# The Form of (Co)homology

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- Prof. Dr. Dmitry Logachev - ICE-UFAM

For you who gaze long into an abyss:

> Man sagt von Gott: „Namen nennen Dich nicht". Das gilt von Mir: kein Begriff drückt Mich aus, nichts, was man als mein Wesen angibt, erschöpft Mich; es sind nur Namen. Gleichfalls sagt man von Gott, er sei vollkommen und habe keinen Beruf, nach Vollkommenheit zu streben. Auch das gilt allein von Mir.
>
> Eigner bin Ich meiner Gewalt, und Ich bin es dann, wenn Ich Mich als Einzigen weiß. Im Einzigen kehrt selbst der Eigner in sein schöpferisches Nichts zurück, auf welchem er geboren wird. Jedes höhere Wesen über Mir, sei es Gott, sei es der Mensch, schwächt das Gefühl meiner Einzigkeit und erbleicht erst vor der Sonne dieses Bewußtseins. Stell Ich auf Mich, den Einzigen, meine Sache, dann steht sie auf dem Vergänglichen, dem sterblichen Schöpfer seiner, der sich selbst verzehrt, und Ich darf sagen:
>
> Ich hab mein Sach auf Nichts gestellt.
>
> —Max Stirner, Der Einzige und sein Eigenthum

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*[. . .] la pulsion de connaissance dans mon travail de mathématicien était de la même nature que la pulsion amoureuse. Les paroles et les images qui me venaient spontanément, voulant évoquer la pulsion de découverte dans son essence, étaient paroles et images de l'amour charnel que me soufflait Éros.*

—Alexander Grothendieck, *La Clef des Songes ou Dialogue avec le Bon Dieu*

Firstly and foremostly, I would like to express my gratitude to the ones that stoked my drives towards creation. I must, however, defer to the possibility that I might be able to finish this work (although, perhaps in an unsatisfactory form, I would claim) mainly or, even solely, due to the fear of not experiencing later such social interactions. Indeed, such fear of disrupting my inertia woes me since I've been.

However, now and henceforth, I must not fear the inertia nor the endless strive towards the non existent absolute incontingent truth for mathematics is not the ascetic *Wille zur Wahrheit* anymore (as it was unveiled incoherent by Stirner and, later, Nietszche): it's *Wille zur Kunst*. It's not about the absolute incontingent truth, the *Form of the Good*, *Ding an sich*, *absolute Geist* or God. It's about art. It's the pure manifestation of *Wille*, where *Trieb* and, in particular, *Wille zur Macht* can emanate almost freely, where Eros fend off Thanatos in the realm of ceasing and becoming. I, therefore, should not care about truth. For as long as my body shows deference to the old moral of *Wille zur Wahrheit*, my conniving mind will thrall the flesh in order to extract such moral pleasures. I, now, delight in the mundane trivialities and embrace the absurd.

I take, thereby, mathematics as an art and not as a science referring to objects with metaphysical content. Mathematics by itself as it is now is indeed lifeless and sinful as Brouwer concluded from Schopenhauer metaphysics of *Wille*: it's pure meaningless language; linguistic confusions travestied in incontingent truths; it's just a convention; an extrapolation of the finitistic mundane in order to engender a meaning. However, once one realises about the aesthetical value of human creativity, embracing the senses and the now or *Einzige* (which is the only incontingent truth that one can ever know), mathematics becomes colourful once again. It's certainly a lie. Nevertheless, as long as my body believes in it, there's no need to cease the masochism.

I'm, thereby, grateful to those who indirectly or directly contributed to such *Selbtsaufhebung* (which certainly won't be the last) by discussing mathematics passionately in my presence. If I have to cite names, I would include firstly Odilon O. Luciano, Hugo L. Mariano, Antônio P. Franco Filho and Dmitry Logachev.

I'm also indebted to my advisor, Ivan Shestakov for accepting the responsibility of advising me, including therein the several documents that were to be signed. The few informal conversations that we had were also pleasant.

Furthermore, the ones who frequented regularly or irregularly the few seminar attempts that I performed should also be recalled here. As my social interactions are minimal, such few moments were certainly memorable to me, even though they would be considered too mundane for any other regular social being.

Albeit the disagreements, my parents where also supportive. In any case, I certainly hold such disagreements and agreements dear to me as I would miss the merry and also the solemn moments equally.

Regarding the technicalities contained herein, I'm firstly grateful to the support of the MathOverflow community; in particular, the Homotopy Theory community therein and the Homotopy Theory chat room. Also, the nLab ([nLab]) should be mentioned as it was and is my main well of knowledge along my higher category theoretical life.

I should too show my gratitude to the conversations with Peter Arndt in his three visits to São Paulo, wherein I was able to annoy him with trivial questions.

Regarding further specfic technicalities contained herein, I thank the jury for the suggestions and Prof. Pierre Deligne for sending a scanned copy of his

letter to Soulé cited herein.

# Resumo

YAMAUTI, F. G. **The Form of (Co)homology**. 2019. xiv+249 f. Dissertação (Mestrado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2019.

Este trabalho é um panorama sobre algumas das principais idéias presentes na teoria dos motivos. Abordando via ponto de vista histórico, o autor convida o leitor a uma dialética em direção à Idéia que pervade as teorias da cohomologia presentes na geometria algébrica: o motivo. Percorrendo desde o sonho sobre cohomologias provindo das Conjecturas de Weil seguida pela imediata frustração das Conjecturas Standard ao jubilo de suprassumir até a teoria de $\mathbf{A}^1$-homotopia de Voevodsky, o autor infunde misticismo na epifania de Grothendieck sobre uma estrutura eterna e imanente engendrando todo o mundo (co)homológico.

**Palavras-chave**: Geometria Algébrica, Motivos, Homologia, Cohomologia, Teoria de Homotopia Motívica, Teoria de Homotopia.

# Abstract


YAMAUTI, F. G. **The Form of (Co)homology**. 2019. xiv+249 f. Dissertação (Mestrado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2019.

This work is a survey in some of the main ideas in the theory of motives. Approaching from an historical point of view, the author invites the reader into a dialectics towards the Idea that pervades cohomology theories in algebraic geometry: the motive. From the dream of those cohomologies arising from Weil's Conjectures and forthwith frustration by the Standard Conjectures to the elation of sublating to Voevodsky's $\mathbf{A}^1$-homotopy theory, the author infuses mysticism in Grothendieck's epiphany of an eternal immanent structure engendering all the (co)homological world.



**Keywords**: Algebraic geometry, Motives, Homology, Cohomology, Motivic Homotopy Theory, Homotopy Theory.

# Contents

**0 Introduction** **1**

**1 Preliminaries, Conventions and Notations** **5**

1.1 Some Remarks on $\infty$-Categories . . . . . . . . . . . . . . . . . . 9
1.2 Localisations and Presentability . . . . . . . . . . . . . . . . . . 10
1.3 Stable $\infty$-Categories . . . . . . . . . . . . . . . . . . . . . . . 17

## I The Cave 27

**2 The Weil's Conjectures** **29**

2.1 Rational Points . . . . . . . . . . . . . . . . . . . . . . . . . 29
2.2 Weil's Proposal . . . . . . . . . . . . . . . . . . . . . . . . . 31
2.3 The Riemann's Hypothesis . . . . . . . . . . . . . . . . . . . 34

## II The Shadows 43

**3 Étale** **45**

3.1 The Étale, fppf and fpqc Topologies . . . . . . . . . . . . . . . 50
3.2 Flat Descent and Hilbert 90 . . . . . . . . . . . . . . . . . . . 54
3.3 Galois Descent . . . . . . . . . . . . . . . . . . . . . . . . . 59
3.4 Étale Fundamental Group . . . . . . . . . . . . . . . . . . . . 63
3.5 Analytification . . . . . . . . . . . . . . . . . . . . . . . . . 73
3.6 Smooth and Proper Base Change . . . . . . . . . . . . . . . . . 76
3.7 Artin's Comparison Theorem . . . . . . . . . . . . . . . . . . 78

3.8 $\ell$-adic sheaves . . . . . . . . . . 79

**4 $K$-Theory** **85**
4.1 Quillen's $Q$ Construction . . . . . . . . . . 86
4.2 Waldhausen's $S$ Construction . . . . . . . . . . 105

**III Freedom and The Sun** **119**

**5 Pure Motives** **121**
5.1 Weil Cohomology Theory . . . . . . . . . . 123
5.2 The Category of Pure Motives . . . . . . . . . . 131
5.3 Grothendieck's Standard Conjectures . . . . . . . . . . 147
5.4 Motives over finite fields . . . . . . . . . . 151

**6 Mixed Motives** **155**
6.1 Beilinson and Lichtenbaum's Idea . . . . . . . . . . 156
6.2 Bloch's Higher Chow Groups . . . . . . . . . . 164
6.3 Suslin's Algebraic Singular Homology . . . . . . . . . . 174
6.4 Voevodsky's Idea . . . . . . . . . . 182

**7 Motivic Homotopy Theory** **193**
7.1 Nisnevich Topology . . . . . . . . . . 194
7.2 cd-Structures . . . . . . . . . . 199
7.3 Motivic Spaces . . . . . . . . . . 202
7.4 Motivic Spectra . . . . . . . . . . 214

**Bibliography** **229**

# Chapter 0

# Introduction

The title of this work refers to the concept of Form as defined in Plato's Theory of Forms. The notion of Form can be portrayed by the Allegory of the Cave. Briefly, imagine people live under the earth in a cavelike dwelling. There, since childhood, they were forced to gaze at a wall with necks and legs shackled such that they were only able to see what occurs in front of their faces. Behind them, it was settled a fire and a low wall in between, allowing them to only see the shadows of what pass just above the low wall. Just behind the low wall, hidden people manipulate puppets, casting the shadows of these puppets alone to the wall in front of the prisoners.

Unaware of the ulterior reality behind them, the prisoners have their reality constrained by the shadows, the unique epistemic tool in hand.

In Plato's Theory of Forms, universal objects (the Forms) exist in an abstract world. The shadows are the instances that one perceive by the senses. These senses, according to Plato, detract the rational being from apprehending the knowledge about the Forms. The Sun outside of the cave, which forms the means to know the reality beyond the cave, is the absolute Form named by Plato as the Form of the Good.

The setting of this work follows the analogy by identifying the shadows with the instantiations of the universal (co)homological objects, technically called realisations. The universal substances that supervene upon all these (co)homological instances were called motives by Grothendieck in reference to the intention of creation, the *motif*, which is subsequently conceived or actualised by a creator without losing its essence or attributes.

The Sun, in this case, can be argued to be abstract homotopy theory or, rather, the homotopy nature that permeates the motivic world.

Fortunately, here one don't need to understand all the realisations in order to obtain properties and, in some restricted cases, one can even construct explicitly the category of motives. Thus, destroying the senses by committing suicide is not necessary here in order to know a universal object. Indeed, sometimes one can even conceive a motive as the extension of its instances.

The theory of motives originally arose from the need to construct cohomology theories for smooth projective varieties satisfying a topological property, the Lefschetz fixed point formula. These were required in order to solve the Weil Conjectures, which stated relations between $X(\mathbf{F}_q)$ for $q = p^n$ and the topology of $X(\mathbf{C})$.

Such Weil cohomology theories, despite generally being strictly algebraic, carried topological information contained in the complex points. Even more impressively, the coefficients used (*e.g.* , $\mathbf{Q}_\ell$ for each $\ell \neq p$) seemed to not matter whatsoever. Such indifference, culminated in several comparison theorems between each of these cohomologies.

Using intersection theory, Grothendieck himself was able to conjecturally construct the smallest category containing all the the characteristics shared by those cohomologies after being evaluated by each $X$: the category of pure motives. However, the universality of such construction depended intrinsically on conjectures which are, still nowadays, far from being proved.

After several years of development in algebraic $K$-theory and the theory of perverse sheaves, the field bourgeoned again through Beilinson insight of using the $K$-theory and the Atiyah spectral sequence converging to singular (co)homology. That idea of viewing motivic (co)homology as some sort of algebraic singular (co)homology was old, but soon abandoned due to the lack of a good Weil cohomology theory with cofficients in $\mathbf{Q}$ as noticed by Serre.

Finally, due to Voevodsky's Grothendieckian puerile rebellion of using the simple and general in order to elegantly solve the particular, motivic homotopy theory was conceived. That theory canonically dealt with algebraic objects by enhancing the theory of schemes with higher geometric information and applying the obvious topological construction to such objects. Such simplicity overseen for decades was admired even by Deligne who described his astonishment:"When I first saw the basic definitions in motivic cohomology I thought, 'This is much too naïve to possibly work'.", "I was wrong, and Voevodsky, starting from those 'naïve ideas, has given us extremely powerful tools..

The work contained herein was initially intended to be a survey passing through all the main (co)homology theories in algebraic geometry. The author is extremely unsatisfied with how far such aim was achieved. Due to enormous gaps in certain chapters, several of them have been omitted, including some sections of the chapters herein contained. If, anyone, by any chance, is reading or intend to read this survey, there's some chance that a better version is available at the moment and, therefore, asking the author for an updated version might be a better choice [1].

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# Chapter 1

# Preliminaries, Conventions and Notations

This chapter is entirely optional to the one knowledgeable about the basics of algebraic geometry. The less knowledgeable but cunning reader will soon realise that this chapter must be used as an appendix despite its location.

This chapter covers some prerequisites regarding kinds of morphisms. It, although insufficiently for the newcomer, recalls about some definitions in the theory of schemes in order the settle down some notations.

First and foremost, everything carried in this work can be accomplished in the formalisms of Grothendieck-Zermelo Universes. Being enough to fix at least two universes $\mathscr{U} \in \mathscr{V}$ where $\omega \in \mathscr{U}$ in order to give soundness to the definition of locally small, not necessarily small categories and the free cocompletion. One, however, can assume less in exchange for some properties usually not present in the algebro-geometric world by iterating Fefferman's trick countably many times [FK69].

Recall that Fefferman's trick consists in defining a conservative extension $\mathbf{ZFC}/S$ of $\mathbf{ZFC}$ by adding a constant symbol $S$ such that $S$ is transitive and supertransitive; withal, $S$ is required to reflect all formulas $\phi$ of the language $L(\mathbf{ZF})$, *i.e.* ,

$$\forall s_1 \in S \forall s_2 \in S \dots \forall s_n \in S(\phi(s_1, s_2, \dots, s_n) \longleftrightarrow \phi^S(s_1, s_2, \dots, s_n))$$

for every $\phi \in L(\mathbf{ZF})$. The possible discrepancies from

$$\mathbf{ZFC} + \text{"}\kappa \text{ strongly innacesssible"}$$

lies in the nonexistence of replacement for arbitrary functional relations $f : M \to M$ (not necessarily definable) and the fact that **ZFC** cannot possibly prove that $V_\kappa$ is a model thereof. Such restrictions are, however, extremely unlikely to happen in the algebro-geometric setting. Indeed, the totality of the algebro-geometric concepts conceived in the SGA's and EGA's can be formalised through Finite Order Arithmetic ([McL19]) [1].

In any case, by working with acessible functors and locally presentable categories, one can preserve all the usual constructions (Kan extensions, adjoints and limits) without the dependence on the universe as a parameter [Low14]. As most constructions can usually be carried in combinatorial model categories, the previous result applies. If one still unsafe, an overkill solution would be to require a linear diagram (indexed by a proper class) of elementary embeddings between universes. Such assumption is stronger than the Universe Axiom and strictly weaker than the existence of a Mahlo cardinals [htt].

Along this work, $X$, $Y$, $Z$, ... will be used to denote spaces (topological spaces, manifolds, schemes, varieties etc). $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$, ... liftings (or models) of schemes viewed as special fibers. $S$, $T$ will usualy denote a base space. $X/S$ will denote a space over a base space $S$ (*i.e.* , a morphism $f_S : X \to S$).

The letters $A$, $B$, $C$, ... will be usually used to denote rings, which will be assumed to be commutative with unity $1 \neq 0$ unless mentioned otherwise. When the base space is $\mathrm{Spec}(A)$, $X/\mathrm{Spec}(A)$ will be abbreviated to $X/A$. Pullback (or base-change) of $X/A$ along $\mathrm{Spec}(f)$ for $f : A \longrightarrow B$ an $A$-algebra will often be denoted by $X \otimes_A B$, $f^*X$ or $X \otimes_f A$ instead of $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ or $\mathrm{Spec}(f)^*X$. $X(S)$ will denote the $S$-points of $X$. $X(A)$ will be used instead of $X(\mathrm{Spec}(A))$. For $f : X \longrightarrow Y$, $f^\# : \mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X$ might denote the respective morphism of sheaves whenever it's clear that it cannot be the motivic functor $f^\#$ for smooth $f$. $k$ will usually denote a base field, while $F$ (*resp.*, $\Lambda$) will usually denote the field (*resp.*, ring) of coefficients and $K$ will denote a local or

[1] However, anything beyond the Zariski site and coherent cohomology on Noetherian schemes will demand at least Third Order Arithmetic ([McL12])

global field such that $k$ is a closed point when $\mathrm{char}(k) \neq 0$. $R(X)$ will denote the field of rational functions for $X$ integral

$\mathscr{C}, \mathscr{D}, \mathscr{E}, \ldots$ will denote categories. $\mathscr{C}_{/A}$ (*resp.*, ${}_{A\backslash}\mathscr{C}$) will denote the slice categories (*resp.*, coslice categories). $\widehat{\mathscr{C}}$ will denote the free cocompletion of $\mathscr{C}$.

A variety $X/k$ will be simply separated $k$-scheme of finite type (being irreducible nor reduced is required).

The notation for morphisms of sites will follow the geometric notation used in [SGA4-I], which reverses the order. More precisely, a morphism of sites $f : \mathscr{C} \longrightarrow \mathscr{D}$ will consist in a morphism $f^{-1} : \mathscr{D} \longrightarrow \mathscr{C}$ of categories and, hence, induce a geometric morphism $f_* : \mathbf{Sh}(\mathscr{C}) \longrightarrow \mathbf{Sh}(\mathscr{D})$ in accordance with the case of topological spaces. [2]

**Definition 1.0.1.** Let $\mathscr{C}$ be a category and $U \in \mathrm{Ob}(\mathscr{C})$. A **sieve** $S$ is a subfunctor

$$S \hookrightarrow h_U$$

such that, for every $V, W \in \mathrm{Ob}(\mathscr{C})$ and $W \xrightarrow{f} V$, $S(V)$ contains the composite $W \xrightarrow{f} V \to U$.

**Definition 1.0.2.** Let $\mathscr{C}$ be a category. A **Grothendieck topology on** $\mathscr{C}$ is a functor

$$J : \mathscr{C}^{op} \to \mathbf{Cat}$$

that associates to every $U \in \mathrm{Ob}(\mathscr{C})$ a collection of sieves $J(U) \in \mathrm{Sub}_{\widehat{\mathscr{C}}}(h_U)$ and for every morphism $U \xrightarrow{f} V$ the functor

$$f^* : \mathrm{Sub}_{\widehat{\mathscr{C}}}(h_U) \longrightarrow \mathrm{Sub}_{\widehat{\mathscr{C}}}(h_U)$$

given by $f^* = Y(f)^*$. Furthermore, $J$ must satisfy the following conditions.

1. $h_U \xrightarrow{1_{h_U}} h_U \in J(U)$ ;
2. If $S \in J(U)$, $R \hookrightarrow h_U$ and, for every $f : V \to U \in R$, $f^*R \in J(V)$, then $R \in J(U)$.

[2] If $f : X \longrightarrow Y$ is a morphism of spaces, the notation $f_* : \mathbf{Sh}(X) \longrightarrow \mathbf{Sh}(Y)$ is used

A **covering sieve** $S$ is a sieve such that there exists $U \in \mathrm{Ob}(\mathcal{C})$ and $S \in J(U)$.

**Definition 1.0.3.** A **site** $(\mathcal{C}, J)$ is an ordered pair composed by a category $\mathcal{C}$ and a Grothendieck topology $J$.

**Definition 1.0.4.** Let $(\mathcal{C}, J)$ be a site. A $J$ **sheaf** $F$ is a presheaf $F \in \widehat{\mathcal{C}}$ such that for every $U \in \mathrm{Ob}(\mathcal{C})$ and covering sieve $S \in J(U)$ the diagram

$$\begin{array}{ccc} S & \longrightarrow & F \\ \big\downarrow & \nearrow & \\ h_U & & \end{array}$$

has a unique lifting $h_U \longrightarrow F$. If the lifting is not unique $F$ is called a **separated presheaf**.

The category of sheaves on $(\mathcal{C}, J)$ is denoted by $\mathbf{Sh}(\mathcal{C}, J)$.

**Proposition 1.0.1.** *Let $(\mathcal{C}, J)$ be a site and $F \in \widehat{\mathcal{C}}$. If $\mathcal{C}$ have pullbacks, then $F$ is a sheaf (resp., separated presheaf) iff for every $U \in \mathrm{Ob}(\mathcal{C})$ and covering sieve $S = \{U_i \to U\}_i \in J(U)$ the diagram*

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

*is an equaliser (resp., $F(U) \longrightarrow \prod_i F(U_i)$ is a monomorphism).*

**Definition 1.0.5.** Let $S$ be a scheme and $J$ a Grothendieck topology on $\mathbf{Sch}_{/S}$. The **petit $J$ site over $S$ (*resp.*., gros $J$ site over $S$)** is defined as the subcategory $S_J \hookrightarrow \mathbf{Sch}_{/S}$ such that $\mathrm{Ob}(S_J) := \{f \in \mathbf{Sch}(X, S) | \, (f \in J(X)) \wedge (X \in \mathrm{Ob}(\mathbf{Sch}))\}$ with the induced topology (*resp.*., as the site $(\mathbf{Sch}_{/S}, J)$).

Let $J_\tau$ be a Grothendieck topology on $\mathbf{Sch}_{/S}$ where $\tau$ is some kind of covering or morphism. The notation $S_\tau$ (*resp.*, $(\mathbf{Sch}_{/S})_{tau}$) will be used for the petit $J_\tau$ site (*resp.*. gros $J_\tau$ site).

*Remark* 1.0.1. The above definition implicitly assumes that $J_\tau$ satisfies the axioms of a Grothendieck topology. Furthermore, the definition of the petit site for general $\tau$ except $\tau = \mathrm{Zar}, \text{ét}, \mathrm{Nis}$ is ill behaved as there's no guarantee that the morphisms in $S_\tau$ will be of type $\tau$. Furthermore, there's also no guarantee that $S_\tau$ is closed under pullbacks (*e.g.* , $S_{\mathrm{fppf}}$ is not closed under pullbacks.

## § 1.1 Some Remarks on ∞-Categories

The language of ∞-categories will be freely used mainly in the last chapters. The reader unfamiliar with the basic definitions in higher category theory is advised to consult [HTT], [Cis19], [nLab] or the background reading in [Sch]. For the basics in stable higher category theory and higher algebra, the reader may consult [HA], [nLab] or [Sch]. Some of the definitions and main theorems will be recalled in the following sections of this chapter in order to set the main notations.

As usual, the name "∞-category" will mean $(\infty,1)-$category. A choice of a model for ∞-categories is usually not required as the space of $(\infty, n)$-categories is of the form $B(\mathbf{Z}/2)^n$. However, for convenience, an ∞-category will be a quasi-category (see, for instance, [HTT]).

Let $\mathscr{M}$ be a (simplicial) model category. $\mathscr{M}^{\mathrm{fib}}$ will denote the (simplicial) category of fibrant objects. $\mathscr{M}^{\mathrm{cof}}$ will denote the (simplicial) category of cofibrant objects. $\mathscr{M}^{\circ}$ will denote the (simplicial) category of objects which are both fibrant and cofibrant. $L^H\mathscr{M}$ will denote the simplicial localisation (also called Dwyer-Kan localisation or hammock localisation).

The following facts being recalled in this paragraph follows from [DK80c], [DK80a], [DK80b] and [Maz16]. Recall that every ∞-category may be presented by a category with weak equivalences. Recall also that model categories (*resp.*, simplicial model categories) present a class of ∞-categories by the coherent nerve of the simplicial localisation (*resp.*, the coherent nerve of the simplicial localisation of the underlying model category or, equivalently, the coherent nerve of the simplicial subcategory of fibrant-cofibrant objects). A Quillen adjunction (*resp.*, simplicial Quillen adjunction) between model categories (*resp.*, simplicial model categories) induces an adjunction between the underlying ∞-categories. The analogous assertion for (simplicial) Quillen equivalences is also true. For such reason, the language of (simplicial) model categories will be used interchangeably with the machinery of ∞-categories.

Whenever convenient, the prefix ∞ will be dropped when the context in clearly ∞-categorical. For instance, a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ will be simply called cocomplete or declared to preserve colimits instead of being called $\infty$-cocomplete or declared to preserve $\infty$-colimits.

Let $X, Y \in \mathrm{Ob}(\widehat{\Delta})$. $X \star Y$ denotes the join, $X^{\triangleleft}$ (*resp.*. , $X^{\triangleright}$) denotes the left (*resp.*, right) cone.

$\mathbf{Grpd}_\infty$ will denote the $\infty$-category of spaces (*i.e.* , the $\infty$-category of Kan complexes or, equivalently the simplicial localisation of any category of topological spaces [3] with respect to the Quillen model structure). For every, $\infty$-category, $\mathbf{PSh}_\infty(\mathscr{C}) := \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathbf{Grpd}_\infty)$. For every abelian category $\mathscr{A}$, $D(\mathscr{A})$ (*resp.*, $D^b(\mathscr{A})$, $D^+(\mathscr{A})$, *resp.*$D^-(\mathscr{A})$, *resp.*$D^b(\mathscr{A})$ will denote the (*resp.*, bounded , *resp.*, bounded below , *resp.*, bounded above) derived category of $\mathscr{A}$ as a linear stable $\infty$-category obtained from the DG-nerve of the underlying DG-category.

## § 1.2 Localisations and Presentability

Here it's recalled some concepts in higher category theory which will be used mainly in chapter. The majority of this section follows closely [HTT] and [nLab].

**Definition 1.2.1.** Let $\mathscr{C}$ be an $\infty$-category and $W \subset \mathscr{C}_1$ a class of morphisms. A **localisation of $\mathscr{C}$ on $W$** consists in an $\infty$-category $\mathscr{C}[W^{-1}]$ and a functor

$$L : \mathscr{C} \longrightarrow \mathscr{C}[W^{-1}]$$

which is initial with respect to functors $f : \mathscr{C} \longrightarrow \mathscr{D}$ such that $f(W)$ consists of equivalences.

When $L : \mathscr{C} \longrightarrow \mathscr{C}[W^{-1}]$ is a localisation, an abuse of notation will be committed by calling $\mathscr{C}[W^{-1}]$ a **localisation of $\mathscr{C}$ on $W$**.

**Theorem 1.2.1.** *Let $\mathscr{C}$ be a small $\infty$-category and $W \subset \mathscr{C}_1$ a set of morphisms. There exists a unique (up to equivalence) localisation*

$$L : \mathscr{C} \longrightarrow \mathscr{C}[W^{-1}].$$

[3] Topological spaces, compactly generated topological spaces, CW complexes etc

*Proof.* This follows from [Cis19, Prop. 7.1.3] and [Cis19, Rem. 7.1.5]. See also [HA, Cons. 4.1.7.1] and [HA, Prop. 4.1.7.2]. □

*Remark* 1.2.1. Notice that even when $\mathcal{C}$ is not small and $W \subset \mathcal{C}_1$ is a class, the localisation $\mathcal{C}[W^{-1}]$ exists in a higher universe.

It's not true, however, that one can choose $\mathcal{C}[W^{-1}]$ to be locally $\mathcal{U}$-small whenever $\mathcal{C}$ is $\mathcal{U}$-small.

**Definition 1.2.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\infty$-categories. A functor $L : \mathcal{C} \longrightarrow \mathcal{D}$ is a **reflective localisation** if there exists a fully faithful functor $i : \mathcal{D} \hookrightarrow \mathcal{C}$ and an adjunction.

$$\mathcal{D} \underset{i}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{C}.$$

When $L : \mathcal{C} \longrightarrow \mathcal{D}$ is a reflective localisation, an abuse of notation will be committed by calling $\mathcal{D}$ a **reflective localisation of** $\mathcal{C}$.

*Remark* 1.2.2. In [HTT, Def. 5.2.7.2], the name "localisation" is given instead of "reflective localisation".

*Remark* 1.2.3. Notice that a reflective localisation is always a localisation on the class $S_L := \{f \in \mathcal{C}_1 | \ f \text{ is an equivalence}\}$ by [HTT, Prop. 5.2.7.12]

*Remark* 1.2.4. Notice that in the above definition it's unnecessary to assume that $i$ fully faithful by [Cis19, Prop. 7.1.17].

**Definition 1.2.3.** Let $\kappa$ be a regular cardinal. An $\infty$-category $\mathcal{J}$ is a **$\kappa$-filtered $\infty$-category** (*resp.*, **$\kappa$-sifted $\infty$-category**) if every colimits of diagrams

$$\mathcal{J} \longrightarrow \mathbf{Grpd}_\infty$$

commutes with $\kappa$-small limits (*resp.*, $\kappa$-small products).

When $\kappa = \aleph_0$, $\mathcal{J}$ is simply called a **filtered $\infty$-category** (*resp.*, **sifted $\infty$-category**).

*Remark* 1.2.5. The above definition coincides with the usual ones (respectively, [HTT, Def. 5.3.1.7] and [HTT, Def. 5.5.8.1]) by, respectively, [HTT, Prop 5.3.3.3] and [HTT, Lem. 5.5.8.11].

**Definition 1.2.4.** Let $\kappa$ be a regular cardinal. An $\infty$-category $\mathcal{C}$ is a **$\kappa$-accessible $\infty$-category** if there exists a small $\infty$-category $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ such that

$$\mathcal{C} \cong \mathrm{Ind}_\kappa(\mathcal{C}_0)$$

, where $\mathrm{Ind}_\kappa(\mathcal{D})$ denotes the full $\infty$-subcategory of $\mathbf{PSh}_\infty(\mathcal{D})$ generated by $\kappa$-filtered colimits of representable presheaves. $\mathcal{C}$ will be simply called an **accessible $\infty$-category** if $\mathcal{C}$ is $\kappa$-accessible for some regular cardinal $\kappa$. If $\kappa = \aleph_0$, $\mathcal{C}$ will be called a **finitely accessible $\infty$-category** .

$\mathcal{C}$ is a **locally $\kappa$-presentable $\infty$-category** (or **$\kappa$-compactly generated $\infty$-category**) if, furthermore, it's cocomplete. Similarly, $\mathcal{C}$ will be simply called an **locally presentable $\infty$-category** if $\mathcal{C}$ is $\kappa$-accessible for some regular cardinal $\kappa$ and, If $\kappa = \aleph_0$, $\mathcal{C}$ will be called a **locally finitely presentable $\infty$-category** (or **compactly generated $\infty$-category**).

Let $\mathcal{D}$ be an $\infty$-category. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an **$\kappa$-accessible functor** if $\mathcal{C}$ and $\mathcal{D}$ are $\kappa$-accessible and $F$ commutes with $\kappa$-filtered colimits.

A $L : \mathcal{C} \longrightarrow \mathcal{D}$ a reflective localisation a **$\kappa$-accessible localisation** if the right adjoint $i$ is $\kappa$-accessible. Similarly, $L$ will be simply called a **accessible localisation** if $L$ is $\kappa$-accessible for some regular cardinal $\kappa$.

*Remark* 1.2.6. As in the 1-categorical case, for every set of regular cardinals $R$, there exists a regular cardinal $\lambda_R > \kappa$ such that every, for every $\kappa \in R$, $\kappa$-accessible $\infty$-category is also $\lambda_R$-acessible. Again, as in the 1-categorical case, every locally $\kappa$-presentable category is also $\lambda$-accessible for every regular cardinal $\lambda > \kappa$.

*Remark* 1.2.7. In [HTT, Def. 5.5.0.1], the name "presentable" is given instead of "locally presentable".

**Theorem 1.2.2** ([HTT, Prop. 5.4.2.2]). *Let $\mathcal{C}$ be an $\infty$-category and $\kappa$ a regular cardinal. The following are equivalent:*

*(i) $\mathcal{C}$ is $\kappa$-accessible;*

*(ii) $\mathcal{C}$ is locally small, has all $\kappa$-filtered colimits, the $\infty$-subcategory of $\mathcal{C}_\kappa \hookrightarrow \mathcal{C}$ of $\kappa$-compact objects is essentially small and generates $\mathcal{C}$ under $\kappa$-filtered colimits;*

*(iii)* $\mathscr{C}$ *has all* $\kappa$*-filtered colimits and is generated under* $\kappa$*-filtered colimits by an essentially small* $\infty$*-subcategory* $\mathscr{C}_0 \hookrightarrow \mathscr{C}$ *consisting of* $\kappa$*-compact objects* [4]*.*

**Theorem 1.2.3.** *Let* $\mathscr{C}$ *be an* $\infty$*-category. The following are equivalent*

*(i)* $\mathscr{C}$ *is locally presentable;*

*(ii) There exists a regular cardinal* $\kappa$ *such that* $\mathscr{C}$ *is* $\kappa$*-accessible and the subcategory of* $\kappa$*-compact objects has all* $\kappa$*-small colimits;*

*(iii) There exists a small* $\infty$*-category such that* $\mathscr{C}$ *is an accessible localisation of* $\mathbf{PSh}_{\infty}(\mathscr{D})$.

*(iv)* $\mathscr{C}$ *can be presented by a left proper combinatorial simplicial model category;*

*(v)* $\mathscr{C}$ *can be presented by a simplicial combinatorial model category;*

*(vi)* $\mathscr{C}$ *can be presented by a combinatorial model category.*

*Proof.* The first four assertion are covered by [HTT, Thm. 5.5.1.1] and [HTT, Prop. A.3.7.6] . The last assertions follows from Dugger's theorem ([Dug01, Thm. 1.1]), which states that every combinatorial model category is Quillen equivalent to a left Bousfield localisation of a presheaf category with the projective global model structure. In particular, since every left Bousfield localisation of a left proper combinatorial simplicial model category is also a left proper combinatorial simplicial model category ([HTT, Prop. A.3.7.3]), every combinatorial model category admits a Quillen equivalent left proper combinatorial model category which makes it a simplicial model category ([Dug01, Cor. 1.2]). Since the simplicial localisation of simplicial model category is equivalent to the simplicial localisation of its underlying model category and Quillen equivalent model categories have equivalent simplicial localisations, the result follows. □

Analogously to the 1-categorical case, there's a version of Adjoint Functor Theorem.

[4] Recall that an object $X \in \mathrm{Ob}(\mathscr{C})$ is $\kappa$-compact if the corepresentable functor $\mathscr{C}(X, -)$ commutes with $\kappa$-filtered colimits.

**Theorem 1.2.4** (Adjoint Functor Theorem, [HTT, Thm. 5.5.2.9]). *Let $\mathscr{C}$ and $\mathscr{D}$ be locally presentable $\infty$-categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor.*

*(i) $F$ is a left adjoint iff it commutes with small colimits.*

*(ii) $F$ is a right adjoint iff it's accessible and commutes with small limits*

Motivated by the Adjoint Functor Theorem, the following notation will be introduced.

**Definition 1.2.5.** $\mathscr{P}r^L$ (*resp.*, $\mathscr{P}r^R$) will denote the $\infty$-category of locally presentable $\infty$-categories and colimiit preserving functors (*resp.*, the category of accessible limit preserving functors).

*Remark* 1.2.8. By the Adjoint Functor Theorem, $\mathscr{P}r^L \cong (\mathscr{P}r^R)^{\mathrm{op}}$. Therefore, colimits in $\mathscr{P}r^L$ (*resp.*, $\mathscr{P}r^R$) are computed by limits in $\mathscr{P}r^R$ (*resp.*, $\mathscr{P}r^L$).

**Proposition 1.2.1.** *The inclusions $\mathscr{P}r^L \hookrightarrow \mathbf{Cat}_\infty$ and $\mathscr{P}r^R \hookrightarrow \mathbf{Cat}_\infty$ creates small limits. In particular, $\mathscr{P}r^L$ and $\mathscr{P}r^R$ are cocomplete.*

*Proof.* This follows from [HTT, Prop. 5.5.3.13] and [HTT, Thm. 5.5.3.18]. □

**Theorem 1.2.5.** *$\mathscr{P}r^L$ admits a unique closed monoidal structure such that for every $\mathscr{C},\mathscr{D} \in \mathrm{Ob}(\mathscr{P}r^L)$ there exists $\mathscr{C} \otimes \mathscr{D} \in \mathrm{Ob}(\mathscr{P}r^L)$ and functor*

$$\mathscr{C} \times \mathscr{D} \longrightarrow \mathscr{C} \otimes \mathscr{D}$$

*which commutes with small colimits in each argument and induces an isomorphism*

$$\mathscr{P}r^L(\mathscr{C} \otimes \mathscr{D}, \mathscr{E}) \xrightarrow{\sim} \mathscr{P}r^L(\mathscr{C} \times \mathscr{D}, \mathscr{E})$$

*Furthermore, this symmetric monoidal structure*

*Proof.* □

**Definition 1.2.6.** Let $\mathscr{C}$ be an $\infty$-category and $S \subset \mathscr{C}_1$. An object $X \in \mathrm{Ob}(\mathscr{C})$ is $S$-local if

$$\mathscr{C}(Z,X) \xrightarrow[\sim]{g^*} \mathscr{C}(Y,X)$$

is a (weak) equivalence for every $g : Y \longrightarrow Z \in S$.

Conversely, $f \in \mathscr{C}_1$ is a $S$-equivalence if $\mathscr{C}(f, W)$ is a (weak) equivalence for every $S$-local $W$.

*Example* 1.2.1. Let $L : \mathscr{C} \longrightarrow \mathscr{D}$ be a localisation with $L \dashv i$ and $S_L := \{f \in \mathscr{C}_1 | \ f \text{ is an equivalence}\}$. The $S_L$-local objects are exactly the ones in the essential image of $i$ and the class of $S_L$-equivalences is exactly $S_L$.

By abstracting the properties of the class $S_L$ in the above example, one can conceive the following definition.

**Definition 1.2.7.** Let $\mathscr{C}$ be a cocomplete $\infty$-category. A class $S \subset \mathscr{C}_1$ is a **strongly saturated class** if it's closed under pushouts along arbitrary morphisms, has all small colimits and satisfies the 2-out-of-3 property. Explicitly, it satisfies

(i) For every $f : X \longrightarrow Y \in S$, $g : X \longrightarrow X' \in \mathscr{C}(X, X')$ and a pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' \end{array}$$

, $f' \in S$;

(ii) $S \hookrightarrow \mathscr{C}_1 \cong \mathrm{Fun}(\Delta^1, \mathscr{C})$ as a subcategory is closed under small colimits;

(iii) For every (homotopy) commutative triangle

$$\begin{array}{ccccc} & & Y & & \\ & {}^{f}\nearrow & & \searrow^{g} & \\ X & & \xrightarrow{h} & & Z \end{array}$$

in $\mathscr{C}$, whenever two elements of $\{f, gh\}$ belong to $S$, $\{f, g, h\} \subset S$.

The smallest strongly saturated class generated by $S$ will be called the **strongly saturated class generated by** $S$ and it will be denoted by $\overline{S}$.

*Remark* 1.2.9. Notice that $\overline{S}$ always exists since arbitrary intersections of strongly saturated classes are strongly saturated

*Remark* 1.2.10. A strongly saturated class $S$, albeit a stronger concept, satisifies the analogous axioms required in the definition of a saturated class (also called weakly saturated in [HTT]) in the theory of cofibrantly generated model categories (*i.e.* , closure under pushouts along arbitrary morphisms, transfinite compositions and retracts)

**Proposition 1.2.2.** *Let $\mathscr{C}$ be a cocomplete $\infty$-category and $S_0 \subset \mathscr{C}_1$. The class of $S_0$-equivalences is strongly saturated.*

*In particular, if there exists a reflective localisation $L : \mathscr{C} \longrightarrow \mathscr{D}$, the class $S_L$ is strongly saturated.*

*Proof.* Let $S_X := \{f \in \mathscr{C}_1 | \mathscr{C}(f, X) \text{ is an equivalence}\}$. Since $\mathscr{C}(-, X) : \mathscr{C} \longrightarrow \mathbf{Grpd}_\infty^{\mathrm{op}}$ preserves colimits, the class of all equivalences in any $\infty$-category is strongly saturated and the inverse image of a strongly saturated class along a colimit preserving functor is also strongly saturated, $S_X$ is strongly saturated. Since being strongly saturated is closed under intersections, the class of $S_0$-equivalences $\bigcap_{X \ S_0\text{-local}} S_X$ is strongly saturated.

The particular case follows by noticing that $S_L$ is already the class of $S_L$-equivalences.

Alternatively, the particular case follows by noticing that $S_L$ is the inverse image of all the equivalences along $L$ and $L$ preserves colimits. □

As stated in Proposition 1.2.2, every reflective localisation engenders a strongly saturated class. For the locally presentable case, one can assert the converse: a strongly saturated class $S$ engenders a localisation whenever $\mathscr{C}$ is locally presentable and $S$ is generated by a set. Indeed, the following states that there's a correspondence between strongly saturated classes generated by a set and reflective localisations.

**Theorem 1.2.6.** *Let $\mathscr{C}$ be a locally presentable $\infty$-category. If $S \subset \mathscr{C}_1$ be an essentially small class (i.e. , $S$ is essentially small as a subcategory of* $\mathrm{Fun}(\Delta^1, \mathscr{C})$*) and $\mathscr{C}_S$ denotes the subcategory of $S$-local objects, then the inclusion $\mathscr{C}_S \xhookrightarrow{i} \mathscr{C}$ has a left adjoint $L$ and defines an accessible localisation*

$$\mathscr{C}_S \underset{i}{\overset{L}{\underset{\perp}{\leftrightarrows}}} \mathscr{C}.$$

*In particular,* $\mathscr{C}_S \cong \mathscr{C}[S^{-1}] \cong \mathscr{C}[S_L^{-1}]$, $S_L$ *consists of* $S$*-equivalences and* $\mathscr{C}_S$ *is locally presentable. Furthermore,* $S_L = \overline{S}$.

*Conversely, for every accessible localisation* $L\colon \mathscr{C} \longrightarrow \mathscr{D}$ *with right adjoint* $i\colon \mathscr{D} \longrightarrow \mathscr{C}$, *there exists a set* $S \subset S_L \subset \mathscr{C}_1$ *such that* $i$ *is equivalent to the inclusion the inclusion* $\mathscr{C}_S \xrightarrow{\ i\ } \mathscr{C}.$ *and* $S_L = \overline{S}$

*In particular given sets* $S, T \hookrightarrow \mathscr{C}_1$, $\mathscr{C}[S^{-1}] \cong \mathscr{C}[T^{-1}]$ *iff* $\overline{S} = \overline{T}$.

*Moreover, under the equivalence between locally presentable categories by* $N((-)^\circ)$, *left Bousfield localisations of the underlying left proper combinatorial simplicial model categories are equivalent to accessible localisations of locally presentable ∞-categories* [5]

*Proof.* The first assertions follows from [HTT, Prop. 5.5.4.15]. For the converse, notice that the equivalences in $\mathscr{D}$ form a strongly saturated class generated by a set and, then, the subcategory of Fun($\Delta^1$,$\mathscr{C}$) generated by $S_L$ is locally presentable. Hence, by [HTT, Prop. 5.5.4.16], the pullback of the equivalences, which is exactly $S_L$ is also a strongly saturated class generated by a set.

By [HTT, Lem 5.5.4.14], there exists a set $S \subset \mathscr{C}_1$ such that $S_L = \overline{S}$.

The equivalence between accessible localisations and left Bousfield localisations follows from [HTT, Prop. A.3.7.8] □

## § 1.3 Stable ∞-Categories

Here it's recalled some concepts in the theory of stable ∞-categories which will be used mainly in the last chapter.

Stable ∞-categories are, on one side, the analogous of abelian categories in the ∞-categorical context. On the other side, they generalise the definition of spectra in algebraic topology, which, once restricted to connective objects, are

[5]Explicitly, if $\mathscr{C} \cong N(\mathscr{M}^\circ)$ for some left proper combinatorial simplicial model category $\mathscr{M}$ and $L : \mathscr{C} \longrightarrow \mathscr{D}$ is an accessible localisation, then there exists a left proper combinatorial simplicial model category $\mathscr{N}$ such that $L$ is induced by applying $N((-)^\circ)$. Conversely, if $L : \mathscr{M} \longrightarrow \mathscr{N}$ is a left Bousfield localisation and $\mathscr{M}$ is a left proper combinatorial simplicial model category, then $N(L^\circ) : N(\mathscr{M}^\circ) \longrightarrow N(\mathscr{N}^\circ)$ defines an accessible localisation.

exactly the $E_\infty$-algebras (*i.e.* , commutative monoids up to coherent homotopy). More generally, one should think about stable ∞-categories as linearisations, tangent spaces or, equivalently, as some category of modules. Regarding the latter point of view, one should have in mind the 1-categorical example of the fibration $\mathbf{Mod}_B \cong \mathbf{Ab}((\mathbf{Alg}_A)_{/B}) \mapsto B \in \mathrm{Ob}(\mathbf{Alg}_A)$ (where the equivalence comes from square-zero extensions) which is the 1-categorical analogue of the fiberwise stabilisation of the fibration $(\mathbf{Alg}_A)_{/B} \mapsto B \in \mathrm{Ob}(\mathbf{Alg}_A)$.

**Definition 1.3.1.** Let $\mathcal{C}$ be an ∞-category. A **zero object in** $\mathcal{C}$ is an object $0 \in \mathrm{Ob}(\mathcal{C})$ which is simultaneously initial and final.

**Definition 1.3.2.** Let $\mathcal{C}$ be an ∞-category. $\mathcal{C}$ is **pointed** if there exists some zero object $0 \in \mathrm{Ob}(\mathcal{C})$.

**Definition 1.3.3.** Let $\mathcal{C}$ be a pointed ∞-category with zero object $0 \in \mathrm{Ob}(\mathcal{C})$. A **triangle in** $\mathcal{C}$ is a diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

in $\mathcal{C}$ which is commutative up to homotopy (*i.e.* , it's a morphism $\Delta^1 \times \Delta^1 \longrightarrow \mathcal{C}$ with $0$ on the vertex $(0,1)$).

**Definition 1.3.4.** Let $\mathcal{C}$ be a ∞-category with final object $1 \in \mathrm{Ob}(\mathcal{C})$ and $f : X \longrightarrow Y$ be a morphism in $\mathcal{C}$. A **fiber (or kernel) of** $f$ is an object $\mathrm{fib}(f) \in \mathrm{Ob}(\mathcal{C})$ such that there exists a diagram

$$\begin{array}{ccc} \mathrm{fib}(f) & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ 1 & \longrightarrow & Y \end{array}$$

which is a pullback square. When $\mathcal{C}$ is pointed, such diagram, which is a triangle, will be called a **fiber sequence**. Dually, a **cofiber (or cokernel) of** $f$ is an object $\mathrm{cofib}(f) \in \mathrm{Ob}(\mathcal{C})$ such that there exists a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \ulcorner & \downarrow \\ 1 & \longrightarrow & \mathrm{cofib}(f) \end{array}$$

which is a pushout square. When $\mathcal{C}$ is pointed, such diagram, which is a triangle, will be called a **cofiber sequence**.

**Definition 1.3.5.** Let $\mathcal{C}$ be an $\infty$-category. $\mathcal{C}$ is a **stable $\infty$-category** if it's pointed, has fibers (kernels) and cofibers (cokernels) and the fiber sequences are exactly the cofiber sequences.

*Remark* 1.3.1. The last requirement is analogous to the condition that every monomorphism is a kernel and every epimorphism is a cokernel, which is required to make an additive category into an abelian category.

**Definition 1.3.6.** Let $\mathcal{C}$ be an $\infty$-category with a final object $1 \in \mathrm{Ob}(\mathcal{C})$ and $X \in \mathrm{Ob}(\mathcal{C})$. A **looping of** $X$ is defined as an object $\Omega X := \mathrm{fib}(X \longrightarrow 0) \in \mathrm{Ob}(\mathcal{C})$. Explicitly, it's an object $\Omega X$ such that the diagram

$$\begin{array}{ccc} \Omega X & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ 1 & \longrightarrow & X \end{array}$$

is a pullback square. Dually, a **suspension of** $X$ is defined as an object $\Omega X := \mathrm{cofib}(0 \longrightarrow X) \in \mathrm{Ob}(\mathcal{C})$ Explicitly, it's an object $\Sigma X$ such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & 1 \\ \downarrow & \ulcorner & \downarrow \\ 1 & \longrightarrow & \Sigma X \end{array}$$

is pushout square.

**Lemma 1.3.1.** *Let $\mathcal{C}$ be an $\infty$-category with a final object $1 \in \mathrm{Ob}(\mathcal{C})$ such that every object $X \in \mathrm{Ob}(\mathcal{C})$ has a looping and suspension. There's an adjunction*

$$\mathcal{C} \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{C}$$

*. If $\mathcal{C}$, furthermore, is stable, then the adjunction is an equivalence.*

*Proof.* The first assertion follows directly from the universal property of (co)limits. The second assertion follows from the fact that every cofiber sequence is a fiber sequence. □

**Proposition 1.3.1.** *Let $\mathcal{C}$ be a pointed $\infty$-category. If $\mathcal{C}$ has cofibers and $\Sigma$ is an equivalence, then $\mathcal{C}$ has finite coproducts. In particular, if $\mathcal{C}$ is stable, then $\mathcal{C}$ has finite products and finite coproducts which coincide (i.e. , it has biproducts).*

*Proof.* Firstly, notice that $\mathrm{cofib}(0 \longrightarrow Y) \cong Y$ and $\mathrm{cofib}(\Sigma^{-1}X \longrightarrow 0) \cong X$. Since cofib commutes with colimits in $\widehat{\Delta}(\Delta^1, \mathcal{C})$ and

$$(0 \longrightarrow Y) \sqcup (\Sigma^{-1}X \longrightarrow 0) \cong \Sigma^{-1}X \overset{0}{\longrightarrow} Y,$$

$$\mathrm{cofib}(\Sigma^{-1}X \overset{0}{\longrightarrow} Y) \cong Y \coprod X.$$

If $\mathcal{C}$ is stable, then $\mathcal{C}^{\mathrm{op}}$ is also stable and, hence, $\mathcal{C}^{\mathrm{op}}$ has finite coproducts. Therefore, $\mathcal{C}$ has finite products. □

*Remark* 1.3.2. Once one admits that the category of spectra **Sp** is the $\infty$-categorical analogous (up to nonconnective and nongrouplike objects) of the category **Ab**, one can identify stable $\infty$-categories as the $\infty$-categorical analogous of abelian categories.

**Proposition 1.3.2.** *Let $\mathcal{C}$ be a pointed $\infty$-category. If $\mathcal{C}$ has cofibers and $\Sigma$ is an equivalence, then* $\mathrm{Ho}(\mathcal{C})$ *is* **Ab***-enriched. In particular, $\mathcal{C}$ is additive and, therefore, any stable $\infty$-category has an homotopy category which is additive.*

*Proof.* Since $\Sigma$ is an equivalence, let $\Sigma^{-1}$ denotes an inverse. By the definition of $\Sigma$, for every $n \in \omega$,

$$\mathcal{C}(X,Y) \cong \mathcal{C}(\Sigma^n \Sigma^{-n} X, Y) \cong \Omega^n \mathcal{C}(\Sigma^{-n} X, Y)$$

. Hence, $\mathcal{C}(X,Y)$ is a $n$-loop space for every $n \in \omega$ and $X, Y \in \mathrm{Ob}(\mathcal{C})$. In particular, for $n = 2$, $\pi_0(\mathcal{C}(X,Y)) \cong \pi_2(\mathcal{C}(\Sigma^{-2}X, Y))$, which implies that

$\pi_0(\mathcal{C}(X,Y))$ is an abelian group. Notice that, by functoriality, the group structure arising from $\pi_1(\mathcal{C}(\Sigma^{-1}X,Y))$ is also commutative.

The bilinearity follows trivially from the fact that $\Omega$ commutes with product.

□

The previous result can be enhanced to the ∞-setting. As in the case of abelian categories, where a category with finite coproducts, finite products, and a zero object is canonically enriched over commutative monoids, there's an ∞-categorical enrichement over any stable ∞-category according to the following theorem.

**Theorem 1.3.1.** *Let $\mathcal{C}$ be an ∞-category. If $\mathcal{C}$ is stable, then it's ∞-enriched over* **Sp**.

*Proof.* The complete proof can be found in [GH15, Ex. 7.4.14] and [HA, Prop. 4.8.2.18]. The definition of ∞-enrichment exceeds the scope of this exposition. □

**Definition 1.3.7.** Let $\mathcal{C}$ be an additive category. $\mathcal{C}$ is a **triangulated category** if there exists an equivalence $(-)[1] : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ and there's a collection of diagrams $T$ of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

called **distinguished triangles** such that it satisfies the following axioms

**TRI**:

i) $T$ is closed under isomorphism.

ii) For every $X \in \mathrm{Ob}(\mathcal{C})$,

$$X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow X[1]$$

is a distinguished triangle;

iii) For every $u : X \longrightarrow Y \in \mathcal{C}(X,Y)$, there exists $Z \in \mathrm{Ob}(\mathcal{C})$ such that

$$X \xrightarrow{u} Y \longrightarrow Z \longrightarrow X[1]$$

is a distinguished triangle.

**TRII**: A diagram

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is a distinguished triangle iff

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

is a distinguished triangle.

**TRIII**: For every commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow f & & \downarrow g & & \vdots h & & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

such that the horizontal lines are distinguished triangles, there exists $h$ making the above diagram commutative.

**TRIV**: For every triple of distinguished triangles

$$X_1 \xrightarrow{u_3} X_2 \xrightarrow{v_3} Z_3 \xrightarrow{w_3} X_1[1],$$

$$X_2 \xrightarrow{u_1} X_3 \xrightarrow{v_1} Z_1 \xrightarrow{w_1} X_2[1]$$

and

$$X_1 \xrightarrow{u_2} X_3 \xrightarrow{v_2} Z_2 \xrightarrow{w_2} X_1[1]$$

, there exists two morphisms and a distinguished triangle

$$Z_3 \xrightarrow{m_1} Z_2 \xrightarrow{m_3} Z_1 \xrightarrow{v_3[1]w_1} Z_3[1]$$

such that

$$\begin{array}{ccccccc} X_1 & \xrightarrow{u_3} & X_2 & \xrightarrow{v_3} & Z_3 & \xrightarrow{w_3} & X_1[1] \\ \downarrow 1_{X_1} & & \downarrow u_1 & & \downarrow m_1 & & \downarrow 1_{X_1[1]} \\ X_1 & \xrightarrow{u_2} & X_3 & \xrightarrow{v_2} & Z_2 & \xrightarrow{w_2} & X_1[1] \end{array}$$

and

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{u_2} & X_3 & \xrightarrow{v_2} & Z_2 & \xrightarrow{w_2} & X_1[1] \\
\downarrow{\scriptstyle u_3} & & \downarrow{\scriptstyle 1_{X_3}} & & \downarrow{\scriptstyle m_3} & & \downarrow{\scriptstyle u_3[1]} \\
X_2 & \xrightarrow{u_1} & X_3 & \xrightarrow{v_1} & Z_1 & \xrightarrow{w_1} & X_2[1]
\end{array}$$

are commutative.

One of the main importances of stable $\infty$-categories is that they lift the inhenrently ill behaved theory of triangulated categories to the $\infty$-categorical world.

**Theorem 1.3.2.** *Let $\mathcal{C}$ be a pointed $\infty$-category. If $\mathcal{C}$ has cofibers and $\Sigma$ is an equivalence, then* $\mathrm{Ho}(\mathcal{C})$ *is triangulated. In particular, the homotopy category of every stable $\infty$-category is triangulated.*

*Proof.* By the previous proposition, $\mathrm{Ho}(\mathcal{C})$ is additive.

Let shift functor $(-)[1]$ be given by $\mathrm{Ho}(\Sigma)$. A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\mathrm{Ho}(\mathcal{C})$ will consist in a diagram

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle g} & & \downarrow \\
0 & \longrightarrow & Z & \xrightarrow{h'} & W
\end{array}$$

such that each square is a pushout (and, consenquently, by pasting, also the rectangle) and $h = eh'$, where $e : W \xrightarrow{\sim} \Sigma X$.

**TRII** follows by noticing that permuting the arrows to 0 in the pushout square

$$\begin{array}{ccc}
X & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Y
\end{array}$$

is the inversion on the (abelian) group $\pi_0(\mathcal{C}(\Sigma X, Y)) \cong \pi_1(\mathcal{C}(X, Y))$ and by considering the vertical rectangle on the pasting of pushout squares

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow \\ 0 & \longrightarrow & Z & \xrightarrow{h'} & W \\ & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & V. \end{array}$$

**TRIII** is trivial and **TRIV** follows again from pasting pushout squares (see [HA, Thm. 1.1.2.14] for a proof) □

Recall that Grothendieck abelian categories are linear analogues of topos: any Grothendieck abelian category is an accessible left exact localisation of a category of linear presheaves. Analogously, locally presentable stable ∞-categories are ∞-categorical version of Grothendieck abelian categories as the following express

**Proposition 1.3.3** ([HA, Prop.1.4.4.9]). *Let $\mathscr{C}$ be a locally presentable stable ∞-category. There exists a small ∞-category $\mathscr{C}_0$ such that*

$$\mathscr{C} \underset{\longleftrightarrow}{\overset{L}{\underset{\perp}{\longleftarrow}}} \mathrm{Fun}(\mathscr{C}_0^{\mathrm{op}}, \mathbf{Sp}).$$

In order to recover an abelian category $\mathscr{A}$ from its derived category $D(\mathscr{A})$, one needs to axiomatise a notion of complex in degree 0 in the stable ∞-category $D(\mathscr{A})$. In order to do so, one need to define the notions of $t$-structure and heart of $t$-structure

**Definition 1.3.8**. Let $\mathscr{T}$ be a triangulated category. A ***t*-structure on** $\mathscr{T}$ consists in a pair $(\mathscr{T}_{\leq 0}, \mathscr{T}_{\geq 0})$ of replete full subcategories of $\mathscr{T}$ satisfying

(i) For every $X \in \mathrm{Ob}(\mathscr{T}_{\geq})$ and $Y \in \mathrm{Ob}(\mathscr{T}_{\leq 0})$, $\mathscr{T}(X, Y[-1]) = 0$;

(ii) $\mathscr{T}_{\leq 0}[-1] \hookrightarrow \mathscr{T}_{\leq 0}$ and $\mathscr{T}_{\geq 0}[1] \hookrightarrow \mathscr{T}_{\geq 0}$;

(iii) For every $X \in \mathrm{Ob}(\mathscr{T})$, there exists a distinguished triangle

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow X'[1]$$

such that $X' \in \mathrm{Ob}(\mathscr{T}_{\geq 0})$ and $X'' \in \mathrm{Ob}(\mathscr{T}_{\leq 0}[-1])$.

Let $\mathcal{C}$ be a stable $\infty$-category. A ***t*-structure on** $\mathcal{C}$ is a $t$-structure on $\mathrm{Ho}(\mathcal{C})$. $\mathcal{C}_{\leq 0}$ and $\mathcal{C}_{\leq 0}$ will denote the full subcategories of $\mathcal{C}$ containing, respectively, all the objects and morphism of $\mathrm{Ho}(\mathcal{C})_{\leq 0}$ and $\mathrm{Ho}(\mathcal{C})_{\leq 0}$

The notation $\mathscr{T}_{\leq n} := \mathscr{T}_{\leq 0}[n]$ and $\mathscr{T}_{\geq n} := \mathscr{T}_{\geq 0}[n]$ will be used.

If $\mathcal{C}$ is a stable $\infty$-category, the inclusions $\mathcal{C}_{\leq n} \hookrightarrow \mathcal{C}$ and $\mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}$ have, respectively, a left adjoint and a right adjoint denoted by $\tau_{\leq n} : \mathcal{C} \longrightarrow \mathcal{C}_{\leq n}$ and $\tau_{\geq n} : \mathcal{C} \longrightarrow \mathcal{C}_{\geq n}$

**Definition 1.3.9.** Let $\mathcal{C}$ be a stable $\infty$-category with $t$-structure $(\mathcal{C}_{\leq 0}, \mathcal{C}_{\geq 0})$. The **heart of** $\mathcal{C}$, $\mathcal{C}^{\heartsuit}$, is defined as $\mathcal{C}^{\heartsuit} := \mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq 0}$.

**Theorem 1.3.3** ([BBD82, Thm. 1.3.6]). *Let $\mathcal{C}$ be a stable $\infty$-category. $\mathcal{C}^{\heartsuit} \cong N(\mathrm{Ho}(\mathcal{C}^{\heartsuit}))$ and $\mathrm{Ho}(\mathcal{C}^{\heartsuit})$ is an abelian category.*

# Part I

# The Cave

# Chapter 2

# The Weil's Conjectures

## § 2.1 Rational Points

The main concern of Diophantine problems lies in the computation of the zeroes of a system of polynomial equations $f_1, f_2 \dots, f_m \in K[x_1, \dots, x_n]$ for $K$ some number field. Such problems can be translated into the geometric setting by defining

$$V = V(f_1, \dots, f_m) \hookrightarrow \mathbf{A}^n_K \hookrightarrow \mathbf{P}^n_K$$

and taking the closure $X$ of $V$ inside $\mathbf{P}^n_K$, *i.e.* , projectivising $V$. Projectivising is not strictly necessary, though it usually makes the problems easier by allowing better tools, such as a nice intersection theory. The Diophantine problem, now, can be stated in more generality as: compute the rational points of an arbitrary variety $X$, $X(K)$ or, analogously, the integral points of $X$, $\mathscr{X}(\mathscr{O}_K)$ [1].

The geometric problem of deciding the existence of integral points is extremely difficult in full generality and, in general, for any variety $X$, undecidable [Poo03; Poo08]; this decidability problem was listed as the Hilbert's tenth problem. The case of rational points still open and it's expected to be also undecidable. One may even restrict to smooth varieties and still obtain

[1]One, instead, usually starts with a variety $X/K$ (*resp.*, $X/\mathbf{F}_q$ for $q = p^n$) satisfying certain properties and proves the existence and unicity of a variety $\mathscr{X}/\mathscr{O}_K$ with generic fiber $\mathscr{X}_\eta = X$ (*resp.*, with fiber $\mathscr{X} \otimes_{\mathbf{Z}} \mathbf{F}_q = X$) satisfying the same properties.

an equivalent problem [Poo03, Prop 12.1]; however, for smooth projective varieties, no such equivalence is known [Poo08, Question 12.3].

Even simple problems such as determining whether $X(K)$ is finite or not can be quite difficult. For instance, any abelian variety $A$ satisfies that $A(K)$ is a finitely generated abelian group and, in particular, any Jacobian $J$ satisfies that $J(K)$ is finitely generated. However the problem of determining the finitude of $X(K)$ for $X$ an arbitrary smooth projective curve of genus $g > 1$, also known as Mordell's conjecture was only solved in 1983 by Faltings [Fal83, Satz 7], whereby he received a Fields medal. Actually, even the problem of computing efficiently such rational points for higher genus smooth projective curves is not known.

The problem, however, is decidable for **R**, **C** and finite fields [Poo03, Table 1]. Furthermore, it's well known that local solutions (*i.e.* , solutions at the prime completions of $K$ for both archimedean and nonarchimedean places) may glue to global solutions in some special cases; such remarkable property is called Hasse's local-global principle. That principle indirectly and directly pervades the field of number theory and, in particular, conspicuously, the Langlands' program, motivating, hence, reciprocity laws [Wym72]. The first instance of such laws, the law of quadratic reciprocity, was studied by Fermat, Euler, Legendre and, finally, proven completely by Gauss. In more generality, they are, in summary, simply laws for determining at modulo which primes the polynomials defining a variety split completely (and, in particular, they give, through Hensel's lifting, all local solutions, $X(K_v)$).

The local-global principle is true in some simple and yet important cases, such as conics (where the respective reciprocity law is the quadratic reciprocity) [2]) or, more generally, quadrics [Ser70a, Chap. IV] and Severi-Brauer varieties [Poo17, Thm 4.5.11]. However, it also fails for simple examples such as arbitrary elliptic curves and, more generally, abelian varieties; there, such property is weakly controlled by the Tate-Shafarevich group or, in full generality, for any variety, by the Brauer-Manin obstruction.

The Weil's conjectures, which is the main topic of this chapter, concern

[2]Actually, one usually proves Hasse-Minkowski's theorem for quadratic forms using Hilbert's reciprocity. However, for the case of conics, by results of Gauss and Legendre, one might reduce it to a problem of quadratic residues.

about the first order approximation of local solutions, *i.e.* , $X(\mathbf{F}_q)$ for $\mathbf{F}_q = \mathcal{O}_K/\mathfrak{p}_v$, $q = p^n$, and $p$ some prime number. Actually, it's even more restricted: they concern about only the cardinality of $X(\mathbf{F}_q)$, $|X(\mathbf{F}_q)|$. A better motivation using the Riemann Hypothesis will follow in the section section 2.3. From this point of view, the local-global principle holds by definition since the (*resp.*, completed) global $\zeta$-function turns set-theoretical partitions of closed points (*resp.*, places) into products.

An introductory reference to rational points is [Zha01]. A more complete and technical reference is [Poo17].

## § 2.2 Weil's Proposal

Weil's seminal paper [Wei49] begins quite mysteriously with the computations of solutions $\mod p^n$ to equations of the form

$$\sum_{i=0}^{r} a_i x_i^{n_i} - b = 0$$

To motivate the problem, Weil recalls some instances of this problem dating back to Gauss. Of particular importance, was Gauss' solution to the cases $y^2 = ax^4 - b$ in $\mathbf{F}_p$. Such solutions have as consequence the truth of what is now known as the Riemann's hypothesis for the curve $X/\mathbf{F}_p$ defined by $X = V(y^2 - ax^4 + b)$.

After a laborious computation under such strained motivation, Weil finally reveals in the last page that such computations were simply a nontrivial example, along with others that he claims not being able to discuss there, of conjectures concerning $|X(\mathbf{F}_{q^m})|$. Weil claimed that the conjecture was known for smooth projective curves; however, he was unable to decide the truth for higher dimensional varieties.

More precisely, Weil suggested some relations to determine each $N_m = |X(\mathbf{F}_{q^m})|$ for every $m \geq 1$ by employing the Betti numbers of $X(\mathbf{C})$ viewed as a smooth complex manifold.

It's easier to illustrate the conjecture by first focusing the attention to specific examples.

Let $X = \mathbf{P}^d$. Since

$$X(\mathbf{F}_{q^m}) = (\mathbf{A}^{d+1}(\mathbf{F}_{q^m}) \setminus \{0\})/\mathbf{G}_m(\mathbf{F}_{q^m}) = (\mathbf{F}_{q^m}^{d+1} \setminus \{0\})/\mathbf{F}_{q^m}^{\times},$$

it implies that

$$N_m = \frac{(q^m)^{d+1} - 1}{q^m - 1} = \sum_{i=0}^{d} q^{im}$$

On the other side,

$$\mathrm{H}^i_{\mathrm{Betti}}(X, \mathbf{Q}) = \mathrm{H}^i_{\mathrm{sing}}(X(\mathbf{C}), \mathbf{Q}) \cong \begin{cases} \mathbf{Q} & \text{if i is even and } i \leq 2d, \\ 0 & \text{if i is odd.} \end{cases}$$

Let $X = \mathrm{Gr}(j, \mathbf{Z}^r)$. Since

$$X(\mathbf{F}_{q^m}) = \{(v_1, v_2, \cdots, v_j) \in (\mathbf{F}_{q^m}^r)^j \mid v_1, v_2, \cdots, v_j \text{ linearly independent}\}/\mathrm{GL}_j(\mathbf{F}_{q^m}),$$

where $\mathrm{GL}_j(\mathbf{F}_{q^m})$ acts diagonally; it implies that

$$\begin{aligned} N_m = |X(\mathbf{F}_{q^m})| &= \frac{|\{(v_1, v_2, \ldots, v_j) \in (\mathbf{F}_{q^m}^r)^j \mid v_1, v_2, \ldots, v_j \text{ linearly independent}\}|}{|\mathrm{GL}_j(\mathbf{F}_{q^m})|} \\ &= \frac{\prod_{i=1}^{j}((q^m)^r - (q^m)^{i-1})}{\prod_{i=1}^{j}((q^m)^j - (q^m)^{i-1})} = \frac{\prod_{i=1}^{j}((q^m)^{r-i+1} - 1}{\prod_{i=1}^{j}((q^m)^{j-i+1} - 1)} \\ &= \binom{r}{j}_{q^m} = \sum_{i=0}^{j(r-j)} p_{r,j}(i) q^{im} \end{aligned}$$

, where $p_{r,j}(i)$ denotes the number of partitions of $i$ into at most $r - j$ subsets of cardinality at most $j$.

On the other side, by an easy and yet boringly laborious computation using Schubert cells,

$$\mathrm{H}^i_{\mathrm{Betti}}(X, \mathbf{Q}) = \mathrm{H}^i_{\mathrm{sing}}(X(\mathbf{C}), \mathbf{Q}) \cong \mathbf{Q}^{p_{n,k}(i)}.$$

More generally, for a variety $X$ with a filtration into subvarieties

$$X_0 \subset X_1 \subset \ldots X_i \subset X_{i+1} \subset \ldots X_n = X$$

such that $X_{j+1} - X_j = \coprod_{l=1}^{n_j} \mathbf{A}^{d_{jl}}$, the above relation between the Betti numbers and $N_m$ will hold. Such phenomena is, actually, very simple since the homotopy type of $X(\mathbf{C})$ can be generated by gluing iteratively the same cells of the mentioned filtration $X_{j+1}(\mathbf{C}) - X_j(\mathbf{C}) = \coprod_{l=1}^{n_j} \mathbf{A}^{d_{jl}}(\mathbf{C})$.

However, not every $X$ have such kinds of algebraic cellular decompositions. For instance, any $X$ having $\mathrm{H}_{\mathrm{Betti}}(X, \mathbf{Q})$ nonzero outside of even degree cannot enjoy such property as each $\mathbf{A}^{r_{jl}}(\mathbf{C})$ is an even dimensional real manifold.

An elliptic curve $E$, for instance, is such that $E(\mathbf{C}) \cong \mathbf{C}^2/\Lambda$ where $\mathbf{Z}^2 \cong \Lambda \subset \mathbf{C}$ is some lattice. Hence, $\mathrm{H}^1_{\mathrm{Betti}}(E, \mathbf{Q}) = \mathbf{Q}^2$. That is, two 1-dimensional real manifolds generate the first cohomology. As such real manifolds are not a priori algebraically detectable (as they are not complex manifolds nor the real points of some variety), a more ingenuous relation must be required.

Luckily, however, the odd dimensional cohomology interact nontrivially with the even dimensional cohomology by cohomological operations, *e.g.* , the cup product. It's, thus, reasonable to fathom such information inside the algebraic pieces that supposedly generate the even dimensional Betti cohomology.

Such relations were already known before Weil for some non-trivial cases.

**Theorem 2.2.1** (1934 Hasse, [Has34, §11]). *Let $E/\mathbf{F}_q$ be an elliptic curve. Then*

$$N_m = |E(\mathbf{F}_{q^m})| = 1 - (\alpha^m + \beta^m) + q^m$$

*for $\alpha$ and $\beta$ algebraic integers such that*

$$|\alpha| = |\beta| = \sqrt{q}$$

.

*Remark* 2.2.1. Notice that in the above theorem the norm was not specified. Indeed, it's remarkable that one can actually take any embedding $K \hookrightarrow \mathbf{C}$ for some number field $K$ containing $\alpha$ and $\beta$.

**COROLLARY 2.2.1** (Hasse's Bound, [Has34, §12]). *Let $E/\mathbf{F}_q$ be an elliptic curve. Then*

$$|E(\mathbf{F}_{q^m}) - 1 - q| \leq 2\sqrt{q}$$

In the above theorem proved by Hasse, the algebraic integers $\alpha$ and $\beta$ correspond to the generators of $\mathrm{H}^1_{\text{Betti}}(E, \mathbf{Q}) = \mathbf{Q}^2$. According to Weil computations in [Wei49], that pattern occurs in more generality. Generally, for $X$ some smooth projective variety, any generator of $\mathrm{H}^i_{\text{Betti}}(X, \mathbf{Q})$ must contribute with some algebraic integer $(-1)^i \alpha_{ij}^m$ in $N_m$ and must satisfy $|\alpha_{ij}| = p^{\frac{i}{2}}$.

More concisely,

$$N_m = 1 + \sum_{i=1}^{2d-1} (-1)^i \sum_{j=1}^{b_i} \alpha_{ij}^m + q^{dm}$$

, where $b_i = \dim_{\mathbf{Q}} \mathrm{H}^i_{\text{Betti}}(X, \mathbf{Q})$ are the Betti numbers.

Weil, however, stated the conjecture by assembling all $N_m$'s into a power series. The above expression translated into that language consist in exactly the Riemann Hypothesis for varieties over finite fields.

## § 2.3 THE RIEMANN'S HYPOTHESIS

As stated previously, the search for rational points as a motivation is a little bit unsatisfactory for $X(\mathbf{F}_{q^m})$ yields only a first order approximation of solutions over extensions of $\mathbf{Q}_p$.

A more satisfactory motivation, which was indeed what mainly motivated Weil and its predecessors, is the generalisation of the Riemann hypothesis for arbitrary varieties.

Recall the basic setting of the Riemann's hypothesis.

**DEFINITION 2.3.1.** The *Riemann's $\zeta$-function* is defined as

$$\zeta(s) := \sum_{i=0}^{\infty} n^{-s}$$

for $\Re(s) > 1$.

**PROPOSITION 2.3.1** (Euler's Product Formula, [Rie60], see also [Edw01, Ch. 1]).

$$\zeta(s) = \prod_{p\ prime} \frac{1}{1 - p^{-s}}$$

*for* $\Re(s) > 1$.

