the closure of the image of $\mathcal{P}$ and $\mathcal{M}$ is a Moebius strip, and as $\mathcal{P}(\mathbb{S}^2\backslash D)$ is a disk D', we obtain

$$\overline{\mathcal{P}} = D' \cup \mathcal{M} \cong \mathbb{RP}^2.$$

We read equation (1.4) as giving us a Moebius strip from a unit vector field with Poincaré index equal to 2. A natural question to pose is: Can one always find a Moebius strip when given a vector field with even Poincaré index at its singularities? In search for an answer to this question, we obtained our first result.

Theorem A. *Let $V_{k,2-k}$ be an area-minimizing unit vector field on $\mathbb{S}^2\backslash\{N,S\}$. If the Poincaré index around the singularity N (or S) is $k \in 2\mathbb{Z}\backslash\{0,2\}$, then the topological closure of the image of $V_{k,2-k}\big(\mathbb{S}^2\backslash\{N,S\}\big)$ is a minimally immersed Klein bottle in $T^1\mathbb{S}^2(1)$.*

The minimally immersed Klein bottle of Theorem A is obtained by gluing together two immersed Moebius strips that appear when restricting to each hemisphere along their common boundary. It is possible to make this collage because the Poincaré index k is even. In this way, the techniques we employ here cannot be directly applied to get similar results for odd Poincaré indexes.

The second result we established is a relationship between the Clifford Torus and the

North-South and South-North unit vector fields. The Clifford Torus in $\mathbb{S}^3(2)$ is given by

$$T^2 = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{S}^3(2) \ : \ x_1^2 + x_4^2 = 2 \quad \text{and} \quad x_2^2 + x_3^2 = 2\right\}.$$

It is not difficult to parameterize the Clifford Torus T^2 see (4.11), in Section 4.1 to obtain

$$\mathrm{vol}(T^2) = 8\pi^2. \tag{1.5}$$

In [BCJ08], the authors computed the volume of the North-South unit vector field $\overrightarrow{NS}$ as being as given by (1.2); therefore, one sees that

$$\mathrm{vol}\left(\overrightarrow{NS} \cup \overrightarrow{SN}\right) = 4\pi^2. \tag{1.6}$$

We remark that the volume of the Clifford Torus (1.5) is twice the volume of $\overrightarrow{NS} \cup \overrightarrow{SN}$ (1.6) and twice the volume of $\overrightarrow{NS}$ on $\mathbb{S}^3$, (1.1). Moreover, since the image of $\overrightarrow{NS}$ is a cylinder with a two-fiber boundary in $\mathbb{RP}^3(2)$ (see (1.3)), we get that the topological closure of $\overrightarrow{NS} \cup \overrightarrow{SN}$ is a torus in $\mathbb{RP}^3(2) \cong \mathrm{T}^1\mathbb{S}^2$. These facts contributed to finding the following result.

Theorem B. *Let T^2 be the Clifford Torus in $\mathbb{S}^3(2)$. If ϕ is a restriction of the Euler parametric representation of* $\mathbf{SO}(3)$*, then*

$$\phi(T^2) = \textit{topological closure of } \left(\overrightarrow{NS} \bigcup \overrightarrow{SN}\right),$$

where $\overrightarrow{NS}$, $\overrightarrow{SN}$ are the North-South and South-North unit vector fields defined on $\mathbb{S}^2 \setminus \{N, S\}$, respectively. A restriction of the Euler parametric representation of $\mathbf{SO}(3)$ *is defined in* (4.1)*, Section 4.1.*

The techniques used here open the way for a new class of examples of minimal surfaces within $\mathbb{S}^3$.

We organize this thesis as follows. In the second chapter we recall some results about the volume of a unit vector field and we report on the most recent result about area-minimizing unit vector fields on the antipodally punctured round 2-sphere, [BCGN21]. The third and fourth chapters are devoted to prove Theorem A and B, respectively. The appendices are dedicated to recall the necessary background material,including the definitions of the Poincare index of a vector field and the Poincare-Hopf theorem .

Chapter 2

Preliminaries

In this chapter we lay down some preliminaries to fix the notation used throughout the manuscript. All concepts and facts presented in this chapter are already present in the literature and we provide appropriate references for them. We start by defining the volume of a unit vector field on a closed oriented Riemannian manifold. Then we move on to recall some results involving area-minimizing unit vector fields, which is the main object of study in this chapter. In the last subsection, we state the most recent result about area-minimizing unit vector fields on the antipodally punctured round 2-sphere (see Theorem 2.10).

2.1 The volume of a unit vector field

Let M be a closed oriented Riemannian manifold and V a unit vector field on M. Consider the unit tangent bundle T^1M equipped with the Sasaki metric. The Sasaki metric is defined by declaring the orthogonal complement of the vertical distribution to be the horizontal distribution given by the Levi-Civita connection ∇, more details in [Sas58, KS75, CM12, DP12]. The ***volume of a unit vector field*** V is defined (see [GZ86]) as the volume of the submanifold $V(M)$, the image of the immersion $V : M \to T^1M$,

$$\mathrm{vol}(V) := \mathrm{vol}(V(M)).$$

Let $\{e_1, \cdots, e_n\}$ be an orthonormal local frame on M and denote by ν_M the volume form of M written with respect to it. The formula for the volume of the unit vector field V is given by

$$\begin{aligned} \mathrm{vol}(V) &= \int_M \sqrt{\det(\mathrm{I} + (\nabla \mathrm{V})(\nabla \mathrm{V})^*)}\nu_M \\ &= \int_M \big(1 + \sum_j^n ||\nabla_{e_j} V||^2 + \sum_{j_1<j_2} ||\nabla_{e_1} V \wedge \nabla_{e_1} V||^2 + \cdots \\ &\quad + \cdots + \sum_{j_1<\cdots<j_{n-1}} ||\nabla_{e_1} V \wedge \cdots \wedge \nabla_{e_{j_{n-1}}} V||^2\big)^{\frac{1}{2}}, \end{aligned} \tag{2.1}$$

where I is the identity, and ∇V is considered as an endomorphism of the tangent space with adjoint operator $(\nabla V)^*$, see [Joh88]. Intuitively speaking, one hopes that the *visually best* *organized* unit vector fields on M are rewarded with minimum possible volume. If a unit vector field is parallel, i.e. $\nabla V = 0$, equation (2.1) implies there is a trivial minimum

$$\mathrm{vol}(V) = \mathrm{vol}(M).$$

A closed Riemannian manifold does not always admit a globally defined parallel vector field. In fact, this is often the case because in order for a vector field to be parallel, it needs to determine two mutually orthogonal complementary totally geodesic foliations. Hence, one expects that the symmetries of volume-minimizing unit vector fields have interesting properties. In regards to the volume-minimizing unit vector fields, the first closed Riemannian manifold that was successfully studied was the unit round 3-sphere. We state this result in full during Subsection 2.1.1, see Theorem 2.2.

On a Riemannian manifold, the critical points of the volume functional restricted to vector fields with length one are not always vector fields of minimal volume, on the other hand, the volume-minimizers are critical points of the volume functional. The upcoming result appears in the proof of Theorem A below.

Theorem 2.1 (Gil-Medrano and Llinhares-Fuster, [GMLF02])**.** *An element $V \in \mathcal{X}^1(M)$ is a critical point of the volume functional restricted to $\mathcal{X}^1(M)$ if and only if $V : M \to (T^1M, g^{Sas})$ is a minimal immersion, where g^{Sas} is Sasaki mectric.*

2.1.1 The Hopf vector field: a volume-minimizing vector field on $\mathbb{S}^3$

In 1931, Heinz Hopf discovered several fibrations of spheres by great subspheres [Hop31]. The fibration of $\mathbb{S}^{2n+1}$ by great circles $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1}$, the fibration of $\mathbb{S}^{4n+3}$ by great 3-spheres $\mathbb{S}^3 \hookrightarrow \mathbb{S}^{4n+3}$, and the fibration of $\mathbb{S}^{15}$ by great 7-spheres $\mathbb{S}^7 \hookrightarrow \mathbb{S}^{15}$ are all named after Hopf. The Hopf fibrations have many physical applications, including magnetic monopoles [Nak90], rigid body mechanics [MR99] and quantum information theory [MD01].

We now recall how to construct the fibration of $\mathbb{S}^{2n+1}$ by great circles, $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1}$. Consider the sphere $\mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. Although there are different ways to identify $\mathbb{R}^{2n+2}$ and $\mathbb{C}^{n+1}$, each one giving a different fibration of the sphere, they are all congruent to one another. Consider all complex lines in $\mathbb{C}^{n+1}$ passing through origin. The intersection of a given complex line with the sphere $\mathbb{S}^{2n+1}$ is called a fiber. Since each point in $\mathbb{S}^{2n+1}$ belongs to only one of these lines, there is but a single fiber at each point. On the other hand, since each complex line is a 2-dimensional linear subspace of $\mathbb{R}^{2n+2}$, the fibers are all great circles and therefore totally geodesic submanifolds of $\mathbb{S}^{2n+1}$. This process defines the Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1}$. If one denotes by J the complex structure on $\mathbb{C}^{n+1}$, each complex line defining a fiber and containing $x \in \mathbb{S}^{2n+1}$ has $\{x, Jx\}$ as a generator set in $\mathbb{R}^{2n+2}$. Note that the vector Jx is tangent to $\mathbb{S}^{2n+1}$ because x is normal to $\mathbb{S}^{2n+1}$. Hence, for all $x \in \mathbb{S}^{2n+1}$, the vector Jx is tangent to the fiber and the orthogonal distribution $(Jx)^\perp$ is not integrable.

In order to define the Hopf fibration $\mathbb{S}^3 \hookrightarrow \mathbb{S}^{4n+3}$, consider $\mathbb{S}^{4n+3} \subset \mathbb{H}^{n+1}$, where $\mathbb{H}^{n+1}$ is quaternion space. In analogy with the above described fibration, we define the fibers as being the intersection of a quaternionic line that contain the origin with the sphere $\mathbb{S}^{4n+3}$. The quaternionic lines are real subspaces of dimension 4 of $\mathbb{R}^{4n+4}$. The orthogonal distribution to the distribution defined by the fibers of $\mathbb{S}^3 \hookrightarrow \mathbb{S}^{4n+3}$ is not integrable.

Do note that the construction of the Hopf fibration $\mathbb{S}^7 \hookrightarrow \mathbb{S}^{15}$ does not follow along the same steps as the two other Hopf fibrations presented above. For more details about Hopf fibrations, we refer the reader to [Hop31, CM12, BGN12].

We call a unit vector field H tangent to the fibres of a Hopf fibration a ***Hopf vector field***.

Theorem 2.2 (Gluck and Ziller, [GZ86]). *The unit vector fields of minimum volume on $\mathbb{S}^3$ are precisely the Hopf vector fields and no others.*

In fact, for the Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1}$, the volume of the Hopf vector field H is

$$\operatorname{vol}(H) = 2^n \operatorname{vol}\left(\mathbb{S}^{2n+1}\right), \quad n \in \mathbb{N}. \tag{2.2}$$

The method that goes into proving the Theorem 2.2 depends on the so-called *calibrated geometries* of Federer and Havey-Lawson ([HL82]), and it does not work in higher dimensions, as remarked [GZ86, p. 180]. For the Hopf vector fields to be minimizers of the volume functional, a necessary condition is that they be critical and stable. However, David Johnson showed that the Hopf vector fields on $\mathbb{S}^{2n+1}$ are unstable for $n > 1$ and he proved the following.

Theorem 2.3 (Johnson, [Joh88]). *The Hopf fibration on the round sphere $\mathbb{S}^5$ is not local minimum of the volume functional.*

For the instability of Hopf flows on spheres of radius r, see [GMLF01].

2.1.2 The Pontryagin vector field: an area-minimizing vector field on $\mathbb{S}^2 \backslash \{p\}$

A ***Pontryagin field*** on $\mathbb{S}^n$ is any unit vector field $\mathcal{P}$ defined on a dense open subset U such that the closure of $\mathcal{P}(U)$ is the n-dimensional generalized Pontryagin cycle of $T^1\mathbb{S}^n$. This cycle is the set of all unit vectors obtained by parallel translating a given vector $v_0 \in T^1_{p_0}\mathbb{S}^n$ along great circles on $\mathbb{S}^n$ passing through p_0 [Ped93] and [BGM10]. The resulting vector field has a singularity at $-p_0$ of Poincaré index 0 or 2 depending on the dimension n of the sphere, see Figure 2.1. Sharon Petersen, in her Ph.D. thesis [Ped93], exhibits a unit vector field of small volume, converging to a vector field with one singularity on every odd dimensional sphere. This result suggests the possibility that there exist no unit vector fields of minimum volume on $\mathbb{S}^n$, for $n \geq 5$. She herself conjectured the latter, and moreover, that the limiting vector field with one singularity is of minimum volume in its homology class in the unit tangent bundle. More precisely, she showed that $\mathcal{P}(U)$ is a minimal submanifold of $T^1\mathbb{S}^n(1)$ and conjectured that, for odd dimensional $\mathbb{S}^n$ with $n \geq 5$, the infimum is not reached by any globally defined unit vector field, but does have the volume of the Pontryagin vector fields.

