On the Homotopy Types

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Dissertação apresentada ao
Instituto de Matemática e Estatística
da
Universidade de São Paulo
para
obtenção do título de
Mestre em Ciências

Programa: Mestrado em Matemática
Orientador: Prof. Dr. Hugo Luiz Mariano

Durante o desenvolvimento deste trabalho o autor recebeu auxílio financeiro da CAPES

São Paulo, Dezembro de 2021

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Esta é a versão original da dissertação elaborada pelo
candidato Thiago Alexandre, tal como
submetida à Comissão Julgadora.

Agradecimentos

Durante os dois anos em que me dediquei a este trabalho, muitas pessoas queridas me apoaiaram. A primeira delas, sem dúvida, foi meu orientador Hugo Luiz Mariano, pelo seu encorajamento, atenção, dedicação e paciência. Depois, não poderia deixar de agradecer aos meus pais, por todo tipo de suporte e auxílio. Aos amigos, também devo minha gratidão, pelas conversas, trocas de ideias e afeto. Em especial, agradeço à Rebeca.

Resumo

ALEXANDRE, THIAGO. 2021. 183 f. Dissertação (Mestrado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2021.

Esta dissertação trata sobre os fundamentos da homotopia seguindo as ideias dos manuscritos *Les Dérivateurs* e *Pursuing Stacks* de Grothendieck. Em particular, discutimos como o formalismo dos derivadores nos pertmite pensar os tipos de homotopia intrinsicamente, ou, mesmo como um conceito primitivo para a matemática, para os quais os conjuntos são um caso particular. Mostramos como a teoria das categorias é naturalmente estendida à álgebra homotópica, entendida aqui como o formalismo dos derivadores. Em seguida, provamos em detalhes um teorema de Heller e Cisinski, caracterizando a categoria dos tipos de homotopia com uma propriedade universal adequada na linguagem dos derivadores, que por sua vez, estende a propriedade universal de Yoneda da categoria dos conjuntos para as categorias co-completas. A partir desse resultado, propomos uma redefinição sintética da categoria dos tipos de homotopia. Isso estabelece uma explicação conceitual matemática para as ligações entre teoria homotópica dos tipos, ∞-categorias e álgebra homotópica, e também para o recente programa de refundamentação das matemáticas via teoria homotópica dos tipos idealizado por Voevodsky. Nesse sentido, a pesquisa sobre os fundamentos da teoria da homotopia reflete em uma discussão sobre os fundamentos das matemáticas. Também expomos a teoria dos ∞-grupoides de Grothendieck-Maltsiniotis e a célebre Hipótese da Homotopia conjecturada por Grothendieck, que afirma a equivalência (homotópica) entre os espaços e os ∞-grupoides. Tal conjectura, se demonstrada, forneceria uma paisagem estritamente algébrica dos espaços.

Palavras-chave: Álgebra homotópica, Derivadores, Fundamentos da Teoria da Homotopia, Categorias superiores, Cohomologia.

Abstract

ALEXANDRE, THIAGO. 2021. 183 f. Dissertação (Mestrado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2021.

This dissertation is concerned with the foundations of homotopy theory following the ideas of the manuscripts *Les Dérivateurs* and *Pursuing Stacks* of Grothendieck. In particular, we discuss how the formalism of derivators allows us to think about homotopy types intrinsically, or, even as a primitive concept for mathematics, for which sets are a particular case. We show how category theory is naturally extended to homotopical algebra, understood here as the formalism of derivators. Then, we proof in details a theorem of Heller and Cisinski, characterizing the category of homotopy types with a suitable universal property in the language of derivators, which extends the Yoneda universal property of the category of sets with respect to the cocomplete categories. From this result, we propose a synthetic re-definition of the category of homotopy types. This establishes a mathematical conceptual explanation for the the links between homotopy type theory, ∞-categories and homotopical algebra, and also for the recent program of re-foundations of mathematics via homotopy type theory envisioned by Voevodsky. In this sense, the research on foundations of homotopy theory reflects in a discussion about the re-foundations of mathematics. We also expose the theory of Grothendieck-Maltsiniotis ∞-groupoids and the famous Homotopy Hypothesis conjectured by Grothendieck, which affirms the (homotopical) equivalence between spaces and ∞-groupoids. This conjectured, if proved, provides a strictly algebraic picture of spaces.

Keywords: Homotopical algebra, Derivators, Foundations of Homotopy Theory, Higher categories, Cohomology.

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Introduction

This dissertation exposes, through the language of *derivators* introduced by Grothendieck in [11] and, independently, by the formalism of *homotopy theories* introduced by Heller in [9], an elementary and axiomatic study of homotopy theory. Following the ideas presented in the two inspiring manuscripts [10] and [11] of Grothendieck, and also some recent developments of these ideas due to Maltsiniotis and Cisinski, we establish the categorical foundations for an intrinsic study of the category Hot of homotopy types, i.e., the traditional homotopy category of CW-complexes, thought as a primitive concept.

In synthesis, we define the category Hot of homotopy types, via the formalism of derivators, free of any description of the objects and arrows of Hot. Such descriptions, may it be CW-complexes, topological spaces, simplicial sets, etc., will be understood here as modelizations of the category Hot. But, how can we think about the homotopy types as a primitive concept? In order to answer that question, we begin by comparing two classical homotopy invariants with two classical set-theoretic invariants. After all, the objects and arrows of the category Ens of sets are regarded as the primitive concept of all mathematics. Yet, as we shall see throughout this dissertation, there are strong mathematical evidences indicating that sets may be only shadows of a more elementary primitive concept, and this is precisely the one of homotopy type.

The first homotopy invariant of topological spaces in history is no doubt the Euler characteristics χ. Starting from a comparison with finite sets, there is only one function of the form

$$\mathfrak{c} : Ens_\omega \longrightarrow \mathbb{Z},$$

where Ens_ω denotes the category of finite sets, satisfying the following three conditions:

(i). For every singleton E, $\mathfrak{c}(E) = 1$,

(ii). $\mathfrak{c}(\varnothing) = 0$

(iii). If $A \subseteq E$ is any inclusion of finite sets, then $\mathfrak{c}(E) = \mathfrak{c}(A) + \mathfrak{c}(E/A) - 1$, where E/A is the usual quotient set, which can be defined as the inductive limit of the diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & E \\
*
\end{tikzcd}$$

in the category Ens_ω, with $*$ denoting any representation of the point and $A \to *$ denoting the unique existent function from A to $*$.

Now, a finite CW-complex E admits a cellular decomposition of closed immersions of the the form

$$E_{-1} = \varnothing \hookrightarrow E_0 \hookrightarrow .... \hookrightarrow E_{n-1} \hookrightarrow E_n = E$$

for some $n \in \omega$, where each inclusion $E_{i-1} \hookrightarrow E_i$ is a relative CW-complex, i.e., E_i is obtained from E_{i-1} by gluing i-cells, which means that E_i is the inductive limit of a digram of the form

$$\begin{tikzcd}
\coprod S^{i-1} \arrow[r] \arrow[d] & E_{i-1} \\
\coprod D^i
\end{tikzcd}$$

where $D^i = \{x \in \mathbb{R}^i : \|x\| \leq 1\}$ and $S^{i-1} = \partial D^i = \{x \in \mathbb{R}^i : \|x\| = 1\}$. Supposing that finite CW-complexes are models for finite homotopy types, then the Euler characteristics could be defined as the unique function of the form

$$\chi : \mathsf{Hot}_\omega \longrightarrow \mathbb{Z},$$

where Hot_ω denotes the homotopy category of *finite* homotopy types, satisfying the following conditions: for any *contractible* finite CW-complex E, $\chi(E) = 1$, $\chi(\varnothing) = 0$, and for any inclusion $A \subseteq X$ of relative finite CW-complexes, $\chi(E) = \chi(A) + \chi(E/A) - 1$. The uniqueness of such a function can be verified easily by induction on the dimension of the finite CW-complex E, but the existence is more delicate.