**PROPOSITION 2.3.2** (Analytic continuation, [Rie60], see also [Edw01, Ch. 1]). *$\zeta(s)$ have a meromorphic continuation to* $\mathbf{C}$ *with a unique pole at* $s = 1$*, which is simple and has residue* $1$.

**DEFINITION 2.3.2.** The *completed $\zeta$-function* or *Riemann's $\xi$-function* is defined as

$$\xi(s) := \pi^{\frac{-s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$$

*REMARK* 2.3.1. The function $\xi(s)$ completes the $\zeta(s)$ by adding a term corresponding to the unique nonarchimedean place of the number field $\mathbf{Q}$.

*REMARK* 2.3.2. In [Rie60], Riemann had actually defined an entire function

$$\xi(t) := \Pi(\frac{s}{2})(s-1)\pi^{\frac{-s}{2}}\zeta(s) = \frac{s}{2}(s-1)\pi^{\frac{-s}{2}}\Gamma(\frac{s}{2})\zeta(s)$$

for $s = \frac{1}{2} + it$ and $t \in \mathbf{C}$. It's customary, however, to denote the above entire function on $s \in \mathbf{C}$ by $\xi(s)$ and, instead, $\xi(t)$ by $\Xi(t)$ (*e.g.* , [Edw01]).

In arithmetic geometry, however, in order to treat every place (archimedean and nonarchimedean) uniformly, one usually follows the convention in DEFINITION 2.3.2 ([Ser70b, §3]).

**PROPOSITION 2.3.3** (Functional Equation, [Rie60], see also [Edw01, Ch. 1]). *The function $\xi(s)$ is entire of $s = 0, 1$, where it has poles of order* $1$. *Furthermore, it satisfies*

$$\xi(s) = \xi(1-s).$$

**PROPOSITION 2.3.4** (Critical Strip, [Rie60], see also [Edw01, Ch. 1]). *The function $\zeta(s)$ satisfies $\zeta(-2n) = 0$ for every $n \in \omega$; such zeroes are called trivial zeroes. Furthermore, any other zeroes lies in the critical strip,* $0 \leq \Re(s) \leq 1$

Behold, now, the Riemann's hypothesis.

**Conjecture 2.3.1** (Riemann's Hypothesis, [Rie60], see also [Edw01]). *All the nontrivial zeroes of the function $\zeta(s)$ lies on the critical line $\Re(s) = \frac{1}{2}$.*

A straightforward generalisation for global fields can be given by using the nonarchimedean places of global field instead of ordinary prime integers. Firstly, notice that every prime ideal in the ring of integers of a number field must be maximal for the ring of integers $\mathcal{O}_K$ has Krull dimension 1. Furthermore, as $\mathcal{O}_K$ is of finite type, $\mathcal{O}_{K_v}/\mathfrak{p}_v$ must be a finite field. Hence the following definition seems sound.

**Definition 2.3.3.** Let $K$ be a global field. The *zeta function of K* is defined as

$$\zeta_K(s) := \prod_{I \text{ integral ideal}} N(I)^{-s} = \prod_{\mathfrak{p}_v \text{ nonarchimedean place}} \frac{1}{1 - N(\mathfrak{p}_v)^{-s}}.$$

*Remark* 2.3.3. Notice that $\zeta_{\mathbb{Q}}(s) = \zeta(s)$

*Remark* 2.3.4. Notice that, in the function field case, every place, whether finite or infinite, is necessarily nonarchimedean. That immediately implies that, for $K$ a function field, $\zeta_K(s)$ doesn't need a completion as in the ordinary case (Definition 2.3.2).

A further step can be made in generalising global fields to arbitrary schemes $X$ by using the closed points of $X$, $|X|$, instead of the places of $\mathcal{O}_K$. Also, observe that the residue field $\kappa_x$ at every point of a scheme of finite type is a finite field.

**Definition 2.3.4.** Let $X$ be a scheme of finite type. The *zeta function of X* is defined as

$$\zeta_X(s) := \prod_{x \in |X|} \frac{1}{1 - N(x)^{-s}}$$

, where $N(x) := |\kappa_x|$.

*Remark* 2.3.5. $\zeta_X(s)$ is absolutely convergent for $\Re(s)$ sufficiently large.

*Remark* 2.3.6. Notice that $\zeta_{\mathrm{Spec}(\mathcal{O}_K)}(s) = \zeta_K(s)$.

The reader should recall that the case of function fields is identical to the case of curves by the correspondence between smooth projective curves and its field of rational functions [3] . Hence, $\zeta_{R(C)}(s) = \zeta_C(s)$ for any curve $C/k$.

Henceforth, the standard notion will be used: $X_0$ denotes a scheme over $\mathbf{F}_q$, $X := X_0 \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q$, $\deg(x) := [\kappa_x : \mathbf{F}_q]$ and $q_x := q^{\deg(x)}$.

Weil's original proposal, as already mentioned in the previous section, was based on a power series called the local $\zeta$-function.

**Definition 2.3.5.** Let $X_0/\mathbf{F}_q$. The *local $\zeta$ function* is defined as

$$Z(X_0, t) := \exp(\sum_{m=1}^{\infty} \frac{N_m}{m} t^m) \in \mathbf{Q}[[t]].$$

*Remark* 2.3.7. $Z(X_0, t)$ converges for $|t| \in \mathbf{C}$ sufficiently small.

**Lemma 2.3.1.**

$$Z(X_0, t) = \prod_{x \in |X_0|} \frac{1}{1 - t^{\deg(x)}}.$$

*Proof.* Firstly, notice that $Z(X_0, t)$ is the unique function satisfying

$$Z(X_0, 0) = 1;$$

$$\frac{d \log(Z(X_0, t))}{dt} = \sum_{m=1}^{\infty} N_m t^{m-1}.$$

Furthermore, by noticing that an $\mathbf{F}_{q^n}$-point $\mathrm{Spec}(\mathbf{F}_q) \to X_0$ factors through $\mathrm{Spec}(\mathcal{O}_{X,x})$,

$$N_m = \sum_{\deg(x) | m} \deg(x). \tag{2.1}$$

Now, the equalities

[3]For every curve $C/k$ one can associates $R(C)$ which is an extension of $k(x) = R(\mathbf{P}^1_k)$. Reciprocally, every finite extension $K$ of $k(x)$ corresponds to a branched finite covering of $\mathbf{P}^1_k$ by associating the normalisation of $\mathbf{P}^1_k$ in $K$.

$$\frac{d}{dt}(\log(\prod_{x\in|X_0|}\frac{1}{1-t^{\deg(x)}})) = [\prod_{x\in|X_0|}(1-t^{\deg(x)})][\sum_{x\in|X_0|}\prod_{y\in|X_0\backslash\{x\}|}\frac{\frac{d}{dt}(\frac{1}{1-t^{\deg(x)}})}{1-t^{\deg(y)}}]$$
$$= [\prod_{x\in|X_0|}(1-t^{\deg(x)})][\sum_{x\in|X_0|}\prod_{y\in|X_0\backslash\{x\}|}\frac{-\frac{-\deg(x)t^{\deg(x)-1}}{(1-t^{\deg(x)})^2}}{1-t^{\deg(y)}}]$$
$$= \sum_{x\in|X_0|}\prod_{x\in|X_0|}(1-t^{\deg(x)})\frac{-\deg(x)t^{\deg(x)-1}}{\prod_{y\in|X_0|}1-t^{\deg(y)}(1-t^{\deg(x)})^2}$$
$$= \sum_{x\in|X_0|}-\frac{-\deg(x)t^{\deg(x)-1}}{1-t^{\deg(x)}}$$
$$= \sum_{k\geq 1, x\in|X_0|}\deg(x)t^{k\deg(x)-1}$$
$$= \sum_{\deg(x)|m}\sum_{x\in|X_0|}\deg(x)t^{m-1}$$
$$\overset{2.1}{=} \sum_{m=1}^{\infty}N_m t^{m-1}$$

hold.

That, forthwith, implies the required equality.

□

**Corollary 2.3.1.**

$$\zeta_{X_0}(s) = Z(X_0, q^{-s}).$$

Originally, the Weil's conjectures were stated in [Wei49] in the following form

**Conjecture 2.3.2** (Weil's Conjectures). *Let $X_0/\mathbf{F}_q$ be a smooth projective variety.*

- *(Rationality)* $Z(X_0,t) = \frac{\prod_{i=0}^{d}P_{2i+1}(t)}{\prod_{i=0}^{d}P_{2i}(t)} = \prod_{i=0}^{2d}P_i(t)^{(-1)^{i+1}}$, *where* $P_i \in \mathbf{Q}[t]$, $P_0(t) = 1-t$, $P_{2d}(t) = 1-q^d t$, $P_i(t) = \prod_{j=1}^{b_i(X_0)}(1-\alpha_{ij}t)$ *and* $\alpha_{ij}$ *are algebraic integers for* $1 \leq i \leq 2d-1$ *and* $1 \leq j \leq b_i(X_0)$*;*

- *(Riemann's Hypothesis)* $|\alpha_{ij}| = q^{\frac{i}{2}}$ *for* $1 \leq i \leq 2d-1$ *and* $1 \leq i \leq 2d-1$*;*

- *(Functional Equation)* $Z(X_0, \frac{1}{q^d t}) = \pm q^{d\frac{\chi(X_0)}{2}} t^{\chi(X_0)} Z(X_0, t)$, *where* $\chi(X_0) = \sum_{i=0}^{2d} (-1)^i b_i(X_0)$;
- *(Continuity of Betti Numbers) Suppose* $X_0/\mathbf{F}_q$ *is a special fiber of some smooth projective variety* $\mathscr{X}/K$ *for* $K$ *some number field. After stating the analogous above conjectures for every projective variety over a finite field (be it smooth or not),* $b_i(\mathscr{X}_s) = \dim_{\mathbf{Q}} \mathrm{H}^i_{\mathrm{Betti}}(\mathscr{X}_\eta(\mathbf{C}), \mathbf{Q})$ *for all except at most a finite number of closed points* $s \in \mathrm{Spec}(\mathscr{O}_K)$.

*Remark* 2.3.8. Notice the assertion that non-trivial zeroes of $\zeta_{X_0}(s)$ lies on $\Re(s) = \frac{i}{2}$ for $1 \leq i \leq 2d-1$ is implied by $|\alpha_{ij}| = q^{\frac{i}{2}}$ for every $i$ and $j$ satisfying $1 \leq i \leq 2d-1$ $1 \leq i \leq 2d-1$. That explains the putative name Riemann's hypothesis in this case.

In [Wei56], a proposal to solve such conjectures was explicitly stated. The idea was using the Lefschetz's trace formula in order to express $N_m$.

Recall the ordinary Lefschetz's trace formula.

**Theorem 2.3.1** (Lefschetz's Trace Formula). *Let* $\mathscr{L}$ *be locally constant sheaf of fields,* $X$ *a* $d$*-dimensional closed real smooth manifold and* $f : X \to X$ *a morphism.*

*If* $\Gamma_f$ *intersects transversally* $\Delta_X$ *in* $X \times X$ *(or, equivalently, the fixed points* $x$ *of* $f$ *satisfy* $\det(1 - df_x) \neq 0$*), then*

$$X^f := \mathrm{Fix}(f) = [\Gamma_f].[\Delta_X] = \sum_{i=0}^{d} (-1)^i \,\mathrm{Tr}(f \,|\, \mathrm{H}_i(X, \mathscr{L})).$$

*Or, equivalently,*

$$X^f := \mathrm{Fix}(f) = [\Gamma_f].[\Delta_X] = \sum_{i=0}^{d} (-1)^i \,\mathrm{Tr}(f \,|\, \mathrm{H}^i(X, \mathscr{L})).$$

*Remark* 2.3.9. The usual proof of the above theorem requires Poincaré duality, which usually assumes $\mathscr{L}$-orientability of a connected (not necessarily closed) smooth manifold $X$ for $\mathscr{L}$ a locally constant sheaf of commutative rings. That is, however, unnecessary by performing a twisting with the orientation sheaf $\mathscr{O}$ [4] defined as the locally constant sheaf $\mathscr{O}(U) = \mathrm{H}_d(X, X \setminus U, \mathscr{L}) \overset{excision}{\cong} \mathrm{H}_d(\mathbf{R}^d, \mathbf{R}^d \setminus \{0\}, \mathscr{L})$ for a contractible open covering.

[4] In this case, that sheaf is actually equivalent to the absolute dualizing sheaf $\omega_X$

More precisely, Poincaré duality becomes

$$(-) \cap [X] \colon \mathrm{H}^i_c(X, \mathscr{M}) \xrightarrow{\sim} \mathrm{H}_{d-i}(X, \mathscr{M} \otimes_{\mathscr{L}} \mathbb{O}).$$

for $\mathscr{M}$ a $\mathscr{L}$-module.

Weil's suggested solution lied in the idea of using $f$ as the geometric or relative Frobenius morphism. Recall that $\mathrm{Fr}_{X_0} \colon X_0 \longrightarrow X_0$, the absolute Frobenius morphism, is defined as the identity on topological spaces and $\mathrm{Fr}^{\#}_{X_0}(f) = f^q \in \Gamma(U, \mathscr{O}_{X_0})$. The geometric Frobenius over $X$ is, then, defined as $\mathrm{Fr} := \mathrm{Fr}_{X/\overline{\mathbf{F}}_q} = \mathrm{Fr}_{X_0} \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q$.

Notice that $|X| \cong X_0(\overline{\mathbf{F}}_q)$ as sets by recalling that $X/\overline{\mathbf{F}}_q$ is of finite type and, hence, have all residue fields isomorphic to some finite extension of $\mathbf{F}_q$ [5]. Then the action of the generator $\phi \in \mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$, which is given by $\phi(x) = x^q$, on $X_0(\overline{\mathbf{F}}_q)$ by $\mathbf{Sch}(\mathrm{Spec}(\phi), X_0)$ coincides with the action of Fr on $|X|$. Hence $X^{\mathrm{Fr}^m} \cong X_0(\mathbf{F}_{q^m})$.

Likewise, the closed points of $X_0$ correspond to orbits of Fr on $X$. Each orbit passing through $x$ have cardinality exactly equals to $\deg(x)$. So $X_0$ should be informally viewed as the homotopy quotient

$$"X//\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q) := X \times_{B\,\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)} E\,\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)".$$

The equality

$$X^{\mathrm{Fr}^m} \cong N_m$$

, therefore, suggests that

$$N_m = [\Gamma_{\mathrm{Fr}^m}].[\Delta_X] = \sum_{i=0}^{d} (-1)^i \,\mathrm{Tr}(\mathrm{Fr}^m \mid \mathrm{H}^i(X, \mathscr{F}))$$

should somehow make sense.

By taking Lefschetz's trace formula trace formula into account, the local $\zeta$ function can be rewritten in the more modern form as stated by Grothendieck's followers.

**Lemma 2.3.2.** *Let $K$ be a field, $V$ be a $K$-vector space and $F \colon V \longrightarrow V$ a morphism.*

[5] That follows forthwith by applying Zariski's lemma

$$t\frac{d}{dt}(\log(\det(1 - Ft|V)^{-1})) = \sum_{i>0} \mathrm{Tr}(F^i|V)t^i.$$

*Proof.* It shall be proved by induction on the dimension of $V$. For $\dim(V) = n = 1$ and $F = \lambda \in K \cong V$, the result follows from the computation

$$\begin{aligned} t\frac{d}{dt}(\log(\det(1 - \lambda t|V)^{-1})) &= t\frac{\frac{d}{dt}(\frac{1}{1-\lambda t})}{\frac{1}{1-\lambda t}} = \lambda t \frac{1-\lambda t}{(1-\lambda t)^2} = \frac{\lambda t}{1 - \lambda t} \\ &= \lambda t \sum_{i\geq 0} \lambda^i t^i = \sum_{i>0} \lambda^i t^i = \sum_{i>0} \mathrm{Tr}(\lambda^i|V)t^i. \end{aligned}$$

Suppose the validity for some $n \in \omega$. By enlarging $K$ if necessary, one may suppose $K \cong \overline{K}$ and, hence, $F$ has an eingenvector in $V$. Let $W \hookrightarrow V$ be the $F$-invariant subspace containing such eigenvector and

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$$

the respective short exact sequence. Since both sides of the equation are multiplicative for short exact sequences and $\dim(V/W) = n - 1$, induction applies. □

**Corollary 2.3.2** (Grothendieck's Cohomological Interpretation).

$$Z(X_0, t) = \prod_i \det(1 - \mathrm{Fr}\, t|\, \mathrm{H}^i(X, \mathscr{F}))^{(-1)^{i+1}}$$

To engender a meaningful statement, however, one must search for a cohomology satisfying the Lefschetz's trace formula. Such cohomology should, therefore, behave as singular cohomology and yet be sufficiently algebraic in order to accept the Frobenius as input.

Any such cohomology, need to be of characteristic 0 in order to yield the correct values of $|X(\mathbf{F}_{q^m})|$. Serre proved that no such cohomology could be conceived with values in $\mathbf{Q}$, $\mathbf{Q}_p$ or $\mathbf{R}$ for $k$ such that $\mathbf{F}_{p^2} \hookrightarrow k$ ([Gro68, §1.7]). Indeed, let $E/k$ be an elliptic curve. $\mathrm{End}(E)$ must be $\mathbf{Z}$, an order in an imaginary quadratic field $\mathbf{Q}(\sqrt{-D})$ or an order in a quaternion algebra over $\mathbf{Q}$. If $\mathrm{char}(k) \neq 0$, the Frobenius endomorphism excludes the first possibility.

If, furthermore, $E$ is supersingular, then $\mathrm{End}(E)$ is a maximal order in a quaternion algebra $D$ over $\mathbf{Q}$ such that $D$ is ramified (*i.e.* , non-split [6]) exactly at $p$ and $\infty$ ([Deu41]). Since it's expected, by 2.2.1, that $\mathrm{H}^1(E,\mathbf{Q}) \cong \mathbf{Q}^2$ and $\mathrm{End}(E)$ acts on $\mathrm{H}^1(E,\mathbf{Q})$, it yields, after tensoring by $\mathbf{Q}$, a morphism $D \longrightarrow \mathrm{End}_{\mathbf{Q}}(\mathbf{Q}^2)$, which is impossible [7]. Since tensoring with $\mathbf{Q}_p$ and $\mathbf{R}$ does not split $D$, no such cohomology can also exist with coefficients in $\mathbf{Q}_p$ nor $\mathbf{R}$.

In 1958, Serre made an abortive attempt by using the cohomology with values in the sheaf of Witt vectors ([Ser58c] and [Ser58b]). That attempt resembled a primitive version of the yet non-existent crystalline cohomology, but lacked the required Grothendieck topology. [8].

In 1960, Dwork proved the Rationality Conjecture using $p$-adic analysis, which, a priori, was devoid of any cohomological machinery ([Dwo60]). That likely caused some discouragement in a search for the so desired cohomology. Later, however, such proof was reinterpreted in Monsky-Washnitzer cohomology.

Only in 1974, Grothendieck and Artin conceived a cohomology enjoying such properties: the étale cohomology, the content of the next chapter.

In this chapter, most of the historical remarks were taken from [Oor14].

[6]Recall that a quaternion algebra $D$ over a global field $K$ is split over a place $v$ iff $D \otimes K_v \cong \mathrm{End}_{K_v}(K_v^2)$ otherwise it's a division algebra of dimension 4.

[7]Since $D$ is simple, the $\mathrm{Ker}(D \longrightarrow \mathrm{End}_{\mathbf{Q}}(\mathbf{Q}^2)) \cong 0$ and, therefore, since both $\mathbf{Q}$-vector spaces have dimension 4, it must be an isomorphism.

[8]Notice that crystalline cohomology, on the other side, satisfy the Lefschetz's trace formula. However, it was only conceived 8 years later.

# Part II

# The Shadows

# Chapter 3

# Étale

Étale cohomology was the first Weil Cohomology Theory conceived. The legend tells that, during the first exposition of the 1958 Séminaire Chevalley (which in that year was mainly focused on Chow groups) given by Serre [Ser58a], Grothendieck forthwith noticed the potential of the étale topology after beholding the properties of principal $G$-bundles defined by means of isotrivialisations [1] (*i.e.* , finite étale coverings) instead of Zariski local trivialisations.

It was well known that for smooth varieties over $\mathbf{C}$ defining a vector bundle as a regular morphism $V \longrightarrow X$ such that $V \times_X U \cong \mathbf{A}^n \times_{\mathrm{Spec}(k)} X$ for some Zariski covering was compatible with the analytic topology.

**Theorem 3.0.1** (1952 Weil, [Wei55]). *Let $X/\mathbf{C}$ be a smooth variety and $V \longrightarrow X$ a regular morphism with fibers isomorphic to $\mathbf{A}^n_{\mathbf{C}}$. $V \longrightarrow X$ is locally trivial in the Zariski topology iff it's locally trivial in the analytic topology.*

In parallel, it was known by Serre and probably inspired him to generalise the setting of Lang and Tate in [LT58]. In there, they have studied principal $A$-bundles for $A/k$ an abelian variety over a field $k$ under the name of principal homogeneous spaces, where they were mainly interested in $\mathrm{H}^1(\mathrm{Gal}(L/k), A(L))$, which classifies principal homogeneous spaces having at least one $L$-point [2].

[1] Serre was mainly concerned with the base $\mathrm{Spec}(k)$ for $k$ algebraically closed. Therefore, no mention to the internal symmetries of $\mathrm{Spec}(k)$ by requiring separability in the condition of non ramification was elicited

[2] The main motivation of Lang and Tate being the case of a genus 1 curve $C/\overline{k}$, which can be seen as a principal homogeneous space under the action of $J(C)$. In this case, the corresponding class $\mathrm{H}^1(G_k, J(C)(\overline{k}))$ has an $L$-point when it factors through $\mathrm{H}^1(\mathrm{Gal}(L/k), J(C)(L))$

As noticed by Serre, desirable examples of principal $G$-bundles in the analytic case are not locally trivial in the algebraic setting. For instance, a finite étale covering nor the canonical projection $G \longrightarrow G/H$ for some algebraic group $G$ and subgroup $H$ are locally trivial in the Zariski topology. To remedy that problem, including forcefully the finite étale coverings is a natural choice and, indeed, Serre redefined a principal $G$-bundle as an isotrivial $G$-bundle. An immediate consequence is that over $\mathbf{C}$ the analytification of an isotrivial principal $G$-bundle is a (locally trivial in the analytic topology) principal $G^h$-bundle. A more interesting consequence is that isotriviality already implies that the action of $G$ is free and transitive on the fibers. Indeed, $P \longrightarrow X$ is a principal $G$-bundle iff $G$ acts freely transitively and there are local sections after lifting to a finite étale covering.

A lot of cases, however, are principal $G$-bundles already locally trivial. For instance, principal $\mathbf{G}_m$, $\mathbf{G}_a$, $\mathrm{SL}_n$, $\mathrm{Sp}_n$ and the already mentioned $\mathrm{GL}_n$-bundles are always (Zariski) locally trivial. In fact, an algebraic group satisfying such property in necessarily connected and linear [3]. Also, In particular, any connected solvable linear group satisfies that property by noticing that it's is closed under extensions (and, in particular, extensions by $\mathbf{G}_m$ and $\mathbf{G}_a$). More generally, Grothendieck proved that the semi-simple groups satisfying such property are exactly the products of $\mathrm{SL}_n$'s and $\mathrm{Sp}_n$'s. That would explain why no one objected Weil's definition since then. However, fundamental examples such as $\mathrm{PGL}_n$ for $n \geq 2$ and $\mathrm{Spin}_n$ for $n \geq 7$ already fail.

In this setting, Serre developed a Galois theory for finite étale coverings of varieties $X/k$ for $k$ algebraically closed. Furthermore, he defined the first cohomology for any variety $X/k$ and algebraic group $G/k$ by setting

$$\widetilde{\mathrm{H}}^1(X, G) := \{\text{equivalence classes of principal } G\text{-bundles}\}$$

. In particular, as a subgroup lies the first derived functor of $\mathrm{H}^0(X, G_X) = \Gamma(X, G_X)$ where $G_X(U) := \mathbf{Sch}_k(U, G)$ is the constant sheaf with value $G$ on $X$, which classifies equivalence classes of (Zariski) locally trivial principal $G$-bundles. Moreover, such $\widetilde{\mathrm{H}}^1(X, G)$ behaved as a first derived functor somehow.

**Theorem 3.0.2** (1958 Serre, [Ser58a]). *Let $k$ be an algebraically closed field,*

[3] Even better, $G$ satisfies that special property iff $\mathrm{GL}_n \longrightarrow \mathrm{GL}_n/G$ is locally trivial

*$X/k$ a variety and*

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

*a short exact sequence of algebraic groups in $\mathbf{Sch}_k$.*

*There's a long exact sequence*

$$1 \to \mathrm{H}^0(X, H_X) \to \mathrm{H}^0(X, G_X) \to \mathrm{H}^0(X, (G/H)_X) \to \widetilde{\mathrm{H}}^1(X, H) \longrightarrow \widetilde{\mathrm{H}}^1(X, G).$$

*If, withal, $H$ in an invariant subgroup of $G$, there's a long exact sequence*

$$1 \to \mathrm{H}^0(X, H_X) \to \mathrm{H}^0(X, G_X) \to \mathrm{H}^0(X, (G/H)_X) \longrightarrow \longrightarrow \widetilde{\mathrm{H}}^1(X, H) \longrightarrow \widetilde{\mathrm{H}}^1(X, G) \longrightarrow \widetilde{\mathrm{H}}^1(X, G/H)$$

*extending the previous one. In both cases, if $H$, $G$ and $G/H$ satisfies the special property that every principal bundle is locally trivial, then the above long exact sequences coincides with the long exact sequence of sheaves after applying the derived functors to the sequence*

$$1 \longrightarrow H_X \longrightarrow G_X \longrightarrow (G/H)_X \longrightarrow 1.$$

Grothendieck, when present in the talk, could forthwith see from the beginning the potentiality of creating $\widetilde{\mathrm{H}}^i(X, G)$ for $i > 1$. Serre, at first, was completely sceptical. Grothendieck, as always, rebelliously and confidently accepted the challenge and even before the official publication of Serre's exposition ([Ser58a]) could construct such cohomology groups.

Serre himself acknowledge him in [Ser58a]:

*"On peut se demander s'il est possible de définir des groupes de cohomologie supérieurs $\widetilde{\mathrm{H}}^q(X, G)$ qui permettent d'étendre la suite exact de la proposition 13 en toute dimension. GROTHENDIECK a montré que c'est bien le cas (non publié), et il semble même que ces nouveaux groupes de cohomologie, lorsque $G$ est fini fournissent la 'vraie cohomologie' nécessaire pour la démonstration des conjecture de Weil. Voir à cet sujet l'introduction de [Gro60]."*

The development, though, of the étale topology as a Grothendieck topology following the conception of a topoi started only in the fall of 1961 during Grothendieck's visit in Harvard, where Artin, Zariski and Mumford where present at the time. The field burgeoned quickly from the lectures of Artin in the spring of 1962 [Art62] to the breathtaking 1963/1964 Séminaire de Géométrie Algébrique du Bois Marie [SGA4-I; SGA4-II; SGA4-III]. The notion of profinite étale fundamental group $\pi^1_{\text{ét}}(X,x)$ was, however, settled before in [SGA1] using finite étale coverings and, later, enlarged by means of arbitrary étale coverings (*i.e.* , only locally quasi-finite instead of finite or, equivalently proper) in [SGA3-II, Exp. X.6] leading to the prodiscrete étale fundamental group. Later the notion of étale fundamental group was generalised to étale homotopy types by Artin and Mazur (announced for the first time in [AM67]; see also [AM86], based on a 1965/66 Harvard seminar) thorugh an à la Čech approach introduced earlier by Lubkin ([Lub67]), and Verdier ([SGA4-II, Exp. V]) following an idea of Cartier. Later it was extended by [Ler79] to arbitrary Galois topoi, Friedlander ([Fri82]) to simplicial schemes and, recently, Hoyois ([Hoy18]) generalised the previous ones to the $(\infty,1)$-case. Even more recently, some shortcomings of the étale site were corrected by Bhatt-Scholze using the pro-étale topology ([BS15]) following Noohi's ideas of an infinite Galois theory ([Noo08]), which obliterated the need of profinite (*resp.*., prodiscrete) spaces and elegantly rendered, instead, a topological homotopy type generalising the previous ones.

It turned out, that, indeed, the étale topology was the correct one as the homotopy type of $X(\mathbf{C})$ for $X/\mathbf{C}$ of locally finite type could be recovered from the profinite étale homotopy types (*resp.*., prodiscrete étale homotopy types) after profinite (*resp.*., prodiscrete) completion. Such fact is remarkable as it's known by an example of Serre ( 3.0.5) that there are smooth projective varieties $X/\mathbf{C}$ such that, for $\sigma \in \mathrm{Aut}(\mathbf{C})$, $\sigma^* X(\mathbf{C})$ has a different fundamental group from the one of $X(\mathbf{C})$.

As it was already mentioned. The Zariski topology is insufficient to capture all the properties coming from the analytic structure of the complex points.

In algebraic topology the cohomology of constant sheaf $\mathbf{Z}$, which is exactly the integral singular cohomology $\mathrm{H}^\bullet_{\mathrm{sing}}(-,\mathbf{Z})$, gives a great amount of information about topological spaces. In the Weil's Conjectures, less is need. Indeed, no torsion information is need, only the Betti numbers are required. However, even in this case, the Zariski topology does not suffice.

**Theorem 3.0.3** (Grothendieck). *Let $X$ be an irreducible topological space, $A$ a ring and $\Lambda$ a constant sheaf of $A$-algebras on $X$.*

$$H^i(X,\Lambda) \cong 0$$

*for* $i \geq 0$.

*Proof.* Since every non-empty open $U$ of $X$ is dense and, hence, connected, $\Gamma(X,\Lambda) \longrightarrow \Lambda(U)$ is an isomorphism, *i.e.* , $\Lambda$ is flabby. As every flabby sheaf is acyclic, the result follows. □

The Zariski topology in known, however, to behave properly for (quasi)coherent sheaves. One may try, therefore, to compute the Betti numbers by

$$b_n := \sum_{i+j=n} h^{i,j}$$

and

$$h^{i,j} := \dim_k \mathrm{H}^i(X,\Omega^j_X)$$

for $X/k$ a smooth projective variety as Serre proposed in [Ser56a, §4]. These numbers satisfies a kind o Poincaré duality $b_n = b_{2d_X-n}$ if $d_X := \dim_k(X)$, but in general fail to satisfy other desired properties. It was known from an example by Igusa that it doesn't work in generality.

**Theorem 3.0.4** (Igusa, [Igu55]). *For any prime $p$, there exists a projective smooth variety $X/k$ of dimension $\frac{1}{2}(p-1)+1$ (resp., 2)* [4]*, whenever $p > 2$ (resp., $p = 2$), such that* $\mathrm{char}(k) = p$ *and $g < h^{0,1}$, where $g = \dim_k \mathrm{Pic}^0(X)$* [5]*. In particular, $h^{0,1} \neq h^{(1,0)}$. Furthermore, one can explicitly compute the Betti numbers defined by the $\zeta$ function of $X$ and, as a consequence, they differ from $b_n = \sum_{i+j=n} h^{i,j}$.*

In fact, the situation was worser than one may expect. Indeed, Serre constructed different embeddings $\sigma,\sigma'\colon K \hookrightarrow \mathbf{C}$ and a smooth projective variety $V/K$ such that $\pi_1(V \otimes_\sigma \mathbf{C}(\mathbf{C})) \not\cong \pi_1(V \otimes_{\sigma'} \mathbf{C}(\mathbf{C}))$. The weird part is that he also proved that the Betti numbers, the real ones, must always coincide ([Ser56b, §4 Prop. 12]). Indeed, that follows from Artin's comparison theorem.

[4] Indeed, when $\mathrm{char}(k) = p > 2$ (*resp.*, $p = 2$), $X/k$ is the quotient by an automorphism of order $p$ of a finite étale covering $A_1 \times_k A_2 \twoheadrightarrow X$ such that $A_1/k$ is the Jacobian of a plane curve (*resp.*, an elliptic curve) and $A_2/k$ is an elliptic curve with points of order $p$

[5] $\mathrm{Pic}^0(X)/k$ is non-reduced here and, therefore, it's not an abelian variety.

**THEOREM 3.0.5** (Serre, [Ser64]). *Let* $p = -1 \mod 4$ *be a prime,* $k = \mathbf{Q}(\sqrt{-p})$, $h = |\operatorname{Cl}(\mathcal{O}_k)|$, $K/k$ *a field extension such that* $[K\colon k] = h$ *(i.e. , the absolute class field or Hilbert class field of* $k$*, its maximal unramified abelian extension),* $h > 1$ *and* $\gcd(h, p-1) = 1$. *There exists a smooth projective variety* $V/K$ *and embeddings* $\sigma, \sigma'\colon K \hookrightarrow \mathbf{C}$ *such that*

$$\pi_1(V \otimes_\sigma \mathbf{C}(\mathbf{C})) \not\cong \pi_1(V \otimes_{\sigma'} \mathbf{C}(\mathbf{C})).$$

Still, however, there should exist an isomorphism

$$\pi_1(V \otimes_\sigma \mathbf{C}(\mathbf{C}))^\wedge \cong \pi_1(V \otimes_{\sigma'} \mathbf{C}(\mathbf{C}))^\wedge$$

between the profinite completions by the comparison theorem between the analytic and étale site, which, in particular, identifies finite étale coverings on the algebraic side with ones on analytic side.

## § 3.1 The Étale, fppf and fpqc Topologies

**DEFINITION 3.1.1.** Let $S$ be a scheme. The **fp topology on** $\mathbf{Sch}_{/S}$ is the Grothendieck topology generated by the pretopology where coverings of $X/S$ consist of jointly surjective collections of morphisms $\{U/S \longrightarrow X/S\}_{U/S \in \mathcal{U}}$ such that $\coprod_{U/S \in \mathcal{U}} U/S \longrightarrow X/S$ is flat.

*REMARK* 3.1.1. The fp topology is finer than the Zariski topology.

**DEFINITION 3.1.2.** Let $S$ be a scheme. The **fpqc topology on** $\mathbf{Sch}_{/S}$ is the Grothendieck topology generated by the pretopology where coverings of $X/S$ consist of jointly surjective collections of morphisms $\{U/S \longrightarrow X/S\}_{U/S \in \mathcal{U}}$ such that $\coprod_{U/S \in \mathcal{U}} U/S \longrightarrow X/S$ is fpqc.

*REMARK* 3.1.2. Recall that $f\colon Y/S \longrightarrow X/S$ being fpqc does not equate to being faithfully flat and quasi-compact despite the putative name fpqc (fidèlement plat et quasi-compact). Instead, one requires that $f$ is fpqc iff it's faithfully flat and every open affine $U/S \hookrightarrow X/S$ can be covered by open affines $\{V/S \hookrightarrow Y/S\}_{V/S \in \mathcal{V}}$ such that $\mathcal{V}$ is finite.

*Remark* 3.1.3. Notice that sheaffication does not always exist in the fpqc topology ([Wat75]). That is due to set-theoretically issues concerning the nonexistence of a set of fpqc coverings such that any other fpqc covering has a refinement belonging to this set. For instance, a field has arbitrarily many transcendental extensions by adding variables. That, actually, happens more generally for any non-zero ring ([Sta18, 0BBK, Lemma 33.9.14]).

*Remark* 3.1.4. The fpqc topology is coarser than the fp topology and finer than the Zariski topology.

**Definition 3.1.3.** Let $S$ be a scheme. The **fppf topology on $\mathbf{Sch}_{/S}$** is the Grothendieck topology generated by the pretopology where coverings of $X/S$ consist of jointly surjective collections of morphisms $\{U/S \longrightarrow X/S\}_{U/S\in\mathcal{U}}$ such that $\coprod_{U/S\in\mathcal{U}} U/S \longrightarrow X/S$ is fppf.

*Remark* 3.1.5. Recall that $f\colon Y/S \longrightarrow X/S$ is fppf (fidèlement plat de présentation finie) iff it's faithfully flat and locally of finite presentation.

*Remark* 3.1.6. The fppf topology is coarser than the fpqc topology and finer than the Zariski topology.

**Definition 3.1.4.** Let $S$ be a scheme. The **étale topology on $\mathbf{Sch}_{/S}$** is the Grothendieck topology generated by the pretopology where coverings of $X/S$ consist of jointly surjective collections of morphisms $\{U/S \longrightarrow X/S\}_{U/S\in\mathcal{U}}$ such that $\coprod_{U/S\in\mathcal{U}} U/S \longrightarrow X/S$ is étale.

*Remark* 3.1.7. The étale topology is coarser than the fppf topology and finer than the Zariski topology.

**Definition 3.1.5.** Let $A \longrightarrow B$ be a faithfully flat $A$-algebra. The **Amitsur augmented cosimplicial object of $A \longrightarrow B$** (*resp.*, **Amitsur cochain complex of $A \longrightarrow B$**), $B^\bullet$, is defined by applying the global sections functor $\Gamma$ to the augmented Čech $\infty$-groupoid of $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ (*resp.*, by applying the global sections functor and, then, inducing the underlying cochain complex by $d = \sum_i (-1)^i d_i^\#$). Explicitly,

$$A \longrightarrow B \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; B \otimes_A B \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; B \otimes_A B \otimes_A B \ldots$$

, where the cofaces of $B^\bullet$ are given by

$$d_i^\#(x_0 \otimes \cdots \otimes x_n) = x_0 \otimes \ldots \otimes x_{i-1} \otimes 1 \otimes x_{i+1} \ldots \otimes x_n$$

and the codegeneracies by

$$s_i^\#(x_0 \otimes \cdots \otimes x_n) = x_0 \otimes \ldots x_{i-1} \otimes x_i x_{i+1} \otimes x_{i+2} \cdots \otimes x_n$$

.

**PROPOSITION 3.1.1.** *Let $A \longrightarrow B$ be faithfully flat and $M$ an $A$-module. The complex $M \otimes_A B^\bullet$ is exact. Equivalently, the cosimplicial module $M \otimes_A B_\bullet$ is a resolution of $M$. In particular, the Amitsur complex is exact.*

*PROOF.* Since $B$ is faithfully flat as an $A$-algebra, it's enough to prove after applying $(-) \otimes_A B$. Now consider the homotopy $h(x_0, \ldots, x_n) = \sum_i (-1)^i s_i(x)$ given by the codegeneracies $s_i(x_0, \ldots, x_n) = x_0 \otimes \ldots x_{i-1} \otimes x_i x_{i+1} \otimes x_{i+2} \cdots \otimes x_n$. Notice that the homotopy only works because $\Delta^\#_{\mathrm{Spec}(B)} \colon B \otimes_A B \longrightarrow B$ is a section of the faces of the underlying cosimplicial object tensored by $B$. □

**LEMMA 3.1.1.** *Let $S$ be a scheme and $F \in \mathrm{Ob}(\widehat{\mathbf{Sch}_{/S}})$. $F \in \mathbf{Sh}((\mathbf{Sch}_{/S})_{fpqc})$ iff $F \in \mathbf{Sh}((\mathbf{Sch}_{/S})_{Zar})$ and $F$ satisfies the sheaf condition for fpqc morphisms of the form $X \longrightarrow Y$ such that $X$ and $Y$ are affines.*

*PROOF.* Let $f \colon Y/S \longrightarrow X/S$ be fpqc and $F \in \mathbf{Sh}((\mathbf{Sch}_{/S})_{\mathrm{Zar}})$. Since $F$ is a Zariski sheaf and being fpqc is preserved by base change, it's possible to assume that $X$ is affine. Let, then, $X$ be affine. Since $X$ is affine and $f$ is fpqc, $Y$ must have a finite collection of affines $\mathcal{V}$ with images covering $X$. One can refine further to a fpqc covering by affines

$$g \colon \coprod_{U/S \in \mathcal{U}} U/S \longrightarrow X/S$$

such that $\mathcal{V} \subset \mathcal{U}$ and any $U \in \mathcal{U}$ is an affine of $Y$. Since $g$ is a refinement of $f$ by open affines of $Y$ and $F$ satisfies Zariski descent, $F$ satisfies the sheaf condition for $g$ iff it satisfies it for $f$.

For every $U \in \mathcal{U}$, let $s_U \in F(U)$ and suppose that the collection $\{s_U | \; U \in \mathcal{U}\}$ equalise on double fiber products. In particular, the collection of local sections $\{s_V \mid V \in \mathcal{V}\}$ also equalise on double fiber products and, since $F$ satisfies the sheaf condition for the fpqc affine covering $g_{|\coprod_{V/S\in\mathcal{V}} V/S}$, there exists a unique global section $s \in F(X)$ such that $s_{|V} = s_V$ for every $V \in \mathcal{V}$.

Let $U \in \mathcal{U}$ and

$$g_U\colon \coprod_{V/S\in\mathcal{V}} (V \times_X U)/S \longrightarrow U/S$$

. Since $g_U$ is an fpqc affine covering of $U$ and

$$(s_{|U})_{|V\times_X U} = (s_{|V})_{|V\times_X U} = (s_V)_{|V\times_X U} = (s_U)_{|V\times_X U}$$

, $s_{|U} = s_U$ and, therefore, $F$ satisfies the sheaf condition for $g$.

The converse is trivial and, therefore, the lemma follows.

□

**Theorem 3.1.1.** *If $f\colon X \longrightarrow S$ is fpqc, then $f$ is an universal effective epimorphism, i.e. , for every $g\colon S' \longrightarrow S$, $g^*(f)$ is an effective epimorphism.*

*Proof.* Notice that being fpqc is preserved by base change. It's, therefore, enough, to prove that it's an effective epimorphism. Being an effective epimorphism is equivalent to the sheaf condition applied to the covering $f$ for every representable presheaf $h_Y$. Since representable presheaves are Zariski sheaves, it's enough to check for affines $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ by Lemma 3.1.1. Being an effective morphism means that

$$A \longrightarrow B \rightrightarrows B \otimes_A B$$

is limit cone. As it's enough to check after a faithfully base change, one may apply $(-) \otimes_A B$ to the above diagram.

□

**Corollary 3.1.1.** *Let $S$ be a scheme. The site $(\mathbf{Sch}_{/S})_{fpqc}$ is subcanonical.*

*Proof.* Recall that, for a fixed covering $X \longrightarrow S$, the sheaf condition for $h_X$ where $X$ varies is equivalent to $X \longrightarrow S$ being an effective epimorphism. □

**Corollary 3.1.2.** *Let $S$ be a scheme. The sites $(\mathbf{Sch}_{/S})_{fppf}$ and $(\mathbf{Sch}_{/S})_{ét}$ are subcanonical.*

*Proof.* The fpqc topology is finer than the étale topology. □

**Proposition 3.1.2.** *The site $(\mathbf{Sch}_{/S})_{fp}$ is not subcanonical.*

*Proof.* This example is due to Vistoli in [Vis05, Rem. 2.56]. Let $C/k$ be an integral smooth curve, $k$ algebraically closed, $U_x = \operatorname{Spec}(\mathscr{O}_{C,x})$

$$U = \coprod_{x \in |X|} U_x \longrightarrow C$$

be the obvious faithfully flat covering. Each summand $U_x$ has only two points $x$ and $\eta_x$ (corresponding to the restriction of the generic point $\eta$ of $C$).

Let $X := \operatorname{colim}_{x \in |X|}(\operatorname{Spec}(R(C) \longrightarrow V_x$, which exists since it's a composition of pushouts along open immersions. Notice that $U_x \times_C U_y \cong \operatorname{Spec}(R(C))$ when $x \neq y$ and $V_x \times_C V_x \cong V_x$. Then the canonical inclusions $(U_x \hookrightarrow X)_{x \in |X|} \in \prod_{x \in |X|} h_X(U_x) \cong h_X(U)$ agree along the two restrictions to $\prod_{x,y \in |X|} h_X(U_x \times_C U_y) \cong h_X(U \times_C U)$.

If $f \in h_X(C)$ such that it restricts to the inclusions $U_x \hookrightarrow C$, then $f(x) = x$ for each $x \in |X|$ and $f([\eta_x]) = \eta$. However such $f$ cannot be continuous since every subset of $X$ containing only closed points will be closed, whereas, in $C$, only the finite sets of closed points are closed. □

## § 3.2 Flat Descent and Hilbert 90

Descent is the method of descending properties. It was used when proving the exactness of the Amitsur complex in Proposition 3.1.1 impliclty by descending the exactness. More generally, when one has a covering $\{U \longrightarrow S\}_{U \in \mathcal{U}}$ such that some property holds locally for each $U \in \mathcal{U}$, it's usual to inquire when such property holds for $S$. Let $S' := \coprod_{U \in \mathcal{U}} U$ and $f \colon S' \longrightarrow S$ be the canonical morphism. Let $\mathscr{E}$ and $\mathscr{E}$ be categories fibered on some subcategory $\mathbf{Sch}_{/S}$ and $\mathbf{Sch}_{/S'}$, where one should think about it as $\mathscr{E}$ the category of objects having a certain property and $\mathscr{E}'$ as the category of objects satisfying some kind of (higher) sheaf condition, which is also called descent datum. The descent problem can be formulated as: when

$$f^*\colon \mathscr{E}(X/S) \longrightarrow \mathscr{E}'(X \times_S S'/S')$$

is an equivalence? Where $f^*$ associates a global object (morphism) to its restrictions. An object or morphism that succeeds in descending (*i.e.* , comes from the essential image of $f^*$) is said to have a an effective descent datum.

One is often concerned about descending properties of morphisms or objects. The following is an example of descent for faithfully flat morphisms.

**Theorem 3.2.1** (Flat Descent for Properties of Morphisms, [EGAIV-2]). *Let $f\colon X \longrightarrow Y$ be a morphism of $S$-scheme and $g\colon S' \longrightarrow S$ a faithfully flat morphism that is quasi-compact or locally of finite presentation. If the pullback $g^*(f)\colon X \times_S S' \longrightarrow X \times_S S'$ satisfy one of the following properties.*

1. *separated;*
2. *quasi-compact;*
3. *proper;*
4. *affine;*
5. *finite;*
6. *quasi-finite;*
7. *flat;*
8. *smooth;*
9. *unramified;*
10. *étale;*
11. *immersion;*
12. *closed immersion;*
13. *surjective;*
14. *injective;*

15. *bijective.*

*Then, so does $g^*(f)$.*

*Remark* 3.2.1. The list contained in proposition above could be made is longer. Actually, any of the usual properties of morphisms of schemes should be there except projectivity and quasi-projectivity.

Another interesting example illustrating descent follows.

**Definition 3.2.1.** Let $f\colon S' \longrightarrow S$ be a morphism of schemes and $\mathscr{F} \in \mathrm{Ob}(\mathbf{QCoh}(S'))$. A **descent datum for a quasi-coherent sheaf $\mathscr{F}$ over a covering $f\colon S' \longrightarrow S$** is the data of the 2-sheaf condition of the local section $\mathscr{F}$ of the 2-presheaf $\mathbf{QCoh}(-)\colon \mathbf{Sch}_{/S} \longrightarrow \mathbf{Grpd}_2$ for the covering $S'$. More explicitly, it consists of isomorphisms

$$d_1^*\mathscr{F} \xrightarrow[\varphi]{\sim} d_0^*\mathscr{F}$$

satisfying the cocycle condition

$$d_2^*\varphi \circ d_0^*\varphi = d_1^*\varphi$$

, where $d_i$ are the faces of the (augmented) Čech $\infty$-groupoid of the covering $S' \longrightarrow S$.

The **category of quasi-coherent sheaves on $S'$ with descent data over $S$** is denoted by $\mathbf{QCoh}(S'/S)$.

**Theorem 3.2.2** (fpqc Descent for $\mathbf{QCoh}$). *Every descent data for quasi-coherent sheaves is effective over an fpqc covering, i.e. , the morphism*

$$f^*\colon \mathbf{QCoh}(S) \longrightarrow \mathbf{QCoh}'(S'/S)$$

*is an equivalence.*

*Proof.* By Lemma 3.1.1, one may reduce the case to open affines. Let, then, $S' = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and $\mathscr{F}$ be the underlying sheaf of the $B$-module $M'$. A descent data is equivalent to the 2-truncated cosimplicial module (*resp.*., ring) $M'^\bullet$ over the 2-truncated Amitsur cosimplicial object $B^\bullet$

$$M'^0 \rightrightarrows M'^1 \rightrightarrows\!\!\!\rightarrow M'^2$$

$$B \rightrightarrows B \otimes_A B \rightrightarrows\!\!\!\rightarrow B \otimes_A B \otimes_A B$$

$$S' \leftleftarrows S' \times_S S' \leftleftarrows\!\!\!\leftarrow S' \times_S S' \times_S S'$$

, such that $(d_i^\#)^* M'^0 \longrightarrow M'^1$ and $(d_i^\#)^* M'^1 \longrightarrow M'^2$ (*i.e.* , the base change of the coface maps) are isomorphisms.

In this context,

$$f^*(M) = M \otimes_A B^\bullet = (M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B \rightrightarrows\!\!\!\rightarrow M \otimes_A B \otimes_A B \otimes_A B).$$

Notice, also, that the underlying complex $M \longrightarrow f^*(M)$ (*i.e.* , after extending by the nerve and defining $d = \sum_i (-1)^i d_i$) is exact by PROPOSITION 3.1.1. By defining $G(M')$ as the limit of $M'^\bullet$ (*i.e.* , the equaliser of the first cofaces), that implies the canonical morphism $M \longrightarrow G(f^*(M))$ is an isomorphism.

Now, one must prove that the (adjoint) canonical morphism

$$(f^*(G(M')))^0 = G(M') \otimes_A B \longrightarrow M'$$

is an isomorphism. One can check that after a faithfully flat base change $A \longrightarrow A'$. Choosing $A' = B$, yields the diagonal $\Delta_{S'}^\# \colon B \otimes_A B \longrightarrow B$ as a section of the restrictions. Hence, one can restrict to the case where $A \longrightarrow B$ has a section. In this case, applying $(-) \otimes_B A$ gives

$$G(M') \longrightarrow M' \otimes_B A.$$

Since $B$ is faithfully flat as an $A$-algebra and applying again $(-) \otimes_A B$ engender a morphism isomorphic to the identity $1_{M'}$, $G(M') \longrightarrow M' \otimes_B A$ must also be an isomorphism.

A more detailed proof can be found in [GW10, Thm 14.66]. A short and but not too detailed proof can be found in [SGA4$\frac{1}{2}$, Thm 1.4.5].

□

**COROLLARY 3.2.1** (Hilbert 90). *Let $L/k$ be a finite Galois extension of fields.*

$$\mathrm{H}^1(G_k, L^\times) \cong 0.$$

*Proof.* Consider the fpqc covering $\operatorname{Spec}(L) \longrightarrow \operatorname{Spec}(k)$. Notice that $L \cong k[x]/(f(x))$ such that $f(x)$ is irreducible and $f(x) = \prod_i f_i(x)$ in $L$, where $f_i(x)$ is linear. Hence

$$\varphi\colon L \otimes_k L \cong L[x]/(f(x)) \cong \bigoplus_i L[x]/((f_i(x))) \cong \bigoplus_{g \in G} L$$

, where the explicit isomorphism is given by $\varphi(a,b) = (a.g(b))_{g \in G}$ (see Example 3.3.1 for further details).

A descent data for a vector space $V'$ over $L$ along $L/k$ is, therefore, equivalent to an action of $G$

$$\sigma\colon G \longrightarrow \operatorname{End}(V)$$

which satisfies $\sigma_g(av) = g(a)\sigma_g(v)$ (*i.e.* , semilinear) and the cocycle condition $\sigma_{gh} = \sigma_g \circ \sigma_h$ for every $g, h \in G$, $a \in L$ and $v \in V$.

Let $V'$ be 1-dimensional. The action is, then, determined by a map

$$c\colon G \longrightarrow L^\times$$

such that $\sigma_g(v) = c(g)v$ for a fixed $v \in V \setminus \{0\}$. The cocycle condition is equivalent to

$$c(gh)v = \sigma_{gh}(v) = \sigma_g(\sigma_h(v)) = \sigma_g(c(h)v) = g(c(h))c(g)v$$

.Hence, the cocycle condition for $\sigma$ is equivalent to $c$ satisfying the cocycle condition for group cohomology

$$c(gh) = c(g)g(c(h)).$$

Equivalently, a descent data for $V'$ is equivalent to a choice of a cocycle $c$ of group cohomology.

By fpqc descent ( 3.2.2), one can descend $V'$ to $V := (V')^G$. Then there exists $b \in L^\times$ such that $bv \in V'$ is $G$-invariant. Hence

$$bv = \sigma_g(bv) = g(b)\sigma_g(v) = g(b)c(g)v$$

holds. Therefore

$$c(g) = \frac{b}{g(b)}$$

, *i.e.* , $c$ is a coboundary.

□

## § 3.3 Galois Descent

Galois' decent here means decent along $G$-torsors where $G$ is a finite group. One usually requires such torsor to be faithfully flat, fpqc or fppf in order to deduce nice properties. That is probably one of the oldest kind of descents which was known since the study of Riemann surfaces and ramifications in global fields under different disguises. As will be shown, such $G$-torsors under mild assumptions form an initial category in the category of finite étale coverings and, therefore, suffice to compute the étale fundamental group, which is the group of automorphisms of the fiber functor.

**Definition 3.3.1.** A **Galois covering** $Y \longrightarrow X$ **with group** $G$ is a morphism $Y \longrightarrow X$ that is a $G$-torsor for $G$ a finite group, *i.e.* , the canonical morphism

$$Y \times_X G_X \xrightarrow{\Delta_Y \times 1_{G_X}} Y \times_X Y \times_X G_X \xrightarrow{1_Y \times \mu} Y \times_X Y$$

is an isomorphism, where $\mu\colon Y \times G_X \longrightarrow Y$ is an action of $X$-schemes and $G_X$ is the constant group scheme with value $G$ or, equivalently, $G_X = \coprod_{g \in G} X$.

*Remark* 3.3.1. When $X$ and $Y$ are irreducible, the above definition is equivalent to a $G$-action of $X$-schemes such that $R(Y)/R(X)$ is a Galois extension with group $G$.

*Example* 3.3.1. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. It's a Galois covering iff $B$ is an $A$-algebra and

$$B \otimes_A B \xrightarrow{1_B \otimes \mu^\#} B \otimes_A B \otimes_A \prod_{g \in G} A \xrightarrow{m \otimes 1} B \otimes_A \prod_{g \in G} A \cong \prod_{g \in G} B$$ is an isomorphism, where $\mu^\#\colon B \longrightarrow \prod_{g \in G} A \otimes_A B \cong \prod_{g \in G} B$ and $m\colon B \otimes_A B \longrightarrow B$ is the product of $B$ as an $A$-algebra.

Equivalently, the morphism

$B \otimes_A B \xrightarrow{f} \prod_{g \in G} B$

is an isomorphism, where $f(b,b') = (bg(b'))_{g \in G}$ and $g(x)$ is defined as $\pi_g(\mu^\#(x))$.

*Remark* 3.3.2. The morphism $\pi_g(\mu^\#(x))$ in Example 3.3.1 gives an action of $G$ on $B$ by $A$-algebra automorphisms.

**Definition 3.3.2.** Let $X \in \mathrm{Ob}(\mathbf{Sch})$. A **geometric point** is a morphism $\mathrm{Spec}(k) \longrightarrow X$ for $k$ algebraically closed or separably closed.

The following notation will be used. Let $\mathrm{Spec}(k) \longrightarrow X$ be a geometric point with image $x \in X_{\mathrm{Zar}}$. $\mathrm{Spec}(k) \longrightarrow X$ will be denoted by $\overline{x}$, $k$ will be denoted by $\kappa(\overline{x})$, $\overline{\kappa(x)}$ or $\kappa(x)^{\mathrm{sep}}$. When one needs to emphasise that $k$ can only be separably closed, the notation $x^{\mathrm{sep}}$ will be used instead of $\overline{x}$ (*resp.*, $\kappa(x^{\mathrm{sep}})$ or $\kappa(x)^{\mathrm{sep}}$ instead of $\kappa(\overline{x})$ or $\overline{\kappa(x)}$).

The following abuse of notation will occur. $\overline{x}$ (*resp.*, $x^{\mathrm{sep}}$) might denote either $\kappa(\overline{x})$ (*resp.*, $\kappa(x^{\mathrm{sep}})$) or the corresponding to $X$.

**Definition 3.3.3.** Let $\mathbf{FÉt}_X$ be the site induced by subcategory of $X_{\text{ét}}$ (*i.e.* , a morphism of sites $X_{\text{ét}} \longrightarrow \mathbf{FÉt}_X$) consisting only of finite étale morphisms and $\overline{x}$ a geometric point. The **fiber functor** is defined as

$$F^{fin}{}_{\overline{x}} \colon \mathbf{FÉt}_X \xrightarrow{(-)\times_X \overline{x}} \mathbf{FÉt}_{\overline{x}} \xrightarrow{U} \mathbf{Fin} \hookrightarrow \mathbf{Set}$$

and

$$F^{disc}{}_{\overline{x}} \colon X_{\text{ét}} \xrightarrow{(-)\times_X \overline{x}} \mathbf{FÉt}_{\overline{x}} \xrightarrow{U} \mathbf{Set}$$

, where $U$ is the forgetful functor. The abuse of notation $F_{\overline{x}} := F^{fin}{}_{\overline{x}}$ and $F_{\overline{x}} := F^{disc}{}_{\overline{x}}$ will be used whenever it's clear that the underlying site is $\mathbf{FÉt}_X$ or $X_{\text{ét}}$.

**Lemma 3.3.1.** *Let $S \in \mathrm{Ob}(\mathbf{Sch})$, $X \in \mathrm{Ob}(\mathbf{Sch}_{/S})$ $S$-étale, $Y \in \mathrm{Ob}(\mathbf{Sch}_{/S})$ connected, $f,g \colon Y \longrightarrow X \in \mathbf{Sch}_{/S}(Y,X)$ and $\overline{y}$ a geometric point of $Y$. If $f(\overline{y}) = g(\overline{y}) \in X(\kappa(x))$ and $X$ is $S$-separated, then $f = g$. In particular, sections of a separated étale morphism $s_1, s_2 \colon U \to T$ (e.g. , a finite étale morphism) coinciding in a geometric point must be equal whenever $U$ is connected.*

*Proof.* Since $X$ are separated, the morphism $\Delta_{X/S}\colon X \longrightarrow X \times_S X$ is a closed immersion. Since étale morphisms are preserved by base-change, $\Delta_{X/S}$ is also étale and, hence, open. Therefore, the image of $\Delta_{X/S}$ is clopen and, then induces a splitting

$$X \times_S X \cong X \sqcup X'$$

for some $X' \in \mathrm{Ob}(\mathbf{Sm}_{/S})$. By the assumption that

$$(f,g)\colon Y \longrightarrow X \times_S X$$

has image intersecting the diagonal $\Delta_{X/S}(X) = X \hookrightarrow X \sqcup X'$ and the connectivity of $Y$, one can conclude that the image of $(f,g)$ is actually $X$. □

**Lemma 3.3.2.** *Let $Y \longrightarrow X$ be a finite étale morphism, $Y$ connected and $\overline{x}$ a geometric point.*

$$|\operatorname{Aut}_X(Y)| \leq |F^{fin}{}_{\overline{x}}(Y)|.$$

*Furthermore, equality holds iff $Y \longrightarrow X$ is Galois with group* $\operatorname{Aut}_X(Y)$.

*Proof.* Let $G = \operatorname{Aut}_X(Y)$ and

$$p\colon Y \times_X G_X \longrightarrow Y \times_X Y$$

the canonical action. After applying the fiber functor, the above results in

$$F(p)\colon F_{\overline{x}}(Y) \times G \longrightarrow F_{\overline{x}}(Y) \times F_{\overline{x}}(Y).$$

By the connectedness of $Y$, any two automorphisms agreeing on a point lying over $\overline{x}$ must be equal (Lemma 3.3.1) and, hence, $F(p)$ is injective. Notice that $F_{\overline{x}}$ reflects monomorphisms and isomorphisms (also, epimorphisms). That, immediatly, implies the result.

□

**Proposition 3.3.1.** *Let $f\colon Y \longrightarrow X$ be fpqc and $Y$ connected. $f\colon Y \longrightarrow X$ is Galois with group $G$ iff $f$ is finite, étale of degree $|G|$ and $G \cong$* $\operatorname{Aut}_X(Y)$.

*Proof.* The implication $\Rightarrow$ follows from 3.2.1 by applying the covering $f$ itself.

The implication $\Leftarrow$ follows from Lemma 3.3.2 by noticing that $G \hookrightarrow \mathrm{Aut}_X(Y)$ is a monorphism of finite groups of the same cardinality. □

**Corollary 3.3.1.** *Let $L$ and $k$ be fields, $X = \mathrm{Spec}(k)$ and $Y = \mathrm{Spec}(L)$. $Y \longrightarrow X$ is a Galois covering with group $\mathrm{Gal}(L/k)$ iff $L/k$ is a finite Galois extension.*

*Proof.* That follows immediately for simple extensions. Consider

$$L \otimes_k L \cong L[x]/(f(x)) \cong \bigoplus_i L[x]/((f_i(x))^{e_i})$$

, where $L \cong k[x]/(f(x))$ for $f$ monic irreducible and $f$ splits into irreducibles $\prod_i f_i^{e_i}$ inside $L$.

The right hand side is isomorphic to a finite sum of $L$'s iff $e_i = 1$ and $f_i$ is linear for every $i$. Equivalently, $L/k$ is normal and separable, *i.e.* , Galois.

However, by Proposition 3.3.1, $L/k$ must be separable and, therefore, simple. □

**Theorem 3.3.1** (Galois' Descent). *Let $Y \longrightarrow X$ be a Galois covering with group $G$, $F \in \widehat{(\mathbf{Sch}_{/X})_\tau}$, or $F \in \widehat{X_\tau}$ and $Y \longrightarrow X$ of type $\tau$. If $F$ preserves finite limits, then $F$ satisfies the sheaf condition for $Y \longrightarrow X$, i.e. ,*

$$F(X) \longrightarrow F(Y) \rightrightarrows F(Y \times_X Y)$$

*is a limit cone, iff the restriction $F(X) \longrightarrow F(Y)$ is isomorphic to the canonical inclusion $F(Y)^G \longrightarrow F(Y)$.*

*Proof.* Since $Y \longrightarrow X$ is a Galois covering with group $G$, the diagram

$$\begin{array}{ccccc} Y \times_X Y & \rightrightarrows & Y & \longrightarrow & X \\ \downarrow{\scriptstyle\sim} & & \| & & \| \\ \prod_{g\in G} Y \cong Y \times_X G_X & \underset{\mu}{\overset{\pi_Y}{\rightrightarrows}} & Y & \longrightarrow & X \end{array}$$

is commutative. By applying $F$ and using that it preserves finite limits, one obtains the commutative diagram

$$\begin{array}{ccccc} F(X) & \longrightarrow & F(Y) & \rightrightarrows & F(Y \times_X Y) \\ \| & & \| & & \downarrow \sim \\ F(X) & \longrightarrow & F(Y) & \underset{F(\mu)}{\overset{F(\pi_Y)}{\rightrightarrows}} & \prod_{g \in G} F(Y) \end{array}$$

where $F(\pi_1)(s) = (s)_{g\in G}$ and $F(\mu)(s) = (g(s))_{g\in G}$.

Therefore lower row is an limit cone iff $F(X) \cong F(Y)^G$. □

## § 3.4 Étale Fundamental Group

The étale fundamental group was first defined in [SGA1] and later enlarged in [SGA3-II, Exp. X.6]. However, it was already clear in [Ser58a] how to define it properly (up to non-separability of field extensions, *i.e.* , the internal symmetries captured in the definition of étale morphism).

In algebraic topology, one defines the fundamental group by using 1-dimensional paths, which, up to homotopy, are 1-dimensional real manifolds. Such naive definition is impossible for the algebro-geometric world, which is only capable to see the topology of the complex points. However, it was well known (when the respective space is connected, locally path-connected and semi-locally simply connected) that the fundamental group could be defined as the group of automorphisms of all non-ramified coverings or, equivalently, as the group of automorphisms of the universal covering space, which corresponds bijectively to the fiber thereof.

The correspondence between smooth projective curves over $\mathbf{C}$ (Riemann surfaces) and finite extensions of $R(\mathbf{P}^1_{\mathbf{C}}) = \mathbf{C}(x)$ or, more generally, between unramified coverings of smooth projective curves $X/\mathbf{C}$ and extensions of $R(X)$ was also well known. Under that correspondence, the Galois group of the field extensions is exactly the group of automorphisms of the covering. It was, therefore, easy to guess the correct definition.

The remarkable fact about the profinite étale fundamental group $\widehat{\pi}_1^{\text{ét}}(X/k, \overline{x})$ and the prodiscrete étale fundamental group $\pi_1^{\text{ét}}(X/k, \overline{x})$ (for $X$ smooth over a field $k$ of characteristic 0) are such that, after profinite completion and prodiscrete completion, respectively, recover the topological fundamental group of

$X(\mathbf{C})$ even though the topological homotopy type of $\sigma^*X(\mathbf{C})$ for different embeddings $\sigma\colon k \hookrightarrow \mathbf{C}$ differ ([Ser64]).

**Definition 3.4.1.** Let $X$ be a scheme, $\overline{x}$ and $\overline{y}$ be geometric points of $X$. The **profinite étale fundamental group of $X$ (*resp.*, prodiscrete étale fundamental group)**, $\widehat{\pi}_1^{\text{ét}}(X,\overline{x},\overline{y})$ (*resp.*, $\pi_1^{\text{ét}}(X,\overline{x},\overline{y})$), is defined by

$$\widehat{\pi}_1^{\text{ét}}(X,\overline{x},\overline{y}) := \mathrm{Iso}(F_{\overline{x}}^{fin}, F_{\overline{y}}^{fin}).$$

(*resp.*,

$$\pi_1^{\text{ét}}(X,\overline{x},\overline{y}) := \mathrm{Iso}^{grpd}(F_{\overline{x}}^{disc}, F_{\overline{y}}^{disc}$$

, where $\mathrm{Iso}^{grpd}(F',F'')$ denotes a natural isomorphism in the groupoid completion of the category of fiber functors). When $\overline{x} = \overline{y}$, $\widehat{\pi}_1^{\text{ét}}(X,\overline{x},\overline{y})$ (*resp.*, $\pi_1^{\text{ét}}(X,\overline{x},\overline{y})$) will be denoted by $\widehat{\pi}_1^{\text{ét}}(X,\overline{x})$ (*resp.*, $\pi_1^{\text{ét}}(X,\overline{x})$).

*Remark* 3.4.1. $\widehat{\pi_1}^{\text{ét}}(X,\overline{x},\overline{y})$ and its prodiscrete version can be endowed canonically with a topology by setting the induced topology on the subspace

$$\widehat{\pi}_1^{\text{ét}}(X,\overline{x},\overline{y}) \hookrightarrow \Pi_{Z\in \mathrm{Ob}(\mathbf{FÉt}_X)} \mathrm{Iso}(F_{\overline{x}}(Z), F_{\overline{y}}(Z)).$$

**Proposition 3.4.1.** *Let $X$ be a scheme and $\overline{x}$ a geometric point. The profinite étale fundamental group $\widehat{\pi}_1^{\text{ét}}(X,\overline{x})$ is a profinite topological group and the prodiscrete étale fundamental group $\pi_1^{\text{ét}}(X,\overline{x})$ is a prodiscrete topological group.*

*Proof.* By the previous remark, $\widehat{\pi}_1^{\text{ét}}(X,\overline{x})$ is a closed subgroup of a compact Hausdorff totally disconnected group (*i.e.* , a closed subgroup of profinite group and, hence, also profinite).

By [Moe89, p. 1.4] and taking the cofiltered diagram [6] induced by the projection of image of $\pi_1^{\text{ét}}(X,\overline{x}) \hookrightarrow \Pi_{Z\in \mathrm{Ob}(\mathbf{FÉt}_X)} \mathrm{Iso}(F_{\overline{x}}(Z), F_{\overline{y}}(Z))$, $\pi_1^{\text{ét}}(X,\overline{x})$ can be realised as a cofiltered limit of localic discrete groups with surjective transition morphisms. Since $\pi_1^{\text{ét}}(X,\overline{x})$ is a closed subgroup of a Hausdorff, it's Hausdorff and, hence, sober. Since the category of sober topological spaces embeds fully faithfully into the category of locales and the limit exists

[6] By the latter proved Lemma 3.4.1, a choice of elements $\xi_i \in F_{\overline{x}}(Y_i)$ for every Galois covering $Y_i$ and choosing a suitable subcategory of elements of the fiber functor, one can take an initial cofiltered subcategory of $X_{\text{ét}}$.

in topological spaces (and is equal to the one in sober topological spaces), $\pi_1^{\text{ét}}(X,\overline{x})$ must be a topological group and, therefore, a prodiscrete topological group. □

**Definition 3.4.2.** Let $X$ be a scheme, $\overline{x}$ and $\overline{y}$ be geometric points of $X$. $\overline{x}$ *is an étale specialisation of* $\overline{y}$ *(resp.,* $\overline{y}$ *is an étale generalisation of* $\overline{x}$*),* $\overline{y} \rightsquigarrow \overline{x}$, whenever every étale neighborhood of $\overline{x}$ contains $\overline{y}$, *i.e.* , for every $\overline{x} \to U \to X$ étale neighborhood of $\overline{x}$, there's a lifting

$$\begin{array}{ccccc} & & U & & \\ & \nearrow & \downarrow & \nwarrow & \\ \overline{x} & \longrightarrow & X & \longleftarrow & \overline{y} \end{array}$$

to or, equivalently,

$$\mathbf{Sh}(X_{\text{ét}})(\overline{y}, \operatorname{Spec}(\mathscr{O}_{X_{\text{ét}},x})) = \mathbf{Sh}(X_{\text{ét}})(\overline{y}, \operatorname{Spec}(\mathscr{O}^{sh}_{X_{\text{Zar}},x}))$$

is non-empty.

**Theorem 3.4.1** ([SGA4-II, Éxp. VIII.7, Thm. 7.9], [SGA4-II, Éxp. IX.2, Prop. 2.13]). *Let* $X \in \operatorname{Ob}(\mathbf{Sch})$. *The geometric points and specialisations are in bijective correspondence with, respectively, points of the topos* $\mathbf{Sh}(\mathbf{FÉt})$ *(resp.,* $\mathbf{Sh}(X_{\text{ét}})$*) and natural transformations between such points.*

*Let* $X$ *be also topologically locally Noetherian (or, more generally, locally have a finite number of irreducible components)* $\overline{x},\overline{y}$ *be geometric points of* $X$*.*

$$\widehat{\pi}_1^{\text{ét}}(X,\overline{x},\overline{y}) \cong \mathbf{Point}(\mathbf{Sh}(X_{\text{ét}}))^{grpd}(\overline{x},\overline{y})$$

*(resp.,*

$$\pi_1^{\text{ét}}(X,\overline{x},\overline{y}) \cong \mathbf{Point}(\mathbf{Sh}(\mathbf{FÉt}_X))^{\mathbf{Grpd}}(\overline{x},\overline{y})$$

*, where* $\mathscr{C}^{grpd}$ *denotes the groupoid completion of a category). Explicitly, there exists a bijective correspondence with finite ordered sequences* $(\overline{p}_0, \overline{p}_1, \cdots \overline{p}_n)$ *of geometric points of* $X$ *together with specialisations* $\overline{p}_i \rightsquigarrow \overline{p}_j$ *such that* $\overline{p}_0 := \overline{x}$, $\overline{p}_n := \overline{x}$, $\overline{p}_n := \overline{y}$ *and* $j = i+1, 1-1$ *(i.e. paths of specialisations and generalisation)*

*Proof.* A point in a topoi $\mathbf{Sh}(\mathcal{C},J)$ [7] is determined by it a left exact cocontinuous (fiber) functor $x^* : \mathbf{Sh}(\mathcal{C},J) \longrightarrow \mathbf{Set}$. Restricting to the Yoneda embedding and assuming that $\mathcal{C}$ is closed by finite limits, a point becomes equivalent to a (fiber) functor

$$\mathcal{C} \longrightarrow \mathbf{Set}$$

which is left exact and sends $J$-covering families to jointly epimorphic families. For every point $x$,

$$\mathcal{U}_x := \mathcal{C}_{/x^*} \cong (\widehat{\mathcal{C}^{\mathrm{op}}}_{/x^*})^{\mathrm{op}}$$

denotes the category of neighbourhoods of $x$. Since $x^*$ preserves finite limits and $\mathcal{C}$ is finitely complete, $\mathcal{U}_x$ is cofiltered [8] and, therefore, the category of points is a subcategory

$$\mathbf{Point}(\mathbf{Sh}(\mathcal{C},J)) \hookrightarrow \mathrm{Pro}(\mathcal{C})$$

given by diagrams of the form $\mathcal{U}_x \longrightarrow \mathcal{C}$, where $(U,u) \in \mathrm{Ob}(\mathcal{U}_x) \mapsto U$. Equivalently, every fiber functor $x^*$ is ind-(co)representable by the presheaf on $\mathcal{C}^{\mathrm{op}}$ (or copresheaf on $\mathcal{C}$)

$$\mathrm{colim}_{(U,u)\in\mathcal{U}_x^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}}(-,U)$$

, where $u \in x^*U$, *i.e.* , $(U,u) \in \mathcal{C}_{/x^*}$.

If $X \in \mathrm{Ob}(\mathcal{C})$. A fiber functor over $y^* : \mathbf{Sh}(\mathcal{C},J)/X \longrightarrow \mathbf{Set}$ corresponds exactly to the data of a fiber functor $x^* : \mathcal{C} \longrightarrow \mathbf{Set}$ together with an element $s \in x^*X$ by precomposing with the inverse image of the étale morphism of topos $X^* = (-) \times X : \mathbf{Sh}(\mathcal{C},J) \longrightarrow \mathbf{Sh}(\mathcal{C},J)/Y$ and taking $s$ as the unique element of $y^*(1_X) \hookrightarrow y^*(X \times X \twoheadrightarrow X) = x^*(X)$.