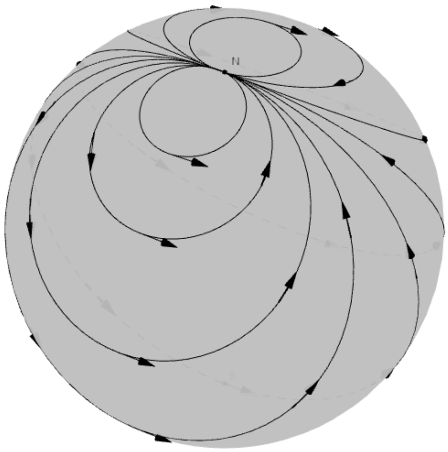

Figure 2.1: *The flow of the Pontryagin vector field V_2 on $\mathbb{S}^2\backslash\{N\}$ with Poincaré index 2*

Consider the stereographic projection centered at the south pole S

$$\begin{aligned} \varphi : \mathbb{S}^n\backslash\{N\} &\longrightarrow \mathbb{R}^n \\ (x_1, \cdots, x_{n+1}) &\longmapsto \left(\frac{x_1}{1-x_{n+1}}, \cdots, \frac{x_n}{1-x_{n+1}}\right) \end{aligned}$$

and its inverse

$$\begin{aligned} \varphi^{-1} : \mathbb{R}^n &\longrightarrow \mathbb{S}^n\backslash\{N\} \\ (z_1, \cdots, z_n) &\longmapsto \left(\frac{2z_1}{\sum_{i=1}^n z_i^2+1}, \cdots, \frac{2z_n}{\sum_{i=1}^n z_i^2+1}, \frac{\sum_{i=1}^n z_i^2-1}{\sum_{i=1}^n z_i^2+1}\right). \end{aligned}$$

Let $\{w_1, w_2, \cdots, w_k\}$ denote the canonical basis of $\mathbb{R}^k$. Since the stereographic projection φ preserves angles, one can compute the parallel transport on $\mathbb{R}^n$. Let $\mathcal{P}_i$ be the Pontryagin field obtained from parallel transporting $v_i = d\varphi_0^{-1}(w_i) \in T_S^1\mathbb{S}^n$. At a point $x = (rx_1, \cdots, rx_{n+1}) \in \mathbb{S}^n(r)$, $\mathcal{P}_i$ has the following expression

$$\mathcal{P}_i(x) = \frac{d\varphi_{\varphi(x)}^{-1}(w_i)}{||d\varphi_{\varphi(x)}^{-1}(w_i)||}.$$

Computing the derivatives of φ^{-1}, we obtain

$$\begin{aligned} \mathcal{P}_1(x) &= \left(1-\frac{x_1^2}{1-x_{n+1}}\right)w_1 + \left(-\frac{x_1x_2}{1-x_{n+1}}\right)w_2 + \cdots + \left(-\frac{x_1x_n}{1-x_{n+1}}\right)w_n + x_1w_{n+1}, \\ \mathcal{P}_2(x) &= \left(-\frac{x_1x_2}{1-x_{n+1}}\right)w_1 + \left(1-\frac{x_2^2}{1-x_{n+1}}\right)w_2 + \left(-\frac{x_2x_3}{1-x_{n+1}}\right)w_3 + \cdots \\ &\qquad + \cdots + \left(-\frac{x_1x_n}{1-x_{n+1}}\right)w_n + x_2w_{n+1}, \\ &\vdots \\ \mathcal{P}_n(x) &= \left(-\frac{x_1x_n}{1-x_{n+1}}\right)w_1 + \cdots + \left(-\frac{x_{n-1}x_n}{1-x_{n+1}}\right)w_{n-1} + \left(1-\frac{x_n^2}{1-x_{n+1}}\right)w_n + x_nw_{n+1}. \end{aligned}$$

In particular, if $n = 2$, the Pontryagin field $\mathcal{P}_2$ at a point $(rx_1, rx_2, rx_3) \in \mathbb{S}^2(r)$ is given by

$$\mathcal{P}_2(rx_1, rx_2, rx_3) = -\frac{x_1 x_2}{1 - x_3} w_1 + \left(1 - \frac{x_2^2}{1 - x_3}\right) w_2 + x_2 w_3. \tag{2.3}$$

In 2010, Vincent Borrelli and Olga Gil-Medrano proved that the Pontryagin vector field given by (2.3) is area-minimizing on the 2-sphere.

Theorem 2.4 (Borrelli and Gil-Medrano, [BGM10]). *Among unit vector fields on $\mathbb{S}^2(r)$ with one singularity those of least area are the Pontryagin ones and no others.*

Moreover, they proved that any great 2-sphere is a minimal surface in the Berger 3-sphere and, as a consequence, they obtained that the images of Pontryagin fields are minimal submanifolds of $T^1\mathbb{S}^2(r)$.

Theorem 2.5 (Borrelli and Gil-Medrano, [BGM10]). *The only minimal surfaces in $T^1\mathbb{S}^2(r)$ homeomorphic to the projective plane arising from vector fields without boundary are Pontryagin cycles.*

2.1.3 Area-minimizing unit vector fields on $\mathbb{S}^2 \backslash \{N, S\}$

In 2008, Fabiano Brito, Pablo Chacón and David Johnson established an explicit relationship between the volume of a unit vector field and its Poincaré index around isolated singularities. Throughout, if V is a vector field with an isolated singularity at p, $I_V(p)$ stands for the Poincaré index of V around p.

Theorem 2.6 (Brito, Chacón and Johnson, [BCJ08]). *Let $M = \mathbb{S}^n \backslash \{N, S\}$, $n = 2$ or 3, be the standard Euclidean sphere with two antipodal points N and S removed. Let V be a unit vector field defined on M. Then,*

i) for $n = 2$, $\operatorname{vol}(V) \geq \frac{1}{2}(\pi + |I_V(N)| + |I_V(S)| - 2) \operatorname{vol}(\mathbb{S}^2)$,

ii) for $n = 3$, $\operatorname{vol}(V) \geq (|I_V(N)| + |I_V(S)|) \operatorname{vol}(\mathbb{S}^3)$.

Theorem 2.6 has been extended to odd dimensional spheres $\mathbb{S}^{2n+1}$ as follows:

Theorem 2.7 (Brito, Gomes and Gonçalves, [BGG19]). *If V is a unit vector field on $\mathbb{S}^{2n+1} \backslash \{\pm p\}$, then*

$$\operatorname{vol}(V) \geq \frac{\pi}{4} \operatorname{vol}(\mathbb{S}^{2n})(|I_V(p)| + |I_V(-p)|).$$

Before turning to reporting the most recent result about area-minimizing unit vector fields on the antipodally punctured round 2-sphere, we exhibit the unit vector fields that are area-minimizing within each index class.

Let $\mathbb{S}^2 \backslash \{N, S\}$ be the Euclidean sphere with two antipodal points N and S removed. Denote by g the usual metric on $\mathbb{S}^2$ inherited from $\mathbb{R}^3$ and by ∇ the Levi-Civita connection associated to g. Consider the following oriented orthonormal frame $\{e_1, e_2\}$ on $\mathbb{S}^2 \backslash \{N, S\}$, given by

$$e_1(p) = \frac{1}{\sqrt{x^2+y^2}}(-y, x, 0), \tag{2.4}$$

$$e_2(p) = \left(\frac{xz}{\sqrt{x^2+y^2}}, \frac{yz}{\sqrt{x^2+y^2}}, -\sqrt{x^2+y^2} \right), \tag{2.5}$$

where $p = (x, y, z) \in \mathbb{S}^2 \backslash \{N, S\}$. We have that e_1 is tangent to the parallels and e_2 to the meridians. Let k be an integer number and define the angle function as

$$\begin{aligned} \theta_k : \mathbb{S}^2 \backslash \{N, S\} &\longrightarrow \mathbb{R} \\ p &\longmapsto \theta_k(p) = (k-1)t + \frac{\pi}{2}, \end{aligned} \tag{2.6}$$

where $t \in [0, 2\pi)$ is the *longitude* coordinate of $p = (x, y, z)$ in $\mathbb{S}^2 \backslash \{N, S\}$. Note that if $\{e_1, e_2\}$ is the oriented orthonormal frame defined by (2.4) and (2.5),

$$d\theta_k(p)(e_1) = \frac{k-1}{\sqrt{x^2+y^2}} \quad \text{and} \quad d\theta_k(p)(e_2) = 0. \tag{2.7}$$

Definition 2.8. *For $k \in \mathbb{Z}$, define the unit vector field $V_{k,2-k}$ at $p \in \mathbb{S}^2 \backslash \{N, S\}$ by*

$$V_{k,2-k}(p) = \cos(\theta_k(p))\, e_1(p) + \sin(\theta_k(p))\, e_2(p),$$

where θ_k is the angle function and $\{e_1, e_2\}$ is the oriented orthonormal frame defined by (2.4) *and* (2.5)*.*

Remark. *Definition 2.8 coincides with the one in [BCGN21]. In either case, the defined vector fields differ from the Pontryagin fields in that the latter are vector fields with only one singularity. Throughout, we consider the $I_{V_{k,2-k}}(N) = k \in \mathbb{Z}^+$ and $I_{V_{k,2-k}}(S) = 2-k$.*

Consider the parametrization of the 2-sphere given by spherical coordinates:

$$\begin{aligned} (0, \pi] \times [0, 2\pi) &\longrightarrow \mathbb{R}^3 \\ (\alpha, \beta) &\longmapsto (\cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha), \end{aligned} \tag{2.8}$$

where α is the latitude and β is the longitude. Given $p = (\cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha) \in \mathbb{S}^2 \backslash \{N, S\}$, we have $\theta_k(p) = (k-1)\beta$ and $e_1(p) = (-\sin\beta, \cos\beta, 0)$. For a fixed parallel α, consider the curve μ on $\mathbb{S}^2 \backslash \{N, S\}$ given by

$$\mu(t) = \left(\cos(\alpha)\cos\left(\frac{t}{\cos(\alpha)} + \beta\right), \cos(\alpha)\sin\left(\frac{t}{\cos(\alpha)} + \beta\right), \sin(\alpha) \right).$$

Since $\mu(0) = p$, and

$$\mu'(t) = \frac{1}{\cos(\alpha)}\left(-\sin\left(\frac{t}{\cos(\alpha)} + \beta\right), \cos\left(\frac{t}{\cos(\alpha)} + \beta\right), 0\right),$$

one concludes that

$$\mu'(0) = \frac{1}{\cos(\alpha)}\left(-\sin(\beta), \cos(\beta), 0\right) = \frac{1}{\cos(\alpha)}e_1(p) \in T_p\mathbb{S}^2\backslash\{N,S\}.$$

Moreover,

$$d\theta_p(e_1) = \frac{d}{dt}(\theta \circ \mu)(0) = \frac{d}{dt}\left(\frac{(k-1)t}{\cos(\alpha)} + (k-1)\beta + \frac{\pi}{2}\right) = \frac{k-1}{\cos(\alpha)}.$$

The previous computation shows that θ_k depends only on the longitude coordinate. The vector field $V_{k,2-k}$ winds $k-1$ times around the parallel α at a constant angle speed with respect to the referential $\{e_1, e_2\}$. The function θ_k defined in (2.6) is the angle between a given parallel and the unit vector field $V_{k,2-k}$. The angle function θ_k changes as k changes. For example, the angle θ_3 between $V_{3,-1}$ and a parallel is twice the angle θ_2 between $V_{2,0}$ and the same parallel. This is to be expected because $V_{3,-1}$ has a singularity with Poincaré index 3 (i.e., with four *petals*, see Figure 2.4), whereas $V_{2,0}$ has a singularity with Poincaré index 2 (i.e., with two *petals*, see Figure 2.1). Also, $V_{1,1}$ is the South-North vector field, which forms an angle $\theta_1 = \pi/2$ with each parallel, see Figure 2.3.

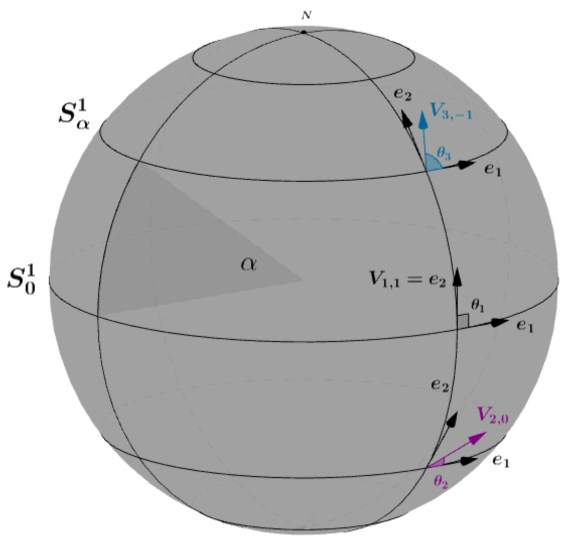

Figure 2.2: *Visual representation of some of the unit vector fields* $V_{k,2-k}$ *on* $\mathbb{S}^2\backslash\{N,S\}$

Let V be a unit vector field tangent to $\mathbb{S}^2\backslash\{N,S\}$ and consider the oriented orthonormal local frame $\{V^\perp, V\}$ on $\mathbb{S}^2\backslash\{N,S\}$. Until the end of this chapter, let $M = \mathbb{S}^2\backslash\{N,S\}$. In this case, the equation for the volume of V (2.1) reduces to

$$\mathrm{vol}(V) = \int_M \sqrt{1+\gamma^2+\delta^2}\,\nu_M, \tag{2.9}$$

where $\gamma := g(\nabla_V V, V^\perp)$ and $\delta := g(\nabla_{V^\perp} V^\perp, V)$ are the geodesic curvatures associated to V and $V^\perp$, respectively.
Let $\mathbb{S}^1_\alpha$ be the parallel of $\mathbb{S}^2$ at latitude $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and S^1_β be the meridian of $\mathbb{S}^2$ at longitude $\beta \in [0, 2\pi)$.