In fact, it was only after the works of Noether that mathematicians discovered that the Euler characteristics is nothing but a shadow of an enhanced homotopy invariant, namely, the homology groups $H_i(E)$ associated to a CW-complex E, from where we deduce the Euler characteristics by the well known formula:

$$\chi(E) = \sum_{n\in\omega}(-1)^n dimH_n(E).$$

The numbers $b_n(E) = dimH_n(E)$ are called the Betti numbers of E, and we have $H_n(E) = 0$ for n larger enough. More generally, the Euler characteristics $\chi(C_*)$ of a (homological) bounded complex of abelian groups

$$... \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to ...$$

is defined by the formula

$$\chi(C_*) = \sum_{n\in\omega}(-1)^n dimH_n(C_*)$$

where $H_n(C_*) = Ker(d_n)/Im(d_{n+1})$ is the usual n-th homology group of C_*. The existence and uniqueness of the Euler-characteristics function χ is then transposed to this higher level of invariants, i.e., the existence and uniqueness of homology theories. For CW-complexes there exists a well known method to compute these homology groups, which is the cellular homology. For a general topological space, we can use the traditional singular homology as a method of computation. Yet, these are methods of computation, and should not be confused (as we shall see in this dissertation) with *the* (co)homology of spaces.

As an analogy with sets, there is also a difference between *the* free abelian group $\mathbb{Z}[E]$ of a set E, and the methods of computation of the group $\mathbb{Z}[E]$ (by generators and relations, for example). In categorical language, existence and uniqueness of these free abelian groups $\mathbb{Z}[E]$, for E varying through the category of sets, is a formal consequence of the following universal property: the category Ens is co-complete, i.e., it admits all inductive limits, and for any (locally small) co-complete category $\mathcal{C}$, the functor

$$\underline{Hom}_!(Ens, \mathcal{C}) \longrightarrow \mathcal{C}, \quad F \mapsto F(*)$$

is an equivalence of categories, where $\underline{Hom}_!(Ens, \mathcal{C})$ denotes the category of functors from Ens to $\mathcal{C}$ commuting with inductive limits. It follows from the previous universal property of the category of sets that, given any object $G : e \to \mathcal{C}$ of $\mathcal{C}$ (viewed as a functor from the terminal category e, with just one object and one arrow, to $\mathcal{C}$), there exists a (essentially unique) functor

$G_! : Ens \to \mathcal{C}$, commuting with inductive limits, such that $G_!(*) \cong G$ in $\mathcal{C}$. In particular, taking $\mathcal{C}$ as being the category Ab of abelian groups, which is co-complete, there exists a (essentially unique) functor $\mathbb{Z}[-] : Ens \to Ab$, commuting with inductive limits, such that $\mathbb{Z}[*] \cong \mathbb{Z}$ in Ab, and $\mathbb{Z}[-]$ is precisely the free abelian group functor.

With the notations of the previous paragraph, we can define the homology groups $H_n(E)$ of a set E as being the homology groups of the abelian group complex concentrated in degree zero:

$$... \to 0 \to 0 \to \mathbb{Z}[E] \to 0 \to ...$$

In this case, we have $H_0(E) \cong \mathbb{Z}[E]$ and $H_n(E) = 0$ for $n \geq 0$. Now, the universal property of the category of sets comes from the fact that every set E is trivially isomorphic to a coproduct of the point:

$$E \cong \coprod_E *.$$

Hence, we can define $\mathbb{Z}[E]$ as the being the complex:

$$\mathbb{Z}[E] =_{df} \bigoplus_E \mathbb{Z}.$$

If E is a finite set, say, of cardinality p, for $p \in \omega$, then E is isomorphic to a coproduct of p-copies of the point $*$, and we have

$$\chi(\mathbb{Z}[E]) = \sum_{n \in \omega} (-1)^n dim H_n(E) = dim \mathbb{Z}[E] = p.$$

which means that the cardinality $\mathfrak{c}(E)$ of E can be redefined as the Euler-characteristics of the homology complex of E [1].

Then, the existence and uniqueness of the cardinality function for finite sets is a consequence of the existence and uniqueness of the homology theory of sets, which is nothing but the abelianization of sets. Yet, the same strategy

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[1] Here, it's crucial to make a remark. The notion of finiteness for homotopy types does not necessarily agrees with the set-theoretic notion of finiteness. Actually, it is well known by algebraic topologists and group cohomologists that the classifying space BG of any *non-trivial* finite group G is a CW-complex of infinite dimension. Conversely, the closed interval $[0, 1]$ is an infinite set, yet, $[0, 1]$ is contractible, and hence, homotopically equivalent to the point, which implies that $[0, 1]$ is a finite homotopy type.

can not be applied for the homotopy types, and this for two reasons:

1. The category Hot is not co-complete.
2. For a higher homotopy type X which is not necessarily a set (since sets are actually the 0-homotopy types), the homology $H_*(X)$ of X can not, in general, be represented by a single abelian group, but as an object in the category $Comp(Ab)$ of all complexes of abelian groups. Since homology is homotopy invariant, it should actually define a functor

   $$\mathsf{Hot} \longrightarrow D(Ab),$$

   where $D(Ab)$ is the usual derived category of abelian groups, obtained formally from $Comp(Ab)$ inverting all the arrows which induce isomorphisms on the homologies. Yet, nor only the categories Hot and $D(Ab)$ are not co-complete, but the homology functor $X \mapsto H_*(X)$ does not preserves inductive limits.

Therefore, we can not reply to homotopy types the same arguments we used for the the homology of sets. Yet, once we regard that inductive and projective limits are shadows of left and right Kan extensions, we can extend category theory to Grothendieck-Heller homotopical algebra, and develop a formalism of homotopical Kan extensions in order to surpass the previous obstructions, for, Hot and $D(Ab)$ are both homotopically co-complete in this previous sense, and homotopical co-completeness recovers the usual notion of co-completeness when we restrict homotopical algebra to category theory. This formalism of homotopy Kan extensions is the theory of derivators.

The derivator **Hot** corresponds to the universal co-complete homotopy theory, like the category of sets *Ens* is the universal co-complete category. Since the derivator $\mathscr{D}(Ab)$ is also co-complete, there exists a universal morphism of derivators

$$Sing : \mathbf{Hot} \longrightarrow \mathscr{D}(Ab)$$

which defines the singular homology when we evaluate $Sing$ at the terminal category:

$$Sing_e : \mathsf{Hot} = \mathbf{Hot}(e) \longrightarrow \mathscr{D}(Ab)(e) = D(Ab).$$

We recall that the objects of $D(Ab)$ are the same of $Comp(Ab)$, and it follows from the universal property of localization of categories the existence of a unique functor

$$D(Ab) \longrightarrow Ab^{\mathbb{Z}}$$

such that the diagram

$$\begin{tikzcd}
Comp(Ab) \arrow[r, "E"] \arrow[d] & Ab^{\mathbb{Z}} \\
D(Ab) \arrow[ur] &
\end{tikzcd}$$

commutes, where E is the functor $C \mapsto (H_n(E))_{n\in\mathbb{Z}}$ from $Comp(Ab)$ to $Ab^{\mathbb{Z}}$. Then, for every pair $(\mathcal{M}, W)$ consisting of a locally small category $\mathcal{M}$ and a class of arrows $W \subseteq Fl(\mathcal{M})$ such that $W^{-1}\mathcal{M} \simeq \mathsf{Hot}$, we have a commutative diagram of the form

$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d] & Ab^{\mathbb{Z}} \\
\mathsf{Hot} \arrow[r] & D(Ab). \arrow[u]
\end{tikzcd}$$

In particular, taking $\mathcal{M} = Top$ and W as being the class of *weak homotopy types* in Top, we have a canonical commutative digram

$$\begin{tikzcd}
Top \arrow[r] \arrow[d] & Ab^{\mathbb{Z}} \\
\mathsf{Hot} \arrow[r] & D(Ab) \arrow[u]
\end{tikzcd}$$

which recovers the traditional singular homology of topological spaces.

The language of derivators will be presented in the first chapter of this dissertation, and we also show how Quillen homotopical algebra (based on model categories) can be extended to derivators. Several formal properties of derivators are also proved.