In particular, for $\mathcal{C} = X_{\text{ét}}$, points are given by cofiltered diagram of étale coverings. Since the underlying (spatial) locale of $\mathbf{Sh}(X_{\text{ét}})$ is $X_{\mathrm{Zar}}$, any point $x$ on $\mathbf{Sh}(X_{\text{ét}})$ induces a point on $x_0 \in |X|$, and $x^*(F)$ is non-empty iff

[7] Recall that the category of points of a topoi $\mathbf{Sh}(\mathcal{C},J)$ is defined by $\mathbf{Point}(\mathbf{Sh}(\mathcal{C},J)) := \mathbf{Topoi}(\mathbf{Set},\mathbf{Sh}(\mathcal{C},J))$, which is equivalent to internally flat functors $\mathcal{C} \longrightarrow \mathbf{Set}$ sending $J$-covering families to jointly epimorphic families.

[8] A category $\mathcal{I}$ is cofiltered iff, for every $D\colon \mathcal{I} \longrightarrow \mathbf{Set}$, $\mathrm{colim}\, D$ commutes with finite limits

${x_0}^*(\mathrm{Im}(F \to 1))$ is non-empty iff $x_0 \in |U|$, where $U$ is the (spatial) sublocale associated to the subframe $\mathrm{Im}(F \to 1)$. Let $U$ be an affine (Zariski) open neighborhood of $x_0 \in |X|$. The subdiagram consisting of étale coverings of $U$ covering the point $x_0$ of the pro-object associated to $x^*$ is is initial (or cofinal), and, hence, such diagram is isomorphic to the one associated to $x^*$. A point $x$, then, is equivalent to an ind-object given by all the local étale algebras $A \longrightarrow B_i$, where $\mathrm{Spec}(A) = \mathscr{O}_{U,x_0}$. Every such algebra is of finite presentation (*i.e.* , compact in the category of $A$-algebras) and, hence, its actual colimit, $A^{sh}$, is isomorphic to the respective ind-object. Restricting, further to the fiber, a point $x$ is equivalent to a colimit of finite separable extensions of $\kappa(x_0)$ and, therefore, it's equivalent to a choice of separable closure for $\kappa(x_0)$.

A natural transformation between points $x_* \Rightarrow y_*$ is exactly a natural transformation between $y^* \Rightarrow x^*$. Let

$$\mathrm{colim}_i\, \mathbf{Sh}(X_{\text{ét}})(Y_i, -)$$

and

$$\mathrm{colim}_j\, \mathbf{Sh}(X_{\text{ét}})(X_j, -)$$

be, respectively, the ind-representable associated to $y^*$ and $x^*$. Since an étale neighborhood is locally of finite presentation (hence, in particular, each $X_j$ is so),

$$\begin{aligned}\mathrm{Nat}(y^*, x^*) &\cong \lim_i \mathrm{colim}_j\, X_{\text{ét}}(X_{\text{ét}}(Y_i, -), X_{\text{ét}}(X_j, -)) \cong \lim_i \mathrm{colim}_j\, \mathbf{Sch}_{/X}(X_j, Y_i) \cong \\ &\cong \mathbf{Sch}_{/X}(\lim_j X_j, \lim_i Y_i) = \mathbf{Sch}_{/X}(\mathrm{Spec}(\mathscr{O}^{\text{ét}}_{X,x}), \mathrm{Spec}(\mathscr{O}^{\text{ét}}_{X,y}))\end{aligned}$$

. Therefore $x_* \Rightarrow y_*$ is exactly an étale specialisation.

For the second paragraph of the theorem, notice that, in each connected component of $X$, any two geometric points are linked by a zig-zag of specialisations and generalisations (*i.e.* , the connected components of $X$ are étale path-connected when $X$ locally have a finite number of irreducible components; in particular, when $X$ is locally topologically Noetherian). Now, notice that every natural transformation between points of the topoi restrict to a natural isomorphism on the full subcategory of the locally constant finite étale sheaves. The finite assumption (or constructible) is essential since one needs to all the sections of the stalk to a unique étale neighborhood (whereas, generally, one can only lift a finite number of elements of the stalk to a local section). □

**Proposition 3.4.2.** *Let $X$ be a scheme and $\overline{x}$ and $\overline{y}$ be geometric points of $X$. Every $p \in \widehat{\pi}_1^{\text{ét}}(X,\overline{x},\overline{y})$ induces a canonical isomorphism*

$$p_*\colon \widehat{\pi}_1^{\text{ét}}(X,\overline{x}) \xrightarrow{\ \sim\ } \widehat{\pi}_1^{\text{ét}}(X,\overline{y}).$$

*In particular, if $X$ is connected and topologically locally Noetherian (or, more generally, locally with a finite number of irreducible components), $\widehat{\pi}_1^{\text{ét}}(X,\overline{x}) \cong \widehat{\pi}_1^{\text{ét}}(X,\overline{y})$ for any two geometric points $\overline{x}$ and $\overline{y}$.*

*Proof.* The first assertion is trivial. The second assertion means that if $X$ is (Zariski) connected and topologically locally Noetherian, then it's étale path-connected; *i.e.* , for every geometric points $x$ and $y$, there exists a finite sequence of geometric points $z_0, z_1 \cdots, z_n$ and étale specialisations $z_i \rightsquigarrow z_j$, where $z_0 := x$, $z_n := y$ and $j = i+1, i-1$ (*i.e.* , an ordered sequence of étale specialisations and generalisations starting with $x$ and ending with $y$). That follows by noticing that, in the above case, connected components are clopen and a constructible set closed under (Zariski) specialisations and (Zariski) generalisations must be clopen. □

**Lemma 3.4.1.** *Let $X$ be a connected scheme and $Y \longrightarrow X$ a connected finite étale covering. There exists a Galois covering $Z \longrightarrow X$ and a morphism $p \in \mathbf{FÉt}_X(Z,Y)$, i.e. , Galois coverings are initial in $\mathbf{FÉt}_X$.*

*Proof.* Let $Z := Y^{\times_X n}$ with the canonical action of $\Sigma_n$, where $n = |Y_{\overline{x}}|$. Consider the decomposition into connected components $Y^{\times_X n} \cong \coprod_{U \in \mathcal{U}} U$. Let $U \in \mathcal{U}$ containing some $(y_1, \ldots, y_n) \in Y^{\times_X n}$ pairwise distinct and $G \hookrightarrow \Sigma_n$ be the subgroup such that $g(U) = U$ for every $g \in G$. Since the image of any inclusion in the diagonal $Y^{\times_X n-1} \longrightarrow Y^{\times_X n}$ is clopen (because it's étale and $Y \to X$ is separated) and $U$ is connected, $U$ cannot intersect such images and, then, any $(y_1, \ldots, y_n) \in U$ must be pairwise distinct. Hence, $G$ acts transitively. Since $\Sigma_n$ acts freely, $G$ also does so and, therefore, $U \longrightarrow X$ is a Galois covering with group $G$.

□

*Remark* 3.4.2. In the proof of the above theorem (Lemma 3.4.1), the apparently unmotivated trick can be justified by the following. Let $X \in \mathrm{Ob}(\mathbf{Top})$ be path-connected and semi-locally simply connected, $Y \in \mathrm{Ob}(\mathbf{Top})$, $p : Y \longrightarrow X$ a path-connected finite covering space, $G := \pi_1(X)$ and $H := p_*(\pi_1(Y))$, which is necessarily open of finite index. A Galois covering dominating $Y$ corresponds exactly to open (of finite index) normal subgroups of $G$, $N$, contained in $H$ (by taking the Galois covering $\widetilde{X}/N$).

Let $x \in X$ be a fixed point. There's an isomorphism

$$Y_x \cong G/H$$

and

$$\Sigma_n \cong \mathrm{Aut}_{\mathbf{Set}}(G/H) \hookrightarrow (Y^{\times_X n})_x \cong \mathrm{End}_{\mathbf{Set}}(G/H)$$

, where $n := [G\colon H]$. The action of $G$ on $Y_x$ defines a morphism

$$G \longrightarrow \Sigma_n$$

, which factors through $\prod_{o \in \mathcal{O}_G} \Sigma_{n_o} \hookrightarrow \Sigma_n$ for $\mathcal{O}_G$ the orbits of $G$ and $n_o := |o|$ (in particular, $\sum_{o \in \mathcal{O}_G} n_o = n$). Choosing a connected component, $U$, of $Y^{\times_X n}$ containing points with distinct coordinates (*i.e.* , something that factors the $G$-action through $\Sigma_n \hookrightarrow \mathrm{End}_{\mathbf{Set}}(n)$) is equivalent to choosing an orbit $o_U$ such that $U_x \cong o_U$ and, therefore, an epimorphism

$$\rho_U : G \twoheadrightarrow \Sigma_{o_U}$$

. Taking $N := \mathrm{Ker}(\rho_U)$ elicits the fact that $U \cong \widetilde{X}/N$ and $U_x \cong G/N$.

**Theorem 3.4.2.** *Let $X$ be a scheme and $\overline{x}$ be a geometric point. The functor $F_{\overline{x}}$ is pro-representable by Galois coverings.*

*Proof.* Every geometric point $\overline{x}$ defines a point $\overline{x}^* \dashv \overline{x}_*$ of the topos $\mathbf{Sh}(\mathbf{FÉt}_X)$ such that $F_{\overline{x}} = \overline{x}^*|_{\mathbf{FÉt}_X}$. Since $x^*$ preserves finite limits and is accessible, it must be pro-representable.

The explicit pro-object representing such functor follows by applying the previous lemma (Lemma 3.4.1). Notice that disjoint unions of Galois coverings are initial among all finite étale coverings and that for a Galois covering $Y \longrightarrow X$ with Galois group $G$, $\mathrm{Aut}(Y_{\overline{x}}) \cong G$. One can suppose, also, that $X$ is connected and that every Galois covering is connected by restricting to the connected component of the point $x$.

Let, then, if $(Y_i \longrightarrow X)_{i\in I}$ is an initial cofiltered sequence of Galois coverings with Galois group $G_i$ .The pro-object $(Y_i \longrightarrow X)_{i\in I}$ represents $F_{\overline{x}}$ and the profinite group $\lim_{i\in I} G_i$ computes to $\widehat{\pi}_1(X,\overline{x}) \cong \lim_{i\in I} G_i$. Such initial cofiltered sequence exists by choosing a non-full subcategory of the Galois objects in $\mathbf{FÉt}_X$ given by choosing $\xi_i \in F_{\overline{x}}(Y_i)$ and the non-full subcategory $I \hookrightarrow \int^{\mathbf{FÉt}_X} F_{\overline{x}}$ (*i.e.* , a subcategory of the category of elements) consisting of Galois objects $Y_i$ and morphisms $(Y_i,\xi_i) \longrightarrow (Y_i,\xi_j)$. □

**Corollary 3.4.1.** *Let $X$ be a scheme and $\overline{x}$ a geometric point. The étale fundamental group $\widehat{\pi}_1^{\text{ét}}(X,\overline{x})$ is a given by a cofiltered limit of $\mathrm{Aut}_X(Y_i)$ for every $Y_i$ Galois and, therefore, since $\mathrm{Aut}(Y_i) \cong \widehat{\pi}_1^{\text{ét}}(X,\overline{x})/N_i$ for some $N_i$ open (of finite index) normal subgroup and the $N_i$'s spans all such subgroups $\lim_I \mathrm{Aut}_X(Y_i)$ is canonically identified with the profinite completion of $\widehat{\pi}_1^{\text{ét}}(X,\overline{x})$.*

*Remark* 3.4.3. Recall that the pro-category of finite groups $\mathrm{Pro}(\mathbf{Grp}_{fin})$ can be embedded in **Top**. Indeed,

$$\mathbf{Fin} \hookrightarrow \mathbf{Set} \hookrightarrow \mathbf{Top}$$

, given by considering the discrete topology, yields, by cocompactness of $\mathbf{Fin} \hookrightarrow \mathbf{Top}$, an equivalence between Pro(**Fin**) and the full subcategory of topological spaces can be presented as cofiltered limits in **Top** of finite discrete spaces.

One can, then, canonically identify Pro(**Fin**) and $\mathrm{Pro}(\mathbf{Grp}_{fin})$ with subcategories of **Top**.

*Remark* 3.4.4. The previous remark fails if one considers Pro(**Set**) and Pro(**Grp**) since there is no cocompactness in **Top**. One should, instead, consider the category of locales or topos instead of **Top** and the subcategory of Pro(**Grp**) with cofiltered diagrams containing only surjective morphisms ([Moe89]).

**DEFINITION 3.4.3.** Let $X$ be a scheme and $\overline{x}$ be a geometric point. The **universal covering space of** $X$, $\widetilde{X}$, is the pro-object that pro-represents $F_{\overline{x}}$.

**COROLLARY 3.4.2.** *Let $X$ be a scheme and $\overline{x}$ a geometric point. By extending $F_{\overline{x}}$ to pro-objects,*

$$\widehat{\pi}_1^{\text{ét}}(X,\overline{x}) \cong F_{\overline{x}}(\widetilde{X}).$$

*REMARK* 3.4.5. Recall that $F_{\overline{x}}$ factors through **Fin** $\hookrightarrow$ **Set** and **Fin** $\hookrightarrow$ Pro(**Fin**) $\hookrightarrow$ **Top**(REMARK 3.4.3).

**LEMMA 3.4.2.** *Let $X$ be a scheme, $\overline{x}$ a geometric point of $X$ and $Y \in \mathrm{Ob}(\mathbf{FÉt}_X)$. The group $\widehat{\pi}_1^{\text{ét}}(X,\overline{x})$ acts continuously on $F_{\overline{x}}(Y)$.*

*PROOF.* Let $G := \widehat{\pi}_1^{\text{ét}}(X,\overline{x})$. Notice that a topological group acts continuously on a discrete set iff the stabilisers are open. Since $F_{\overline{x}}(Y)$ is finite, $G \longrightarrow \mathrm{Aut}(F_{\overline{x}}(Y))$ has finite kernel iff $G$ acts continuously. Since $G$ is profinite such kernel must contain a normal subgroup of finite index, *i.e.* , $G$ acts continuously iff it acts through a finite quotient.

Let $Z$ be a finite Galois covering of $X$ with group $H$ dominating $Y$ (which exists by LEMMA 3.4.1). By naturality of $\mathrm{Aut}(F_{\overline{x}}) = G$, the action of $G$ on $F_{\overline{x}}(Z)$ restricts to an action of $F_{\overline{x}}(Y)$ and, since $G$ acts as $H$ on $F_{\overline{x}}(Z)$, it must also act though $H$ on $F_{\overline{x}}(Y)$, *i.e.* , $G$ acts through a finite group. □

**THEOREM 3.4.3.** *Let $X$ be connected topologically locally Noetherian scheme and $\overline{x}$ a geometric point of $X$. There's an equivalence*

$$\mathbf{FÉt}_X \underset{N}{\overset{(-)_{\overline{x}}}{\rightleftarrows}} \mathbf{Fin}_{\widehat{\pi}_1^{\text{ét}}(X,\overline{x})}$$

*, where $\mathbf{Fin}_{\widehat{\pi}_1^{\text{ét}}(X,\overline{x})}$ is the category of $\widehat{\pi}_1^{\text{ét}}(X,\overline{x})$-sets and $N(S)$ is the unique finite étale covering such that $F_{\overline{x}}(N(S)) \cong S$.*

*PROOF.* The same argument given in the proof of pro-representability gives the functor $N$ by associating a connected $\widehat{\pi}_1^{\text{ét}}(X,\overline{x})$-set, which is always of the form $\widehat{\pi}_1^{\text{ét}}(X,\overline{x})/H$ for some open non-necessarily normal subgroup, to a connected finite étale covering $X_H$ such that $\widehat{\pi}_1^{\text{ét}}(X_H,\overline{x}) = H$, which is necessarily given by $\tilde{X}/H$ (such quotient is representable by an ordinary finite covering since there's some Galois covering of the form $X_N$ above $\tilde{X}/H$ for some open normal subgroup $N \subset H$ and, hence, $\tilde{X}/H \cong X_N/(H/N)$). □

**Corollary 3.4.3.** *Let $k$ be a field and $\overline{x}$ be a geometric point of $Spec(k)$, i.e. , a choice of $k^{\text{sep}}$. There's an equivalence of topos*

$$BG_k \underset{N}{\overset{(-)_{\overline{x}}}{\rightleftarrows}} \mathbf{Sh}(\mathrm{Spec}(k)_{\acute{e}t})$$

*, where $N(S) := BG_k((-)_{\overline{x}}, S)$. Furthermore, it restricts to the equivalence*

$$\mathbf{Mod}_{G_k} \underset{N}{\overset{(-)_{\overline{x}}}{\rightleftarrows}} \mathbf{Ab}(\mathbf{Sh}(\mathrm{Spec}(k)_{\acute{e}t}))$$

*, where $\mathbf{Mod}_{G_k}$ is the category of discrete $G$-modules and $N(M) := \mathbf{Mod}_G((-)_{\overline{x}}, M)$.*

**Lemma 3.4.3.** *Let $X$ be a scheme and $F \in \mathbf{Sh}(X_{\text{ét}})$. If $F$ is locally constant, then it's representable. If, furthermore, $F$ is finite (i.e. , locally constant constructible), then it's representable by a a finite étale morphism.*

*Proof.* Let $F_{|U}$ be constant (*resp.*, finite constant) for some étale covering $U \in \mathcal{U}$. By refining further, one may assume that each $U \in \mathcal{U}$ is connected. Let $X_U := \coprod_{s \in F_{|U}} U \longrightarrow U$. Then $h_{X_U}$ represents $F_{|U}$. Therefore $F_{|U}$ representable by $X_U$ for every $U \in \mathcal{U}$.

Let $U, V \in \mathcal{U}$. Notice that $F_{|U \times_X V}$ is representable by $X_U \times_X U \times_X V$ and $X_V \times U \times_X V$. Then there exists an isomorphism $\varphi_{UV} \colon X_U \times_X U \times_X V \longrightarrow X_V \times_X U \times_X V$ which satisfies the cocycle condition. Hence, the $\varphi_{UV}$'s for every $U, V \in \mathcal{U}$ gives a descent datum. As the category of finite étale coverings (*resp.*, finite étale coverings) is finitely cocomplete, the coequaliser exists and, therefore, represents $F$.

See also [SGA4-III, Exp. IX, Lem. 2.2]. □

**Theorem 3.4.4.** *Let $X$ be a connected scheme and $\overline{x}$ a geometric point. There are equivalences*

$$\mathbf{Fin}_{\widehat{\pi}_1^{\text{ét}}(X,\overline{x})} \underset{N}{\overset{(-)_{\overline{x}}}{\rightleftarrows}} \mathrm{Loc}^{fin}(X_{\acute{e}t})$$

*, where $\mathbf{Fin}_{\widehat{\pi}_1^{\text{ét}}(X,\overline{x})}$ denotes the category of finite $\widehat{\pi}_1^{\text{ét}}(X, \overline{x})$-sets and $\mathrm{Loc}^{fin}(X_{\text{ét}})$ the category of locally constant sheaves on $X_{\text{ét}}$ with finite stalks, and*

$$\mathbf{Mod}^{fin}_{\widehat{\pi}_1^{\text{ét}}(X,\overline{x})} \underset{N}{\overset{(-)_{\overline{x}}}{\rightleftarrows}} \mathbf{Ab}(\text{Loc}^{fin}(X_{\acute{e}t}))$$

*, where $\mathbf{Mod}_{\widehat{\pi}_1^{\text{ét}}(X,\overline{x})}$ denotes the category of finite $\widehat{\pi}_1^{\text{ét}}(X,\overline{x})$-modules.*

*Proof.* That follows forthwith from Lemma 3.4.3. □

*Example* 3.4.1. Let $k$ be a field and $\overline{x}$ be a geometric point. As it was already implicitly mentioned, $\widehat{\pi}_1^{\text{ét}}(\text{Spec}(k),\overline{x}) \cong G_k$

*Example* 3.4.2. Let $X$ be a normal integral scheme and $\overline{x}$ a geometric point. The generic point of $X$ determines entirely its étale fundamental group by $\widehat{\pi}_1^{\text{ét}}(X,\overline{x}) \cong \text{Gal}(R(X)^{\text{unr}}/R(X))$, where $R(X)^{\text{unr}}$ denotes the maximal unramified extension of $R(X)$.

*Example* 3.4.3. Let $A/k$ be an abelian variety. By [Mum12, Theorem (Serre-Lang), p.155], every étale covering of $A$ by a variety is a separable isogeny and, by [Mum12, Remark, p.157], for every isogeny $f: A' \longrightarrow A$, there's an isogeny $g: A \longrightarrow A$ and $n \in \omega$ such that $fg = n_X$, where $n_X$ denotes the multiplication by $n$. That, forthwith, implies that $\{n_X: A \longrightarrow A\}_{n\in\omega}$ is cofinal in $A_{\text{ét}}$. Therefore $\widehat{\pi}_1^{\text{ét}}(A,0) \cong \prod_\ell T_\ell(A)$, where $T_\ell(A) = \lim_{n\in\omega} A[\ell^n]$ is the $\ell$-adic Tate module.

*Example* 3.4.4. Analogously to the previous example, the collection of multiplications $n: \mathbf{G}_m \longrightarrow \mathbf{G}_m$ is also cofinal in $(\mathbf{G}_m)_{\text{ét}}$. Therefore, $\widehat{\pi}_1^{\text{ét}}(\mathbf{G}_m,1) \cong \lim_{n\in\omega} \mu_n \cong \lim_{n\in\omega} \mu_{\ell^n} \cong \prod_\ell T_\ell(\mathbf{G}_m) \cong \prod_\ell \mathbf{Z}_\ell \cong \widehat{Z}$.

Over a field $k$ such that $\text{char}(k) \neq \ell$, the underlying $G_k$-module structure on the pro-$\ell$ part of $\widehat{\pi}_1^{\text{ét}}((\mathbf{G}_m)_{k^{\text{sep}}},1)$, $T_\ell((\mathbf{G}_m)_{k^{\text{sep}}})$, is usually denoted by $\mathbf{Z}_\ell(1)$ and called the $\ell$-adic Tate twist or $\ell$-adic Tate motive. That representation is simply the $\ell$-adic cyclotomic character $\chi_\ell$, which is defined by $\lim_{n\in\omega} \chi_{\ell^n}$ and $g(\zeta_{\ell^n}) = \zeta_{\ell^n}^{\chi_{\ell^n}(g)}$.

## § 3.5 Analytification

**Definition 3.5.1.** Let $X/\mathbf{C}$ be a scheme and $\Phi_X: \mathbf{An}_\mathbf{C} \longrightarrow \mathbf{Set}$ the functor $\mathbf{RS}_\mathbf{C}(-,X)$, where $\mathbf{An}_\mathbf{C}$ is the category of complex analytic spaces and $\mathbf{RS}_\mathbf{C}$ is the category of ringed spaces in $\mathbf{C}$-algebras, be representable. The **analytification of** $X$, $X^{\text{an}}$, is the complex analytic space representing $\Phi_X$.

*Remark* 3.5.1. The above definition canonically yields a functor in the full subcategory of $\mathbf{Sch}_{\mathbf{C}}$ consisting of $X/\mathbf{C}$ such that $\Phi_X$ is representable. Furthermore, by the equivalence $\mathbf{RS}_{\mathbf{C}}(X^{\mathrm{an}}, X) \cong \mathbf{An}_{\mathbf{C}}(X^{\mathrm{an}}, X^{\mathrm{an}})$, the identity $1_{X^{\mathrm{an}}}$ defines a canonical morphism $\varphi_X \colon X^{\mathrm{an}} \longrightarrow X$.

**Theorem 3.5.1** ([SGA1, Exp. XII, Thm. 1.1]). *Let $X/\mathbf{C}$ be locally of finite type. The functor $\Phi_X$ is representable and $\varphi\colon X^{\mathrm{an}} \longrightarrow X$ induces a bijection between $|X^{\mathrm{an}}|$, the underlying set of $X^{\mathrm{an}}$, and $X(\mathbf{C})$. Furthermore, the morphism*

$$\varphi_x^\# \colon \mathscr{O}_{X,\varphi(x)} \longrightarrow \mathscr{O}_{X^{\mathrm{an}},x}$$

*is local and*

$$\widehat{\varphi_x^\#} \colon \widehat{\mathscr{O}_{X,\varphi(x)}} \longrightarrow \widehat{\mathscr{O}_{X^{\mathrm{an}},x}}$$

*is an isomorphism. In particular, $\varphi$ is flat.*

*Sketch.* Given a subscheme $Y \hookrightarrow X$, one can represents it. If $Y$ is open, $X \times_X Y^{\mathrm{an}}$ represents $\Phi_Y$. If $Y$ is closed, $V(\mathscr{I}_Y \mathscr{O}_{X^{\mathrm{an}}})$ represents $\Phi_Y$.

If $X = \mathbf{A}^n_{\mathbf{C}}$, then $X^{\mathrm{an}} = \mathbf{C}^n$. Hence, every affine scheme of finite type is representable. By gluing the analytification of a covering of $X$ by affine schemes, $\Phi_X$ becomes representable. □

Given the above theorem, for $X/\mathbf{C}$ locally of finite type, the underlying topological space of $X^{\mathrm{an}}$ will be denoted by $X(\mathbf{C})$.

**Definition 3.5.2.** Let $S$ be an analytic space (*resp.*, topological space). The **étale topology on** $(\mathbf{An}_{\mathbf{C}})_{/S}$ **(*resp.*, Top)** is the Grothendieck topology generated by the pretopology where coverings consist of jointly surjective collections of morphisms $\{U \longrightarrow S\}_{U \in \mathscr{U}}$ such that $\coprod_{U \in \mathscr{U}} U \longrightarrow S$ is a local isomorphism.

*Remark* 3.5.2. Let $S$ be an analytic space and $X$ be a topological space. Notice that every local isomorphism of topological spaces $X \longrightarrow |S|$ induces a structure of analytic space over $X$. The petit topoi $\mathbf{Sh}(S_{\text{ét}})$, therefore, is canonically isomorphic to $\mathbf{Sh}(|S|_{\text{ét}})$.

**Theorem 3.5.2** (Riemann's Existence Theorem or Grauert-Remmert's Theorem, [SGA1, Exp. XII, Thm. 5.1] or [SGA4-III, Exp. XI, Thm. 4.3.(iii)]). *Let $X(\mathbf{C})$ be locally of finite type. The restriction to $X_{\text{ét}}$ of the analytification functor*

$$(-)^{\text{an}}_{|X_{\text{ét}}} : X_{\text{ét}} \xrightarrow{\sim} X^{\text{an}}_{\text{ét}}$$

*is fully faithfull morphism of sites and*

$$(-)^{\text{an}}_{|\mathbf{FÉt}_{X^{\text{ét}}}} : \mathbf{FÉt}_{X^{\text{ét}}} \xrightarrow{\sim} \mathbf{FÉt}_{X^{\text{an}}}$$

*is an equivalence of categories, where $\mathbf{FÉt}_{X^{\text{an}}}$ denotes the category of finite étale coverings of $X^{\text{an}}$, i.e. , local isomorphism with finite fibers.*

**Lemma 3.5.1.** *Let $X$ be an analytic space and $\delta^{-1}\colon \operatorname{Open}(|X|) \longrightarrow |X|_{\text{ét}} \cong X_{\text{ét}}$ be the inclusion of sites. $\delta$ induces an equivalence of topos*

$$\mathbf{Sh}(X_{\text{ét}}) \cong \mathbf{Sh}(|X|_{\text{ét}}) \underset{\delta_*}{\overset{\delta^*}{\rightleftarrows}} \mathbf{Sh}(|X|)$$

*. In particular, for $X\mathbf{C}$ of finite type, there's an equivalence of topos*

$$\mathbf{Sh}(X^{\text{an}}_{\text{ét}}) \cong \mathbf{Sh}(X(\mathbf{C})_{\text{ét}}) \underset{\delta_*}{\overset{\delta^*}{\rightleftarrows}} \mathbf{Sh}(X(\mathbf{C}))$$

*Proof.* Let $Y \longrightarrow X$ be a local isomorphism and $\mathcal{V}$ be a covering of $Y$ and $\mathcal{U}$ be a covering of $f(Y)$ and $i\colon \mathcal{V} \longrightarrow \mathcal{U}$ a function such that every point of $Y$ has a neighbourhood $V \in \mathcal{V}$ such that $f_{|V}\colon V \longrightarrow U$ is an isomorphism. The canonical morphism $\coprod_{V\in\mathcal{V}} V \longrightarrow X$ is isomorphic to $\coprod_{U\in\mathcal{U}} \coprod_{V\in i^{-1}(U)} U \longrightarrow X$, which is a covering by open sets of $X$. □

**Corollary 3.5.1.** *Let $X/\mathbf{C}$ be locally of finite type connected and $x \in X^{\text{an}}$. There exists an isomorphism $\widehat{\pi}_1^{\text{ét}}(X, \varphi_X(x)) \cong \widehat{\pi_1(X^{\text{an}}, x)}^{profin}$ (resp., $\pi_1^{\text{ét}}(X, \varphi_X(x)) \cong \widehat{\pi_1(X^{\text{an}}, x)}^{prodisc}$), where the topology on $\pi_1(X^{\text{an}}, x)$ is the one induced by the subgroups of finite index.*

## § 3.6 Smooth and Proper Base Change

The Proper Base Change theorem guarantees that one can compute the cohomology of the fiber of a proper morphism by base changing the cohomology sheaf to the fiber.

It's a lot more easier to restated the assertion in a $\infty$-categorical language as will be seen in the last chapter. In this case, it simply states that the mates of a morphism between adjunctions induces an isomorphism. In order to restate it in the 1-categorical version one must derive.

**Theorem 3.6.1** ((Grothendieck) Proper Base Change, [SGA4-III, Exp. XII, Thm. 5.1]). *Let $f\colon X \longrightarrow S$ be a proper morphism of schemes, $g\colon S' \longrightarrow S$ a morphism of schemes and*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & \lrcorner & \big\downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

*be a pullback diagram.*

1. *If $F \in \mathbf{Sh}(X_{\text{ét}})$, then $\varphi\colon g^* f_* F \longrightarrow f'^* g'_* F$ is an isomorphism;*
2. *If $F \in \mathbf{Grp}(\mathbf{Sh}(X_{\text{ét}}))$ (resp.., $F \in \mathrm{Ind}(\mathbf{Grp}(\mathbf{Sh}(X_{\text{ét}})))$), then*

$$\varphi^1\colon g^*(R^1 f_* F) \longrightarrow (R^1 f'^*) g'_* F$$

*is a monomorphism (resp., an isomorphism);*

3. *If $F \in \mathbf{Ab}(\mathbf{Sh}(X_{\text{ét}}))$ and $F$ is torsion, then*

$$\varphi^q\colon g^*(R^q f_* F) \longrightarrow (R^q f'^*) g'_* F$$

*is an isomorphism for every $q \geq 0$*

**Corollary 3.6.1.** *Let $f\colon X \longrightarrow S$ be a proper morphism of schemes, $\xi \longrightarrow S$ a geometric point.*

1. *If* $F \in \mathbf{Sh}(X_{\text{ét}})$*, then* $\varphi\colon f_*F_\xi \longrightarrow \mathrm{H}^0_{\text{ét}}(X_\xi, F_{|X_\xi})$ *is an isomorphism;*

2. *If* $F \in \mathbf{Grp}(\mathbf{Sh}(X_{\text{ét}}))$ *(resp.., $F \in \mathrm{Ind}(\mathbf{Grp}(\mathbf{Sh}(X_{\text{ét}})))$), then* $\varphi^1\colon R^1 f_*F_\xi \longrightarrow \mathrm{H}^1_{\text{ét}}(X_\xi, F_{|X_\xi})$ *is a monomorphism (resp., an isomorphism);*

3. *If* $F \in \mathbf{Ab}(\mathbf{Sh}(X_{\text{ét}}))$ *and $F$ is torsion, then* $\varphi^1\colon R^q f_*F_\xi \longrightarrow \mathrm{H}^q_{\text{ét}}(X_\xi, F_{|X_\xi})$ *is an isomorphism for every* $q \geq 0$

Similarly, the Smooth Base Change guarantees that, for torsion sheaves $\mathbf{Z}/\ell$ having characteristic coprime to all the residue fields of the geometric objects involved, deriving and base changing commutes.

**Theorem 3.6.2** (Smooth Base Change, [SGA4-III, Exp. XVI, Cor. 1.2]). *Let* $f\colon X \longrightarrow S$ *be quasi-compact and quasi-separated,* $g\colon S' \longrightarrow S$ *a smooth morphism of schemes, $P$ a set of primes such that* $\mathrm{char}(\kappa(s)) \notin P$ *for every* $s \in S$ *and*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & \lrcorner & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

*be a pullback diagram.*

1. *If* $F \in \mathbf{Sh}(X_{\text{ét}})$*, then* $\varphi\colon g^*f_*F \longrightarrow f'^*g'_*F$ *is an isomorphism;*

2. *If* $F \in \mathrm{Ind}_P(\mathbf{Grp}(\mathbf{Sh}(X_{\text{ét}})))$ [9]*, then* $\varphi^q\colon g^*(R^q f_*F) \longrightarrow (R^q f'^*)g'_*F$ *is an isomorphism for* $q = 0, 1$*;*

3. *If* $F \in \mathbf{Ab}_P(\mathbf{Sh}(X_{\text{ét}}))$ *and $F$ is torsion, then* $\varphi^q\colon g^*(R^q f_*F) \longrightarrow (R^q f'^*)g'_*F$ *is an isomorphism for every* $q \geq 0$

The following theorem proves that étale cohomology coincides with the ordinary cohomology of the topological space induced by the complex points.

[9] $\mathrm{Ind}_P(\mathbf{Grp})$ denotes the category of ind-$P$-groups, *i.e.* , filtered colimits of finite groups of order belonging to $P$

## § 3.7 Artin's Comparison Theorem

**Theorem 3.7.1** (Artin's Comparison Theorem, [SGA4-III, Exp. XI, Thm. 4.4]). *Let $X/\mathbf{C}$ be a smooth scheme and $\varepsilon^{-1} = (-)^{\mathrm{an}}_{|X_{\text{ét}}}$.*

*(i) There's an equivalence between the category of finite locally constant torsion sheaves over $X_{\text{ét}}$ and the category of finite locally constant torsion sheaves over $X^{\mathrm{an}}_{\text{ét}}$;*

*(ii) Let $F$ be a finite locally constant torsion sheaf over $X^{\mathrm{an}}_{\text{ét}}$. Then*

$$R^q\varepsilon_* F \cong 0$$

*if $q > 0$;*

*(iii) The morphism induced by the above equivalence*

$$\mathrm{H}^q_{\text{ét}}(X, F) \xrightarrow{\sim} \mathrm{H}^q_{\text{ét}}(X^{\mathrm{an}}, F)$$

*is an isomorphism for every $q \geq 0$. In particular,*

$$\mathrm{H}^q_{\text{ét}}(X, \mathbf{Z}/n) \xrightarrow{\sim} \mathrm{H}^q_{\text{ét}}(X^{\mathrm{an}}, \mathbf{Z}/n)$$

*is an isomorphism for every $q \geq 0$.*

*Sketch.* $(i)$ follows from 3.5.2 and $(ii) \Rightarrow (iii)$ follows from the Leray spectral sequence applied to the morphism of sites $\varepsilon$ .

Notice that $R^\bullet\varepsilon_* F$ is the sheaffication of the presheaf $Y \to X \mapsto \mathrm{H}^\bullet(Y, F)$. Since all the topos in question have enough points, it's enough to verify the assertion stalkwise.

One, then should prove that for every $\xi \in \mathrm{H}^q_{\text{ét}}(X^{\mathrm{an}}, F)$ and $x \in X(\mathbf{C})$, there exists an étale covering $f\colon Y \longrightarrow X$ with image covering $x$ such that $f^(\xi) = 0$. The argument follows by dévissage using induction on $n$ for a sequence of elementary fibrations $X = X_0 \to X_1 \to \cdots \to X_n$. The Leray spectral sequence, then, reduces the problema to $X_1$. □

*Remark* 3.7.1. By Lemma 3.5.1, for every $X/\mathbf{C}$ locally of finite type and $F$ finite locally constant torsion sheaf over $X_{\text{ét}}$,

$$\mathrm{H}^q_{\text{ét}}(X,F) \xrightarrow{\sim} \mathrm{H}^q_{\text{sing}}(X(\mathbf{C}),F)$$

for every $q \geq 0$, where, on the right side, $F$ was identified with its equivalent sheaf over $\mathrm{Open}(X(\mathbf{C}))$.

One can generalise the previous result to a global version which also guarantees that the comparison between étale and Betti are compatible with base changing.

**Theorem 3.7.2** (General Comparison Theorem, [SGA4-III, Exp. XVI, Thm. 4.1]). *Let $X/\mathbf{C}$ and $Y/\mathbf{C}$ be of locally finite type, $f: X \longrightarrow S$ be of finite type and consider the diagram*

$$\begin{array}{ccc} \mathbf{Sh}(X^{\text{an}}_{\text{ét}}) & \underset{\varepsilon_*}{\overset{\varepsilon^*}{\rightleftarrows}} & \mathbf{Sh}(X_{\text{ét}}) \\ \downarrow f_* & & \downarrow f^{\text{an}}_* \\ \mathbf{Sh}(S^{\text{an}}_{\text{ét}}) & \underset{\varepsilon_*}{\overset{\varepsilon^*}{\rightleftarrows}} & \mathbf{Sh}(S_{\text{ét}}) \end{array}$$

. *Suppose that $f$ is proper or $F \in \mathbf{Sh}(X_{\text{ét}})$ is constructible.*

1. *$\varphi: \varepsilon^* f_* F \longrightarrow f'^* \varepsilon_* F$ is an isomorphism;*
2. *If $F \in \mathrm{Ind}(\mathbf{Grp}^{fin}(\mathbf{Sh}(X_{\text{ét}})))$, where $\mathbf{Grp}^{fin}$ denotes the category of finite groups, then $\varphi^q: \varepsilon^*(R^q f_* F) \longrightarrow (R^q f'^*)\varepsilon_* F$ is an isomorphism for $q = 0,1$;*
3. *If $F \in \mathbf{Ab}_P(\mathbf{Sh}(X_{\text{ét}}))$ and $F$ is torsion, then $\varphi^q: \varepsilon^*(R^q f_* F) \longrightarrow (R^q f'^*)\varepsilon_* F$ is an isomorphism for every $q \geq 0$*

## § 3.8 $\ell$-adic sheaves

**Definition 3.8.1.** Let $A$ be a commutative ring and $X \in \mathrm{Ob}(\mathbf{Sch})$. A sheaf $F$ of sets (*resp.*, of groups, *resp.*, of $A$-modules) over $X$ is **constructible** if for every open affine $U \hookrightarrow X$, there exists a decomposition of $U$ into a union of reduced locally closed constructible subschemes $U_i$ such that $F_{|U_i}$ is locally constant of finite (*resp.*, finite , *resp.*, of finite presentation) value.

**Definition 3.8.2.** Let $\Lambda$ be a Noetherian ring, $\mathfrak{m} \subset \Lambda$ an ideal and $X \in \mathrm{Ob}(\mathbf{Sch})$. A **constructible $\mathfrak{m}$-adic sheaf** is a cofiltered diagram indexed by $n \in \omega$ such that

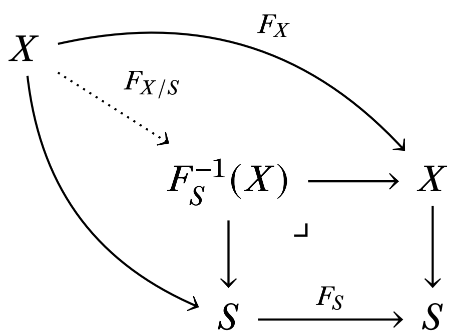
(i) $mathscrF_n$ is a constructible sheaf on $X_{\text{ét}}$ of $\Lambda/\mathfrak{m}^n$-modules;

(ii) The morphism $\mathscr{F}_n \longrightarrow \mathscr{F}_m$ evaluated in $m < n$ induces an isomorphism $\mathscr{F}_n \otimes_{\Lambda/\mathfrak{m}^n} \Lambda/\mathfrak{m}^m \xrightarrow{\sim} \mathscr{F}_m$.

$\mathscr{F}$ is a **lisse $\mathfrak{m}$-adic sheaf** If, furthermore, $\mathscr{F}_n$ is locally constant for every $n \in \omega$.

The following exposition on the relative Frobenius follows closely [SGA5, Éxp. XV, §2].

**Definition 3.8.3.** Let $S \in \mathrm{Ob}(\mathbf{Sch}_{\mathbf{F}_q})$ be a scheme and $X \longrightarrow S$ be a morphism. The **relative Frobenius of** $X/S$ is defined as the unique morphism $F_{X/S} \colon X \longrightarrow F_S^{-1}(X)$ induced by the universal property of the pullback, *i.e.* , the unique morphism such that the diagram




is commutative, where $F_X$ and $F_S$ denotes the absolute Frobenius induced by taking the $q$-th power on the structural sheaves and the identity on the topological spaces.

Let $S \in \mathrm{Ob}(\mathbf{Sch}_{\mathbf{F}_q})$. When $U \longrightarrow S$ is an étale morphism, the relative Frobenius is radicial surjective (since $F_U$ and $F_S$ are and being radicial is preserved by base change). In particular, since $U/S$ and $F_S^{-1}(U)/S$ are étale, $F_{U/S}$ is an étale radicial surjection and, hence, an isomorphism. Let $\mathscr{F} \in \mathrm{Ob}(\mathbf{Sh}(S_{\text{ét}}))$. Then, the isomorphism $F_{U/S}$ induces an isomorphism

$$\mathscr{F}(F_{-/S}) : (F_S)_*\mathscr{F} \xrightarrow{\sim} \mathscr{F}$$

in the étale topoi $\mathbf{Sh}(S_{\text{ét}})$. By inverting $F_{-/S}$ and composing with the counit $\epsilon : (F_S)^*(F_S)_* \Rightarrow 1_{\mathbf{Sh}(S_{\text{ét}})}$, one defines

$$\mathrm{Fr}_{\mathscr{F}/S} := \varepsilon_{\mathscr{F}} \circ \mathscr{F}(F_{-/S})^{-1} : (F_S)^*\mathscr{F} \longrightarrow \mathscr{F}$$

, which is an isomorphism since the counit is a natural equivalence by topological invariance of the étale site ([EGAIV-4, Thm. 8.1.2]).

Let $\mathscr{F} = h_X$ for $X/S$ étale. Applying

$$\mathrm{Fr}_{h_X/S} : (F_S)^* h_X \cong h_{F_S^{-1}(X)} \longrightarrow h_X$$

to $X$ leads to the conclusion that $\mathrm{Fr}_{h_X/S}$ is given exactly by the morphism $(F_{X/S})^{-1}$. Extending, then, by colimits of representables, one reaches to the natural isomorphism

$$\mathrm{Fr}_{-/S} : F_S \Longrightarrow 1_{\mathbf{Sh}(S_{\text{ét}})}$$

given objectwise by $\mathrm{Fr}_{\mathscr{F}/S}$. By noticing that $\mathrm{Fr}_{-/S}$ preserves any module structure, it follows that the same holds for sheaves of $\Lambda$-modules (or even complexes of $\Lambda$-modules) for any constant coefficient ring $\Lambda$ instead of only sheaves of sets.

Now, notice that

$$R\Gamma(S_{\text{ét}}, \mathscr{F}) \xrightarrow{(F_S)^*} R\Gamma(S_{\text{ét}}, (F_S)^*\mathscr{F}) \xrightarrow{R\Gamma(S, \mathrm{Fr}_{\mathscr{F}/S})} R\Gamma(S_{\text{ét}}, \mathscr{F})$$

is the identity since the same is true for the 0-th derived functor and for the representable $h_S$, and one can extend, again, by functoriality (simply notice that the above isomorphism is a natural isomorphism in $\mathscr{F}$ and that $\Gamma(S_{\text{ét}}, \mathscr{F}) = \mathbf{Sh}(S_{\text{ét}})(h_S, \mathscr{F})$).

Let $X_0/\mathbf{F}_q$ and $\mathscr{F}_0$ defined over it. Let also $X/\overline{\mathbf{F}}_q$ and $\mathscr{F}$ be the respective pullbacks. There are two Frobenius acting on $X$. The geometric Frobenius defined by $\phi_n := F_{X_0} \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q$ for $q = p^n$ and the arithmetic Frobenius induced by $\mathrm{Frob}_q\, 1_X \otimes_{\mathbf{F}_q} F_{\mathrm{Spec}(\overline{\mathbf{F}}_q)}$. By the definition

$$\phi_n \,\mathrm{Frob}_q = \mathrm{Frob}_q\, \phi_n.$$

By [SGA5, Éxp. XV, §2, n.3, Prop.3],

$$\mathrm{Fr}_{\mathscr{F}\otimes_\Lambda \mathscr{F}'/S\times_{\mathbf{F}_q}S'} = 1_{(F_S)^*\mathscr{F}} \otimes \mathrm{Fr}_{\mathscr{F}'/S'} \circ \mathrm{Fr}_{\mathscr{F}/X} \otimes 1_{\mathscr{F}'}$$

and it induces a factorisation of the respective endomorphism on $R\Gamma((S \times_{\mathbf{F}_q} S')_{\text{ét}}, \mathscr{F} \otimes_\Lambda \mathscr{F}')$. As shown above, such endomorphism is necessarily the identity. In particular, for $S = X_0$ and $S' = \mathrm{Spec}(\overline{\mathbf{F}}_q)$, one deduces that the geometric Frobenius acts as the inverse of the arithmetic Frobenius on $R\Gamma(X_{\text{ét}}, \mathscr{F} \otimes_\Lambda \Lambda_{\mathrm{Spec}(\overline{\mathbf{F}}_q)})$.

Now, let $X/\mathbf{F}_q$. For every closed point $x \in |X|$ and $d_x := [\kappa(x) \colon \mathbf{F}_q]$ one can consider the induced morphism

$$\mathrm{Fr}_{\mathscr{F}/X} : (F_X^{d_x})^*\mathscr{F} = \mathscr{F} \longrightarrow \mathscr{F}$$

, which, after pullback along $x \hookrightarrow X$, induces a morphism denoted by

$$\mathrm{Fr}_{\mathscr{F}_x} \colon \mathscr{F}_x \longrightarrow \mathscr{F}_x.$$

Now, by étale descent, $\mathscr{F}_x$ is entirely determined by the $\mathrm{Gal}(\overline{\kappa(x)}/\kappa(x))$-module $\mathscr{F}_{\overline{x}}$.

Again, notice that the absolute Frobenius acts trivially on the cohomology and, therefore, $\mathrm{Frob}_q^{-1} = \phi_{d_x}$ as actions on cohomology. By the previous discussion, then, under the correspondence of $\mathrm{Gal}(\overline{\kappa(x)}/\kappa(x))$-modules and $\mathbf{Sh}(x_{\text{ét}})$, $\mathrm{Fr}_{\mathscr{F}_x} = \phi_{d_x}$, *i.e.* , $\mathrm{Fr}_{\mathscr{F}_x}$ acts exactly as the inverse of the generator of $\mathrm{Gal}(\overline{\kappa(x)}/\kappa(x))$.

When $X/\overline{K}$, where $K$ is a local field of characteristic 0, one can still induce a Frobenius action without lifting $X$ to $\mathcal{O}_K$ by using, instead, the conjugacy classes with a choice of $\mathrm{Frob}_k \in \mathrm{Gal}_K$ such that $\mathrm{Frob}_k$ is the Frobenius on the residue field $G_k \cong \widehat{\mathbf{Z}}$. Since the trace is invariant by inner automorphisms, the trace of a conjugacy class is well defined on the cohomology.

More generally, one can proceed in the same way in any global field.

The following is the statement proved by Deligne claiming the resolution of the Riemann's Hypothesis.

**Theorem 3.8.1** (Riemann's Hypothesis, [Del74a, Thm. 1.3]). *Let $q = p^n$ for some prime $p$ and $X_0/\mathbf{F}_q$ a smooth projective variety. For every $i$, the characteristic* *polynomial*

$$\det(1 - t\phi_n | \mathrm{H}^i_{\text{ét}}(X, \mathbf{Q}_\ell))$$

*has integral coefficients independent of* $\ell$ ($\ell \neq p$)*. The complex roots* $\alpha$ *of this polynomial have absolute value* $|\alpha| = p^{\frac{i}{2}}$

*Remark* 3.8.1. Notice that the characteristic polynomial of the Frobenius is the characteristic polynomial of the arithmetic Frobenius by the previous discussions.

In analogy with the Riemann's Hypothesis over Finite Fields, one can indagate what happens to the norm the eigenvalues of the Frobenius when $X_0/\mathbf{F}_q$ is not proper.

Indeed, in this case, there's a filtration of the constant $\ell$-adic sheaf $\mathbf{Z}_\ell$ on $X_{\text{ét}}$ such that the graded pieces have a fixed norm for the eigenvalues.

**Theorem 3.8.2** (Relative Riemann's Hypothesis, [Del80, Thm. 1 (3.3.1)]). *Let* $f : X_0 \longrightarrow S_0 \in \mathbf{Sch}_{\mathbf{F}_q}$ *be a morphism of finite type and* $\mathscr{F}$ *a mixed sheaf of weight* $\leq n$ *over* $X := X_0 \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q$*. Then, for every* $i$*,* $R^i f_! \mathscr{F}_0$ *over* $S_0$ *is mixed of weight* $\leq n + i$

Such filtration and the comparisons with Betti cohomology motivated Grothendieck to formulate results and conjectures in Hodge Theory which to outsiders of Grothendieck's school appeared as extremely mysterious. Indeed, Hodge Theory had been dominated by transcendental methods for such a long time until Grothendieck envisioned a way of transferring results from the étale side to the Betti side.

# Chapter 4

# $K$-Theory

On 16 April 1953, in a letter to Borel, Serre had conjectured a Riemann-Roch's theorem for higher dimensional varieties ([Ser03]). That generalization, nowadays known as Hirzebruch-Riemann-Roch's theorem, was proved by F. Hirzebruch in his Habilitationsschrift, which was later published 1956 ([Hir62]). Serre already on December of 1953 at the Séminaire Bourbaki ([Col54]) presented part of the results in [Hir62] (although, not yet what Hirzebruch-Riemann-Roch's theorem).

In 1957, Grothendieck proved a vast generalisation, now knowns as Grothendieck-Riemann-Roch's theorem. The result was, for the first time, publicly presented later by Borel and Serre in a Princeton seminar ([BS58]).

In 1959, Atiyah and Hirzebruch proved an analogous of the Grothendieck-Riemann-Roch's theorem for differentiable manifolds and topological $K$-theory ([AH59]). Part of the results were already exposed earlier at the Séminaire Bourbaki ([Hir60]) on February 1959 [1].

During 1966 and 1967, the results were generalised further by Grothendieck and his school on the Séminaire de Géométrie Algébrique du Bois-Marie ([SGA6]).

The exposition, herein, starts with Quillen's later definition of the $Q$ construction. Other historically important definitions might be found in [Wei13].

[1] It's mentioned in [Hir60], that despite being Hirzebruch the one who presented the results, all the results on that éxpose were mainly due to Atiyah.

## § 4.1 Quillen's $Q$ Construction

**Definition 4.1.1.** A **Quillen exact category** $\mathscr{M}$ consists in an additive category $\mathscr{M}$ and a collection $\mathscr{E}$ of sequences

$$0 \longrightarrow M' \overset{i}{\rightarrowtail} M \overset{j}{\twoheadrightarrow} M'' \longrightarrow 0$$

, called (short) exact sequences. The morphism $i$ (*resp.*, $j$) is called an admissible monomorphism (*resp.*, admissible epimorphism). $\mathscr{E}$ must satisfy the following properties.

1. Any sequence in $\mathscr{M}$ isomorphic to some sequence in $\mathscr{E}$ is also in $\mathscr{E}$ . Furthermore a split sequence is an exact sequence, *i.e.* , for any $M, M'' \in \mathrm{Ob}(\mathscr{M})$, the sequence

$$0 \longrightarrow M' \overset{1_{M'}\oplus 0}{\rightarrowtail} M' \oplus M'' \overset{\pi_{M'}}{\twoheadrightarrow} M' \longrightarrow 0$$

   is in $\mathscr{E}$. Moreover, for any sequence in $\mathscr{E}$, $i \cong \mathrm{Ker}(j)$ and $j \cong \mathrm{Coker}(i)$

2. The class of admissible epimorphisms is closed under composition and base change by arbitrary morphisms. Dually, the class of admissible monomorphisms is closed under composition and cobase change by arbitrary morphisms.

3. Let $j : M \longrightarrow M'' \in \mathscr{M}(M, M'')$ such that $j$ has a kernel. If there exists $f \in \mathscr{M}(N, M)$ such that $jf$ is an admissible epimorphism, then $j$ is an admissible epimorphism. Dually, let $i : M' \longrightarrow M \in \mathscr{M}(M', M)$ such that $i$ has a cokernel. If there exists $g \in \mathscr{M}(M, N)$ such that $ig$ is an admissible monomorphism, then $i$ is an admissible monomorphism.

Let $\mathscr{M}'$ be a Quillen exact category. An **exact functor** $F$ is a functor $F\mathscr{M} \longrightarrow \mathscr{M}'$ such that $F$ sends exact sequences to exact sequences.

**Theorem 4.1.1** (Quillen-Gabriel's Embedding Theorem). *Let $\mathscr{M}$ be an additive full subcategory of an abelian category $\mathscr{A}$ that is closed by extensions. Then $\mathscr{M}$ together with the exact sequences of $\mathscr{A}$ having underlying objects in $\mathscr{M}$ is a Quillen exact category.*

*Conversely, let $\mathscr{M}$ be a Quillen exact category. The subcategory $\mathscr{A} \hookrightarrow \mathbf{Ab}(\widehat{\mathscr{M}})$ consisting of left exact functors is an abelian category. Furthermore, the corestriction of the Yoneda embedding $\mathscr{M} \hookrightarrow \mathscr{A}$ exhibits $\mathscr{M}$ as a full subcategory of $\mathscr{A}$ closed under extensions .*

*Example* 4.1.1. Any additive category is canonically exact by considering only the split sequences to be exact sequences.

*Example* 4.1.2. Let $\mathscr{M}$ be exact. $\mathrm{Ch}(\mathscr{M})$ is exact by defining the exact sequences to be index-wise exact sequences. Furthermore $\mathrm{Ch}^b(\mathscr{M})$ is an full and exact subcategory of $\mathrm{Ch}(\mathscr{M})$

*Example* 4.1.3. Every abelian category is canonically a Quillen exact category by declaring an exact sequence to be an exact sequence in the usual sense. In particular, $\mathbf{QCoh}(X)$ and $\mathbf{Coh}(X)$ is a Quillen exact category for every scheme $X$.

*Example* 4.1.4. The category $\mathbf{Vect}(X) \hookrightarrow \mathbf{QCoh}(X)$ consisting of vector bundles of finite rank can be endowed with a structure of a Quillen exact category for every scheme $X$ by using 4.1.1. One can choose $\mathscr{E}$ to consist in the exact sequences of $\mathbf{Mod}_{\mathscr{O}_X}$ which are objectwise in $\mathrm{Ob}(\mathbf{Vect}(X))$. Conversely one can define $\mathscr{E}$ by collecting only exact sequences of $\mathbf{Vect}(X)$, which, in general, differ from the exact sequences of the abelian category $\mathbf{Mod}_{\mathscr{O}_X}$. Analogously, one can give different structures of exact categories for $\mathbf{QCoh}(X)$ and $\mathbf{Coh}(X)$

Notice that extensions of vector bundles are vector bundles by checking locally and considering the fact that finitely generated projective modules are closed under extension.

**Definition 4.1.2** (Quillen's $Q$ Construction). Let $\mathscr{M}$ be an exact category. The category $Q\mathscr{M}$ is defined by $\mathrm{Ob}(Q\mathscr{M}) := \mathrm{Ob}(\mathscr{M})$ and setting $\mathscr{M}(M, M')$ to be composed of diagrams

$$\begin{tikzcd} & N \arrow[dl, two heads, "j"'] \arrow[dr, tail, "i"] & \\ M & & M' \end{tikzcd}$$

, where $i$ is an admissible epimorphism, $i$ an admissible monomorphism and $N \in \mathrm{Ob}(\mathscr{M})$.

Furthermore, composition is defined by the pullback diagram

$$\begin{tikzcd} N \times_{M'} N' \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & N' \arrow[r, tail, "i'"] \arrow[d, two heads, "j'"'] & M'' \\ N \arrow[r, tail, "i"'] \arrow[d, two heads, "j"] & M' & \\ M. & & \end{tikzcd}$$

**Lemma 4.1.1.** *Let $\mathscr{M}$ be an exact category and $M, M' \in \mathrm{Ob}(\mathscr{M})$. Consider the triples $(i_1 : M_1 \rightarrowtail M', i_2 : M_2 \rightarrowtail M', \theta)$ consisting of subobjects such that there exists an admissible monomorphism $c : M_1 \rightarrowtail M_2$ over $M'$ and an isomorphism $\theta : M \xrightarrow{\sim} M_2/M_1$. There's a bijective correspondence between such triples and $Q\mathscr{M}(M, M')$.*

*Proof.* Given such triple, the diagram

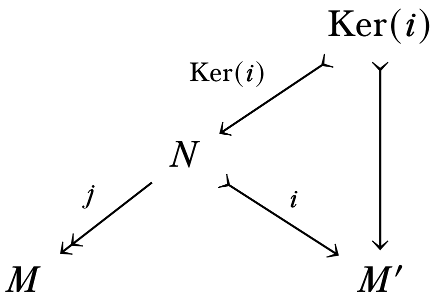
$$\begin{tikzcd} & M_2 \arrow[dl, two heads, "j"'] \arrow[dr, tail, "\theta\,\mathrm{Coker}(c)"] & \\ M & & M' \end{tikzcd}$$

defines a morphism in $Q\mathscr{M}(M, M')$.

Conversely, any diagram

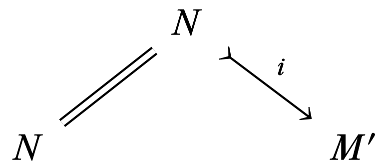
$$\begin{tikzcd} & N \arrow[dl, two heads, "j"'] \arrow[dr, tail, "i"] & \\ M & & M' \end{tikzcd}$$

in $\mathscr{M}$, by taking the the kernel and using its universal property, induces a diagram

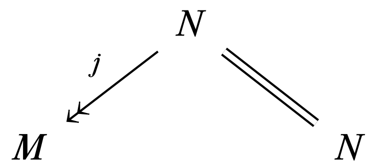



. Since $j$ is an admissible epimorphism, $\operatorname{Ker}(i)$ is an admissible monorphism. As admissible monomorphisms are closed under composition, $i \operatorname{Ker}(i)$ is an admissible monomorphism. □

Let $\mathscr{M}$ be an exact category $M, M', N \in \operatorname{Ob}(\mathscr{M})$. If $i : N \rightarrowtail M'$ is an admissible monomorphism, then let $i_!$ be the morphism




in $Q\mathscr{M}(M, M')$. Dually, if $j : N \twoheadrightarrow M$ is an admissible epimorphism, then let $j^!$ be the morphism




in $Q\mathscr{M}(M, M')$.

**Definition 4.1.3.** Let $\mathscr{M}$ be an exact category. A morphism of the form $i_!$ is called **injective**. Dually, a morphism of the form $j^!$ is called **surjective**.

**Lemma 4.1.2.** *Let $\mathscr{M}$ be an exact category $M, M', N \in \operatorname{Ob}(\mathscr{M})$ and $u = (j : N \twoheadrightarrow M, i : N \rightarrowtail M') \in Q\mathscr{M}(M, M')$.*

1. $u = i_! j^!$;

2. *The above decomposition is unique up to unique isomorphism;*

3. *If $i$ and $i'$ are composable admissible monomorphisms, then $(i'i)_! = i'_! i_!$;*

4. *If $j$ and $j'$ are composable admissible epimorphisms, then $(j'j)^! = j^!(j')^!$;*

5. $1_M = (1_M)_! = (1_M)^!$

6. *If*

$$\begin{array}{ccc} N & \overset{i}{\rightarrowtail} & M' \\ {\scriptstyle j}\downarrow & \lrcorner \quad \ulcorner & \downarrow{\scriptstyle j'} \\ M & \underset{i'}{\rightarrowtail} & N' \end{array}$$

*is a bicartesian (i.e. , pushout and pullback) square in $\mathscr{M}$ such that the horizontal (resp., vertical) arrows are admissible monomorphisms (resp., admissible epimorphisms), then $u = (j')^! i'_!$.*

*Furthermore, $Q\mathscr{M}$ is initial among the categories together with functors $(-)_!$ and $(-)^!$ from subcategories of $\mathscr{M}$ satisfying the above properties.*

*Proof.* Notice the pullback square

$$\begin{array}{ccccc} N & = & N & \overset{i}{\rightarrowtail} & M' \\ \| & \lrcorner & \| & & \\ N & = & N & & \\ {\scriptstyle j}\downarrow & & & & \\ M. & & & & \end{array}$$

For the equality on assertion of the bicartesian square, notice the pullback diagram

$$\begin{array}{ccccc} N & \overset{i}{\rightarrowtail} & M' & = & N' \\ \downarrow{\scriptstyle j} & \lrcorner & {\scriptstyle j'}\downarrow & & \\ M & \underset{i'}{\rightarrowtail} & N' & & \\ \| & & & & \\ M. & & & & \end{array}$$

□

**Corollary 4.1.1.** *Let $Q$ induces a functor from small exact categories to small categories such that every morphism is a monomorphism.*

*Furthermore, $Q\mathscr{M} \cong Q\mathscr{M}^{\mathrm{op}}$.*

**COROLLARY 4.1.2.** *Let $\mathscr{M}$ be an exact category. A morphism in $Q\mathscr{M}$ is an isomorphism iff it's surjective and injective. In particular, $i$ and $j$ induces an isomorphism $u$ iff $i$ and $j$ are isomorphisms. In this case, let $\theta := ij^{-1}$, $\theta^! = \theta_!^{-1}$.*

**DEFINITION 4.1.4.** Let $\mathscr{M}$ be a small exact category and

$$0 \longrightarrow M' \overset{i}{\rightarrowtail} M \overset{j}{\twoheadrightarrow} M'' \longrightarrow 0$$

be an arbitrary exact sequence in $\mathscr{M}$. The **Grothendieck group of $\mathscr{M}$** is defined as

$$K_0(\mathscr{M}) := \{[M] |\, M \in \mathrm{Ob}(\mathscr{M})\}/\{[M'].[M''] - [M]\}$$

, where $[-]$ denotes the isomorphism class in $\mathscr{M}$.

*REMARK* 4.1.1. Notice that the above definition implies that $K_0(\mathscr{M})$ is abelian by using property (1) of exact categories.

**THEOREM 4.1.2** (Quillen, [Qui73, Thm. 2.1]). *Let $\mathscr{M}$ be a small exact category.*

$$\pi_1(B(Q\mathscr{M}), 0) \cong K_0(\mathscr{M}).$$

*PROOF.* Firstly, notice that, for every category $\mathscr{C}$, there exists an equivalence between $F : \mathscr{C} \longrightarrow \mathbf{Set}$ functors inverting every morphism and unramified coverings of $B(\mathscr{C})$ by applying the comma category (also called Grothendieck construction or category of elements of $F$) $1/F \longrightarrow \mathscr{C}$, where 1 is the constant functor sending all morphisms to $1_1$.

Since 0 is both initial and final in $\mathscr{M}$, $B(Q\mathscr{M})$ is 0-connected and, hence, $K_0(\mathscr{M})$-sets are equivalent to unramified coverings of $B(Q\mathscr{M})$.

Every $K_0(\mathscr{M})$-set $S$ induces a functor $F_S$ defined, for every $M \in \mathrm{Ob}(Q\mathscr{M})$, by $F_S(M) = S$, $F(i_!) = 1_S$ and $F(j^!) =$ (multiplication by $[\mathrm{Ker}(j)]$). Equivalently, it corresponds to the unramified covering with fiber $S$ and holonomy around $u = i_! j^!$ given by multiplication with $[\mathrm{Ker}(j)]$.

Let $i_M : 0 \rightarrowtail M$ and $j_M : M \twoheadrightarrow 0$ be the universal morphisms. Let

$$0 \longrightarrow M' \overset{i}{\rightarrowtail} M \overset{j}{\twoheadrightarrow} M'' \longrightarrow 0$$

be an exact sequence in $\mathscr{M}$. The bicartesian square

$$\begin{array}{ccc} M' & \overset{i}{\rightarrowtail} & M \\ {\scriptstyle j_{M'}}\downarrow & & \downarrow{\scriptstyle j} \\ 0 & \underset{i_{M''}}{\rightarrowtail} & M'' \end{array}$$

implies that $j^!(i_{M''})_! = i_! j^!_{M'}$. Hence, $F(j^!) = F(j^!_{M'})$ and, then,

$$F(j^!_M) = F((j_{M''}j)^!) = F(j^! j^!_{M''}) = F(j^!)F(j^!_{M''}) = F(j^!_{M'})F(j^!_{M''})$$

, which implies that there's indeed an action of $K_0(\mathscr{M})$ on $F(0) = S$.

□

The above theorem motivates the following definition

**Definition 4.1.5.** Let $\mathscr{M}$ be an exact category and $i \in \omega$. The ***i*-th *K*-group of** $\mathscr{M}$ is defined as

$$K_i(\mathscr{M}) := \pi_{i+1}(B(Q\mathscr{M}), 0).$$

*Remark* 4.1.2. The above definition defines $K_i(\mathscr{M})$ as the $\pi_i$ group completion $\Omega_0(B(Q\mathscr{M}))$ of the monoid object $0 \in \mathrm{Ob}(Q\mathscr{M})$ since $Q\mathscr{M}$ is 0-connected.

**Lemma 4.1.3.** *$K_I$ induces a functor $K_i : \mathbf{Cat}_{\mathrm{ex}} \longrightarrow \mathbf{Ab}$, where $\mathbf{Cat}_{\mathrm{ex}}$ denotes the category of small exact categories. Furthermore, $K_0$ commutes with finite limits and small filtered colimits.*

*Proof.* The first property is obvious as $Q$, the nerve $N$ and geometric realisation (from $\widehat{\Delta}$ to compactly generated topological spaces) commutes with finite limits. The second property follows by noticing that there's a weak equivalence between $N(\mathrm{colim}_{i\in I}\mathscr{C}_i)$ and $\mathrm{colim}_{i\in I} N(\mathscr{C}_i)$ for $I$ a filtered set. That follows by noticing that any $\Delta^n \to N(\mathrm{colim}_{i\in I}\mathscr{C})$ lifts to some $N(\mathscr{C})_i$ and any other lifting to some $N(\mathscr{C}_j)$ eventually becomes equal for some $k > i, j$. More generally, let $K$ be any Kan fibrant simplicial set with a finite number $m$ of non-degenerated simplices $\Delta^{d_i}$, which can be lifted through the same procedure to some $N(\mathscr{C}_{i_l})$. After choosing a $k > i_1, \cdots i_m$ equating every lift in $N(\mathscr{C}_k)$, the result follows. In particular, by taking $K = \mathbf{S}^n$, one obtains a weak equivalence. □

The following theorems proved by Quillen in his seminal article [Qui73] are used to derive properties of the functor $K$.

**Theorem 4.1.3** (Quillen's Theorem A [Qui73, Thm. 1.A]). *Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor. If $|N(F/d)| \in \mathrm{Ob}(\mathbf{Grpd}_\infty)$ is contractible for every $d \in \mathrm{Ob}(\mathscr{D})$, then $|N(F)|$ induces a weak equivalence in $\mathbf{Grpd}_\infty$. Equivalently, the functor $|N(F)|$ is a final $\infty$-functor (i.e. , precomposition with $|N(F)|$ preserves and reflects $\infty$-colimits).*

*Proof.* The proof consists in constructing a zig-zag

$$\mathscr{D}^{\mathrm{op}} \longleftarrow (-)^{\mathrm{op}}/F \longrightarrow \mathscr{C}$$

of weak equivalences after applying $|N(-)|$. Where $(-)^{\mathrm{op}}$ denotes the "contravariant inversion functor" $\mathscr{D}^{\mathrm{op}} \longrightarrow \mathscr{D}$ and $(-)^{\mathrm{op}}/F$ the obvious analogue of the comma category. □

**Theorem 4.1.4** (Quillen's Theorem B [Qui73, Thm. 1.B]). *Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor. If for every $f : d_1 \longrightarrow d_2 \in \mathscr{D}(d_1, d_2)$, the functor $f_* : F/d_1 \longrightarrow F/d_2$ induces a weak equivalence $|N(f_*)|$ in $\mathbf{Grpd}_\infty$, then the diagram*

$$\begin{array}{ccc} |N(F/d_1)| & \longrightarrow & |N(\mathscr{C})| \\ \big\downarrow p & \lrcorner & \big\downarrow |N(F)| \\ |N(\mathscr{D}/d_2)| & \longrightarrow & |N(\mathscr{D})| \end{array}$$

*is an $\infty$-pullback square, where $p(c, F(c) \to d_1) = (F(c) \to d_1)$. In particular, there exists a fiber sequence*

$$\begin{array}{ccc} |N(F/d_1)| \cong |N(F/d_2)| & \longrightarrow & |N(\mathscr{C})| \\ \big\downarrow & \lrcorner & \big\downarrow |N(F)| \\ 1 & \longrightarrow & |N(\mathscr{D})|. \end{array}$$

*Proof.* The proof follows by pasting $\infty$-pullback squares by using squares containing a variant of a comma category as in 4.1.3. □

For every exact category $\mathscr{M}$, let $\mathscr{E}$ denotes the exact category of exact sequences by defining an exact sequence to be a pointwise exact sequence in $\mathscr{M}$. Consider the exact functors $s, e, t : \mathscr{E} \longrightarrow \mathscr{M}$ defined by setting

$$0 \longrightarrow sE \rightarrowtail tE \twoheadrightarrow qE' \longrightarrow 0$$

in $\mathscr{M}$ for every $E \in \mathrm{Ob}(\mathscr{E})$.

**THEOREM 4.1.5.** *Let $\mathscr{M}$ be an exact category. The functor*

$$(s,q) : \mathscr{E} \longrightarrow \mathscr{M} \times \mathscr{M}$$

*induces an homotopy equivalence $B(Q\mathscr{E}) \cong B(Q\mathscr{M}) \times B(Q\mathscr{M})$ in $\mathbf{Grpd}_\infty$.*

*PROOF.* Let $M,N \in \mathrm{Ob}(Q\mathscr{M})$ and $\mathscr{C}'' \hookrightarrow \mathscr{C}' \hookrightarrow (s,q)/(M,N)$ be, respectively, the category of, simultaneously, surjective and injective morphisms and the category of surjective morphisms. By taking a factorization into injective-surjective, one can construct left adjoints to the inclusions using the property (3) and preservation of admissible monomorphisms (*resp.*, admissible epimorphisms) under pushouts (*resp.*, pullbacks). That gives deformation retract of $B(Q((s,q)/(M,N))$ to the the final object. By 4.1.3, the result follows. □

**COROLLARY 4.1.3.** *Let $\mathscr{M}$ and $\mathscr{M}'$ be exact categories and*

$$0 \longrightarrow F' \rightarrowtail F \twoheadrightarrow F'' \longrightarrow 0$$

*an exact sequence of exact functors $\mathscr{M} \longrightarrow \mathscr{M}'$. There exists an isomorphism*

$$F_* \cong F'_* + F''_* : K_i(\mathscr{M}) \longrightarrow K_i(\mathscr{M}')$$

*for every $i \in \omega$*

*PROOF.* Notice that the functor $S : \mathscr{N} \times \mathscr{N} \longrightarrow \mathscr{E}$ given by setting $S(M,M') :=$

$$0 \longrightarrow M' \xrightarrow{1_{M'} \oplus 0} M' \oplus M'' \xrightarrow{\pi_{M'}} M' \longrightarrow 0$$

is a section of $(s,q)$ and, hence, by the previous theorem, induces an isomorphism $S_* : K_i(\mathscr{N}) \times K_i(\mathscr{N}) \xrightarrow{\sim} K_i(\mathscr{E})$. Then $t_* \cong s_* + q_*$. Therefore, by taking $\mathscr{N} := \mathbf{Cat}_{\mathrm{ex}}(\mathscr{M},\mathscr{M}')$ (which is exact), the claim follows.