Lemma 2.9. *Let $\theta \in [0, \pi/2]$ be the oriented angle from e_1 to V. If $V = (\cos\theta)e_1 + (\sin\theta)e_2$ and $V^\perp = (-\sin\theta)e_1 + (\cos\theta)e_2$, then along a parallel $\mathbb{S}^1_\alpha$, it holds the following*

$$1+\gamma^2+\delta^2 = 1 + (\tan\alpha + d\theta(e_1))^2 + d\theta(e_2)^2.$$

Lemma 2.9 allows us to rewrite the volume functional as an integral depending on the latitude α and the derivatives of θ

$$\mathrm{vol}(V) = \int_M \sqrt{1 + (\tan\alpha + d\theta(e_1))^2 + d\theta(e_2)^2}\,\nu_M. \tag{2.10}$$

We now turn to report the result in [BCGN21]. It says that the volume of a unit vector field V is bounded below by the length of an ellipse naturally associated to it.

Theorem 2.10. *Let V be a unit vector field on $\mathbb{S}^2 \setminus \{N, S\}$. If $k = \max\{I_V(N), I_V(S)\}$, $k \in \mathbb{Z}\setminus\{0,2\}$, then*

$$\mathrm{vol}(V) \geq \pi L(\varepsilon_k),$$

where $L(\varepsilon_k)$ is the length of the ellipse $\frac{x^2}{k^2} + \frac{y^2}{(k-2)^2} = 1$.

For proving the Theorem 2.10, we use Lemma 2.9 and equation (2.10), more details see [BCGN21].

Corollary 2.11. *For $k \in \mathbb{Z}\setminus\{0,2\}$, the unit vector field $V_{k,2-k}$ on $\mathbb{S}^2 \setminus \{N, S\}$ is area-minimizing if*

$$\mathrm{vol}(V_{k,2-k}) = \pi L(\varepsilon_k),$$

where $L(\varepsilon_k)$ is the length of the ellipse $\frac{x^2}{k^2} + \frac{y^2}{(k-2)^2} = 1$.

Proof. In the case when $k > 2$, the ellipse ε_k can be parametrized by $\mu(t) = (k\cos t, (k -$

$2)\sin t)$. We use this parametrization to compute the length of ε_k,

$$\begin{aligned} L(\varepsilon_k) &= \int_0^{2\pi} \|\mu'(t)\|^{\frac{1}{2}}\, dt \\ &= \int_0^{2\pi} \sqrt{k^2\sin^2(t) + (k-2)^2\cos^2(t)}dt \\ &= \int_0^{2\pi} \sqrt{k^2\sin^2(t) + (k-2)^2(1-\sin^2(t))}dt \\ &= \int_0^{2\pi} \sqrt{(k-2)^2 + (k^2-(k-2)^2)\sin^2(t)}dt \\ &= \int_0^{2\pi} \sqrt{(k-2)^2 + 4(k-1)\sin^2(t)}dt \\ &= 4\int_0^{\frac{\pi}{2}} \sqrt{(k-2)^2 + 4(k-1)\sin^2(t)}dt. \end{aligned} \tag{2.11}$$

Using Lemma 2.9 and (2.7), we compute

$$\begin{aligned} \mathrm{vol}(V_{k,2-k}) &= \int_M \sqrt{1 + (\tan\alpha + d\theta(e_1))^2 + d\theta(e_2)^2}\,\nu_M \\ &= \int_M \sqrt{1 + \left(\tan\alpha + \frac{k-1}{\cos\alpha}\right)^2}\,\nu_M \\ &= \int_M \sqrt{1 + \left(\frac{\sin\alpha + k-1}{\cos\alpha}\right)^2}\,\nu_M \\ &= \int_M \sqrt{\frac{1+(k-1)^2+2(k-1)\sin\alpha}{\cos^2\alpha}}\,\nu_M \\ &= \lim_{\alpha_0\to-\frac{\pi}{2}} \int_{\alpha_0}^{\frac{\pi}{2}}\int_0^{2\pi} \sqrt{\frac{1+(k-1)^2+2(k-1)\sin\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)}{\cos^2\alpha}}\cos\alpha d\beta d\alpha \\ &= 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1+(k-1)^2+2(k-1)\sin\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)}\,d\alpha. \end{aligned}$$

Letting $t := \frac{\alpha}{2} + \frac{\pi}{4}$, we obtain

$$\mathrm{vol}(V_{k,2-k}) = 4\pi\int_0^{\frac{\pi}{2}} \sqrt{(k-2)^2 + 4(k-1)\sin^2 t}\, dt = \pi L(\varepsilon_k),$$

where $L(\varepsilon_k)$ is the length of the ellipse $\dfrac{x^2}{k^2} + \dfrac{y^2}{(k-2)^2} = 1$ obtained in (2.11).

□

Combining these results with Theorem 2.1, we conclude:

Corollary 2.12. *For $k \in 2\mathbb{Z}\backslash\{0,2\}$, the image of the unit vector field $V_{k,2-k}$ on $\mathbb{S}^2\backslash\{N,S\}$ is minimal surface in $T^1\mathbb{S}^2$.*

Example 2.1. *The South-North $\overrightarrow{SN}$ unit vector field on $\mathbb{S}^2\backslash\{N,S\}$ has the Poincaré index 1 at both the singularities N and S.*

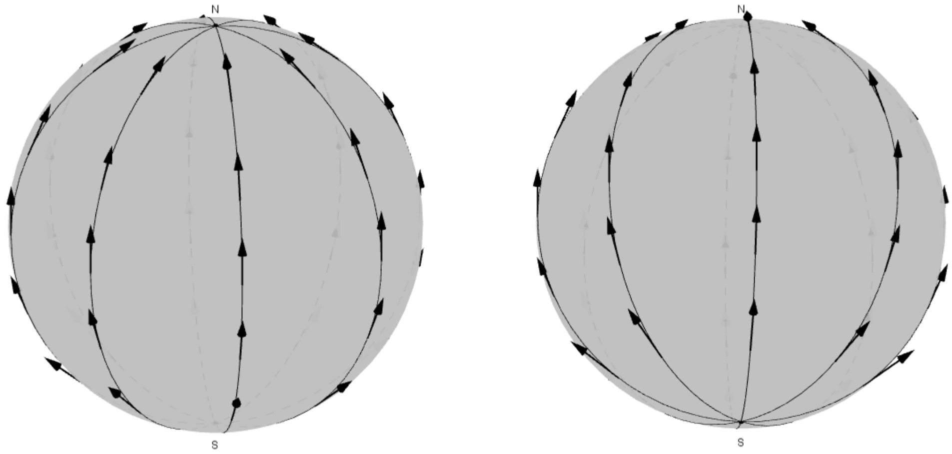

Figure 2.3: *The flow of the unit vector field $V_{1,1}$ on $\mathbb{S}^2\backslash\{N,S\}$, i.e., $k=1$*

From Theorem 2.10,

$$\mathrm{vol}(V_{1,1}) = \pi\int_0^{2\pi} dt = 2\pi^2 = \frac{1}{2}\pi\,\mathrm{vol}(\mathbb{S}^2).$$

Example 2.2. *Consider the unit vector field $V_{3,-1}$ on $\mathbb{S}^2\backslash\{N,S\}$.*

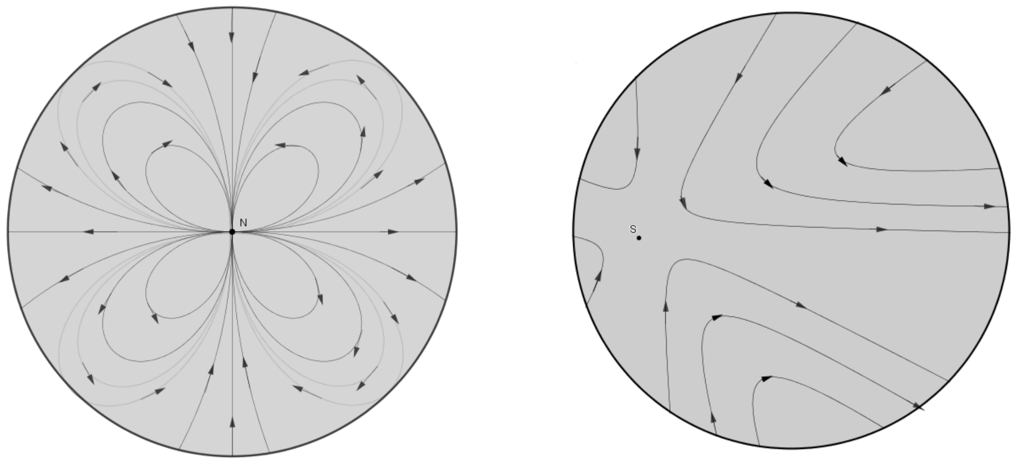

Figure 2.4: *The flow of the unit vector field $V_{3,-1}$ on $\mathbb{S}^2\backslash\{N,S\}$, i.e., $k=3$*

From Theorem 2.10,

$$\mathrm{vol}(V_{3,-1}) \geq 4\pi\int_0^{\frac{\pi}{2}} \sqrt{1+8\sin^2 t}\,dt \sim \frac{1}{3}\pi^2\,\mathrm{vol}(\mathbb{S}^2)$$

Example 2.3. *Consider the unit vector field $V_{4,-2}$ on $\mathbb{S}^2\backslash\{N,S\}$.*

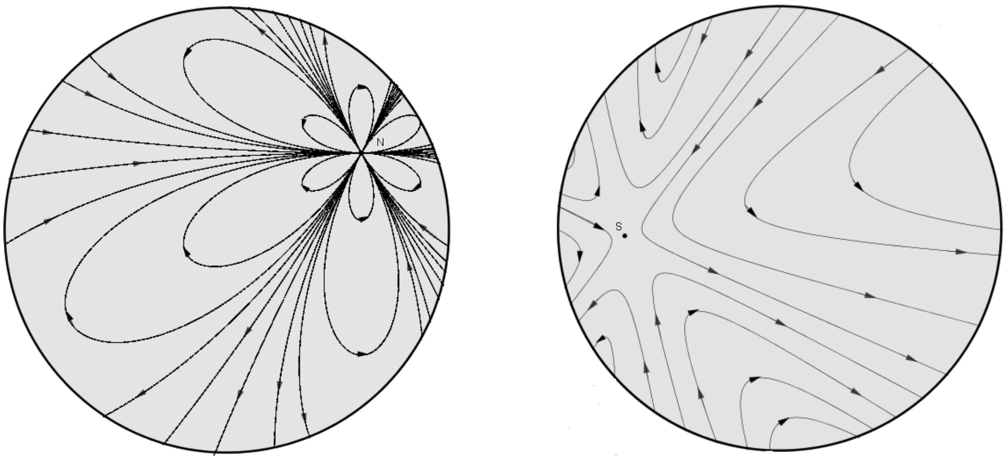

Figure 2.5: *The flow of the unit vector field* $V_{4,-2}$ *on* $\mathbb{S}^2\backslash\{N,S\}$, *i.e.*, $k=4$

From Theorem 2.10,

$$\operatorname{vol}(V_{4,-2}) \geq 4\pi \int_0^{\frac{\pi}{2}} \sqrt{4+12\sin^2 t}\, dt \sim \frac{3}{2}\pi \operatorname{vol}(\mathbb{S}^2)$$

Chapter 3

Minimally immersed Klein bottles in $T^1\mathbb{S}^2$

In this chapter we prove that the topological closure of the area-minimizing unit vector field $V_{k,2-k}$ of Definition 2.8 is the image of an immersed Klein bottle in $T^1\mathbb{S}^2$. The minimally immersed Klein bottle is obtained by gluing together two immersed Moebius strips that appear when restricting to each hemisphere along their boundary. The techniques we employ here cannot be directly be applied to get similar results for odd Poincaré indexes.