In the the second chapter, we follow Grothendieck, Heller and Cisinski, to expose a proof of the universal property of the derivator **Hot** (in place of the category Hot). The category Hot of homotopy types can be re-defined formally from **Hot**. We also give a conceptual definition of the (co)homology of homotopy types, proving its existence and uniqueness, and showing how to interpret singular (co)homology in the language of derivators, for, in the theory of derivators, there is a precise mathematical meaning to the heuristics '(co)homology is the abelianization of homotopy'. Finally, we finish the second chapter with a quick exposition of the beautiful theory of Grothendieck test categoris, indicating the crucial theorems.

The last chapter exposes the homotopy conjecture of Grothendieck, still open, which affirms that, up to homotopy, spaces are equivalent to ∞-groupoids. This is an algebraic-geometric statement, since the ∞-groupoids defined by Grothendieck-Maltsiniotis are strict algebraic structures.

Before the treatise exposition part of this dissertation, which covers the chapters 1, 2 and 3, we expose a mathematical history of the category of homotopy types. This chapter is crucial for this dissertation since it is a retrospective (with comments and even sketches of proofs) of the main theorems which allows us to make a complete axiomatic presentation of homotopy theory, and we recommend to read this chapter before the others.

Prerequisites

Except for the mathematical-historical chapter of this dissertation, where some terminology of topos theory and algebraic topology are mentioned (like homotopy groups of a topological space and cohomology groups of a topos), we assume only that the reader has familiarity with categorical language, more precisely, we assume familiarity with the contents of Exposé I of [1], [27], or [40], which includes functors, natural transformations, adjunctions, inductive and projective limits [2], presheaves, and the Lemma of Yoneda. Some background in Kan extensions may also be helpful, but not strictly necessary.

General conventions

Given a category C, we always denote by $Hom_C(U, V)$ the class of arrows of type $U \to V$ in C. If f is an arrow, then $dom(f)$ (resp. $codom(f)$) designates the domain (resp. codomain) of f. The class of objects (resp. arrows) in C, will always be denoted by $Ob(C)$ (resp. $Fl(C)$). Given a set E, we denote by $\mathfrak{c}(E)$ its cardinal.

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[2] Also called respectively by colimits and limits in the mathematical literature on this subject.

We fix once and for all three Grothendieck universes U, V and W, such that $\omega \in \mathsf{U} \in \mathsf{V} \in \mathsf{W}$. For a reference of Grothendieck universes, see [1]. Throughout all this dissertation, the term 'set' will be reserved for the sets which are isomorphic to an element of U, and we say 'small set' to emphasize this convention. The term 'class' makes reference to possible large classes, i.e., sets which are not necessarily isomorphic to an element of U. From this convention, U is a class.

A category C is called small if $Ob(C)$ e $Fl(C)$ are sets, and locally small if for all pair (X, Y) of objects in C, $Hom_C(X, Y)$ is a set. We adopt the terminology of [1] and say that a category C is a U-category if it is locally small and $Ob(C) \subseteq \mathsf{U}$. The usual categories of mathematical practice that we eventually invoke (topological spaces, groups, abelian groups, etc.) will be considered as U-categories.

We denote by Ens the category of sets U-sets, which will be called just by 'the category of sets'.

As it is usual, we will always identify a partially ordered set $(I, \leq)$ with a (small) category, denoted just by I, where the objects are the elements of I, and given any pair (x, y) of objects in I, then $Hom_I(x, y) = \{(x, y)\}$ for $x \leq y$, and $Hom_I(x, y) = \varnothing$ otherwise. Hence, there is exactly one arrow $x \to y$ for $x, y \in I$ with $x \leq y$. In particular, we identify every set E with it's respective discrete category, also denoted by E, such that the objects are the elements of E, and for every pair (x, y) of objects in E, we have $Hom_E(x, y) = \{1_x\}$ for $x = y$ and $Hom_E(x, y) = \varnothing$ otherwise.

A commutative square

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

in a category C will be called *cartesian* (resp. *cocartesian*), if the object X' (resp. Y) with the structural arrows f' and g' (resp. f and g) represents the projective (resp. inductive) limit [3] of the diagram

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[3] We say that an object of C *represents* the projective (resp. inductive) limit, and not that the projective (resp. inductive) limit exists, because, rigorously speaking, the prejective (resp. inductive) limit always exists as a presheaf over C. The point is not about existence but about the represent-ability of the projective (resp. inductive) limit functor over C. See Exposé I of [1] for more details.

$$\begin{tikzcd}
 & Y' \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

(resp.

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \\
X &
\end{tikzcd}$$

This terminology is also known in the English mathematical literature as *pullback* (resp. *pullshout*) squares.

Notations

The category (resp. the 2-category) of small categories will be denoted by Cat (resp. by $\mathcal{C}at$), and the 2-category of locally small categories will be denoted by $\mathcal{C}AT$. The symbol e (resp. $\varnothing$) denotes the terminal (resp. the initial object) in Cat. Therefore, $\varnothing$ is the empty category and e is the category with just one object $*$, called *the point*, and just one arrow. In other words, e is nothing but the discrete category of the set $\{\varnothing\}$. Given a category C which admits a terminal (resp. initial) object, we denote by e_C (resp. $\varnothing_C$) the initial (resp. terminal) object in C. When the context is clear, we use the notation $*$ (resp. 0) to indicate the object e_C (resp. $\varnothing_C$) in C.

Given a category A, the symbol A^o denotes the dual category of A.

Given a functor $u : A \to B$, we denote by $u^o : A^o \to B^o$ the respective induced functor on the dual categories, called the dualized functor.

Throughout all this dissertation, the symbol by A/b (resp. $b \setminus A$) denotes the comma category relative to a functor $u : A \to B$ and an object $b \in Ob(B)$, which is defined in the following way:

- The objects of A/b (resp. $b \setminus A$) are pairs of the form (a, φ) (resp. (φ, a)) where $a \in Ob(A)$ and $\varphi : u(a) \to b$ (resp. $\varphi : b \to u(a)$) is an arrow in B.

- The arrows of A/b (resp. of $b \setminus A$), say, from an object (a, φ) to an object (a', φ') (resp. from (φ, a) to (φ', a')) are the arrows of the form $\phi : a \to a'$ in

A such that the triangle

$$\begin{tikzcd}
u(a) \arrow[rr, "u(\phi)"] \arrow[dr, "u(\varphi)"'] & & u(a') \arrow[dl, "u(\varphi')"] \\
& b &
\end{tikzcd}$$

(resp.

$$\begin{tikzcd}
& b \arrow[dl, "u(\varphi')"'] \arrow[dr, "u(\varphi)"] & \\
u(a) \arrow[rr, "u(\phi)"'] & & u(a)
\end{tikzcd}$$

commutes. Hence, $b \setminus A = (A^o/b)^o$, where the comma A^o/b is considered in relation to the dual functor $u^o : A^o \to B^o$.

We also denote always by $\zeta(u, b) : A/b \to A$ (resp. $\xi(u, b) : b \setminus A \to A$) the forgetful functor $(a, \varphi) \mapsto a$ (resp. $(\varphi, a) \mapsto a$) from A/b to A (resp. from $b \setminus A$ to A), and by $i_{B,b} : e \to B$ (resp. or, just $b : e \to B$, when the context is clear) the only functor from e to B sending the point to b. In particular, taking $u = 1_B$, we have the usual comma and co-comma categories B/b and $b \setminus B$.

The functor $u/b : A/b \to B/b$ (resp. $u \setminus B : b \setminus A \to b \setminus B$) denotes the evident functor $(a, \varphi) \mapsto (u(a), \varphi)$ (resp. $(\varphi, a) \mapsto (\varphi, u(a))$) from A/b to B/b (resp. from $b \setminus A$ to $b \setminus B$).

The symbol A_b denotes the fiber of the category A at the point $b \in Ob(B)$, which is defined as the subcategory of A formed by the objects $a \in Ob(A)$ such that $u(a) = b$, and by the arrows $\varphi : a' \to a$ in A such that $u(\varphi) = 1_b$. We define the evident functors:

$$\Theta_b : A_b \to A/b, \quad a \mapsto (a, 1_b)$$

and

$$\Xi_b : A_b \to b \setminus A, \quad a \mapsto (1_b, a)$$

and the inclusion functors

$$j_b : A_b \longrightarrow A$$

from A_b to A.