□

**COROLLARY 4.1.4.** *Let $(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ be a ringed space and $K_i(\mathscr{X}) := K_i(\mathbf{Vect}(\mathscr{X}))$ for every $i \in \omega$. The tensor product of $\mathscr{O}_{\mathscr{X}}$-modules induces a structure of $K_0(\mathscr{X})$-module over $K_i(\mathscr{X})$.*

*Proof.* By the previous corollary, notice that $(V \otimes (-))_* \cong (V' \otimes (-))_* + (V'' \otimes (-))_*$ for every short exact sequence of vector bundles

$$0 \longrightarrow V' \rightarrowtail V \twoheadrightarrow V'' \longrightarrow 0.$$

□

**Theorem 4.1.6** (Dévissage, [Qui73, Thm. 5.4]). *Let $\mathscr{A}$ be an abelian category and $\emptyset \neq \mathscr{B} \hookrightarrow \mathscr{A}$ a full subcategory closed under taking subobjects. If, for every $M \in \mathrm{Ob}(\mathscr{A})$, there exists a finite filtration*

$$0 \cong M_0 \hookrightarrow M_1 \hookrightarrow \cdots \hookrightarrow M_n \cong M$$

*such that $M_{j+1}/M_j \in \mathrm{Ob}(\mathscr{B})$ for every $0 \leq j \geq n$, then the inclusion $\mathscr{B} \hookrightarrow \mathscr{B}$ induces an homotopy equivalence $B(Q\mathscr{B}) \cong B(Q\mathscr{B})$. In particular, $K_i(\mathscr{B}) \cong K_i(\mathscr{A})$ for every $i \in \omega$.*

**Theorem 4.1.7** (Localisation, [Qui73, Thm. 5.4]). *Let $\mathscr{A}$ be an abelian category and $E : \mathscr{B} \hookrightarrow \mathscr{A}$ be a Serre subcategory (i.e. , a non-empty full subcategory closed under extensions) with localisation functor $S : \mathscr{A} \twoheadrightarrow \mathscr{A}/\mathscr{B}$. There exists a functorial on $\mathscr{A} \times \mathscr{B}$ fiber sequence*

$$\begin{array}{ccc} B(Q\mathscr{B}) & \xrightarrow{E_*} & B(Q\mathscr{A}) \\ \downarrow & \lrcorner & \downarrow S_* \\ 1 & \longrightarrow & B(Q(\mathscr{A}/\mathscr{B})) \end{array}$$

*in $\mathbf{Grpd}_\infty$. In particular, it induces a long exact sequence of homology for the $K_i$'s.*

*Sketch.* □

**Definition 4.1.6.** Let $X$ be a scheme. The ***i*-th *K*-group of $X$** is defined as $K(X) := Q\mathbf{Vect}(X)$ for $\mathbf{Vect}(X)$ with the exact category structure coming from the abelian category $\mathbf{Mod}_{\mathscr{O}_X}$. If, furthermore, $X$ is coherent (*i.e.* , $OX_X$ is coherent), the ***i*-th *G*-group of $X$** is defined as $G(X) := Q\mathbf{Coh}(X)$ with the exact category structure coming from the abelian category $\mathbf{Mod}_{\mathscr{O}_X}$.

When $X = \mathrm{Spec}(A)$, the notations $K_i(A) := K_i(\mathrm{Spec}(A))$ and $G_i(A) := G_i(\mathrm{Spec}(A))$ will be used.

The notations $K(X) := B(Q\mathbf{Vect}(X))$, $G(X) := B(Q\mathbf{Coh}(X))$, $K(A) := B(Q\mathbf{Vect}(\mathrm{Spec}(A)))$ and $G(A) := B(Q\mathbf{Coh}(\mathrm{Spec}(A)))$ will also be used.

*Remark* 4.1.3. For non-coherent $X$, a sheaf of $\mathcal{O}_X$-modules which is finitely presented may not be coherent. Indeed, being finitely presented and coherent coincides for $X$ coherent.

The usual definition of $G$-theory usually assumes $X$ at least locally Noetherian (Quillen assumes in [Qui73] that $X$ is Noetherian and separated), however such assumption is usually not necessary

*Remark* 4.1.4. The category of coherent sheaves is abelian for any ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ [2] and it's not Grothendieck unless $\mathcal{X} = \emptyset$ (since infinite coproducts do not exist).

Notice that the above choice of exact category structure will also coincide with the one coming from the abelian structure of $\mathbf{Coh}(X)$ [3].

*Remark* 4.1.5. Notice that the category of vector bundles $\mathbf{Vect}(X)$ is not abelian in general. Even for a projective smooth curve $X/k$ such that $k$ is algebraically closed, **AB2** may fail despite the existence of kernels and cokernels [4].

*Remark* 4.1.6. For $X$ coherent, the inclusion $\mathbf{Vect}(X) \hookrightarrow \mathbf{Coh}(X)$ makes sense and is exact. It's, therefore, reasonable to apply the $Q$-construction on $\mathbf{Vect}(X)$.

Notice, however, that the above choice of exact category structure will not coincide with the one coming from exact sequences (in the ordinary sense of ) of $\mathbf{Vect}(X)$.

**Theorem 4.1.8** (Resolution Theorem). *Let $\mathcal{M}$ be an exact category and $\mathcal{P} \hookrightarrow \mathcal{M}$ a full subcategory closed under extensions and containing $0 \in \mathrm{Ob}(\mathcal{M})$. Suppose that the following holds for $\mathcal{P}$*

1. *For every exact sequence*

$$0 \longrightarrow M' \overset{i}{\rightarrowtail} M \overset{j}{\twoheadrightarrow} M'' \longrightarrow 0$$

[2]The same assertion is incorrect for quasi-coherent sheaves unless $\mathcal{X}$ is a scheme, in which case it's even Grothendieck abelian and, therefore, has enough injectives (which differ from the injectives in $\mathbf{Mod}_{\mathcal{O}_X}$ even for affine $X$).

[3]It will not, however, coincide with the exact category structure coming from the abelian category $\mathbf{QCoh}(X)$ for some scheme $X$

[4]The inclusion $\mathbf{Vect}(X) \hookrightarrow \mathbf{Mod}_{\mathcal{O}_X}$, in general, does not commute with cokernels

*in $\mathcal{M}$ such that $M' \in \mathrm{Ob}(\mathcal{P})$, $M' \in \mathrm{Ob}(\mathcal{P})$.;*

2. *For any $M'' \in \mathrm{Ob}(\mathcal{P})$, there exists an exact sequence as above such that $M \in \mathrm{Ob}(\mathcal{P})$.*

*The inclusion $\mathcal{P} \hookrightarrow \mathcal{M}$ induces an homotopy equivalence*

$$B(Q\mathcal{P}) \xrightarrow{\sim} B(Q\mathcal{M})$$

*in* $\mathbf{Grpd}_\infty$.

*Proof.* Firstly, notice that $Q\mathcal{P}$ is not a full subcategory of $Q\mathcal{M}$. Let $\mathcal{C}$ be the full subcategory of $Q\mathcal{M}$ such that $\mathrm{Ob}(\mathcal{C}) := \mathrm{Ob}(Q\mathcal{P})$. There's a factorisation

$$Q\mathcal{P} \xrightarrow{G} \mathcal{C} \xrightarrow{F} Q\mathcal{M}$$

Let $P \in \mathrm{Ob}(\mathcal{C})$. The category $G/P$ is equivalent to the category of pairs $(M_0, M_1) \in \mathrm{Ob}(\mathcal{M} \times \mathcal{M})$ such that $M_0 \rightarrowtail M_1$ is a morphism of subobjects of $P$ (in $\mathcal{M}$) and $M_0/M_1 \in \mathrm{Ob}(\mathcal{P})$. A morphism $(M_0, M_1) \to (M_0', M_1')$ in this category is equivalent to morphisms $M_0' \rightarrowtail M_0 \rightarrowtail M_1 mono M_1'$. Since for every $(M_0, M_1) \in \mathrm{Ob}(G/P)$, by property (1), $M_0, M_1 \in \mathrm{Ob}(\mathcal{P})$, one can consider the canonical zig zag

$$(M_0, M_1) \rightarrowtail (0, M_1) \leftarrowtail (0, 0)$$

in $G/P$. That induces a deformation retract of $B(G/P)$. Therefore, by 4.1.3, $B(Q\mathcal{P}) \cong B(\mathcal{C})$.

Analogously, let $M \in \mathrm{Ob}(Q\mathcal{M})$ and consider $M/F$. Let $P \in \mathrm{Ob}(Q\mathcal{P})$ and notice that every morphism $u : M \longrightarrow P$ in $Q\mathcal{M}$ is of the form

$$\begin{array}{ccc} & \overline{P} & \\ {}^{j}\swarrow & & \searrow^{i} \\ M & & P \end{array}$$

with $\overline{P} \in\in \mathrm{Ob}(Q\mathcal{P})$. By property (2), there exists $P_0 \in\in \mathrm{Ob}(Q\mathcal{P})$ such that $j_0 : P_0 \twoheadrightarrow M$ in $\mathcal{M}$. Since $\mathcal{P}$ is closed under extensions and $\mathrm{Ker}(\overline{P} \twoheadrightarrow M) \in \mathrm{Ob}(\mathcal{P})$ by property (1), $P \times_M P_0 \in \mathrm{Ob}(\mathcal{P})$. Since the projections from $P \times_M P_0$ to $P$ and $P_0$ are $\mathcal{M}$-admissible epimorphisms, one can consider the following zig-zag

$$u \xleftarrow{i} j^! \xleftarrow{\pi_P} (j\pi_P)^! = (j\pi_{P_0})^! \xrightarrow{\pi_{P_0}} j_0^!$$

in $M/F$, where $\pi_{(-)}$ denotes such projections. That induces a deformation retract of $B(M/F)$. Therefore, by 4.1.3, $B(\mathscr{C}) \cong B(Q\mathscr{M})$. □

**Corollary 4.1.5.** *Let $\mathscr{M}$ be an exact category and $\mathscr{P} \hookrightarrow \mathscr{M}$ a full subcategory closed under extensions and containing $0 \in \mathrm{Ob}(\mathscr{M})$. For every $n \in \omega$, let $\mathscr{P}_n$ denotes the full subcategory of $\mathscr{M}$ consisting of objects that admits a $\mathscr{P}$-resolution of length $n$. There are canonical equivalences*

$$B(Q\mathscr{P}) = B(Q\mathscr{P}_0) \xrightarrow{\sim} B(Q\mathscr{P}_1) \xrightarrow{\sim} \cdots \xrightarrow{\sim} B(Q\mathscr{P}_n) \xrightarrow{\sim} \cdots.$$

*In particular,*

$$B(Q\mathscr{P}) \xrightarrow{\sim} B(Q(\mathrm{colim}_{n\in\omega}\,\mathscr{P}_n)).$$

*Proof.* Apply the previous theorem to $\mathscr{P}_n \hookrightarrow \mathscr{P}_{n+1}$ for every $n \in \omega$ □

**Corollary 4.1.6.** *Let $X$ be a coherent and regular scheme such that*

$$n := \sup\{\dim(\mathscr{O}_{X,x}) | x \in |X|\} \in \omega.$$

*There exists an isomorphism $K_i(X) \cong G_i(X)$ for every $i \in \omega$.*

*Proof.* Locally on $\mathrm{Spec}(A)$, every $A$-module has projective dimension bounded by $n$. Since every coherent $A$-module is finitely presented and, then, of finite type, every projective module is locally free. Now, notice that every $A$-module of finite type has a resolution by projective $A$-modules of finite type. Therefore, one may take $\mathscr{P} := \mathbf{Vect}(X)$, $\mathscr{M} := \mathbf{Coh}(X)$ and apply the previous corollary. □

Let $f : X \longrightarrow Y$ be a morphism in **Sch**. The induced functor

$$f^* : \mathbf{Vect}(Y) \longrightarrow \mathbf{Vect}(X)$$

is exact and, hence, it induces a morphism

$$f^* : K(Y) \longrightarrow K(X).$$

Analogously, when $X$ is exact and $f$ is flat, there's a morphism

$$f^* : G(Y) \longrightarrow G(X).$$

That, however, holds in more generality by the following proposition.

**Corollary 4.1.7** (Contravariance of $G$-theory for morphisms of finite Tor-dimension). *Let $X$ and $Y$ be coherent schemes. Let $f : X \longrightarrow Y$ be a morphism of finite* Tor *dimension (i.e. , there exists $d \in \omega$, the* Tor *dimension of $\mathscr{O}_X$, such that $\mathrm{Tor}_i^{f^{-1}(\mathscr{O}_Y)}(\mathscr{O}_X, \mathscr{F}) \cong 0$ for $i > d$ and $\mathscr{F} \in \mathrm{Ob}(\mathbf{Mod}_{f^{-1}(\mathscr{O}_Y)})$) and $\mathscr{L} \hookrightarrow \mathbf{Coh}(Y)$ be the full category consisting of $\mathscr{F} \in \mathbf{Coh}(Y)$ such that $L_i f^* \mathscr{F} \cong \mathrm{Tor}_i^{f^{-1}(\mathscr{O}_Y)}(\mathscr{O}_X, f^{-1}(\mathscr{F})) \cong 0$ for $i > 0$. If every $\mathscr{F} \in \mathrm{Ob}(\mathbf{Coh}(Y))$ is quotient of an object in $\mathscr{L}$, there exists a canonical morphism*

$$f^* : G(Y) \longrightarrow G(X).$$

*Proof.* Consider the zig-zag

$$\mathbf{Coh}(Y) \hookleftarrow \mathscr{L} \xrightarrow{f^*} \mathbf{Coh}(X)$$

and notice that $f^* : \mathscr{L} \longrightarrow \mathbf{Coh}(X)$ is exact.

By applying the Resolution Theorem ( 4.1.8) to the inclusion $\mathscr{L} \hookrightarrow \mathbf{Coh}(Y)$, the result follows. □

**Corollary 4.1.8.** *Let $X$ and $Y$ be coherent schemes. Let $f : X \longrightarrow Y$ be a proper morphism and $\mathscr{L} \hookrightarrow \mathbf{Coh}(X)$ be the full category consisting of $\mathscr{F} \in \mathbf{Coh}(Y)$ such that $R^i f_* \mathscr{F} \cong 0$ for $i > 0$. If every $\mathscr{F} \in \mathrm{Ob}(\mathbf{Coh}(X))$ is subobject of an object in $\mathscr{L}$, there exists a canonical morphism*

$$f_* : G_i(X) \longrightarrow G_i(Y)$$

*for every $i \in \omega$.*

*Proof.* Consider the zig-zag

$$\mathbf{Coh}(X) \hookleftarrow \mathscr{L} \xrightarrow{f_*} \mathbf{Coh}(Y)$$

and notice that $f_* : \mathscr{Z} \longrightarrow \mathbf{Coh}(X)$ is well defined and exact since $f$ is proper [5].

By applying the Resolution Theorem ( 4.1.8) to the inclusion $\mathscr{Z}^{\mathrm{op}} \hookrightarrow \mathbf{Coh}(Y)^{\mathrm{op}}$, the result follows. )) □

**Corollary 4.1.9.** *$G$-theory is contravariant for flat (e.g. , étale) and morphisms with codomain possesing an ample line bundle (e.g. , the codomain is a quasi-projective variety). Furthermore, it's covariant for finite proper morphisms (e.g. , closed immersions) and proper morphisms with domain possessing an ample line bundle (e.g. , the domain is a quasi-projective variety) and codomain Noetherian.*

*Furthermore, in the above cases, $(-)_*$ and $(-)^*$ commutes with composition.*

*Proof.* All assertions are trivial except for the one of proper morphisms $f : X \longrightarrow Y$ with $X$ possessing an ample line bundle and $Y$ Noetherian. Let $\mathscr{L}$ be a very ample $f$-relative line bundle (which exists by tensoring the ample line bundle on $X$ with itself). Since $\mathscr{L}$ is very ample, let $(\mathscr{O}_X)^{nd} \twoheadrightarrow \mathscr{L}^{\otimes n}$ be an epimorphism and consider the short exact sequence of vector bundles

$$0 \longrightarrow \mathscr{O}_X \longrightarrow (\mathscr{L}^{\otimes n})^{nd} \longrightarrow \mathscr{E} \longrightarrow 0$$

, where the monomorphism was induced by the above epimorphism after dualizing and tensoring by $\mathscr{L}^{\otimes n}$. After applying $\otimes_{\mathscr{O}_X}\mathscr{F}$ for $F \in \mathrm{Ob}(\mathbf{Coh}(X))$, one obtains

$$0 \longrightarrow \mathscr{F} \longrightarrow (\mathscr{F}(n))^{nd} \longrightarrow \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{E} \longrightarrow 0.$$

By Serre's Theorem ([EGAIII-1, Thm. 2.2.1]), $R^i f_*(F(n)) \cong 0$ for every $i > 0$ and $n > N$ for some fixed $N \in \omega$. By increasing $n$, if necessary, one can assumes that $\mathscr{F} \in \mathrm{Ob}(\mathscr{Z})$ (using the notation of the previous corollary), which proves the claim. □

As in the usual algebraic (co)homologies , $G$-theory also enjoys proper-base change.

**Proposition 4.1.1** (Proper Base-change [Qui73, Prop. 7.2.11]). *Let he diagram*

[5] Notice that, for $f$ proper and $\mathscr{F}$ coherent, $R^i\mathscr{F}$ is coherent and it vanishes for large enough $i$.

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & \lrcorner & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

*be a pullback square of schemes possessing an ample line bundle. If $f$ is proper, $g$ has finite* Tor*-dimension and $S'$ and $X$ are* Tor*-independent over $S$ (i.e. , $\mathrm{Tor}_i^{\mathcal{O}_{S,s}}(\mathcal{O}_{S',s'}, \mathcal{O}_{X,x})$ for every $i > 0$ and points $x, s$ and $s'$ satisfying $f(x) = s = g(s')$), then*

$$g^* f_* = f'_*(g')^*.$$

**Proposition 4.1.2.** *Let $X$ be coherent. A nilpotent closed immersion $i : Z \hookrightarrow X$ induces an isomorphism*

$$i_* : G_i(X) \xrightarrow{\sim} G_i(X)$$

*for every $i \in \omega$*

*Proof.* Notice that $\mathcal{F}/\mathcal{I}^n\mathcal{F} \in \mathrm{Ob}(\mathbf{Coh}(Z))$ for every $\mathcal{F} \in \mathrm{Ob}(\mathbf{Coh}(X))$ and $n > 0$, where $V(\mathcal{I}) = Z$. Therefore, by dévissage ( 4.1.6), the result follows. □

**Proposition 4.1.3.** *Let $X$ be a coherent scheme, $i : Z \hookrightarrow X$ a closed immersion and $j : U \hookrightarrow X$ the complementary open immersion. The diagram*

$$\begin{array}{ccc} G(Z) & \xrightarrow{i_*} & G(X) \\ \downarrow{\scriptstyle j_*} & \lrcorner & \downarrow{\scriptstyle f} \\ 1 & \longrightarrow & G(U) \end{array}$$

*is a pullback square in $\mathbf{Grpd}_\infty$. In particular, $G$ satisfies Mayer-Vietories.*

*Proof.* Let $\mathbf{Coh}_Z(X)$ denotes the category of coherent sheaves with support in $Z$. $\mathbf{Coh}_Z(X) \hookrightarrow \mathbf{Coh}(X)$ is a Serre subcategory and $j_* : \mathbf{Coh}(U) \longrightarrow \mathbf{Coh}(X)$ induces and factors through the equivalence $\mathbf{Coh}(U) \xrightarrow{\sim} \mathbf{Coh}(X)/\mathbf{Coh}_Z(X)$. Now, notice that $i_* : G(Z) \longrightarrow B(Q\mathbf{Coh}_Z(X))$ is an isomorphism by dévissage ( 4.1.6) as one can define $\mathcal{F}/\mathcal{I}^n\mathcal{F} \in \mathrm{Ob}(\mathbf{Coh}(Z))$ for every $\mathcal{F} \in \mathrm{Ob}(\mathbf{Coh}(X))$ and $n > 0$, where $V(\mathcal{I}) = Z$. □

**Theorem 4.1.9.** *Let $A$ be a coherent commutative ring. The following assertions are true*

1. *If $A$ is Noetherian, $G(A[t]) \cong G(A)$;*

2. *$G_i(A[t,t^{-1}]) \cong G_i(A) \oplus G_{i-1}(A)$ for every $i \in \omega$.*

*Proof.* (1) follows from [Qui73, Thm. 6.7].

For (2), let $\mathcal{B} \hookrightarrow \mathbf{Mod}^{\mathrm{fp}}(A)$ be the Serre subcategory (of the category of finitely presented $A$-modules) consisting of $A$-modules where $t$ is nilpotent. By dévissage ( 4.1.6), $G(A) \cong B(Q\mathcal{B})$. By the Localisation Theorem ( 4.1.7) applied to $\mathcal{B} \hookrightarrow \mathbf{Mod}^{\mathrm{fp}}(A)$, there exists a fiber sequence

$$\begin{array}{ccc} G(A) \cong B(Q\mathcal{B}) & \longrightarrow & G(A[t]) \cong G(A) \\ \downarrow & \lrcorner & \downarrow \\ 1 & \longrightarrow & G(A[t,t^{-1}]) \end{array}$$

, which induces a long exact sequence

$$\cdots \longrightarrow G_i(A) \longrightarrow G_i(A) \longrightarrow G(A[t,t^{-1}]) \longrightarrow \cdots$$

Now, notice that the morphism $A[t,t^{-1}] \longrightarrow A$ such that $t \mapsto 1$ has finite Tor dimension (actually, Tor dimension 1) and, therefore, by the functoriality of $G$, splits the above long exact sequence.

□

**Corollary 4.1.10.** *Let $X$ be a coherent scheme, $G(X \times \mathbf{A}^1) \cong G(X)$. In particular, if $X$ is regular such that* $\sup\{\dim(\mathscr{O}_{X,x}) | x \in |X|\} \in \omega$, $K(X \times \mathbf{A}^1) \cong K(X)$.

**Corollary 4.1.11** (Homotopy invariance of $G$-theory). *Let $X$ be a coherent scheme and $p : V \longrightarrow X$ a flat morphism with fibers isomorphic to $\mathbf{A}^n$ for some $n \in \omega$ which may varies (e.g. , $X$ is not connected). The morphism $p$ induces an isomorphism*

$$p^* : G(X) \xrightarrow{\sim} G(V).$$

*Proof.* See [Qui73, Prop. 7.4.1]. By applying Proposition 4.1.3 for $Z \hookrightarrow X$ closed immersion and $P \times_X Z \hookrightarrow P$, one can use induction on the chains of closed subschemes to conclude that both fibration sequences are isomorphic. The unique nontrivial case which induction doesn't cover is when $X$ is irreducible. Suppose, hence, $X$ irreducible. Noticing that $G_i(\kappa(\eta_X)) \cong \operatorname{colim}_Z G_i(X \setminus Z)$, where $\eta_X$ is a generic point of $X$. That reduces the assertion to $G_i(\kappa(\eta)[t_1, \cdots, t_n]) \cong G_i(\kappa(x))$. □

For every scheme $X$ and $p \in \omega$, let $X_p$ denotes the subset of $X$ containing only points with codimension $p$. If, furthermore, $X$ is coherent, let $\mathbf{Coh}_p(X) \hookrightarrow \mathbf{Coh}(X)$ denotes the Serre subcategory such that $\mathscr{F} \in \operatorname{Ob}(\mathbf{Coh}_p(X))$ iff $\operatorname{supp}(\mathscr{F})$ has codimension ($= \inf\{\dim(\mathscr{O}_{X,\eta}) | (\eta \in \operatorname{supp}(\mathscr{F})) \wedge (\eta \text{ is the generic point of irreducible component})\}) \geq p$. The inclusions

$$\cdots \hookrightarrow \mathbf{Coh}_{p+1}(X) \hookrightarrow \mathbf{Coh}_p(X) \hookrightarrow \cdots \hookrightarrow \mathbf{Coh}_0(X) \cong \mathbf{Coh}(X)$$

induces a filtration of $\mathbf{Coh}(X)$, which engenders the following spectral sequence.

**Theorem 4.1.10** (Brown-Gersten-Quillen Spectral Sequence [Qui73, Thm. 7.5.4, Prop. 7.5.6, Prop. 7.5.8]). *Let $X$ be Noetherian. The codimension filtration induces a spectral sequence*

$$E_1^{p,q}(X) \cong \coprod_{x \in X_p} K_{-p-q}(\kappa(x)) \Rightarrow G_{-p-q}(X)$$

*, which converges when $X$ is of finite Krull dimension. This spectral sequence $E(X)$ is functorial and contravariant for flat morphisms. Furthermore, the functor $E$ commutes with filtered colimits composed of flat affine transition morphisms.*

*If, furthermore, $X$ satisfies one of the equivalent assertions*

1. *For every $p \in \omega$, the inclusion $i_p : \mathbf{Coh}_{p+1}(X) \hookrightarrow \mathbf{Coh}_p(X)$ induces an the $0$ morphism after applying $K_i(-)$ for every $i \in \omega$;*

2. *For every $q \in \omega$ and $p \neq 0$, $E_2^{p,q}(X) \cong 0$ and the morphism $G_{-q}(X) \xrightarrow{\sim} E_2^{0,q}(X)$ is an isomorphism;*

3. *For every $n \in \mathbf{Z}$, the sequence*

$$G_n(X) \longrightarrow \coprod_{x\in X_0} K_n(\kappa(x)) \xrightarrow{d_1} \coprod_{x\in X_1} K_{n-1}(\kappa(x)) \xrightarrow{d_1} \cdots$$

*is exact, where the first morphism is induced by the $G_n(\mathrm{Spec}(\kappa(x)) \to X)$'s and $d_1$ denotes the differential of $E_1(X)$.*

*, then*

$$E_2^{p,q}(X) \cong \mathrm{H}^p(X_{\mathrm{Zar}}, \mathscr{K}_{-q}(X))$$

*, where $\mathscr{K}_i(X)$ denotes the sheaf of $i$-th $K$-groups of $X$.*

One important consequence of the above spectral sequences is how it relates the Chow group with $K$-theory. The following result generalises, in particular, the case of line bundles $\mathrm{Pic}(X) \cong \mathrm{H}^1(X_{\mathrm{Zar}}, \mathscr{O}_X^\times) \cong \mathrm{H}^1(X_{\mathrm{Zar}}, \mathscr{K}_1(\mathscr{O}_X))$ to more general cycles.

**Theorem 4.1.11** (Quillen-Bloch's Formula, [Qui73, Thm. 7.5.19]). *Let $k$ be a field and $X/k$ be a regular scheme of finite type. There exists an isomorphism*

$$\mathrm{H}^i(X_{\mathrm{Zar}}, \mathscr{K}_i(X)) \cong \mathrm{CH}^i(X)$$

*, where $\mathscr{K}_i(X)$ denotes the sheaf of $i$-th $K$-groups of $X$.*

*Proof.* That follows by noticing that $E_X^{p,-p}(X) \cong \mathrm{CH}^p(X)$ and that the $X$ satisfies one of the equivalent conditions contained in 4.1.10. □

**Theorem 4.1.12** (Projective Bundle Formula, [Qui73, Thm 8.2.1, Prop. 7.4.3]). *Let $X$ be a quasi-compact scheme and $\mathscr{E}$ a locally free sheaf of finite rank $n$ of $\mathscr{O}_X$-modules. There exists an isomorphism*

$$p : K_i(X)^n \xrightarrow{\sim} K_i(\mathbf{P}(\mathscr{E}))$$

*given by $p(\zeta_j) = \xi^{j-1} f^*(\zeta_j)$ for $1 \leq j \leq n$, where $\xi = [\mathscr{O}_{\mathbf{P}(\mathscr{E})}(-1)] \in K_0(\mathbf{P}(\mathscr{E}))$ and $f : \mathbf{P}(\mathscr{E}) \longrightarrow X$ is the structural morphism.*

*Equivalently, the canonical $K_0(X)$-module morphism*

$$g : K_0(\mathbf{P}(\mathscr{E})) \otimes_{K_0(X)} K_i(X) \xrightarrow{\sim} K_i(\mathbf{P}(\mathscr{E}))$$

*given by* $g(\eta \otimes \zeta) = \eta f^* \zeta$ *is an isomorphism.*

*If furthermore,* $X$ *is Noetherian, the analogous isomorphisms*

$$p : G_i(X)^n \xrightarrow{\sim} G_i(\mathbf{P}(\mathscr{E}))$$

*and*

$$g : K_0(\mathbf{P}(\mathscr{E})) \otimes_{K_0(X)} G_i(X) \xrightarrow{\sim} G_i(\mathbf{P}(\mathscr{E}))$$

*are true.*

*Proof.* The equivalent version follows from [SGA6, Exp. VI, Thm. 1.1], where it's proved that $K_0(\mathbf{P}(\mathscr{E})) \cong K_0(X)[1, \xi, \cdots, \xi^{n-1}]$. □

## § 4.2 Waldhausen's $S$ Construction

In [Wal85], Waldhausen generalised Quillen's $K$-theory to more general categories with an additional homotopical flavour. Such homotopical flavour allows, forthwith, an application of the same construction to categories of complexes such as the derived category and, immediatelly, gives an $\Omega$-spectrum structure to $K$-theory. In [TT90], Thomason and Trobaugh used this construction to define a finer notion of $K$-theory (and $G$-theory) of schemes, which "locally" coincides with Quilen's. Analogously, also, one can define the Waldhausen $K$-theory of pointed finitely cocomplete $\infty$-categories, which lifts the domain of Waldhausen $K$-theory of suitable categories to the $\infty$-categorical setting.

**Definition 4.2.1.** Let $\mathscr{C}$ be a category and $\mathrm{cof}(\mathscr{C}) \hookrightarrow \mathscr{C}$ a subcategory. A **category with cofibrations** is a pair $(\mathscr{C}, \mathrm{cof}(\mathscr{C}))$ such that $\mathscr{C}$ is pointed (by a zero object $0 \in \mathrm{Ob}(\mathscr{C})$) and $\mathrm{cof}(\mathscr{C})$, the **category of cofibrations in** $\mathscr{C}$, satisfies the following

1. Every isomorphism of $\mathscr{C}$ is a cofibration;

2. For every $X \in \mathrm{Ob}(\mathcal{C})$, $0 \longrightarrow A$ is a cofibration;

3. The pushout of a cofibration along any morphism exists and is a cofibration.

Cofibrations will be denoted by the symbol $\rightarrowtail$ and quotients by cofibrations will denoted by the symbol $\twoheadrightarrow$ and called simply quotients.

A **cofibration sequence** is any sequence of the form

$$A \rightarrowtail X \twoheadrightarrow X/A$$

for $A, X, X/A \in \mathrm{Ob}(\mathcal{C})$.

Let $\mathrm{Quot}(\mathcal{C})$ denote the set of quotients. A **category with bifibrations** is category with cofibrations $(\mathcal{C}, \mathrm{cof}(\mathcal{C}))$ such that $(\mathcal{C}^{\mathrm{op}}, \mathrm{Quot}(\mathcal{C})^{\mathrm{op}})$ is category with cofibrations and cofibration sequences of $\mathcal{C}$ coincides with cofibration sequences of $\mathcal{C}^{\mathrm{op}}$ (after reversing the direction of the arrows).

*Remark* 4.2.1. The above definition of category with bifibration is slightly different from the one in [TT90, p. 1.2.2], where, in addition, the canonical morphism $X \sqcup Y \longrightarrow X \times Y$ must be an isomorphism

**Definition 4.2.2.** Let $\mathcal{C}$ be a category, $\mathrm{cof}(\mathcal{C}) \hookrightarrow \mathcal{C}$ a subcategory and $\mathrm{w}(\mathcal{C}) \hookrightarrow \mathcal{C}$ a subcategory. A **Waldhausen category** is a triple $(\mathcal{C}, \mathrm{cof}(\mathcal{C}), \mathrm{w}(\mathcal{C}))$ such that $(\mathcal{C}, \mathrm{cof}(\mathcal{C}))$is a category with cofibrations and $\mathrm{w}(\mathcal{C})$, the **category of weak equivalences in** $\mathcal{C}$, satisfies the following

1. Every isomorphism of $\mathcal{C}$ is a weak equivalence;

2. Consider the commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow\!\!\prec & A & \longrightarrow & Y \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ X' & \longleftarrow\!\!\prec & A' & \longrightarrow & Y' \end{array}$$

   in $\mathcal{C}$ such that the vertical arrows are weak equivalences. The canonical morphism (induced by the universal property of the pushout) $X \sqcup_A Y \xrightarrow{\sim} X' \sqcup_{A'} Y'$ is a weak equivalence.

Weak equivalences will be denoted by the symbol $\xrightarrow{\sim}$.

A **biWaldhausen category** is a Waldhausen category $(\mathcal{C},\mathrm{cof}(\mathcal{C}),\mathrm{w}(\mathcal{C}))$ such that $(\mathcal{C},\mathrm{cof}(\mathcal{C})$ is a category with bifibrations and $(\mathcal{C}^{\mathrm{op}},\mathrm{Quot}(\mathcal{C})^{\mathrm{op}},\mathrm{w}(\mathcal{C})^{\mathrm{op}})$ is a Waldhausen category.

*Remark* 4.2.2. The above definition of category with bifibration is slightly different from the one in [TT90, Def. 1.2.4], where, in addition, the canonical morphism $X \sqcup Y \longrightarrow X \times Y$ must be an isomorphism.

**Definition 4.2.3.** Let $(\mathcal{C},\mathrm{cof}(\mathcal{C}),\mathrm{w}(\mathcal{C}))$ and $(\mathcal{D},\mathrm{cof}(\mathcal{D}),\mathrm{w}(\mathcal{D}))$ be Waldhausen categories (*resp.*, biWaldhausen categories) and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor. $F$ is an **exact functor** if it preserves cofibrations and weak equivalences and $F(X \sqcup_A Y) \cong F(X) \sqcup_{F(A)} F(Y)$ for every zig zag $X \longleftarrow\!\!\prec A \longrightarrow Y$ in $\mathcal{C}$ (*resp.*, $F$ and $F^{\mathrm{op}}$ are exact functors between Waldhausen categories).

*Example* 4.2.1. Let $\mathcal{M}$ be an exact category. If $\mathrm{cof}(\mathcal{M})$ denotes the subcategory of admissible monomorphisms, then $\mathrm{Quot}(\mathcal{M})$ is the category of admissible epimorphisms and $\mathcal{M}$ is a biWaldhausen category by defining weak equivalences to be isomorphisms.

Furthermore, a functor $F : \mathcal{M} \longrightarrow \mathcal{N}$ between exact categories is exact iff it's exact as a functor between biWaldhausen categories.

**Definition 4.2.4.** Let $(\mathcal{C},\mathrm{cof}(\mathcal{C}),\mathrm{w}(\mathcal{C}))$ be a Waldhausen category and $P$ be a partially ordered set viewed as a category. A **$P$-gapped object of** $\mathcal{C}$ is a functor $A : P \times P \longrightarrow \mathcal{C}$ such that

1. $A(i,i) \cong 0$ for every $i \in P$;

2. $A(i,j) \rightarrowtail A(i,k) \twoheadrightarrow A(j,k)$ is a cofibration sequence for every $i \leq j \leq k$.

The **category of $P$-gapped objects in** $\mathcal{C}$ is the category $\mathrm{Gap}_P(\mathcal{P})$ consisting of $P$-gapped functors and natural transformations.

Let $P = [n]$, where $[n] := \{0 \leq 1 \leq \cdots \leq n-1 \leq n\}$. The **Waldhausen $S$-construction of** $\mathcal{C}$ is the space $S(\mathcal{C}) := |N(S_\bullet(\mathcal{C}))|$ [6], where $S_\bullet(\mathcal{C})$ is the simplicial Kan complex such that $S_n(\mathcal{C}) := \mathrm{Core}(\mathrm{Gap}_{[n]}(\mathcal{C}))$, $\mathrm{Core}(-)$ is the

[6] Recall that for each bismplicial set, the realisation is given by either realising each coordinate individually or by realising the diagonal

functor that forgets all natural transformations that are not objectwise weak equivalences and the simplicial structure is induced by setting $\varphi^* A(i, j) := A(\varphi(i), \varphi(j))$ for every $\varphi \in \Delta([m], [n])$ .

Analogously, one can define the $S$-construction for any pointed $\infty$-category $\mathcal{C}$.

**Definition 4.2.5.** Let $\mathcal{C}$ be a pointed $\infty$-category which admits finite colimits and $P$ be a partially ordered set viewed as a category. A **$P$-gapped object of** $\mathcal{C}$ is a functor $A : P \times P \longrightarrow \mathcal{C}$ such that

1. $A(i, i) \cong 0$ for every $i \in P$;

2. The diagram

$$\begin{array}{ccc} A(i,j) & \longrightarrow & A(i,k) \\ \downarrow & \ulcorner & \downarrow \\ A(j,j) \cong 0 & \longrightarrow & A(j,k) \end{array}$$

   is a pullback square (*i.e.* , a cofibration sequence) for every $i \leq j \leq k$.

The **category of $P$-gapped objects in** $\mathcal{C}$ is the category $\mathrm{Gap}_P(\mathscr{D})$ consisting of $P$-gapped functors and natural transformations.

Let $P = [n]$, where $[n] := \{0 \leq 1 \leq \cdots \leq n-1 \leq n\}$. The **Waldhausen $S$-construction of** $\mathcal{C}$ is the space $S(\mathcal{C}) := |S_\bullet(\mathcal{C})|$, where $S_\bullet(\mathcal{C})$ is the simplicial Kan complex such that $S_n(\mathcal{C}) := \mathrm{Core}(\mathrm{Gap}_{[n]}(\mathcal{C}))$, $\mathrm{Core}(-)$ is the functor that forgets all non-invertible morphisms and the simplicial structure is induced by setting $\varphi^* A(i, j) := A(\varphi(i), \varphi(j))$ for every $\varphi \in \Delta([m], [n])$ .

**Definition 4.2.6.** Let $(\mathcal{C}, \mathrm{cof}(\mathcal{C}), \mathrm{w}(\mathcal{C}))$ be a Waldhausen category. $\mathcal{C}$ is a good Waldhausen category if there exists a (simplicial) model category $(\mathscr{M}, \mathrm{cof}(\mathscr{M}), \mathrm{fib}(\mathscr{M}), \mathrm{w}(\mathscr{M}))$ and a fully faithful functor $i : \mathcal{C} \hookrightarrow \mathscr{M}$ such that

1. The essential image $i(\mathcal{C})$ is contained in the category of cofibrant objects;

2. $i$ preserves cofibrations and weak equivalences;

3. The essential image$i(\mathscr{C})$ is closed under weak equivalences (*i.e.* , if $X \in \mathrm{Ob}(\mathscr{C})$ is such that $i(X)$ is weak equivalent to $Y \in \mathrm{Ob}(\mathscr{M})$, then $Y \in \mathrm{Ob}(i(\mathscr{C}))$).;

4. $i$ preserves finite colimits;

Let $(\mathscr{C}, \mathrm{cof}(\mathscr{C}), \mathrm{w}(\mathscr{C}))$ be a biWaldhausen category. $\mathscr{C}$ is a good biWaldhausen category if the dual of the above axioms (for fibrations) also holds for $\mathrm{Quot}(\mathscr{C})$ instead of $\mathrm{cof}(\mathscr{C})$.

**Lemma 4.2.1.** *Let $(\mathscr{C}, \mathrm{cof}(\mathscr{C}), \mathrm{w}(\mathscr{C}))$ be a good Waldhausen category. The simplicial localisation (Hammock localisation or Dwyer-Kan localisation) $N((L^H\mathscr{C})^{\mathrm{fib}})$ is a pointed $\infty$-category with finite $\infty$-colimits. Furthermore, the pushout squares in $\mathscr{C}$ weak equivalent to one with at least one morphism being a cofibration are exactly the pushout squares in $L^H\mathscr{C}$.*

*If, furthermore, $(\mathscr{C}, \mathrm{cof}(\mathscr{C}), \mathrm{w}(\mathscr{C}))$ is a good biWaldhausen category, then $N(L^H\mathscr{C})$ is a stable $\infty$-category. Furthermore, the pullback squares in $\mathscr{C}$ weak equivalent to one with at least one morphism being a quotient are exactly the pullback squares in $L^H\mathscr{C}$.*

*If $F : \mathscr{C} \longrightarrow \mathscr{D}$ is an exact functor between good Waldhausen categories (resp., biWaldhausen categories), then the localisation $N(L^H F)$ is a morphism of $\infty$-categories that commutes with cofiber sequences (resp., cofiber and fiber sequences, i.e. , an exact functor between stable $\infty$-categories).*

*Proof.* Recall that an homotopy pushout coincides with the ordinary pushout whenever one of the morphisms is a cofibration and all the objects are cofibrant. Dually, an homotopy pullback coincides with the ordinary pullback whenever one of the morphisms is a cofibration and all the objects are cofibrant.

Since every object in a biWaldhausen category if both fibrant and cofibrant, the cofiber sequences are the fiber sequences, which implies that $L^H\mathscr{C}$ is stable whenever $\mathscr{C}$ is biWaldhausen.

The other statements follows from [TV04]. □

In both the cofibration and the $\infty$-case, notice that $S_n(\mathscr{C})$ consist in diagrams of the form

$$\begin{array}{ccccccc}
0 \cong A(0,0) & \rightarrowtail & A(0,1) & \rightarrowtail & \cdots & \rightarrowtail & A(0,n) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 \cong A(1,1) & \rightarrowtail & \cdots & \rightarrowtail & A(1,n) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \cdots & \rightarrowtail & A(0,n) \\
 & & & & & & \downarrow \\
 & & & & & & 0 \cong A(n,n)
\end{array}$$

, where each square defines a fibration sequence (or an $\infty$-pushout square).

Since $A(0,i) \rightarrowtail A(0,j) \twoheadrightarrow A(i,j)$ is a cofibration sequence for every $i \leq j$, the data

$$0 \cong A(0,0) \rightarrowtail A(0,1) \rightarrowtail A(0,2) \rightarrowtail \cdots \rightarrowtail A(0,n)$$

is functorially equivalent to the above diagram by taking quotients (or cofibers).That implies the following lemma.

**Lemma 4.2.2.** *Let $\mathcal{C}$ be an $\infty$-category (resp., Waldhausen category). The functor*

$$\mathrm{Gap}_{[n]}(\mathcal{C}) \longrightarrow \mathrm{Fun}(\Delta^{n-1}, \mathcal{C})$$

*(resp.,*

$$\mathrm{Gap}_{[n]}(\mathcal{C}) \longrightarrow \mathrm{Fun}^{\mathrm{cof}}(\Delta^{n-1}, N(\mathcal{C}))$$

*, where* $\mathrm{Fun}^{\mathrm{cof}}$ *denotes the functors sending every morphism to cofibrations) given by forgetting $A(i,j)$ for $i \neq 0$ is an equivalence of $\infty$-categories.*

*Remark* 4.2.3. Notice, however, that one cannot simply work with $\mathrm{Fun}(\Delta^{n-1}, \mathcal{C})$ without remembering the choices of cofibration sequences (or $\infty$-pushout squares) as the face morphism $S_n(\mathcal{C}) \longrightarrow S_{n-1}(\mathcal{C})$ sends

$$X_1 \rightarrowtail X_2 \rightarrowtail \cdots \rightarrowtail X_n$$

to

$$X_2/X_1 \rightarrowtail \cdots \rightarrowtail X_n/X_1$$

The space of choices is contractible. However, one still need to make a choice in order to define $S_\bullet(\mathscr{C})$.

**Proposition 4.2.1.** *Let $\mathscr{C}$ be a good Waldhausen category. The canonical inclusion morphism*

$$S_n(\mathscr{C}) \longrightarrow S_n(N(L^H\mathscr{C}^{fib}))$$

*is an equivalence in* $\mathbf{Grpd}_\infty$ *for every $n \in \omega$.*

*Proof.* See Notice that $S_n(\mathscr{C})$ consists of sequences of morphisms which are cofibrations, whereas $S_n(N(L^H\mathscr{C}^{fib}))$ consists of sequences which are only weakly equivalent to sequences of cofibrations. See [BM08, Thm 2.9] and [BGT13, Cor. 7.7] □

By taking into account the above two lemmas, for good Waldhausen categories, it's enough to work with pointed and finitely cocomplete ∞-categories. So, henceforth, both definitions will be used interchangeably whenever convenient.

**Definition 4.2.7.** Let $\mathscr{C}$ be a pointed finitely cocomplete ∞-category. The **Waldhausen *K*-theory of** $\mathscr{C}$ is the space $K(\mathscr{C}) := \Omega_0|S_\bullet(\mathscr{C})|$, where 0 denotes the zero object of $\mathscr{C}$, which is contained in $S_0(\mathscr{C})$.

*Remark* 4.2.4. Notice that $S_0(\mathscr{C})$ is the space of zero objects of $\mathscr{C}$. Since it's contractible, $|S_\bullet(\mathscr{C})|$ is 0-connected and, therefore, the choice of base point for the loop space is irrelevant.

**Theorem 4.2.1** (Additivity Theorem, [Wal85, Thm. 1.4.2]). *Let $\mathscr{C}$ be a pointed finitely cocomplete ∞-category, $\mathscr{A},\mathscr{B} \hookrightarrow \mathscr{C}$ subcategories preserving all finite colimits and $E(\mathscr{A},\mathscr{C},\mathscr{B}) \hookrightarrow \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$ the category of cofiber sequences of the form $A \rightarrowtail C \twoheadrightarrow B$, where $A \in \mathrm{Ob}(\mathscr{A})$, $B \in \mathrm{Ob}(\mathscr{B})$ and $C \in \mathrm{Ob}(\mathscr{C})$. Let $s,t$ and $q$ denotes, respectively, the projections $E(\mathscr{A},\mathscr{C},\mathscr{B}) \longrightarrow \mathscr{C}$ such that every cofiber sequence $E \in \mathrm{Ob}(E(\mathscr{C}))$ can be written as*

$$sE \rightarrowtail tE \twoheadrightarrow qE.$$

*The morphism*

$$(s,q) : |S_\bullet(E(\mathscr{C}))| \xrightarrow{\sim} |S_\bullet(\mathscr{A})| \times |S_\bullet(\mathscr{B})|$$

*an homotopy equivalence of spaces.*

**Corollary 4.2.1.** *Let $\mathscr{C}$ and $\mathscr{D}$ be a pointed finitely cocomplete $\infty$-categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor preserving finite colimits. The projections*

$$K(\mathscr{C}) \longleftarrow K(\mathscr{C} \times \mathscr{D}) \longrightarrow \mathscr{D}$$

*induces an equivalence*

$$K(\mathscr{C} \times \mathscr{D}) \xrightarrow{\sim} K(\mathscr{C} \times \mathscr{D})$$

*by the universal property of products.*

*Proof.* That follows from the Additivity Theorem applied to the canonical inclusions $\mathscr{C} \hookrightarrow \mathscr{C} \times \mathscr{D}$ and $\mathscr{D} \hookrightarrow \mathscr{C} \times \mathscr{D}$ given by setting $0$ one of the coordinates. □

The following gives an alternative proof for the structure of an (connective) $\Omega$-spectrum in $K(\mathscr{C})$.

**Corollary 4.2.2.** *$K(\mathscr{C})$ is a group-like $E_\infty$-space. In particular, $K(\mathscr{C})$ is a connective $\Omega$-spectrum.*

*Proof.* The coproduct $\vee$ in $\mathscr{C}$ commutes with finite colimits and, hence, induces a morphism

$$m : K(\mathscr{C}) \times K(\mathscr{C}) \cong K(\mathscr{C} \times \mathscr{C}) \longrightarrow K(\mathscr{C}).$$

Since the product on $\mathscr{C}$ is coherently commutative and associative, $m$ will also be.

Since $K_0(\mathscr{A})$ is an abelian group, $K(\mathscr{C})$ is group-like. □

*Remark* 4.2.5. The above structure of (connective) $\Omega$-spectrum in $K(\mathscr{C})$ concides with the one of the next theorem.

Notice that $\mathrm{sk}_0(|S_\bullet(\mathcal{C})|) \cong S_0(\mathcal{C})$ and that $\mathrm{sk}_1(|S_\bullet(\mathcal{C})|)$ is obtained from $S_0(\mathcal{C})$ by attaching $S_1(\mathcal{C}) \times \Delta^1$. Since $S_0(\mathcal{C})$ is contractible, $\mathrm{sk}_1(|S_\bullet(\mathcal{C})|) \cong \mathbf{S}^1 \wedge \mathrm{Core}(\mathcal{C})$. Therefore, one obtains a morphism

$$\Sigma\,\mathrm{Core}(\mathcal{C}) \longrightarrow |S_\bullet(\mathcal{C})|$$

. By taking adjoints, one obtains

$$\mathrm{Core}(\mathcal{C}) \longrightarrow \Omega_0|S_\bullet(\mathcal{C})|$$

, which embeds $\mathrm{Core}(\mathcal{C})$ into a group completion.

By iterating the Waldhausen's $S$-construction, one obtains the morphisms

$$\mathrm{Core}(\mathcal{C}) \longrightarrow \Omega|S_\bullet(\mathcal{C})| \longrightarrow \Omega^2|S_\bullet S_\bullet(\mathcal{C})|$$

**Theorem 4.2.2.** *The morphisms*

$$\Omega^n|S^n_\bullet(\mathcal{C})| \xrightarrow{\sim} \Omega^{n+1}|S^{n+1}_\bullet(\mathcal{C})|$$

*are equivalences in* $\mathbf{Grpd}_\infty$ *for* $n > 0$. *In particular,* $K(\mathcal{C})$ *has the structure of an* $\Omega$*-spectrum.*

*Proof.* See [Wal85, Prop. 15. 3]. □

*Remark* 4.2.6. The above structure of $\Omega$-spectrum in $K(\mathcal{C})$ concides with the one of the previous theorem.

**Theorem 4.2.3** ([Wal85, 1.9. Appendix]). *Let* $\mathcal{M}$ *be an exact category and* $K^Q(\mathcal{M}) := \Omega_0 BQ(\mathcal{M})$. *There exists a natural equivalence*

$$K^Q(\mathcal{M}) \cong K(\mathcal{M}).$$

*Proof.* (Sketch) The main idea relies on applying barycentric subdivision to the simplicial category $S_\bullet(\mathcal{M})$ and considering, instead of $Q\mathcal{M}$, the bicategory of squares such that vertical morphisms are isomorphisms. Now, after applying the nerve in the horizontal direction, one obtains an equivalent simplicial category. One just have, then, to notice that composable morphisms in $Q\mathcal{M}$ are given by a diagram of the form $S_n(\mathcal{C})$ for some $n \in \omega$. □

**THEOREM 4.2.4** (Gillet-Waldhausen). *Let $\mathscr{M}$ be an exact category such that there exists an abelian category $\mathscr{A}$ and fully faithful functor $\mathscr{M} \hookrightarrow \mathscr{A}$ satisfying the following*

1. *$\mathscr{M}$ is closed under extensions in $\mathscr{A}$;*
2. *Every sequence of $\mathscr{M}$ which is exact in $\mathscr{A}$ is, also, exact in $\mathscr{M}$.*

*Let $\mathrm{Ch}^b(\mathscr{M})$ be the biWaldhausen with $\mathrm{cof}(\mathrm{Ch}^b(\mathscr{M}))$ the objectwise admissible monomorphisms and $\mathrm{w}(\mathrm{Ch}^b(\mathscr{M}))$ the quasi-isomorphisms. Let, moreover, $\mathrm{Ch}^b(\mathscr{M})_{split}$ be as $\mathrm{Ch}^b(\mathscr{M})$, but with $\mathrm{cof}(\mathrm{Ch}^b(\mathscr{M})_{split})$ the objectwise split monomorphisms with quotients lying in $\mathrm{Ch}^b(\mathscr{M})_{split}$. The inclusion functors induces homotopy equivalences of spectra*

$$K(\mathrm{Ch}^b(\mathscr{M})_{split}) \xrightarrow{\sim} K(\mathrm{Ch}^b(\mathscr{M})) \xleftarrow{\sim} K(\mathscr{M}).$$

*In particular, if $\mathscr{M}$ is a good biWaldhausen category, there's an equivalence, $K(D^b(\mathscr{M})) \cong K(\mathscr{M})$, where $D^b(\mathscr{M})$ is viewed as a stable $\infty$-category.*

*PROOF.* See [TT90, Thm. 1.11.7]. □

One can generalise Quillen's $K$-theory (and $G$-theory) to a derived version which coincides locally with it and enjoys additional properties. Henceforth, in this section, Quillen's $K$-theory (*resp.*, $G$-theory) will be denoted by $K^Q(-)$ (*resp.*, $G^Q(-)$).

**DEFINITION 4.2.8.** Let $X$ be a scheme. The **$K$-theory of $X$** is defined as the spectrum $K(X) := K(\mathbf{Perf}(X))$, where $\mathbf{Perf}(X)$ is the category of perfect complexes of finite (global) $Tor$-amplitude (*i.e.* , complexes $E$ of $\mathscr{O}_X$-modules which are locally quasi-isomorphic to a bounded complex of vector bundles and there exists $a \leq b \in \mathbf{Z}$ such that $\pi_{-k}(E \otimes \mathscr{F}) \cong 0$ for every $a \leq k \leq b$).

Let $Y \hookrightarrow X$ be a closed immersion. The **$K$-theory of $X$ with support in $Y$** is defined as the spectrum $K_Y(X) := K(\mathbf{Perf}_Y(X))$, where $\mathbf{Perf}_Y(X) \hookrightarrow \mathbf{Perf}(X)$ is the full subcategory of complexes which are acyclic on $X \setminus Y$

*REMARK* 4.2.7. If $X$ is quasi-compact, the finite Tor-amplitude assumption is irrelevant as locally every perfect complex has finite Tor-amplitude and, for $X$ quasi-compact, local and global finite Tor-amplitude agree.

**Definition 4.2.9.** Let $X$ be a scheme. The **$G$-theory of $X$** is defined as the spectrum $G(X) := G(\mathbf{pCoh}(X))$, where $\mathbf{pCoh}(X)$ is the category of pseudo-coherent complexes of (globally) bounded cohomology (*i.e.* , complexes $E$ of $\mathscr{O}_X$-modules which are locally quasi-isomorphic to a bounded above complex of vector bundles and there exists $a \leq b \in \mathbf{Z}$ such that $\pi_{-k}(E) \cong 0$ for every $a \leq k \leq b$).

Let $Y \hookrightarrow X$ be a closed immersion. The **$K$-theory of $X$ with support in $Y$** is defined as the spectrum $G_Y(X) := K(\mathbf{pCoh}_Y(X))$, where $\mathbf{pCoh}_Y(X) \hookrightarrow \mathbf{pCoh}(X)$ is the full subcategory of complexes which are acyclic on $X \setminus Y$.

**Theorem 4.2.5** ([TT90, Prop. 3.10]). *Let $X$ be a scheme with an ample family of line bundles and $\mathscr{P} \hookrightarrow \mathbf{Perf}(X)$ be the subcategory of strict perfect complexes. There exists an equivalence*

$$K^Q(X) \cong K(\mathscr{P}) \cong K(X)$$

*of spectra.*

*Proof.* The first isomorphism follows from Gillet-Waldhausen's Theorem, while the second isomorphism follows by noticing that, for $X$ quasi-compact and quasi-separated with an ample family of line bundles, every perfect complex $E$ is quasi-isomorphic to a strict perfect one. □

**Theorem 4.2.6** ([TT90, Cor. 3.13]). *Let $X$ be a Noetherian scheme. There exists an equivalence*

$$G^Q(X) \cong G(X)$$

*of spectra.*

*Proof.* That follows from Gillet-Waldhausen's Theorem and the fact that, for $X$ Noetherian, $E$ is pseudo-coherent iff $E$ has coherent cohomology and it's (globally) cohomologically bounded above (and therefore, $E$ is quasi-isomorphic to a strict bounded complex of coherent $\mathscr{O}_X$-modules). □

Thomason and Trobaugh with the aid of these new $K$-groups were able to generalise the Mayer-Vietories (Proposition 4.1.3) and the Localisation Theorem for $K$-theory (and not only $G$-theory).

**Theorem 4.2.7** (Localisation Theorem, [TT90, Thm. 7.4]). *Let $X$ be quasi-compact and quasi-separated, $j : U \hookrightarrow X$ an open immersion with $U$ quasi-compact and $iY \hookrightarrow X$ the complementary closed immersion. Let also $Z \hookrightarrow X$ be a closed subscheme with quasi-compact complementar. There are fiber sequences*

$$\begin{array}{ccc} K_Y(X) & \longrightarrow & K(X) \\ \downarrow & \lrcorner & \downarrow j^* \\ 1 & \longrightarrow & K(U) \end{array}$$

*and*

$$\begin{array}{ccc} K_{Y\cap Z}(X) & \longrightarrow & K_Z(X) \\ \downarrow & \lrcorner & \downarrow j^* \\ 1 & \longrightarrow & K_{U\cap Z}(U) \end{array}$$

*of spectra.*

*Remark* 4.2.8. In [TT90], Bass' $K$-theory $K^B(X)$ was used. One should notice, however, that the truncation of $K^B(X)$ engendering a connective spectrum is equivalent to $K(X)$. In other words, $K^B(X)$ only contributes to negative $K$-theory.

By pasting squares with two sides fiber sequences such that the fibers are equivalent, one can obtain a Mayer-Vietories $\infty$-pullback square.

**Corollary 4.2.3** (Mayer-Vietories, [TT90, Thm. 8.1]). *Let $X$ be a quasi-separated scheme, $U \hookrightarrow X$ and $V \hookrightarrow X$ open immersions with $U$ and $V$ quasi-compact. Let also $Z \hookrightarrow U \cap V$ be a closed subscheme with $(U \cap V) \setminus Z$ quasi-compact. The squares*

$$\begin{array}{ccc} K(U\cap V) & \longrightarrow & K(U) \\ \downarrow & \lrcorner & \downarrow j^* \\ K(V) & \longrightarrow & K(U\cap V) \end{array}$$

*and*

$$\begin{array}{ccc} K_Z(U\cap V) & \longrightarrow & K_{U\cap Z}(U) \\ \downarrow & \lrcorner & \downarrow j^* \\ K_{V\cap Z}(V) & \longrightarrow & K_{U\cap V\cap Z}(U\cap V) \end{array}$$

*are $\infty$-pullback squares in the category of spectra* $\mathbf{Sp}$.

In [TT90], the authors were also able to generalise the Brown-Gersten spectral sequence to the Nisnevich topology and the relative support case.

**Theorem 4.2.8** (Brown-Gersten spectral sequence[TT90, Thm. 10.3, Cor. 10.5, Thm. 10.8, Cor. 10.10 ]). *Let $X$ be a Noetherian scheme of finite Krull dimension, $Y \hookrightarrow X$ a closed immersion and $\tau \in \{\mathrm{Nis}, \mathrm{Zar}\}$. The augmentation is a natural homotopy equivalence*

$$K(X) \xrightarrow{\sim} \mathrm{H}(X_\tau, \mathscr{K})$$

*and*

$$K_Y(X) \xrightarrow{\sim} \mathrm{H}(X_\tau, \mathscr{K}_{(-)\cap U}) \cong \mathrm{H}_Y(X_\tau, \mathscr{K})$$

*In particular, the hypercohomology spectral sequence induces strongly convergent spectral sequences*

$$E_2^{p,q} \cong \mathrm{H}^p(X_\tau, \mathscr{K}_q) \Rightarrow K_{-p+q}(X)$$

*and*

$$E_2^{p,q} \cong \mathrm{H}^p_Y(X_\tau, \mathscr{K}_q) \cong \mathrm{H}^p(X_\tau, (\mathscr{K}_Y)_q) \Rightarrow (K_Y)_{-p+q}(X)$$

# Part III

# Freedom and The Sun

# Chapter 5

# Pure Motives

The theory of pure motives arose quite naturally from the comparisons after tensoring by some ring of periods between the plethora of Weil cohomology theories. Such comparisons were not only restricted to isomorphisms of ordinary modules over some ring. Indeed, a geometric structure embodied by means of some action or representation was lively present in such cohomologies. Those structures despite their evident differences were somehow magically preserved and transformed by such comparisons leading to natural transformations between functors (after tensoring by some ring of periods).

Something was lurking behind those structured modules that the cohomology defined. Something must be lying deeper in order to assemble all these unexpected relations. That form instantiating to all reasonable cohomology theories $\mathrm{H}_W(X, F)$ functorially and naturally was called a motive.

The comparisons are listed in the following for the convenience of the reader.

Let $k$ be a field of characteristic $p$, $K := \mathrm{Frac}(W(k))$ whenever $p \neq 0, X/k$ a proper smooth variety and $\mathscr{X}/\mathscr{O}_K$ a proper smooth model.

$$\mathrm{H}^i_{\text{ét}}(X \otimes_k \overline{k}, \mathbf{Q}_\ell) \cong \mathrm{H}^i_{\mathrm{Betti}}(X, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$$

when $\ell \neq p$.

$$\mathrm{H}^i_{\mathrm{dR}}(X, k) \otimes_k \mathbf{C} \cong \mathrm{H}^i_{\mathrm{Betti}}(X, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$$

when $p = 0$.

$$\mathrm{H}^i_{\mathrm{cris}}(X, W(k)) \otimes_{W(k)} B_{\mathrm{cris}} \cong \mathrm{H}^i_{\text{ét}}(\mathcal{X}_\eta \otimes_K \overline{K}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}}$$

when $p \neq 0$.

$$\mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}_\eta) \otimes_K B_{\mathrm{dR}} \cong \mathrm{H}^i_{\text{ét}}(\mathcal{X}_\eta \otimes_K \overline{K}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}$$

when $p \neq 0$.

In each of these comparisons, besides the $A$-module structure for the respective period ring $A$, additional structures are preserved. Such structures are entirely determined by a gerbe called the Motivic Galois Group, which should have an underlying groupoid with $F$-points for each suitable $F$-valued Weil cohomology theory and morphisms for each comparison morphism between such cohomologies. That motivic action should instantiate to the respective actions that determine the respective structures : Galois action in the $\ell$-adic and $p$-adic case; Hodge filtration in the Betti and de Rham case; and the Hodge-Tate filtration in the $p$-adic étale case. These compatibilities between the inertia groups of the gerbe, however, depend on several conjectures such as the Mumford-Tate Conjecture and the Hodge Conjecture. The Hodge Conjecture, for instance, guarantees that the motivic Galois group acts on the Betti side as the Mumford-Tate group or, equivalently, that the motivic action is completely determined by the Hodge filtration; whereas the Mumford-Tate conjecture guarantees that such group under the comparison with the étale side is actually the absolute Galois group $G_k$.

When $X$ is not proper, an additional structure called weight filtration must also be taken into consideration. Such filtration will be instantiated to the respective weight filtrations on each Weil cohomology theory. It's completely characterised as the maximal increasing filtration having graded pieces of pure weight. Such weights are also entirely determined by the motivic action. For instance, in the $\ell$-adic case, the Riemann's Hypothesis guarantee the purity of $\mathrm{H}^i_{\text{ét}}(X, \mathbf{Q}_\ell)$ for $X/k$ smooth projective by declaring the weight to be exactly the exponent of the eingenvalues of the Frobenius. In fact, the origin of the theory of weights came historically from the behaviour of such eigenvalues and the analogous theory for the other cohomologies was induced by the known comparisons morphisms.

One, therefore, may consider pure motives by restricting to the motives generated by the several $\mathrm{H}_W(X,F)$ for each $X/k$ smooth projective. Such motives are indeed the main substance of the general theory as any other (mixed) motive can be generated by assembling such pure motives in several nontrivial ways through extensions and colimits [1].

The idea of a motif was firstly described by Grothendieck in a letter to Serre on 16 August 1964 ([Gro04] or [Ser91]). Grothendieck, himself, actually, never officially published on the topic. He, however, around 1965-1970, wrote a manuscript ([Gro70]) describing axiomatically the main properties that a category of motives shoud enjoy. There, he even described the structure of weight filtration that characterises the category of (geometric) mixed motives.

In the spring of 1967, Grothendieck himself presented his ideas in lectures at the Institut des Hautes Études Scientifiques (I.H.É.S.). These lectures resulted in three exposition by attendants: respectively, Manin ([Man68]), Demazure at the Séminaire Bourbaki presented in November of 1969 ([Dem71]) and Kleiman at the $5^{th}$-Nordic Summer-School in Mathematics in Oslo ([Kle72]).

As for the name *motif* coined by Grothendieck himself, Yuri Manin explains (by citing H. Read, "A concise history of modern painting") in the first paper about motives ever published ([Man68]):

*" For the reasoning behind the use of the word cf. Herbert Read's remark ([6], French p. 16): "Cézanne's method of painting was first to choose his 'motif' a landscape, a person to be portrayed, a still-life; then to bring into being his visual apprehension of this* **motif***; and in this process to lose nothing of the vital intensity that the* **motif** *possessed in its actual existence." (In order to preserve the vital intensity of a motif, realization should obviously be a functor. Yu. M.) "*

## § 5.1 Weil Cohomology Theory

A Weil cohomology theory consists in the data of a cohomology which should satisfy enough properties in order to prove the Lefschetz trace formula. These properties alone determine what structures should be present in the category of motives (for instance, a symmetric monoidal structure)

[1]One, however, cannot split every (mixed) motive into pure ones.

This section is based on de Jong list of exercises on Weil cohomology theories [Jon07].

**DEFINITION 5.1.1.** Let $k$ be an algebraically closed field and $F$ be a field of characteristic 0. A **Weil cohomology theory** $\mathrm{H}_W$ **with coefficients in** $F$ consists in the following data.

1. A functor

$$\mathrm{H}_W^\bullet : \mathbf{SmProj}_k^{\mathrm{op}} \longrightarrow \mathbf{Mod}_F^{\mathbf{Z}}$$

   . For $f$ a morphism in $\mathbf{SmProj}_k$, $\mathrm{H}_W^i(f)$ will be denoted by $f^*$ for every $i$.

2. For every $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$, $\mathrm{H}_W^\bullet(X)$ a structure of commutative graded algebra. The product on $\mathrm{H}_W^\bullet$, which will be denoted by $\cup$, will be called **cup product**.

3. (Tate Twist) For every $1 \in \mathbf{Z}$, $F(1) \in \mathbf{Mod}_F$, which will be called **Tate twist**. When $V \in \mathbf{Mod}_F$, the notation $V(n) := V \otimes_F F(1)^{\otimes n}$ will be used.

4. (Trace Morphism) For every $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$ of dimension $d$, a morphism

$$\mathrm{Tr}_X : \mathrm{H}_W^{2d}(X)(d) \longrightarrow F$$

   , which will be called **trace morphism**

5. (Cycle Morphism) For every $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$ and $i$, a morphism

$$\gamma_X^i : \mathcal{Z}^i(X) \longrightarrow \mathrm{H}_W^{2i}(X)(i)$$

   [2]

   , which will be called **cycle class morphism**.

[2] $\mathcal{Z}^i(X)$ denotes the abelian group of cyles of codimension $i$.

That data must satisfy the following axioms

1. (Finiteness) For every $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$ of dimension $d$, $\mathrm{H}^i_W(X)$ is finite dimensional for every $i \geq 0$ and $\mathrm{H}^i_W(X) \cong 0$ for every $i \notin [0, 2d]$.

2. (Künneth Formula) $H^\bullet_W$ is a strong symmetric monoidal functor, *i.e.* , for every $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$ and $Y \in \mathrm{Ob}(\mathbf{SmProj}_k)$, the projections induces an isomorphism

$$\mathrm{H}^\bullet_W(X) \otimes \mathrm{H}^\bullet_W(Y) \xrightarrow{\pi_X^*(-) \cup \pi_Y^*(-)} \mathrm{H}^\bullet(X \times_k Y)$$

3. (Poincaré Duality) For every $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$ of dimension $d$,

$$\mathrm{H}^i_W(X) \otimes_K \mathrm{H}^{2d-i}_W(X)(d) \xrightarrow{(-)\cup(-)} \mathrm{H}^{2d}(X \times_k Y)(d) \xrightarrow{\mathrm{Tr}_X} F.$$

   is a perfect duality.

4. (Multiplicativity of Tr)For every $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$ of dimension $d_X$, $Y \in \mathrm{Ob}(\mathbf{SmProj}_k)$ of dimension $d_Y$, $\alpha \in \mathrm{H}^{2d_X}_W(X)(d_X)$ and $\beta \in \mathrm{H}^{2d_Y}_W(Y)(d_Y)$,

$$\mathrm{Tr}_{X \times_k Y}(\pi_X^*(\alpha) \cup \pi_Y^*(\beta)) = \mathrm{Tr}_X(\alpha)\,\mathrm{Tr}_Y(\beta).$$

5. (Multiplicativity of $\gamma$) For every $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$, $Y \in \mathrm{Ob}(\mathbf{SmProj}_k)$, closed immersions $Y \hookrightarrow X$ and $W \hookrightarrow Y$,

$$\gamma_{X \times_k Y}(Z \times_k W) = \pi_X^*(\gamma_X(Z)) \cup \pi_Y^*(\gamma_Y(W)).$$

6. (Naturality of $\gamma$) $\gamma_{(-)} : \mathscr{Z}^\bullet(-) \longrightarrow \mathrm{H}^{2\bullet}()(\bullet)$ is a natural transformation with respect to the (partially defined [3]) pullback. Explicitly, let $f : X \longrightarrow Y$ be a morphism in $\mathbf{SmProj}_k$ satisfying one of the following

   - $f$ is flat;

[3]The pullback of cycles is not well defined. The scheme theoretic inverse image $f^{-1}(Z)$ of a closed subscheme $Z \hookrightarrow X$ is a closed subscheme since closed immersions are stable by pullbacks. However the codimension of $f^{-1}(Z)$ may differ from the one of $Z$. Also, in general, it will not preserve adequate equivalence relations on cycles.

- $f^*$ with the domain $\mathcal{Z}^\bullet(Y)$ restricted to the class of closed subschemes generically transverse to $f$ (*i.e.* , closed subschemes $Z$ such that $f^{-1}(Z)$ is generically reduced and $\operatorname{codim}_X(f^{-1}(Z)) = \operatorname{codim}_Y(Z)$);
- $f^*$ with the domain $\mathcal{Z}^\bullet(Y)$ restricted to the class of closed subschemes that are Cohen-Macauley.

, then

$$f^*\gamma_Y = \gamma_X f^*.$$

Even more explicilty, let $Z \hookrightarrow Y$ be a closed integral subscheme and $f$ satisfying one of the above. If $k := \dim f^{-1}(Z) = \dim Z + (\dim X - \dim Y)$, then $\gamma_Y(Z) = \sum_i n_i\gamma_X(Z_i)$ for $[f^{-1}(Z)]_k = \sum_i n_i Z_i$.

7. (Naturality of $\gamma$ with respect to pushfowards) Let $f : X \longrightarrow Y$ be a morphism in $\mathbf{SmProj}_k$. For every $\alpha \in \mathrm{H}^{2c}_W(Y)(c)$ and $\beta \in \mathcal{Z}^c(X)$,

$$\mathrm{Tr}_X(\gamma_X(\beta) \cup f^*(\alpha)) = \mathrm{Tr}_Y(\gamma_Y(f_*(\beta)) \cup \alpha).$$

More explicitly, for $Z \hookrightarrow X$ a closed integral subscheme of codimension $c$ and $m := [R(Z) : R(f(Z))]$ [4], $\mathrm{Tr}_X(\gamma_X(Z) \cup f^*(\alpha)) = m\,\mathrm{Tr}_Y(\gamma_Y(f(Z)) \cup \alpha)$.

8. (Calibration) $\mathrm{Tr}_{\mathrm{Spec}(k)}(\gamma_{\mathrm{Spec}(k)}(\mathrm{Spec}(k))) = 1$.

*Remark* 5.1.1. Notice that the data of a cycle class morphism $\gamma_X$ and a trace morphism $\mathrm{Tr}_X$ for every $X \in \mathbf{SmProj}_k$ are equivalent as long as one requires that $\mathrm{Tr}_X$ and $\gamma_X^{d_X}$ are isomorphisms. Indeed, one can define $\gamma_X(Z) := i^*((\mathrm{Tr}_Z)^{-1}(1))$ for $i : Z \hookrightarrow X \in \mathbf{SmProj}_k(Z,X)$ and $\mathrm{Tr}_X(\alpha) := (\gamma_X^i)^{-1}$ whenever $Z$ is smooth. If $Z$ is not smooth, then one can apply the same procedure to the smooth locus of $Z$.

*Remark* 5.1.2. Notice that axioms 4 and 5 are equivalent. Furthermore, they, respectively, state, by assuming the Künneth's Formula, that Tr and $\gamma$ are

[4]Notice that the degree is finite since all varieties are locally of finite type over $k$.

compatible with the monoidal structure. Explicitly, $\gamma_{X\times_k Y} \cong \gamma_X \otimes_F \gamma_Y$ and $\mathrm{Tr}_{X\times_k Y} \cong \mathrm{Tr}_X \otimes_F \mathrm{Tr}_Y$.

*Remark* 5.1.3. Notice that, by Poincaré duality, one can define pushfoward morphisms

$$f_* : \mathrm{H}^\bullet_W(X) \longrightarrow \mathrm{H}^{\bullet+2(d_Y-d_X)}_W(Y)(d_Y - d_X)$$

which satisfies the projection formula $f_*(f^*(\alpha)\cup\beta) = \alpha\cup f_*(\beta)$. Such formula expresses that $(\mathrm{H}^\bullet_W)^\vee$ is a left $\mathrm{H}^\bullet_W$-module.

Furthermore, the axiom 7 implies the naturality of $\gamma$ with respect to pushfowards of morphisms, clarifying its, a priori, unjustified name. Explicitly, $f_*\gamma_X = \gamma_Y f_*$.

*Remark* 5.1.4. Notice that the data of a commutative graded product is irrelevant once one assumes that the isomorphism in Künneth's Formula

$$\mathrm{H}^\bullet_W(X)\otimes \mathrm{H}^\bullet_W(Y) \xrightarrow{K_{X,Y}} \mathrm{H}^\bullet(X\times_k Y)$$

gives $\mathrm{H}^\bullet_W$ the structure of a strong symmetric monoidal functor (*i.e.* , it commutes with the monoidal structure and the braidings). That induces

$$\alpha\cup\beta := (\Delta_X)^*(K_{X,X}(\alpha\otimes\beta))$$

for every $\alpha,\beta \in \mathrm{H}^\bullet_W(X)$ and, therefore, one can produce such product from the other data. In this case, commutativity follows from the fact that the transposition $t_{X,Y} : X\times_k Y \longrightarrow Y\times_k X$ (*i.e.* , the braiding structure in the symmetric monoidal category $\mathbf{SmProj}_k$) in sent to the braiding in $\mathbf{Mod}^{\mathbf{Z}}_F$, which is defined by the Koszul rule.

Recall the definition of an adequate equivalence relation on cycles

**Definition 5.1.2.** Let $k$ be an algebraically closed field and $\sim$ an equivalence relation in $\mathcal{Z}^\bullet(-)$. $\sim$ is an **adequate equivalence relation** if the following holds for all $X,Y\in \mathrm{Ob}(\mathbf{SmProj}_k)$

1. If $\alpha,\beta,\alpha',\beta' \in \mathcal{Z}^\bullet(X)$ and $\alpha'\sim\alpha$ and $\beta'\sim\beta$, then $\alpha+\beta\sim\alpha'+\beta'$;

2. If $\alpha, \beta \in \mathcal{Z}^\bullet(X)$, then there exists $\alpha' \sim \alpha$ and $\beta' \sim \beta$ such that $\alpha'$ and $\beta'$ intersects properly (*i.e.* each summand intersects properly with each summand);

3. If $\alpha \in \mathcal{Z}^\bullet(X)$ and $\beta \in \mathcal{Z}^\bullet(X \times_k Y)$ are such that $\pi_X^*(\alpha)$ intersects $\beta$ properly, then $\alpha \sim 0$ implies that $(\pi_Y)_*(\pi_X^*(\alpha).\beta) \sim 0$.