3.1 Minimally immersed Moebius strips

Consider the oriented orthonormal frame $\{u_1, u_2\}$ on $\mathbb{S}^2\backslash\{N, S\}$ given by the rotation of the frame $\{e_1, e_2\}$ defined in (2.4) and (2.5) by the angle corresponding to the longitude coordinate $t \in [0, 2\pi)$. Specifically, if $p \in \mathbb{S}^2\backslash\{N, S\}$,

$$u_1(p) = \sin(t)e_1(p) + \cos(t)e_2(p) \tag{3.1}$$
$$u_2(p) = -\cos(t)e_1(p) + \sin(t)e_2(p). \tag{3.2}$$

Recall from Definition 2.8 that

$$V_{k,2-k}(p) = \cos\left((k-1)t + \frac{\pi}{2}\right) e_1(p) + \sin\left((k-1)t + \frac{\pi}{2}\right) e_2(p).$$

Using the trigonometric identities

$$\begin{cases} \cos\left((k-1)t + \frac{\pi}{2}\right) e_1(p) = -\sin((k-1)t)e_1(p) = \cos(kt)\sin(t)e_1(p) - \cos(t)\sin(kt)e_1(p) \\ \sin\left((k-1)t + \frac{\pi}{2}\right) e_2(p) = \cos((k-1)t)e_2(p) = \cos(kt)\cos(t)e_2(p) + \sin(kt)\sin(t)e_2(p), \end{cases}$$

we deduce

$$\begin{aligned} V_{k,2-k}(p) &= \cos(kt)\left(\sin(t)e_1(p) + \cos(t)e_2(p)\right) + \sin(kt)\left(-\cos(t)e_1(p) + \sin(t)e_2(p)\right) \\ &= \cos(kt)u_1(p) + \sin(kt)u_2(p), \end{aligned} \tag{3.3}$$

for all $p \in \mathbb{S}^2\backslash\{N, S\}$. The vector field $V_{k,2-k}$ winds k times around the parallel α at a constant angle speed with respect to the referential $\{u_1, u_2\}$.
The next definition is an alternative to Definition 2.8 that exploits (3.3).

Definition 3.1. *Let D_r^2 be the disk of radius r centered at the origin in $\mathbb{C}$. Define the unit vector field $\vec{v}_k$ as*

$$\vec{v}_k(z) = \frac{z^k}{||z^k||},$$

where $k \in \mathbb{Z}$ and $z \in D_r^2 \backslash \{0\}$.

Given $z \in D_r^2 \backslash \{0\}$, one can write it as $z = r\cos(t) + ir\sin(t)$, where $t \in [0, 2\pi)$. In polar coordinates, Definition 3.1 becomes

$$\vec{v}_k(z) = \cos(kt) + i\sin(kt).$$

Let $T^1 D^2_{\pi/2}$ be the unit tangent bundle of $D^2_{\pi/2}$ equipped with the Sasaki metric. In this case, $T^1 D^2_{\pi/2}$ is the total space of a Riemannian S_1^1-fibration over a $D^2_{\pi/2}$, where S_1^1 is the circle of radius 1. The unit vector field $\vec{v}_k$ is a section of the unit tangent bundle $T^1 D^2_{\pi/2} \backslash \{0\}$, the image of which can be explicitly written as

$$\vec{v}_k\big(D^2_{\pi/2} \backslash \{0\}\big) = \Big\{(r\cos t, r\sin t, \cos(kt), \sin(kt)) \in D^2_{\pi/2} \times \ S_1^1 \ : \ 0 < r \leq \frac{\pi}{2}, \ \ 0 \leq t < 2\pi\Big\}. \tag{3.4}$$

The topological closure of $\vec{v}_k\big(D^2_{\pi/2} \backslash \{0\}\big)$ is by definition

$$\overline{\vec{v}_k\big(D^2_{\pi/2} \backslash \{0\}\big)} = \vec{v}_k\big(D^2_{\pi/2} \backslash \{0\}\big) \bigcup \partial\big(\vec{v}_k(D^2_{\pi/2} \backslash \{0\})\big),$$

where

$$\partial\big(\vec{v}_k(D^2_{\pi/2} \backslash \{0\})\big) = \big\{(0, 0, \cos(kt), \sin(kt)) \ : \quad 0 \leq t \leq 2\pi\big\}.$$

Lemma 3.2. *If $k \in 2\mathbb{Z} \backslash \{0\}$, then $\vec{v}_k(D^2_{\pi/2} \backslash \{0\})$ is a ruled surface in $\mathbb{R}^4$.*

Proof. Define the following sets

$$\begin{aligned}
A_t &:= \Big\{\Big(-\frac{\pi}{2}\cos(t), -\frac{\pi}{2}\sin(t), \cos(kt), \sin(kt)\Big) \in \mathbb{R}^4 \ : \ -\pi \leq t \leq \pi\Big\}, \\
B_t &:= \Big\{\Big(\frac{\pi}{2}\cos(t), \frac{\pi}{2}\sin(t), \cos(kt), \sin(kt)\Big) \in \mathbb{R}^4 \ : \ -\pi \leq t \leq \pi\Big\}, \\
O_t &:= \big\{(0, 0, \cos(kt), \sin(kt)) \in \mathbb{R}^4 \ : \ -\pi \leq t \leq \pi\big\}.
\end{aligned}$$

We are going to show that

$$\vec{v}_k(D^2_{\pi/2} \backslash \{0\}) = A_t O_t \bigcup O_t B_t,$$

where

$$A_tO_t := \left\{(0,0,\cos(kt),\sin(kt)) + a\left(-\frac{\pi}{2}\cos(t), -\frac{\pi}{2}\sin(t), \cos(kt), \sin(kt)\right) \;:\; t\in[-\pi,\pi],\;\; a\in[0,1]\right\},$$
$$O_tB_t := \left\{\left(\frac{\pi}{2}\cos(t), \frac{\pi}{2}\sin(t), \cos(kt), \sin(kt)\right) + b\,(0,0,\cos(kt),\sin(kt)) \;:\; -\pi\le t\le\pi,\;\; b\in[0,1]\right\}.$$

Given $q \in \vec{v}_k(D^2_{\pi/2}\backslash\{0\})$, there exist $(r,t)\in\left(0,\frac{\pi}{2}\right]\times[0,2\pi)$ such that

$$\begin{aligned} q &= \big(r\cos(t), r\sin(t), \cos(kt), \sin(kt)\big) \\ &= (0,0,\cos(kt),\sin(kt)) + \left(\frac{-2r}{\pi}\right)\left(-\frac{\pi}{2}\cos(t), -\frac{\pi}{2}\sin(t), \cos(kt), \sin(kt)\right). \end{aligned}$$

Letting $a := \dfrac{-2r}{\pi}$, we clearly see that $q \in A_tO_t$, and hence, we get the inclusion

$$\vec{v}_k(D^2_{\pi/2}\backslash\{0\}) \subset A_tO_t \bigcup O_tB_t.$$

We now show that A_tO_t and O_tB_t are subsets of $\vec{v}_k(D^2_{\pi/2}\backslash\{0\})$. A point q_1 belongs to A_tO_t if and only if there exists an $a\in[0,1]$ such that

$$\begin{aligned} q_1 &= (0,0,\cos(kt),\sin(kt)) + a\left(-\frac{\pi}{2}\cos(t), -\frac{\pi}{2}\sin(t), \cos(kt), \sin(kt)\right) \\ &= a\left(-\frac{\pi}{2}\cos(t), -\frac{\pi}{2}\sin(t), \cos(kt), \sin(kt)\right) + (1-a)\,(0,0,\cos(kt),\sin(kt)). \end{aligned}$$

We observe that

$$\left(-\frac{\pi}{2}\cos(t), -\frac{\pi}{2}\sin(t)\right) = \left(\frac{\pi}{2}\cos(\pi+t), \frac{\pi}{2}\sin(\pi+t)\right). \tag{3.5}$$

Since k is even, $k = 2n$ for some $n\in\mathbb{Z}$, and we have

$$\big(\cos\big(k(\pi+t)\big), \sin\big(k(\pi+t)\big)\big) = (\cos(2n\pi+kt), \sin(2n\pi+kt)) = (\cos(kt),\sin(kt)). \tag{3.6}$$

From equations (3.5) and (3.6), it follows that

$$q_1 = \left(\frac{a\pi}{2}\cos(\pi+t), \frac{a\pi}{2}\sin(\pi+t), \cos\big(k(\pi+t)\big), \sin\big(k(\pi+t)\big)\right). \tag{3.7}$$

Note that $0\le a\le 1$ implies $0 \le \dfrac{a\pi}{2} \le \frac{\pi}{2}$, and consequently, letting $r' := \dfrac{a\pi}{2}$, we get $0\le r'\le \dfrac{\pi}{2}$. Thus q_1 gets rewritten as

$$q_1 = \big(r'\cos(\pi+t), r'\sin(\pi+t), \cos\big(k(\pi+t)\big), \sin\big(k(\pi+t)\big)\big).$$

Further letting $t' := \pi + t$, since $-\pi \leq t \leq \pi$,

$$-\pi \leq t' - \pi \leq \pi \implies 0 \leq t' \leq 2\pi. \tag{3.8}$$

Therefore, $q_1 = (r'\cos(t'), r'\sin(t'), \cos(kt'), \sin(kt')) \in \vec{v}_k(D^2_{\pi/2}\backslash\{0\})$, and $A_tO_t \subset \vec{v}_k(D^2_{\pi/2}\backslash\{0\})$, as desired. An analogous computation shows that if $q_2 \in O_tB_t$, $q_2 \in \vec{v}_k(D^2_{\pi/2}\backslash\{0\})$, thereby implying

$$B_tO_t \bigcup O_tA_t \subset \vec{v}_k(D^2_{\pi/2}\backslash\{0\}). \tag{3.9}$$

Note however, that the union is not disjoint as, e.g., $\left(-\frac{\pi}{4}, 0, 1, 0\right)$ belongs to the intersection when taking $a = 1/2$, $b = -a$ and $t = 0$. $\square$

One can realize $\vec{v}_k\left(D^2_{\pi/2}\backslash\{0\}\right)$ geometrically as a surface in $\mathbb{R}^3$ given by the image of

$$\begin{aligned} f : [0, \pi/2] \times [0, 2\pi] &\longrightarrow \mathbb{R}^3 \\ (r, t) &\longmapsto (r\sin(t), 2\cos(kt) + r\cos(t)\cos(kt), 2\sin(kt) + r\cos(t)\sin(kt)). \end{aligned} \tag{3.10}$$

As an example, fix $k = 4$. In order to draw $\vec{v}_k\left(D^2_{\pi/2}\backslash\{0\}\right)$, we split the interval $[0, 2\pi]$ in four quarters and draw them successively in four separate steps (see Figures 3.1, 3.2, 3.3 and 3.4). The visualization of this construction is also available at https://www.geogebra.org/m/cm28qe5k.

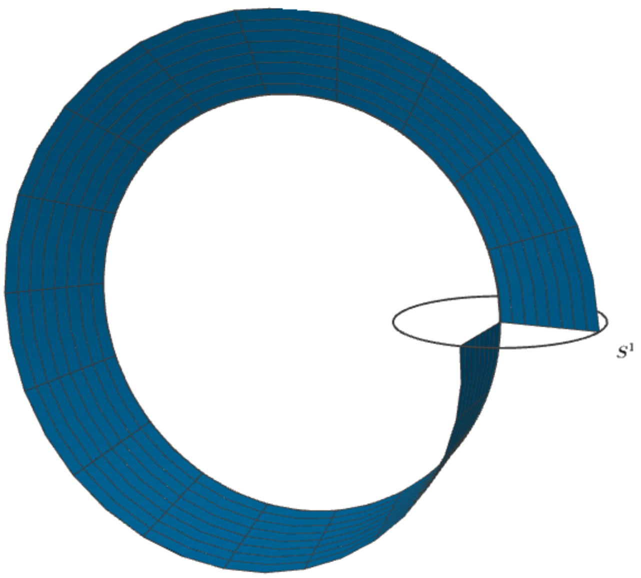

Figure 3.1: *Visualizing* $\vec{v}_4\left(D^2_{\pi/2}\backslash\{0\}\right)$*, the image of* (3.10) *restricted to* $0 \leq t \leq \pi/2$.

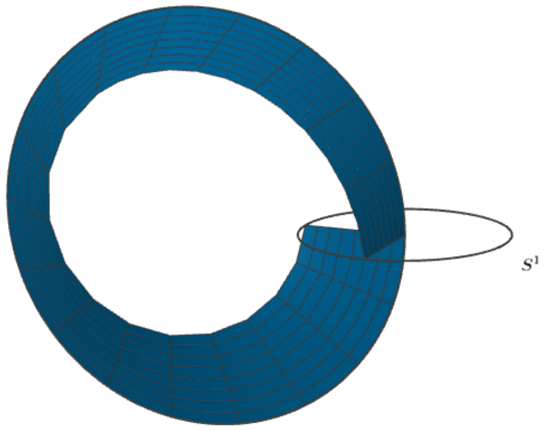

Figure 3.2: *Visualizing* $\vec{v}_4\big(D^2_{\pi/2}\backslash\{0\}\big)$*, the image of* (3.10) *restricted to* $\pi/2 \le t \le \pi$*.*

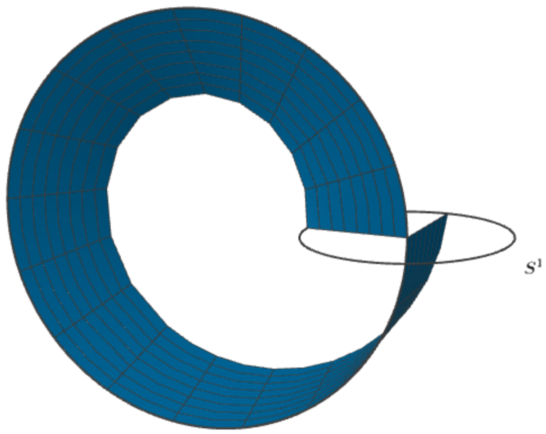

Figure 3.3: *Visualizing* $\vec{v}_4\big(D^2_{\pi/2}\backslash\{0\}\big)$*, the image of* (3.10) *restricted to* $\pi \le t \le 3\pi/2$*.*

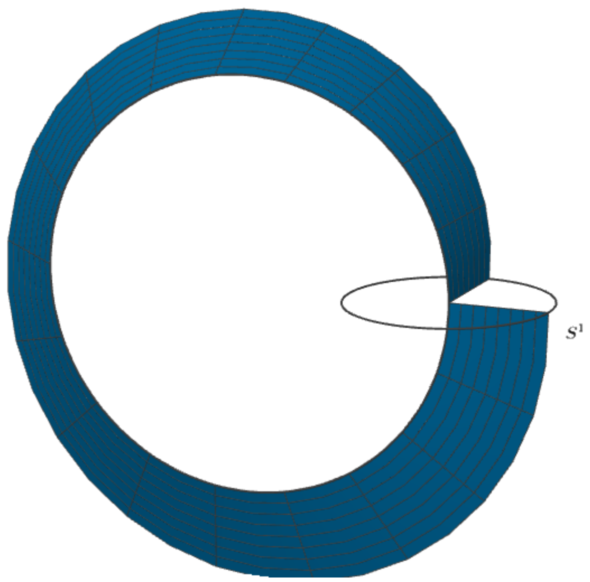

Figure 3.4: *Visualizing* $\vec{v}_4\big(D^2_{\pi/2}\backslash\{0\}\big)$*, the image of* (3.10) *restricted to* $3\pi/2 \le t \le 2\pi$*.*

Ultimately, the full image of (3.10) with $0 \le t \le 2\pi$ glues together an immersed Moebius strip in $\mathbb{R}^4$ plotted in Figure 3.5, where the four Figures 3.1, 3.2, 3.3, and 3.4 appear simultaneously, glued along the white central circle.