Other notations will be explained along the text.

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A Mathematical History of Homotopy Types

In this chapter we make a great flyover throughout the mathematical history of homotopy types, highlighting the main concepts and results from where this dissertation begins. This follows a very digressive but enlightening path, in contrast to the other three chapters of this dissertation, which have a treatise style of exposition. We remark that the following historical introduction is not dispensable, since several concepts and results employed in the other three chapters are already presented or even justified here, and we recommend to the reader to read this chapter before the others.

The Homotopy Hypothesis

For every topological space X we can associate a small groupoid $\Pi_1(X)$, called the fundamental groupoid of X, described in the following way: the objects are the points of X and the arrows are the homotopy classes of paths between points fixing the boundary. The assignment $X \mapsto \Pi_1(X)$, from topological spaces to small groupoids, is functorial, and hence, there exists a functor

$$\Pi_1 : Top \longrightarrow Gpd,$$

where Top denotes the usual category of topological spaces and Gpd denotes the category of small groupoids. A morphism $f : X \to Y$ between topological spaces is called a 1-equivalence if $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection ($\pi_0(X)$ being the set of path connected components of X) and $\pi_1(f;x) : \pi_1(X,x) \to \pi_1(Y, f(x))$ is an isomorphism of groups for every $x \in X$ ($\pi_1(X,x)$ being the fundamental group of the pointed topological space (X,x)). We denote by Hot_1 the localization of the category of topological spaces by 1-equivalences, named the category of 1-homotopy types.

On the other hand, given a small groupoid G, we can define $\pi_0(G)$ as the set of connected components of G (in the categorical sense) and for each $x \in Ob(G)$, we can define $\pi_1(G,x)$ as the group $Aut_G(x)$, which is nothing but $Hom_G(x,x)$. With the above notations, we say that a morphism $f : G \to H$ between small groupoids is a 1-equivalence if $\pi_0(f) : \pi_0(G) \to \pi_0(H)$ is a bijection and $\pi_1(f,x) : \pi_1(G,x) \to \pi_1(H, f(x))$ is an isomorphism of groups for every $x \in Ob(G)$. We can verify that a morphism of groupoids is a 1-equivalence precisely when it is an equivalence of categories. We denote by $\mathsf{Ho}(Gpd)$ the localization of the category of groupoids by 1-equivalences of groupoids.

Clearly, for every topological space X we have $\pi_0(X) \cong \pi_0(\Pi_1(X))$ and $\pi_1(X, x) \cong \pi_1(\Pi_1(X), x)$ for every $x \in X$. Moreover, an arrow $f : X \to Y$ of topological spaces is a 1-equivalence if, and only if, $\Pi_1(f) : \Pi_1(X) \to \Pi_1(Y)$ is an equivalence of groupoids. Then, by the universal property of localization, the functor Π_1 induces a functor

$$\overline{\Pi}_1 : \mathsf{Hot}_1 \longrightarrow \mathsf{Ho}(Gpd)$$

such that the diagram

$$\begin{tikzcd}
Top \arrow[r, "\Pi_1"] \arrow[d] & Gpd \arrow[d] \\
\mathsf{Hot}_1 \arrow[r, "\overline{\Pi}_1"'] & \mathsf{Ho}(Gpd)
\end{tikzcd}$$

commutes, where the vertical arrows are the respective localizing functors. We can also proof that $\overline{\Pi}_1$ is an equivalence of categories, which means that 1-homotopy types are groupoids up to equivalence, and the study of 1-homotopy types is the study of the category $\mathsf{Ho}(Gpd)$.

In 1983, Grothendieck starts to write a letter in response to Quillen, which later would become the starting point of the deep and inspiring *Pursing Stacks* manuscript ([10]), where he proposes a generalization of the previous result: if 1-homotopy types are groupoids, then homotopy types should be equivalent to ∞-groupoids. This is in fact suggested by the proper definition of the category Hot of homotopy types when modelised by topological spaces. Traditionally, Hot is defined as the category where the objects are CW-complexes and the morphisms are continuous functions between CW-complexes up to homotopy equivalence. After a well known theorem proved by Whitehead, we have that Hot is equivalent the localization of the category of topological spaces by ∞-equivalences, i.e., continous functions $f : X \to Y$ such that $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection and $\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is a group isomorphism for every $n > 0$ and $x \in X$. Then, Grothendieck conjectured the existence of a category ∞-*Gpd* of ∞-groupoids, and a functor

$$\Pi_\infty : Top \longrightarrow \infty\text{-}Gpd$$

associating to each topological space X its fundamental ∞-groupoid $\Pi_\infty(X)$, which should, by localization, imply in an equivalence of categories:

$$\overline{\Pi}_\infty : \mathsf{Hot} \longrightarrow \mathsf{Ho}(\infty\text{-}Gpd).$$

Then, the study of homotopy types would be the study of ∞-groupoids up to equivalence. This conjecture is called the *Homotopy Hypothesis*.

Intuitively, the fundamental ∞-groupoid $\Pi_\infty(X)$ of a topological space X should be described in the following way: the objects of $\Pi_\infty(X)$ are the points of X, the 1-arrows, are homotopy classes of paths between points, the 2-arrows are homotopy classes of arrows between 1-arrows, the 3-arrows are homotopy classes of arrows between 2-arrows and so on. Hence, there should exists a notion of ∞-category behind the one of ∞-groupoids which is a generalization of the usual notion of 1-category.

Grothendieck never defined exactly the notions of ∞-groupoids and ∞-categories. After the circulation and transmission of the ideas presented in *Pursuing Stacks* several definitions of ∞-groupoids and ∞-categories were proposed, for example, by Joyal and Lurie ([26] and [32]), Batanin ([43] and [44]), Segal ([28]), and, in the spirit of Grothendieck himself (based on a reading interpretation of *Pursuing Stacks*), by Maltsiniotis ([14] and [39]). Comparison between these different models of ∞-groupoids is a very recent program of research, and it is not known if they are theoretic-equivalent. Indeed, the orientation of thought which guides the theory of ∞-groupoids is the following: the syntax of ∞-groupoids is the one of homotopy theory itself, and to provide a model (a semantic interpretation) for ∞-groupoids corresponds to provide a model for homotopy types.

Depending of the definition we use for ∞-groupoids, the *Homotopy Hypothesis* is a proven theorem, but for the Grothendieck-Maltsiniotis model of ∞-groupoids, it is still a conjecture. Yet, in the paper [14], where Malstsiniotis proposes a precise definition of ∞-groupoids, he also suggests a program to proof the *Homotopy Hypothesis* through the theory of test categories, another beautiful insight of Grothendieck's view of homotopical algebra.

(Pre)sheaves

Given a small category I and a locally small category C, we denote by C^I the category of functors from I to C, with morphisms being natural transformations, and we denote by $C(I)$ the category C^{I^o}, which is the category of presheaves over I at values in C.

If $u : I \to J$ is a morphism between small categories, then we can for the functor

$$u^* : C^J \longrightarrow C^I, \quad F \mapsto F \circ u$$

called the inverse image of u induced by C. For every $j \in Ob(I)$, we denote by I/j the category of pairs (i, φ) where $i \in Ob(I)$ and $\varphi \in Hom_J(u(i), j)$. Well understood, a morphism of I/j, say, from (i, φ) to (i', φ'), is an arrow $\alpha : i \to i'$ of I such that $\varphi' \circ u(\alpha) = \varphi$. The dual construction $j \setminus I$ of I/j can be defined by the formula $j \setminus I =_{df} (I^o/j)^o$. With the previous notations, we have the commutative squares

$$\begin{tikzcd}
I/j \arrow[r, "\zeta(u,j)"] \arrow[d, "p_{I/j}"'] & I \arrow[d, "u"] \\
e \arrow[r, "j"'] & J
\end{tikzcd}$$

and

$$\begin{tikzcd}
j \setminus I \arrow[r, "\xi(u,j)"] \arrow[d, "p_{j\setminus I}"'] & I \arrow[d, "u"] \\
e \arrow[r, "j"'] & J.
\end{tikzcd}$$

