Weakenings of compactness and normality on Isbell-Mrówka spaces, Hyperspaces of Vietoris and Abelian groups

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To my mother Tânia and my father Alberto.

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Resumo

Vinicius de Oliveira Rodrigues. **Enfraquecimentos de compacidade e normalidade em espaços de Isbell-Mrówka, hiperespaços de Vietoris e grupos Abelianos**. Tese (Doutorado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022.

Nós fornecemos um exemplo de espaço topológico Tychonoff, almost-normal não normal e exploramos almost-normalidade restrita aos espaços de Isbell-Mrówka. Seguindo essa linha de estudo, estudamos almost disjoint families fortemente $\aleph_0$-separadas comparando elas ao que se sabe sobre almost disjoint families normais e pseudonormais. Definimos uma nova família de conjuntos especiais de números reais relacionadas a esses problemas que chamamos de weak λ-sets. Esse estudo explora algumas questões de Paul Szeptycki e Sergio García-Balan.

Nós exploramos as perguntas de John Ginsburg sobre pseudocompacidade e compacidade enumerável de hiperespaços de Vietoris. Em particular, obtivemos um exemplo de um subespaço de $\beta\omega$ contendo ω cujas todas potências menores do que a característica cardinal $\mathfrak{h}$ são enumeravelmente compactas, mas cujo hiperespaço de Vietoris não é pseudocompacto. Também exploramos essas perguntas restritas a espaços de Isbell-Mrówka, provando que a existência de uma MAD family cujo hiperespaço de Vietoris de seu espaço de Isbell-Mrówka não é pseudocompacto é equivalente ao número de Baire de ω^* ser menor ou igual à $\mathfrak{c}$. Também obtivemos um exemplo consistente de um espaço de Isbell-Mrówka deste tipo de cardinalidade $\omega_2 < \mathfrak{c}$.

Finalmente, utilizamos forcing para obter uma classificação para grupos Abelianos de não torção de cardinalidade $\leq 2^{\mathfrak{c}}$ que admitem uma topologia enumeravelmente compacta Hausdorff contendo sequências convergentes, parcialmente respondendo uma questão de Dikranjan and Shakhmatov.

Palavras-chave: Espaços de Isbell-Mrówka. Grupos Topológicos. Hiperespaços de Vietoris. Compacidade Enumerável. Pseudocompacidade. Combinatória Infinita. Topologia Geral.

Abstract

Vinicius de Oliveira Rodrigues. **Weakenings of compactness and normality on Isbell-Mrówka spaces, Hyperspaces of Vietoris and Abelian groups**. Thesis (Doctorate). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2022.

We provide an example of a Tychonoff almost-normal topological space which is not normal and explore almost-normality in the realm of Isbell-Mrówka spaces. Following this line, we study strongly $\aleph_0$-separated almost disjoint families by comparing them with what is known about normal and pseudonormal almost disjoint families. We define a new family of special sets of reals related to these problems which we called weak λ-sets. This study explores some questions of Paul Szeptycki and Sergio García-Balan.

We explore John Ginsburg's questions on pseudocompact and countably compact Vietoris hyperspaces. In particular, we provide an example of a subspace of $\beta\omega$ containing ω whose every power below the cardinal characteristic $\mathfrak{h}$ is countably compact, but whose Vietoris hyperspace fails to be pseudocompact. We explore the converse implications in this class of spaces. We also study these questions in the realm of Isbell-Mrówka spaces, proving that the existence of a MAD family whose Vietoris hyperspace of its Isbell-Mrówka space is not pseudocompact is equivalent to the Baire number of ω^* being less or equal to $\mathfrak{c}$. We also provide a consistent example of such an Isbell-Mrówka space of cardinality $\omega_2 < \mathfrak{c}$.

Finally, we force a classification of non-torsion Abelian groups of size $\leq 2^{<\mathfrak{c}}$ that admit a Hausdorff countably compact group topology containing convergent sequences, partially answering a question of Dikranjan and Shakhmatov.

Keywords: Isbell-Mrówka spaces. Topological Groups. Hyperspaces of Vietoris. Countably Compactness. Pseudocompacity. Infinitary Combinatorics. General Topology.

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Chapter 0

Preliminary material

0.1 Introduction

0.1.1 About this thesis

This thesis contains a big part of the material I have worked with during my PhD. The main objects of study here are countably compact topological groups, Isbell-Mrówka spaces and pseudocompact hyperspaces of Vietoris.

Instead of just presenting the results of our own, I also tried to include background material to make this thesis more self-contained than our papers. However, we believe that it is important to be really clear about which results are an original contribution I coauthored during my PhD and which are not. Not every result I coauthored is part of the thesis: some results I coauthored appeared or are going to appear in the thesis of other people, so they are not part of this thesis. Also, there are results I coauthored that I feel that I did not contribute enough to make them part of this thesis.

Thus, the results which I coauthored and are part of this thesis are marked with an asterisk (*). This marking means that I am a coauthor of the result, that they are not appearing in the thesis of other people and that I have made a contribution to them. This marking does not imply that I claim I am the "main contributor" of the result, but it does mean I have contributed.

The original results of this thesis appear or are very strongly based on the following published or accepted papers:

- Yasser F. Ortiz-Castillo, Vinicius O. Rodrigues and Artur H. Tomita. "Small cardinals and the pseudocompactness of hyperspaces on subspaces of $\beta\omega$". In: *Topology and its Applications* 246 (2018). pp. 9-21.
- Vinicius O. Rodrigues and Artur H. Tomita. "Small MAD families whose Isbell-Mrówka space has pseudocompact hyperspace". In: *Fundamenta Mathematicae* 247 (2019), pp. 99-108.
- Matheus K. Bellini, Ana Carolina Boero, Irene Castro-Pereira, Vinicius O. Rodrigues

and Artur H. Tomita. “Countably compact group topologies on non-torsion Abelian groups of size continuum with non-trivial convergent sequences”. In: *Topology and its Applications* 267 (2019) p. 106894.

- Vinicius O. Rodrigues and Victor S. Ronchim. “Almost-normality of Isbell-Mrówka spaces”. In: *Topology and its Applications* 288 (2021). p. 107470
- O. Guzmán, M. Hrušák, Vinicius O. Rodrigues, S. Todorčević and A. H. Tomita. “Maximal almost disjoint families and pseudocompactness of hyperspaces”. In: *Topology and its Applications* 305 (2022) p. 107872.
- Vinicius O. Rodrigues, Victor S. Ronchim and Paul Szeptycki. “Special sets of reals and weak forms of normality on Isbell-Mrówka spaces”. In: *Commentationes Mathematicae Universitatis Carolinae* (2022, accepted).

The papers that I coauthored and appear in the references but are not listed above do not contain results that are part of this thesis. Of course, I have contributed to their results, but they will appear on the thesis of other people or have already appeared. Some of their results will be stated without proof in this thesis and are used to motivate some of the problems we have worked with, but since they are not part of this thesis they will not be marked with an asterisk, as explained above.

The end of definitions are marked with a $\square$ symbol. This does not have any deep meaning, it only marks the end of the definition to avoid confusion since I am using the same font for the definitions and for the main text of the thesis. Some texts, such as [50], do the same. I do the same for theorems stated with no proof.

In this preliminary chapter we will fix the notation common to all chapters and state (often without proof) some basic results used in General Topology, Set Theory and Logic. We present some proofs for the convenience of the reader, but we assume that the reader has some experience with the basics of General Topology (up to the construction of $\beta\omega$ using ultrafilters and basic metrization theorems) and set theory (including the basics on Forcing and Iterated Forcing). Most of the nontrivial results will be reviewed in this preliminary chapter, but the reader which is not used to the material presented here may find some difficulty. Our base theory is ZFC, so all our results will be of relative consistency with ZFC and we do not keep track of the use of the Axiom of Choice. Not all background material is presented in this chapter: some of it is presented as they are needed.

Regarding forcing, we have decided to include few to no background material about it in this thesis since we believe it would take too many pages and would possibly feel very incomplete and unhelpful. We follow the basic literature as [49], [50] and [47] with the main difference that when working with iterated forcing, we consider $\mathbb{P}_\alpha$ to be a complete suborder of $\mathbb{P}_\beta$ whenever $\alpha < \beta$, as in [3], so elements of $\mathbb{P}_\alpha$ are functions whose domain is a subset of α. This means that we work with domains (as done in [3]) instead of supports (as done in [49]).

We have organized the structure of this thesis as follows:

- Chapter 1 contains background material about almost disjoint families, Isbell-Mrówka spaces and their relations with some cardinal characteristics of the continuum.

- Chapter 2 contains original results about weakenings of normality on Isbell-Mrówka spaces along with background material.
- Chapter 3 contains background material about problems concerning weakenings of compacity on Vietoris Hyperspaces along with original results, mostly about subspaces of $\beta\omega$.
- Chapter 4 contains original results related to the problems presented on Chapter 3 about Isbell-Mrówka spaces.
- Chapter 5 contains original results about countably compact Abelian topological groups, along with background material.

0.2 Ordered sets and trees

As a shorthand, in this thesis, "iff" stands for "if, and only if,".

In this section we fix the basic language to talk about orders and trees.

Definition 0.2.1. Let X be a set and R be a binary relation on X.

- We say that R is *reflexive* iff for every $x \in X$, xRx.
- We say that R is *irreflexive* iff for every $x \in X$, $\neg(xRx)$.
- We say that R is *transitive* iff for every $x, y, z \in X$, if xRy and yRz then xRz.
- We say that R is *symmetric* iff for every $x, y \in X$, if xRy then yRx.
- We say that R is *anti-symmetric* iff for every $x, y \in X$, if xRy and yRx then $x = y$.
- We say that R is *total*, or *linear* iff for every $x, y \in X$, xRy or yRx or $x = y$.

□

Definition 0.2.2. Let X be a set, $x \in X$ and R be a binary relation on X. We define $\mathrm{pred}_R(x) = \{y \in X : yRx\}$. We read it as *the set of R-predecessors of x.* □

Definition 0.2.3. Let X be a set and R be a binary relation on X.

- We say that (X, R) is a *pre-ordered set*, or a *pre-order* (p.o.), iff R is reflexive and transitive on X. In this case, we say that R is a pre-order on X.
- We say that (X, R) is a *partially ordered set*, or a *partial order*, iff R is an anti-symmetric pre-order on X. In this case, we say that R is a partial order on X.
- We say that (X, R) is a *strict partially ordered set*, or a *strict partial order*, iff R is a irreflexive pre-order on X. In this case, we say that R is a strict partial order on X.
- We say that (X, R) is a *linearly ordered set*, or a *linear order*, iff R is a total partial order on X. In this case, we say that R is a linear order on X.
- We say that (X, R) is a *strict linearly ordered set*, or a *strict linear order*, iff R is a total strict partial order on X. In this case, we say that R is a strict linear order on X.

□

We expect the reader to be used to the relations between partially ordered sets/linearly ordered sets and their strict versions.

Now we define some special subsets of p.o.'s.

Definition 0.2.4. Let $(P, \leq)$ be a p.o. A set $D \subseteq \mathbb{P}$ is *dense* iff for every $p \in \mathbb{P}$ there exists $d \in D$ such that $d \leq p$.

A set $G \subseteq \mathbb{P}$ is a *filter* iff is *closed upwards* (for every $p \in \mathbb{P}$ and $q \in G$, if $q \leq p$ then $p \in G$) and for every $p, q \in G$ there exists $r \in G$ such that $r \leq p, q$.

Two elements p, q of $\mathbb{P}$ are said to be *incompatible* iff there is no $r \in \mathbb{P}$ such that $r \leq p, q$. In this case, we write $p \perp q$. If p, q are not incompatible, we write $p \not\perp q$ and say that they are *compatible*.

A set $A \subseteq \mathbb{P}$ is an *antichain* iff every two distinct elements of A are incompatible. □

Now we define some special elements of p.o.'s

Definition 0.2.5. Let $(P, \leq)$ be a p.o., $p \in P$ and $a \in A$.

- We say that p is an upper bound of A iff for all $a \in A$, $a \leq p$.
- We say that p is a lower bound of A iff for all $a \in A$, $p \leq a$.
- We say that p is a maximum of A iff p is an upper bound of A and $p \in A$. Notice that if $(P, \leq)$ is a partial order then there exists at most one maximum of A. In this case, we denote it by $\max A$.
- We say that p is a minimum of A iff p is a lower bound of A and $p \in A$. Notice that if $(P, \leq)$ is a partial order then there exists at most one minimum of A. In this case, we denote it by $\max A$.
- We say that p is a supremum of A iff p is a minimum element of the collection of the upper bounds of A and $p \in A$. Notice that if $(P, \leq)$ is a partial order then there exists at most one supremum of A. In this case, we denote it by $\sup A$. Also, notice that every maximum element of A is a supremum of A.
- We say that p is a infimum of A iff p is a maximum element of the collection of the lower bounds of A and $p \in A$. Notice that if $(P, \leq)$ is a partial order then there exists at most one infimum of A. In this case, we denote it by $\inf A$. Also, notice that every minimum element of A is a infimum of A.
- We say that p is maximal in A if $p \in A$ and for all $b \in A$, if $a \leq b$ then $b \leq a$.
- We say that p is minimal in A if $p \in A$ and for all $b \in A$, if $b \leq a$ then $a \leq b$.

□

We also expect the reader to be used with well orders.

Definition 0.2.6. An *well ordered set*, or an *well order*, is a linearly ordered set $(X, \leq)$ such that every nonempty subset of X has a minimum element. □

Now we define what a tree is along with its associated concepts.

Definition 0.2.7. A tree is a strict partially ordered set $(T, <)$ such that for every $x \in T$, $\mathrm{pred}_<(x)$ is well ordered by $<$ (or, more formally, to the restriction of $<$ to $\mathrm{pred}_<(x)$). $\square$

As it is usual in mathematics, we usually identify T with $(T, <)$ when no confusion arises.

Definition 0.2.8. Let $(T, <)$ be a tree. We define:

- For $x \in T$, the *height of x in T*, denoted by $\mathrm{ht}_T(x)$, is the order type of $\mathrm{pred}_<(x)$.
- For an ordinal α, the *αth level of T* is $\mathrm{Lev}_T(\alpha) = \{x \in T : \mathrm{ht}_T(x) = \alpha\}$.
- The *height of T* is $\mathrm{ht}(T) = \min\{\alpha : \mathrm{Lev}_T(\alpha) = \emptyset\}$.
- For $x \in T$, the *set of successors of x in T* is $\mathrm{succ}_T(x) = \{y \in T : x < y \text{ and } \mathrm{ht}_T(y) = \mathrm{ht}_T(x) + 1\}$.
- T is *rooted* if $|\mathrm{Lev}_T(0)| = 1$. In this case, we say that the unique element in $\mathrm{Lev}_T(0)$ is the *root* of T.

$\square$

0.3 On basic General Topology

The set of the natural numbers is identified with the first infinite ordinal, ω. So 0 is officially a natural number. Ordinals are identified with cardinals, so $\omega = \aleph_0$. $\mathfrak{c}$ is the cardinality of the continuum.

We follow most definitions from S. Willard's "General Topology" [70] and R. Engelking's "General Topology" [24].

There is a lot of ambiguity regarding the definitions of the separation axioms in the literature as they are not really standard. For instance, the two books we just mentioned do not agree on them. Thus, we officially define:

Definition 0.3.1. Let X be a topological space. Then X is...

1. ...T_0, if for every $x, y \in X$, if $x \neq y$ then there exists an open set U such that ($x \in U$ and $y \notin U$) or ($y \in U$ and $x \notin U$) (that is, no two distinct points are exactly in the same open sets),
2. ...T_1, if for every $x, y \in X$, if $x \neq y$ then there exists an open set U such that $x \in U$ and $y \notin U$,
3. ...T_2, or *Hausdorff*, if for every $x, y \in X$ there exist two disjoint open sets U, V such that $x \in U$ and $y \in V$,
4. ...T_3, if for every $x \in X$ and closed set $F \subseteq X$ with $x \notin F$ there exist two disjoint open sets U, V such that $x \in U$ and $F \subseteq V$,
5. ...*regular*, if it is T_3 and T_1,

6. ...$T_{3\frac{1}{2}}$, if points can be separated from closed sets by functions, that is, for every closed set $F \subseteq X$ and $x \in X \setminus F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $F \subseteq f^{-1}[\{1\}]$,
7. ...*completely regular*, or *Tychonoff*, if X is $T_{3\frac{1}{2}}$ and T_1,
8. ...T_4, if for every two disjoint closed sets F, K there exist two disjoint open sets U, V with $F \subseteq U$, $K \subseteq V$,
9. ...*normal*, if it is T_4 and T_1.

□

In this document, a compact space is not automatically assumed to be Hausdorff (which agrees with [70] but not with [24]).

Definition 0.3.2. A topological space X is said to be *compact* iff every open cover of X contains a finite subcover. □

We will use Alexander's subbase theorem.

Theorem 0.3.3 (Alexander's subbase theorem, [1])**.** Let X be a topological space. If there is a subbase $\mathcal{B}$ of X such that every open cover $\mathcal{U} \subseteq \mathcal{B}$ has a finite subcover, then X is compact. □

Sketches of proofs can also be found in [70, Problem 17S] or [24, Problem 3.12.2].

Regarding sequences and families, limit points, cluster points and accumulation points, the terminology adopted is:

Definition 0.3.4. We use the following conventions for functions:

1. A function is a set of ordered pairs f such that for all a, b, c, if $(a, b), (a, c) \in f$, then $b = c$. The domain of f, denoted by $\operatorname{dom} f$, is the set of all a's for which there exists b such that $(a, b) \in f$. The *range* of f, denoted by $\operatorname{ran} f$, is the set of all b's for which there exists a such that $(a, b) \in f$. The expression $f : A \to B$ means that "f is a function, $\operatorname{dom} f = A$ and $\operatorname{ran} f \subseteq B$". We read it as f is a function from A into B. If, additionally, $\operatorname{ran} f = B$, we say that f is *onto* B.
2. Let f be a function. If $a \in \operatorname{dom} f$, $f(a)$ is the unique element b such that $(a, b) \in f$. Given sets A and B, we define $f[A] = \{f(a) : a \in A \cap \operatorname{dom} f\}$, $f^{-1}[B] = \{a \in \operatorname{dom} f : f(a) \in B\}$. Notice that this makes sense even if $A \not\subseteq \operatorname{dom} f$ and $B \not\subseteq \operatorname{ran} f$.
3. Let A be a function of domain I. The terminology $(A_i : i \in I)$ denotes a *family* (which is the same as the function A). The terminology $\{A_i : i \in I\}$ denotes the range of a family A. The term *collection* means the same as "set".
4. A *sequence* is a family whose domain is ω. So sequences have 0 in their domain.

□

Definition 0.3.5. Let X be a topological space. Then:

1. Let $f : \omega \to X$ be a sequence. We say that x is a *cluster point* or an *accumulation point* for f iff for every open neighborhood U of x, $\forall n \in \omega\, \exists m \geq n\, f(m) \in U$, that is, $\{m \in \omega : f(m) \in U\}$ is infinite.

2. Let $f : \omega \to X$ be a sequence. We say that x is a *limit point* of f, or simply a *limit* of f, iff for every open neighborhood U of x, the set $\{m \in \omega : f(m) \in U\}$ is cofinite, that is, there exists $n \in \omega$ such that for every $m \geq n$, $f(m) \in U$. In this case, we say that f *converges* to x. If X is Hausdorff, f has at most one limit point, and we denote it by $\lim f$ or $\lim_{n\to\infty} f(n)$.

3. If $A \subseteq X$, we say that x is a *cluster point* or an *accumulation point* for A iff for every open neighborhood U of x, $A \cap U \setminus \{x\} \neq \emptyset$.

4. If $A \subseteq X$, we say that x is an ω-*cluster point* or an ω-*accumulation point* for A iff for every open neighborhood U of x, $|A \cap U| \geq \omega$.

5. Let $(U_n)_{n\in\omega}$ be a sequence of open subsets of X. We say that $x \in X$ is a *cluster point*, or *accumulation point* for $(U_n)_{n\in\omega}$ iff for every open neighborhood V of x, the set $\{n \in \omega : U_n \cap V \neq \emptyset\}$ is infinite.

$\square$

We mention in advance that almost disjoint families are not families in our sense. Notice that for T_1 spaces, the concepts of ω-accumulation point and accumulation point are the same.

We use the following convention for products:

Definition 0.3.6. Let $(X_i : i \in I)$ be a family of topological spaces and $X = \prod_{i\in I} X_i$ its cartesian product. The natural projection from X onto X_i is denoted by π_i, unless confusion arises (we will use an ad hoc notation when needed).

The product topology is the Tychonoff topology. So basic open sets are of the form $\bigcap_{i\in F} \pi_i^{-1}[U_i]$, where $F \subseteq I$ is finite and U_i is an open subset of X_i for each $i \in F$. $\square$

Countably compact spaces are also not automatically assumed to be T_1:

Definition 0.3.7. A topological space X is said to be *countably compact* iff one of the three equivalent conditions hold:

- Every countable open cover of X has a finite subcover,
- every sequence on X has an accumulation point, or
- every infinite subset of X has an ω-accumulation point

In particular, if X is T_1, X is countably compact iff every countable subset of X has an accumulation point. $\square$

Perfect mappings may be useful when dealing with compact and countably compact spaces.

Definition 0.3.8. Let X, Y be topological spaces and $f : X \to Y$ onto Y.

We say that f is *perfect* iff f is continuous, closed, and for every $y \in Y$, $f^{-1}[\{y\}]$ is compact.

We say that f is *quasi-perfect* iff f is continuous, closed and for every $y \in Y$, $f^{-1}[\{y\}]$ is countably compact. $\square$

Proposition 0.3.9 (This is essentially [24, p. 3.7.2])**.** Let X, Y be topological spaces and let $f : X \to Y$ be onto. Then:

1) If f is perfect and K is a compact subset of Y, then $f^{-1}[K]$ is compact.

2) If f is quasi-perfect and K is a countably compact subset of Y, then $f^{-1}[K]$ is countably compact.

Proof. We prove 2) since 1) is similar. Suppose f is quasi-perfect and let K be a countably compact subset of Y. Let $\mathcal{U}$ be a countable open cover of $L = f^{-1}[K]$ by open subsets of X. We claim that $\{Y \setminus f[X \setminus \bigcup \mathcal{U}'] : \mathcal{U}' \in [\mathcal{U}]^{<\omega}\}$ is a countable open cover of K. To see that, fix $k \in K$. Since $f^{-1}[\{k\}]$ is countably compact and is covered by $\mathcal{U}$, there exists $\mathcal{U}' \in [\mathcal{U}]^{<\omega}$ such that $f^{-1}[\{k\}] \subseteq \bigcup \mathcal{U}'$. This implies that $Y \setminus f[X \setminus \bigcup \mathcal{U}']$.

So there exists $l \in \omega$ and $\mathcal{U}_0, \dots, \mathcal{U}_l \in [\mathcal{U}]^{<\omega}$ such that $\{Y \setminus f[X \setminus \bigcup \mathcal{U}_i] : i < l\}$ covers K. We claim that $L \subseteq \bigcup\{\bigcup \mathcal{U}_i : i < l\}$: given $a \in L$, $f(a) \in K$, so there exists $i < l$ such that $f(a) \in Y \setminus f[X \setminus \bigcup \mathcal{U}_i]$. This implies that $a \notin (X \setminus \bigcup \mathcal{U}_i)$, so $a \in \bigcup \mathcal{U}_i$, as intended. $\square$

Locally compact spaces are officially defined as follows:

Definition 0.3.10. Let X be a topological space. We say that X is *locally compact* iff every point of X has a compact neighborhood basis. $\square$

We have the following result, which we state without proof:

Lemma 0.3.11 ([70, Theorem 18.2])**.** Let X be a Hausdorff space. Then X is locally compact iff every point of X has a compact neighborhood. $\square$

Regarding pseudocompactness, there are two very popular definitions. However, they are only equivalent for $T_{3\frac{1}{2}}$. We will deal with spaces which are not $T_{3\frac{1}{2}}$, so making an initial distinction is important.

Definition 0.3.12. Let X be a topological space. Then:

1. We say that X is *pseudocompact* iff every continuous function $f : X \to \mathbb{R}$ is bounded.

2. We say that X is *G-pseudocompact* [32], or *feebly compact* [46] iff every sequence of nonempty open subsets of X has an accumulation point.

$\square$

Their equivalence is left as an exercise to the reader:

Proposition 0.3.13. Let X be a $T_{3\frac{1}{2}}$ space. Then X is pseudocompact iff X is feebly compact.

There is also the standard notion of sequential compactness.

Definition 0.3.14. Let X be a topological space. We say that X is sequentially compact iff for every sequence $f : \omega \to X$ there exists a strictly increasing $k : \omega \to \omega$ such that $f \circ k : \omega \to X$ converges (i.e., there exists $x \in X$ such that x is a limit point of $f \circ k$).

Equivalently, X is sequentially compact iff for every sequence $f : \omega \to X$ there exists $A \in [\omega]^\omega$ and $x \in X$ such that for every open neighborhood V of x, the set $\{n \in A : f(n) \notin V\}$ is finite. □

Relative sequential compactness and countable compactness are going to be useful concepts for us.

Definition 0.3.15. Let X be a topological space and $Y \subseteq X$. We say that Y is *relatively countably compact* in X iff for every countable open cover $\mathcal{U}$ of X there exists a finite $\mathcal{U}' \subseteq \mathcal{U}$ such that $Y \subseteq \bigcup \mathcal{U}'$. □

Proposition 0.3.16. Let X be a topological space and $Y \subseteq X$. The following are equivalent:

a) Y is relatively countably compact in X

b) every infinite subset of Y has an ω-accumulation point in X

c) every sequence in Y has an accumulation point in X.

Proof. a) implies b): Let A be an infinite subset of Y. Without loss of generality A is countable. Suppose by contradiction that A has no ω-accumulation point in X. For each $F \in [A]^{<\omega}$, let $U_F = \operatorname{int}_X(X \setminus (A \setminus F)) = \operatorname{int}_X(F \cup (X \setminus A))$. $\mathcal{U} = \{U_F : F \in [A]^{<\omega}\}$ is a collection of open sets. We claim that it covers X: for each $x \in X$, there exists an open neighborhood V of X such that $F = V \cap A$ is finite, but then $V \subseteq F \cup (X \setminus A)$, so $x \in U_F$.

There exists a finite $\mathcal{F} \subseteq [A]^{<\omega}$ such that $Y \subseteq \bigcup\{U_F : F \in \mathcal{F}\}$. But then $Y \subseteq (\bigcup F) \cup X \setminus A$, so $A \subseteq \bigcup \mathcal{F}$, so A is finite, a contradiction.

b) implies a): Let $f : \omega \to Y$ be a sequence. If the range of f is finite, there exists $y \in Y$ such that $f^{-1}[\{y\}]$ is infinite, and it is clear then that y is an accumulation point of f. So suppose $\operatorname{ran} f$ is infinite. By b), $\operatorname{ran} f$ has an ω-accumulation point $x \in X$. We claim that x is an accumulation point of f. Let V be an open neighborhood of x and $m \in \omega$. Since $V \cap \operatorname{ran} f$ is infinite, there exists $y \in \operatorname{ran} f \cap V$ such that there exists $n \geq m$ such that $f(n) = y$. This completes the proof.

c) implies a): We show that not a) implies not c). Suppose that $\{U_n : n \in \omega\}$ is a countable open cover of X such that for every $m \in \omega$, $Y \setminus \bigcup_{n \leq m} U_n \neq \emptyset$. Fix $f(n) \in Y \setminus \bigcup_{n \leq m} U_n$. We claim that f has no accumulation point in x: there exists $k \in \omega$ such that $x \in U_k$, but $\{n \in \omega : f(n) \in U_k\} \subseteq k$ is finite. □

Since for T_1 spaces an accumulation point is the same as an ω-accumulation point, we get:

Corollary 0.3.17. Let X be an T_1 topological space and $Y \subseteq X$. Then Y is relatively countably compact in X iff every infinite subset of Y has an accumulation point in X.

Regarding relative sequential compactness:

Definition 0.3.18. Let X be a topological space and $Y \subseteq X$. We say that Y is relatively sequentially compact in X iff every sequence in Y has a convergent subsequence, that is, iff for every $f : \omega \to Y$ there exists an strictly increasing $k : \omega \to \omega$ and $x \in X$ such that $f \circ k$ converges to x.

Equivalently, Y is relatively sequentially compact in X iff for every sequence $f : \omega \to Y$ there exists $A \in [\omega]^\omega$ and $x \in X$ such that for every open neighborhood V of x, the set $\{n \in A : f(n) \notin V\}$ is finite. □

The following should be clear from the previous definitions and propositions:

Corollary 0.3.19. Every sequentially compact topological space is countably compact. Moreover, for every topological space X and $Y \subseteq X$, if Y is relatively sequentially compact in X, then it is relatively countably compact in X.

There are some relations between these concepts and pseudocompactness. This is a version of [43, Lemma 2.2].

Proposition 0.3.20. Let X be a topological space and $D \subseteq X$ be dense. Then:

a) If D is relatively countably compact in X, then X is feebly compact

b) If, additionally, X is T_1 and D is open and discrete, then X pseudocompact implies that D is relatively countably compact in X

Proof. a) Let $(U_n : n \in \omega)$ be a sequence of nonempty open subsets of X. For each $n \in \omega$, fix $d_n \in D \cap U_n$. Let x be an ω-accumulation point of $\{d_n : n \in \omega\}$ in X. Given an open neighborhood V of x, $\{n \in \omega : U_n \cap V \neq \emptyset\}$ contains $\{n \in \omega : d_n \in V\}$, which is infinite.

b) Suppose that D is open and discrete. Suppose that D is not relatively countably compact in X. Let $A \subseteq D$ be set with no accumulation points in X. Then A is closed in X. So A is a discrete clopen, so every function from A into $\mathbb{R}$ is continuous and can be extended to a continuous function from X into A. Since A is infinite, the proof is complete. □

0.4 Filters, ultrafilters and filter-limits

Our basic definitions regarding filters and ideals are:

Definition 0.4.1. Let X be a nonempty set. A *filter* on X is a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ such that:

1. $X \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$,
2. For every $A \in \mathcal{F}$ and $B \subseteq X$, if $A \subseteq B$ then $B \in \mathcal{F}$, and
3. For every $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$.

□

Dually, we define:

Definition 0.4.2. Let X be a nonempty set. An *ideal* on X is a collection $\mathcal{I} \subseteq \mathcal{P}(X)$ such that:

1. $X \notin \mathcal{I}, \emptyset \in \mathcal{I}$,
2. For every $A \in \mathcal{I}$ and $B \subseteq X$, if $B \subseteq A$ then $B \in \mathcal{I}$, and
3. For every $A, B \in \mathcal{I}$, $A \cup B \in \mathcal{I}$.

□

Some authors allow $\emptyset$ to be in filters and X to be in ideals, so $\mathcal{P}(X)$ is both an ideal and a filter on their texts. They usually name what we are naming filters/ideals as proper filters/proper ideals. We will not use their approach.

The dual filters/ideals are defined in the following proposition, which is left to the reader:

Proposition 0.4.3. Let X be a nonempty set, $\mathcal{F}$ be a filter on X and $\mathcal{I}$ be an ideal on X. Then:

1. The collection $\mathcal{F}^d = \{X \setminus A : A \in \mathcal{F}\}$ is an ideal on X. This ideal is the *dual ideal* of $\mathcal{F}$,
2. The collection $\mathcal{I}^d = \{X \setminus A : A \in \mathcal{I}\}$ is a filter on X. This filter is the *dual filter* of $\mathcal{I}$,
3. $(\mathcal{I}^d)^d = \mathcal{I}$ and $(\mathcal{F}^d)^d = \mathcal{F}$. □

We also define:

Definition 0.4.4. Let X be a nonempty set, $\mathcal{F}$ be a filter on X and $\mathcal{I}$ be an ideal on X. Then:

1. We say that $\mathcal{F}$ is *free* iff it contains the cofinite subsets of X.
2. We say that $\mathcal{I}$ is *free* iff it contains the finite subsets of X.
3. We say that $\mathcal{F}$ is an *ultrafilter* iff for every $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$ (or, equivalently, if $\mathcal{F}$ is not contained in a distinct larger filter on X)
4. We say that $\mathcal{I}$ is a *maximal ideal* iff for every $A \subseteq X$, either $A \in \mathcal{I}$ or $X \setminus A \in \mathcal{I}$ (or, equivalently, if $\mathcal{I}$ is not contained in a distinct larger ideal on X)

□

Before we continue, we need to defined the relation "almost contained":

Definition 0.4.5. Suppose A, B are sets. We define:

- $A \subseteq^* B$ iff $A \setminus B$ is finite,
- $A \subsetneq^* B$ iff ($A \subseteq^* B$ and $B \not\subseteq^* A$), and
- $A =^* B$ iff $A \subseteq^* B$ and $B \subseteq^* A$.

□

Ultrafilters and maximal ideals exist by the Zorn's Lemma. Moreover, the following holds, which is also left to the reader:

Definition 0.4.6. Let X be a nonempty set and let $\mathcal{C} \subseteq \mathcal{P}(X)$.

1. There exists a ultrafilter on X containing $\mathcal{C}$ iff for every finite nonempty subset $\mathcal{C}' \subseteq \mathcal{C}, \bigcap \mathcal{C}' \neq \emptyset$. In this case, there also exists the smallest filter containing $\mathcal{C}$, which is given by $\{A \subseteq X : \exists B \in \mathcal{C}\, B \subseteq A\}$.

2. There exists a free ultrafilter on X containing $\mathcal{C}$ iff for every finite nonempty subset $\mathcal{C}' \subseteq \mathcal{C}, |\bigcap \mathcal{C}'| = \infty$. In this case, there also exists the smallest free filter containing $\mathcal{C}$, which is given by $\{A \subseteq X : \exists B \in \mathcal{C}\, B \subseteq^* A\}$.

3. There exists a maximal ideal on X containing $\mathcal{C}$ iff for every finite nonempty subset $\mathcal{C}' \subseteq \mathcal{C}, \bigcup \mathcal{C}' \neq X$. In this case, there also exists the smallest ideal containing $\mathcal{C}$, which is given by $\{A \subseteq X : \exists B \in \mathcal{C}\, A \subseteq B\}$.

4. There exists a free maximal ideal on X containing $\mathcal{C}$ iff for every finite nonempty subset $\mathcal{C}' \subseteq \mathcal{C}$, $X \setminus \bigcup \mathcal{C}'$ is infinite. In this case, there also exists the smallest free ideal containing $\mathcal{C}$, which is given by $\mathcal{F} = \{A \subseteq X : \exists B \in \mathcal{C}\, A \subseteq^* B\}$.

□

The fixed ultrafilters are defined by the following proposition, which is left to the reader.

Proposition 0.4.7. Let X be a topological space. For every $x \in X$, the set $\mathcal{U}_x = \{A \subseteq X : x \in A\}$ is a ultrafilter. We say that a ultrafilter $\mathcal{U}$ on X is *fixed* if there exists $x \in X$ such that $\mathcal{U} = \mathcal{U}_x$ (or, equivalently, $\{x\} \in \mathcal{U}$). Every ultrafilter is either free or fixed. ω^* is the set of free ultrafilters on ω. □

This implies that every filter is contained in a ultrafilter and that every ideal is contained in a maximal ideal.

Now we define $\mathcal{F}$-limits:

Definition 0.4.8. Let X be a topological space, $f : \omega \to X$ be a sequence, $x \in X$ and $\mathcal{F}$ be a filter on ω. We say that x is a $\mathcal{F}$*-limit* for f iff for every open neighborhood V of x, the set $\{n \in \omega : f(n) \in V\} \in \mathcal{F}$. □

We leave the following basic, known results to the reader:

Proposition 0.4.9. Let X be a Hausdorff topological space, $f : \omega \to X$ be a sequence, $x_0, x_1 \in X$ and $\mathcal{F}$ be a filter on ω. If x_0 and x_1 are $\mathcal{F}$-limits of f, then $x_0 = x_1$. Thus, if there exists $x \in X$ such that x is an $\mathcal{F}$-limit of f, we define $\mathcal{F}$-$\lim f$ as this unique x. □

Proposition 0.4.10. Let X, Y be topological spaces, $f : \omega \to X$ be a sequence, $x \in X$, $g : X \to Y$ be a continuous function and $\mathcal{F}$ be a filter on ω. If x is a $\mathcal{F}$-limit for f, then $g(x)$ is a $\mathcal{F}$-limit for $g \circ f$. So, if X, Y are Hausdorff, $x = \mathcal{F}$-$\lim f$ implies $g(x) = \mathcal{F}$-$\lim g \circ f$. □

Proposition 0.4.11. Let $(X_i : i \in I)$ be a family of topological spaces, $X = \prod_{i \in I} X_i$, $f : \omega \to X$ be a sequence, $x \in X$, and $\mathcal{F}$ be a filter on ω. Then x is a $\mathcal{F}$-limit for f iff for every $i \in I$ $\pi_i(x)$ is an $\mathcal{F}$-limit for X_i. So, if X_i is Hausdorff for every $i \in I$, it follows that f has a $\mathcal{F}$-limit iff $f_i = \pi_i \circ f$ has a $\mathcal{F}$-limit for each $i \in I$, and, in this case, $\mathcal{F}\text{-}\lim f = (\mathcal{F}\text{-}\lim f_i : i \in I)$. $\square$

Free filter limits are an extension of the natural notion of convergence of sequences, and fixed ultrafilters limits are rather uninteresting. We leave the proof to the reader.

Proposition 0.4.12. Let X be a topological space, $x \in X$ and $\mathcal{F}$ be a free filter on ω. If x is a limit point for f, then x is a $\mathcal{F}$-limit of f. If $\mathcal{F}$ is the filter of the cofinite subsets of ω, the converse also holds. $\square$

Proposition 0.4.13. Let X be a topological space, $x \in X$, $f : \omega \to X$ and $n \in \omega$. Then x is a $\mathcal{U}_n$-limit if and only if every open set containing x also contains $f(n)$ ($\mathcal{U}_n$ is the fixed ultrafilter generated by n). In particular, $f(n)$ is the only $\mathcal{U}_n$-limit if X is T_1, and, following our convention, if X is Hausdorff we write $\mathcal{U}_n\text{-}\lim f = f(n)$. $\square$

Finally, we define filter accumulation points of sets:

Definition 0.4.14. Let X be a topological space, $x \in X$, $\mathcal{F}$ be a filter on ω and $(A_n : n \in \omega)$ be a sequence of subsets of X. We say that x is an *$\mathcal{F}$-cluster point*, or an *$\mathcal{F}$-accumulation point* of $(A_n : n \in \omega)$ iff for every neighborhood V of x, $\{n \in \omega : V \cap A_n \neq \emptyset\} \in \mathcal{F}$. $\square$

Now we relate ultrafilter limits to compactness-like properties. We prove the following known propositions for the sake of completeness:

Proposition 0.4.15. Let X be a compact topological space, $f : \omega \to X$ and $\mathcal{U}$ be an ultrafilter on ω. Then f has an $\mathcal{U}$-limit.

Proof. Let $\mathcal{C} = \{\operatorname{cl} f[A] : A \in \mathcal{U}\}$. $\mathcal{C}$ is a collection of closed sets with the finite intersection property, so there exists $x \in \bigcap \mathcal{C}$ by the compactness of X. We claim that x is a $\mathcal{U}$-limit of f.

Let V be an open neighborhood of x. We must see that $\{n \in \omega : f(n) \in V\} \in \mathcal{U}$. If not, $A = \{n \in \omega : f(n) \notin V\} \in \mathcal{U}$ since $\mathcal{U}$ is an ultrafilter. Then $\operatorname{cl} f[A] \in \mathcal{C}$, so $x \in \operatorname{cl} f[A]$, but $\operatorname{cl} f[A] \cap V = \emptyset$ and $x \in V$, a contradiction. $\square$

Regarding countable compactness:

Proposition 0.4.16. Let X be a topological space, $x \in X$ and $f : \omega \to X$ be given. Then x is an ω-accumulation point of f iff there exists a free ultrafilter $\mathcal{U}$ such that x is a $\mathcal{U}$-limit of f.

Proof. For the "if" part, suppose there exists $\mathcal{U} \in \omega^*$ such that x is a $\mathcal{U}$-limit of f. Given a open neighborhood V of x, the set $\{n \in \omega : f(n) \in V\}$ is in $\mathcal{U}$, therefore is infinite.

For the converse, suppose x is a $\mathcal{U}$-limit of f. For each open neighborhood V of x, let $A_V = \{n \in \omega : f(n) \in V\}$. A_V is infinite. Notice that if V_0, V_1 are two open neighborhoods of x, $A_{V_0 \cap V_1} = A_{V_0} \cap A_{V_1}$, therefore $\{A_V : V \text{ is an open neighborhood of } x\}$ is contained in some free ultrafilter $\mathcal{U}$. It follows that x is a $\mathcal{U}$-limit of x. $\square$

Corollary 0.4.17. A topological space X is countably compact iff for every $f : \omega \to X$ there exists $\mathcal{U} \in \omega^*$ such that f has a $\mathcal{U}$-limit.

Finally, we have analogous results for feeble compactness:

Proposition 0.4.18. Let X be a topological space, $x \in X$ and $U = (U_n : n \in \omega)$ be a sequence of nonempty open subsets of X. Then x is an ω-accumulation point of U iff there exists a free ultrafilter $\mathcal{U}$ such that x is a $\mathcal{U}$-accumulation point of U.

Proof. For the "if" part, suppose there exists $\mathcal{U} \in \omega^*$ such that x is a $\mathcal{U}$-accumulation point of U. Given a open neighborhood V of x, the set $\{n \in \omega : U_n \cap V \neq \emptyset\}$ is in $\mathcal{U}$, therefore is infinite.

For the converse, suppose x is a $\mathcal{U}$-limit of f. For each open neighborhood V of x, let $A_V = \{n \in \omega : U_n \cap V \neq \emptyset\}$. A_V is infinite. Notice that if V_0, V_1 are two open neighborhoods of x, $A_{V_0 \cap V_1} \subseteq A_{V_0} \cap A_{V_1}$, therefore $\{A_V : V \text{ is an open neighborhood of } x\}$ is contained in some free ultrafilter $\mathcal{U}$. It follows that x is a $\mathcal{U}$-accumulation point of U. $\square$

Corollary 0.4.19. A topological space X is G-pseudocompact iff for every sequence $(U_n : n \in \omega)$ of nonempty open subsets of X there exists $\mathcal{U} \in \omega^*$ such that $(U_n : n \in \omega)$ has a $\mathcal{U}$-accumulation point.

With these results in mind, the following definition makes sense:

Definition 0.4.20. Let X be a topological space and $\mathcal{U} \in \omega^*$.

- We say that X is $\mathcal{U}$-compact iff every sequence on X has an $\mathcal{U}$-limit.
- We say that X is $\mathcal{U}$-pseudocompact iff every sequence of nonempty open subsets of X has an $\mathcal{U}$-accumulation point.

$\square$

We note that free ultrafilters on ω are sometimes denoted by letters such as p, q, r. Thus, $\mathcal{U}$-compactness is often referred to as p-compactness. In this thesis we stick to the notation above.

The following corollary is clear from our previous results.

Corollary 0.4.21. Let X be a topological space and $\mathcal{U} \in \omega^*$. Then:

1. If X is compact, then X is $\mathcal{U}$-compact.
2. If X is *$\mathcal{U}$-compact*, then X is $\mathcal{U}$-pseudocompact and countably compact.
3. If X is *$\mathcal{U}$-pseudocompact*, then X is feebly compact.

Thus, we have the following diagram:

$$
\begin{tikzcd}
& \text{compact} \arrow[dl] \arrow[dr] & \\
\mathcal{U}\text{-compact} \arrow[rr] \arrow[d] & & \mathcal{U}\text{-pseudocompact} \arrow[d] \\
\text{countably compact} \arrow[rr] & & G\text{-pseudocompact} \arrow[d, shift left] \\
& & \text{pseudocompact} \arrow[u, dashed, shift left, "T_{3\frac{1}{2}}"]
\end{tikzcd}
$$

How far is a countably compact space to being $\mathcal{U}$-compact for some $\mathcal{U} \in \omega^*$? The following theorem gives us an idea. We present the proof for the sake of completeness since the reference assumes that X is Hausdorff (although the proof is the same).

Theorem 0.4.22 ([33, Theorem 2.6.])**.** Let X be a topological space. Then the following are equivalent:

a) There exists $\mathcal{U} \in \omega^*$ such that X is $\mathcal{U}$-compact,

b) all the powers of X are countably compact,

c) $X^{2^{\mathfrak{c}}}$ is countably compact,

d) $X^{|X|^\omega}$ is countably compact.

Proof. a) implies b): Let $\mathcal{U} \in \omega^*$ be such that X is $\mathcal{U}$-compact. By Proposition 0.4.11, $\mathcal{U}$-compactness is productive, so all the powers of X are $\mathcal{U}$-compact, thus, countably compact.

b) implies c) and d): trivial.

c) implies a): we show that not a) implies not c). Enumerate $\omega^* = \{\mathcal{U}_\alpha : \alpha < 2^{\mathfrak{c}}\}$. For each $\alpha < \mathfrak{c}$, let $f_\alpha : \omega \to X$ have no $\mathcal{U}$-limit. Let $f : \omega \to X^{2^{\mathfrak{c}}}$ be such that $\pi_\alpha \circ f = f_\alpha$ for each $\alpha < 2^{\mathfrak{c}}$. Then by Proposition 0.4.11 f has no $\mathcal{U}$-limit for any $\mathcal{U} \in \omega^*$. Thus, by Proposition 0.4.16, f has no accumulation point.

d) implies a): Let $\kappa = |X|^\omega$. Write $X^\omega = \{f_\alpha : \alpha < \kappa\}$. Let $f : \omega \to X^\kappa$ be such that $\pi_\alpha \circ f = f_\alpha$ for each $\alpha < \kappa$. Since X^κ is countably compact, f has an accumulation point x. By Proposition 0.4.16, x is an $\mathcal{U}$-limit of f for some $\mathcal{U} \in \omega^*$. Then X is $\mathcal{U}$-compact by Proposition 0.4.11. ☐

In particular, this theorem shows that the "degree of countable compactness" of a space X trivializes at $\min\{|X|^\omega, 2^{\mathfrak{c}}\}$. Pseudocompactness if a bit simpler:

Proposition 0.4.23. Let X be a topological space. Then X^ω is pseudocompact iff all powers of X are pseudocompact.

Proof. We prove the non-trivial direction. Since projections are continuous, if X^ω is pseudocompact then every finite power of X is pseudocompact. Now if κ is infinite, suppose $f : X^\kappa \to \mathbb{R}$ is unbounded. WLOG f is positive. For each n, there exists a finite $F_n \subseteq \kappa$ and a family $(U^n_\alpha : \alpha \in F_n)$ of nonempty open subsets of X such that $f\left[\bigcap_{\alpha\in F_n} \pi_\alpha^{-1}[U^n_\alpha]\right] \subseteq (n, +\infty)$. For each $\alpha \in \kappa \setminus F$, fix $x_\alpha \in X$. Let $F = \bigcup_{n\in\omega} F_n$. Let $\rho : X^F \to X^\kappa$ be such that $\rho(x)|F = x$ and $\rho(x)(\alpha) = x_\alpha$ for each $\alpha \in X \setminus F$. ρ is continuous. We claim that $f\circ\rho$ is unbounded: given n, let $\bar{x} \in \bigcap_{\alpha\in F_n} \pi_\alpha^{-1}[U^n_\alpha]$. Let $x = \bar{x}|F$. Since $F_n \subseteq F$, $\rho(x) \in \bigcap_{\alpha\in F_n} \pi_\alpha^{-1}[U^n_\alpha]$ as well, so $f \circ \rho(x) \in (n, +\infty)$. But then X^F is not pseudocompact, a contradiction. □

Proposition 0.4.24. Let X be a topological space. Then X^ω is feebly compact iff all powers of X are pseudocompact.

Proof. We prove the non-trivial direction. Since projections are continuous, if X^ω is pseudocompact then every finite power of X is pseudocompact. Now if κ is infinite, suppose $(W_n : n \in \omega)$ is a sequence of nonempty open subsets of X^κ.

By π_α, we denote the projection of X^κ into the coordinate α. For each n, fix a finite $F_n \subseteq \kappa$ and a family $(U^n_\alpha : \alpha \in F_n)$ of nonempty open subsets of X such that $\bigcap_{\alpha\in F_n} \pi_\alpha^{-1}[U^n_\alpha] \subseteq W_n$. Let $F = \bigcup_{n\in\omega} F_n$. Let ρ_α be the projection of X^F into the α'th coordinate for each $\alpha \in \omega$. Let $W'_n = \bigcap_{\alpha\in F_n} \rho_\alpha^{-1}[U^n_\alpha]$. Since X^F is pseudocompact, there exists an accumulation point $x \in X^F$ of the latter. Then if $\bar{x} \in X^\kappa$ is such that $\bar{x}|F = x$, it follows that $\bar{x}$ is an accumulation point of $(W_n : n \in \omega)$. □

0.5 Cardinal characteristics of the continuum and Martin's Axiom

Cardinal characteristics of the continuum are cardinals between ω_1 and $\mathfrak{c}$ that capture some combinatorical aspects that distinguish ω from $\mathfrak{c}$. For a very complete survey on them, see [10]. Here, we are only going to define and state the basic facts about some of them that we are going to use.

Definition 0.5.1. A collection $\mathcal{A} \subseteq [\omega]^\omega$ is *centered* iff every finite nonempty subcollection $\mathcal{A}' \subseteq \mathcal{A}$ has infinite intersection.

A *pseudointersection* of $\mathcal{A}$ is an infinite set P such that $P \subseteq^* A$ for every $A \in \mathcal{A}$. □

It is clear that every $\mathcal{A} \subseteq [\omega]^\omega$ which admits a pseudointersection is centered. However, the converse is not true: free ultrafilters are centered, but...

Lemma 0.5.2. Free ultrafilters do not admit pseudointersections.

Proof. Let $\mathcal{U}$ be a free ultrafilter. Suppose P is a pseudointersection of $\mathcal{U}$. Either $P \in \mathcal{U}$ or $P \notin \mathcal{U}$.

First, suppose $P \in \mathcal{U}$. Write $P = P_0 \cup P_1$, where P_0, P_1 are disjoint and infinite. Since $\mathcal{U}$ is an ultrafilter, either P_0 or P_1 are in $\mathcal{U}$. Let i be such that $P_i \in \mathcal{U}$. Then $P \subseteq^* P_i$, which means that $P_{1-i} = P \setminus P_i$ is finite, a contradiction.

Now suppose that $P \notin \mathcal{U}$. Then $\omega \setminus P \in \mathcal{U}$, so $P \subseteq^* \omega \setminus P$, a contradiction. □

Thus, we define:

Definition 0.5.3. $\mathfrak{p}$ is the smallest cardinal for which there exist $\mathcal{A} \subseteq [\omega]^\omega$ such that $\mathcal{A}$ is centered but does not admit a pseudointersection. $\mathfrak{p}$ is called the ***pseudointersection number***. □

Since free ultrafilters qualifies as such $\mathcal{A}$'s, we have that $\mathfrak{p}$ is well defined and $\mathfrak{p} \leq \mathfrak{c}$. However, $\mathfrak{p} > \omega$, since if $\mathcal{A} = \{A_n : n \in \omega\}$ is centered, by choosing $x_n \in \bigcap_{i \leq n} A_i \setminus n$, $P = \{x_i : i \in \omega\}$ is a pseudointersection of $\mathcal{A}$. Thus, we have:

Proposition 0.5.4. $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c}$.

Another important cardinal characteristic is $\mathfrak{h}$. We need some definitions:

Definition 0.5.5. Let $A \subseteq [\omega]^\omega$. We say that A is...

a) ...open, if for every $a \in A$ and every $b \in [\omega]^\omega$, if $b \subseteq^* a$ then $b \in A$.

b) ...dense, if for every $b \in [\omega]^\omega$ there exists $a \in A$ such that $a \subseteq^* b$.

□

The reader may verify that these open sets really define a topology in $[\omega]^\omega$ where the dense sets of this topology are exactly the dense sets defined above. This topology does not have good separating properties: it is not even T_0 since $=^*$-sets cannot be distinguished, and if we identify them, the resulting topology is not T_1 since if $a \subseteq^* b$, all open sets containing b also contain a.

Definition 0.5.6. The *$\subseteq^*$-topology of* $[\omega]^\omega$ is the topology of $[\omega]^\omega$ described above. □

The following lemma is just an observation, and usually used as the definition of "open dense" without referring to topological notions (as in [10]). We leave it to the reader.

Lemma 0.5.7. Let $A \subseteq [\omega]^\omega$. A is an open dense set iff A is open and for every $b \in [\omega]^\omega$ there exists $a \in A$ such that $a \subseteq b$. □

Then $\mathfrak{h}$ is defined as the Baire number of $[\omega]^\omega$ with this topology, that is...

Definition 0.5.8. The *distributivity number* $\mathfrak{h}$ is the smallest cardinality of a collection of open dense sets of $[\omega]^\omega$ whose intersection is empty. □

Proposition 0.5.9. $\mathfrak{h}$ is well defined and $\mathfrak{p} \leq \mathfrak{h} \leq \mathfrak{c}$

Proof. First, we see that $\mathfrak{h}$ is well defined and that $\mathfrak{h} \leq \mathfrak{c}$. Let $\mathcal{U}$ be a free ultrafilter. For each $x \in \mathcal{U}$, let $D_x = \{a \in [\omega]^\omega : a \cap x =^* \emptyset \text{ or } a \subseteq^* x\}$. Each D_x is open and dense. We claim that $\bigcap_{x \in \mathcal{U}} D_x = \emptyset$. Fix $b \in [\omega]^\omega$, then either $b \in \mathcal{U}$ or $b \notin \mathcal{U}$.

Case $b \in \mathcal{U}$: There exists $b_0 \in \mathfrak{U}$ such that $b_0 \subseteq b$ and $b \setminus b_0$ is infinite. Then $b \notin D_{b_0}$.

Case $b \notin \mathcal{U}$: Let $b_0 = \omega \setminus b$. Then $b_0 \in \mathfrak{U}$. Let $b_1 \subseteq b$ be an infinite set such that $b \setminus b_1$ is infinite. Let $b_2 = b_0 \cup b_1$. Then b_2 is in $\mathcal{U}$ and $b \notin D_{b_2}$.

Now we show that $\mathfrak{p} \leq \mathfrak{h}$. Let $(D_\alpha : \alpha < \mathfrak{h})$ be a collection of open dense subsets of $[\omega]^\omega$ with empty intersection. Construct a $\subseteq^*$-decreasing sequence $(a_\alpha : \alpha < \mathfrak{h})$ such that $a_0 = \omega$, $a_{\alpha+1} \in D_\alpha$ and a_γ is a pseudointersection of the preceding a_α's. There is no pseudointersection a of $\{a_\alpha : \alpha < \mathfrak{h}\}$ or we would have, since each D_α is open, that $a \in \bigcap_{\alpha<\mathfrak{h}} D_\alpha$. □

In fact, $\mathfrak{h}$ has the following property:

Proposition 0.5.10. Let $\kappa < \mathfrak{h}$ and $(\mathcal{D}_\alpha : \alpha < \kappa)$ be a family of open dense subsets of $[\omega]^\omega$. Then $\bigcap_{\alpha<\kappa} D_\alpha$ is open and dense.

Proof. It is straightforward to see that any intersection of open sets is open. We verify that in this case, the intersection is dense. Suppose not. Then there exists $U \in [\omega]^\omega$ such that for all $B \in D = \bigcap_{\alpha<\kappa} D_\alpha$, $B \not\subseteq^* U$. Let $g : U \to \omega$ a bijection. Let $D'_\alpha = \{g[B] : B \in D_\alpha \text{ and } B \subseteq U\}$. We claim that each D'_α is dense and that $\bigcap_{\alpha<\kappa} D'_\alpha = \emptyset$, which is a contradiction.

D'_α **is dense for every** $\alpha < \kappa$**:** Suppose $C \subseteq \omega$ is infinite. Then $g^{-1}[C] \subseteq U$ is infinite. Since D_α is open and dense, there exists $B \in D_\alpha$ such that $B \subseteq g^{-1}[C] \subseteq A$, so $g[B] \subseteq C$.

$\bigcap_{\alpha<\kappa} D'_\alpha$ **is empty:** Let A be given and let $E = g^{-1}[A] \subseteq U$. There exists $\alpha < \kappa$ such that $g^{-1}[A] \notin D_\alpha$, so $A = g[g^{-1}[A]] \notin D'_\alpha$ or we would have $g^{-1}[A] \notin D_\alpha$. □

Now we define unbounded and dominant families along with their cardinal characteristics.

Definition 0.5.11. Let N be an infinite countable set. Suppose $f, g \in \omega^N$. We define:

- $f <^* g$ if, and only if $\{n \in N : g(n) \leq f(n)\}$ is finite.
- $f \leq^* g$ if, and only if $\{n \in N : g(n) < f(n)\}$ is finite.

□

Definition 0.5.12. Let N be an infinite countable set.

- We say that $\mathcal{D} \subseteq \omega^N$ is a *dominant family* iff for every $f \in \omega^N$ there exists $g \in \mathcal{D}$ such that $f <^* g$ (or, equivalently, $f \leq^* g$).
- We say that $\mathcal{B} \subseteq \omega^N$ is an *unbounded family* iff for every $f \in \omega^N$ there exists $g \in \mathcal{B}$ such that $g \not<^* f$ (or, equivalently, $g \not\leq^* f$).
- $\mathfrak{d}$, the *dominating number*, is the least size of a dominating family on ω^ω.
- $\mathfrak{b}$, the *unbounding number*, is the least size of an unbounded family on ω^ω.

□

It should be clear that, despite the fact that $\mathfrak{b}$ and $\mathfrak{d}$ are defined for $N = \omega$, the set N does not matter. Moreover, unbounded families and dominating families are not families (i.e., functions/indexed sets). Also, the following should be obvious:

Lemma 0.5.13. Every dominating family is unbounded, thus, $\mathfrak{b} \leq \mathfrak{d}$. □

Another cardinal characteristic which will be useful for us is the splitting number.

Definition 0.5.14. Let $A, B \in [\omega]^\omega$. We say that B *splits* A iff both $A \setminus B$ and $A \cap B$ are infinite. A *splitting family* is a collection $\mathcal{S} \subseteq [\omega]^\omega$ such that for every $A \in [\omega]^\omega$ there exists $B \in \mathcal{S}$ such that B splits A. The *splitting number*, denoted by $\mathfrak{s}$, is the least size of a splitting family. □

It should be noted that, formally, splitting families are also not families.

Finally, we define the cardinal characteristic $\mathfrak{par}$.

Definition 0.5.15. Let $f : [\omega]^2 \to 2$ and $A \in [\omega]^\omega$. We say that A is *homogeneous* for f iff $f|[A]^2$ is constant. We say that A is *almost homogeneous* for f if there exists a finite set F such that $f|[A \setminus F]^2$ is constant.

$\mathfrak{par}$ is the least cardinality of a collection $\mathcal{A} \subseteq 2^{[\omega]^2}$ such that there is no $A \in [\omega]^\omega$ which is almost homogeneous for all $f \in \mathcal{A}$. □

By Ramsey's theorem, given $f : [\omega]^2 \to 2$, the set $\{A \in [\omega]^\omega : A \text{ is almost homogeneous for } f\}$ is open and dense, thus the following should be clear:

Proposition 0.5.16. $\mathfrak{h} \leq \mathfrak{par}$. □

Moreover, the following holds. For a proof see the reference.

Proposition 0.5.17 ([10, Theorem 3.5.]). $\mathfrak{par} = \min\{\mathfrak{b}, \mathfrak{s}\}$. □

A very important set-theoretic statement is *Martin's Axiom*.

Definition 0.5.18. Let κ be a cardinal and $\mathbb{P}$ a p.o.. $\mathrm{FA}_{\mathbb{P}}(\kappa)$ is the following statement: For every collection $\mathcal{D}$ of at most κ dense subsets of $\mathbb{P}$ there exists a filter $G \subseteq P$ such that for every $D \in \mathcal{D}$, $G \cap D \neq \emptyset$. □

Definition 0.5.19. A p.o. $\mathbb{P}$ is said to have the *countable chain condition* (c.c.c.) iff every antichain of $\mathbb{P}$ is countable. In general, we say that a $\mathbb{P}$ has the *κ-c.c.* (for a cardinal κ) iff every antichain of $\mathbb{P}$ has size less than κ. So c.c.c. means ω_1-c.c. □

Definition 0.5.20. $\mathrm{MA}(\kappa)$ means $\forall \mathbb{P}\,(\mathbb{P}$ is a p.o. with the c.c.c. $\to \mathrm{FA}_{\mathbb{P}}(\kappa))$.

Martin's axiom is the sentence $\forall \kappa < \mathfrak{c}\ \mathrm{MA}(\kappa)$.

$\mathfrak{m}$ is the smallest cardinal for which $\mathrm{MA}(\kappa)$ fails. □

Lemma 0.5.21. For every p.o. $\mathbb{P}$, for every countable collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$ and for every $p \in \mathbb{P}$ there exists a filter G of $\mathbb{P}$ such that $p \in G$ and for every $D \in \mathcal{D}$, $G \cap D$. In particular, for every p.o. $\mathbb{P}$, $\mathrm{FA}_{\mathbb{P}}(\omega)$.

Proof. Let $\mathfrak{D} = \{D_n : n \in \omega\}$ be a countable collection of dense subsets of $\mathbb{P}$ and $p \in \mathbb{P}$. Recursively choose a decresing sequence $(p_n : n \in \omega)$ such that $p_0 = p$ and $p_{n+1} \in D_n$. Let $G = \{q \in \mathbb{P} : \exists n \in \omega\, p_n \leq q\}$. □

Proposition 0.5.22. $\omega_1 \leq \mathfrak{m} \leq \mathfrak{p}$.

Proof. $\omega_1 \leq \mathfrak{m}$ follows from the previous lemma.

$\mathfrak{m} \leq \mathfrak{p}$ is known as Solovay's lemma. Suppose $\kappa < \mathfrak{m}$. We show that $\kappa < \mathfrak{p}$. Let $\mathfrak{A} \subseteq [\omega]^\omega$ be centered. Consider $\mathbb{P} = [\omega]^{<\omega} \times [\mathfrak{A}]^{<\omega}$ ordered by $(s, F) \leq (s', F')$ iff $s' \subseteq s$, $F' \subseteq F$ and $\forall n \in s \setminus s' \,\forall A \in F'\, n \in A$.

$\mathbb{P}$ has the c.c.c. since two elements of $\mathbb{P}$ with the same first coordinates are compatible (just take the union of the second coordinates). For each n, let $D_n = \{(s, F) \in \mathbb{P} : \exists m \geq n\, m \in s\}$ and, for each $A \in \mathcal{A}$, let $E_A = \{(s, F) \in \mathbb{P} : A \in F\}$. The reader may verify that each of these sets are dense.

Let G be a filter intersecting each E_A, D_n. Let $P = \bigcup\{s : \exists F\, (s, F) \in G\}$. s is infinite since $G \cap D_n \neq \emptyset$ for each $n \in \omega$. To see that $P \setminus A$ is finite for each $A \in \mathcal{A}$, fix $A \in \mathcal{A}$ and then fix $(s', F') \in G \cap E_A$. We claim that $P \setminus A \subseteq s'$. Fix $(s, F) \in G$. We must see that for every $n \in \omega$, if $n \in s \setminus A$ then $n \in s'$. Fix n. There exists $(s'', F'') \in G$ such that $(s'', F'') \leq (s', F'), (s, F)$. Since $(s'', F'') \leq (s, F)$, we have that $n \in s''$. Since $(s'', F'') \leq (s', F')$, if by contradiction we had $n \notin s'$, this would give us $n \in s'' \setminus s' \subseteq A$, a contradiction. $\square$

$\mathfrak{m} \leq \mathfrak{p}$ can be also seen as a consequence of Bell's Theorem, which we state below without proof.

Definition 0.5.23. Let $\mathbb{P}$ be a p.o. and $A \subseteq \mathbb{P}$. We say that A is *centered* iff for every $n \in \omega$ and $p_0, \ldots, p_n \in A$ there exists $p \in A$ such that $p \leq p_i$ for every $i \leq n$.

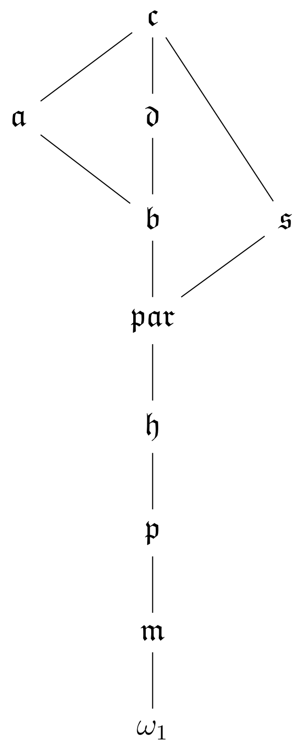
We say that $\mathbb{P}$ is σ-*centered* if there exists a sequence $(A_n : n \in \omega)$ of centered subsets of $\mathbb{P}$ such that $\mathbb{P} = \bigcup_{n \in \omega} A_n$. $\square$

It is worth comparing definitions 0.5.1 and 0.5.23. If we consider $[\omega]^\omega$ to be ordered by $\subseteq^*$, both definitions of *centered* coincide.

It is also clear that every σ-centered poset has the countable chain condition. Now we are ready to state Bell's Theorem. For a proof, see [49, Theorem III.3.61].

Theorem 0.5.24 (Bell's Theorem [4])**.** $\mathfrak{p}$ is the minimum cardinal κ for which there exists a σ-centered pre-order $\mathbb{P}$ such that $\mathrm{FA}_{\mathbb{P}}(\kappa)$ does not hold.

For a quick reference we draw the following diagram. The cardinal $\mathfrak{a}$ will be introduced in Definition 1.1.6 and is the smallest size of a MAD family.



0.6 The compactification of Stone-Čech of discrete spaces

Definition 0.6.1. Let X be a topological space. A *compactification* of X is a pair (Y, e) where Y is a compact Hausdorff topological space, $e : X \to Y$ is a topological embedding and $e[X]$ is dense in Y. $\square$

Clearly, if X admits a compactification, then X is a Tychonoff topological space. The converse is true: in this case, not only X admits a compactification, but a maximal one in some sense. This is what we call the Stone-Čech compactification of X. We will not treat the general case here. We refer to [69] for a complete treatment on the basics of this subject.

Proposition 0.6.2. Let X be a Tychonoff topological space. There exists a compactification $(\beta X, e)$ of X such that for every compact Hausdorff space K and every continuous $f : X \to K$, there exists a continuous function $\bar{f} : \beta X \to K$ such that $f = \bar{f} \circ e$. $\square$

Such a compactification is unique up to an homeomorphism preserving X, so we usually call it βX, the Stone-Čech compactification of X.

Lemma 0.6.3. Let X be a Tychonoff topological space. Suppose (Y, e), (Y', e') are two compactifications of X with the property from Proposition 0.6.2. Then there exists an homeomorphism $a : Y \to Y'$ such that $a \circ e = e'$.

Proof. There exists a continuous function $a : Y \to Y'$ such that $a \circ e = e'$. There exists a continuous function $a' : Y' \to Y$ such that $a \circ e' = e$. Notice that $a' \circ a|e[X]$ and

$a \circ a'|e'[X]$ are the identity on $e[X]$, $e'[X]$ which are dense in Y, Y', so $a' \circ a$ and $a \circ a'$ are the identity. So a is the homeomorphism we were looking for. $\square$

There is a nice construction for βD if D is a discrete nonempty space.

Definition 0.6.4. Let D be a nonempty discrete space. βD be the set of all ultrafilters on D. For each $d \in D$, let $e(d)$ be the fixed ultrafilter generated by $\{d\}$.

For each $A \subseteq D$, let $\widehat{A} = \{\mathcal{U} \in \beta D : A \in \mathcal{U}\}$. We consider βD have the topology generated by $\{\widehat{A} : A \subseteq D\}$. $\square$

Proposition 0.6.5. With the notation of the previous definition, the following is true:

a) $\{\widehat{A} : A \subseteq D\}$ is a basis of clopens for βD, and for each $A \subseteq D$, $\widehat{A} = \operatorname{cl} e[A]$.

b) βD is Hausdorff,

c) βD is compact,

d) e is a topological embedding whose range is dense.

e) $(\beta D, e)$ is the Stone-Čech compactification of D.

Proof. a) To see that this is a basis for βD, it suffices to prove that $\widehat{A} \cap \widehat{B} = \widehat{A \cap B}$. This follows from the fact that in a filter, $A \in \mathcal{U}$ and $B \in \mathcal{U} \leftrightarrow A \cap B \in \mathcal{U}$. To see that each $\widehat{A}$ is clopen, notice that its complement is $\widehat{D \setminus A}$. For the last claim, it is clear that $e[A] \subseteq \widehat{A}$, so $\operatorname{cl} e[A] \subseteq \widehat{A}$. For the converse, notice that if $\mathcal{U} \notin \operatorname{cl} e[A]$, then there exists $B \in \mathcal{U}$ such that $\widehat{B} \cap e[A] = \emptyset$, which implies that $A \cap B = \emptyset$. So $\mathcal{U} \notin \widehat{B}$.

b) Given distinct $\mathcal{U}, \mathcal{V} \in \beta D$, let A, B be complementary sets such that $A \in \mathcal{U}$, $B \in \mathcal{V}$. Then $\widehat{A}$, $\widehat{B}$ are disjoint open neighborhoods of $\mathcal{U}$, $\mathcal{V}$.

c) Let $\mathcal{F}$ be a collection of subsets of D such that $\{\widehat{A} : A \in \mathcal{F}\}$ covers βD. We must see that there is a finite subcover. For each $A \in \mathcal{F}$, let $A' = D \setminus A$. Then $\beta D \setminus \widehat{A} = \widehat{A'}$. This implies that $\bigcap\{\widehat{A'} : A \in \mathcal{F}\} = \emptyset$. This means that no single ultrafilter contains the set $\{A' : A \in \mathcal{F}\}$, so this collection does not have the finite intersection property. Thus, there exists a finite $\mathcal{G} \subseteq \mathcal{F}$ such that $\bigcap\{A' : A \in \mathcal{G}\} = \emptyset$, so $\bigcup\{A : A \in \mathcal{G}\} = D$. Since $\mathcal{G}$ is finite, for every ultrafilter $\mathcal{U}$ there exists $A \in \mathcal{G}$ such that $A \in \mathcal{U}$, that is, $\mathcal{U} \in \widehat{A}$.

d) e is clearly injective. We verify that e is continuous by verifying that for every $A \subseteq D$, $e^{-1}[\widehat{A}] = A$. Just notice that for every n, $e(n) \in \widehat{A} \leftrightarrow A \in e(n) \leftrightarrow n \in A$. This also shows that $e : X \to e[D]$ is open, since $e[A] \cap e[D] = \widehat{A} \cap e[D]$.

To see that e is dense, notice that if $A \subseteq D$ is such that $\widehat{A}$ is nonempty, then $A \neq \emptyset$, and that if $d \in A$, then $e(d) \in \widehat{A}$.

e) Let K be a compact Hausdorff space and $f : D \to K$. Define $g : \beta D \to K$ by $g(\mathcal{U}) \in \bigcap\{\operatorname{cl}_K f[A] : A \in \mathcal{U}\}$. To see that this is well defined, we must see that this intersection is a singleton. It is nonempty since it has the finite intersection property and K is empty. To see that it is a singleton, let $x \in \bigcap\{\operatorname{cl}_K f[A] : A \in \mathcal{U}\}$ and $y \in K$, $y \neq X$. Let V be an open neighborhood of x such that $y \notin \operatorname{cl} V$. Let $A = f^{-1}[V]$, $B = f^{-1}[K \setminus V]$. $A \cup B = D$, so one of them is in $\mathcal{U}$. It is not B, since x is not in $\operatorname{cl} f[f^{-1}[B] \subseteq K \setminus V$. So it is A. But y is not in $\operatorname{cl} f[A] \subseteq \operatorname{cl} V$.

We must see that $g \circ e = f$. Just notice that given n, $g(e(n))$ is the only element of $\mathrm{cl}_K f[\{n\}] = \{f(n)\}$.

Finally, we must see that g is continuous. So fix $\mathcal{U}$ and let U be an open neighborhood of $y = g(\mathcal{U})$. Let V be an open neighborhood of y such that $\mathrm{cl}\, V \subseteq U$. Let $A = f^{-1}[V]$, $B = f^{-1}[K \setminus V]$. As before, $B \notin \mathcal{U}$, so $A \in \mathcal{U}$. We claim that $g[\widehat{A}] \subseteq U$: if $\mathcal{V} \in \widehat{A}$, then $A \in \mathcal{V}$, so $g(\mathcal{V}) \in \mathrm{cl}\, f[A] \subseteq \mathrm{cl}\, V \subseteq U$, as intended. $\square$

The notation $\widehat{A}$ was defined just for this proof. We will usually drop it and just write $\mathrm{cl}\, e[A]$. In fact, one usually identifies D with $e[D]$, so we just write $\mathrm{cl}\, A$. No confusion arises from this since cl_D is almost meaningless in our context since D is discrete, so cl "should" mean $\mathrm{cl}_{\beta D}$.

Definition 0.6.6. Let D be a discrete space. Then $D^* = \beta D \setminus D$ is the set of free ultrafilters on D, and, when $A \subseteq D$ and a fixed D is clear from the context, we write A^* to denote $\mathrm{cl}\, A \setminus D = \{\mathcal{U} \in D^* : A \in D\}$. $\square$

Note that the notation makes sense since $\mathrm{cl}\, A \subseteq \beta D$ is homeomorphic to βA.

0.7 Baire numbers

We expect the reader is familiar with the celebrated Baire theorems:

Theorem 0.7.1 (Baire's Theorem)**.** Let X be a locally compact Hausdorff topological space, or a completely metrizable topological space. Then a countable intersection of open dense subsets is non-empty (in fact, dense).

We refer to [24] or [70] for proofs.

So, what is the least cardinality for which there is a collection of open dense sets of this size with empty intersection? This is what is usually defined as the Baire number.

Definition 0.7.2. Let X be a topological space such that for every $x \in X$, $\mathrm{cl}\{x\}$ has empty interior. Then we define the *Baire number* of X, which is denoted by $n(X)$, as the minimum cardinality of a collection of open dense sets of X whose intersection is empty. $\square$

The condition we stated above (for every $x \in X$, $\mathrm{cl}\{x\}$ has empty interior) is exactly the condition we need in order for $n(X)$ to be well defined. Notice that for T_1 spaces this holds if and only if X has no isolated points.

With this notation, it is clear that $\mathfrak{h}$ is the Baire number of $[\omega]^\omega$ with the $\subseteq^*$-topology.

Another very important Baire number is $n(\omega^*)$. The result below is stated without proof in the reference.

Proposition 0.7.3. [2] $n(\omega^*)$ is the first cardinal κ for which $\neg\, \mathrm{FA}_{[\omega]^\omega}(\kappa)$.

Proof. Let $\mathfrak{m}^*$ be the first cardinal κ for which $\neg\, \mathrm{FA}_{[\omega]^\omega}(\kappa)$. To see that $m^* \leq n(\omega^*)$, let λ be a cardinal such that $\lambda < m^*$. We show that $\lambda < n(\omega^*)$. Let $(U_\alpha : \alpha < \lambda)$ be a collection of open dense subsets of ω^* such that $\bigcap_{\alpha<\lambda} D_\alpha = \emptyset$.

For each $\alpha < \lambda$, let $D_\alpha = \{A \subseteq \omega : A^* \subseteq U_\alpha\}$. Each D_α is clearly open. It is also dense, since if $B \subseteq \omega$ is infinite, $B^* \cap U_\alpha$ is a nonempty open set, so there exists $A^* \subseteq B^* \cap U_\alpha$ where $A \subseteq \omega$ is infinite. Then $A \subseteq^* B$ and $A \in D_\alpha$. Now suppose by contradiction that there is a filter on $([\omega]^\omega, \subseteq^*)$ such that $\mathcal{U} \cap D_\alpha$ is nonempty for every α. $\mathcal{U}$ is clearly a free filter on ω. We can extend it to a free ultrafilter $\mathcal{U}' \in \omega^*$. For each α, since $\mathcal{U}' \cap D_\alpha \neq \emptyset$, there is $A^* \subseteq U_\alpha$ such that $\mathcal{U}' \in A^* \subseteq U_\alpha$. But then $\mathcal{U}' \in \bigcap_{\alpha<\lambda} U_\alpha$, as intended.

For the converse, let $\lambda < n(\omega^*)$. We show that $\lambda < m^*$.

Let $(D_\alpha : \alpha \subseteq \omega)$ be a collection of open dense subsets of $([\omega]^\omega, \subseteq^*)$. We must show that there exists a filter intersecting each of them. For each α, let $U_\alpha = \bigcup\{A^* : A \in D_\alpha\}$. U_α is clearly a open subset of ω^*. It is also dense: given $B \in [\omega]^\omega$, there exists $A \in D_\alpha$ such that $A \subseteq B$, so $A^* \subseteq B^* \cap U_\alpha$.

Let $\mathcal{U} \in \bigcap_{\alpha<\lambda} U_\alpha$. Then $\mathcal{U}$ is a filter on $([\omega]^\omega, \subseteq^*)$. We must see that $\mathcal{U} \cap D_\alpha \neq \emptyset$ for each $\alpha < \lambda$. Fix α. We know that $\mathcal{U} \in U_\alpha$, so there exists $A \in D_\alpha$ such that $\mathcal{U} \in A^*$. But then $A \in \mathcal{U}$, so $\mathcal{U} \in D_\alpha \neq \emptyset$, as intended. □

Proposition 0.7.4 ([2]). The following is true:

a) $n(\omega^*) \leq 2^{\mathfrak{c}}$,

b) $\mathfrak{p} < n(\omega^*)$,

c) $\mathfrak{h} \leq n(\omega^*)$.

Proof. We use the preceding proposition. a) This is obvious since $|\omega^*| = 2^{\mathfrak{c}}$.

b) Suppose $\mathcal{D}$ is a collection of $\mathfrak{p}$ dense subsets of $[\omega]^\omega$. Enumerate them as $\{D_\alpha : \alpha < \mathfrak{p}\}$. Recursively, construct a $\subseteq^*$-decreasing sequence $(a_\alpha : \alpha < \mathfrak{p})$ such that $a_0 = \omega$ and $a_{\alpha+1} \in D_\alpha$. This is possible by the definition of $\mathfrak{p}$. Let $\mathcal{U}$ be a free ultrafilter containing $\{a_\alpha : \alpha < \mathfrak{p}\}$. Then $\mathcal{U}$ intersects every element of $\mathcal{D}$.

c) We show that if $\kappa < \mathfrak{h}$, then $\kappa < \mathfrak{m}^*$ Let $\mathcal{D}$ be a nonempty collection of dense sets. For each $D \subseteq [\omega]^\omega$, let $D' = \{a \in [\omega]^\omega : \exists d \in D\, a \subseteq^* d\}$. It is clear that D' is open and dense whenever D is dense. Let $a \in \bigcap\{D' : D \in \mathcal{D}\}$. Let $G = \{b \in [\omega]^\omega : a \subseteq^* b\}$. Then G intersects every $D \in \mathcal{D}'$. □

0.8 (κ, A)-compactness and pseudocompactness

A natural generalization of feeble compactness is the following definition. It was originally defined in [31]. A newer reference is Section 3.4. of [30].

Definition 0.8.1. Let X be a topological space, κ be a cardinal (finite or infinite) and $A \subseteq \omega^*$ be nonempty. We say that X is (κ, A)-*pseudocompact* iff for every family $(U_\alpha : \alpha < \kappa)$ where each $U_\alpha = (U_\alpha(n) : n \in \omega)$ is a sequence of nonempty open subsets of X, there exists $\mathcal{U} \in A$ such that U_α has an $\mathcal{U}$-accumulation point for every $\alpha < \kappa$. □

It is worth comparing this definition with Corollary 0.4.19.

The following are direct consequences of the definition and from Corollary 0.4.19.

Proposition 0.8.2 ([30]). Let X be a topological space. Let κ, λ be cardinals such that $\kappa \geq \lambda$. Let $A, B \subseteq \omega^*$ be nonempty such that $B \subseteq A$. Then:

1. For every $\mathcal{U} \in A$, if X is $\mathcal{U}$-pseudocompact then X is (κ, A)-pseudocompact.
2. If X is (κ, A)-pseudocompact, then X is (λ, B)-pseudocompact.
3. X is feebly compact iff X is $(1, \omega^*)$-pseudocompact □

The following proposition makes this notion really interesting for us. For a proof, see the reference.