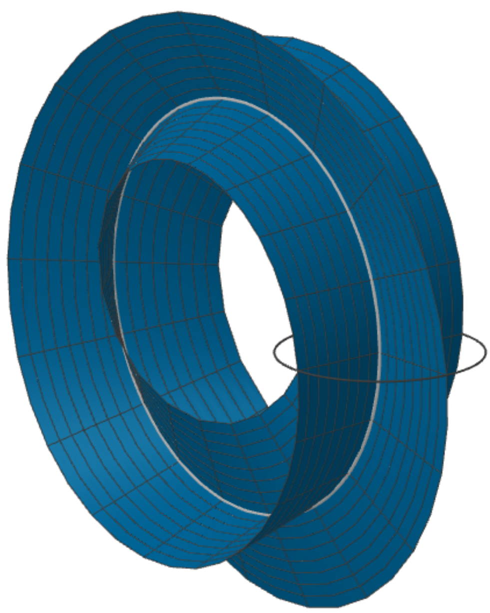

Figure 3.5: *An immersed Moebius strip in* $\mathbb{R}^4$ *given by the image of* (3.10) *when* $k = 4$*.*

In fact, the following lemma shows that the image of (3.10) is an immersed Moebius strip, regardless of the value of k.

Lemma 3.3. *If* $k \in 2\mathbb{Z}\backslash\{0\}$*, then the topological closure of* $\vec{v}_k\big(D^2_{\pi/2}\backslash\{0\}\big)$ *is the image of an immersed smooth Moebius strip with boundary* $\vec{v}_k\big(\partial(D^2_{\pi/2}\backslash\{0\})\big)$*.*

Proof. Fix $k \in 2\mathbb{Z}\backslash\{0\}$ and consider the smooth immersion

$$\begin{array}{rcl} \varphi : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [0,\pi] & \longrightarrow & \mathbb{R}^4 \\ (\rho, t) & \longmapsto & (\rho\cos(t), \rho\sin(t), \cos(kt), \sin(kt)). \end{array}$$

In order to prove the result, we verify the following conditions:

i) The image of φ is the topological closure of $\vec{v}_k\big(D^2_{\pi/2}\backslash\{0\}\big)$, i.e., $\mathrm{Im}\varphi = \overline{\vec{v}_k\big(D^2_{\pi/2}\backslash\{0\}\big)}$. Indeed, the topological closure of $\vec{v}_k\big(D^2_{\pi/2}\backslash\{0\}\big)$ is a subset of the image of φ, see (3.4). Conversely, let $p \in \mathrm{Im}\varphi$, that is, there are $(\rho, t) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0,\pi]$ such that

$$p = (\rho\cos t, \rho\sin t, \cos(kt), \sin(kt)).$$

If $\rho \in [0, \frac{\pi}{2}]$, φ shares the same formula as v_k, thus implying $p \in \overline{\vec{v}_k\big(D^2_{\pi/2}\backslash\{0\}\big)}$ trivially. On the other hand, if $\rho \in [-\frac{\pi}{2}, 0)$, by letting $r := -\rho$, we get $r \in (0, \frac{\pi}{2}]$. If $\vartheta := t + \pi$, then $\vartheta \in [\pi, 2\pi]$, and

$$\begin{aligned} \rho\cos t &= -r\cos t = -r\cos(\vartheta - \pi) = -r\cos\vartheta\cos\pi + r\sin\vartheta\sin\pi = r\cos\vartheta, \\ \rho\sin t &= -r\sin t = -r\sin(\vartheta - \pi) = -r\cos\vartheta\sin\pi - r\cos\pi\sin\vartheta = r\sin\vartheta. \end{aligned}$$

Moreover, if $k = 2n$,

$$\begin{aligned} \cos(kt) &= \cos(2n(\vartheta - \pi)) = \cos(k\vartheta)\cos(2n\pi) + \sin(k\vartheta)\sin(2n\pi) = \cos(k\vartheta), \\ \sin(kt) &= \sin(2n(\vartheta - \pi)) = \sin(k\vartheta)\cos(2n\pi) - \cos(k\vartheta)\sin(2n\pi) = \sin(k\vartheta). \end{aligned}$$

Therefore, $\mathrm{Im}\varphi \subset \overline{\vec{v}_k\big(D^2_{\pi/2}\backslash\{0\}\big)}$, as claimed.

ii) The image of the smooth immersion φ is a Moebius strip in $\mathbb{R}^4$.

The reverse identification of the two opposite edges of the domain of φ is given by the relation $\varphi(\rho, \pi) = \varphi(-\rho, 0)$ that holds for all $\rho \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

iii) The boundary of the smoothly immersed Moebius strip is $\vec{v}_k\big(\partial(D^2_{\pi/2}\backslash\{0\})\big)$. Indeed,

$$\partial(\mathrm{Im}\varphi) = \{(0, 0, \cos(kt), \sin(kt)) \ : \ 0 \leq t \leq \pi\} = \partial(\vec{v}_k(D^2_{\pi/2}\backslash\{0\})).$$

Since $\vec{v}_k$ is a section of the unit tangent bundle of $D^2_{\pi/2}\backslash\{0\}$, we have that $\vec{v}_k\big(\partial(D^2_{\pi/2}\backslash\{0\})\big) = \partial\big(\vec{v}_k(D^2_{\pi/2}\backslash\{0\})\big)$.

$\square$

Remark 3.4. *A minimally embedded Moebius strip appears when restricting to Lemma 3.3 for $k = \pm 2$.*

Consider the decomposition $\mathbb{S}^2 = \mathbb{S}^2_+ \cup \mathbb{S}^2_-$, where $\mathbb{S}^2_+$ and $\mathbb{S}^2_-$ are respectively the northern and southern hemispheres. The next propositions establish a relationship between the unit vector fields $\vec{v}_k$ and $V_{k,2-k}$, when $k \in 2\mathbb{N}\backslash\{0,2\}$.

Proposition 3.5. *If the Poincaré index k at the singularity $N \in \mathbb{S}^2_+$ is an even number greater than 2, then the topological closure of $V_{k,2-k}(\mathbb{S}^2_+\backslash\{N\})$ in $T^1\mathbb{S}^2_+$ is the image of an immersed Moebius strip with boundary $V_{k,2-k}\big(\partial(\mathbb{S}^2_+\backslash\{N\})\big)$.*

Proof. At the singularity of the North pole N, assume the canonical identification $T_N\mathbb{S}^2_+ \cong \mathbb{R}^2$. Let $D^2_{\pi/2}$ be the disk of radius $\frac{\pi}{2}$ centered at the origin in $\mathbb{R}^2$. The map

$$\psi : D^2_{\pi/2} \times S^1 \to T^1\mathbb{S}^2_+, \quad (X,\sigma) \mapsto \left(\exp_N(X), \frac{d\exp_N(\sigma)}{||d\exp_N(\sigma)||}\right),$$

where $\exp_N : T^1_N\mathbb{S}^2_+ \to \mathbb{S}^2_+$ is the exponential map, defines a diffeomorphism between the unit tangent bundles $T^1D^2_{\pi/2}$ and $T^1\mathbb{S}^2_+$. Hence, the result follows from Lemma 3.3. □

An analogous statement holds for the southern hemisphere.

Proposition 3.6. *If Poincaré index $2-k$ at the singularity $S \in \mathbb{S}^2_-$ is a strictly negative even number, then the topological closure of $V_{k,2-k}(\mathbb{S}^2_-\backslash\{S\})$ in $T^1\mathbb{S}^2_-$ is the image of an immersed Moebius strip with boundary $V_{k,2-k}\big(\partial(\mathbb{S}^2_-\backslash\{S\})\big)$.*

The intersection of these Moebius strips is given by the central circle as in Figure 3.5.

***Proof of Theorem A**.* A smooth immersed Klein bottle in $T^1\mathbb{S}^2(1)$ is obtained by gluing together the two Moebius strips of Proposition 3.5 and Proposition 3.6 along their respective boundaries. It follows from Theorem 2.10 that $V_{k,2-k}$ is an area-minimizing unit vector field in its topological conjugation class. It follows from Corollary 2.12, the section seen as a surface in $T^1\mathbb{S}^2(1)$ is geometrically minimal, i.e., it has zero mean curvature. Therefore, the topological closure of $V_{k,2-k}$ is a minimal surface in $T^1\mathbb{S}^2(1)$. □

Chapter 4

Minimally embedded Clifford Torus in $T^1\mathbb{S}^2$

In this chapter we establish a relationship between the North-South and South-North unit vector fields and a Clifford Torus. More specifically, we prove that the closure of the union of the images of the North-South and the South-North vector fields in $T^1\mathbb{S}^2(1)$ is an embedded Clifford Torus.

4.1 The closure of the North-South and South-North union

In 1975, W. Klingenberg and S. Sasaki showed that the unit tangent bundle $T^1\mathbb{S}^2(r)$ is isometric to the projective space $\mathbb{R}\mathrm{P}^3$ via the Euler parametric representation (4.1) of $\mathbf{SO}(3)$. The complete proof can be found in [BGM10, KS75]. At this point, we recall the maps and isometries that will be needed throughout this section. Let $\mathbf{SO}(3)$ be the special orthogonal group equipped with the metric $\frac{1}{2}\langle\cdot,\cdot\rangle$ given by

$$\langle A, B\rangle = tr(A^tB),$$

where $A, B \in \mathfrak{so}(3) := T_I SO(3)$, and I is the identity matrix.
Let g denote the standard metric on $\mathbb{R}^n$, and consider the Euler parametric representation of $\mathbf{SO}(3)$

$$\begin{aligned} \Phi : \left(\mathbb{S}^3(2), g\right) &\longrightarrow \left(\mathbf{SO}(3), \frac{1}{2}\langle\cdot,\cdot\rangle\right) \\ (x_1,x_2,x_3,x_4) &\longmapsto \frac{1}{4}\begin{pmatrix} x_1^2+x_2^2-x_3^2-x_4^2 & 2x_1x_4+2x_2x_3 & -2x_1x_3+2x_2x_4 \\ -2x_1x_4+2x_2x_3 & x_1^2-x_2^2+x_3^2-x_4^2 & 2x_1x_2+2x_3x_4 \\ 2x_1x_3+2x_2x_4 & -2x_1x_2+2x_3x_4 & x_1^2-x_2^2-x_3^2+x_4^2 \end{pmatrix}, \end{aligned} \tag{4.1}$$

and the diffeomorphism

$$\begin{aligned} \psi : \left(T^1\mathbb{S}^2(1), g^{Sas}\right) &\longrightarrow \left(\mathbf{SO}(3), \frac{1}{2}\langle\cdot,\cdot\rangle\right) \\ (\mathrm{x},\mathrm{v}) &\longmapsto (\mathrm{x},\mathrm{v},\mathrm{x}\wedge\mathrm{v}). \end{aligned}$$

The projective space can be defined as $\mathbb{R}\mathrm{P}^3(2) = \mathbb{S}^3(2)/\mathbb{Z}_2$. The map Φ induces an isometry $\overline{\Phi}$ between $(\mathbb{R}\mathrm{P}^3(2), \overline{g})$ and $\left(\mathbf{SO}(3), \frac{1}{2}\langle\cdot,\cdot\rangle\right)$. Ultimately, the desired isometry is given by

$$\begin{tikzcd}
(\mathbb{R}\mathrm{P}^3(2), \overline{g}) \arrow[rr, "\psi^{-1} \circ \overline{\Phi}"] \arrow[dr, "\overline{\Phi}"'] & & \big(T^1\mathbb{S}^2(1), g^{Sas}\big) \\
& \big(\mathbf{SO}(3), \frac{1}{2}\langle \cdot, \cdot \rangle\big) \arrow[ur, "\psi^{-1}"'] &
\end{tikzcd} \tag{4.2}$$