The category C is co-complete (resp. complete) precisely when the inverse image functors

$$u^* : C^J \longrightarrow C^I$$

admit a left (resp. right) adjoint

$$u_! : C^I \longrightarrow C^J, \qquad u_* : C^I \longrightarrow C^J.$$

called the left (resp. right) Kan extension of u. In this case, the canonical morphism $p_I : I \to e$ (i.e., the unique existent functor from an arbitrary small category I to the point category e) induces a functor $(p_I)_! : C^I \to C$ (resp. $(p_I)_* : C^I \to e$), for $C^e \cong C$, and we can define the inductive (resp. projective) limit of a diagram $F : I \to C$, as

$$\varinjlim_I F =_{df} (p_I)_!(F), \qquad \varprojlim_I F =_{df} (p_I)_*(F).$$

Moreover, given a diagram $F : I \to C$, we can compute, for each $j \in Ob(J)$, the value $u_!(F)_j$ by the canonical isomorphisms

$$u_!(F)_j \cong \varinjlim_{I/j} \zeta(u,j)^*(F), \qquad u_*(F)_j = \varprojlim_{j\backslash I} \xi(u,j)^*(F).$$

Dualizing the previous notations, we have, for the presheaves over C, also an inverse image functor

$$u^* : C(J) \longrightarrow C(I), \quad F \mapsto F \circ u^o$$

derived by composition from $u : I \to J$, and the respective left and right Kan extensions:

$$u_! : C(I) \longrightarrow C(J), \qquad u_* : C(I) \longrightarrow C(J),$$

whenever C is co-complete and complete. In this case, the Kan extensions are pointwise determined respectively by the isomorphisms

$$u_!(F)_j \cong \varinjlim_{j\backslash I} \xi(u,j)^*(F) \qquad u_*(F)_j \cong \varprojlim_{I/j} \zeta(u,j)^*(F).$$

The category of sets Ens is complete and co-complete. Moreover, the category Ens satisfies the following universal property: given any co-complete category C, the functor

$$\underline{Hom}_!(Ens, C) \longrightarrow C, \quad F \mapsto F(*)$$

is an equivalence of categories, because for each set E, we have a canonical isomorphism

$$E \cong \coprod_E *.$$

Therefore, every functor $F : Ens \to C$, commuting with inductive limit, is complete determined by the value $F(*)$ through the isomorphisms:

$$F(E) \cong F(\coprod_E *) \cong \coprod_E F(*).$$

More generally, given any small category A, we can form the category of presheaves

$$\widehat{A} =_{df} Ens(A) = \underline{Hom}(A^o, Ens),$$

and we have the canonical Yoneda functor

$$h : A \longrightarrow \widehat{A}, \quad a \mapsto (h_a : b \mapsto Hom_A(b,a))$$

which satisfies the following property: for each object X of $\widehat{A}$, the function

$$Hom_{\widehat{A}}(h_a, X) \longrightarrow X_a, \quad f \mapsto f_\flat = f_a(Id_a)$$

is a natural bijection (with relation to a and X), where the inverse image is defined as following: for each $s \in X_a$, we define $s^\sharp : h_a \to X$ as the natural transformation assigning to each $b \in Ob(B)$ to the function

$$(s^\sharp)_b : h_a(b) = Hom_A(b,a) \longrightarrow X_b, \quad \varphi \mapsto \varphi^*(s),$$

where φ^* denotes the map $X_\varphi : X_a \to X_b$ induced from the arrow $\varphi : b \to a$ applying the presheaf X. With the previous notations, we can verify easily the equations $(f_\flat)^\sharp = f$ and $(s^\sharp)_\flat = s$. This preceding fact is the celebrate *Lemma of Yoneda*. The immediate consequence of the *Lemma of Yoneda* is the that, the Yoneda functor $h : A \to \widehat{A}$, is actually faithful fully.

If C is a co-complete (resp. complete) category, then we can also verify that C^I is co-complete (resp. complete) for every small category I. Then, the presheaf categories $\widehat{A}$ are always complete and co-complete. In particular, if $u : A \to B$ is any functor between small categories, then u induces (by composition) an inverse image

$$u^* : \widehat{B} \longrightarrow \widehat{A}$$

which admits a left (resp. right) Kan extension

$$u_! : \widehat{A} \longrightarrow \widehat{B}, \qquad u_* : \widehat{A} \longrightarrow \widehat{B}.$$

The Yoneda functor $h : A \to \widehat{A}$ is in fact the universal co-completation of A. We explain the previous assertion. We say that a co-complete locally small category $\mathcal{C}$ is a co-completation of A if there exists a functor $\Phi : A \to \mathcal{C}$. To say that $\widehat{A}$ is the universal co-completation of A means that the functor

$$\underline{Hom}_!(\widehat{A}, \mathcal{C}) \longrightarrow \underline{Hom}(A, \mathcal{C}), \quad F \mapsto F \circ h$$

is an equivalence of categories. The proof of the universal property of the

Yoneda immersion $h : A \to \widehat{A}$ have two steps. First, we verify that h is dense in $\widehat{A}$, i.e., given a presheaf $X : A^o \to Ens$, we can form the category A/X of elements of X, which is composed by the objects (a, s), where $a \in Ob(A)$ and $s \in X_a$, and arrows $\varphi : (a, s) \to (b, t)$, where $\varphi : a \to b$ is an arrow of A such that $\varphi^*(t) = s$ (again, φ^* denotes the map $X_\varphi : X_b \to X_a$). Therefore, the category A/X is small and there exists an evident forgetful functor

$$pr_{A/X} : A/X \longrightarrow A, \quad (a, s) \mapsto a.$$

Composing $pr_{A/X}$ with the Yoneda immersion $h : A \to \widehat{A}$, we have a functor

$$\mathcal{U}_X : A/X \longrightarrow \widehat{A}, \quad (a, s) \mapsto h_a,$$

and, for each object (a, s) of A/X, there is a canonical arrow $s^\sharp : h_a \to X$, defined in the proof of the *Lemma of Yoneda*. Clearly, any morphism $\varphi : (a, s) \to (b, t)$ in A/X induces a commutative triangle

$$\begin{tikzcd} h_a \arrow[rr, "h_\varphi"] \arrow[dr, "s^\sharp"'] & & h_b \arrow[dl, "t^\sharp"] \\ & X & \end{tikzcd}$$

since, by the *Lemma of Yoneda*, the equation $\varphi^*(t) = s$ is equivalent to the equation $t^\sharp \circ h_\varphi = s^\sharp$. Therefore, there is a canonical morphism

$$\varinjlim_{A/X} \mathcal{U}_X \longrightarrow X,$$

and we can proof without difficult that this above arrow is actually an isomorphism, i.e.,

$$X \cong \varinjlim_{(a,s) \in A/X} h_a.$$

In particular, if $X = h_b$ for some $b \in Ob(A)$, then $A/X = A/h_b \simeq A/b$, and

$$h_b \cong \varinjlim_{(a,\varphi) \in A/b} h_a.$$

From the density of the Yoneda immersion $h : A \to \widehat{A}$, it follows that any functor $F : \widehat{A} \to \mathcal{C}$, commuting with inductive limits, is essentially determined by the values $F(h_a)$ for $a \in Ob(A)$. This fact allows us to define the quasi-inverse image of the functor

$$\underline{Hom}_!(\widehat{A}, \mathcal{C}) \longrightarrow \underline{Hom}(A, \mathcal{C}), \quad F \mapsto F \circ h.$$

Namely, to each functor $\Phi : A \to \mathcal{C}$, we can assign the functor

$$\Phi_\sharp : \widehat{A} \longrightarrow \mathcal{C}, \quad X \mapsto \varinjlim_{(a,s) \in A/X} \Phi_a.$$

Note that, if $f : X \to Y$ is an arrow in $\widehat{A}$, then f induces an evident functor

$$A/f : A/X \longrightarrow A/Y, \quad (a, s) \mapsto (a, f_a(s)),$$

from where we deduce the functoriality of $\Phi_\sharp$. Moreover, it follows from the construction $\Phi_\sharp$ that $\Phi_\sharp$ commutes with inductive limits and $\Phi_\sharp(h_a)$ is naturally isomorphic to Φ_a for every $a \in Ob(A)$. The functors $\Phi \mapsto \Phi_\sharp$ and $F \mapsto F \circ h$ are proven to be quasi-inverse one each other, and they define a canonical equivalence of categories

$$\underline{Hom}_!(\widehat{A}, \mathcal{C}) \simeq \underline{Hom}(A, \mathcal{C}).$$

A locally small category $\mathcal{E}$ is called a sheaf category (or, a Grothendieck topos, or even just a topos), if it admits finite projective limits, and there exists a functor $a : \widehat{C} \to \mathcal{E}$, for some small category C, which commutes with finite projective limits and admits a faithful fully right adjoint $\imath : \mathcal{E} \to \widehat{C}$. In particular, every presheaf category, i.e., every category of the form $\widehat{A}$ for some small category A, is a topos. For the general theory of toposes, including the notions of geometric morphisms, topos of sheaves over the open subsets of a topological space and cohomology of toposes, which will be mentioned freely in this mathematical-historical chapter, we indicate the reader to see [1] and [7]. The latter furnishes a very concise exposition of all the terms and results that we going to use.