Let $\sim_1$ and $\sim_2$ be equivalence relations. $\sim_1$ is finer than $\sim_2$ if $\alpha \sim_1 0$ implies that $\alpha \sim_2 0$ for every $\alpha \in \mathcal{Z}^\bullet(X)$ and every $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$.

*Example* 5.1.1. The rational equivalence relation given by $Z \sim_{rat} 0$ if there exists $H \in \mathcal{Z}^\bullet(X \times_k \mathbf{P}^1_k)$ such that

$$(\pi_X)_*((X \times_k \{0\} - X \times_k \{\infty\}).H) = Z$$

is adequate. Furthermore, it's the finest adequate equivalence relation.

*Example* 5.1.2. The algebraic equivalence relation given by $\alpha \sim_{alg} 0$ if there exists a smooth projective curve $C/k$ with $k$-points $x, x' : \mathrm{Spec}(k) \longrightarrow C$ and $H \in \mathcal{Z}^\bullet(X \times_k C)$ such that

$$(\pi_X)_*((X \times_k \{x\} - X \times_k \{x'\}).H) = \alpha$$

is adequate. Furthermore, evidently, it's coarser than $\sim_{\mathbf{Q}}$

*Example* 5.1.3. The $\tau$-equivalence relation given by $Z \sim_\tau 0$ if there exists $m \in \mathbf{Z}^\times$ such that

$$mZ \sim_{alg} 0$$

is adequate. Furthermore, evidently, it's coarser than $\sim_{alg}$

*Example* 5.1.4. For every Weil cohomology theory $\mathrm{H}^\bullet_W$ the homological equivalence relation given by $\alpha \sim_{hom} 0$ if

$$\gamma_X(\alpha) = 0$$

is adequate. Furthermore, $\sim_{hom}$ is coarser than $\sim_\tau$.

*Example* 5.1.5. The numerical equivalence relation given by $\alpha \sim_{num} 0$ if, for every $\beta \in \mathcal{Z}^{d_X - d_Z}(X)$ that intersects it $\alpha$ properly,

$$\deg(\alpha.\beta) = 0$$

is an adequate equivalence relation. Furthermore, it's the coarsest non-zero adequate equivalence relation.

Since rational equivalence relation is finer than the homological one, the following is true.

**Proposition 5.1.1.** *Let $k$ be an algebraically closed field and $F$ be a field of characteristic $0$ and $\mathrm{H}_W$ a Weil cohomology theory with coefficients in $F$. The cycle class map respect rational equivalences and, hence, induces a ring morphism*

$$\gamma_X : \mathrm{CH}^i(X) \longrightarrow \mathrm{H}^{2i}_W(X)(i).$$

**Proposition 5.1.2.** *Let $k$ be an algebraically closed field and $F$ be a field of characteristic $0$, $\mathrm{H}_W$ a Weil cohomology theory with coefficients in $F$ and $n \in \omega$. $\mathrm{H}^\bullet_W(\mathbf{P}^n_k)$ is concentrated in even degrees and*

$$\bigoplus_i H^{2i}_W(\mathbf{P}^n_k)(i) \cong F[H]/(H^{n+1})$$

*, where $H \in \mathrm{H}^2_W(\mathbf{P}^n_k)(1)$ is the cycle class of an hyperplane at infinity. Furthermore, $\mathrm{Tr}_{\mathbf{P}^n_k}(H^n) = 1$.*

*Proof.* (Sketch) Since $\mathbf{P}^n_k$ admits a stratification by affine spaces $\mathbf{A}^n_k \sqcup \mathbf{A}^{n-1}_k \sqcup \cdots \sqcup \mathrm{Spec}(k)$. The Chow group $\mathrm{CH}(\mathbf{P}^n_k)$ is generated by the classes $[\overline{\mathbf{A}^i_k}] = [\mathbf{P}^{i-1}_k]$ of the respective cells. That, forthwith, implies that

$$\mathrm{CH}(\mathbf{P}^n_k) \cong \mathbf{Z}[H]/[H^{n+1}]$$

, where $H$ is the class of the cell $[\overline{\mathbf{A}^{n-1}_k}] = [\mathbf{P}^{n-1}_k]$.

Now, one must only show that $\gamma([H])$ generates $\mathrm{H}^\bullet_W(\mathbf{P}^n_k)$. For $n = 1$, that follows by computing the self-intersection $[\Delta_{\mathbf{P}^1_k}].[\Delta_{\mathbf{P}^1_k}]$ and noticing that $\chi(X) = \deg(\Delta_X.\Delta_X)$. For arbitrary $n$, consider the morphism

$$\sigma : (\mathbf{P}^1_k)^n \longrightarrow \mathbf{P}^n_k$$

defined by $\sigma(x) = (\sigma_i(x))$ were $\sigma_i$ are the elementary symmetric polynomials. That factors through the isomorphism

$$(\mathbf{P}^1_k)^n/\Sigma_n \xrightarrow{\sim} \mathbf{P}^n_k$$

, defining, then, by pullback, a monomorphism

$$\mathrm{H}^\bullet_W(\mathbf{P}^n_k) \hookrightarrow (\mathrm{H}^\bullet_W(\mathbf{P}^1_k)^{\otimes n})^{\Sigma_n}$$

since $\sigma$ is dominant. Since $\mathrm{H}^\bullet_W(\mathbf{P}^1_k)^{\otimes n})^{\Sigma_n}$ is isomorphic to the collection of symmetric polynomials in $F[\gamma_{\mathbf{P}^1_k}([H_1]),\dots,\gamma_{\mathbf{P}^1_k}([H_n])]$ such that $H_i^2 = 1$, one can create any such polynomial using $F$-linear combinations of elementary symmetric polynomials. Now notice that $\sigma^{-1}(\{x_n = 0\}) = \coprod_i \{x_i = 0\}$ and, therefore, $\sigma^*(\gamma_{\mathbf{P}^n_k}([H])) = \sum_i H_i$ and

$$\sigma^*(\gamma_{\mathbf{P}^n_k}([H])^j) = \sigma_j(\gamma_{\mathbf{P}^1_k}([H_1]),\dots,\gamma_{\mathbf{P}^1_k}([H_n])),$$

proving that the above morphism is also surjective.

□

**Definition 5.1.3.** Let $k$ be an algebraically closed field and $F$ be a field of characteristic $0$ and $\mathrm{H}_W$ a Weil cohomology theory with coefficients in $F$. A **Chern class for** $\mathrm{H}_W$ consists of the following data for every $X, Y \in \mathrm{Ob}(\mathbf{SmProj}_k)$, $f : X \longrightarrow Y$, $\mathscr{L} \in \mathrm{Pic}(X)$ and $\mathscr{E} \in \mathbf{Vect}(X)$.

1. A morphism $c_1 : \mathrm{Pic}(X) \longrightarrow \mathrm{H}^2_W(X)(1)$ such that $c_1(\mathscr{L}) = \gamma_X(D_1) - \gamma_X(D_2)$ for $\mathscr{L} = \mathscr{O}_X(D_1 - D_2)$ and $f^*c_1(\mathscr{L}) = c_1(f^*\mathscr{L})$.
2. A morphism $c : K^0(X) \longrightarrow \bigoplus_i \mathrm{H}^{2i}_W(X)(i)$ such that $c(\mathscr{L}) = 1 + c_1(\mathscr{L})$ and $f^*c(\mathscr{E}) = c(f^*\mathscr{E})$.

**Theorem 5.1.1.** *Let $k$ be an algebraically closed field and $F$ be a field of characteristic $0$ and $\mathrm{H}_W$ a Weil cohomology theory with coefficients in $F$. There exists a unique Chern class for $\mathrm{H}_W$.*

*Proof.* (Sketch) The proof is identical to the one defining a Chern class for coherent sheaves with values in the Chow group. By taking the cycle morphism, one obtains the above Chern classes.

More explicitly, recall that every projective variety has an ample line bundle and, then, every quasi-coherent sheaf $\mathscr{F}$ is a quotient of a sum of line bundles. If $\mathscr{F}$ is coherent, the kernel of such epimorphism will be of finite type and, therefore, one can obtain a finite long exact sequence

$$0 \longrightarrow \bigoplus_i \mathscr{L}_{0,i} \longrightarrow \cdots \longrightarrow \bigoplus_j \mathscr{L}_{n,i} \longrightarrow \mathscr{F} \longrightarrow 0.$$

For a locally free sheaf of finite type $\mathscr{E}$, that implies that

$$[\mathscr{E}] = \sum_i (-1)^i \sum_j [\mathscr{L}_{i,j}] \in K_0(X).$$

Then, one defines

$$c(\mathscr{E}) = \prod_i (-1)^i \prod_j (1 + c_1(\mathscr{L}_{i,j}))$$

and

$$c_1(\mathscr{O}_X(D)) = \gamma_X(D)$$

for each Weil divisor $D \in \mathrm{Pic}(X)$. □

## § 5.2 The Category of Pure Motives

The category of pure motives should be universal with respect to all reasonable realisations, *i.e.* , Weil cohomology theories. More precisely, there should exist an inclusion $M : \mathbf{SmProj}_k \longrightarrow \mathbf{Mot}(k,F)$ such that for every Weil cohomology theory $\mathrm{H}_W : \mathbf{SmProj}_k^{\mathrm{op}} \longrightarrow \mathbf{Mod}_F^{\mathbf{Z}}$, there exists a lifting $r_W : \mathbf{Mot}(k,F) \longrightarrow \mathbf{Mod}_F^{\mathbf{Z}}$ such that the diagram

$$\begin{array}{ccc} \mathbf{Mot}(k,F) & \overset{r_W}{\dashrightarrow} & \mathbf{Mod}_F^{\mathbf{Z}} \\ {\scriptstyle M}\uparrow & \nearrow_{\mathrm{H}_W} & \\ \mathbf{SmProj}_k^{\mathrm{op}} & & \end{array}$$

is commutative.

To satisfy such requirement, one must understand what structures $\mathbf{Mot}(k,F)$ is supposed to have. Firstly, by the Künneth formula, it should be a symmetric monoidal category. Since every cohomology group is a commutative graded $F$-algebra, it also should be $F$-linear. As the tensor product of vector spaces commutes with finite (co)limits, the monoidal product in $\mathbf{Mot}(k,F)$ should also be exact. Furthermore, $\mathrm{End}(\mathrm{H}^\bullet_W(\mathrm{Spec}(k))) = \mathrm{End}(F) \cong F$.

**Definition 5.2.1.** Let $\mathcal{C}$ be a category and $F$ a field. $\mathcal{C}$ is **a $F$-tensor category** if it's symmetric monoidal with unity $1$ and monoidal product $\otimes$, $F$-linear (*i.e.* , enriched in $\mathbf{Mod}_F$), $\mathrm{End}(1) \cong F$, $\otimes$ is $F$-linear in each argument and exact (*i.e.* , preserves finite limits and finite colimits).

Therefore, $\mathbf{Mot}(k,F)$ must at least be an $F$-linear tensor category.

As $\mathbf{Mod}_F^{\mathbf{Z}}$ is abelian and the finite (co)limits in $\mathbf{SmProj}_k$ are preserved by any known $\mathrm{H}^\bullet_W$ [5], finite (co)limits must exist in $\mathbf{Mot}$(k, F) and $r_W$ should create them. Therefore $\mathbf{Mot}(k,F)$ must at least be abelian when restricted to the morphisms coming from $\mathbf{SmProj}_k$.

As implicitly mentioned above, $\mathbf{Mot}(k,F)$ should have at least the morphisms of $\mathbf{SmProj}_k^{\mathrm{op}}$. However, such morphisms are not enough. Instead, Poincaré duality, Künneth formula, the cycle class map and the initiality of the category of pure motives imply the diagram

$$\begin{array}{ccccc}
\mathrm{H}_W^{2d_Y+2r}(X\times_k Y)(d_Y+r) & \xrightarrow{\sim} & \bigoplus_i \mathrm{H}_W^{2d_Y-i}(Y)(d_Y)\otimes_F \mathrm{H}_W^{i+2r}(X)(r) & \xrightarrow{\sim} & \bigoplus_i \mathrm{H}^i_W(Y)^\vee \otimes_F \mathrm{H}_W^{i+2r}(X)(r) \\
\uparrow{\scriptstyle \gamma_{X\times_k Y}} & & & & \downarrow{\scriptstyle \sim} \\
\mathcal{Z}^{d_Y+r}(X\times_k Y) & & \longrightarrow & & \bigoplus_i \mathbf{Mod}_F(\mathrm{H}^i_W(Y),\mathrm{H}_W^{i+2r}(X)(r)) \\
& \searrow & & \nearrow & \\
& & \mathbf{Mot}(k,F)(M(Y),M(X)[2r](r)) & &
\end{array}$$

, which implies that every $\alpha \in \mathcal{Z}^{d_Y+r}(X\times_k Y)$ induces a pullback morphism

$$\alpha^* : \mathrm{H}^\bullet_W(Y) \longrightarrow \mathrm{H}_W^{\bullet+2r}(X)(r).$$

. Dually, through Poincaré duality, it also induces a pushforward morphism

[5] That follows from Mayer-Vietoris and Poincaré duality

$$\alpha_* : \mathrm{H}^\bullet_W(X) \longrightarrow \mathrm{H}^{\bullet+2(d_Y-2d_X)+2r}_W(Y)(d_Y - d_X + r).$$

Notice that $(\alpha^*)^\vee = \alpha_*$ and, dually, $(\alpha_*)^\vee = (\alpha)_*$, where $(-)^\vee$ is the Poincaré dual.

The following shows the that a Weil cohomology theory can be naturally extended to be functorial on correspondences.

**Lemma 5.2.1.** *Let $f : X \longrightarrow Y \in \mathbf{SmProj}_k(X,Y)$.*

$$(\Gamma_f)_* \cong f_*$$

*and*

$$(\Gamma_f)^* \cong f^*$$

*Proof.* It's enough to prove one of the isomorphisms. The other one will follow by duality. Notice that $\Gamma_f = (f \times_k 1_Y)^*(\Delta_Y)$. Then

$$\gamma_{X\times_k Y}(\Gamma_f) = \gamma_{X\times_k Y}((f \times_k 1_Y)^*(\Delta_Y)) \in \mathrm{H}^{2d_Y}_W(X \times_k Y)(d_Y)$$

, which, under Künneth's Formula, becomes

$$\begin{aligned}\gamma_{X\times_k Y}(\Gamma_f) &= (f^* \otimes 1_{H^\bullet_W(Y)})(\gamma_{Y\times_k Y}(\Delta_Y)) \\ &\in \bigoplus_i \mathrm{H}^{2d_X-i}_W(X)(d_X) \otimes_F \mathrm{H}^{i+2(d_Y-d_X)}_W(Y)(d_Y - d_X).\end{aligned}$$

Notice that $\gamma_{Y\times_k Y}(\Delta_Y) = \sum_j f_{2d_Y-j} \otimes e_j$ for some orthogonal (with respect to Poincaré duality) basis of $\mathrm{H}^i_W(Y)$ such that $e_j^* = \mathrm{Tr}(f_{2d_Y-j} \cup (-))$. Let $\xi \in \mathrm{H}^i_W(X)$. Then

$$(\Gamma_f)_*(\xi) = \sum_j \mathrm{Tr}(f^*(f_{2d_Y-j}) \cup \xi) e_j = \sum_j \mathrm{Tr}(f_{2d_Y-j} \cup f_*(\xi)) e_j = \sum_j e_j^*(f_*(\xi)) e_j = f_*(\xi).$$

□

Recall that given an adequate equivalence relation $\sim$ on cycles, $\mathcal{Z}^\bullet_\sim(-)$

becomes a graded ring and one can extend the partially defined pullback of cycles to the whole domain. Actually, even more, analogously to the case of a Weil cohomology, one can define pullbacks

$$\beta^* : \mathcal{Z}^\bullet(X) \longrightarrow \mathcal{Z}^{\bullet+r}(Y)$$

for every correspondence $\beta \in \mathcal{Z}_\sim^{d_X+r}(X \times_k Y)$ by setting $\beta^*(\alpha) := (\pi_Y)_*(\pi_X^*(\alpha).\beta)$. In particular, one can extend $f^*$ to

$$f^* : \mathcal{Z}^\bullet(Y) \longrightarrow \mathcal{Z}^\bullet(X)$$

by setting $f^*(\alpha) := (t_{X,Y}(\Gamma_f))^*$.

The following states that such extension is compatible with the cycle class morphism. In particular, the composition of cycles (which will be recalled later) and, in particular, morphisms is compatible with $(-)_*$ and, hence, also with $(-)^*$.

**Lemma 5.2.2.** *Let* $\alpha \in \mathcal{Z}^{d_Y+r}(X \times_k Y)$.

$$\alpha^*(\beta) = (\pi_X)_*(\pi_Y^*(\beta) \cup \gamma_{X\times_k Y}(\alpha))$$

*for every* $\beta \in \mathrm{H}_W^\bullet(Y)$.

*Proof.* By linearity, it's enough to prove for $\gamma_{X\times_k Y}(\alpha) = a \otimes b$ under the identification by the Künneth formula.

$$(\pi_X)_*(\pi_Y^*(\beta) \cup \gamma_{X\times_k Y}(\alpha)) = (\pi_X)_*(\pi_Y^*(\beta) \cup \pi_Y^*(a) \cup \pi_X^*(b))$$

$$= (\pi_X)_*(\pi_Y^*(\beta) \cup \pi_Y^*(a)) \cup b = \mathrm{Tr}(\beta \cup a)b = \alpha^*(\beta)$$

□

One can, by the above discussion, therefore, conclude that at least correspondences should induce morphisms between pure motives. That motivates the following

**Definition 5.2.2.** Let $k$ be an algebraically closed field, $F$ be a field of characteristic 0 and $\sim$ an adequate equivalence relation on cycles. The **cat-**

**egory of cohomological correspondences over $k$ with coefficients in $F$**, $\mathbf{Corr}_{\sim}(k,F)$, is defined by setting $\mathrm{Ob}(\mathbf{Corr}_{\sim}(k,F)) := \mathrm{Ob}(\mathbf{SmProj}_k)$ and declaring, for every $X, Y \in \mathrm{Ob}(\mathbf{SmProj}_k)$ irreducibles, $\mathbf{Corr}(k,F)(X,Y) := \bigoplus_{r\in\mathbf{Z}} \mathcal{Z}_{\sim}^{d_X+r}(X \times_k Y)$ and extending by $\mathbf{Z}$-linearity decomposing into the irreducible components together with the respective multiplicity. The composition is performed by composing spans, *i.e.* , considering the diagram

$$\begin{array}{ccccccc} & & & X \times_k Y \times_k Z & & & \\ & & \swarrow^{\pi_{XY}} & & \searrow^{\pi_{YZ}} & & \\ & X \times_k Y & & & & Y \times_k Z & \\ \swarrow^{\pi_X} & & \searrow^{\pi_Y} & & \swarrow^{\pi_Y} & & \searrow^{\pi_Z} \\ X & & & Y & & & Z \end{array}$$

, $\alpha \in \mathbf{Corr}(k,F)(X,Y)$ and $\beta \in \mathbf{Corr}(k,F)(Y,Z)$,

$$\beta \circ \alpha := \pi_{YZ}^*(\pi_{XY}^*(\alpha).\pi_{YZ}^*)$$

, which exists by the flatness of $\pi_{YZ}$.

*Remark* 5.2.1. The above definition engenders canonically a faithfull embedding $M : \mathbf{SmProj}_k^{\mathrm{op}} \longrightarrow \mathbf{Corr}(k,F)$ such that $M(f : X \longrightarrow Y) := \Gamma_f \in \mathbf{Corr}_{\sim}(k,F)(X,Y)$, where $\Gamma_f \hookrightarrow X \times_k Y$ denotes the graph of $f$. Dually, $M : \mathbf{SmProj}_k \longrightarrow \mathbf{Corr}_{\sim}(k,F)$ such that $M(f : X \longrightarrow Y) := (t_{X,Y})_*(\Gamma_f) \in \mathbf{Corr}_{\sim}(k,F)(X,Y)$.

*Remark* 5.2.2. Analogously, one can define the category of homological correspondences by setting $\mathbf{Corr}(k,F)(X,Y) := \bigoplus_{r\in\mathbf{Z}} \mathcal{Z}^{d_Y-r}(X \times_k Y))$, where $X$ and $Y$ are irreducible, and extending linearly as in the cohomological case. That would make the functor $M$ covariant. However, as the main concern comes from interpreting Weil cohomology theories instead of Borel-Moore homologies (as in the case of mixed motives), Grothendieck's cohomological notation will be used.

*Remark* 5.2.3. The category $\mathbf{Corr}_{\sim}(k,F)$ is a $F$-tensor category by setting $M(X) \otimes M(Y) := M(X \times_k Y)$ and $1 = M(\mathrm{Spec}(k))$ as the monoidal unit.

The category $\mathbf{Mod}_F$ is also idempotent complete or pseudo-abelian. Therefore $\mathbf{Mot}(k,F)$ must also be. That notion will be recalled in the following.

**Definition 5.2.3.** Let $\mathscr{C}$ be a category , $A \in \mathrm{Ob}(\mathscr{C})$ and $e \in \mathscr{C}(A,A)$ be an idempotent (*i.e.* , $e^2 = e$). $e$ **splits** or $e$ is a **splitting idempotent** if there exist $B \in \mathrm{Ob}(\mathscr{C})$ and a retract pair $(r : A \longrightarrow B, i : B \longrightarrow A)$ (*i.e.* , $ri = 1_B$) such that $e = ir$.

Furthermore, $\mathscr{C}$ is **idempotent complete (or Karoubi complete, or Cauchy complete)** if every idempotent morphism splits.

**Definition 5.2.4.** Let $\mathscr{C}$ be an **Ab**-enriched category. $\mathscr{C}$ is pseudo-abelian if it's pre-additive (*i.e.* , it has a zero object 0) and idempotent complete.

**Proposition 5.2.1.** *Let $\mathscr{C}$ be a pre-additive category. $\mathscr{C}$ is idempotent complete iff every idempotent has a kernel. In particular, every idempotent $e \in \mathscr{C}(A,A)$ induces a splitting $A \cong \mathrm{Ker}(e) \oplus \mathrm{Ker}(1-e)$*

**Theorem 5.2.1.** *Let $\mathscr{C}$ be a category. There exists an idempotent category $\mathscr{C}^{\#}$ together with a fully faithfull embedding $i_{\mathscr{C}} : \mathscr{C} \hookrightarrow \mathscr{C}^{\#}$ that satisfies the universal property: for every functor $j_{\mathscr{D}} : \mathscr{C} \longrightarrow \mathscr{D}$ such that $\mathscr{D}$ is idempotent complete, there exists a unique functor $j^{\#}_{\mathscr{D}} : \mathscr{C}^{\#} \longrightarrow \mathscr{D}$ (up to natural equivalence) such that the diagram*

$$\begin{array}{ccc} \mathscr{C} & \xhookrightarrow{i_{\mathscr{C}}} & \mathscr{C}^{\#}. \\ & \searrow^{j_{\mathscr{D}}} & \downarrow{\scriptstyle j^{\#}_{\mathscr{D}}} \\ & & \mathscr{D} \end{array}$$

*is commutative.*

*Proof.* Define $\mathrm{Ob}(\mathscr{C}^{\#})$ to consist in pairs $(A,e)$ such that $A \in \mathscr{C}$ and $e \in \mathscr{C}(A,A)$ is idempotent. The morphisms are defined by $\mathscr{C}^{\#}((A,e_A),(B,e_B)) := e_B \circ \mathscr{C}(A,B) \circ e_A$ or, equivalently, as morphisms $f : A \longrightarrow B$ such that $f = e_B f e_A$. The identity is defined as $1_{(A,e)} := e$.

The fully faithfull embedding $i_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathscr{C}^{\#}$ is defined by $i_{\mathscr{C}}(A) = (A, 1_A)$.

Let $e : A \longrightarrow A$ be idempotent. The pair of morphisms $(e : (A,1_A) \longrightarrow (A,e_A), e : (A,e) \longrightarrow (A,1_A))$ defines a retraction pair since $ee1_A = e$, $1_A ee = e$ and $e^2 = e = 1_{(A,e)}$.

Let $j_{\mathscr{D}} : \mathscr{C} \longrightarrow \mathscr{D}$ be fully faithfull and $\mathscr{D}$ be idempotent complete. Using the axiom of choice, define $j^{\#}_{\mathscr{D}}((A,e))$ as the underlying object of a splitting of $F(e) : F(A) \longrightarrow F(A)$ by choosing one. Now, define $F_{\mathscr{D}(f:(A,e_A)\longrightarrow(B,e_B))}$ by

$$\begin{array}{ccc} j_{\mathscr{D}}(A) & \xrightarrow{j_{\mathscr{D}}(f)} & j_{\mathscr{D}}(B) \\ \uparrow & & \uparrow \\ j^{\#}_{\mathscr{D}}((A,e_A)) & \xrightarrow{j^{\#}_{\mathscr{D}}(f)} & j^{\#}_{\mathscr{D}}((B,e_B)) \end{array}$$

As every splitting is unique up to isomorphism in $\mathscr{D}$, $j^{\#}_{\mathscr{D}}$ is unique up to up equivalence. □

**Proposition 5.2.2.** *Let $\mathscr{C}$ be a category and $F$ a field. If $\mathscr{C}$ is a symmetric monoidal category (resp.., $F$-tensor category), then $\mathscr{C}^{\#}$ is a symmetric monoidal category (resp.., $F$-tensor category) by setting $(A,e_A) \otimes (B,e_B) := (A \otimes B, e_A \otimes e_B)$ for every $A,B \in \mathrm{Ob}(\mathscr{C})$ and $e_A, e_B$ idempotents.*

In summary, $\mathbf{Mot}(k,F)$ should at least be an abelian $F$-tensor idempotent complete category such that $\mathbf{Mot}(k,F)(M(X),M(Y))$ contains at least the free $F$-module generated by the correspondences between $X$ and $Y$ for every $X,Y \in \mathrm{Ob}(\mathbf{SmProj}_k)$. That motivates the following definition.

**Definition 5.2.5.** Let $k$ be an algebraically closed field, $F$ be a field of characteristic 0 and $\sim$ be a relation on cycles. The category of $\sim$ effective pure motives over $k$ with coefficients in $F$, $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k,F)$, is defined as $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k,F) := \mathbf{Corr}_{\sim}(k,F)^{\#}$

The above definition is not enough as every cohomology theory comes with a Tate twist and, in particular, a dual induced by Poincaré duality. It's, then, necessary to impose the existence of a Tate twist in $\mathbf{Mot}(k,F)$. The canonical choice which is compatible with all the the well known Weil cohomology theories would be choosing an object that under the universal property maps to $\check{\mathrm{H}}^2_W(\mathbf{P}^1_k)$. In all the known cohomologies $\mathrm{H}^2_W(\mathbf{P}^1_k)$ is generated by the compactly supported cohomology class of $\mathbf{A}^1_k \hookrightarrow \mathbf{P}^1_k$, which, in the particular case of Betti cohomology, is exactly the Poincaré dual of the 2-cell $\mathbf{A}^1_{\mathbf{C}}(\mathbf{C}) \cong \mathbf{P}^1_{\mathbf{C}}(\mathbf{C}) \setminus \{\infty\} \hookrightarrow \mathbf{P}^1_{\mathbf{C}}(\mathbf{C})$, where $\infty$ is any rational point. As in the known Weil cohomologies (actually, in all of them) $\mathrm{H}^{\bullet}_W(\mathbf{P}^1_k) \cong \mathrm{H}^1_W(\mathbf{P}^1_k) \oplus \mathrm{H}^2_W(\mathbf{P}^1_k) \cong F \oplus F(-1)$. That motivates the following definition.

**Definition 5.2.6.** Let $k$ be an algebraically closed field,$F$ be a field of characteristic 0, $\sim$ be a relation on cycles, $\infty : \mathrm{Spec}(k) \longrightarrow \mathbf{P}^1_k$ a rational point,

$f : \mathbf{P}^1_k \longrightarrow \mathrm{Spec}(k)$ is the structural morphism and $e := \infty \circ f : \mathbf{P}^1_k \longrightarrow \mathbf{P}^1_k$. The **Lefschetz motive** $\mathbb{L}$ is the object $\mathbb{L} := \mathrm{Ker}(e^*) = (M(\mathbf{P}^1_k), 1_{M(\mathbf{P}^1_k)} - e^*) = (M(\mathbf{P}^1_k), \mathbf{P}^1_k \times_k \{\infty\})$. The suggestive notation $F(-1) := \mathbb{L}$ will also be used

*Remark* 5.2.4. The name Lefschetz motive probably comes from the Lefschetz operator, which is exactly the Poincaré dual of the hyperplane at infinity $H$ in $\mathbf{P}^1_k$. After base change and analytification, that Poincaré dual becomes exactly the Fubini-Study metric of the Kåhler manifold $P^1_{\mathbf{C}}(\mathbf{C})$. More generally, $\mathrm{H}^{2d_X}_W(X) \cong F(-d_X)$ should be viewed as the free $F$-module generated by the Poincaré dual of the hyperplane at infinity $X \cap H$ for an embedding $X \hookrightarrow \mathbf{P}^n_k$ and $\mathrm{Tr}_X(-d_X)$ as the integration of that Poincaré dual, which is a normalised volume form.

*Remark* 5.2.5. Notice that $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k,F)(\mathbb{L}^{\otimes d}, M(X)) \cong \mathcal{Z}^d_{\sim}(X)$. In particular, $\mathbf{Mot}^{\mathrm{eff}}_{\sim}(k,F)(M(X) \otimes_F \mathbb{L}^{\otimes m}, M(Y) \otimes_F \mathbb{L}^{\otimes n}) \cong \mathcal{Z}^{d_X - n+m}_{\sim}(X \times_k Y)$.

In order to infuse Poincaré duality in $\mathbf{Mot}(k,F)$, one must invert $\mathbb{L}$ in order to obtain an object that maps to the Tate twist $F(1)$.

**Definition 5.2.7.** Let $k$ be an algebraically closed field, $F$ be a field of characteristic 0 and $\sim$ be a relation on cycles. The category of $\sim$ pure motives over $k$ with coefficients in $F$ is the localisation $\mathbf{Mot}_{\sim}(k,F) := \mathbf{Mot}^{\mathrm{eff}}_{\sim}(k,F)[\mathbb{L}^{-1}]$. Explicitly $M(i) \in \mathrm{Ob}(\mathbf{Mot}_{\sim}(k,F))$ if $M \in \mathbf{Mot}^{\mathrm{eff}}_{\sim}(k,F)$ and $i \in \mathbf{Z}$, and $\mathbf{Mot}_{\sim}(k,F)(M(i), N(j)) := \mathrm{colim}_n \mathbf{Mot}^{\mathrm{eff}}_{\sim}(k,F)(M \otimes_F \mathbb{L}^{n-i}, N \otimes_F \mathbb{L}^{n-j})$. The notation $M := M(0) \in \mathrm{Ob}(\mathbf{Mot}_{\sim}(k,F))$ will be used.

The **Tate motive** $F(1)$ is defined as the motive $\mathbb{L}^{-1}$. The notation $F(n) := F(1)^{\otimes n}$ will be used.

*Remark* 5.2.6. Notice that the notation $M(i)$ is consistent with the Tate twist $F(1)$ in the sense that $M(i) \cong M(0) \otimes F(i)$.

*Remark* 5.2.7. Notice that $\mathbf{Mot}_{\sim}(k,F)(M(X)(m), M(Y)(n)) \cong \mathcal{Z}^{d_X+n-m}(X \times_k Y)$. In particular, $\mathbf{Mot}_{\sim}(k,F)((M(X),p)(m),(M(Y),q)(n)) \cong q\mathcal{Z}^{d_X+n-m}(X \times_k Y)p$

*Remark* 5.2.8. In $\ell$-adic cohomology and $p$-adic cohomology, the Tate twist is exactly the $\ell$-adic cyclotomic character (*resp.*, $p$-adic étale) $\pi_1^{\text{ét}}(\mathbf{G}_{mk}) \cong \mu_{\ell^\infty}$. In Betti cohomology, it's the integral Hodge structure $\frac{1}{2\pi i}\mathbf{Z}$ that comes from the integration $\mathrm{H}^2_{\mathrm{Betti}}(\mathbf{P}^1) \cong \mathrm{H}^1_{\mathrm{Betti}}(\mathbf{G}_m, \mathbf{Z}) \cong \mathrm{H}^1_{\mathrm{sing}}(\mathbf{S}^1, \mathbf{Z}) \longrightarrow 2\pi i\mathbf{Z}$. In de Rham cohomology, it's the trivial module $k$.

By Poincaré duality, $\mathbf{Mot}_\sim(k, F)$ should also have duals and, therefore, should be rigid. Recall the definition of a rigid (or autonomous) monoidal category.

**Definition 5.2.8.** Let $(\mathscr{C}, \otimes, 1)$ be a monoidal category and $A \in \mathrm{Ob}(\mathscr{C})$. A **left dual of** $A$ is a triple $(\eta_A : 1 \to A^\vee \otimes A, \varepsilon_A : A \otimes A^\vee \to 1, A^\vee) \in \mathscr{C}(1, A^\vee \otimes A) \times \mathscr{C}(A \otimes A^\vee, 1) \times \mathrm{Ob}(\mathscr{C})$ such tha object $A^\vee \in \mathrm{Ob}(\mathscr{C})$ and

$$\begin{array}{ccc} A & \xrightarrow{1_A \otimes \eta_A} & A \otimes A^\vee \otimes A \\ & \searrow^{=} & \downarrow{\scriptstyle \varepsilon_A \otimes 1_A} \\ & & A \end{array}$$

. Dually, a **right dual of** $A$ is a triple $(\eta_A : 1 \to A \otimes A^\vee, \varepsilon_A : A^\vee \otimes A \to 1, A^\vee) \in \mathscr{C}(1, A^\vee \otimes A) \times \mathscr{C}(A \otimes A^\vee, 1) \times \mathrm{Ob}(\mathscr{C})$ that is a let dual of $\mathscr{A}^\vee$.

$\mathscr{C}$ is a **left rigid (or autonomous) monoidal category** (*resp.*, **right rigid (or autonomous) monoidal category**) if every every object $A \in \mathrm{Ob}(\mathscr{C})$ has a (strong) left dual (*resp.*, right dual).

$\mathscr{C}$ is a **rigid (or autonomous) monoidal category** if it's left rigid and right rigid.

*Remark* 5.2.9. Notice that, a priori, in a rigid monoidal category $\mathscr{C}$, $A \in \mathrm{Ob}(\mathscr{C})$ might have non isomorphic left and right duals. However, if $\mathscr{C}$ is braided, such duals will obviously be isomorphic. In particular, in a symmetric monoidal category (*e.g.* , the category of motives $\mathbf{Mot}_\sim(k, F)$), left duals and right duals are isomorphic.

*Remark* 5.2.10. Notice that every rigid symmetric monoidal category $\mathscr{C}$ must be closed monoidal (*i.e.* , $(-) \otimes A$ has a right adjoint $[A, (-)]$, the internal hom) by setting

$$[A, B] := A^\vee \otimes B$$

for every $A, B \in \mathrm{Ob}(\mathscr{C})$. That, forthwith, implies that $(-) \otimes A$ commutes with small colimits

In particular, $\mathbf{Mot}_{\sim}(k,F)$ must be closed symmetric and, therefore, the tensor product $\otimes$ must be more than exact (as required in the definition of a tensor $F$-linear category). It must be a bifunctor commuting with all small colimits in each argument and not only finite colimits.

*Example* 5.2.1. Let $\mathcal{C} := \mathbf{Mod}_F^{\mathbf{Z}}$, *i.e.* , the category of $\mathbf{Z}$-graded $F$-vector spaces. It's a rigid symmetric monoidal category by setting $V^\vee := \mathbf{Mod}_F^{\mathbf{Z}}(V,F) := \bigoplus_{i\in\mathbf{Z}}\bigoplus_{k\in\mathbf{Z}}\mathbf{Mod}_F(V_k,(F)_{k+i})$, *i.e.* , the grading of $V^\vee$ is such that $V_i^\vee = V_{-i}$ since $F$ is concentrated in degree 0.

Given the validity of Poincaré duality in $\mathbf{Mot}_{\sim}(k,F)$, one obtains a structure of idempotent complete rigid $F$-tensor category on $\mathbf{Mot}_{\sim}(k,F)$ by setting

$$(M(X),e)(m)^\vee := (M(X),t_{X,X}(e))(d_X - m)$$

with the unit $\eta$ and counit $\varepsilon$ given by $\eta = [\Delta_X] = \varepsilon$.

By all the previous discussion and the fact that $\mathbf{Mod}_F^{\mathbf{Z}}$ is idempotent complete, one can concludes the following.

**Theorem 5.2.2** (Universal property of pure motives). *Let $k$ be an algebraically closed field, $F$ be a field of characteristic 0 and $\sim$ a equivalence relation finer or equal to $\sim_{hom}$ . Let also $H_W : \mathbf{SmProj}_k^{\mathrm{op}} \longrightarrow \mathbf{Mod}_F^{\mathbf{Z}}$ be a Weil cohomology theory with coefficients in $F$. There exists a unique $F$-linear tensor lifting $r_W : \mathbf{Mot}_{\sim}(k,F) \longrightarrow \mathbf{Mod}_F^{\mathbf{Z}}$ such that the diagram*

$$\begin{array}{ccc} \mathbf{Mot}_{\sim}(k,F) & \overset{r_W}{\dashrightarrow} & \mathbf{Mod}_F^{\mathbf{Z}} \\ {\scriptstyle M}\big\uparrow & \nearrow_{\mathrm{H}_W} & \\ \mathbf{SmProj}_k^{\mathrm{op}} & & \end{array}$$

*is commutative.*

*Furthermore, when $\sim = \sim_{rat}$,*

One feature contained in every rigid monoidal category, is the presence of a trace morphism.

**Definition 5.2.9.** Let $(\mathcal{C},\otimes,1)$ be a monoidal category, $A \in \mathrm{Ob}(\mathcal{C})$ and $f : A \longrightarrow A \in \mathcal{C}(A,A)$. The **trace of** $f$ is defined as the morphism $\mathrm{tr}(f) \in \mathcal{C}(1,1)$

$$1 \xrightarrow{\eta_A} A^\vee \otimes A \xrightarrow{1_{A^\vee}\otimes f} A^\vee \otimes A \xrightarrow[\sim]{B_{A^\vee,A}} A \otimes A^\vee \xrightarrow{\varepsilon_A} 1$$

, where $B_{A,B} : A \otimes B \overset{\sim}{\longrightarrow} B \otimes A$ denotes the braiding.

*Remark* 5.2.11. Notice that in a $F$-linear tensor category $\mathscr{C}$, $\mathrm{tr}(f) \in \mathrm{End}(1) \cong F$.

*Example* 5.2.2. Let $\mathscr{C} := \mathbf{Mod}_F$. The trace of $f \in \mathbf{Mod}_F(V,V)$ is the usual trace of linear morphisms.

*Example* 5.2.3. Let $\mathscr{C} := \mathbf{Mod}_F^{\mathbf{Z}}$. The trace of

$$f \in \mathbf{Mod}_F^{\mathbf{Z}}(V,V) := \bigoplus_{i\in\mathbf{Z}}(\bigoplus_{k\in\mathbf{Z}} \mathbf{Mod}_F(V_k, V_{k+i}))$$

is exactly

$$\mathrm{tr}(f) = \sum_{i\in\mathbf{Z}}(-1)^i\,\mathrm{tr}(f^i)$$

, where $f^i : V_i \longrightarrow V_i$ is the $i$-th component of the 0-th degree part of $f$, $f_0 \in \bigoplus_{k\in\mathbf{Z}} \mathbf{Mod}_F(V_k, V_k)$. Notice that that occurs specifically because the braiding

$$((B_{V,W})_i)_k : (V\otimes W)_k \cong \bigoplus_{l\in\mathbf{Z}}(V_l \otimes_F W_{k-l}) \longrightarrow \bigoplus_{l\in\mathbf{Z}}(W_l \otimes_F V_{k+i-l}) \cong (W\otimes V)_{k+i}$$

of $\mathbf{Mod}_F^{\mathbf{Z}}$ is defined by the Koszul rule $B_{V,W}(v_i \otimes w_j) = (-1)^{i+j} w_j \otimes v_i$ for $v_i \in V_i$ and $w_j \in W_j$.

**Lemma 5.2.3.** *Let $\mathscr{C}$ and $\mathscr{D}$ be rigid symmetric monoidal categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a symmetric monoidal functor (i.e. , $F$ preserves the monoidal structure $\otimes$ and the braidings $B_{A,B}$). $F$ commutes with* tr*, i.e. ,*

$$F(\mathrm{tr}_{\mathscr{C}}(f)) = \mathrm{tr}_{\mathscr{D}}(F(f)).$$

*Proof.* Since $F$ commutes with $\otimes$ and the braidings $B_{A,B}$, it also commutes with duals. Therefore it must commute with traces. □

The existence of a rigid structure on $\mathbf{Mot}_\sim(k,F)$ automatically gives a Lefschetz's Trace Formula.

**Theorem 5.2.3** (Lefschetz's Trace Formula). *Let $k$ be an algebraically closed field and $F$ be a field of characteristic $0$. Let also $H^\bullet_W$ be a Weil cohomology theory with coefficients in $F$, $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$ of dimension $d$, $\alpha \in \mathcal{Z}^{d_X+r}_\sim(X \times_k Y)$ and $\beta \in \mathcal{Z}^{d_Y-r}_\sim(Y \times_k X)$. The identity*

$$\deg(\alpha.(t_{X,Y})_*(\beta)) = \sum_{i=0}^{2d}(-1)^i \operatorname{tr}((\beta \circ \alpha)^i | \operatorname{H}^i_W(Y))$$

*is true. In particular, for $f : X \longrightarrow Y = X$,*

$$\deg(\Gamma_f.\Delta_X) = \sum_{i=0}^{2d}(-1)^i \operatorname{tr}((f^*)^i | \operatorname{H}^i_W(X))$$

*and*

$$\deg(\Delta_X.(t_{X,X})_*(\Gamma_f)) = \sum_{i=0}^{2d}(-1)^i \operatorname{tr}((f_*)^i | \operatorname{H}^i_W(X)).$$

*Proof.* By the previous lemma and remark, it's enough to prove that

$$\operatorname{tr}_{\mathbf{Mot}_\sim(k,F)}(\beta \circ (\alpha)) = \deg(\alpha.(t_{X,Y})_*(\beta)$$

Since

$$\deg(\alpha.(t_{X,Y})_*(\beta)) = (\beta \circ \alpha).\delta_Y$$

always holds ([Ful98, Ex.16.1.3]). It's enough to prove that

$$\operatorname{tr}_{\mathbf{Mot}_\sim(k,F)}(\alpha) = \deg(\alpha.\Delta_X)$$

for $\alpha \in \mathcal{Z}^{d_X+r}_\sim(X \times_k X)$.

Notice that the composition

$$1 \xrightarrow{\eta} M(X)(d_X) \otimes M(X) \xrightarrow{1\otimes\alpha\circ\beta} M(X)(d_X) \otimes M(X) \xrightarrow[\sim]{B_{M(X)(d_X),M(X)}} M(X) \otimes M(X)(d_X) \xrightarrow{\varepsilon} 1$$

is given by $\deg((t_{X,X})_*(\alpha))$ since the unit $\eta$ is given by $\Delta_X$, tensoring is intersection, the braiding is given by the transposition $(t_{X,X})_*$ and the counit, or evaluation, $\epsilon$, is $\deg(-)$.

□

The category of motives $\mathbf{Mot}(k,F)$ should comes with a realisation functor $r_W := H^\bullet_W : \mathbf{Mot}(k,F) \longrightarrow \mathbf{Mod}^{\mathbf{Z}}_F$, which shares several similarities with the fiber functor $F_{\overline{x}}$. In both cases, usually some group acts by automorphisms. In the case of $F_{\overline{x}}$, it's the étale fundamental group $\pi^1_{\text{ét}}(X,\overline{x})$, which acts on finite étale coverings. When $X$ is connected, every finite étale covering can, then, be recovered from a $\pi^1_{\text{ét}}(X,\overline{x})$-set. In this case, $F_{\overline{x}}$ becomes the forgetful functor $\mathbf{Set}_{\pi^1_{\text{ét}}(X,\overline{x})} \longrightarrow \mathbf{Set}$ forgetting the respectives $\pi^1_{\text{ét}}(X,\overline{x})$ actions. More generally, for base points $x,y \in X$, $\pi^1_{\text{ét}}(X,\overline{x},\overline{y})$ acts on $X \longrightarrow Y$ by comparing the fiber over $x$ to the fiber over $y$. Still, more generally, one can consider the automorphisms of all fibers over all points of $X$ using the étale fundamental groupoid $\Pi^1_{\text{ét}}(X)$.

Fix a coefficient field $F$. Analogously, in the case of the known Weil cohomologies with coefficients in $F$, the realisation functors have automorphisms, which, as in the case of $F_{\overline{x}}$, determines completely the structure of the cohomology (*e.g.* , the structure of a $G_k$-module determines completely the $\ell$-adic cohomology). More generally, comparisons between realisations act as $\pi^1_{\text{ét}}(X,\overline{x},\overline{y})$. Still, more generally, all the comparisons between known Weil cohomologies assemble to a groupoid $\mathcal{G}_F$. That groupoid has points corresponding to $r_W$'s and morphisms corresponding to each comparison. Now, one can vary the coefficient field $F$ and, instead, use Weil cohomologies with coefficient field $F'$, creating another groupoid $\mathcal{G}_{F'}$. The morphism

$$r_W \mapsto r_W \otimes_F F'$$

defines a new Weil cohomology over $F'$. Therefore, one can assemble everything to a presheaf of groupoids

$$\mathcal{G} : {}_{F\backslash}\mathbf{Field} \longrightarrow \mathbf{Grpd}$$

such that $\mathcal{G}(F) := \mathcal{G}_F$. All the know Weil cohomologies become equivalent after base changing by a large enough period ring, which usually can be chosen to be a field by taking the ring of fractions. It's, therefore, reasonable to expect that $\mathcal{G}(F') \cong BG$ for $F'$ sufficiently large (*i.e.* , large enough to make all the Weil cohomologies with coefficients in $F'$ isomorphic) and $G$ some group.

More generally, one can define Weil cohomologies with coefficients in a quasicoherent sheaf over $S \in \mathbf{Sch}_F$. A suitable definition would imply the existence of a new restriction morphism

$$r_W \mapsto r_W \otimes_F \mathcal{O}_S.$$

That would allow realisation functors with values in $\mathbf{QCoh}(S)$ instead of $\mathbf{Mod}_F$ and extend the domain of $\mathcal{G}$ to a new presheaf of groupoids

$$\mathcal{G} : \mathbf{Sch}_F \longrightarrow \mathbf{Grpd}$$

, which, by the same argument as above, should be a gerbe.

All these analogies and constructions motivate the following definition.

**Definition 5.2.10.** Let $\mathcal{C}$ be a category and $F$ a field. $\mathcal{C}$ is a **Tannakian category** f it's a rigid $F$-linear tensor category such that there exists an $F$-scheme $S$ and a $F$-linear exact tensor functor $\omega : \mathcal{C} \longrightarrow \mathbf{QCoh}(S)$. In this case, $r$ is a **fiber functor**.

If, furthermore, $S$ can be chosen to be $\mathrm{Spec}(F)$, $\mathcal{C}$ is a **neutral Tannakian category**.

It's remarkable that that definition alone requires the respective presheaf of groupoids to be a gerbe as proved by Deligne.

**Theorem 5.2.4** (Tannaka Duality/ Reconstruction Theorem, [Del90, Thm. 1.12] ). *Let $\mathcal{C}$ be a category and $F$ a field. If $\mathcal{C}$ is a $F$-linear Tannakian category, the functor*

$$\mathrm{FIB}(\mathcal{C}) : \mathbf{Sch}_F \longrightarrow \mathbf{Grpd}$$

*defined by*

$$\mathrm{FIB}(\mathscr{C})(S) = \{\omega : \mathscr{C} \longrightarrow \mathbf{QCoh}(S) | \ \omega \text{ is a fiber functor}\}$$

*is an affine gerbe over* $\mathrm{Spec}(F)$ *in the fpqc topology. Furthermore,* $\mathscr{C} \cong \mathrm{Rep}(\mathrm{FIB}(\mathscr{C}))$ *and, under that equivalence, a fiber functor*

$$\omega : \mathscr{C} \longrightarrow \mathbf{QCoh}(S)$$

*is the forgetful functor*

$$\mathrm{Rep}(\mathrm{FIB}(\mathscr{C})) \longrightarrow \mathbf{QCoh}(S).$$

*If* $\mathscr{C}$ *is also neutral with fiber functor*

$$\omega : \mathscr{C} \longrightarrow \mathbf{Mod}_F$$

, $\mathrm{FIB}(\mathscr{C}) \cong BG$ *such that* $G$ *is the affine flat* $F$*-group scheme defined by* $G(B) = \mathrm{Aut}^{\otimes}(\omega \otimes B)$ *for every* $F$*-algebra* $B$*. Furthermore,* $\mathscr{C} \cong \mathrm{Rep}(G)$ *and, under that equivalence,* $\omega$ *is the forgetful functor* $\mathrm{Rep}(G) \longrightarrow \mathbf{Mod}_F$.

It turns out that the resulting category $\mathbf{Mot}_{\sim}(k,F)$, despite being $F$-linear rigid tensor idempotent complete category, it may fail to be abelian, which is a requirement in the definition of a Tannakian category.

**Theorem 5.2.5** ([Sch94, Cor. 3.5]). *Let* $k$ *be a field not contained in any* $\overline{\mathbf{F}}_p$ *for every prime* $p$*. The category* $\mathbf{Mot}_{rat}(k,\mathbf{Q})$ *is not abelian.*

The failure of $\mathbf{Mot}_{\sim}(k,F)$ being, in general, abelian is certainly not the only issue as the following theorem elicits.

**Theorem 5.2.6** (Deligne, [Del90, Thm. 7.1]). *Let* $F$ *be a field of characteristic* $0$ *and* $\mathscr{C}$ *be an abelian* $F$*-linear rigid tensor category. The following are equivalent*

1. $\mathscr{C}$ *is Tannakian;*

2. *For every* $A \in \mathrm{Ob}(\mathscr{C})$, $\dim(A) := \mathrm{tr}(1_A)$ *is a non-negative integer.*

3. *For every* $A \in \mathrm{Ob}(\mathscr{C})$, $\bigwedge^n(A) \cong 0$ *for some* $n \in \omega$*;*

Indeed,

$$r_W(\dim(M(X))) = \dim(r_W(M(X))) = \dim(\mathrm{H}_W(X)) = \sum_i (-1)^i \dim_F(\mathrm{H}^i_W(X))$$

and, then, there's no hope that $\mathbf{Mot}_\sim(k,F)$ will ever be Tannakian as the rank will be a sum containing non-positive terms. For instance, in the case of a genus $g > 1$ smooth projective curve $C$ over $k$, $r_W(\mathrm{rank}(M(C))) = 2 - 2g < 0$.

If one could split $M(X)$ into $\bigoplus_i M^i(X)$ such that $r_W(M^i(X)) = \mathrm{H}^i_W(X)$, the canonical solution would be to modify the braiding in $\mathbf{Mot}_\sim(k,F)$. If

$$(\phi_{M,N})^{i,j} : M_i \otimes_F N_j \longrightarrow N_j \otimes_F M_i$$

denotes each component of the braiding in $\mathbf{Mot}_{\sim(k,F)}$ under such splitting, then an immediate solution would be new braiding given by $(\widehat{\phi_{M,N}})^{i,j} = (-1)^{i+j}\phi_{M,N}$. Such definition, if possible would give

$$r_W(\dim(M(X))) = \sum_i \dim_F(\mathrm{H}^i_W(X))$$

, which is certainly positive.

When $k$ is a finite field, such choice is possible by the work of Deligne on the Weil's Conjectures. Given $X \in \mathbf{SmProj}_k$, with polynomials

$$P_i(X,t) = \det(1 - \mathrm{Fr}_X\, t|\, \mathrm{H}^i_{\text{ét}}(X, \mathbf{Q}_\ell))$$

composing the local $\zeta$ function, one can define projectors

$$p_i := \gamma_{P_i(X,\mathrm{Fr}_X)} \in \mathcal{Z}^{d_X}(X \times_k X)$$

by noticing that the determinant makes sense in any symmetric monoidal category and interpreting $\mathrm{Fr}_X$ as a correspondence.

Such projectors satisfy $\sum_{i=0}^{2d_X} p_i = 1$ and, hence, splits $M(X)$ into pieces $M^i(X) := \mathrm{Im}(1 - p_i)$.

Whenever such splitting is possible, it will be assumed that the category $\mathbf{Mot}_\sim(k,F)$ has the respective modified braiding, which allows it to possibly be Tannakian. Notice that such splitting implies that the category $\mathbf{Mot}_\sim(k,F)$

is semi-simple.

The failure of being abelian can be corrected once one takes $\sim = \sim_{num}$. In this case, even more can be reached, $\mathbf{Mot}_{num}(k,F)$ becomes semi-simple.

**Theorem 5.2.7** (Jannsen's Semisimplicity Theorem, [Jan92, Thm. 1]). *Let $k$ be a field and $F$ a field of characteristic 0. The following are equivalent*

1. *$\mathbf{Mot}_{\sim}(k,F)$ is a semi-simple abelian category;*

2. *$\mathcal{Z}^{d_X}_{\sim}(X \times_k X)$ is a finite dimensional, semi-simple $F$ algebra for every $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$;*

3. *$\sim = \sim_{num}$.*

Still, by considering $\sim_{num}$, there's no obvious universal property for $\mathbf{Mot}_{\sim}(k,F)$ as $\sim_{hom}$ is finer than $\sim_{num}$ (*i.e.* , $\mathcal{Z}^{\bullet}_{hom}(-) \longrightarrow \mathcal{Z}^{\bullet}_{num}(-)$ goes in wrong direction). Also semi-simplicity does not imply the existence of a splitting of $M(X)$ into $M^i(X)$'s as above. Indeed, one can obtain a splitting $M(X) \cong \bigoplus_i M_i$ such that $\mathbf{Mot}_{\sim}(k,F)(M_i,M_j) \cong \delta_{ij}F$. However such splitting will not be finer enough in order to guarantee that $r_W(M_i) \cong \mathrm{H}^i_W(X)$. Of course, once such splitting into cohomological degrees becames available, there's no further obstruction to the property of being Tannakian.

**Corollary 5.2.1.** *Let $k$ be an algebraically closed field, $F$ be a field of characteristic 0. If $\mathbf{Mot}_{num}(k,F)$ admits a splitting into cohomological degrees for some Weil cohomology $\mathrm{H}^{\bullet}_W$, then $\mathbf{Mot}_{num}(k,F)$ is a semi-simple Tannakian category.*

## § 5.3 Grothendieck's Standard Conjectures

The existence of a genuine category of pure motives depends intrinsically on difficult conjectures proposed by Grothendieck and Bombieri around 1965 and only officially proposed in 1968 by Grothendieck in the International Colloquium in Bombay (which was published in 1969 in [Gro69]).

Such conjectures were originally motivated to be solved in order to achieve a general and elegant proof of the Weil's Conjectures and, even more, a genuine category of pure motives. In the period of Grothendieck's proposal, what remained to be proved was Riemann's Hypothesis over finite fields and the integrality of coefficients in the Rationality Conjecture.

Here, for the convenience of the reader, such conjectures will be listed.

Recall that any $\alpha \in \mathcal{Z}^{d_X+r}(X \times_k Y)$, splits

$$\alpha^* : \mathrm{H}^\bullet_W(X) \longrightarrow \mathrm{H}^{\bullet+2r}_W(Y)(r)$$

into $\alpha^* = \sum_i (\alpha^*)^i \in \mathrm{H}^{2d_X-i}(X)(d_X) \otimes_F \mathrm{H}^{i+2r}(Y)(r)$.

Let $\alpha = [\Delta_X]$, $\pi_i := (\alpha^*)^i$. Then $[\Delta_X] = \sum_i \pi_i$ and $\pi_i : \mathrm{H}^\bullet_W(X) \to \mathrm{H}^i_W(X) \hookrightarrow \mathrm{H}^\bullet_W(X)$ is the $i$-th projection.

**Conjecture 5.3.1** ($C(X)$ or of Weak Lefschetz Type). *Let $k$ be an algebraically closed field, $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$, $F$ be a field of characteristic $0$ and Weil cohomology theory $\mathrm{H}_W$ with coefficients in $F$. The projections $\pi_i$ are algebraic (i.e. , $\pi_i = \Pi_i^*$ for some $\Pi_i \in cyc^{d_X}(X \times_k X)$).*

When $C(X)$ holds for every $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$, then one can obtain a splitting into cohomological degrees in $\mathbf{Mot}_\sim(k,F)$ for every adequate equivalence relation $\sim$. Explicitly, $(X,p)(m)_i = (X,\pi_i p)(m)$. That, forthwith, implies the following.

**Corollary 5.3.1.** *Let $k$ be an algebraically closed field, $F$ be a field of characteristic $0$. If $C(X)$ holds for every $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$ and some Weil cohomology theory $\mathrm{H}_W$, $\mathbf{Mot}_{num}(k,F)$ is semi-simple Tannakian.*

Let $\xi = \gamma_X([H]) \in \mathrm{H}^2_W(X)(1)$, where $[H]$ denotes the class of the hyperplane at infinity (*i.e.* , the class of the very ample line bundle of $X$). Let

$$L_X : \mathrm{H}^\bullet_W(X) \longrightarrow \mathrm{H}^{\bullet+2}_W(X)(1)$$

denotes the morphism $L_X(\alpha) = \alpha \cup \xi$. Consider the induced commutative square

$$\begin{array}{ccc} \mathrm{H}_W^{2i}(X)(i) & \xrightarrow{L_X^{d_X-2i}(i)} & \mathrm{H}_W^{2d_X-2i}(X)(d_X-i) \\ \gamma_X^i \uparrow & & \gamma_X^{d_X-i} \uparrow \\ \mathcal{Z}_{hom}^i(X)\otimes_{\mathbf{Z}} F & \longrightarrow & \mathcal{Z}_{hom}^{d_X-i}(X)\otimes_{\mathbf{Z}} F \end{array}$$

for $2i \leq d_X$.

The Hard Lefschetz's Theorem for Kähler manifolds motivates the following analogous assertion.

**Conjecture 5.3.2** ($A(X)$ or of Lefschetz Type). *Let $k$ be an algebraically closed field, $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$, $F$ be a field of characteristic 0 and Weil cohomology theory $\mathrm{H}_W$ with coefficients in $F$.*

*(a) $L_X^{d_X-i}$ is an isomorphism for $i \leq d_X$;*

*(b) $L_X^{d_X-2i}(i)$ induces an isomorphism (or, equivalently, an epimorphism) $\mathcal{Z}^i(X) \xrightarrow{\sim} \mathcal{Z}^{d_X-i}(X)$ for $2i \leq d_X$.*

As in Hodge Theory for Kähler manifolds, one can define the operator $\Lambda_X$ as an adjoint to $L_X$. Let $P^i(X) := \mathrm{Ker}(L_X^{d_X-i+1})$ be the primitive elements. $A(X)(b)$ implies that $\mathrm{H}_W^i(X) \cong P^i(X) \oplus L_X(-1)(coh_W^{i-2}(X)(-1))$. Proceeding inductively,

$$\mathrm{H}_W^i(X) \cong \bigoplus_{j\geq \max\{i-d_X,0\}} L_X^j(-j)(P^{i-2j}(X)(-j)).$$

One, then, can define $\Lambda_X$ to be the projection on $j \geq \max\{i-d_X, 1\}$, *i.e.* ,

$$\Lambda_X(\alpha) = \sum_{j\geq \max\{i-d_X,1\}} L_X^{j-1}(-j)(p_{i-2j}(\alpha))$$

, where $p_j : \mathrm{H}_W^i(X) \twoheadrightarrow P^{i-2j}(X)(-j)$ denotes the projection.

Analogously, one can also define a Hodge star $\star$ satisfying the identity $\Lambda_X = \star L_X \star$.

In any of these constructions, the square

$$\begin{array}{ccc} \mathrm{H}^i_W(X) & \xrightarrow{L_X^{d_X-i}} & \mathrm{H}^{2d_X-i}_W(X)(d_X-i) \\ \downarrow{\Lambda_X} & & \downarrow{L_X(d_X-i)} \\ \mathrm{H}^{i-2}_W(X)(-1) & \xrightarrow{L_X^{d_X-i+2}(-1)} & \mathrm{H}^{2d_X-i+2}_W(X)(d_X-i+1) \end{array}$$

is commutative. In fact, one could define $\Lambda_X$ by using the above square.

**Conjecture 5.3.3** ($B(X)$). *Let $k$ be an algebraically closed field, $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$, $F$ be a field of characteristic $0$ and Weil cohomology theory $\mathrm{H}_W$ with coefficients in $F$. $\Lambda_X$ is algebraic.*

It's clear that $B(X) \Rightarrow A(X)$ by the previous diagram since $\Lambda_X^{d_X-i}$ would be algebraic and also define an inverse for $L_X^{d_X-i}$. Furthermore, $A(X \times_k X) \Rightarrow B(X)$.

Since the $\mathbf{Q}$-algebra of endomorphisms generated by $\Lambda_X$ and $L_X$ contains $\pi_i$, one, moreover, obtains $B(X) \Rightarrow C(X)$.

Let $\mathcal{Z}^i_{prim}(X) := P^2i(X)(i) \cap \gamma^i_X(\mathcal{Z}^i(X))$. In analogy with the polarisation in Hodge Theory, one can define the bilinear form

$$Q(\alpha,\beta) := (-1)^i \operatorname{Tr}_X(L_X^{n-2i}(\alpha \cup \beta))$$

on $\mathcal{Z}^i_{prim}(X)$. In Hodge Theory of Kähler manifolds, the Hodge Index Theorem, implies that $Q$ is positive definite. That motivates the following

**Conjecture 5.3.4** ($Hdg(X)$ or of Hodge Type). *Let $k$ be an algebraically closed field, $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$, $F$ be a field of characteristic $0$ and Weil cohomology theory $\mathrm{H}_W$ with coefficients in $F$. $Q$ is positive definite.*

The argument of Weil using the Castelnuovo-Weil Positivity Theorem ($\sigma(\xi \circ \xi') > 0$) in order to prove the Riemann's Hypothesis for smooth projective curves over finite fields was extended by Serre in [Ser60] to the transcendental case of smooth projective varieties over $\mathbf{C}$. Applying the same technique, one can prove the following.

**Theorem 5.3.1** ([Gro69, Rem. 4.(2)], [Ser60, Thm. 1]). *Let $k$ be an algebraically closed field, $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$, $F$ be a field of characteristic $0$ and Weil cohomology theory $\mathrm{H}_W$ with coefficients in $F$. Let, also, $f : X \longrightarrow X \in$*

$\mathbf{SmProj}_k(X,X)$ *such that* $f^*(\xi) = q.\xi$ *for some* $q \in \mathbf{Q}$. $B(X) + Hdg(X \times_k X)$ *implies that* $|\alpha| = q^{\frac{i}{2}}$ *for every eigenvalue* $\alpha$ *of* $(f^*)^i$, *i.e.* , $P_i(\alpha) = \det(1 - (f^*)^i \alpha | \mathrm{H}^i_W(X))^{(-1)^{i+1}} = 0$. *In particular, for* $k = \mathbf{F}_q$, *it implies the Riemann Hypothesis over finite fields.*

**Theorem 5.3.2** ([Gro69, Rem. 4.(3)]). *Let* $k$ *be an algebraically closed field,* $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$, $F$ *be a field of characteristic* 0 *and Weil cohomology theory* $\mathrm{H}_W$ *with coefficients in* $F$.

$$Hdg(X) + A(X)(a) \Rightarrow (\sim_{num} = \sim_{hom} \Leftrightarrow A(X)).$$

## § 5.4 Motives over finite fields

Tate's Conjecture states that every algebraic cycle on $X$ for $X/k$ comes from a a $G_k$-invariant cohomology class in the $\ell$-adic cohomology of $X$. Of course, every cycle $Z \hookrightarrow X$ is defined over $k$ and, therefore, is $G_k$-invariant. The converse is extremely non-trivial.

**Conjecture 5.4.1** (Tate's Conjecture, [Tat65, Conj. 1]). *Let* $k$ *be a finite field,* $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$ *and* $\ell \neq 0$ *in* $k$ . *The cycle class morphism induces an isomorphism*

$$\mathcal{Z}^i_{hom}(X) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell \overset{\gamma_X \otimes \mathbf{Q}_\ell}{\longrightarrow} \mathrm{H}^2_{\text{ét}}\, i(X \otimes_k \overline{k}, \mathbf{Q}_\ell(i))^{G_k}$$

*Remark* 5.4.1. Tate's Conjecture usually is stated in a stronger form, which implies semi-simplicity of the Frobenius and, sometimes, that homological equivalence coincides with numerical equivalence.

Once one supposes Tate's Conjeture and the semi-simplicity of the Frobenius action, the category $\mathbf{Mot}_{hom}(k,F)$ becomes generated by motives of abelian varieties and 0-dimensional smooth schemes.

**Theorem 5.4.1**. *Let* $k$ *be a finite field such that* $|k| = p^n = q$ *for some prime* $p$. *If the Tate's Conjecture is true and the Frobenius acts semi-simply on the* $\ell$*-adic cohomology, then* $\mathbf{Mot}^{eff}_{hom}(k,F)$ *is a semi-simple abelian tensor* $F$*-linear category* *with simple objects being Artin motives (i.e. , motives $M(X)$ for $X/k$ a 0-dimension smooth $k$-schemes) and $M(C)$ for every smooth projective curve $C/k$.*

*In particular, $\mathbf{Mot}_{hom}(k,F)$ is generated by Artin motives, Tate motives and motives of smooth projective curves over $k$ and $\mathbf{Mot}_{hom}(\overline{k},F)$ is generated by Tate motives and motives of smooth projective curves over $k$.*

*Proof.* (Sketch) See [Mil94, Prop. 2.26, Rem. 2.7]. If the Frobenius acts semi-simply and Tate's conjecture is true, then $\sim_{hom}=\sim_{num}$ (where $\mathrm{H}^{\bullet}_{W} = \mathrm{H}^{\bullet}_{\text{ét}}((-),\mathbf{Q}_{\ell}(-))$). Hence, by Jannsen's semisimplicity theorem, $\mathbf{Mot}_{hom}(k,F)$ is semi-simple.

By the Tate's Conjecture, the $\ell$-adic realisation $r_{\ell}\otimes\mathbf{Q}_{\ell} : \mathbf{Mot}_{hom}(k,F)\otimes\mathbf{Q}_{\ell} \longrightarrow \mathbf{Mod}_{G_k}$ is fully faithful.

Since $r_{\ell}\otimes\mathbf{Q}_{\ell}$ is fully faithful, simple motives are entirely determined by its Weil $q$-numbers (*i.e.* , the eigenvalues of the Frobenius acting on the cohomology from where the motive came from) so that, whenever its Weil $q$-numbers are conjugate by the action of $G_{\mathbf{Q}}$, they are, actually, isomorphic.

More precisely, let $M,N \in \mathrm{Ob}(\mathbf{Mot}_{hom}(k,F))$ be simple motives with conjugate Weil $q$-numbers $\alpha$ and $\beta$ respectively. Then $\mathbf{Mod}_{G_k}(r_{\ell}(M)\otimes\mathbf{Q}_{\ell}, r_{\ell}(N)\otimes\mathbf{Q}_{\ell})^{G_{\mathbf{Q}}} \neq 0$. Hence, by Tate's Conjecure, $\mathbf{Mod}_{hom}(k,F)(M,N) \neq 0$. Since $M$ and $N$ are simple, every morphism is an isomorphism and, therefore, $M \cong N$.

Reciprocally, since Tate twist multiplies the Weil $q$-numbers by $q$, every Weil $q$-number may be assumed to be an algebraic integer. If $\alpha$ is an integral Weil $q$-number, by Honda-Tate theory ([Tat71, Thm. 1]) [6], one can find an abelian variety $A/k'$ with $[k':k] = m \geq 0$ and a Weil $q$-number $\pi_A \in \mathbf{Q}[\pi_A] \in \mathrm{End}^0_k(A)$ [7] that is conjugate to $\alpha$ and $|\pi_A| = q^{\frac{1}{2}}$. The characteristic polynomial $P(M^1(A),t)$ of the (conjugate of the) Frobenius $\pi_A$ is of degree $2g$ and it's also a power the minimal polynomial of $\pi_A$. Since the Weil restriction $\mathrm{Res}_{k'/k}(A)$ is again an abelian variety, motives are semisimple and

$$P(M^1(\mathrm{Res}_{k'/k}(A)),t) = P(M^1(A),t^m) = P(M^1(A)^{\otimes m},t)$$

, it implies that $M^1(A)^{\otimes m}$ should have a simple factor with eingenvalue conjugate to $\pi_A$.

[6]Honda-Tate theory gives an equivalence between isogenous abelian varieties and the characteristic polynomials of the Frobenius appearing in the local $\zeta$ function.

[7]Notice that $\mathbf{Q}[\pi_A]$ is a field.

Now, notice that for curves $C, C' \in \mathbf{SmProj}_k$, $\mathbf{Mot}(k,F)(M^1(C), M^1(C')) \cong \mathbf{SmProj}_k(J(C), J(C')) \otimes \mathbf{Q}$, which implies that the category composed of $M^1(C)$'s is equivalent to the category of Jacobians with isogenies as morphisms. Since every abelian variety $A$ is an abelian subvariety of a Jacobian $J(C)$, by Poincaré's Reducibility Theorem, there exists an isogeny $A \times_k A' \longrightarrow J(C)$. Therefore, the category of motives generated by abelian varieties is equivalent to the category of motives generated by smooth projective curves. □

# Chapter 6

# Mixed Motives

The theory of pure motives is inherently dependent on the Grothendieck's Standard Conjectures. As such conjectures were and still nowadays out of reach, it's reasonable to generate alternatives. One such surrogate is the derived category of motives.

Mixed motives are supposed to be the objects that generalises pure motives by allowing arbitrary smooth schemes instead of only proper smooth schemes. That allows in particular quasi-projective varieties such as pointed smooth projective curves which are, for instance, simple objects not present in the pure version.

The name mixed, as opposed to pure, comes from an additional parameter: the weight. Pure motives always have only one weight that usually comes from a shift of the degree of the cohomology. For instance, $\mathrm{H}^i(X,F)$ have pure weight equals to $i$ whenever $X$ is proper smooth. The weight does not always comes from the degree of the cohomology as $\mathrm{H}^1(\mathbf{G}_m,F) \cong \mathrm{H}^2(\mathbf{P}^1,F)$, for instance, have weight 2 by applying Mayer-Vietoris for the covering $\mathbf{A}^1 \sqcup \mathbf{A}^1$ consisting of the two hemispheres of the sphere $\mathbf{P}^1(\mathbf{C})$.

If one, however, allows for nonproper schemes, some fixed degree of the cohomology may acquire more than one weight.

Let, for instance, $K$ be a field of characteristic 0, $X/K$ be proper smooth,$Y \hookrightarrow X$ be a closed immersion and $U := X \setminus Y$, which is clearly non-proper. The long exact sequence of the pair $(X,U)$

$$\cdots \longrightarrow \mathrm{H}^i_{\mathrm{Betti}}(X,U,F) \longrightarrow \mathrm{H}^i_{\mathrm{Betti}}(X,F) \longrightarrow \mathrm{H}^i_{\mathrm{Betti}}(U,F) \longrightarrow \mathrm{H}^{i+1}_{\mathrm{Betti}}(X,U,F) \longrightarrow \cdots$$

$$0 \longrightarrow \frac{\mathrm{H}^i_{\mathrm{Betti}}(X,F)}{\mathrm{H}^i_{\mathrm{Betti}}(X,U,F)} \longrightarrow \mathrm{H}^i_{\mathrm{Betti}}(U,F) \longrightarrow \mathrm{Ker}(H^{i+1}(X,U,F) \to \mathrm{H}^{i+1}_{\mathrm{Betti}}(X,F)) \longrightarrow 0$$

. The left term is a quotient of an object of pure weight $i$, whereas the right term is a subobject of an object of pure weight $i+1$. Then $\mathrm{H}^i_{\mathrm{Betti}}(U,F)$ is an extension of objects with different weights. Hence it must somehow consist of at least a combination of weights $i$ and $i+1$. Such extension gives an increasing filtration $0 \subset W_0 \subset W_1 = \mathrm{H}^i_{\mathrm{Betti}}(U,F)$, where $\mathrm{Gr}^W_j$ has weight $j$. Therefore, the underlying motive of $\mathrm{H}^i_{\mathrm{Betti}}(U,F)$ must be mixed of at least weight $i$ and $i+1$. If $X/\mathbf{C}$ is, for instance, a smooth projective curve and $Y$ a finite number of points, then the left and right sides are indeed pure of weight $i$ and $i+1$.

More generally, every mixed motive must be a colimit of extensions by pure motives.

Historically, mixed motives were already known to somehow exist by Grothendieck in his personal manuscript dating around 1965-1970 ([Gro70]) were he describes a theory weights.

Officially, however, a theory of weights presented by means of mixed Hodge structures only appeared in Deligne's Doctorat d'État ([Del71]), which was published in 1971 [1]. In a subsequent work, ([Del74b]) expanding his thesis on Mixed Hodge Theory, the name 1-motive ([Del74b, Def. 10.1.2]) was coined to objects that should present the Betti realisation of mixed $\mathrm{H}^1$'s. Even the name *motivic* $\mathrm{H}^1$ of $X_0$ ([Del74b, Def. 10.3.4]), denoted by $\mathrm{H}^1_m(X_0)(1)$, was attributed to the appropriate mixed Hodge structure (which is a 1-motive) of an algebraic curve $X_0/k$ over an algebraically closed field $k$.