Proposition 0.8.3 ([30, Theorems 3.4.8, 3.4.9]). Let X be a topological space and $\kappa \leq \omega$. Then X^κ is feebly compact iff there exists a nonempty $A \subseteq \omega^*$ such that X is (κ, A)-pseudocompact (or, equivalently, iff X is (κ, ω^*)-pseudocompact).

Recall that the "degree of pseudocompactness" of a topological space collapses at ω (see propositions 0.4.23 and 0.4.24). However, (κ, ω^*)-pseudocompactness does not: it is clear that if X is (κ, ω^*)-pseudocompact for every κ, then there exists $\mathcal{U}$ such that X is $\mathcal{U}$-pseudocompact. However, there exists an example of a Tychonoff topological space whose all powers are pseudocompact but it is not $\mathcal{U}$-pseudocompact for any $\mathcal{U}$ [30, Example 3.4.10] (we will construct another example in this thesis).

0.9 Selective Ultrafilters

In this section we review some basic facts about selective ultrafilters, the Rudin-Keisler order and some lemmas we will use in this thesis.

Definition 0.9.1. A *selective ultrafilter* (on ω), also called *Ramsey ultrafilter*, is a free ultrafilter $\mathcal{U}$ on ω such that for every partition $(A_n : n \in \omega)$ of ω by nonempty sets, either there exists n such that $A_n \in \mathcal{U}$ or there exists $B \in \mathcal{U}$ such that $|B \cap A_n| = 1$ for every $n \in \omega$. □

The following proposition is well known. We provide [47] as a reference for the equivalence between a) and c), but we provide a proof for the equivalence for a) and b) the sake of completeness.

Proposition 0.9.2. Let $\mathcal{U}$ be a free ultrafilter on ω. Then the following are equivalent:

a) $\mathcal{U}$ is a selective ultrafilter,

b) for every function $f : \omega \to \omega$, there exists $A \in \mathcal{U}$ such that f is either constant or one-to-one on A,

c) for every function $f : [\omega]^2 \to 2$ there exists $A \in \mathcal{U}$ such that f is constant on $[A]^2$.

Proof. a) implies b): if $\operatorname{ran} f$ is finite, there exists n such that $f^{-1}[\{n\}] \in \mathcal{U}$ since $\mathcal{U}$ is an ultrafilter. If $\operatorname{ran} f$ is infinite, write $\operatorname{ran} f = \{a_n : n \in \omega\}$ injectively. Let $A_n = f^{-1}[\{n\}]$.

Consider the partition $(A_n : n \in \omega)$ of ω. If one of the A_n's is in $\mathcal{U}$ we are done. If not, there exists $B \in \mathcal{U}$ such that $|B \cap A_n| = 1$ for every n, so $f|B$ is injective.

b) implies c): Let $(A_n : n \in \omega)$ be a partition of ω by nonempty sets. Suppose that no A_n is in $\mathcal{U}$. Let $f : \omega \to \omega$ be such that $f(m) = n$ iff $m \in A_n$. Since $A_n \notin \mathcal{U}$ for all $n \in \omega$, f has no constant subsequence whose domain is in $\mathcal{U}$. So there exists $B \in \mathcal{U}$ such that $f|B$ is injective. This implies that for all $n \in \omega$, $|B \cap A_n| \leq 1$. Since the A_n's are pairwise disjoint we may expand B to a set B' (which is also in $\mathcal{U}$) such that $|B' \cap A_n| = 1$ for every $n \in \omega$.

For a) iff c), see [47, Lemma 9.2] □

Now we aim to define the Rudin-Keisler order.

Definition 0.9.3. Let $\mathcal{U}$ be a filter on ω and $f : \omega \to \omega$. We define $f_*(\mathcal{U}) = \{A \subseteq \omega : f^{-1}[A] \in \mathcal{U}\}$. □

The following is easy to verify and we leave it to the reader:

Lemma 0.9.4. Let $\mathcal{U}$ be a filter on ω. Then:

1. $f_*(\mathcal{U})$ is a filter,
2. if $\mathcal{U}$ is an ultrafilter, so is $f_*(\mathcal{U})$,
3. if $f, g : \omega \to \omega$, then $(f \circ g)_* = f_* \circ g_*$,
4. $(\mathrm{id}_\omega)_*$ is the identity over the set of all filters,
5. if $f : \omega \to \omega$ is bijective, then $(f^{-1})_* = (f_*)^{-1}$,
6. if $f : \omega \to \omega$ is finite-to-one and $\mathcal{U}$ is a free ultrafilter, so is $f_*(\mathcal{U})$. □

Now we define the Rudin-Keisler order for filters.

Definition 0.9.5. Let $\mathcal{U}$ and $\mathcal{V}$ be filters on ω We say that $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{V}$ iff there exists $f \in \omega$ such that $f_*(\mathcal{V}) = \mathcal{U}$.

The *Rudin-Keisler order* is the set of all free ultrafilters over ω ordered by $\leq_{\mathrm{RK}}$, that is, $(\omega^*, \leq_{\mathrm{RK}})$. We say that two free ultrafilters $\mathcal{U}$ and $\mathcal{V}$ are *equivalent* iff $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{V}$ and $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$, and we write $\mathcal{U} =_{\mathrm{RK}} \mathcal{V}$. □

$(\omega^*, \leq_{\mathrm{RK}})$ is easily seen to be a preorder.

Proposition 0.9.6. $(\omega^*, \leq_{\mathrm{RK}})$ is a preorder, that is, $\leq_{\mathrm{RK}}$ is reflexive and transitive. Thus, $=_{\mathrm{RK}}$ is an equivalence relation over ω^*.

We leave the proof of the following well known proposition to the reader. The reader may look at Exercises 7.11 and 7.12 of [47] for hints.

Proposition 0.9.7. The following are true:

1. Let $\mathcal{U}, \mathcal{V} \in \omega^*$. Then $\mathcal{U} =_{\mathrm{RK}} \mathcal{V}$ if, and only if there exists a bijection $f : \omega \to \omega$ such that $f_*(\mathcal{U}) = \mathcal{V}$.
2. The selective ultrafilters are exactly the minimal elements of $(\omega^*, \leq_{\mathrm{RK}})$.

Chapter 1

Introduction to Isbell-Mrówka spaces

The aim of this chapter is to introduce almost-disjoint families and Isbell-Mrówka spaces, discuss their most basic properties, some concepts related to them we are going to be working with and to mentioning enough material to motivate them. We refer to [40] for a very complete survey on this field of study.

1.1 Almost disjoint families

An almost disjoint family is a natural generalization of the concept of disjoint families. They are collections of subsets of ω which are "disjoint mod $=^*$".

Definition 1.1.1. Let N be an infinite countable set. An *almost disjoint family* on N is a infinite collection $\mathcal{A}$ of infinite subsets of N such that for all two distinct $a, b \in \mathcal{A}$, $a \cap b$ is finite.

An *almost disjoint family* is an almost disjoint family on ω.

□

So an almost disjoint family is not a family since it is not a function.

Notice that by using a bijection between N and ω is is possible to "copy" almost disjoint families on N to ω maintaining all the relevant combinatorical facts, so the choice of N is not really relevant. However, sometimes it is useful to consider an specific N distinct from ω because then we are able to define almost disjoint families that interact with some additional structure (for instance, if $N = 2^{<\omega}$, N has a natural tree structure). However, every such additional structure may be copied by a bijection.

Thus, for "general" results on almost disjoint families, we will always write "almost disjoint family" in the statements (so $N = \omega$). This makes the statements of the results less clogged, and the analogous theorems for general N's usually follow trivially by fixing a bijective function between N and ω and copying all the relevant structure.

We only allow almost disjoint families to be infinite to avoid some pathological examples and some trivialities. Some authors admit finite almost disjoint families.

The most basic example of an almost disjoint family is an infinite partition of ω by infinite subsets (e.g., biject $\omega \times \omega$ with ω and consider the columns or rows). However, there is not much fun in that. The first question which may come to mind is whether there almost disjoint families which are very different from partitions. The answer is certainly yes. For a start, it is clear that partitions of ω are at most countable, however, we have the following:

Proposition 1.1.2. There exists an almost disjoint family of size $\mathfrak{c}$.

Proof. We sketch two proofs for this proposition. First, let $N = \mathbb{Q} \subseteq \mathbb{R}$. For each $x \in \mathbb{R} \setminus \mathbb{Q}$, let a_x be the range of a sequence of elements of $\mathbb{Q}$ converging to x. It is clear that $\{a_x : x \in \mathbb{R} \setminus \mathbb{Q}\}$ is an almost disjoint family on $\mathbb{Q}$ of size $\mathfrak{c}$.

For the second example, let $N = 2^{<\omega}$. Let $X \subseteq 2^\omega$ be infinite. For each $x \in 2^\omega$, let $a_x = \{x|n : n \in \omega\}$. It is clear that $\{a_x : x \in X\}$ is an almost disjoint family of size $|X|$. □

According to [40], it is not known who was the first person to prove the previous proposition, but that it is sometimes attributed to Sierpínski [65]. However, it has been known a lot earlier, at least implicitly.

The second example will be very important for us. Thus, we define:

Definition 1.1.3. A *branching family*, or an *almost disjoint family of branches* is an almost disjoint family of the form $\mathcal{A}_X$ for some $X \subseteq 2^\omega$, as in the previous proposition.

That is, for each $X \subseteq 2^\omega$, the branching family of X is the set $\mathcal{A}_X = \{a_x : x \in X\}$, where for each $x \in 2^\omega$, $a_x = \{a|n : n \in \omega\}$. A branching family (or an almost disjoint family of branches) is an almost disjoint family of the form $\mathcal{A}_X$, for some infinite $X \subseteq 2^\omega$. □

As we have just seen, there are almost disjoint families which are fundamentally different from partitions since they have distinct cardinalities. But uncountability is far from being the only thing that partitions of ω may fail to be. Another property is maximality. Partitions are always maximal as partitions: if P is a infinite partition of ω of infinite sets, there is no way to add a new infinite set to P so that it remains a partition. However, there is always a way to add a new member which makes it an almost disjoint family: just select one point from each element of P and cook them into a new element. Thus, partitions of ω always fail to be maximal as almost disjoint families.

Definition 1.1.4. Let N an infinite countable set. We say that an almost disjoint family $\mathcal{A}$ is *maximal* on N, or a *MAD family* on N (***M**aximal **A**lmost **D**isjoint family*) iff for every almost disjoint family $\mathcal{B}$ on N, if $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A} = \mathcal{B}$.

Equivalently, an almost disjoint family $\mathcal{A}$ on N is a MAD family on N iff for every $x \in [N]^\omega$ there exists $a \in \mathcal{A}$ such that $a \cap x$ is infinite.

A MAD family is a MAD family on ω. □

We leave the proof of the equivalence as an exercise to the reader. We do the same with the following proposition, which is a straightforward application of Zorn's Lemma.

Proposition 1.1.5. Every almost disjoint family is contained in a MAD family. In particular, there are MAD families of size $\mathfrak{c}$. □

In fact, no countable almost disjoint family is a MAD family.

Definition 1.1.6. The *almost disjointness number* $\mathfrak{a}$ is the least size of a MAD family. □

One should be attempted to define $\mathfrak{a}_N$ for an infinite countable N, but it should be clear that these cardinals are all the same.

The proposition below can be found in [10, Proposition 8.4]. We prove it for the sake of completeness.

Proposition 1.1.7. $\mathfrak{b} \leq \mathfrak{a}$.

Proof. Let $\mathcal{A}$ be a MAD family (on ω) of size $\mathfrak{a}$. Choose arbitrary distinct sets $(a_n : n \in \omega)$. Define recursively $b_n = (a_n \cup \{n\}) \setminus \bigcup_{i<n} b_i$. Then for each $n \in \omega$, $b_n =^* a_n$ and $(b_n : n \in \omega)$ is a partition of ω. For each n, let $f_n : b_n \to \omega \times \{n\}$ be a bijection. Let $f = \bigcup_{n \in \omega} f_n$. Then $f : \omega \to \omega \times \omega$ is a bijection.

Let $\mathcal{A}' = \mathcal{A} \setminus \{a_n : n \in \omega\}$. For each $a \in \mathcal{A}'$, let $f_a : \omega \to \omega$ be such that $f_a(n) > \sup f_n[a \cap b_n]$.

Let $g : \omega \to \omega$ be arbitrary. We show that g does not dominate $\{f_a : a \in \mathcal{A}'\}$, so $\mathfrak{b} \leq |\mathcal{A}'| \leq \mathfrak{a}$.

g is infinite, so $|f^{-1}[g] \cap a| = \omega$ for some $a \in \mathcal{A}$. Since $X \cap (\{n\} \times \omega)$ is finite for every $n \in \omega$, such an a is in $\mathcal{A}'$. So $g \cap f[a]$ is infinite. For each $n \in \operatorname{dom} g \cap f[a]$ (which is infinite) it follows that $g(n) < f_a(n)$. □

We finish this section by discussing ideals related to almost disjoint families. There are two standard ways to associate an almost disjoint family to an ideal.

Definition 1.1.8. Let $\mathcal{A}$ be an almost disjoint family. We define $\mathcal{I}(\mathcal{A}) = \{x \subseteq \omega : \exists \mathcal{A}' \in [\mathcal{A}]^{<\omega} x \subseteq^* \bigcup \mathcal{A}'\}$ and $\mathcal{J}(\mathcal{A}) = \{x \subseteq \omega : |\{a \in \mathcal{A} : |a \cap x| = \omega\}| < \omega\}$.

We define the *positive sets* with relation to these ideals as $\mathcal{I}^+(\mathcal{A}) = \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$ and $\mathcal{J}^+(\mathcal{A}) = \mathcal{P}(\omega) \setminus \mathcal{J}(\mathcal{A})$. □

Lemma 1.1.9. Let $\mathcal{A}$ be an almost disjoint family. Then:

1) $\mathcal{I}(\mathcal{A}) \subseteq \mathcal{J}(\mathcal{A})$

2) $\mathcal{I}(\mathcal{A})$ and $\mathcal{J}(\mathcal{A})$ are non maximal free ideals

3) $\mathcal{I}(\mathcal{A}) = \mathcal{J}(\mathcal{A})$ iff $\mathcal{A}$ is MAD.

Proof. 1) Let $x \in \mathcal{I}(\mathcal{A})$. Let $\mathcal{A}'$ be a finite subset of x such that $x \subseteq^* \bigcup \mathcal{A}'$. Given $a \in \mathcal{A} \setminus \mathcal{A}'$, $a \cap x$ is finite since $a \cap x \subseteq^* \bigcup_{b \in \mathcal{A}'} a \cap b$, which is a finite union of finite sets.

2) Both sets are clearly closed by finite unions and downwards. They are nonempty since $[\omega]^{<\omega}$ are contained in both, and ω is not in any of them (it is easy to see that it is not in $\mathcal{J}(\mathcal{A})$, so it is not in $\mathcal{I}(\mathcal{A})$ by 1)). To see that $\mathcal{J}(\mathcal{A})$ is not maximal, let $(a_n : n \in \omega)$, $(b_n : n \in \omega)$ be two injective sequence into $\mathcal{A}$ with disjoint ranges. Let $f : \omega \to \omega$ be such that for every $n \in \omega$, $f^{-1}[\{n\}]$ is infinite. Consider $X = \{x_m : m \in \omega\}$, where $x_m \in a_{f(m)} \setminus \bigcup_{k<m} b_k$. Then neither x or $\omega \setminus x$ are in $\mathcal{J}(\mathcal{A})$.

3) Suppose $\mathcal{A}$ is not MAD. then cleary any infinite x such that $x \cap a = \emptyset$ for every $a \in \mathcal{A}$ is in $\mathcal{J}(\mathcal{A})$ but not in $\mathcal{I}(\mathcal{A})$. Conversely, suppose $\mathcal{A}$ is MAD. Let $x \in \mathcal{J}(\mathcal{A})$ and let $\mathcal{A}' = \{a \in \mathcal{A} : |a \cap x| < \omega\}$, which is finite. We claim that $a \setminus \bigcup \mathcal{A}'$ is finite: if it is not, there exists $b \in \mathcal{A}$ intersecting it infinitely, and such an b is not in $\mathcal{A}'$, a contradiction. □

Now we define completely separable almost disjoint families.

Definition 1.1.10. An almost disjoint family $\mathcal{A}$ is *completely separable* iff for every $x \in \mathcal{J}^+(\mathcal{A})$ there exists $a \in \mathcal{A}$ such that $a \subseteq x$. □

Completely separable MAD families were defined in [37] who showed that they exist under MA. It is not known if such families exist in ZFC, and this is listed as problem 19 in the "Twenty problems in set-theoretic topology" chapter of Open Problems in General Topology II [45]. We know that such a family exist in almost every known model of Set Theory with choice. In particular, we know that if $\mathfrak{s} \leq \mathfrak{a}$ then there is such a family [53] and if $\mathfrak{c} < \aleph_\omega$ then there is such a family [64]. However, we do know that completely separable almost disjoint families exist in ZFC [27].

We can also use ideals to define tightness.

Definition 1.1.11. Let $\mathcal{A}$ be an almost disjoint family. We say that $\mathcal{A}$ is *tight*, also called ω-MAD, iff for every sequence $(x_n : n \in \omega)$ of elements of $\mathcal{I}^+(\mathcal{A})$ there exists $a \in \mathcal{A}$ such that $|x_n \cap a|$ is infinite for every $n \in \omega$. □

It is clear that every tight family is MAD. Moreover, such a family $\mathcal{A}$ is Cohen indestructible, that is, its maximality is preserved by one (equivalently, any quantity) of Cohen reals. For a proof, see [41]. It is not known if such families exist in ZFC.

For more about Cohen-indestructible MAD families, tight families and completely separable almost disjoint families see the Section 4 of [40].

1.2 Isbell-Mrówka spaces

The first uses of spaces homeomorphic to Isbell-Mrówka spaces were due to J. R. Isbell (as atribbuted by Gillman and Jerison) and S. Mrówka ([58]). A Isbell-Mrówka space is associated to an almost disjoint family, and its topological properties depend on the combinatoric properties of the family.

Definition 1.2.1. Let N be an infinite countable set such that $N \cap [N]^\omega = \emptyset$ (such as ω or $2^{<\omega}$). The *Isbell-Mrówka space associated to* $\mathcal{A}$, also called denoted by $\Psi(\mathcal{A})$, is the set $N \cup \mathcal{A}$ topologized by:

$$\{\{n\} : n \in N\} \cup \bigcup_{a \in \mathcal{A}} \{\{a\} \cup (a \setminus F) : F \in [a]^{<\omega}\} \tag{1.1}$$

□

The reader may verify that the expression above really defines a basis for a topology in $\Psi(\mathcal{A})$. Also, $\Psi(\mathcal{A})$ should really be $\Psi(\mathcal{A}, N)$, but if no confusion arises and N is clear from the context we just write $\Psi(\mathcal{A})$.

The following fact should then be clear:

Proposition 1.2.2. Let N be an infinite countable set such that $N \cap [N]^\omega = \emptyset$ and $\mathcal{A}$ be an almost disjoint family on N. Then:

a) N is open and discrete.

b) For each $a \in \mathcal{A}$, $\{\{a\} \cup (a \setminus F) : F \in [a]^{<\omega}\}$ is a local basis of open compact sets for a.

c) $\mathcal{A}$ is closed and discrete.

d) $\Psi(\mathcal{A})$ is Hausdorff.

e) $\Psi(\mathcal{A})$ is locally compact.

f) $\Psi(\mathcal{A})$ is Tychonoff.

g) $\Psi(\mathcal{A})$ is not countably compact.

Proof. a) is clear since each $\{n\}$ is a basic open set.

b) This set is the collection of all basic open sets which has a as a member, so it is a local basis for a. Each such set is compact, since if $\mathcal{U}$ is a collection of open basic sets containing a set of the form $\{a\} \cup (a \setminus F)$, some element of $\mathcal{U}$ has the point a as a member, so this element must be of the form $\{a\} \cup (a \setminus F')$, which covers all but finitely many points of $\{a\} \cup (a \setminus F)$.

c) $\mathcal{A}$ is closed because N is open, and it is discrete since for each $a \in \mathcal{A}$, $(\{a\} \cup a) \cap \mathcal{A} = \{a\}$.

d) Given distinct $a, b \in \mathcal{A}$, $a \cap b$ is finite. Then $\{a\} \cup (a \setminus (a \cap b))$ and $\{b\} \cup (b \setminus (a \cap b))$ are disjoint open sets separating a from b. Given distinct $n, m \in N$, $\{n\}$, $\{m\}$ are disjoint open sets separating n from m. Finally, given $n \in N$ and $a \in \mathcal{A}$, $\{n\}$ and $a \setminus \{n\}$ are two disjoint open sets separating n from a.

e) This follows from b).

f) This follows from d) and e).

g) This follows from c). □

It should be noted that the statement of the previous theorem is a bit cumbersome due to the "Let N be an infinite countable set such that $N \cap [N]^\omega = \emptyset$". This may look unnecessary, and, in fact, it is: clearly, given such an N and an almost disjoint family $\mathcal{A}$ on

N, if we fix a bijection $f : N \to \omega$ we can use it to construct an almost disjoint family $\mathcal{A}'$ on ω and an homeomorphism from $\Psi(\mathcal{A})$ onto $\Psi(\mathcal{A}')$. Thus, to talk about general results, we may just state results about almost disjoint families on ω and their Isbell-Mrówka spaces. As we conventioned that an "almost disjoint family" is an "almost disjoint family on ω", this can make statements a bit simpler. We are going to adopt this convention from now on.

As we have mentioned, the topological properties of $\Psi(\mathcal{A})$ often depend on combinatorical properties of $\mathcal{A}$ and vice-versa. As a first (important) well known example, we have:

Proposition 1.2.3. Let $\mathcal{A}$ be an almost disjoint family. The following are equivalent:

a) $\mathcal{A}$ is a MAD family,

b) $\Psi(\mathcal{A})$ is pseudocompact.

Proof. a) implies b): Suppose by contradiction that $f : \Psi(\mathcal{A}) \to \mathbb{R}$ is continuous and unbounded. We may suppose that $|f| \geq 0$. For ever $k \in \omega$, let $n_k \in \omega \cap f^{-1}[(k, +\infty)]$. Let $b = \{n_k : k \in \omega\}$. Since b is infinite, $|b \cap a| = \omega$ for some $a \in \mathcal{A}$. It follows that $a \in \operatorname{cl} b$, so $a \in \operatorname{cl}\{n_k : k \geq m\}$ for every $m \in \omega$, so $f(a) \in f[\operatorname{cl}\{n_k : k \geq m\}] \subseteq \operatorname{cl} f[\{n_k : k \geq m\}] \subseteq \operatorname{cl} f[f^{-1}[(m, +\infty)]] \subseteq \operatorname{cl}(m, +\infty) = [m, +\infty)$ for every $m \in \omega$, so $f(a) \geq m$ for every $m \in \omega$, a contradiction.

b) implies a): Suppose that $\mathcal{A}$ is not a MAD family. We show that $\Psi(\mathcal{A})$ is not pseudocompact. Suppose that $b \in [\omega]^\omega$ is such that $b \cap a$ is finite for every $a \in \mathcal{A}$. It is clear that b is a clopen discrete subset of $\Psi(\mathcal{A})$. Let $f : \Psi(\mathcal{A}) \to \mathbb{R}$ be such that $f|b$ is unbounded and $f|(\Psi(\mathcal{A}) \setminus b)$ is constant. □

1.3 Refining almost disjoint families and base trees

An important feature of $\mathfrak{h}$, which may be used as an alternative definition of $\mathfrak{h}$ (in fact, this was its original definition) is that it is the least height of a base tree. This fact is known as base tree lemma or base matrix lemma, and was first explored in [2]. A more modern reference is [10, Theorem 6.20]. This section could be in Chapter 0, but we decided to put this here since MAD families were only introduced in this chapter.

First, we define refinements of almost disjoint families and shattering families.

Definition 1.3.1. Let $\mathcal{A}, \mathcal{B}$ be almost disjoint families. We say that $\mathcal{A} \preceq \mathcal{B}$ ($\mathcal{A}$ *refines* $\mathcal{B}$) iff for every $a \in \mathcal{A}$ there exists $b \in \mathcal{B}$ such that $a \subseteq^* b$.

A family $(\mathcal{A}_\alpha)_{\alpha<\kappa}$ of MAD families is a *shattering family* iff for every $x \in [\omega]^\omega$ there exists $\alpha < \kappa$ and two distinct $a, a' \in \mathcal{A}_\alpha$ such that $|a \cap x| = |a' \cap x| = \omega$. □

The lemma below is easy and left to the reader.

Lemma 1.3.2. Suppose $(\mathcal{A}_\alpha : \alpha < \kappa)$ is a shattering family and that $(\mathcal{B}_\alpha : \alpha < \kappa)$ is a family of MAD families such that for every $\alpha < \kappa$, $B_\alpha \preceq A_\alpha$. Then $(\mathcal{B}_\alpha : \alpha < \kappa)$ is a shattering family.

We are going to be refining many MAD families into a single one. The following basic well known lemma relates almost disjoint families with the structure of $[\omega]^\omega$, which gives us a step towards this direction. We leave the proof to the reader.

Proposition 1.3.3. Consider $[\omega]^\omega$ partially ordered by $\subseteq^*$. Then...

- ... the (maximal) antichains of $[\omega]^\omega$ are precisely the (maximal) almost disjoint families on ω,
- ... every dense subset of $[\omega]^\omega$ contains a MAD family, and
- for every MAD family $\mathcal{A}$, the set $\mathcal{A}_\downarrow = \{b \in \mathcal{A} : \exists a \in \mathcal{A}\, b \subseteq^* a\}$ is open and dense in the topology generated by $\subseteq^*$. $\square$

Now we get:

Proposition 1.3.4. Suppose $\kappa < \mathfrak{h}$ and let $(\mathcal{A}_\alpha : \alpha < \kappa)$ be a family of MAD families. Then there exists a MAD family $\mathcal{A}$ such that $\mathcal{A} \preceq \mathcal{A}_\alpha$ for every $\alpha < \kappa$, thus, in particular, this family of MAD families is not shattering. Moreover, there exists a shattering family of length $\mathfrak{h}$. Thus, $\mathfrak{h}$ is the smallest cardinality of a shattering family.

Proof. By Proposition 0.5.10, $D = \bigcap_{\alpha<\kappa}(\mathcal{A}_\alpha)_\downarrow$ is dense, so there exists a MAD family $\mathcal{A}$ contained in D. Now clearly $\mathcal{A} \preceq \mathcal{A}_\alpha$ for every $\alpha < \kappa$. In particular, for every $a \in \mathcal{A}$ and $\alpha < \kappa$, there exists $b \in \mathcal{A}_\alpha$ such that $a \subseteq^* b$, thus, if $b' \in \mathcal{A}$ is not b, we get $|a \cap b| < \omega$. Thus, $(\mathcal{A}_\alpha : \alpha < \kappa)$ is not shattering.

For the second claim, let $(D_\alpha : \alpha < \mathfrak{h})$ be a collection of open dense subsets of $[\omega]^\omega$ with empty intersection. For each α, let $\mathcal{B}_\alpha$ be a MAD family contained in D_α. We claim that $(\mathcal{B}_\alpha)_{\alpha<\mathfrak{h}}$ is shattering. Let $x \in [\omega]^\omega$ be given. There exists α such that $x \notin (\mathcal{B}_\alpha)_\downarrow$. Since $\mathcal{B}_\alpha$ is MAD, there exists $a \in \mathcal{B}_\alpha$ such that $x \cap a$ is infinite, however, $x \not\subseteq^* a$, that is, $x \setminus a$ is infinite. Let $a' \in \mathcal{B}_\alpha$ be such that $(x \setminus a) \cap a'$ is infinite. Then $a' \neq a$ are in $\mathcal{B}_\alpha$ and $|a \cap x| = |a' \cap x| = \omega$. $\square$

Now we are ready to start discussing base trees.

Definition 1.3.5. A *base tree* is a subset $T \subseteq [\omega]^\omega$ satisfying:

- $(T, \supsetneq^*)$ is a tree,
- $T \subseteq [\omega]^\omega$ is dense,
- T is rooted in ω, and
- α with $0 < \alpha < \mathrm{ht}(T)$, $\mathrm{Lev}_T(\alpha)$ is a MAD family (this follows from the previous bullets).

We say that a base tree T is *sharp* if:

- for every $a \in [\omega]^\omega$ there exists $b \in T$ such that $b \subseteq a$, and
- for every $a \in T$ and $b \in \mathrm{succ}_T(a)$, $b \subseteq a$.

We say that a base tree T is *κ-branching* if for every $a \in T$, $|\operatorname{succ}_T(a)| \geq \kappa$. $\square$

The terms “sharp” and “κ-branching” are terms I invented to use in this thesis to try to unify the treatment of base trees, since their definition may be slightly different depending on the author.

There is a lower bound for the size of a base tree.

Lemma 1.3.6. Let T be a base tree. Then every point of T has a successor. In particular, $\mathrm{ht}(T)$ is a limit ordinal. Moreover, $\mathfrak{h} \leq \operatorname{cf} \mathrm{ht}(T)$.

Proof. Let $a \in T$. Let b, c be two infinite disjoint subsets of a. There exists $b' \in T$ such that $b' \subseteq^* b$. Then $b' \subsetneq^* a$, so a has a successor.

Now we verify that $\mathfrak{h} \leq \operatorname{cf} \mathrm{ht}(T)$. Let $\kappa = \operatorname{cf} \mathrm{ht}(T)$. Let $(\delta_\alpha : \alpha < \kappa)$ be a cofinal sequence in $\mathrm{ht}(T)$. For each α, $D_\alpha = \mathrm{Lev}_T(\delta_\alpha)_\downarrow$ is open and dense. Suppose by contradiction that there exists $x \in \bigcap_{\alpha<\kappa} D_\alpha$. There exists $y \in T$ such that $y \subseteq x$. Let α be such that $\delta_\alpha > \mathrm{ht}_T(y)$. Since $x \in D_\alpha$, there exists $y' \in \mathrm{Lev}_T(\delta_\alpha)$ such that $x \subseteq^* y'$. Let $y'' \in \mathrm{Lev}_T(\mathrm{ht}_T(y))$ be such that $y' \subsetneq^* y''$. It follows that $y \subseteq^* y''$, so $y = y''$ since they are in the same level. But then $y \subseteq x \subseteq^* y' \subsetneq^* y'' = y$, so $y \setminus y$ is infinite, a contradiction. $\square$

We can sharpen any base tree by trimming its leaves.

Lemma 1.3.7. Let T be a base tree. There exists a sharp base tree T' and an isomorphism $f : T \to T'$ such that for every $A \in T$, $A =^* f(A)$.

Proof. Let $\gamma = \mathrm{ht}(T)$. For every $\alpha < \gamma$, let β_α, n_α be the unique ordinals such that $\alpha = \omega.\beta_\alpha + n_\alpha$ and $n_\alpha < \omega$. If $a \in T$ is in a successor level ($n_{\mathrm{Lev}_T(a)} > 0$), denote by a^- the predecessor of a.

We define $f_n : \{a \in T : n_{\mathrm{Lev}_T(a)} = n\} \to [\omega]^\omega$ by induction.

We define $f_0(a) = a$ for every a such that $n_{\mathrm{Lev}_T(a)} = 0$ (that is, for every a in a limit level). Having defined f_n, we define $f_{n+1}(a) = (a \cap f_n(a^-)) \setminus (n+1)$.

Let $f = \bigcup_{n\in\omega} f_n$. Then $f : T \to [\omega]^\omega$. Let $T' = \operatorname{ran} f$.

Claim 1: $f(\omega) = \omega$. Since $\mathrm{ht}(\omega) = 0$, $f(\omega) = \omega$.

Claim 2: $f(a) =^* a$ for every $a \in T$. We already know that $f(\omega) = \omega$. Also, $f(a) = a$ whenever a is in a limit level. Now suppose that this is true for a’s in some level α. We show that this is true for the a’s in the level $\alpha + 1$. Let a be in the level $\alpha + 1$. Then $f(a) = (a \cap f(a^-)) \setminus (n_{\mathrm{Lev}\,-_T(a)}) =^* a \cap a^- =^* a$.

Claim 3: f is an isomorphism and T' is a tree of the same height as T: The first part follows from Claim 2. The second part follows from the fact that f is an isomorphism.

Claim 4: if $b \in T'$ and $d \in \mathrm{succ}_T(b)$, then $d \subseteq b$: Let $a, c \in T$ be such that $f(a) = b$, $f(c) = d$. Then $c' = a$. So $d = f(c) \subseteq c \cap f(a) = c \cap b \subseteq b$.

Claim 5: for every $b \in T'$ there exists $c \in T'$ such that $a \subsetneq b$: We know that there exists $a \in T$ such that $a \subseteq^* b$. Then $f(a) \subseteq^* b$. Let $n > \sup f(a) \setminus b$. Let $\alpha = \mathrm{Lev}_T(a) = \mathrm{Lev}_{T'}(f(a)) + n$, so $n_\alpha \geq n$. Since $\mathrm{Lev}_T(\alpha)$ is a MAD family, there exists c in it such that $c \cap f(a)$ is infinite. There exists $e \in \mathrm{Lev}_T(a)$ such that $c \subseteq^* e$, so $e \cap f(a)$ is infinite, which implies that $f(a) = e$, so $c \subseteq^* f(a)$. By Claim 4, since n is finite, $c \subseteq f(a)$, and by the definition of f, $c \cap n = \emptyset$, so $c \setminus b \subseteq f(a) \setminus (b \cup n) = \emptyset$, as intended. $\square$

Now we prove a version of the celebrated Base Tree Lemma.

Theorem 1.3.8 (Base Tree Lemma, [2])**.** There exists a $\mathfrak{c}$-branching sharp base tree of height $\mathfrak{h}$. More specifically, given a shattering family $(\mathcal{A}_\alpha : \alpha < \mathfrak{h})$, there exists a $\mathfrak{c}$-splitting sharp base tree such that for every $\alpha < \mathfrak{h}$, $\mathrm{Lev}_T(\alpha) \preceq \mathcal{A}_\alpha$.

Proof. We construct what will become the levels $(L_\alpha : \alpha < \mathfrak{h})$ of T by induction on α. To help us in our construction, we also define $(L'_\alpha : \alpha < \mathfrak{h}$ and α is limit$)$. At successor stages $\alpha + 1$ we must take care of refining A_α and of the $\mathfrak{c}$-splitting. At limit stages we guarantee the sharp denseness. Thus, we need the following:

1) $L_0 = \{\omega\}$ and for every $\alpha < \mathfrak{h}$ such that $\alpha > 0$, L_α is a MAD family.
2) For every $\alpha < \mathfrak{h}$ and every $a \in L_{\alpha+1}$ there exists $b \in L_\alpha$ such that $a \subseteq b$ and such that $|b \setminus a| = \omega$.
3) For every $\alpha, \beta < \mathfrak{h}$ with $\beta < \alpha$ and every $a \in L_\alpha$ there exists $b \in L_\beta$ such that $a \subsetneq^* b$.
4) For every $\alpha < \mathfrak{h}$ and every $b \in L_\alpha$, $|\{a \in L_{\alpha+1} : a \subseteq b\}| = \mathfrak{c}$.
5) For every $\alpha < \mathfrak{h}$, $L_{\alpha+1} \preceq \mathcal{A}_\alpha$.
6) For every limit $\alpha < \mathfrak{h}$, $L'_\alpha \preceq L_\beta$ for every $\beta < \mathfrak{h}$ such that $\beta > 0$.
7) For every limit $\alpha < \mathfrak{h}$, $L_\alpha \preceq L'_\alpha$.
8) For every limit $\alpha < \mathfrak{h}$ and $x \in [\omega]^\omega$ such that $|\{a \in L'_\alpha : |a \cap x| = \omega\}| = \mathfrak{c}$, there exists $y \in L_\alpha$ such that $y \subseteq x$.

Suppose that this construction can be carried out. Let $T = \bigcup_{\alpha<\mathfrak{h}} L_\alpha$. Conditions 1), 3), 6) and 7) guarantee that $(T, \supsetneq^*)$ is a tree with $\mathrm{Lev}_T(\alpha) = L_\alpha$. To see that, first we show, by induction by α, that for every $a \in L_\alpha$, $\mathrm{pred}_T(a)$ is well ordered and of type α.

- $\mathrm{pred}_T(\omega) = \emptyset$ by 3).
- If this is true for α, it is true for $\alpha + 1$: suppose $a \in L_{\alpha+1}$. By 3), there exists $b \in L_\alpha$ such that $a \subsetneq^* b$. We already know that $\mathrm{pred}_T(b)$ has type α and it is clear that $\mathrm{pred}_T(a) \supset \mathrm{pred}_T(b) \cup \{b\}$, so all we need to do is to show that, in fact equality holds. So suppose $a \supsetneq^* c$ and that $c \in T$. Let β be such that $c \in L_\beta$. If $\beta > \alpha + 1$, by 3) there exists $a' \in L_{\alpha+1}$ with $c \subsetneq^* a'$, but then $a \subsetneq^* c \subsetneq^* a'$, so, by 1), $a = a'$ and we derive that $a \setminus a$ is infinite, a contradiction. If $\beta = \alpha + 1$, by 1) we get that $c = a$, so $a \setminus a$ is infinite, a contradiction. If $\beta = \alpha$ then by 1) $c = b$ since both almost contain a. Finally, if $\beta < \alpha$, there exists $b' \in L_\alpha$ such that $a \subsetneq^* b \subsetneq^* b'$, so both c, b' almost contain a, therefore, by 1), $c = b'$.
- If this is true for $\beta < \gamma$ where $\gamma < \mathfrak{h}$ is limit: suppose $a \in L_\gamma$. By 7), there exists $a' \in L'_\gamma$ such that $a \subseteq^* a'$, and, by 6), for every $\beta < \gamma$ there exists $b_\beta \in L_\beta$ such that $b \subsetneq^* a' \subsetneq^* a$. We claim that $\{b_\beta : \beta < \gamma\}$ is a $\supsetneq^*$ chain of type γ and that this is precisely $\mathrm{pred}_T(a)$.

$\{b_\beta : \beta < \gamma\}$ is a $\supsetneq^*$ chain of type γ: suppose $\beta < \beta' < \gamma$. We must show that $b_{\beta'} \subsetneq^* b_\beta$. By 3) there exists $c \in L_\beta$ such that $b'_\beta \subsetneq c$. Thus, $a' \subseteq^* c, b_\beta$, so, by 1), $c = b_\beta$.

This is precisely $\mathrm{pred}_T(a)$: given $\beta < \gamma$, $b_\beta \supsetneq b_{\beta+1} \supsetneq a' \subseteq a$. Conversely, suppose $c \in T$ is such that $c \subsetneq a$. There exists $\beta < \mathfrak{h}$ such that $c \in L_\beta$. If $\beta > \alpha$, by 3) there exists $c' \in L_\alpha$ with $a \subsetneq c \subsetneq c'$, so by 1) $a = c'$ and we get a contradiction. If $\beta = \alpha$, by 1) it follows that $a = c$ and we get a contradiction. Finally, if $\beta < \alpha$, we have that $a \subsetneq c, b_\beta$, so $c = b_\beta$ by 1).

We also have that if $\beta < \alpha$, $L_\alpha \cap L_\beta = \emptyset$: suppose by contradiction that $a \in L_\alpha \cap L_\beta$. By 3), there exists $b \in L_\beta$ such that $a \subsetneq b$, so by 1), $a = b$ and we get a contradiction.

Thus, T is a tree and L_α is the α'th level of T. By 1), T is rooted and by 4), it is $\mathfrak{c}$-splitting. Moreover, if $a \in L_{\alpha+1}$, $b \in L_\alpha$ and $a \subsetneq^* b$, by 2) there exists $b' \in L_\alpha$ such that $a \subset b'$, and by 1) it follows that $b = b'$, so $a \subseteq b$.

It only remains to see that T is $\supseteq$-dense. Let $x \in [\omega]^\omega$ be given. By 8), it suffices to see that there exists some limit $\alpha < \mathfrak{h}$ such that $\{a \in L'_\alpha : |a \cap x| = \omega\}| = \mathfrak{c}$.

By 5), $(L_{\alpha+1} : \alpha < \mathfrak{h})$ is a shattering family.

Claim: For all $\beta, \alpha < \mathfrak{h}$, for all $b \in L_\beta$ and for all $y \in [\omega]^\omega$, if $\beta < \alpha$ and $y \cap b$ is infinite, then there exists $a \in L_\alpha$ such that $a \subsetneq^* b$ and $y \cap a$ is infinite.

Proof of the claim: fix α, β, y, b. Since L_α is a MAD family, there exists $a \in L_\alpha$ such that $|a \cap b \cap x| = \omega$. Since T is a tree, there exists $b' \in L_\beta$ such that $a \subsetneq^* b'$. Since $b \cap b'$ is infinite, it follows from 1) that $b = b'$ so we are done.

Now we construct a increasing sequence $(\alpha_n : n \in \omega)$ of ordinals $< \mathfrak{h}$ and sets $a_s \in L_{\alpha_{|s|}}$ for $s \in 2^{<\omega}$ such that for every $s \in 2^{<\omega}$, $|a_s \cap x| = \omega$ and for every $s, t \in 2^{<\omega}$ such that $s \subseteq t$, $a_t \subsetneq a_s$.

To see that this is possible, first let $a_\emptyset = \omega$ and $\alpha_0 = 0$. Having defined $a_{|s|}$ for every $s \in 2^n$, fix $s \in 2^n$. Since $(L_{\alpha+1} : \alpha < \mathfrak{h})$ is shattering (by 5)), there exists $\beta_s < \mathfrak{h}$ such that there exists two distinct $b_{s^\frown(0)}, b_{s^\frown(1)}$ in L_{β_s+1}, such that for $i < 2$, $b_{s^\frown(i)} \cap x \cap a_s$ is infinite. Let $\alpha_{n+1} = \max(\{\beta_s + 1 : s \in 2^n\} \cup \{\alpha_n\}) + 1$. By the Claim, for each $t \in 2^{n+1}$, since $\beta_{t|n} < \alpha_{n+1}$ there exists $a_t \in L_{\alpha_{n+1}}$ such that $a_t \subsetneq^* b_{t|n}$ and $|a_t \cap x \cap a_{t|n}|$ is infinite. In particular, $|a_t \cap x|$ is infinite and $a_t \cap a_{t|n}$ is infinite, and the latter implies that $a_t \subsetneq^* a_{t|n}$.

Now let $\gamma = \sup\{\alpha_n : n \in \omega\}$, which is a limit ordinal, and for each $f \in 2^\omega$ let P_f be a pseudointersection of $(a_{f|n} \cap x : n \in \omega)$. Since $\mathcal{A}'_\gamma$ is a MAD family, there exists $a_f \in \mathcal{A}'_\gamma$ such that $|a_f \cap P_f| = \omega$. If $f \neq g$, let n be the first such that $f(n) \neq g(n)$. Then $a_{f|(n+1)} \cap a_{g|(n+1)}$ is finite, $a_f \cap a_{f|(n+1)}$ is infinite and $a_g \cap a_{g|(n+1)}$ is infinite, which implies, by the tree structure, that $a_f \subsetneq^* a_{f|(n+1)}$ and $a_g \subsetneq^* a_{g|(n+1)}$ are distinct. Thus, $f \to a_f$ is injective. Moreover, for each such f, $a_f \cap x$ is infinite, so we are done.

Construction:

Let $L_0 = \{\omega\}$. Suppose we have constructed L_β for every $\beta < \alpha$.

If $\alpha = 1$, fix an arbitrary $a \in \mathcal{A}_0$, let $\mathcal{B}$ be a MAD family on a and let $L_0 = (\mathcal{A}_0 \setminus \{a\}) \cup \mathcal{B}$.

If $\alpha = \beta + 1$, it suffices to take care of 1), 2), 4) and 5) since 3) will follow from the others plus the inductive hypothesis. Let L' be a MAD family refining both $\mathcal{A}_\alpha$ and L_β. Given $a \in L'$ there exists an unique $b_a \in L_\beta$ such that $a \subseteq^* b_a$. Let $L'' = \{a \cap b_a : a \in L'\}$. Then L'' is a MAD family refining both $\mathcal{A}_\alpha$ and L_α as well, with the additional property that for every $a \in L''$ there exists $b \in L'$ such that $a \subseteq b$. For each $a \in L''$, let $\mathcal{B}_a$ be MAD family of cardinality $\mathfrak{c}$ on a. Let $L_{\alpha+1} = \bigcup_{a \in L''} \mathcal{B}_a$.

If α is limit, let L'_α be a MAD family which is a common refinement of $(L_\beta : \beta < \alpha)$. Let $\mathcal{U} = \{x \in [\omega]^\omega : |\{a \in L'_\alpha : |a \cap x| = \omega\}| = \mathfrak{c}\}$. By an easy recursion, define an injective function $F : \mathcal{U} \to L'_\alpha$ such that for every $x \in \mathcal{U}$, $|F(x) \cap x| = \omega$. Let $L_\alpha = (L'_\alpha \setminus F[\mathcal{U}]) \cup \{F(x) \cap x : x \in \mathcal{U}\} \cup \{x \setminus F(x) : x \in \mathcal{U} \text{ and } |x \setminus F(x)| = \omega\}$. We leave the (easy) details to the reader. □

Finally, we get the following corollary, which is a rephrasing of Lemma 3.4. of [2].

Corollary 1.3.9 ([2, Lemma 3.4.]). $\mathfrak{h} = n(\omega^*)$ if, and only if there exists a base tree of height $\mathfrak{h}$ with no chains of size $\mathfrak{h}$.

Proof. First, suppose that $\mathfrak{h} = n(\omega^*)$. Then $\mathrm{FA}_{[\omega]^\omega}(\mathfrak{h})$ fails by Proposition 0.7.3, thus there exists a family $(\mathcal{A}_\alpha : \alpha < \mathfrak{h})$ of maximal antichains of $[\omega]^\omega$ (i.e., MAD families) with no free ultrafilter intersecting them all.

This family is shattering, for if not, there would exist $x \in [\omega]^\omega$ intersecting exactly one element of each $\mathcal{A}_\alpha$, which implies that $x \in (A_\alpha)_\downarrow$ for each $\alpha < \mathfrak{h}$, so any free ultrafilter containing x would give us a contradiction.

By the Base Tree Lemma, there exists a base tree of height $\mathfrak{h}$ such that its levels $(L_\alpha : \alpha < \mathfrak{h})$ are such that $L_{\alpha+1} \preceq \mathcal{A}_\alpha$ for each $\alpha < \mathfrak{h}$. Such a tree cannot have a chain of size $\mathfrak{h}$, for if it had, this chain would intersect every level, so any free ultrafilter containing this chain would generate a contradiction.

Conversely, suppose $\mathfrak{h} < n(\omega^*)$. Let T be a base tree of height $\mathfrak{h}$. To see that T has a chain of size $\mathfrak{h}$, just notice that a free ultrafilter $\mathcal{U}$ intersecting every level of T is such that $\mathcal{U} \cap T$ is a chain of size $\mathfrak{h}$ of T. □

Chapter 2

Weakenings of Normality in Isbell-Mrówka spaces

2.1 Introduction

In this chapter we study weakenings of normality in Isbell-Mrówka spaces. The new results (marked with an asterisk *) presented in this chapter are in our published paper [61] or in [62]). Normality is a topological property which is not as well behaved as some of the most known weaker standard separation properties, such as regularity, Tychonoffness and Hausdorffness: a subspace of a normal space does not need to be normal, and the product of two normal spaces may fail to be normal.

Normality is related to metrization results: every metrizable space is normal. The converse is not true, but if we add some extra conditions we have some metrization theorems which guarantee the converse. We will mention one of these results in this section to help us to introduce our problems.

In general, Isbell-Mrówka spaces do not need to be normal. Countable Isbell-Mrówka spaces are metrizable (thus, normal), but these are the only metrizable Isbell-Mrówka spaces. The proof of this well known fact is easy, but we add it here for the convenience of the reader.

Proposition 2.1.1. Let $\mathcal{A}$ be an almost disjoint family. Then $\Psi(\mathcal{A})$ is metrizable iff $\mathcal{A}$ is countable.

Proof. Suppose $\mathcal{A}$ is countable. Then $\Psi(\mathcal{A})$ is a countable first countable T_3 space, thus, a second countable T_3 space. This implies that $\Psi(\mathcal{A})$ is metrizable by Urysohn's metrization Theorem (see e.g. [70, p. 23.1.]).

Conversely, if $\mathcal{A}$ is uncountable, it is clear that $\Psi(\mathcal{A})$ is not Lindelöf since any covering of $\Psi(\mathcal{A})$ by basic open sets does not have a countable subcover since each basic open subset of $\Psi(\mathcal{A})$ has at most one point from $\mathcal{A}$. However, $\Psi(\mathcal{A})$ is separable. It is well known that separable metrizable spaces are Lindelöf (e.g. [70, p. 16.11]), thus, $\Psi(\mathcal{A})$ is not metrizable. $\square$

Thus, it is natural to ask: is there a normal uncountable Isbell-Mrówka space? There are some immediate well known restrictions:

Proposition 2.1.2. Let $\mathcal{A}$ be an almost disjoint family. Then:

i) If $\mathcal{A}$ is a MAD family, then $\Psi(\mathcal{A})$ is not normal.

ii) If $2^{|\mathcal{A}|} > \mathfrak{c}$, then $\Psi(\mathcal{A})$ is not normal. In particular, if $|\mathcal{A}| = \mathfrak{c}$ then $\Psi(\mathcal{A})$ is not normal.

Proof. i) We have already seen that if $\mathcal{A}$ is a MAD family, then $\Psi(\mathcal{A})$ is a pseudocompact non countably compact space (it is pseudocompact by Proposition 1.2.3 and is not countably compact since $\mathcal{A}$ is an infinite closed discrete subspace). Since pseudocompact normal spaces are countably compact (as a direct consequence of Tietze's extension Theorem [70, p. 15.8]), $\Psi(\mathcal{A})$ is not normal.

ii) This can be seen as a consequence of Jones's lemma, but we prove it directly. Suppose $\Psi(\mathcal{A})$ is normal. For each $S \subseteq \mathcal{A}$, let U_S, V_S be two disjoint open subsets of $\Psi(\mathcal{A})$ such that $U_S \cap V_S = \emptyset$, $S \subseteq U_S$ and $\mathcal{A} \setminus S \subseteq V_S$. We claim that the function $S \to U_S \cap \omega$ is injective, which completes the proof since this implies that $2^{|\mathcal{A}|} \leq 2^\omega$: if $S \neq S'$, WLOG $S \setminus S' \neq \emptyset$, so fix $a \in S \setminus S'$. Then $a \in U_S \cap V_{S'}$, so $\omega \cap U_S \cap V_{S'}$ is a nonempty subset of $U_S \cap \omega$ disjoint from $U_{S'} \cap \omega$. □

As a corollary from ii), it follows that:

Corollary 2.1.3. CH implies that there are no uncountable normal Isbell-Mrówka spaces. □

Since CH is consistent with ZFC, this implies that it is consistent that every uncountable Isbell-Mrówka space is not normal. This does not answer the question of whether it is consistent with ZFC if there is an uncountable normal Isbell-Mrówka space.

The problem of whether such an space exists is closely related to the problem of the existence of a normal separable non-metrizable Moore space. To state this problem we need some well used definitions.

Definition 2.1.4. Let X be a topological space, $x \in X$ and $\mathcal{U}$ be a cover of X. The *star with respect to* $\mathcal{U}$ *around* x is the set $\mathrm{St}(x, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : x \in U\}$.

Moreover, if $V \subseteq X$, the *star with respect to* $\mathcal{U}$ *around* V is the set $\mathrm{St}(V, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap V \neq \emptyset\}$. □

Definition 2.1.5. Let X be a topological space. A *development* for X is a sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X such that for every x, $\{\mathrm{St}(x, \mathcal{U}_n) : n \in \omega\}$ is a neighborhood basis for x, that is, for every open neighborhood U of x there exists $n \in \omega$ such that $\mathrm{St}(x, \mathcal{U}_n) \subseteq U$.

We say that a development for X $(\mathcal{U}_n)_{n\in\omega}$ is a *strong development* iff for every $x \in X$ and for every open neighborhood U of x there exists $n \in \omega$ and an open neighborhood V of x such that $\mathrm{St}(V, \mathcal{U}_n) \subseteq U$.

In case there exists a (strong) development for X, we say that X is *(strongly) developable.* □

Now we can state Moore's Metrization Theorem. We will not present a proof of this result. See [24, p. 5.4.2.] for a proof.

Theorem 2.1.6 (Moore's Metrization Theorem)**.** A topological space is metrizable iff it is T_0 and strongly developable.

It is natural to ask if the "strongly" hypothesis can be removed. The answer is negative, as we shall see. Now this is a good moment to introduce Moore spaces.

Definition 2.1.7. A *Moore space* is a developable regular space. □

Thus, we could ask if every Moore space is metrizable. The answer is negative by the following example.

Proposition 2.1.8. Every Isbell-Mrówka space is a Moore space.

Proof. We have already seen that Isbell-Mrówka spaces are T_3. We show that they are developable.

Let $\mathcal{A}$ be an almost disjoint family. For each $n \in \omega$, let $\mathcal{U}_n = \{\{k\} : k \in \omega\} \cup \{\{a\} \cup (a \setminus n) : a \in \mathcal{A}\}$. For each n, it is clear that $\mathcal{U}_n$ is an open cover of $\Psi(\mathcal{A})$.

If $n \in \omega \subseteq \Psi(\mathcal{A})$, then $\mathrm{St}(n, \mathcal{U}_{n+1}) = \{n\}$ is contained in any set which has n as a point.

If $a \in \mathcal{A}$, $\{\mathrm{St}(a, \mathcal{U}_n) : n \in \omega\} = \{\{a\} \cup (a \setminus n) : n \in \omega\}$ is an open basis for a, so we are done. □

So uncountable Isbell-Mrówka spaces are examples of Moore separable non-metrizable spaces. What happens if, in addition, we require that the space is normal? This is what is known as the normal separable non-metrizable Moore space problem. It is also motivated by Bing's Metrization Theorem, which states that a topological space is metrizable iff it is a collectionwise normal Moore space. (see [24, p. 5.4.1.]). The answer turns out to be independent of ZFC. We will study the proof of this previously well known result in the next section and extract some tools from it which will be useful to get to new results.

We proceed to formally state Bing's Metrization Theorem without proof along with some basic lemmas about some concepts related to it.

Definition 2.1.9. Let X be a topological space and $\mathcal{C} \subseteq \mathcal{P}(X)$.

- $\mathcal{C}$ is said to be *locally finite* iff for every $x \in X$ there exists an open neighborhood U of x such that $|\{F \in \mathcal{C} : F \cap U \neq \emptyset\}| < \omega$
- $\mathcal{C}$ is said to be *discrete* iff for every $x \in X$ there exists an open neighborhood U of x such that $|\{F \in \mathcal{C} : F \cap U \neq \emptyset\}| \leq 1$.

□

The following is well known and left as an exercise.

Lemma 2.1.10. Let X be a topological space. The union of a locally finite collection of closed set is closed. □

The following lemma is easy, follows from Lemma 2.1.10 and is left to the reader.

Lemma 2.1.11. Let X be a topological space and $\mathcal{C}$ be a locally finite collection of pairwise disjoint nonempty closed sets. Let $f : \mathcal{C} \to X$ be a choice function, that is, a function satisfying $f(F) \in F$ for every $F \in \mathcal{C}$. Then $\operatorname{ran} f$ is closed.

Now we define what collectionwise normal means.

Definition 2.1.12. Let X be a topological space. We say that X is *collectionwise normal* iff X is T_1 and for every discrete collection $\mathcal{C}$ of closed sets there exists a family of pairwise disjoint open sets $(U_F : F \in \mathcal{C})$ such that for every F in $\mathcal{C}$, $F \subseteq U_F$. □

Normality has this property up to certain point.

Proposition 2.1.13. Let X be a T_4 topological space. Then for every countable discrete collection $\mathcal{C}$ of closed sets there exists a family of pairwise disjoint open sets $(U_F : F \in \mathcal{C})$ such that for every F in $\mathcal{C}$, $F \subseteq U_F$.

Proof. We prove the infinite case, the finite case is analogous. Enumerate $\mathcal{C} = \{F_n : n \in \omega\}$ injectively. Let U_0, V_0 be two disjoint open sets separating F_0 from $\bigcup_{i \geq 1} F_i$.

Suppose we have defined U_n, V_n for every $n < m$ for some $m \in \omega$ in a way such that for every $n \leq m$, $F_n \subseteq U_n$, $\bigcup_{i \leq n} F_i \subseteq V_n$, $U_n \cap V_n = \emptyset$, and that for every $n < n' \leq m$, $U_{n'} \subseteq V_n$ and $V_{n'} \subseteq V_n$.

We show how to define U_m, V_m if $m > 0$. Let V, U be two disjoint open sets separating F_m from $\bigcup_{i \geq m+1} F_i$. Let $U_m = V_{m-1} \cap U$ and $V_m = V_{m-1} \cap V$.

Now let $U_{F_n} = U_n$ for each $n \in \omega$ and we are done. □

Now we state a version of Bing's theorem.

Theorem 2.1.14 (Bing's Metrization Theorem [9])**.** Every collectionwise normal Moore space is metrizable. □

2.2 More on Normality

In this section we study the known result which related the normal Isbell-Mrówka spaces with the normal separable non-metrizable Moore space problem and extract some of the tools used in the proof.

First, we state some basic definitions for studying normality-like properties in Isbell-Mrówka spaces.

Definition 2.2.1. Let $\mathcal{A}$ be an almost disjoint family and $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$.

- A *partitioner* for $\mathcal{A}$ is a subset $X \subseteq \omega$ such that for every $a \in \mathcal{A}$, either $a \subseteq^* X$ or $a \cap X =^* \emptyset$.
- Given a partitioner X for $\mathcal{A}$, we say that X *separates* $\mathcal{B}$ *from* $\mathcal{C}$ iff for every $a \in \mathcal{B}$, $a \subseteq^* X$ and for every $a \in \mathcal{C}$, $a \cap X =^* \emptyset$.

- Given $X \subseteq \omega$, we say that X *weakly separates* $\mathcal{B}$ *from* $\mathcal{C}$ iff for every $a \in \mathcal{B}$, $|a \cap X| = \omega$ and for every $a \in \mathcal{C}$, $a \cap X =^* \emptyset$.
- We say that $\mathcal{B}, \mathcal{C}$ *can be separated* iff there exists a partitioner X which separates $\mathcal{B}$ from $\mathcal{C}$.
- We say that $\mathcal{B}, \mathcal{C}$ *can be weakly separated* iff there exists $X \subseteq \omega$ which weakly separates $\mathcal{B}$ from $\mathcal{C}$.

□

Moreover, we will adopt the following definition as a shorthand.

Definition 2.2.2. Let $\mathcal{A}$ be an almost disjoint family. We say that $\mathcal{A}$ is *normal* iff $\Psi(\mathcal{A})$ is normal (as a topological space). □

Some sets are trivially partitioners. We leave the following examples as an easy warm-up exercise to the reader.

Proposition 2.2.3. Let $\mathcal{A}$ be an almost disjoint family and $X, Y \subseteq \omega$. Then:

- ω, $\emptyset$ are partitioners.
- If X, Y are partitioners, so are $X \cup Y$, $X \cap Y$ and $X \setminus Y$.
- If X is a partitioner and $X =^* Y$, then Y is a partitioner.
- Let $\mathcal{A}' \subseteq \mathcal{A}$ be finite. Then $\bigcup \mathcal{A}'$ is a partitioner separating $\mathcal{A}'$ from $\mathcal{A} \setminus \mathcal{A}'$.

Thus, the collection of all partitioners is a $=^*$-closed algebra containing all finite unions of subsets of $\mathcal{A}$.

On the other hand, sometimes we know that certain subcollections of $\mathcal{A}$ cannot be weakly separated. We state and prove the following folkore result.

Proposition 2.2.4. Let $\mathcal{A}$ be a MAD family and $\mathcal{A}' \subseteq \mathcal{A}$ be such that $|\mathcal{A}'| = \omega$. Then $\mathcal{A}'$ and $\mathcal{A} \setminus \mathcal{A}'$ cannot be weakly separated.

Proof. Let $(a_n : n \in \omega)$ be an injective enumeration of $\mathcal{A}'$. Working towards a contradiction, let X be a partitioner for $\mathcal{A}'$, $\mathcal{A} \setminus \mathcal{A}'$. Define a strictly increasing sequence $(x_n : n \in \omega)$ such that $x_n \in (a_n \setminus \bigcup_{m<n} a_m) \cap X$. Then $\{x_n : n \in \omega\} \cap a = \emptyset$ for every $a \in \mathcal{A}$, thus $\mathcal{A}$ is not MAD. □

Partitioners encode the clopen subsets of the Isbell-Mrówka spaces.

Proposition 2.2.5. Let $\mathcal{A}$ be an almost disjoint family. Then:

1. If $C \subseteq \Psi(\mathcal{A})$ is clopen, then $C \cap \omega$ is a partitioner separating $\mathcal{A} \cap C$ from $\mathcal{A} \setminus C$.
2. If $X \subseteq \omega$ is a partitioner separating $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, then $X \cup \{a \in \mathcal{A} : a \subseteq^* X\}$ is a clopen set separating $\mathcal{B}$ from $\mathcal{C}$.

Proof. 1. We only need to show that C is a partitioner since the rest is clear. Let $a \in \mathcal{A}$ be given. If $a \notin C$, then since $\Psi(\mathcal{A}) \setminus C$ is open, $a \subseteq^* \Psi(\mathcal{A}) \setminus C$, so $a \cap \omega \cap C =^* \emptyset$. If $a \in C$, then since C is open, $a \subseteq^* C \cap \omega$.

2. We only need to see that $C = X \cup \{a \in \mathcal{A} : a \subseteq^* X\}$ is a clopen since the rest is clear. It is clear that C is open. We show that C is closed. Suppose that $a \in \mathcal{A}$ is in $\operatorname{cl} C$. We must see that $a \in C$. Since X is a partitioner, it suffices to see that $|a \cap X| = \omega$. But this is true since $\omega = |(\{a\} \cup a) \cap C| = |a \cap X|$. $\square$

This relation is useful to prove normality-related theorems. The following is also folklore:

Proposition 2.2.6. Let $\mathcal{A}$ be an almost disjoint family. $\mathcal{A}$ is normal iff for every $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ can be separated.

Moreover, in this case, every two closed disjoint subsets of $\Psi(\mathcal{A})$ can be separated by clopens.

Proof. Suppose $\Psi(\mathcal{A})$ is normal. Let $\mathcal{B}$ be given. Since $\mathcal{A}$ is normal, there exists open sets U, V such that $\mathcal{B} \subseteq U$, $\mathcal{A} \setminus \mathcal{B} \subseteq V$ and $U \cap V = \emptyset$. By the previous proposition it suffices to see that $C = \Psi(\mathcal{A}) \setminus V$ is a clopen, since it contains $\mathcal{B}$ and is disjoint from $\mathcal{A} \setminus \mathcal{B}$. It is clearly closed. To see that it is open, just notice that given $a \in C \cap \mathcal{A} = \mathcal{B}$, $a \subseteq^* U \subseteq C$.

Conversely, suppose that for every $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ can be separated. Let F, K be two closed disjoint subsets of $\Psi(\mathcal{A})$. By the previous proposition, there exists a clopen set C separating $\mathcal{B} = F \cap \mathcal{A}$ from $\mathcal{C} = K \cap \mathcal{A}$. Now $D = C \cup (F \cap \omega)$ is also clopen and separates F from K. To see that it is closed, just notice that if $a \in \operatorname{cl} D$, then $a \cap (F \cap \omega)$ is infinite, thus $a \in \operatorname{cl} F = F$, so $a \in \mathcal{B} \subseteq C$. $\square$

An Isbell-Mrówka space which is a counterexample would have to be an uncountable Isbell-Mrówka normal space, which does not exist under CH as we have mentioned above.

Before we state the result relating these two objects we will talk about a last ingredient which we still haven't talked about. A class of special sets of reals.

Proposition 2.2.7. Let $X \subseteq 2^\omega$. We say that X is a *Q-set* iff it is uncountable and every subset of X is a G_δ (or, equivalently, an F_σ) in the relative subspace topology.

It is clear that these objects also do not exist under CH since for every uncountable $X \subseteq 2^\omega$, by CH there are only $\omega_1 = \mathfrak{c}$ many G_δ relative subsets of X and $2^\mathfrak{c}$ subsets of X.

The following theorem, due to F. Tall [67], appears with this form in [39]. The proof that is sketched there is partially attributed to folklore. We will dedicate the rest of this section to prove this result in a very detailed way. The techniques we will use, which appear in these two papers, will be useful in the next sections of this chapter.

Theorem 2.2.8. The following are equivalent:

a) There exists a Q-set,

b) There exists an uncountable almost disjoint family $\mathcal{A}$ such that $\Psi(\mathcal{A})$ is normal, and

c) There exists a separable normal Moore space which is not metrizable.

□

To study this theorem we need some tools which relate subsets of the reals (2^ω) to almost disjoint families.

Every element of an almost disjoint family is an element of $[\omega]^\omega \subseteq \mathcal{P}(\omega) \approx 2^\omega$ by using the identification between a subset of ω and its characteristic function, so every almost disjoint family can be seen as a set of reals.

Conversely, given a set of reals X, one has the almost disjoint family of branches $\mathcal{A}_X$ as defined in Definition 1.1.3.

We will discuss on how the properties of these objects behave and transform into one another.

The next result, which is probably folklore, appears in [62].

Proposition 2.2.9 (Probably folklore)**.** Given $X \subseteq 2^\omega$ and $Y \subseteq X$. The following are equivalent:

1. $\mathcal{A}_Y$ and $\mathcal{A}_{X\setminus Y}$ can be separated in $\mathcal{A}_X$;
2. Y and $X \setminus Y$ are F_σ in X.

Proof. (1) implies (2): Let $Z \subseteq 2^{<\omega}$ be a partitioner separating $\mathcal{A}_Y$ from $\mathcal{A}_{X\setminus Y}$ such that for all $y \in Y$ and $x \in X \setminus Y$, $a_y \subseteq^* Z$ and $a_x \cap Z =^* \emptyset$. It follows that:

$$Y = \{y \in X : a_y \subseteq^* Z\} = \bigcup_{n\in\omega} \bigcap_{m\geq n} \underbrace{\{y \in X : y|m \in Z\}}_{\text{closed in } X}.$$

Notice that $Z_0 = 2^{<\omega} \setminus Z$ is a partitioner for $\mathcal{A}_Y$ and $\mathcal{A}_{X\setminus Y}$ such that $A_x \subseteq^* Z_0$ iff $x \in X \setminus Y$, one concludes that $X \setminus Y$ is also an F_σ set of X.

(2) implies (1): for $F \subseteq 2^\omega$, we denote $\hat{F} = \{x|n : n \in \omega, x \in F\}$.

Write $Y = \bigcup_{n\in\omega} F_n$ and $X \setminus Y = \bigcup_{n\in\omega} G_n$, where F_n and G_n are closed in X. We proceed by a standard shoelace argument. Define $J_0 = \widehat{F}_0$, $K_0 = \widehat{G}_0 \setminus \widehat{F}_0$, and, recursively, $J_n = \widehat{F}_n \setminus \left(\bigcup_{i<n} \widehat{G}_i\right)$, $K_n = \widehat{G}_n \setminus \left(\bigcup_{i\leq n} \widehat{F}_i\right)$ for $n > 0$. Let $J = \bigcup_{n\in\omega} J_n$. It follows that $J \cap K_m = \emptyset$ for all $m \in \omega$. We claim that J is a partitioner separating $\mathcal{A}_Y$ from $\mathcal{A}_{X\setminus Y}$.

If $a_x \in \mathcal{A}_Y$, then $x \in Y$ so there exists a $n \in \omega$ such that $x \in F_n$. Since $\bigcup_{i<n} G_i$ is closed, there exists $k \in \omega$ that $\{f \in 2^\omega : x|_k \subseteq f\} \cap \bigcup_{i<n} G_i = \emptyset$. Hence, $a_x \subseteq^* J_n \subseteq J$. Similarly, if $a_x \in \mathcal{A}_{X\setminus Y}$, then $a_x \cap J =^* \emptyset$. □

From this, it follows that:

Proposition 2.2.10. Let $X \subseteq 2^\omega$ be a uncountable set. Then $\mathcal{A}_X$ is normal iff X is a Q-set.

Proof. First, suppose X is a Q-set. By Proposition 2.2.6, it suffices to see that for every $\mathcal{A}' \subseteq \mathcal{A}_X$, $\mathcal{A}'$ and $\mathcal{A}_X \setminus \mathcal{A}'$ can be separated. Fix $\mathcal{A}'$, which is of the form $\mathcal{A}_Y$ for some

$Y \subseteq X$. Then $\mathcal{A}_X \setminus \mathcal{A}_Y = \mathcal{A}_{X \setminus Y}$. Since both Y and $X \setminus Y$ are relative F_σ subsets of X, it follows from Proposition 2.2.9 that $\mathcal{A}'$ and $\mathcal{A}_X \setminus \mathcal{A}'$ can be separated.

Conversely, suppose that $\mathcal{A}_X$ is normal. We must see that every subset of X is a relative F_σ. Let $Y \subseteq X$ be given. Since $\mathcal{A}_X$ is normal, it follows from Proposition 2.2.6 that $\mathcal{A}_Y$ and $\mathcal{A}_X \setminus \mathcal{A}_Y = \mathcal{A}_{X \setminus Y}$ can be separated. Thus, by Proposition 2.2.9, Y is a relative F_σ subspace of X. □

In particular, this shows that if a Q-set exists, then a normal uncountable Isbell-Mrówka space exists. Now we prove the converse. First, we have the following lemma:

Lemma 2.2.11. Let $\mathcal{A}$ be an uncountable normal almost disjoint family. Then for every $X \subseteq^* \omega$, $\{\chi_a : a \subseteq^* X \text{ and } a \in \mathcal{A}\}$ is an F_σ subset of $\{\chi_a : a \in \mathcal{A}\}$.

Proof. Let $Q = \{\chi_a : a \in \mathcal{A}\}$. Notice that $Q \setminus \{\chi_a : a \subseteq^* X \text{ and } a \in \mathcal{A}\} = \{\chi_a : (\forall n\, \exists m \geq n\, m \in a \setminus X) \text{ and } a \in \mathcal{A}\} = Q \cap \bigcap_{n\in\omega} \bigcup_{m\geq n} \{f \in 2^\omega : f(m) = 1 \text{ and } m \notin X\}$. which is a G_δ subset of X since for each $m \in \omega$, $\{f \in 2^\omega : f(m) = 1 \text{ and } m \notin X\}$ is an open subset of 2^ω since it is either $\{f \in 2^\omega : f(m) = 1\}$ or $\emptyset$. □

Now, the result on normality.

Proposition 2.2.12. Let $\mathcal{A}$ be an uncountable normal almost disjoint family. Then $\mathcal{A}$ is a Q-set of $\mathcal{P}(\omega)$, that is, $\{\chi_a : a \in \mathcal{A}\}$ is a Q-set.

Proof. Let $\mathcal{B} \subseteq \mathcal{A}$ be given. Let $Q = \{\chi_a : a \in \mathcal{A}\}$. By Proposition 2.2.6, let X be a partitioner separating $\mathcal{B}$ from $\mathcal{A} \setminus \mathcal{B}$. Then $\{\chi_a : a \in \mathcal{B}\}$ is an F_σ subset of $\{\chi_a : a \in \mathcal{A}\}$ by Lemma 2.2.11. □

The converse of the previous proposition is consistently not true. Of course, it is vacuously true under CH, but in [54] A. Miller has used forcing to contruct a MAD family which is a Q-set (that is, a MAD family $\mathcal{A}$ such that $\{\chi_a : a \in \mathcal{A}\}$ is a Q-set). By Proposition 2.1.2, no MAD family is normal.

Now there is just one piece missing for the classical result.

Proposition 2.2.13. Suppose there is a separable uncountable normal Moore space which is not metrizable. Then there is an uncountable normal almost disjoint family.

Proof. Let X be such an space and let N be a countable dense subset of X. We may suppose that $[N]^\omega \cap N = \emptyset$ since if this is not true, biject X with $|X|$ in a way that N is sent onto ω and copy the topological structure of X to $|X|$.

By Bing's metrization theorem, since X is not metrizable, X is not collectionwise normal, so since $\mathcal{C}$ is normal it follows from Lemma 2.1.11 and Proposition 2.1.13 that X contains an uncountable closed discrete subspace Y. By removing N, we may suppose that $Y \cap N = \emptyset$.

For each $y \in Y$, let a_y be a sequence of elements of N converging to y. This is possible since X is first countable and N is dense. Then $\{a_y : y \in \omega\}$ is an uncountable almost disjoint family over N. We must verify that it is normal. We aim to apply Proposition 2.2.6.

So let $Z \subseteq Y$. Since Y is closed and discrete, both $Y \setminus Z$ and Z are closed subsets of X, thus there exists disjoint open sets U, V such that $Z \subseteq U$, $Y \setminus Z \subseteq V$. Let $A = N \cap U$. We claim that A is a partitioner separating $\{a_y : y \in Z\}$ from its complement $\{a_y : y \in Y \setminus Z\}$.

If $y \in Z$, then $a_y \subseteq^* U \cap N = A$ since it is a sequence converging to y and U is an open neighborhood of y.

If $y \in Y \setminus Z$, then $a_y \subseteq^* V$ since it is a sequence converging to y and U is an open neighbordhood of y, and V is disjoint from $U \cap N$ so the proof is complete. $\square$

Thus, Theorem 2.2.8 follows from Proposition 2.2.10 (a implies b), Proposition 2.2.13 (c implies b), 2.2.12 (b implies a) and from the that uncountable Isbell-Mrówka spaces are non metrizable Moore spaces (b implies c) as we have seen in the previous section.

As a corollary from these same results, we also get the following:

Corollary 2.2.14. Let κ be an infinite cardinal. There is a normal almost disjoint family of size κ iff there is a Q-set of size κ. $\square$

It is natural to ask what is the least cardinality of a set which is not a Q-set. Following [14], we define:

Definition 2.2.15. $\mathfrak{q}$ is defined as $\min\{|X| : X \subseteq 2^\omega$ is uncountable and not a Q-set$\}$. $\square$

Since no almost disjoint family of cardinality $\mathfrak{c}$ is normal, there are no Q-sets of cardinality $\mathfrak{c}$. So $\mathfrak{q}$ is well defined and $\omega_1 \leq \mathfrak{q} \leq \mathfrak{c}$.

In fact, it is known that $\mathfrak{p} \leq \mathfrak{q} \leq \mathfrak{b}$. We will show that $\mathfrak{p} \leq \mathfrak{q}$ in this section with a known proof, but, again, the techniques we will use will be used to prove some new results later. In the next section we will show that $\mathfrak{q} \leq \mathfrak{b}$.

We aim to apply Bell's theorem. Thus, we define some orders:

Definition 2.2.16. Let $A, X \subseteq 2^\omega$ be such that $A \subseteq X$. We define $P(A, X)$ as the following set:

$$\{r \in [\omega \times (2^{<\omega} \cup A)]^{<\omega} : \forall n \in \omega\, \forall x \in A\, \forall s \in 2^{<\omega}\, (n, x) \in r \text{ and } (n, s) \in r \rightarrow s \not\subseteq x\}$$

We order $P(A, X)$ by the reverse inclusion: so $\emptyset$ is the maximum element and $r \leq r'$ iff $r' \subseteq r$. $\square$

Lemma 2.2.17. Let $A, X \subseteq 2^\omega$ be such that $A \subseteq X$. Then $P(A, X)$ is σ-centered.

Proof. Notice that for each finite $S \in [\omega \times 2^{<\omega}]^{<\omega}$, the set $\{r \in P(A, X) : r \cap (\omega \times 2^{<\omega}) = S\}$ is centered: if $r_1, \ldots, r_n$ are members of this set, then $r_1 \cup \cdots \cup r_n$ is a member of this set as well. $\square$

The following is easy and follows directly from the finiteness of the conditions.

Lemma 2.2.18. Let $A, X \subseteq 2^\omega$ be such that $A \subseteq X$. Let $x \in A$. Then $D_x = \{r \in P(A, X) : \exists n \in \omega\, (n, x) \in r\}$ is dense. $\square$

Lemma 2.2.19. Let $A, X \subseteq 2^\omega$ be such that $A \subseteq X$. Let $x \in X \setminus A$ and $n \in \omega$. Then $E^n_x = \{r \in P(A, X) : \exists s \in 2^{<\omega}\ s \subseteq x \text{ and } (n, s) \in r\}$ is dense.

Proof. Fix r. Let k be large enough so that $x|k \neq y|k$ for every y such that $(n, y) \in r$. Let $s = x|k$. Let $r' = r \cup \{(n, s)\}$. □

Proposition 2.2.20. Let $A, X \subseteq 2^\omega$ be such that $A \subseteq X$. If there exists a filter G on $P(A, X)$ such that $G \cap D_x \neq \emptyset$ for all $x \in A$ and $G \cap E^n_x \neq \emptyset$ for all $x \in X \setminus A$, $n \in \omega$ (as defined in the previous two lemmas), then A is a F_σ subset of X.

Proof. For each $n \in \omega$, let $U_n = \{x \in 2^\omega : \exists s \in 2^{<\omega}, \exists r \in G\ (n, s) \in r\}$. U_n is clearly open. We claim that $X \cap \bigcap_{n \in \omega} U_n = X \setminus A$.

$\supseteq$: fix $x \in X \setminus A$ and $n \in \omega$. There exists $r \in E^n_x \cap G$, so there exists $s \in \omega$ such that $s \subseteq x$ and $(n, s) \in r$. This implies that $x \in U_n$.

$\subseteq$: Suppose that $x \in A$. We show that $x \notin \bigcap_{n \in \omega} U_n$. There exists $r \in G \cap D_x$. There exists n such that $(n, x) \in r$. We claim that $x \notin U_n$. Suppose by contradiction that $x \in U_n$. Then there exists $r' \in G$ and $s \in 2^{<\omega}$ such that $(n, s) \in r'$ and $s \subseteq x$. Since G is a filter, there exists $r'' \leq r, r'$ in G. So $(n, s) \in r''$ and $(n, x) \in r''$, a contradiction by the definition of $P(A, X)$. □

Then the following follows easily from Bell's theorem:

Corollary 2.2.21. Let $A, X \subseteq 2^\omega$ be such that $A \subseteq X$. Suppose $|X| < \mathfrak{p}$. Then A is a F_σ subset of X. □

Corollary 2.2.22. $\mathfrak{p} \leq \mathfrak{q}$. □

The consistency of $\mathfrak{p} < \mathfrak{q}$ was implicitly proved in [23], as explained in [14].

2.3 On the pseudonormality of Isbell-Mrówka spaces

In this section we recall some other classical results using the tools discussed in the previous section.

Pseudonormality is a natural weakening of normality.

Definition 2.3.1. Let X be a topological space. We say that X is *pseudonormal* iff it is T_1 and for every closed sets F, K, if F is countable and $F \cap K = \emptyset$ then there exists two open sets U, V such that $U \cap V = \emptyset$, $F \subseteq U$ and $K \subseteq V$. □

In the previous section we saw that Q-sets are closely related to the normality of Isbell-Mrówka spaces. In this section we obtain folklore results stating that λ-sets have similar properties with respect to pseudonormality.

Definition 2.3.2. Let $X \subseteq 2^\omega$. We say that X is a λ-*set* iff X is uncountable and every countable subset of X is a relative G_δ. □

Of course, every Q-set is a λ-set. Again, we use the following as a shorthand:

Definition 2.3.3. Let $\mathcal{A}$ be an almost disjoint family. We say that $\mathcal{A}$ is pseudonormal iff $\Psi(\mathcal{A})$ is pseudonormal. □

Now we derive results analogous to the previous section.

Proposition 2.3.4. Let $\mathcal{A}$ be an almost disjoint family. $\mathcal{A}$ is pseudonormal iff for every countable $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ can be separated.

Moreover, in this case, every two closed disjoint subsets of $\Psi(\mathcal{A})$ can be separated by clopens when one of them is countable.