Derivators

In the previous section of this introduction, we showed that any locally small category C determines a 2-functor

$$C : \mathcal{C}at^o \longrightarrow \mathcal{C}AT, \quad u \mapsto u^*, \quad \alpha \mapsto \alpha^*$$

where $C(I) = \underline{Hom}(I^o, C)$ (the category of presheaves over I at values in C). Moreover, the category C is complete (resp. co-complete) precisely when the 2-strict functor

$$C : \mathcal{C}at \longrightarrow \mathcal{C}AT, \quad I \mapsto C(I), \quad u \mapsto u^*, \quad \alpha \mapsto \alpha^*$$

admits right (resp. left) Kan extensions u_* (resp. $u_!$) for each 1-arrow u of $\mathcal{C}at$.

A derivator is a generalization of the theory of Kan extensions in the previous sense. Essentially, a prederivator $\mathscr{D}$ is just a 2-functor of the form

$$\mathscr{D} : \mathcal{C}at^o \longrightarrow \mathcal{C}AT.$$

We say that $\mathscr{D}$ is a left (resp. right) derivator, when the inverse images

$$u^* : \mathscr{D}(J) \longrightarrow \mathscr{D}(I)$$

for any arrow $u : I \to J$ of Cat, admits a right (resp. left) adjoint

$$u_* : \mathscr{D}(I) \longrightarrow \mathscr{D}(J) \qquad u_! : \mathscr{D}(I) \longrightarrow \mathscr{D}(J),$$

satisfying a list of axioms (which are related to the computations of these left and right Kan extensions). Hence, we can define, for each canonical arrow $p_I : I \to e$ and object F of $\mathscr{D}(I)$, the homology (resp. the cohomology):

$$\mathsf{H}_*^{\mathscr{D}}(I; F) =_{df} (p_I)_!(F), \qquad \mathsf{H}^*_{\mathscr{D}}(I; F) = (p_I)_*(F).$$

With the previous notations, we say that $\mathsf{H}_*^{\mathscr{D}}(I; F)$ (resp. $\mathsf{H}^*_{\mathscr{D}}(I; F)$) is the homology (resp. cohomology) of I with coefficients in F for the derivator $\mathscr{D}$ [4].

---

[4] There exists also an interpretation of these homologies and cohomologies as homotopy inductive limits and homotopy projective limits. Yet, this not agree necessarily with the traditional notation in the Grothendieck formalism of derivators, which are 1-contravariant

We remark that the inductive (resp. projective) limits of a locally small category C are shadows of the homology (resp. cohomology) of the prederivator $C : I \mapsto C(I) = \underline{Hom}(I^o, C)$. In fact, for each small category I, we have $C(I^o) = C^I$, and the inverse image functor

$$p^*_{I^o} : C \longrightarrow C(I^o) = C^I$$

coincides with the constant diagram functor

$$(-)_I : C \longrightarrow C^I.$$

Hence, any diagram $F : I \to C$ is an object of $C(I^o)$, and the inductive (resp. projective) limit $\varinjlim_I$ (resp. $\varprojlim_I$), is the left (resp. right) adjoint, of the functor $(-)_I$, from where we have

$$\varinjlim_I F \cong (p_{I^o})_!(F) = \mathsf{H}^C_*(I^o; F)$$

(resp.

$$\varprojlim_I F \cong (p_{I^o})_*(F) = \mathsf{H}^*_C(I^o; F).$$

Given a derivator $\mathscr{D}$, the category $\mathscr{D}(e)$ is called the category of absolute coefficients of $\mathscr{D}$.

The main examples of derivators are the ones induced from a pair (C, W) where C is a locally small category (usually supposed to be both complete and co-complete) and W is a class of arrows in C (usually supposed to be a class of weak equivalences). Define, for each small category I, the class W_I of arrows f in $C(I)$ such that $f_i \in W$ for every $i \in Ob(I)$. Then, we can form the prederivator

$$\mathbf{Ho}_W C : I \longmapsto (W_I)^{-1}C(I)$$

---

and also 2-contravariant functors from $\mathcal{C}at$ to $\mathcal{CAT}$. In the mathematical literature about derivators, it is also very common to define derivators and 1-contrvariant and 2-covariant functors, in order to formalize the notions of homotopy projective and inductive limits. These two formalisms are in fact equivalent. Here, we follow the Grothendieck terminology, and interpret (homotopy) projective and inductive limits as shadows of the (co)homology of derivators, which are also shadows of the Kan extensions. Actually, as we going to see, all the theory of derivators can be even reduced (by duality arguments) to the study of left derivators. In other words, we choose a cohomological orientation instead of homological orientation.

associated to the localizer (C, W). When $\mathbf{Ho}_W C$ is a derivator, we say that (C, W) is a *Grothendieck localizer.*

The language of derivators was introduced by Grothendieck in [11] as *le lit à deux places* (using a Grothendieck expression for topos theory) in which places both homotopy theory and homological algebra.

For instance, let's consider that homotopy theory is formalized by localizers. A modelizer (C, W) is a localizer such that $W^{-1}C$ is equivalent to the *homotopy category* Hot of CW-complexes. On the other hand, homological algebra can be formalized in the language of triangulated categories introduced by Verdier in [36]. Yet, these respective languages revealed to be insufficient for the study of homotopy theory and homological algebra. The former presents two problems: first, the localized categories $W^{-1}C$ are not complete and co-complete in general, and there are several different localizers (C, W) such that $W^{-1}C \simeq \mathsf{Hot}$. Therefore, we can not have an intrinsic definition of the category Hot only in the language of localizers. The latter is also insufficient for the following two reasons exposed in [37]:

(i). The category derived category $D(\mathcal{A})$ of a Grothenndieck abelian category lacks sufficient information to determine some homotopy inductive limits, as the cone of a morphism, for example.

(ii). The derived category $D(\mathcal{A})$ of a Grothendieck abelian category is not the universal triangulated category associated to $\mathcal{A}$.

The second problem above was solved by Keller in [37] replacing the derived category $D(\mathcal{A})$ by the derivator $\mathscr{D}(\mathcal{A})$ associated to the localizer $(Comp(\mathcal{A}), W_{qis})$, where $Comp(\mathcal{A})$ is the category of complexes in the abelian category $\mathcal{A}$ and W_{qis} is the class of quasi-isomorphisms in $Comp(\mathcal{A})$ (arrows which induce an isomorphism on the cohomology objects). We can verify that $\mathscr{D}(\mathcal{A})$ is a triangulated derivator, which is essentially a derivator such that $\mathscr{D}(\mathcal{A})(I)$ is a triangulated category for every small category I. The abelian category $\mathcal{A}$, being complete and co-complete, defines a derivator $\mathcal{A} : I \mapsto \mathcal{A}(I)$, also denoted by $\mathcal{A}$, and the canonical functor $\mathcal{A} \to Comp(\mathcal{A})$, carrying each object A of $\mathcal{A}$ to the complex

$$... \to 0 \to 0 \to A \to 0 \to 0 \to ...$$

concentrated in degree zero, induces a morphism of derivators

$$\mathcal{A} \longrightarrow \mathscr{D}(\mathcal{A}).$$

Then, from the arguments presented in [37], one can proof that $\mathscr{D}(\mathcal{A})$ is essentially the universal triangulated derivator associated to $\mathcal{A}$ in a suitable sense.