Later, on 3 May 1963, in a letter to Illusie ([Gro63] and [Jan94, Appendix]), Grothendieck suggests a theory of mixed motives based on Deligne's 1-motives. There he envisions a theory over an arbitrary general base.

## § 6.1 Beilinson and Lichtenbaum's Idea

**Theorem 6.1.1** (Atiyah-Hirzebruch's Spectral Sequence). *Let $E$ be a generalized Eilenberg-Steenrood cohomology theory and $F \hookrightarrow P \twoheadrightarrow X$ a fibration sequence.*

[1] Deligne's, however, completed his Doctorat d'État in 1972 at Université Paris-Sud XI - Orsay.

*There's a spectral sequence*

$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{sing}}(X, E^q(F)) \Longrightarrow E^{p+q}(P).$$

*In particular,*

$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{sing}}(X, E^q(1)) \Longrightarrow E^{p+q}(X)$$

*for any point* $1 \to X$.

As it was already mentioned, motivic cohomology should be the analogous of singular cohomology in the algebraic setting. It's, therefore, reasonable to use such spectral sequence applied to some algebraic $E$ in order to engender an algebraic version of $\mathrm{H}_{\mathrm{sing}}(-, \mathbf{Z})$. Belinson's idea, which was explicit stated in a letter to Soulé ([Bei82]) and later published in [Bei87a, §5.10] and [Bei87b, §5.10], was exploiting such spectral sequence for the case $E = K$, the Quillen's algebraic K-theory. Indeed, such choice was quite natural as the Atiyah-Hirzebruch's spectral sequence was mainly motivated by complex K-theory and higher algebraic K-theory itself was motivated by Atiyah and Hizerbruch's work on higher complex K-theory and their success on proving the differentiable version Grothendieck-Riemann-Roch's Theorem. Furthermore, Gillet, recently, was able to extended Grothendieck-Riemann-Roch to higher algebraic K-theory, so the analogy was getting stronger than ever.

In Beilinson own words ([Bei82]):

"But before presenting it, I want to tell a little hand-waving philosophy. Note that the situation in topology is quite clear: first you define very simply the cohomology groups of a space, which are quite manageable; secondly you define K-theory, which becomes more or less manageable only after you prove Bott's theorem: one has Atiyah-Hirzebruch spectral sequence, and so on. In algebraic geometry the situation is far more confusing. First, you have no cohomology. I hope very much , that in fact it exists, and may be defined by elementary means – one has to construct "universal" cohomology theory. It should be the cohomology of some complexes of sheaves $\Gamma(i)$ on the Zariski site, satisfying something like Gillet's axioms."

Let $X$ be a compact Hausdorff topological space [2].The Atiyah-Hirzebruch's spectral sequence for the complex K-theory becomes

$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{sing}}(X, \mathbf{Z}(\frac{-q}{2})) \Longrightarrow \tilde{K}^{p+q}(X)$$

, where $\mathbf{Z}(\frac{-q}{2})$ equals to $\mathbf{Z}$ when $q$ is even and 0 otherwise [3].

After applying $(-)\otimes_{\mathbf{Z}}\mathbf{Q}$, that sequence degenerates to the Chern character

$$\mathrm{ch} : K^\bullet(X) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \mathrm{H}^\bullet_{\mathrm{sing}}(X, \mathbf{Q})$$

.

Or, more explicitly,

$$\mathrm{ch} : K^i(X) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \prod_j \mathrm{H}^{2j+i}_{\mathrm{sing}}(X, \mathbf{Q})$$

for $i = 0,1$ [4].

The formalism of perverse $\ell$-adic sheaves and mixed Hodge structures were already well known before the letter to Soulé, *i.e.* ., the codomain of the $\ell$-adic and Betti realisations [5] were well understood in this period. Furthermore, from the theory of derived categories applied to such objects, one could extract an abelian category using a nontrivial t-structure (the perverse t-structure) which rendered well behaved objects. Indeed, the whole theory of perverse sheaves was motivated by a search for a good cohomology for stratified singular spaces over $\mathbf{C}$ [6], which should satisfy, for instance, Poincaré duality.

Hence, analogously, one could use of the derived category as a way to understand motives. That is, if one cannot explicitly construct the tensor abelian category of motives due to the lack of knowledge about the Standard Conjectures, then one might as well proceed with the category of motives using the analogous procedure that was applied to the codomain of the realisations

[2]Recall that K-theory can be defined for noncompact spaces. However, it's not representable by $BU \times \mathbf{Z}$. Actually, using Grothendieck construction for the category of vector bundles doesn't even engender a generalized cohomology theory.

[3]Notice that $\tilde{K}^q(1) := \tilde{K}_{-q}(1) = \mathbf{Z}(\frac{-q}{2})$

[4]Recall that Bott periodicity implies that every $K^i$ for arbitrary integer $i$ is isomorphic to one with $i = 0,1$

[5]Mixed Hodge modules were conceived latter, though

[6]This good cohomology is Goresky-Macpherson's intersection cohomology, which is analogous to the cohomology of the constant sheaf or, equivalently, the singular cohomology in the smooth case

in the hope of extracting such well behaved abelian category.

As mentioned by Beilinson ([Bei82]), Gillet made an abstract K-theoretic formalism containing the necessary pieces to prove Grothendieck-Riemann-Roch for higher algebraic K-theory ([Gil81]). Such axiomatic formalism contained complexes of sheaves $\Gamma(i) \in \mathrm{D}(X_{\mathrm{Zar}})$ satisfying nice properties [7] that were, in its turn, motivated by the formalism of perverse sheaves.

These $\Gamma(i)$'s were assumed to satisfy $\Gamma(0) = \mathbf{Z}$, $\Gamma(1) = \mathbf{G}_m[-1]$ and $\Gamma(i) = 0$ for $i < 0$. The first condition declares that the coefficients are integral, whereas the second condition declares that $\Gamma(1)$ is the Tate twist [8].

In this setting, the expected Atiyah-Hirzebruch could be translated as

$$E_2^{p,q} = \mathrm{H}^p(X, \mathbf{Z}(\frac{-q}{2})) \Longrightarrow K^{p+q}(X)$$

for smooth schemes $X$. The usual convention, though, is to make the change of variables $p \mapsto p - q$ and use the homological notation.

$$E_2^{p,q} = \mathrm{H}^{p-q}(X, \Gamma(q)) \Longrightarrow K^{-p-q}(X).$$

In this setting, after applying $(-) \otimes_{\mathbf{Z}} \mathbf{Q}$, that sequence should analogously degenerate to the Chern character

$$\mathrm{ch} : K_i(X) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \prod_j \mathrm{H}^{2j-i}(X, \Gamma(j) \otimes_{\mathbf{Z}} \mathbf{Q})$$

such that

$$\mathrm{Gr}^i_\gamma(K_j(X) \otimes \mathbf{Q}) \cong \mathrm{H}^{2i-j}(X, \mathbf{Q}(i)).$$

By noticing that $\mathrm{H}^2(\mathrm{Spec}(k)_t, \Gamma(1)) = \mathrm{H}^1(\mathrm{Spec}(k)_t, \mathbf{G}_m) = 0$ for any field $k$ (by the usual Hilbert 90) and that the Brown-Gersten-Quillen spectral sequence determines, under the Gersten conjecture, $K(X)$ from the $K(\kappa(x))$'s for $x$ some point in $X$, one may conjecture the following.

**Conjecture 6.1.1** (Beilinson-Soulé). *For $i \geq 1$, $\Gamma(i)$ is acyclic outside of $[1, i]$.*

[7] Properties such as the Poincaré duality for smooth schemes, the projection formula, $\mathbf{A}^1$-homopy invariance, existence of a Gysin's long exact sequence and a projective bundle formula

[8] Recall that $\mathrm{H}_1(\mathbf{G}_m, F)$ is the Tate twist.

Furthermore, Beilinson when investigating such universal cohomology was mainly motivated by the Quillen-Lichtenbaum conjecture, which is essentially an $\ell$-adic version of the Atiyah-Hirzebruch spectral sequence for finite type affine schemes. More precisely, Beilinson was interested in the K-theory with finite coefficients $K_i(X) \otimes_{\mathbf{Z}} \mathbf{Z}/\ell$ and its relation to étale cohomology with coefficients in a $\ell$-torsion sheaf mediated by some type of Atiyah-Hirzebruch's spectral sequence.

As the $\Gamma(i)$'s are expected to be an integral version in the Zariski topology of the Tate twists $(\mathbf{Z}/\ell)_t(i)$ in the étale topology, one should suppose that quotienting $\Gamma(i)$ modulo $\ell$ should somehow gives $(\mathbf{Z}/\ell)_t(i)$. The issue, however, is that they are located in different topos and $(\mathbf{Z}/\ell)_t(i)$ is not acyclic outside of $[1,i]$ when $i \geq 1$ [9]. Therefore, one might conjecture the following.

**Conjecture 6.1.2** (Beilinson-Lichtenbaum's conjectures). *Let $X$ be smooth and defined over $\mathbf{Z}[\frac{1}{\ell}]$. Consider the obvious geometric morphism of ringed topos $\pi : X_t \longrightarrow X_{\text{Zar}}$. Then*

$$\Gamma(i) \otimes^L \mathbf{Z}/\ell \cong \tau_{\leq i} R\pi_*(\mathbf{Z}/\ell)_t(i)$$

*, where $\tau_{\leq i}$ truncates the cohomologies of degree greater than $i$.*

*Remark* 6.1.1. Usually such conjectured is stated using the notation $\mu_{\ell,t}^{\otimes i}$ instead of $(\mathbf{Z}/\ell)_t(i)$. Notice, however, that $\mu_{\ell,t} \cong (\mathbf{Z}/\ell)_t(1)$ by Kummer's exact sequence;

Also, notice that $\ell$ must indeed be invertible in $A$ in order to avoid ramifications, which would collapse all roots of unity to a unique one.

*Remark* 6.1.2. Notice that Beilinson-Lichtenbaum conjectures assumes an $\mathbf{Z}/\ell$ version of the Beilinson-Soulé conjecture due to the truncation.

It was also conjectured that, for smooth $X$, $\mathrm{H}^i(\Gamma(i))$ should be a sheaf of $K_i^{\text{Milnor}}$- functors. In particular,

$$\mathrm{H}^i(\mathrm{Spec}(k), \Gamma(i)) \cong K_i^{\text{Milnor}}(k)$$

for any field $k$.

[9] Any proper smooth geometrically connected curve $C$ over a field, for instance, satisfy $\mathrm{H}^2_t(C_t, (\mathbf{Z}/\ell(1)) \cong \mathbf{Z}/ell$

Independently, Lichtenbaum ([Lic84, §1, §2]) noticed that the values of the local $\zeta$-functions $Z(X, q^{-s})$ at $s = 0$ and $s = 1$ were dependent on, respectively, the étale cohomology with coefficients in $\mathbf{Z}$ and $\mathbf{G}_m$. More precisely, the lowest coefficient of the Laurent series around the poles $s = 0$, which is always simple, and $s = 1$, which is not necessarily simple, could be computed by the respective cohomologies.

That motivated a search for complexes of étale sheaves that would play the same role for $s$ any non-negative integer. Lichtenbaum, therefore, conjectured (in [Lic84, §5]) the existence of complexes of sheaves $\Gamma_L(i) \in D(X_t)$ with $\Gamma(0) = \mathbf{Z}$, $\Gamma(1) = \mathbf{G}_m[-1]$, satisfying analogous properties by replacing algebraic K-theory with étale K-theory and the analogous Beilinson-Soulé conjecture. Such complexes of sheaves, in analogy with Kummer's exact sequence should relate to the $\ell$-torsion sheaves by the (co)fiber sequence [10]

$$\Gamma_L(i) \xrightarrow{\ell} \Gamma_L(i) \longrightarrow (\mathbf{Z}/\ell)_t(i) \longrightarrow \Gamma_L(i)[1]$$

. Hence $\Gamma_L(i)$ should be thought as the integral version of the $i$-th Tate twist in the $\ell$-torsion case. That is, quotient it modulo $\ell$ should be the $i$-th Tate twist.

An analogous of the Hilbert 90 was also supposed.

**Conjecture 6.1.3** (Hilbert 90).

$$R^{i+1}\pi_*\Gamma_L(i) \cong 0.$$

*Remark* 6.1.3. As $\Gamma(1) = \mathbf{G}_m[-1]$, for $k$ a field and $X = \operatorname{Spec}(k)$, the above conjecture is essentially the ordinary Hilbert 90 $\mathrm{H}^1(\operatorname{Spec}(k), \mathbf{G}_m) \cong 0$.

Furthermore, it generalises other known K-theoretic forms of the Hilbert 90 such as Mercuriev-Suslin's version of Hilbert 90 for $K_2$.

The correspondence with Beilinson's definition up to some minor details [11] can de stablished by setting

$$\Gamma(i) := \tau_{\leq i} R\pi_*\Gamma_L(i)$$

[10] Or, it should relate $\ell$-adic sheaves after taking the cofiltered limit by the powers of $\ell$.

[11] Lichtenbaum, for instance, does not require that $\mathrm{H}^i(X, \Gamma(i)) \cong \mathscr{K}_i^M(X)$ for arbitrary smooth $X$

and the following proposition.

**PROPOSITION 6.1.1.** *Let* $\Gamma(i) := \tau_{\leq i} R\pi_* \Gamma_L(i)$. *If theres a (co)fiber sequence*

$$\Gamma_L(i) \xrightarrow{\ell} \Gamma_L(i) \longrightarrow (\mathbf{Z}/\ell)_t(i) \longrightarrow \Gamma_L(i)[1]$$

*and Hilbert 90 holds, then the* $\Gamma(i)$*'s satisfy Beilinson-Lichtenbaum's conjecture*

*PROOF.* By applying $R\pi_*$, one obtains the (co)fiber sequence

$$\Gamma(i) \xrightarrow{\ell} \Gamma(i) \longrightarrow R\pi_*(\mathbf{Z}/\ell)_t(i) \longrightarrow \Gamma_L(i)[1]$$

Since $\Gamma_L(i)$ was assumed to satisfy Hilbert 90, one can apply $\tau_{\leq i}$ and still obtains a (co)fiber sequence [12]

$$\tau_{\leq i}\Gamma(i) \xrightarrow{\ell} \tau_{\leq i}\Gamma(i) \longrightarrow \tau_{\leq i} R\pi_*(\mathbf{Z}/\ell)_t(i) \longrightarrow \Gamma_L(i)[1]$$

. As $\Gamma(i)$ was already defined as an $i$-th truncation, one can rewrites

$$\Gamma(i) \xrightarrow{\ell} \Gamma(i) \longrightarrow \tau_{\leq i} R\pi_*(\mathbf{Z}/\ell)_t(i) \longrightarrow \Gamma_L(i)[1].$$

Since $\Gamma(i) \otimes^L_{\mathbf{Z}} \mathbf{Z}/\ell$ is the homotopy (co)kernel or (co)fiber of the multiplication by $\ell$,

$$\Gamma(i) \otimes^L \mathbf{Z}/\ell \cong \tau_{\leq i} R\pi_*(\mathbf{Z}/\ell)_t(i)$$

, which is exactly the Beilinson-Lichtenbaum conjecture.

□

A summary of the required properties is summarised is the followings.

**CONJECTURE 6.1.4** (Beilinson's Universal Cohomology). *Let* $X \in \mathrm{Ob}(\mathbf{Sch})$. *For every* $i \in \mathbf{Z}$ *nonnegative, there is a complex of sheaves* $\Gamma(i) \in D(X_{Zar})$ *satisfying the following.*

1. $\Gamma(0) = \mathbf{Z}$ *and* $\Gamma(1) = \mathbf{G}_m[-1]$;
2. *For every* $i$ *and* $j$, *there's a product morphism* $\Gamma(i) \otimes^L_{\mathbf{Z}} \Gamma(j) \longrightarrow \Gamma(i+j)$;

[12] If $X \to Y \to Z \to X[1]$ is a (co)fiber sequence in an stable $\infty$-category and $\mathrm{H}^{i+1}(X) \cong 0$, then $\tau_{\leq i}X \to \tau_{\leq i}Y \to \tau_{\leq i}Z \to \tau_{\leq i}X[1]$ is a (co)fiber sequence

3. *(Beilinson-Soulé) For* $i \geq 1$, $\Gamma(i)$ *is acyclic outside* $[1,i]$;

4. *For every* $i$, $\Gamma(i) \otimes^L \mathbf{Z}/\ell \cong \tau_{\leq i} R\pi_*(\mathbf{Z}/\ell)_t(i)$;

5. *For every* $i$ *and* $j$, $\mathrm{Gr}^i_\gamma(K_j(X) \otimes_{\mathbf{Z}} \mathbf{Q}) \cong \mathrm{H}^{2i-j}(X, \Gamma(i) \otimes_{\mathbf{Z}} \mathbf{Q})$;

6. *For every* $i$ *and smooth* $X$, $\Gamma(i) \cong \mathscr{K}_i^{Milnor}(X)$ [13].

**Conjecture 6.1.5** (Lichtenbaum's Universal Cohomology). *Let* $X \in \mathrm{Ob}(\mathbf{Sch})$. *For every* $i \in \mathbf{Z}$ *nonnegative, there is a complex of sheaves* $\Gamma_L(i) \in D(X_t)$ *satisfying the following.*

1. $\Gamma_L(0) = \mathbf{Z}$ *and* $\Gamma_L(1) = \mathbf{G}_m[-1]$;

2. *For every* $i$ *and* $j$, *there's a product morphism* $\Gamma_L(i) \otimes^L_{\mathbf{Z}} \Gamma_L(j) \longrightarrow \Gamma(i+j)$;

3. *(Beilinson-Soulé) For* $i \geq 1$, $\Gamma_L(i)$ *is acyclic outside* $[1,i]$;

4. *(Hilbert 90) For every* $i$, $R^{i+1}\pi_*\Gamma_L(i) \cong 0$;

5. *(Kummer's Exact Sequence)For every* $i$, *there's a (co)fiber sequence*

$$\Gamma_L(i) \xrightarrow{\ell} \Gamma_L(i) \longrightarrow (\mathbf{Z}/\ell)_t(i) \longrightarrow \Gamma_L(i)[1] \ ;$$

6. *For every* $i$ *and* $j$, $\mathrm{Gr}^i_\gamma(K^t_j(X) \otimes_{\mathbf{Z}} \mathbf{Q}) \cong \mathrm{H}^{2i-j}(X, \Gamma_L(i) \otimes_{\mathbf{Z}} \mathbf{Q})$;

7. *For every* $i$ *and smooth* $X$, $\Gamma_L(i) \cong \mathscr{K}_i^{Milnor}(X)$ [14].

**Theorem 6.1.2.** *If there are* $\Gamma_L(i)$*'s defining a Lichtenbaum's Universal Cohomology, then the assignment*

$$\Gamma(i) := \tau_{\leq i} R\pi_*\Gamma_L(i)$$

*defines a Beilinson's Universal Cohomology.*

[13] Originally, Beilinson in [Bei82] also required the $\Gamma(i)$'s to satisfy Gillet's axioms in [Gil81]

[14] Originally, Lichtenbaum in [Lic84] required such property for only the spectrum of a field

## § 6.2 Bloch's Higher Chow Groups

The first candidate for a Beilinson-Lichtenbaum's cohomology theory (which will actually be a Borel-Moore homology theory [15]) was proposed in 1986 in [Blo86] by Bloch. Bloch developed a theory of higher Chow groups $\mathrm{CH}^i(X,n)$ generalising the case $\mathrm{CH}^i(X) \cong \mathrm{CH}^i(X,0)$. The main motivation was to generalise the Baum-Fulton-Macpherson version of the Riemann-Roch Theorem, which implies the existence of the isomorphism on the right side of

$$\bigoplus_i \mathrm{Gr}^i_\gamma(G_0(X)) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} G_0(X) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} \bigoplus_i \mathrm{CH}^i(X)$$

for $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ of finite type and equidimensional.

The generalisation would imply an extension of the above to

$$\bigoplus_i \mathrm{Gr}^i_\gamma(G_n(X)) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} G_n(X) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} \bigoplus_i \mathrm{CH}^i(X,n)$$

For $S \in \mathrm{Ob}(\mathbf{Sch})$, let $\Delta^n_S = \mathrm{Spec}(\mathbf{Z}[x_0,\dots,x_n]/(\sum_i x_i - 1)) \times S \cong \mathbf{A}^n_S$. Let $S = \mathrm{Spec}(k)$ for some field and $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ of finite type locally equidimensional. Let

$$\mathcal{Z}^\bullet(X,n) \hookrightarrow \mathcal{Z}(\Delta^n_X) = \mathcal{Z}(X \times_k \Delta^n_k)$$

be a subgroup. The main aim consists in engendering a simplicial object $\mathcal{Z}^i(X,\bullet) =$

$$\mathcal{Z}^i(X,0) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathcal{Z}^i(X,1) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathcal{Z}^i(X,2)\cdots$$

for every $i \in \omega$ by defining morphisms between the $\Delta^m_k$'s for every morphism in $\Delta$. Let $\varphi : [m] \to [n] \in \Delta([m],[n])$. As in topology, $\varphi^\#_*(x_i) := \sum_{j\in\varphi^{-1}(i)} x_j$. That defines the structure of a cosimplicial $X$-scheme $\Delta^\bullet_X : \Delta \longrightarrow \mathbf{Sch}_X$.

Now, one must define $\mathcal{Z}^\bullet(X,n) \hookrightarrow \mathcal{Z}(\Delta^n_X)$ functorially enough. If one wishes to be able to restrict cycles to the faces of $\Delta^m_X \hookrightarrow \Delta^n_X$, one must require, at least, that the cycles intersect the faces properly.

[15] In the case of smooth schemes, Poincaré duality ensures that it defines a cohomology theory.

**Definition 6.2.1.** Let $k$ be a field, $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ be of finite type and $i \in \omega$. The **simplicial object of $i$-dimensional algebraic cycles on $X$** is defined objectwise as the abelian subgroup $\mathcal{Z}_i(X,n) \hookrightarrow \mathcal{Z}_{i+n}(\Delta^n_X)$ consisting of $\alpha = \sum_j n_j Z_j$ such that, for every face $\Delta^m_X \hookrightarrow \Delta^n_X$ and every $j$, $Z_j$ intersects $\Delta^m_X$ properly inside $\Delta^n_X$, *i.e.* ,

$$\dim(Z_j \cap \Delta^m_X) \leq m + i.$$

Let, furthermore, $X$ be locally equidimensional. The **simplicial object of algebraic cycles of codimension $i$ on $X$** is defined, likewise, by requiring that $\alpha = \sum_j n_j Z_j \in \mathcal{Z}^i(X,n) \hookrightarrow \mathcal{Z}^i(\Delta^n_X)$ satisfies

$$\mathrm{codim}_{\Delta^m_X}(Z_j \cap \Delta^m_X) \geq i$$

for every $\Delta^m_X \hookrightarrow \Delta^n_X$ and $j$.

*Remark* 6.2.1. In tho above definition, $\mathcal{Z}_i(X,\bullet)$ is indeed a simplicial abelian group. The condition of intersecting properly guarantees that, for $Z \in \mathcal{Z}_{i+n}(\Delta^n_X)$, $[Z \cap \Delta^{n-1}_X] \in \mathcal{Z}_{i+n-1}(\Delta^{n-1}_X)$. That occurs because, even though $\Delta^n_X$ may not be regular (in which case, the reverse inequality in the above definitions would be true), $\Delta^m_X \hookrightarrow \Delta^n_X$ is a regular immersion of codimension $n-m$ (*i.e.* , the underlying sheaf of ideals is locally generated by a regular sequence of length $n-m$) by noticing that the subsequence of $(x_0, x_1, \cdots, x_n)$ are always regular in $A[x_0, x_1, \cdots, x_n]/(\sum_i x_i - 1)$.

For a regular sequence $(a_1, a_2, \cdots, a_n)$, a regular closed immersion

$$V(a_1, ..., a_n) = Z \hookrightarrow X$$

and $W \hookrightarrow X$ closed integral subscheme intersecting properly

$$V(a_{i_1}, a_{i_2}, \cdots, a_{i_n})$$

for every sequence $i$, the intersection of $W \cap \bigcap_i V(a_i)$ does not depend on the order of the $a_i's$ and, by Krull's Principal Ideal Theorem, the dimension decreases by 1 after intersecting $W$ with each $V(a_j)$. That ensures that the above inequalities in the definition are actually equalities and, furthermore, any diagram with face morphisms commuting in $\Delta$ still commutes after applying

$\mathcal{Z}_i(X, \bullet)$.

The case of degeneracies is trivial since the projections $\Delta_X^{n+1} \twoheadrightarrow \Delta_X^n$ have affine fibers (*i.e.* , of the form $\mathbf{A}_X^1$) and, hence, are flat.

The above remark ensures that for every $i \in \omega$,

$$\mathcal{Z}_i(X, \bullet) : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Ab}$$

is a simplicial abelian group. Now, it's possible to apply the functor $\pi_n(-)$.

**Definition 6.2.2.** Let $k$ be a field, $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ be of finite type and $i, n \in \omega$. The **$n$-th Chow group of dimension $i$ of $X$** is defined as $\mathrm{CH}_i(X, n) = \pi_i(\mathcal{Z}_i(X, \bullet))$.

Let, furthermore, $X$ be locally equidimensional. The **$n$-th Chow group of dimension $i$ of $X$** is defined as $\mathrm{CH}_i(X, n) = \pi_i(\mathcal{Z}_i(X, \bullet))$

*Remark* 6.2.2. Notice that, by the Dold-Kan correspondence, one can, instead, define $\mathrm{CH}_i(X, n)$ (*resp.*, $\mathrm{CH}^i(X, n)$) by taking the underlying chain complex of $\mathcal{Z}_i(X, \bullet)$ (*resp.*, $\mathcal{Z}^i(X, \bullet)$). That was, indeed, the original definition in [Blo86]

**Proposition 6.2.1.** *Let $k$ be a field, $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ be of finite type and $i \in \omega$. There's a natural isomorphism*

$$\mathrm{CH}_i(X, 0) \cong \mathrm{CH}_i(X).$$

*Proof.* Notice that $\mathrm{CH}_i(X, 0) \hookrightarrow \mathcal{Z}_i(X)$ is defined by the quotient with cycles of the form $Z(1) - Z(0)$ for $Z \hookrightarrow X \times_k \mathbf{A}_k^1$ closed integral subscheme of dimension $i+1$ such that $Z$ intersects $\{0\}, \{1\} \hookrightarrow \mathbf{A}_X^1$ properly. The abelian subgroup defined by such differences is equal to the abelian group of cycles rationally equivalent to 0. □

The following guarantees the localisation long exact sequence and, in particular, the Mayer-Vietoris's long exact sequence.

**Theorem 6.2.1** (Bloch's Moving Lemma, [Blo94, Thm. 0.1]). *Let $k$ be a field, $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ be quasi-projective of finite type, $j : Z \hookrightarrow X$ a closed immersion of codimension $d$ and $i : U \hookrightarrow X$ the complementary open immersion. The induced morphism*

$$\mathcal{Z}^i(X,\bullet)/\mathcal{Z}^{i-d}(Z,\bullet) \longrightarrow \mathcal{Z}^i(U,\bullet)$$

*is an homotopy equivalence for every $i \geq d$.*

*Remark* 6.2.3. A wrong proof was given in [Blo86, Thm. 3.3] as noticed by Suslin. It was, however, later reproved in [Blo94, Thm. 0.1].

*Remark* 6.2.4. Notice that by taking the Quillen model structure on $\widehat{\Delta}$, every object is cofibrant and $j_* : \mathcal{Z}_i(Z,\bullet) \longrightarrow \mathcal{Z}_i(X,\bullet)$ is a monomorphism and, hence, a cofibration. The quotient above, then, is a an homotopy quotient (*i.e.* , $\infty$-coequaliser in $\mathbf{Grpd}_\infty$).

**Corollary 6.2.1** (Localisation, [Blo94, Cor. 0.2]). *Let $k$ be a field, $X \in$ $\mathrm{Ob}(\mathbf{Sch}_k)$ be quasi-projective of finite type, $j : Z \hookrightarrow X$ a closed immersion of codimension $d$ and $i : U \hookrightarrow X$ the complementary open immersion. There exists a long exact sequence*

$$\cdots \longrightarrow \mathrm{CH}^{i+d}(X,1) \xrightarrow{i^*} \mathrm{CH}^{i+d}(U,1) \longrightarrow \mathrm{CH}^i(Y,0) \xrightarrow{j_*} \mathrm{CH}^{i+d}(X,0) \xrightarrow{i^*} \mathrm{CH}^{i+d}(U,0).$$

*Proof.* That follows by the above remark and the long exact sequence associated to a cofiber sequence. □

**Corollary 6.2.2** (Mayer-Vietoris's Sequence). *Let $k$ be a field, $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ be quasi-projective of finite type and $U \sqcup V \twoheadrightarrow X$ a Zariski covering. The square*

$$\begin{array}{ccc} \mathcal{Z}^i(U,\bullet) & \longrightarrow & \mathcal{Z}^i(U,\bullet) \oplus \mathcal{Z}^i(V,\bullet) \\ \downarrow & \ulcorner & \downarrow \\ 1 & \longrightarrow & \mathcal{Z}^i(V,\bullet) \end{array}$$

*is an $\infty$-pushout square in $\mathbf{Grpd}_\infty$ (and, hence, also an $\infty$-pullback square [16]) for every $i \in \omega$.*

**Corollary 6.2.3.** *Let $k$ be a field and $i, n \in \omega$. The application $\mathcal{Z}^i(-,\bullet)$ defines functors*

$$\mathcal{Z}^i(-,\bullet) : (\mathbf{Sm}_k^{eq})^{\mathrm{op}} \longrightarrow \mathbf{Grpd}_\infty$$

[16] That occurs because $\mathbf{Ab}^{\Delta^{op}}$ (after simplicial localisation) is a stable $\infty$-category

,

$$\mathcal{Z}^i(-,\bullet) : (\mathbf{Sm}_k^{eq})^{\mathrm{op}} \longrightarrow D^-(\mathbf{Ab})$$

*and*

$$\mathrm{CH}^i(-,n) : (\mathbf{Sm}_k^{eq})^{\mathrm{op}} \longrightarrow \mathbf{Ab}$$

*, where* $\mathbf{Sm}_k^{eq}$ *denotes the category of smooth locally equidimensional* $k$*-schemes of finite type.*

*More generally,* $\mathcal{Z}_i(-,\bullet)$ *is contravariant for arbitrary morphisms of locally equidimensional* $k$*-schemes of finite type such that the codomain is smooth.*

*Furthermore,* $\mathcal{Z}_i(-,\bullet)$ *is contravariant for flat morphisms and locally complete intersections (i.e. , locally of finite type and, locally, factors as the composition of a closed immersion defined by a regular sequence followed by projection from an affine space) with fibers of fixed dimension between* $k$*-schemes of finite type. Moreover, it's covariant for proper morphisms.*

*Proof.* Contravariance for flat morphisms is trivial as the pullback of cycles intersecting properly still intersect properly.

Covariance for proper morphisms follows from [Blo86, Prop. 1.3].

The case of locally complete intersections follows by noticing that one can pullback along closed regular immersions as in Remark 6.2.1.

The case of a smooth codomain follows from an argument due to Levine. Let $f : Y \longrightarrow X$ be a morphism between schemes locally equidimensional and of finite type over $k$. Suppose that $X$ is smooth. Let $\mathcal{U}$ be a Zariski covering of $X$. Let $\mathcal{V}$ be the induced Zariski covering by applying $f^{-1}(-)$. Let $\mathcal{Z}^i(\mathcal{U},\bullet)$ denotes the realisation of the simplicial $\infty$-groupoid (or bisimplicial abelian group) given by $\mathcal{Z}^i(\check{C}^\bullet(\mathcal{U}),\bullet)$, where $\check{C}(-)$ is the Čech $\infty$-groupoid. Similarly, let $\mathcal{Z}^i(\mathcal{U},\bullet)_f$ denote the realisation of $\mathcal{Z}^i(\check{C}^\bullet(\mathcal{U}),\bullet)_f$, where the lower $f$ denotes the subobject of $\mathcal{Z}^i(\check{C}^\bullet(\mathcal{U}),\bullet)$ consisting of cycles that after pullbacking by $f_{|\check{C}^\bullet(\mathcal{U})}$ still intersect the faces properly.

The inclusion

$$\mathcal{Z}^i(\mathcal{U},\bullet)_f \hookrightarrow \mathcal{Z}^i(\mathcal{U},\bullet)$$

is a weak equivalence (see [Lev98, Part I, Ch. II, §3.5, Thm.3.5.14] for $U/k$ smooth essentially of finite type affine (or projective); see, also, for $U/k$ quasi-projective smooth [Lev94, Cor. 4.8] in the rational case). Furthermore the morphisms $\mathcal{Z}^i(X,\bullet) \xrightarrow{\sim} \mathcal{Z}^i(\mathcal{U},\bullet)$ and $\mathcal{Z}^i(Y,\bullet) \xrightarrow{\sim} \mathcal{Z}^i(\mathcal{V},\bullet)$ induced by pullback along the open immersions defined by the coverings are also weak equivalences by Mayer-Vietoris. Now, the morphism

$$\mathcal{Z}^i(\check{C}^\bullet(\mathcal{U}),\bullet)_f \xrightarrow{f^*} \mathcal{Z}^i(\check{C}^\bullet(\mathcal{V}),\bullet)$$

induced by the restrictions of $f^*$ is actually defined since pullbacks of cycles in $\mathcal{Z}^i(\check{C}^\bullet(\mathcal{U}),\bullet)_f$ intersects the faces properly by definition. One, then, can consider the induced zig-zag.

□

*Remark* 6.2.5. The case of contravariant functoriality over a smooth codomain in the above theorem was proved for equidimensional quasi-projective schemes over $k$ in [Blo86, Thm. 4.1]. However, again, the proof was wrong due to its reliance on an argument used to prove the Moving Lemma. According to Levine ([Lev98, Part I, Ch. II, §2.1, Rem. 2.1.7.(i)]), the proof was corrected in a private communication with Bloch. A proof for for quasi-projective schemes over $k$, after applying $(-) \otimes_{\mathbf{Z}} \mathbf{Q}$, was given by Levine in [Lev94, Cor. 4.9].

Notice that $f^*(\alpha)$ may not intersect the faces properly and, therefore, the contravariant functoriality relies on finding an equivalent cycle up to coboundaries that intersect the faces properly.

The desired isomorphism

$$\mathrm{Gr}^i_\gamma(G_n(X)) \xrightarrow{\sim} \mathrm{CH}_n(X, 2i-n)$$

, as in the ordinary case, should be induced by the Chern character $\mathrm{ch}_X$. One, therefore, should construct a higher Chern classes

$$c_{p,q} : G_{2q-p}(X) \longrightarrow \mathrm{CH}_q(X, p-2q)$$

. The canonical way of proceeding further is, then, by proving a Projective Bundle Formula and extracting the Chern classes from it.

**Theorem 6.2.2** (Projective Bundle Formula, [Blo86, Thm. 7.1]). *Let $k$ be a field, $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ be locally equidimensional quasi-projective of finite type, $\mathscr{E}$ be a locally free sheaf of $\mathscr{O}_X$-modules of rank $n \in \omega$, $\pi : \mathbf{P}(\mathscr{E}) \longrightarrow X$ and $m \in \omega$. Let*

$$\alpha_i : \mathrm{CH}^{j-i}(X,m) \longrightarrow \mathrm{CH}^j(\mathbf{P}(\mathscr{E}),m)$$

*denotes the morphism $\alpha_i(\zeta) = \pi^*(\zeta).\xi^i$ for every $i,j \in \omega$, where $\xi = c_1(\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)) \in \mathrm{CH}^1(X) \cong \mathrm{CH}^1(X,0)$. There's an isomorphism*

$$\bigoplus_{i=0}^{n-1} \alpha_i : \bigoplus_{i=0}^{n-1} \mathrm{CH}^{j-i}(X,m) \xrightarrow{\sim} \mathrm{CH}^j(\mathbf{P}(\mathscr{E}),m)$$

*for every $j \in \omega$.*

*In particular, when, furthermore, $X$ is smooth,*

$$\mathrm{CH}^\bullet(\mathbf{P}(\mathscr{E}),m) \cong \mathrm{CH}^\bullet(X,m) \otimes_{\mathrm{CH}^\bullet(X,0)} \mathrm{CH}^\bullet(\mathbf{P}(\mathscr{E}),0).$$

*Proof.* The particular case when $X$ smooth follows from the general one by noticing that $\mathrm{CH}^\bullet(\mathbf{P}(\mathscr{E}),0) \cong \mathrm{CH}^\bullet(X,0)[1,\xi,\cdots,\xi^{n-1}]$ when $X$ is smooth.[17] □

For simplicity assume that $X$ is smooth. For $m = 0$, the Chern classes $c_{2q,q} := c_i : K_0(X) \cong G_0(X) \longrightarrow \mathrm{CH}^q(X,0)$
, then, can be extracted by the identity

$$\sum_{n-1} \pi^*(c_i(\mathscr{E}))\xi^i = 0.$$

More generally, by a technique of Grothendieck in [Gil81], one can define $B\,\mathrm{GL}_n := N((\mathrm{GL}_n)_S) \in \mathrm{Ob}(\mathbf{Sch}_S^{\Delta^{\mathrm{op}}})$ (*i.e.* , the nerve of the internal category $(\mathrm{GL}_n)_S$ in **Sch**) and a universal projective bundle $\mathbf{P}(E\,\mathrm{GL}_n) := \mathbf{P}_S^{n-1}//(\mathrm{GL}_n)_S \in \mathrm{Ob}(\mathbf{Sch}^{\Delta^{\mathrm{op}}})$ (*i.e.* , the nerve of the action $\mathrm{GL}_n \times_S \mathbf{P}_S^{n-1} \to \mathbf{P}_S^{n-1}$). [18] One can also prove a Projective Bundle Formula in this case and produce universal Chern classes

[17] In general, it certainly fails. Notice that even $\mathrm{Pic}(X) \neq \mathrm{CH}^1(X)$ when $X$ is not smooth as the equivalence between Cartier divisors and Weil divisors fail.

[18] Actually, assuming $\mathbf{A}_S^1$-equivalence, $\mathbf{P}(E\,\mathrm{GL}_n)$ corresponds to the projectivisation of the universal bundle $E\,\mathrm{GL}_n$.

$$C_i \in \mathrm{CH}^i(B\,\mathrm{GL}_n, 0)$$

, where $\mathcal{Z}^i(B\,\mathrm{GL}_n, m) := \bigoplus_{m=r-l} \mathcal{Z}^i((B\,\mathrm{GL}_n)_l, r)$. Gillet in [Gil81], proves that those nontrivially define morphisms

$$C_i : K_m(X,Y) \longrightarrow \mathrm{CH}^i(X,Y,m)$$

, where $\mathrm{CH}^i(X,Y,m) := \pi_m(\mathrm{fib}(\mathcal{Z}^i(X,\bullet) \to \mathcal{Z}^i(Y,\bullet)))$, $X/k$ is smooth and $Y \hookrightarrow X$ is a closed immersion. More generally, they define morphisms

$$C_i : K_m^Z(X, Y_1, \cdots, Y_j) \longrightarrow (\mathrm{CH}^i)^Z(X, Y_1, \cdots, Y_j, m)$$

where $(-)^Z$ denotes taking $\pi_m$ with respect to $\Gamma_Z$ of the respective functors, $Y_i \hookrightarrow X$ regular closed subschemes intersecting transversally and $Z \hookrightarrow X$ a closed subscheme.

Now, one can construct a multiplicative Chern character

$$\mathrm{ch}_X : K_m^Z(X, Y_1, \cdots, Y_j) \longrightarrow \bigoplus_i (\mathrm{CH}^i)^Z(X, Y_1, \cdots, Y_j, m) \otimes_{\mathbf{Z}} \mathbf{Q}$$

and, for the sake of applying the results of [Gil81], it's convenient to replace the above with $\tau_X := \mathrm{Td}(X).\,\mathrm{ch}_X$ , which, by taking $j = 0$ and $Z \hookrightarrow X$ equidimensional, instantiates, by the Localisation Theorem (of K-theory and higher Chow goups), to a morphism

$$\tau : G_m(Z) \longrightarrow \bigoplus_i \mathrm{CH}^i(X,m) \otimes_{\mathbf{Z}} \mathbf{Q}$$

Bloch was, then, able to construct a cycle class morphism which inverted the above one , proving, then, the following theorem.

**Theorem 6.2.3** (Higher Grothendieck-Riemann-Roch, [Blo86, Thm. 9.1], [Lev94, Thm. 3.1], [Lev97, Cor. 8.2]). *Let $k$ be a field, $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ be locally equidimensional quasi-projective of finite type. There's an isomorphism*

$$\tau : G_n(X) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} \bigoplus_i \mathrm{CH}^i(X,n)$$

*, which restricts to an isomorphism*

$$G_n(X)_{(i)} \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathrm{Gr}^i_\gamma(G_n(X)) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} \mathrm{CH}^i(X,n)$$

*for very $m \in \omega$. In particular, when $X$ is regular, there are isomorphisms*

$$\mathrm{ch}_X : K_n(X) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} \bigoplus_i \mathrm{CH}^i(X,n)$$

*and*

$$K_n(X)^{(i)} \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathrm{Gr}^i_\gamma(K_n(X)) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} \mathrm{CH}^i(X,n).$$

*Furthermore, one can refine the last isomorphism to*

$$\begin{aligned} K_p(X)^{(q)}[\frac{1}{(d_X+p-1)!}] \cong \mathrm{Gr}^q_\gamma(K_p(X))[\frac{1}{(d_X+p-1)!}] \cong \\ \cong \mathrm{CH}^q(X,p)[\frac{1}{(d_X+p-1)!}] \end{aligned}$$

*for every $p,q \in \omega$, where $d_X$ is the dimension of $X$.*

*Remark* 6.2.6. Notice that [Blo86, Thm. 9.1] depends on the Localisation Theorem and the contravariant functoriality for morphisms with smooth codomain, both of which had erroneous proofs in [Blo86]. However, assuming the validity of such results, Bloch's proof still valid. As such results were later proved to be correct, the above result follows.

Levine, in [Lev94, Thm. 3.1] was able to obtain the above result without expliciting constructing Chern classes. Instead, he constructed directly a cycle class isomorphism using a cubical version of $\mathcal{Z}^\bullet(X,\bullet)$, which is isomorphic to the usual one.

*Remark* 6.2.7. The above isomorphism was originally using $\tau$ was originally given in [Blo86, Thm. 9.1] in order to use directly the results in [Gil81]. Still, it's harmless to omit $\mathrm{Td}(X)$ when $X$ is smooth since $\mathrm{Td}(X)$ is invertible in $\mathrm{CH}^\bullet(X)$.

By Beilinson's idea exposed in the previous section, one is immediately motivated to set

$$\mathrm{H}_p^{BM}(X,\mathbf{Z}(q)) := \mathrm{CH}_q(X,2p-q)$$

and, in particular, when $X$ is regular,

$$\mathrm{H}^p(X,\mathbf{Z}(q)) := \mathrm{CH}^q(X,p-2q).$$

A lot later, the above result was refined to an Atiyah-Hirzebruch's spectral sequence, usually called homotopy (co)niveau spectral sequence, which should degenerates to the above Higher Grothendieck- Riemann-Roch's Theorem [19]

**Theorem 6.2.4** (Homotopy Niveau/Coniveau Spectral Sequence). *Let $S$ be a regular Noetherian scheme of dimension at most $1$ and $X \in \mathrm{Ob}(\mathbf{Sch}_{/S})$ of finite type. There exists a spectral sequence*

$$E^2_{p,q} = \mathrm{H}_p^{BM}(X,\mathbf{Z}(\frac{-q}{2})) \Longrightarrow G_{p+q}(X).$$

*In particular, when $X$ is regular, there exists a spectral sequence*

$$E_2^{p,q} = \mathrm{H}^p(X,\mathbf{Z}(\frac{-q}{2})) \Longrightarrow K_{-p-q}(X).$$

*Proof.* See [BL95, Cor. 1.3.4] for the case $X = \mathrm{Spec}(k)$, where $k$ is a field. See [FS02, Thm. 13.6] for smooth schemes of finite type over a field and for equidimensional smooth quasi-projectives over a field. For the general case, see [Lev01, §8]. See also [Lev08, Thm. 6.4.2, Thm. 11.3.2] for an axiomatic approach which instantiates to the case of smooth schemes over a perfect field □

The Higher Grothendieck-Riemann-Roch's Theorem guarantees that

$$\mathrm{CH}^i(\mathrm{Spec}(k),n) \cong K_i(k)^{(n)} \otimes_{\mathbf{Z}} \mathbf{Q} \cong K_i^M(k) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

In fact, one can refine this result without killing the torsion.

**Theorem 6.2.5** ([NS89, Thm 4.9] [Tot92, Thm. 1]). *Let $k$ be a field. There exists an isomorphism*

[19] Apparently no one has proved until now that the homotopy niveau/coniveau spectral sequence, after applying $(-) \otimes_{\mathbf{Z}} \mathbf{Q}$, degenerates to $\tau$ or $\mathrm{ch}_X$.

$$K_n^M(k) = \mathrm{CH}^n(\mathrm{Spec}(k), n)$$

*for every $n \in \omega$.*

## § 6.3 Suslin's Algebraic Singular Homology

A second proposal for a motivic cohomology was given by Suslin. His idea presented in the 1987 conference on algebraic $K$-theory at Luminy was to use the Dold-Thom theorem in order to define a strictly algebraic version of the singular cohomology. The

Recall the Dold-Thom theorem.

**Theorem 6.3.1** (Dold-Thom). *For every $X \in \mathrm{Ob}(\mathbf{Top})$, let $\mathrm{Sym}^n(X) := X^n/\Sigma_n$. The free topological commutative monoid*

$$\mathrm{Sym}^\infty(X) := \mathrm{colim}_{n\in\omega}\, \mathrm{Sym}^n(X)$$

*satisfies*

$$\tilde{\mathrm{H}}_i^{\mathrm{sing}}(X, \mathbf{Z}) \cong \pi_i(\mathrm{Sym}^\infty(X)).$$

*for every $i \in \omega$.*

*Remark* 6.3.1. The Dold-Thom theorem can interpreted as the claim that abelianising either in the level of spaces or in the level of complexes (via the Dold-Kan correspondences) gives the same result after taking homotopy groups. In the level of spectra, the analogous correspondence is the free $H\mathbf{Z}$-module $H\mathbf{Z} \wedge \Sigma^\infty X_+$ generated by $X \in \mathrm{Ob}(\mathbf{Top})$.

Let $\mathbf{Z}[X]$ denotes the free topological abelian group. $\mathrm{Sym}^\infty(X) \hookrightarrow \mathbf{Z}[X]$ is a weak equivalence, as one can choose homotopy inverses in a connected topological monoid. In fact, $\mathrm{Sym}^\infty(X)$ has the weak homotopy type of a product of Eilenberg-Mclane spaces, so that it's actually an strict $\infty$-groupoid. Since it is abelian, by delooping, one can be endow $\mathrm{Sym}^\infty(X)$ with a $E_\infty$-space structure which is strictly associative.

For the unreduced cohomology, one must, instead, consider $X_+$ since $\mathrm{H}(X) \cong \tilde{\mathrm{H}}(X_+)$. In this case, one obtains

$$\mathrm{Sym}^\infty(X_+) \cong \coprod_{d\in\omega} \mathrm{Sym}^d(X).$$

Taking into account the above remark, one can, instead, use the group completion $\Omega B(\mathrm{Sym}^\infty(X_+))$. In analogy with the complex $K$-theory spectrum, one can even obtain

$$\Omega B(\coprod_{d\in\omega} \mathrm{Sym}^d(X)) \cong \mathbf{Z} \times \mathrm{colim}_{n\in\omega}\, \Omega B(\mathrm{Sym}^d(X)) \cong \mathbf{Z} \times \Omega B(Sym^\infty(X))$$

That last identity will, however, not be relevant here.

Rewriting the Dold-Thom theorem in $\widehat{\Delta}$ for the unreduced case leads to $\Omega B(\mathrm{Sing}(\coprod_{d\in\omega} \mathrm{Sym}^d(X))) \cong \Omega B(\coprod_{d\in\omega} \mathrm{Sing}(\mathrm{Sym}^d(X)))$ [20].

Let $(-)^+$ denotes the group completion. One, then, obtains

$$\pi_i((\coprod_{d\in\omega} \mathrm{Sing}(\mathrm{Sym}^d(X)))^+) \cong \mathrm{H}_i^{\mathrm{sing}}(X).$$

If $X \in \mathbf{Sch}_{/S}$, one can define the singular complex functor $\mathrm{Sing} : \mathbf{Sch}_{/S} \longrightarrow \widehat{\Delta}$ analogously by setting $\mathrm{Sing}(X)_n = \mathbf{Sch}_{/S}(\Delta^n_S, X)$, where

$$\Delta^n_S = \mathrm{Spec}(\mathbf{Z}[x_0, \ldots, x_n]/(\sum_i x_i - 1)) \times S$$

. One is, forthwith, led to the following definition.

**Definition 6.3.1.** Let $k$ be a field, $X \in \mathbf{Sch}_k$ of finite type. The **algebraic singular cohomology of $X$** is defined as

$$\mathrm{H}_i^{\mathrm{sing}}(X) := \pi_i((\coprod_{d\in\omega} \mathrm{Sing}(\mathrm{Sym}^d(X)))^+).$$

*Example* 6.3.1. Let $S \in \mathrm{Ob}(\mathbf{Sch})$. Notice that, for every $n \in \omega$,

[20]Recall that $\mathrm{Sing} : \mathbf{Top} \longrightarrow \widehat{\Delta}$ is the functor given on objects by $\mathrm{Sing}(X) = \mathbf{Top}(\Delta^n_{top}, X)$, *i.e.* , it's the right adjoint (the nerve in the adjunction realisation/nerve) to the left Kan extension of $[n] \mapsto \Delta^n_{top}$ along the Yoneda embedding.

$$\mathrm{Sym}^n((\mathbf{G}_m)_S) \cong \mathbf{A}_S^{n-1} \times_S (\mathbf{G}_m)_S \cong \mathrm{Spec}(\mathcal{O}_S[x_1, \dots, x_n, t]/(x_n t - 1)).$$

In particular, for $S = \mathrm{Spec}(\mathbf{C})$

$$\mathrm{H}_i^{\mathrm{sing}}(X, \mathbf{Z}) \cong \begin{cases} \mathbf{Z} \oplus \mathbf{C}^\times & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

The previous example shows that the natural request for an isomorphism

$$\mathrm{H}_i^{\mathrm{sing}}(X, \mathbf{Z}) \cong \mathrm{H}_i^{\mathrm{sing}}(X(\mathbf{C}), \mathbf{Z})$$

, where the right side denotes the ordinary (topological) singular homology, fails miserably. Still, as expected from observing the good behaviour of étale cohomology for torsion sheaves, one can obtain the following.

**Theorem 6.3.2** ([SV96, Cor 7.8]). *Let $k$ be an algebraically closed field, $X \in$ $\mathrm{Ob}(\mathbf{Sch}_k)$ be separated of finite type and $n$ coprime to* $\mathrm{char}(k)$. *If* $\mathrm{char}(k) = 0$ *or $k$ admits resolution of singularities, then there's an isomorphism*

$$\mathrm{H}^\bullet_{\mathrm{sing}}(X, \mathbf{Z}/n) \cong \mathrm{H}^\bullet_{qfh}(X, \mathbf{Z}/n) \cong \mathrm{H}^\bullet_{\mathrm{\acute{e}t}}(X, \mathbf{Z}/n).$$

[21]

And, then, by applying the Artin's Comparison Isomorphism, also, the following.

**Theorem 6.3.3** ([SV96, Thm. 8.3]). *Let $X \in \mathrm{Ob}(\mathbf{Sch}_{\mathbf{C}})$ be separated of finite type. The canonical morphism of sites $\varepsilon : X(\mathbf{C})_{\mathrm{\acute{e}t}} \longrightarrow X_{\mathrm{\acute{e}t}}$ induces isomorphisms*

$$\mathrm{H}_\bullet^{\mathrm{sing}}(X, \mathbf{Z}/n) \xrightarrow{\sim} \mathrm{H}_\bullet^{\mathrm{sing}}(X(\mathbf{C}), \mathbf{Z}/n)$$

*and*

$$\mathrm{H}^\bullet_{\mathrm{sing}}(X, \mathbf{Z}/n) \xrightarrow{\sim} \mathrm{H}^\bullet_{\mathrm{sing}}(X(\mathbf{C}), \mathbf{Z}/n)$$

[21] The $qfh$ topology is the topology generated by quasi-finite universal topological epimorphisms. A topological epimorphism is a quotient morphism in the category of topological spaces. Notice that $qfh$ is finer than the étale topology.

Let $k$ be a field, $S, X \in \mathrm{Ob}(\mathbf{Sch}_k)$ of finite type and $X \longrightarrow S \in \mathbf{Sch}_k(X, S)$. Let

$$c_{equi}(X, 0) : \mathbf{Nor}_k^{\mathrm{op}} \longrightarrow \mathbf{Ab}$$

22

be the presheaf defined as the subobject $c_{equi}(X,0)(Y) \hookrightarrow \mathcal{Z}^{\bullet}(Y \times_k X)$ generated by integral closed subschemes $Z \hookrightarrow Y \times_k X$ such that the projection $(\pi_Y)_{|Z}$ is finite and surjective onto an irreducible component of $Y$, where $\mathbf{Nor}_k$ denotes the category of normal connected schemes over $k$.

The functor $c_{equi}(X, 0)$ enjoys a special property of having transfers, which is related intimately with the $qfh$-topology.

**Definition 6.3.2.** A **topological epimorphism** is a morphism $f : X \longrightarrow Y \in \mathbb{T}(X, Y)$ such that $f$ is surjective and $Y$ has the quotient topology (*i.e.* , the coarsest topology making $f$ continuous). A morphism $f : X \longrightarrow Y \in \mathbf{Sch}_{/S}(X, Y)$ is a **topological epimorphism** if the induced morphism of topological spaces is a topological epimorphism.

Let $S$ be a scheme. The ***h*-topology on $S$** is the Grothendieck pretopology on $\mathbf{Sch}_{/S}$ generated by finite families $\{p_U : U \longrightarrow X\}_{U \in \mathcal{U}}$ of $S$-morphisms of finite type such that $\coprod_{U \in \mathcal{U}} p_U$ topological epimorphisms. Explicitly, a morphism $U \longrightarrow X \in \mathbf{Sch}_{/S}(U, X)$ is a universal topological epimorphism if, for every $Z \longrightarrow X \in \mathbf{Sch}_{/S}(Z, X)$, the induced morphism

$$Z \times_X U \longrightarrow Z \in \mathbf{Sch}_{/S}(Z \times_X U, Z)$$

is a topological epimorphism. Such coverings will be also called ***h*-coverings**.

Let $S$ be a scheme. The ***qfh*-topology on $S$** is the Grothendieck pretopology on $\mathbf{Sch}_{/S}$ generated by quasi-finite $h$-coverings. Such coverings will be also called ***qfh*-coverings**.

*Remark* 6.3.2. Since every étale covering is a quasi-finite topological epimorphism and being quasi-finite étale is preserved by base-change. There are morphisms of sites

[22] The apparently weird notation will be justifies in the next section.

$$(\mathbf{Sch}_{/S})_h \xrightarrow{\alpha} (\mathbf{Sch}_{/S})_{qfh} \xrightarrow{\beta} (\mathbf{Sch}_{/S})_{\text{ét}}.$$

The following are fundamental properties of the topologies $h$ and $qfh$.

**Theorem 6.3.4** (Voevodsky, [SV96, Appendix]). *Let $S \in \mathrm{Ob}(\mathbf{Sch})$ be Noetherian and $A \in \mathrm{Ob}(\mathscr{Ab})$. The constant Zariski sheaf $A$ is a $h$-sheaf and, hence, also a $qfh$-sheaf. Furthermore, $\beta_*(A) \cong A$ and $R^i\beta_*(A) \cong 0$ for every $i > 0$.*

*If $S$ is Noetherian excellent, $(\beta\alpha)_*(\mathbf{Z}/n) \cong \mathbf{Z}/n$ and $R^i(\beta\alpha)_*(\mathbf{Z}/n) \cong 0$ for every $n, i > 0$.*

*If $S$ is Noetherian, $\alpha_*(\mathbf{Z}/n) \cong \mathbf{Z}/n$ and $R^i\alpha_*(\mathbf{Z}/n) \cong 0$ for every $n, i > 0$.*

*In particular, there are isomorphisms*

$$\mathrm{Ext}_{\text{ét}}(\mathscr{F}, \mathbf{Z}/n) \cong \mathrm{Ext}_{qfh}(\beta^*\mathscr{F}, \mathbf{Z}/n)$$

*and*

$$\mathrm{Ext}_{qfh}(\mathscr{G}, \mathbf{Z}/n) \cong \mathrm{Ext}_h(\alpha^*\mathscr{G}, \mathbf{Z}/n)$$

*for every $\mathscr{F} \in \mathbf{Sch}((\mathbf{Sch}_{/S})_{\text{ét}})$, $\mathscr{G} \in \mathbf{Sch}((\mathbf{Sch}_{/S})_{qfh})$ and $n, i > 0$.*

**Definition 6.3.3.** Let $S \in \mathrm{Ob}(\mathbf{Sch})$ and $p_U : U \longrightarrow X_{U \in \mathscr{U}}$ an $h$-covering in $(\mathbf{Sch}_{/S})_h$. $p_U : U \longrightarrow X_{U \in \mathscr{U}}$ is in **normal form** if each $p_U$ can factorised as

$$U = \coprod_{V \in \mathscr{V}} V \to W \twoheadrightarrow X_Z \to X$$

such that the morphisms are, respectively, a Zariski covering, a finite surjective morphism and a Blow up along a closed subscheme $Z \hookrightarrow X$.

**Theorem 6.3.5.** *Every $h$-covering of a reduced Noetherian excellent scheme admits a refinement to a normal form.*

The main property of $qfh$-sheaves is that they admit transfers morphisms

$$\mathrm{Tr}_{X/S} : \mathscr{F}(X) \longrightarrow \mathscr{F}(S)$$

for finite surjective morphisms $p : X \longrightarrow S$ such that $X$ is integral and $S$ is normal. Such morphisms should have $p^*$ as sections and should compatibly descend cycles on $X$ which are finite and surjective onto an irreducible component of $S$, *i.e.* , they should somehow define make the sheaf enriched over some relative finite correspondences.

The name *transfer* and the notation Tr were suggestively adopted from Galois theory. For instance, if $p$ is a Galois covering with finite group $G$ acting on $X$, one can define transfers by

$$\mathrm{Tr}_{X/S}(a) = \sum_{g\in G} g^*(a)$$

, which by Galois descent, *i.e.* , $\mathscr{F}(X)^G \cong \mathscr{F}(S)$, and the invariance og $\sum_{g\in G} g^*(a)$ implies that it descends to $\mathscr{F}(S)$. In general, one must factorise by the normalisation $Y \longrightarrow S$ of $p$ in a normal extension $L/R(S)$ such that $R(X) \hookrightarrow L$ and define Tr as the above for $G = \mathrm{Gal}(R(Y)/R(S))$. Then one can take into account the inseparable part of the extension $R(X)/R(S)$ in order to descend the sections more.

**Theorem 6.3.6** ([SV96, Thm. 6.7]). *Let $k$ be a field such that* $\mathrm{char}(k) = p$ *and $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ separated. There exists a unique extension of $c_{equi}(X,0)$ to a $qfh$-sheaf such that there's an isomorphism*

$$\mathbf{Z}[X]_{qfh}[\frac{1}{p}] \xrightarrow{\sim} c_{equi}(X,0)[\frac{1}{p}]$$

*given by the $f \mapsto \Gamma_f$, where $p = 1$ when* $\mathrm{char}(k) = 0$ *are $\mathbf{Z}[X]_{qfh}$ denotes the $qfh$-sheaffication of the presheaf $\mathbf{Z}[h_X]$.*

*Remark* 6.3.3. Notice that the above theorem implies that $c_{equi}(X,0)[\frac{1}{p}]$ extends uniquely to a sheaf with transfers.

Now, one can define the Suslin complex of the presheaf $c_{equi}(X,0)$ as the simplicial abelian group as

$$L(X)_k := L_{\mathbf{A}^1}(c_{equi}(X,0))_k := c_{equi}(X,0)(\bullet \times \Delta_k).$$

23

[23] That is simply the $L_{\mathbf{A}^1}$-localisation which will be defined in the next section and next chapter.

**Theorem 6.3.7.** *Let $k$ be a field such that $\mathrm{char}(k) = p$ and $S \in \mathrm{Ob}(\mathbf{Sch}_k)$ be a normal connected scheme and $Z \in \mathrm{Ob}(\mathbf{Sch}_k)$ of finite type such that any finite subset is contained in an affine open set. There exists an isomorphism*

$$c_{equi}^{eff}(Z,0)(S)[\frac{1}{p}] \xrightarrow{\sim} \mathbf{Sch}_k(S, \coprod_{d \in \omega} \mathrm{Sym}^d(Z))[\frac{1}{p}].$$

*In particular, $L(Z)[\frac{1}{p}] \cong (\coprod_{d\in\omega}(\mathrm{Sym}^d(\mathrm{Sing}(Z))))^+[\frac{1}{p}]$ and, since $\Delta_k^n$ satisfies the assumptions of $S$ for any $n$, there exists an isomorphism*

$$\mathrm{H}_i^{\mathrm{sing}}(Z, \mathbf{Z}/m) \cong \mathrm{H}^i(L(Z) \otimes \mathbf{Z}/m)$$

*for every $m$ coprime to $p$*

**Corollary 6.3.1.**

$$\mathrm{H}_i^{\mathrm{sing}}(X, \mathbf{Z}/n) \cong \mathrm{H}_{qfh}(X, \mathbf{Z}/m)$$

*for every $i \in \omega$, $m$ coprime to $\mathrm{char}(k)$ and $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ of finite type such that any finite subset is contained in an affine open set.*

One can also define a non-proper version of $c_{equi}(X,0)$. Let $k$ be a field and $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ of finite type. Let

$$z_{equi}(X,0) : \mathbf{Nor}_k^{\mathrm{op}} \longrightarrow \mathbf{Ab}$$

[24]

be the presheaf defined as the subobject $z_{equi}(X,0)(Y) \hookrightarrow \mathcal{Z}^\bullet(Y \times_k X)$ generated by integral closed subschemes $Z \hookrightarrow Y \times_k X$ such that the projection $(\pi_Y)_{|Z}$ is quasi-finite and surjective onto an irreducible component of $Y$ (*i.e.* , the projection does not need to be proper; only quasi-finite), where $\mathbf{Nor}_k$ denotes the category of connected normal schemes over $k$.

Now, analogously ,let

$$L^{BM}(X)_k := L_{\mathbf{A}^1}(z_{equi}(X,0))_k := z_{equi}(X,0)(\bullet \times \Delta_k)$$

The following definition is non-standard.

[24]The apparently weird notation, again, will be justifies in the next section.

**Definition 6.3.4.** Let $k$ be a field and $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ of finite type and $A \in \mathrm{Ob}(Ab)$. The **Suslin's algebraic singular Borel-Moore homology** is defined as

$$\mathrm{H}_i^{\mathrm{sing}}{}_{BM}(X,A) := \mathrm{H}_i(L^{BM}(X) \otimes_{\mathbf{Z}} A).$$

As in the case $c_{equi}(X,0)$, there's an analogous extension to a $qfh$-sheaf which computes the algebraic Borel-Moore singular homology. Indeed, any sheaf with transfer satisfies that $\mathrm{Ext}_{qfh}$ and $\mathrm{Tor}_{qfh}$ computes, respectively, the homology of $L_{\mathbf{A}^1}(-) \otimes^L (-)$ and cohomology of $RHom(L_{\mathbf{A}^1}(-),(-))$ whenever the cofficients are $\mathbf{Z}/n$ for $n$ coprime to $\mathrm{char}(k)$.

**Theorem 6.3.8** (Suslin-Voevodsky). *Let $k$ be a field such that $\mathrm{char}(k) = p$ and $X \in \mathrm{Ob}(\mathbf{Sch}_k)$. There exists a unique extension of $z_{equi}(X,0)[\frac{1}{p}]$ to a $qfh$-sheaf, where $p = 1$ if $\mathrm{char}(k) = 0$. Furthermore*

$$\mathrm{H}_i^{\mathrm{sing}}{}_{BM}(X,\mathbf{Z}/n) := \mathrm{H}_i(z_{equi}(X,0) \otimes_{\mathbf{Z}} \mathbf{Z}/m)$$

Let $X/k$ be equidimensional of dimension $d_X$. Notice, that there's a canonical morphism

$$L^{BM}(X)_\bullet \longrightarrow \mathcal{Z}^{d_X}(X,\bullet)$$

since every integral subscheme $W \hookrightarrow X \times \Delta^n_k$ which is quasi finite over $\Delta^k_n$ must intersect it properly.

Suslin proved that it induces an quasi-isomorphism under suitable suppositions.

**Theorem 6.3.9** (Suslin, [Sus00, Thm. 3.2]). *Let $k$ be a field and $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ an equidimensional scheme of dimension $d_X$. If $\mathrm{char}(k) = 0$ and $X$ is quasiprojective, there's a weak equivalence*

$$L^{BM}(X)_\bullet \xrightarrow{\sim} \mathcal{Z}^{d_X}(X,\bullet).$$

*In particular, there's an isomorphism*

$$\mathrm{H}_\bullet^{\mathrm{sing}}{}_{BM}(X,\mathbf{Z}/m) \cong \mathrm{CH}^{d_X}(X,\bullet) \otimes \mathbf{Z}/m$$

*for every m.*

*If* $\mathrm{char}(k) = p \neq 0$ *and X is affine, the above morphisms are also a quasi-isomorphism and isomorphism respectively whenever m is coprime to p.*

*Proof.* The first case follows from [Sus00, Thm. 3.2]. The affine case seems to be unpublished, but is mentioned in notes of Levine. □

Using the previous theorems, including the compatibility of $qfh$ and ét, Suslin was able to prove the following

**Theorem 6.3.10** (Suslin, [Sus00, Thm. 4.2]). *Let k be a field and* $X \in \mathrm{Ob}(\mathbf{Sch}_k)$ *an equidimensional scheme of dimension* $d_X$. *If* $\mathrm{char}(k) = 0$ *and X is quasiprojective. There's an isomorphism*

$$\mathrm{CH}^i(X, n, \mathbf{Z}/m) \cong \mathrm{H}^{2(d_X - i)+n}_{c,\text{ét}}(X, \mathbf{Z}/m(d-i))^\vee.$$

*If* $\mathrm{char}(k) = p \neq 0$ *and X is affine, the above morphisms is an isomorphism whenever m is coprime to p.*

*Proof.* Only the first case follows from [Sus00, Thm. 4.2]. Again, the affine case semms to be unpublished. □

## § 6.4 Voevodsky's Idea

The construction (or maybe a sketch) of the derived category of motives $DM_\tau(S)$ was given in Voevodsky's Ph.D. thesis ([Voe92, Def. 4.1]). Given a site $(\mathcal{C}, \tau)$ with an internal interval $I \in \mathrm{Ob}(\mathcal{C})$, one can consider the category $D^b(\mathbf{Sh}_\tau)[S_I^{-1}]$, where $S_I$ is the strongly saturated class generated by the projections $X \times I \twoheadrightarrow X$ for every $X \in \mathrm{Ob}(\mathcal{C})$. There are localisation functors

$$D^b(\widehat{\mathcal{C}}) \xrightarrow{L_\tau} D^b(\mathbf{Sh}_\tau) \xrightarrow{L_I} D^b(\mathbf{Sh}_\tau)[S_I^{-1}]$$

with right adjoints given by including the local objects. The $S_I$-local objects, in particular, are complexes of sheaves $F$ that satisfies $F(X \times I) \cong F(X)$.

Voevodsky, in his thesis, applied the above construction for $\mathcal{C} = \mathbf{Sch}_{/S}$ and $\tau = h, qfh$, which are topologies introduced by him therein [25]. His thesis, despite containing the main ideas for a theory of the derived categories of mixed motives, also contained several minor mistakes that were corrected in later articles.

As in the case of pure motives $\mathbf{Mot}_{\sim}(k, F)$, one must consider correspondences. In the relative case, it will be considered instead of mere $\mathrm{H}^{\bullet}_{W}$, sheafs of the form $Rf_*F$ for some coefficient $F$ and $f : X \longrightarrow S$. As the main aim consists in constructing a derived category of mixed motives, as opposed to an abelian category of mixed motives (which is a lot harder), one should consider complexes of sheaves that have cohomology of the form $Rf_*F$.

The main idea of Voevodsky was to consider, instead of arbitrary correspondences, only the finite corresponences, *i.e.* , multivalued functions. In fact, such restriction suffice and, as known for centuries, the theory of multivalued functions is very well behaved (*e.g.* , in the case of smooth projective curves over an algebraically closed field of characteristic 0, they correspond to finite coverings that are étale outside a finite number of points). Indeed, the composition of correspondences requires some moving lemma for the respective adequate equivalence relation. That limits all the theory to only smooth schemes (which usually have a good intersection theory). Now, when restricting to finite correspondences, the composition is trivial and does not relies on any kind moving lemma, which would usually require smoothness.

**Definition 6.4.1.** Let $k$ be a field and $X, Y \in \mathrm{Ob}(\mathbf{Sch}_k)$. The **finite correspondences between $X$ and $Y$** is the abelian subgroup $c(X, Y) \hookrightarrow \mathcal{Z}(X, Y)$ generated by integral closed subschemes $Z \hookrightarrow X \times_k Y$ such that the projection $\pi_{X|Z}$ is fiinite and surjective onto an irreducible component.

Let $\Lambda$ be a commutative ring. The **category of finite correspondences with coefficients in** $\Lambda$ is defined as the category $\mathbf{Corr}_{fin}(k, \Lambda)$ such that $\mathrm{Ob}(\mathbf{Corr}_{fin}(k, \Lambda)) := \mathrm{Ob}(\mathbf{Sm}_k)$ and $\mathbf{Corr}_{fin}(k, \Lambda)(X, Y) := c(X, Y) \otimes_{\mathbf{Z}} \Lambda$.

The **category of effective geometric motives with coefficients in** $\Lambda$ is defined as the idempotent completion $DM^{eff}_{gm}(k, \Lambda) := (K^b(\mathbf{Corr}_{fin}(k, \Lambda))[S^{-1}])^{\#}$,

[25] The $h$-topology consists of coverings given by universal topological epimorphisms, where a topological epimorphism is defined as an quotient morphism in the category of topological spaces. Similarly, the $qfh$-topology is defined by the quasi-finite $h$-coverings. Later a better version, the $cdh$-topology, will be introduced.

where $S$ is the strongly saturated class generated by the following

1. ($\mathbf{A}^1_k$-invariance) $[X \times_k \mathbf{A}^1_k] \longrightarrow [X]$ for every $X \in \mathrm{Ob}(\mathbf{Sch}_k)$;
2. (Mayer-Vietoris) $[U \cap V] \overset{j_U \otimes -j_V}{\rightarrow} [U] \otimes [V] \overset{i_U \otimes -i_V}{\rightarrow} [X]$ for every Zariski covering $\{U,V\}$ of $X \in \mathrm{Ob}(\mathbf{Sch}_k)$, *i.e.* , it's the complex induced by the Čech complex of the Zariski covering $\{U,V\}$ .

Notice that there exists a morphism

$$M^{eff}_{gm} : \mathbf{Sm}_k \longrightarrow DM^{eff}_{gm}(k,\Lambda)$$

given by $M(X) = [X]$ and $M(f) = \Gamma_f$.

**Proposition 6.4.1** ([Voe00, Prop. 2.1.4, Cor. 4.2.6]). *Let $k$ be a field, $F$ a field of characteristic $0$ and $\mathbf{Mot}^{eff}_{rat}(k,F)$ denotes the opposite of the (co)homological category of effective pure motives with coefficients in $F$. There exists a morphism*

$$M^{eff}_{rat}(k,F) \longrightarrow DM^{eff}_{gm}(k,F)$$

*defined by the complex concentrated in degree $0$ such that $M(X) = M^{eff}_{gm}(X)$ for every $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$.*

*Furthermore, if $k$ admits resolution of singularities (e.g. , $chara(k) = 0$), the above functor is fully faithful and any triangle in the essential image objectwise of the form $M(X)$ splits.*

*Proof.* That follows by the non-trivial fact that the cokernel of the morphism

$$c(X \times_k \mathbf{A}^1_k, Y) \longrightarrow c(X,Y)$$

defined on generators by $Z(1) - Z(0)$ for each $Z(t) \hookrightarrow X \times_k \{t\} \times_k Y$ coincides with $\mathrm{CH}(X \times_k Y)$ for every $X, Y \in \mathrm{Ob}(\mathbf{SmProj}_k)$. Now, $\mathbf{A}^1_k$-homotopy invariance implies that the morphism

$$c(X,Y) \longrightarrow DM^{eff}_{gm}(k,F)(M^{eff}_{gm}(X), M^{eff}_{gm}(Y))$$

factors through the above mentioned cokernel.