Proof. Suppose $\Psi(\mathcal{A})$ is pseudonormal. Let $\mathcal{B}$ be given. Since $\mathcal{A}$ is pseudonormal, there exists open sets U, V such that $\mathcal{B} \subseteq U$, $\mathcal{A} \setminus \mathcal{B} \subseteq V$ and $U \cap V = \emptyset$. By Proposition 2.2.5 it suffices to see that $C = \Psi(\mathcal{A}) \setminus V$ is a clopen, since it contains $\mathcal{B}$ and is disjoint from $\mathcal{A} \setminus \mathcal{B}$. It is clearly closed. To see that it is open, just notice that given $a \in C \cap \mathcal{A} = \mathcal{B}$, $a \subseteq^* U \subseteq C$.

Conversely, suppose that for every countable $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ can be separated. Let F, K be two closed disjoint subsets of $\Psi(\mathcal{A})$ with F countable. By Proposition 2.2.5, there exists a clopen set C separating $\mathcal{B} = F \cap \mathcal{A}$ from $\mathcal{C} = K \cap \mathcal{A}$. Now $D = C \cup (F \cap \omega)$ is also clopen and separated F from K. To see that it is closed, just notice that if $a \in \operatorname{cl} D$, then $a \cap (F \cap \omega)$ is infinite, thus $a \in \operatorname{cl} F = F$, so $a \in \mathcal{B} \subseteq C$. □

It follows that MAD families cannot be pseudonormal.

Lemma 2.3.5. Let $\mathcal{A}$ be a MAD family. Then $\mathcal{A}$ is not pseudonormal.

Proof. This follows from the previous proposition and from Proposition 2.2.4. □

Proposition 2.3.6. Let $X \subseteq 2^\omega$ be a uncountable set. Then $\mathcal{A}_X$ is pseudonormal iff X is a λ-set.

Proof. First, suppose X is a λ-set. By Proposition 2.3.4, it suffices to see that for every countable $\mathcal{A}' \subseteq \mathcal{A}_X$, $\mathcal{A}'$ and $\mathcal{A}_X \setminus \mathcal{A}'$ can be separated. Fix a countable $\mathcal{A}'$, which is of the form $\mathcal{A}_Y$ for some countable $Y \subseteq X$. Then $\mathcal{A}_X \setminus \mathcal{A}_Y = \mathcal{A}_{X \setminus Y}$. Since both Y and $X \setminus Y$ are relative F_σ subsets of X, it follows from Proposition 2.2.9 that $\mathcal{A}'$ and $\mathcal{A}_X \setminus \mathcal{A}'$ can be separated.

Conversely, suppose that $\mathcal{A}_X$ is pseudonormal. We must see that every countable subset of X is a relative G_δ. Let $Y \subseteq X$ be given. Since $\mathcal{A}_X$ is pseudonormal, it follows from Proposition 2.3.4 that $\mathcal{A}_Y$ and $\mathcal{A}_X \setminus \mathcal{A}_Y = \mathcal{A}_{X \setminus Y}$ can be separated. Thus, by Proposition 2.2.9, Y is a relative G_δ subspace of X. □

Proposition 2.3.7. Let $\mathcal{A}$ be an uncountable pseudonormal almost disjoint family. Then $\mathcal{A}$ is a λ-set of $\mathcal{P}(\omega)$, that is, $\{\chi_a : a \in \mathcal{A}\}$ is a λ-set.

Proof. Let $Q = \{\chi_a : a \in \mathcal{A}\}$. Let $\mathcal{B} \subseteq \mathcal{A}$ be a countable set. By Proposition 2.3.4, let X be a partitioner separating $\mathcal{B}$ from $\mathcal{A} \setminus \mathcal{B}$. Then $\{\chi_a : a \in \mathcal{B}\}$ is an G_δ subset of $\{\xi_a : a \in \mathcal{A}\}$ by Lemma 2.2.11. $\square$

As with Q-sets, the converse of the previous proposition is consistently not true. Recall that as we have already mentioned [54] A. Miller has used forcing to contruct a MAD family which is a Q-set (that is, a MAD family $\mathcal{A}$ such that $\{\chi_a : a \in \mathcal{A}\}$ is a Q-set). In particular, this set is a λ-set, and it is not pseudonormal since no MAD family is pseudonormal.

Recall that $\mathfrak{q}$ is the smallest uncountable size of a non Q-set. It may be tempting to define the smallest uncountable cardinality for a non λ-set, but it is not necessary since, as we shall expose, this cardinal is known to be $\mathfrak{b}$. We will prove the first half of this folklore result now, but we will leave the other half to the next section to avoid writing the same argument twice. This result appears in [22, Section 9].

First, we define:

Definition 2.3.8. Let $X \subseteq 2^\omega$. We say that X is a λ'-set iff it is uncountable and for every countable $N \subseteq 2^\omega$, $X \cup N$ is also a λ-set. $\square$

Proposition 2.3.9. There is a λ-set of size $\mathfrak{b}$ with is not a λ'-set. In particular, there exists a non λ-set of size $\mathfrak{b}$.

Proof. Recall that the Baire space ω^ω is characterized by being a completely metrizable, zero-dimensional separable topological space such that every compact set has empty interior (see [48, Theorem 7.7]). Let $N = \{f \in 2^\omega : \exists n \forall m \geq n f(m) = 0\}$ and $W = 2^\omega \setminus N$. We claim that W is homeomorphic to ω^ω.

To see that, first notice that N is countable, so W is a G_δ subspace of the completely metrizable space 2^ω, thus, it is also completely metrizable. It is zero-dimensional since 2^ω is zero-dimensional. Finally, let K be a compact subset of W. Suppose that it has a nonempty interior V. Then $V = W \cap U$ for some open set U of 2^ω. Since U is open, $V = U \setminus (U \setminus V)$ and $U \setminus V \subseteq N$ has empty interior, it follows that $\mathrm{cl}_{2^\omega} V = \mathrm{cl}_{2^\omega} U$, so $U \subseteq K$. However, N is dense, so $U \cap N \neq \emptyset$ and $N \cap K = \emptyset$, a contradiction.

Let $\phi : \omega^\omega \to W$ be an homeomorphism.

Let $(f_\alpha : \alpha < \mathfrak{b})$ be a $<^*$-unbounded tower, that is, an enumeration of an unbounded family such that if $\alpha < \beta$ then $f_\alpha \leq^* f_\beta$. This is possible: fix any enumerated unbounded family $(g_\alpha : \alpha < \mathfrak{b})$ and recursively define an $<^*$-increasing family $(f_\alpha : \alpha < \mathfrak{b})$ such that for every α, $f_\alpha <^* g_\alpha$, which is possible by the definition of $\mathfrak{b}$. Let $X = \{f_\alpha : \alpha < \mathfrak{b}\}$. X is clearly uncountable. Given a countable $I \subseteq b$, there exists $\gamma < \mathfrak{b}$ such that $I \subseteq \gamma$ (by the regularity of $\mathfrak{b}$). By the previous part of this result, $\phi[\{f_\alpha : \alpha \in I\}]$ is a G_δ subset of $\phi[\{f_\alpha : \alpha < \gamma\}]$, so $\{f_\alpha : \alpha \in I\}$ is a subset of $\{f_\alpha : \alpha < \gamma\}$. Moreover, $\{f_\alpha : \alpha \geq \gamma\} = X \cap \{f \in \omega^\omega : f \geq^* f_\gamma\} = X \cap \bigcup_{h =^* f_\gamma} \bigcap_{n \in \omega} \{f \in \omega^\omega : f(n) \geq f_\gamma(n)\}$ is an F_σ subset of X, thus, $\{f_\alpha : \alpha < \gamma\}$ is a G_δ subset of X. Since a relative G_δ of a G_δ is a G_δ, then $\{f_\alpha : \alpha \in I\}$ is a G_δ-subset of X. Since I is arbitrary, this shows that every countable subset of X is a relative G_δ of X. Thus, $\phi[X]$ is a λ-set.

We show that $\phi[X] \cup N$ is not a λ-set by showing that N is not a G_δ subset of $\phi[X] \cup N$. We show that $\phi[X]$ is not a relative F_σ of $\phi[X] \cup N$. Suppose it is. Then $\phi[X]$ can be written as $\bigcup_{n\in\omega} F_n$, where each F_n is a closed subset of $\phi[X] \cup N$. For each n, $F_n = K_n \cap (\phi[X] \cup N)$ for some closed (compact) subset K_n of ω^ω. Notice that $K_n \cap N = \emptyset$ for each n, so K_n is a closed compact subset of W. For each n, let $L_n = \phi^{-1}[K_n]$. L_n is a compact subset of ω^ω and $\{f_\alpha : \alpha < \mathfrak{b}\} \subseteq \bigcup_{n\in\omega} L_n$. However, since each L_n is compact, it is pseudocompact, thus there exists an $h_n \in \omega^\omega$ such that L_n is $<$-bounded by h_n. Let $h \in \omega^\omega$ be such that $h \geq^* h_n$ for every $n \in \omega$. Then h bounds $\{f_\alpha : \alpha < \mathfrak{b}*\}$, a contradiction since this is an unbounded family. □

2.4 Almost-normality of Isbell-Mrówka spaces

Almost-normality is a weakening of normality which was proposed in [66].

Definition 2.4.1. Let X be a topological space. We say that $F \subseteq X$ is *regularly closed* iff $F = \operatorname{cl}\operatorname{int} F$.

We say that X is *almost-normal* iff it is T_1 and for every regularly closed set F and every closed set K such that $F \cap K = \emptyset$, there exists open sets U, V such that $F \subseteq U$, $K \subseteq V$ and $U \cap V = \emptyset$. □

Simmilarly to the previous sections, we adopt the following definition as a shorthand.

Definition 2.4.2. Let $\mathcal{A}$ be an almost disjoint family. We say that $\mathcal{A}$ is *almost-normal* iff $\Psi(\mathcal{A})$ is almost-normal. □

There are not many examples of almost-normal spaces which are not normal, so we consider that every new example is interesting. In particular, in [29], S. A. Garcia-Balan and P. Szeptycki asked the following questions:

Problem 2.4.3. Is there an almost-normal almost disjoint family which is not normal?

We will partially answer this question in this section.

More strongly, they asked:

Problem 2.4.4. Is there an almost-normal MAD family which is not normal?

This problem is still open. More broadly, they asked:

Problem 2.4.5. Are almost-normal pseudocompact spaces countably compact?

One of the reasons that make this question interesting is the well-known result that states that every normal pseudocompact space is countably compact.

We will start by answering the last question negatively. First, recall the following definition:

Definition 2.4.6. Let X be a topological space. We say that X is *extremally disconnected* iff the closure of every open subset is open. □

The following is well known, but we prove it here for the convenience of the reader.

Proposition 2.4.7. $\beta\omega$ is extremally disconnected.

Proof. Let $U \subseteq \beta\omega$ be open. Let $(A_\alpha : \alpha \in I)$ be a family of subsets of ω such that $U = \bigcup_{i\in I} \operatorname{cl} A_i$. We claim that $\operatorname{cl} U = \operatorname{cl}\left(\bigcup_{\alpha\in I} A_i\right)$, which is open.

$\subseteq$: $j \in I$, $\operatorname{cl} A_j \subseteq \operatorname{cl}\left(\bigcup_{i\in I} A_i\right)$, thus, $U \subseteq \operatorname{cl}\left(\bigcup_{i\in I} A_i\right)$ which implies $\operatorname{cl} U \subseteq \operatorname{cl}\left(\bigcup_{i\in I} A_i\right)$.

$\supseteq$ is clear since $\operatorname{cl} U$ is closed and for each $i \in I$, $A_i \subseteq U$. □

Proposition 2.4.8. Let X be an extremally disconnected topological space. Then every dense subspace of X is extremally disconnected.

Proof. Let $D \subseteq X$ be dense. Let U be an open subset of D. We must see that $\operatorname{cl}_D U$ is dense in D.

$U = V \cap D$ for some open subset V of X, and $\operatorname{cl}_D U = \operatorname{cl}_X(U) \cap D$. It suffices to see that $\operatorname{cl}_X(U) \cap D = \operatorname{cl}_X(V) \cap D$. $\subseteq$ is clear. To verify $\supseteq$, let $d \in D \cap \operatorname{cl}_X(V)$. Let W be an open neighborhood of d in X. Then $W \cap V \neq \emptyset$. But D is dense, so $W \cap V \cap D = W \cap U \neq \emptyset$. This shows that $d \in \operatorname{cl}_X(U)$, as intended. □

The following is easy:

Proposition 2.4.9. Every extremally disconnected T_1 space is almost-normal.

Proof. Let F be a regularly closed subset of X and K be a closed subset of X such that $F \cap K = \emptyset$. Since $F = \operatorname{cl} \operatorname{int} F$, F is a clopen set, so we are done. □

Thus, to answer Problem 2.4.5, it suffices to construct a subspace of $\beta\omega$ containing ω (which is extremally disconnected, thus, almost-normal) which is pseudocompact but not countably compact. We construct such a space for the sake of completeness:

Proposition 2.4.10 (*). There exists a subspace of $\beta\omega$ containing ω which is almost-normal, pseudocompact but not countably compact.

Proof. By the previous discussion it suffices to contruct a subspace of $\beta\omega$ containing ω which is pseudocompact but not countably compact.

Let $(P_n : n \in \omega)$ be a partition of ω into pairwise disjoint infinite sets. For each $n \in \omega$, let $\mathcal{V}_n$ be a free ultrafilter such that $P_n \in \mathcal{V}_n$. Let $F = \{\mathcal{V}_n : n \in \omega\}$. F is infinite and discrete since given n, $\{\mathcal{V}_n\} = F \cap \operatorname{cl} P_n$.

Given $A \in [\omega]^\omega$, let $\mathcal{Q}_A \in \omega^*$ be defined as follows:

(1) If there exists $n \in \omega$ such that $A \in \mathcal{V}_n$, let $\mathcal{Q}_A = \mathcal{V}_n$, for any such n (e.g. the least such n), or

(2) if for all $n \in \omega$ $A \notin \mathcal{V}_n$, let $\mathcal{Q}_A \in \omega^*$ be any free ultrafilter such that $A \in \mathcal{Q}_A$.

Let $\mathcal{U}_n$ be the principal ultrafilter generated by $\{n\}$ and $N = \{\mathcal{U}_n : n \in \omega\}$. In any case, $A \in \mathcal{Q}_A$. Let $X = N \cup \{\mathcal{Q}_A : A \in [\omega]^\omega\}$ and notice that, for each $n \in \omega$, $\mathcal{Q}_{P_n} = \mathcal{V}_n$ by (1). Hence, $F \subseteq X$.

$\underline{X \text{ is pseudocompact}}$: since N is dense in X, by Proposition 0.3.20 it suffices to see that every sequence $f : \omega \to N$ has an accumulation point. By passing to a subsequence, we can suppose f is either constant or injective. Constant sequences converge, so suppose f is injective. Let $g : \omega \to \omega$ be such that $f(n) = \mathcal{U}_{g(n)}$. Let $A = \operatorname{ran}(g)$. We claim q_A is an accumulation point of f. Given a basic nhood $\operatorname{cl} B \ni \mathcal{Q}_A$, we know $B \cap A \in \mathcal{Q}_A$ is infinite, so it follows that $g^{-1}[A \cap B] \subseteq \{n \in \omega : f(n) \in \operatorname{cl} B\}$ is also infinite. Since B is arbitrary, the proof is complete.

$\underline{X \text{ is not countably compact}}$: we know F is an infinite discrete subspace of X (since it is in $\beta\omega$). Thus, it suffices to show that F is closed in X. We show $X \setminus F$ is open in X. Clearly, every point of N is in the interior of $X \setminus F$ since N is open. If $A \in [\omega]^\omega$ and $\mathcal{Q}_A \notin F$, then (2) holds, so $\mathcal{Q}_A \in \operatorname{cl} A$ and $F \cap \operatorname{cl} A = \emptyset$, that is, $\mathcal{Q}_A \in X \cap \operatorname{cl} A \subseteq X \setminus F$. □

Now we aim to partially answer Problem 2.4.3 by mimicking the techniques used in the classic results of normality and pseudonormality. The following result appears in our paper [61].

Proposition 2.4.11 (*). Let $\mathcal{A}$ be an almost disjoint family. $\mathcal{A}$ is almost-normal iff for every regularky closed set F, $F \cap \mathcal{A}$ and $\mathcal{A} \setminus F$ can be separated.

Moreover, in this case, closed sets are separated from regular closed sets by clopen sets.

Proof. Suppose $\Psi(\mathcal{A})$ is almost-normal. Let F be given. Since $\mathcal{A}$ is almost-normal, there exists open sets U, V such that $F \subseteq U$, $\mathcal{A} \setminus F \subseteq V$ and $U \cap V = \emptyset$. By Proposition 2.2.5 it suffices to see that $C = \Psi(\mathcal{A}) \setminus V$ is a clopen, since it contains $\mathcal{A} \cap F$ and is disjoint from $\mathcal{A} \setminus F$. It is clearly closed. To see that it is open, just notice that given $a \in \mathcal{A} \cap F$, $a \subseteq^* U \subseteq C$.

Conversely, suppose that for every regular closed set F, $\mathcal{A} \cap F$ and $\mathcal{A} \setminus F$ can be separated. Let F, K be two closed disjoint subsets of $\Psi(\mathcal{A})$ with F regularly closed. By Proposition 2.2.5, there exists a clopen set C separating $\mathcal{B} = F \cap \mathcal{A}$ from $\mathcal{C} = \mathcal{A} \setminus F$. Now $D = (C \cup (F \cap \omega)) \setminus K$ is also clopen containing F. To see that it is closed, just notice that if $a \in \operatorname{cl} D$, then $a \cap (F \cap \omega)$ is infinite, thus $a \in \operatorname{cl} F = F$, so $a \in \mathcal{B} \subseteq C$ and $a \cap K$ is finite. □

Of course, we could try to understand what are the regularly closed sets of a Isbell-Mrówka space. It turns out that their characterization if somewhat simple. This is probably folklore.

Lemma 2.4.12. Let N be an infinite countable set such that $[N]^\omega \cap N = \emptyset$. Let $\mathcal{A}$ be an almost disjoint family over N. Then $F \subseteq \mathcal{A}$ is a regular closed set if, and only if there exists $W \subseteq N$ such that $F = \operatorname{cl}(W)$ (notice that $\operatorname{cl} W = W \cup \{a \in A : |a \cap W| = \omega\}$).

Proof. Suppose F is a regularly closed set. Let $W = N \cap \operatorname{int}(F)$. Since N is dense and $\operatorname{int}(F)$ is open, it follows that $\operatorname{cl} W = \operatorname{cl}(N \cap \operatorname{int}(F)) = \operatorname{cl} \operatorname{int} F = F$.

Conversely, if $W \subseteq N$, $\operatorname{cl}\operatorname{int}\operatorname{cl} W \subseteq \operatorname{cl} W$ holds (as always). Suppose $x \in \operatorname{cl} W$. If $x \in N$, then $x \in \operatorname{int}\operatorname{cl} W$, so $x \in \operatorname{cl}\operatorname{int}\operatorname{cl} W$. If $x \in \mathcal{A}$, then $|x \cap W| = \omega$. Thus, $|x \cap \operatorname{int}\operatorname{cl} W| = \omega$, which implies that $x \in \operatorname{cl}\operatorname{int}\operatorname{cl} W$. □

By combining the last two results, we get:

Corollary 2.4.13 (*). Let N be an infinite countable set such that $N \cap [N]^\omega = \emptyset$ and $\mathcal{A}$ be an almost disjoint family over N. Then $\mathcal{A}$ is almost normal iff for every $W \subseteq N$, the sets $\{a \in \mathcal{A} : |a \cap W| = \omega\}$ and $\{a \in \mathcal{A} : |a \cap W| < \omega\}$ can be separated. That is, iff every pair of weakly separated sets is separated.

In [61] we defined a candidate for the equivalent of a Q-set for almost-normality.

Definition 2.4.14 (*). Let $X \subseteq 2^\omega$. We say that X is an *almost Q-set* iff it is uncountable and for every $W \subseteq 2^{<\omega}$, $[W]_X = \{x \in X : \forall m \in \omega\, \exists n \geq m\, (x|_n \in W)\}$ (which is $\{x \in X : |a_x \cap W| = \omega\}$) is an F_σ in X. □

Notice that for every $X \subseteq 2^\omega$ and $W \subseteq 2^{<\omega}$, $[W]_X$ is a G_δ subset of X.

The relationship between almost Q-sets and almost-normality is similar to the relationship between Q-sets and normality:

Proposition 2.4.15 (*). Let $X \subseteq 2^\omega$ be a uncountable set. Then $\mathcal{A}_X$ is almost-normal iff X is an almost Q-set.

Proof. First, suppose X is an almost Q-set. By Corollary 2.4.13, $W \subseteq 2^{<\omega}$, $\mathcal{B} = \{a \in \mathcal{A}_X : |a \cap W| = \omega\}$ and $\mathcal{C} = \{a \in \mathcal{A}_X : |a \cap W| < \omega\}$ can be separated. Let $Y = \{x \in X : a_x \in \mathcal{B}\} = [W]_X$. Since both Y and $X \setminus Y$ are relative F_σ subsets of X, it follows from Proposition 2.2.9 that $\mathcal{B}$ and $\mathcal{C}$ can be separated.

Conversely, suppose that $\mathcal{A}_X$ is almost-normal. Fix $W \subseteq 2^{<\omega}$. Let $\mathcal{B} = \{a \in \mathcal{A}_X : |a \cap W| = \omega\} = \{a_x : x \in [W]_X\}$ and $\mathcal{C} = \mathcal{A} \setminus \mathcal{B}$ Since $\mathcal{A}_X$ is almost-normal, it follows from Corollary 2.4.13 that $\mathcal{B}$ and $\mathcal{C}$ can be separated. Thus, by Proposition 2.2.9, $[W]_X$ is a relative G_δ subspace of X. □

We do not know if an almost-normal almost disjoint family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is an almost Q-set of $\mathcal{P}(\omega)$ (this would be an result analogous to propositions 2.2.12 and 2.3.7).

It is possible to directly construct almost Q-sets which are not Q-sets by forcing. We have done that in our paper [61]. We obtained better results later, as we shall see, but we will show how to do it here as well.

Theorem 2.4.16 (*). Suppose that $X \subseteq 2^\omega$ is infinite, and let $\kappa = \mathfrak{c}$. Then there exists a c.c.c. forcing notion $\mathbb{P}$ with a dense subset of size κ such that $\mathbb{P} \Vdash \check{X}$ is an almost-Q set.

Proof. We will proceed by iterated forcing assuming the existence of a countable transitive model M, and prove the theorem relativized to M.

We recursively construct, working in M, a finitely supported κ-stage iterated forcing construction $(\langle(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi) : \xi \leq \kappa\rangle, \langle(\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_\xi, \mathring{\mathbb{1}}_\xi) : \xi < \kappa\rangle)$.

Each $\mathring{\mathbb{Q}}_\xi$ will be forced by $\mathbb{P}_\xi$ to have size $\leq \kappa$ and to have the c.c.c., therefore for each $\xi \leq \kappa$, $\mathbb{P}_\xi$ will have a dense subset $\mathbb{P}'_\xi$ of cardinality at most κ which will have the c.c.c. as well.

Fix a function f from κ onto $\kappa \times \kappa$ such that if $f(\xi) = (\zeta, \mu)$, then $\zeta \leq \xi$. We will use f as a bookkeeping device. Let $N = 2^{<\omega}$.

Suppose we have constructed $(\langle(\mathbb{P}_\zeta, \leq_\zeta, \mathbb{1}_\zeta) : \zeta \leq \xi\rangle, \langle(\mathring{\mathbb{Q}}_\zeta, \mathring{\leq}_\zeta, \mathring{\mathbb{1}}_\zeta) : \zeta < \xi\rangle)$ for some $\xi < \kappa$. We must determine $(\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_\xi, \mathring{\mathbb{1}}_\xi)$. Suppose that for each stage $\zeta < \xi$ we have also listed all $\mathbb{P}'_\zeta$-nice names for subsets of $\check{N}$ as $(\tau^\mu_\zeta : \mu < \kappa)$. This is possible since $|\mathbb{P}'_\zeta| \leq \kappa = \kappa^\omega$ and has since $\mathbb{P}_\zeta$ has the countable chain condition. List all $\mathbb{P}'_\xi$-nice names for subsets of $\check{N}$ as $(\tau^\mu_\xi : \mu < \kappa)$ as well.

Let $f(\xi) = (\zeta, \mu)$. Since $\zeta \leq \xi$, the name τ^μ_ζ is a nice $\mathbb{P}'_\xi$-name for a subset of $\check{N}$. Let $(\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_\xi, \mathring{\mathbb{1}}_\xi)$ be an appropriate triple of $\mathbb{P}_\xi$-names such that $\mathbb{P}_\xi \Vdash (\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_\xi, \mathring{\mathbb{1}}_\xi) = P\left(\left[\tau^\mu_\zeta\right]_{\check{X}}, \check{X}\right)$. For the definition of $P(A, X)$, see Definition 2.2.16.

Let $\mathbb{P} = \mathbb{P}_\kappa$.

Let G be $\mathbb{P}$-generic over M. We claim X is an almost Q-set in $M[G]$. It is uncountable since $\mathbb{P}$ preserves cardinals. Now let W be a subset of N in $M[G]$. Since $\mathrm{cf}(\kappa)^M > \omega$, There exists $\zeta < \kappa$ such that $W \in M[G_\zeta]$, where $G_\zeta = G \cap \mathbb{P}_\zeta$. There exists $\mu < \kappa$ such that $W = \mathrm{val}(\tau^\mu_\zeta, G_\zeta)$. Let ξ be such that $f(\xi) = (\zeta, \mu)$. Then, since $W = \mathrm{val}(\tau^\mu_\zeta, G_\xi)$ it follows from the choice of $\mathring{\mathbb{Q}}_\xi$, $M[G_{\xi+1}]$ contains a $P([W]_X, X)$-generic filter over $M[G_\xi]$, so, by Proposition 2.2.20 in $M[G_{\xi+1}]$, $[W]_X$ is an F_σ-subset of X, hence, the same happens in $M[G]$. □

Corollary 2.4.17 (*)**.** The following are relatively consistent with ZFC:

1. CH + There exists an almost-normal almost disjoint family which is not normal.
2. There exists an almost-normal almost disjoint family of size $\omega_1 < \mathfrak{c}$ which is not normal.

Proof. For 1., apply the previous proposition assuming $\mathfrak{c} = \kappa = \omega_1 = |X|$. For 2. assume, for concreteness, that in the ground model, $|X| = \omega_1 < \mathfrak{c} = \omega_2 < 2^{\omega_1} = \omega_3$.

In both examples the value of $\mathfrak{c}$ does not change by the preservation of cardinals and by counting nice names. $\Psi(\mathcal{A}_X)$ is not normal by Proposition 2.1.2 since we would have $2^{|\mathcal{A}_X|} = 2^{|X|} = 2^{\omega_1} \leq \mathfrak{c}$, contradicting $2^{\omega_1} = 2^{\mathfrak{c}} > \mathfrak{c}$ in the first case, and $2^{\omega_1} \geq \omega_3 > \mathfrak{c}$ in the second case. This last $\geq$ inequality holds since $\mathbb{P}$ preserves cardinals due to the countable chain condition. □

After obtaining this result we turned our attention to anoter class of special subsets of the reals, the σ-sets, which are easily seen to be contained in the class of the almost Q-sets.

Definition 2.4.18. Let $X \subseteq 2^\omega$. We say that X is a σ-set iff it is uncountable and every relative F_σ subset of X is a G_δ. □

It is clear that every σ-set is an almost Q-set and a λ-set. The existence of σ-sets is independent from ZFC: they exist under CH since CH implies the existence of Sierpiński sets and every such set is a σ-set (see [57, Theorem 2.2., Theorem 4.2.]), and there is a model with no σ-sets [55, Theorem 22]. Since under CH there are no Q-sets, this gives another proof for item 1. of the previous corollary.

Moreover, we proved the rather surprising that almost Q-sets are in fact σ-sets. We will need the following result from our paper [62].

Proposition 2.4.19 (*). Let $K \subset \mathcal{A}_X$. The following are equivalent:

(1) There exists $W \subset 2^{<\omega}$ such that $K = \mathrm{cl}(W) \cap \mathcal{A}_X$;

(2) $\{x \in X : a_x \in K\}$ is G_δ in X.

Proof. (1) $\implies$ (2): Let $W \subset 2^{<\omega}$ such that $K = \mathrm{cl}(W) \cap \mathcal{A}_X$. It follows that:

$$\{x \in X : a_x \in K\} = \{x \in X : |a_x \cap W| = \omega\} = \bigcap_{n\in\omega} \bigcup_{m\geq m} \underbrace{\{x \in X : x|_m \in W\}}_{\text{open set in } X}.$$

Thus, this is a G_δ-set of X.

(2) $\implies$ (1): Let $Z = \{x \in X : a_x \in K\}$. Suppose Z is a G_δ of X. Write $Z = \bigcap_{n\in\omega} U_n$, where each U_n is an open subset of X and $U_n \subseteq U_m$ whenever $n \geq m$.

For each n, write $U_n = \bigcup\{[s] : s \in L_n\}$, where L_n is a countable subset of $2^{<\omega}$ such that for all $s, t \in L_n$, s, t are incompatible and $|s|, |t| > n$.

Let $W = \bigcup\{L_n : n \in \omega\}$. We claim that $\mathrm{cl}\, W \cap \mathcal{A}_X = K$.

Suppose $a \in \mathrm{cl}(W) \cap \mathcal{A}_X$. Let $x \in X$ be such that $a = a_x$. It suffices to see that $x \in U_n$ for every $n \in \omega$. Fix n. Since $a_x \in \mathrm{cl}(W)$, there exists infinitely many $m \in \omega$ such that $x|_m \in W = \bigcup_{k\in\omega} L_k$. Since all members of L_k are pairwise incompatible, for each k, $x|_m \in L_k$ for at most one m. So there exists $m \in \omega$ and $k \geq n$ such that $x|_m \in L_k$, so $x \in U_k \subseteq U_n$.

On the other hand, if $a \in K$, let $x \in Z$ be such that $a = a_x$. Then $x \in U_n$ for all $n \in \omega$, that is, for each $n \in \omega$ there exists $s_n \in L_n$ such that $s_n \subseteq x$. Since $|s_n| > n$ for each n and $s_n \in W$, this implies that $x \in \mathrm{cl}(W)$. □

Now we prove our result.

Proposition 2.4.20 (*). Let $X \subseteq 2^\omega$ be uncountable. The following are equivalent:

a) X is an almost Q-set,

b) $\mathcal{A}_X$ is almost-normal, and

c) X is a σ-set.

Proof. It only remains to show that b) implies c).

Let $G \subseteq X$ be a relative G_δ subset of X. We show that it is a relative F_σ. Let $K = \{a_x : x \in G\}$. Then $G = \{x \in X : a_x \in K\}$ is a G_δ relative to X. Thus, by the previous proposition, there exists $W \subseteq 2^{<\omega}$ such that $K = \mathrm{cl}(W) \cap \mathcal{A}_X$. By Corollary 2.4.13, there exists a partitioner separating $K = \mathcal{A}_G$ from $\mathcal{A}_X \setminus K = \mathcal{A}_{X\setminus G}$. Now it follows from Proposition 2.2.9 that K is an F_σ subset of X. □

We end this section by proving the folklore result that the least uncountable cardinality of a non σ-set (or λ-set) is $\mathfrak{b}$. This result appears in [22].

Proposition 2.4.21 (Folklore)**.** $\mathfrak{b}$ the least uncountable cardinality of a non σ-set and of a non λ-set.

Proof. We have already seen that there is a non λ-set of size $\mathfrak{b}$ on Proposition 2.3.9, so it suffices to prove that if $X \subseteq 2^\omega$ has cardinality $< \mathfrak{b}$, then X is a σ-set. Fix X and let $F \subseteq X$ be an F_σ set, and write it as $\bigcup_{n\in\omega} F_n$.

Since X is metrizable, each F_n is a G_δ. Write, for each $n \in \omega$, $F_n = \bigcap_{m\in\omega} U_{n,m}$, where each $U_{n,m}$ is open and $U_{n,m} \subseteq U_{n,m'}$ whenever $m' \leq m$. We will show that $F = \bigcup_{n\in\omega}\bigcap_{m\in\omega} U_{n,m}$ is a G_δ.

For each $x \in X \setminus F$, let $f_x : \omega \to \omega$ be a function such that $x \notin \bigcup_{n\in\omega} U_{n,f_x(n)}$. Let $f \in \omega^\omega$ be such that $f \geq^* f_x$ for every $X \setminus F$. Consider $G = \bigcap\{\bigcup_{n\in\omega} U_{n,h(n)} : h \in \omega^\omega \text{ and } h =^* f\}$. It is clear that $F \subseteq G$. We must see that if $x \in X \setminus F$ then $x \notin G$, which is clear since there exists $h \in \omega^\omega$ such that $h =^* f$ and $h \geq f_x$. □

2.5 Weak λ-set and strongly $\aleph_0$-separated almost disjoint families

In [29], S. A. García-Balan and P. Szeptycki were interested in constructing an almost disjoint (or MAD) family which is almost normal but not normal. As we have already mentioned, the problem of whether there is an almost-normal MAD family remains open. To study this problem, they defined the following:

Definition 2.5.1. Let $\mathcal{A}$ be an almost disjoint family. We say that $\mathcal{A}$ is strongly $\aleph_0$-separated iff for every two disjoint countable subsets of $\mathcal{A}$ can be separated. □

They showed that every almost-normal almost disjoint family has this property. We reproduce a version of their proof:

Proposition 2.5.2 ([29, Lemma 3.2.])**.** Every almost-normal almost disjoint family is strongly $\aleph_0$-separated.

Proof. Let $\mathcal{A}$ be an almost-normal almost disjoint family. Let $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ be countable and disjoint. We can suppose that they are nonempty. By Corollary 2.4.13, it suffices to construct a set $W \subseteq \omega$ for every $a \in \mathcal{B}$, $|a \cap W| = \omega$ and for every $a \in \mathcal{C}$, $|a \cap W| < \omega$. Enumerate $\mathcal{B} = \{b_n : n \in \omega\}$ in a way such that for every $b \in \mathcal{B}$, $|\{n \in \omega : b_n = b\}| = \omega$. Enumerate $\mathcal{C} = \{c_n : n \in \omega\}$. For each $n \in \omega$, fix $x_n \in b_n \setminus (\bigcup_{i<n} c_n \cup n)$. Let $W = \{x_n : n \in \omega\}$. □

They proved that under CH there is a strongly $\aleph_0$-separated MAD family [29, Lemma 3.3.]. In this section we further explore this class of almost disjoint families.

First, we introduce a new class of sets of reals which will have about the same relations to strongly $\aleph_0$-separatedness as Q-sets have with normality.

Definition 2.5.3 (*). Let $X \subseteq 2^\omega$. We say that X is a *weak λ-set* iff it is uncountable and for every pair of disjoint countable subsets of X, A and B, there exists $G \subseteq X$ such that $A \subseteq G$, $B \cap G = \emptyset$ and G is both a G_δ and a F_σ.

We say that X is an *weak λ'-set* iff for every countable subset N of 2^ω, $X \cup N$ is a weak λ-set. □

It is readily seen that every λ-set is a weak λ-set. Moreover, these sets are contained in a known larger class of special subsets of reals:

Definition 2.5.4. Let $X \subseteq 2^\omega$. We say that X is *perfectly meager* iff for every perfect subset P of 2^ω, $X \cap P$ is meager in P. □

Recall that P is perfect means that P is closed and has no isolated points.

Proposition 2.5.5 (*). Every weak λ-set is perfectly meager.

Proof. Let X be a weak λ subset of 2^ω. Fix a perfect set P. Write $X \cap P = Y \cup C$ where $Y = \{x \in X \cap P : \text{for every open nhood } U \text{ of } x, |U \cap X \cap P| \geq \omega_1\}$ and $C = X \cap P \setminus Y$. Notice that C is countable since the space is second countable, and that Y is dense in itself and nonempty. It suffices to show that Y is meager in P.

It is straightforward to construct two countable disjoint subsets $F, K \subset Y$ such that $\overline{Y} = \overline{F} = \overline{K}$.

Since X is a weak λ-set, there exist sequences of open sets A_n, B_n $(n \in \omega)$ such that:

1. $F \subset \bigcap_{n\in\omega} A_n$;
2. $K \subset \bigcap_{n\in\omega} B_n$;
3. $\bigcap_{n\in\omega} A_n \cap X \cap \bigcap_{n\in\omega} B_n = \emptyset$;
4. $X \cap \bigcap_{n\in\omega} B_n = X \setminus \bigcap_{n\in\omega} A_n$.

For each $n \in \omega$, let $G_n = A_n \cup (2^\omega \setminus \overline{Y})$. Then each G_n is an open dense set, furthermore, $G_n \cap P$ is dense in P because:

$$\overline{G_n \cap P} = \underbrace{\overline{A_n \cap P}}_{\supseteq \overline{F} = \overline{Y}} \cup \overline{P \setminus \overline{Y}} = P.$$

Since $G \cap P = \bigcap_{n\in\omega} G_n \cap P$ is a dense G_δ in P, it is comeager in P and we have that:

$$Y \cap G \subset Y \cap \bigcap_{n\in\omega} A_n \subset Y \setminus \bigcap_{n\in\omega} B_n = \bigcup_{n\in\omega} Y \setminus B_n.$$

Notice that for each n, $\overline{Y \setminus B_n}$ has empty interior since $\overline{Y \setminus B_n} \subset \overline{Y} \setminus B_n$ and B_n is dense in $\overline{Y}$. Thus $Y \cap G$ is meager in P. But also, $Y \setminus G = Y \setminus (P \cap G)$ is meager in P. Hence, Y is meager in P. □

These sets characterize the strongly $\aleph_0$-separatedness of branching spaces:

Proposition 2.5.6 (*)**.** Let $X \subseteq 2^\omega$ be uncountable. Then $\mathcal{A}_X$ is strongly $\aleph_0$-separated iff X is a weak λ-set.

Proof. First, suppose X is a weak λ-set. Fix disjoint countable subsets $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}_X$, which is of the form $\mathcal{B} = \mathcal{A}_B$ and $\mathcal{C} = \mathcal{A}_C$ for some countable disjoint subsets B, C of X. There exists a G_δ-F_σ set G separating B from C. Then it follows from Proposition 2.2.9 that $\mathcal{A}_G$ and $\mathcal{A}_{X\setminus G}$ can be separated, thus, that $\mathcal{B}$ and $\mathcal{C}$ can be separated.

Conversely, suppose that $\mathcal{A}_X$ is strongly $\aleph_0$-separated. Fix B, C two disjoint countable subsets of X. Then $\mathcal{A}_B$ and $\mathcal{A}_C$ can be separated. Thus, there exists a partitioner P separating $\mathcal{A}_B$ from $\mathcal{A}_C$. There exists an unique $Y \subseteq X$ be such that $\mathcal{A}_Y$ and $\mathcal{A}_{X\setminus Y}$ are separated by P. For this Y, it follows that $A \subseteq Y$ and $B \cap Y = \emptyset$. By Proposition 2.2.9, Y is a relative G_δ-F_σ subspace of X. □

And, as with Q-sets and normality, strongly $\aleph_0$-separated almost disjoint families are naturally weak λ-sets:

Proposition 2.5.7 (*)**.** Let $\mathcal{A}$ be an uncountable strongly $\aleph_0$-separated almost disjoint family. Then $\mathcal{A}$ is a weak λ-set of $\mathcal{P}(\omega)$, that is, $\{\chi_a : a \in \mathcal{A}\}$ is a weak λ-set.

Proof. Let $\mathcal{B}, \mathcal{C}$ be two given disjoint countable subsets of $\mathcal{A}$. Let $Q = \{\xi_a : a \in \mathcal{A}\}$. let X be a partitioner separating $\mathcal{B}$ from $\mathcal{C}$. Then $\{\chi_a : a \subseteq^* X\}$ and $\{\chi_a : a \subseteq^* P \setminus X\}$ are complementary F_σ subsets of $\{\xi_a : a \in \mathcal{A}\}$ by Lemma 2.2.11. □

In [26], forcing was used to construct a consistent example of a Q-set X which is concentrated on a countable dense subset F of 2^ω (F is in the ground model). The proof actually shows that X is concentrated in every dense subset F' of F which is in the ground model (we say that a $Y \subseteq X$ is concentrated in X iff whenever U is an open set containing Y, $X \setminus U$ is countable). Therefore, by letting F_0, F_1 be two disjoint dense subsets of F in the ground model, we get the set $Y = F_0 \cup F_1 \cup X$ is perfectly meager and not a weak λ-set (in the forcing extension). This latter fact holds since it is easy to verify that a weak λ-set cannot be concentrated on two pairwise disjoint countable subsets. This discussion yields the following proposition:

Proposition 2.5.8 (*)**.** It is consistent that there exists a perfectly meager set which is not a weak λ-set and it is consistent that the class of weak λ-sets along with the countable sets is not an ideal.

However, we do not know if λ-sets and weak λ-sets are the same.

Problem 2.5.9. Is there, at least consistently, a weak λ-set that is not a λ-set? In the negative case, is there a weak λ-set that is not weak λ'-set?

The following diagram illustrated the relation between these special subsets of reals known so far. For the undefined notions, see [57].

$$
\begin{tikzcd}
 & Q\text{-set} \arrow[d] & & & \\
\text{almost } Q\text{-set} \arrow[r, leftrightarrow] & \sigma\text{-set} \arrow[d] & & & \\
\text{Sierpinski set} \arrow[ur] \arrow[dr] & \lambda\text{-set} \arrow[r] & \text{weak } \lambda\text{-set} \arrow[r] & \text{perfectly meager set} \arrow[r] & s_0\text{-set} \\
 & \lambda'\text{-set} \arrow[u] \arrow[r] & \text{weak } \lambda'\text{-set} \arrow[u] & &
\end{tikzcd}
$$

We could also ask that is the least cardinality of a non weak λ-set. First we note the following easy known fact:

Proposition 2.5.10. The least cardinality of a non perfectly meager set is $\mathrm{non}(\mathcal{M})$.

Proof. Let κ be this cardinal.

If we define $\mathrm{non}(\mathcal{M}, X)$ as the least cardinality of a non meager subset of x, then $\mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{M}, P)$ for every nonempty perfect subset P of 2^ω. Thus, if X is a non perfectly meager subset of size κ, let P be a perfect set such that $X \cap P$ is not meager. Then we have that $\kappa = |X| \geq |X \cap P| \geq \mathrm{non}(\mathcal{M}, P) = \mathrm{non}(\mathcal{M})$.

On the other hand, if X is a nonmeager set, then X is not perfectly meager (let $P = 2^\omega$). Thus, $\mathrm{non}(\mathcal{M}) \geq \kappa$. $\square$

Thus, the least uncountable cardinality of a non weak λ-set is a cardinal between $\mathfrak{b}$ and $\mathrm{non}(\mathcal{M})$.

The following diagram describes the relations we have done so far for almost disjoint families of branches. The double arrows are the results that hold in ZFC, the dashed arrows are consistent implications which are consistently false and the dotted arrows are implications that remain unknown. The first line stands for the least size of a set of reals which does not have the respective property.

$$
\begin{tikzcd}
\text{Cardinal :} & \mathrm{non}(\mathcal{M}) \arrow[d, leftrightarrow] \arrow[r, phantom, "\geq"] & ? \arrow[d, leftrightarrow] \arrow[r, phantom, "\geq"] & \mathfrak{b} \arrow[d, leftrightarrow] \arrow[r, phantom, "="] & \mathfrak{b} \arrow[d, leftrightarrow] \arrow[r, phantom, "\geq"] & \mathfrak{q} \arrow[d, leftrightarrow] \\
X \subseteq 2^\omega : & \text{Perfectly meager} \arrow[r, dashed, bend left, "\not"] & \text{weak } \lambda \arrow[l, Leftarrow] \arrow[r, dotted, bend left, "?"] \arrow[d, leftrightarrow] & \lambda\text{-set} \arrow[l, Leftarrow] \arrow[r, dashed, bend left, "\not"] \arrow[d, leftrightarrow] & \sigma\text{-set} \arrow[l, Leftarrow] \arrow[r, dashed, bend left, "\not"] \arrow[d, leftrightarrow] & Q\text{-set} \arrow[l, Leftarrow] \arrow[d, leftrightarrow] \\
\Psi(\mathcal{A}_X) : & & \aleph_0\text{-separated} & \text{pseudonormal} \arrow[l, Leftarrow] & \text{almost-normal} \arrow[l, Leftarrow] & \text{normal} \arrow[l, Leftarrow]
\end{tikzcd}
$$

In the same sense, the diagram below summarizes the known implications and the open questions concerning these properties of $\Psi(A)$ and properties of A as a subset of $[\omega]^\omega$.

$$\begin{tikzcd}
\mathcal{A} \subseteq [\omega]^\omega : & \text{weak } \lambda\text{-set} \arrow[d, dotted, shift right, "?"'] \arrow[dr, dashed, bend right, "/" marking] & \lambda\text{-set} \arrow[l, Leftarrow] \arrow[d, dashed, shift right, "/" marking] \arrow[dr, dashed, bend right, "/" marking] & \sigma\text{-set} \arrow[l, Leftarrow] \arrow[d, dashed, shift right, "/" marking, "?"] \arrow[dr, dashed, bend right, "/" marking] & Q\text{-set} \arrow[l, Leftarrow] \arrow[d, dashed, shift right, "/" marking] \\
\Psi(\mathcal{A}) : & \aleph_0\text{-separated} \arrow[u, Rightarrow, shift right] & \text{pseudonormal} \arrow[l, Leftarrow] \arrow[u, Rightarrow, shift right] & \text{almost-normal} \arrow[l, dotted, "?"] \arrow[u, dotted, shift right] \arrow[ll, Rightarrow, bend left] & \text{normal} \arrow[l, Leftarrow] \arrow[u, Rightarrow, shift right]
\end{tikzcd}$$

2.6 On Luzin Families

As we have mentioned in the introduction of this chapter, almost disjoint families of size $\mathfrak{c}$ are not normal, and the existence of almost disjoint families of size ω_1 which are normal is independent of ZFC. With this in mind, a natural question is whether there exists a normal almost disjoint family of size ω_1. The answer is positive and is due to Luzin.

Definition 2.6.1. A *Luzin family* is an almost disjoint family $\mathcal{A}$ which can be injectively enumerated as $(a_\alpha : \alpha < \omega_1)$ so that for every finite $s \subseteq \omega$ and for every $\alpha < \omega_1$, the set $\{\beta < \alpha : a_\beta \cap a_\alpha \subseteq s\}$ is finite.

A *Luzin* family* is an almost disjoint family $\mathcal{A}$ which can be injectively enumerated as $(a_\alpha : \alpha < \omega_1)$ so that for every $n \in \omega$ and for every $\alpha < \omega_1$, the set $\{\beta < \alpha : |a_\beta \cap a_\alpha| < n\}$ is finite. □

Of course, every Luzin* family is a Luzin family. Luzin families fail to be normal badly:

Proposition 2.6.2 (Luzin)**.** Let $\mathcal{A}$ be a Luzin family. Then no pair of disjoint uncountable subsets of $\mathcal{A}$ can be separated.

Proof. Let $\mathcal{B}, \mathcal{C}$ be two uncountable disjoint subsets of $\mathcal{A} = \{a_\alpha : \alpha < \omega\}$, assuming that this enumeration satisfies the definition of a Luzin family. There exists two uncountable disjoint sets I, J such that $\mathcal{B} = \{a_\alpha : \alpha \in I\}, \mathcal{C} = \{a_\alpha : \alpha \in J\}$. Suppose that there exists a partitioner X separating $\mathcal{B}, \mathcal{C}$. There exists $n \in \omega$ and uncountable sets $I' \subseteq I$, $J' \subseteq J$ such that for every $\alpha \in I'$, $a_\alpha \subseteq X \cup n$, and for every $\alpha \in J'$, $a_\alpha \cap X \subseteq n$. Let α be such that $\alpha \in I'$ and that $|J' \cap \alpha| = \omega$. Then if $J' \cap \alpha\{\beta < \alpha : a_\gamma \cap a_\alpha \subseteq n\}$, a contradiction. To see that, notice that if $\beta \in J' \cap \alpha$, then $a_\alpha \cap a_\beta \subseteq (X \cup n) \cap a_\beta \subseteq n$. □

It is not difficult to construct a Luzin* family in ZFC:

Proposition 2.6.3 (Luzin)**.** Luzin* families exist.

Proof. Let $(a_n : n \in \omega)$ enumerate a partition of ω by infinite sets. Having defined $(a_\beta : \beta < \alpha)$ for some $\alpha \in [\omega, \omega_1)$, we define a_α as follows: fix a bijection $f : \omega \to \alpha$. Let

$b_n = a_{f(n)}$ for each $n \in \omega$. Choose $x_n \in b_n \setminus (\bigcup_{i<n} b_i \cup n)$. Let $a_\alpha = \{x_n : n \in \omega\}$. Such an a_α is readily seen to work. □

Since Luzin families cannot be normal, it would be interesting to know which weakenings of normality they can be.

Following this direction, to get to new results we will use the following definition from [42].

Definition 2.6.4. Given a property P of almost disjoint families and an almost disjoint family $\mathcal{A}$, we say that $\mathcal{A}$ *is potentially* P if there exists a c.c.c. forcing notion $\mathbb{P}$ such that $\mathbb{P} \Vdash \check{\mathcal{A}}$ is P. □

In particular, they showed that $\mathcal{A}$ is potentially normal iff $\mathcal{A}$ has no n-Luzin gap (see their paper for the definition).

We can ask if there is a nice characterization for potentially almost-normal almost disjoint families. We don't have a answer for this question. However, we have the following:

Proposition 2.6.5 (*). There exists a Luzin* family which is not potentially almost-normal.

Proof. Let $(a_n : n \in \omega)$ be a partition of ω into infinite sets. For each n, let X_n be an infinite subset of A_{2n} such that $A_{2n} \setminus A_n$ is infinite. Let $X = \bigcup_{n\in\omega} X_n$. For each infinite countable ordinal α, let $\phi_\alpha : \omega \to \alpha$ be a bijection.

We will inductively define $(a_\alpha : \alpha < \omega)$ such that for all $\alpha < \omega_1$:

(i) $a_\alpha \in [\omega]^\omega$ and $a_\alpha \cap a_\beta$ is finite for every $\beta < \alpha$,

(ii) $\forall n \in \omega \, \{\beta < \alpha : |a_\beta \cap a_\alpha| < n\}$ is finite,

(iii) if α is odd, then $a_\alpha \cap X = \emptyset$, and

(iv) if α is even, then X splits a_α, that is, both $a_\alpha \setminus X$ and $a_\alpha \cap X$ are infinite.

The items (i) and (ii) guarantees that $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ is a Luzin* family, (iii) and (iv) guarantees that X is such that $\{\alpha < \omega_1 : |a_\alpha \cap X| = \omega\}$ is the set of even countable ordinals.

Notice that (i)-(iv) hold for $\alpha \in \omega$. Having constructed a_β for $\beta < \alpha$ for some infinite $\alpha < \omega_1$, we construct a_α as follows: for each n, let $s_n \subseteq a_{\phi_\alpha(n)} \setminus \bigcup_{i<n} a_{\phi_\alpha(i)}$ such that $|s_n| = n$. If α is odd, we choose s_n such that $s_n \cap X = \emptyset$, which is possible by (iii) and (iv). If α is even and $\phi_\alpha(n)$ is even, we choose $s_n \subseteq X$, which is possible by (iv), and if $\phi_\alpha(n)$ is odd, we choose s_n such that $X \cap s_n = \emptyset$, which is possible by (iii). It is clear that by letting $a_\alpha = \bigcup\{s_n : n \in \omega\}$, (i)-(iv) are satisfied.

We claim that $\mathcal{A}$ is not potentially almost-normal: if $V[G]$ is a c.c.c. forcing extension of V, $\mathcal{A}$ is still a Luzin* family in $V[G]$ (since c.c.c. forcings preserve cardinals) and $\{a_\alpha : \alpha < \omega \text{ is even}\} \cup X$ is a regular closed subset of $\Psi(\mathcal{A})$ that cannot be separated from the closed set $\{a_\alpha : \alpha < \omega \text{ is odd}\}$ since that would imply the existence of a partitioner for the uncountable sets $\{a_\alpha : \alpha < \omega \text{ is even}\} \cup X$ and $\{a_\alpha : \alpha < \omega \text{ is odd}\}$, violating the fact that $\mathcal{A}$ is Luzin*. □

Such a set X does not exist for every Luzin family. For instance, in Example 2.10 [29], CH is used to construct a MAD Luzin family $\mathcal{A}$ for which for every $X \subseteq \omega$, $\{a \in \mathcal{A} : |a \cap X| = \omega\}$ is either finite or co-countable. It is not clear for us if that Luzin family is potentially almost-normal.

We consider the two following questions interesting:

Problem 2.6.6. Is it consistent that there is an almost-normal Luzin-family? What about a potentially almost-normal one?

Problem 2.6.7. What is a nice characterization of potentially almost-normal almost disjoint families?

We note that for any Luzin family $\mathcal{A}$ and for any uncountable set $\mathcal{B} \subseteq \mathcal{A}$ whose complement is also uncountable, we can add, by a c.c.c. forcing, a set X such that $\mathcal{B} = \{a \in A : |a \cap X| = \omega\}$, thus:

Proposition 2.6.8 (*). Every Luzin family is potentially not almost-normal.

Proof. Let $\mathcal{A}$ be a Luzin family, let $\mathcal{B} \subseteq \mathcal{A}$ be a an uncountable set whose complement in $\mathcal{A}$ is also uncountable. Consider Solovay's poset for adding a set X almost disjoint with $\mathcal{A} \setminus \mathcal{B}$, i.e., $\mathbb{P} = [\omega]^{<\omega} \times [\mathcal{A} \setminus \mathcal{B}]^{<\omega}$ ordered by $(s, A) \leq (s', A')$ ($\leq$ means stronger) iff $s \supseteq s'$, $A \supseteq A'$ and $\forall n \in s \setminus s'\,(n \notin \bigcup A')$. For more information see the proof of 0.5.22. $\square$

Notice that $\mathcal{A}$ may potentially have a property P and potentially have property $\neg P$, as above. This is not a contradiction.

Now we aim to show that every almost disjoint family is potentially pseudonormal.

The following definition is from [42].

Definition 2.6.9. Let $\mathcal{A}$ be an almost disjoint family and $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$. We define:

$$\mathcal{S}_{\mathcal{B},\mathcal{C}} = \{(s, \mathcal{F}, \mathcal{G}) \in \omega^{<\omega} \times [\mathcal{B}]^{<\omega} \times [\mathcal{C}]^{<\omega} : (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq |s|\}.$$

We order $\mathcal{S}_{\mathcal{B},\mathcal{C}}$ by letting $(s, \mathcal{F}, \mathcal{G}) \leq (s', \mathcal{F}', \mathcal{G}')$ iff:

- $s' \subseteq s$,
- $\mathcal{F}' \subseteq \mathcal{F}$,
- $\mathcal{G}' \subseteq \mathcal{G}$,
- $\forall n \in |s| \setminus |s'| (n \in \bigcup \mathcal{F}' \rightarrow s(n) = 1)$ and
- $\forall n \in |s| \setminus |s'| (n \in \bigcup \mathcal{F}' \rightarrow s(n) = 0)$.

$\square$

As observed by the authors of [42], the following hold:

Proposition 2.6.10. Let $\mathcal{A}$ be an almost disjoint family and $\mathcal{B}, \mathcal{C}$ be disjoint subsets of $\mathcal{A}$. The following sets are dense:

1. $E_b = \{(s, \mathcal{F}, \mathcal{G}) \in \mathbb{S}_{\mathcal{B},\mathcal{C}} : b \in \mathcal{F}\}$ for $b \in \mathcal{B}$,
2. $E'_c = \{(s, \mathcal{F}, \mathcal{G}) \in \mathbb{S}_{\mathcal{B},\mathcal{C}} : c \in \mathcal{G}\}$ for $c \in \mathcal{C}$,
3. $D_n = \{(s, \mathcal{F}, \mathcal{G}) \in \mathbb{S}_{\mathcal{B},\mathcal{C}} : n \in |s|\}$ for $n \in \omega$.

Moreover, if G is a filter in $\mathbb{S}_{\mathcal{B},\mathcal{C}}$ intersecting all of these sets, the set $X = \{n \in \omega : \exists (s, \mathcal{F}, \mathcal{G}) \in G \, n \in |s| \text{ and } s(n) = 1\}$ is such that for every $b \in \mathcal{B}$, $b \subseteq^* X$ and for every $c \in \mathcal{C}$, $c \cap X =^* \emptyset\}$.

Proof. E_b is dense for $b \in \mathcal{B}$: Let $(s, \mathcal{F}, \mathcal{G})$ be given. Suppose $b \notin \mathcal{F}$. Let $n \geq |s|$ be such that $b \cap \bigcup \mathcal{G} \subseteq n$. Let $s' : n \to 2$ be such that $s \subseteq s'$ and for $m \in n \setminus |s|$, $s'(m) = 1$ iff $m \in \bigcup F$. Then $(s', \mathcal{F} \cup \{b\}, \mathcal{G}) \in \mathbb{S}_{\mathcal{B},\mathcal{C}}$ and $(s', \mathcal{F} \cup \{b\}, \mathcal{G}) \leq (s, \mathcal{F}, \mathcal{G})$.

E'_c is dense for $c \in \mathcal{C}$: Let $(s, \mathcal{F}, \mathcal{G})$ be given. Suppose $c \notin \mathcal{G}$. Let $n \geq |s|$ be such that $c \cap \bigcup \mathcal{F} \subseteq n$. Let $s' : n \to 2$ be such that $s \subseteq s'$ and for $m \in n \setminus |s|$, $s'(m) = 1$ iff $m \in \bigcup F$. Then $(s', \mathcal{F}, \mathcal{G} \cup \{c\}) \in \mathbb{S}_{\mathcal{B},\mathcal{C}}$ and $(s', \mathcal{F}, \mathcal{G} \cup \{c\}) \leq (s, \mathcal{F}, \mathcal{G})$.

D_n is dense for $n \in \omega$: Let $(s, \mathcal{F}, \mathcal{G})$ be given. Suppose $|s| \leq n$. Let $s' : n + 1 \to 2$ be such that $s \subseteq s'$ and for $m \in (n + 1) \setminus |s|$, $s'(m) = 1$ iff $m \in \bigcup F$. Then $(s', \mathcal{F}, \mathcal{G}) \in \mathbb{S}_{\mathcal{B},\mathcal{C}}$ and $(s', \mathcal{F}, \mathcal{G}) \leq (s, \mathcal{F}, \mathcal{G})$.

For the last claim, first suppose that $b \in \mathcal{B}$. Let $(s, \mathcal{F}, \mathcal{G}) \in G \cap E_b$. We claim that if $n \geq |s|$ then $n \in X$. To see that, fix such an n. Let $(s', \mathcal{F}', \mathcal{G}') \in D_n \cap G$. Let $(s'', \mathcal{F}'', \mathcal{G}'') \leq (s, \mathcal{F}, \mathcal{G}), (s', \mathcal{F}', \mathcal{G}')$. Then $n \in |s''|$ since $(s'', \mathcal{F}'', \mathcal{G}'') \leq (s', \mathcal{F}', \mathcal{G}')$ and $s''(n) = 1$ since $(s'', \mathcal{F}'', \mathcal{G}'') \leq (s, \mathcal{F}, \mathcal{G})$, $n \in |s''| \setminus |s|$ and $n \in b \subseteq \bigcup F$.

Now suppose that $c \in \mathcal{C}$. Let $(s, \mathcal{F}, \mathcal{G}) \in G \cap E'_c$. We claim that if $n \geq |s|$ then $n \notin X$. To see that, fix such an n. Let $(s', \mathcal{F}', \mathcal{G}') \cap G$ such that $n \in |s'|$ be given. We must see that $s'(n) = 0$. Let $(s'', \mathcal{F}'', \mathcal{G}'') \leq (s, \mathcal{F}, \mathcal{G}), (s', \mathcal{F}', \mathcal{G}')$. Then $n \in |s''|$ since $(s'', \mathcal{F}'', \mathcal{G}'') \leq (s', \mathcal{F}', \mathcal{G}')$ and $s''(n) = 0$ since $(s'', \mathcal{F}'', \mathcal{G}'') \leq (s, \mathcal{F}, \mathcal{G})$, $n \in |s''| \setminus |s|$ and $n \in c \subseteq \bigcup G$. ☐

The following easy fact was not observed by the authors of [42] (but was not used as well).

Proposition 2.6.11 (*). Let $\mathcal{A}$ be an almost disjoint family and $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$. Suppose either $\mathcal{B}$ of $\mathcal{C}$ is countable. Then $\mathcal{S}_{\mathcal{B},\mathcal{C}}$ is σ-centered.

Proof. Just notice that if $s' \in \omega^{<\omega}$, $\mathcal{F}' \in [\mathcal{B}]^{<\omega}$ and $\mathcal{G}' \in [\mathcal{C}]^{<\omega}$, then $\{(s, \mathcal{F}, \mathcal{G}) \in \mathcal{S}_{\mathcal{B},\mathcal{C}} : s = s' \text{ and } \mathcal{F} = \mathcal{F}'\}$ and $\{(s, \mathcal{F}, \mathcal{G}) \in \mathcal{S}_{\mathcal{B},\mathcal{C}} : s = s' \text{ and } \mathcal{G} = \mathcal{G}'\}$ are centered. ☐

Therefore all we need to do to make an almost disjoint family pseudonormal is to iterate this notion. Concretely:

Theorem 2.6.12 (*). Suppose that $\mathcal{A}$ is an almost disjoint family, and let $\kappa = \mathfrak{c}$. Then there exists a c.c.c. forcing notion $\mathbb{P}$ with a dense subset of size κ such that $\mathbb{P} \Vdash \check{A}$ is pseudonormal.

Proof. We will proceed by iterated forcing assuming the existence of a countable transitive model M, and prove the theorem relativized to M.

We recursively construct, working in M, a finitely supported κ-stage iterated forcing construction $(\langle(\mathbb{P}_\xi, \leq_\xi, \mathbb{1}_\xi) : \xi \leq \kappa\rangle, \langle(\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_\xi, \mathring{\mathbb{1}}_\xi) : \xi < \kappa\rangle)$.

Each $\mathring{\mathbb{Q}}_\xi$ will be forced by $\mathbb{P}_\xi$ to have size $\leq \kappa$ and to have the c.c.c., therefore for each $\xi \leq \kappa$, $\mathbb{P}_\xi$ will have a dense subset $\mathbb{P}'_\xi$ of cardinality at most κ which will have the c.c.c. as well.

Fix a function f from κ onto $\kappa \times \kappa$ such that if $f(\xi) = (\zeta, \mu)$, then $\zeta \leq \xi$. We will use f as a bookkeeping device.

Suppose we have constructed $(\langle(\mathbb{P}_\zeta, \leq_\zeta, \mathbb{1}_\zeta) : \zeta \leq \xi\rangle, \langle(\mathring{\mathbb{Q}}_\zeta, \mathring{\leq}_\zeta, \mathring{\mathbb{1}}_\zeta) : \zeta < \xi\rangle)$ for some $\xi < \kappa$. We must determine $(\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_\xi, \mathring{\mathbb{1}}_\xi)$. Suppose that for each stage $\zeta < \xi$ we have also listed all $\mathbb{P}'_\zeta$-nice names τ for subsets of $\omega \check{\times} \mathcal{A}$ such that $\mathbb{P}_\zeta \Vdash \tau : \omega \to \mathcal{A}$ as $(\tau^\mu_\zeta : \mu < \kappa)$. This is possible since $|\mathbb{P}'_\zeta| \leq \kappa = \kappa^\omega$ and has since $\mathbb{P}_\zeta$ has the countable chain condition. List all $\mathbb{P}'_\xi$-nice names for subsets $\omega \check{\times} \mathcal{A}$ such that $\mathbb{P}'_\xi \Vdash \tau : \omega \to \mathcal{A}$ as $(\tau^\mu_\xi : \mu < \kappa)$ as well.

Let $f(\xi) = (\zeta, \mu)$. Since $\zeta \leq \xi$, the name τ^μ_ζ is a nice $\mathbb{P}'_\xi$-name for a subset of $\omega \check{\times} \mathcal{A}$ such that $\mathbb{P}_\xi \Vdash \tau^\mu_\eta : \omega \to \check{A}$. Let $(\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_\xi, \mathring{\mathbb{1}}_\xi)$ be an appropriate triple of $\mathbb{P}_\xi$-names such that $\mathbb{P}_\xi \Vdash (\mathring{\mathbb{Q}}_\xi, \mathring{\leq}_\xi, \mathring{\mathbb{1}}_\xi) = \mathbb{S}_{\operatorname{ran}\tau^\mu_\eta, \check{A}\setminus\operatorname{ran}\tau^\mu_\eta}$.

Let $\mathbb{P} = \mathbb{P}_\kappa$.

Let G be $\mathbb{P}$-generic over M. We claim $\mathcal{A}$ is pseudocompact $M[G]$. In order to see that, let $\mathcal{B}$ be a countable subset of $\mathcal{A}$ relative to $M[G]$ and $\mathcal{C} = \mathcal{A}\setminus\mathcal{B}$. We must see that $\mathcal{B}, \mathcal{C}$ can be separated. We can suppose that $\mathcal{B} \neq \emptyset$. Since $\operatorname{cf}(\kappa)^M > \omega$, There exists $\zeta < \kappa$ such that $\mathcal{B} \in M[G_\zeta]$, where $G_\zeta = G \cap \mathbb{P}_\zeta$. There exists $\mu < \kappa$ such that $\mathcal{B} = \operatorname{ran}\operatorname{val}(\tau^\mu_\zeta, G_\zeta)$. Let ξ be such that $f(\xi) = (\zeta, \mu)$. Then, since $\mathcal{B} = \operatorname{ran}\operatorname{val}(\tau^\mu_\zeta, G_\xi)$ and $\mathcal{C} = \mathcal{A} \setminus \operatorname{ran}\operatorname{val}(\tau^\mu_\zeta, G_\xi)$ by the choice of $\mathring{\mathbb{Q}}_\xi$, $M[G_{\xi+1}]$ contains a $\mathbb{S}_{\mathcal{B},\mathcal{C}}$-generic filter over $M[G_\xi]$, so, by Proposition 2.6.10 in $M[G_{\xi+1}]$, $\mathcal{B}$ and $\mathcal{C}$ can be separated in $M[G_{\xi+1}]$, and therefore in $M[G]$. $\square$

Chapter 3

The Pseudocompactness of Hyperspaces of Vietoris

In the first four sections this chapter we introduce Vietoris hyperspaces, some of its basic properties and introduce some problems and known results. We present our new results in sections 5 to 8.

There are problems regarding Isbell-Mrówka spaces which are related to the problems we present in this chapter, but we postpone the discussion about them to the next chapter.

3.1 The Hyperspace of Vietoris

The study of topologies on subsets of $\mathcal{P}(X)$, where X is a topological space, began more than ninety years ago. According to [52], the first step towards topologizing such a collection of subsets was due to by Hausdorff (as in the first edition of [36], 1917) who defined a metric on the set of all nonempty closed subsets of X when X is a bounded metric space.

Definition 3.1.1. Let X be a topological space. $\exp(X)$ is the set of all nonempty closed subsets of X. $\mathcal{K}(X)$ is the set of all nonempty closed compact subsets of X. □

Some authors use $\mathrm{CL}(X)$ or 2^X instead of $\exp(X)$. In this thesis we will adopt the notation $\exp(X)$, which is also used by [24].

Notice that if X is a compact Hausdorff space, $\mathcal{K}(X) = \exp(X)$, and if X is a Hausdorff space, $\mathcal{K}(X) \subseteq \exp(X)$.

Definition 3.1.2 (Hausdorff's Metric, [36])**.** Let (X, d) be a metric space. The *Hausdorff's metric* on $\exp(X)$ is the metric $\bar{d} : \exp(X) \to \mathbb{R}^+$ defined by:

$$\bar{d}(F, K) = \max\{\sup\{\inf d(x, y) : y \in K, x \in F\}, \sup\{\inf d(x, y) : x \in F, y \in k\}\} \tag{3.1}$$

□

The reader may verify that $\bar{d}$ is indeed a metric for $exp(X)$. One could also have used $\mathcal{P}(X) \setminus \{\emptyset\}$ instead of $\exp(X)$. This would give us a pseudometric where $\bar{d}(F) = \bar{d}(\operatorname{cl} F)$ for every $F \in \mathcal{P}(X)$, and, by quotienting equivalent points, one gets a metric space isometric to $(\exp(X), \bar{d})$.

For a general topological space one defines the Vietoris topology of $\exp(X)$ as follows:

Definition 3.1.3. Let X be a T_1 topological space. For each set $A \subseteq X$, we define:

$$\begin{aligned} A^+ &= \{F \in \exp(X) : F \subseteq A\} \\ A^- &= \{F \in \exp(X) : F \cap A \neq \emptyset\} \end{aligned} \tag{3.2}$$

The *Vietoris topology* of $\exp(X)$, also known as *finite topology*, is the topology in $\exp(X)$ generated by the sets of the form U^+, U^-, where U ranges over the open subsets of X.

The space $\exp(X)$ with the Vietoris topology is called the *Vietoris hyperspace* of X.
□

This definition does not require X to be T_1 to make sense. However, we will restrict ourselves to the study of T_1 spaces since a lot of basic results regarding the Vietoris hyperspace do not hold for non-T_1 spaces. In particular, it does not need to contain a copy of X if it is not T_1, so the terminology "hyperspace" does not make much sense.

The following notation, also used in [52], is very useful:

Definition 3.1.4. Let X be a topological space, $n \geq 0$ and $A_0, \ldots, A_n$ be subsets of X. Then:

$$\begin{aligned} \langle A_0, \ldots, A_n \rangle &= \{F \in \exp(X) : F \subseteq \bigcup_{i \leq n} A_i \text{ and } \forall i \leq n\, A_i \cap F \neq \emptyset\} \\ &= \left(\bigcup_{i \leq n} A_i \right)^+ \cap \bigcap_{i \leq n} A_i^- \end{aligned} \tag{3.3}$$

□

The reader may verify the following often used facts:

Proposition 3.1.5. Suppose X is a T_1 space. Then:

a) The sets of the form $\langle U_0, \ldots, U_n \rangle$ with $n \in \omega$ and $U_0, \ldots, U_n$ open subsets of X form a basis for the Vietoris hyperspace of X, and:

b) Whenever $n \in \omega$ and $A_0, \ldots, A_n$ are subsets of X, $\operatorname{cl}_{\exp(X)} \langle A_0, \ldots, A_n \rangle = \langle \operatorname{cl}_X A_0, \ldots, \operatorname{cl}_X A_n \rangle$. [52, Lemma 2.3.2]

[52] uses a) to define the Vietoris topology.

The relation between the Hausdorff's metric and the Vietoris topology is given by the following proposition, as mentioned in [52]. Their proof uses uniformities, but the interested reader may find a straightforward direct proof.

Proposition 3.1.6. Suppose (X, d) is a compact (and therefore bounded) metric space. Then the topology generated by the Hausdorff's metric on $\exp(X)$ coincides with the Vietoris topology. □

There are many relations between the topological properties of X and of $\exp(X)$. We cite some basic results regarding separation axioms from Ernest Michael's paper.

Proposition 3.1.7 ([52])**.** Let X be a T_1 space. Then:

a) $\exp(X)$ is T_1. [52, Lemma 4.9.2]

b) X is regular iff $\exp(X)$ is Hausdorff. [52, Lemma 4.9.3]

c) X is normal iff $\exp(X)$ is regular iff $\exp(X)$ is Tychonoff. [52, Lemma 4.9.5]

The proofs are straightforward and can be verified in the reference. We make a note regarding item c). The implication X normal $\implies \exp(X)$ Tychonoff may look nontrivial and [52] uses uniformities in their proof. So we sketch an alternative proof. First, we need the following lemma whose proof is left to the reader as an exercise.

Lemma 3.1.8 ([52, Proposition 4.7])**.** Let X be a T_1 space; Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ be the extended real line (which is homeomorphic to $[0, 1]$). Let $f : X \to \bar{\mathbb{R}}$. Define $f^+ : \exp(X) \to \bar{\mathbb{R}}$ be given by $f^+(F) = \sup\{f(x) : x \in F\}$, and $f^- : \exp(X) \to \bar{\mathbb{R}}$ be given by $f^-(F) = \inf\{f(x) : x \in F\}$. Then f^+, f^- are continuous. □

Sketch of proof of c). Let $F \in \exp(X)$ and $\mathcal{F} \subseteq \exp(X)$ be a closed set not containing F. Let $\langle U_0, \dots U_n\rangle$ be a basic open set (as in 3.1.5 a)) containing the point F disjoint from $\mathcal{F}$.

For each $i \leq n$, fix $x_i \in U_i \cap F$. Since X is completely regular, for each $i \leq n$, fix $f_i : X \to [0, 1]$ a continuous function such that $X \setminus U_i \subseteq f_i^{-1}[\{0\}]$ and $f_i(x_i) = 1$. By Lemma 3.1.8, f_i^+ is continuous for each $i \leq n$.

Let $U = \bigcup_{i\leq n} U_i$. Since $F \subseteq U$ and X is normal, let $g : X \to [0, 1]$ be a continuous function such that $X \setminus U \subseteq g^{-1}[\{0\}]$, $F \subseteq g^{-1}[\{1\}]$. By Lemma 3.1.8, g^- is continuous.

Let $h : \exp(X) \to [0, 1]$ be given by $h = g^-. \prod_{i\leq n} f_i^+$. h is continuous, $h(F) = 1$ and:

$$\begin{aligned} \mathcal{F} \subseteq \exp(X) \setminus \langle U_0, \dots U_n\rangle &= (X \setminus U)^- \cup \bigcup_{i\leq n} (X \setminus U_i)^+ \\ &\subseteq {g^-}^{-1}[\{0\}] \cup \bigcup_{i\leq n} {f_i^+}^{-1}[\{0\}] \subseteq h^{-1}[\{0\}]. \end{aligned} \tag{3.4}$$

□

The density of $\exp(X)$ and X is the same. We leave the (easy) details for a proof of the following known proposition to the reader.

Proposition 3.1.9. Let X be a T_1 topological space. Then:

a) If $D \subseteq X$ is dense in X, then $[D]^{<\omega} \subseteq \exp(X)$ is dense in $\exp(X)$.

b) If $E \subseteq \exp(X)$ is dense in $\exp(X)$ and $f : E \subseteq X$ is a choice function for E, then $f[E]$ is dense in X.

c) $d(X) = d(\exp(X))$

d) X is separable iff $\exp(X)$ is separable. [52, Lemma 4.5.1]

Since X is T_1, there is a straightforward way to visualize X inside of $\exp(X)$. More generally, we have the following proposition, which we also prove for the sake of completeness since I could not find a reference containing a proof of c) in English.

Proposition 3.1.10. Let X be a T_1 space. Then:

a) The mapping $e : X \to \exp(X)$ given by $e(x) = \{x\}$ is a topological embedding, [52, Lemma 2.4]

b) for each $n \geq 1$, the mapping $e_n : X^n \to \exp(X)$ given by $e(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$ is continuous [52, Lemma 2.4.3], and

c) if X is Hausdorff, $e_n[X] = [X]^{\leq n} \setminus \{\emptyset\}$ is closed for every $n \geq 1$, and e_n is also closed (thus, $e_n : X^n \to [X]^{\leq n} \setminus \{\emptyset\}$ is a perfect mapping) [28].

Proof. a) This function is clearly injective. For the continuity, it suffices to see that if U is a open subset of X, then $e^{-1}[U^+]$ and $e^{-1}[U^-]$ are both open. But these sets coincide with U. Moreover, $e[U] = U^+ \cap e[X]$, so e is open.

b) Let $U \subseteq X$ be open. We must see that $e_n^{-1}[U^+]$ and $e_n^{-1}[U^-]$ are closed. Notice that $e_n^{-1}[U^+] = \prod_{i\leq n} U$ and $e_n^{-1}[U^-] = \bigcup_{i\leq n} \prod_{j\leq n} Z_{ij}$, where $Z_{ij} = U$ if $i = j$, and $Z_{ij} = X$ if $i \neq j$.

c) First, let us see that the range is closed. Suppose K is a closed set with more than n elements. Choose $x_0, \ldots, x_n \in K$. Since X is Hausdorff, there are open sets $V_0, \ldots, V_n$ such that $x_i \in V_i$ and $V_i \cap V_j = \emptyset$ whenever $i \neq j$, $i, j \leq n$. Then clearly every element of $e_n \left[\prod_{i\leq n} V_i\right]$ has at least $n + 1$ elements, at least one in each V_i.

Now let F be a closed subset of X^n. We must see that $e_n[F]$ is closed in $e_n[X]$ Suppose $K \in \operatorname{cl} e_n[F] \subseteq e_n[X^n]$. We will see that $K \in e_n[F]$. Write $K = \{x_0, \ldots, x_m\}$, where $m \leq n - 1$ and $(x_i)_{i\leq m}$ is injective. Let $(W_i)_{i\leq m}$ be a collection of pairwise disjoint open subsets of X such that $x_i \in W_i$ for each $i \leq m$.

Claim: The sets of the form $\langle U_0, \ldots, U_m \rangle$, where U_i is an open subset of x and $x_i \in U_i \subseteq W_i$ for each $i \leq m$ form a local basis for K.

Proof of the claim. Suppose $K \in \langle V_0, \ldots, V_k \rangle$, where $V_j \subseteq X$ is open for each $j \leq k$. For each $i \leq m$, let $J_i = \{j \leq k : x_i \in V_j\}$. Notice that $\bigcup_{i\leq n} J_i = k + 1$. Now let $U_i = \bigcap (\{W_i\} \cup \bigcup \{V_j : j \in J_i\})$. It is easy to verify that $K \in \langle U_0, \ldots, U_m \rangle \subseteq \langle V_0, \ldots, V_k \rangle$. □

Let $P = \{(U_0, \dots, U_m) : \forall i \leq m\, x_i \in U_i \subseteq W_i\}$. Order P by letting $(U'_0, \dots, U'_m) \leq (U_0, \dots, U_m)$ iff $\forall i \leq m\, U_i \subseteq U'_i$. $(P, \leq)$ is clearly a partial ordered set directed upwards.

For each $U = (U_0, \dots, U_m) \in P$ there exists $y^U \in F$ such that $e_n(y^U) \in \langle U_i \rangle_{i \leq m}$. Let $\sigma(U) : n \to m + 1$ be the (unique, necessarily onto) funcion such that $y^U_j \in U_{\sigma(j)}$.

Since $(m + 1)^n$ is finite and $(P, \leq)$ is directed upwards, there exists $\bar{\sigma} \in (m + 1)^n$ such that $\mathcal{A}_{\bar{\sigma}} = \{U \in P : \sigma(U) = \bar{\sigma}\}$ is cofinal (upwards) in P (working towards a contradiction notice that if for each $\bar{\sigma}$ where exists $U_{\bar{\sigma}}$ with no larger element in $\mathcal{A}_{\bar{\sigma}}$, the elements of the (finite) family $(U_{\bar{\sigma}} : \bar{\sigma} \in (m + 1)^n)$ has no common larger element).

Define $y \in X^n$ given by $y_j = x_{\bar{\sigma}(j)}$. Since $\bar{\sigma}$ is onto $m + 1$, $e_n(y) = K$. We will show that $y \in F$ by showing that $y \in \operatorname{cl} F$, which completes the proof.

Let $A = \prod_{j<n} A_j$ be a basic open neighborhood of y. By shrinking A, we can suppose that $A_j = A_{j'}$ if $\sigma(j) = \sigma(j')$ (since this implies that $y_j = y_{j'}$) and that $A_j \subseteq W_{\bar{\sigma}(j)}$ (since $y_j = x_{\bar{\sigma}(j)}$). For each $i \leq m$, let $U'_i = A_{\sigma(j)}$ where $\sigma(j) = i$. By the cofinality of $\mathcal{A}_{\bar{\sigma}}$, there exists $U \geq U'$ in $\mathcal{A}_{\bar{\sigma}}$. Then for each $j < n$, $y^U(j) \in U_{\bar{\sigma}(j)} \subseteq U'_{\bar{\sigma}(j)} = A_j$, that is, $y^U \in A \cap F$.

□

Regarding subspaces, we have the following straightforward lemma which is left to the reader:

Lemma 3.1.11. Let X be a T_1 topological space. Let $F \in \exp(X)$. Then $F^+ = \{K \in \exp(X) : K \subseteq F\}$ with the subspace topology is the same topological space as $\exp(F)$ with the Vietoris topology (where $F \subseteq X$ has the subspace topology). □

3.2 Compactness-like properties in the Vietoris hyperspace

The topological properties of $\exp(X)$ often depend on the topological properties of X (and vice versa). One of the central results of this field of research is the following, due to L. Vietoris. Due to the centrality of this result, we present a proof for the sake of completeness. This proof is sketched in [52]. We note that it works even for non T_1-spaces.