Independently, in an attempt to give a precise mathematical answer to the question '*What is a homotopy theory*?', Heller introduces in [9] the notion of *homotopy theories*, which are essentially the derivators in the sense of [11].

Heller exposes a deep and clarifying reflection in the introduction of [9] concerning homotopy theory, and explaining the reasons why a homotopy theory can not be codified as a localizer (C, W) or even as a model category in the sense of Quillen ([4]]). If we think a homotopy theory as a localizer (C, W), for example, then, since several different localizers may define equivalent localizations, the *homotopy theory* **H** defined by a localizer (C, W) should be independent of the localizer (C, W), which is only one among many other presentations of **H**. In analogy with topos theory ([1]), a site $(\mathcal{S}, J)$ defines a topos $\mathcal{E}$, which is the topos $Sh(\mathcal{S}, J)$ of sheaves over the site $(\mathcal{S}, J)$, but there is no canonical site to define $\mathcal{E}$, and $\mathcal{E}$ may be defined as the 'same' topos of sheaves for many other different sites. Similarly, a localizer (C, W) defines a homotopy theory **H**, which is the derivator $\mathbf{Ho}_W C$, but there is no canonical localizer to define **H**, and **H** may be defined as the 'same' homotopy theory for many other localizers. Following this analogy, we have that the axioms of derivator (see (1.3.9)) are to the Giraud axioms of a topos (see *Théorème 1.2.* of [1]) as the Grothendieck localizers (C, W) (including the model categories) are to sites, and, going further in this analogy (as we shall see in the second chapter of this dissertation), the homotopy theory **Hot** of CW-complexes is to the topos of sets *Ens*.

Yet, the formalism of derivators is beyond homotopy theory, and contemplates the deep heart of the foundations of category theory itself. Folllowing Grothendieck, a derivator is a theory of coefficients (homological, cohomoloigcal, homotopical, etc.) over *Cat*, where *Cat* is visualized as a category of *geometric and spatial* objects. In other words, Grothendieck considers the category *Cat* as being *the* category of spaces. A derivator $\mathscr{D}$ assigns to each small category I (or, space) a 'derived' category $\mathscr{D}(I)$ of coefficients of type $\mathscr{D}$ over I. The idea of Grothendieck, to establish the axioms of derivators, is that $\mathscr{D}(I)$ should depend (up to equivalence), only of the topos $\widehat{I}$ of presheaves over I. In particular, if $u : I \to J$ is a morphism of small categories, then the inverse image functor

$$u^* : \mathscr{D}(J) \longrightarrow \mathscr{D}(I)$$

should admit respectively a left and right adjoint

$$u_! : \mathscr{D}(I) \longrightarrow \mathscr{D}(J), \qquad u_* : \mathscr{D}(I) \longrightarrow \mathscr{D}(J)$$

as we already stated. Then, investigating the contravariant 2-functors of the form

$$\mathscr{D} : \mathcal{T}OP \longrightarrow \mathcal{C}AT$$

where $\mathcal{T}OP$ denotes the 2-category of toposes and geometric morphisms, and restricting them to the full 2-subcategory $\mathcal{TOPCAT}$ of $\mathcal{T}OP$ formed by the *categorical toposes*, i.e., toposes of the form $\widehat{I}$ for a some small category I, Grothendieck achieves the formalism of derivators.

One of the main aspects of the language of derivators relies on its conceptual enlightenment of several constructions in mathematics as universal constructions. For instance, the homotopy theory of CW-complexes defines a derivator **Hot**, which will be formally defined in this dissertation using the the theory of fundamental localizers (a concrete definition of **Hot** is presented in Chap. II of [9], p. 25-31). It was already remarked by Grothendieck in [11] and by Heller in [9], that **Hot** is the acts on each derivator in an appropriate sense, and hence, it should be the universal (or, canonical) derivator. Yet, a complete proof of this fact depends, as we shall see, of a characterization of the ∞-equivalences as the minimal class of arrows in Cat satisfying three elementary conditions (which are precisely the axioms of fundamental localizers). Using this minimality condition, and an explicit description of the left Kan extensions of **Hot** (that are presented by Maltsiniotis in the third chapter of [13]), Cisinski proves this universal property in [17].

Historical comparisons

Let Top be the category of topological spaces, CW be the category of CW-complexes, Cat be the category of small categories, and $\widehat{\Delta}$ be the category of simplicial sets. We recall that the category of simplicial sets $\widehat{\Delta}$ is the category of presheaves over the small category Δ of standard simplexes, which by it's turn, is the category formed by the non-empty ordered finite sets

$$[n] = \{0 < 1 < ... < n\}, \qquad n \in \omega$$

as objects, and non-decreasing functions as arrows, with the usual composition of functions. We also denote by Δ_n an object $[n]$ of Δ, identifying it with the small category generated by the graph

$$0 \to 1 \to ... \to n.$$

We recall that the class W_{Top} of topological weak equivalences in Top is the class of arrows $f : X \to Y$ in Top such that

$$\pi_0 : \pi_0(X) \longrightarrow \pi_0(Y)$$

is a bijection (where $\pi_0(X)$ denotes the set of path connected components of X) and

$$\pi_n(f, x) : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$$

is a group isomorphism for every $n \geq 1$ and every $x \in X$ (where $\pi_n(X, x)$ is the usual n-th homotopy group of the pointed space (X, x)).

Theorem (Whitehead) - A morphism of CW-complexes is a weak equivalence of topological spaces if, and only if, it is a homotopical equivalence. In particular, the inclusion functor $i : \mathsf{CW} \hookrightarrow Top$ induces a functor

$$\bar{i} : \mathsf{Hot} \longrightarrow (W_{Top})^{-1} Top$$

which is an equivalence of categories.

After the formalism of model categories introduced by Quillen, we know that this classical Whitehead theorem is a consequence of the *Fundamental Theorem of Homotopical Algebra*, due to Quillen (see (1.1.45) of this dissertation) and from the fact that the CW-complexes are the cofibrant-fibrant objects in the usual model category structure over topological spaces (see (1.1.60) for a description of this model category). For the Whitehead's original proof, we indicate [33].

There exists a functor

$$r : \Delta \longrightarrow Top,$$

called the topological realization functor, which is defined in the following way: for each object Δ_n of Δ, we define

$$r(\Delta_n) = |\Delta_n| = \{x \in [0,1]^{n+1} : \sum_{i=0}^{n} x_i = 1\}$$

with the topology induced from the usual topology of $\mathbb{R}^{n+1}$, and for each arrow $\varphi : \Delta_n \to \Delta_m$ of Δ, we define $r(\varphi) = |\varphi| : |\Delta_n| \to |\Delta_m|$ as the continuous application such that,

$$(|\varphi|(x))_i = \sum_{j \in \varphi^{-1}(i)} x_j, \qquad 0 \leq i \leq m.$$

Since Top is co-complete, it follows from the universal property of $\widehat{\Delta}$ as the free co-completation of the small category Δ, that the functor $r : \Delta \to Top$ can be extended to a (essentially unique) functor

$$|?| : \widehat{\Delta} \longrightarrow Top$$

which commutes with inductive limits, and is left adjoint to the functor

$$Simp : Top \longrightarrow \widehat{\Delta}, \quad X \mapsto ([n] \mapsto Hom_{Top}(|\Delta_n|, X)).$$

We call $Simp$ (resp. $|-|$) the *simplicial* (resp. *topological*) *realization functor*. One of the main properties of the topological realization is that for every simplicial set X, the topological space $|X$ is actually a CW-complex. Hence, the topological realization functor

$$|?| : \widehat{\Delta} \longrightarrow Top$$

factors through the category of CW-complexes

$$\widehat{\Delta} \xrightarrow{|-|} \mathsf{CW} \hookrightarrow Top.$$

Defining the class W_s of arrows $f : X \to Y$ in $\widehat{\Delta}$ such that $|f| \in W_{Top}$, called the class of simplicial weak equivalences, we have the following:

Theorem (Milnor, Theorem 11.4. of [25]) - The topological realization

$$|?| : \widehat{\Delta} \longrightarrow Top$$

and the simplicial realization

$$Simp : Top \longrightarrow \widehat{\Delta}$$

are equivalence between the localizers (Top, W_{Top}) and $(\widehat{\Delta}, W_{\widehat{\Delta}})$, quasi-inverse one each other. In particular, they induce an equivalence of categories:

$$(W_{Top})^{-1}Top \simeq (W_s)^{-1}\widehat{\Delta}.$$

We remark that in virtue of the Whitehead's Theorem, an arrow of simplicial sets $f : X \to Y$ is a simplicial weak equivalence if, and only if, $|f| : |X| \to |Y|$ is a homotopy equivalence.