The second assertion follows from [Voe00, Cor . 4.2.6]. □

For every $[X] \in K^b(\mathbf{Corr}_{fin}(k,\Lambda))$, let

$$[\tilde{X}] := \mathrm{fib}([X] \longrightarrow [\mathrm{Spec}(k)])[-1]$$

. Analogously to the pure case, if there exists a point $x : X(k)$, the induced morphism $x_* : [\mathrm{Spec}(k)] \longrightarrow [X]$ splits $[X]$ as

$$[X] \cong [\tilde{X}] \otimes [\mathrm{Spec}(k)]$$

in $K^b(\mathbf{Corr}_{fin}(k,\Lambda))$ since $f_*x_* = 1_{[\mathrm{Spec}(k)]}$, where $f : X \longrightarrow \mathrm{Spec}(k)$ is the structural morphism. In particular, if $\Lambda(1) := M_{gm}^{\tilde{eff}}(\mathbf{P}^1_k)[-2]$, there's an isomorphism
$M_{gm}^{eff}(\mathbf{P}^1_k) \cong \Lambda \oplus \Lambda(1)[2]$

**Definition 6.4.2.** Let $k$ be a field and $\Lambda$ be a commutative ring. The **category of geometric motives with coefficients in** $\Lambda$ is defined as the Spanier-Whitehead category of $DM_{gm}^{eff}(k,\Lambda)$. Explicitly, $DM_{gm}(k,\Lambda)$ is defined as the category such that $\mathrm{Ob}(DM_{gm}(k,\Lambda)) := \mathbf{Z} \times \mathrm{Ob}(DM_{gm}^{eff}(k,\Lambda))$ and, for each $M(p), N(q) \in \mathrm{Ob}(DM_{gm}(k,\Lambda))$,

$$DM_{gm}(k,\Lambda)(M(p),N(q)) := \mathrm{colim}_{n\in\omega} DM_{gm}^{eff}(k,\Lambda)(M\otimes\Lambda(p+n),N\otimes\Lambda(q+n))$$

*Remark* 6.4.1. The category $DM_{gm}(k,\Lambda)$ is again stable by noticing that the braiding $B_{\Lambda(1),\Lambda(1)}$ is isomorphic to the $1_{\mathbf{Z}(1)}$ (for details see the next chapter on the stable motivic category), which is implied by the above proposition.

**Definition 6.4.3.** Let $k$ be a field, $\Lambda$ a commutative ring and $X \in \mathrm{Ob}(\mathbf{Sm}_k)$. The **motivic cohomology of $X$ with coefficients in** $\Lambda$ is defined by

$$\mathrm{H}^p(X,\mathbf{Z}(q)) := DM_{gm}(k,\Lambda)(M_{gm}(X),\Lambda(q)[p])$$

.

**Definition 6.4.4.** Let $S \in \mathrm{Ob}(\mathbf{Sch})$, $k$ a field and $x \in S(k)$. A **fat point of $S$ over** $x$ is defined as a point $\tilde{x} \in S(A)$ for some $A$ discrete valuation ring with special point $\mathrm{Spec}(k)$ (*i.e.* , $A/\mathfrak{p}_A \cong k$ for the unique non-zero prime ideal $\mathfrak{p}_A$ of $A$) such that $\tilde{x} : \mathrm{Spec}(A) \longrightarrow S$ sends the generic point of $\mathrm{Spec}(A)$ to a generic point of $S$ (*i.e.* , to the generic point of some integral closed subscheme) and the triangle

$$\begin{array}{ccc} & & \mathrm{Spec}(k) \\ & \swarrow & \downarrow x \\ \mathrm{Spec}(A) & \xrightarrow{\tilde{x}} & S \end{array}$$

commutes.

The following will be required in order to pullback cycles to fat points without "degenerating" the fiber over the special point.

**Lemma 6.4.1** (Flat Extension). *Let $S \in \mathrm{Ob}(\mathbf{Sch})$, $X \longrightarrow S \in \mathrm{Ob}(\mathbf{Sch}_{/S})$, $A$ a discrete valuation ring, $\eta \hookrightarrow \mathrm{Spec}(A)$ the generic point and $f : \mathrm{Spec}(A) \longrightarrow S \in \mathrm{Ob}(\mathbf{Sch}_{/S})$. If $Z \hookrightarrow X$ is a closed immersion, then there exists a unique flat $\phi_f(Z) \in \mathrm{Ob}(\mathbf{Sch}_{/\mathrm{Spec}(A)})$ together with a closed immersion*

$$\phi_f(Z) \hookrightarrow Z \times_S \mathrm{Spec}(A)$$

*such that*

$$\phi_f(Z)_\eta \xrightarrow{\sim} (Z \times_S \mathrm{Spec}(A))_\eta = Z_\eta$$

*is an isomorphism.*

*Proof.* That follows from [EGAIV-2, Prop. 2.8.5]. Indeed, one can simply take the $\phi_f(Z) = \overline{Z_\eta}$, where $\overline{Z_\eta}$ denotes the scheme-theoretic image of the morphism

$$Z_\eta \to Z \times_S \mathrm{Spec}(A).$$

□

**Definition 6.4.5.** Let $S \in \mathrm{Ob}(\mathbf{Sch})$, $f : X \longrightarrow S \in \mathrm{Ob}(\mathbf{Sch}_{/S})$ and $\Lambda$. a commutative ring A **relative cycle of $X$ over $S$ with coefficients in $\Lambda$** is a locally finite formal sum $\alpha = \sum_i n_i \eta_i$ such that $n_i \in \Lambda$ and $\eta_i$ are generic points of $X$ satisfying the following

(i) $f(\eta_i) \in S$ is a generic point of $S$ for every $i$;

(ii) For every field $k$, $x \in S(k)$, $i$ and $\tilde{x}_1, \tilde{x}_2$ fat points over $x$,

$$\phi_{\tilde{x}_1}(Z_i)_s \cong \phi_{\tilde{x}_2}(Z_i)_s,$$

where $Z_i := \overline{\eta_i}$ and $s$ denotes the special point of the discrete valuation rings associated to $\tilde{x}_1$ and $\tilde{x}_2$.

A **relative cycle of $X$ over $S$ of dimension $r$ with coefficients in $\Lambda$** (*resp.*, **equidimensional relative cycle of $X$ over $S$ of dimension $r$ with coefficients in $\Lambda$** ; *resp.*, **proper relative cycle of $X$ over $S$ with coefficients in $\Lambda$**) is a locally finite formal sum $\alpha = \sum_i n_i \eta_i$ that is a relative cycle of $X$ over $S$ with coefficients in $\Lambda$ such that $\dim(Z_i) = r$ for every $i$ (*resp.*,$\mathrm{supp}(\alpha) \to S$ is equidimensional of dimension $r$ ; *resp.*, $\mathrm{supp}(\alpha) \to S$ is proper). The $\Lambda$-module of relative cycle of $X$ over $S$ of dimension $r$ with coefficients in $\Lambda$ (*resp.*, **equidimensional relative cycle of $X$ over $S$ of dimension $r$ with coefficients in $\Lambda$** ; *resp.*, **proper relative cycle of $X$ over $S$ with coefficients in $\Lambda$**) will be denoted by $\mathcal{Z}(X/S, r, \Lambda)$ (*resp.*, $\mathcal{Z}_{equi}(X/S, r, \Lambda)$; *resp.*, $\mathcal{Z}_{prop}(X/S, r, \Lambda)$). When $\Lambda = \mathbf{Z}$, it will be omitted.

*Remark* 6.4.2. Informally, a relative cycle over $X/S$ is simply a cycle over $X$ which lies above a cycle of $S$ and has a unique (and, therefore, well-defined) specialisation.

One of the main problems of relative cycles is the lack of explicit functoriality when fixing the coefficient ring $\Lambda$. Indeed, once one chooses $\Lambda = \mathbf{Z}$, the specialisations (*i.e.* , pullback to a fiber) may fail to have coefficients in $\mathbf{Z}$ whenever $S$ does not have closed points with residue fields of characteristic 0 ([SV00, Ex. 3.5.10]).

Sometimes even worser pathologies occur. The effective cycles $\mathcal{Z}^{eff}(X/S, r)$ (*i.e.* , such that each $n_i > 0$ for every $i$) does not always generate all the cycles when $S$ is not unibranch ([SV00, Ex. 3.4.7]). Furthermore, the support of a non effective cycle $\alpha = \sum_i n_i \eta_i \in \mathcal{Z}(X/S, r)$ may fail to have support of dimension $r$, *i.e.* , $\overline{\eta_i}/S$ may have different dimension from $\overline{\eta_j}/S$ ([SV00, Ex 3.1.9])

Recall that the pullback of $\alpha = \sum_i n_i \eta_i \in \mathcal{Z}(X/S, r, \Lambda)$ along $T \longrightarrow S$ i defined by $\alpha_T := \sum_i n_i [Z_i \times_S T]_{X \times_S T}$, where $Z_i = \overline{\eta_i}$ are flat over $S$ and $[Z]_Y := \sum n_i \ell(\mathcal{O}_Y, \eta_i)\eta_i$.

**Definition 6.4.6** ([CD12, Def. 8.1.17]). Let $p : \tilde{S} \longrightarrow S$ be a birational morphism and $C \hookrightarrow S$ a minimal closed subset such that the restriction $\tilde{S} \setminus (\tilde{S} \times_S C) \xrightarrow{\sim} S \setminus C$ is an isomorphism. Let $\alpha = n_i \eta_i \in \mathcal{Z}(S)$ and $Z_i = \overline{\eta_i}$. The **strict transform of** $Z_i$ **along** $p$ is defined as the schematic closure of the morphism $(Z_i \setminus (Z_i \times_S C)) \times_S \tilde{S} \longrightarrow X \times_S \tilde{S}$. Similarly, the **strict transform of** $\alpha$ **along** $p$ is defined as

$$\tilde{\alpha} := \sum_i n_i [Z_i]_{X \times_S \tilde{S}}.$$

**Theorem 6.4.1** (Gruson-Raynaud, "Platiffication",[SV00, Thm. 2.2.2]). *Let $p : \tilde{S} \longrightarrow S$ be a morphism of Noetherian schemes such that $p$ is flat over an open $U \hookrightarrow S$. There exists a closed $Z \hookrightarrow S$ disjoint of $U$ such that the proper transform of $\tilde{S}$ along the blow-up $S_Z \longrightarrow S$ is flat over $S_Z$.*

The above theorem allows to transform cycles while preserving flatness and other properties such as equidimensionality. In particular, it's intimately related to transfers of arithmetic importance such as the ones defined for qfh sheaves in the previous section.

**Definition 6.4.7** ([SV00, Lem .3.3.9]). Let $S \in \mathrm{Ob}(\mathbf{Sch})$, $f : X \longrightarrow S \in \mathrm{Ob}(\mathbf{Sch}_{/S})$ of finite type $r \in \omega$ and $\Lambda$. a commutative ring. The following notations will be used

$$z(X/S, r, \Lambda) \hookrightarrow \mathcal{Z}(X/S, r, \Lambda);$$

$$c(X/S, r, \Lambda) \hookrightarrow \mathcal{Z}_{prop}(X/S, r, \Lambda);$$

$$z_{equi}(X/S, r, \Lambda) \hookrightarrow \mathcal{Z}_{equi}(X/S, r, \Lambda);$$

$$c_{equi}(X/S, r, \Lambda) \hookrightarrow \mathcal{Z}^{prop}_{equi}(X/S, r, \Lambda).$$

in order to denote the respective subgroups of cycles $\alpha = \sum_i n_i \eta_i$ that satisfies one of the following equivalent assertions

1. For any $T/S$ Noetherian scheme, $\alpha_T \in \mathcal{Z}(X \times_S T/T, r, \Lambda)$;
2. For any point $s \in S$, $\alpha_s \in \mathcal{Z}(X_s, \Lambda)$;
3. For any point $s \in S$, there exists a separable extension $k/\kappa(s)$ such that $\alpha \otimes_{\kappa(s)} \in \mathcal{Z}(X \times_S \mathrm{Spec}(k), r, \Lambda)$.

**Definition 6.4.8.** Let $S$ be a Noetherian scheme of finite Krull dimension and $\Lambda$ a commutative ring. The **category of finite $S$-correspondences with coefficients in** $\Lambda$ is defined as the category $\mathbf{Corr}(S)$ such that $\mathrm{Ob}(\mathbf{Corr}(S)) := \mathrm{Ob}(\mathbf{Sm}^{ftsep}_{/S})$ and $\mathbf{Corr}(S)(X,Y) := c_{equi}(X \times_S Y/X, 0, \Lambda)$ for every $X, Y \in \mathrm{Ob}(\mathbf{Corr}(S))$, where $\mathbf{Sm}^{ftsep}_{/S}$ denotes the subcategory consisting of $S$-schemes separated of finite type.

**Definition 6.4.9.** Let $S$ be a Noetherian scheme of finite Krull dimension, $\Lambda$ a commutative ring and $\tau$ a topology on $\mathbf{Sm}^{ftsep}_{/S}$. The **category of $\tau$-sheaves of $\Lambda$-modules with transfers** is the subcategory $\mathbf{Sh}^{tr}_{\tau}(S) \hookrightarrow \mathbf{Sh}((\mathbf{Sm}^{ftsep}_{/S})_{\tau})$ consisting of sheaves $F$ tha are in the essential image of the forgetful functor by applying the forgetful functor $\widehat{\mathbf{Corr}(S)} \longrightarrow \widehat{\mathbf{Sm}^{ftsep}_{/S}}$.

**Definition 6.4.10.** Let $S \in \mathrm{Ob}(\mathbf{Sch})$ be a Noetherian scheme of finite Krull dimension and $\Lambda$ a commutative ring. The **derived category of effective mixed motives with coefficients in** $\Lambda$ is the $\Lambda$-linear stable $\infty$-category is defined as $DM^{eff}(S,\Lambda) := D_{\mathbf{A}^1}(\mathbf{Sh}^{tr}_{\mathrm{Nis}}(S,\Lambda))$, where $D_{\mathbf{A}^1}$ denotes the localisation with respect to the strongly saturated class $S_{\mathbf{A}^1}$ generated by the projections $F \times_S \mathbf{A}^1_S \longrightarrow F$.

Analogously, the **derived category of mixed motives with coefficients in** $\Lambda$ is the $\Lambda$-linear stable $\infty$-category is defined as $DM(S,\Lambda) := D_{\mathbf{A}^1}(\mathbf{Sh}^{tr}_{Nis}(S,\Lambda))$

The following theorem destroy any hope in recovering an abelian category of mixed motives with coefficients in $\mathbf{Z}$ over an arbitrary field $k$ from the philosophy of Deligne-Beilinson-Lichtenbaum.

**Definition 6.4.11.** Let $k$ be a field. A **reasonable $t$-structure on** $DM^{eff}_{gm}(k,\mathbf{Z})$ is a $t$-structure $\tau = (\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ such that

(i) $\mathscr{D}^{\leq 0}$ and $\mathscr{D}^{\geq 0}$ are closed under the Tate twist functors $(-)(m)$ for every $m \in \mathbf{Z}$;

(ii) For every affine $k$-scheme $X$ of dimension $n \in \omega$,

$$^{\tau}\,\mathrm{H}^{i}(M_{gm}(X)) \cong 0$$

for $i \notin [0, n]$ and

$$^{\tau}\,\mathrm{H}^{i}(M^{c}_{gm}(X)) \cong 0$$

for $i \notin [n, 2n]$.

**Theorem 6.4.2** (Voevodsky, [Voe00, Prop. 4.3.8]). *Let $k$ be a field. If there exists a conic $X/k$ that has no $k$-points, then $DM^{eff}_{gm}(k, \mathbf{Z})$ does not have a reasonable $t$-structure.*

In the case of $k$ of characteristic 0, Beilinson also proved that anyone is far from obtaining a category of mixed motives even with coefficients in $\mathbf{Q}$ since it becomes contingent on the truth of the Grothendieck's Standard Conjectures.

**Definition 6.4.12.** Let $k$ be a field of characteristic $p$. If $p \neq 0$, let $r = r_{\ell}$ for a prime $\ell \neq p$. If $p = 0$, let $r = r_i$ for every $i : k \hookrightarrow \mathbf{C}$ or $r = r_{\ell}$ for any prime $\ell$. A **motivic $t$-structure on** $DM_{gm}(k, F)$ is a $t$-structure $\mu$ such that $^{\mu}\,\mathrm{H}^{\bullet}(-) = \tau_{\geq \bullet}\tau_{\leq \bullet} \cong \tau_{\leq \bullet}\tau_{\geq \bullet}$ is conservative and, furthermore, the monoidal structure $\otimes$ and every $r$ are $t$-exact $t$-exact.

**Theorem 6.4.3** (Beilinson, [Bei10]). *Let $k$ be a field of characteristic $p$. If $p \neq 0$, let $r = r_{\ell}$ for a prime $\ell \neq p$. If $p = 0$, let $r = r_i$ for every $i : k \hookrightarrow \mathbf{C}$ or $r = r_{\ell}$ for any prime $\ell$. Let, also, $\mu$ denotes a motivic $t$-structure and $\mathscr{MM}_k := DM_{gm}(k, \mathbf{Q})^{\heartsuit}$. The restriction $r$ of the heart*

$$r : \mathscr{MM}_k \longrightarrow \mathbf{Vect}_F$$

*is conservative and induces the structure of a Tannakian $F$-category.*

*Furthermore, if $\mu$ exists, the conjectures of type $A(X)(a)$ (and, consequently, $C(X)$) are true. Moreover, let $n \in \omega$ and $W_n(DM_{gm}(k,\mathbf{Q})) \hookrightarrow DM^{gm}(k,\mathbf{Q})$ be defined as the subcategory generated under extensions and colimits by $M(X)(m)$ for every $X$ smooth $k$-variety such that $d_X \leq n$. That defines an increasing filtration. If $p = 0$, the motivic $t$-structure $\mu$ is the unique $t$-structure compatible with the filtration $W_\bullet(DM^{gm}(k,\mathbf{Q}))$ such that, for every $n > 0$,*

$$(W_{n+1}(DM_{gm}(k,\mathbf{Q}))/W_n(DM_{gm}(k,\mathbf{Q})))^{\heartsuit}$$

*contains all objects $M(X)[n+1]$ such that $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$.*

*In addition, if $p = 0$ and $\mu$ exists, the above filtration, induces a filtration on every object $M \in \mathrm{Ob}(\mathscr{MM}_k)$ such that, if $M$ is irreducible (and, hence, simple by the semi-simplicity mentioned above), $W_n(M) = M$ and $W_{n-1}(M) = 0$ iff $M \cong {}^{\mu}\mathrm{H}^i(M(X)(m))$ for some $X \in \mathrm{Ob}(\mathbf{SmProj}_k)$ and $i, m$ satisfying $n = i - 2m$.*

*Furthermore, if $p = 0$ and $\mu$ exists, then $\mathscr{MM}_k$ is semi-simple and all Grothendieck's Standard Conjectures are true.*

*Remark* 6.4.3. In summary, the above theorem states that the motivic $t$-structure $\mu$ is the unique $t$-structure compatible with the weight filtration for every realisation and that any such $t$-structure induces conservative $\ell$-adic and Betti realisations. Furthermore, the existence of such $\mu$ guarantees all the Grothendieck's Standard Conjectures. In particular, the subcategory of pure motives will be equivalent to $\mathbf{Mot}_{num}(k,\mathbf{Q}) \cong \mathbf{Mot}_{hom}(k,\mathbf{Q})$ and, by semi-simplicity, the motivic Galois groups of the Betti realisation will be pro-reductive. [26].

In summary, the field of motives still largely open and conjectural. Only particular cases of the abelian category of mixed motives have been produced (*e.g.* , mixed Tate motives). Still, even the existence of an abelian category of effective geometric Artin motives (*i.e.* , motives generated by 0-dimensional schemes) with coefficients in $\mathbf{Z}$ is widely open. The mysticism, therefore, shall remain.

[26]The categories of representation on an affine group scheme $G$ over a field $k$ is a neutral Tannakian category such that it's semi-simple iff $G$ is pro-reductive

# Chapter 7

# Motivic Homotopy Theory

Along the development of the derived category of mixed motives by Voevodsky Suslin and Friedlander during 1992-1995, an homotopy theory was conspicuously being elicited. Such homotopy theory was so naive and yet momentously important. It simply and straightforwardly copied all the definitions arising in ordinary homotopy theory by inserting the algebraic analogous input. That homotopy theory was used for the first time in its stable version in [Voe96] by means of algebraic cobordism in order to solve the Milnor's Conjecture. A systematic study was, however, only published later, in 1999, by Morel and Voevodsky in [MV99] despite its citation in [Voe96]. Earlier, nevertheless, at the International Congress of Mathematicians in 1998,[Voe98], Voevodsky presented an overview about their new theory. In [Voe98], Voevodsky attributed the creation of the unstable version to Morel.

The genesis of motivic homotopy theory, however, explicitly started earlier. Morel, when writing his Habilitation, was interested in defining a $\mathbf{A}^1_k$-homotopy theory $\mathcal{H}(k)$ for smooth schemes over a field $k$ such that

$$\mathcal{H}(k)(X, B\,\mathrm{GL})$$

Recall that ordinary homotopy theory starts with the $\infty$-category of spaces or $\infty$-groupoids $\mathbf{Grpd}_\infty$. In order to create generalised cohomology theories, one can stabilise such category to obtain the category of spectra $\mathbf{Sp}$.

$$\mathbf{Sp} \underset{\Omega_\infty}{\overset{\Sigma_\infty}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Grpd}_\infty$$

Proceeding further, one can construct canonically, for any ring $A$, the Eilenberg-Maclane spectrum $HA$ and define singular homology by $\pi_\bullet(\Sigma^\infty_+(-)\wedge HA)$. The $\infty$-category where the complexes computing singular homology lies can be, then, engendered by using the Dold-Kan correspondence or, equivalently, noticing that for any ring $A$ there's an equivalence $\mathbf{Mod}_{HA} \cong \mathrm{Ch}(\mathbf{Mod}_A)$, which can be further enhanced to an equivalence $\mathbf{Mod}_{HA} \cong \mathbf{dgAlg}_A$ of symmetric monoidal $\infty$-categories whenever $A$ is comutative.

The topoi theoretical principle dictates that a category of varying sets should behave like **Set**. Analogously, in the $\infty$-categorical context, a category of varying spaces should behave like $\mathbf{Grpd}_\infty$. One, thus, is immediately forced to consider the $\infty$-category $\mathbf{Sh}_\infty((\mathscr{S}_{/S})_\tau)$ for some full subcategory $\mathscr{S} \hookrightarrow \mathbf{Sch}$, $S \in \mathrm{Ob}(\mathscr{S})$ and $\tau$ some topology in $\mathscr{S}$. Then, motivated by the contractibility of $\mathbf{A}^1_S$ according to the usual cohomologies, one can invert the collection of projections $S_{\mathbf{A}^1} = \{X \times_S \mathbf{A}^1_S \twoheadrightarrow X \mid X \in \mathrm{Ob}(\mathscr{S}_{/S})\}$. Such step was not necessary in the ordinary case since the topological line is already contractible. The category $\mathscr{H}(S) := \mathbf{Sh}_\infty((\mathscr{S}_{/S})_\tau)[S^{-1}_{\mathbf{A}^1}]$ will be the so called category of motivic spaces, which can be summarised by the following localisations

$$\mathrm{Fun}(\mathscr{S}_{/S}, \mathbf{Grpd}_\infty) \xrightarrow{L_\tau} \mathbf{Sh}_\infty((\mathscr{S}_{/S})_\tau) \xrightarrow{L_{\mathbf{A}^1}} \mathscr{H}(S)$$

One, then, must make a choice of model spaces $\mathscr{S} \hookrightarrow \mathbf{Sch}$ and a Grothendieck topology $\tau$. As already mentioned before, $\mathscr{S} = \mathbf{Sm}$ would be the best choice when taking into account intersection theory. While $\tau = \mathrm{Nis}$ would be a good choice given its the already mentioned connection to descent in K-theory and its relation to cd-structures as will be shown.

## § 7.1 Nisnevich Topology

Here, it's recalled some properties of the Nisnevich topology. The Nisnevich topology was introduced in [Nis89] due to its good behaviour in K-theory. It resembles the Zariski topology and also the étale topology simultaneously by forcing an étale covering to contain open immersions covering the generic points of irreducible subschemes.

**Definition 7.1.1.** Let $S$ be a scheme. The **Nisnevich topology on $\mathbf{Sch}_{/S}$** is the Grothendieck topology generated by the pretopology where coverings consist of jointly surjective collections of morphisms $\{U \longrightarrow S\}_{U\in\mathcal{U}}$ such that $\coprod_{U\in\mathcal{U}} U \longrightarrow S$ is étale and, for every $x \in |S|$, there exists $x_U \in U \in \mathcal{U}$ such that $\kappa(x_U) \cong \kappa(x)$.

**Definition 7.1.2.** Let $S$ be a scheme. The **small Nisnevich site** $S_{\mathrm{Nis}}$ is the category $S_{\text{ét}}$ with the Nisnevich topology.

**Definition 7.1.3.** Let $S$ be a scheme and $f : X \longrightarrow S$ an étale morphism. A **splitting sequence for** $f$ is a finite nested sequence of closed subschemes

$$\emptyset = Z_{n+1} \hookrightarrow Z_n \hookrightarrow Z_{n-1} \hookrightarrow \cdots \hookrightarrow Z_0 = X$$

for $0 \geq i \leq n$ such that $f$ has a section over $Z_i \setminus Z_{i+1}$ for every $i$.

**Lemma 7.1.1.** *([BH18, Lem. A.1]) Let $S$ be a quasi-compact and quasi-separated scheme and $f : X \longrightarrow S$ be an étale morphism. If $f$ is a Nisnevich covering, than $f$ admits a splitting sequence. Furthermore, such splitting sequence may be taken to be composed of only finitely presented closed subschemes.*

*Proof.* The proof relies on showing, by Zorn's lemma, that the set

$$\Phi := \{Z | (\ Z \hookrightarrow S \text{ closed subscheme}) \\ \wedge (f|_{f^{-1}(Z)} \text{does not admit a splitting sequence})\}$$

is empty by showing that there's no minimal element.

Firstly, in order to apply Zorn's lemma, one should notice that, by [EGAIV-3, Prop. 8.6.3] and [EGAIV-3, Thm.8.8.2 (i)], $\Phi$ has an infimum for every cofiltered subset.

Let $Z \in \Phi$. Notice that $Z \neq \emptyset$ and, then, $Z$ has a point $x$ corresponding to a minimal prime. Hence, $\mathrm{Spec}(\mathcal{O}_{Z,x})$ is Henselian since

$$(\mathscr{O}_{Z,x})_{\mathrm{red}} = \kappa(x)$$

, which is Henselian. As $f$ is Nisnevich, there exists a lifting of $x$ and, since $\mathrm{Spec}(\mathscr{O}_{S,x})$ is Henselian, the lifting extends to an open neighborhood $U$ of $x$. Let

$$W := Z \setminus U$$

. Since $Z \in \Phi$, $W \in \Phi$, which implies that $\Phi$ has no minimal element. Therefore, $\Phi = \emptyset$.

If one defines

$$\Phi := \{Z | (\, Z \hookrightarrow S \text{ closed subscheme}) \wedge (f|_{f^{-1}(Z)} \text{does not admit a f.p. splitting seq.})\}$$

and, furthermore, chooses $U$ to be quasi-compact, $W$ can be endowed with a finitely presented structure by using the fact that $Z$ is quasi-separated and quasi-compact. As such choice is always possible, the result follows. □

*Remark* 7.1.1. In [MV99, Lem 3.1.5], Morel and Voevodsky proved the above result only for Noetherian schemes. The result was generalised later in [BH18, Lem. A.1].

**Theorem 7.1.1.** *Let $X$ be a scheme. Consider the following collections of étale morphisms on the category $X_{\text{ét}}$.*

1. *Nisnevich coverings;*

2. *Zariski open coverings and* $\{\mathrm{Spec}(A_f) \sqcup \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)\}$ *for every étale $A$-algebra $B$ and $f \in A$;*

3. *Finite collections of étale morphisms $\{U \longrightarrow X\}_{U \in \mathcal{U}}$ such that $\coprod_{U \in \mathcal{U}} U \longrightarrow X$ has a splitting sequence;*

4. *Finite collections of étale morphisms $\{U \longrightarrow X\}_{U \in \mathcal{U}}$ such that $\coprod_{U \in \mathcal{U}} U \longrightarrow X$ has a finitely presented splitting sequence;*

5. *$\emptyset$ and $\{j : U \hookrightarrow X, p : V \to X\}$ for every open immersions $j$ and $p$ étale such that $(X \setminus U)_{\mathrm{red}} \times_S V \longrightarrow (X \setminus U)_{\mathrm{red}}$ is an isomorphism;*

(1) *and* (2) *generate the same Grothendieck topology. For $S$ quasi-compact and quasi-separated schemes,* (1) – (5) *generate the same Grothendieck topology.*

*Proof.* (2) ⊂ (1) is trivial. The converse follows by covering $X$ with affine schemes (over $\mathrm{Spec}(\mathbf{Z})$) and noticing that only an open immersion can cover an irreducible component. Explicitly, let $C$ be an étale $A$-algebra. Notice that, as in the proof of Lemma 7.1.1, there exists a section defined over some $\mathrm{Spec}(A_f) \hookrightarrow \mathrm{Spec}(A)$, where $f \in A$. Since any morphism between affines is separated, by [EGAIV-3, Cor. 17.9.3], any section of an étale morphism

$$\mathrm{Spec}(C) \longrightarrow \mathrm{Spec}(A)$$

, must be a clopen immersion. That implies the splitting

$$\mathrm{Spec}(C) := \mathrm{Spec}(A_f) \sqcup \mathrm{Spec}(B).$$

The equivalences (1) ≅ (3) and (1) ≅ (4) follows from Lemma 7.1.1.

(5) ⊂ (1) is trivial. Now, it shall be proved descent over (5) implies descent over (1) by induction on the length of the splitting sequence. For $n = 0$, the result is trivial. Let the result holds for $n - 1$. Let

$$f : Y = \coprod_{W \in \mathcal{U}} W \longrightarrow X$$

be a Nisnevich covering in $\mathbf{Sm}_{/S}$ and

$$\emptyset = Z_{n+1} \hookrightarrow Z_n \hookrightarrow Z_{n-1} \hookrightarrow \cdots \hookrightarrow Z_0 = X$$

a splitting of $f$ of length $n$ and $F$ a sheaf for (5). By definition, there's a section $s$ of

$$f_{|Y \times_X Z_n} : Y \times_X Z_n \longrightarrow Z_n$$

. Let $U := X \setminus Z_n$ and $V = Y \setminus (Y \times_{Z_n} X \setminus s(Z_n))$ . By [EGAIV-3, Cor. 17.9.3], $s$ is an open immersion and, therefore, the corestriction

$$s : Z_n \longrightarrow s(Z_n)$$

is an isomorphism. That, immediately, implies that $\{j : U \hookrightarrow X, p : V \to X\}$ is of the form (5). Hence, $f_{|Y \times_X U} : Y \times_X U \longrightarrow U$ has splitting sequence of length $n-1$ and, then $F$ satisfies descent for it. Since $F$ satisfies descent for $f_{|Y \times_X U}$ and $U \sqcup V \longrightarrow X$, a simple diagram chasing implies that $F$ satisfies descent for $f$. □

*Remark* 7.1.2. The above proof of the equivalence between (1) and (5) works for $F$ satisfying only excision for (5). In particular, it proves that excision for (5) implies descent for (1).

**Proposition 7.1.1.** *Let $X$ be a quasi-compact and quasi-separated scheme. Following the notation of (5) 7.1.1, the diagram*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

*is a pushout square in $\mathbf{Sh}(X_{\text{Nis}})$. Or, equivalently, $F \in \mathbf{Sh}(X_{\text{Nis}})$ satisfies excision for such squares.*

*Proof.* One must check that sections of $F$ agreeing on $U \times_X V$, also agree on $(U \sqcup V) \times_X (U \sqcup V)$. The unique non-trivial task consists in showing that a section defined over $V$ is compatible with itself after restriction to $V \times_X V$. That follows by covering $V \times_X V$ by $V \times_X V \times_X U$ and the diagonal $\Delta_{V/X}$. □

**Corollary 7.1.1.** *Let $X$ be a quasi-compact and quasi-separated scheme. Following the notation of (5) in 7.1.1. A presheaf $F \in \widehat{X_{\text{Nis}}}$ satisfies excision for (5) iff it satisfies Nisnevich descent.*

The above results will be generalised to the $\infty$-categorical context in the next section.

## § 7.2 cd-Structures

The theory of cd-structure gives a convenient formalism for descent in Grothendieck topologies. In particular, it shows that excision and Čech descent are equivalent under reasonable hypothesis which are satisfied, for instance, by the Nisnevich topology and the cdh-topology. When restricting to Noetherian schemes of finite Krull dimension, even hyperdescent becomes equivalent to excision when a Grothendieck topology is generated by a reasonable cd-structure.

**Definition 7.2.1.** Let $\mathcal{C}$ be a small category and $0 \in \mathrm{Ob}(\mathcal{C})$ an initial object. A **cd structure on** $\mathcal{C}$ is a collection $P$ of commutative squares such that if $Q \in P$ and $Q' \cong Q$, then $Q' \in P$. A square $Q \in P$ will be called a **distinguished square of** $P$.

Let $P$ be a cd structure on $\mathcal{C}$. The **Grothendieck topology generated by** $P$ is the coarsest Grothendieck topology $\tau_P$ such that the empty sieve covers 0 and, for every square

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

in $P$, the sieve generated by $j$ and $p$ is a covering sieve.

**Definition 7.2.2.** Let $S$ be a scheme. The **Zariski cd-structure on** $\mathbf{Sm}_S$ is the collection $P_{\mathrm{Zar}}$ containing squares of the form

$$\begin{array}{ccc} U \cap V & \hookrightarrow & V \\ \downarrow & & \downarrow j \\ U & \xhookrightarrow{i} & X \end{array}$$

for every open immersions $i$ and $j$. A square of such form will be called a **Zariski square**.

The **Nisnevich cd-structure on** $\mathbf{Sm}_S$ is the collection $P_{\mathrm{Nis}}$ containing squares of the form

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \overset{j}{\hookrightarrow} & X \end{array}$$

for every open immersions $j$ and $p$ étale such that $(X \backslash U)_{\text{red}} \times_S V \longrightarrow (X \backslash U)_{\text{red}}$ is an isomorphism. A square of such form will be called a **Nisnevich square**.

*Remark* 7.2.1. Notice that the topology $\tau_{P_{\text{Zar}}}$ is the Zariski topology and the topology $\tau_{P_{\text{Nis}}}$ is equivalent to the Nisnevich topology whenever $X$ is quasi-compact and quasi-separated by 7.1.1.

**Definition 7.2.3.** Let $\mathscr{C}$ be a small category with an initial object $0 \in \text{Ob}(\mathscr{C})$ and $P$ a cd-structure. $P$ is a **complete cd-structure** if

1. 0 is strictly initial (*i.e.* , any morphism $X \longrightarrow 0 \in \mathscr{C}(X,0)$ is an isomorphism);
2. For any square $Q$ of the form

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \overset{j}{\longrightarrow} & X \end{array}$$

   and morphism $X' \longrightarrow X$, the square $Q' := Q \times_X X'$ exists and is a distinguished square of $P$.

*Remark* 7.2.2. By [Voe10a, Lem. 2.5], the above definition is slightly stronger than the original definition by Voevodsky ([Voe10a, Def. 2.3]).

**Definition 7.2.4.** Let $\mathscr{C}$ be a small category with an initial object $0 \in \text{Ob}(\mathscr{C})$ and $P$ a cd-structure. $P$ is a **regular cd-structure** if, for any distinguished square $Q \in P$ of the form

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \overset{j}{\longrightarrow} & X \end{array}$$

1. $Q$ is a pullback square;

2. $j$ is a monomorphism;

3. The induced square

$$\begin{array}{ccc} W & \longrightarrow & V \\ \big\downarrow{\scriptstyle \Delta_{W/U}} & & \big\downarrow{\scriptstyle \Delta_{V/X}} \\ W \times_U W & \longrightarrow & V \times_X V \end{array}$$

   exists and is a distinguished square of $P$.

*Remark* 7.2.3. By [Voe10a, Lem 2.11], the above definition is slightly stronger than the original definition by Voevodsky ([Voe10a, Def. 2.10]).

**Theorem 7.2.1** (Voevodsky). *Let $\mathcal{C}$ be a small $\infty$-category with an initial object $0 \in \mathrm{Ob}(\mathcal{C})$, $P$ a cd-structure on $\mathcal{C}$ and $F \in \mathrm{Ob}(\mathbf{PSh}_\infty(\mathcal{C}))$. If $P$ is complete, then $F$ satisfies excision for distinguished squares of $P$ implies that $F$ satisfies (Čech) $\tau_P$-descent.*

*If $P$ is regular and every $\tau_P$-covering has a finite refinement, then $F$ satisfies $\tau_P$-descent implies that $F$ satisfies excision for distinguished squares of $P$.*

*In particular, if $P$ is complete and regular, excision for distinguished squares of $P$ is equivalent to $\tau_P$-descent.*

*Proof.* The first part follows from [Voe10a, Prop. 5.8]. The second part follows from [Voe10a, Lem. 5.3]. A more self-contained proof can be found in [AHW17, Thm. 3.2.5].

□

**Corollary 7.2.1.** *Let $S$ be quasi-separated and quasi-compact. An $\infty$-presheaf $F \in \mathrm{Ob}(\mathrm{Fun}(\mathcal{S}_{\mathrm{Nis}})^{\mathrm{op}}, \mathbf{Grpd}_\infty))$ satisfies Čech descent for the Nisnevich topology iff it satisfies excision for Nisnevich squares.*

*Proof.* That follows from the above theorem and 7.1.1. □

In [Voe10a, Def. 2.22], Voevodsky defines the notion of a **bounded cd-structure**, which codifies the dimension of $\mathcal{C}$ as a site according to the following theorem.

**THEOREM 7.2.2.** *Let $\mathcal{C}$ be a small $\infty$-category with an initial object* $0 \in \mathrm{Ob}(\mathcal{C})$ *and $P$ a cd-structure on $\mathcal{C}$. If $P$ is complete, regular and bounded,* $\mathbf{Sh}_\infty(\mathcal{C}_{\tau_P})$ *is locally of finite cohomological dimension.*

*PROOF.* That follows from [Voe10a, Thm. 2.26]. □

**THEOREM 7.2.3.** *Let $\mathcal{C}$ be a small $\infty$-category with an initial object* $0 \in \mathrm{Ob}(\mathcal{C})$ *and $P$ a cd-structure on $\mathcal{C}$. If $P$ is complete, regular and bounded, then $tau_P$-hyperdescent is equivalent to excision for distinguished squares of $P$.*

*PROOF.* That follows from [Voe10a, Lem. 3.5]. □

**COROLLARY 7.2.2.** *Let $S$ be Noetherian of finite Krull dimension. An $\infty$-presheaf* $F \in \mathrm{Ob}(\mathrm{Fun}((S_{\mathrm{Nis}})^{\mathrm{op}}, \mathbf{Grpd}_\infty))$ *satisfies hyperdescent iff it satisfies excision for Nisnevich squares. In particular, the petit $\infty$-topos* $\mathbf{Sh}_\infty(S_{\mathrm{Nis}})$ *is hypercomplete whenever $S$ is Noetherian of finite Krull dimension.*

**COROLLARY 7.2.3.** *Let $S$ be Noetherian of finite Krull dimension $d$. The petit $\infty$-topos* $\mathbf{Sh}_\infty(S_{\mathrm{Nis}})$ *is locally of homotopy dimension $\leq d$.*

*PROOF.* That follows from that fact that an $\infty$-topos is locally of cohomological dimension $\leq d$ iff its hypercompletion is locally of homotopy dimension $\leq d$, COROLLARY 7.2.2 and the fact that $\mathbf{Sh}_\infty(S_{\mathrm{Nis}})$ is locally of cohomological dimension $\leq d$ ([MV99, Prop. 3.1.18] or the proof of [Voe10a, Thm. 2.26] applied to the standard density structure as in [Voe10b, Prop. 2.10]) □

## § 7.3 MOTIVIC SPACES

Given that, over quasi-compact quasi-separated schemes, all the definitions of Nisnevich topology are equivalent ( 7.1.1) and excision for Nisnevich squares is equivalent to Nisnevich descent for both presheaves of sets and $\infty$-presheaves. It's reasonable work only with quasi-compact and quasi-separated schemes or, instead, use excision for Nisnevich square and work with arbitrary $S$-smooth schemes for some base $S$. Let $S$ be quasi-compact and quasi-separated. If one defines $\mathbf{Sm}^{\mathrm{fp}}_{/S}$ to be the category of finitely presented smooth $S$-schemes, every $X \in \mathrm{Ob}(\mathbf{Sm}^{\mathrm{fp}}_{/S})$ will be quasi-compact and quasi-separated. However, any Nisnevich ∞-sheaf over $\mathbf{Sm}^{\mathrm{fp}}_{/S}$, by right Kan extension, can be extended to an $\tau_{P_{\mathrm{Nis}}}$-∞-sheaf over $\mathbf{Sm}_{/S}$ ([Hoy14, Prop. C.5.(3)]). For this reason, when $S$ is quasi-separated and quasi-compact, the choice of working with $\mathbf{Sm}^{\mathrm{fp}}_{/S}$ or $\mathbf{Sm}_{/S}$ is almost always irrelevant. Therefore, henceforth, $\mathbf{Sm}_{/S}$ will denote the category of finitely presented smooth $S$-schemes and $S$ will be assumed to be quasi-compact and quasi-separated

**Definition 7.3.1.** Let $S$ be a scheme and $S_{\mathbf{A}^1}$ the strongly saturated class generated by $\{X \times_S \mathbf{A}^1_S \longrightarrow X\}_{X \in \mathrm{Ob}(\mathbf{Sm}_{/S})}$. The **∞-category of motivic spaces** $\mathcal{H}(S)$ is defined as

$$\mathcal{H}(S) := \mathbf{Sh}_\infty((\mathbf{Sm}_{/S})_{\mathrm{Nis}})[S_{\mathbf{A}^1}^{-1}].$$

*Remark* 7.3.1. The ∞-category of motivic spaces $\mathcal{H}(S)$ was originally defined only for $S$ Noetherian of finite Krull dimension in [MV99]. One, however, can generalise most of the results to $S$ quasi-separated and quasi-compact whenever anything of strength similar to hyperdescent is not required.

The localisations

$$\mathrm{Fun}(\mathcal{S}_{/S}, \mathbf{Grpd}_\infty) \xrightarrow{L_{\mathrm{Nis}}} \mathbf{Sh}_\infty((\mathcal{S}_{/S})_\tau) \xrightarrow{L_{\mathbf{A}^1}} \mathcal{H}(S)$$

can be explicitly computed. The first localisation, $L_{\mathrm{Nis}}$ is simply sheaffication and, therefore, can be computed by transfinitely iterating the $(-)^+$ functor [1]. Notice that it's left exact since it's a topological localisation and, in particular, engenders an ∞-topoi.

**Definition 7.3.2.** Let $S$ be scheme, $F, G \in \mathrm{Ob}((\mathbf{Sm}_{/S})_{\mathrm{Nis}})$ and $f, g \in\in (\mathbf{Sm}_{/S})_{\mathrm{Nis}}(F, G)$. An **elementary $\mathbf{A}^1$-homotopy from $f$ to $g$** is a morphism

$$H : F \times \mathbf{A}^1_S \longrightarrow G$$

such that $Hi_0 = f$ and $Hi_1 = g$, where $i_0, i_1 : S \longrightarrow \mathbf{A}^1_S$ are the morphisms induced by $i_0^\#, i_1^\# : \mathcal{O}_S[T] \longrightarrow \mathcal{O}_S$ sending $T$ to, respectively, 0 and 1. In this case, $f$ and $g$ will be called $\mathbf{A}^1$-homotopic.

[1] Recall that the functor $(-)^+$ can be defined as $F(X) = \mathrm{colim}_R\, \widehat{\mathcal{C}}(R, F)$, where $R$ denotes a covering sieve of $X$, and, in the 1-categorical case engenders, from a presheaf, a separated presheaf

$f$ is a **strict $\mathbf{A}^1$-homotopy equivalence** if there exists $h \in (\mathbf{Sm}_{/S})_{\text{Nis}}(G, F)$ such that $fh$ and $hf$ are, respectively, $\mathbf{A}^1$-homotopic to $1_F$ and $1_G$.

Let $\Delta^n_S := S \times \text{Spec}(\mathbf{Z}[x_0, \ldots, x_n]/(\sum_i x_i - 1)) \cong \mathbf{A}^n_S$. The second localisation, $L_{\mathbf{A}^1}$, can be described by the following proposition.

**Proposition 7.3.1.**

$$L_{\mathbf{A}^1}(F) = \text{colim}_{n \in \Delta^{\text{op}}} F(- \times \Delta^n_S)$$

*Proof.* See [MV99, p. 2.3]. □

**Corollary 7.3.1.** *$L_{\mathbf{A}^1}$ preserves finite products and small colimits.*

*Proof.* Notice that it's a left adjoint and $\Delta^{\text{op}}$ is sifted. □

One, however, should notice that, after the $\mathbf{A}^1_S$-localisation, the resulting $\infty$-category is not a Grothendieck $\infty$-topoi.

**Proposition 7.3.2.** *Let $S$ be a scheme. The $\infty$-category $\mathcal{H}(S)$ is locally presentable, however it's not not an $\infty$-topoi. Or, equivalently, not all colimits are Van Kampen.* [2]

*Proof.* The claim of local presentability follows by noticing that the second localisation is accessible. The second claim follows from [Elm+18, Rem. 3.1.7]. The 0-truncated group object $L_{\text{Nis}}(\mathbf{Z}[(\mathbf{G}_m)_S])$ is $S_{\mathbf{A}^1}$-local, however it's not loop space, *i.e.* , it's not equivalent to its group completion, which implies that the simplicial colimit of the bar construction is not Van Kampen. □

For every morphism $f : S' \longrightarrow S$, $f^{-1} : (\mathbf{Sm}_{/S})_{\text{Nis}} \longrightarrow (\mathbf{Sm}_{/S'})_{\text{Nis}}$ does not commutes with pullbacks ([MV99, Ex. 3.1.19]). Still, it induces a geometric morphism

$$\mathbf{Sh}_\infty((\mathbf{Sm}_{/S'})_{\text{Nis}}) \underset{f_*}{\overset{f^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Sh}_\infty((\mathbf{Sm}_{/S})_{\text{Nis}})$$

between $\infty$-topos. When $f$ is smooth, the forgetful functor induces an additional left adjoint functor

[2]Recall that locally presentable $\infty$-category is an $\infty$-topoi iff every colimit is Van Kampen

$$\mathbf{Sh}_\infty((\mathbf{Sm}_{/S'})_{\mathrm{Nis}}) \underset{f_*}{\overset{f^\#}{\underset{\perp}{\longleftarrow}} \overset{f^*}{\longleftarrow} \underset{\perp}{\longrightarrow}} \mathbf{Sh}_\infty((\mathbf{Sm}_{/S})_{\mathrm{Nis}})$$

, which satisfies the following

1. $f_\#(X) = X$ for any $X \in \mathbf{Sm}_{/S'}$;
2. (Projection Formula) $f_\#(F \times_{S'} f^*(G)) \longrightarrow f_\#(F) \times_S G$.

Since $f^*(X \times_{S'} \mathbf{A}^1_{S'}) = X \times_S \mathbf{A}^1_S$, the functor $f^*$ preserves $S_{\mathbf{A}^1}$-equivalences. Conversely, the functor $f_*$ preserves $S_{\mathbf{A}^1}$-local objects. Furthermore, if $f$ is smooth, by the projection formula, $f_\#$ preserves $S_{\mathbf{A}^1}$-equivalences and $f^*$ preserves $S_{\mathbf{A}^1}$-local objects. Therefore, the above adjunction descends to

$$\mathcal{H}(S') \underset{f_*}{\overset{f^\#}{\underset{\perp}{\longleftarrow}} \overset{f^*}{\longleftarrow} \underset{\perp}{\longrightarrow}} \mathcal{H}(S)$$

It's not true, however, that $f_*$ preserves ([MV99, Ex. 3.2.11]) $S_{\mathbf{A}^1}$-equivalences nor that $f^*$ preserves $S_{\mathbf{A}^1}$-local objects ([MV99, Ex. 3.2.10]).

For every locally presentable $\infty$-category $\mathcal{C}$ with a final object $1 \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{C}_\bullet$ be the $\infty$-category of pointed objects, *i.e.* , the $\infty$-category consisting of pairs $(X, x)$ for every $X \in \mathrm{Ob}(\mathcal{C})$ and point $x : 1 \longrightarrow X$. Notice that, by the Adjoint Functor Theorem, there's an adjunction

$$\mathcal{C}_\bullet \underset{\perp}{\overset{(-)_+}{\longleftarrow}\longrightarrow} \mathcal{C}$$

between locally presentable $\infty$-categories induced by forgetting the point and the functor $X_+ = (X \sqcup 1, 1)$ since the forgetful functor is accessible and preserves small limits. Whenever the underlying point is known, the notation $(X, x)$ will be shortened to simply $X$.

The $\infty$-category $\mathcal{H}(S)_\bullet$ becomes symmetric monoidal by using the localisation of the smash product on $\mathrm{Fun}((\mathbf{Sm}_{\S})^{\mathrm{op}}, \mathbf{Grpd}_\infty)_\bullet$ ([MV99, Lem 3.2.13])

$$((F,x)\wedge_S (G,y))(X) := (F(X),x(X)) \wedge (G(X),y(X)))$$

, where the latter $(-)\wedge(-)$ denotes the smash product on $(\mathbf{Grpd}_\infty)_\bullet$. Equivalently, let $F \vee_S G$ denotes the coproduct in $\mathbf{Sh}_\infty((\mathbf{Sm}_{/(-)})_{\mathrm{Nis}})_\bullet$ (*resp.*, $\mathcal{H}(S)_\bullet$) and, then, define $F\wedge_S G$ as the cofiber

$$\begin{array}{ccc} F\vee_S G & \xrightarrow{f} & F\times G \\ \downarrow & \ulcorner & \downarrow \\ S & \longrightarrow & F\wedge_S G \end{array}$$

in $\mathbf{Sh}_\infty((\mathbf{Sm}_{/(-)})_{\mathrm{Nis}})_\bullet$ (*resp.*, $\mathcal{H}(S)_\bullet$). Sometimes the wedge product of $F$ with itself will be denoted by $F^n$ instead of $F^{\wedge n}$. Such wedge product gives a universal symmetric monoidal structure ([Rob15, Cor. 2.32] and [Rob15, Cor. 2.37]), which satisfies

$$(F\times_S G)_+ \cong F_+ \wedge_S G_+$$

Since $f^*$ commutes with finite limits, it's also a symmetric monoidal functor.

Since $f_\#$ commutes with colimits, it satisfies the analogous pointed formulas

1. $f_\#(X_+) = X_+$ for any $X\in \mathbf{Sm}_{/S'}$;
2. (Projection Formula) $f_\#(F\wedge_{S'} f^*(G)) \longrightarrow f_\#(F)\wedge_S G$.

**Theorem 7.3.1.** *Let $f : X \longrightarrow S$ and $g : Y \longrightarrow X$ be morphisms of schemes. The following canonical morphisms are isomorphisms of functors between values of $\mathbf{Sh}_\infty((\mathbf{Sm}_{/(-)})_{\mathrm{Nis}})$, $\mathbf{Sh}_\infty((\mathbf{Sm}_{/(-)})_{\mathrm{Nis}})_\bullet$ ,$\mathcal{H}(-)$ and $\mathcal{H}(S)_\bullet$.*

1. $f^*g^* \xrightarrow{\sim} (gf)^*$;
2. $(gf)_* \xrightarrow{\sim} g_*f_*$;
3. *If $f\in \mathrm{Ob}(\mathbf{Sm}_{/S})$ and $g\in \mathrm{Ob}(\mathbf{Sm}_{/X})$, then $(gf)_\# \xrightarrow{\sim} f_\# g_\#$.*

*Proof.* For Nisnevich sheaves, that follows from [MV99, Prop. 2.1.57].

For motivic spaces, that follows from [MV99, Prop. 2.3.17]. □

Recall the definition of the Thom space in clasical homotopy theory. Let $V \longrightarrow X$ be a smooth vector bundle with zero section $i : X \longrightarrow V$, by choosing a metric, let $D(V)$ the underlying disk bundle and $S(V)$ the sphere bundle. In classical homotopy theory, as $D(V) \cong V$ and $S(V) \cong V \setminus i(X)$, $\mathrm{Th}(V) := D(V)/S(V) \cong V/(V \setminus i(V))$. In the category of motivic spaces, due to the $\mathbf{A}^1$-localisation, one can copy the same definition.

**Definition 7.3.3.** Let $S$ be a scheme, $X \in \mathrm{Ob}(\mathbf{Sm}_{/S})$ and $\mathscr{E}$ quasi-coherent sheaf of $\mathscr{O}_X$-modules (*e.g.* , the dual of a sheaf associated to a vector bundle $\mathbf{V}(\mathscr{E})$). The **Thom space of** $\mathscr{E}$ is defined as the pointed motivic space

$$\mathrm{Th}(\mathscr{E}) := (\mathbf{V}(\mathscr{E})/\mathbf{V}(\mathscr{E})^{\times}, [\mathbf{V}(\mathscr{E})^{\times}])$$

, where $\mathbf{V}(\mathscr{E})^{\times} := \mathbf{V}(\mathscr{E}) \setminus 0(X)$, $0$ denotes the zero section and $[-]$ denotes the equivalence class.

More generally, let $p : V \longrightarrow X \in \mathbf{Sm}_{/X}$ and $s : X \longrightarrow V$ a section of $p$ which is a closed immersion (*e.g.* , $p$ is separated). The **Thom suspension functor** is defined as

$$\Sigma_{V/X} := p_{\#}s_* : \mathscr{H}(X) \longrightarrow \mathscr{H}(X)$$

In this case, $\Sigma^{\mathbf{V}(\mathscr{E})/X}(X) = \mathrm{Th}(\mathscr{E})$. For this reason, $\Sigma_{V/X}(X)$ will be denoted by $\mathrm{Th}(V)$.

*Remark* 7.3.2. The above equality between definitions of $\Sigma_{V/X}$ follows from a theorem that will be proved later, the localisation/gluing theorem ( 7.3.2), applied to $F = X$, $i = s$, $Z = X$ and $j : V \setminus X \hookrightarrow V$. More precisely, the square

$$\begin{array}{ccc} V \setminus X & \hookrightarrow & V \\ \| & \ulcorner & \downarrow \\ V \setminus X & \longrightarrow & s_*(X) \end{array}$$

is cocartesian and, hence $s_*(X) \cong V/(V \setminus X)$ and, therefore, $p_{\#}s_*(X) \cong \mathrm{Th}(V)$.

**Lemma 7.3.1.** *Let $S$ be a scheme and $V \in \mathbf{Sm}_{/S}$. The functor $\Sigma^{V/S}$ satisfies*

$$\Sigma_{V/S}(F \wedge_S G) = \Sigma_{V/S}(F) \wedge_S G$$

. *In particular,*

$$\Sigma_{V/S}(-) = \mathrm{Th}(V) \wedge_S (-).$$

The category of pointed motivic spaces $\mathcal{H}(S)_\bullet$ features, besides the simplicial circle $(\mathbf{S}^1, 1)$ [3], an internal algebraic circle: $((\mathbf{G}_m)_S, 1)$. That additional internal circle will correspond to Tate twists and it will give rise to the weight filtration of motives by using the corresponding Postnikov tower.

**Definition 7.3.4.** Let $S$ be a scheme. The **Tate circle** is the pointed motivic space $T := \mathrm{Th}(\mathbf{A}^1_X) \in \mathrm{Ob}(\mathcal{H}(S)_\bullet)$.

The following notations will be used

$$\mathbf{S}^{p,q} := (\mathbf{S}^{p-q}, 1) \wedge_S ((\mathbf{G}_m)_S, 1)^{\wedge q};$$

$$\Sigma^{p,q} := \Sigma^{\mathbf{S}^{p,q}/S}$$

for $p, q \in \omega$.

The following proposition illustrates that the Thom space behaves as in classical homotopy theory

**Proposition 7.3.3.** *Let $S$ be a scheme, $X, X' \in \mathrm{Ob}(\mathbf{Sm}_{/S})$, $V, V'$ vector bundles over, respectively $X$ and $X'$. Let, moreover, $V \boxtimes V'$ denotes the respective vector bundle over $X \times_S X'$. The following holds in $\mathbf{Sh}_\infty((\mathbf{Sm}_{/S})_{\mathrm{Nis}})_\bullet$*

1. *$V \longrightarrow X$ is a strict $\mathbf{A}^1$-homotopy equivalence;*
2. *$\mathrm{Th}(V \boxtimes V') \cong \mathrm{Th}(V) \wedge_S \mathrm{Th}(V')$;*
3. *$\mathrm{Th}(\mathcal{O}^n_X) = \mathrm{Th}(\mathbf{A}^n_X) = \Sigma^n_T X_+$.*

[3] Any object $X$ of $\mathbf{Grpd}_\infty$ can be viewed inside $\mathcal{H}(S)$ by noticing that $X = \mathrm{colim}_I\, 1$ for some $I$ and defining $X = \mathrm{colim}_I\, S$ inside $\mathcal{H}(S)$. Equivalently, it's induced by the value on $X$ of the canonical inverse image of the unique geometric morphism from $\mathbf{Sh}_\infty((\mathbf{Sm}_{/S})_{\mathrm{Nis}})$ to the terminal $\infty$-topos $\mathbf{Grpd}_\infty$

*Proof.* The first assertion follows trivially by choosing the zero section as an homotopy inverse and locally on $Spec(A) \hookrightarrow X$, the morphism

$$h : A[x_1, \ldots, x_n] \longrightarrow A[x_1, \ldots, x_n] \otimes_A A[x]$$

given by $h(x_i) x_i \otimes x$.

For the first isomorphism, notice that for $F, G, H, K \in \mathrm{Ob}(\mathbf{Sh}_\infty((\mathbf{Sm}_{/S})_{\mathrm{Nis}}))$

$$(F/H) \wedge_S (G/K) \cong (F \times_S G)/((F \times_S K) \sqcup_{F \times_S G} (H \times_S G))$$

since it's true in $(\mathbf{Grpd}_\infty)_\bullet$. Furthermore, the Zariski square

$$\begin{array}{ccc} V^\times \times_S (V')^\times & \longrightarrow & V^\times \times_S V' \\ \downarrow & \lrcorner & \downarrow \\ V \times_S (V')^\times & \longrightarrow & (V \boxtimes_S V')^\times \end{array}$$

implies that

$$V \times_S (V')^\times \sqcup_{V^\times \times_S (V')^\times} V^\times \times_S V' \cong (V \boxtimes_S V')^\times$$

in $\mathbf{Sh}_\infty((\mathbf{Sm}_{/S})_{\mathrm{Nis}})_\bullet$.

Analogously, for the second isomorphism, notice that

$$(F/H) \wedge_S (G, y) \cong (F \times_S G)/((F \times_S y) \sqcup_{F \times_S G} (H \times_S G)).$$

□

**Proposition 7.3.4.** *Let $S$ be a scheme, $X \in \mathrm{Ob}(\mathbf{Sm}_{/S})$ and $\mathscr{E}$ quasi-coherent sheaf of $\mathscr{O}_X$-modules. There exists an isomorphism*

$$\mathrm{Th}(\mathscr{E}) \xrightarrow{\sim} \mathbf{P}(\mathscr{E} \oplus \mathscr{O}_X)/(\mathbf{P}(\mathscr{E} \oplus \mathscr{O}_X) \setminus i0(X))$$

*in $\mathbf{Sh}_\infty((\mathbf{Sm}_{/S})_{\mathrm{Nis}})_\bullet$ and an isomorphism*

$$\mathbf{P}(\mathscr{E} \oplus \mathscr{O}_X)/\mathbf{P}(\mathscr{E}) \xrightarrow{\sim} \mathrm{Th}(\mathscr{E})$$

*in $\mathscr{H}(S)_\bullet$.*

*Proof.* Recall that the projective closure of a vector bundle $\mathbf{V}(\mathscr{E})$ is the canonical (affine and dominant) open immersion

$$i : \mathbf{V}(\mathscr{E}) \hookrightarrow \mathbf{P}(\mathscr{E} \oplus \mathscr{O}_X)$$

with place at infinity

$$\infty : \mathbf{P}(\mathscr{E}) \hookrightarrow \mathbf{P}(\mathscr{E} \oplus \mathscr{O}_X)$$

([EGAII, Prop. 8.4.2]). Now, notice that the diagram

$$\begin{array}{ccc} \mathbf{V}(\mathscr{E}) \setminus 0(X) & \longrightarrow & \mathbf{V}(\mathscr{E}) \\ \downarrow & \ulcorner & \downarrow \\ \mathbf{P}(\mathscr{E} \oplus \mathscr{O}_X) \setminus i0(X) & \longrightarrow & \mathbf{P}(\mathscr{E} \oplus \mathscr{O}_X) \end{array}$$

is Zariski square and, therefore, by excision, a ($\infty$)-pushout square of Nisnevich sheaves. Since coequalisers commute with colimits,

$$\mathrm{Th}(\mathscr{E}) \cong \mathbf{P}(\mathscr{E} \oplus \mathscr{O}_X)/(\mathbf{P}(\mathscr{E} \oplus \mathscr{O}_X) \setminus i0(X))$$

. Since $i0(X)$ and $\infty(\mathbf{P}(\mathscr{E}))$ have disjoint images, there's a morphism

$$\mathbf{P}(\mathscr{E} \oplus \mathscr{O}_X)/\mathbf{P}(\mathscr{E}) \longrightarrow \mathrm{Th}(\mathscr{E})$$

Notice that the closed immersion $\infty$ is the canonical section of the affine surjective morphism

$$\mathbf{P}(\mathscr{E} \oplus \mathscr{O}_X) \setminus i0(X) \longrightarrow \mathbf{P}(\mathscr{E})$$

induced by, respectively, the projection $\mathscr{E} \oplus \mathscr{O}_X \twoheadrightarrow \mathscr{E}$ and inclusion $\mathscr{E} \hookrightarrow \mathscr{E} \oplus \mathscr{O}_X$ ([EGAII, Prop. 8.3.5]). Therefore, $\infty$ is an $\mathbf{A}^1$-equivalence. □

**Corollary 7.3.2.** *Let $S$ be a scheme.*

$$\mathbf{P}^{n+1}_S/\mathbf{P}^n_S \cong T^{\wedge n+1}$$

*in $\mathscr{H}(S)_\bullet$ for every $n \in \omega$. In particular,*

$$T \cong (\mathbf{P}^1_S, \infty) \cong (\mathbf{P}^1_S, 1) \cong (\mathbf{P}^1_S, 0)$$

*in* $\mathcal{H}(S)_\bullet$.

**Proposition 7.3.5.** *Let $S$ be a scheme. There's an isomorphism*

$$(\mathbf{A}^{n+1}_S \setminus \{0\}, 1) \cong (\mathbf{S}^1)^{\wedge n} \wedge_S ((\mathbf{G}_m)^1_S)^{\wedge n+1} = \mathbf{S}^{2n+1,n+1}$$

*in the $\infty$-category $\mathbf{Sh}_\infty((\mathbf{Sm}_{/S})_{\mathrm{Nis}})_\bullet$ for every $n \in \omega$.*

*Proof.* See [MV99, Ex. 3.2.20] for the result in $\mathbf{Sh}_\infty((\mathbf{Sm}_{/S})_{\mathrm{Nis}})_\bullet$.

One can prove it easily, however, in $\mathcal{H}(S)_\bullet$. Firstly, notice that

$$\mathbf{P}^{n+1}_S/\mathbf{P}^n_S \cong \mathbf{A}^{n+1}_S/(\mathbf{A}^{n+1}_S \setminus \{0\})$$

in $\mathbf{Sh}_\infty((\mathbf{Sm}_{/S})_{\mathrm{Nis}})_\bullet$. The result follows by considering the $\infty$-pushout square

$$\begin{array}{ccc} (\mathbf{A}^{n+1}_S \setminus \{0\}, 1) & \longrightarrow & (\mathbf{A}^{n+1}_S, 1) \\ \downarrow & \ulcorner & \downarrow \\ 1 & \longrightarrow & \Sigma(\mathbf{A}^n_S \setminus \{0\}, 1) \end{array}$$

in $\mathcal{H}(S)_\bullet$ and Corollary 7.3.2.

□

**Proposition 7.3.6.** *Let $S$ be a scheme. There's an isomorphism*

$$T \cong \mathbf{S}^1 \wedge_S (\mathbf{G}_m)_S$$

*in the $\infty$-category $\mathcal{H}(S)_\bullet$. In particular, $\mathbf{S}^{2,1} \cong T$ and $\Sigma^{2,1} \cong \Sigma_T$.*

*Proof.* That follows by noticing that the diagram

$$\begin{array}{ccc} (\mathbf{G}_m)_S & \longrightarrow & \mathbf{A}^1_S \\ \downarrow & \ulcorner & \downarrow \\ \mathbf{A}^1_S & \longrightarrow & \mathbf{P}^1_S \end{array}$$

is a Zariski square and, hence, also a Nisnevich square. Therefore, by excision, it's a ($\infty$-)pushout square.

Since $\mathbf{A}^1_S$ is contractible, the diagram

$$\begin{array}{ccc} (\mathbf{G}_m)_S & \longrightarrow & S \\ \downarrow & \ulcorner & \downarrow \\ S & \longrightarrow & \mathbf{P}^1_S \end{array}$$

is equivalent to the previous one and, hence, also a pushout square. Therefore, by COROLLARY 7.3.2, $T \cong \mathbf{P}^1_S$ and, hence, $\Sigma((\mathbf{G}_m)_S, 1) \cong T$. □

Let the diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & \lrcorner & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a pullback square in **Sch** such that $f$ is smooth (and, hence, also $f'$). Consider the respective diagram

$$\begin{array}{ccc} \mathcal{H}(X) & \xrightarrow{(g')^*} & \mathcal{H}(X') \\ \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle (f')_\#} \\ \mathcal{H}(S) & \xrightarrow{g^*} & \mathcal{H}(S'). \end{array}$$

By 7.3.1, there's an isomorphism

$$\begin{aligned} (f')_\#(g')^*(f^* f_\#) &= (f')_\#((g')^* f^*) f_\# \cong (f')_\#(f g')^* f_\# = \\ &= (f')_\#(g f')^* f_\# \cong (f')_\#(f')^* g^* f_\# \end{aligned}$$

in Nisnevich sheaves or motivic spaces. Then, by the previous isomorphism, there exists a morphism

$$(f')_\#(g')^* \xrightarrow{\eta} (f')_\#(g')^*(f^* f_\#) \cong (f')_\#(f')^* g^* f_\# \xrightarrow{\varepsilon} g^*(f)_\#$$

in Nisnevich sheaves or motivic spaces.

**LEMMA 7.3.2** (Smooth Base-Change/Beck-Chevalley). *The morphism*

$$(f')_\#(g')^* \xrightarrow{\sim} g^*(f)_\#$$

*and its dual*

$$f^* g_* \xrightarrow{\sim} (g')_* (f')^*$$

*are isomorphisms.*

*Proof.* See [Kha18, Prop. 2.5.9]. That follows by noticing that, in the first isomorphism, all the functors commute with colimits and that the category of motivic spaces $\mathcal{H}(S)$ is generated under sifted colimits by representable sheaves of affines in $\mathbf{Sm}_{/S}$. □

Let $S$ be a scheme, $i : Z \hookrightarrow S \in \mathrm{Ob}(\mathbf{Sm}_{/S})$ a closed immersion, $j : U \hookrightarrow S \in \mathrm{Ob}(\mathbf{Sm}_{/S})$ be the complementary open immersion and $F \in \mathrm{Ob}(\mathcal{H}(S))$. By Smooth Base-Change, there's an isomorphism $i_* i^*(F) \cong U$ and, hence, the diagram

$$\begin{array}{ccc} j_\# j^*(F) & \xrightarrow{\varepsilon} & F \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\ j_\# j^* i_* i^*(F) & \xrightarrow{\varepsilon} & i_* i^*(F) \end{array}$$

is isomorphic to the diagram

$$\begin{array}{ccc} j_\# j^*(F) & \xrightarrow{\varepsilon} & F \\ \downarrow & & \downarrow{\scriptstyle \eta} \\ j_\#(U) & \longrightarrow & i_* i^*(F) \end{array}$$

**Theorem 7.3.2** (Gluing/Localisation). *Let $S$ be a scheme, $i : Z \hookrightarrow S \in \mathrm{Ob}(\mathbf{Sm}_{/S})$ a closed immersion, $j : U \hookrightarrow S \in \mathrm{Ob}(\mathbf{Sm}_{/S})$ be the complementary open immersion and $F \in \mathrm{Ob}(\mathcal{H}(S))$. The diagram*

$$\begin{array}{ccc} j_\# j^*(F) & \xrightarrow{\varepsilon} & F \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \eta} \\ j_\#(U) & \longrightarrow & i_* i^*(F) \end{array}$$

*is a pushout square in $\mathcal{H}(S)$.*

*Proof.* For $S$ Noetherian of finite Krull dimension see [MV99, Thm. 3.2.21]. For general $S$, see [Hoy14, Prop. C.10] or [Kha18, Thm. 3.2.2] □

*Remark* 7.3.3. The Localisation Theorem holds when restricted to the small Nisnevich ∞-topoi without requiring $\mathbf{A}^1$-localisation.

**Theorem 7.3.3** (Purity). *Let $S$ be a scheme, $Z, X \in \mathrm{Ob}(\mathbf{Sm}_{/S})$ and $i : Z \hookrightarrow X$ a closed immersion in $\mathbf{Sm}_{/S}$. There exists an isomorphism*

$$X/(X \setminus i(Z)) \cong \mathrm{Th}(N_{X/Z})$$

*in $\mathcal{H}(S)$.*

*Proof.* For $S$ Noetherian of finite Krull dimension, see [MV99, Prop. 2.23]. For general $S$, see [Hoy14, Appendix A] □

**Theorem 7.3.4.** *Let $S$ be Noetherian of finite Krull dimension, $Z, X \in \mathrm{Ob}(\mathbf{Sm}_{/S})$, $i : Z \hookrightarrow X$ a closed immersion in $\mathbf{Sm}_{/S}$, $p : X_Z \twoheadrightarrow X$ the blow up of $i$ and $U = X \setminus i(Z) \cong X_Z \setminus (X_Z \times_X i(Z))$. The diagram*

$$\begin{array}{ccc} Z \times_X X_Z & \longrightarrow & X_Z/U \\ \downarrow & \ulcorner & \downarrow \\ Z & \longrightarrow & X/U \end{array}$$

*is a pushout square in $\mathcal{H}(S)$. Furthermore, the induced diagram*

$$\begin{array}{ccc} \Sigma(Z \times_X X_Z)_+ & \longrightarrow & \Sigma(X_Z)_+ \\ \downarrow & \ulcorner & \downarrow \\ \Sigma Z_+ & \longrightarrow & \Sigma X_+ \end{array}$$

*is a also a pushout square in $\mathcal{H}(S)$.*

*Proof.* See [MV99, Prop. 3.2.29] and [MV99, Rem. 3.2.30]. □

## § 7.4 Motivic Spectra

In order to proceed the analogy with ordinary homotopy theory, one must apply a stabilisation procedure

$$\mathbf{Sp} \underset{\Omega_\infty}{\overset{\Sigma_\infty}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Grpd}_\infty.$$

Still, in this case, as already mentioned, there are two circles. One coming from the external structure by enrichment over $\mathbf{Grpd}_\infty$, $(\mathbf{S}^1, 1) \in \mathcal{H}(S)_\bullet$, and the other coming from the internal structure of $\mathbf{Sch}_{/S}$, $((\mathbf{G}_m)_S), 1) \in \mathrm{Ob}(\mathcal{H}(S)_\bullet)$. So, in order to proceed the analogy, one must develop an enriched process of stabilisation for symmetric monoidal $\infty$-categories.

**Definition 7.4.1.** Let $\mathcal{D}^\otimes$ be a symmetric monoidal $\infty$-category and $X \in \mathrm{Ob}(\mathcal{D}^\otimes)$. $X$ is **$\otimes$-invertible** if there exists an object $X^* \in \mathrm{Ob}(\mathcal{D}^\otimes)$ such that $X \otimes X^* \cong X^* \otimes X \cong 1_{\mathcal{D}^\otimes}$.

$X$ is **$\otimes$-symmetric** if there exists a 2-morphism (or, equivalenty, 2-isomorphism) between the cyclic permutation of $X \otimes X \otimes X$ and the identity $1_{X \otimes X \otimes X}$.

**Lemma 7.4.1** ([Voe98, Lem 4.4]). *Let $S$ be a scheme. The pointed motivic space $T \in \mathrm{Ob}(\mathcal{H}(S)_\bullet)$ is symmetric with respect to the monoidal structure $\wedge_S$.*

**Definition 7.4.2** ([Rob15, §2.2.1]). Let $\mathcal{C}$ be an $\infty$-category, and $F \dashv G : \mathcal{C} \longrightarrow \mathcal{C}$ an adjunction. The $(F, G)$**-stabilisation of** $\mathcal{C}$ is defined as

$$\mathrm{Stab}_{(F,G)}(\mathcal{C}) := \lim(\cdots \xrightarrow{G} \mathcal{C} \xrightarrow{G} \mathcal{C})$$

in $\mathbf{Cat}_\infty$.

The functor defined by the restriction of the limit cone to the diagram evaluated at $0 \in \mathrm{Ob}(\omega^{\mathrm{op}})$ will be suggestively denoted by

$$\Omega^\infty_{\mathcal{C}} : \mathrm{Stab}_{(F,G)}(\mathcal{C}) \longrightarrow \mathcal{C}.$$

In the case of $\mathcal{C} = \mathcal{D}^\otimes$ and $F = (-) \otimes X$ for some $\mathcal{D}^\otimes$ symmetric monoidal $\infty$-category and $X \in \mathrm{Ob}(\mathcal{D}^\otimes)$, the notation

$$\mathrm{Stab}_X(\mathcal{D}^\otimes) := \mathrm{Stab}_{((-)\otimes X, G)}(\mathcal{D}^\otimes)$$

*Example* 7.4.1. Let $\mathcal{C}$ be a finitely complete pointed $\infty$-category with fibers and cofibers. Consider the adjunction $\Sigma \dashv \Omega : \mathcal{C} \longrightarrow \mathcal{C}$. The $(\Sigma, \Omega)$-stabilisation is the exactly the ordinary stabilisation of finitely complete $\infty$-categories, which coincides with the category of ($\Omega$-)spectrum objects $\mathbf{Sp}(\mathcal{C}) \cong \mathbf{Sp}(\mathcal{C}_\bullet) \cong \mathrm{Stab}(\mathcal{C})$.