Define the following functions that are the first and second columns of the Euler parametric representation (4.1)

$$\begin{aligned}
\mathrm{x} : \mathbb{R}^4 &\longrightarrow \mathbb{R}^3 \\
\mathrm{x}(x_1, x_2, x_3, x_4) &= \frac{1}{4}(x_1^2 + x_2^2 - x_3^2 - x_4^2,\ \ -2x_1x_4 + 2x_2x_3,\ \ 2x_1x_3 + 2x_2x_4),
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\mathrm{v} : \mathbb{R}^4 &\longrightarrow \mathbb{R}^3 \\
\mathrm{v}(x_1, x_2, x_3, x_4) &= \frac{1}{4}(2x_1x_4 + 2x_2x_3,\ \ x_1^2 - x_2^2 + x_3^2 - x_4^2,\ \ -2x_1x_2 + 2x_3x_4).
\end{aligned} \tag{4.4}$$

Note that the third column in (4.1) is the cross product of the vectors x and v, that is,

$$(\mathrm{x} \times \mathrm{v})(x_1, x_2, x_3, x_4) = \frac{1}{4}(-2x_1x_3 + 2x_2x_4,\ \ 2x_1x_2 + 2x_3x_4,\ \ x_1^2 - x_2^2 - x_3^2 + x_4^2). \tag{4.5}$$

Moreover, in the case when $(x_1, x_2, x_3, x_4) \in \mathbb{S}^3(2)$, the vectors $\mathrm{x}(x_1, x_2, x_3, x_4)$, $\mathrm{v}(x_1, x_2, x_3, x_4)$, and $\mathrm{x} \times \mathrm{v}(x_1, x_2, x_3, x_4)$ are unit vectors since they land in $\mathbf{SO}(3)$.

Definition 4.1. *The restriction of the Euler parametric representation of* $\mathbf{SO}(3)$ *is*

$$\begin{aligned}
\phi : \mathbb{S}^3(2) &\longrightarrow M_{3\times 2}(\mathbb{R}) \\
(x_1, x_2, x_3, x_4) &\longmapsto \Big(\mathrm{x}(x_1, x_2, x_3, x_4)^T \quad \mathrm{v}(x_1, x_2, x_3, x_4)^T\Big).
\end{aligned} \tag{4.6}$$

Consider $T^1\mathbb{S}^2$ seen as the set of matrices $\begin{pmatrix} x_1 & x_4 \\ x_2 & x_5 \\ x_3 & x_6 \end{pmatrix} \in M_{3\times 2}(\mathbb{R})$ such that

$$x_1^2 + x_2^2 + x_3^2 = x_4^2 + x_5^2 + x_6^2 = 1 \quad \text{and} \quad x_1x_4 + x_2x_5 + x_3x_6 = 0. \tag{4.7}$$

Proposition 4.2. *If ϕ is the restriction of Euler parametric representation of* $\mathbf{SO}(3)$ *of Definition 4.1, then $\phi\big(\mathbb{S}^3(2)\big)$ is a subset of $T^1\mathbb{S}^2$.*

Proof. It follows from the fact that the Euler parametric representation (4.1) takes values in $\mathbf{SO}(3)$. $\square$

The Clifford Torus in $\mathbb{S}^3(2)$ is given by

$$T^2 = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{S}^3(2) \ : \ x_1^2 + x_4^2 = 2 \quad \text{and} \quad x_2^2 + x_3^2 = 2\right\}. \tag{4.8}$$

It follows from Proposition 4.2 that $\phi\left(T^2\right) \subset T^1\mathbb{S}^2$.
The image of the North-South and South-North unit vector fields on $\mathbb{S}^2(1)\backslash\{N, S\}$ in $T^1\mathbb{S}^2$ considered as above are respectively

$$\overrightarrow{NS} = \left\{ \begin{pmatrix} x & \dfrac{zx}{\sqrt{1-z^2}} \\ y & \dfrac{zy}{\sqrt{1-z^2}} \\ z & -\sqrt{1-z^2} \end{pmatrix} \in M_{3\times 2}(\mathbb{R}) \ : \ x^2 + y^2 + z^2 = 1 \text{ and } z \neq \pm 1 \right\},$$

$$\overrightarrow{SN} = \left\{ \begin{pmatrix} x & \dfrac{-zx}{\sqrt{1-z^2}} \\ y & \dfrac{-zy}{\sqrt{1-z^2}} \\ z & \sqrt{1-z^2} \end{pmatrix} \in M_{3\times 2}(\mathbb{R}) \ : \ x^2 + y^2 + z^2 = 1 \text{ and } z \neq \pm 1 \right\}.$$

Moreover, in the topological closure of their union

$$\overline{\overrightarrow{NS} \bigcup \overrightarrow{SN}} = \left(\overrightarrow{NS} \bigcup \overrightarrow{SN}\right) \bigcup \partial\left(\overrightarrow{NS} \bigcup \overrightarrow{SN}\right),$$

we have that

$$\partial\left(\overrightarrow{NS} \bigcup \overrightarrow{SN}\right) = \partial\left(\overrightarrow{NS}\right) \bigcup \partial\left(\overrightarrow{SN}\right) = \left\{ \begin{pmatrix} 0 & \cos(v) \\ 0 & \sin(v) \\ \mp 1 & 0 \end{pmatrix} \ : \ v \in [0, 2\pi] \right\}.$$

Theorem B. *Let T^2 be the Clifford Torus in $\mathbb{S}^3(2)$ (4.8). If ϕ is the restriction (4.1) of the Euler parametric representation of $\mathbf{SO}(3)$, then*

$$\phi\left(T^2\right) = \textit{topological closure of } \left(\overrightarrow{NS} \bigcup \overrightarrow{SN}\right), \tag{4.9}$$

where $\overrightarrow{NS}$, $\overrightarrow{SN}$ are, respectively, the images of the North-South and South-North unit vector fields on $\mathbb{S}^2\backslash\{N, S\}$ in $T^1\mathbb{S}^2$.

Proof. For the first inclusion in (4.9), $\phi\left(T^2\right) \subset \overline{\overrightarrow{NS} \bigcup \overrightarrow{SN}}$, we just need to prove that v defined in (4.4) is a multiple of $\overrightarrow{NS}$ or $\overrightarrow{SN}$. Note that the cross product between the vector fields $\overrightarrow{SN}$ or $\overrightarrow{NS}$ and any position vector on the sphere is parallel to the xy-plane, that is, its z-coordinate is zero (see Figure 4.1).

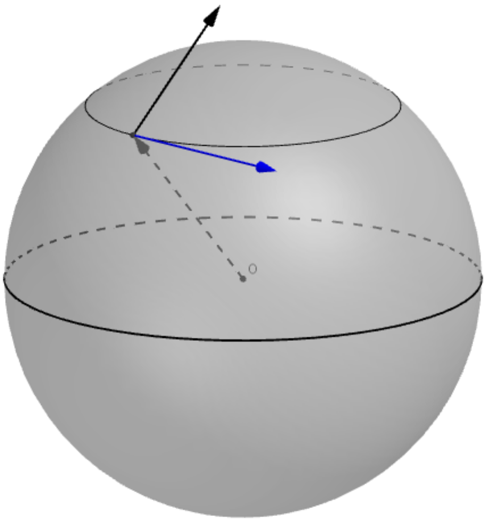

Figure 4.1: *In blue, the cross product between the vector field pointing north and the position vector.*

Indeed, let $p = (x, y, z) \in \mathbb{S}^2(1)$, and $\overrightarrow{OP} = (x, y, z)$ be its corresponding position vector in $\mathbb{R}^3$. The cross product between the vectors $\overrightarrow{OP}$ and $\overrightarrow{SN}$ at p is given by

$$\overrightarrow{OP} \times \overrightarrow{SN} = \left(y\sqrt{1-z^2} + \frac{yz^2}{\sqrt{1-z^2}}, \ \frac{-xz^2}{\sqrt{1-z^2}} - x\sqrt{1-z^2}, \ 0 \right).$$

For each $(x_1, x_2, x_3, x_4) \in T^2$, we have that

$$\Phi(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1^2 + x_2^2 - x_3^2 - x_4^2 & 2x_1x_4 + 2x_2x_3 & -2x_1x_3 + 2x_2x_4 \\ -2x_1x_4 + 2x_2x_3 & x_1^2 - x_2^2 + x_3^2 - x_4^2 & 2x_1x_2 + 2x_3x_4 \\ 2x_1x_3 + 2x_2x_4 & -2x_1x_2 + 2x_3x_4 & 0 \end{pmatrix} \in SO(3).$$

In particular,

$$(\mathrm{x} \times \mathrm{v})(x_1, x_2, x_3, x_4) = (-2x_1x_3 + 2x_2x_4, 2x_1x_2 + 2x_3x_4, 0) \in \mathbb{R}^3.$$

Therefore, v is parallel to the vector field $\overrightarrow{SN}$, thus implying

$$\phi\left(T^2\right) \subset \overline{\overrightarrow{SN} \bigcup \overrightarrow{NS}}. \tag{4.10}$$

For the converse inclusion in (4.9), we restrict our attention to prove that $\left(\overrightarrow{SN} \cup \partial\big(\overrightarrow{SN}\big)\right) \subset \phi\big(T^2\big)$, because the North-South part will follow from a similar argument. We start by proving that the topological closure of $\overrightarrow{SN}$ is a subset of $\phi(T^2)$. Consider the following parametrization of the Clifford Torus T^2 (4.8),

$$\begin{aligned} \Upsilon : [-\pi, \pi] \times [-\pi, \pi] \ &\longrightarrow \ T^2 \\ (a, b) \ &\longmapsto \ \left(\sqrt{2}\cos a, \sqrt{2}\cos b, \sqrt{2}\sin b, \sqrt{2}\sin a\right). \end{aligned} \tag{4.11}$$

Thus, letting $p = \big(\sqrt{2}\cos(a), \sqrt{2}\cos(b), \sqrt{2}\sin(b), \sqrt{2}\sin(a)\big) \in T^2$ for some $(a,b) \in [-\pi,\pi]\times[-\pi,\pi]$, we compute

$$\phi(p) = \frac{1}{4}\begin{pmatrix} 2(\cos^2(a)+\cos^2(b)-\sin^2(b)-\sin^2(a)) & 4\cos(a)\sin(a)+4\cos(b)\sin(b) \\ -4\cos(a)\sin(a)+4\cos(b)\sin(b) & 2(\cos^2(a)-\cos^2(b)+\sin^2(b)-\sin^2(a)) \\ 4(\cos(a)\sin(b)+\cos(b)\sin(a)) & -4\cos(a)\cos(b)+4\sin(b)\sin(a) \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} \cos(2a)+\cos(2b) & \sin(2a)+\sin(2b) \\ \sin(2b)-\sin(2a) & \cos(2a)-\cos(2b) \\ 2\sin(a+b) & -2\cos(a+b) \end{pmatrix}. \tag{4.12}$$

Picking values in $[0,\pi/4]$ and $(x,y,z) \in \mathbb{S}^2(1)$, set

$$a := \frac{1}{2}\arccos\left(x+\frac{yz}{\sqrt{1-z^2}}\right) = \frac{1}{2}\arcsin\left(-y+\frac{xz}{\sqrt{1-z^2}}\right), \tag{4.13}$$
$$b := \frac{1}{2}\arcsin\left(x-\frac{yz}{\sqrt{1-z^2}}\right) = \frac{1}{2}\arcsin\left(y+\frac{xz}{\sqrt{1-z^2}}\right). \tag{4.14}$$

It follows then from (4.13) and (4.14) that

$$\cos(2a) = x+\frac{yz}{\sqrt{1-z^2}}, \qquad \sin(2a) = -y+\frac{xz}{\sqrt{1-z^2}},$$
$$\cos(2b) = x-\frac{yz}{\sqrt{1-z^2}}, \qquad \sin(2b) = y+\frac{xz}{\sqrt{1-z^2}};$$

therefore,

$$\frac{\cos(2a)+\cos(2b)}{2} = x \quad \text{and} \quad \frac{\sin(2b)-\sin(2a)}{2} = y.$$

In so, for each $(x,y,z) \in \mathbb{S}^2(1)$, we have that

$$1-z^2 = \left(\frac{\cos(2a)+\cos(2b)}{2}\right)^2 + \left(\frac{\sin(2b)-\sin(2a)}{2}\right)^2$$
$$= \frac{1+\cos(2(a+b))}{2} = \cos^2(a+b),$$

which, in order, implies $z^2 = \sin^2(a+b)$. Hence, there are two possible values for z. If $z = -\sin(a+b)$, then we would have points in $\phi(T^2)$ that are not in $\overrightarrow{SN}\bigcup\overrightarrow{NS}$, which cannot be due to our first inclusion (4.10). Consequently, $z = \sin(a+b)$, and

$$\overrightarrow{SN} \subset \phi\left(T^2\right).$$

It remains to be seen that the set $\partial\left(\overrightarrow{SN}\right) = \left\{\begin{pmatrix} 0 & \cos(v) \\ 0 & \sin(v) \\ 1 & 0 \end{pmatrix} \;:\; v \in [0,2\pi]\right\}$ is also a subset of $\phi(T^2)$. It is enough to consider $a := \frac{\pi}{2} - b$, for then

$$\begin{aligned}\sin(2a) &= \sin(\pi - 2b) = \sin(\pi)\cos(2b) - \sin(2b)\cos(\pi) = \sin(2b),\\ \cos(2a) &= \cos(\pi - 2b) = \cos(\pi)\cos(2b) + \sin(2b)\sin(\pi) = -\cos(2b).\end{aligned}$$

One thus concludes that there exists $p \in T^2$ such that

$$\phi(p) = \frac{1}{2}\begin{pmatrix}\cos(2a) + \cos(2b) & \sin(2a) + \sin(2b)\\ \sin(2b) - \sin(2a) & \cos(2a) - \cos(2b)\\ 2\sin(a+b) & -2\cos(a+b)\end{pmatrix} = \begin{pmatrix}0 & \sin(2a)\\ 0 & \cos(2a)\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & \cos(\frac{\pi}{2} - 2a)\\ 0 & \sin(\frac{\pi}{2} - 2a)\\ 1 & 0\end{pmatrix}.$$

By the same line of reasoning, one can prove that the closure of $\overrightarrow{NS}$ is subset $\phi\left(T^2\right)$ and the result follows. □

Using the isometry (4.2), we can identify the antipodal points of $\phi(T^2)$. Therefore, the topological closure of $\overrightarrow{NS} \bigcup \overrightarrow{SN}$ is an embedded Clifford Torus in $T^1\mathbb{S}^2(1)$, which is the content of the following corollary.