Theorem 3.2.1 ([68])**.** Let X be a T_1 topological space. Then X is compact if, and only if $\exp(X)$ is compact.

Proof. Suppose $\exp(X)$ is compact. Let $\mathcal{U}$ be an open cover of X. Then $\{U^- : U \in \mathcal{U}\}$ is an open cover of $\exp(X)$. So there exists an finite set $\mathcal{U}' \subseteq \mathcal{U}$ such that $\{U^- : U \in \mathcal{U}'\}$ covers $\exp(X)$. But then $F = X \setminus \bigcup \mathcal{U}'$ must be empty, since if it is not empty, then $F \in \exp(X) \setminus \bigcup \{U^- : U \in \mathcal{U}'\}$.

Conversely, suppose X is compact. By Alexander's subbasis lemma, in order to verify that $\exp(X)$ is compact, it suffices to verify that covers of the type $\{U^+ : U \in \mathcal{U}\} \cup \{V^- : V \in \mathcal{V}\}$ have finite subcovers, where $\mathcal{U}, \mathcal{V}$ are collections of open subsets of X.

Case 1: $\mathcal{V}$ covers X. In this case, let $\mathcal{V}'$ be a finite subcover of X. Then $\{V^- : V \in \mathcal{V}'\}$ covers $\exp(X)$, for if $K \in \exp(X)$, there exists $V \in \mathcal{V}'$ such that $K \cap V \neq \emptyset$. **Case 2:** $K = X \setminus \bigcup \mathcal{V}$ is nonempty. In this case, K does not intersect any $V \in \mathcal{V}$, so there must exist $U \in \mathcal{U}$ such that $K \subseteq U$. Thus $\{U\} \cup \mathcal{V}$ covers X, so there exists a finite subset $\mathcal{V}'$ of $\mathcal{V}$ such that $\{U\} \cup \mathcal{V}'$ covers X. Then $\{U^+\} \cup \bigcup\{V^- : V \in \mathcal{V}'\}$ covers $\exp(X)$: given $F \in \exp(X)$, if F intersects no $V \in \mathcal{V}'$, then $F \subseteq K \subseteq U$, so $K \in U^+$. □

It is natural, then, to ask if there are similar relations between X and $\exp(X)$ regarding generalizations of the notion of compactness.

The following lemma gives a condition which implies compactness. I do not know if this is stated elsewhere.

Proposition 3.2.2. Let K be a locally compact space, L be a compact space and $f : K \to L$ be a continuous closed map for such there exists $A \subseteq L$ such that $f^{-1}[A]$ is compact and $f|_{X \setminus f^{-1}[A]}$ is injective. Then K is compact.

Proof. Since K is locally compact and $f^{-1}[A]$ is compact, there exists an open $U \subseteq K$ such that $f^{-1}[A] \subseteq U$ and $\operatorname{cl} U$ is compact. $K \setminus U$ is homeomorphic (by f) to a closed set of the compact space L, so $K \setminus U$ is compact. But $\operatorname{cl} U$ is also compact, so the union K is compact. □

Regarding local compactness, we have the following:

Proposition 3.2.3 (This is essentially Lemma 4.3. of [52])**.** Let X be a T_1 topological space. Then:

a) If $F \in \exp(X)$, then F^+ is compact iff F is compact.

b) If X is locally compact, $F \in \exp(X)$ and F is compact, then F has a compact neighborhood in $\exp(X)$.

c) Suppose X is a locally compact Hausdorff space and $F \in \exp(X)$. If F has a compact neighborhood in $\exp(X)$, then F is compact.

Proof. a) This follows from 3.1.11 and 3.2.1.

b) Since X is locally compact, F is compact and a finite union of compact sets is compact, there exists an open set U such that $F \subseteq U$ and $\operatorname{cl} U$ is compact. Then by a), $\operatorname{cl} U^+ = \operatorname{cl}\langle U \rangle = \langle \operatorname{cl} U \rangle$ is a closed neighborhood of F (the last equality follows from Proposition 3.1.5 b)).

c) Suppose F has a compact neighborhood. There exist open $U_0, \dots, U_n$ such that $F \in \langle U_0, \dots, U_n \rangle$ and $L = \operatorname{cl}\langle U_0, \dots, U_n \rangle = \langle \operatorname{cl} U_0, \dots, \operatorname{cl} U_n \rangle$ is compact. It suffices to show that $K = \bigcup_{i \leq n} \operatorname{cl} U_i$ is compact. For each $i \leq n$ fix $u_i \in U_i \cap K$. Since K is closed and X is locally compact, K is locally compact. Let $f : K \to L$ be given by $f(x) = \{x, u_0, \dots, u_n\}$. Notice that f is the composition of the closed continuous functions (with closed range) $a : K \to K^{n+2}$ given by $a(x) = (x, u_0, \dots, u_n)$ and $e_{n+2} : X^{n+2} \to \exp(X)$ given by $e_{n+2}(v_0, \dots, v_{n+1}) = \{v_0, \dots, v_{n+1}\}$, so f is a closed continuous function. Let $A = \{u_0, \dots, u_n\}$. Then $f^{-1}[A] = \{u_0, \dots, u_n\} = A$ is finite, and therefore compact, and $f|_{K\setminus A}$ is injective, so by Proposition 3.2.2, K is compact and the proof is complete. □

Corollary 3.2.4. Let X be a Hausdorff space. The following are equivalent:

a) X is compact Hausdorff,

b) $\exp(X)$ is compact Hausdorff,

c) $\exp(X)$ is locally compact Hausdorff.

Proof. a) implies b): Since X is compact and Hausdorff, X is regular. Now b) follows from 3.2.1 and Proposition 3.1.7 b).

b) implies c): trivial.

c) implies a): First, we verify that X is locally compact. Let $x \in X$ be given. $\{x\} \in \exp(X)$, so there exists a open neighborhood $\langle U_0, \dots, U_n\rangle$ of $\{x\}$ such that $\operatorname{cl}\langle U_0, \dots, U_n\rangle$ is compact. Then $x \in U = \bigcap_{i\leq n} U_i$. It is clear that $\{x\} \in U^+ \subseteq \langle U_0, \dots, U_n\rangle$, so $\operatorname{cl} U^+ = (\operatorname{cl} U)^+$ is compact. But this implies that $\operatorname{cl} U$ is compact, as intended.

Now, since X is locally compact, $X \in \exp(X)$ and $\exp(X)$ is locally compact, it follows from item c) of the previous proposition that X is compact. □

We note that Proposition 4.4.1 of [52] has problems: it states that X is locally compact iff $\exp(X)$ is locally compact. But this is not true, any locally compact Hausdorff space X which is not compact (such as $\mathbb{R}$ or Isbell-Mrówka spaces) illustrates that the statement is false. This is probably a typo. We believe the author intended to write $\mathcal{K}(X)$ instead of $2^X (= \exp(X))$.

3.3 Countable compactness in the Vietoris hyperspace

Motivated by the the results from the previous section, John Ginsburg has explored, in [32], the countable compactness and the pseudocompactness of the Vietoris Hyperspaces. In this section we discuss his results on countable compactness.

Theorem 3.3.1 ([32, Theorem 2.1.])**.** Let X be a topological space. Let $\mathcal{U} \in \omega^*$. If X is $\mathcal{U}$-compact, then $\exp(X)$ is $\mathcal{U}$-compact. The converse holds if X is Hausdorff.

Proof. First, suppose $\exp(X)$ is $\mathcal{U}$-compact and that X is Hausdorff. Since $\mathcal{U}$-compactness is hereditary for closed subspaces and preserved by homeomorphisms, the conclusion follows from Proposition 3.1.10.

Conversely, suppose X is $\mathcal{U}$-compact. Let $(F_n : n \in \omega)$ be a sequence in $\exp(X)$. Let $L = \{x \in X : x \text{ is an } \mathcal{U}\text{-accumulation point of } (F_n : n \in \omega)\}$. If $f : \omega \to X$ is such that $f(n) \in F_n$, it follows that the $\mathcal{U}$-limits of f are in L, so $L \neq \emptyset$. Moreover, L is closed: if $x \notin L$, there exists an open neighborhood V of x such that $\{n \in \omega : V \cap F_n \neq \emptyset\} \notin \mathcal{U}$,so $V \cap L \neq \emptyset$.

So $L \in \exp(X)$. We claim that L is an $\mathcal{U}$-limit of X. It suffices to verify that if W is a subbasic open neighborhood of L in $\exp(X)$, then $\{n \in \omega : F_n \in W\} \in \mathcal{U}$.

If $W = V^+$ for some open subset V of X containing L: suppose by contradiction that $A = \{n \in \omega : F_n \setminus V\} \in \mathcal{U}$. Let $f : \omega \to X$ be such that $f(n) \in F_n \setminus V$ for each $n \in A$ and $f(n) \in F_n$ for every $n \in \omega$. Let x be a $\mathcal{U}$-limit of f. Then $x \in L$ by the definition of L, but $x \notin V$ (or we would have that $\{n \in \omega : f(n) \in V\} \in \mathcal{U}$, but $\{n \in \omega : f(n) \in V\} \cap A = \emptyset$).

If $W = V^-$ for some open subset V of X intersecting L: let $x \in V \cap L$. By the definition of L, x is a $\mathcal{U}$-accumulation point of $(F_n : n \in \omega)$. Since $F_n \cap V \neq \emptyset$ is equivalent to $F_n \in V^-$, it follows that $\{n \in \omega : F_n \in V^-\} \in \mathcal{U}$. □

Corollary 3.3.2 ([32, Corollary 2.3.])**.** Let X be a topological space. If all powers of X are countably compact, then $\exp(X)$ is countably compact.

Proof. This follows from the previous theorem and from Theorem 0.4.22. □

Corollary 3.3.3 ([32, Corollary 2.3.])**.** If X is Hausdorff and $\exp(X)$ is countably compact, then for every $n \in \omega$, X^n is countably compact.

Proof. By Proposition 3.1.10, $Y = [X]^{\leq n} \setminus \{\emptyset\}$ is a closed subset of $\exp(X)$, and $e_n : X^n \to Y$ is a perfect mapping. Since Y is closed in the countably compact space $\exp(X)$, Y is countably compact. So by Proposition 0.3.9, X^n is countably compact. □

3.4 Ginsburg's results on the pseudocompactness of hyperspaces

As we have seen in the previous section, in [32] John Ginsburg has explored countable compactness of Vietoris hyperspaces. He also tried to obtain analogous results for pseudocompactness.

The first question that may come into mind is what is the "correct" definition to work with when dealing with pseudocompactness on Vietoris hyperspaces. In Chapter 0 we introduced two definitions: the non existence of unbounded continuous functions, which we called pseudocompactness, and the existence of accumulation points for sequences of nonempty open sets, which is called feebly compactness (see Definition 0.3.12). However, we only know that they are equivalent as long as the topological space is $T_{3\frac{1}{2}}$ (Proposition 0.3.13), and by Proposition 3.1.7 c), $\exp(X)$ is Tychonoff iff X is normal, which may look like an issue. However, the following proposition removes this problem if we restrict ourselves to the realm of Tychonoff spaces:

Proposition 3.4.1 ([32, Proposition 2.6.])**.** Let X be a Tychonoff space. Then $\exp(X)$ is pseudocompact iff $\exp(X)$ is feebly compact.

Proof. For the nontrivial direction, suppose $\exp(X)$ is not feebly compact. Then there is a sequence $(B_n)_{n\in\omega}$ of nonempty basic open sets with no accumulation point. We can write

$B_n = \langle G_0^n, \dots, G_{t_n}^n \rangle$, where each $t_n \in \omega$ and G_i^n is a open subset of X. For each $n \in \omega$ and $i \leq t_n$, choose $p_i^n \in G_i^n$. Let $F_n = \{p_i^n : i \leq t_n\}$ for each $n \in \omega$. Then $F_n \in B_n$.

Since X is Tychonoff and each F_n is finite, then for each n there exists a continuous $f_n : X \to [0, 1]$ such that $F_n \subseteq f^{-1}[\{1\}]$ and $X \setminus \bigcup \bigcup_{i \leq t_n} G_i^n \subseteq f^{-1}[\{0\}]$.

Again, since X is Tychonoff, for each $n \in \omega$ and $i \leq t_n$ there exists a continuous $g_i^n : X \to [0, 1]$ such that $g_i^n(p_i^n) = 1$ and $X \setminus G_i^n \subseteq g_i^{n-1}[\{0\}]$. The functions $(g_i^n)^+$ and $(f_n)^-$ ($n \in \omega, i \leq t_n$) from $\exp(X)$ into $[0, 1]$ defined in Lemma 3.1.8 are continuous. For each n, let $\phi_n : \exp(X) \to [0, 1]$ be given by $\phi_n = (f_n)^- . \prod_{i \leq t_n} (g_i^n)^+$. Then for each $n \in \omega$, $\phi_n(F_n) = 1$ and $\phi_n(F) = 0$ for every $F \in \exp(X) \setminus B_n$.

Consider the function $H : \exp(X) \to \mathbb{R}$ be defined by $H(F) = \sum_{n \in \omega} n\phi_n$. Since $(B_n)_{n \in \omega}$ has no accumulation point, H is well defined and is continuous. H is unbounded since $H(F_n) \geq n\phi(F_n) = n$ for each $n \in \omega$. So $\exp(X)$ is not pseudocompact. □

Regarding the "ultrafilter version" of pseudocompactness and finite powers of X, Ginsburg proved the following results. We refer to his paper for proofs.

Theorem 3.4.2 ([32, Theorem 2.4])**.** Let X be a T_1 space and $\mathcal{U} \in \omega^*$. Then X is $\mathcal{U}$-pseudocompact iff $\exp(X)$ is $\mathcal{U}$-pseudocompact. □

Theorem 3.4.3 ([32, Theorem 2.5])**.** Let X be a regular space. If $\exp(X)$ is feebly compact, then all finite powers of X are feebly compact. □

Theorem 3.4.4 ([32, Corollary 2.7])**.** Let X be a Tychonoff space. If $\exp(X)$ is pseudocompact, then all finite powers of X are pseudocompact. □

We can summarize Ginsburg's results listed in this thesis in the following table. In the table below, X is a Tychonoff space and $\mathcal{U} \in \omega^*$. CC stands for "countably compact", $\mathcal{U}$C stands for $\mathcal{U}$-compact, PC stands for "pseudocompact" and $\mathcal{U}$PC stands for $\mathcal{U}$-pseudocompact. Also, n is quantified in ω and κ over the ordinals.

Ultrafilter version	$\exp(X)$ is...	all powers are...
X $\mathcal{U}$C iff $\exp(X)$ $\mathcal{U}$C	$\exp(X)$ CC $\to \forall n\, X^n$ CC	$\forall \kappa\, X^\kappa$ CC $\to \exp(X)$ CC
X $\mathcal{U}$PC iff $\exp(X)$ $\mathcal{U}$PC	$\exp(X)$ PC $\to \forall n\, X^n$ PC	??

To complete the analogy, by propositions 0.4.23 and 0.4.24, we would need a theorem such as "if X is Tychonoff and that X^ω is pseudocompact, then $\exp(X)$ is pseudocompact". **It turns out that this result is false**, as firstly proved in [43], but J. Ginsburg did not know about this at that time. We will discuss more about that in the next sections.

Ginsburg also provided an example of a Tychonoff space whose every finite power is pseudocompact but $\exp(X)$ is not pseudocompact. Then, he explicitly asked:

Problem 3.4.5 (Citing [32])**.** "In light of the results (...) it is natural to ask whether there is any relation between the pseudocompactness (countable compactness) of X^ω and that of $\exp(X)$. It would also be interesting to characterize those spaces X whose hyperspaces are countably compact (pseudocompact). The author [J. Ginsburg] has been unable to resolve these questions, and leaves them open to the reader."

There are several results on this problem, but it is still open. There is currently no characterization of the spaces whose hyperspace is pseudocompact, or countably compact.

It is worth mentioning that in [16], J. Cao, T. Nogura and A. H. Tomita gave a partial answer to the question about the relation of the pseudocompactness of X^ω and that of $\exp(X)$. We refer to their paper for a proof.

Theorem 3.4.6 ([16, Theorem 3.1.]). Let X be a homogeneous regular space. If $\exp(X)$ is pseudocompact, then X^ω is pseudocompact.

3.5 Some lemmas needed for new results

For the rest of this chapter we prove some new results on hyperspaces of subspaces of $\beta\omega$. The results displayed in this Chapter are mainly based on [59] by myself, A. H. Tomita and Y. F. Ortiz-Castillo.

As we have mentioned in the previous section, while studying Ginsburg's questions on the pseudocompactness of hyperspaces, I. Martinez-Ruiz, F. Hernandez-Hernandez and M. Hrušak provided, in [43], a $T_{3\frac{1}{2}}$ topological space whose all powers are pseudocompact and its hyperspace is not pseudocompact. More explicitly, they proved:

Proposition 3.5.1 ([43, Theorem 5.1.]). There exists a subspace of $\beta\omega$ containing ω whose all powers are pseudocompact but its Vietoris hyperspace is not.

In Section 6 we will improve this result by making many powers become countably compact. In Section 7 we will explore a condition that makes the hyperspace of such a space pseudocompact. In Section 8 we explore consequences of the hyperspace of such a space being pseudocompact.

The following two lemmas will be useful. They are probably folklore and are easy, but we write proofs for the sake of completeness.

First, we do know how to take limits of increasing sequences.

Lemma 3.5.2. Let X be a T_1 topological space. Let $F = (F(n) : n \in \omega)$ be a increasing sequence of nonempty closed sets. Then F converges to $\operatorname{cl} \bigcup_{n\in\omega} F(n)$.

Proof. Let $A = \operatorname{cl} \bigcup_{n\in\omega} F_n$. First, suppose that U is open and that $A \in U^+$. It is clear that $F(n) \in U^+$ for every n. Now suppose that U is open and that $A \in U^-$. Then $U \cap \operatorname{cl} \bigcup_{n\in\omega} F_n \neq \emptyset$, thus, $U \cap \bigcup_{n\in\omega} F_n \neq \emptyset$. So there exists N such that $U \cap F_N \neq \emptyset$. It is clear that for all $n \geq N$, $F_n \in U^-$. □

Second, we do know how to get accumulation points for a union of two sequences if at least one of them converges.

Lemma 3.5.3. Let X be a T_1 topological space. Let $F = (F(n) : n \in \omega)$, $K = (K(n) : n \in \omega)$ be two sequences in $\exp(X)$. Suppose that A is a limit of the sequence F and that B is an accumulation point of K. Let $C = F \cup K$ be the sequence given by $F(n) \cup K(n)$ for every $n \in \omega$. Then $A \cup B$ is an accumulation point of C.

Proof. Of course, $A \cup B \in \exp(X)$. Let $\langle V_0, \dots, V_i \rangle$ be a basic open neighborhood of $A \cup B$. WLOG there exists $j \leq i$ such that $A \in \langle V_0, \dots, V_j \rangle$ and such that $A \cap V_l = \emptyset$ for all $l > j$ such that $l \leq i$. There exists $N \in \omega$ such that $F(n) \in \langle V_0, \dots, V_j \rangle$ for every $n \geq N$.

We have that $B \in \langle V_{j+1}, \dots, V_i \rangle$, thus, $I = \{n \in \omega : K(n) \in \langle X, V_{j+1}, \dots, V_i \rangle\}$ is infinite. Now, if $n \in I \setminus N$, then $F(n) \cup K(n) \in \langle V_0, \dots, V_i \rangle$. □

Thus, it would be useful to split a sequence into a growing part and "something else". We do have something like that for sequences of finite sets. This is possibly folklore, but we could not find a reference anywhere, so we proved it in our paper [59, Lemma 4.4].

Definition 3.5.4 (*). Suppose C is a sequence of finite sets. Given $X \in [\omega]^\omega$, a *nice split* of C over X is a pair (U, D) such that $U|X$ is increasing, $D|X$ is pairwise disjoint and, for every $n \in X$, $U(n) \cap D(n) = \emptyset$ and $C(n) = U(n) \cup D(n)$. □

Lemma 3.5.5 (*). For every sequence of finite sets C and every $Y \in [\omega]^\omega$, there exists $X \in [Y]^\omega$ such that C admits a nice split over X.

Proof. Let Y and C be given. Recursively, we choose $x_n \in Y$ and a decreasing sequence $J_n \in [Y]^\omega$ such that:

1. $J_{n+1} \cap x_{n+1} = \emptyset$,
2. $x_0 \in Y$, $x_{n+1} \in J_n$ for each $n \in \omega$, and
3. For all $t \in C(x_n)$, either $\forall j \in J_n \, t \in C(j)$ or $\forall j \in J_n \, t \notin C(j)$.

This is possible since each $C(n)$ is finite. Let $X = \{x_n : n \in \omega\}$ and, for each $n \in \omega$, let $U(x_n) = \{t \in C(x_n) : \forall j \in J_n(t \in C(j))\}$ and $D(x_n) = \{t \in C(x_n) : \forall j \in J_n(t \notin C(j))\}$ □

So, knowing more about accumulation points of pairwise disjoint sequences of finite sets is needed. For that sake, in this chapter we will adopt the following definition:

Definition 3.5.6 (*). Let X be any set. A *block sequence* on X is a sequence $F : \omega \to [X]^{<\omega} \setminus \{\emptyset\}$ of pairwise disjoint sets.

We say that a block sequence F on X is *increasing* iff for every n, $|F(n)| \leq |F(n+1)|$ and $\{|F(n)| : n \in \omega\}$ is unbounded. □

Summing up, we get:

Proposition 3.5.7 (*). Let X be a T_1 topological space and let $D \subseteq X$ be dense. Suppose that all block sequences of D have an accumulation point in $\exp(X)$. Then X is feebly compact.

Proof. Since $[D]^{<\omega} \setminus \{\emptyset\}$ is dense, it follows from Proposition 0.3.20 that it suffices to show that every sequence $C : \omega \to [D]^{<\omega} \setminus \{\emptyset\}$ has an accumulation point in $\exp(X)$. By passing to a subsequence and by Lemma 3.5.5, it suffices to consider the case where there exists a pair of sequences (U, D) where for every n, $C(n) = U(n) \cup D(n)$ and $U(n) \cap D(n) = \emptyset$ where U is increasing and D is pairwise disjoint. If all $U(n)$'s are empty, then $C = D$ has an accumulation point by hypothesis. If some $U(n)$ is nonempty then cofinally many is nonempty, so we may now assume that all $U(n)$'s are nonempty by passing to a subsequence. If only finitely many $D(n)$'s are nonempty, so by passing to a subsequence we get $C = U$, which converges. Now if infinitely many D's are nonempty the conclusion follows from the previous lemma by passing to a subsequence. □

Now we start studying accumulation points of block sequences. The following lemma gives us equivalent conditions for a block sequence having an ultrafilter limit in the hyperspace of a subspace of $\beta\kappa$ containing κ.

Lemma 3.5.8 (*). Suppose $\kappa \subseteq X \subseteq \beta\kappa$ and let $\mathcal{U}$ be a free ultrafilter over ω. Let F be a block sequence on κ. Let $\mathcal{G} = \prod_{n\in\omega} F(n)$ and $Z_{\mathcal{U}} = Z = \{\mathcal{U}\text{-}\lim g : g \in \mathcal{G}\}$.

Then the following are equivalent:

1) F has a $\mathcal{U}$-limit in $\exp(X)$,

2) $Z \subseteq X$ and $\mathcal{U}$-$\lim F = \mathrm{cl}_X Z$, and

3) $Z \subseteq X$.

Moreover, Z discrete, and, if F is increasing, then $|Z| = \mathfrak{c}$.

Proof. Z is well-defined since $\beta\omega$ is compact (see Proposition 0.4.18). We start by the last claim. So suppose F is increasing.

By using a tree argument, it is not hard to construct a collection $\{g_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathcal{G}$ such that if for all α, β, if $\alpha < \beta < \mathfrak{c}$ then $g_\alpha[\omega] \cap g_\beta[\omega]$ is finite. Thus $\{\mathrm{cl}\, g_\alpha[\omega]^* : \alpha < \mathfrak{c}\}$ is a pairwise disjoint family of closed sets. Clearly, $\mathcal{U}$-$\lim g_\alpha \in g_\alpha[\omega]^*$ for every $\alpha < \mathfrak{c}$, so we are done.

Now we show that Z is a discrete subspace of $\beta\kappa$. Let $z \in Z$. Fix $g \in \mathcal{G}$ such that $\mathcal{U}$-$\lim g = z$.

Then z belongs to the clopen set $W = \mathrm{cl}_{\beta\kappa}\{g(n) : n \in \omega\} \subseteq \beta\kappa$. We claim that $Z \cap W = \{z\}$. For suppose $h \in \mathcal{G}$. Observe since the sets $\{n \in \omega : g(n) = h(n)\}$ and $\{n \in \omega : g(n) \neq h(n)\}$ are complementary, one of them is in $\mathcal{U}$. If the former is in $\mathcal{U}$, then $\mathcal{U}$-$\lim h = z$. If the latter is in $\mathcal{U}$, let $A = \{n \in \omega : g(n) \neq h(n)\}$. Since the elements of the range of F are pairwise disjoint, it follows that $\{g(n) : n \in \omega\} \cap \{h(n) : n \in A\} = \emptyset$, so $\mathrm{cl}\{g(n) : n \in \omega\} \cap \mathrm{cl}\{h(n) : n \in A\} = \emptyset$. But $\mathcal{U}$-$\lim h \in \mathrm{cl}\{h(n) : n \in A\}$, so $\mathcal{U}$-$\lim h \notin \mathrm{cl}\{g(n) : n \in \omega\} = W$.

1) implies 2): Suppose that the $\mathcal{U}$-limit of the sequence F exists in $\exp(X)$ and call it K.

First, we show that $K \subseteq \mathrm{cl}_{\beta\kappa}(Z)$. Fix $x \in K$. Let U be a basic open neighborhood of x, so $U = \mathrm{cl}\, A$ for some $A \in x$. Notice that $K \in \{K' \in \exp(X) : K' \cap (U \cap X) \neq \emptyset\}$. Since $K = \mathcal{U}$-$\lim F$, we have that $B = \{n \in \omega : F(n) \cap A \neq \emptyset\} \in \mathcal{U}$. Let $g \in \mathcal{G}$ be such that $g(n) \in A \cap F(n)$ for every $n \in B$. Thus, $\mathcal{U}$-$\lim g \in (\mathrm{cl}_{\beta\kappa} A) \cap Z = U \cap Z$, so $U \cap Z \neq \emptyset$, as intended.

Now we show that $Z \subseteq K$ (which implies that $Z \subseteq X$). Suppose by contradiction that there exists $z \in Z \setminus K$, and fix z. Fix $g \in \mathcal{G}$ such that $\mathcal{U}$-$\lim g = z$. Since Z is discrete, there exists a basic clopen neighborhood $U = \mathrm{cl}_{\beta\kappa} A$ (for some $A \subseteq \omega$ with $A \in z$) such that $U \cap Z = \{z\}$. This implies that $\{n \in \omega : F(n) \subseteq \beta\kappa \setminus U\} \subseteq \{n \in \omega : g(n) \notin U\} \notin \mathcal{U}$, but this is a contradiction since $K = K \setminus \{z\} \subseteq \mathrm{cl}_{\beta\kappa}(Z) \setminus \{z\} \subseteq \mathrm{cl}_{\beta\omega}(Z \setminus \{z\}) \subseteq \beta\kappa \setminus U$.

Thus, $K \subseteq (\mathrm{cl}_{\beta\kappa} Z) \cap X = \mathrm{cl}_X Z$, and $Z \subseteq K$, which implies that $\mathrm{cl}_X(Z)$ since K is closed in X.

2) implies 3) is trivial.

3) implies 1): Let $K = \mathrm{cl}_X(Z)$. We claim that $Z = \mathcal{U}\text{-}\lim F$. Let $U = \mathrm{cl}_{\beta\kappa} A$ be a basic clopen set of $\beta\kappa$, where $A \subseteq \kappa$. We must see that if $K \subseteq U$, then $\{n \in \omega : F(n) \subseteq A\} \in \mathcal{U}$ and that if $K \cap U \neq \emptyset$, then $\{n \in \omega : F(n) \cap A \neq \emptyset\} \in \mathcal{U}$.

First, suppose $K \cap U \neq \emptyset$. Then $K \cap (U \cap X) \neq \emptyset$, so $Z \cap (U \cap X) = Z \cap U \neq \emptyset$. Let $z \in Z \cap U$. Let $g \in \mathcal{G}$ be such that $\mathcal{U}\text{-}\lim g = z$. Then $\{n \in \omega : g(n) \in A\} \in \mathcal{U}$, which implies that $\{n \in \omega : F(n) \cap A \neq \emptyset\} \in \mathcal{U}$.

Now suppose that $K \subseteq U$. Suppose by contradiction that $B = \{n \in \omega : F(n) \nsubseteq A\} \in \mathcal{U}$. Let $g \in \mathcal{G}$ be such that for each $n \in B$, $g(n) \notin A$ (equivalently, $g(n) \notin U$). Let $z = \mathcal{U}\text{-}\lim g$. Then $z \in Z \setminus U$, so $K \nsubseteq U$, a contradiction. $\square$

3.6 Subspaces of $\beta\omega$ whose hyperspace is not pseudocompact

In this section we improve Proposition 3.5.1 by providing spaces with the same properties plus having many powers being countably compact.

Theorem 3.6.1 (*). Let μ and λ be two uncountable cardinals such that:

a) $\omega_1 \leq \mu \leq \mathfrak{c} \leq \lambda \leq 2^{\mathfrak{c}}$, $\lambda^{<\mu} = \lambda$ and $\mathrm{cf}(\lambda) \geq \mu$, and

b) for every infinite cardinal $\kappa < \mu$ and every $Y \subseteq [\omega^*]^{<\lambda}$, $(\beta\omega \setminus Y)^\kappa$ is countably compact.

Then there exists $X \subseteq \beta\omega$ such that $\omega \subseteq X$, X^κ is countably compact for every $\kappa < \mu$ and $\exp(X)$ is not pseudocompact.

Proof. Let $C = (C_n : n \in \omega)$ be a increasing block sequence on ω. Let $\mathcal{G} = \prod_{n\in\omega} C_n$. Enumerate $\mathcal{G} = \{g_\alpha : \alpha < \mathfrak{c}\}$. For every $\alpha < \mathfrak{c}$ let $\mathcal{G}_\alpha = \{g_\beta : \beta < \alpha\}$.

Let $\mathcal{F} = \{(\lambda^\kappa)^\omega : \kappa \text{ is an infinite cardinal and } \kappa < \mu\}$. Enumerate $\mathcal{F} = \{f_\alpha : \alpha < \lambda\}$. This is possible since $\lambda^{<\mu} = \lambda$. We may suppose that for each $f \in \mathcal{F}$, $\{\alpha < \lambda : f_\alpha = f\}| = \lambda$. For each $\alpha < \lambda$, let $\kappa_\alpha = \mathrm{dom}\, f_\alpha(n)$ (of course, this does not depend on n) and $\zeta_\alpha = \sup\{f_\alpha(n)(\beta) + 1 : n \in \omega, \beta < \kappa_\alpha\}$. Notice that $\zeta_\alpha < \lambda$ since $\mathrm{cf}(\lambda) \geq \mu > \kappa_\alpha, \omega$.

Recursively, we will define, for each $\alpha < \lambda$, ordinals δ_α and ϵ_α, sets $X_\alpha = \{x_\xi : \delta_\alpha \leq \xi < \epsilon_\alpha\} \subseteq \beta\omega$, $Y_\alpha \subseteq \beta\omega$, sequences $\hat{f}_\alpha : \omega \to \left(\bigcup_{\beta<\alpha} X_\beta\right)^{\kappa_\alpha}$ and collections of ultrafilters P_α satisfying:

1. $\delta_0 = 0$, $\epsilon_0 = \omega$ and $x_n = n$ for every $n < \omega$,

2. $\delta_\alpha = \sup\{\epsilon_\beta : \beta < \alpha\}$ for each $\alpha < \lambda$,
3. The family $(\epsilon_\alpha : \alpha < \lambda)$ is strictly increasing sequence of ordinals $< \lambda$,
4. $0 < |\epsilon_\alpha \setminus \delta_\alpha| \leq \kappa_\alpha$ for each $\alpha < \lambda$,
5. $x_\xi \neq x_{\xi'}$ whenever $\xi \neq \xi'$,
6. $\left(\bigcup_{\beta\leq\alpha} X_\alpha\right) \cap \left(\bigcup_{\beta\leq\alpha} Y_\alpha\right) = \emptyset$ for every $\alpha < \lambda$,
7. if $\zeta_\alpha < \sup\{\epsilon_\beta : \beta < \alpha\}$, then $\hat{f}_\alpha(n)(\xi) = x_{f_\alpha(n)(\xi)}$ for each $\xi < \kappa_\alpha$, $n < \omega$ and $\alpha < \lambda$,
8. $\hat{f}_\alpha$ has an accumulation point in $\left(\bigcup_{\beta\leq\alpha} X_\beta\right)^{\kappa_\alpha}$.

Also, if $\lambda = \mathfrak{c}$:

9. $|Y_\alpha| \leq |\alpha|.|\epsilon_\alpha|$ for every $\alpha < \mathfrak{c}$,
10. $P_\alpha = \{\mathcal{U} \in \omega^* : \exists g \in \mathcal{G}_\alpha \mathcal{U}\text{-}\lim g \in X_\alpha)\}$ for every $\alpha < \mathfrak{c}$, and
11. $\forall\alpha < \mathfrak{c}\,\forall\mathcal{U} \in P_\alpha\,\exists y \in Y_\alpha \exists h \in \mathcal{G}\,\mathcal{U}\text{-}\lim h = y$.

And, if $\mathfrak{c} < \lambda \leq 2^{\mathfrak{c}}$:

9'. $|Y_\alpha| \leq \mathfrak{c}$ for every $\alpha < \lambda$,

10'. $P_\alpha = \{\mathcal{U} \in \omega^* : \mathcal{U} \notin \bigcup_{\beta<\alpha} P_\beta \,\text{and}\, \exists g \in \mathcal{G}\,\mathcal{U}\text{-}\lim g \in X_\alpha)\}$ for every $\alpha < \mathfrak{c}$, and

11'. $\forall\alpha < \lambda\,\forall\mathcal{U} \in P_\alpha\,\exists y \in Y_\alpha\,\exists h \in \mathcal{G}\,(\mathcal{U}\text{-}\lim h = y)$.

Clearly, by setting $\delta_0 = 0$, $\epsilon_0 = \omega$, $x_n = n$ for each $n < \omega$, $X_0 = \omega$, $Y_0 = \emptyset$, $P_0 = \emptyset$ and $\hat{f}_0$ be a constantly equal to 0, then all the preceding clauses hold for the basis of the induction.

Suppose we have defined ϵ_β, δ_β, X_β, Y_β, $(x_\gamma : \gamma < \beta)$, Y_β and P_β for every $\beta < \alpha$ for some α such that $0 < \alpha < \lambda$. We show how to define ϵ_α, δ_α, X_α, Y_α, $(x_\gamma : \gamma < \alpha)$, Y_α and P_α.

Let δ_α be defined by 2., and $\hat{f}_\alpha$ be defined by 7. (if $\zeta_\alpha \geq \sup\{\epsilon_\beta : \beta < \alpha\}$, let $\hat{f}_\alpha$ be the constant sequence equal to $(0)_{\alpha<\kappa_\alpha}$. Notice that $\bigcup_{\beta<\alpha}[\delta_\beta, \epsilon_\beta) = \sup\{\epsilon_\beta : \beta < \alpha\}$ (by 1. to 4. for $\beta < \alpha$). Now we verify that this supremum is smaller than λ. First, notice that if $\mu < \lambda$, then $\left|\bigcup_{\beta<\alpha}[\delta_\beta, \epsilon_\beta)\right| \leq \sum_{beta<\alpha} \kappa_\beta \leq \mu.|\alpha| < \lambda$. Second, if $\mu = \lambda$, then $\mu = \lambda = \mathfrak{c}$, thus, $\mathfrak{c} \geq \mathrm{cf}\,\mathfrak{c} = \mathrm{cf}\,\lambda \geq \mu = \mathfrak{c}$, so $\mathfrak{c}(= \lambda = \mu)$ is regular, which implies that $\sup\{\epsilon_\beta : \beta < \alpha\} < \lambda = \mu = \mathfrak{c}$.

In any case, $\left|\bigcup_{\beta<\alpha} X_\beta\right| = |\sup_{\beta<\alpha} \epsilon_\beta| < \lambda$. Now we aim to show that $\bigcup_{\beta<\alpha} Y_\beta$ also has size less than λ. If $\lambda = \mathfrak{c}$, it follows from 9. that $\left|\bigcup_{\beta<\alpha} Y_\beta\right|| \leq \sum_{\beta<\alpha} |\beta|.|\epsilon_\beta| \leq |\alpha|.|\sup\{\epsilon_\beta : \beta < \alpha\}| < \lambda$. If $\mathfrak{c} < \lambda$< it follows from 9'. that this union has size $\leq \mathfrak{c}.|\alpha| < \lambda$.

Case 1: $\zeta_\alpha \geq \sup\{\epsilon_\beta : \beta < \alpha\}$ or $\hat{f}_\alpha$ already has an accumulation point in $\left(\bigcup_{\beta<\alpha} X_\beta\right)^{\kappa_\alpha}$. In this case, let $\epsilon_\alpha = \delta_\alpha + 1$ and $x_{\delta_\alpha} \in \beta\omega \setminus \bigcup_{\beta<\alpha} X_\beta \cup Y_\beta$. It is clear then that all items up to 8. hold. We will define Y_α and verify the other items later.
Case 2: not Case 1. Temporally write $Y = \bigcup_{\beta<\alpha} Y_\beta$. We have seen that $|Y| < \lambda$, so, by item

6. and b) from the hypothesis, $\hat{f}_\alpha$ has an accumulation point $(u_\delta)_{\delta\in\kappa_\alpha} \in (\beta\omega \setminus Y)^{\kappa_\alpha}$. Let $X_\alpha = \{u_\delta : \delta < \kappa_\alpha\} \setminus \bigcup_{\beta<\alpha} X_\beta$. Since $\hat{f}_\alpha$ has no accumulation point in $\bigcup_{\beta<\alpha} X_\beta$, it follows that X_α is nonempty. Let $\epsilon_\alpha = \delta_\alpha + |X_\alpha|$ (ordinal sum) and write $X_\alpha = \{x_\xi : \delta_\alpha \leq \xi < \epsilon_\alpha\}$. It is clear then that all items up to 8. hold.

If $\lambda = \mathfrak{c}$, P_α is given by 10., so $|P_\alpha| \leq \left|\bigcup_{g\in\mathcal{G}_\alpha}\{\mathcal{U} \in \omega^* : \mathcal{U}\text{-}\lim g \in X_\alpha\right| \leq |\alpha|.|X_\alpha| = |\alpha|.|\epsilon_\alpha| < \mathfrak{c}$ (this happens since if $\mathcal{U}, \mathcal{V} \in \omega^*$ and $g : \omega \to \omega$ is injective, then $\mathcal{U}$-$\lim g \neq \mathcal{V}$-$\lim g$ since if $A \in \mathcal{U} \setminus \mathcal{V}$, then the first is in $\operatorname{cl} g[A]$ and the latter is $\operatorname{cl} g[\omega \setminus A]$, which are disjoint). By Lemma 3.5.8, the set Z as in the statement of the lemma has size $\mathfrak{c}$, so there exists $h_\mathcal{U} \in \mathcal{G}$ so that $\mathcal{U} - \lim h_\mathcal{U} \notin \bigcup_{\beta<\alpha} X_\beta$. Let $Y_\alpha = \{\mathcal{U} - \lim h_\mathcal{U} : \mathcal{U} \in P_\alpha\}$ and notice that 9.-11. hold.

Now we show that the sequence C has no accumulation point on $\exp(X)$, which completes the proof by Proposition 0.3.20. If such a set exists, by Lemma 3.5.8 the set $Z = \{\mathcal{U}\text{-}\lim g : g \in \mathcal{G}\}$ is contained in X Let α be the first ordinal such that $Z \cap X_\alpha \neq \emptyset$.

If $\lambda = \mathfrak{c}$, there exists $\beta < \lambda = \mathfrak{c}$ such that $\mathcal{U}$-$\lim g_\beta \in Z \cap X_\alpha$. Let $\gamma = \max\{\alpha, \beta\} + 1$. By 10., $\mathcal{U} \in P_\gamma$. So $Z \cap Y_\gamma \neq \emptyset$, which violates the fact that $Z \subseteq X$.

If $\mathfrak{c} < \lambda$, there exists $g \in \mathcal{G}$ such that $\mathcal{U}$-$\lim g \in Z \cap X_\alpha$. By 10', $\mathcal{U} \in P_\alpha$. So $Z \cap Y_\gamma \neq \emptyset$, which violates the fact that $Z \subseteq X$. □

The following very similar lemmas are used when applying the previous theorem.

Lemma 3.6.2 (*). Let $Y \subseteq \omega^*$ be such that $|Y| < \mathfrak{c}$. Let $\kappa < n(\omega^*)$. Then $(\beta\omega \setminus Y)^\kappa$ is countably compact.

Proof. Let Y and κ be given. Let $f : \omega \to (\beta\omega \setminus Y)^\kappa$ be given.

For each $\alpha < \kappa$, let $f_\alpha = \pi_\alpha \circ f : \omega \to \beta\omega \setminus Y$ and $D_\alpha = \{A \in [\omega]^\omega : f_\alpha|A \text{ is constant or injective }, f_\alpha[A] \text{ is discrete } and Y \cap \operatorname{cl} f_\alpha[A] = \emptyset\}$.

D_α is dense: given $B \in [\omega]^\omega$, there exists $A_0 \in [B]^\omega$ such that $f_\alpha|B_0$ is either constant or injective. There exists $B_1 \in [B_0]^\omega$ such that $f[B_1]$ is discrete (this is obvious if $f_\alpha|B_0$ is constant, and, if $f_\alpha|B_0$ is injective, this follows from the fact that every infinite Hausdorff space has an infinite discrete subspace).

Now recall that if Z is an infinite countable discrete subspace of $\beta\omega$, then $\operatorname{cl} Z$ is homeomorphic to βA. Thus, if Z_0, Z_1 are subsets of Z, $\operatorname{cl} Z_0 \cap \operatorname{cl} Z_1 = \operatorname{cl}(Z_0 \cap Z_1)$.

Let $\mathcal{A}$ be an almost disjoint family of size $\mathfrak{c}$ on B_1. If f is injective and $f[B_1]$ is discrete and disjoint from Y, each element of Y is in $\operatorname{cl} f[A]$ for at most one $A \in \mathcal{A}$. So there is an $A \in \mathcal{A}$ such that $\operatorname{cl} f[A] \cap Y = \emptyset$. Thus, $A \in D_\alpha$.

Now let $\mathcal{U}$ be a free ultrafilter such that for every $\alpha < \kappa, \mathcal{U} \cap D_\alpha \neq \emptyset$ (by $\mathrm{FA}_{[\omega]^\omega}(\kappa)$). Since $\beta\omega^\kappa$ is compact, there exist a $\mathcal{U}$-limit $(x_\alpha)_{\alpha<\kappa}$. We must show that for each α, $x_\alpha \notin Y$. But this is true since $x_\alpha = \mathcal{U}$-$\lim f_\alpha \in \operatorname{cl} f[A]$, where $A \in D_\alpha \cap \mathcal{U}$. □

Lemma 3.6.3 (*). Let $Y \subseteq \omega^*$ be such that $|Y| < 2^\mathfrak{c}$. Let $\kappa < \mathfrak{h}$. Then $(\beta\omega \setminus Y)^\kappa$ is countably compact.

Proof. Let Y and κ be given. Let $f : \omega \to (\beta\omega \setminus Y)^\kappa$ be given.

As in the previous lemma, for each $\alpha < \kappa$, let $f_\alpha = \pi_\alpha \circ f : \omega \to \beta\omega \setminus Y$ and $D_\alpha = \{A \in [\omega]^\omega : \exists F \in [\omega]^{<\omega} f_\alpha|(A \setminus F)$ is constant or injective , $f_\alpha[A \setminus F]$ is discrete $and Y \cap \operatorname{cl} f_\alpha[A \setminus F] = \emptyset\}$. As before, D_α is dense and open. Thus, let $A \in \bigcap_{\alpha<\mathfrak{h}} D_\alpha$.

For each α, let $B_\alpha = \{\mathcal{U} \in \omega^* : \mathcal{U}\text{-lim } f_\alpha \in Y \text{ and } A \in \mathcal{U}\}$. B_α has cardinality at most $|Y|$. Thus, $\bigcup_{\alpha<\kappa} B)\alpha$ has cardinality less than $2^{<\mathfrak{c}}$, so there exists $\mathcal{U}$ such that $A \in \mathcal{U}$ and for every $\alpha < \mathfrak{c}, \mathcal{U}\text{-lim } f_\alpha \notin Y$. This shows that $\mathcal{U}\text{-lim } f \in (\beta\omega \setminus Y)^\kappa$. □

We now can prove concrete results which improve Proposition 3.5.1, such as:

Theorem 3.6.4 (*)**.** We have the following examples:

1. Let $\lambda = 2^\theta$ for some cardinal θ such that be such that $\mathfrak{h} \leq \theta \leq \mathfrak{c}$. There exists a subspace X of $\beta\omega$ containing ω such that $|X| = \lambda$, X^κ is countably compact for every $\kappa < \mathfrak{h}$ and $\exp(X)$ is not pseudocompact.

2. Suppose $2^\mathfrak{p} = \mathfrak{c}$. Then there exists a subspace X of $\beta\omega$ containing ω such that $|X| = \mathfrak{c}$, $X^\mathfrak{p}$ is countably compact and $\exp(X)$ is not pseudocompact.

Proof. For the first example we apply Theorem 3.6.1 using $\mu = \mathfrak{h}$ and λ. It is clear that $\omega_1 \leq \mu \leq \mathfrak{c} \leq \lambda \leq 2^\mathfrak{c}$. Moreover if $\kappa < \mathfrak{h}$, $\lambda^\kappa = 2^{\theta.\kappa} = 2^\theta = \lambda$ and $\operatorname{cf}(\lambda) \geq \theta \geq \mathfrak{h}$, so a) holds. b) follows from 3.6.3.

For the second example we apply Theorem 3.6.1 using $\mu = \mathfrak{p}^+$ and $\lambda = \mathfrak{c}$. It is clear that $\omega_1 \leq \mu \leq \mathfrak{c} \leq \lambda \leq 2^\mathfrak{c}$, and $\mathfrak{c}^\mathfrak{p} = 2^\mathfrak{p} = \mathfrak{c}$ by hypothesis, which also implies that $\operatorname{cf}(\mathfrak{c}) = \operatorname{cf}(2^\mathfrak{p}) \geq \mathfrak{p}^+$ by König's Lemma, so a) holds. b) follows from 3.6.2. □

Corollary 3.6.5 (*)**.** There exists a $T_{3\frac{1}{2}}$ topological space X such that X^ω is countably compact and $\exp(X)$ is not pseudocompact. Thus, the countable compactness of X^ω does not imply the pseudocompactness of $\exp(X)$.

3.7 A condition which implies the pseudocompactness of $\exp(X)$

As we already discussed in the previous section, we already knew, due to Proposition 3.5.1, that the pseudocompactness of X^ω does not imply the pseudocompactness of $\exp(X)$, not even for Tychonoff spaces. Now we know that the countable compactness of X^ω does not imply the pseudocompactness of $\exp(X)$ as well, by Corollary 3.6.5, which we have proved in [59]. In this section we prove a sufficient condition for the pseudocompactness of $\exp(X)$ which applies to subspaces of $\beta\omega$ containing ω. This condition involves the use of the concept of (κ, A)-pseudocompactness. For more about that, see Chapter 0.

Theorem 3.7.1 (*)**.** Suppose X is T_1. Let $D \subseteq X$ be a dense subset of X. If $D^\mathfrak{c}$ is relatively countably compact in $X^\mathfrak{c}$, then $[D]^{<\omega} \setminus \{\emptyset\}$ is relatively countably compact in $\exp(X)$. Thus, $\exp(X)$ is feebly compact.

Proof. For the last sentence, just notice that $[D]^{<\omega} \setminus \{\emptyset\}$ is a dense subset of $\exp(X)$ and apply Proposition 0.3.20.

Now let $f : \omega \to [D]^{<\omega} \setminus \{\emptyset\}$ be a sequence. Let $\mathcal{G} = \prod_{n\in\omega} f(n)$. Then $|G| \leq \mathfrak{c}$. Write $\mathcal{G} = \{g_\alpha : \alpha < \mathfrak{c}\}$. Let $g : \omega \to D^{\mathfrak{c}}$ be given by $g(n)(\alpha) = g_\alpha(n)$, so $\pi_\alpha \circ g = g_\alpha$. Since $D^{\mathfrak{c}}$ is relatively countably compact in $X^{\mathfrak{c}}$, g has an accumulation point $x = (x_\alpha)_{\alpha<\mathfrak{c}}$. Let $\mathcal{U}$ be a free ultrafilter such that x is a $\mathcal{U}$-limit point of g.

Let $K = \operatorname{cl}\{x_\alpha : \alpha < \mathfrak{c}\}$. We claim that K is a $\mathcal{U}$-limit of f, which completes the proof. Let U be a subbasic open neighborhood of F, so U is of the form W^+, where $K \subseteq W$ and W is open, or U is of the form W^-, where $W \cap F \neq \emptyset$ and W is open.

First, assume that $K \in W^-$. So $\operatorname{cl}\{x_\alpha : \alpha < \mathfrak{c}\} \cap W \neq \emptyset$. Since W is open this implies that there exists $\alpha < \mathfrak{c}$ such that $x_\alpha \in W$. Thus, $\{n \in \omega : g_\alpha(n) \in W\} \in \mathcal{U}$, and this set is contained in $\{n \in \omega : f(n) \cap W \neq \emptyset\} = \{n \in \omega : f(n) \in W^-\}$, so the latter is in $\mathcal{U}$ too.

Now assume that $K \in W^+$. Suppose by contradiction that $\{n \in \omega : f(n) \notin W^+\} \notin \mathcal{U}$. So its complement $A = \{n \in \omega : f(n) \setminus W \neq \emptyset\}$ is in $\mathcal{U}$. Let $h \in \mathcal{G}$ be such that $h(n) \in f(n) \setminus W$ for every $n \in A$. $h = g_\alpha$ for some $\alpha < \mathfrak{c}$, so $x_\alpha \in X \setminus W$, contradicting the fact that $K \subseteq W$. $\square$

We note the following, which I believe to be an original result.

Lemma 3.7.2 (*). Let X be a topological space. Let $D \subseteq X$. Let κ be a cardinal. Then:

1. If D is dense and D^κ is relatively countably compact in X^κ, then X is (κ, ω^*)-pseudocompact.
2. If D is open and discrete and X is (κ, ω^*)-pseudocompact, then D^κ is relatively countably compact in X^κ.

Proof. 1. D^κ is relatively countably compact in X^κ. Let $(U_\alpha : \alpha < \kappa)$ be a family such that for each $\alpha < \kappa$, U_α is a sequence of nonempty open subsets of X. For each α, let $d_\alpha : \omega \to D$ be such that $d_\alpha(n) \in U_\alpha(n)$ for every $n \in \omega$. Let $d : \omega \to D^\kappa$ be given by $d(n)(\alpha) = d_\alpha(n)$, so $\pi_\alpha \circ d = d_\alpha$. Since D^κ is relatively countably compact in X^κ, there exists an ω-accumulation point $(x_\alpha)_{\alpha<\kappa} \in X^\kappa$ for d. Let $\mathcal{U}$ be a free ultrafilter such that x is an $\mathcal{U}$-limit of d. We claim that x_α is an $\mathcal{U}$-accumulation point of U_α for each $\alpha < \kappa$. Fix α. Given an open neighborhood V of x_α, let $W = \pi_\alpha^{-1}[W]$. W is an open neighborhood of x, thus $\{n \in \omega : d(n) \in W\} = \{n \in \omega : d_\alpha(n) \in V\} \in \mathcal{U}$, but this set is contained in $\{n \in \omega : U_\alpha(n) \cap V \neq \emptyset\}$, so the latter is also in $\mathcal{U}$.

2. Let $d : \omega \to D^\kappa$ be a sequence. For each $\alpha < \kappa$< let $d_\alpha = \pi_\alpha \circ d$. Let $U_\alpha = (\{d_\alpha(n)\} : n \in \omega)$ for each $\alpha < \kappa$. Then each U_α is a sequence of nonempty open sets. Let $\mathcal{U}$ be a free ultrafilter such that for every α, U_α has some $\mathcal{U}$-accumulation point x_α. We claim that $x = (x_\alpha : \alpha < \kappa)$ is a $\mathcal{U}$-limit point of d. It suffices to see that for each α, x_α is an $\mathcal{U}$-limit point of d_α. Fix α and an open neighborhood V of x_α. We must see that $\{n \in \omega : d_\alpha(n) \in V\} \in \mathcal{U}$. But this set is precisely $\{n \in \omega : U_n(\alpha) \cap V \neq \emptyset\}$, which is in $\mathcal{U}$.

$\square$

Thus, we obtain:

Corollary 3.7.3 (*)**.** Let X be a subspace of $\beta\omega$ containing ω and κ be a cardinal. The following are equivalent:

a) ω^κ is relatively countably compact in X^κ, and

b) X is (κ, ω^*)-pseudocompact

Moreover, if these conditions hold for $\kappa = \mathfrak{c}$, then $[\omega]^{<\omega} \setminus \{\emptyset\}$ is relatively countably compact in $\exp(X)$, and $\exp(X)$ is pseudocompact.

Proof. This follows from Theorem 3.7.1 and from Lemma 3.7.2. □

It is consistent that, in some sense, Corollary 3.7.3 is the best result we can have. If $\mathfrak{h} = \mathfrak{c}$, Theorem 3.6.4 says we have an space X between ω and $\beta\omega$ which is (κ, ω^*)-pseudocompact for every $\kappa < \mathfrak{c}$ but its hyperspace is not (in fact, X^κ is countably compact for every $\kappa < \mathfrak{c}$). We can prove the same result by using selective ultrafilters. First we need a lemma.

The following lemmas, which we state without proof in [59] as Lemma 3.4. and Corollary 3.5., were probably already well known. We prove it here for the sake of completeness.

Lemma 3.7.4. Let $h : \omega \to \omega, \mathcal{U}$ be an ultrafilter. Then, computing limits within $\beta\omega$, it follows that $\mathcal{U} - \lim h$=$h_*[\mathcal{U}] = \{A \subseteq \omega : h^{-1}[A] \in \mathcal{U}\}$, and that $\mathcal{U} - \lim h \in \omega$ iff $h|B$ is constant for some $B \in \mathcal{U}$.

Proof. For the first part: let A^* be a open neighborhood of $\mathcal{V} = \mathcal{U}\text{-}\lim h$. So $A \in \mathcal{V}$, therefore $h^{-1}[A] \in \mathcal{U}$. Denote by $[n]$ the fixed ultrafilter generated by $\{n\}$. We must see that $\{n \in \omega : [h(n)] \in A^*\} \in \mathcal{U}$, but this set is $\{n \in \omega : A \in [h(n)]\} = \{n \in \omega : h(n) \in A\} = h^{-1}[A]$.

For the second part: if $\mathcal{V} \in \omega$, there exists n such that $\{n\} \in \mathcal{V} = h_*(\mathcal{U})$, so $B = h^{-1}[\{n\}] \in \mathcal{U}$. Conversely, if for some n, if there exists $B \in \mathcal{U}$ such that $h|B$ is constant to some n, then $h^{-1}[\{n\}] \in \mathcal{U}$, then $\{n\} \in \mathcal{V}$. □

Lemma 3.7.5. Let $h_0, h_1 : \omega \to \omega$. Let $\mathcal{U}_0, \mathcal{U}_1$ be two incomparable selective ultrafilters such that are not Rudin-Keisler equivalent and suppose that for each $i < 2$ and $n_0 \in \omega$, $\{n \in \omega : h(n) = n_0\} \notin \mathcal{U}_i$ (that is, h_i is not constant mod $\mathcal{U}_i$). Then $\mathcal{U}_0\text{-}\lim h_0 \neq \mathcal{U}_1\text{-}\lim h_1$.

Proof. Let $\mathcal{V}_i = (h_i)_*(\mathcal{U}_i) = \{A \subseteq \omega : (h_i)^{-1}[A] \in \mathcal{U}_i\}$. Since h_i is not constant in elements of $\mathcal{V}_i$ and by Lemma 0.9.4, $\mathcal{V}_i$ is a free ultrafilter. Moreover, $\mathcal{V}_i \leq_{\mathrm{RK}} \mathcal{U}_i$, so by the minimality of selective ultrafilters, $\mathcal{V}_i =_{\mathrm{RK}} \mathcal{U}_i$. So $\mathcal{V}_0 \neq \mathcal{V}_1$.

By the previous lemma, $\mathcal{V}_i$ is $\mathcal{U}_i$-$\lim h_i$ for each $i < 2$. □

Now we are ready to prove our result.

Theorem 3.7.6 (*). Assume that $\mathfrak{c}$ is regular and that there exists there exists $\mathfrak{c}$ pairwise RK-incomparable selective ultrafilters. Then there exists $X \subseteq \beta\omega$ containing ω and a set $A \subseteq \omega^*$ of selective ultrafilters with $|A| \leq \mathfrak{c}$ which is (κ, A)-pseudocompact for every $\kappa < \mathfrak{c}$ and $\exp(X)$ is not pseudocompact.

Proof. Enumerate ω^ω as $(f_\alpha : \alpha < \mathfrak{c})$. Let $C : \omega \to \omega$ any increasing block sequence and $\mathcal{G} = \prod_{n\in\omega} C(n)$. Fix $i : \mathfrak{c} \to \mathfrak{c} \times \mathfrak{c}$ onto such that for every $\mu < \mathfrak{c}$, if $i(\mu) = (\xi_\mu, \eta_\mu)$, then $\xi_\mu \leq \mu$. For each ultrafilter $\mathcal{W}$, let $Z_\mathcal{W} = \{\mathcal{W} - \lim h : h \in \mathcal{G}\}$.

Given an ultrafilter $\mathcal{U}$ and $\alpha < \mathfrak{c}$, let $S_\alpha(\mathcal{U}) = \{\mathcal{U}\text{-}\lim f_\beta : \beta \leq \alpha\}$ and $S(\mathcal{U}) = \{\mathcal{U}\text{-}\lim f : f \in \omega^\omega\}$.

Recursively, we will define, for $\alpha < \mathfrak{c}$, selective ultrafilters $\mathcal{U}_\alpha$, subsets X_α of $\beta\omega$, sets of free ultrafilters $P_\alpha = \{\mathcal{V}(\alpha, \mu) : \mu < \mathfrak{c}\}$ and $\mathcal{W}_\alpha \in \beta\omega$ such that:

1. $X_0 = \omega \cup S_0(\mathcal{U}_0)$, $X_\alpha = S_\alpha(\mathcal{U}_\alpha)$ for $\alpha \in [1, \mathfrak{c})$.
2. For every $\alpha < \mathfrak{c}$, $X_\alpha \cap \{\mathcal{W}_\xi : \xi < \mathfrak{c}\} = \emptyset$,
3. $\{\mathcal{W} \in \omega^* : Z_\mathcal{W} \cap X_\alpha = \emptyset\} \subseteq P_\alpha$, and
4. $\mathcal{W}_\alpha \in Z_{\mathcal{V}_{i(\alpha)}} \setminus \left(\bigcup_{\beta<\alpha} X_\beta\}\right)$.

Let $\mathcal{U}_0$ be an arbitrary selective ultrafilter and X_0 be given by 1.. Since $|X_0| = \omega$ and $\{\mathcal{W} \in \omega^* : Z_\mathcal{W} \cap X_0 = \emptyset\} = \bigcup_{g\in\mathcal{G}}\{\mathcal{W} \in \omega^* : \mathcal{W}\text{-}\lim g \in X_0\}$, the latter has cardinality $\leq \mathfrak{c}$. Therefore we can enumerate ultrafilters $\mathcal{P}_0 = \{\mathcal{V}_{0,\mu} : \mu < \mathfrak{c}\}$ containing this set. Let $\mathcal{W}_0 \in Z_{\mathcal{V}_{i(0)}} \setminus X_0$, as in 4..

Suppose X_β, $P_\beta = \{\mathcal{V}(\beta, \mu) : \mu < \mathfrak{c}\}, \mathcal{U}_\beta$ and $\mathcal{W}_\beta$ have been defined for all $\beta < \alpha$ for some $\alpha \in [1, \mathfrak{c})$.

By the previous two lemmas, $S(\mathcal{U}) \cap \omega^* \cap S(\mathcal{U}') = \emptyset$ whenever $\mathcal{U}, \mathcal{U}'$ are two RK-distinct selective ultrafilters. Thus, there exists a selective ultrafilter $\mathcal{U}$ such that $S(\mathcal{U}) \cap \{\mathcal{W}_\beta : \beta < \alpha\} = \emptyset$. Denote such $\mathcal{U}$ by $\mathcal{U}_\alpha$. X_α is defined as in 1. Let $P_\alpha \supseteq \{\mathcal{W} \in \omega^* : Z_\mathcal{W} \cap X_\alpha \neq \emptyset\}$ have size $\mathfrak{c}$, and enumerate it as $\{\mathcal{V}_{(\alpha,\mu)} : \mu < \mathfrak{c}\}$. Finally, choose $\mathcal{W}_\alpha$ as in 4. to end the construction.

Let $X = \bigcup X_n : n \in \omega$. It is clear that X is a subspace of $\beta\omega$ containing ω. Let $A = \{\mathcal{U}_\delta : \delta < \mathfrak{c}\}$.

X is (κ, A)-pseudocompact for every $\kappa < \mathfrak{c}$: Let $(U_\alpha : \alpha < \kappa)$ be a family of sequences of nonempty open subsets of X. For each α, let $f_\alpha : \omega \to X$ be such that for every $\alpha < \kappa$ and $n \in \omega$, $g_\alpha(n) \in U_\alpha(n)$. It suffices to see that there exists $\mathcal{U} \in A$ such that $\mathcal{U}\text{-}\lim g_\alpha \in X$ for every $\alpha < \kappa$. Since $\mathfrak{c}$ is regular, there exists $\delta < \mathfrak{c}$ such that $\{g_\alpha : \alpha < \kappa\} \subseteq \{f_\beta : \beta < \delta\}$. Thus, $\mathcal{U}_\delta\text{-}\lim g_\beta \in X_\delta$ for every $\beta < \delta$.

$\exp(X)$ is not pseudocompact: we show that C has no accumulation point. Suppose it has. Then there exists a free ultrafilter $\mathcal{W}$ such that C has a $\mathcal{W}$-limit in $\exp(X)$. This implies that $Z_\mathcal{W} \subseteq X$. Fix α such that $Z_\mathcal{W} \cap X_\alpha \neq \emptyset$. Then $\mathcal{W} = \mathcal{V}_{(\alpha,\mu)}$ for some $\mu < \mathfrak{c}$. Fix $\theta < \mathfrak{c}$ such that $i(\theta) = (\alpha, \mu)$. Then $\mathcal{W} = \mathcal{V}_{i(\theta)}$, so $\mathcal{W}_\theta \in Z_\mathcal{W} \setminus X$, a contradiction. $\square$

3.8 Consequences of the pseudocompactness of the hyperspace

In this section we investigate what happens if $\exp(X)$ is pseudocompact when $\omega \subseteq X \subseteq \beta\omega$. We know that if X is $(\mathfrak{c}, \omega^*)$-pseudocompact when $\omega \subseteq X \subseteq \beta\omega$, then $\exp(X)$ is pseudocompact (Corollary 3.7.3). Some natural questions regarding this corollary are: does the converse hold? Does the pseudocompactness of the Vietoris hyperspace of such an X imply the (κ, ω^*)-pseudocompactness of some κ? Which κ's?

As a first result, we have:

Theorem 3.8.1 (*). Let X be a subspace of $\beta\omega$ containing ω. If $\exp(X)$ is pseudocompact, then for every $\kappa < \mathfrak{par}$ X is (κ, ω^*)-pseudocompact.

Proof. Fix κ. Let $g : \omega \to \omega^\kappa$ be a sequence. For each $\alpha < \kappa$, let $g_\alpha = \pi_\alpha \circ g$, so $g_\alpha : \omega \to \omega$.

Since $\kappa < \mathfrak{par}$, there exists $A \subseteq \omega$ such that for every $\alpha < \kappa$ there exists $F \in [\omega]^{<\omega}$ such that $g_\alpha|(A \setminus F)$ is constant or injective. Let $j : \omega \to A$ strictly increasing and let $g'_\alpha = g_\alpha \circ j$. It suffices to show that there exists a strictly increasing sequence $l : \omega \to \omega$ and a free ultrafilter $\mathcal{U}$ such that $\mathcal{U}$-$\lim g'_\alpha \circ l \in X$ for every $\alpha < \kappa$.

Since $\mathfrak{b} \geq \mathfrak{par} > \kappa$, there exists $a : \omega \to \omega$ such that $a >^* g_\alpha$ for every $\alpha < \kappa$.

For each $m \in \omega$, let

$$I_m = \{\alpha < \kappa : \forall n \geq m\, a(n) > g_\alpha(n) \,\text{and}\, g_\alpha|(A \setminus m) \text{ is strictly increasing}\}.$$

I_m increases with m.

Now we define a strictly increasing sequence $l : \omega \to \omega$ such that $l(0) = 0$ and for all $\alpha \in I_{l(k)}$, $a(l(k)) < g_\alpha(l(k+1)) < a(l(k+1))$. To see that this is possible, define $l(k+1) = 2l(k) + 1$. Then, if $\alpha \in I_{l(k)}$, we have that for every $i \in \omega$, $i \leq g'_\alpha(l(k) + i)$ (by an easy induction), so $a(l(k)) < a(l(k)) + 1 \leq g'_\alpha(2l(k) + 1) = g'_\alpha(l(k+1)) < a(l(k+1))$.

Now define an increasing block sequence C by $C(0) = [0, a(0)]$ and $C(k+1) = [a(l(k)) + 1, a(l(k+1))]$ for every $k \in \omega$. Since $\exp(X)$ is pseudocompact there exists $\mathcal{U} \in \omega^*$ such that C has an $\mathcal{U}$-limit. Notice that for every α such that g'_α is almost injective, there exists m $g_\alpha(l(k)) \in C(k)$ for every $k \geq m$. Thus, $\mathcal{U}$-$\lim g'_\alpha \circ l \in X$. $\square$

Corollary 3.8.2 (*). Suppose X is such that $\omega \subseteq X \subseteq \beta\omega$. If $\exp(X)$ is pseudocompact then X^ω is pseudocompact.

Can we improve the previous result by using some cardinal larger than $\mathfrak{par}$? We will show that we cannot improve it too much.

Theorem 3.8.3 (*). Let $\kappa, \theta \leq \mathfrak{c}$ be infinite cardinals. We give θ the discrete topology. Suppose that there exists $\mathcal{A} \subseteq [\theta]^\omega$ such that $|\mathcal{A}| \leq \kappa$ and that for every block sequence $C = (C(n) : n \in \omega)$ on θ there exists $E \in [\omega]^\omega$ and $B \in \mathcal{A}$ such that $|B \cap \bigcup_{n \in E} C_n| < \omega$.

Then there exists X such that $\theta \subseteq X \subseteq \beta\theta$, $\exp(X)$ is pseudocompact and X is not (κ, ω^*)-pseudocompact.

Proof. Enumerate all block sequences on θ as $(C_\alpha : 0 < \alpha < \mathfrak{c})$. This is possible since $\theta^\omega = \mathfrak{c}^\omega = \mathfrak{c}$. For each $\alpha \in [1, \mathfrak{c})$, let $G_\alpha = \prod_{n\in\omega} C(n)$. For each $A \in \mathcal{A}$, let $f_A : \omega \to A$ be a bijection. Let $X_0 = \theta$. Recursively, for $\alpha \in [1, \mathfrak{c})$, we define $X_\alpha \subseteq \beta\theta$, a free ultrafilter $\mathcal{U}_\alpha$ on ω. $P_\alpha \subseteq \beta\omega$ and $Y_\alpha \subseteq \beta\theta$ satisfying:

1. $X_\alpha = \{\mathcal{U}_\alpha\text{-}\lim g : g \in \mathcal{G}_\alpha\}$,
2. $P_\alpha = \left\{\mathcal{V} \in \omega^* : \mathcal{V} \notin \bigcup_{0<\beta<\alpha} P_\beta \text{ and } \exists A \in \mathcal{A}\, (\mathcal{V}\text{-}\lim f_A \in X_\alpha)\right\}$,
3. $\left(\bigcup_{\beta\leq\alpha} X_\beta\right) \cap \left(\bigcup_{0<\beta\leq\alpha} Y_\beta\right) = \emptyset$ (for every $\alpha \in [1, \mathfrak{c})$),
4. $\forall \mathcal{V} \in P_\alpha\, \exists B \in \mathcal{A}\, \mathcal{V}\text{-}\lim f_B \in Y_\alpha$, and
5. $|Y_\alpha| \leq \mathfrak{c}$.

Suppose we have defined $X_\beta, \mathcal{U}_\beta, P_\beta$ and Y_β for every β such that $0 < \beta < \alpha$ for some $\alpha < \mathfrak{c}$.

There exists $E \in [\omega]^\omega$ and $B \in \mathcal{A}$ such that $|B_n \cap \bigcup_{n\in E} C_\alpha(n)| < \omega$. Notice that given $g \in \mathcal{G}_\alpha$ and $z \in \beta\theta$, there exists at most one free ultrafilter $\mathcal{W}$ such that $\mathcal{W}\text{-}\lim g = z$, therefore, $Y = \{\mathcal{W} \in \omega^* : \exists g \in \mathcal{G}_\alpha (\mathcal{W}\text{-}\lim g \in \bigcup_{0<\beta<\alpha} Y_\beta)\}$ has cardinality at most $\mathfrak{c}$. Let $\mathcal{U}_\alpha \in \omega^* \setminus Y$ be such that $E \in V_\alpha$.

Let X_α be as in 1. and P_α as in 2. Let $Y_\alpha = \{\mathcal{V}\text{-}\lim f_B : \mathcal{V} \in P_\alpha\}$. 1, 2, 4 and 5 clearly hold. To see that 3 holds, we first check that $Y_\alpha \cap X_\alpha = \emptyset$. Given $g \in \mathcal{G}_\alpha$ and $\mathcal{V} \in P_\alpha$, it follows that $\mathcal{U}_\alpha\text{-}\lim g_\alpha \in \mathrm{cl}_{\beta\theta}(g[E])$ and $\mathcal{V}\text{-}\lim f_B \in \mathrm{cl}_{\beta\theta}(B)$. Since $g[E] \cap B$ is finite and g is injective, it follows that $\mathcal{V}\text{-}\lim f_B \neq \mathcal{U}_\alpha\text{-}\lim g$. This proves $Y_\alpha \cap X_\alpha = \emptyset$. Now suppose that $\gamma < \alpha$ and $Y_\alpha \cap X_\gamma \neq \emptyset$. Fix x in this intersection. Since $x \in Y_\alpha$, there exists $\mathcal{V} \in P_\alpha$ such that $x = \mathcal{V}\text{-}\lim f_B$ and $\mathcal{V} \notin \bigcup_{0<\beta<\alpha} P_\beta$. In particular, $\mathcal{V} \notin \bigcup_{0<\beta<\gamma} P_\beta$. Since $\mathcal{V}\text{-}\lim f_B = x \in X_\gamma$ and $B \in \mathcal{A}$, then $\mathcal{V} \in P_\gamma \cap P_\alpha$, a contradiction.

Let $X = \bigcup_{\alpha<\mathfrak{c}} X_\alpha$. By construction, $\{\mathcal{U}_\alpha\text{-}\lim g : g \in \mathcal{G}_\alpha\} \subseteq X$, so by Lemma 3.5.8, $\exists \mathcal{U}_\alpha\text{-}\lim C_\alpha \in \exp(X)$, so by Proposition 3.5.7, $\exp(X)$ is pseudocompact. To see that X is not (κ, ω^*)-pseudocompact, it suffices to show that for every free ultrafilter $\mathcal{V}$ there exists $B \in \mathcal{A}$ such that $\mathcal{V}\text{-}\lim f_B \in X$, so fix a free ultrafilter $\mathcal{V}$. Suppose that there exists $A \in \mathcal{A}$ such that $\mathcal{V}\text{-}\lim f_A \in X$ (the other case is trivial). Let α be the least ordinal for which there exists $A \in \mathcal{A}$ such that $\mathcal{V}\text{-}\lim f_\alpha \in X_\alpha$. Then $\mathcal{V} \notin P_\beta$ for $\beta < \alpha$, thus, $\mathcal{V} \in P_\alpha$. Therefore, by 4, there exists $B \in \mathcal{A}$ such that $\mathcal{V}\text{-}\lim f_B \in Y_\alpha$. Since $X \cap Y_\alpha = \emptyset$, we are done. □

Now we apply the previous theorem with $\theta = \omega$ and $\kappa = \mathfrak{b}$. In particular, this shows that the converse of Corollary 3.7.3 cannot hold.

Theorem 3.8.4 (*). There exists a space X such that $\omega \subseteq X \subseteq \beta\omega$, $\exp(X)$ is pseudocompact and X is not $(\mathfrak{b}, \omega^*)$-pseudocompact.

Proof. We aim to apply Theorem 3.8.3 for $\theta = \omega$ and $\kappa = \mathfrak{b}$. Let $\mathcal{B}$ be an unbounded family of strictly increasing sequences. We claim that $\mathfrak{A} = \{g[\omega] : g \in \mathcal{B}\}$ works.

Let C be a block sequence of ω. Let $f : \omega \to \omega$ be given by $f(m) = 1+\max\left(\bigcup_{k\leq 2m} C_k\right)$ (for $m \in \omega$). There exists $g \in \omega$ such that $N = \{m \in \omega : g(m) \geq f(m)\}$ is infinite. We claim that $L = \{n \in \omega : C_n \cap g[\omega] = \emptyset\}$ is infinite, which concludes the proof.

Fix $m \in N$. We claim that for every $p \geq m$, $g(p) \notin \bigcup_{k\leq 2m} C_k$: for suppose $g(p) \in \bigcup_{k\leq 2m} C_k$. Then $g(m) < g(p) < f(m)$.

Thus, $Z = \{i \leq 2m : C_i \cap g[\omega] \neq \emptyset\} = \{i \leq 2m : C_i \cap g[m] \neq \emptyset\}$ which has cardinality at most m since C is a sequence of pairwise disjoint sets.

Therefore $|L \cap (2m+1)| = |(2m+1) \setminus Z| \leq 2m+1-m = m+1$. Since N is infinite, $|L| = \omega$, as intended. □

As another application we have the following example, which shows that for compactifications of larger discrete spaces such failure for a converse of Corollary 3.7.3 occurs at ω_1.

Theorem 3.8.5 (*). Suppose θ is a cardinal such that $\omega_1 \leq \theta \leq \mathfrak{c}$. There exists a space X such that $\theta \subseteq X \subseteq \beta\theta$, $\exp(X)$ is pseudocompact and X is not (ω_1, ω^*)-pseudocompact.

Proof. We apply Theorem 3.8.3 with θ as the given θ and $\kappa = \omega_1$. Let $\mathcal{A}$ be a partition of ω_1 into ω_1 subsets of cardinality ω_1.

Let C be a block sequence on θ. Let $E = \omega$. Since $\bigcup_{n\in\omega} C(n)$ is countable, there exists $A \in \mathcal{A}$ such that $A \cap \bigcup_{n\in\omega} C_n$ is empty. □

We end this chapter by proving a result similar to Theorem 3.8.1 for $\exp(X)$ instead of X.

Theorem 3.8.6 (*). Let X be a subspace of $\beta\omega$ containing ω. If $\exp(X)$ is pseudocompact, then for every $\kappa < \mathfrak{h}$ $\exp(X)$ is (κ, ω^*)-pseudocompact.

Proof. Fix $\kappa < \mathfrak{h}$. Let $E = [\omega]^{<\omega} \setminus \{\emptyset\}$. Since E is a dense subset of $\exp(X)$, it suffices to show that E^κ is relatively countably compact in $\exp(X)^\kappa$. Let $(A_\alpha)_{\alpha<\kappa}$ be a family of κ sequences of elements of E.

For each $\alpha < \kappa$, let

$$\mathcal{U}_\alpha = \{Y \in [\omega]^\omega : \exists m \in \omega\, A_\alpha \text{ admits a nice split over } Y \setminus m\}.$$

By Lemma 3.5.5, each $\mathcal{U}_\alpha$ is a open dense subset of $[\omega]^\omega$. Since $\kappa < \mathfrak{h}$, there exists $I \in \bigcap\{\mathcal{U}_\alpha : \alpha < \kappa\}$. For each α, fix m^0_α, U_α and D_α such that (U_α, D_α) is a nice split of $\mathcal{A}_\alpha$ over $I \setminus m^0_\alpha$.

For each $\alpha < \kappa$, let

$$\mathcal{V}_\alpha = \{J \in [I]^\omega : D_\alpha|(J \setminus m^0_\alpha) \text{ is eventually empty or eventually not empty}\}$$

Each $\mathcal{V}_\alpha$ is open and dense, so fix $J \in \bigcap_{\alpha<\kappa} \mathcal{V}_\alpha$. For each α, let $m^1_\alpha \geq m^0_\alpha$ be such that either $\forall n \in J \setminus m^1_\alpha D_\alpha(n) = \emptyset$ or $\forall n \in J \setminus m^1_\alpha D_\alpha(n) \neq \emptyset$. Let $T_0 = \{\alpha < \kappa : \forall n \in J \setminus m^1_\alpha D_\alpha(n) \neq \emptyset\}$ and $T_1 = \{\alpha < \kappa : \forall n \in J \setminus m^1_\alpha D_\alpha(n) = \emptyset\}$.

For each $\alpha \in T_0$, define $f_\alpha, g_\alpha : \omega \to \omega$ such that for all $n \in J \setminus m^1_\alpha$, $f_\alpha(n) = \min D_\alpha(n)$ and $g_\alpha(n) = \max D_\alpha(n)$. Now notice that, for each $\alpha \in T_0$,

$$\mathcal{W}_\alpha = \{Z \in [J]^\omega : f_\alpha|z \text{ and } g_\alpha|Z \text{ are eventually strictly increasing}\}$$

is open and dense. So let $Z \in \bigcap\{\mathcal{W}_\alpha : \alpha \in T_0\}$. For each $\alpha \in T_0$, let $m_\alpha \geq m^1_\alpha$ be such that $f_\alpha|(J \setminus m_\alpha)$ and $g_\alpha|(J \setminus m_\alpha)$ are both strictly increasing. For $\alpha \in T_1$, let $m_\alpha = m^1_\alpha$.

Let $j : \omega \to Z$ be a strictly increasing bijection. For each $\alpha < \kappa$ let $D'_\alpha = D_\alpha \circ j$, $U'_\alpha = U_\alpha \circ j$, $A'_\alpha = A_\alpha \circ j$, $m'_\alpha = j^{-1}(m_\alpha)$. For $\alpha \in T_0$, let $f'_\alpha = f_\alpha \circ j$. Then (D'_α, U'_α) is a nice split of A'_α over $\omega \setminus m'_\alpha$. Also, if $n \in \omega \setminus m'_\alpha$ and $\alpha \in T_0$, then $D_\alpha(n) \neq \emptyset$, $f'_\alpha(n) = \min D'_\alpha(n) \leq \max D'_\alpha(n) = g'_\alpha(n)$. If $n \in \omega \setminus m'_\alpha$ and $\alpha \in T_1$, then $D'_\alpha(n) = \emptyset$. Finally, notice that $f'_\alpha|(\omega \setminus m'_\alpha)$ and $g'_\alpha|(\omega \setminus m'_\alpha)$ are strictly increasing.

Since $\mathfrak{h} \leq \mathfrak{b}$, there exists $a : \omega \to \omega$ such that $a \geq^* f'_\alpha, g'_\alpha$ for every $\alpha \in T_0$.