When $\mathscr{C}$ is locally presentable and $F : \mathscr{C} \longrightarrow \mathscr{C}$ preserving small colimits. By the Adjoint Functor Theorem, there exists a right adjoint $G : \mathscr{C} \longrightarrow \mathscr{C}$, which, dually, preserves small limits. Since $\mathscr{P}r^R \cong (\mathscr{P}r^L)^{\mathrm{op}}$, limits in $\mathscr{P}r^R$ are isomorphic to colimits in $\mathscr{P}r^L$ by taking adjoints. Now, by the fact that $\mathscr{P}r^R$ has all limits (and, also, colimits) and $\mathscr{P}r^R \hookrightarrow \mathbf{Cat}_\infty$ preserves limits ([HTT, Prop. 5.5.3.13]), one can compute $\mathrm{Stab}_{(F,G)}(\mathscr{C})$ through the colimit

$$\mathrm{colim}(\mathscr{C} \xrightarrow{F} \mathscr{C} \xrightarrow{F} \mathscr{C} \longrightarrow \cdots)$$

in $\mathscr{P}r^L$ [4].

Furthermore, $\mathrm{Stab}_{(F,G)(\mathscr{C})}$ will be again locally presentable by the already mentioned fact that $\mathscr{P}r^R \cong (\mathscr{P}r^L)^{\mathrm{op}}$ creates all limits in $\mathbf{Cat}_\infty$.

In general, when $\mathscr{C}$ is not locally presentable, the above colimit will still exist in $\mathbf{Cat}_\infty$. However, it will almost never coincides with $\mathrm{Stab}_{(F,G)}(\mathscr{C})$. In fact, even when $\mathscr{C}$ is locally presentable, it will not, in general, coincides with $\mathrm{Stab}_{(F,G)}(\mathscr{C})$ as it will be the analogous of a Spanier-Whitehead category, which already in the case of $(\mathbf{Grpd}_\infty)_\bullet$ and $\Sigma \dashv \Omega$ fails to be stable.

Notice that in this case, one has a functor

$$\Sigma^\infty_{\mathscr{C}} : \mathrm{Stab}_{(F,G)}(\mathscr{C}) \longrightarrow \mathscr{C}$$

defined by the restriction of the colimit cocone to the diagram evaluated at $0 \in \mathrm{Ob}(\omega)$.

The universal property of the colimit induces a unit

$$1_{\mathscr{C}} \longrightarrow \Omega^\infty_{\mathscr{C}}\Sigma^\infty_{\mathscr{C}}$$

Dually, the universal property of the colimit, induces a counit

$$\Sigma^\infty_{\mathscr{C}}\Omega^\infty_{\mathscr{C}} \longrightarrow 1_{\mathrm{Stab}_{(F,G)(\mathscr{C})}}$$

, and, therefore an adjunction $\Sigma^\infty_{\mathscr{C}} \dashv \Omega^\infty_{\mathscr{C}}$.

Notice that the universal property of $\mathrm{Stab}_{(F,G)}(\mathscr{C})$, there's a unique morphism

[4]Notice that the inclusions $\mathscr{P}r^L \hookrightarrow \mathbf{Cat}_\infty$ and $\mathscr{P}r^R \hookrightarrow \mathbf{Cat}_\infty$ only preserves limits. They, in general, do not preserves colimits. Indeed, small coproducts in $\mathscr{P}r^L$ coincides with small products in $\mathbf{Cat}_\infty$. Analogously, the copower (tensor) in $\mathscr{P}r^L$ coincides with presheaves, *i.e.* , up to opposition, it's almost the power (cotensor).

$$G : \mathrm{Stab}_{(F,G)}(\mathscr{C}) \longrightarrow \mathrm{Stab}_{(F,G)}(\mathscr{C})$$

such that the diagram

$$\begin{array}{ccc}
\mathrm{Stab}_{(F,G)}(\mathscr{C}) & \overset{G}{\dashrightarrow} & \mathrm{Stab}_{(F,G)}(\mathscr{C}) \\
\downarrow{\scriptstyle \Omega^{\infty}_{\mathscr{C}}} & & \downarrow{\scriptstyle \Omega^{\infty}_{\mathscr{C}}} \\
\mathscr{C} & \xrightarrow{\quad G \quad} & \mathscr{C}
\end{array}$$

is commutative. Dually, when $\mathscr{C}$ is locally presentable, there's a unique morphism

$$F : \mathrm{Stab}_{(F,G)}(\mathscr{C}) \longrightarrow \mathrm{Stab}_{(F,G)}(\mathscr{C})$$

such that the diagram

$$\begin{array}{ccc}
\mathrm{Stab}_{(F,G)}(\mathscr{C}) & \overset{G}{\dashrightarrow} & \mathrm{Stab}_{(F,G)}(\mathscr{C}) \\
\downarrow{\scriptstyle \Sigma^{\infty}_{\mathscr{C}}} & & \downarrow{\scriptstyle \Sigma^{\infty}_{\mathscr{C}}} \\
\mathscr{C} & \xrightarrow{\quad G \quad} & \mathscr{C}
\end{array}$$

is commutative. The following guarantees that when $F = (-) \otimes X$ for some $X$ symmetric, the induced morphism by $F$ and, dually $G$ is an isomorphism.

**Proposition 7.4.1.** *Let $\mathscr{D}^{\otimes}$ be a locally presentable symmetric monoidal $\infty$-category and $X \in \mathrm{Ob}(\mathscr{D}^{\otimes})$ $\otimes$-symmetric. The morphism*

$$(-) \otimes X : \mathrm{Stab}_X(\mathscr{D}^{\otimes}) \longrightarrow \mathrm{Stab}_X(\mathscr{D}^{\otimes})$$

*is an isomorphism.*

*Proof.* That is a particular instance of [Rob15, Prop. 2.19]. Notice that the morphism

$$(-) \otimes X : \mathrm{Stab}_X(\mathscr{D}^{\otimes}) \longrightarrow \mathrm{Stab}_X(\mathscr{D}^{\otimes})$$

induces on the base of the colimit cocones

$$N(\omega) \star \Delta^0 \longrightarrow \mathscr{D}^{\otimes}$$

the following diagram

$$\begin{array}{ccccccc} & & \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \cdots \\ & {}^{(-)\otimes X}\nearrow & B_{X,X}\Uparrow & {}^{(-)\otimes X}\nearrow & B_{X,X}\Uparrow & {}^{(-)\otimes X}\nearrow & \\ \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \cdots \end{array}$$

, where $B_{X,X} : \Delta^1 \times \Delta^1 \longrightarrow \mathbf{Cat}_\infty$ is the 2-morphism induced by the braidings

$$B_{X,X} : X \otimes X \longrightarrow X \otimes X$$

in $\mathscr{D}^{\otimes}$. It's easier to notice the above by letting $Y \in \mathrm{Ob}(\mathscr{D}^{\otimes})$ and, instead, induce a morphism

$$(-) \otimes Y : \mathrm{Stab}_X(\mathscr{D}^{\otimes}) \longrightarrow \mathrm{Stab}_X(\mathscr{D}^{\otimes})$$

, in which case the morphism in the base of the cocone becomes

$$\begin{array}{ccccccc} & & \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \cdots \\ & {}^{(-)\otimes Y}\nearrow & B_{X,Y}\Uparrow & {}^{(-)\otimes Y}\nearrow & B_{X,Y}\Uparrow & {}^{(-)\otimes Y}\nearrow & \\ \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \cdots . \end{array}$$

Now notice that, the composition of the 2-morphisms satisfies $((-) \otimes X \circ B_{X,Y}) \circ (B_{X,Y} \circ (-) \otimes X) \cong \sigma_{X,X,Y}$, where $\sigma_{(X,Y,Z)} : \Delta^1 \times \Delta^1 \longrightarrow \mathbf{Cat}_\infty$ is the 2-morphism induced by the cyclic permutation

$$\sigma_{(X,Y Z)} : X \otimes Y \otimes Z \longrightarrow Z \otimes X \otimes Y$$

for every $X, Y, Z \in \mathrm{Ob}(\mathscr{D}^{\otimes})$ and the compositions of 2-morphisms with 1-morphisms is induced by the degenerated 2-cells $\Delta^1 \to \Delta^1 \times \Delta^1$ including, respectively, $\Delta^1$ on the upper horizontal and lower horizontal. Again, it's easier to see the above by including a $(-) \otimes Z$ in the above diagram instead of a $(-) \otimes X$ in the horizontal 1-morphisms localted on the extreme right.

Since $X$ is symmetric, the first morphism of diagrams becomes isomorphic to

$$\begin{array}{ccccccc} & & \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \cdots \\ & \nearrow^{(-)\otimes X} & \Uparrow 1_{(-)\otimes X} & \nearrow^{(-)\otimes X} & \Uparrow 1_{(-)\otimes X} & \nearrow^{(-)\otimes X} & \\ \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \mathscr{D}^{\otimes} & \xrightarrow{(-)\otimes X} & \cdots \end{array}$$

□

**Corollary 7.4.1.** *Let $S$ be a scheme. The functor*

$$(-) \wedge_S T : \mathrm{Stab}_T(\mathscr{H}(S)_\bullet) \longrightarrow \mathrm{Stab}_T(\mathscr{H}(S)_\bullet)$$

*is an isomorphism.*

**Theorem 7.4.1.** *Let $\mathscr{D}^{\otimes}$ be a locally presentable symmetric monoidal $\infty$-category and $X \in \mathrm{Ob}(\mathscr{D}^{\otimes})$. There exists a locally presentable symmetric monoidal $\infty$-category $\mathscr{D}^{\otimes}[X^{-1}]$ and a morphism*

$$i : \mathscr{D}^{\otimes} \longrightarrow \mathscr{D}^{\otimes}[X^{-1}]$$

*such that $i(X)$ is $\otimes$-invertible and $i$ is initial with respect this property* [5].

*Proof.* That follows from the particular case of [Rob15, Prop. 2.9 (1)] and the diagram [Rob15, p. 2.38], which claims an adjunction. □

**Theorem 7.4.2.** *Let $\mathscr{D}^{\otimes}$ be a locally presentable symmetric monoidal $\infty$-category and $X \in \mathrm{Ob}(\mathscr{D}^{\otimes})$ such that $X$ is symmetric. The canonical morphism*

$$\mathscr{D}^{\otimes}[X^{-1}] \xrightarrow{\sim} \mathrm{Stab}_X(\mathscr{D}^{\otimes})$$

*induced by the universal property of $\mathscr{D}^{\otimes}[X^{-1}]$ is an isomorphism.*

*Proof.* In the 1-categorical case, that follows from the more general result in [Voe98, Thm. 4.3]. In the $\infty$-categorical case, that is the particular instance of [Rob15, Cor. 2.22].

The result follows from noticing that the morphism

$$\mathscr{D}^{\otimes}[X^{-1}] \longrightarrow \mathrm{Stab}_X(\mathscr{D}^{\otimes})$$

[5]The initiality is taken with respect to any functor that inverts $X$ and has as codomain a locally presentable symmetric monoidal $\infty$-category

, as its a reflector, factors as

$$\mathcal{D}^{\otimes}[X^{-1}] \longrightarrow \mathrm{Stab}_X(\mathcal{D}^{\otimes})[X^{-1}] \longrightarrow \mathrm{Stab}_X(\mathcal{D}^{\otimes})$$

, where the second morphism is the counit of the adjunction.

Since $(-)[X^{-1}]$ is a left adjoint, it commutes with colimits and, therefore,

$$\mathrm{Stab}_X(\mathcal{D}^{\otimes})[X^{-1}] \cong \mathrm{Stab}_X(\mathcal{D}^{\otimes}[X^{-1}])$$

, which, by the invertibility of $X$ in $\mathcal{D}^{\otimes}[X^{-1}]$, implies that the first morphism is an isomorphism.

Again, as $X$ is already invertible in $\mathrm{Stab}_X(\mathcal{D}^{\otimes})$, the second morphism is also an isomorphism and, therefore, the composition is an isomorphism. □

**Corollary 7.4.2.** *Let $S$ be a scheme. The categories $\mathcal{H}(S)_\bullet[T^{-1}]$, $\mathcal{H}(S)_\bullet[(\mathbf{S}^1)^{-1}]$ are a symmetric monoidal stable locally presentable $\infty$-categories.*

*Furthermore,*

$$\mathcal{H}(S)_\bullet[(\mathbf{S}^1)^{-1}] \cong \mathrm{Stab}(\mathcal{H}(S)_\bullet) \cong \mathrm{Stab}(\mathcal{H}(S)) \cong \mathbf{Sp}(\mathcal{H}(S)_\bullet).$$

*Proof.* Everything follows from the previous theorem and Lemma 7.4.1. However, one can easily see that $\mathcal{H}(S)_\bullet[T^{-1}]$ is stable by simply noticing that $\Omega$ also has an inverse since $\Sigma \cong \Sigma_T$. □

**Proposition 7.4.2** ([Rob15, Cor. 2.23]). *Let $\mathcal{D}^{\otimes}$ be a stable locally presentable symmetric monoidal $\infty$-category and $X \in \mathrm{Ob}(\mathcal{D}^{\otimes})$ such that $X$ is symmetric. The $\infty$-category $\mathcal{D}^{\otimes}[X^{-1}]$ is also stable locally presentable symmetric monoidal $\infty$-category*

*Proof.* That follows by noticing that the functor $(-) \otimes X$ is exact since it already preserves all small colimits. Now one should notice that the inclusion of the subcategory of $\mathcal{P}r^L$ consisting of locally presentable stable $\infty$-categories creates all colimits. By the previous theorem, $\mathcal{D}^{\otimes}[X^{-1}]$ is given by a colimit in $\mathcal{P}r^L$ □

**Corollary 7.4.3.** *Let $S$ be a scheme. There are equivalences*

$$\mathcal{H}(S)_\bullet[T^{-1}] \cong (\mathcal{H}(S)_\bullet[(\mathbf{S}^1)^{-1}])[((\mathbf{G}_m)_S,1)^{-1}] \cong \cong (\mathcal{H}(S)_\bullet[(\mathbf{S}^1)^{-1}])[((\mathbf{G}_m)_S,1)^{-1}].$$

Now, it's clear how to define the analogous version of stabilisation (and, consequently, spectra) in the motivic context.

**Definition 7.4.3.** Let $S$ be a scheme. The **stable $\infty$-category of motivic spaces of $S$** (or **$\infty$-category of motivic spectra**) is defined as $\mathcal{SH}(S) := \mathcal{H}(S)_\bullet[T^{-1}]$.

**Theorem 7.4.3** (Universal Property of $\mathcal{SH}(S)$, [Rob15, Cor 1.2, Cor. 2.39]). *Let $S$ be a scheme. The functor*

$$M : \mathbf{Sm}_{/S}^{\times_S} \longrightarrow \mathcal{SH}(S)^{\wedge_S}$$

*induced by all the localisations is a monoidal functor which is initial with respect to functors*

$$F : \mathbf{Sm}_{/S}^{\times_S} \longrightarrow \mathcal{D}^{\otimes}$$

*such that*

*(i)* $\mathcal{D}^{\otimes}$ *is a pointed locally presentable symmetric monoidal $\infty$-category;*

*(ii)* $F$ *satisfies Nisnevich descent and* $\mathbf{A}^1_S$*-invariance;*

*(iii)* $\mathrm{cofib}(F(S) \xrightarrow{F(\infty)} F(\mathbf{P}^1_S))$ *is invertible*

*, i.e. , if $F$ satisfies the above properties, there exists up to isomorphism, a unique monoidal colimit preserving functor* $\tilde{F} : \mathcal{SH}(S)^{\wedge_S} \longrightarrow \mathcal{D}^{\otimes}$.

*Furthermore, any such* $\mathcal{D}^{\otimes}$ *is already stable.*

*Sketch.* Notice, firstly, that $\mathrm{cofib}(S \xrightarrow{F(\infty)} \mathbf{P}^1_S)) \cong \mathbf{P}^1_S/\mathbf{P}^0_S \cong T$ in $\mathcal{SH}(S)$ by the previous results and the fact that the $T$-stabilisation preserves colimits since $\Sigma^\infty_T$ is a left adjoint. Then, since the morphism $\tilde{F}$ is monoidal and preserves colimits, it must satisfy $F(T) \cong \mathbf{S}^1 \otimes F(((\mathbf{G}_m)_S, 1))$. In particular, since $\mathbf{S}^1$ is invertible in $\mathscr{SH}(S)$, it must also be in $\mathscr{D}^{\otimes}$, which, therefore, implies that $\mathscr{D}^{\otimes}$ must be stable.

Now, notice that any such functor can be induced by extending the Yoneda embedding through colimits (or left Kan extensions). All these extension are monoidal since the induced monoidal structure on presheaves (by extending through colimits) makes it universal among all colimit preserving monoidal functors. □

*Remark* 7.4.1. The above theorem guarantees the existence of several realisation functors. For instance, since

$$\mathrm{cofib}(1 \xrightarrow{F(\infty)} \mathbf{P}^1_{\mathbf{C}}(\mathbf{C})) \cong \mathbf{S}^1$$

in $\mathbf{Sp}$, $\mathbf{A}^1_{\mathbf{C}}(\mathbf{C}) = \mathbf{C}$ is already contractible in $\mathbf{Grpd}_\infty$ and every scheme locally of finite type over $\mathbf{C}$ already satisfies étale descent for the analytic topology (by the results on the analytification in the Étale chapter), there exists Betti realisation functors

$$r_{\mathrm{Betti}} : \mathscr{SH}(\mathbf{C}) \longrightarrow \mathbf{Sp}$$

and

$$r_{\mathrm{Betti}} : \mathscr{SH}(\mathbf{C}) \longrightarrow D(\mathbf{C}, \Lambda).$$

More generally, whenever $S$ is quasi-compact quasi-separated and smooth over $\mathbf{C}$ (actually, with a little more work any $S$ having a good enough stratification would suffice), there exists relative Betti realisation functors

$$r_{\mathrm{Betti}} : \mathscr{SH}(S) \longrightarrow \mathscr{SH}(S(\mathbf{C}))$$

and

$$r_{\mathrm{Betti}} : \mathscr{SH}(S) \longrightarrow D(S(\mathbf{C}), \Lambda)$$

, where $\Lambda$ is a commutative ring and $\mathscr{SH}(S(\mathbf{C}))$ is constructed similarly by using the usual analytic topology.

The definition of $\mathrm{Stab}_{(F,G)}(\mathscr{C})$ as a limit can be explicilty described by writing it as a composition of equaliser and products. Or even better, in the case of locally presentable locally cartesian closed ∞-categories, writing it in the internal language. Indeed, $\mathrm{Stab}_{(F,G)}(\mathscr{C})$ consists in $\mathbf{Z}$-indexed sequences of paths $p_i : \Delta^1 \longrightarrow \mathscr{C}$ together with objects $X_i \in \mathrm{Ob}(\mathscr{C})$ such that

$$p_i : X_i \xrightarrow{\sim} G(X_{i+1})$$

. A formal justificative of the above assertion can be found in [Rob15, Prop. 2.13] when $\mathscr{C}$ can be presented by a combinatorial simplicial model category and computing the homotopy limit using fibrant objects and fibrations.

Now, the next step is defining a motivic version of $H\mathbf{Z}$, the Eilenberg-Maclane spectrum. Since $\mathscr{SH}(S)$ is stable and $\mathscr{H}(S)$ is tensored over $\mathbf{Grpd}_\infty$, $\mathscr{SH}(S)$ is also tensored over $\mathbf{Sp}$. One, therefore, could define $H\mathbf{Z}_n$ by $K(\mathbf{Z}, n) \otimes S$. That definition, however, fails as there are no non-trivial morphisms of the form

$$T \wedge_S K(\mathbf{Z}, n) \longrightarrow K(\mathbf{Z}, n+1).$$

By the Dold Thom theorem, in ordinary homotopy theory, following the discussion at the beginning of the section on Suslin's algebraic singular homology, one can define

$$K(\mathbf{Z}, n) := \coprod_{d \in \omega} (\mathrm{Sym}^d(\mathbf{S}^n))^+.$$

In analogy with the ordinary case and the success of Suslin's algebraic singular homology, one can define for $S = \mathrm{Spec}(k)$ and $k$ a field,

$$H\mathbf{Z}_{2n,n} := (\mathrm{Sym}^\infty(T^{\wedge n}))^+ \cong L(T^{\wedge n})$$

in $\mathscr{H}(k)_\bullet$ or $\mathscr{SH}(k)$, where

$$\mathrm{Sym}^n(X/Z) := \mathrm{cofib}\Big(\coprod_{1 \le j \le d} X \times_k \cdots \times_k X \times_k Z \times_k X \times_k \cdots \times_k X \to X^d \twoheadrightarrow \mathrm{Sym}^d(X)\Big)$$

in $\mathscr{H}(k)$ for every closed immersion $i : Z \hookrightarrow X$ in $\mathbf{Sm}_k$ such that the above morphism is induced on the $j$-th coordinate by $i$ and $L$ is defined similarly.

More generally, one can, then, define

$$H\mathbf{Z}_{p,q} := (\mathrm{Sym}^{\infty}(\mathbf{S}^{p,q}))^{+}$$

in $\mathcal{H}(S)_{\bullet}$ or $\mathcal{SH}(S)$ for a scheme $S$. In the case of a field $S = \mathrm{Spec}(k)$, then

$$H\mathbf{Z}_{p,q} \cong L(\mathbf{S}^{p,q}).$$

By noticing, now, that the canonical that there are canonical morphisms

$$(\mathrm{Sym}^{m}(X))^{+} \wedge_S (\mathrm{Sym}^{n}(Y))^{+} \longrightarrow (\mathrm{Sym}^{n+m}(X \wedge_S Y))^{+}$$

in $\mathcal{H}(S)_{\bullet}$ for every $X, Y \in \mathrm{Ob}(\mathbf{Sm}_{/S})$. In particular, $H\mathbf{Z}$ becames a $\Sigma_T$-spectrum by the precomposition with the canonical inclusion

$$T \wedge_S H\mathbf{Z}_{p,q} \to \mathrm{Sym}^{\infty}(T) \wedge_S H\mathbf{Z}_{p,q} \to H\mathbf{Z}_p + 2, q + 1.$$

It's extremely non-trivial to check that the adjoint morphism is an isomorphism in $\mathcal{H}(S)_{\bullet}$. The following was proved by Voeovodsky.

**Theorem 7.4.4** ([Voe98, Thm. 6.2]). *Let $k$ be a field such that* $\mathrm{char}(k) = 0$ *and $S \in \mathrm{Ob}(\mathbf{Sm}_k)$ a variety. The adjoint morphism*

$$H\mathbf{Z}_{p,q} \longrightarrow \Omega_T H\mathbf{Z}_{p+2,q+1}$$

*is an $\mathbf{A}^1_S$-equivalence in $\mathbf{Sh}_{\infty}((\mathbf{Sm}_{/S})_{\mathrm{Nis}})_{\bullet}$.*

In particular, by noticing that the inclusions $\mathbf{Grpd}_{\infty} \hookrightarrow \mathcal{H}(S)$ and $\mathbf{Sp} \hookrightarrow \mathcal{SH}(S)$ are fully faithful and $\mathbf{S}^{n,n} \cong \mathbf{S}^{n}$, one obtains,

$$H\mathbf{Z}_{n,0} \cong K(\mathbf{Z},, n).$$

and, therefore an isomorphism

$$H\mathbf{Z}_{\bullet,0} \cong (H\mathbf{Z})^{top}$$

in $\mathscr{SH}(S)$. In particular, for $S = \mathrm{Spec}(\mathbf{C})$ and the previous remark of the last theorem,

$$r_{\mathrm{Betti}}(H\mathbf{Z}_{\bullet,0} \cong (H\mathbf{Z})^{top}$$

and, more generally,

$$r_{\mathrm{Betti}}(H\mathbf{Z}) \cong (H\mathbf{Z})^{top}$$

, where $r_{\mathrm{Betti}}(\Omega_T) = \Omega^2$.

Furthermore, by the definition of Suslin's algebraic homology,, for $S = \mathrm{Spec}(k)$,

$$\mathrm{H}_i^{\mathrm{sing}}(X,\mathbf{Z}) \cong \mathscr{H}(k)_\bullet(\mathbf{S}^i, L(X_+)) = \pi_{i,0}^{\mathbf{A}_k^1}(X)$$

Recall, now, how to define a cohomology and homology from any motivic spectrum. Let $E \in \mathscr{SH}(k)$. The cohomology and homology, respectively, of $E$ are defined by

$$E^{p,q}(X,x) := \mathscr{SH}(k)(\Sigma_T^\infty(X,x), \Sigma^{p,q}E)$$

and

$$E_{p,q}(X,x) := \mathscr{SH}(k)(\Sigma^{p,q}1, E \wedge_k \Sigma_T^\infty(X,x)).$$

By noticing that $\Omega^n L(((\mathbf{G}_m)_k, 1)^{\wedge n})$ is, by Dold-Kan correspondence, exactly the complex defining motivic cohomology, one, then, obtains from the identity $\mathrm{Sym}^n((\mathbf{G}_m)_S) \cong \mathrm{Sym}^n((\mathbf{G}_m)_S) \cong \mathbf{A}_S^{n-1} \times_S (\mathbf{G}_m)_S$.

$$\mathrm{H}^p(X, \mathbf{Z}(q)) \cong H\mathbf{Z}^{p,q}(X).$$

More generally, the following is true

**Theorem 7.4.5** ([RØ06, Thm. 2]). *Let $k$ be a field such that* $\mathrm{char}(k) = 0$. *There's an equivalence of linear stable $\infty$-categories* $\mathbf{Mod}_{H\mathbf{Z}}(\mathscr{SH}(k)) \cong DM(k, \mathbf{Z})$ *given on the presentations by a zig-zag of Quillen equivalences.*

*Remark* 7.4.2. As mentioned at the end of [RØ06], there's also an equivalence

$\mathbf{Mod}_{H\mathbf{Q}}(\mathscr{SH}(k)) \cong DM(k,\mathbf{Q})$ when $k$ is a perfect field, which was obtained by using de Jong' alterations

Recall that $\Omega B(\coprod_{n\in\omega} BGL_n(A)) \cong \mathbf{Z}\times BGL_\infty^+$ since the plus construction initial with respect to $H$-spaces for any associative ring with unity $A$. Recall also that (Thomason-Trobaugh) $K$-theory defines a Nisnevich $\infty$-sheaf $\mathscr{K}$ (since it's Nisnevich excisive is the small Nisnevich site of any quasi-compact and quasi-separated scheme) and it's $\mathbf{A}^1$-invariant on coherent regular schemes. Furthermore, it coincides with Quillen's $K$-theory in the case of $X$ with an ample family of line bundles, which, in particular, implies the result for affine schemes and, then, the induced sheaves. These facts, hence, implies that whenever $S$ is quasi-compact quasi-separated, coherent and regular there exists a morphism of motivic spaces

$$L_{\mathrm{Nis}}L^1_{\mathbf{A}}(\mathbf{Z}\times BGL_\infty) \longrightarrow \mathscr{K}$$

in $\mathscr{H}(S)$ [6]. Now, the localisation $L^1_{\mathbf{A}}$ induces an isomorphism

$$L^1_{\mathbf{A}}(BGL_\infty(A)) \longrightarrow L^1_{\mathbf{A}}(BGL_\infty(A)^+)$$

since $L^1_{\mathbf{A}}(BGL(A))$ is an $H$-space. Therefore the morphism

$$L_{\mathrm{Nis}}L^1_{\mathbf{A}}(\mathbf{Z}\times BGL_\infty) \longrightarrow \mathscr{K}$$

is an isomorphism. In general, when dealing with non-regular $S$ one must consider Weibel's homotopy invariant $K$-theory $KH := L_{\mathbf{A}^1_S}(K^B)$, where $K^B$ is the nonconnective extension of $K$ defined by Bass-Thomason-Trobaugh.

Notice now that the Projective Bundle Formula for $K$-theory implies that $\mathbf{Z}\times BGL_\infty$ is $T$-periodic by using the Bott element

$$\beta = 1 - H^{-1} \in K_0(\mathbf{P}^1_S) \cong \pi_0(K)$$

, where $H$ denotes the class of the hyperplane (in this case, a point).

In this case, the morphism

$$\beta : \mathbf{P}^1_S \longrightarrow K \cong \mathbf{Z}\times BGL_\infty$$

[6] The localisations will often be omitted for convenience.

allows to define the structural morphisms of the spectrum objectwise defined as $\mathbf{Z} \times BGL_\infty$ by smashing with $\beta$, *i.e.* ,

$$\beta \wedge (-) : \Sigma_T(\mathbf{Z} \times BGL_{infty}) \longrightarrow \mathbf{Z} \times BGL_\infty$$

.

One can also identify $\mathbf{Z} \times BGL_\infty$ with Grassmanians and, even, better.

**Theorem 7.4.6** ([GS09, Prop. 4.2]). *Let $S$ be Noetherian of finite Krull dimension. The composition*

$$\Sigma_T^\infty \mathbf{S}^1 \longrightarrow \Sigma_T^\infty T \longrightarrow \Sigma_T^\infty(\mathbf{P}_S^\infty)_+ \longrightarrow K$$

*is equal to $\beta$. In particular, there exists a morphism*

$$\Sigma_T^\infty(\mathbf{P}_S^\infty)_+[\frac{1}{\beta}] \xrightarrow{\sim} \mathbf{Z} \times BGL_\infty$$

*, which is an isomorphism in $\mathscr{SH}(S)$.*

# Bibliography


[Rie60] B. Riemann. "Über die Anzahl der Primzahlen unter einer gegebenen Größe". In: *Monatsberichte der Königlichen Preußischen Akademie der Wissenschaften zu Berlin. Aus dem Jahre 1859 Monatsberichte der Königlichen Preußischen Akademie der Wissenschaften zu Berlin. Aus dem Jahre 1859*. Berlin: Berlin-Brandenburgische Akademie der Wissenschaften, 1860, pp. 671–680.

[Has34] H. Hasse. "Abstrakte begründung der komplexen multiplikation und riemannsche vermutung in funktionenkörpern". In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 10.1 (June 1934), pp. 325–348. DOI: 10.1007/bf02940685. URL: https://doi.org/10.1007%2Fbf02940685.

[Deu41] M. Deuring. "Die Typen der Multiplikatorenringe elliptischer Funktionenkörper". In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg* 14.1 (Oct. 1941), pp. 197–272. DOI: 10.1007/bf02940746. URL: https://doi.org/10.1007%2Fbf02940746.

[Wei49] A. Weil. "Numbers of solutions of equations in finite fields". In: *Bulletin of the American Mathematical Society* 55.5 (May 1949), pp. 497–509. ISSN: 0002-9904. DOI: 10.1090/s0002-9904-1949-09219-4. URL: http://dx.doi.org/10.1090/S0002-9904-1949-09219-4.

[Col54] Collectif. "TRAVAUX D'HIRZEBRUCH SUR LA TOPOLOGIE DES VARIÉTÉS". fr. In: *Séminaire Bourbaki : années 1951/52 - 1952/53 - 1953/54, exposés 50-100*. Séminaire Bourbaki 2. talk:88. Société Mathématique de France, 1954. URL: http://www.numdam.org/item/SB_1951-1954__2__351_0/.

[Igu55] J.-I. Igusa. "On Some Problems in Abstract Algebraic Geometry". In: *Proceedings of the National Academy of Sciences of the United States of America* 41.11 (1955), pp. 964–967. ISSN: 00278424. URL: http://www.jstor.org/stable/89189.

[Wei55] A. Weil. *Fibre spaces in algebraic geometry (Notes by A. Wallace, 1952)*. Department of Mathematics, University of Chicago, 1955.

[Ser56a] J.-P. Serre. "COHOMOLOGIE ET GÉOMÉTRIE ALGÉBRIQUE". In: *Proceedings of the International Congress of Mathematicians, 1954, Amsterdam*. Ed. by J. C. H. Gerretsen and J. De Groot. Vol. III. Groningen and Amsterdam: Erven P. Noordhoff N.V. and North-Holland Publishing Co., 1956, pp. 515–520. URL: https://www.mathunion.org/fileadmin/ICM/Proceedings/ICM1954.3/ICM1954.3.ocr.pdf.

[Ser56b] J.-P. Serre. "Géométrie algébrique et géométrie analytique". In: *Annales de l'Institut Fourier* 6 (1956), pp. 1–42. DOI: 10.5802/aif.59. URL: http://www.numdam.org/item/AIF_1956__6__1_0.

[Wei56] A. Weil. "Abstract versus classical algebraic geometry". In: *Proceedings of the International Congress of Mathematicians, 1954, Amsterdam*. Ed. by J. C. H. Gerretsen and J. De Groot. Vol. III. Groningen and Amsterdam: Erven P. Noordhoff N.V. and North-Holland Publishing Co., 1956, pp. 550–558. URL: https://www.mathunion.org/fileadmin/ICM/Proceedings/ICM1954.3/ICM1954.3.ocr.pdf.

[BS58] A. Borel and J.-P. Serre. "Le théorème de Riemann-Roch". In: *Bulletin de la Société Mathématique de France* 86 (1958), pp. 97–136. DOI: 10.24033/bsmf.1500. URL: http://www.numdam.org/item/BSMF_1958__86__97_0.

[LT58] S. Lang and J. Tate. "Principal Homogeneous Spaces Over Abelian Varieties". In: *American Journal of Mathematics* 80.3 (1958), pp. 659–684. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2372778.

[Ser58a] J.-P. Serre. "Espaces fibrés algébriques". In: *Séminaire Claude Chevalley* 3 (1958). talk:1, pp. 1–37. URL: http://www.numdam.org/item/SCC_1958__3__A1_0.

[Ser58b] J.-P. Serre. "Quelques Proprietes des Varietes Abeliennes en Caracteristique p". In: *American Journal of Mathematics* 80.3 (1958), pp. 715–739. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2372780.

[Ser58c] J.-P. Serre. "Sur la topologie des variétés algébriques en caractéristique $p$". In: *Symposium internacional de topología algebraica.* Mexico City: Universidad Nacional Autónoma de México and UNESCO, 1958, pp. 24–53.

[AH59] M. F. Atiyah and F. Hirzebruch. "RIEMANN-ROCH THEOREMS FOR DIFFERENTIABLE MANIFOLDS". In: *Bulletin of the American Mathematical Society* 65.4 (1959), pp. 276–281. DOI: bams/1183523205. URL: https://doi.org/.

[Dwo60] Bernard Dwork. "On the Rationality of the Zeta Function of an Algebraic Variety". In: *American Journal of Mathematics* 82.3 (1960), pp. 631–648. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2372974.

[Gro60] A. Grothendieck. "The cohomology theory of abstract algebraic varieties". In: *Proceedings of the International Congress of Mathematicians (Edinburgh, 1958).* Ed. by J. A. Todd. London and New York: Cambridge University Press, 1960. Chap. One-hour Addresses. URL: https://www.mathunion.org/fileadmin/ICM/Proceedings/ICM1958/ICM1958.ocr.pdf.

[Hir60] F. Hirzebruch. "A RIEMANN-ROCH THEOREM FOR DIFFERENTIABLE MANIFOLDS". en. In: *Séminaire Bourbaki : années 1958/59 - 1959/60, exposés 169-204.* Séminaire Bourbaki 5. talk:177. Société Mathématique de France, 1960. URL: http://www.numdam.org/item/SB_1958-1960__5__129_0/.

[Ser60] Jean-Pierre Serre. "Analogues Kählériens de Certaines Conjectures de Weil". In: *Annals of Mathematics* 71.2 (Mar. 1960), pp. 392–394. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970088.

[EGAIII-1] A. Grothendieck. "Éléments de géométrie algébrique : III. Étude cohomologique des faisceaux cohérents, Première partie". In: *Publications Mathématiques de l'IHÉS* 11 (1961), pp. 5–167. URL: http://www.numdam.org/item/PMIHES_1961__11__5_0.

[EGAII] A. Grothendieck. "Éléments de géométrie algébrique : III. Étude cohomologique des faisceaux cohérents, Seconde partie". In: *Publications Mathématiques de l'IHÉS* 8 (1961), pp. 5–222. URL: http://www.numdam.org/item/PMIHES_1961__8__5_0.

[Art62] M. Artin. "Grothendieck Topologies". Seminar Notes, Harvard University, Department of Mathematics. 1962.

[Hir62] F. Hirzebruch. *Neue Topologische Methoden in der Algebraischen Geometrie.* Vol. NEUE FOLGE. Ergebnisse der Mathematik und Ihrer Grenzgebiete HEFT 9. Berlin Heidelberg: Springer-Verlag, 1962. DOI: 10.1007/978-3-662-41083-7. URL: https://doi.org/10.1007%2F978-3-662-41083-7.

[Gro63] A. Grothendieck. "A letter from Grothendieck to Illusie". Buffalo, May 1963. URL: http://matematicas.unex.es/~navarro/res/illusie.pdf.

[Ser64] J.-P. Serre. "Exemples de variétés projectives conjuguées non homéomorphes". In: *COMPTES RENDUS HEBDOMADAIRES DES SÉANCES DE L'ACADÉMIE DES SCIENCES* 258 (1964), pp. 4194–4196. DOI: https://gallica.bnf.fr/ark:/12148/bpt6k4011c/f1832.item. URL: https://gallica.bnf.fr/ark:/12148/bpt6k4011c/f1832.item.

[EGAIV-2] A. Grothendieck. "Éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, Seconde partie". In: *Publications Mathématiques de l'IHÉS* 24 (1965), pp. 5–

231. URL: http://www.numdam.org/item/PMIHES_1965__24__5_0.

[Tat65] J. T. Tate. "Algebraic Cycles and Poles of Zeta Functions". In: *Arithmetical Algebraic Geometry. Proceedings of a Conference Held at Purdue Univiversity, December 5-7, 1963*. Ed. by O. F. G. Schilling. New York: Harper & Row, 1965, pp. 93–110.

[EGAIV-3] A. Grothendieck. "Éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, Troisième partie". In: *Publications Mathématiques de l'IHÉS* 28 (1966), pp. 5–255. URL: http://www.numdam.org/item/PMIHES_1966__28__5_0.

[AM67] M. Artin and B. Mazur. "Homotopy of Varieties in the Etale Topology". In: *Proceedings of a Conference on Local Fields - NUFFIC Summer School held at Driebergen (The Netherlands) at 1966*. Ed. by T.A. Springer. Berlin Heidelberg: Springer-Verlag, 1967, pp. 1–15. DOI: 10.1007/978-3-642-87942-5_1. URL: https://doi.org/10.1007%2F978-3-642-87942-5_1.

[EGAIV-4] A. Grothendieck. "Éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie". In: *Publications Mathématiques de l'IHÉS* 32 (1967), pp. 5–361. URL: http://www.numdam.org/item/PMIHES_1967__32__5_0.

[Lub67] S. Lubkin. "ON A CONJECTURE OF ANDRE WEIL". In: *American Journal of Mathematics* 89.2 (1967), pp. 443–548. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2373129.

[Gro68] A. Grothendieck. "Crystals and the De Rham Cohomology of Schemes". In: *Dix exposés sur la cohomologie des schémas*. Vol. 3. Adv. Stud. Pure Math. Notes by I. Coates and O. Jussila. Amsterdam: North-Holland, 1968, pp. 306–358.

[Man68] Yu. I. Manin. "Correspondences, motifs and monoidal transformations". In: *Matematicheskii Sbornik, Novaya Seriya* 77(119).4 (1968), pp. 475–507.

[FK69] S. Feferman and G. Kreisel. "Set-Theoretical foundations of category theory". In: *Reports of the Midwest Category Seminar III.* Ed. by S. Maclane. Berlin and Heidelberg: Springer-Verlag, 1969, pp. 201–247. ISBN: 978-3-540-36150-3.

[Gro69] A. Grothendieck. "Standard conjectures on algebraic cycles. (International Colloquium, Tata Institute of Fundamental Research, Bombay, 1968)". In: *Algebraic Geometry.* London: Oxford University Press, 1969, pp. 193–199.

[SGA3-II] M. Artin et al. *Schémas en Groupes (SGA 3). Séminaire de Géometrie Algébrique du Bois Marie 1962/64.* Tome II (Groupes de Type Multiplicatif, et Structure des Schémas en Groupes Généraux). Ed. by M. Demazure and A. Grothendieck. Vol. 2. Lecture Notes in Mathematics 152. Berlin and Heidelberg: Springer-Verlag, 1970. ISBN: 978-3-540-05180-0. DOI: 10.1007/BFb0059005.

[Ser70a] J.-P. Serre. *Cours d'Arithmétique.* Paris: Presses Universitaires de France, 1970.

[Ser70b] J.-P. Serre. "FACTEURS LOCAUX DES FONCTIONS ZÊTA DES VARIÉTÉS ALGÉBRIQUES (DÉFINITIONS ET CONJECTURES)". fr. In: *Séminaire DELANGE-PISOT-POITOU. Théorie des nombres* 11.2 (May 1970). talk:19. URL: http://www.numdam.org/item/SDPP_1969-1970__11_2_A4_0/.

[SGA6] P. Berthelot et al. *Théorie des Intersections et Théorème de Riemann-Roch (SGA 6). Séminaire de Géométrie Algébrique du Bois Marie 1966-67.* Lecture Notes in Mathematics 225. Berlin and Heidelberg: Springer-Verlag, 1971. ISBN: 978-3-540-36936-3. DOI: 10.1007/BFb0066283.

[Del71] Pierre Deligne. "THÉORIE DE HODGE, II". fr. In: *Publications mathématiques de l'IHÉS* 40 (1971), pp. 5–57. URL: http://www.numdam.org/item/PMIHES_1971__40__5_0.

[Dem71] M. Demazure. "MOTIFS DES VARIÉTÉS ALGÉBRIQUES. Vol. 1969/70, Exposés 364-381". fr. In: *Séminaire Bourbaki*. Ed. by N. Bourbaki. Vol. 180. Lecture Notes in Mathematics 12. talk:365. Berlin and Heidelberg: Springer-Verlag, 1971. Chap. 2, pp. 19–38. URL: http://www.numdam.org/item/SB_1969-1970__12__19_0.

[SGA1] A. Grothendieck and M. Raynaud. *Revêtements Étales et Groupe Fondamental (SGA 1). Séminaire de Géométrie Algébrique du Bois Marie 1960-61*. Lecture Notes in Mathematics 224. Berlin and Heidelberg: Springer-Verlag, 1971. ISBN: 978-3-540-36910-3. DOI: 10.1007/BFb0058656.

[Tat71] J. Tate. "CLASSES D'ISOGÉNIE DES VARIÉTÉS ABÉLIENNES SUR UN CORPS FINI (d'après T. Honda)". fr. In: *Séminaire Bourbaki. Vol. 1968/69. Exposés 347 - 363*. Ed. by N. Bourbaki. Vol. 179. Lecture Notes in Mathematics. Springer-Verlag, 1971, pp. 95–110. URL: http://www.numdam.org/item/SB_1968-1969__11__95_0.

[SGA4-I] M. Artin et al. *Théorie des Topos et Cohomologie Etale des Schémas (SGA 4). Séminaire de Géométrie Algebrique du Bois Marie 1963-1964*. Tome I Theorie des Topos (Exposés I à IV). Lecture Notes in Mathematics 269. Berlin and Heidelberg: Springer-Verlag, 1972. ISBN: 978-3-540-37549-4. DOI: 10.1007/BFb0081551.

[SGA4-II] M. Artin et al. *Théorie des Topos et Cohomologie Etale des Schémas (SGA 4). Séminaire de Géométrie Algebrique du Bois Marie 1963-1964*. Tome 2 (Exposés V à VIII). Lecture Notes in Mathematics 270. Berlin and Heidelberg: Springer-Verlag, 1972. ISBN: 978-3-540-37987-4. DOI: 10.1007/BFb0061319.

[Kle72] Steven L. Kleiman. "Motives. Proceedings of the 5th Nordic Summer-School in Mathematics, Oslo, August 5-25, 1970". In: *Algebraic geometry, Oslo 1970*. Ed. by F. Oort. Groningen: Wolters-Noordhoff Publishing, 1972. Chap. 5, pp. 53–82.

[Wym72] B. F. Wyman. “What is a Reciprocity Law?” In: *The American Mathematical Monthly* 79.6 (1972), pp. 571–586. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/2317083.

[SGA4-III] M. Artin et al. *Théorie des Topos et Cohomologie Etale des Schémas (SGA 4). Séminaire de Géométrie Algebrique du Bois Marie 1963-1964.* Tome 3 Theorie des Topos (Exposés IX à XIX). Lecture Notes in Mathematics 305. Berlin and Heidelberg: Springer-Verlag, 1973. ISBN: 978-3-540-38124-2. DOI: 10.1007/BFb0070714.

[Qui73] D. Quillen. “Higher algebraic $K$-theory: I”. In: *Algebraic $K$-Theory I. Proceedings of the Conference Held at the Seattle Research Center of Battelle Memorial Institute, August 28 - September 8, 1972.* Ed. by H. Bass. Lecture Notes in Mathematics 341. Berlin and Heidelberg: Springer-Verlag, 1973, pp. 85–147. ISBN: 978-3-540-06434-3. DOI: 10.1007/BFb0067048.

[Del74a] P. Deligne. “La conjecture de Weil.I”. In: *Publications Mathématiques de l’IHÉS* 43 (1974), pp. 273–307. URL: http://www.numdam.org/item/PMIHES_1974__43__273_0.

[Del74b] P. Deligne. “THÉORIE DE HODGE, III”. fr. In: *Publications mathématiques de l’IHÉS* 44 (1974), pp. 5–77. URL: http://www.numdam.org/item/PMIHES_1974__44__5_0.

[Wat75] William C. Waterhouse. “Basically bounded functors and flat sheaves.” In: *Pacific Journal of Mathematics* 57.2 (1975), pp. 597–610. URL: https://projecteuclid.org:443/euclid.pjm/1102906018.

[SGA4$\frac{1}{2}$] P. Deligne. *Cohomologie étale. Séminaire de Géométrie Algébrique du Bois-Marie SGA* 4$\frac{1}{2}$. Vol. 569. Lecture Notes in Mathematics. Berlin and Heidelberg: Springer-Verlag, 1977.

[SGA5] A. Grothendieck et al. *Cohomologie l-adique et Fonctions L. Séminaire de Géométrie Algébrique du Bois Marie 1965-66 SGA 5.* Ed. by L. Illusie. Lecture Notes in Mathematics 589. Springer-Verlag, 1977.

[Ler79] O. Leroy. *GROUPOÏDE FONDAMENTAL ET THÉORÈME DE VAN KAMPEN EN THÉORIE DES TOPOS.* Cahiers Mathématiques Montpellier 17. Pl. Eugène Bataillon, Montpellier, France: Université des Sciences et Techniques du Languedoc, U.E.R. de Mathématiques, 1979, pp. iii+65.

[Del80] P. Deligne. "La conjecture de Weil.II". In: *Publications Mathématiques de l'IHÉS* 52 (1980), pp. 137–252. URL: http://www.numdam.org/item/PMIHES_1980__52__137_0.

[DK80a] W. G. Dwyer and D. M. Kan. "CALCULATING SIMPLICIAL LOCALIZATIONS". In: *Journal of Pure and Applied Algebra* 18 (1 July 1980), pp. 17–35. ISSN: 0022-4049. DOI: https://doi.org/10.1016/0022-4049(80)90113-9. URL: http://www.sciencedirect.com/science/article/pii/0022404980901139.

[DK80b] W. G. Dwyer and D. M. Kan. "FUNCTION COMPLEXES IN HOMOTOPICAL ALGEBRA". In: *Topology* 19 (4 1980), pp. 427–440. ISSN: 0040-9383. DOI: https://doi.org/10.1016/0040-9383(80)90025-7. URL: http://www.sciencedirect.com/science/article/pii/0040938380900257.

[DK80c] W. G. Dwyer and D. M. Kan. "SIMPLICIAL LOCALIZATIONS OF CATEGORIES". In: *Journal of Pure and Applied Algebra* 17 (3 June 1980), pp. 267–284. ISSN: 0022-4049. DOI: https://doi.org/10.1016/0022-4049(80)90049-3. URL: http://www.sciencedirect.com/science/article/pii/0022404980900493.

[Gil81] H. Gillet. "Riemann-Roch theorems for higher algebraic $K$-theory". In: *Advances in Mathematics* 40 (3 June 1981), pp. 203–289. ISSN: 0001-8708. DOI: https://doi.org/10.1016/S0001-8708(81)80006-0. URL: http://www.sciencedirect.com/science/article/pii/S0001870881800060.

[Bei82] A. A. Beilinson. "Lettre de Beilinson à Soulé". Preprint, June 12, 2004, K-theory Preprint Archives. Nov. 1982. URL: https://faculty.math.illinois.edu/K-theory/0694/.

[BBD82] A. A. Beilinson, J. Bernstein, and P. Deligne. "Faisceaux pervers". In: *analyse et topologie sur les espaces singuliers, CIRM, 6-10 juillet 1981*. Vol. I. Astérisque 100. Société Mathématique de France, 1982, pp. 5–171.

[Fri82] Eric M. Friedlander. *Étale Homotopy of Simplicial Schemes*. Annals of Mathematics Studies 104. Princeton, New Jersey; Tokyo: Princeton University Press; University of Tokyo Press, 1982, pp. vii+190. ISBN: 0-691-08288-X. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=676809.

[Fal83] G. Faltings. "Endlichkeitssätze für abelsche Varietäten über Zahlkörpern." ger. In: *Inventiones mathematicae* 73 (1983), pp. 349–366. URL: http://eudml.org/doc/143051.

[Lic84] S. Lichtenbaum. "VALUES OF ZETA-FUNCTIONS AT NON-NEGATIVE INTEGERS". In: *Number Theory Noordwijkerhout 1983. Proceedings of the Journées Arithmétiques held at Noordwijkerhout, The Netherlands July 11–15, 1983*. Ed. by H. Jager. Vol. 1068. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1984. Chap. 11, pp. 127–138. ISBN: 9783540389064. DOI: 10.1007/bfb0099447. URL: http://dx.doi.org/10.1007/BFb0099447.

[Wal85] F. Waldhausen. "Algebraic $K$-theory of spaces". In: *Algebraic and Geometric Topology. Proceedings of a Conference held at Rutgers University, New Brunswick, USA July 6–13, 1983*. Ed. by A. Ranicki, N. Levitt, and F. Quinn. Vol. 1126. Lecture Notes in Mathematics. Berlin and Heidelberg: Springer-Verlag, 1985, pp. 318–419. ISBN: 9783540394136. DOI: 10.1007/bfb0074449. URL: http://dx.doi.org/10.1007/BFb0074449.

[AM86] M. Artin and B. Mazur. *Etale Homotopy*. Ed. by A. Dold and B. Eckmann. 2nd ed. Vol. 100. Lecture Notes in Mathematics. Berlin Heidelberg: Springer-Verlag, 1986. DOI: 10.1007/bfb0080957. URL: https://doi.org/10.1007%2Fbfb0080957.

[Blo86] S. Bloch. "Algebraic cycles and higher $K$-theory". In: *Advances in Mathematics* 61 (3 Oct. 1986), pp. 267–304. ISSN: 0001-8708.

DOI: https://doi.org/10.1016/0001-8708(86)90081-2. URL: http://www.sciencedirect.com/science/article/pii/0001870886900812.

[Bei87a] A. A. Beilinson. "Height pairing between algebraic cycles". In: *K-Theory, Arithmetic and Geometry. Seminar, Moscow University, 1984–1-986.* Ed. by Yu. I. Manin. Vol. 1289. Lecture Notes in Mathematics. Berlin and Heidelberg: Springer-Verlag, 1987. Chap. 1, pp. 1–26. ISBN: 9783540480167. DOI: 10.1007/bfb0078364. URL: http://dx.doi.org/10.1007/BFb0078364.

[Bei87b] A. A. Beilinson. "Height pairing between algebraic cycles. Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference on Algebraic Geometry, August 18-24, 1985, Arcata, California". In: *Current Trends in Arithmetical Algebraic Geometry.* Ed. by K. A. Ribet. Vol. 67. Contemporary Mathematics. Providence, RI: American Mathematical Society, 1987. Chap. 1, pp. 1–24. DOI: 10.1090/conm/067/902590. URL: https://doi.org/10.1090/conm/067/902590.

[Moe89] I. Moerdijk. "Prodiscrete groups and Galois toposes". In: *Nederl. Akad. Wetensch. Indag. Math.* 51.2 (1989), pp. 219–234. ISSN: 0019-3577. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=1005054.

[NS89] Yu. P. Nesterenko and A. A. Suslin. "Homology of the general linear group over a local ring and the Milnor *K*-theory". Russian. In: *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 53.1 (1989), pp. 121–146. ISSN: 0373-2436.

[Nis89] Ye. A. Nisnevich. "The Completely Decomposed Topology on Schemes and Associated Descent Spectral Sequences in Algebraic K-Theory". In: *Algebraic K-Theory: Connections with Geometry and Topology.* Ed. by J. F. Jardine and V. P. Snaith. Vol. 279. NATO ASI Series. Springer Netherlands, 1989, pp. 241–342. ISBN: 9789400923997. DOI: 10.1007/978-94-009-2399-7_11. URL: http://dx.doi.org/10.1007/978-94-009-2399-7_11.

[Del90] P. Deligne. "Catégories tannakiennes". In: *The Grothendieck Festschrift. A Collection of Articles Written in Honor of the 60th Birthday of Alexander Grothendieck*. Ed. by P. Cartier et al. Vol. II. Modern Birkhäuser Classics 88. Boston, MA: Birkhäuser, 1990, pp. 111–195. ISBN: 9780817645670. DOI: 10.1007/978-0-8176-4575-5_3. URL: http://dx.doi.org/10.1007/978-0-8176-4575-5_3.

[TT90] R. W. Thomason and T. Trobaugh. "Higher Algebraic $K$-Theory of Schemes and of Derived Categories". In: *The Grothendieck Festschrift. A Collection of Articles Written in Honor of the 60th Birthday of Alexander Grothendieck*. Ed. by P. Cartier et al. Vol. III. Modern Birkhäuser Classics 88. Boston, MA: Birkhäuser, 1990, pp. 247–435. ISBN: 978-0-8176-4568-7. DOI: 10.1007/978-0-8176-4576-2_10.

[Ser91] J.-P. Serre. "Motifs". In: *JOURNÉES ARITHMÉTIQUES de LUMINY, 17-21 JUILLET 1989*. Ed. by G. Lachaud. Vol. 198-200. Astérisque. Société Mathématique de France, 1991. Chap. 27, pp. 333–349. DOI: 10.24033/ast.105.

[Jan92] U. Jannsen. "Motives, numerical equivalence, and semi-simplicity." In: *Inventiones mathematicae* 107 (3 1992), pp. 447–452. URL: http://eudml.org/doc/143974.

[Tot92] B. Totaro. "Milnor $K$-theory is the simplest part of algebraic $K$-theory." English. In: *K-Theory. An Interdisciplinary Journal for the Development, Application, and Influence of K-Theory in the Mathematical Sciences* 6.2 (1992), pp. 177–189. ISSN: 0920-3036; 1573-0514/e.

[Voe92] V. Voevodsky. "Homology of schemes and covariant motives". <p>Thesis (Ph.D.)–Harvard University</p>. PhD thesis. 1992, p. 64. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004%5C&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation%5C&res_dat=xri:pqdiss%5C&rft_dat=xri:pqdiss:9228294.

[Blo94] S. Bloch. “THE MOVING LEMMA FOR HIGHER CHOW GROUPS”. In: *Journal of Algebraic Geometry* 3 (3 1994), pp. 537–568. ISSN: 1056-3911.

[Jan94] Uwe Jannsen. “Motivic sheaves and filtrations on Chow groups”. In: *Motives. Summer Research Conference on Motives, July 20-August 2, 1991, University of Washington, Seattle, Washington.* Ed. by U. Jannsen, S. L. Kleiman, and J.-P. Serre. Vol. 55.1. Proceedings of Symposia in Pure Mathematics. Providence, RI: American Mathematical Society, 1994, pp. 245–302. DOI: 10.1090/pspum/055.1/1265529. URL: https://doi.org/10.1090/pspum/055.1/1265529.

[Lev94] M. Levine. “Bloch’s higher Chow groups revisited”. English. In: *K-theory. Strasbourg, 1992.* Ed. by C. Kassel, J.-L Loday, and N. Schappacher. Vol. 226. Astérisque. Paris: Société Mathématique de France, 1994, pp. 235–320.

[Mil94] J. S. Milne. “Motives over Finite Fields”. In: *Motives. Summer Research Conference on Motives, July 20-August 2, 1991, University of Washington, Seattle, Washington.* Ed. by U. Jannsen, S. L. Kleiman, and J.-P. Serre. Vol. 55.1. Proceedings of Symposia in Pure Mathematics. Providence, RI: American Mathematical Society, 1994, pp. 401–459. DOI: 10.1090/pspum/055.1/1265529. URL: https://doi.org/10.1090/pspum/055.1/1265529.

[Sch94] A. J. Scholl. “Classical Motives”. In: *Motives. Summer Research Conference on Motives, July 20-August 2, 1991, University of Washington, Seattle, Washington.* Ed. by U. Jannsen, S. L. Kleiman, and J.-P. Serre. Vol. 55.1. Proceedings of Symposia in Pure Mathematics. Providence, RI: American Mathematical Society, 1994, pp. 163–187. DOI: 10.1090/pspum/055.1/1265529. URL: https://doi.org/10.1090/pspum/055.1/1265529.

[BL95] S. Bloch and S. Lichtenbaum. “A Spectral Sequence for Motivic Cohomology”. Preprint, March 3, 1995, K-theory Preprint

Archives, http://www.math.uiuc.edu/K-theory/0062/. 1995. URL: `http://www.math.uiuc.edu/K-theory/0062/`.

[SV96] A. Suslin and V. Voevodsky. "Singular homology of abstract algebraic varieties". In: *Inventiones Mathematicae* 123 (1 Dec. 1996), pp. 61–94. ISSN: 1432-1297. DOI: `10.1007/bf01232367`. URL: `http://dx.doi.org/10.1007/BF01232367`.

[Voe96] V. Voevodsky. *The Milnor Conjecture.* preprint, Max-Planck-Institute for Mathematics, Bonn. Dec. 1996. URL: `https://faculty.math.illinois.edu/K-theory/0170/`.

[Lev97] M. Levine. "Lambda-operations, $K$-theory and motivic cohomology". In: *Algebraic $K$-theory (Toronto, ON, 1996).* (volume in memory of R. Thomason). Ed. by V. P. Snaith. Vol. 16. Fields Institute Communications. Providence, RI: American Mathematical Society, 1997, pp. 131–184.

[Ful98] W. Fulton. *Intersection Theory.* 2nd ed. Berlin and Heidelberg: Springer-Verlag, 1998. ISBN: 978-0-387-98549-7. DOI: `10.1007/978-1-4612-1700-8`.

[Lev98] M. Levine. *Mixed motives.* Vol. 57. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 1998, pp. x+515. ISBN: 0-8218-0785-4. DOI: `10.1090/surv/057`. URL: `https://doi.org/10.1090/surv/057`.

[Voe98] V. Voevodsky. "$\mathbf{A}^1$-Homotopy Theory". In: *Proceedings of the International Congress of Mathematicians Berlin 1998 August 18-27.* Ed. by G. Fischer and U. Rehmann. Vol. 1. Extra Volume ICM 1998. Bielefeld: Documenta Mathematica Journal of the Deutsche Mathematiker-Vereinigung, 1998. Chap. Invited One-Hour Plenary Lectures, pp. 579–604. URL: `https://www.mathunion.org/fileadmin/ICM/Proceedings/ICM1998.1/ICM1998.1.ocr.pdf`.

[MV99] F. Morel and V. Voevodsky. "$A^1$-homotopy theory of schemes". en. In: *Publications Mathématiques de l'IHÉS* 90 (1999), pp. 45–143. URL: `http://www.numdam.org/item/PMIHES_1999__90__45_0`.

[Sus00] A. A. Suslin. "Higher Chow Groups and Étale Cohomology". In: *Cycles, Transfers, and Motivic Homology Theories. (AM-143)*. Vol. 143. Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 2000, pp. 239–254. ISBN: 9780691048154. URL: http://www.jstor.org/stable/j.ctt7tcnh.8.

[SV00] A. A. Suslin and V. Voevodsky. "Relative cycles and Chow sheaves". In: *Cycles, Transfers, and Motivic Homology Theories. (AM-143)*. Vol. 143. Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 2000, pp. 10–86. URL: http://www.jstor.org/stable/j.ctt7tcnh.

[Voe00] Vladimir Voevodsky. "Triangulated categories of motives over a field". In: *Cycles, Transfers, and Motivic Homology Theories.* Ed. by V. Voevodsky, A. Suslin, and E. M. Friedlander. Vol. 143. Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 2000, pp. 188–238. URL: http://www.jstor.org/stable/j.ctt7tcnh.

[Dug01] D. Dugger. "Combinatorial Model Categories Have Presentations". In: *Advances in Mathematics* 164 (1 Dec. 2001), pp. 177–201. ISSN: 0001-8708. DOI: https://doi.org/10.1006/aima.2001.2015. URL: http://www.sciencedirect.com/science/article/pii/S0001870801920158.

[Edw01] H. M. Edwards. *Riemann's Zeta Function.* Reprint of the 1974 original [Academic Press, New York; MR0466039 (57 #5922)]. 31 East 2nd Street,Mineola, NY 11501: Dover Publications, Inc., 2001, pp. xiv+315. ISBN: 0-486-41740-9.

[Lev01] M. Levine. "Techniques of localization in the theory of algebraic cycles". In: *Journal of Algebraic Geometry* 10.2 (2001), pp. 299–363. ISSN: 1056-3911.

[Zha01] S.-W. Zhang. "Geometry of algebraic points". In: *First International Congress of Chinese Mathematicians (Beijing, 1998)*. Ed. by L. Yang and S.-T. Yau. Vol. 20. AMS/IP Studies in Advanced

Mathematics. American Mathematical Society, 2001, pp. 185–198.

[FS02] E. M. Friedlander and A. Suslin. "The spectral sequence relating algebraic $K$-theory to motivic cohomology". en. In: *Annales scientifiques de l'École Normale Supérieure*. 4th ser. 35.6 (2002), pp. 773–875. DOI: 10.1016/s0012-9593(02)01109-6. URL: http://www.numdam.org/item/ASENS_2002_4_35_6_773_0.

[Poo03] B. Poonen. *Hilbert's tenth problem over rings of number-theoretic interest*. Arizona Winter School on "Number theory and logic", March 15–19, 2003, Tucson, Arizona. Feb. 2003. URL: https://math.mit.edu/~poonen/papers/aws2003.pdf.

[Ser03] J.-P. Serre. "Lettre à Armand Borel". In: *Œuvres - Collected Papers 1949 - 1959*. Vol. 1. Springer Collected Works in Mathematics. Berlin, Heidelberg: Springer-Verlag, 2003. Chap. 20, pp. 243–250.

[Gro04] A. Grothendieck. "A. Grothendieck, 16 août 1964". In: A. Grothendieck and J.-P. Serre. *GROTHENDIECK-SERRE CORRESPONDENCE. Bilingual Edition*. Ed. by P. Colmez and J.-P. Serre. Providence, RI: American Mathematical Society, Société Mathématique de France, 2004. Chap. 61, pp. 172–175.

[TV04] B. Toën and G. Vezzosi. "A remark on $K$-theory and $S$-categories". In: *Topology* 43 (4 July 2004), pp. 765–791. ISSN: 0040-9383. DOI: https://doi.org/10.1016/j.top.2003.10.008. URL: http://www.sciencedirect.com/science/article/pii/S0040938303000806.

[Vis05] A. Vistoli. "Grothendieck topologies, fibered categories and descent theory". In: *Fundamental Algebraic Geometry Grothendieck's FGA Explained*. Vol. 123. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 2005, pp. 1–104.

[RØ06] O. Röndigs and P. A. Østvær. "Motives and modules over motivic cohomology". In: *Comptes Rendus Mathematique* 342 (10 May 2006), pp. 751–754. ISSN: 1631-073X. DOI: https://doi.

org/10.1016/j.crma.2006.03.013. URL: http://www.sciencedirect.com/science/article/pii/S1631073X06001233.

[Jon07] A. J. de Jong. *Weil cohomology theories.* 2007. URL: http://math.columbia.edu/~dejong/seminar/note_on_weil_cohomology.pdf.

[BM08] A. J. Blumberg and M. A. Mandell. "The localization sequence for the algebraic $K$-theory of topological K-theory". In: *Acta Mathematica* 200.2 (2008), pp. 155–179. ISSN: 0001-5962. DOI: 10.1007/s11511-008-0025-4. URL: http://dx.doi.org/10.1007/s11511-008-0025-4.

[Lev08] M. Levine. "The homotopy coniveau tower". In: *Journal of Topology* 1.1 (2008), pp. 217–267. ISSN: 1753-8416. DOI: 10.1112/jtopol/jtm004. URL: https://doi.org/10.1112/jtopol/jtm004.

[Noo08] B. Noohi. "Fundamental groups of topological stacks with the slice property". In: *Algebraic & Geometric Topology* 8.3 (Aug. 2008), pp. 1333–1370. DOI: 10.2140/agt.2008.8.1333. URL: https://doi.org/10.2140%2Fagt.2008.8.1333.

[Poo08] B. Poonen. "Undecidability in Number Theory". In: *Notices of the AMS* 55.3 (2008). URL: https://www.ams.org/notices/200803/.

[GS09] D. Gepner and V. Snaith. "On the motivic spectra representing algebraic cobordism and algebraic $K$-theory". In: *Documenta Mathematica* 14 (2009), pp. 359–396. ISSN: 1431-0635.

[HTT] J. Lurie. *Higher Topos Theory.* Vol. 189. Annals of Mathematics Studies 170. Princeton, NJ: Princeton University Press, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.

[Bei10] A. Beilinson. "Remarks on Grothendieck's standard conjectures". In: (June 2010). eprint: 1006.1116. URL: https://arxiv.org/pdf/1006.1116.

[GW10] U. Görtz and T. Wedhorn. *Algebraic Geometry. Part I: Schemes. With Examples and Exercises.* Advanced Lectures in Mathematics. Vieweg+Teubner Verlag, 2010.

[Voe10a] Vladimir Voevodsky. "Homotopy theory of simplicial sheaves in completely decomposable topologies". In: *Journal of Pure and Applied Algebra* 214 (8 2010), pp. 1384–1398. ISSN: 0022-4049. DOI: https://doi.org/10.1016/j.jpaa.2009.11.004. URL: http://www.sciencedirect.com/science/article/pii/S0022404909002631.

[Voe10b] Vladimir Voevodsky. "Unstable motivic homotopy categories in Nisnevich and cdh-topologies". In: *Journal of Pure and Applied Algebra* 214 (8 Aug. 2010), pp. 1399–1406. ISSN: 0022-4049. DOI: https://doi.org/10.1016/j.jpaa.2009.11.005. URL: http://www.sciencedirect.com/science/article/pii/S0022404909002643.

[CD12] D.-C. Cisinski and F. Déglise. "Triangulated categories of mixed motives". Version 3. In: (Dec. 2012). eprint: 0912.2110. URL: https://arxiv.org/pdf/0912.2110.

[McL12] C. McLarty. "Zariski cohomology in second order arithmetic". Version 2. In: (July 2012). eprint: 1207.0276. URL: https://arxiv.org/abs/1207.0276.

[Mum12] D. Mumford. *Abelian Varieties.* Ed. by R. A. Rao, A. Hogadi, and R. Munshi. 2nd. Vol. 13. Tata Institute of Fundamental Research Publications. New Delhi: Hindustan Book Agency, 2012.

[BGT13] A. J. Blumberg, D. Gepner, and G. Tabuada. "A universal characterization of higher algebraic $K$-theory". In: *Geometry & Topology* 17.2 (Apr. 2013), pp. 733–838. ISSN: 1465-3060. DOI: 10.2140/gt.2013.17.733. URL: http://dx.doi.org/10.2140/gt.2013.17.733.

[Wei13] C. A. Weibel. *The $K$-book. An introduction to algebraic $K$-theory.* Graduate Studies in Mathematics 145. Providence, RI: American Mathematical Society, 2013, pp. xii+618. ISBN: 978-0-8218-9132-2.

[Hoy14] M. Hoyois. "A quadratic refinement of the Grothendieck–Lefschetz–Verdier trace formula". In: *Algebraic & Geometric Topology* 14.6 (Dec. 2014), pp. 3603–3658. ISSN: 1472-2747. DOI: 10.2140/agt.2014.14.3603. URL: http://dx.doi.org/10.2140/agt.2014.14.3603.

[Low14] Z. L. Low. "Universes for category theory". Version 2. In: (Nov. 2014). eprint: 1304.5227. URL: https://arxiv.org/pdf/1304.5227.

[Oor14] F. Oort. "The Weil conjectures". In: *Nieuw Archief voor Wiskunde (5)* 15.3 (Sept. 2014), pp. 211–219. URL: http://www.nieuwarchief.nl/serie5/toonnummer.php?deel=15&nummer=3&taal=1.

[BS15] B. Bhatt and P. Scholze. "The pro-étale topology for schemes". In: *De la Géométrie Algébrique aux Formes Automorphes (I). Une collection d'articles en l'honneur du soixantième anniversaire de Gérard Laumon.* Ed. by J.-B. Bost et al. Vol. 369. Société Mathématique de France, 2015, pp. 99–201. ISBN: 978-2-85629-805-3.

[GH15] D. Gepner and R. Haugseng. "Enriched ∞-categories via non-symmetric ∞-operads". In: *Advances in Mathematics* 279 (2015), pp. 575–716. ISSN: 0001-8708. DOI: https://doi.org/10.1016/j.aim.2015.02.007. URL: http://www.sciencedirect.com/science/article/pii/S0001870815000535.

[Rob15] M. Robalo. "*K*-theory and the bridge from motives to noncommutative motives". In: *Advances in Mathematics* 269 (Jan. 2015), pp. 399–550. ISSN: 0001-8708. DOI: https://doi.org/10.1016/j.aim.2014.10.011. URL: http://www.sciencedirect.com/science/article/pii/S0001870814003570.

[Maz16] A. Mazel-Gee. "Quillen adjunctions induce adjunctions of quasicategories". In: *New York Journal of Mathematics* 22 (Jan. 2016), pp. 57–93. ISSN: 1076-9803. URL: http://nyjm.albany.edu:8000/j/2016/22_57.html.

[AHW17] A. Asok, M. Hoyois, and M. Wendt. “Affine representability results in $\mathbf{A}^1$ -homotopy theory, I: Vector bundles”. In: *Duke Mathematical Journal* 166.10 (July 2017), pp. 1923–1953. ISSN: 0012-7094. DOI: 10.1215/00127094-0000014x. URL: http://dx.doi.org/10.1215/00127094-0000014X.

[HA] J. Lurie. “Higher Algebra”. Sept. 2017. URL: http://www.math.harvard.edu/~lurie/papers/HA.pdf.

[Poo17] B. Poonen. *Rational points on varieties.* Vol. 186. American Mathematical Society, 2017.

[BH18] T. Bachmann and M. Hoyois. “Norms in motivic homotopy theory”. Version 4. In: (Nov. 2018). eprint: 1711.03061. URL: https://arxiv.org/pdf/1711.03061.

[Elm+18] E. Elmanto et al. “Motivic infinite loop spaces”. Version 4. In: (Sept. 2018). eprint: 1711.05248. URL: https://arxiv.org/pdf/1711.05248.

[Hoy18] M. Hoyois. “Higher Galois theory”. In: *Journal of Pure and Applied Algebra* 222 (7 July 2018), pp. 1859–1877. ISSN: 0022-4049. DOI: https://doi.org/10.1016/j.jpaa.2017.08.010. URL: http://www.sciencedirect.com/science/article/pii/S002240491730186X.

[Kha18] A. A. Khan. “The Morel-Voevodsky localization theorem in spectral algebraic geometry”. Version 3. In: (July 2018). eprint: 1610.06871. URL: https://arxiv.org/pdf/1610.06871.

[Sta18] The Stacks project authors. *The Stacks project.* https://stacks.math.columbia.edu. 2018.

[Cis19] D.-C. Cisinski. *Higher Categories And Homotopical Algebra.* Vol. 180. Cambridge studies in advanced mathematics. Cambridge, NY: Cambridge University Press, 2019, pp. xviii+430. ISBN: 978-1-108-47320-0. DOI: 10.1017/9781108588737. URL: https://doi.org/10.1017/9781108588737.

[McL19] C. McLarty. "THE LARGE STRUCTURES OF GROTHENDIECK FOUNDED ON FINITE ORDER ARITHMETIC". In: *The Review of Symbolic Logic* (Aug. 2019), pp. 1–30. ISSN: 1755-0211. DOI: 10.1017/s1755020319000340. URL: http://dx.doi.org/10.1017/S1755020319000340.

[htt] Joel David Hamkins (https://mathoverflow.net/users/1946/joel-david-hamkins). *Large categories vs.* U*-categories: why is the loss of category-theoretic information inessential?* MathOverflow. URL:https://mathoverflow.net/q/279696 (version: 2017-08-28). eprint: https://mathoverflow.net/q/279696. URL: https://mathoverflow.net/q/279696.

[nLab] *nLab*. URL: https://ncatlab.org/nlab/show/HomePage.

[Sch] U. Schreiber. *Introduction to Stable Homotopy Theory*. URL: https://ncatlab.org/nlab/show/Introduction+to+Stable+Homotopy+Theory.

[Gro70] A. Grothendieck. "MOTIFS". 1965–1970. URL: https://webusers.imj-prg.fr/~leila.schneps/grothendieckcircle/motives.pdf.