Corollary 4.3. *If $\overrightarrow{NS}$, $\overrightarrow{SN}$ are the North-South and South-North unit vector fields defined on $\mathbb{S}^2 \backslash \{N, S\}$, then the topological closure of the union of their images in $T^1\mathbb{S}^2(1)$ is an embedded Clifford Torus.*

One can use the matrix (4.12) to locate the points in the domain of the Clifford Torus parametrization, in the 2-sphere, and in the unit tangent bundle of the 2-sphere. We summarize this location in the following Table 1 and Figure 4.2.

TABLE 1. Location of the points in T^2, $\mathbb{S}^2(1)$ and $T^1\mathbb{S}^2(1)$.

Values in $[-\pi, \pi] \times [-\pi, \pi]$	**Points in** $\mathbb{S}^2(1)$**,** $z_{\mathrm{x}} = \sin(a+b)$	**Points in** $T^1\mathbb{S}^2$, $z_{\mathrm{v}} = -\cos(a+b)$
$a + b = 2\pi$	Equator	$\overrightarrow{NS}$ (North-South)
$a + b = 3\pi/2$	Singularity S	$\partial(\overrightarrow{NS})$ (Boundary of North-South)
$a + b = \pi$	Equator	$\overrightarrow{SN}$ (South-North)
$a + b = \pi/2$	Singularity N	$\partial(\overrightarrow{SN})$ (Boundary of South-North)
$a + b = 0$	Equator	$\overrightarrow{NS}$ (North-South)
$a + b = -\pi/2$	Singularity S	$\partial(\overrightarrow{NS})$ (Boundary of North-South)
$a + b = -\pi$	Equator	$\overrightarrow{SN}$ (South-North)
$a + b = -3\pi/2$	Singularity N	$\partial(\overrightarrow{SN})$ (Boundary of South-North)
$a + b = -2\pi$	Equator	$\overrightarrow{NS}$ (North-South)

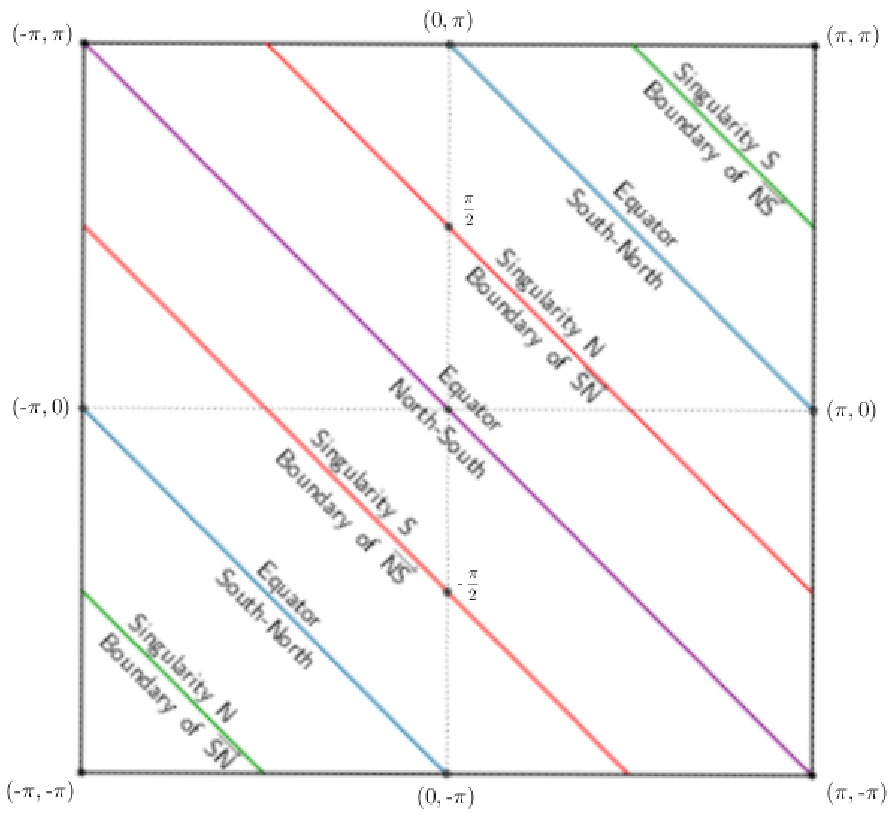

Figure 4.2: *Location of the points in T^2, $\mathbb{S}^2(1)$ and $T^1\mathbb{S}^2(1)$.*

Chapter 5

Appendices

The main objective of this chapter is defining two topological invariants: the Poincaré index of a vector field and the Euler characteristic of a manifold M, and to state the Poincaré-Hopf Theorem that relates them.

5.1 Appendix A: The Poincaré index of a vector field

Let M be a smooth manifold and V a vector field on M. A point $p \in M$ is said to be a ***singular point*** or a ***singularity*** of V if $V(p) = 0$, and a ***regular point***, otherwise. If there is $r > 0$ such that $V(y) \neq 0$ for all $y \in B_r(p)\backslash\{p\}$, where $B_r(p)$ is a ball of radius r in a coordinate chart centered at p, we say that p is an ***isolated singularity*** of V.

From a local point of view, it is interesting to see how V behaves around its singularities. Indeed, if we try and draw a vector field on some compact surface, one first needs to determine patterns around the singularities and then smoothly interpolate the rest of the field. Locally each singularity can be of the type: sink, source, spiral or circulation, see Figures 5.3, 5.2, 5.6 and 5.9, respectively. However, as one tries and interpolate them, one quickly discovers that the topology of the manifold limits the possibilities. For example, we defined the Pontryagin vector field on the round 2-sphere in (2.3) (see Figure 2.1), which is a vector field with only one singularity. This cannot be done on the torus, which, in contrast, has a vector field without singularities, an object that cannot be built on the round 2-sphere due to the Hairy ball Theorem. On the other hand, one can construct vector fields with exactly two singularities on both surfaces, but one needs to be careful as certain patterns around these singularities such as a saddle (Figure 5.9) plus a source (Figure 5.2) cannot exist on the sphere due to the Poincaré-Hopf theorem that we state in the next subsection (see Theorem 5.8.

In order to investigate the relation between V and the topology of M, we must quantify the directional change of V around its singularities. First, assume that $M = \mathbb{R}^n$ and V has an isolated singularity at the origin. The directional variation of V around 0 is measured by the ***Gauss map***

$$\begin{aligned} G : \mathbb{S}_r &\longrightarrow \mathbb{S}^n \\ p &\longmapsto \frac{V(p)}{||V(p)||} \end{aligned}$$

where $\mathbb{S}_r$ is small sphere of radius r around at the origin. The Gauss map takes the vector given by the field V at a given point p into a unit vector in $\mathbb{R}^{n+1}$. Here, we choose a small enough radius r such that V has no singularities inside $\mathbb{S}_r$, except at the origin. The ***Poincaré index of a vector field*** V at 0 is defined as the degree of the Gauss map and is denoted by $I_V(0)$. Note that if there is a different suitable r' other than r, then $\frac{V(p)}{||V(p)||}$ extends to the annulus bounded by the two spheres, ultimately implying that the definition of the Poincaré index does not depend on the choice of radius.

In the two-dimensional case, $I_V(0)$ counts the number of times V rotates completely while going once around the circle counterclockwise. One counterclockwise rotation of V has ***positive direction*** and counts $+1$, whereas a clockwise rotation is ***negative*** and counts -1, see figure below.

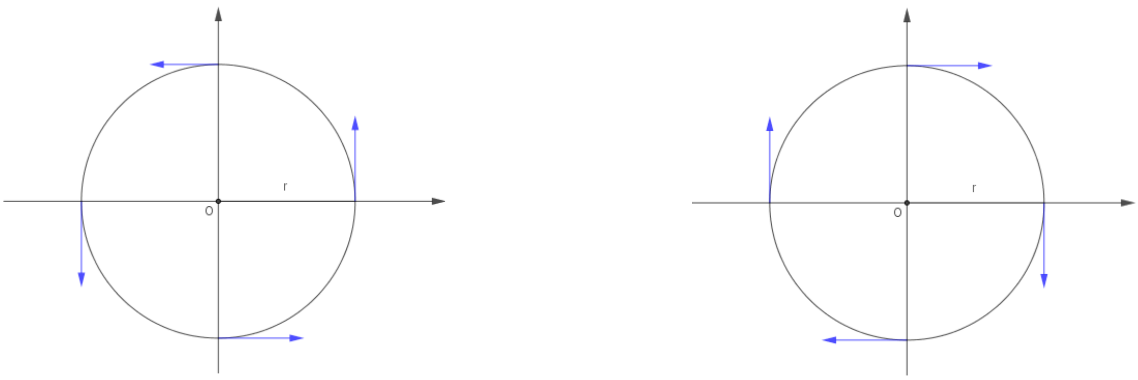

Figure 5.1: *The positive and negative direction on S^1, respectively.*

In the upcoming examples, let $\{c_1, c_2\}$ be the orthomormal canonical base of $\mathbb{R}^2$. Below, we present the flow of certain vector fields with one singularity.

Example 5.1. *Let V be the vector field on $\mathbb{R}^2$ defined by $V(x, y) := xc_1 + yc_2$. The Poincaré index of V at the origin is* 1*.*

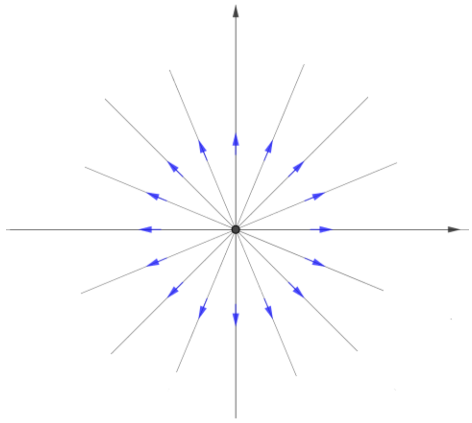

Figure 5.2: *The flow of the vector field* $V(x,y) = xc_1 + yc_2$ *on* $\mathbb{R}^2$

Example 5.2. *Let* V *be the vector field on* $\mathbb{R}^2$ *defined by* $V(x,y) := -xc_1 - yc_2$*. The Poincaré index of* V *at the origin is* 1*.*

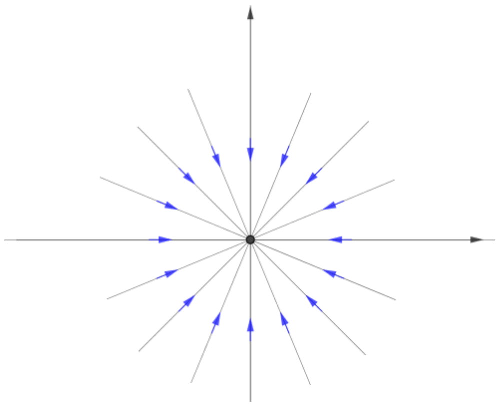

Figure 5.3: *The flow of the vector field* $V(x,y) = -xc_1 - yc_2$ *on* $\mathbb{R}^2$

Example 5.3. *Let* V *be the vector field on* $\mathbb{R}^2$ *defined by* $V(x,y) := xc_1 + 2yc_2$*. The Poincaré index of* V *at the origin is* 1*.*