For each $m \in \omega$, let

$$I_m = \{\alpha \in T_0 : \forall n \geq m \, (a(n) > f'_\alpha(n), g'_\alpha(n) \text{ and } m \geq m'_\alpha\}.$$

I_m is a growing sequence of subsets of T_0. Recursively, we define a strictly increasing sequence $l : \omega \to \omega$ satisfying that $l(0) = 0$ and, for every $k \in \omega$ and for every $\alpha \in I_{l(k)}$, $a(l(k)) < f'_\alpha(l(k+1)) \leq g'_\alpha(l(k+1)) < a(l(k+1))$.

To see that this is possible, define, for instance, recursively, $l(0) = 0$ and $l(k+1) = 2a(l(k)) + 1$. This function l works: first, notice that, inductively, for every natural i, $i \leq f'_\alpha(l(k) + i)$ since $f'_\alpha|(\omega \setminus l_k)$ is strictly increasing. Thus, $a(l(k)) < a(l(k)) + 1 \leq f'_\alpha(2a(l(k)) + 1) = f'_\alpha(l(k+1))$. Since $\alpha \in I_{l(k)}$ and l is strictly increasing, $l(\alpha) \geq m_\alpha$, so $f'_\alpha(l(k+1)) \leq g'_\alpha(l(k+1)) < a(l(k+1))$.

Let $C_0 = [0, a_0] \subseteq \omega$. For each $k \in \omega$, let $C_{k+1} = [a(l(k)) + 1, a(l(k+1))]$. Then $C = (C_k : k \in \omega)$ is a block sequence on ω.

For each $\alpha \in I_{l(k)}$ and $k \in \omega$, $D'_\alpha(l(k)) \subseteq C_k$. Since $\exp(X)$ is pseudocompact, there exists a free ultrafilter $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$-$\lim C$ exists in $\exp(X)$. By Lemma 3.5.8, given $\alpha \in T_0$ and $f \in \prod_{k\in\omega} D'_\alpha(l(k))$, $\mathcal{U}$-$\lim f \in X$. So, by the same lemma, there exists a $\mathcal{U}$-$\lim D'_\alpha \circ l$. Since $\lim U'_\alpha \circ l$ is eventually increasing and A'_α is eventually $U'_\alpha \cup D'_\alpha$, A'_α has a $\mathcal{U}$-limit by lemmas 3.5.3 and 3.5.2. $\square$

Chapter 4

Results on the pseudocompactness of Hyperspaces of Isbell-Mrówka spaces

4.1 Introduction

According to [43], in a private conversation J. Cao and T. Nogura explicitly asked whether $\exp(X)$ is pseudocompact for some/every Isbell-Mrówka space X.

These spaces are natural candidates for exploring Ginsburg's questions regarding the pseudocompactness of hyperspaces (Problem 3.4.5) since $\Psi(\mathcal{A})$ is pseudocompact iff $\Psi(\mathcal{A})^\omega$ is pseudocompact iff $\mathcal{A}$ is MAD. We will not prove this result since we will improve it in the next section.

Proposition 4.1.1 ([43, Lemma 2.4.])**.** Let $\mathcal{A}$ be an almost disjoint family. The following are equivalent:

a) $\mathcal{A}$ is a MAD family,

b) $\Psi(\mathcal{A})$ is pseudocompact

c) $\Psi(\mathcal{A})^\omega$ is pseudocompact.

Thus, we have the following easy result.

Corollary 4.1.2. Let $\mathcal{A}$ be an almost disjoint family. If $\exp(\Psi(A))$ is pseudocompact, then $\mathcal{A}$ is MAD and $\Psi(\mathcal{A})^\omega$ is pseudocompact.

Proof. This follows from Proposition 4.1.1 and Theorem 3.4.4. $\square$

Thus, in the realm of Isbell-Mrówka spaces, Ginsburg's questions translate as:

Problem 4.1.3. What is the relationship between the maximality of an almost disjoint family and the pseudocompactness of the Vietoris hyperspace of its associated Isbell-Mrówka space?

This question may sound a bit vague, but we may extract some more precise questions from it. As a shorthand, we use the following definition:

Definition 4.1.4. Let $\mathcal{A}$ be a MAD family. We say that $\mathcal{A}$ is *pseudocompact* iff $\exp(\Psi(\mathcal{A}))$ is pseudocompact. $\square$

Problem 4.1.5 (Problem A)**.** Is every MAD family pseudocompact?

Problem 4.1.6 (Problem B)**.** Is there a pseudocompact MAD family?

Problem 4.1.7 (Problem C)**.** What are the possible cardinalities of pseudocompact/non pseudocompact MAD families?

It turned out that Problem A is independent of ZFC as proved in [43], as stated below. Below, b) is a rephrasing of the original statement of their paper.

Theorem 4.1.8. The following is true:

a) If $\mathfrak{p} = \mathfrak{c}$, then every MAD family is pseudocompact [43, Theorem 3.2.].

b) Suppose that there exists a base tree with no chains of size $\mathfrak{c}$. Then there exists a MAD family which is not pseudocompact [43, Theorem 4.2.].

Proof. We only prove item b), as we will improve item a) in the next section, by following its original proof. Our MAD family will be an almost disjoint family on $N = 2^{<\omega}$. First, fix a $\mathfrak{c}$-splitting sharp base tree $(T, \supsetneq^*)$ of height $\mathfrak{h}$ (see Theorem 1.3.8). For every $X \subseteq 2^{<\omega}$, let $\pi(X) = \{n \in \omega : X \cap 2^n \neq \emptyset\}$.

Enumerate all the subsets of N as $\{X_\alpha : \omega \leq \alpha < \mathfrak{c}\}$. Recursively we define $(a_\alpha : \alpha < \mathfrak{c})$ and $f : \mathfrak{c} \to 2$ such that:

1. For every $\alpha < \mathfrak{c}$ and $\beta < \alpha$, $a_\alpha \cap a_\beta$ is finite.
2. $f[\omega] = \{1\}$,
3. for $\alpha \in [\omega, \mathfrak{c})$, $f(\alpha) = 1$ if, and only if for all $\beta < \alpha$, $X_\alpha \cap a_\beta$ is finite,
4. for $\alpha \in [\omega, \mathfrak{c})$, if $f(\alpha) = 1$ then $|X_\alpha \cap a_\alpha| = \omega$,
5. for $\alpha \in [\omega, \mathfrak{c})$, if $f(\alpha) = 1$, if a_α is either a finite chain or an antichain of $2^{<\omega}$ and $\pi(a_\alpha) \in T$,
6. for $\alpha, \beta \in [\omega, \mathfrak{c})$, if $\alpha < \beta$ and $f(\alpha) = f(\beta) = 1$, then $\pi(\alpha) \neq \pi(\beta)$

To see that this can be done: first, we construct $(a_n : n \in \omega)$. These first ω steps do not really need to be defined separately, but it makes it easier to see that the final almost disjoint family is infinite. Let $\mathcal{B}$ be a countable almost disjoint family of ω. Let $g : \omega \to \{0\}$ be constant. Enumerate $\mathcal{B} = \{b_n : n \in \omega\}$. Let, for each n, $b'_n \in T$ be such that $b'_n \subseteq b_n$. Let $a_n = \{g|k : k \in b'_n\}$.

Now we treat the case $\alpha > \omega$ supposing that $\alpha < \mathfrak{c}$ and that we have already constructed the $(a_\alpha : \alpha < \beta)$ and $f|\alpha$. Define $f(\alpha)$ as in 3.. If $f(\alpha) = 0$, define $a_\alpha = \emptyset$. If not, let c be an infinite subset of x_α which is a chain or an antichain. $\pi(c)$ is infinite, thus, there exists $y \in T$ such that $y \subseteq \pi(c)$. Moreover, since $|\operatorname{succ}_T(c)| = \mathfrak{c}$, there exists $d \subseteq c$ in t such that $d \neq \pi(a_\beta)$ for every $\beta < \alpha$. Let $a_\alpha = \{s \in c : \operatorname{dom} s \in d\}$.

Let $\mathcal{A} = \{a_\alpha : f(\alpha) = 1\}$. Then $\mathcal{A}$ has the following properties:

a) $\mathcal{A}$ is a mad family on $2^{<\omega}$,

b) $\pi : \mathcal{A} \to T$ is injective, and

c) Every element of $\mathcal{A}$ is either a chain or an antichain of $2^{<\omega}$

We claim that $\mathcal{A}$ is not pseudocompact. For each n, let $C_n = 2^n$ and let $C = (C_n : n \in \omega)$. We claim that C has no accumulation point in $\exp(\Psi(\mathcal{A}))$. Suppose there is such an accumulation point F. First, notice that $F \cap 2^{<\omega}$ is empty, for if s is in this set, $\{s\}^-$ would be a neighborhood of F intersecting only one element of C, a contradiction.

Case 1: $|F| < \mathfrak{c}$: In this case, there exists $h \in 2^\omega$ such that $B = \{h|n : n \in \omega\}$ is such that $B \cap x$ is finite for every $x \in F$. This happens because every element of $\mathcal{A}$ is either a chain or an antichain. Thus, $F \in (\Psi(\mathcal{A}) \setminus \operatorname{cl} B)^+$, but this open set is disjoint from all the elements of C, a contradiction.

Case 2: $|F| = \mathfrak{c}$: In this case, there exists two distinct $a, b \in F$ such that $\pi(a), \pi(b)$ are in the same level of T, thus, $\pi(a) \cap \pi(b) \subseteq 2^{<k}$ for some $k \in \omega$. This implies that for all $l \geq k$, $C_l \notin (\{a\} \cup a)^- \cap (\{b\} \cup b)^-$, but the latter is a open neighborhood of F, a contradiction. $\square$

The original statement in [43] uses $\mathfrak{h} < \mathfrak{c}$, and they mention that this hypothesis can “clearly be weakened to the existence of such a tree with no branches of size $\mathfrak{c}$”.

Notice that all the MAD families of which the previous theorem talks about have cardinality $\mathfrak{c}$. a) only says something about objects of size $\mathfrak{c}$ since $\mathfrak{p} = \mathfrak{c}$ implies $\mathfrak{a} = \mathfrak{c}$, and the example constructed in b) has size $\mathfrak{c}$ by the following easy lemma which is probably folklore.

Lemma 4.1.9. Let $\mathcal{A}$ be a MAD family on $2^{<\omega}$ whose every element is a chain or an antichain of $2^{<\omega}$. Then $|\mathcal{A}| = \mathfrak{c}$.

Proof. For each $f \in 2^\omega$, there exists B_f in $\mathcal{A}$ whose intersection with $\{f|n : n \in \omega\}$ is infinite. This is easily seen to define an injective function. $\square$

Thus, Problem A is independent of ZFC. In Section 2 we will explore more about that by showing that a positive answer is equivalent to $n(\omega^*) > \mathfrak{c}$.

Problem B is an open problem (in the sense that we do not if it is true, false or independent of ZFC) which was stated in [43], [40], [38] and in our paper [34]. Thus, it is interesting to know new consistent examples of pseudocompact MAD families.

Problem C is open as well. In Section 4.3 we will show that in the Cohen model there exists a pseudocompact MAD family of size ω_1, and that it is consistent that there exists a non pseudocompact MAD family of size $\omega_2 < \mathfrak{c}$. We do not know if it is consistent that there exists a MAD family of size ω_1 which is not pseudocompact.

4.2 Conditions for every MAD family being pseudocompact

In this section we improve the results of Theorem 4.1.8 from [43] by showing that "Every MAD family is pseudocompact" is equivalent to $n(\omega^*) > \mathfrak{c}$. These results appear in Section 2 of our paper [34], but we use a slightly different approach here.

First, we prove two very similar lemmas about relative sequentially compactness. The second lemma is the one that is more interesting for us, but we also prove the first since it does not follow directly from the second. Even though they are easy I did not find a reference for them.

Lemma 4.2.1 (*). Let $\kappa < \mathfrak{h}$. Let $(X_\alpha : \alpha < \kappa)$ be a family of topological spaces, and $(Y_\alpha : \alpha < \kappa)$ be a family such that $Y_\alpha \subseteq X_\alpha$ for each $\alpha < \kappa$. Let $Y = \prod_{\alpha<\kappa} Y_\alpha$ and $X = \prod_{\alpha<\kappa} X_\alpha$. Suppose that for each $\alpha < \kappa$, Y_α is relatively sequentially compact in X_α. Then Y is relatively sequentially compact in X.

Proof. Let $f : \omega \to Y$ be given. Let $f_\alpha = \pi_\alpha \circ f$ be the αth coordinate function of f.

For each $\alpha < \kappa$, let $D_\alpha = \{A \subseteq \omega : f_\alpha|A \text{ converges}\}$. D_α is an open dense subset of $[\omega]^\omega$ since Y_α is relatively sequentially compact in X_α. Let $A \in \bigcap_{\alpha<\kappa} D_\alpha$. Then $f|A$ converges. □

Lemma 4.2.2 (*). Let $\kappa < n(\omega^*)$. Let $(X_\alpha : \alpha < \kappa)$ be a family of topological spaces, and $(Y_\alpha : \alpha < \kappa)$ be a family such that $Y_\alpha \subseteq X_\alpha$ for each $\alpha < \kappa$. Let $Y = \prod_{\alpha<\kappa} Y_\alpha$ and $X = \prod_{\alpha<\kappa} X_\alpha$. Suppose that for each $\alpha < \kappa$, Y_α is relatively sequentially compact in X_α. Then Y is relatively countably compact in X.

Proof. Let $f : \omega \to Y$ be given. Let $f_\alpha = \pi_\alpha \circ f$ be the αth coordinate function of f.

For each $\alpha < \kappa$, let $D_\alpha = \{A \subseteq \omega : f_\alpha|A \text{ converges}\}$. D_α is an open dense subset of $[\omega]^\omega$ since Y_α is relatively sequentially compact in X_α. By $\mathrm{FA}_{[\omega]^\omega}(\kappa)$, there exists a free ultrafilter $\mathcal{U}$ intersecting each D_α (see Lemma 0.7.3). For each α, fix $A_\alpha \in D_\alpha \cap \mathcal{U}$ and x_α such that $f_\alpha|A_\alpha$ converges to x_α.

For each $\alpha < \kappa$, since $f_\alpha|A_\alpha$ converges to x_α and $A_\alpha \in \mathcal{U}$, it follows that x_α is an $\mathcal{U}$-limit of f_α. Thus, by Proposition 0.4.11, x is an $\mathcal{U}$-limit of f, as intended. □

Thus, we get:

Theorem 4.2.3 (*). Suppose $\mathcal{A}$ is a MAD family. Then:

a) ω^κ is relatively sequentially compact in $\Psi(\mathcal{A})^\kappa$ for every $\kappa < \mathfrak{h}$,

b) ω^κ is relatively countably compact in $\Psi(\mathcal{A})^\kappa$ for every $\kappa < \mathfrak{n}(\omega^*)$, and

c) $\Psi(\mathcal{A})$ is (κ, ω^*)-pseudocompact for every $\kappa < n(\omega^*)$.

Proof. To prove a) and b), it suffices, by lemmas 4.2.1 and 4.2.2, to see that ω is relatively sequentially compact in $\Psi(\mathcal{A})$.

Let $f : \omega \to \omega$ be given. f has a constant subsequence or f has a injective subsequence, so WLOG f is injective. Since $\mathcal{A}$ is maximal, there exists $a \in \mathcal{A}$ such that $a \cap f[\omega]$ is infinite. Then $f|f^{-1}[a]$ converges to $a \in \mathcal{A}$.

c) follows from b) and Lemma 3.7.2 □

Now we are already able to conclude our equivalence.

Corollary 4.2.4 (*). $n(\omega^*) > \mathfrak{c}$ iff every MAD family is pseudocompact.

Proof. The "only if" part follows from item c) of the previous theorem and from Theorem 3.7.1.

The "if" part follows from 4.1.8 b) and Corollary 1.3.9 since if $n(\omega^*) \leq \mathfrak{c}$, then either $\mathfrak{h} < \mathfrak{c}$, or $\mathfrak{h} = n(\omega^*) = \mathfrak{c}$ and in both cases we have base trees with no branches of cardinality $\mathfrak{c}$. □

Since $\mathfrak{p} < n(\omega^*)$, this result implies Theorem 4.1.8 a).

Summing up, we get our result Theorem 2.4. of [34].

Corollary 4.2.5 (*). The following are equivalent:

a) $\mathrm{FA}_{[\omega]^\omega}(\mathfrak{c})$,

b) Every MAD family is pseudocompact,

c) $\mathfrak{h} = \mathfrak{c}$ and every base tree has a cofinal branch, and

d) $n(\omega^*) > \mathfrak{c}$.

Proof. The equivalence between b) and d) is the previous corollary, and the equivalence between a) and d) follows from Proposition 0.7.3.

c) implies d) by Corollary 1.3.9, and if either $\mathfrak{h} < \mathfrak{c}$ or there exists a base tree with no cofinal branch, then there is a base tree with no branch of cardinality $\mathfrak{c}$, so the negation of b) follows from 4.1.8 b). □

We end this section by proving the consistency of $\mathfrak{p} < \mathfrak{c}$ plus $n(\omega^*) > \mathfrak{c}$, which shows that the result above is really a strenghtening of Theorem 4.1.8 a). We will need the following:

Lemma 4.2.6 (*). Let $\mathcal{A}$ be an almost disjoint family, $\mathcal{U}$ be a free ultrafilter and let F be a block sequence on ω. Then if for every $f \in \prod_{n\in\omega} F_n$ there exists $A \in \mathcal{A}$ and $B \in \mathcal{U}$ such that $f[B] \subseteq A$, then F has a $\mathcal{U}$-limit.

Proof. Let $P = \prod_{n\in\omega} F_n$. Given $f \in P$, fix $B_f \in \mathcal{U}$ and $a_f \in \mathcal{A}$ such that $f[B_f] \subseteq a_f$. Let $\mathcal{B} = \{a_f : f \in P\}$. We claim that $\mathcal{B} = \mathcal{U}\text{-lim}\, F$.

To verify the claim, it suffices to verify the $\mathcal{U}$-limit condition for sub-basic sets, so let $U \subset \Psi(\mathcal{A})$ be open.

If $\mathcal{B} \in U^-$, then there exists $f \in P$ with $a_f \in U$. Since U is open, $a_f \subseteq^* U$. Then $f[B_f] \subseteq^* U$. So $B_f \subseteq^* \{n \in \omega : f(n) \in U\} \subseteq \{n \in \omega : F_n \in U^-\}$. Since $B_f \in \mathcal{U}$ and $\mathcal{U}$ is a free ultrafilter, it follows that $\{n \in \omega : F_n \in U^-\} \in \mathcal{U}$.

If $\mathcal{B} \in U^+$, suppose by contradiction that $\{n \in \omega : F_n \in U^+\} \notin \mathcal{U}$. Then $I = \{n \in \omega : F_n \setminus U \neq \emptyset\} \in \mathcal{U}$. Let $f \in P$ be such that for each $n \in I$, $f(n) \in F_n \setminus U$. Then $f[I \cap B_f] \subseteq^* a_f$ and $f[I \cap B_f] \setminus U$ is infinite, so $a_f \setminus U$ is infinite. On the other hand, since $\mathcal{B} \in U^+$ we have $a_f \in U$, but U is open, so $a_f \subseteq^* U$, a contradiction. $\square$

We recall the definition of a Suslin tree.

Definition 4.2.7. A *Suslin tree* is a tree $(T, <)$ of height ω_1 whose all chains and antichains (in the reverse order) are countable. In particular, all levels are countable, thus $|T| = \omega_1$.

We say that a Suslin tree T is *well pruned* iff $|\operatorname{Lev}_T(0)| = 1$ and for every $x \in T$, $|\{y \in T : x \leq y\}| = \omega_1$. $\square$

The existence of a Suslin tree is independent of the axioms of ZFC. In particular, $\mathfrak{m} > \omega_1$ implies that there is no Suslin tree, and $\diamondsuit$ implies the existence of a Suslin tree. For modern proofs, see [49].

The lemma below is left to the reader, who may also read the reference.

Lemma 4.2.8 ([49, Lemma III.5.2.26, Lemma III.5.2.27])**.** Every Suslin tree contains a well pruned Suslin tree. More specifically, if T is a Suslin tree, the set $\{x \in T : |\{y \in T : x \leq y\}| = \omega_1\}$ is a Suslin tree. $\square$

The lemma below is well known, but we did not find a reference.

Lemma 4.2.9. Suppose T is a tree of height $\leq \mathfrak{p}$ whose levels have cardinality $\leq \mathfrak{c}$. Then T is isomorphic to a subset of the order $([\omega]^\omega, \supsetneq^*)$.

Proof. Let $\mathcal{T}_\alpha = \{p \in T : \operatorname{Lev}_T(p) < \alpha\}$ for each $\alpha < \operatorname{ht}(T)$. Recursively we define $T'_\alpha \subseteq [\omega]^\omega$ and $f_\alpha : T_\alpha \to T'_\alpha$ for $\alpha < \operatorname{ht}(T)$ such that:

i) If $\beta < \alpha$ then $f_\beta \subseteq f_\alpha$,

ii) $f_\alpha : T_\alpha \to T'_\alpha$ is an isomorphism

Then by setting $T' = T'_{\operatorname{ht}(T)}$ and $f' = f'_{\operatorname{ht}(T)}$, it is clear that $f : T \to T'$ is the desired isomorphism.

To see that such construction can be carried out, at step 0 we set $f_0 = T'_0 = \emptyset$ and at limits we just take unions. If T'_α, f'_α have been defined, we proceed as follows:

If $\alpha = 0$, we let T'_1 be a (possibly finite) almost disjoint family of size $|\operatorname{Lev}_T(0)|$ and f_0 be any bijection between $\operatorname{Lev}_T(0)$ and T'_1.

If $\alpha = \beta + 1$, for each $p \in \operatorname{Lev}_T(\beta)$ let $\mathcal{A}_p$ be an almost disjoint family (possible finite, or even empty) on $f_{\beta+1}(p)$ of size $|\operatorname{succ}_T(p)|$. Let $f_p : \operatorname{succ}_T(p) \to \mathcal{A}_p$ be a bijection. Let $T'_{\alpha+1} = T'_\beta \cup \bigcup_{p \in \operatorname{Lev}_T(\beta+1)} \mathcal{A}_p$, $f_{\alpha+1} = f_\beta \cup \bigcup_{p \in \operatorname{Lev}_T(\beta+1)} f_p$.

If α is limit, we let $p \sim q$ for $p, q \in \operatorname{Lev}_\alpha(T)$ iff $\{r \in T : r \leq p\} = \{r \in T : r \leq q\}$. It is clear that $\sim$ is an equivalence relation. For each equivalence class $\mathcal{C}$, let $P_{\mathcal{C}}$ be a pseudointersection of $\{f_\alpha(r) : r \leq p\}$ where p is any member of $\mathcal{C}$. For each $\mathcal{C}$, let $\mathcal{A}_\mathcal{C}$ be a (possible finite) almost disjoint family on $P_\mathcal{C}$ of size $|\mathcal{C}|$, and let $f_\mathcal{C} : \mathcal{C} \to \mathcal{A}_\mathcal{C}$ be a bijection. $T'_{\alpha+1} = T'_\beta \cup \bigcup_{\mathcal{C} \in \mathrm{Lev}_\alpha(T)/\sim} \mathcal{A}_\mathcal{C}$, $f_{\alpha+1} = f_\beta \cup \bigcup_{\mathcal{C} \in \mathrm{Lev}_\alpha(T)/\sim} f_\mathcal{C}$. □

The proposition below also is well known (see, for instance, [25, Lemma 2.]). We prove it for the sake of completeness.

Proposition 4.2.10. Let T be a well pruned Suslin tree. Consider forcing with the reverse order of T. Then T has the c.c.c., $T \Vdash \mathfrak{p} = \omega_1$ and forcing with T does not add reals (nor new sequences into the ground model).

Proof. Denote the order of T by $\prec$ and the forcing order by $\leq$, which is given by $p \leq q$ iff $q \preceq q$. T has the countable chain conditions since it does not contain antichains.

T does not add reals: we work with a countable transitive model M and with a T-generic filter G over M. Let $f \in M[G] \cap M^\omega$. Let τ be such that $\tau_G = f$. There exists $p \in G$ such that $p \Vdash \tau : \omega \to \mathbf{V}$. Let $p' \leq p$ be arbitrary.

For each $n \in \omega$, let $D_n = \{q \leq p' : \exists x \in M\, q \Vdash \check{x} = \tau(\check{n})\}$. It is clear that each D_n is dense below p', so it contains a antichain $\mathcal{A}_n$ which is maximal below p. Let $\alpha = \sup \bigcup \{\mathrm{Lev}_T(q) : q \in \bigcup_{n\in\omega} \mathcal{A}_n\} + 1$. Since each $\mathcal{A}_n$ is countable, $\alpha < \omega_1$. Let $r \leq p'$ be such that $\mathrm{Lev}_T(r) \geq \alpha$, which exists since T is well pruned. Since for each n, $\mathcal{A}_n$ is a maximal antichain below p', there exists $p_n \in \mathcal{A}_n$ such that p_n and r are compatible. Since we are in a tree, this implies that $r \leq p_n$. Thus, there exists (an unique) $x_n \in M$ such that $r \Vdash \tau(n) = x_n$. By substitution in M, there exists $g : \omega \to M$ in M such that $g(n) = x_n$ for every $n \in \omega$, and $r \Vdash \check{g} = \tau$.

We have shown that the set $\{r \leq p : \exists g \in M\, r \Vdash \check{g} = \tau\}$ is dense below p, thus, there exists such an r in G, so there exists $g \in M$ such that $g = \tau_G = f$, as intended.

$T \Vdash \mathfrak{p} = \omega_1$: By the previous proposition, we may suppose that T is a subset of $[\omega]^\omega$ ordered by $\supsetneq^*$. Again, we use a countable transitive model M and a generic filter G. Since each level of T is a maximal antichain, a generic filter of T is a chain of T intersecting every level of T, so it is not in M (as we already knew). Since T has the countable chain condition, it preserves ω_1, so G is a centered family of $[\omega]^\omega$ in $M[G]$ of size ω_1. It cannot have a pseudointersection in $M[G]$: if it had a pseudointersection P, we would have $P \in M$ since T does not add reals, but now $G = \{p \in T : P \subseteq^* p\} \in M$, a contradiction. □

Finally, we get to our consistency result. The model below has already been studied, but the conclusion that all MAD families are pseudocompact in it was not known prior to our work. We prove all the needed properties for the sake of completeness.

Proposition 4.2.11 (*). It is consistent that $\mathfrak{p} = \omega_1 < \mathfrak{c}$ and all MAD families are pseudocompact. More specifically, starting with a model of $\mathfrak{p} = \mathfrak{c}$ in which there exists a Suslin tree, forcing with a well pruned Suslin tree generated a model where $\omega_1 < \mathfrak{p} = \omega_1$, the value of $\mathfrak{c}$ is preserved and all MAD families are pseudocompact.

Proof. Start with a model M of $\omega_1 < \mathfrak{p} = \mathfrak{c}$ + there exists a Suslin Tree. Let S be a well pruned Suslin tree of M and let G be S generic over M. Again, $\leq$ denoted the reverse order of S. We have just proved that S forces $\mathfrak{p} = \omega_1$, and since no reals are added and cardinals are preserved, $\mathfrak{c}$ remains the same.

Suppose $\mathcal{A}$ is a MAD family in $M[G]$.

Claim: there exists a MAD family $\mathcal{B} \in M$ such that for every $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ such that $B \subseteq^* A$.

Proof of the claim. Let $\mathring{\mathcal{A}}$ be a name for $\mathcal{A}$ and let $p \in S$ be such that $p \Vdash \mathring{\mathcal{A}}$ is a MAD family. If $t \leq p$, let $\mathcal{A}_t = \{A \in [\omega]^\omega : t \Vdash \check{A} \in \mathring{\mathcal{A}}\}$. Each of there sets is an almost disjoint family. In M, for each $t \leq p$ let $\mathcal{B}_t$ be a MAD family containing $\mathcal{A}_t$.

Since $|S| = \omega_1 < \mathfrak{h}$, there exists $\mathcal{B}$ refining $\{\mathcal{B}_t : t \leq p\}$, that is, for every $B \in \mathcal{B}$ and for every $t \leq p$, there exists $A \in \mathcal{B}_t$ such that $B \subseteq^* A$.

We show $\mathcal{B}$ is as intended: given $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $|B \cap A| = \omega$. Since forcing with Suslin trees do not add reals, there exists $t \leq p$ such that $t \Vdash A \in \mathring{\mathcal{A}}$, so $A \in \mathcal{A}_t$. There exists $A' \in \mathcal{B}_t$ such that $B \subseteq^* A'$. Since $A', A \in \mathcal{B}_t$, it follows that $A = A'$, which completes the proof of the claim. $\square$

Let $F \in M[G]$ by a block sequence of ω. By Proposition 3.5.7, it suffices to show that F has an accumulation point in $\exp(\Psi(\mathcal{A}))$. Since forcing with S does not add reals, $F \in M$. Working in M, since $\mathfrak{p} = \mathfrak{c}$ we have $\mathrm{FA}_{[\omega]^\omega}(\mathfrak{c})$, so there exists a free ultrafilter $\mathcal{U}$ intersecting each of the open dense sets $D_f = \{I \in [\omega]^\omega : \exists A \in \mathcal{B}\, f[I] \subseteq^* A\}$ for $f \in P$.

In $M[G]$, $\mathcal{U}$ is still a free ultrafilter and for every $f \in \prod_{n\in\omega} F_n$ there is $I \in p$ such that $f[I]$ is contained in an element of $\mathcal{A}$, so by Lemma 4.2.6, there exists $\mathcal{U}$-$\lim F$. $\square$

4.3 A pseudocompact MAD family in the Cohen model

In this section we discuss the existence of Cohen indestructible MAD families which also have their pseudocompactness preserved. We will show that when we add ω_1 Cohen reals such a MAD family exist, and that it also exists under CH. This section is based on our paper [63].

We start by defining the following forcing notion:

Definition 4.3.1. Let $\mathcal{A}$ be an almost disjoint family. We define $\mathbb{Q}(\mathcal{A}) = [\omega]^{<\omega} \times [\mathcal{A}]^{<\omega}$. We define $(s, F) \leq (s', F')$ iff:

- $s' \subseteq s$,
- $F' \subseteq F$, and
- For all $a \in F$, $a \cap (s \setminus s') = \emptyset$.

The maximum element of $\mathbb{Q}(\mathcal{A})$ is $\mathbb{1} = (\emptyset, \emptyset)$. $\square$

We leave the following easy lemma to the reader:

Lemma 4.3.2. Let $\mathcal{A}$ be an almost disjoint family. Then $(\mathbb{Q}(\mathcal{A}), \leq, \mathbb{1})$ is a σ-centered forcing poset with no atoms. □

This forcing notion is related to Solovay's theorem [50, Theorem 2.15]. We will iterate it as sketched in [35, p. 428].

Notice that if $\mathcal{A}$ is countable, then $\mathbb{Q}(\mathcal{A})$ is a countable forcing poset with no atoms, thus, $\mathbb{Q}(\mathcal{A})$ is equivalent to Cohen forcing. The next lemma is easy and left to the reader as well. We will not use it since we will prove it again for the iteration, but it serves as a warm up to the reader.

Lemma 4.3.3. Let $\mathcal{A}$ be an almost disjoint family. Then the following sets are dense:

- $\{(s, F) \in \mathbb{Q}(\mathcal{A}) : a \in F\}$, for $a \in \mathcal{A}$, andt
- $\{(s, F) \in \mathbb{Q}(\mathcal{A}) : \exists m > n\, m \in s \cap X\}$, for $m \in \omega$ and $X \in \mathcal{I}^+(\mathcal{A})$.

If G is a filter intersecting all these sets, then $b = \bigcup\{s : (s, F) \in G\}$ is such that $\mathcal{A} \cup \{b\}$ is an almost disjoint family and $b \cap X$ is infinite for every $X \in \mathcal{I}^+(\mathcal{A})$.

The proposition below also contains a definition. This iteration is not new, however we verify all the details for the sake of completeness.

Proposition 4.3.4. Let $\mathcal{A}$ be an infinite countable almost disjoint family. There exists an iterated forcing construction with finite supports $((\mathbb{P}_\alpha, \leq_\alpha, \mathbb{1}_\alpha)_{\alpha \leq \omega_1}, (\mathring{\mathbb{Q}}_\alpha, \mathring{\leq}_\alpha, \mathring{\mathbb{1}}_\alpha)_{\alpha < \omega_1})$ and and families $(\mathring{\mathcal{A}}_\alpha)_{\alpha \leq \omega_1}$ and $(\dot{a}_\alpha)_{\alpha < \omega_1}$ such that:

i) For each $\alpha < \omega_1$, $\dot{a}_\alpha$ is a $\mathbb{P}_{\alpha+1}$-name.

ii) For each $\alpha \leq \omega_1$, $\mathring{\mathcal{A}}_\alpha$ is a $\mathbb{P}_\alpha$ name and $\mathring{\mathcal{A}}_\alpha = \check{\mathcal{A}} \cup \{(\dot{a}_\beta, \mathbb{1}_\alpha) : \beta < \alpha\}$.

iii) For each $\alpha \leq \omega_1$, $\mathbb{1}_\alpha \Vdash_\alpha \mathring{\mathcal{A}}_\alpha$ is an almost disjoint family.

iv) For each $\alpha < \omega_1$, $\mathbb{1}_\alpha \Vdash \mathring{\mathbb{Q}}_\alpha = \mathbb{Q}(\mathring{\mathcal{A}}_\alpha)$,

v) For each $\alpha < \omega_1$, $\mathbb{1}_{\alpha+1} \Vdash \dot{a}_\alpha = \bigcup\{s : \exists F\, \exists p \in \mathring{G}_{\alpha+1}\, (s, F) = \mathrm{val}(p(\alpha), \mathring{G}_\alpha)\}$.

Such a iterated forcing structure, which is equivalent to adding ω_1 Cohen reals, along with the families of names, is called *ω_1-Cohen expansion of $\mathcal{A}$*.

Proof. We may prove the existence of such a family in some countable transitive model M.

For the basis, we define $\mathbb{P}_0$ to be the trivial order and $\mathring{\mathcal{A}}_0 = \mathrm{check}(\mathcal{A}, \mathbb{P}_0)$. Then ii) and iii) clearly follow.

For the step, suppose we have defined $(\mathbb{P}_\beta, \leq_\beta, \mathbb{1}_\beta)_{\beta \leq \alpha}$, $(\mathring{\mathbb{Q}}_\beta, \mathring{\leq}_\beta, \mathring{\mathbb{1}}_\beta)_{\beta < \alpha}$, $(\mathring{\mathcal{A}}_\beta)_{\beta \leq \alpha}$ and $(\dot{a}_\beta)_{\beta < \alpha}$. We must define $\mathbb{Q}_\alpha$, $\mathbb{P}_{\alpha+1}$, $\dot{a}_\alpha$, and $\mathring{\mathcal{A}}_{\alpha+1}$.

We define $\mathbb{Q}_\alpha$ by 4), by the use of maximal principle, then $\mathbb{P}_{\alpha+1}$ by the definitions of iterated forcing constructions. Then we define $\dot{a}_\alpha$ by 5) and $\mathring{\mathcal{A}}_{\alpha+1}$ by 2). Then 3) follows by density arguments as in the one step block. We will write them for completeness.

Fix a filter $\mathbb{P}_{\alpha+1}$-filter $G_{\alpha+1}$ generic over M. By ii), $\mathrm{val}(\mathring{\mathcal{A}}_{\alpha+1}, G_{\alpha+1}) = \mathrm{val}(\mathring{\mathcal{A}}_\alpha, G_\alpha) \cup \{\mathrm{val}(\dot{a}_\alpha, G_{\alpha+1})\}$. We already know that $\mathrm{val}(\mathring{\mathcal{A}}_\alpha, G_\alpha)$ is an almost disjoint family, so it suffices to prove that $a = \mathrm{val}(\dot{a}_\alpha, G_{\alpha+1})$ is infinite and almost disjoint from every element of $\mathrm{val}(\mathring{\mathcal{A}}_\alpha, G_\alpha)$.

a is infinite: given $n \in \omega$, let $D_n = \{p \in \mathbb{P}_{\alpha+1} : \exists m > n\, p|\alpha \Vdash \text{“}\exists s\, \exists F\, p(\alpha) = (s, F) \wedge n \in s\text{”}\}$. We claim that D_n is dense: given $q \in \mathbb{P}_{\alpha+1}$, $q|\alpha \Vdash q(\alpha) \in \mathbb{Q}(\mathring{\mathcal{A}}_\alpha$. Thus, $q|\alpha \Vdash \exists m \in \check{\omega} \setminus \check{n}\, \exists s\, \exists F\, p(\alpha) = (s, F)$ and $m \in \omega \setminus \bigcup F$. Fix names $\dot{s}$ and $\dot{F}$ such that $q|\alpha \Vdash (\dot{s}, \dot{F}) = p(\alpha$. This implies that there exists $p' \leq q|\alpha$ in $\mathbb{P}_\alpha$ and $m > n$ such that $p' \Vdash \check{m} \notin \bigcup \dot{F}$, so $p' \Vdash (\{\check{m}\} \cup \dot{s}, \dot{F}) \in \mathbb{Q}(\mathring{\mathcal{A}}_\alpha)$, thus there exists a name $p(\alpha)$ such that $p = p' \cup \{(\alpha, p(\alpha)\} \in \mathbb{P}_{\alpha+1}$ and $p' \Vdash p(\alpha) = (\dot{s} \cup \{\check{m}\}, \dot{F})$. It is clear that $p \leq q$ and $p \in D_n$.

Since D_n is dense, there exists $q \in G_{\alpha+1} \cap D_n$. Then by v), $a = \bigcup\{s : \exists F\, \exists p \in G_{\alpha+1}\, (s, F) = \mathrm{val}(p(\alpha), G_\alpha)\}$. Since $q \in D_n$, there exists $m > n$ such that $q|\alpha \Vdash \exists s\, \exists F q(\alpha) = (s, F)$ and $m \in s$. Thus, in $M[G_\alpha]$, there exists s, F such that $\mathrm{val}(q(\alpha), G_\alpha) = (s, F)$ and $m \in s$. By absoluteness, $m \in a$.

$a \cap \mathrm{val}(\dot{a}_\beta, G_{\alpha+1})$ is finite for every $\beta < \alpha$: fix β. Let $D_\beta = \{p \in \mathbb{P}_{\alpha+1} : \exists s \in [\omega]^{<\omega}\, p|\alpha \Vdash \text{“}\exists F\, (s, F) = p(\alpha)$ and $\dot{a}_\beta \in F\text{”}\}$. We claim that D_β is dense. Fix $q \in \mathbb{P}_{\alpha+1}$. We know that $q|\alpha \Vdash q(\alpha) \in \mathbb{Q}(\mathring{\mathcal{A}}_\alpha)$, thus there exists $p' \leq q|\alpha$, a name $\dot{F}$ and $s \in [\omega]^{<\omega}$ such that $p' \Vdash q(\alpha) = (\check{s}, \dot{F})$. Let $p(\alpha)$ be a name such that $p = p' \cup \{(\alpha, p(\alpha)\} \in \mathbb{P}_{\alpha+1}$ and that $p \Vdash p(\alpha) = (\check{s}, \dot{F} \cup \{\dot{a}_\beta\})$. It is clear that $p \leq q$ and that $p \in D_\beta$.

Now let $q \in D_\beta \cap G_{\alpha+1}$. There exists $s \in [\omega]^{<\omega}$ such that $p|\alpha \Vdash \text{“}\exists F\, (s, F) = p(\alpha)$ and $\dot{a}_\beta \in F\text{”}$, so fix s, F such that $\mathrm{val}(\dot{a}_\beta, G_\alpha) \in F$ and $(s, F) = \mathrm{val}(p(\alpha), G_\alpha)$. We claim that $a \cap \mathrm{val}(\dot{a}_\beta, G_{\alpha+1}) \subseteq s$. To see that, fix $n \in a \cap \mathrm{val}(\dot{a}_\beta, G_{\alpha+1})$. Since $n \in a$, there exists s' such that $n \in s'$, F' and $p' \in G_{\alpha+1}$ such that $(s', F') = \mathrm{val}(p'(\alpha), G_\alpha)$. Since $p, p' \in G_{\alpha+1}$, there exists $p'' \in G_{\alpha+1}$ such that $p'' \leq p, p'$. Thus, $p''|\alpha \Vdash p''(\alpha) \leq p(\alpha), p'(\alpha)$. Let $(s'', F'') = \mathrm{val}(p''(\alpha), G_\alpha)$. We have that $n \in s'$, so $n \in s''$. but $(s'', F''') \leq (s, F)$ so since $\mathrm{val}(\dot{a}_\beta, G_\alpha) \in F$ we have that $n \notin s$ implies $n \notin \mathrm{val}(\dot{a}_\beta, G_\alpha)$, a contradiction.

$a \cap b$ is finite for every $b \in \mathcal{A}$: very similar and left to the reader.

For a limit step $\alpha \leq \omega_1$, we define $\mathbb{P}_\alpha = \bigcup_{\beta<\alpha} \mathbb{P}_\beta$ and $\mathring{\mathcal{A}}_\alpha$ as in ii). We must see that iii) follows: let G_α be $\mathbb{P}_\alpha$-generic over M. Let $\beta < \alpha$ be given. First, we must se that $\mathrm{val}(\dot{a}_\beta, G_\alpha)$ is limit. We know that $\mathrm{val}(\dot{a}_\beta, G_\alpha) = \mathrm{val}(\dot{a}_\beta, G_{\beta+1})$, which we have already seen to be infinite. Now given $\beta' < \beta < \alpha$, we must see that $\mathrm{val}(\dot{a}_{\beta'}, G_\alpha) \cap \mathrm{val}(\dot{a}_\beta, G_\alpha)$ is finite. But this is the same as $\mathrm{val}(\dot{a}_{\beta'}, G_{\beta+1}) \cap \mathrm{val}(\dot{a}_\beta, G_{\beta+1})$ which has also already seen to be finite. $\square$

As a foundational remark, the global axiom of choice is not needed in the preceding discussion. By reflection arguments or by a careful observation about the behaviour of the rank function in the proof of the Maximal Principle, the whole construction can be made withing a suitable V_δ for some limit ordinal δ.

Proposition 4.3.5. Let $\mathcal{A}$ be an infinite countable almost disjoint family. Let $((\mathbb{P}_\alpha, \leq_\alpha, \mathbb{1}_\alpha)_{\alpha\leq\omega_1}, (\mathring{\mathbb{Q}}_\alpha, \mathring{\leq}_\alpha, \mathring{\mathbb{1}}_\alpha)_{\alpha<\omega_1})$, $(\mathring{\mathcal{A}}_\alpha)_{\alpha\leq\omega_1}$ and $(\mathring{a}_\alpha)_{\alpha<\omega_1}$ be a ω_1 Cohen expansion of $\mathcal{A}$. Then:

1. For every $\alpha < \omega_1$, for every $p \in \mathbb{P}_\alpha$ and for every $\mathbb{P}_\alpha$-name τ, if $p \Vdash_\alpha \tau \in [\omega]^\omega \setminus \mathcal{I}(\mathring{\mathcal{A}}_\alpha)$ then $p \Vdash_{\alpha+1} |\tau \cap \dot{a}_\alpha| = \omega$.

2. $\mathbb{1}_{\omega_1} \Vdash \mathring{\mathcal{A}}_{\omega_1}$ is a tight MAD family.

Proof. 1. We employ a countable transitive model M. Fix α. Let p, τ be given. Let $G = G_{\alpha+1}$ be $\mathbb{P}_{\alpha+1}$-generic over M such that $p \in G$. Let $x = \mathrm{val}(\tau, G_{\alpha+1}) = \mathrm{val}(\tau, G_\alpha)$, $\mathcal{B} = \mathrm{val}(\mathcal{A}_\alpha, G_\alpha) = \mathrm{val}(\mathcal{A}_\alpha, G_{\alpha+1})$ and $a = \mathrm{val}(\dot{a}_\alpha, G)$. Since $p \in G_\alpha$, x is an infinite subset of ω not almost contained in any finite union of elements of $\mathcal{B}$. We want to show that $|x \cap a| = \omega$.

For each n, let $D_n = \{q \in \mathbb{P}_{\alpha+1} : q \leq p \,\mathrm{and}\, \exists m > n\, q|\alpha \Vdash \text{“}\exists s\, \exists F\, p(\alpha) = (s, F)\,\mathrm{and}\, \check{m} \in s \cap \tau''\}$. We will show that D is dense below p. Fix $r \leq p$. WLOG $\alpha \in \mathrm{dom}\, r$. We know that $r|\alpha \Vdash_\alpha r(\alpha) \in \mathbb{Q}(\mathring{\mathcal{A}}_\alpha)$. Thus, fix names $\dot{s}, \dot{F}$ such that $r|\alpha \Vdash r(\alpha) = (\dot{s}, \dot{F})$. Since $r|\alpha \leq p$, $r|\alpha \Vdash \exists m > \check{n}\, m \in \tau \setminus \bigcup \dot{F}$. Fix $m > n$ and $q' \leq r|\alpha$ in $\mathbb{P}_\alpha$ such that $q' \Vdash_\alpha \check{m} \in \tau \setminus \bigcup \dot{F}$. Let $q(\alpha)$ be a $\mathbb{P}_\alpha$-name such that $q = q' \cup \{(\alpha, q(\alpha)\} \in \mathbb{P}_{\alpha+1}$ and $q' \Vdash_\alpha q(\alpha) = (\dot{s} \cup \{\check{m}\}, \dot{F})$. Then clearly $q \leq r$ and $q \in D_n$.

Now let $q \in D \cap G$. Fix $m > n$ such that $q|\alpha \Vdash \exists s\, \exists F\, p(\alpha) = (s, F)$. Notice that $q|\alpha \in G_\alpha$, so v) of Proposition 4.3.4 it follows that $m \in a \cap x$.

2. We employ a countable transitive model M again. Let G be $\mathbb{P}$-generic over M. Let $x = (x_n : n \in \omega) \in M[G] \cap ([\omega]^\omega)^\omega$ be such that $x_n \in \mathcal{I}^+(\mathcal{A})$ for every $n \in \omega$. By general properties of finitely supported iterations, $x \in M[G_\alpha]$ for some $\alpha < \omega_1$. For each n, let τ_n be a $\mathbb{P}_\alpha$-name such that $x_n = \mathrm{val}(\tau, G_\alpha) = \mathrm{val}(\tau, G)$. By the truth lemma, there exists $p_n \in G_\alpha$ such that $p_n \Vdash_\alpha \tau_n \in I^+(\mathring{\mathcal{A}}_\alpha)$. Then $p_n \Vdash_{\alpha+1} |\tau_n \cap \dot{a}_\alpha| = \omega$. Since $p_n \in G_{\alpha+1}$, it follows that $\mathrm{val}(\tau_n, G_{\alpha+1}) \cap \mathrm{val}(\dot{a}_\alpha, G_{\alpha+1}) = x \cap \mathrm{val}(\dot{a}_\alpha, G_{\alpha+1})$ is infinite, so we are done. ☐

We will prove that this MAD family is pseudocompact. Before that, we need the two lemmas about accumulation points. We start with a simple one:

Lemma 4.3.6 (*)**.** Let $\mathcal{A}$ be an almost disjoint family and let $X = \exp(\Psi(\mathcal{A}))$. Suppose $C = (C_n : n \in \omega)$ is a block on ω. If there exists $F \in [\mathcal{A}]^{<\omega}$ such that $\{n \in \omega : C_n \subseteq \bigcup F\}$ is infinite, then C has an accumulation point in X.

Proof. Set $I = \{n \in \omega : C_n \subseteq \bigcup F\}$ and enumerate F as $\{a_0, \dots, a_k\}$.

We will show that there exists $J \in [I]^\omega$ such that for every $a \in F$, $\{n \in J : a \cap C_n \neq \emptyset\}$ is either J or $\emptyset$. Recursively, we define a decreasing sequence $I_n \in [I]^\omega$ for $n \leq k+1$ as follows: Let $I_0 = I$. After defining I_n for $n < k+1$, let $I_{n+1} = \{m \in I_n : a_n \cap C_m \neq \emptyset\}$ if this set is infinite. Otherwise, let $I_{n+1} = \{m \in I_n : a_n \cap C_m = \emptyset\}$. Finally, let $J = I_{k+1}$.

Let $K = \{a \in F : \{n \in J : a \cap C_n \neq \emptyset\} = J\}$. K is not empty, for if it was, then given $n \in J$, $C_n \cap \bigcup F = \emptyset$, but $n \in I$, so $C_n = \emptyset$, a contradiction. Also, notice that if $n \in J$ and $a \in F \setminus K$, then $C_n \cap a = \emptyset$. So $C_n \subseteq \bigcup K$.

We claim that $(C_n : n \in J)$ converges to K: given open subsets $U_0, \dots, U_l$ of $\Psi(\mathcal{A})$ such that $K \in \langle U_0, \dots, U_l \rangle$, there exists $M \in \omega$ such that for every $a \in K$, $a \setminus M \subseteq \bigcup_{i \leq l} U_i$ (because $K \subseteq \bigcup_{i \leq l} U_i$ and K is finite). Also, for each $i \leq l$, there exists $N_i \in \omega$ and $a_i \in K$ such that $a_i \setminus N_i \subseteq U_i$. Finally, since the C_n's are pairwise disjoint, there exists m_0 such that if $n \geq m_0$, then $C_n \cap \max\{N_0, \dots, N_l, M\} = \emptyset$. So if $n \in J \setminus m_0$, it follows that

$\emptyset \neq C_n \cap a_i \setminus N_i \subseteq C_n \cap U_i$ for each $i \leq k$ and that $C_n \subseteq (\bigcup K) \setminus M \subseteq \bigcup_{i\leq l} U_i$.
□

Now we state a more complicated lemma.

Lemma 4.3.7 (*). Let $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ be a MAD family and let C be a block sequence on ω. Suppose that there exists $\gamma < \omega_1$ and $I \in [\omega]^\omega$ such that:

(i) For every $\xi < \gamma$, $\{n \in I : C_n \cap a_\xi \neq \emptyset\}$ is either finite or cofinite on I, and,

(ii) $\{\{n \in I : a_\xi \cap C_n \neq \emptyset\} : \gamma \leq \xi < \omega_1\}$ is centered.

Define:

- $\mathcal{A}_0 = \{a_\xi : \xi < \gamma \text{ and } \{n \in I : C_n \cap a_\xi \neq \emptyset\} \text{ is cofinite in } I\}$
- $\mathcal{A}_1 = \mathcal{A}_0 \cup \{a_\xi : \gamma \leq \xi < \omega_1\}$

Then $\mathcal{A}_1$ is an accumulation point of C (in fact, of $C|I$) in $\exp(\Psi(\mathcal{A}))$.

Proof. Let $J = \{\xi < \gamma : |\{n \in I : C_n \cap a_\xi = \emptyset\}| < \omega\}$. Then $\mathcal{A}_0 = \{a_\xi : \xi \in J\}$ and $\mathcal{A}_1 = \{a_\xi : \xi \in J \cup [\gamma, \omega_1)\}$. Suppose $\langle U_0, \dots U_k\rangle$ is a neighborhood of $\mathcal{A}_1$, where $U_0, \dots, U_k$ are open subsets of $\Psi(\mathcal{A})$. For each $i \leq k$, there exists $N_i \in \omega$ and $\xi_i \in J \cup [\gamma, \omega_1)$ such that $a_{\xi_i} \setminus N_i \subseteq U_i$. Let $K = \{\xi_i : i \leq k\} \cap [\gamma, \omega_1)$. By (ii), $\{n \in I : \forall \xi \in K\, a_\xi \cap C_n \neq \emptyset\}$ is infinite.

Since if $i \leq k$ and $\xi_i < \gamma$ then $\xi_i \in J$, it follows that $\{n \in I : a_{\xi_i} \cap C_n \neq \emptyset\}$ is cofinite on I, so $\{n \in I : \forall i \leq k\, a_{\xi_i} \cap C_n \neq \emptyset\} = \bigcap_{i\leq k}\{n \in I : a_{\xi_i} \cap C_n \neq \emptyset\}$ is infinite.

Let $\tilde{I} = \bigcap_{i\leq k}\{n \in I : a_\xi \cap C_n \neq \emptyset\} \setminus \max\{N_i : i \leq k\}$. Notice that if $l \in \tilde{I}$, then $\forall i \leq k$, $C_l \cap U_i \neq \emptyset$, so all that is left to see is that $\{n \in \tilde{I} : C_n \setminus \bigcup_{i\leq k} U_i \neq \emptyset\}$ is finite.

Suppose by contradiction that it is infinite. Since the C_n's are pairwise disjoint, $\bigcup_{n\in\tilde{I}} C_n \setminus \bigcup_{i\leq k} U_i$ is infinite, therefore there exists α such that $a_\alpha \cap \left(\bigcup_{n\in\tilde{I}} C_n \setminus \bigcup_{i\leq k} U_i\right)$ is infinite. If $\alpha \in \gamma \setminus J$, then, by (i), $\{n \in I : a_\alpha \cap C_n \neq \emptyset\}$ is finite, which implies $a_\alpha \cap \left(\bigcup_{n\in\tilde{I}} C_n \setminus \bigcup_{i\leq k} U_i\right)$ is finite, a contradiction. Thus, $a_\alpha \in \bigcup_{i\leq k} U_i$. Since the latter is open, it follows that $a_\alpha \subseteq^* \bigcup_{i\leq k} U_i$ so, again, $a_\alpha \cap \left(\bigcup_{n\in\tilde{I}} C_n \setminus \bigcup_{i\leq k} U_i\right)$ is finite, another contradiction. □

In order to apply the previous lemma, we need a special set $I \in [\omega]^\omega$. It will be useful to have a standard candidate for an I that only depends on a given a countable almost disjoint family, an enumeration of it and a sequence of pairwise disjoint nonempty finite sets of naturals. First, we define a pseudointersection operator.

Definition 4.3.8. Let $A = (b_n : n \in \omega)$ be centered countable family of elements of $[\omega]^\omega$. Let $\mathrm{Pseudo}(A) = \{\min(\bigcap_{k\leq n} b_k \setminus n) : k \leq n\}$. □

Notice that $\mathrm{Pseudo}(A)$ is really a pseudointersection of $\{b_n : n \in \omega\}$ and that $\mathrm{Pseudo}(A)$ is absolute for transitive models of ZFC. Now we present the default candidate for an I.

Definition 4.3.9. Given a infinite countable ordinal γ, a bijection $f : \omega \to \gamma$, a family $A = (a_\alpha : \alpha < \gamma)$ of distinct elements whose image is an almost disjoint family, $C = (C_n : n \in \omega)$ a block sequence on ω, one recursively defines:

- $I_0(A, C, f) = \omega$,
- $I_{m+1}(A, C, f) = \{n \in I_m : A_{f(m)} \cap C_n \neq \emptyset\}$, if $\{n \in I_m : A_{f(m)} \cap C_n \neq \emptyset\}$ is infinite,
- $I_{m+1}(A, C, f) = I_m \setminus \{n \in I_m : A_{f(m)} \cap C_n \neq \emptyset\}$ otherwise,
- $I(A, C, f) = \mathrm{Pseudo}(I_m(A, C, F) : m \in \omega)$.

□

Now we are able to prove our result.

Theorem 4.3.10 (*)**.** Let $\mathcal{A}$ be an infinite countable almost disjoint family. Let $((\mathbb{P}_\alpha, \leq_\alpha, \mathbb{1}_\alpha)_{\alpha \leq \omega_1}, (\mathring{\mathbb{Q}}_\alpha, \mathring{\leq}_\alpha, \mathring{\mathbb{1}}_\alpha)$ $(\mathring{\mathcal{A}}_\alpha)_{\alpha \leq \omega_1}$ and $(\mathring{a}_\alpha)_{\alpha < \omega_1}$ be a ω_1 Cohen expansion of $\mathcal{A}$. Then:

$$\mathbb{1}_{\omega_1} \Vdash_{\omega_1} \mathring{\mathcal{A}}_{\omega_1} \text{ is a MAD pseudocompact family.}$$

Proof. We prove this by using a countable transitive model M. Let G be $\mathbb{P}_{\omega_1}$-generic over M. Let $\mathcal{B} = \mathrm{val}(\mathring{\mathcal{A}}_{\omega_1}, G)$.

Let $C = (C_n : n \in \omega) \in M[G]$ be a block sequence on ω. By Proposition 3.5.7, it suffices to show that C has an accumulation point in the hyperspace.

Case 1: There exists $F \in [\omega_1]^{<\omega} \setminus \{\emptyset\}$ such that $I = \{n \in \omega : C_n \subseteq \bigcup_{\alpha \in F} a_\alpha\}$ is infinite. Working in $M[G]$, it follows by Lemma 4.3.6, that C has a convergent subsequence.

Case 2: For every $F \in [\omega_1]^{<\omega} \setminus \{\emptyset\}$, the set $\{n \in \omega : C_n \setminus \bigcup_{\alpha \in F} a_\alpha \neq \emptyset\}$ is cofinite. In this case, by general properties of finitely supported iterated forcing, there exists an infinite $\mu < \omega_1$ such that $C \in M[G_\mu]$.

Let $f : \omega \to \mu$ be any bijection in M and let $I = I((a_\beta : \beta < \mu), C, f)$.

Claim: For every $K \in [[\mu, \omega_1[]^{<\omega}$, $\bigcap_{\xi \in K}\{n \in I : a_\xi \cap C_n \neq \emptyset\}$ is infinite.

Proof of the claim: Write $K = \{\mu_1, \ldots, \mu_k\}$, where $\mu_1 < \cdots < \mu_k$. Working in $M[G_{\mu_1}]$, write, $\mu_1 = \bigcup_{m \in \omega} F_m$, where for each m, $F_m \subseteq \mu_1$ is finite and $F_m \subseteq F_{m+1}$. Since for every m the set $\{n \in \omega : C_n \setminus \bigcup_{\alpha \in F_m} a_\alpha \neq \emptyset\}$ is cofinite, we may recursively choose a strictly growing sequence $n_m \in I$ and a sequence k_m such that $k_m \in C_{n_m} \setminus \bigcup_{\xi \in F_m} a_\xi$. Since the C_n's are pairwise disjoint, $X = \{k_m : m \in \omega\}$ is infinite and $\{X\} \cup \{a_\xi : \xi < \mu_1\}$ is an almost disjoint family, which implies, by Proposition 4.3.5 1. that $X \cap a_{\mu_1}$ is infinite. Let $I_1 = \{n \in I : a_{\mu_1} \cap C_n \neq \emptyset\} \in M[G_{\mu_1+1}]$. Since each k_m belong to a different C_n, the set I_1 is infinite. Now we recursively repeat the argument for $n + 1 \leq k$ to get I_{n+1} by using I_n in the place of I, μ_{n+1} in the place of μ_1. Notice that $I_k \subseteq \bigcap_{\xi \in K}\{n \in I : a_\xi \cap C_n \neq \emptyset\}$, which proves the claim.

Working in $M[G]$, it follows from Lemma 4.3.7, that the sequence C has an accumulation point. $\square$

It is clear that if M also satisfies ¬CH, then so does $M[G]$, therefore we get the following corollary since the Cohen model has $\mathfrak{b} = \mathfrak{h} = \omega_1$:

Corollary 4.3.11 (*). Con(ZFC)→ Con(ZFC+ there exists a mad family $\mathcal{A}_0$ of cardinality ω_1 and a tight MAD family $\mathcal{A}_1$ of cardinality $\mathfrak{c} > \omega_1$ such that $\exp(\Psi(\mathcal{A}_0))$ is pseudocompact but $\exp(\Psi(\mathcal{A}_1))$ is not.)

We can use CH to construct something similar, as we shall see. The next other construction does not appear in our paper [63]. It appeared in an early preprint of that paper, but it ended up not being published due to the similarity with the previous example. However, I decided to add it to this thesis. This construction is due to myself and Artur Tomita.

Let $\mathbb{C}$ denote a poset for adding one Cohen real, that is, any countable poset with no atoms, such as $2^{<\omega}$ or Fn$(\omega, 2)$. $\mathbb{C}_\kappa$ denotes a poset for adding κ Cohen reals with finite supports, such as Fn$(\kappa, 2)$.

Theorem 4.3.12 (*). [ZFC+CH] Let:

a) $((\dot{C}_\gamma, p_\gamma) : \omega \leq \gamma < \omega_1)$ be a listing of all pairs $(\dot{C}, p)$ such that:

- $\dot{C}$ is a $\mathbb{C}$-nice name for a subset of $(\omega \times \check{[\omega]}^{<\omega})$,
- $p \in \mathbb{C}$,
- $p \Vdash \dot{C}$ is a block sequence on ω.

b) $(f_\gamma : \omega \leq \gamma < \omega_1) \in M$ be such that $f_\gamma : \omega \to \gamma$ is bijective,

c) $\mathcal{A}'$ be an infinite countable almost disjoint family.

Then there exists tight MAD family $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ containing $\mathcal{A}'$ such that, by letting $\mathcal{A}|\gamma = (a_\xi : \xi < \gamma)$

For all $\beta < \omega_1$, for all infinite $\gamma \leq \beta$, and for all $F \in [\beta]^{<\omega}$, if for all $J \in [\beta]^{<\omega}$...

$$\ldots p_\gamma \Vdash \left|\left\{n \in I(\check{\mathcal{A}}|\gamma, \dot{C}_\gamma, \check{f}_\gamma) : \forall \xi \in \check{F}\, (\dot{C}_\gamma(n) \cap a_\xi \neq \emptyset) \text{ and } \dot{C}_\gamma(n) \setminus \bigcup_{\xi \in J} a_\xi \neq \emptyset\right\}\right| = \omega,$$

then...

$$\ldots p_\gamma \Vdash |\{n \in I(\check{\mathcal{A}}|\gamma, \dot{C}_\gamma, \check{f}_\gamma) : \forall \xi \in F \cup \{\check{\beta}\}\, (\dot{C}_\gamma(n) \cap a_\xi \neq \emptyset)\}| = \omega.$$

Proof. Again, we employ a countable transitive model M in the proof. So working in M:

Enumerate all sequences $X = (X_n : n \in \omega)$ of elements of $[\omega]^\omega$ as $(X^\alpha : \alpha < \omega_1)$.

Enumerate $\mathcal{A}' = \{a_n : n \in \omega\}$. Given $\beta \in [\omega, \omega_1)$, suppose we have defined $(a_\xi : \xi < \beta)$ such that:

a) $\{a_\xi : \xi < \beta\}$ is an almost disjoint family,

b) for all $\beta' < \beta$, for every infinite $\gamma \leq \beta'$, and for every $F \in [\beta']^{<\omega}$, if for all $J \in [\beta']^{<\omega}$...

   ...$p_\gamma \Vdash |\{n \in I(\check{\mathcal{A}|}\gamma, \dot{C}_\gamma, \check{f}_\gamma) : \forall\xi \in \check{F}\,(\dot{C}_\gamma(n) \cap a_\xi \neq \emptyset)$ and $\dot{C}_\gamma(n) \setminus \bigcup_{\xi \in J} a_\xi \neq \emptyset\}| = \omega$,

   then $p_\gamma \Vdash |\{n \in I(\check{\mathcal{A}|}\gamma, \dot{C}_\gamma, \check{f}_\gamma) : \forall\xi \in \check{F} \cup \{\check{\beta'}\}\,(\dot{C}_\gamma(n) \cap a_\xi \neq \emptyset)\}| = \omega$, and

c) If $X^\beta(n) \in \mathcal{I}^+(\mathcal{A}|\beta)$ for each $n \in \omega$, then $|a_\beta \cap X^\beta(n)| = \omega$ for every $n \in \omega$.

Notice that a) and c) implies that the final object is a tight MAD family.

We must define a_β. If the hypotheses of both b) and c) fail, just let a_β be an element almost disjoint with a'_β for every $\beta' < \beta$. If only the hypothesis of c) hold, let $f : \omega \to \omega$ be such that for every m, $f^{-1}[\{m\}]$ is infinite and let $a_\beta = \{x_n : n \in \omega\}$ where $x_n \in X^\beta(f(n)) \setminus \bigcup_{i<n} a_{g_\beta(i)}$.

The only the hypothesis of b) hold, we proceed as follows:

Working in M, suppose $\{(r, F, \gamma, l) : l \in \omega, r \leq p_\gamma, F \in [\beta]^{<\omega}, \gamma < \beta, \forall J \in [\beta]^{<\omega}\,(p_\gamma \Vdash |\{n \in I(\check{\mathcal{A}|}\gamma, \dot{C}_\gamma, \check{f}_\gamma) : \forall\xi \in \check{F}\,(\dot{C}_\gamma(n) \cap a_\xi \neq \emptyset)$ and $\dot{C}_\gamma(n) \setminus \bigcup_{\xi \in J} a_\xi \neq \emptyset\}| = \omega)\}$ is nonempty and enumerate it as $\{(r_m, F_m, \gamma_m, l_m) : m \in \omega\}$.

For every $m \in \omega$, there exists $s_m \leq r_m$, $n_m, k_m > l_m$ such that $s_m \Vdash \check{n}_m \in I(\check{\mathcal{A}|}\gamma_m, \dot{C}_{\gamma_m}, \check{f}_{\gamma_m}), \forall\xi \in \check{F}_m\, \dot{C}_{\gamma_m}(\check{n}_m) \cap \check{a}_\xi \neq \emptyset$ and $\check{k}_m \in \dot{C}_{\gamma_m}(n_m) \setminus \bigcup_{i \leq m} \check{a}_{\gamma_i}$.

k_m may be picked greater than l_m since $r_m \leq p_{\gamma_m} \Vdash$ (the $\dot{C}_{\gamma_m}(n)$'s are pairwise disjoint). Let $a_\beta = \{k_m : m \in \omega\}$. If the preceding set is empty, just let a_β be an infinite subset of ω almost disjoint from every a_ξ ($\xi < \beta$). This makes a_β satisfy b).

If both hypothesis of b) and c) holds, define a^0_β as in the case where only the hypothesis from b) hold, a^1_β as in the case where only the hypothesis from c) hold and let $a_\beta = a^0_\beta \cup a^1_\beta$. This completes the construction. $\square$

Theorem 4.3.13 (*). [ZFC+CH] Let $\mathcal{A}'$ be an infinite countable almost disjoint family. There exists a tight MAD family $\mathcal{A}$ containing $\mathcal{A}'$ such that in every Cohen extension, $\exp(\Psi(\mathcal{A}))$ remains pseudocompact.

Proof. Working in a ctm M, let $((\dot{C}_\alpha, p_\alpha) : \omega \leq \alpha < \omega_1)$, $(f_\gamma : \omega \leq \gamma < \omega_1)$ and $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\}$ be as in the previous theorem. We claim that in every Cohen extension, $\exp(\Psi(\mathcal{A}))$ is pseudocompact. Suppose κ is an infinite cardinal and that G is $\mathbb{C}_\kappa$-generic over M. Suppose by contradiction that, in $M[G]$, $\exp(\Psi(\mathcal{A}))$ is not pseudocompact. Then, in $M[G]$, there exists a block sequence on ω, $C : \omega \to [\omega]^\omega$, with no accumulation point in $\exp(\Psi(\mathcal{A}))$. By Lemma 4.3.6, for every $J \in [\omega_1^M]^{<\omega}$, $\{n \in \omega : C_n \subseteq \bigcup_{\alpha \in J} a_\alpha\}$ is finite.

Let $S \subseteq \kappa$ be infinite countable such that $Q_0 = \mathrm{Fn}(S, 2)$ and $H_0 = G \cap Q$ are such that $C \in M[H_0]$. Let $Q_1 = \mathrm{Fn}(\kappa \setminus I) \cap G$ and $H = G \cap Q_1$. Then $M[H_0][H_1] = M[G]$. Since $Q_0 \approx \mathbb{C}$, there exists a generic filter K over $\mathbb{C}$ such that $M[K] = M[H_0]$.

There exists $\dot{C} \in M^{\mathbb{C}}$ such that $\dot{C}_K = C$ and such that $\dot{C}$ is a nice name for a subset of $\omega \times \check{[\omega]}^{<\omega}$. There exists $p \in K$ such that:

1. $p \Vdash \dot{C} : \check{\omega} \to [\omega]^{<\omega}\check{} \setminus \{\emptyset\}$ is a sequence of pairwise disjoint sets, and
2. $\forall J \in [\omega_1^M]^{<\omega}\, (p \Vdash |\{n \in \check{\omega} : \dot{C}_n \subseteq \bigcup_{\alpha \in \check{J}} a_\alpha\}| < \omega)$.

So, there exists $\gamma \in [\omega, \omega_1)$ such that $(\dot{C}, p) = (\dot{C}_\gamma, p_\gamma)$. Working on $M[G]$, we aim to get a contradiction by appling Lemma 4.3.7 by letting I be $I(\mathcal{A}|\gamma, C, f_\gamma)$. We already know that (i) holds. Since being centered for transitive models of ZFC, we may verify (ii) holds on $M[K]$. So let $F \in [[\gamma, \omega_1)]^{<\omega}$ and write $P = \{\alpha_0, \ldots, \alpha_l\}$ with $\alpha_0 < \cdots < \alpha_l$. For $i \leq l$, let $P_i = \{\alpha_0, \ldots, \alpha_i\}$. We proceed by induction for $i \leq l$ to show that:

$$p_\gamma \Vdash |\{n \in I(\check{\mathcal{A}|}\gamma, \dot{C}_\gamma, \check{f}_\gamma) : \forall \xi \in \check{P}_i\, (\dot{C}_\gamma(n) \cap a_\xi \neq \emptyset)\}| = \omega.$$

which will complete the proof.

To see that it holds for $i = 0$, let $\beta = \alpha_0$. Then $\forall J \in [\beta]^{<\omega}\, (p_\gamma \Vdash |\{n \in I(\check{\mathcal{A}|}\gamma, \dot{C}_\gamma, \check{f}_\gamma) : \forall \xi \in \check{\emptyset}\, (\dot{C}_\gamma(n) \cap a_\xi \neq \emptyset)$ and $\dot{C}_\gamma(n) \setminus \bigcup_{\xi \in J} a_\xi \neq \emptyset\}| = \omega)$ is logically equivalent to $\forall J \in [\beta]^{<\omega}\, (p_\gamma \Vdash |\{n \in I(\check{\mathcal{A}|}\gamma, \dot{C}_\gamma, \check{f}_\gamma) : \dot{C}_\gamma(n) \setminus \bigcup_{\xi \in J} a_\xi \neq \emptyset\}| = \omega)$ which holds, by 2.. Therefore:

$$p_\gamma \Vdash |\{n \in I(\check{\mathcal{A}|}\gamma, \dot{C}_\gamma, \check{f}_\gamma) : \forall \xi \in \{\check{\alpha}_0\}\, (\dot{C}_\gamma(n) \cap a_\xi \neq \emptyset)\}| = \omega$$

Now, suppose we have proved or claim for some $i < l$. We prove it for $i + 1$. This time, let $\beta = \alpha_{i+1}$. We already know that $p_\gamma \Vdash |\{n \in I(\check{\mathcal{A}|}\gamma, \dot{C}_\gamma, \check{f}_\gamma) : \forall \xi \in \check{P}_i\, (\dot{C}_\gamma(n) \cap a_\xi \neq \emptyset)\}| = \omega$. Again, by 2., it follows that $\forall J \in [\beta]^{<\omega}\, (p_\gamma \Vdash |\{n \in I(\check{\mathcal{A}|}\gamma, \dot{C}_\gamma, \check{f}_\gamma) : \forall \xi \in \check{P}_i\, (\dot{C}_\gamma(n) \cap a_\xi \neq \emptyset)$ and $\dot{C}_\gamma(n) \setminus \bigcup_{\xi \in J} a_\xi \neq \emptyset\}| = \omega)$, which implies that:

$$p_\gamma \Vdash |\{n \in I(\check{\mathcal{A}|}\gamma, \dot{C}_\gamma, \check{f}_\gamma) : \forall \xi \in \check{P}_i \cup \{\check{\alpha}_{i+1}\}\, (\dot{C}_\gamma(n) \cap a_\xi \neq \emptyset)\}| = \omega,$$

completing the proof.

□

4.4 An example of non pseudocompact small MAD family

All our examples of non pseudcompact MAD families we have discussed so far had cardinality $\mathfrak{c}$. Thus, it is natural to ask if it is consistent that there exists a non pseudocompact MAD family of size $< \mathfrak{c}$.

In particular, it would be interesting to know if ZFC implies that every MAD family of size ω_1 is pseudocompact. We do not know the answer for this question. In this section, we provide a consistent example of a MAD family of size $\omega_2 < \mathfrak{c}$ which is not pseudocompact. These techniques cannot be easily modified to make it an example of size ω_1.

The construction in this section appears in Section 3 of our paper [34].

The example we construct will be a MAD family over the countably infinite set $\triangle = \{(n,m) \in \omega \times \omega : m \leq n\}$. The elements of our MAD family will be graphs of partial functions. The result will follow from the theorem below:

Theorem 4.4.1 (*). It is consistent with $\mathfrak{c} > \omega_2$ that there is a MAD family $\mathcal{A}$ of size ω_2 on Δ consisting of partial functions, and there are MAD families $(\mathcal{A}_\alpha : \alpha < \omega_1)$ on ω, such that

1. $\forall s \in \mathcal{A}\, \exists \alpha < \omega_1\, \mathrm{dom}(s) \in \mathcal{A}_\alpha$,
2. $s \neq t \in \mathcal{A} \Rightarrow \mathrm{dom}(s) \neq \mathrm{dom}(t)$, and
3. for every family $\mathcal{F}$ of ω_1-many partial functions below the diagonal there is a total function below the diagonal almost disjoint from all elements of $\mathcal{F}$.

We shall postpone the proof of the theorem and first show that it suffices to prove the desired result. To see that such a $\mathcal{A}$ is not pseudocompact we mimic, in some sense, the proof of 4.1.8 b).

Theorem 4.4.2 (*). It is relatively consistent with ZFC that there is a non-pseudocompact MAD family $\mathcal{A}$ of size $< \mathfrak{c}$.

Proof. Assume that $\mathfrak{c} > \omega_2$ and there exist $\mathcal{A}$ and $(\mathcal{A}_\alpha : \alpha < \omega_1)$ as in Theorem 4.4.1. We shall show that $\exp(\Psi(\mathcal{A}))$ is not pseudocompact.

Let $F = (F_n : n \in \omega)$ be the sequence of elements of $\exp(\Psi(\mathcal{A}))$ given by $F_n = \{(n,m) : m \leq n\}$. We claim that F has no accumulation point in $\exp(\Psi(\mathcal{A}))$, which completes the proof by Proposition 0.3.20. Suppose L is such an accumulation point. Then, since F is a sequence of pairwise disjoint finite subsets of $\triangle$ and for each pair (n,m) with $m \leq n$, $\{(n,m)\}^-$ is open, $L \subseteq \mathcal{A}$.

If $|L| < \omega_2$, there exists a total function f below the diagonal which is almost disjoint from every element of L. Then $L \in (\Psi(\mathcal{A}) \setminus \mathrm{cl}\, f)^+$ but $F_n \notin (\Psi(\mathcal{A}) \setminus \mathrm{cl}\, f)^+$ for every $n \in \omega$, a contradiction.

Now suppose $|L| = \omega_2$. There exists $\alpha < \omega_1$ such that there exists two distinct $s, t \in \mathcal{A}$ such that $\mathrm{dom}\, s, \mathrm{dom}\, t \in \mathcal{A}_\alpha$. Since s, t are distinct, it follows that $\mathrm{dom}(s) \neq \mathrm{dom}(t)$, and since $\mathcal{A}_\alpha$ is an almost disjoint family, $\mathrm{dom}\, s \cap \mathrm{dom}\, t \subseteq k$ for some $k \in \omega$. Then

$$L \in (\{s\} \cup \{s \setminus \{(n,m) : m \leq n < k\}\})^- \cap (\{t\} \cup \{t \setminus \{(n,m) : m \leq n < k\}\})^-,$$

but no element of the sequence F is a member of the latter open set. $\square$

Now we recall the definition of the *Mathias Forcing* [51].

Definition 4.4.3. Let $\mathcal{A}$ be a (possibly finite) almost disjoint family. The *Mathias forcing* $\mathbb{M}(\mathcal{A})$ associated with $\mathcal{A}$ is defined as follows: the base set is $2^{<\omega} \times [\mathcal{A}]^{<\omega}$. Given $p \in \mathbb{M}(\mathcal{A})$, we write $p = (s_p, F_p)$ and $n_p = |s_p|$. We call s_p the *stem of* p and F_p the *side condition of* p. The *length of* p is $\mathrm{len}(p) = n_p$.

We order $\mathbb{M}(\mathcal{A})$ by letting $p = (s_p, F_p) \leq q = (s_q, F_q)$ iff

1. $s_q \subseteq s_p$ (hence $n_q \leq n_p$), $F_q \subseteq F_p$, and
2. if $B \in F_q$, then $B \cap s_p^{-1}(1) \subseteq n_q$.

□

If $G \subseteq \mathbb{M}(\mathcal{A})$ is a generic filter over V, the *generic real of* $\mathbb{M}(\mathcal{A})$ is defined denoted by $A_{gen} = \{i \mid \exists (s, F) \in G\, (s(i) = 1)\}$ when no confusion arises. More formally, we may define the following:

Definition 4.4.4. Let $\mathcal{A}$ be an almost disjoint family. $\dot{A}_{gen}$ is a name satisfying $\mathbb{M}(\mathcal{A}) \Vdash \dot{A}_{gen} = \{i : \exists(s, F) \in \mathring{G}(s(i) = 1)\}$. □

Of course, $\dot{A}_{gen}$ depends on $\mathcal{A}$, but we use this notation when no confusion arises. It also depends on a choice, since more than one $\mathbb{M}(\mathcal{A})$-name satisfies such an expression.

The following lemma is well-known and easy to prove. We leave it as an exercise to the reader.

Lemma 4.4.5. Let $\mathcal{A}$ be an almost disjoint family and consider the forcing poset $\mathbb{M}(\mathcal{A})$. $\mathcal{A}$ may be finite (or even empty), as long as $\omega \setminus \bigcup \mathcal{A}$ is not finite. Then:

1. $\mathbb{1} \Vdash \dot{A}_{gen} \in [\omega]^{\omega}$,
2. $\mathbb{1} \Vdash \forall a \in \check{\mathcal{A}}\, |a \cap \dot{A}_{gen}| < \omega$,
3. $\mathbb{1} \Vdash \forall x \in \mathcal{I}(\mathcal{A})^{+}(x \in V \to |x \cap \dot{A}_{gen}| = \omega)$, and
4. If $\mathcal{A}$ is finite (or even empty), $\mathbb{1} \Vdash |\omega \setminus \bigcup \check{\mathcal{A}} \cup \dot{A}_{gen}| = \omega$.

The following definitions will be very useful in this section:

Definition 4.4.6. Fun denotes the set of all functions $f : \omega \longrightarrow \omega$ such that $f \subseteq \Delta$.

PFun is the set of all functions g such that there is $A \in [\omega]^{\omega}$ for which $g : A \longrightarrow \omega$ and $g \subseteq \Delta$. □

Note that if $f, g \in \mathrm{PFun}$ then f and g are almost disjoint if and only if the set $\{n \in \mathrm{dom}\,(f) \cap \mathrm{dom}\,(g) \mid f\,(n) = g\,(n)\}$ is finite.

Definition 4.4.7 (*). $\mathfrak{ie}$ is the smallest size of a family $\mathcal{F} \subseteq \mathrm{PFun}$ such that for every $g \in \mathrm{Fun}$ there is $f \in \mathcal{F}$ such that $|f \cap g| = \omega$. $\mathfrak{ie}$ stands for **i**nfinitelly **e**qual. □

Now we define the following forcing poset, which is analogous to Mathias Forcing.

Definition 4.4.8 (*). Let $X \in [\omega]^{\omega}$ and $\mathcal{B} \subseteq \mathrm{PFun}$. Define the forcing $\mathbb{E}_{\Delta}(\mathcal{B}, X)$ as the set of all $p = (s_p, n_p, F_p)$ with the following properties:

1. $n_p \in \omega$, $F_p \in [\mathcal{B}]^{<\omega}$.
2. $s_p : X \cap n_p \longrightarrow \omega$ and $s_p \subseteq \Delta$.
3. $2\,|F_p| \leq n_p$.

Given $p = (s_p, n_p, F_p) \in \mathbb{E}_\Delta(\mathcal{B}, X)$, we call s_p the *stem of* p, n_p the *length of* p and F_p the *side condition of* p.

Let $p = (s_p, n_p, F_p)\,, q = (s_q, n_q, F_q) \in \mathbb{E}_\Delta(\mathcal{B})\,,$ we define $p \leq q$ iff the following conditions hold:

1. $n_q \leq n_p$, $F_q \subseteq F_p$ and $s_q \subseteq s_p$.
2. For every $f \in F_q$ and $i \in \operatorname{dom} f \cap (X \cap (n_p \setminus n_q))$, $s_p(i) \neq f(i)$.

$\mathbb{E}_\Delta$ denotes $\mathbb{E}_\Delta(\mathrm{Fun}, \omega)$. □

If $G \subseteq \mathbb{E}_\Delta(\mathcal{B}, X)$ is a $\mathbb{E}_\Delta(\mathcal{B}, X)$-generic filter, over V the *generic real of* $\mathbb{E}_\Delta(\mathcal{B}, X)$ is defined as $f_{gen} = \bigcup\{s \mid \exists\,(s, n, F) \in G\}$ when no confusion arises. As before, more formally, we define the following:

Definition 4.4.9 (*). Let $X \in [\omega]^\omega$ and $\mathcal{B} \subseteq \mathrm{PFun}$. $\dot{f}_{gen}$ is a name satisfying $\mathbb{E}_\Delta(\mathcal{B}, X) \Vdash \dot{f}_{gen} = \bigcup\{s : \exists(s, n, F) \in \mathring{G}\}$. □

As before, $\dot{f}_{gen}$ depends on X, $\mathcal{B}$ and on a choice, but we use this notation when no confusion arises.

The analogue of Lemma 4.4.5 is the following:

Lemma 4.4.10 (*). Let $X \in [\omega]^\omega$, $\mathcal{B} \subseteq \mathrm{PFun}$. Then:

1. $\mathbb{E}_\Delta(\mathcal{B}, X) \Vdash \dot{f}_{gen} : \check{X} \longrightarrow \omega$ and $\dot{f}_{gen} \subseteq \Delta$.
2. $\mathbb{E}_\Delta(\mathcal{B}, X) \Vdash \dot{f}_{gen}$ is almost disjoint from every element of $\check{\mathcal{B}}$.
3. If $g \in \mathrm{PFun}$ is such that $\operatorname{dom}(g) \subseteq X$ and $g \in \mathcal{I}(\mathcal{B})^+$, then $\mathbb{E}_\Delta(\mathcal{B}, X) \Vdash |\dot{f}_{gen} \cap g| = \omega$ (where $\mathcal{I}(\mathcal{B}) = \{A \subseteq \triangle : \exists \mathcal{B}' \in [\mathcal{B}]^{<\omega} A \subseteq^* \mathcal{B}'\}$ is the free ideal generated by $\mathcal{B}$).

Let $\mathbb{P}$ be a partial order. Recall that a set $L \subseteq \mathbb{P}$ is *linked* if every $p, q \in L$ are compatible. $\mathbb{P}$ is σ-linked if $\mathbb{P}$ is the union of countably many linked sets. The following estabilishes that $\mathbb{E}_\Delta(\mathcal{B}, X)$ is σ-linked:

Lemma 4.4.11 (*). Let $X \in [\omega]^\omega$ and $\mathcal{B} \subseteq \mathrm{Fun}$. Let $p = (s_p, n_p, F_p)$, $q = (s_q, n_q, F_q) \in \mathbb{E}_\Delta(\mathcal{B}, X)$. If $s_p = s_q$ and $4\,|F_p|\,, 4\,|F_q| \leq n_p$ then $r = (s_p, n_p, F_p \cup F_q)$ extends both p and q.

Proof. Let $p = (s_p, n_p, F_p)\,, q = (s_q, n_q, F_q) \in \mathbb{E}_\Delta(\mathcal{B}, X)$ with $s = s_p = s_q$. We first find a finite partial function $t \subseteq \Delta$ with the following properties:

1. $s \subseteq t$.
2. For every $f \in F_p \cup F_q$ and $i \in \operatorname{dom}(t) \setminus \operatorname{dom}(s)$, we have that $t(i) \neq f(i)$.
3. $|t| \geq 2\,|F_p \cup F_q|$.

We can find such t since $4\left|F_p\right|, 4\left|F_q\right| \leq n_p$. It follows that $r=\left(t, \operatorname{dom}(t), F_p \cup F_q\right)$ is an extension of both p and q. □

Lemma 4.4.12 (*). $\mathbb{E}_{\Delta}(\mathcal{B}, X)$ is σ-linked.

Proof. For every $n \in \omega$ and $s : X \cap n \rightarrow \omega$ with $s \subseteq \Delta$, define

$$L(s, n)=\left\{q \mid \exists p \leq q\ p=\left(s_p, n_p, F_p\right)\ n_p=n,\ s_p=s \text{ and } 4\left|F_p\right| \leq n_p\right\}.$$

Clearly each $L(s, n)$ is linked by the previous lemma and

$$\mathbb{E}_{\Delta}(\mathcal{B}, X)=\bigcup\left\{L(s, n) : n \in \omega, s \subseteq \Delta, s \in \omega^{X \cap n}\right\}.$$

□

In particular, this implies that $\mathbb{E}_{\Delta}(\mathcal{B}, X)$ has the countable chain condition.

Due to the previous lemmas, the following definition comes handy:

Definition 4.4.13 (*). Let $\mathcal{B} \subseteq$ Fun and $X \in[\omega]^{\omega}$. We say that $p=\left(s_p, n_p, F_p\right) \in \mathbb{E}_{\triangle}(\mathcal{B}, X)$ has the *four property* iff $4|F_p| \leq n_p$. □

The following result was inspired by Lemma 5.1 of A. Miller's [56]:

Proposition 4.4.14 (*). Let $n \in \omega$, $s : n \longrightarrow \omega$ with $s \subseteq \Delta$. Let $D \subseteq \mathbb{E}_{\Delta}$ be an open dense set. There is an antichain $Z \in[D]^{<\omega}$ such that for every $p=\left(s, n, F_p\right) \in \mathbb{E}_{\Delta}$, there is $q \in Z$ such that p and q are compatible.

Proof. Let $A=\left\{r_m \mid m \in \omega\right\} \subseteq D$ be a maximal antichain (note that A is countable since $\mathbb{E}_{\Delta}$ is σ-linked and therefore c.c.c.), let $k=\frac{n}{2}$ in case n is even and $k=\frac{n-1}{2}$ in case n is odd.

Assume the proposition is false, so for every $m \in \omega$, there is $p_m=\left(s, n, F_m\right) \in \mathbb{E}_{\Delta}$ such that $p_m \perp r_i$ for each $i \leq m$. As $\left|F_m\right| \leq k$ we can assume that each F_m has size k, let $F_m=\left\{f_i^m\right\}_{i<k}$. We may view $B=\left\{F_m \mid m \in \omega\right\}$ as a subset of Fun^k. Since Fun^k is a compact space, we can find an accumulation point $\left(g_i\right)_{i<k}$ of B. Let $F=\left\{g_i\right\}_{i<k}$ and $p=(s, n, F)$. Since A is a maximal antichain, there is $j \in \omega$ such that p and r_j are compatible. Let $q=(t, l, G)$ be a common extension of both of them. Since F is an accumulation point of B, there is $m>l, j$ such that $f_i^m | l=g_i | l$ for every $i<k$. Let $\overline{p}_m=\left(t, l, F_m\right)$ and note that $\overline{p}_m \leq p_m$. It follows that $\overline{p}_m$ and q are compatible, in particular, p_m and q are compatible, which implies that p_m and r_j are compatible, which is a contradiction. □

Now we aim to define an iterated forcing notion.

For the rest of the section, we fix a family of sets $\left(D_\gamma \mid \gamma \in \omega_1\right)$, sets H and E and a function R with the following properties:

1. $\{H, E\} \cup\left\{D_\gamma \mid \gamma \in \omega_1\right\}$ is a partition of ω_2 of pairwise distinct sets,
2. For every $\gamma \in \omega_1$, we have that $\left|D_\gamma\right|=|H|=|E|=\omega_2$.

3. $R : \bigcup_{\gamma\in\omega_1} D_\gamma \to H$ is a bijective function such that $\alpha < R(\alpha)$ for every $\alpha \in \bigcup_{\gamma\in\omega_1} D_\gamma$.

Then we define a finite support iteration by the following proposition:

Proposition 4.4.15 (*). There exists an iterated forcing construction with finite supports $((\mathbb{P}_\alpha, \leq_\alpha, \mathbb{1}_\alpha)_{\alpha\leq\omega_2}, (\mathring{\mathbb{Q}}_\alpha, \mathring{\leq}_\alpha, \mathring{\mathbb{1}}_\alpha)_{\alpha<\omega_2})$ and families $(\dot{\mathcal{A}}^\xi_\gamma : \gamma < \omega_1, \xi \leq \omega_2)$, $(\dot{A}_\xi : \xi \in \bigcup_{\gamma<\omega_1} D_\gamma)$, $(\dot{\mathcal{B}}_\xi : \xi < \omega_2)$ and $(\dot{f}_\eta : \eta \in H)$ such that:

i) For each $\alpha \in E$, $\mathbb{P}_\alpha \Vdash \mathring{\mathbb{Q}}_\alpha = \mathbb{E}_\Delta$.

ii) For each $\gamma \in \omega_1$ and $\xi \in D_\gamma$, $\dot{A}_\xi$ is a $\mathbb{P}_{\xi+1}$-name.

iii) For each $\gamma \in \omega_1$ and $\xi \leq \omega_2$, $\dot{\mathcal{A}}^\xi_\gamma$ is a $\mathbb{P}_\xi$ name and $\dot{\mathcal{A}}^\xi_\gamma = \{(\dot{A}_\beta, \mathbb{1}_\beta) : \beta \in D_\gamma \cap \xi\}$.

iv) For each $\gamma \in \omega_1$ and $\xi \leq \omega_2$, $\mathbb{P}_\xi \Vdash \dot{\mathcal{A}}^\xi_\gamma$ is an almost disjoint family.

v) For each $\gamma \in \omega_1$ and $\xi \in D_\gamma$, $\mathbb{P}_\xi \Vdash \mathring{\mathbb{Q}}_\xi = \mathbb{M}(\dot{\mathcal{A}}^\xi_\gamma)$.

vi) For each $\gamma \in \omega_1$ and $\xi \in D_\gamma$, $\dot{A}_\xi$ evaluates as the $\mathbb{M}(\dot{\mathcal{A}}^\xi_\gamma)$ generic real, that is:

$$\mathbb{P}_{\xi+1} \Vdash \dot{A}_\xi = \{i : \exists p \in \mathring{G}_{\xi+1}\, \exists (s, F)\ (p(\xi) = (s, F) \text{ and } s(i) = 1)\}.$$

vii) For each $\xi \in H$, $\dot{f}_\xi$ is a $\mathbb{P}_{\xi+1}$-name.

viii) For each $\xi \leq \omega_2$, $\dot{\mathcal{B}}_\xi$ is a $\mathbb{P}_\xi$ name and $\dot{\mathcal{B}}_\xi = \{(\dot{f}_\eta, \mathbb{1}_\eta) : \eta \in H \cap \xi\}$.

ix) For each $\xi \leq \omega_2$, $\mathbb{P}_\xi \Vdash \dot{\mathcal{B}}_\xi \subseteq \mathrm{PFun}$.

x) For each $\xi \in H$ and every $\beta < \omega_2$, if $R(\beta) = \xi$ then $\mathbb{P}_\xi \Vdash \mathring{\mathbb{Q}}_\xi = \mathbb{E}_\triangle(\dot{\mathcal{B}}_\xi, \dot{A}_\beta)$.

xi) For each $\xi \in H$ and every $\beta < \omega_2$, if $R(\beta) = \xi$, $\dot{f}_\xi$ evaluates as the $\mathbb{E}_\triangle(\dot{\mathcal{B}}_\xi, \dot{A}_\beta)$ generic real, that is:

$$\mathbb{P}_{\xi+1} \Vdash \dot{f}_\xi = \bigcup\{s : \exists p \in \mathring{G}_{\xi+1}\, \exists n\, \exists F\, p(\xi) = (s, n, F)\}. \quad \square$$

Such iterated forcing construction with finite supports $((\mathbb{P}_\alpha, \leq_\alpha, \mathbb{1}_\alpha)_{\alpha\leq\omega_2}, (\mathring{\mathbb{Q}}_\alpha, \mathring{\leq}_\alpha, \mathring{\mathbb{1}}_\alpha)_{\alpha<\omega_2})$ and families $(\dot{\mathcal{A}}^\xi_\gamma : \gamma < \omega_1, \xi \leq \omega_2)$, $(\dot{A}_\xi : \xi \in \bigcup_{\gamma<\omega_1} D_\gamma)$, $(\dot{\mathcal{B}}_\xi : \xi < \omega_2)$ and $(\dot{f}_\eta : \eta \in H)$ exist by standard iterated forcing techniques, similarly as done in Proposition 4.3.4. For the rest of this section, we fix such families.

We will need to develop some notation and combinatorial tools for our forcing in order to prove the main result. First, some basic notation:

Definition 4.4.16 (*). Let $\alpha < \omega_2$. + Given $p \in \mathbb{P}_\alpha$ and $\dot{x}$ is a $\mathbb{P}_\alpha$-name such that $p \Vdash_\alpha \dot{x} \in \dot{\mathbb{Q}}_\alpha$, we denote by $p^\frown\dot{x}$ a condition $r \in \mathbb{P}_{\alpha+1}$ such that $\operatorname{dom} r = \operatorname{dom} p \cup \{\alpha\}$, $p \subseteq r$ and $p \Vdash_\alpha p(\alpha) = \dot{x}$. We note that every pair of conditions r, r' satisfying this are such that $r \leq r'$ and $r' \leq r$, so $p^\frown\dot{x}$ is well defined mod $\mathbb{P}_{\alpha+1}$-equivalence. We may use the axiom of choice to fix one of them. $\square$

Definition 4.4.17 (*). Let $\alpha \leq \omega_2$. We say that a condition $p \in \mathbb{P}_\alpha$ is *pure* iff there is $n \in \omega$ such that for every $\xi \in \operatorname{dom} p$:

1. If there exists $\gamma \in \omega_1$ such that $\xi \in D_\gamma$, then there exists $s_\xi \in 2^n$ and $J_\xi \in [D_\gamma \cap \xi]^{<\omega}$, $J_\xi \subseteq \operatorname{dom}(p)$ such that $p|\xi \Vdash_\xi p(\xi) = (\check{s}_\xi, \{(\dot{A}_\eta, \mathbb{1}_\xi) \mid \eta \in J_\xi\})$.
2. If $\xi \in H$, let $\beta = R^{-1}(\xi) \in \bigcup_{\gamma<\omega_1} D_\gamma$. Then $\beta \in \operatorname{dom} p$ and there is $z_\xi : s_\beta^{-1}(1) \longrightarrow \omega$ with $z_\xi \subseteq \Delta$ and $J_\xi \in [H \cap \xi]^{<\omega}$, $J_\xi \subseteq \operatorname{dom}(p)$ such that $4|J_\xi| \leq n$ and $p|\xi \Vdash_\xi p(\xi) = (\check{z}_\xi, \check{n}, \{(\dot{f}_\eta, \mathbb{1}_\xi) \mid \eta \in J_\xi\})$ (where s_β is defined as in 1.).
3. If $\xi \in E$, then there is $m_\xi, k_\xi \in \omega$, with $4k_\xi \leq m_\xi$, $z_\xi : m_\xi \to \omega$ with $z_\xi \subseteq \Delta$ and $\rho_0, \ldots, \rho_{k_\xi-1}$ $\mathbb{P}_\xi$-names for elements of ω^ω such that by letting $\dot{J} = \{(\rho_0, \mathbb{1}_\xi), \ldots, (\rho_{k_\xi-1}, \mathbb{1}_\xi)\}$ we have $p|\xi \Vdash_\xi p(\xi) = (\check{z}_\xi, \check{m}_\xi, \dot{J})$.

Given a pure condition p, $\operatorname{len}(p)$ denotes the size of the first coordinate of p. We call n *the height of* p. In case there is more than one such n (i.e. $\operatorname{dom} p \subseteq E$), the height of p is 0. $\square$

An important difference between points 2. and 3. is that in point 3. we may have $m_\xi \neq n$. One of the purposes of pure conditions is to avoid (as much as possible) the use of names by using "real" objects.

Lemma 4.4.18 (*). For every $\alpha \leq \omega_2$ and $u \in \omega$, pure conditions of height $\geq u$ are dense in $\mathbb{P}_\alpha$.

Proof. We prove the lemma by induction on α. The cases where $\alpha = 0$ or α is limit are straightforward, so we focus on the successor case. Assume the lemma is true for α, we will prove it is also true for $\alpha + 1$. Let $p \in \mathbb{P}_{\alpha+1}$, we may assume that $\alpha \in \operatorname{dom}(p)$.

Case. $\alpha \in E$.

Fix a name $\dot{L}$, $\dot{z}$, $\dot{m}$ be $\mathbb{P}_\alpha$-names such that $p|\alpha \Vdash_\alpha p(\alpha) = (\dot{z}, \dot{m}, \dot{L})$.

Let $p_0 \leq p|\alpha$ be an element of $\mathbb{P}_\alpha$ and $m_\alpha \in \omega$ be such that $p_0 \Vdash_\alpha \max\{\dot{m}, 4|\dot{L}|\} = \check{m}_\alpha$. There exists a name $\dot{z}_1$ such that $p_0 \Vdash_\alpha (\dot{z}_1, \check{m}_\alpha, \dot{L}) \in \mathring{Q}_\alpha$ and$(\dot{z}_1, \check{m}_\alpha, \dot{L}) \leq (\dot{z}_1, \check{m}_\alpha, \dot{L})$.

Fix $z_\alpha \in \omega^{m_\alpha}$ with $z_\alpha \subseteq \triangle$ and $p_1 \leq p_0$ with $p_1 \in \mathbb{P}_\alpha$ such that $p_1 \Vdash_\alpha \dot{z}_1 = \check{z}_\alpha$. Then there exists $p_2(\alpha)$ such that $p_2 = p_1 \cup \{(\alpha, p_2(\alpha))\} \in \mathbb{P}_{\alpha+1}$ and $p_1 \Vdash p_2(\alpha) = (\check{z}_\alpha, \check{m}_\alpha, \dot{L})$. It is clear that $p_2 \leq p$.

Let $p_3 \leq p_1 = p_2|\alpha$, $k_\alpha \in \omega$ and $\rho_0, \ldots, \rho_{k_\alpha-1}$ be such that $p_3 \Vdash |\dot{L}| = \check{k}_\alpha$ and $\dot{L} = \{\rho_0, \ldots, \rho_{\check{k}_\alpha-1}\}$. Let $\dot{J} = \{(\rho_0, \mathbb{1}_\xi), \ldots, (\rho_{k_\alpha-1}, \mathbb{1}_\xi)\}$ It is clear that $p_3 \Vdash \dot{L} = \dot{J}$. Of course, $4k_\alpha \leq m_\xi$. By the inductive hypothesis there exists a pure condition of height $\geq u$, $p_4 \in \mathbb{P}_\alpha$, such that $p_4 \leq p_3$. Let $q = p_4 \cup \{(\alpha, p_2(\alpha))\}$ and we are done.

Case. $\alpha \in D_\gamma$ (for some $\gamma \in \omega_1$).

First, we find $p_1 \leq p|\alpha$ such that there are $m \in \omega$, $s \in 2^m$ and $J_\alpha \in [D_\gamma \cap \alpha]^{<\omega}$ such that $p_1 \Vdash p(\alpha) = (s, \{(\dot{A}_\eta, \mathbb{1}_\alpha) : \eta \in J_\alpha\})$. To see that this is possible, let σ be the $\mathbb{P}_\alpha$-name given by $\sigma = \{\mathrm{op}(\check{\eta}, \dot{A}_\eta) : \eta \in D_\gamma \cap \alpha\}$. Then:

$$p|\alpha \Vdash_\alpha \exists J_\alpha \in ([D_\gamma \cap \alpha]^{<\omega})\check{}\; \exists m \in \check{\omega}\, \exists s \in 2^m\, p(\alpha) = (s, \sigma[J_\alpha]).$$

So there exists $p_1 \leq p|\alpha$ fixing J_α, m and s, as intended.

We may assume that $J_\alpha \subseteq \mathrm{dom}\,(p_1)$. By the inductive hypothesis, let $q \leq p_1$ be a pure condition of height $n \geq \max\{u, m\}$. Let $s_\alpha \in 2^n$ such that $s_\alpha|m = s$ and $s_\alpha\,(i) = 0$ for every $i \in [m, n)$. Let $\bar{q} \in \mathbb{P}_{\alpha+1}$ such that the following holds:

1. $\bar{q}|\alpha = q$.
2. $\bar{q}|\alpha \Vdash_\alpha \bar{q}(\alpha) = (s_\alpha, \{(\dot{A}_\eta, \mathbb{1}_\alpha) : \eta \in J_\alpha\})$.

It is easy to see that q is a pure extension of p.

Case. $\alpha \in H$.

Let $\beta = R^{-1}(\alpha)$. Let $\dot{s}, \dot{m}, \dot{L}$ be $\mathbb{P}_\alpha$-names such that $p|\alpha \Vdash p(\alpha) = (\dot{s}, \dot{m}, \dot{L})$. There exists names $\dot{x}$, $\dot{s}'$ and $\dot{m}'$ such that $p|\alpha \Vdash \dot{x} = (\dot{s}', \dot{m}', \dot{L}) \leq (\dot{s}, \dot{m}, \dot{L})$ and $4|\dot{L}| < \dot{m}'$. Now, similarly to the previous case, we find $p_1 \leq p|\alpha$ such that there are $m \in \omega$, a partial function $s : m \to \omega$ with $s \subseteq \Delta$ and $J_\alpha \in [H \cap \alpha]^{<\omega}$ such that $4|J_\alpha| < m$ and:

$$p_1 \Vdash \dot{x} = (\check{s}, \check{m}, \{(\dot{f}_\eta, \mathbb{1}_\alpha) : \eta \in J_\alpha\}).$$

Without loss of generality, we may assume $J_\alpha \cup \{\beta\} \subseteq \mathrm{dom}\,(p_1)$. By the inductive hypothesis, let $q \leq p_1$ be a pure condition of height $n \geq \max u, m$ witnessing that q is pure.

Notice that $p_1 \Vdash \mathrm{dom}\, s = \dot{A}_\beta \cap \check{m} \subseteq \dot{A}_\beta \cap \check{n} = \check{s}_\beta^{-1}[\{1\}]$. Let $z_\alpha : \check{s}_\beta^{-1}[\{1\}] \to \omega$ be such that $z_\alpha \subseteq \Delta$, $z_\alpha|m = s$ and $z_\alpha\,(i) \neq z_\xi\,(i)$ for every $i \in [m, n)$ and $\xi \in J_\alpha$. Define $\overline{q} \in \mathbb{P}_{\alpha+1}$ such that the following holds:

1. $\overline{q}|\alpha = q$.
2. $q \Vdash \overline{q}(\alpha) = (\check{z}_\alpha, \check{n}, \dot{L})$.

It is easy to see that $\overline{q}$ is a pure extension of p. □

Lemma 4.4.19 (*). Let $\alpha \leq \omega_2$, $p \in \mathbb{P}_\alpha$ a pure condition and $m \in \omega$. There is $q \in \mathbb{P}_\alpha$ with the following properties:

1. $q \leq p$.
2. q is pure.
3. If $\beta \in \mathrm{dom}\,(q)$ then $m \leq \mathrm{len}\,(q\,(\beta))$.

Proof. We prove the lemma by induction on α. The cases where $\alpha = 0$ or α is limit are straightforward, so we focus on the successor case. Assume the lemma is true for α, we will prove it is also true for $\alpha + 1$. Let $p \in \mathbb{P}_{\alpha+1}$, we may assume that $\alpha \in \mathrm{dom}\,(p)$. Let n be the height of p.

Case. $\alpha \in E$.

Consider, for the coordinate α, z_α,m_α, k_α, $\rho_0, \ldots, \rho_{k_\alpha - 1}$ and $\dot{J}$ as in the definition of pure condition for p. In case that $m \leq m_\alpha$, we apply the inductive hypothesis to $p|\alpha$ and we are done. Assume that $m_\alpha < m$. By the inductive hypothesis, we may find $q \leq p|\alpha$ such that the following holds:

1. $q \in \mathbb{P}_\alpha$ is pure and $\delta = R^{-1}(\alpha) \in \operatorname{dom} q$.
2. If $\beta \in \operatorname{dom}(q)$ then $\operatorname{len} q(\beta) \geq m$.
3. For every $j < k_\xi$ there is $w_j : m \to \omega$ such that $q \Vdash \rho_j | m = w_j$.

We now define $z'_\alpha : m \to \omega$, with $z'_\alpha \subseteq \Delta$ such that $z_\alpha \subseteq z'_\alpha$ and $z'_\alpha(i) \neq w_j(i)$ for every $i \in [m_\alpha, m)$ and $j < n$. Let $\dot{x}$ be a $\mathbb{P}_\alpha$-name such that $q \Vdash \dot{x} = (\check{z}'_\alpha, \check{m}, \dot{J})$. It is clear that $q^\frown \dot{x}$ has the desired properties.

Case. $\alpha \in D_\gamma$ (for some $\gamma \in \omega_1$).

Consider, for the coordinate α, s_α and J_α for p as in the definition of pure condition. By the inductive hypothesis, we may find $q \leq p|\alpha$ in $\mathbb{P}_\alpha$ such that the following holds:

1. q is pure.
2. If $\beta \in \operatorname{dom}(p)$ then $\operatorname{len} q(\beta) \geq \max\{m, n\}$.

Let k be the height of q. Now we define $z : k \longrightarrow 2$ such that $s_\alpha \subseteq z$ and $z(i) = 0$ for every $i \in [n, k)$. Let $\dot{x}$ be a $\mathbb{P}_\alpha$-name such that $q \Vdash \dot{x} = (\check{z}, \{(\dot{A}_\eta, \mathbb{1}_\alpha) : \eta \in J_\alpha\})$. It is clear that $q^\frown \dot{x}$ has the desired properties.

Case. $\alpha \in H$.

Consider, for the coordinate α, n and z_α and J_α and n for p as in the definition of pure condition. Let $\beta = R^{-1}(\alpha)$. By the inductive hypothesis, we may find $q \leq p|\alpha$ in $\mathbb{P}_\alpha$ such that the following holds:

1. q is pure.
2. If $\delta \in \operatorname{dom}(p)$ then $\operatorname{len} q(\delta) \geq \max\{m, n\}$.

Note that $\beta \in \operatorname{dom} q$. Let k be the height of q. Let $s_\beta \in 2^k$ be as in the definition of pure condition for the coordinate β of q. Now we define $z : s_\beta^{-1}[\{1\}] \to \omega$ such that $z_\alpha \subseteq z$ and $z(i) \neq z_\xi(i)$ for every $\xi \in J_\alpha$, $i \in \operatorname{dom} z_\xi \cap s_\beta^{-1} \cap [n, k)$. Let $\dot{x}$ be a $\mathbb{P}_\alpha$-name such that $q \Vdash \dot{x} = (\check{z}, \check{n}, \{(\dot{f}_\eta, \mathbb{1}_\alpha) : \eta \in J_\alpha\})$. It is clear that $q^\frown \dot{x}$ has the desired properties. $\square$

Definition 4.4.20. Let $\alpha \leq \omega_2$ and $p \in \mathbb{P}_\alpha$ a pure condition. We say that p has the *descending condition* if for every $\beta_1, \beta_2 \in \operatorname{dom}(p) \cap E$, if $\beta_1 < \beta_2$, then $\operatorname{len}(p(\beta_1)) \geq \operatorname{len}(p(\beta_2))$. $\square$

Using the previous lemma and induction, we get the following:

Lemma 4.4.21 (*). For every $\alpha \leq \omega_2$, the pure conditions with the descending condition are dense.

Proof. We prove the lemma by induction on α. The cases where $\alpha = 0$ or α is limit are straightforward, so we focus on the successor case. Assume the lemma is true for α, we will prove it is also true for $\alpha + 1$. Let $p \in \mathbb{P}_{\alpha+1}$ be a pure condition, we may assume that $\alpha \in \operatorname{dom}(p)$. In case $\alpha \notin E$, there is nothing to do, so assume that $\alpha \in E$.

Let $p(\alpha) = \left(s, n, \dot{J}\right)$. By the inductive hypothesis and Lemma 4.4.19 we can find $q \in \mathbb{P}_\alpha$ such that $q \leq p|\alpha$, q is pure with the descending condition and all the stems in q have size larger than n. It is clear that $q^\frown \left(s, n, \dot{J}\right)$ is the condition we are looking for. □

Although pure conditions are nice to work with, we will need to deal with non-pure conditions for some arguments. We will develop the tools needed in order to do this.

Definition 4.4.22 (*). Given $A \in [E]^{<\omega}$, a function $K : A \longrightarrow \omega^{<\omega}$ is said to be *suitable* if $K(\alpha) \subseteq \Delta$ for every $\alpha \in A$. In this case, we say that a condition $q \in \mathbb{P}_{\omega_2}$ *follows* K if the following holds:

1. $A \subseteq \operatorname{dom}(q)$.
2. If $\alpha \in A$, then $q|\alpha \Vdash \exists F\, q(\alpha) = \left(\check{K(\alpha)}, |\check{K(\alpha)}|, F\right)$.

□

Definition 4.4.23 (*). Let $A \in [E]^{<\omega}$. We say that $p \in \mathbb{P}_\alpha$ has the A*-descending condition* if the following holds:

1. For every $\beta_1, \beta_2 \in (\operatorname{dom}(p) \setminus A) \cap E$, if $\beta_1 < \beta_2$, then $p|\beta_2 \Vdash \operatorname{len}(p(\beta_1)) \geq \operatorname{len}(p(\beta_2))$.
2. For every $\beta_1, \beta_2 \in \operatorname{dom}(p) \cap H$, if $\beta_1 < \beta_2$, then $p|\beta_2 \Vdash \operatorname{len}(p(\beta_1)) \geq \operatorname{len}(p(\beta_2))$.
3. For every $\gamma \in \omega_1$ and for every $\beta_1, \beta_2 \in \operatorname{dom}(p) \cap D_\gamma$, if $\beta_1 < \beta_2$, then $p|\beta_2 \Vdash \operatorname{len}(p(\beta_1)) \geq \operatorname{len}(p(\beta_2))$.
4. If $\beta = \min(\operatorname{dom}(p)) \setminus A$, then there exists n such that $p|\beta = \mathbb{1}_\beta \Vdash \check{n} = \operatorname{len}(p(\beta))$ and, for every $\eta \in \operatorname{dom}(p) \setminus A$, we have that $p|\eta \Vdash \operatorname{len}(p(\beta)) \geq \operatorname{len}(p(\eta))$.

□

Notice that this new notion does not clash with our previous terminology, since pure conditions with the descending condition satisfy the $\emptyset$-descending condition. We now introduce the following notions:

Definition 4.4.24 (*). Let $\alpha \leq \omega_2$, $A \in [E \cap \alpha]^{<\omega}$ and $K : A \longrightarrow \omega^{<\omega}$ be suitable. We define $\mathbb{P}_\alpha^K$ as the set of all $p \in \mathbb{P}_\alpha$ such that the following conditions hold:

1. p follows K.
2. p satisfies the A-descending condition.
3. For every $\beta \in \operatorname{dom}(p) \cap (H \cup E)$ $p|\beta \Vdash p(\beta)$ has the four property.

□

The following result is similar to Lemma 4.4.19:

Lemma 4.4.25 (*). Let $\alpha \leq \omega_2$, $A \in [E \cap \alpha]^{<\omega}$, $K : A \longrightarrow \omega^{<\omega}$ be suitable, $p \in \mathbb{P}_\alpha^K$ and $m \in \omega$. There is q such that the following holds:

1. $q \in \mathbb{P}_\alpha^K$.
2. $\mathrm{dom}\,(q) = \mathrm{dom}\,(p)$.
3. $q \leq p$.
4. If $\beta \in A$, then $q\,(\beta) = p\,(\beta)$.
5. If $\beta \in \mathrm{dom}\,(q) \setminus A$ then $q|\beta \Vdash \mathrm{len}(q(\beta)) = \max\{m, \mathrm{len}(p(\beta))\}$.

Proof. Note that the last point already implies that q satisfies the A-descending condition. We proceed by induction, the cases $\alpha = 0$ and α is limit are immediate. Assume the lemma is true for α, we will now prove it for $\alpha + 1$. We may assume that $\alpha \in \mathrm{dom}(p)$.

Case. $\alpha \notin H \cup E$.

Note that in particular, $\alpha \notin A$. Let $\dot{s}$, $\dot{F}$ be $\mathbb{P}_\alpha$-name such that $p|\alpha \Vdash p(\alpha) = (\dot{s}, \dot{F}$. By the inductive hypothesis, there is $q \leq p|\alpha$ as in the lemma. Let $\dot{k}$ be a $\mathbb{P}_\alpha$-name for a natural number such that $q \Vdash \dot{s} : \dot{k} \to 2$. Let $\dot{z}$ be a $\mathbb{P}_\alpha$-name such that q forces the following:

1. $\mathrm{dom}(\dot{z}) = \max\{\check{m}, \dot{k}\}$.
2. $\dot{s} \subseteq \dot{z}$.
3. If $i \in \mathrm{dom}(\dot{z}) \setminus \mathrm{dom}(\dot{s})$, then $\dot{z}(i) = 0$.

Notice that if $\alpha = \min(\mathrm{dom}\,p) \setminus A$, then we may choose $\dot{k} = \check{k}$ for some $k \in \omega$, so by letting $n = \max\{\check{m}, \check{k}\}$ it follows that $q \Vdash \mathrm{dom}\,\dot{z} = \check{n}$. Let $\dot{x}$ be a $\mathbb{P}_\alpha$-name such that $q \Vdash \dot{x} = (\dot{z}, \dot{F})$. It is clear that $q^\frown\dot{x}$ is the condition we are looking for.

Case. $\alpha \in H$.

Let $\gamma \in \omega_1$ and $\beta \in D_\gamma$ be such that $R(\beta) = \alpha$. By the inductive hypothesis there exists $q \leq p|\alpha$ as in the lemma. Fix names $\dot{s}, \dot{k}, \dot{F}$ such that $q \Vdash p(\alpha) = (\dot{s}, \dot{k}, \dot{F})$ and $\dot{n}$ a $\mathbb{P}_\alpha$-name such that $q \Vdash \dot{n} = \max\{\dot{k}, \check{m}\}$. Let $\dot{z}$ be a $\mathbb{P}_\alpha$-name for a partial function forced by q to have the following properties:

1. $\dot{z} \subseteq \Delta$.
2. $\dot{s} \subseteq \dot{z}$.
3. $\mathrm{dom}(\dot{z}) = \dot{A}_\beta \cap \dot{n}$.
4. For all $i \in \mathrm{dom}(\dot{z})$, if $i \notin \mathrm{dom}\,(\dot{s})$. then $\dot{z}(i) = \min\left\{j \in \omega : \forall g \in \dot{F}\,(g(i) \neq j)\right\}$.

Notice that if $\alpha = \min \mathrm{dom}\,p \setminus A$, then we may choose $\dot{k} = \check{k}$ for some $k \in \omega$, so by letting $n = \max\{\check{m}, \check{k}\}$ it follows that $q \Vdash \dot{n} = \check{n}$. Let $\dot{x}$ be a $\mathbb{P}_\alpha$-name such that $q \Vdash \dot{x} = (\dot{z}, \dot{n}, \dot{F})$. It is clear that $q^\frown\dot{x}$ is the condition we are looking for.

Case. $\alpha \in E$ and $\alpha \notin A$.

By the inductive hypothesis, there exists $q \in \mathbb{P}_\alpha$ such that $q \leq p|\alpha$ satisfying the lemma. Let $\dot{s}$, $\dot{k}$ and $\dot{F}$ such that $q| \Vdash p(\alpha) = (\dot{s}, \dot{k}, \dot{F})$ and $\dot{n}$ a $\mathbb{P}_\alpha$-name such that $q \Vdash \dot{n} = \max\{\dot{k}, \check{m}\}$.

Let $\dot{z}$ be a $\mathbb{P}_\alpha$-name for a function forced by q to have the following properties:

1. $\dot{z} \subseteq \Delta$.
2. $\dot{s} \subseteq \dot{z}$.
3. $\mathrm{dom}(\dot{z}) = \dot{n}$.
4. For all $i \in \mathrm{dom}(\dot{z})$, if $i \notin \mathrm{dom}\,(\dot{s})$. then $\dot{z}(i) = \min\left\{j \in \omega : \forall g \in \dot{F}\,(g(i) \neq j)\right\}$.

Notice that if $\alpha = \min \mathrm{dom}\, p \setminus A$, then we may choose $\dot{k} = \check{k}$ for some $k \in \omega$, so by letting $n = \max\{\check{m}, \check{k}\}$ it follows that $q \Vdash \dot{n} = \check{n}$. Let $\dot{x}$ be a $\mathbb{P}_\alpha$-name such that $q \Vdash \dot{x} = (\dot{z}, \dot{n}, \dot{F})$. It is clear that $q^\frown \dot{x}$ is the condition we are looking for.

Case. $\alpha \in E$ and $\alpha \in A$.

Let $A_1 = A \setminus \{\alpha\}$ and $K_1 = K|A_1$. By the inductive hypothesis applied to $p|\alpha$ and K_1, let $q \leq p|\alpha$ be as in the lemma. It is easy to see that $q^\frown p\,(\alpha)$ has the desired properties. $\square$

We will need the following result, which is the generalization of Proposition 4.4.14 for the iteration:

Lemma 4.4.26 (*). Let $\alpha \leq \omega_2$, $A \in [E \cap \alpha]^{<\omega}$, $K : A \longrightarrow \omega^{<\omega}$ be suitable and $p \in \mathbb{P}_\alpha^K$. Let $D \subseteq \mathbb{P}_\alpha$ be open and dense below p. Then there is q with the following properties:

1. $q \in \mathbb{P}_\alpha^K$
2. $q \leq p$.
3. If $\beta \in A$, then $q(\beta) = p(\beta)$.
4. There is an antichain $L \in [D]^{<\omega}$ such that for every $r \leq q$, if r follows K, then r is compatible with an element of L.

Proof. We prove the lemma by induction on α. The case $\alpha = 0$ is clear, so we focus on the successor and limit stages. We start by the successor case $\alpha + 1$:

Case. $\alpha \notin A$.

WLOG $\alpha \in \mathrm{dom}\, p$ since if it not, it suffices to prove the result for $p \cup \{(\alpha, \dot{\mathbb{1}}_\alpha)\}$.

Define $\overline{D}$ as the set of all $q \in \mathbb{P}_\alpha$ below $p|\alpha$ for which there exists $\overline{q} \in \mathbb{P}_{\alpha+1}$ with the following properties:

1. $\overline{q}|\alpha = q$.
2. $\overline{q} \in D$.

3. $q \Vdash \overline{q}(\alpha) \leq p(\alpha)$.
4. There is $m_q \in \omega$ such that $q \Vdash \operatorname{len}(\overline{q}(\alpha)) = \check{m}_q$.

It is easy to see that $\overline{D}$ is open and dense below p.

By the inductive hypothesis there is $\overline{p} \leq p|\alpha$ as in the lemma. Let $L \in \left[\overline{D}\right]^{<\omega}$ be an antichain such that for every $q \leq \overline{p}$, if q follows K, then q is compatible with an element of L. Let $L = \{q_i \mid i < k\}$ for some $k \in \omega$. For every $i < k$, fix $\overline{q}_i \in D$ as in the definition of $\overline{D}$. Let $\beta_0 = \min((p)) \setminus A$. We now find $m \in \omega$ such that $m > \operatorname{len}((\beta_0))$ as well as$m > m_{q_i}$ for every $i < k$. Since L is an antichain, we can find, by the antichain principle, a $\mathbb{P}_\alpha$-name $\dot{x}$ for an element of $\dot{\mathbb{Q}}_\alpha$ with the following properties:

1. $q_i \Vdash \dot{x} \leq \overline{q}_i(\alpha)$ for every $i < k$.
2. $p|\alpha \Vdash \dot{x} \leq p(\alpha)$ and $\operatorname{len} p(\alpha) = 2\check{m}$.
3. In case $\alpha \in H \cup E$, $p|\alpha \Vdash \dot{x}$ has the four property.

We now apply Lemma 4.4.25 to find p_1 with the following properties:

1. $p_1 \in \mathbb{P}^K_\alpha$.
2. $p_1 \leq \overline{p}$.
3. $\operatorname{dom}(p_1) = \operatorname{dom}(\overline{p})$.
4. If $\gamma \in A$, then $p_1(\gamma) = \overline{p}(\gamma)$.
5. If $\beta \in \operatorname{dom}(p_1) \setminus A$, then $p_1|\beta \Vdash \operatorname{len}(p_1(\beta)) = \max\{2\check{m}, \operatorname{len}(\overline{p}(\beta))\}$.

Let $q = p_1^\frown \dot{x}$. We claim that q has the desired properties.

The only nontrivial property to verify that $q \in \mathbb{P}^K_{\alpha+1}$ is the A-descending condition. Note that p_1 forces that the length of the stem of $\dot{x}$ is m and the length of the stem in all the elements of $\operatorname{dom}(p_1) \setminus A$ is at least m, so it follows that q has the A-descending condition.

Clearly $q \leq p$ and if $\beta \in A$, then $q(\beta) = p(\beta)$.

Finally, let $L_1 = \{\overline{q}_i \mid i < k\} \subseteq D$ and let $r \leq q$ be a condition following K. We need to prove that r is compatible with an element of L_1. Since $r|\alpha \leq q|\alpha = p_1 \leq \overline{p}$ and it follows K, we know there is $q_i \in L$ such that $r|\alpha$ and q_i are compatible. We claim that r and $\overline{q}_i$ are compatible.

Let $r_1 \in \mathbb{P}_\alpha$ be a common extension of both $r|\alpha$ and q_i. Define $\overline{r} = r_1^\frown r(\alpha)$, we will prove that $\overline{r}$ extends both r and $\overline{q}_i$. Clearly $\overline{r} \leq r$. In order to show that $\overline{r} \leq \overline{q}_i$, we only need to prove that $r_1 \Vdash r(\alpha) \leq \overline{q}_i(\alpha)$. Since $r_1 \leq q_i$, we have that $r_1 \Vdash \dot{x} \leq \overline{q}_i(\alpha)$. We also know that $r|\alpha \Vdash r(\alpha) \leq \dot{x}$, so $r_1 \Vdash r(\alpha) \leq \overline{q}_i(\alpha)$ and we are done.

Case. $\alpha \in A$ (in particular, $\alpha \in E$).

For this case, we employ a countable transitive model M as a ground model. Let $s = K(\alpha)$ and $n = |s|$. We have that $p|\alpha \Vdash p(\alpha) = (\check{s}, \check{n}, \dot{F})$ for some $\dot{F}$.

Let $\overset{\circ}{\overline{D}}$ be the $\mathbb{P}_\alpha$-name given by $\overset{\circ}{\overline{D}} = \{(\dot{x}, q) : q \leq p|\alpha \, \text{and} \, p^\frown \dot{x} \in D\}$. We claim that $p|\alpha \Vdash \overset{\circ}{\overline{D}} \subseteq \mathbb{E}_\triangle$ is open and dense below $p(\alpha)\}$.

To see that, let G be $\mathbb{P}_\alpha$-generic over M with $p|\alpha \in G$. Let $\overline{D} = \overset{\circ}{\overline{D}}[G] = \{\dot{x}[G] : \exists q \leq p|\alpha \, (q \in G \, \text{and} \, q^\frown \dot{x} \in D)\}$. We verify that $\overline{D} \subseteq \mathbb{E}_\Delta = \mathring{\mathbb{Q}}_\alpha[G]$ is open dense below $p(\alpha)[G]$.

Fix $t \in \mathbb{E}_\Delta$. Let $\dot{t}$ be such that $\dot{t}[G] = t$. There exists $p' \leq p|\alpha$ in G such that $p' \Vdash \dot{t} \in \mathbb{E}_\Delta \, \text{and} \, \dot{t} \leq p(\alpha)$. Then $D' = \{s \in \mathbb{P}_\alpha : s \leq p' \, \text{and} \, \exists \dot{x} \, (s^\frown \dot{x} \in D \, \text{and} \, s \Vdash \dot{x} \leq \dot{t})\}$ is dense below p': given $s' \leq p'$, $s'^\frown \dot{t} \leq p$, so there exists $s'' \leq s'^\frown t$ in D, thus $s''|\alpha \in D'$. Now let $s \in D' \cap G$, letting $\dot{x}$ be as in the definition of D'. Then it is clear that $\dot{x}[G] \in \overline{D}$ and $\dot{x}[G] \leq t$.

By Proposition 4.4.14 and the maximal principle, there exists a name $\dot{Z}$ such that:

$$p|\alpha \Vdash \dot{Z} \in [\overset{\circ}{\overline{D}}]^{<\omega} \text{ is an antichain such that for every } x \in \mathbb{E}_\triangle \text{ such that}$$
$$\text{stem}(x) = \check{s} \, \text{and} \, \text{len}(x) = \check{n} \text{ there exists } z \in Z \text{ such that } z \not\perp x.$$

Define B as the set of all $r \leq p$ in $\mathbb{P}_\alpha$ such that there exists $k \in \omega$ and a sequence $(\dot{x}_i : i < k)$ such that $r \Vdash \dot{Z} = \{(\dot{x}_i \ \mathbb{1}_\alpha) : i < k\}$ and for all $i < k$, $r \frown \dot{x}_i \in D$.

It is easy to see that B is an open dense subset of $\mathbb{P}_\alpha$ and we leave it to the reader.

Let $K_1 = K|\alpha$. We apply the inductive hypothesis with $p|\alpha$, B and K_1. In this way, there are q and L with the following properties:

1. $q \leq p|\alpha$.
2. $q \in \mathbb{P}_\alpha^{K_1}$.
3. If $\beta \in A \setminus \{\alpha\}$, then $q(\beta) = p(\beta)$.
4. $L \in [B]^{<\omega}$ is an antichain.
5. For every $q' \leq q$, if q' follows K_1, then q_1 is compatible with an element of L.

For each $r \in L$, let k_r and $(x_i^r : i < k_r)$ be as in the definition of B.

Define $L_1 = \{r^\frown \dot{x}_i^r : r \in L \, \text{and} \, i < k_r\}$. Note that L_1 is a finite antichain of D. Define $\overline{q} = q^\frown p\,(\alpha)$, we claim that $\overline{q}$ and L_1 have the desired properties.

Clearly $\overline{q} \in \mathbb{P}_{\alpha+1}^K$. Now, let $q_1 \leq \overline{q}$ that follows K. Since $q_1|\alpha \leq \overline{q}|\alpha = q$ and $q_1|\alpha$ follows K_1, we know that there is $r \in L$ compatible with $q_1|\alpha$. Let $q_2 \leq q_1|\alpha, r$ and note that $q_2 \Vdash \dot{Z} = \{(\dot{x}_i^r, \mathbb{1}_\alpha) : i < k_r\}$, hence WLOG there exists $i < r_r$ such that q_2 forces that $q_1(\alpha)$ and $\dot{x}_i^r$ are compatible (recall that $q_1(\alpha)$ is forced to be of the form $(s, n, \dot{J})$ since q_1 follows K). It follows that q_1 and $r^\frown \dot{x}_i^r$ are compatible.

Finally, we consider the case when α is a limit ordinal and the proposition is true for every $\beta < \alpha$. This case is similar to the one where $\alpha \notin A$.

First we find $\beta < \alpha$ such that $A, \mathrm{dom}(p) \subseteq \beta$, so $p \in \mathbb{P}_\beta$. Define $\overline{D}$ as the set of all $q \in \mathbb{P}_\beta$ such that there is $\overline{q} \in \mathbb{P}_\alpha$ with $\overline{q} \leq p$ such that there exists $\overline{q} \in \mathbb{P}_\alpha$ satisfying the following properties:

1. $\overline{q}|\beta = q$.
2. $\overline{q} \in D$.
3. $\overline{q}|[\beta, \alpha)$ has the descending condition.
4. There is $n_{\overline{q}}$ such that for every $\xi \in \mathrm{dom}(\overline{q}) \setminus \beta$, the condition $\overline{q}|\xi \Vdash \mathrm{len}(\overline{q}(\xi)) \leq n_{\overline{q}}$.

It is easy to see that $\overline{D}$ is a subset of $\mathbb{P}_\beta$ which is open and dense below p (it is dense by Lemma 4.4.21). By the induction hypothesis, there are $q \leq p$ in $\mathbb{P}^K_\beta$ and an antichain $L = \{q_i \mid i < k\} \subseteq \overline{D}$ such that for every $r \leq q$ in $\mathbb{P}_\beta$ that follows K, r is compatible with an element of L. For every $i < k$, choose $\overline{q}_i \in D$ witnessing that $q_i \in \overline{D}$. Find $n \in \omega$ such that $n > n_{\overline{q}_i}$ for every $q_i \in L$. By Lemma 4.4.25, we may assume that all of the stems in $\mathrm{dom}\,(q) \setminus A$ are forced to be larger than $2n$. Let $B_i = \mathrm{dom}(\overline{q}_i)$ for every $i < k$. We now define a condition $\widehat{q} \in \mathbb{P}_\alpha$ with the following properties (which is possible by the antichain principle):

1. $\widehat{q}|\beta = q$.
2. $\mathrm{dom}(\widehat{q}) = \mathrm{dom}\,(q) \cup \bigcup_{i<k} B_i$.
3. For every $i < k$ and $\xi \in B_i \setminus \beta$, we have that $\overline{q}_i|\xi \Vdash \widehat{q}(\xi) \leq \overline{q}_i(\xi)$.
4. For every $\xi \in [\beta, \alpha) \cap \mathrm{dom}\,\widehat{q}$, $p|\xi \Vdash \mathrm{len}(\widehat{q}(\xi)) = 2\check{n}$.
5. For every $\xi \in [\beta, \alpha) \cap \mathrm{dom}\,p$, $p|\xi \Vdash \widehat{q}(\xi) \leq p(\xi)$.
6. If $\xi \in (\mathrm{dom}(\widehat{q}) \setminus \beta) \cap (H \cup E)$ then $\widehat{q}|\xi \Vdash \widehat{q}(\xi)$ has the four property.

Let $L_1 = \{\overline{q}_i \mid i < k\}$, we will show that $\widehat{q}$ and L_1 have the desired properties. It is easy to see that $\widehat{q} \in \mathbb{P}^K_\alpha$. Now, fix $r \leq \widehat{q}$ that follows K. Clearly, $r|\beta$ extends q and follows K, so there is $i < k$ such that q_i is compatible with $r|\beta$. Let $s \in \mathbb{P}_\beta$ be such that $s \leq q_i, r|\beta$. Then $\overline{s} = s \cup (r|[\beta, \alpha)) \leq r, \overline{q}_i$: to see that $\overline{s} \leq r_i$, if $\xi \in B_i \setminus \beta$ and we already know that $\overline{s}|\xi \leq \overline{q}_i|\xi$, then by 3. we have that $r|\xi \Vdash r(\xi) \leq \widehat{q}(\xi) \leq \overline{q}_i(\xi)$. $\square$

We can now prove the following:

Proposition 4.4.27 (*)**.** There is a model of ZFC such that:

1. $\mathfrak{c} = \omega_3$.
2. $\mathfrak{ie} = \omega_2$.
3. There are families $\{\mathcal{A}_\gamma \mid \gamma \in \omega_1\}$, $\mathcal{B} = \{f_\alpha \mid \alpha \in \omega_2\}$ such that:
   (a) $\mathcal{A}_\gamma \subseteq [\omega]^\omega$ is a MAD family of size ω_2 (for every $\gamma \in \omega_1$).
   (b) $\mathcal{B} \subseteq \mathrm{PFun}$ is a MAD family.
   (c) If $\pi : \mathrm{PFun} \longrightarrow [\omega]^\omega$ is the function defined by $\pi(f) = \mathrm{dom}(f)$, then $\pi|\mathcal{B} : \mathcal{B} \longrightarrow \bigcup_{\gamma\in\omega_1} \mathcal{A}_\gamma$ is bijective.

Proof. We employ a ground countable transitive model M such that $M \models \mathfrak{c} = \omega_3$ and we will force with $\mathbb{P}_{\omega_2}$ within M. Let $G \subseteq \mathbb{P}_{\omega_2}$ be a generic filter. By counting nice names, it is easy to see that $M[G] \models \mathfrak{c} = \omega_3$ since for every $\kappa < \omega_2$, $P_\alpha \Vdash |Q_\alpha| = \mathfrak{c} = \omega_2$ (by induction). For every $\gamma \in \omega_1$, let $\mathcal{A}_\gamma = \{A_\xi : \xi \in D_\gamma\}$, where $A_\xi = \dot{A}_\xi[G]$. We have the following:

Claim (1). Let $\gamma \in \omega_1$.

1. $\mathcal{A}_\gamma \subseteq [\omega]^\omega$ is a MAD family of size ω_2.
2. For every $X \in M[G]$, if $X \in \mathcal{I}(\mathcal{A}_\gamma)^+$ then $|\{\xi \in D_\gamma : |X \cap A_\xi| = \omega\}| = \omega_2$.

The claim follows easily by Lemma 4.4.5. A more interesting fact is the following:

Claim (2). $M[G] \models \bigcap_{\gamma\in\omega_1} \mathcal{I}(\mathcal{A}_\gamma) = [\omega]^{<\omega}$.

Let $\dot{X}$ be a $\mathbb{P}_{\omega_2}$-name for an infinite subset of ω. Let $N \in M$ be a countable elementary submodel of $H((2^{\omega_3})^+)$ (of M) such that $\dot{X}, \mathbb{P}_{\omega_2} \in N$ and let $\gamma' \in \omega_1 \setminus N$. We will show that $\dot{X}$ is forced to be in $\mathcal{I}(\mathcal{A}_{\gamma'})^+$. In fact, we will prove that $\dot{X}$ will have infinite intersection with every element of $\mathcal{A}_{\gamma'}$. Note that $D_\gamma \cap N = \emptyset$ since $\gamma \notin N$ (recall that $(D_\eta : \eta \in \omega_1) \in N$ since $\mathbb{P}_{\omega_2} \in N$).

Let $\xi' \in D_{\gamma'}$, $k' \in \omega$ and $p \in \mathbb{P}_{\omega_2}$. We must find an extension of p forcing that $\dot{X}$ and $\dot{A}_\xi$ intersect beyond k. We may assume that $0, \xi' \in \mathrm{dom}(p)$ and that p is pure and has the descending condition. Let n be the height of p. We may also assume that $n > k'$. Let $B = \mathrm{dom}(p) \cap N$ and $A = B \cap E$. Note that $p \in \mathbb{P}^K_\alpha$, where K is the suitable function on A defined by the stems of p. Let $\mathrm{dom}\,(p) = \{\alpha_0, ..., \alpha_m\}$ where $\alpha_i < \alpha_j$ whenever $i < j$.

For each $i \leq m$, Let s_i be the stem of $p(\alpha_i)$ (which is a real object).

If $\alpha_i \in E$, let m_i, k_i, $\rho_0 \ldots, \rho_{k_i-1}$ be as in 3. of the definition of pure condition for $\xi = \alpha_i$.

Define J^p_i be the J of the definition of pure condition for $\xi = \alpha_i$ when $\alpha_i \in H \cup \bigcup_{\gamma<\omega_1} D_\gamma$.

Subclaim. There is $\overline{p} \in M \cap \mathbb{P}_{\omega_2}$, $\delta_0, \ldots, \delta_m \in N$ and $J^{\overline{p}}_{\delta_i} \in N$ for the $i \leq m$ with $\alpha_i \notin E$ and $\overline{\rho}^i_0, \ldots, \overline{\rho}^i_{k_i-1} \in N$ for the i's such that $\alpha_i \in E \cap N$ (or, equivalently, $\alpha_i \in A$) such the following holds (for every $i \leq m$ where i appears free):

1. $\overline{p}$ is a pure descending condition of height n.
2. $\mathrm{dom}\,(\overline{p}) = \{\delta_0, ..., \delta_m\}$ where $\delta_i < \delta_j$ whenever $i < j$ and $B \subseteq \mathrm{dom}\,(\overline{p})$.
3. $\overline{p} \in \mathbb{P}^K_{\omega_2}$.
4. If $\alpha_i \in B$, then $\delta_i = \alpha_i$.
5. If $\alpha_i \notin B$, then $\delta_i < \alpha_i$.
6. $\alpha_i \in E$ if and only if $\delta_i \in E$.

7. $\alpha_i \in H$ if and only if $\delta_i \in H$.
8. For every $\eta \in N \cap \omega_1$, $\alpha_i \in D_\eta$ if and only if $\delta_i \in D_\eta$.
9. For every $j \leq m$, if $\alpha_i, \alpha_j \in \bigcup_{\eta \in \omega_1} D_\eta$ then α_i, α_j are in the same element of the partition if and only if δ_i, δ_j are in the same element of the partition.
10. If $\alpha_i \in H$, then the following holds:
    (a) $\overline{p}|\delta_i \Vdash \overline{p}\,(\delta_i) = \left(\check{s}_i, \check{n}, \{(\dot{f}_\mu, \mathbb{1}_{\delta_i}) : \mu \in J_i^{\overline{p}}\}\right)$.
    (b) For every $j < i$, we have that $\alpha_j \in J_i^p$ if and only if $\delta_j \in J_i^{\overline{p}}$.
11. If $\alpha_i \in D_\eta$ for some $\eta < \omega_1$, then the following holds:
    (a) $\overline{p}|\delta_i \Vdash \overline{p}\,(\delta_i) = \left(\check{s}_{\alpha_i}, \{(\dot{A}_\mu, \mathbb{1}_{\delta_i}) : \mu \in J_i^{\overline{p}}\}\right)$.
    (b) For every $j < i$, we have that $\alpha_j \in J_i^p$ if and only if $\delta_j \in J_i^{\overline{p}}$.
12. If $\alpha_i \in E$, then the following holds:
    (a) $\overline{p}|\delta_i \Vdash \overline{p}\,(\delta_i) = \left(\check{s}_i, \check{m}_i, \{(\overline{\rho}_j^i, \mathbb{1}_{\delta_i}) : j < k_i\}\right)$.
    (b) We also let $\dot{J}_i^{\overline{p}} = \{(\overline{\rho}_j^i, \mathbb{1}_{\delta_i}) : j < k_i\} \in N$.

The subclaim is almost an immediate consequence of the elementarity of N since forcing statements of sentences relativized to (or absolute with) $H(\lambda)$ are definable within $H(\lambda)$. Point 5. is the only one that requires some extra explanation. For every $\alpha_i \notin B$, we define the following:

1. $\xi_i^0 = \max\,(B) \cap \alpha_i$ (this is well defined since $0 \in B$).
2. $\xi_i^1 = \min\,(M \cap (\omega_2 + 1) \setminus \alpha_i)\,.$

Note that $\xi_i^0, \xi_i^1 \in M$ and $\xi_i^0 < \alpha_i < \xi_i^1$. The claim then follows by applying elementarity and requiring that $\xi_i^0 < \delta_i < \xi_i^1$. Since $\delta_i \in M$ and is smaller that ξ_i^1, it follows that $\delta_i < \alpha_i$.

Let $\overline{p}$ be as in the claim. We now define

$$D = \{r \in \mathbb{P}_{\omega_2} : \exists l_r \in \omega\,(r \Vdash l_r = \min(\dot{X} \setminus n))\}.$$

Clearly $D \subseteq \mathbb{P}_{\omega_2}$ is an open dense subset and $D \in N$. Since $\overline{p} \in \mathbb{P}_{\omega_2}^K$, applying Lemma 4.4.26, there is $q \leq \overline{p}$ as in the lemma. We may even assume that $q \in M$. Note that in general, q might not be pure (we could extend it to a pure condition, but it might not follow K anymore). Let $L \in [D]^{<\omega}$ be such that for every $r \leq q$, if r follows K, then r is compatible with an element of L. Let $Z = \{l_r \mid r \in L\}$ and note that $Z \cap n = \emptyset$. It is clear that if $r \in L$, then $r \Vdash \check{Z} \cap \dot{X} \neq \emptyset$. Let $n_1 = \max(Z) + 1$.

We now define the condition p_Z with the following properties:

1. $\mathrm{dom}(p_Z) = \mathrm{dom}(p)$.
2. For every $i \leq m$, the following holds:

(a) If $\alpha_i \notin D_{\gamma'}$, then $p_Z(\eta) = p(\eta)$. Define $s_i^{pz} = s_i^p$ $J_i^{pz} = J_i^p$ for this case.

(b) If $\alpha_i \in D_{\gamma'}$ with $\alpha_i \neq \xi$, define $s_i^{pz} : n_1 \to 2$ such that $s_i^p \subseteq s_i^{pz}$ and $s_i^{pz}(i) = 0$ for every $i \in [n, n_1)$. Let $p_Z|\alpha_i \Vdash p_Z(\alpha_i) = \left(\check{s}_i^{pz}, \{(\dot{A}_\mu, \mathbb{1}_{\alpha_i}) : \mu \in J_i^p\}\right)$.

(c) If $\alpha_i = \xi$, define $s_i^{pz} : n_1 \to 2$ such that $s_i^p \subseteq s_i^{pz}$ and $s_i^{pz}(i) = 1$ for every $i \in [n, n_1)$. Let $p_Z|\alpha_i \Vdash p_Z(\alpha_i) = \left(s_i^{pz}, \{(\dot{A}_\mu, \mathbb{1}_{\alpha_i}) : \mu \in J_\xi^p\}\right)$.

Note that $p_Z \Vdash \check{Z} \subseteq \dot{A}_\xi$. Since $J_i^p \subseteq \operatorname{dom}(p)$ for every $i \in D_{\gamma'}$, it is follows from (b) that $p_Z \leq p$.

We now will define the condition r as follows. Its existence is not readily seen.

1. $\operatorname{dom}(r) = \operatorname{dom}(p_Z) \cup \operatorname{dom}(q)$.
2. If $\eta \in \operatorname{dom}(q) \setminus \operatorname{dom}(p_Z)$, then $r(\eta) = q(\eta)$.
3. Let $i \leq m$. We have the following:

(a) Assume $\alpha_i \in D_\gamma$ for some $\gamma \notin N$, define $r(\alpha_i) = p_Z(\alpha_i)$ (note that this will be the case when $\gamma' = \gamma$ and that, in this case, $\alpha_i \notin \operatorname{dom} q$ as $q \in N$).

(b) Assume $\alpha_i \in D_\gamma$ with $\gamma \in N$. Let $\zeta_{\delta_i} = \{(\operatorname{op}(\check{\mu}, \dot{A}_\mu), \mathbb{1}_{\delta_i}) : \mu \in D_{\gamma'} \cap \delta_i\}$. There exist $\mathbb{P}_{\delta_i}$-names $\dot{t}_i^q, \dot{J}_i^q$ such that $i \Vdash q(\delta_i) = \left(\dot{t}_i^q, \zeta_{\delta_i}[\dot{J}_i^q]\right)$. Define $r(\alpha_i)$ such that $r|\alpha_i \Vdash r(\alpha_i) = \left(\dot{t}_i^q, \zeta_{\delta_i}[\check{J}_i^{pz} \cup \dot{J}_i^q]\right)$. Note that $\dot{t}_{\delta_i}^q$ is a $\mathbb{P}_{\delta_i}$-name, since $\delta_i \leq \alpha_i$ it is also a $\mathbb{P}_{\alpha_i}$-name, so the definition makes sense.

(c) Assume $\alpha_i \in H$. Let $\zeta_{\delta_i} = \{(\operatorname{op}(\check{\mu}, \dot{f}_\mu), \mathbb{1}_{\delta_i}) : \mu \in H \cap \delta_i\}$. There exist $\mathbb{P}_{\delta_i}$-names $\dot{t}_i^q, \dot{J}_i^q$ and $\dot{m}_i^q$ such that $q|\delta_i \Vdash q(i) = \left(\dot{t}_i^q, \dot{m}_i^q, \zeta_{\delta_i}[\dot{J}_i^q]\}\right)$. Let $r(\alpha_i)$ be such that $r|\alpha_i \Vdash r(\alpha_i) = \left(\dot{t}_i^q, \dot{m}_i^q, \zeta_{\delta_i}[\dot{J}_i^q \cup J_i^{pz}]\right)$.

(d) Assume $\alpha_i \in E$ and $\alpha_i \notin \operatorname{dom}(q)$. Define $r(\alpha_i) = p_Z(\alpha_i)$.

(e) Assume $\alpha_i \in E$ and $\alpha_i \in \operatorname{dom}(q)$ (so $\delta_i = \alpha_i$ and $\alpha_i \in A$). Define $r(\alpha_i)$ such that $r|\alpha_i \Vdash r(\alpha_i) = \left(\check{s}_i^p, \check{m}_i^p, \dot{J}_i^p \cup \dot{J}_i^{\overline{p}}\right)$.

Such an r exists by induction due to the following subclaim:

Subclaim. Let $\eta \in \operatorname{dom}(r)$.

1. $r|\eta \in \mathbb{P}_\eta$.
2. $r|\eta \Vdash r(\eta) \in \mathring{\mathbb{Q}}_\eta$.
3. $r|\eta \leq q|\eta$.
4. $r|\eta \Vdash r(\eta) \leq q(\eta)$.
5. $r|\eta \leq p_Z$.
6. $r|\eta \Vdash r(\eta) \leq p_Z(\eta)$.

We proceed by induction. We assume 1. and 3. and 5. and prove 2. and 4. and 6.. Condition 4. is readily seen to be true once we prove 2., so we only prove 2. and then

6. Furthermore, 2. is trivial whenever $\eta \in \mathrm{dom}\,(q) \setminus \mathrm{dom}\,(p_Z)$, so we focus on the other cases. From now on, $\eta \in \mathrm{dom}(p_Z)$, so we may assume that $\eta = \alpha_i$ for some $i \leq m$.

Case (a). $\alpha_i \in D_\gamma$ for some $\gamma \notin N$.

This is trivial since $r(\alpha_i) = p_Z(\alpha_i)$.

Case (b). $\alpha_i \in D_\gamma$ for some $\gamma \in N$.

Note that $\gamma \neq \gamma'$. We write the relevant names for all the relevant conditions:

- $p_Z|\alpha_i \Vdash p_Z(\alpha_i) = (\check{s}_i^p, \zeta_{\alpha_i}[\check{J}_i^p])$,
- $q|\delta_i \Vdash q(\delta_i) = (\dot{t}_i^q, \zeta_{\delta_i}[\dot{J}_i^q])$,
- $\overline{p}|\delta_i \Vdash \overline{p}(\delta_i) = (\check{s}_i^p, \zeta_{\delta_i}[\dot{J}_i^{\bar{p}}])$, and,
- $r|\alpha_i \Vdash r(\alpha_i) = (\dot{t}_i^q, \zeta_{\alpha_i}[\dot{J}_i^q \cup \check{J}_i^p])$.

Here, the ζ's are names for enumerations of $\{\dot{A}_\mu : \mu \in D_\gamma\}$ with the appropriate domain, analogous to as done before.

It is clear that $r|\alpha_i \Vdash r(\alpha_i) \in \mathring{\mathbb{Q}}_\alpha$. Now we prove that $r|\alpha_i \Vdash r(\alpha_i) \leq p_Z(\alpha_i)$.

Since $q \leq \overline{p}$ we have $r|\alpha_i \leq q|\alpha_i \Vdash \check{s}_i^p \subseteq \dot{t}_i^q$ and that $r|\alpha_i \Vdash \forall a \in \zeta_{\alpha_i}[\dot{J}_i^q]\, a \cap (\dot{t}_i^q)^{-1}[\{1\}] \subseteq |\check{s}_i^p| = \check{n}$. It remains to see that $r|\alpha_i \Vdash \forall a \in \zeta_{\alpha_i}[\check{J}_i^p]\, a \cap (\dot{t}_i^q)^{-1}[\{1\}] \subseteq |\check{s}_i^p| = \check{n}$. It suffices to see that for every $\alpha_j \in J_i^p$ (with $j < i$), $r|\alpha_i \Vdash \dot{A}_{\alpha_j} \cap (\dot{t}_i^q)^{-1}[\{1\}] \subseteq \check{n}$. Fix j.

Let $\dot{m}_i^q, \dot{m}_j^q$ be such that $q|\delta_j \Vdash \dot{t}_j^q : \dot{m}_j^q \to 2$ and $q|\delta_i \Vdash \dot{t}_i^q : \dot{m}_i^q \to 2$. Since $\delta_j \leq \delta_i \leq \alpha_i$ we have that $r|\alpha_i \leq q|\alpha_i \leq q|\delta_i \leq q|\delta_j$. Thus, $r|\alpha_i \Vdash |\dot{t}_i^q| = \dot{m}_i$ and $|\dot{t}_j^q| = \dot{m}_j^q$ as well. Moreover, $r|\alpha_i \Vdash \dot{m}_i^q \geq \check{n}$ since $r|\alpha_i \Vdash \check{s}_i^p \subseteq \dot{t}_i^q$. Since q satisfies the A-descending condition, we know that $q|\delta_i \Vdash \dot{m}_j^q \geq \dot{m}_i^q$. Thus, $r|\alpha_i \Vdash \dot{m}_j{}^q \geq \dot{m}_i^q \geq \check{n}$. In particular, we have that:

$$r|\alpha_i \Vdash \dot{A}_{\alpha_j} \cap (\dot{t}_i^q)^{-1}[\{1\}] = \dot{A}_{\alpha_j} \cap \dot{m}_j^q \cap (\dot{t}_i^q)^{-1}[\{1\}] = (\dot{t}_j^q)^{-1}[\{1\}] \cap (\dot{t}_i^q)^{-1}[\{1\}].$$

Since $q|\delta_i \Vdash \overline{p}|\delta_i$ we have that $q|\delta_i \Vdash \dot{A}_{\delta_i} \cap \dot{A}_{\delta_j} \subseteq \check{n}$. But then, intersecting with $\dot{m}_j^q$ and then with $\dot{m}_i^q$, we have that $q|\delta_i \Vdash (\dot{t}_j^q)^{-1}[\{1\}] \cap (\dot{t}_i^q)^{-1}[\{1\}] \subseteq \check{n}$. Thus:

$$r|\alpha_i \leq q|\alpha_i \leq q|\delta_i \Vdash (\dot{t}_j^q)^{-1}[\{1\}] \cap (\dot{t}_i^q)^{-1}[\{1\}] \subseteq \check{n},$$

as intended.

Case (c). $\alpha_i \in H$.

We write the relevant names for all the relevant conditions:

- $p_Z|\alpha_i \Vdash p_Z(\alpha_i) = (\check{s}_i^p, \check{n}, \zeta_{\alpha_i}[\check{J}_i^p])$,
- $q|\delta_i \Vdash q(\delta_i) = (\dot{t}_i^q, \dot{m}_i^q, \zeta_{\delta_i}[\dot{J}_i^q])$,
- $\overline{p}|\delta_i \Vdash \overline{p}(\delta_i) = (\check{s}_i^p, n, \zeta_{\delta_i}[\dot{J}_i^{\overline{p}}])$, and,
- $r|\alpha_i \Vdash r(\alpha_i) = (\dot{t}_i^q, \dot{m}_i^q, \zeta_{\alpha_i}[\dot{J}_i^q \cup \check{J}_i^p])$.

Here, the ζ's are names for enumerations of $\{\dot{f}_\zeta : \zeta \in H\}$ with the appropriate domain, analogous to as done before.

Since $q \leq \overline{p}$ we have $q|\delta_i \Vdash \check{n} \leq \dot{m}_i^q$ and $\dot{t}_i^q \supseteq \check{s}_i^p$. Furthermore, $q|\delta_i \Vdash 4|\dot{J}_i^q| \leq \dot{m}_i^q$. We also know that $4|J_i^p| \leq n$ since p is pure, hence $q|\delta_i \Vdash_{\delta_i} 4|\dot{J}_i^q|, 4|\check{J}_i^{pz}| \leq \dot{m}_i^q$. Since $r|\alpha_i \leq r|\delta_i \leq q|\delta_i$, $\mathbb{P}_{\delta_i}$ is completely embedded into $\mathbb{P}_{\alpha_i}$ and this last formula is absolute for transitive models of ZFC, we get that $r|\alpha_i \Vdash_{\alpha_i} 4|\dot{J}_i^q|, 4|J_i^{pz}| \leq \dot{m}_i^q$, so $r(\alpha)$ is forced to be a condition by Lemma 4.4.11.

Now we prove that $r|\alpha_i \Vdash r(\alpha_i) \leq p_Z(\alpha_i)$.

Since $q \leq \overline{p}$ we have $r|\alpha_i \leq q|\alpha_i \Vdash (\check{s}_i^p \subseteq \dot{t}_i^q$ and $\check{n} \leq \dot{m}_i^q)$ and that $r|\alpha_i \Vdash \forall f \in \zeta_{\alpha_i}[\dot{J}_i^q]\ f \cap \dot{t}_i^q \subseteq \check{n} \times \check{n}$. It remains to see that $r|\alpha_i \Vdash \forall f \in \zeta_{\alpha_i}[\check{J}_i^p]\ f \cap \dot{t}_i^q \subseteq \check{n} \times \check{n}$. It suffices to see that for every $\alpha_j \in J_i^p$ (with $j < i$), $r|\alpha_i \Vdash \dot{f}_{\alpha_j} \cap \dot{t}_i^q \subseteq \check{n} \times \check{n}$. Fix j.

Since $\delta_j \leq \delta_i \leq \alpha_i$ we have that $r|\alpha_i \leq q|\alpha_i \leq q|\delta_i \leq q|\delta_j$. Thus, $r|\alpha_i \Vdash |\dot{t}_i^q| = \dot{m}_i$ and $|\dot{t}_j^q| = \dot{m}_j^q$ as well. Since q satisfies the A-descending condition, we know that $q|\delta_i \Vdash \dot{m}_j^q \geq \dot{m}_i^q$. Thus, $r|\alpha_i \Vdash \dot{m}_j{}^q \geq \dot{m}_i^q \geq \check{n}$. In particular, we have that:

$$r|\alpha_i \Vdash \dot{f}_{\alpha_j} \cap \dot{t}_i^q = \dot{f}_{\alpha_j} \cap \dot{m}_j^q \cap \dot{t}_i^q = \dot{t}_j^q \cap \dot{t}_i^q.$$

Since $q|\delta_i \Vdash \overline{p}|\delta_i$ we have that $q|\delta_i \Vdash \dot{f}_{\delta_i} \cap \dot{f}_{\delta_j} \subseteq \check{n} \times \check{n}$. But then, intersecting with $\dot{m}_j^q \times \dot{m}_j^q$ and then with $\dot{m}_i^q \times \dot{m}_i^q$, we have that $q|\delta_i \Vdash \dot{t}_j^q \cap \dot{t}_i^q \subseteq \check{n} \times \check{n}$. Thus:

$$r|\alpha_i \leq q|\alpha_i \leq q|\delta_i \Vdash \dot{t}_j^q \cap \dot{t}_i^q \subseteq \check{n} \times \check{n},$$

as intended.

Case (d). $\alpha_i \in E \setminus \mathrm{dom}(q)$.

This is trivial since $r(\alpha_i) = p_Z(\alpha_i)$.

Case (e). $\alpha_i \in \mathrm{dom}\, q \cap E$.

We write the relevant names for all the relevant conditions. Notice that in this case, $\alpha_i = \delta_i$ and $\alpha_i \in A$, so $\overline{p}(\alpha_i) = q(\alpha_i)$.

- $p_Z|\alpha_i \Vdash p_Z(\alpha_i) = (\check{s}_i^p, \check{m}_i^p, \dot{J}_i^p)$,
- $q|\alpha_i \Vdash q(\alpha_i) = (\check{s}_i^p, \check{m}_i^p, \dot{J}_i^{\overline{p}})$,
- $\overline{p}|\alpha_i \Vdash \overline{p}(\alpha_i) = (\check{s}_i^p, \check{m}_i^p, \dot{J}_i^{\overline{p}})$, and,

- $r|\alpha_i \Vdash r(\alpha_i) = (\check{s}_i^p, \check{m}_i^p, \dot{J}_i^{\overline{p}} \cup \dot{J}_i^p)$.

We have that $r|\alpha_i \Vdash 4|\dot{J}_i^p|, 4|\dot{J}_i^{\overline{p}}| \leq \check{m}_i^p$ so $r(\alpha)$ is forced to be a condition by Lemma 4.4.11. Since the stems and lenghts are all the same, it is clear that $r|\alpha_i \Vdash r(\alpha) \leq p_Z(\alpha_i)$.

This completes the proof of the Subclaim.

We have proved that $r \leq q, p_Z$. It is easy to see that r follows K: given $\nu \in A$ we have that $\nu \in E \cap \mathrm{dom}(p) \cap N \subseteq \mathrm{dom}(\overline{p}) \subseteq \mathrm{dom}(q)$, thus we fall into (e), which says that the stems of $r(\nu)$ and $p(\nu)$ are the same, that is, $K(\nu)$.

Thus, there is $r' \in L$ such that r' and r are compatible. Let $\overline{r}$ be a common extension. Then:

1. $\overline{r} \Vdash \dot{X} \cap \check{Z} \neq \emptyset$ since $\overline{r} \leq r'$, as we have previously mentioned, and,
2. $\overline{r} \Vdash \check{Z} \subseteq \dot{A}_{\xi'}$ since $\overline{r} \leq p_Z$.

Since $Z \cap n = \emptyset$ it follows that $\overline{r} \Vdash \dot{A}_\xi \cap \dot{X} \subseteq k$, which is what we wanted to prove. We conclude that $M[G] \models \bigcap_{\gamma\in\omega_1} \mathcal{I}(A_\gamma) = [\omega]^{<\omega}$ and the proof of the Claim is complete.

Recall that $\mathcal{B} = \{f_\alpha \mid \alpha \in H\}$.

Claim. $\mathcal{B}$ is an almost disjoint family of size ω_2 intersecting every element of PFun. Thus, $\mathfrak{ie} \leq \omega_2$ and $\mathcal{B}$ is a MAD family.

It is easy to see that $\mathcal{B}$ is an almost disjoint family of size ω_2. Let $h \in \mathrm{PFun}$ and $A = \mathrm{dom}\,(h)$. By the last claim, there is $\gamma \in \omega_1$ such that $A \in \mathcal{I}(\mathcal{A}_\gamma)^+$, so there exists $\beta \in D_\gamma$ such that $C = A \cap A_\beta$ is infinite and $h \in M[G_\beta]$. Define $h_1 = h|C$ and note that $h_1 \in M[G_{\beta+1}]$. Let $\alpha = R\,(\beta)$ (so $\beta < \alpha$).

In case $h_1 \in \mathcal{I}(\mathcal{B}_\alpha$, there exists n and $\alpha_0, \ldots, \alpha_{n-1} \in H$ such that $h_1 \subseteq^* f_{\alpha_1} \cup \cdots \cup f_{\alpha_{n-1}}$, so clearly h_1 has infinite intersection with an f_{α_i}. In case $h_1 \in \mathcal{I}(\mathcal{B}_\alpha)^+$ we have that $f_\alpha \cap h_1$ is infinite by Lemma 4.4.10.

Finally, we will prove the following:

Claim. $\mathfrak{ie} = \omega_2$.

The remaining inequality follows since we are forcing with $\mathbb{E}_\Delta$ cofinally many times.

Claim. $\pi : \mathcal{B} \to \bigcup_{\gamma\in\omega_1} \mathcal{A}_\gamma$ is bijective.

To see that the codomain is $\bigcup_{\gamma\in\omega_1} \mathcal{A}_\gamma$, notice that $\mathrm{dom}\, f_\xi = A_{R^{-1}(\xi)}$. Given $\beta \in \bigcup_{\gamma<\omega_1} D_\gamma$, $\mathrm{dom}\, f_R(\beta) = A_\beta$, thus π is onto. Finally, if $\xi \neq \xi'$ then $A_{R^{-1}(\xi)} \neq A_{R^{-1}(\xi')}$, so the proof is complete. $\square$

4.5 Further Results on existence of pseudocompact MAD families

The question of whether there exists a pseudocompact MAD family in ZFC is still open. As mentioned, we also do not know if every MAD family of cardinality ω_1 is necessarily pseudocompact.

We do not have many examples so far, so we believe that every new example of a MAD family which is decidedly pseudocompact or not pseudocompact is interesting. In this section we examine some extra conditions which make a MAD family pseudocompact.

The following definition is due to Brendle and Shelah [15].