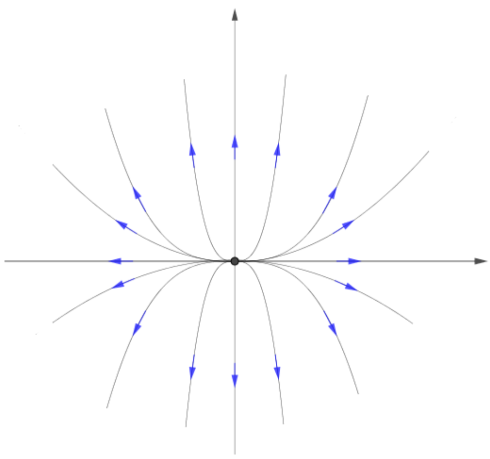

Figure 5.4: *The flow of the vector field* $V(x,y) = xc_1 + 2yc_2$ *on* $\mathbb{R}^2$

Example 5.4. *Let V be the vector field on $\mathbb{R}^2$ defined by $V(x, y) := xc_1 - yc_2$. The Poincaré index of V at the origin is 1.*

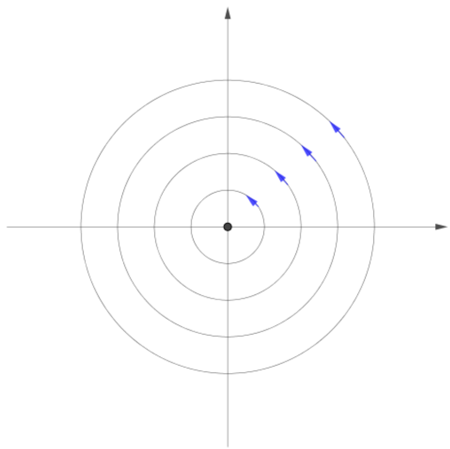

Figure 5.5: *The flow of the vector field $V(x, y) = xc_1 - yc_2$ on $\mathbb{R}^2$*

Example 5.5. *Let V be the vector field on $\mathbb{R}^2$ defined by $V(x, y) := (y - x)c_1 - (y + x)c_2$. The Poincaré index of V at the origin is 1.*

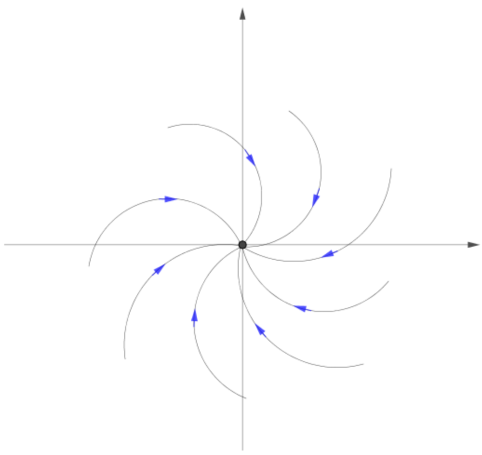

Figure 5.6: *The flow of the vector field $V(x, y) = (y - x)c_1 - (y + x)c_2$ on $\mathbb{R}^2$*

Example 5.6. *Let V be the vector field on $\mathbb{R}^2$ defined by $V(x, y) := x^2c_1 - y^2c_2$. The Poincaré index of V at the origin is -1.*

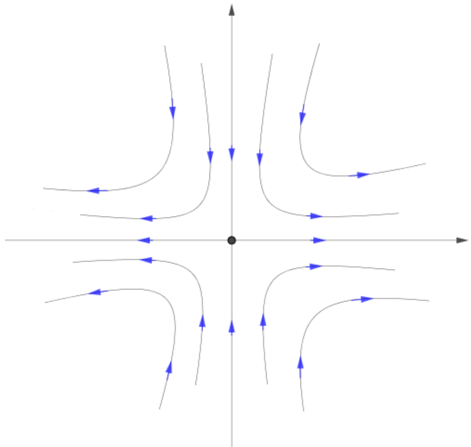

Figure 5.7: *The flow of the vector field* $V(x, y) = x^2 c_1 - y^2 c_2$ *on* $\mathbb{R}^2$

Example 5.7. *Let* V *be the magnetic vector field on* $\mathbb{R}^2$*. The Poincaré index of* V *at the origin is* 2*.*

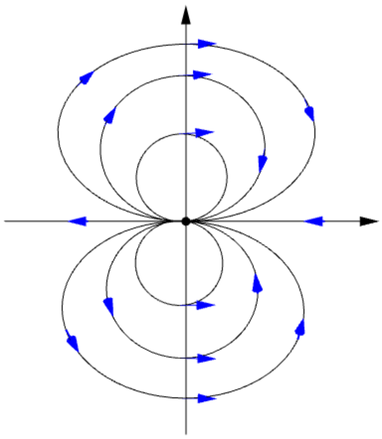

Figure 5.8: *The flow of the magnetic vector field* V *on* $\mathbb{R}^2$

Example 5.8. *Let* V *be the vector field on* $\mathbb{R}^2$ *such that the integral lines of* V *are given by* $r = \cos(2\theta)$*. The Poincaré index of* V *at the origin is* 3*.*

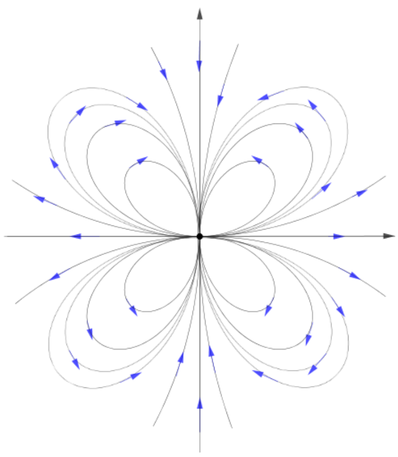

Figure 5.9: *The flow of the vector field* V *such that the integral lines are given by* $r = \cos(2\theta)$

In order to define the Poincaré index of a vector field at an isolated singularity on an arbitrary oriented manifold, one uses a local parametrization. Locally a differentiable manifold of dimension n is a piece of Euclidean space $\mathbb{R}^n$, so we simply see the Poincaré index as though the vector field is defined over the Euclidean space $\mathbb{R}^n$. Explicitly, consider $\varphi : U \subset \mathbb{R}^n \longrightarrow M$ is a local parametrization centered at the origin and such that $\varphi(0) = p \in M$. For each $u \in U$ the derivative $d\varphi_u$ is an isomorphism of $\mathbb{R}^n$ and the tangent space of M at $\varphi(u)$. We define the ***pullback vector field of*** V by

$$\varphi^* V(u) := d\varphi^{-1} V(\varphi(u)).$$

Now, if V has an isolated singularity at p, then $\varphi^* V$ has an isolated singularity at the origin and we define

$$I_V(p) := I_{\varphi^* V}(0).$$

If the reader is interestd in seeing why this definition does not depend on the choice of parametrization, we suggest the references [GP74, dC76]. Some examples of Poincaré indices of unit vector fields on the round 2-sphere were presented in Chapter 2.

5.2 Appendix B: The Euler characteristic

In this subsection we define the Euler characteristic of a smooth, compact and orientable manifold M and we state the Poincaré-Hopf theorem, which was demonstrated in dimension 2 by Henri Poincaré in 1885, and generalized to higher dimensions by Heinz Hopf in 1927, [Hop27].

Leonhard Euler was the first to consider the number $V - A + F$ associated to a given a 2-dimensional polyhedron $P \subset \mathbb{R}^3$, where V is its number of vertices of P, A its number of edges of P and F its number of faces. Thus defined,

$$\chi(P) := V - A + F \tag{5.1}$$

is called the ***Euler characteristic of the polyhedron*** P.

Definition 5.1. *A **finite n-dimensional polyhedron** P in $\mathbb{R}^n$ is a finite collection of n simplices in $\mathbb{R}^n$ such that:*

i) If S is a simplex of P, then every face of S is also a simplex;

ii) If S, and T are simplices of P, then the intersection $S \cap T$ is a common face to S and T, or is empty.

We denote by $|P|$ is the topological space given by the set P with its induced topology from the ambient space.

Definition 5.2. *Let P be a finite n-dimensional polyhedron and denote by n_i its number of i-dimensional simplices. The* ***Euler characteristic of the polyhedron*** *P is defined by*

$$\chi(P) := \sum_i (-1)^i n_i \tag{5.2}$$

Equation (5.2) is clearly a generalization of equation (5.1) for polyhedra of dimension greater than or equal to 2. Given a topological space X, under what conditions can we define its Euler characteristic? To answer this question, we need the following definition.

Definition 5.3. *Let X be a topological space, we say that X is triangularizable if there is a homeomorphism $h : |K| \longrightarrow X$, for some polyhedron K. In this case, (K, h) is said to be a triangulation of X.*

Not every topological space X is triangularizable; however, if X is a smooth manifold, a sufficient condition for X to admit a triangulation is to be compact.

Theorem 5.4. *Let M be a compact smooth manifold, then there is a triangulation (K, h) of M.*

In the case where M is a surface, that is, a smooth manifold of dimension 2, the above result was demonstrated by Whitehead in 1940 and can be found in [Whi40]. For the general case, see [BG05].

Given two different triangulations of the same topological space X, it is reasonable to wonder whether the Euler characteristics of the associated polyhedra are the same. The theorem below, due to Henri Poincaré, shows just that.

Theorem 5.5. *Let (K, h) and (K', h') be two triangulations of the same topological space X. Then, $\chi(K) = \chi(K')$.*

Definition 5.6. *Let X be a triangularizable topological space and fix (K, h) a triangulation of X. The* ***Euler characteristic*** *of X, $\chi(X)$, is defined as $\chi(K)$.*

Theorem 5.7. *Let X and Y be triangularizable topological spaces, if the space X is homeomorphic to the space Y, then $\chi(X) = \chi(Y)$.*

It follows from Theorem 5.7 that the Euler characteristic is a topological invariant.

Example 5.9. *It is known that there is a homeomorphism between the round sphere $\mathbb{S}^2$ and the cube $\partial([0, 1] \times [0, 1] \times [0, 1])$. Consider the triangulation of the cube schematically given in figure 5.10.*

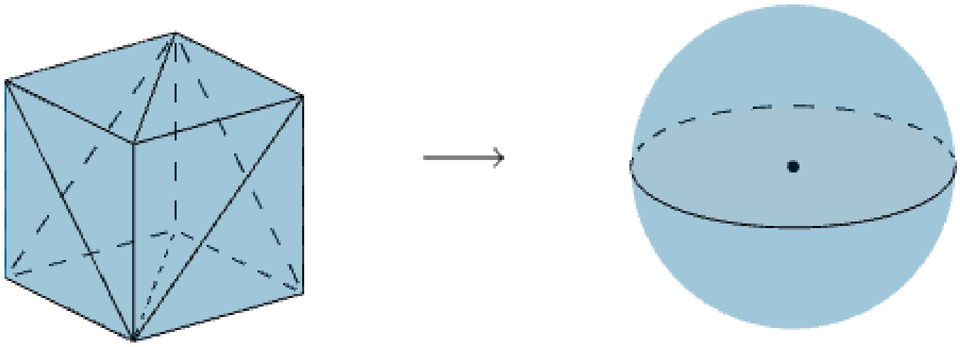

Figure 5.10: *Triangulation of round 2-sphere*

Then,

$$\chi(\mathbb{S}^2) = 8 - 18 + 12 = 2.$$

Example 5.10. *Consider the triangulation of the torus T obtained by identifying the segments of the same name in figure 5.11.*

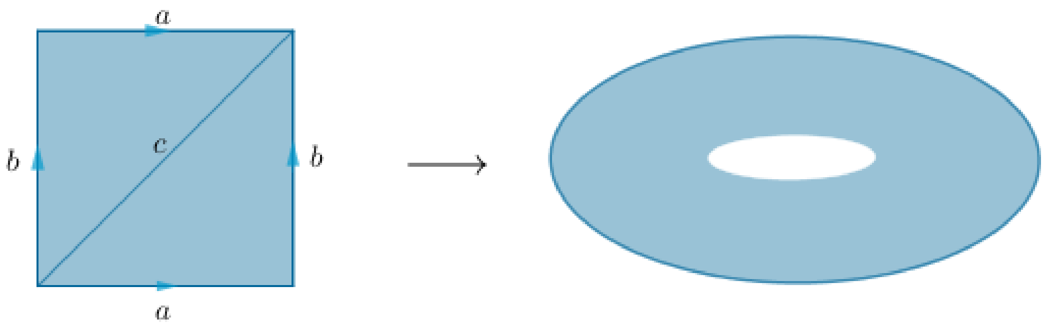

Figure 5.11: *Triangulation of torus T*

Then,

$$\chi(T) = 1 - 3 + 2 = 0.$$

Given M a compact oriented smooth manifold, its Euler characteristic $\chi(M)$ measures an *"obstruction"* for us to construct a continuous vector field on M without singularities. More precisely, we have that there is a continuous vector field V on M, without singularities if and only if $\chi(M) = 0$. This result is a consequence of the Poincaré-Hopf Theorem that we proceed to state.

Theorem 5.8 (Poincaré-Hopf)**.** *Let M be a compact orientable smooth manifold and let $V : M \longrightarrow TM$ be a vector field such that its singularities are isolated. Then the Euler characteristic of M equals the sum of the Poincaré indices of all the singularities of V, that* *is,*

$$\chi(M) = \sum_i I_V(p_i),$$

where $p_i \in M$ *are the isolated singularities of* V*.*

A proof of Poincaré-Hopf Theorem can be found in [GP74, p. 134]. Remarkably, note that the vector field V in Theorem 5.8 is arbitrary as long as its singularities are isolated.

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