Definition 4.5.1. Let $\mathcal{U}$ be a free ultrafilter on ω. We define the *pseudointersection number* of $\mathcal{U}$, denoted by $\mathfrak{p}(\mathcal{U})$ of $\mathcal{U}$ is defined as the minimal size of a subcollection of $\mathcal{U}$ without a pseudointersection in $\mathcal{U}$. □

Now we prove:

Theorem 4.5.2 (*). Suppose $\mathcal{A}$ is a MAD family such that there exists a free ultrafilter $\mathcal{U}$ an such that $|\mathcal{A}| < \mathfrak{p}(\mathcal{U})$. Then $\mathcal{A}$ is pseudocompact.

Proof. Let $\mathcal{A}$ and $\mathcal{U}$ be given. By Lemma 4.2.6, and Proposition 3.5.7, it is sufficient to verify that for every injective sequence $f : \omega \to \omega$ there exists $B \in \mathcal{U}$ and $A \in \mathcal{A}$ such that $f[B] \subseteq A$.

Suppose this is not the case. Then there exists $f : \omega \to \omega$ such that for all $a \in \mathcal{A}$ and $B \in \mathcal{U}$, $f[B] \setminus A$ is infinite. First, notice that given $a \in \mathcal{A}$, there exists $B_a \in \mathcal{U}$ such that $f[B_a] \cap a$ is empty: the sets $\{n \in \omega : f(n) \notin a\}$ and $\{n \in \omega : f(n) \in a\}$ form a partition of ω, so one of them is in $\mathcal{U}$. But the second is not in $\mathcal{U}$ by hypothesis. Let B_a be the first set.

Now let B be a pseudointersection of $\{B_a : a \in \mathcal{A}\}$ in $\mathcal{U}$. It follows that $f[B] \cap a$ is finite for every $a \in \mathcal{A}$, contradicting the maximality of $\mathcal{A}$. □

With the ideas from the previous proof we may also show the following:

Theorem 4.5.3 (*). Suppose that there is an ultrafilter $\mathcal{U}$ such that $\mathfrak{p}(\mathcal{U}) = \mathfrak{c}$. Then every almost disjoint family of size $< \mathfrak{c}$ may be extended to a pseudocompact MAD family.

Proof. Let $\kappa < \mathfrak{c}$ be infinite and $(a_\alpha : \alpha < \kappa)$ be an enumeration of an almost disjoint family $\mathcal{A}'$ of size κ.

It suffices to expand $\mathcal{A}'$ to an almost disjoint family $\mathcal{A}$ such that for every injective $f : \omega \to \omega$ there exists $B \in \mathcal{U}$ such that $f[B] \subseteq a$.

Enumerate all the injective sequences from ω to ω as $(a_\beta : \kappa \leq \beta < \mathfrak{c})$. Define $h(\beta) = 0$ for every $\beta < \kappa$. We define $a_\beta \in [\omega]^\omega$ and $h(\beta) \in \{0, 1\}$ for $\beta \in [\kappa, \mathfrak{c})$ recursively satisfying:

1. $h(\beta) = 1$ iff there exists $\beta' < \beta$ with $h(\beta') = 0$ and $B \in \mathcal{U}$ such that $f_\beta[B] \subseteq a_{\beta'}$,

2. if $h(\beta) = 0$, then there exists $B \in \mathcal{U}$ such that $f_\beta[B] = a_\beta$.
3. if $h(\beta) = 0$, then $a_\beta \cap a'_\beta$ is finite for every $\beta' < \kappa$ such that $h(\beta') = 0$.

Then clearly $\mathcal{A} = \{a_\beta : \beta \in [\kappa, \mathfrak{c}) \text{ and } h(\beta) = 0\}$ works.

To see that such a recursion is possible to be carried out, at stage β assume that for every $\beta' < \beta$ with $h(\beta) = 0$, there is no $B \in \mathcal{U}$ such that $f_\beta[B] \subseteq a_{\beta'}$. Then for every $\beta' < \beta$ with $h(\beta') = 0$, notice that either $f_\beta^{-1}[a_{\beta'}]$ or $f_\beta^{-1}[\omega_{\beta'}]$ are in $\mathcal{U}$. By our hypothesis on β', $f_{\beta^{-1}}[\omega \setminus a_{\beta'}] \in \mathcal{U}$. Let B be a pseudointersection of $\{f_\beta^{-1}[\omega \setminus a_{\beta'}] : \beta' < \beta \text{ and } h(\beta') = 0\}$ and let $f_\beta[B] = a_\beta$, $h(\beta) = 0$. $\square$

It is consistent that there exists a MAD family of size ω_1 and a free ultrafilter $\mathcal{U}$ such that $\mathfrak{p}(\mathcal{U}) = \omega_2$. This result appears in our paper [34], Section 3 and uses techniques of matrix forcing. This result will not appear in this thesis. It is also worth mentioning that $n(\omega^*) > \mathfrak{c}$ implies the existence of selective ultrafilters $\mathcal{U}$ such that $\mathfrak{p}(\mathcal{U}) > \mathfrak{c}$ (see [2, Theorem 3.7.]).

In [18], C. Corral obtained some other results related to these problems. In particular, he proved that PFA implies that there are no almost-normal MAD families.

Chapter 5

Forcing large countably compact Abelian groups

5.1 Topological groups

We expect the reader to be familiar with groups and their basic algebraic related notions.

Proposition 5.1.1. A *group* is a 4-uple $(G, \cdot, e, (.)^{-1})$ where . is a binary operation, $e \in G$ and $(.)^{-1}$ is a unary operation satisfying:

- For every x, y, z, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
- For every $x \in G$, $x \cdot e = e \cdot x = x$
- For every x, $x \cdot x^{-1} = x^{-1} \cdot x = e$.

If, additionally, for every $x, y \in G$, $x.y = y.x$, we say that G is an *Abelian* group.

If G is an Abelian group, it is very common to denote its binary operation by $+$, its inversion operation by $-$, its constant by 0 and $x + (-y)$ by $x - y$. We will use this basic notation with no additional explanations. We also write G instead of the whole 4-uple if no confusion arises. We also expect the reader to be familiar with group homomorphisms, subgroups and quotients.

Some basic notation is:

Definition 5.1.2. Let G be an Abelian group. For $x \in G$ and $n \in \omega$, we recursively define $nx = n.x$ as follows: $0.x = 0$ and $(n+1).x = n.x + x$. In particular, $1.x = x$.

For $n \in \omega$, we define $G[n] = \{x \in G : nx = 0\}$ and $nG = \{nx : x \in G\}$.

If $nG = \{0\}$ for some $n \in \omega \setminus \{0\}$, we say that G is a *torsion group*. In this case, we say that G is of *exponent* n if n is the least n with this property.

G is a *non-torsion group* iff it is not a torsion group.

The *torsion part of* G is the subgroup $t(G) = \bigcup_{n \in \omega} G[n]$. $\square$

It is easy to see that $t(G)$, $G[n]$ and nG are subgroups of G. Also, notice that non-torsion groups are infinite.

Lemma 5.1.3. Let G be an Abelian group and n, d be positive integers such that $d|n$. Then $dG[n] \approx G[n]/G[d]$.

Proof. First, it is clear that if $dx = 0$ then $nx = 0$, so $G[d] \subseteq G[n]$. Let $\phi : G[n] \to dG[n]$ the the homomorphism given by $\phi(g) = dg$. Then ϕ is onto and $\ker \phi = G[d]$, which yields the proposition by the homomorphism theorems. $\square$

Lemma 5.1.4. Let G be an Abelian group. If G is a non-torsion group, then $G/t(G)$ is infinite.

Proof. Let $x \in G \setminus t(G)$. We claim that if $n \neq m$ then $nx + t(G) \neq mx + t(G)$. For suppose there exists $y \in t(G)$ such that $nx + y = mx$. Then $(n - m)x = y$, so there exists $k \neq 0$ such that $k(n - m)x = 0$, a contradiction. $\square$

Definition 5.1.5. A *topological group* is a group G endowed with a topology for which . and $(.)^{-1}$ are continuous operations. $\square$

Every T_0 group is uniformizable, and therefore Tychonoff (see [70, 35F]). Thus, we will usually write "Hausdorff groups" to refer to topological Tychonoff groups.

In this chapter we are going to construct topologies which transforms a group into a Hausdorff groups. The following well known lemma will come handy.

Lemma 5.1.6. Let G be a group and $(H_i : i \in I)$ be a family of topological groups. For each $i \in I$, let $f_i : G \to H_i$ be a group homomorphism. Then the weak topology generated by the family $(f_i : i \in I)$ (that is, the topology generated by the sets of the form $f_i^{-1}[A]$, where $i \in I$ and $A \subseteq H_i$ is open) makes G a topological group. Moreover, if each H_i is Hausdorff and for ever $g \in G$ there exists $i \in I$ such that $f_i(g) \neq e_i$ (where e_i is the neutral element of H_i), then this topology is Hausdorff.

Proof. The binary operation . is continuous: suppose $a.b = c$. Let W be a basic open neighborhood of c. Then $W = \bigcap_{i \in J} f_i^{-1}[A_i]$, where $J \subseteq I$ is finite and for each $i \in J$, $A_i \subseteq H_i$ is open. Fix $i \in J$. Then $f_i(a).f_i(b) = f_i(c)$. Since H_i is a topological group, there exists U_i, V_i open subsets of H_i such that $f_i(A) \in U_i$, $f_i(B) \in V_i$ and $U_i.V_i \subseteq A_i$. Let $U = \bigcap_{i \in J} f_i^{-1}[U_i]$, $V = \bigcap_{i \in J} f_i^{-1}[V_i]$. It is clear that $(x, y) \in U \times V$, then $xy \in W$.

The inversion is continuous: denote x^{-1} by $I_i(x)$ on H_i, and I on G. So I_i is continuous for each $i \in I$. Let $I(a) = b$. Let W be a basic open neighborhood of b. Then $W = \bigcap_{i \in J} f_i^{-1}[A_i]$, where $J \subseteq I$ is finite and for each $i \in J$, $A_i \subseteq H_i$ is open. For each $i \in I$, $f_i(c) = I_i(f_i(a)) \in A_i$, so there exists an open set $U_i \subseteq H_i$ such that $f_i(a) \in U_i$ and $I[U_i] \subseteq A_i$. Let $U = \bigcap_{i \in J} f_i^{-1}[U_i]$. It is clear that $I[U] \subseteq W$.

For the last claim, fix $g \neq h$ in G. Let i be such that $f_i(g.h^{-1}) \neq e_i$. Then $f_i(g) \neq f_i(h)$ Let U, V be two disjoint open sets separating $f_i(g)$, $f_i(h)$ in H_i. Then $f_i^{-1}[U_i]$, $fi^{-1}[V_i]$ are two disjoint open sets separating g, h in G. $\square$

Regarding subspaces, the following is immediate:

Lemma 5.1.7. The subspace of a topological group is a topological group with the subspace topology.

And regarding quotients, we have the following:

Lemma 5.1.8. Let K be a normal subgroup of a topological group G. Consider the natural homomorphism $j : G \to G/K$. Then the quotient topology induced by j makes G/K a topological group.

Proof. . is continuous: Let $a, b \in G$ be given and $c = ab$. Let W' be an open neighborhood of $i(c)$. We must see that there exists open neighborhoods U', V' of a, b (resp.) such that $U'.V' \subseteq W'$. Let $W = j^{-1}[W']$. W is a open neighborhood of c. There exist open neighborhoods U, V of a, b (resp) such that $U.V \subseteq W$.

Notice that $\tilde{U} = j^{-1}[j[U]] = UK = \bigcup\{U.k : k \in K\}$ and $\tilde{V} = j^{-1}[j[V]] = VK = \bigcup\{V.k : k \in K\}$ are open and $j^{-1}[j[\tilde{U}]] = \tilde{U}$, $j^{-1}[j[\tilde{V}]] = \tilde{U}$. Let $U' = j[\tilde{U}]$, $V' = j[\tilde{V}]$. We claim that $U'.V' \subseteq W'$.

Let $j(x) \in U'$, $j(y) \in V'$. Then $x \in \tilde{U}$, $y \in \tilde{V}$. There exists $k_0, k_1 \in K$, $u' \in U$ and $v' \in V$ such that $u = u'k_0$, $v = v'k_1$. We know that $u.v \in W$ So $j(u).j(v) = j(u'k_0).j(v'k_1) = j(u')j(v') \in W'$, as intended.

$(.)^{-1}$ is continuous: let $i : G/K \to G/K$ be given by $i(u) = u^{-1}$. We want to see that i is continuous. Let $a \in G$ be given. Let $b = a^{-1}$ Let W' be an open neighborhood of $j(b) = i(j(a))$. Let $W = j^{-1}[W']$. $x^{-1} \in W$ and W is open, so there exists an open neighborhood U of x such there $i[U] \subseteq W$. Let $\tilde{U} = j^{-1}[j[U]] = U.K$, which is open. Notice that $j^{-1}[j[\tilde{U}] = \tilde{U}$ is open. Let $U' = j[\tilde{U}]$. Clearly, $j(a) \in U$. We only need to see that if $x \in \tilde{U}$, then $i(j(x)) \in W'$, that is, that $i[U'] \subseteq W'$. Given $x \in \tilde{U}$ there exists $x' \in U$ and $k \in K$ such that $x = x'k$. Then $i(j(x)) = i(j(x')) \in W$.

□

Divisible groups will be very important for us.

Definition 5.1.9. An Abelian group G is said to be *divisible* iff for each $g \in G$ and each $n \in \omega \setminus \{0\}$, there exists $x \in G$ such that $nx = g$. □

The proof of the next three results are well known basic results of the theory of divisible groups and can be found in [60].

Proposition 5.1.10. Let G be an Abelian group, H be a subgroup of G, $\tilde{G}$ be a divisible group and $f : H \to \tilde{G}$ be a group homomorphism. There exists a group homomorphism $F : G \to \tilde{G}$ such that $F|H = f$. □

The group $\mathbb{Q}/\mathbb{Z}$ is called the *quasicyclic group.*

Theorem 5.1.11. An Abelian group is divisible if and only if, it is isomorphic to a direct sum of copies of $\mathbb{Q}$ and of quasicyclic groups. □

Theorem 5.1.12. Every Abelian group is isomorphic to a subgroup of a divisible group. □

5.2 Countably compact topological groups

There are three natural questions concerning Hausdorff countably compact Abelian groups with or without non-trivial convergent sequences that can be asked separately or jointly.

(1) What groups admit such topologies?

(2) How large they can be?

(3) Do they exist in ZFC?

Question (1) was solved under Martin's Axiom in [21] for Abelian groups of cardinality $\mathfrak{c}$. To talk about this result we need a basic result:

Proposition 5.2.1 (Folklore)**.** Let G be an infinite Hausdorff pseudocompact group. Then $|G| \geq \mathfrak{c}$

Proof. Since topological groups are homogeneous, G has no isolated points (or G would not be pseudocompact since it would be infinite and discrete). We will inject 2^ω into G.

Recursively construct nonempty open sets U_s for $s \in 2^{<\omega}$ such that $U_\emptyset = G, \operatorname{cl} U_{s^\frown(i)} \subseteq U_s$ and $U_{s^\frown(0)} \cap U_{s^\frown(1)} = \emptyset$. This is possible since G is regular and has no isolated points. For each $f \in 2^\omega$, let $x_f \in \bigcap_{n\in\omega} U_{f|n} = \bigcap_{n\in\omega} \operatorname{cl} U_{f|n}$, which exists since G is pseudocompact (let x_f be an accumulation point of the sequence $(U_{f|n} : n \in \omega)$ of open sets). It is clear that the mapping $f \to x_f$ is injective. □

The mapping $x \to nx$ is clearly seen to be countinuous. Since closed subspaces and countinuous images of countably closed spaces are countably closed we get the following (odd, for now) corollaries:

Corollary 5.2.2. Suppose G is a pseudocompact Hausdorff Abelian torsion group of exponent n. Then dG is either finite or of cardinality $\mathfrak{c}$ for every proper divisor d of n.

Corollary 5.2.3. Suppose G is a countably compact Hausdorff non-torsion Abelian group. $|G/t(G)| \geq \mathfrak{c}$ and for every positive integers n, d such that $d|n$, $dG[n] \approx G[n]/G[d]$ is finite or of cardinality $\geq \mathfrak{c}$.

The interesting things about these easy corollaries is that the converse holds under MA for "small" groups:

Theorem 5.2.4 ([21, p. 3.9])**.** Assume MA. Let G be an Abelian torsion group of cardinality at most $\mathfrak{c}$. The following are equivalent:

1. G admits a pseudocompact group topology,
2. G admits a countably compact group topology;
3. G admits a countably compact group topology without non-trivial convergent sequences;
4. G has exponent n for some $n \in \omega$ and dG is either finite or has cardinality $\mathfrak{c}$ for every proper divisor d of n. □

Theorem 5.2.5 ([21, p. 4.4]). Assume MA. Let G be an Abelian non-torsion group of cardinality at most $\mathfrak{c}$. The following are equivalent:

1. G admits a countably compact group topology,
2. G admits a countably compact group topology without non-trivial convergent sequences, and
3. $|G/t(G)| = \mathfrak{c}$ and for every positive integers n, d such that $d|n$, $dG[n] \approx G[n]/G[d]$ is finite or of cardinality $\mathfrak{c}$.

Together, these two results say that (under MA) any "not-so-large" Abelian group satisfying a necessary very basic and easy-to-prove condition for having a countably compact Hausdorff group topology admits such a topology, and we can even guarantee that it does not have convergent sequences. This partially answers question (1).

Regarding question (2), in [17] A. Tomita and I. Castro-Pereira used some cardinal arithmetic weaker than GCH and the existence of a selective ultrafilter p to classify all the torsion groups that admit a p-compact group topology (without non-trivial convergent sequences). This gave the first arbitrarily large countably compact groups without non-trivial convergent sequences. Their main result is the following:

Theorem 5.2.6. Assume that:

i) $\kappa^\omega = \kappa$ for every infinite cardinal κ such that $\operatorname{cf} \kappa > \omega$,

ii) every cardinal β of countable cofinality is a strong limit cardinal, and

iii) there exists a selective ultrafilter.

Let G be a torsion Abelian group and $\mathcal{U}$ be a selective ultrafilter. The following are equivalent:

(a) G admits a topological group topology that turns G into a $\mathcal{U}$-compact topological group without non-trivial convergent sequences.

(b) G admits a countably compact group topology without non-trivial convergent sequences.

(c) G admits a pseudocompact group topology.

(d) For all but finitely many prime numbers p, $G_p = \bigcup_{k \geq 1} G[p^k]$ is non-trivial. For each prime p, G_p is isomorphic to $\bigoplus_{l \leq l_p} \mathbb{Z}_{p^{t}_{p,i}}^{(\alpha_{p,i})}$ for some positive natural number l_p, naturals $t_{p,0} < \cdots < t_{p,l_p}$ and cardinals $\alpha_{p,i}$ such that for each i, $\alpha_{p,i}$ is finite or there exists $j \geq i$ such that $\alpha_{p,i} \leq \alpha_{p,j} = (\alpha_{p,j})^\omega$.

Regarding non-torsion Abelian groups, we have obtained two results which are worth mentioning. They are not part of this thesis, although I coauthored them. The first result is published in our paper [8].

Theorem 5.2.7 ([8]). Let κ be an infinite cardinal such that $\kappa^\omega = \kappa$ and $\mathcal{U}$ be a selective ultrafilter. Then $\mathbb{Q}^{(\kappa)}$ admits a $\mathcal{U}$-compact Hausdorff group topology with no nontrivial convergent sequences.

The second result is available in our preprint [5].

Theorem 5.2.8 ([5])**.** Assume the existence of $\mathfrak{c}$ pairwise RK-incomparable selecive ultrafilters. Let κ be an infinite cardinal such that $\kappa^\omega = \kappa$. Then the free Abelian group $\mathbb{Z}^{(\kappa)}$ admits a countably compact Hausdorff group topology with no nontrivial convergent sequences whose all finite powers are also countably compact.

Dikranjan and Shakhmatov [19] used a forcing model to classify all Abelian groups of cardinality at most $2^\mathfrak{c}$ that admit a countably compact group topology (without non-trivial convergent sequences).

Question 3 is the best known question in the subject. It has been finally answered by M. Hrusak, U. A. Ramos-Garcia, J. van Mill and S. Shelah in [44]. In their paper, they prove the following:

Theorem 5.2.9. There is a countably compact Hausdorff group topology on the group $([\mathfrak{c}]^{<\omega}, \Delta)$ without non-trivial convergent sequences.

Δ stands for symmetric difference, so this is a boolean Abelian group (a boolean group is a group G such that $2G = \{0\}$, therefore this is a torsion group). The main new ingredient in the ZFC construction is the use of a clever filter which takes care of the combinatorics that guarantee the existence of accumulation points.

This new idea has yet two limitations: first the construction depends on the use of a group of finite order. Second, the example has cardinality $\mathfrak{c}$. It is not yet known if the example could be improved to obtain an example of cardinality strictly greater than $\mathfrak{c}$.

In this thesis, we present our result that (consistently) improves Theorem 5.2.5 by using forcing.

Our first result in this direction, which we will not present the proof in this thesis, but will use many of the tools developed while working on it, is the following:

Theorem 5.2.10 ([6])**.** Assume $\mathfrak{p} = \mathfrak{c}$. Let G be an Abelian non-torsion group of cardinality at most $\mathfrak{c}$. The following are equivalent:

1. G admits a countably compact group topology with non-trivial convergent sequences;
2. G admits a countably compact group topology without non-trivial convergent sequences;
3. $|G/t(G)| = \mathfrak{c}$ and for every positive integers n, d such that $d|n$, $dG[n] \approx G[n]/G[d]$ is finite or of cardinality $\mathfrak{c}$.

The result above was part of old unpublished manuscripts written by A. H. Tomita, I. Castro-Pereira and A. C. Boero during postdoctoral studies of I. Castro-Pereira. During my PhD, myself, my fellow student M. K. Bellini and A. H. Tomita completely rewrote this manuscript, fixing some typos, changing most of the notation and making it more clear, also obtaining some corollaries in the process. Just after doing that, we did the same with the following result (with a different notation) in the PhD thesis of A. C. Boero [12] which was not published in any paper, which weakens the hypothesis of $\mathfrak{p} = \mathfrak{c}$.

Theorem 5.2.11 ([6])**.** Assume the existence of $\mathfrak{c}$ pairwise RK-incomparable selective ultrafilters. Let G be an Abelian non-torsion group of cardinality at most $\mathfrak{c}$. The following are equivalent:

1. G admits a countably compact group topology with non-trivial convergent sequences;

2. G admits a countably compact group topology without non-trivial convergent sequences;

3. $|G/t(G)| = \mathfrak{c}$ and for every positive integers n, d such that $d|n$, $dG[n] \approx G[n]/G[d]$ is finite or of cardinality $\mathfrak{c}$.

The paper [7] appears as a natural follow up to [6] and uses its results and notation, so, although it uses the same techniques as [12], the presentation of the result is different.

Theorem 5.2.11 is not a part of this thesis. Using the techniques revamped in [6], we were able to obtain, by applying forcing, the following classification result, which is published in our paper [7]:

Theorem 5.2.12 (*)**.** It is consistent that for every Abelian non-torsion group G of cardinality at most $2^{\mathfrak{c}}$, the following are equivalent:

1. G admits a countably compact group topology;

2. G admits a countably compact group topology with non-trivial convergent sequences;

3. $|G/t(G)| \geq c$ and for every positive integers n, d such that $d|n$, $dG[n] \approx G[n]/G[d]$ is finite or of cardinality at least $\mathfrak{c}$.

To prove this result, in [7] we relied on results from [6], claiming that the proofs are analogous, as they really are. In this document we will prove [7] directly without relying in this kind of argument.

In particular, this consistently answers Question 24 of Dikranjan and Shakhmatov [20] for countably compact groups of cardinality at most $2^{\mathfrak{c}}$. The question is the following problem:

Problem 5.2.13. Let G be an infinite group admitting a countably compact (or a pseudocompact) group topology. Does G have a countably compact (respectively, pseudocompact) group topology that contains a nontrivial convergent sequence?

Note that this result differs from the results from [19] since we are classifying the Abelian groups which have countably compact Hausdorff group topologies that do have convergent sequences.

The proof of this result is very long and we dedicate the rest of this chapter to it.

5.3 The groups for the immersion

We start by presenting some of the notation that will be used throughout this chapter.

$\mathbb{T}$ is the unit circle $\mathbb{R}/\mathbb{Z}$, where $\mathbb{R}$ is the real additive group and $\mathbb{Z}$ the subgroup of the integers.

Fix a partition $\{P_0, P_1\}$ of $\mathfrak{c}$ such that $|P_0| = |P_1| = \mathfrak{c}$, and $\omega + 1 \subseteq P_1$.

Fix a partition $\{R_0, R_1\}$ be a partition of $2^{\mathfrak{c}} \setminus \mathfrak{c}$ such that $|R_0| = |R_1| = 2^{\mathfrak{c}}$.

Define $\mathbb{U} = \mathbb{Q}^{(R_0)} \oplus \mathbb{Q}^{(R_1)}$ and $\mathbf{U} = (\mathbb{Q}/\mathbb{Z})^{(R_0)} \oplus \mathbb{Q}^{(R_1)}$.

Define $\mathbb{W} = (\mathbb{Q}/\mathbb{Z})^{(P_0 \times \omega)} \oplus \mathbb{Q}^{(P_1)} \oplus \mathbf{U}$.

Also define $\mathbb{X} = \mathbb{Q}^{(P_0 \times \omega)} \oplus \mathbb{Q}^{(P_1)} \oplus \mathbb{U}$.

Fix a partition $\{C_{n,m} : n > 1, m \geq 1\}$ of P_0 of sets cardinality $\mathfrak{c}$, where $C_0 = \bigcup_{m,n>1} C_{n,m}$ is such that $P_0 \setminus C_0$ has cardinality $\mathfrak{c}$. Partition $P_0 \setminus C_0$ as $\bigcup_{n>1} C_{n,1}$, where each $C_{n,1}$ has cardinality $\mathfrak{c}$.

Let $C_1 \subseteq P_1$ be such that $|C_1| = |P_1 \setminus C_1| = \mathfrak{c}$, and partition C_1 as $\{C_{1,m} : m > 1\}$ and let $C_{1,1} = P_1 \setminus C_1$ in a way such that all these sets have cardinality $\mathfrak{c}$ as well.

The main group throughout this chapter will be $\mathbf{X} = \oplus_{n>1,m\geq1}(\mathbb{Q}/\mathbb{Z})^{(C_{n,m}\times m)} \oplus \mathbb{Q}^{(P_1)} \oplus \mathbf{U} \subseteq \mathbb{W}$.

If G is a subset of $\mathbb{X}$ or $\mathbb{W}$ and $E \subseteq 2^{\mathfrak{c}}$, we define $G_E = \{g \in G : \operatorname{supp} g \subseteq (E \times \omega) \cup E\}$.

5.3.1 Structure of the construction

We will use forcing to construct an injective group homomorphism $\Phi : \mathbf{X} \to \mathbb{T}^{\mathfrak{c}}$. The range of this homomorphism will be countably compact and have convergent sequences. Each forcing condition will be a partial countable piece of this homomorphism.

Of course, not every subgroup of $\mathbf{X}$ will be countably compact. However, we will show that if H is a group such that $2^{\mathfrak{c}} \geq |H| \geq |H/t(H)| \geq \mathfrak{c}$ and for all $d, n \in \mathbb{N}$ with $d \mid n$, the group $H[n]/H[d]$ is either finite or has cardinality at least $\mathfrak{c}$, then it is isomorphic to a subgroup of $\mathbf{X}$ that is countably compact and has convergent sequences considering the subspace topology of $\mathbf{X}$ generated by Φ. To show that such a copy exists, we define the concept of *large subgroup of* $\mathbf{X}$, which was inspired by the concept of *nice immersion* from [6]. This concludes the classification.

We will guarantee that the sequences of the form $(\frac{n!}{S}\chi_n : n \in \omega)$ in $G \subseteq \mathbf{X}$, for each positive integer S, converge to 0 (identifying G with its copy in $\mathbf{X}$), where χ_n is the characteristic function of $\{n\} \subseteq P_1$.

5.3.2 More notation

Given $w \in \mathbb{W}$ or $w \in \mathbb{X}$, $x \in (P_0 \times \omega) \cup R_0$ and $y \in P_1 \cup R_1$, we denote by $w(x)$ the x-th coordinate of w and $w(y)$ the y-th of w, so the functions $w \to w(x)$ and $w \to w(y)$ are the natural projections.

We also fix well defined numerators and denominators for fractions: if $r \in \mathbb{Q}/\mathbb{Z}$ is not 0, then $p(r)$ and $q(r)$ are the unique integers p, q such that $q > 0$, $\gcd(p, q) = 1$, $0 \leq p < q$

and $r = \frac{p}{q} + \mathbb{Z}$. Likewise, if $r \in \mathbb{Q}$ is not 0, $p(r)$ and $q(r)$ are the unique integers p, q such that $q > 0$, $\gcd(p, q) = 1$ and $r = \frac{p}{q}$. We define $p(0) = 0$ and $q(0) = 1$.

Given $w \in \mathbb{W}$ (or $w \in \mathbb{X}$), we denote by w^0 and w^1 the unique elements of $\mathbb{W}$ (or $\mathbb{X}$) such that $\operatorname{supp} w^0 \subseteq ((P_0 \times \omega) \cup R_0)$, $\operatorname{supp} w^1 \subseteq P_1 \cup R_1$ and $w = w^0 + w^1$, that is, $w \to w^0$ is the natural projections into $(\mathbb{Q}/\mathbb{Z})^{((P_0\times\omega)\cup R_0)}$ (or $\mathbb{Q}^{((P_0\times\omega)\cup R_0)}$) and $w \to w^1$ is the natural projection into $\mathbb{Q}^{(P_1 \cup R_1)}$. Also, we call $w^{1,0}$ and $w^{1,1}$ the natural projections of w onto $\mathbb{Q}^{(\omega)}$ and $\mathbb{Q}^{((P_1\cup R_1)\setminus\omega)}$, respectively.

We also define $p(w) = \max\{|p(w(z))| : z \in \operatorname{supp} w\}$ and $q(w) = \max\{q(w(z)) : z \in \operatorname{supp} w\}$ if $w \neq 0$. We define $p(0) = 0$ and $q(0) = 1$.

Similarly, given $g : \omega \to \mathbb{W}$ (or $\mathbb{X}$), we define g^0, g^1, $g^{1,0}$ and $g^{1,1}$. So $g = g^0 + g^1 = g^0 + g^{1,0} + g^{1,1}$, $\operatorname{supp} g^0 \subseteq ((P_0 \times \omega) \cup R_0)$, $\operatorname{supp} g^1 \subseteq P_1 \cup R_1$, $\operatorname{supp} g^{1,0} \subseteq \omega$ and $\operatorname{supp} g^{1,1} \subseteq (P_1 \cup R_1) \setminus \omega$; where $\operatorname{supp} h = \bigcup\{\operatorname{supp} h(k) : k \in \omega\}$ for a sequence h.

It will be useful to be able to easily transform an element of $\mathbb{X}$ into an element of $\mathbb{W}$. Thus, given $w \in \mathbb{X}$, we define $[w]$ as the unique element of $\mathbb{W}$ such that for every $x \in (P_0 \times \omega) \cup R_0$, $[w](x) = w(x) + \mathbb{Z}$ and for every $y \in P_1 \cup R_1$, $[w](y) = w(y)$. Clearly, the function $w \to [w]$ is a group homomorphism from $\mathbb{X}$ onto $\mathbb{W}$. Given a function $g : \omega \to \mathbb{X}$, we also define $[g] : \omega \to \mathbb{W}$ be given by $[g](n) = [g(n)]$ for every $n \in \omega$.

5.4 Types of sequences

5.4.1 Associating sequences to a type

In this section we define the 11 types of sequences related to a subgroup G of $\mathbb{W}$ and state the theorem that every sequence is related to one of them.

The set of the sequences which are of one of these 11 types of sequences for G (which will be defined in the following subsections) will be denoted by $\mathcal{H}_G$.

The main result we will state in this section is the following, which, in particular, implies that when working with the existence of accumulation points for a sequence, by passing to a subsequence it is enough to guarantee the existence of an accumulation point for the 11 types and the convergence of the sequence of $(\frac{n!}{S}\chi_n : n \in \omega)$ to 0 for each positive integer S. The proof is the same as the proof of Theorem 3.1. of [6] although the group called $\mathbb{W}$ is different. We will present a proof here.

Theorem 5.4.1 (*). Let $f : \omega \to \mathbb{X}$ be given by $f(n) = n!\chi_n$ for every $n \in \omega$. Let G be a subgroup of $\mathbb{W}$ containing $[\chi_n]$ for every $n \in \omega$. Let $g : \omega \to \mathbb{X}$ with $[g] \in G^\omega$.

Then there exists $h : \omega \to \mathbb{X}$ such that $h \in \mathcal{H}_G$ or $h = 0$ is the constant sequence null sequence, $c \in \mathbb{X}$ with $[c] \in G$, $F \in [\omega]^{<\omega}$, $p_i, q_i \in \mathbb{Z}$ with $q_i \neq 0$ for every $i \in F$, $(j_i : i \in F)$ strictly increasing sequences of natural numbers and $j : \omega \to \omega$ strictly increasing such that

$$g \circ j = h + c + \sum_{i \in F} \frac{p_i}{q_i} f \circ j_i$$

with $q_i \leq j_i(n)$ for each $n \in \omega$ and $i \in F$ (which implies $[\frac{1}{q_i} f \circ j_i] \in G^\omega$ since $q_i|((j_i(n))!)$ for each $i \in F$ and $n \in \omega$). $\square$

We will prove this result in the end of this section after defining the types.

5.4.2 The eleven types

We prove this theorem by pieces. First, we must define the types. We classify them in three groups.

First group

Definition 5.4.2 (*). [The types related to $(R_1 \cup P_1) \setminus \omega$] Let G be a subgroup of $\mathbb{W}$. We define the first three types of sequence (with respect to G) as follows:

Let $g : \omega \to \mathbb{X}$ be such that $[g] \in G^\omega$.

We say that g is *of type 1* iff $\operatorname{supp} g^{1,1}(n) \setminus \bigcup_{m<n} \operatorname{supp} g^{1,1}(m) \neq \emptyset$, for every $n \in \omega$.

We say that g is *of type 2* iff $q(g^{1,1}(n)) > n$, for every $n \in \omega$.

We say that g is *of type 3* iff $\{q(g^{1,1}(n)) : n \in \omega\}$ is bounded and $|p(g^{1,1}(n))| > n$, for every $n \in \omega$. $\square$

Proposition 5.4.3 (*). If $g : \omega \to \mathbb{X}$ with $[g] \in G^\omega$ then there exists a strictly increasing sequence $j : \omega \to \omega$ such that there exists such that $g \circ j$ is of type 1, 2 or 3 or $g^{1,1} \circ j$ is constant.

Proof. **Case 1** $\bigcup_{n\in\omega} \operatorname{supp} g^{1,1}(n)$ is infinite.
By induction, define a strictly increasing sequence $(n_k : k \in \omega)$ of natural numbers such that $\operatorname{supp} g^{1,1}(n_{k+1}) \setminus \bigcup_{l<k+1} \operatorname{supp} g^{1,1}(n_l) \neq \emptyset$. The function $j : \omega \to \omega$ defined by $j(k) = n_k$ for each $k \in \omega$ is strictly increasing and $g \circ j$ is of type 1.

Case 2 $\bigcup_{n\in\omega} \operatorname{supp} g^{1,1}(n)$ is finite and $\{q(g^{1,1}(n)) : n \in \omega\}$ is unbounded.
By induction, define a strictly increasing sequence $\{n_k : k \in \omega\}$ of natural numbers such that $q(g^{1,1}(n_k)) > k$ for each $k \in \omega$. The function $j : \omega \to \omega$ defined by $j(k) = n_k$ for each $k \in \omega$ is strictly increasing and $g \circ j$ is of type 2.

Case 3 $\bigcup_{n\in\omega} \operatorname{supp} g^{1,1}(n)$ is finite, $\{q(g^{1,1}(n)) : n \in \omega\}$ is bounded and $\{|p(g^{1,1}(n))| : n \in \omega\}$ is unbounded.
By induction, define a strictly increasing sequence $(n_k : k \in \omega)$ of natural numbers such that $|p(g^{1,1}(n_k))| > k$ for each $k \in \omega$. The function $j : \omega \to \omega$ defined by $j(k) = n_k$ for each $k \in \omega$ is strictly increasing and $g \circ j$ is of type 3.

Case 4 None of the cases above, that is: $\bigcup_{n\in\omega} \operatorname{supp} g^{1,1}(n)$ is finite, $\{q(g^{1,1}(n)) : n \in \omega\}$ is bounded and $\{|p(g^{1,1}(n))| : n \in \omega\}$ is bounded. There exists a infinite subset A such that $(\operatorname{supp} g^{1,1}(n) : n \in A)$ is constant. There are only finitely many possibilities for the coefficients in this finite support, therefore, there is an infinite subset B of A for which

$(g^{1,1}(n) : n \in B)$ is constant. Let j be an increasing enumeration of B. Let $j : \omega \to B$ be the increasing enumeration of B.

$\square$

Second group

Definition 5.4.4 (*). [The types related to ω] Let $G \subseteq \mathbb{W}$ be a subgroup such that $\{[\chi_n] : n \in \omega\} \subseteq G$. Let $g : \omega \to \mathbb{X}$ be such that $[g] \in G^\omega$. Then we define types 4 to 9 (with respect to G) as follows:

We say that g is *of type 4* iff $q(g^{1,0}(n)) > n$, for every $n \in \omega$.

We say that g is *of type 5* iff $\{q(g(n)) : n \in \omega\}$ is bounded and there exists $M \in \bigcap_{n\in\omega} \operatorname{supp} g^{1,0}(n)$ such that $|p(g(n)(M))| > n$ for every $n \in \omega$.

To define types 6, 7 and 8, suppose g is such that for each $n \in \omega$, there exists $M_n \in \operatorname{supp} g^{1,0}(n) \setminus \bigcup_{m<n} \operatorname{supp} g^{1,0}(m)$ such that

$$\left(\frac{g(n)(M_n)}{M_n!} : n \in \omega\right)$$

is a 1-1 sequence that converges to some $u \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{-\infty, 0, \infty\}$.

We say that g is of *type 6* iff $u = 0$.

We say that g is of *type 7* iff $u \in \mathbb{R} \setminus \mathbb{Q}$.

We say that g is of *type 8* iff u is ∞ or $-\infty$.

We say that g is *of type 9* iff $\left\{\dfrac{g(n)(M)}{M!} : M \in \operatorname{supp} g^{1,0}(n), n \in \omega\right\}$ is finite and $|\operatorname{supp} g^{1,0}(n)| > n$ for every $n \in \omega$. $\square$

Proposition 5.4.5 (*). Let $G \subseteq \mathbb{W}$ be a subgroup such that $[\chi_n] \in G$ for every $n \in \omega$, and let $f : \omega \longrightarrow \mathbb{X}$ be such that $f(n) = n!\chi_n$, for each $n \in \omega$.

Let $g : \omega \to \mathbb{X}_{P_0 \cup R_0 \cup \omega}$ be such that $[g] \in G^\omega$.

Then there exists $c \in \mathbb{X}$ with $[c] \in G$, a finite set F (possibly empty), $p_i, q_i \in \mathbb{Z}$ with $q_i \neq 0$ for every $i \in F$, a sequence h either of type 4, 5, 6, 7, 8 or 9 or $h = h^0$ with $[h] \in G^\omega$ and $j_i : \omega \to \omega$ a strictly increasing sequence for each $i \in F$ such that

$$g \circ j = h + \sum_{i\in F} \frac{p_i}{q_i} f \circ j_i + c$$

with $[h] \in G^\omega$ and $q_i \leq j_i(n)$, for each $n \in \omega$ and $i \in F$.

Proof. **Case 1** $\{q(g^{1,0}(n)) : n \in \omega\}$ is unbounded.
By induction, define a strictly increasing sequence j of elements of ω such that $q(g^{1,0} \circ j(n)) > n$, for each $n \in \omega$. Then $g \circ j$ is of type 4. Let $F = \emptyset$, $c = 0$.

For the next cases, let

$$Q = \left\{ \frac{g(n)(M)}{M!} : M \in \operatorname{supp} g^{1,0}(n), n \in \omega \right\}.$$

Case 2 $\{q(g^{1,0}(n)) : n \in \omega\}$ is bounded and Q is infinite.

There exists $J_0 \in [\omega]^\omega$ and $(M_n : n \in J_0)$ such that $M_n \in \operatorname{supp} g^{1,0}(n)$ and

$$\left(\frac{g(n)(M_n)}{M_n!} : n \in J_0 \right) \to r$$

strictly monotonically (the main fact is that we need a sequence that converges and is one to one), for some $r \in [-\infty, +\infty]$. Then there exists J_1 an infinite subset of J_0 such that one of the following hold:

(a) $M_n \notin \bigcup \{\operatorname{supp} g^{1,0}(m) : m < n, m \in J_1\}$, for every $n \in J_1$

or

(b) there exists $M \in \omega$ such that $M_n = M$, for every $n \in J_1$.

Subcase 2.1 Condition (a) holds and $r \notin \mathbb{Q} \setminus \{0\}$.

If $r = 0$, then $g \circ j$ is of type 6, where $j : \omega \to J_1$ is the order isomorphism. Analogously, if $r \in \mathbb{R} \setminus \mathbb{Q}$, then $g \circ j$ is of type 7 and if $r \in \{-\infty, +\infty\}$, then $g \circ j$ is of type 8. Let $F = \emptyset$, $c = 0$.

Subcase 2.2 Condition (a) holds and $r \in \mathbb{Q} \setminus \{0\}$.

Define j' such that $j'(n) = M_n$ for each $n \in J_1$ and let $\tilde{g} : \omega \to \mathbf{X}$ be such that $\tilde{g}(n) = g(n) - rf \circ j'$ for every $n \in J_1$. Since $(M_n : n \in J_1)$ is injective, there exists a cofinite subset J_2 of J_1 such that $q(r) \leq M_n$ for each $n \in J_2$ and therefore $[\tilde{g}(n)] \in G$ for every $n \in J_2$.

Note that

$$\left(\frac{\tilde{g}(n)(M_n)}{M_n!} : n \in J_2 \right) \to 0.$$

Let $j : \omega \to J_2$ be an order isomorphism. Then $\tilde{g} \circ j$ is of type 6, $[\tilde{g} \circ j] \in G^\omega$ and $g \circ j = \tilde{g} \circ j + rf \circ j' \circ j$. Let $h = \tilde{g} \circ j$, $F = \{0\}$, $j_0 = j' \circ j$, $p_0 = p(r)$, $q_0 = q(r)$ and we are done.

Subcase 2.3 Condition (b) holds. Since

$$\left(\frac{g(n)(M)}{M!} : n \in J_1\right)$$

is injective and $\{q(g(n)(M)) : n \in J_1\}$ is bounded, there exists $J_2 \in [J_1]^\omega$ such that $|p(g(n)(M))| > n$, for every $n \in J_2$. Hence, $g \circ j$ is of type 5, where $j : \omega \to J_2$ is the order isomorphism.

Case 3 $\{q(g^{1,0}(n)) : n \in \omega\}$ is bounded and Q is finite.

Then there exists J_0 a infinite subset of ω such that either

(c) $(|\operatorname{supp} g^{1,0}(n)| : n \in J_0)$ is injective

or

(d) $(|\operatorname{supp} g^{1,0}(n)| : n \in J_0)$ is constant.

Subcase 3.1 If property (c) holds then there exists a strictly increasing sequence $j \in (J_0)^\omega$ such that $|\operatorname{supp} g^{1,0}(j(n))| > n$, for every $n \in \omega$ and $g \circ j$ is of type 9. Let $F = \emptyset$, $c = 0$.

Subcase 3.2 If property (d) holds, then there exists $k \in \omega$ such that $|\operatorname{supp} g^{1,0}(n)| = k$, for every $n \in J_0$ and we may write $\operatorname{supp} g^{1,0}(n) = \{M_0^n, \dots, M_{k-1}^n\}$, where $M_i^n \neq M_{i'}^n$ if $i \neq i'$. Since Q is finite, there exist $J_1 \in [J_0]^\omega$ and $p_0/q_0, \dots, p_{k-1}/q_{k-1} \in \mathbb{Q} \setminus \{0\}$ such that

$$\frac{g(n)(M_i^n)}{M_i^n!} = \frac{p_i}{q_i}$$

for each $n \in J_2$ and $i < k$. By refining J_2 (if necessary), we can suppose that for each $i < k$, $(M_i^n : n \in J_2)$ is either constant or is injective with $q_i \leq M_i^n$ for every $n \in J_2$. Let $F = \{i < k : (M_i^n : n \in J_2)$ is a 1-1 sequence$\}$. Let $j_i' : J_2 \longrightarrow \omega$ such that $j_i'(n) = M_i^n$, for each $i < k$. Note that $\sum_{i \in F} \left[\frac{p_i}{q_i} f\right] \circ j_i'(n) \in G$, for every $n \in J_2$. Fix $n_0 \in J_2$.

$$g(n) - g(n_0) = g^0(n) - g^0(n_0) + g^{1,0}(n) - g^{1,0}(n_0) =$$

$$g^0(n) - g^0(n_0) + \sum_{i<k} \frac{p_i}{q_i} f \circ j_i'(n) - \sum_{i<k} \frac{p_i}{q_i} f \circ j_i'(n_0) =$$

$$g^0(n) - g^0(n_0) + \sum_{i \in k \setminus F} \frac{p_i}{q_i} f \circ j_i'(n) - \sum_{i \in k \setminus F} \frac{p_i}{q_i} f \circ j_i'(n_0) + \sum_{i \in F} \frac{p_i}{q_i} f \circ j_i'(n) - \sum_{i \in F} \frac{p_i}{q_i} f \circ j_i'(n_0) =$$

$$g^0(n) - g^0(n_0) + \sum_{i \in F} \frac{p_i}{q_i} f \circ j_i'(n) - \sum_{i \in F} \frac{p_i}{q_i} f \circ j_i'(n_0)$$

for every $n \in J_2$. The last equality follows from $j_i'(n) = j_i'(n_0)$, for each $i \in k \setminus F$ and $n \in J_2$. Since $[g(n) - g(n_0)]$ and $[\sum_{i \in F} \frac{p_i}{q_i} f \circ j_i'(n) - \sum_{i \in F} \frac{p_i}{q_i} f \circ j_i'(n_0)]$ are elements of G, it follows that $[g^0(n) - g^0(n_0)] \in G$, for every $n \in J_2$.

Let j be the order preserving bijection between ω and J_2, $c_1 = g(n_0)$ and $c_2 = -\sum_{i\in F} \frac{p_i}{q_i} f \circ j'_i(n_0)$. Note that $[c_1], [c_2] \in G$.

Then
$$g \circ j = c_1 + (g - c_1) \circ j = c_1 + (g - c_1)^0 \circ j + \sum_{i\in F} \frac{p_i}{q_i} f \circ j'_i \circ j + c_2$$

with $[(g - c_1)^0 \circ j] \in G^\omega$ and $(g - c_1)^0 \circ j(n) \in \mathbb{X}_{P_0 \cup R_0}$. Let $c = c_1 + c_2$, $h = (g - c_1)^0 \circ j$, $j_i = j'_i \circ j$ and we are done.

□

Third group

Definition 5.4.6 (*). [The types related to $P_0 \cup R_0$] We define types 10 and 11 (with respect to G) as follows: Let $g : \omega \to \mathbb{X}$ be such that $\operatorname{supp} g(n) \subseteq R_0 \cup (P_0 \times \omega)$ for each $n \in \omega$ and $[g] \in G^\omega$.

We say that g is *of type 10* iff $q(g^0(n)) > n$, for every $n \in \omega$.

We say that g is *of type 11* iff the family $\{[g(n)] : n \in \omega\}$ is an independent family whose elements have a fixed order k, for some positive integer k. □

To prove properties relating a sequence to a subsequence of type 11 we will use following concept introduced in [21]:

Definition 5.4.7. Let G be an Abelian group and $n \in \mathbb{N} \setminus \{1\}$. A countably infinite subset S of G is said to be *n-round* if $nS = \{0\}$ and the restriction of the group homomorphism

$$\begin{array}{rccc} \varphi_d : & G & \to & G \\ & x & \mapsto & dx \end{array}$$

to S is finite-to-one for every proper divisor d of n. □

The proof of the following proposition can also be found in [21]:

Proposition 5.4.8. Every infinite set in an Abelian group G such that $nG = \{0\}$ contains an infinite subset of the form $T + z$, where $z \in G$ and T is a d-round subset of G for some divisor d of n.

Proof. Let $A \subseteq G$ be infinite. If A is n-round we are done. If not, there exists a positive integer $d < n$ such that $d \mid n$ and $h \in G$ such that $\{g \in A : dg = h\}$ is infinite. Let d be the smallest such positive integer and fix $g' \in A$. Let $T = A - g'$. Then clearly $dT = \{0\}$.

Suppose $d' \mid d$, where $d' < d$ is a positive integer. Let $h' \in G$. We must see that $B = \{g \in T : d'g = h\}$ is finite. Notice that $B + g \subseteq A$. Moreover, if $g \in B$, $g + g' \in A$ and $d'(g + g') = h + d'g'$. Let $h'' = h + d'g'$. Then B injects into $\{a \in A : d'a = h''\}$, which is finite by the minimality of d. □

From this, we can obtain independent sets. The following is probably folklore.

Lemma 5.4.9. Let G be an Abelian group. Then every infinite d-round subset of G contains an infinite independent subset.

Proof. Let H be a finite torsion subgroup of G. We claim that there exists $g \in T$ such that $\langle g \rangle$ is a direct summand of H. If this is not the case, then for every $g \in T$ there exists a positive integer r such that $rg \in H$. Let r_g be the smallest such r. Since $dT = \{0\}$ it follows that $1 \leq r_g < d$.

We claim that $r_g \mid d$. Suppose not. Write $d = r_g m + n$, where $m \in \mathbb{Z}$ and $0 < n < d$. Clearly, $r_g mg \in H$. Also, $0 = dg = r_g mg + ng \in H$. This implies that $ng \in H$, but $1 \leq n < r_g$, a contradiction.

Since H is finite, there exists $h \in H$ such that $\{g \in T : r_g g = h\}$ is infinite. Thus, there exists a divisor r of d such that $\{g \in T : rg = h\}$ is infinite which shows that T is not d-round, a contradiction.

Now, one can define inductively a independent subset $T' \subseteq T$ as required.

□

Proposition 5.4.10 (*)**.** Let G be a subgroup of $\mathbb{W}$. If $g : \omega \to \mathbb{X}_{P_0 \cup R_0}$ is such that $[g] \in G^\omega$, then there exist $j : \omega \to \omega$ strictly increasing, $c \in \mathbb{X}_{P_0 \cup R_0}$ with $[c] \in G$, and $h : \omega \to \mathbb{X}$ of type 10 or 11, such that either $g \circ j = h + c$, or $[g \circ j]$ is constant.

Proof. By the choice of g, we have $g = g^0$.

Case 1 $\{q(g(n)) : n \in \omega\}$ is unbounded.
By induction, define a strictly increasing sequence j of ω such that $q(g(j(n))) > n$, for each $n \in \omega$. Then $g \circ j$ is of type 10, so let $h = g \circ j$ and $c = 0$.

Case 2 $\{q(g(n)) : n \in \omega\}$ is bounded and $S = \{[g(n)] : n \in \omega\}$ is infinite.
Since $\{q(g(n)) : n \in J_1\}$ is bounded, there exists a natural number $m > 1$ such that $mS = \{0\}$. Thus, by Proposition 5.4.8 there exist $T \subseteq G$ infinite and $z \in G$ such that $T + z \subseteq S$ and T is k-round, for some divisor k' of m. By Lemma 5.4.9, T contains an infinite independent subset T' of some fixed order $k \mid k'$. Let $c \in \mathbb{X}$ such that $[c] = z$ and let $A = \{n \in \omega : [g(n)] - z \in T'\}$. Let j be an increasing enumeration of ω onto A. Then $(g - c) \circ j$ is of type 11, so let $h = (g - c) \circ j$.

Case 3 $\{q(g(n)) : n \in \omega\}$ is bounded and $S = \{[g(n)] : n \in \omega\}$ is finite.

In this case, there exists $c \in \mathbb{X}$ such that $A = \{n \in \omega : [g(n)] = [c]\}$ is infinite. Let j be a order preserving bijection between ω and A. We have that $[g \circ j]$ is constant.

□

5.4.3 Proof of Theorem 5.4.1

Proof. Let $g \in \mathbb{X}^\omega$ be such that $[g] \in G^\omega$.

By Proposition 5.4.3, there exists an increasing sequence j' such that $g \circ j'$ is of type $1, 2, 3$ or $g^{1,1} \circ j'$ is constant. If $g \circ j'$ is of type $1, 2, 3$, let $h = g \circ j'$, $j = j'$, $c = 0$, $F = \emptyset$ and we are done. If not, let $g \circ j'(0) = c_1$. Notice that $[c_1] \in G$ and that $g_1 = (g - c_1) \circ j' : \omega \to X$ is such that $[g_1] \in G^\omega$ and $g_1^{1,0} = g_1$. Also, $g \circ j' = g_1 + c_1$.

Now we apply Proposition 5.4.5 to g_1 and obtain $\tilde{j} : \omega \to \omega$, $c_2 \in \mathbb{X}$ with $[c_2] \in G$, a finite set F, $p_i, q_i \in \mathbb{Z}$ with $q_i \neq 0$ for every $i \in F$, a sequence g_2 of type i with $i \in \{4, \dots, 9\}$ or such that $g_2 = g_2^0$ and $[g_2] \in G^\omega$ and $j'_i : \omega \to \omega$ a strictly increasing sequence for each $i \in F$ such that

$$g_1 \circ \tilde{j} = g_2 + \sum_{i \in F} \frac{p_i}{q_i} f \circ j'_i + c_2 \tag{5.1}$$

with $[g_2] \in G^\omega$ and $q_i \leq j'_i \in G^\omega$ for each $n \in \omega$ and $i \in F$.

Case 1 g_2 is of type i with $i \in \{4, \dots 9\}$.

In this case, let $j = j' \circ \tilde{j}$, $h = g_2$, $c = c_1 + c_2$ and $j_i = j'_i$. Set F and the p_i's and q'_is as above.

Case 2 $[g_2] \in G^\omega$ and $g_2 = g_2^0$.

Apply Proposition 5.4.10 on g_2 to obtain $l : \omega \to \omega$ strictly increasing, $c_3 \in \mathbb{W}$ with $[c_3] \in G$, and $g_3 : \omega \to \mathbb{X}$ of type 10 or 11, such that either $g_2 \circ l = g_3 + c_3$ or $[g_2 \circ l]$ is constant.

Case 2a $g_2 \circ l = g_3 + c_3$. In this case, let $j = j' \circ \tilde{j} \circ l$, $h = g_3$, $c = c_1 + c_2 + c_3$, and $j_i = j'_i \circ l$. Set F and the p_i's, q_i's as above.

Case 2b $[g_2 \circ l]$ is constant. Let $j = j' \circ \tilde{j} \circ l$, $h = 0$, $c = c_1 + c_2 + g_2 \circ l(0)$ and $j_i = j'_i \circ l$. Set F and the p_i's and q_i's as above. By (1) and (2), we are done.

□

The following lemma is easy to verify and left to the reader.

Lemma 5.4.11 (*)**.** Being a sequence of one of the types is absolute for transitive models of ZFC.

5.5 Arc homomorphisms and countable homomorphisms

5.5.1 Countable homomorphisms

In this subsection, we will state a theorem that guarantees (in ZFC) the existence of partial homomorphisms defined on countable subgroups of $\mathbf{X}$. The statement of this theorem is very similar to Proposition 4.3. of [6] and has a completely analogous (long) proof, which we write here for the sake of completeness.

Proposition 5.5.1 (*)**.** Let E be a countable subset of $2^{\mathfrak{c}}$ containing ω, $e \in \mathbf{X}_E$ with $e \neq 0$, a countable family $(g_k : k \in \omega)$ of elements of $\mathcal{H}_{\mathbf{X}_E}$ and A_k infinite subsets of ω for each $k \in \omega$.

Fix a family $(c_k : k \in \omega)$ of elements of $\mathbb{X}$ such that $[c_k] \in \mathbf{X}_E$, c_k is a non torsion element if g_k is of one of types from 1 to 10, and $[c_k]$ has the same order as $[g_k]$ if g_k is of type 11.

Then there exists a homomorphism $\rho : \mathbf{X}_E \to \mathbb{T}$ such that:

1. $\rho(e) \neq 0$,
2. for each $k \in \omega$, there exists $B_k \subseteq A_k$ infinite such that $(\rho([g_k(n)]))_{n \in B_k}$ converges to $\rho([c_k])$, and
3. $\left(\rho\left(\frac{n!}{S}\chi_n\right) : n \in \omega\right)$ converges to $0 \in \mathbb{T}$, for every integer $S > 0$.

We will prove this result in the end of this section.

5.5.2 Arc Homomorphisms

To prove the result in the previous subsection we will need the technical concept of arc homomorphism, which is an approximation of an homomorphism. First, we need some notation:

Definition 5.5.2 (*). Given a subset E of $2^{\mathfrak{c}}$ we define:

- $E^0 = E \cup (P_0 \cup R_0)$,
- $E^1 = E \cup (P_1 \cup R_1)$,
- $\mathbf{X}_E = \{w \in \mathbf{X} : \operatorname{supp} w \subseteq (E \times \omega) \cup E\}$,
- $\mathbf{X}_{(K)} = \{w \in \mathbf{X} : \exists u \in \mathbb{X}, w = [u]$ and Ku has integer coordinates$\}$, and
- $\mathbf{X}_{E,K} = \mathbf{X}_E \cap \mathbf{X}_{(K)}$.

□

Definition 5.5.3. An arc is an subset of $\mathbb{R}/\mathbb{Z}$ of the form $(a, b) + \mathbb{Z}$, where $a < b$ are real numbers. $\mathcal{B}$ is the set of all arcs. □

Definition 5.5.4 (*). Given a positive real ϵ, a subset E of $\mathfrak{c}$ and a positive integer K, an (E, K, ϵ)*-arc homomorphism* is a pair $\phi = (\phi^0, \phi^1)$ where:

a) $\phi^0 : \mathbf{X}_{E^0,K} \to \mathbb{T}$ is a homomorphism,

b) $\phi^1 : \{\frac{1}{K}\chi_\xi : \xi \in E^1\} \to \mathcal{B}$,

c) $\phi^1(\frac{1}{K}\chi_\xi)$ is an arc of length ϵ, for every $\xi \in E^1$.

□

The idea is that ϕ^1 is an approximation of a homomorphism from $\mathbf{X}_{E^1}$ to $\mathbb{T}$. Notice that ϕ^1 can naturally be extended to the whole $\mathbf{X}_{E,K}$ as follows:

Definition 5.5.5 (*). Let $E \subseteq 2^{\mathfrak{c}}$, K be a positive integer, $\epsilon > 0$, $\phi = (\phi^0, \phi^1)$ be an (E, K, ϵ)-arc homomorphism and $w \in \mathbf{X}_{E,K}$. Then we define:

$$\hat{\phi}(w) = \phi^0(w^0) + \sum_{\xi \in \operatorname{supp} w^1} K w(\xi) \phi^1 \left(\frac{\chi_\xi}{K} \right)$$

We define the empty sum of arcs to be the singleton $\{0 + \mathbb{Z}\}$, so if $w \in \mathbf{X}_{E^0,K}$, it follows that $\hat{\phi}(w) = \{\phi^0(w)\}$. □

We will extend these arc homomorphisms while shrinking ϵ to construct homomorphisms. When extending arc homomorphisms, we must be careful to guarantee the convergence of the sequences $([\frac{m!}{S}\chi_m] : m \in \omega)$, where S is a positive integer. Thus, the following notion of extension will be very useful:

Definition 5.5.6 (*). Given $\epsilon', \epsilon > 0$, positive integers K, K', subsets E, E' of $2^\mathfrak{c}$, $\phi = (\phi^0, \phi^1)$ an (E, K, ϵ)-arc homomorphism and $\psi = (\psi^0, \psi^1)$ an (E', K', ϵ')-arc homomorphism, we say that $\psi < \phi$ (ψ *extends* ϕ) iff:

a) $E \subseteq E'$, $K|K'$ and $\epsilon' \leq \epsilon$,

b) $\phi^0 \subseteq \psi^0$,

c) $\operatorname{cl}\left(\frac{K'}{K}\psi^1(\frac{1}{K'}\chi_\xi)\right) \subseteq \phi^1(\frac{1}{K}\chi_\xi)$, for each $\xi \in E^1$.

d) $m!\frac{K'}{K}\psi^1(\frac{\chi_m}{K'})$ is contained in the arc $(-\frac{\epsilon}{2K}, \frac{\epsilon}{2K}) + \mathbb{Z}$, for each $m \in (E' \cap \omega) \setminus E$.

□

Now we state some straightforward properties of arc homomorphisms. We leave the proofs to the reader.

Lemma 5.5.7 (*). Given ϵ_i, K_i, E_i and ϕ_i (E_i, K_i, ϵ_i)-arc homomorphisms ($i = 1, 2$) such that $\phi_2 < \phi_1$, then $\operatorname{cl}\left(\hat{\phi}_2(w)\right) \subseteq \hat{\phi}_1(w)$ for every $w \in \mathbf{X}_{E_1,K_1}$.

Lemma 5.5.8 (*). Given ϕ an (E, K, ϵ)-arc homomorphism and $w, w' \in \mathbf{X}_{E,K}$, $\hat{\phi}(w + w') \subseteq \hat{\phi}(w) + \hat{\phi}(w')$.

The lemmas above show that extension of arc homomorphisms are better approximations of a homomorphism. The following lemma shows how to use a family of arc homomorphisms to define a group homomorphism.

Proposition 5.5.9 (*). Suppose $(\phi_n : n \in \omega)$ is a sequence of (E_n, K_n, ϵ_n)-arc homomorphisms such that: $\phi_{n+1} < \phi_n$ for every $n \in \omega$; $(\epsilon_n : n \in \omega) \to 0$; each positive integer divides cofinitely many elements of $(K_n : n \in \omega)$ (or, equivalently, one K_n); and $\omega \subseteq \bigcup_{n\in\omega} E_n$. Let $E = \bigcup_{n\in\omega} E_n$. Then:

1. For every $w \in \mathbf{X}_E$, $\bigcap\{\hat{\phi}_n(w) : w \in \mathbf{X}_{E_n,K_n}\}$ is a singleton.

2. The function $\psi : \mathbf{X}_E \to \mathbb{T}$ so that $\psi(w) \in \bigcap\{\hat{\phi}_n(w) : w \in \mathbf{X}_{E_n,K_n}\}$ is a group homomorphism.

3. If $\omega \subseteq E$ and E_n is finite for every n, then for every positive integer S, $(\psi\left(\frac{m!}{S}\chi_m\right) : m \in \omega)$ converges to $0 \in \mathbb{T}$.

Proof. (1) Notice that the hypothesis about $(K_n : n \in \omega)$ guarantees that the sets being intersected are nonempty. By Lemma 5.5.7, $\bigcap\{\hat{\phi}_n(w) : w \in \mathbf{X}_{E_n,K_n}\} = \bigcap\{\mathrm{cl}\,\hat{\phi}_n(w) : w \in \mathbf{X}_{E_n,K_n}\}$. Since $(\epsilon_n : n \in \omega) \to 0$, the second intersection is a decreasing family of closed sets with diameter converging to 0. Since $\mathbb{T}$ is a complete metric space it follows that the intersection is a singleton.

(2) Notice that given $w, w' \in \mathbf{X}_E$,

$$\{\psi(w{+}w')\} = \bigcap\{\hat{\phi}_n(w{+}w') : w{+}w' \in \mathbf{X}_{E_n,K_n}\} \subseteq \bigcap\{\hat{\phi}_n(w){+}\hat{\phi}_n(w') : w{+}w' \in \mathbf{W}_{E_n,K_n}\}.$$

Notice that $(\hat{\phi}_n(w){+}\hat{\phi}_n(w') : w{+}w' \in \mathbf{X}_{E_n,K_n})$ is a family with arbitrarily small diameters, so its intersection is at most a singleton, which must be $\{\psi(w+w')\}$. So:

$$\{\psi(w+w')\} = \bigcap\{\hat{\phi}_n(w) + \hat{\phi}_n(w') : w + w' \in \mathbf{X}_{E_n,K_n}\}$$

$$\supseteq \bigcap\{\hat{\phi}_n(w) : w \in \mathbf{X}_{E_n,K_n}\} + \bigcap\{\hat{\phi}_n(w') : w' \in \mathbf{X}_{E_n,K_n}\} = \{\psi(w) + \psi(w')\}.$$

So $\psi(w + w') = \psi(w) + \psi(w')$, as intended.

(3) Let S be given. Fix N such that S divides K_n for every $n \geq N$. Now we show that the sequence converges.

Let $\epsilon > 0$ be given. There exists $t > S, N$ such that $\epsilon_t < \epsilon$ and $E_t \cap \omega \neq \emptyset$. Let $M = \max\{E_t \cap \omega\}$. We claim that if $m > M$ then $\psi\left(\frac{m!}{S}\chi_m\right) \in (-\epsilon, \epsilon) + \mathbb{Z}$, which completes the proof.

If $m > M$, then $m \notin E_t$. Let n be the first natural number such that $m \in E_n$. It follows that $n > t$. By the definition of arc extension and the choice of n, we have:

$$m!\frac{K_n}{K_{n-1}}\hat{\phi}_n\left(\frac{1}{K_n}\chi_m\right) \subseteq \left(-\frac{\epsilon_n}{2K_{n-1}}, \frac{\epsilon_n}{2K_{n-1}}\right) + \mathbb{Z}$$

So multiplying this expression by the integer $\frac{K_{n-1}}{S}$, it follows that:

$$\hat{\phi}_n\left(\frac{m!}{S}\chi_m\right) = m!\frac{K_n}{S}\hat{\phi}_n\left(\frac{1}{K_n}\chi_m\right) \subseteq \left(-\frac{\epsilon_n}{2S}, \frac{\epsilon_n}{2S}\right) + \mathbb{Z} \subseteq (-\epsilon, \epsilon) + \mathbb{Z},$$

so $\psi\left(\frac{m!}{S}\chi_m\right) \in (-\epsilon, \epsilon) + \mathbb{Z}$, as intended. □

Given $w \in \mathbf{X} \setminus \{0\}$, it is easy to construct an arc homomorphism ϕ for which $w \in \mathbf{X}_{E,K}$ and $\hat{\phi}(w) \cap \{0\} = \emptyset$. After doing that, just by shrinking the arcs it is easy to construct a sequence of extensions of ϕ that satisfy the conditions of the previous proposition. A more complicated problem is how to do it while guaranteeing accumulation points for arbitrary sequences and this is where the 11 types play their role.

The lemma below will be used to extend arc homomorphisms. Roughly speaking, given an arc homomorphism, we find a good homomorphism approximated by the arc homomorphism and then use the Lemma 5.5.10 to define the arc homomorphism extension.

Lemma 5.5.10 (*)**.** Suppose that the following are given:

i) $E \in [2^{\mathfrak{c}}]^{<\omega}$,

ii) K is a positive integer,

iii) $\epsilon < \frac{1}{2}$ is a positive real number,

iv) ϕ is an (E, K, ϵ)-arc homomorphism,

v) U is an open arc,

vi) $v \in \mathbf{X}$,

vii) E' is a finite set with $E \subseteq E' \subseteq 2^{\mathfrak{c}}$, and

viii) K' is a multiple of K.

Suppose $v \in \mathbf{X}_{E',K'}$. Then if there exists a homomorphism $\theta : \mathbf{X}_{E',K'} \to \mathbb{T}$ such that:

a) $\phi^0 \subseteq \theta$,

b) $\theta(\frac{1}{K}\chi_\xi) \in \phi^1(\frac{1}{K}\chi_\xi)$ for every $\xi \in E^1$,

c) $\theta(\frac{m!}{K}\chi_m) \in \left(-\frac{\epsilon}{2K}, \frac{\epsilon}{2K}\right) + \mathbb{Z}$ for each $m \in E' \cap \omega \setminus E$, and

d) $\theta\,(v) \in U$,

then there exists $\epsilon' > 0$ with $\epsilon' < \epsilon$ such that for all $\epsilon^* \leq \epsilon'$, there exists an (E', K', ϵ^*)-arc homomorphism $\phi' \leq \phi$ such that $\hat{\phi}'(v) \subseteq U$ if $v^1 \neq 0$ and $(\phi')^0(v) \in U$ if $v^1 = 0$.

Proof. Let $(\phi')^0 = \theta|\mathbf{X}_{E'^0,K'}$.

Given $\xi \in E^1$, let ϵ_ξ be such that the closed arc centered in $\theta(\frac{1}{K}\chi_\xi)$ of length $\frac{K'}{K}\epsilon_\xi$ is contained in $\phi^1(\frac{1}{K}\chi_\xi)$.

Given $m \in (E' \setminus E) \cap \omega$, let ϵ_m be such that the closed arc centered in $\theta(\frac{m!}{K}\chi_m)$ of length $m!\frac{K'}{K}\epsilon_m$ is contained in $\left(-\frac{\epsilon}{2K}, \frac{\epsilon}{2K}\right) + \mathbb{Z}$.

If $v^1 \neq 0$, let $\bar{\epsilon} > 0$ be such that the closed arc centered on $\theta\,(v)$ of length $K'\bar{\epsilon}(\sum_{\xi \in \operatorname{supp} v^1} |v(\xi)|)$ is contained in U. If not, let $\bar{\epsilon} = 1$.

Let $\epsilon' = \min\left(\{\bar{\epsilon}, \frac{\epsilon}{2}\} \cup \{\epsilon_\xi : \xi \in E^1\} \cup \{\epsilon_m : m \in (E' \setminus E) \cap \omega\}\right)$. Given $\epsilon^* \leq \epsilon'$, define, for $\xi \in (E')^1$, $(\phi')^1(\frac{1}{K'}\chi_\xi)$ as the arc centered in $\theta(\frac{1}{K'}\chi_\xi)$ of length ϵ^*. Now it is easy to verify that all the properties we need hold. □

5.5.3 Extension of Arc Homomorphisms

The main results are the following two theorems. The proof is rather long and will be postponed to the end of this subsection since we will need some preliminary lemmas.

Theorem 5.5.11 (*)**.** Let the following be given:

i) $E \in [2^{\mathfrak{c}}]^{<\omega}$,

ii) K positive integers,

iii) ϵ a positive real with $\epsilon < \frac{1}{2}$,

iv) ϕ a (E, K, ϵ)-arc homomorphism,

v) h a sequence of type 1, 2, 3, 4, 5, 6, 7, 8, 9 or 10,

vi) $\gamma > 0$ a real number.

Then there exists a cofinite set $S \subseteq \omega$ such that for all $n \in S$, for all finite E' with $E \subseteq E' \subseteq 2^{\mathfrak{c}}$, for all K' with $K|K'$ such that $\left[\frac{1}{K}h(n)\right] \in \mathbf{X}_{E',K'}$ and for all $U \in \mathcal{B}$ of length $\geq \gamma$:

There exist $\epsilon' \leq \epsilon$ such that for every positive $\epsilon^* \leq \epsilon'$ there exists an (E', K', ϵ^*)-arc homomorphism $\phi' < \phi$ such that $\hat{\phi}'\left(\left[\frac{1}{K}h(n)\right]\right) \subseteq U$. □

The reason type 11 is separate from the others is that its accumulation point have finite order and its proof can be readily presented as it does not require extra preliminaries.

Theorem 5.5.12 (*). Let the following be given:

i) $E \in [2^{\mathfrak{c}}]^{<\omega}$,

ii) K positive integers,

iii) ϵ positive real with $\epsilon < \frac{1}{2}$,

iv) ϕ a (E, K, ϵ)-arc homomorphism,

v) h a sequence of type 11 of order k.

Then there exists a cofinite set $S \subseteq \omega$ such that for all $n \in S$, for all finite E' with $E \subseteq E' \subseteq 2^{\mathfrak{c}}$, for all K' multiple of Kk such that $\left[\frac{1}{K}h(n)\right] \in \mathbf{X}_{E',K'}$ and for all $r \in \mathbb{T}$ such that $kr = 0$:

There exist $\epsilon' \leq \epsilon$ such that for every positive $\epsilon^* \leq \epsilon'$ there exists an (E', K', ϵ^*)-arc homomorphism $\phi' < \phi$ such that $(\phi')^0\left([h(n)]\right) = r$.

Proof. The family $\{[h(n)] : n \in \omega\}$ is an independent family whose elements have order k.

Let $\psi : \mathbf{X}_{E,K} \to \mathbb{T}$ be a homomorphism such that $\psi|\mathbf{X}_{E^0,K} = \phi^0$ and $\psi(\frac{1}{K}\chi_\xi)$ is the center of $\phi^1(\frac{1}{K}\chi_\xi)$ for each $\xi \in E^1$.

Let K' be any multiple of Kk. The group $\mathbf{X}_{E^0,K'}$ is a finite group. Therefore, for all but finitely n the group generated by $[h(n)]$ and $\mathbf{X}_{E^0,K'}$ is a direct sum. Let S be the set of such n's and fix $n \in S$. Notice that $\frac{K'}{K}h(n)(x)$ is an integer for every $x \in \operatorname{supp} h(n)$. Notice that the sum of the group generated by $\langle [h(n)] \rangle$ with $\mathbf{X}_{E,K'}$ is also a direct sum. We can extend ψ to ψ' on $\mathbf{X}_{E,K'} \oplus \langle [h(n)] \rangle$ so that $\psi'([h(n)]) = r$. Let θ be the extension of ψ' defined on $\mathbf{X}_{E',K'}$ using the divisibility of $\mathbb{T}$ in such a way that for every $m \in (E' \setminus E) \cap \omega$, $\psi'(\frac{m!}{K}\chi_m) = 0 + \mathbb{Z}$. Apply Lemma 5.5.10 to obtain the desired arc homomorphism ϕ'. □

Notice that K' did not depend on the choice of $n \in S$.

Before we start proving Theorem 5.5.11, we will prove two lemmas.

The lemma below will be needed for the cases related to types 2, 4, 5, 6, 8 and 10.

Lemma 5.5.13 (*). Let $c, d \in \mathbb{Z} \setminus \{0\}$, $\epsilon > 0$, $a \in \mathbb{T}$ and $B \in \mathcal{B}$ be such that $\delta(B) \geq \epsilon$ and $|d|\epsilon > \gcd(c, d)$. There exists $x \in \mathbb{T}$ such that $dx = a$ and $cx \in B$.

Proof. Let $e = \gcd(c, d)$ and let $c', d' \in \mathbb{Z}$ be such that $c = ec'$ and $d = ed'$. Notice that $\epsilon > \frac{1}{|d'|}$. If $B = \mathbb{T}$ the proof is trivial, so suppose $\delta(B) \geq \epsilon$ and $|d|\epsilon > e$.

Let $a = \tilde{a} + \mathbb{Z}$ for some $\tilde{a} \in \mathbb{R}$. Since $\epsilon > \frac{1}{|d'|}$, there exists $l \in \mathbb{Z}$ such that:

$$\frac{c\tilde{a}}{d} + \frac{l}{d'} + \mathbb{Z} \in B.$$

Since $\gcd(c', d') = 1$, there exists $u, v \in \mathbb{Z}$ such that $uc' + vd' = l$. Now, since $\frac{c}{d} = \frac{c'}{d'}$, it follows that:

$$\frac{c\tilde{a}}{d} + \frac{uc}{d} + \mathbb{Z} \in B.$$

So let $x = \frac{\tilde{a}+u}{d} + \mathbb{Z}$ and we are done. □

The second lemma will be used to treat sequences of type 7 and its proof uses Kronecker's Theorem. Lemma 5.5.16 is a less elaborated version of a construction made on [11], but it is difficult to recognize it inside the construction. Therefore, we present a proof of it. First, we state Kronecker's Theorem.

Theorem 5.5.14 (Kronecker)**.** Let $k \in \omega$ and $\{1, \xi_0, \dots, \xi_{k-1}\}$ be a linearly independent subset of $k+1$ vectors of the vector space $\mathbb{R}$ over the field $\mathbb{Q}$. Then $\{(\xi_0 n, \dots, \xi_{k-1} n) + \mathbb{Z}^k : n \in \mathbb{Z}\}$ is a dense subset of the product of $\mathbb{T}^k$. □

We refer to [13, Theorem 4.13.] for a proof.

If $\xi \in \mathbb{R} \setminus \mathbb{Q}$, then let μ be such that $\{1, \mu, \xi\}$ is linearly independent. Then $\{\mu^{-1}, 1, \xi\mu^{-1}\}$ is linearly independent. Then $\{(\mu^{-1}n, \xi\mu^{-1}n) + \mathbb{Z}^2 : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^2$.

Hence $\{(x, \xi x) + \mathbb{Z}^2 : x \in \mathbb{R}\}$ is a dense subset of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. This implies that given an irrational number ξ and $\epsilon > 0$, there exists a such that $\{(x, \xi x) + \mathbb{Z}^2 : x \in [0, a]\}$ is ϵ-dense.

Definition 5.5.15. Let ξ be an irrational real number and $\epsilon > 0$. $a(\xi, \epsilon)$ is a the least natural number such that $\{(x, \xi x) + \mathbb{Z}^2 : x \in [0, a]\}$ is ϵ-dense. □

It will not be important that we are choosing the smallest such a, but we had to make a choice.

Lemma 5.5.16 (*). Let $c_1, c_2 \in \mathbb{Z} \setminus \{0\}$, $\epsilon > 0$ and $B_1, B_2 \in \mathcal{B}$ be such that $\delta(B_1) \geq \epsilon$ and $\delta(B_2) \geq \epsilon$. Suppose that there exist $\xi \in \mathbb{R} \setminus \mathbb{Q}$ such that $\left|\frac{c_1}{c_2} - \xi\right| < \frac{\epsilon}{2a(\xi, \epsilon/2)}$. Then there exists $x \in \mathbb{T}$ such that $c_1 x \in B_1$ and $c_2 x \in B_2$.

Proof. Let $a = a(\xi, \epsilon/2)$. First, we show that $A = \{(x+\mathbb{Z}, \frac{c_1}{c_2}x+\mathbb{Z}) : x \in [0,a]\}$ is ϵ-dense. Fix a ball B of radius $\geq \epsilon$. Let b be the center of the B and $\tilde{B}$ be the ball centered in b of length $\frac{\epsilon}{2}$. Since $\{(x+\mathbb{Z}, \xi x+\mathbb{Z}) : x \in [0,a]\}$ is $\frac{\epsilon}{2}$-dense, there exists $x \in [0,a]$ such that $(x+\mathbb{Z}, \xi x+\mathbb{Z}) \in \tilde{B}$. Now notice that by applying the triangular inequality,

$$\delta(b, (x+\mathbb{Z}, \frac{c_1}{c_2}x+\mathbb{Z})) \leq \delta(b, (x+\mathbb{Z}, \xi x+\mathbb{Z})) + \delta((x+\mathbb{Z}, \xi x+\mathbb{Z}), (x+\mathbb{Z}, \frac{c_1}{c_2}x+\mathbb{Z}))$$

$$< \frac{\epsilon}{2} + |\frac{c_1}{c_2}x - \xi x| \leq \frac{\epsilon}{2} + |\frac{c_1}{c_2}a - \xi a| \leq \epsilon.$$

Then $\{(c_2 x+\mathbb{Z}, c_1 x+\mathbb{Z}) : x \in [0, \frac{a}{c_2}]\} = A$ is ϵ-dense. Since $\delta(B_2 \times B_1) \geq \epsilon$ it follows that there exists $x \in [0, \frac{a}{c_2}]$ such that $c_i(x+\mathbb{Z}) = (c_i x) + \mathbb{Z} \in B_i$, for $i \in 1, 2$. □

Now we present proof of Theorem 5.5.11. The set of proofs are rather long, however, they are split in 10 separate cases, each related to a type. The reader doesn't need to read all the cases at once.

Proof of Theorem 5.5.11. By shrinking ϵ if necessary (by shrinking the arcs $\phi(\frac{1}{K}\chi_\xi)$) we may suppose that $\gamma \geq \epsilon$. Let $\psi : \mathbf{X}_{E,K} \to \mathbb{T}$ be a homomorphism such that $\psi|\mathbf{X}_{E^0,K} = \phi^0$ and $\psi(\frac{1}{K}\chi_\xi)$ is the center of $\phi^1(\frac{1}{K}\chi_\xi)$ for each $\xi \in E^1$.

We now break the proof in 10 cases, one of each type from 1 to 10. In each case, we aim to apply Lemma 5.5.10.

Proof of type 1 $\operatorname{supp} h^{1,1}(n) \setminus \bigcup_{m<n} \operatorname{supp} h^{1,1}(m) \neq \emptyset$, for every $n \in \omega$.

Notice that the set of natural n's such that $\operatorname{supp} h^{1,1}(n) \setminus E \neq \emptyset$ is cofinite. Let S be the set of such n's and fix $n \in S$. Let E', K' be given.

Fix $\mu \in \operatorname{supp} h^{1,1}(n) \setminus E$. Using the divisibility of $\mathbb{T}$, extend ψ to ψ_1 defined in $\mathbf{X}_{E' \setminus \{\mu\}, K'}$ so that $\psi_1(\frac{m!}{K}\chi_m) = 0 + \mathbb{Z}$, for each $m \in (E' \setminus E) \cap \omega$.

This is possible since the sum $\mathbf{X}_{E,K} \oplus \mathbf{X}_{(E' \setminus E) \cap \omega, K'} \subseteq \mathbf{X}_{E' \setminus \{\mu\}, K'}$ is direct, so we first extend ψ to a ψ' defined on $\mathbf{X}_{E,K} \oplus \mathbf{X}_{E' \cap \omega \setminus E, K'}$ using direct sum properties so that $\psi'(\frac{1}{K'}\chi_m) = 0 + \mathbb{Z}$ for every $m \in (E' \setminus E) \cap \omega$, and then extend ψ' to ψ_1 using divisibility. This kind of argument will be needed repeatedly and will be omitted from now on and the details are left to the reader.

Now, since $\mu \in \operatorname{supp} h(n)$, $\mu \notin E \cup (\operatorname{supp} h(n) \setminus \{\mu\})$ and $\mathbb{T}$ is divisible, we can extend ψ_1 to θ defined in $\mathbf{X}_{E',K'}$ so that $\theta([\frac{1}{K}h(n)]) \in U$. To do this, it suffices to define $\theta\left(\frac{(h(n))(\mu)}{K}\chi_\mu\right)$ such that

$$\theta\left(\frac{(h(n))(\mu)}{K}\chi_\mu\right) + \psi_1\left(\left[\frac{1}{K}h(n) - \frac{(h(n))(\mu)}{K}\chi_\mu\right]\right) \in U.$$

Now we apply Lemma 5.5.10 and we are done. □

Proof of type 2 $q(h^{1,1}(n)) > n$, for every $n \in \omega$.

There are cofinitely many $n \in \omega$ for which there exists $\mu \in \operatorname{supp} h^{1,1}(n)$ such that

$$1 < q(h(n)(\mu))\epsilon. \tag{5.2}$$

Let S be the set of such n's and fix $n \in S$. Let E', K' be given. Notice that Kq divides K'.

Fix such a μ for a fixed n. We abbreviate $p = p(h(n)(\mu))$ and $q = q(h(n)(\mu))$.

Extend ψ to a ψ_1 defined on $\mathbf{X}_{E'\setminus\{\mu\},K'} \oplus \mathbf{X}_{\{\mu\},K}$ so that $\psi(\frac{m!}{K}\chi_m) = 0 + \mathbb{Z}$, for each $m \in \operatorname{supp}(E' \setminus E) \cap \omega$.

Given an open arc B of length $\geq \epsilon$, we may use (5.2) to apply Lemma 5.5.13 using $c = p$, $d = q$, $a = \psi_1(\frac{1}{K}\chi_\mu)$, B and ϵ to obtain u such that $pu \in B$ and $qu = \psi_1(\frac{1}{K}\chi_\mu)$. We apply this using $B = U - \psi_1\left(\left[\frac{1}{K}h(n) - \frac{(h(n))(\mu)}{K}\chi_\mu\right]\right)$ and fix u.

Extend ψ_1 to a ψ_2 defined over $\mathbf{X}_{E'\setminus\{\mu\},K'} \oplus \mathbf{X}_{\{\mu\},Kq}$ by defining $\psi_2(\frac{1}{Kq}\chi_\mu) = u$ and $\psi_2|_{\mathbf{X}_{E'\setminus\{\mu\},K'}} = \psi_1|_{\mathbf{X}_{E'\setminus\{\mu\},K'}}$. Notice that this is really an extension of ψ_1 since

$$\psi_2\left(\frac{1}{K}\chi_\mu\right) = \psi_2\left(\frac{q}{qK}\chi_\mu\right) = q\psi_2\left(\frac{1}{Kq}\chi_\mu\right) = qu = \psi_1\left(\frac{1}{K}\chi_\mu\right).$$

Now let θ be an homomorphism extending ψ_2 to $\mathbf{X}_{E',K'}$, which exists since $\mathbb{T}$ is divisible.

Finally, notice that, by letting $t = \psi_1\left(\left[\frac{1}{K}h(n) - \frac{(h(n))(\mu)}{K}\chi_\mu\right]\right)$, it follows that

$$\theta\left(v\right) = t + \theta\left(\frac{(h(n))(\mu)}{K}\chi_\mu\right) = t + p\theta\left(\frac{1}{qK}\chi_\mu\right) = t + pu \in B + t = U.$$

Now we apply Lemma 5.5.10 and we are done. □

Proof of type 3 $\{q(h^{1,1}(n)) : n \in \omega\}$ is bounded and $|p(h^{1,1}(n))| > n$, for every $n \in \omega$.

Let M be a positive integer such that $\{q(h^{1,1}(n)) : n \in \omega\} \subseteq [-M, M]$. There are cofinitely many $n \in \omega$ for which there exists $\mu \in \operatorname{supp} h^{1,1}(n)$ such that $\frac{\epsilon}{M}|p(h(n)(\mu))| > 1$. Let S be the set of such n's and fix $n \in S$. Let E', K' be given.

Extend $\psi|_{\mathbf{X}_{E\setminus\{\mu\},K'}}$ to ψ_1 defined on $\mathbf{X}_{E'\setminus\{\mu\},K'}$ so that $\psi_1(\frac{m!}{K}\chi_m) = 0 + \mathbb{Z}$, for each $m \in (E' \setminus E) \cap \omega$.

Let $B = U - \psi_1\left(\left[\frac{1}{K}h(n) - \frac{h(n)(\mu)}{K}\chi_\mu\right]\right)$, $p = p(h(n)(\mu))$, $q = q(h(n)(\mu))$. Let A be an open arc of size $\frac{\epsilon}{q}$ such that $qA = \phi^1(\frac{1}{K}\chi_\mu)$. Then, as $\frac{\epsilon}{q} \geq \frac{\epsilon}{M}$, it follows that $pA = \mathbb{T}$, so we may choose $u \in A$ such that $pu \in B$. Notice that $qu \in \phi^1(\frac{1}{K}\chi_\mu)$.

Now define ψ_2 extending ψ_1 defined on $\mathbf{X}_{E'\setminus\{\mu\},K'} \oplus \mathbf{X}_{\{\mu\},Kq}$ satisfying $\psi_2(\frac{1}{qK}\chi_\mu) = u$ and extend it to an homomorphism $\theta : \mathbf{X}_{E',K'} \to \mathbb{T}$ using the divisibility of $\mathbb{T}$.

Now we apply Lemma 5.5.10 and we are done. □

Proof of type 4 $q(h(n)) > n$, for every $n \in \omega$.

There are cofinitely many $n \in \omega$ for which there exists $M \in \operatorname{supp} h^{1,0}(n)$ such that

$$1 < q(h(n)(M))\epsilon. \tag{5.3}$$

Notice that there are cofinitely many such n's. Let S be the set of such n's and fix $n \in S$. Let E', K' be given. Fix M.

If $M \in E$, we proceed exactly as in the case of type 2.

If $M \notin E$, we proceed as follows: we abbreviate $p = p(h(n))(M)$ and $q = q(h(n))(M)$.

Extend ψ to a ψ_1 defined on $\mathbf{X}_{E' \setminus \{M\}, K'}$ so that $\psi_1(\frac{m!}{K}\chi_m) = 0 + \mathbb{Z}$, for each $m \in (E' \setminus E) \cap \omega$, $M \neq m$.

Given an open arc B of length $\geq \epsilon$, we may use (5.3) to apply Lemma 5.5.13 using $c = p$, $d = q$, $a = 0$, B and ϵ to obtain u such that $pu \in B$ and $qu = 0$. We apply this using $B = U - \psi_1\left(\left[\frac{1}{K}h(n) - \frac{(h(n))(M)}{K}\chi_M\right]\right)$ and fix u.

Extend ψ_1 to a ψ_2 defined over $\mathbf{X}_{E' \setminus \{M\}, K'} \oplus \mathbf{X}_{\{M\}, Kq}$ by defining $\psi_2(\frac{1}{Kq}\chi_M) = u$ and $\psi_2|_{\mathbf{X}_{E' \setminus \{M\}, K'}} = \psi_1|_{\mathbf{X}_{E' \setminus \{M\}, K'}}$. Notice that:

$$\psi_2\left(\frac{M!}{K}\chi_M\right) = \psi_2\left(\frac{qM!}{qK}\chi_M\right) = qM!\psi_2\left(\frac{1}{Kq}\chi_M\right) = M!qu = 0 + \mathbb{Z}.$$

Now let θ be an homomorphism extending ψ_2 to $\mathbf{X}_{E', K'}$, which exists since $\mathbb{T}$ is divisible.

Now we apply Lemma 5.5.10 and we are done. $\square$

Proof of type 5 There exists $M \in \bigcap_{n \in \omega} \operatorname{supp} h^{1,0}(n)$ such that $\{q(h(n)) : n \in \omega\}$ is bounded and $|p(g(n)(M))| > n$, for every $n \in \omega$. Fix M. Let Q be a positive integer such that $\{q(h(n)) : n \in \omega\} \subseteq [-Q, Q]$. There are cofinitely many $n \in \omega$ such that $\frac{\epsilon}{QKM!}|p(h(n)(M))| > 1$. Let S be the set of such n's, fix $n \in S$ and let E', K' be given. Let $p = p(h(n)(M))$ and $q = q(h(n)(M))$.

If $M \in E$, we proceed as in Type 3.

If $M \notin E$: Let $E' \supseteq E^* \cup E$ such that $h(n) \in \mathbf{Q}_{E'}$ and K' be such that Kq divides K', $S|K'$ and $\left[\frac{h(n)}{K}\right] \in \mathbf{X}_{E', K'}$.

Extend ψ to ψ_1 defined in $\mathbf{X}_{E' \setminus \{M\}, K'}$ so that $\psi_1(\frac{m!}{K}\chi_m) = 0 + \mathbb{Z}$ for every $m \in E' \cap \omega \setminus (E \cup \{M\})$.

Consider $t = \psi_1\left(\left[\frac{1}{K}h(n) - \frac{h(n)(M)}{K}\chi_M\right]\right)$. Apply Lemma 5.5.13 with a being the middle point of $U - t$, $B = \left(-\frac{\epsilon}{2KM!}, \frac{\epsilon}{2KM!}\right) + \mathbb{Z}$, $d = p$ and $c = q$ to obtain u such that $pu = a$ and $qu \in B$.

Extend ψ_1 to ψ_2 defined in $\mathbf{X}_{E' \setminus \{M\}, K'} \oplus \mathbf{X}_{\{M\}, Kq}$ by letting $\psi_2\left(\frac{1}{Kq}\chi_M\right) = u$. Finally, extend ψ_2 to θ in $\mathbf{X}_{E', K'}$ using the divisibility of $\mathbb{T}$.

Now we apply Lemma 5.5.10 and we are done. $\square$

Recall that for types 6, 7 and 8, h is such that for each $n \in \omega$, there exists $M_n \in \operatorname{supp} h^{1,0}(n) \setminus \bigcup_{m<n} \operatorname{supp} h^{1,0}(m)$ such that

$$\left(\frac{h(n)(M_n)}{M_n!} : n \in \omega\right) \to u \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{-\infty, 0, \infty\}.$$

Proof of type 6 $u = 0$.

There are cofinitely many $n \in \omega$ such that $M_n \notin E$ and $|p(h(n)(M_n))| \leq M_n! q(h(n)(M_n))\epsilon$. Let S be the set of such n's, fix $n \in S$ and let E', K' be given. Let $M = M_n$ $p = p(h(n)(M))$ and $q = q(h(n)(M))$.

Extend ψ to ψ_1 defined in $\mathbf{X}_{E'\setminus\{M\},K'}$ so that $\psi_1(\frac{m!}{K}\chi_m) = 0 + \mathbb{Z}$ for every $m \in E' \cap \omega\setminus(E \cup \{M\})$.

Consider $t = \psi_1\left(\left[\frac{1}{K}h(n) - \frac{h(n)(M)}{K}\chi_M\right]\right)$. Apply Lemma 5.5.13 with $a = 0+\mathbb{Z}$ $B = U - t$, $c = p$ and $d = M!q$ to obtain u such that $pu = a$ and $M!qu \in B$. We may apply the lemma since $\gcd(p, M!q) \leq \min\{p, M!q\} = p \leq M!q\epsilon$

Extend ψ_1 to ψ_2 defined in $\mathbf{X}_{E'\setminus\{M\},K'} \oplus \mathbf{X}_{\{M\},Kq}$ by letting $\psi_2\left(\frac{1}{Kq}\chi_M\right) = u$. Finally, extend ψ_2 to θ in $\mathbf{X}_{E',K'}$ using the divisibility of $\mathbb{T}$.

Now we apply Lemma 5.5.10 and we are done. □

Proof of type 7 u is an irrational number ξ.

Fix $a = a(\xi, \frac{\epsilon}{2K})$. There are cofinitely many $n \in \omega$ such that $M_n \notin E$ and $\left|\frac{h(n)(M_n)}{M_n!} - \xi\right| < \frac{\epsilon}{2Ka}$. Let S be the set of such n's, fix $n \in S$ and let E', K' be given. Let $M = M_n$, $p = p(h(n)(M))$ and $q = q(h(n)(M))$.

Extend ψ to ψ_1 defined in $\mathbf{X}_{E'\setminus\{M\},K'}$ so that $\psi_1(\frac{m!}{K}\chi_m) = 0 + \mathbb{Z}$ for every $m \in E' \cap \omega\setminus(E \cup \{M\})$.

Consider $t = \psi_1\left(\left[\frac{1}{K}h(n) - \frac{h(n)(M)}{K}\chi_M\right]\right)$. Apply Lemma 5.5.16 with $B_1 = U - t$, $B_2 = (-\frac{\epsilon}{2K}, \frac{\epsilon}{2K}) + \mathbb{Z}$, $c_1 = p$ and $c_2 = M!q$ to obtain u such that $pu \in B_1$ and $M!qu \in B_2$.

Extend ψ_1 to ψ_2 defined in $\mathbf{X}_{E'\setminus\{M\},K'} \oplus \mathbf{X}_{\{M\},Kq}$ by letting $\psi_2\left(\frac{1}{Kq}\chi_M\right) = u$. Finally, extend ψ_2 to θ in $\mathbf{X}_{E',K'}$ using the divisibility of $\mathbb{T}$.

Now we apply Lemma 5.5.10 and we are done. □

Proof of type 8 $u \in \{-\infty, \infty\}$.

There are cofinitely many $n \in \omega$ such that $M_n \notin E$ and $\frac{|p(h(n)(M_n))|}{KM_n!}\epsilon \geq 1$. Let S be the set of such n's, fix $n \in S$ and let E', K' be given. Let $M = M_n$ $p = p(h(n)(M))$ and $q = q(h(n)(M))$.

Extend ψ to ψ_1 defined in $\mathbf{X}_{E'\setminus\{M\},K'}$ so that $\psi_1(\frac{m!}{K}\chi_m) = 0 + \mathbb{Z}$ for every $m \in E' \cap \omega\setminus(E \cup \{M\})$.

Consider $t = \psi_1\left(\left[\frac{1}{K}h(n) - \frac{h(n)(M)}{K}\chi_M\right]\right)$. Apply Lemma 5.5.13 with $\frac{\epsilon}{KM!}$, a as the center of $U - t$, $B = (-\frac{\epsilon}{2KM!}, \frac{\epsilon}{2KM!}) + \mathbb{Z}$, $d = p$ and $c = q$ to obtain u such that $pu = a$ and $qu \in B$.

Extend ψ_1 to ψ_2 defined in $\mathbf{X}_{E'\setminus\{M\},K'} \oplus \mathbf{X}_{\{M\},Kq}$ by letting $\psi_2\left(\frac{1}{Kq}\chi_M\right) = u$. Finally, extend ψ_2 to θ in $\mathbf{X}_{E',K'}$ using the divisibility of $\mathbb{T}$.

Now we apply Lemma 5.5.10 and we are done. □

Proof of type 9 $\left\{\dfrac{h(n)(M)}{M!} : M \in \operatorname{supp} h^{1,0}(n), n \in \omega\right\}$ is finite and $|\operatorname{supp} h^{1,0}(n)| > n$ for every $n \in \omega$.

Let

$$L = \min\left\{\frac{|h(n)(M)|}{M!} : M \in \operatorname{supp} h^{1,0}(n), n \in \omega\right\}.$$

There are cofinitely many $n \in \omega$ such that $|\operatorname{supp} h^{1,0}(n) \setminus E|\frac{\epsilon}{K}L \geq 1$. Let S be the set of such n's, fix $n \in S$ and let E', K' be given.

Extend ψ to ψ_1 defined in $\mathbf{X}_{(E'\setminus\omega)\cup E,K'}$. Let $F = \operatorname{supp} h^{1,0}(n) \setminus E$ and

$$w = \sum_{M\in F} \frac{h(n)(M)}{K}\chi_M$$

Consider $t = \psi_1\left(\left[\frac{1}{K}h(n) - w\right]\right)$. For each $M \in F$, let $\epsilon_M = \frac{\epsilon}{q(h(n)(M))M!}$. Notice that by the choice of L and n,

$$\mathbb{T} = \sum_{M\in F} p(h(n)(M))\left(-\frac{\epsilon_M}{2K}, \frac{\epsilon_M}{2K}\right) + \mathbb{Z}.$$

So there exists a family $(y_M : M \in F)$ such that $y_M \in \left(-\frac{\epsilon_M}{2K}, \frac{\epsilon_M}{2K}\right) + \mathbb{Z}$, for every $M \in F$ and such that:

$$\sum_{M\in F} p(h(n)(M))y_M \in U - t$$

Extend ψ_1 to ψ_2 defined in $\mathbf{X}_{E'\setminus F,K'} \oplus \mathbf{X}_{F,Kq}$ by letting $\psi_2\left(\frac{1}{Kq}\chi_M\right) = y_M$ for each $M \in F$, and $\psi_2(\frac{m!}{K}\chi_m) = 0 + \mathbb{Z}$ for every $m \in E^* \cap \omega \setminus \operatorname{supp} h^{1,0}(n)$. Finally, extend ψ_2 to θ in $\mathbf{X}_{E',K'}$ using the divisibility of $\mathbb{T}$.

Now we apply Lemma 5.5.10 and we are done. □

Proof of type 10 $q(h^0(n)) > n$, for every $n \in \omega$.

There are cofinitely many $n \in \omega$ such that there exists $\mu \in \operatorname{supp} h^0(n)$ such that

$$1 < q(h(n)(\mu))\epsilon. \tag{5.4}$$

Let S be the set of such n's, fix $n \in S$ and let E', K' be given. Fix such a μ for the fixed n. We abbreviate $p = p(h(n)(\mu))$ and $q = q(h(n)(\mu))$.

Extend ψ to a ψ_1 defined on $\mathbf{X}_{E' \setminus \{\mu\}, K'} \oplus \mathbf{X}_{\{\mu\}, K}$ so that $\psi(\frac{m!}{K}\chi_m) = 0 + \mathbb{Z}$ for every $(E' \setminus E) \cap \omega$.

Given an open arc B of length $\geq \epsilon$, we may use (5.4) to apply Lemma 5.5.13 using $c = p$, $d = q$, $a = \psi_1(\frac{1}{K}\chi_\mu)$, B and ϵ to obtain u such that $pu \in B$ and $qu = \psi_1(\frac{1}{K}\chi_\mu)$. We apply this using $B = U - \psi_1\left(\left[\frac{1}{K}h(n) - \frac{(h(n))(\mu)}{K}\chi_\mu\right]\right)$ and fix u.

Extend ψ_1 to a ψ_2 defined over $\mathbf{X}_{E' \setminus \{\mu\}, K'} \oplus \mathbf{X}_{\{\mu\}, Kq}$ by defining $\psi_2(\frac{1}{Kq}\chi_\mu) = u$ and $\psi_2|_{\mathbf{X}_{E' \setminus \{\mu\}, K'}} = \psi_1|_{\mathbf{X}_{E' \setminus \{\mu\}, K'}}$. Notice that this is really an extension of ψ_1 since

$$\psi_2\left(\frac{1}{K}\chi_\mu\right) = \psi_2\left(\frac{q}{qK}\chi_\mu\right) = q\psi_2\left(\frac{1}{Kq}\chi_\mu\right) = qu = \psi_1\left(\frac{1}{K}\chi_\mu\right).$$

Now let θ be an homomorphism extending ψ_2 to $\mathbf{X}_{E', K'}$, which exists due to the divisibility of $\mathbb{T}$.

Finally, notice that, by letting $t = \psi_1\left(\left[\frac{1}{K}h(n) - \frac{(h(n))(\mu)}{K}\chi_\mu\right]\right)$, it follows that

$$\theta(v) = t + \theta\left(\frac{(h(n))(\mu)}{K}\chi_\mu\right) = t + p\theta\left(\frac{1}{qK}\chi_\mu\right) = t + pu \in B + t = U.$$

Now we apply Lemma 5.5.10 and we are done. □

This ends the construction of the arc homomorphism for the types 1 to 10. □

5.5.4 Proof of Proposition 5.5.1

Proposition 5.5.17 (*). Let E be a countable subset of $2^{\mathfrak{c}}$ containing ω, $e \in \mathbf{X}_E$ with $e \neq 0$, a countable family $(g_k : k \in \omega)$ of elements of $\mathcal{H}_{\mathbf{X}_E}$ and A_k infinite subsets of ω for each $k \in \omega$.

Fix a family $(c_k : k \in \omega)$ of elements of $\mathbb{X}$ such that $[c_k] \in \mathbf{X}_E$, c_k is a non torsion element if g_k is of one of types from 1 to 10, and $[c_k]$ has the same order as $[g_k]$ if g_k is of type 11.

Then there exists a homomorphism $\rho : \mathbf{X}_E \to \mathbb{T}$ such that:

1. $\rho(e) \neq 0$,
2. for each $k \in \omega$, there exists $B_k \subseteq A_k$ infinite such that $(\rho([g_k(n)]))_{n \in B_k}$ converges to $\rho([c_k])$, and
3. $\left(\rho\left(\frac{n!}{S}\chi_n\right) : n \in \omega\right)$ converges to $0 \in \mathbb{T}$, for every integer $S > 0$.

Proof. Without loss of generality, we can suppose that for every $k' \in \omega$, $|\{k \in \omega : (g_k, c_k, A_k) = (g_{k'}, c_{k'}, A_{k'})\}| = \omega$ (just define a new sequence with the same ranges where this holds).

Let $(E^*_n : n \in \omega)$ be a sequence of finite sets such that $\bigcup_{n\in\omega} E^*_n = E$ and $\operatorname{supp}[c_{g_{t+1}}] \subseteq E^*_t$ for every $t \in \omega$.

Recursively, we define sequences $(K_t : t \in \omega)$ of positive integers, $(E_t : t \in \omega)$ of finite subsets of E, $(\epsilon_t : t \in \omega)$ of positive real numbers less than $\frac{1}{2}$, $(m_t : t \in \omega)$ strictly growing of natural numbers and $(\phi_t : t \in \omega)$ such that, for every $t \in \omega$:

(i) ϕ_t is an (E_t, K_t, ϵ_t)-homomorphism,

(ii) $\phi_{t+1} < \phi_t$,

(iii) $\epsilon_t < \frac{1}{2^t K_t \|c_t\|}$,

(iv) $[c_t] \in \mathbf{X}_{E_t,K_t}$,

(v) $[g_t(m_t)] \in \mathbf{X}_{E_{t+1},K_{t+1}}$,

(vi) $t!$ divides K_t,

(vii) $E^*_t \subseteq E_t$,

(viii) $\hat{\phi}_{t+1}([g_t(m_t)]) \subseteq \hat{\phi}_t([c_t])$,

(ix) $m_t \in A_t$,

(x) $e \in \mathbf{X}_{E_0,K_0}$ and $0 + \mathbb{Z} \notin \operatorname{cl} \hat{\phi}_0(e)$.

We start by showing that ϕ_0, ϵ_0, K_0 and E_0 exists. Let $E_0 = \operatorname{supp} e \cup \operatorname{supp}[c_0] \cup E^*_0$. Let K_0 be such that $[c_0], e \in \mathbf{X}_{E_0,K_0}$. Then we define any homomorphism θ from the subgroup generated by e to $\mathbb{T}$ with $\theta(e) \neq 0$ and extend it to $\mathbf{X}_{E_0,K_0}$ using the divisibility of $\mathbb{T}$. Now we define $\epsilon' > 0$ with $\epsilon' < \min\{\frac{1}{2}, \frac{1}{K_0\|c_0\|}\}$ satisfying that the closed arc centered in $\theta(e)$ of length $\epsilon_0\|e\|$ does not intersect 0.

This guarantees that by defining $\phi^0_0 = \theta|\mathbf{X}_{E^0_0,K_0}$ and $\phi^1_0(\frac{1}{K_0}\chi_\xi)$ as the arc centered in $\theta(\frac{1}{K_0}\chi_\xi)$ of length ϵ', for each $\xi \in E'$, then $\phi_0 = (\phi^0_0, \phi^1_0)$ is an (E_0, K_0, ϵ_0)-arc homomorphism satisfying the needed properties.

Suppose we have defined $(\phi_i, \epsilon_i, E_i, K_i : i \leq t)$ and $(m_i : i < t)$. We show how to define ϕ_{t+1}, E_{t+1}, K_{t+1}, ϵ_{t+1} and m_t.

Depending whether g_t is of one of the first 10 types, or type 11, we apply Theorem 5.5.11 or Theorem 5.5.12 by using $E = E_t$, $K = K_t$, $\epsilon = \epsilon_t$, $\phi = \phi_t$, $h = g_t$. In the first case, we use γ as the size of the arc $\hat{\phi}_t([c_t])$ divided by K_t to obtain S. We choose $m_t \in S \cap A_t$ larger than m_i for every $i < t$. Fix $E_{t+1} = E_t \cup E^*_t \cup \operatorname{supp}[c_{t+1}] \cup \operatorname{supp}[g_t(m_t)]$ and K_{t+1} a multiple of K_t and of $(t+1)!$ (and of the order g_t in case it is of type 11 times K_t) such that $\left[\frac{1}{K_t} g_t(m_t)\right] \in \mathbf{X}_{E_{t+1},K_{t+1}}$. In the first case, let U is an arc such that $K_t U = \hat{\phi}_t([c_t])$ and in the second case let $r = [c_t]$. By the theorems, there exists an arc function ϕ_t of length $\epsilon_t < \frac{1}{2^t K_t \|c_t\|}$ satisfying all we need. Condition (ix) follows from the definition of the hat operator.

For each $k \in \omega$, let $B_i = \{m_t : (g_k, c_k, A_k) = (g_t, c_t, A_t)\}$.

For each $c \in \mathbf{X}_E$, define $\rho(c) \in \bigcap\{\hat{\phi}_t(c) : \ c \in \mathbf{X}_{E_t,K_t}\}$. By Proposition 5.5.9, ρ is an homomorphism for which (3) holds. Condition (1) holds by (x). It remains to see that (2) holds.

Fix $k \in \omega$ and let $\epsilon > 0$ be given. Let $t \in \omega$ such that $\frac{1}{2^t} < \epsilon$. Now, given $n \in B_g$ such that $n \geq m_t$, let $i \in \omega$ such that $n = m_i$ (notice $i \geq t$). We have that $\rho([g(m_i)]) \in \hat{\phi}_{i+1}([g_i(m_i)]) \subseteq \hat{\phi}_i([c_i])$ and $\rho([c_i]) \in \hat{\phi}_i([c_i])$. By itens (i) and (iii), it follows that this open arc has length $< \frac{1}{2^i} < \epsilon$. □

5.6 Forcing homomorphisms defined on larger groups

Now we are ready to define the forcing poset we are going to use.

Definition 5.6.1 (*). We define $\mathcal{P}$ as the set of the tuples of the form $(E, \alpha, \phi, \mathcal{G}, c, A)$ such that:

- E is a countable subset of $2^{\mathfrak{c}}$ containing ω,
- $\alpha < \mathfrak{c}$,
- $\mathcal{G} = (\mathcal{G}_{n,m} : n \geq 1, m > 1)$ is such that each $\mathcal{G}_{n,m}$ is a countable subset of $\mathcal{H}$, where the types are defined with respect to $\mathbf{X}_E$. If $n = 1$, the elements of $\mathcal{G}_{n,m}$ are sequences of types 1-10. If not, they are all of type 11 and order n.
- $A = (A_{n,m,g} : n \geq 1, m > 1, g \in \mathcal{G}_{n,m})$ is such that each $A_{n,m,g}$ is an infinite subset of ω,
- $c = (c_{n,m,g} : n \geq 1, m > 1, g \in \mathcal{G}_{n,m})$ is a family of elements of $\mathbf{X}_E$,
- if $n, m \geq 2$ and $g \in \mathcal{G}_{n,m}$, $c_{n,m,g}$ is an element of order n with $c_{n,m,g} = [\frac{1}{n}\chi_{(\mu,0)}]$ for some $\mu \in C_{n,m}$,
- if $m \geq 2$ and $g \in \mathcal{G}_{1,m}$, $c_{1,m,g} = [\chi_\mu]$ for some $\mu \in C_{1,m}$,
- $\phi : \mathbf{X}_E \to \mathbb{T}^\alpha$ is an homomorphism,
- $(\phi([g(k)]))_{k \in A_{n,m,g}}$ converges to $\phi(c_{n,m,g})$ for each $n \geq 1, m > 1$,
- $(\phi([\frac{n!}{S}\chi_n]))_{n\in\omega}$ converges to $0 \in \mathbb{T}^\alpha$, for every positive integer $S \geq 1$.

We define $(E, \alpha, \phi, \mathcal{G}, c, A) \leq (E', \alpha', \phi', \mathcal{G}', c', A')$ if:

1. $E \supseteq E'$
2. $\alpha \geq \alpha'$
3. $\mathcal{G}_{n,m} \supseteq \mathcal{G}'_{n,m}$ for every $n \geq 1$ and $m > 1$
4. $c_{n,m,g} = c'_{n,m,g}$ for each $n \geq 1$, $m > 1$ and $g \in \mathcal{G}'_{n,m}$
5. $A_{n,m,g} \subseteq^* A'_{n,m,g}$ for each $n \geq 1$, $m > 1$ and $g \in \mathcal{G}'_{n,m}$
6. For every $\xi < \alpha'$ and $a \in X_{E'}$, $\phi(a)(\xi) = \phi'(a)(\xi)$.

Given $p \in \mathcal{P}$, we may denote its components by E^p, α^p, ϕ^p, $\mathcal{G}^p$, c^p and A^p.

If G is a generic filter over $\mathcal{P}$ then the generic homomorphism defined by G is the mapping Φ of domain $\bigcup\{\operatorname{dom}\phi^p : p \in G\}$ into $\mathbb{T}^{\mathfrak{c}}$ defined by $\phi(\cdot)(\xi) = \bigcup\{\phi^p(\cdot)(\xi) : p \in G\}$. In other words, if $p \in G$, $a \in \mathbf{X}_{E^p}$ and $\xi < \alpha^p$, then $\Phi(a)(\xi) = \phi^p(a)(\xi)$. $\square$

Of course, we must see that the generic homomorphisms are really well defined homomorphisms into $\mathbb{T}^{\mathfrak{c}}$. We will see later that by assuming CH in the ground model, $\mathcal{P}$ is ω_1 closed and has the ω_2-c.c., therefore it preserves cardinals and $\mathfrak{c}$. We reserve the rest of this section to prove this fact.

Proposition 5.6.2 (*). Let $e \in \mathbf{X}$ be a non-zero element. Then $\mathcal{C}_e = \{p \in \mathcal{P} : e \in \mathbf{X}_{E^p}, \phi^p(e) \neq 0\}$ is open and dense in $\mathcal{P}$.

Proof. Openness is clear. Fix $p \in \mathcal{P}$. We will define an extension $q \leq p$ that is an element of $\mathcal{C}_e$.

Let $E^q = E^p \cup \operatorname{supp} e$ and $\alpha^q = \alpha^p + 1$. Extend $\phi^p : \mathbf{X}_{E^p} \to \mathbb{T}^{\alpha^p}$ to a homomorphism $\phi : \mathbf{X}_{E^q} \to \mathbb{T}^{\alpha^p}$ using divisibility. Apply Proposition 5.5.1 with $\{(n, m, g) : n \geq 1, m > 1 \text{ and } g \in \mathcal{G}_{p,n,m}\}$, $\{A^p_{n,m,g} : n \geq 1, m > 1 \text{ and } g \in \mathcal{G}^p_{n,m}\}$ and $\{c^p_{n,m,g} : n \geq 1, m > 1 \text{ and } g \in \mathcal{G}^p_{n,m}\}$. Then there exists $\rho : \mathbf{X}_{E^q} \to \mathbb{T}$ such that

1. $\rho(e) \neq 0$,
2. for each (n, m, g) with $n \geq 1$, $m > 1$ and $g \in \mathcal{G}^p_{n,m}$, there exists $B_{n,m,g} \subseteq A^p_{n,m,g}$ infinite such that $(\rho([g(k)]))_{k \in B_{n,m,g}}$ converges to $\rho([c^p_{n,m,g}])$ and
3. $(\rho([\frac{n!}{S}\chi_n]) : n \in \omega)$ converges to $0 \in \mathbb{T}$, for every positive integer S.

Set $\mathcal{G}^q_{n,m} = \mathcal{G}^p_{n,m}$ for each $n \geq 1$ and $m > 1$. Set $c^q_{n,m,g} = c^p_{n,m,g}$, $A^q_{n,m,g} = B_{n,m,g}$ for each $g \in \mathcal{G}^p_{n,m}$ with $n \geq 1$, $m > 1$ and $\phi^q = \phi^\frown\rho$.

Then $q \leq p$ and $q \in \mathcal{C}_e$. $\square$

Proposition 5.6.3 (*). For each $\alpha < \mathfrak{c}$ the set $\mathcal{A}_\alpha = \{p \in \mathcal{P} : \alpha^p > \alpha\}$ is an open dense subset of $\mathcal{P}$.

Proof. Fix $p \in \mathcal{P}$. If $\alpha < \alpha^p$ then $p \in \mathcal{A}_\alpha$. So suppose that $\alpha \geq \alpha^p$.

We will define q. Set $E^q = E^p$, $\alpha^q = \alpha + 1$, $\mathcal{G}^q_{n,m} = \mathcal{G}^p_{n,m}$, $c^q_{n,m,g} = c^p_{n,m,g}$, $A^q_{n,m,g} = A^p_{n,m,g}$. Let $\rho : \mathbf{X}_{E^p} \longrightarrow \{0\}^{[\alpha^p, \alpha^q)}$ and $\phi^q = {\phi^p}^\frown\rho$.

Then $q \leq p$ and $q \in \mathcal{A}_\alpha$. $\square$

Proposition 5.6.4 (*). The partial order $\mathcal{P}$ is ω_1-closed.

Proof. Fix a decreasing sequence $(p_t : t < \omega)$. Write $p_t = (E^t, \alpha^t, \phi^t, \mathcal{G}^t, c^t, A^t)$. We define a common extension r as follows:

Let $E^r = \bigcup\{E^t : t < \omega\}$, $\mathcal{G}^r_{n,m} = \bigcup\{\mathcal{G}^t_{n,m} : t \in \omega\}$ for each $n \geq 1$ and $m > 1$. For each $n \geq 1$, $m > 1$ and $g \in \mathcal{G}^r_{n,m}$, define $c^r_{n,m,g} = c^t_{n,m,g}$ for some (every) t such that $g \in \mathcal{G}^t_{n,m}$ (the value does not depend of t). Fix $A^r_{n,m,g}$ a pseudointersection of $\{A^t_{n,m,g} : t \in \omega\ g \in \mathcal{G}^t_{n,m}\}$.

Let $\alpha^r = \sup\{\alpha^t : t < \omega\}$.

Given $\xi < \alpha^r$ and $a \in \mathbf{X}_{E^r} = \bigcup_{t<\omega} \mathbf{X}_{E^t}$, let $\phi^r(a)(\xi) = \phi^t(a)(\xi)$ for some (every) t such that $a \in X_{E^t}$ and $\alpha^t > \xi$. $\square$

Proposition 5.6.5 (*). Assume CH. Then the partial order $\mathcal{P}$ has the $\mathfrak{c}^+$-cc.

Proof. Fix an arbitrary subset $\mathcal{Q}$ of $\mathcal{P}$ of cardinality $\mathfrak{c}^+$. We show that there $\mathcal{Q}$ has a subset of $\mathfrak{c}^+$-many pairwise compatible elements.

Fix $\mathcal{Q}_0 \subseteq \mathcal{Q}$ of cardinality $\mathfrak{c}^+$ and $\alpha < \mathfrak{c}$ such that $\alpha = \alpha^p = \alpha^q$ for every $p, q \in \mathcal{Q}_0$.

Using the Δ-system Lemma, there exists $\mathcal{Q}_1 \subseteq \mathcal{Q}_0$ of cardinality $\mathfrak{c}^+$ such that $\{E^p : p \in \mathcal{Q}_1\}$ is a Δ-system of root $\tilde{E}$. Furthermore, using CH plus the fact that $\tilde{E}^\omega$ has cardinality at most $\mathfrak{c}$, it follows that there exists $\mathcal{Q}_2 \subseteq \mathcal{Q}_1$ of cardinality $\mathfrak{c}^+$ such that $\phi^p|_{\mathbf{X}_{\tilde{E}}} = \phi^q|_{\mathbf{X}_{\tilde{E}}}$ for every $p, q \in \mathcal{Q}_2$.

For each $p \in \mathcal{Q}_2$, let $J^p = \{(n, m, g) : \ n, m \geq 1, g \in \mathcal{G}^p_{n,m}\}$ for each $p \in \mathcal{Q}_2$. Using the Δ-system Lemma, we can find $\mathcal{Q}_3 \subseteq \mathcal{Q}_2$ of cardinality $\mathfrak{c}^+$ such that $\{J^p : \ p \in \mathcal{Q}_3\}$ is a delta system of root $\tilde{J}$.

Notice that $\mathbf{X}^{\tilde{J}}_{\tilde{E}}$ has cardinality $\mathfrak{c}$, so there exists $\mathcal{Q}_4 \subseteq \mathcal{Q}_3$ of cardinality $\mathfrak{c}^+$ such that for every $p, q \in \mathcal{Q}_4$ and $(n, m, g) \in \tilde{J} = J^p \cap J^q$, $c^p_{n,m,g} = c^q_{n,m,g}$. Similarly, since $([\omega]^\omega)^{\tilde{J}}$ has cardinality $\mathfrak{c}$, there exists $\mathcal{Q}_5 \subseteq \mathcal{Q}_4$ of cardinality $\mathfrak{c}^+$ such that for every $p, q \in \mathcal{Q}_5$ and $(n, m, g) \in \tilde{J} = J^p \cap J^q$, $A^p_{n,m,g} = A^q_{n,m,g}$.

Given $p, q \in \mathcal{Q}_5$, a common extension is given by the element r whose components are defined as follows: $E^r = E^p \cup E^q$, $\alpha^r = \alpha^q = \alpha^p$, $\mathcal{G}^r_{n,m} = \mathcal{G}^p_{n,m} \cup \mathcal{G}^q_{n,m}$, $A^r_{n,m,g} = A^s_{n,m,g}$ and $c^r_{n,m,g} = c^s_{n,m,g}$ if $(n, m, g) \in J^s$ (where $s \in \{p, q\}$).

To define ϕ^r, notice that $\mathbf{X}_{E^p \setminus \tilde{E}} \oplus \mathbf{X}_{\tilde{E}} \oplus \mathbf{X}_{E^q \setminus \tilde{E}} = \mathbf{X}_{E^r}$. Let $\pi_0 : \mathbf{X}_{E^r} \to \mathbf{X}_{E^p \setminus \tilde{E}}$, $\pi_1 : \mathbf{X}_{E^r} \to \mathbf{X}_{\tilde{E}}$, $\pi_2 : \mathbf{X}_{E^r} \to \mathbf{X}_{E^q \setminus \tilde{E}}$ be the projections. Define $\phi^r = \phi^p \circ \pi_0 + \phi^p \circ \pi_1 + \phi^q \circ \pi_2 = \phi^p \circ \pi_0 + \phi^q \circ \pi_1 + \phi^q \circ \pi_2$. $\square$

Proposition 5.6.6 (*). Let g be sequence of one of the types of $\mathbf{X}$ and $m > 1$. If g is of types 1 to 10, let $n = 1$. If g is type 11, let n be the order of g. Then $\mathcal{S}_{n,m,g} = \{p \in \mathcal{P} : \ g \in \mathcal{G}^p_{n,m}\}$ is open and dense in $\mathcal{P}$.

Proof. Let $p \in \mathcal{P}$ be an arbitrary condition. Fix E countable such that $E^p \subseteq E$ and $[g(k)] \in \mathbf{X}_E$ for each $k \in \omega$.

Fix $\mu \in C_{n,m} \setminus E$. We set $E^q = E \cup \{\mu\}$. For each $(m', n') \neq (m, n)$ with $m', n' \geq 1$, define $\mathcal{G}^q_{n',m'} = \mathcal{G}^p_{n',m'}$ and $\mathcal{G}^q_{n,m} = \mathcal{G}^p_{n,m} \cup \{g\}$. Set $\alpha^q = \alpha^p$.

For every $n' \geq 1$, $m' > 1$ and $g' \in \mathcal{G}_{p,n',m'} \setminus \{g\}$, define $c^q_{n',m',g'} = c^p_{n',m',g'}$ and $A^q_{n',m',g'} = A^p_{n',m',g'}$. It remains to define $c^q_{n,m,g}$ and $A^q_{n,m,g}$. Let $c^q_{n,m,g} = \frac{1}{n}\chi_{(\mu,0)}$ if $n > 1$ and χ_μ if $n = 1$.

Extend $\phi_p : \ \mathbf{X}_{E^p} \to \mathbb{T}^{\alpha^p}$ to $\phi : \ \mathbf{X}_E \to \mathbb{T}^{\alpha^p}$ using divisibility. Now, let $A \subseteq \omega$ be an infinite such that the sequence $(\phi([g(k)]) : \ k \in A)$ is convergent, as $\mathbb{T}^{\alpha^p}$ is a compact metric space. Extend ϕ to a homomorphism $\phi^q : \mathbf{X}_{E^q} \to \mathbb{T}^{\alpha^p}$ such that $\phi^q([c^q_{n,m,g}]) = \lim(\phi([g(k)]) : k \in A)$. Set $A^q_{n,m,g} = A$. Then $q \leq p$ and $q \in \mathcal{S}_{n,m,g}$. $\square$

Theorem 5.6.7 (*)**.** Assume CH. Then $\mathcal{P}$ preserves cardinals, $\mathfrak{c}$ and does not add reals. If G is generic over $\mathcal{P}$, then the G-generic homomorphism Φ is a well defined injective homomorphism from $\mathbf{X}$ into $\mathbb{T}^{\mathfrak{c}}$. Moreover, the following holds:

1. For every sequence g of one of the types from 1 to 10 in $\mathbf{X}$ and $m \geq 1$, there exists $\mu \in C_{1,m}$ such that $[\chi_\mu]$ is an accumulation point of $(\Phi([g(k)]) : k \in \omega)$
2. For every sequence g of type 11 and order n in $\mathbf{X}$ and for every $m \geq 1$, there exists $\mu \in C_{n,m}$ such that $[\frac{1}{n}\chi_{\mu,0}]$ is an accumulation point of $(\Phi([g(k)]) : k \in \omega)$.
3. $(\Phi([\frac{n!}{S}\chi_n)])_{n\in\omega}$ converges to $0 \in \mathbb{T}^{\mathfrak{c}}$, for every integer $S > 0$.

Proof. We employ a countable transitive model M in the proof. By CH, propositions 5.6.4 and 5.6.5, $\mathcal{P}$ is ω_1 closed and has the ω_2 chain condition, so $\mathcal{P}$ preserves cardinals, does not add reals and preserves $\mathfrak{c}$. Notice that since being a type is absolute for transitive models of ZFC, the functions of type 1 to 11 are the same in the ground model and in the extension.

Let G be a $\mathcal{P}$-generic filter over M and Φ the associated generic homomorphism.

Φ is well defined: suppose $p, q \in G$, $\xi < \alpha^p \cap \alpha^q$ and $e \in \mathbf{X}_{E^p} \cap \mathbf{X}_{E^q}$. We must see that $\phi^p(e)(\xi) = \phi^q(e)(\xi)$. Since G is a filter, there exists r such that $r \leq p, q$, so $\xi < \alpha^r$, $e \in \mathbf{X}_{E^r}$ and $\phi^p(e)(\xi) = \phi^r(e)(\xi) = \phi^q(e)(\xi)$.

Now we verify that the domain of Φ is $\mathbf{X}$, that the codomain is $\mathbb{T}^{\mathfrak{c}}$ and that Φ is injective at the same time. It is clear that the domain contained in $\mathbf{X}$. Let $e \neq 0$ be an element of $\mathbf{X}$ and $\alpha < \mathfrak{c}$. By propositions 5.6.2 and 5.6.3, $\mathcal{C}_e$ and $\mathcal{A}_\alpha$ are open and dense subsets of $\mathcal{P}$, therefore there exists $p \in G$ such that $\alpha^p > \alpha$ $e \in \mathbf{X}_{E^p}$, $\phi^p(e) \neq 0$. So there exists $\xi < \alpha^p$ such that $\phi^p(e)(\xi) \neq 0$, which implies that $\Phi(e)(\xi) \neq 0$. Moreover, $\alpha \subseteq \operatorname{dom}\Phi(e) \subseteq \mathfrak{c}$. Since α is arbitrary, $\operatorname{dom}\Phi(e) = \mathfrak{c}$.

Φ is an homomorphism: given $e, e' \in \mathbf{X}$, by Proposition 5.6.2 there exists $p \in G$ such that $e, e', e + e' \in \mathbf{X}_{E^p}$. Since ϕ^p is an homomorphism, it follows that $\Phi(e + e') = \phi^p(e + e') = \phi^p(e) + \phi^p(e') = \Phi(e) + \Phi(e')$.

Let g be a type and $m > 1$. If g is of type 1 to 10, let $n = 1$. If g is type 11, let n be the order of g. Then by Proposition 5.6.6, $G \cap \mathcal{S}_{n,m,g} \neq \emptyset$. Fix p in this intersection. We claim $\Phi([c^p_{n,m,g}])$ is an accumulation point of $\Phi([g])$.

We know $\phi_p([c^p_{n,m,g}])$ is the limit of the convergent sequence $(\phi_p([g(k)]) : k \in A^p_{n,m,g})$. Fix F a finite subset of $\mathfrak{c}$ and let α such that F is a subset of α. Let $q \leq p$ such that $\alpha < \alpha^q$ (which exists since $\mathcal{A}_{\alpha+1} \cap G \neq \emptyset$). Then $(\pi_F \circ \Phi([g(k)]) : k \in A^q_{n,m,g})$ converges to $\pi_F \circ \Phi([c^q_{n,m,g}])$. Since $c^q_{n,m,g} = c^p_{n,m,g}$, this concludes that $\Phi([c^p_{n,m,g}])$ is an accumulation point of $(\Phi([g(k)]) : k \in \omega)$.

It remains to see that $(\Phi([\frac{n!}{S}\chi_n]) : n \in \omega)$ is a convergent sequence in $0 \in \mathbb{T}^{\mathfrak{c}}$. Let $\xi < \mathfrak{c}$. Let $p \in G$ such that $\alpha^p > \xi$. Then $(\pi_\xi \circ \Phi([\frac{n!}{S}\chi_n]) : n \in \omega) = (\pi_\xi \circ \phi_p([\frac{n!}{S}\chi_n]) : n \in \omega)$ converges to 0. Since ξ is arbitrary, we are done. $\square$

5.6.1 The subspace topology on large subgroups of $\mathbf{X}$

Of course, not every subgroup of $\mathbf{X}$ is countably compact with the forced topology. However, some of them are if they have enough accumulation points. Thus, we define the concept of *large subgroup* of $\mathbf{X}$.

Definition 5.6.8 (*). Let H be a subgroup of $\mathbf{X}$. Let D be the set of all integers $n > 1$ such that H contains an isomorphic copy of the group $\mathbb{Z}_n^{(\mathfrak{c})}$.

We say that H is a *large subgroup of* $\mathbf{X}$ if $2^{\mathfrak{c}} \geq |H| \geq H/T(H) \geq \mathfrak{c}$, for all $d, n \in \mathbb{N}$ with $d \mid n$ the group $dH[n]$ is either finite or has cardinality at least $\mathfrak{c}$ and there exist $(k_n : n \in D)$ with k_n a positive integer such that:

i) $\{\chi_\mu \in \mathbf{X} : \mu \in C_1\} \subseteq H$, and

ii) $\{[\frac{1}{n}\chi_{(\mu,0)}] \in \mathbf{X} : n \in D,\, \mu \in \bigcup_{n\in D} C_{n,k_n}\} \subseteq H$.

□

Theorem 5.6.9 (*). Consider $\mathbf{X}$ with the group topology in Theorem 5.6.7. If H is a large subgroup of $\mathbf{X}$, then it is countably compact in the subspace topology and has convergent sequences.

Proof. It follows from Theorem 5.6.7 that if S is a positive integer, then the sequence $\left(\frac{1}{S}n!\chi_n : n \in \omega\right)$ converges to the neutral element of $\mathbf{X}$. Since the elements of the sequence are eventually in H and the limit is in H, it follows that H has non-trivial convergent sequences.

Let $g : \omega \to H$. Take any $\tilde{g} : \omega \to \mathbb{X}$ such that $[\tilde{g}] = g$. It follows from Theorem 5.4.1 that there exist $h : \omega \to \mathbb{X}$ such that $h \in \mathcal{H}_H$ or $[h]$ is a constant in H, $c \in \mathbb{X}$ with $[c] \in H$, $F \in [\omega]^{<\omega}$, $p_i, q_i \in \mathbb{Z}$ with $q_i \neq 0$ for every $i \in F$, $(j_i : i \in F)$ strictly increasing sequences of natural numbers and $j : \omega \to \omega$ strictly increasing such that

$$\tilde{g} \circ j = h + c + \sum_{i\in F} \frac{p_i}{q_i} f \circ j_i$$

with $q_i \leq j_i(n)$ for each $n \in \omega$ and $i \in F$, where $f : \omega \to \mathbb{X}$ is given by $f(n) = n!\chi_n$ for every $n \in \omega$.

In the case where $[h]$ is constant, say constantly $v \in \mathbf{X}$, we have that $g \circ j = [\tilde{g}] \circ j = [\tilde{g} \circ j] = v + [c] + \sum_{i\in F}[\frac{p_i}{q_i} f \circ j_i])$ converges to $v + [c]$.

In the case $h \in \mathcal{H}_H$ and h is type 11 of order n, then H contain infinitely many copies of $\mathbb{Z}_n$. Thus, by hypothesis, $n \in D$. Since $n \in D$, it follows from $\mathcal{H}_H \subseteq \mathcal{H}$ that $(h(k) : k \in \omega)$ has an accumulation point $[\frac{1}{n}\chi_{\mu,0}]$ with $\mu \in C_{n,k_n}$. Hence, an accumulation point of h in H. Thus the sequence $(g \circ j(k) : k \in \omega)$ has an accumulation point in $[\frac{1}{n}\chi_{\mu,0}] + [c]$ in H.

In the case $h \in \mathcal{H}_G$ and h is type 1 to 10, it follows from $\mathcal{H}_G \subseteq \mathcal{H}$ that $(h(k) : k \in \omega)$ has an accumulation point $[\chi_\mu]$ with $\mu \in C_1$. Hence, an accumulation point of h in H. Thus the sequence $(g \circ j(k) : k \in \omega)$ has accumulation point $[\chi_\mu] + [c]$ in H. □

5.7 The classification of Abelian groups of cardinality $2^{\mathfrak{c}}$.

5.7.1 Immersions

We change slightly the statement and the notation of Proposition 6.1 in [6] to facilitate the application, but it is implicit in the proof in [6] and will be presented in the next subsection.

We define $\mathbf{A} = (\mathbb{Q}/\mathbb{Z})^{(P_0)} \oplus \mathbb{Q}^{(P_1)} \oplus \mathbf{U}$.

Definition 5.7.1 (*). We say that $\mathbf{W}$ is a *nice subgroup* of $\mathbb{W}_{\mathfrak{c}}$ if there exists a family of positive integers $(n_\xi)_{\xi \in P_0}$ such that

$\mathbf{W} = (\mathbb{Q}/\mathbb{Z})^{(\bigcup_{\xi \in P_0}\{\xi\}\times n_\xi)} \oplus \mathbb{Q}^{(P_1)}$. For this $\mathbf{W}$ we denote $\vec{P_0} = (\bigcup_{\xi \in P_0}\{\xi\} \times n_\xi)$, so $\mathbf{W} = (\mathbb{Q}/\mathbb{Z})^{(\vec{P_0})} \oplus \mathbb{Q}^{(P_1)}$. □

Proposition 5.7.2. Let H be an Abelian group such that $|H| = H/T(H) = \mathfrak{c}$ with H a subgroup of $\mathbf{A}_{\mathfrak{c}}$.

Let D be the set of all integers $n > 1$ such that H contains an isomorphic copy of the group $\mathbb{Z}_n^{(\mathfrak{c})}$.

Then there exists a nice subgroup $\mathbf{W}$ of $\mathbb{W}_{\mathfrak{c}}$, $K_1 \in [P_1]^{\mathfrak{c}}$ with $\omega \subseteq K_1$, a family $(K_n : n \in D)$ of pairwise disjoint elements of $[P_0]^{\mathfrak{c}}$, a family $(z_\xi : \xi \in \bigcup_{n\in D} K_n)$ and a group monomorphism $\phi : \mathbf{A}_{\mathfrak{c}} \to \mathbf{W}$ such that:

a) $\{\chi_\xi \in \mathbf{W}_{P_1} : \xi \in K_1\} \subseteq \phi[H]$,

b) $\{z_\xi \in \mathbf{W}_{P_0} : \xi \in \bigcup_{n\in D} K_n\} \subseteq \phi[H]$,

c) $\mathrm{o}(z_\xi) = n, \ \forall \xi \in K_n, n \in D$ and

d) $\operatorname{supp} z_\xi \subseteq \{\xi\} \times \omega \ \forall \xi \in \bigcup_{n\in D} K_n$.

We say that ϕ is a nice immersion for H. □

The proof of this proposition will be presented in the next subsection. Finally, $\mathbf{W}$ is divisible, so we can extend the isomorphism that Proposition 6.1. gives us to the whole group $\mathbf{A}_{\mathfrak{c}}$.

Proposition 5.7.3 (*). Let H be an Abelian group such that $2^{\mathfrak{c}} \geq |H| \geq H/T(H) \geq \mathfrak{c}$.

Let D be the set of all integers $n > 1$ such that H contains an isomorphic copy of the group $\mathbb{Z}_n^{(\mathfrak{c})}$.

Then there exists a family $(k_n : n \in D)$ of positive integers and a group monomorphism $\varphi : H \to \mathbf{X}$ such that:

i) $\{\chi_\mu \in \mathbf{X}_{P_1 \cup R_1} : \mu \in C_1\} \subseteq \varphi[H]$, and

ii) $\{[\frac{1}{n}\chi_{(\mu,0)}] \in \mathbf{X}_{P_0\cup R_0} : n \in D, \ \mu \in \bigcup_{n\in D} C_{n,k_n}\} \subseteq \varphi[H]$.

Thus, G is isomorphic to a large subgroup of $\mathbf{X}$.

Proof. By theorems 5.1.11 and 5.1.12, we may consider H is a subgroup of $\mathbf{A}$. Then we can fix a subgroup $\tilde{H}$ of H of cardinality $\mathfrak{c}$ such that $r(\tilde{H}) = \mathfrak{c}$ and for every $n \in D$, there exists a copy of $(\mathbb{Z}_n)^{\mathfrak{c}}$ in $\tilde{H}$. By a trivial permutation of coordinates we can assume that $\tilde{H}$ is a subgroup $\mathbf{A}_{\mathfrak{c}}$. Applying Proposition 5.7.2, there exist $\mathbf{W}$ a nice subgroup of $\mathbb{W}_{\mathfrak{c}}$, $K_1 \in [P_1]^{\mathfrak{c}}$ with $\omega \subseteq K_1$, a family $(K_n : n \in D)$ of pairwise disjoint elements of $[P_0]^{\mathfrak{c}}$, a family $(z_\xi : \xi \in \bigcup_{n \in D} K_n)$ and a group monomorphism $\phi : \mathbf{A}_{\mathfrak{c}} \to \mathbf{W}_{\mathfrak{c}} = \mathbf{W}_{P_0} \oplus \mathbf{W}_{P_1}$ such that:

a) $\{(\chi_\xi \in \mathbf{W}_{P_1} : \xi \in K_1\} \subseteq \phi[\tilde{H}]$,

b) $\{z_\xi \in \mathbf{W}_{P_0} : \xi \in \bigcup_{n \in D} K_n\} \subseteq \phi[\tilde{H}]$,

c) $\mathrm{o}(z_\xi) = n, \ \forall \xi \in K_n, n \in D$ and

d) $\operatorname{supp} z_\xi \subseteq \{\xi\} \times \omega, \ \forall \xi \in \bigcup_{n \in D} K_n$.

We can shrink K_n if necessary to find k_n positive integer such that

e) $|\operatorname{supp} z_\xi| = k_n$, for each $n \in D$ and $\xi \in K_n$.

By making some permutation within each $\{\xi\} \times k_n$ we can further assume that

f) $\operatorname{supp} z_\xi = \{\xi\} \times k_n$ for each $n \in D$ and $\xi \in K_n$.

Define $\mathbf{W} = \mathbf{W}_{\mathfrak{c}} \oplus \mathbf{U}$.

We can assume that $\phi : \ \mathbf{A}_{\mathfrak{c}} \to \mathbf{W}$ and extend it to $\phi : \ \mathbf{A} = \mathbf{A}_{\mathfrak{c}} \oplus \mathbf{U} \to \mathbf{W} = \mathbf{W}_{\mathfrak{c}} \oplus \mathbf{U}$, using the identity on $\mathbf{U}$.

Fix σ_n a bijection between K_n and C_{n,k_n} for each $n \in D$ and σ_1 a bijection between K_1 and C_1 with $\sigma_1(k) = k$ for every $k \in \omega$.

Define $\eta : \ \mathbf{W} \longrightarrow \mathbf{X}$ an injective homomorphism such that

- $\eta : \ \mathbf{W}_{\{\xi\} \times k_n} \to \mathbf{X}_{\{\sigma_n(\xi)\} \times k_n}$ is an isomorphism with $\eta(z_\xi) = [\frac{1}{n}\chi_{(\sigma_n(\xi),0)}]$ for each $n \in D$ and $\xi \in K_n$ (this is possible by condition f)),

- $\eta([\chi_\xi]) = [\chi_{\sigma_1(\xi)}]$ for each $\xi \in K_1$,

- η restricted to $\mathbf{U}$ is the identity.

Now, let $\varphi = \eta \circ \phi|_G$. The homomorphism is an embedding, since both η an ϕ are injective homomorphisms.

Applying η in a) it follows that $\{\eta(\chi_\xi) \in \eta[\mathbf{W}_{\mathfrak{c}}] \subseteq \eta[\mathbf{W}] : \xi \in K_1\} \subseteq \varphi[\tilde{H}]$.

Therefore, $\{\chi_\mu \in \mathbf{X} : \ \mu \in C_1\} \subseteq \varphi[\tilde{H}] \subseteq \varphi[H]$ and i) holds.

Likewise, condition ii) holds. $\square$

5.7.2 More on nice subgroups

In this subsection we prove Proposition 5.7.2. We divide it in a few lemmas.

Lemma 5.7.4 (*). There exist $K_1 \in [P_1]^{\mathfrak{c}}$ and a group monomorphism $\tilde{\varphi} : \mathbf{A}_{\mathfrak{c}} \to \mathbf{A}_{\mathfrak{c}}$ such that $\omega \cup \{\omega\} \subseteq K_1$ and $\{(0, \chi_\xi) \in \mathbf{A}_{\mathfrak{c}} : \xi \in K_1\} \subseteq \tilde{\varphi}[G]$.

Proof. Since $|G/T(G)| = \mathfrak{c}$, there exists $W \subseteq G$ such that:

- $|W| = \mathfrak{c}$;
- $w \notin T(G)$, for every $w \in W$;
- $w_1 - w_2 \notin T(G)$, for every $w_1, w_2 \in W$ with $w_1 \neq w_2$.

We will now obtain recursively an independent family $(y_\beta : \beta < \mathfrak{c})$ of elements of W. Let $\gamma < \mathfrak{c}$ and suppose we have $(y_\alpha : \alpha < \gamma)$ an independent family of elements of W. Let $\overline{Y} = \{y_\alpha : \alpha < \gamma\}$; we claim that there exists $y_\gamma \in W$ such that $\langle\{y_\gamma\}\rangle \cap \langle\overline{Y}\rangle = \{0\}$. Suppose that for every $w \in W$ there is $m_w \in \mathbb{Z}\setminus\{0\}$ such that $m_w w \in \langle\overline{Y}\rangle$. Since $|W| = \mathfrak{c}$ and $|\langle\overline{Y}\rangle| < \mathfrak{c}$, there is $\overline{W} \subseteq W$ such that $|\overline{W}| = \mathfrak{c}$ and $m_w w = m_{w'} w'$ for all $w, w' \in \overline{W}$. Furthermore, there exists $\tilde{W} \subset \overline{W}$ such that $|\tilde{W}| = \mathfrak{c}$ and $m_w = m_{w'}$ for all $w, w' \in \tilde{W}$. Call this integer m. We have then that $m(w - w') = 0$ for all $w, w' \in \tilde{W}$.

But this contradicts the fact that the difference between any two distinct elements of W does not belong to $T(G)$. Therefore, there exists $y_\gamma \in W$ such that $\langle y_\gamma\rangle \cap \langle\overline{Y}\rangle = \{0\}$.

So $Y = \{y_\beta : \beta < \mathfrak{c}\} \subseteq W$ is an independent subset of G with cardinality $\mathfrak{c}$. Write $y_\beta = (a_\beta, b_\beta) \in (\mathbb{Q}/\mathbb{Z})^{(P_0)} \oplus \mathbb{Q}^{(P_1)}$, for every $\beta < \mathfrak{c}$. For each $n \in \omega$, define $A_n = \{\beta < \mathfrak{c} : n \cdot a_\beta = 0\}$. Since $|Y| = \mathfrak{c}$, there exists $n \in \omega$ such that $|A_n| = \mathfrak{c}$. Fix such an n and consider the set $\{ny_\beta : \beta \in A_n\}$. Observe that it has cardinality $\mathfrak{c}$, since $|A_n| = \mathfrak{c}$ and $Y \subseteq W$ is an independent subset of G. Besides, $ny_\beta = (0, nb_\beta)$, for every $\beta \in A_n$.

Fix $\{c_\zeta : \zeta \in P_1\}$ a basis of $\mathbb{Q}^{(P_1)}$ as a vector space over $\mathbb{Q}$ containing $\{nb_\beta : \beta \in A_n\}$ and such that $\{c_\zeta : \zeta \in \omega \cup \{\omega\}\} \subseteq \{nb_\beta : \beta \in A_n\}$. Let $\tilde{\varphi} : \mathbf{A}_{\mathfrak{c}} \to \mathbf{A}_{\mathfrak{c}}$ be the group isomorphism given by $\tilde{\varphi}(a, 0) = (a, 0)$, for every $a \in (\mathbb{Q}/\mathbb{Z})^{(P_0)}$ and $\tilde{\varphi}(0, \sum_{\zeta\in F} \alpha_\zeta \cdot c_\zeta) = (0, \sum_{\zeta\in F} \alpha_\zeta \cdot \chi_\zeta)$, for every $F \in [P_1]^{<\omega}$ with $\{\alpha_\zeta : \zeta \in F\} \subseteq \mathbb{Q} \setminus \{0\}$. Pick $K_1 \in [P_1]^{\mathfrak{c}}$ such that and $\{nb_\beta : \beta \in A_n\} = \{c_\zeta : \zeta \in K_1\}$. We have that $\omega \cup \{\omega\} \subseteq K_1$ and $\{(0, \chi_\xi) : \xi \in K_1\} \subseteq \tilde{\varphi}[G]$. $\square$

We fix such a $\tilde{\varphi}$ for the rest of this section.

Lemma 5.7.5 (*). Fix $n \in D$. Then there exists $\{(x_\xi, 0) \in \mathbf{A}_{\mathfrak{c}} : \xi < \mathfrak{c}\} \subseteq \tilde{\varphi}[G]$ with the following properties:

(i) $\mathrm{o}(x_\xi) = n$, for every $\xi < \mathfrak{c}$;

(ii) $\operatorname{supp} x_\xi \cap \operatorname{supp} x_\mu = \emptyset$, for every $\xi, \mu < \mathfrak{c}$ with $\xi \neq \mu$.

Proof. Let $\{(y_\xi, 0) \in \mathbf{A}_{\mathfrak{c}} : \xi < \mathfrak{c}\} \subseteq \tilde{\varphi}[G]$ independent with $\mathrm{o}(y_\xi) = n$, for every $\xi < \mathfrak{c}$ enumerated faithfully.

Case 1: $\mathfrak{c}$ is regular.

Applying the Δ-system lemma for the family $(\operatorname{supp} y_\xi : \xi < \mathfrak{c})$, we obtain $I \in [\mathfrak{c}]^{\mathfrak{c}}$ and $R \in [P_0]^{<\omega}$ such that the family $(\operatorname{supp} y_\xi : \xi \in I)$ is a Δ-system with root R. Fix $J \in [I]^{\mathfrak{c}}$ such that if $\xi, \mu \in J$, then $y_\xi(\zeta) = y_\mu(\zeta)$ for every $\zeta \in R$. Let $\{J_0, J_1\}$ be a partition of J such that $|J_0| = |J_1| = \mathfrak{c}$. Fix $f : \mathfrak{c} \to J_0$ and $g : \mathfrak{c} \to J_1$ bijections and define $x_\xi = y_{f(\xi)} - y_{g(\xi)}$, for every $\xi < \mathfrak{c}$.

Given $\xi < \mathfrak{c}$, we have that $\mathrm{o}(x_\xi) = n$, since $n \cdot x_\xi = n \cdot (y_{f(\xi)} - y_{g(\xi)}) = n \cdot y_{f(\xi)} - n \cdot y_{g(\xi)} = 0 - 0 = 0$ and if m is a non-zero natural number lower than n, then $m \cdot x_\xi = m \cdot y_{f(\xi)} - m \cdot y_{g(\xi)} \neq 0$, since $\mathrm{o}(y_{f(\xi)}) = \mathrm{o}(y_{g(\xi)}) = n$ and $\{y_\xi : \xi < \mathfrak{c}\}$ is an independent subset of G.

Finally, let $\xi, \mu < \mathfrak{c}$ be such that $\xi \neq \mu$. If $\zeta \in \operatorname{supp} x_\xi \cap \operatorname{supp} x_\mu$, then one of the following possibilities occur:

- $\zeta \in \operatorname{supp} y_{f(\xi)} \cap \operatorname{supp} y_{f(\mu)}$;
- $\zeta \in \operatorname{supp} y_{f(\xi)} \cap \operatorname{supp} y_{g(\mu)}$;
- $\zeta \in \operatorname{supp} y_{g(\xi)} \cap \operatorname{supp} y_{f(\mu)}$;
- $\zeta \in \operatorname{supp} y_{g(\xi)} \cap \operatorname{supp} y_{g(\mu)}$.

Thus, $\zeta \in R$ and, therefore, $\zeta \in \operatorname{supp} y_{f(\xi)} \cap \operatorname{supp} y_{f(\mu)} \cap \operatorname{supp} y_{g(\xi)} \cap \operatorname{supp} y_{g(\mu)}$. Since $f(\xi), g(\xi), f(\mu), g(\mu) \in J$, it follows that $y_{f(\xi)}(\zeta) = y_{f(\mu)}(\zeta) = y_{g(\xi)}(\zeta) = y_{g(\mu)}(\zeta)$. So, $x_\xi(\zeta) = x_\mu(\zeta) = 0$. This is a contradiction, as $\zeta \in \operatorname{supp} x_\xi \cap \operatorname{supp} x_\mu$. Therefore, $\operatorname{supp} x_\xi \cap \operatorname{supp} x_\mu = \emptyset$, for every $\xi, \mu < \mathfrak{c}$ with $\xi \neq \mu$.

Case 2: $\mathfrak{c}$ is not regular.

In this case, $\operatorname{cf}(\mathfrak{c}) < \mathfrak{c}$ and $\mathfrak{c}$ is a limit cardinal.

Let $\{I_\alpha : \alpha < \operatorname{cf}(\mathfrak{c})\}$ be a family of pairwise disjoint subsets of $\mathfrak{c}$ such that $\mathfrak{c} = \bigcup_{\alpha < \operatorname{cf}(\mathfrak{c})} I_\alpha$ and $|I_\alpha| = \kappa_\alpha$, for every $\alpha < \operatorname{cf}(\mathfrak{c})$, where:

- $(\kappa_\alpha : \alpha < \operatorname{cf}(\mathfrak{c}))$ is a strictly increasing and cofinal sequence in $\mathfrak{c}$;
- κ_α is a regular cardinal, for every $\alpha < \operatorname{cf}(\mathfrak{c})$;
- $\kappa_\alpha \geq \max\{\omega, |\alpha|, \sup_{\beta<\alpha} \kappa_\beta\}^+$.

For every $\alpha < \operatorname{cf}(\mathfrak{c})$ it is possible to repeat the construction presented in case 1 to obtain $\{(x_\xi, 0) \in \mathbf{A}_\mathfrak{c} : \xi \in I_\alpha\} \subseteq \tilde{\varphi}[G]$ such that $\mathrm{o}(x_\xi) = n$ for every $\xi \in I_\alpha$ and $\operatorname{supp} x_\xi \cap \operatorname{supp} x_\mu = \emptyset$, for every $\xi, \mu \in I_\alpha$ with $\xi \neq \mu$.

Let $J_0 = I_0$ and $\alpha < \operatorname{cf}(\mathfrak{c})$ be an ordinal. Suppose that for each $\beta < \alpha$, there exists $J_\beta \subseteq I_\beta$ such that $|J_\beta| = |I_\beta| = \kappa_\beta$ and $\operatorname{supp} x_\xi \cap \operatorname{supp} x_\mu = \emptyset$, for every $\xi, \mu \in \bigcup_{\beta<\alpha} J_\beta$ with $\xi \neq \mu$. Put $X_\alpha = \bigcup_{\xi \in \bigcup_{\beta<\alpha} J_\beta} \operatorname{supp} x_\xi$. We have that $|X_\alpha| < \kappa_\alpha$ and, therefore, there exists $J_\alpha \subseteq I_\alpha$ such that $|J_\alpha| = |I_\alpha|$ and $\operatorname{supp} x_\xi \cap X_\alpha = \emptyset$, for every $\xi \in J_\alpha$.

Define $J = \bigcup_{\alpha < \operatorname{cf}(\mathfrak{c})} J_\alpha$. It follows that $|J| = \mathfrak{c}$ and $\operatorname{supp} x_\xi \cap \operatorname{supp} x_\mu = \emptyset$, for every $\xi, \mu \in J$ with $\xi \neq \mu$. $\square$

Lemma 5.7.6 (*). For each $n \in D$, there exists $K_n \in [P_0]^{\mathfrak{c}}$ such that $K_m \cap K_n = \emptyset$, for every $m, n \in D$ with $m \neq n$. Moreover, there exists $\{(x_\xi, 0) \in \mathbf{A}_{\mathfrak{c}} : \xi \in \bigcup_{n \in D} K_n\} \subseteq \tilde{\varphi}[G]$ with the following properties:

(i) If $\xi \in K_n$, then $\mathrm{o}(x_\xi) = n$;

(ii) If $\xi, \mu \in \bigcup_{n \in D} K_n$ and $\xi \neq \mu$, then $\operatorname{supp} x_\xi \cap \operatorname{supp} x_\mu = \emptyset$;

(iii) $\xi \in \operatorname{supp} x_\xi$, for every $\xi \in \bigcup_{n \in D} K_n$.

Proof. For each $n \in D$, consider $\{x_\xi^n : \xi < \mathfrak{c}\}$ where $\{(x_\xi^n, 0) \in \mathbf{A}_{\mathfrak{c}} : \xi < \mathfrak{c}\} \subseteq \tilde{\varphi}[G]$ satisfies conditions (i) and (ii) of Lemma 5.7.5. Let $(X_\zeta : \zeta < \mathfrak{c})$ be an enumeration of $\{\{x_\xi^n : \xi < \mathfrak{c}\} : n \in D\}$ such that

$$|\{\zeta < \mathfrak{c} : X_\zeta = \{x_\xi^n : \xi < \mathfrak{c}\}\}| = \mathfrak{c}$$

for every $n \in D$.

Fix $x \in X_0$ and fix $\xi_0 \in \operatorname{supp} x$. Denote x by x_{ξ_0}. Let $\alpha < \mathfrak{c}$ be an ordinal. For each $\beta < \alpha$, suppose defined $\xi_\beta \in P_0$ and $x_{\xi_\beta} \in X_\beta$ with the following properties:

- $\xi_\beta \in \operatorname{supp} x_{\xi_\beta}$, and
- if $\gamma < \beta < \alpha$, then $\operatorname{supp} x_{\xi_\gamma} \cap \operatorname{supp} x_{\xi_\beta} = \emptyset$.

We have that $|\bigcup_{\beta<\alpha} \operatorname{supp} x_{\xi_\beta}| \leq \max\{|\alpha|, \omega\} < \mathfrak{c}$. Since $|X_\alpha| = \mathfrak{c}$ and $\operatorname{supp} x \cap \operatorname{supp} y = \emptyset$ for every x and y distinct elements of X_α, there exists $z \in X_\alpha$ such that $\operatorname{supp} z \cap (\bigcup_{\beta<\alpha} \operatorname{supp} x_{\xi_\beta}) = \emptyset$. Take any $\xi_\alpha \in \operatorname{supp} z$ and write $x_{\xi_\alpha} = z$. By induction, we obtain $\xi_\alpha \in P_0$ and $x_{\xi_\alpha} \in X_\alpha$, for every $\alpha < \mathfrak{c}$.

If $n \in D$, define

$$K_n = \{\xi_\alpha : X_\alpha = \{x_\xi^n : \xi < \mathfrak{c}\}\}$$

Note that $|K_n| = \mathfrak{c}$, since $|\{\alpha < \mathfrak{c} : X_\alpha = \{x_\xi^n : \xi < \mathfrak{c}\}\}| = \mathfrak{c}$ and $\xi_\alpha \neq \xi_\beta$ if α and β are distinct elements of $\mathfrak{c}$. Besides, $K_m \cap K_n = \emptyset$, if $m, n \in D$ and $m \neq n$. If $\xi \in K_n$, then $\mathrm{o}(x_\xi) = n$ and if $\xi, \mu \in \bigcup_{n \in D} K_n$ and $\xi \neq \mu$, then $\operatorname{supp} x_\xi \cap \operatorname{supp} x_\mu = \emptyset$. Finally, if $\xi \in \bigcup_{n \in D} K_n$, then $\xi \in \operatorname{supp} x_\xi$. $\square$

Now we are ready to prove Proposition 5.7.2.

Proof of Proposition 5.7.2. Consider $\{(x_\xi, 0) \in \mathbf{A}_{\mathfrak{c}} : \xi \in \bigcup_{n \in D} K_n\} \subset \tilde{\varphi}[G]$ satisfying conditions (i), (ii) and (iii) of Lemma 5.7.6. Consider also the mapping $\hat{\varphi} : \mathbf{A}_{\mathfrak{c}} \to \mathbb{W}_{\mathfrak{c}}$ defined in the following way:

- Let $\xi \in K_n$, for some $n \in D$. Denote $\operatorname{supp} x_\xi$ by $\{\alpha_0, \alpha_1, ..., \alpha_m\}$, where $\alpha_0 < \alpha_1 < ... < \alpha_m$. The α_i-th summand $\mathbb{Q}/\mathbb{Z}$ of the direct sum $(\mathbb{Q}/\mathbb{Z})^{(P_0)}$ will be mapped identically to the (ξ, i)-th summand of the direct sum $(\mathbb{Q}/\mathbb{Z})^{(P_0 \times \omega)}$. In this case we let $n_\xi = m + 1$.
- If $\mu \in P_0 \setminus \bigcup_{\xi \in \bigcup_{n \in D} K_n} \operatorname{supp} x_\xi$, then the μ-th summand $\mathbb{Q}/\mathbb{Z}$ of the direct sum $(\mathbb{Q}/\mathbb{Z})^{(P_0)}$ will be mapped identically to the $(\mu, 0)$-th summand of the direct sum $(\mathbb{Q}/\mathbb{Z})^{(P_0 \times \omega)}$. In this case, we let $n_\mu = 1$.

- $\hat{\varphi}(0,b) = (0,b)$, for every $b \in \mathbb{Q}^{(P_1)}$.

Let $\vec{n} = (n_\xi : \xi \in P_0)$. The mapping $\hat{\varphi}$ is a group monomorphism into $\mathbf{W} = (\mathbb{Q}/\mathbb{Z})^{(\vec{P}_{0,\vec{n}})} \oplus \mathbb{Q}^{(P_1)}$. Consider $\varphi = \hat{\varphi} \circ \tilde{\varphi}$ and $(y_\xi, 0) = \hat{\varphi}(x_\xi, 0)$, for every $\xi \in \bigcup_{n\in D} K_n$. It follows that φ is a group monomorphism such that

$$\{(0,\chi_\xi) \in \mathbf{W} : \xi \in K_1\} \subset \varphi[G]$$

and

$$\{(y_\xi, 0) \in \mathbf{W} : \xi \in \bigcup_{n\in D} K_n\} \subset \varphi[G].$$

Besides, if $\xi \in K_n$, then $\mathrm{o}(y_\xi) = \mathrm{o}(\hat{\varphi}(x_\xi, 0)) = n$, since $\hat{\varphi}$ is a group monomorphism. Finally, $\operatorname{supp} y_\xi \subset \{\xi\} \times n_\xi$, for every $\xi \in \bigcup_{n\in D} K_n$. □

5.7.3 The classification

Theorem 5.7.7 (*)**.** Consider $\mathbf{X}$ with the topology from Theorem 5.6.7. Let H be a group such that $2^{\mathfrak{c}} \geq |H| \geq H/T(H) = \mathfrak{c}$ and for all $d, n \in \mathbb{N}$ with $d \mid n$, the group $dG[n]$ is either finite or has cardinality at least $\mathfrak{c}$. Then H admits a countably compact group topology with a non-trivial convergent sequence.

Proof. By Proposition 5.7.3, the group H is isomorphic to a large subgroup of $\mathbf{X}$ therefore by Theorem 5.6.9, H admits a countably compact group topology with a convergent sequence. □

Now we can (consistently) answer Dikranjan and Shakhmatov's question for Abelian groups of cardinality $\leq 2^{\mathfrak{c}}$ by restating and proving our main result.

Corollary 5.7.8 (*)**.** Consider the forcing model in Theorem 5.6.7

Let H be a non-torsion Abelian group of size at most $2^{\mathfrak{c}}$. Then the following are equivalent:

1) $2^{\mathfrak{c}} \geq |H| \geq H/T(H) \geq \mathfrak{c}$ and for all $d, n \in \mathbb{N}$ with $d \mid n$, the group $dH[n]$ is either finite or has cardinality at least $\mathfrak{c}$,

2) H admits a countably compact Hausdorff group topology, and

3) H admits a countably compact Hausdorff group topology with non-trivial convergent sequences.

Proof. As it is well known, $2)$ implies $1)$ by Corollary 5.2.3.

By Theorem 5.7.7, if H satisfies $1)$ then $3)$ holds.

Clearly, $3)$ implies $2)$. □

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