Since the category Cat is also co-complete, the inclusion functor

$$i : \Delta \longrightarrow Cat$$

can be extended to a (essentially unique) functor

$$ho : \widehat{\Delta} \longrightarrow Cat,$$

commuting with inductive limits, and left adjoint to the *nerve functor*

$$N : Cat \longrightarrow \widehat{\Delta}, \quad C \mapsto ([n] \mapsto Hom_{Cat}(\Delta_n, Cat)).$$

We remark that, for each small category C and $n \in \omega$, $N(C)_n$ is the set of paths of arrows of the form

$$c_0 \to c_1 \to \ldots \to c_n$$

in C.

There is also a functor

$$i_\Delta : \widehat{\Delta} \longrightarrow Cat, \quad X \mapsto \Delta/X$$

which sends each simplicial set to its category of elements Δ/X. This functor is left adjoint to the evident functor

$$i_{\Delta}^{*} : Cat \longrightarrow \widehat{\Delta}, \quad C \mapsto ([n] \mapsto Hom_{Cat}(\Delta/\Delta_n, C)$$

and may not be confused with the functor $ho : \widehat{\Delta} \to Cat$ which is left adjoint to the nerve functor $N : Cat \to \widehat{\Delta}$. Yet, defining the class W_{Cat} of arrows $u : A \to B$ in Cat such that $N(u) \in W_s$:

$$W_{Cat} = N^{-1}(W_s),$$

called the class of weak homotopy equivalences of small categories, we have the following equalities:

$$W_s = W_{\widehat{\Delta}} =_{df} (i_{\Delta})^{-1}(W_{Cat}) = (i_{\Delta})^{-1}N^{-1}(W_s).$$

Hence, an arrow of simplicial sets $f : X \to Y$ is a weak homotopy equivalence if, and only if, the functor

$$\Delta/f : \Delta/X \longrightarrow \Delta/Y$$

is a weak homotopy equivalence of small categories.

Theorem (Illusie-Quillen, *Corollaire 3.3.1.* of [5]) - The nerve functor $N : Cat \to \widehat{\Delta}$ and the functor $i_{\Delta} : \widehat{\Delta} \to Cat$ are equivalence between the localizers (Cat, W_{Cat}) and $(\widehat{\Delta}, W_{\widehat{\Delta}})$, quasi-inverse one each other. In particular, they induce an equivalence of categories:

$$(W_s)^{-1}\widehat{\Delta} \simeq (W_{Cat})^{-1}Cat.$$

From all the previous comparisons theorems (Whitehead, Milnor and Illusie-Quillen), there is a sequence of functors

$$Cat \xrightarrow{N} \widehat{\Delta} \xrightarrow{|?|} \mathsf{CW} \hookrightarrow Top$$

inducing, after localization, a sequence of functors

$$(W_{Cat})^{-1}Cat \to (W_s)^{-1}\widehat{\Delta} \to \mathsf{Hot} \to (W_{Top})^{-1}Top,$$

which are all equivalences of categories.

Now, every functor $u : A \to B$ between small categories induces a continous function

$$|N(u)| : |N(A)| \longrightarrow |N(B)|$$

and a canonical geometric morphism of toposes

$$\widehat{u} =_{df} (u^*, u_*) : \widehat{A} \longrightarrow \widehat{B},$$

where u^* and u_* can be regarded respectively as the inverse image and the right Kan extension of u induced by the derivator associated to the category of sets Ens. Given a small category A, we call $|N(A)|$ the classifying topological space of A, and we call $\widehat{A}$ the classifying topos of A. On the other hand, every map $\varphi : X \to Y$ of topological spaces also induces a geometric morphism

$$\widetilde{\varphi} : \widetilde{X} \longrightarrow \widetilde{Y},$$

where $\widetilde{X}$ denotes the usual topos of sheaves over the open subsets of X, and for every morphism of simplicial sets $f : X \to Y$, we also have the geometric morphisms

$$\widehat{\Delta}/X \longrightarrow \widehat{\Delta}/Y$$

and

$$\widetilde{|X|} \longrightarrow \widetilde{|Y|}$$

Therefore, the category $\mathcal{TOP}$ of toposes and geometric morphisms is a common place to compare morphisms both in Top and in Cat, for we have the canonical functor

$$Cat \longrightarrow \mathcal{TOP}, \quad A \mapsto \widehat{A}, \quad u \mapsto \widehat{u} = (u^*, u_*),$$

and the functor

$$Top \longrightarrow \mathcal{TOP}, \quad X \mapsto \widetilde{X}, \quad \varphi \mapsto \widetilde{\varphi}.$$

The Artin-Mazur equivalences, introduced in [6], and developed in [7], form a class of geometric morphisms which allows us to compare, via cohomology of toposes, the topological topos

$$\widetilde{|N(A)|}$$

associated to the classifying topoloigical space of a small category A, with its classifying (categorical) topos

$$\widehat{A}.$$

Actually, it follows from [7] that the Artin-Mazur equivalences of toposes satisfy the following conditions:

1. A continuous function $\varphi : X \to Y$ between reasonable topological spaces, like CW-complexes, is a weak equivalence of topological spaces if, and only if, it is an Artin-Mazur equivalence.
2. For every small category A, there exists an Artin-Mazur equivalence
   $$\alpha_A : |\widetilde{N(A)}| \longrightarrow \widehat{A}$$
   which is functorial with respect to the functors between small categories, i.e., every morphism of small categories $u : A \to B$ induces a commutative square
   $$\begin{tikzcd} {|\widetilde{N(A)}|} \arrow[r, "\alpha_A"] \arrow[d, "|\widetilde{N(u)}|"'] & \widehat{A} \arrow[d, "\widehat{u}"] \\ {|\widetilde{N(B)}|} \arrow[r, "\alpha_B"'] & \widehat{B}. \end{tikzcd}$$
   of geometric morphisms between toposes.
3. Every equivalence of toposes is an Artin-Mazur equivalence.
4. Given any commutative triangle
   $$\begin{tikzcd} \mathcal{X} \arrow[rr, "\phi"] \arrow[dr, "\varphi"'] & & \mathcal{Y} \arrow[dl, "\psi"] \\ & \mathcal{E} & \end{tikzcd}$$
   of geometric morphisms between toposes, if two of the arrows φ, ψ and ϕ are Artin-Mazur equivalences, then so is the third.

The previous formal properties of the Artin-Mazur equivalences imply that an arrow $u : A \to B$ of small categories is a weak equivalence if, and only if, $u : A \to B$ is an Artin-Mazur equivalence. In fact, the topological realization $|X|$ of any simplicial set X is a CW-complex. Hence, a morphism of small categories $u : A \to B$ is an element of W_{Cat} precisely when the morphism of toposes

$$\widetilde{|N(u)|} : \widetilde{|N(A)|} \longrightarrow \widetilde{|N(B)|}$$

is an Artin-Mazur equivalence (by the condition (1) of Artin-Mazur equivalences). On the other hand, the conditions (2) and (4) of Artin-Mazur equivalences imply that

$$\widetilde{|N(u)|} : \widetilde{|N(A)|} \longrightarrow \widetilde{|N(B)|}$$

is an Artin-Mazur equivalence if, and only if, the canonical geometric morphism between the classifying toposes:

$$\widehat{u} = (u^*, u_*) : \widehat{A} \longrightarrow \widehat{B}$$

is an Artin-Mazur equivalence, because in the commutative square

$$\begin{tikzcd}
\widetilde{|N(A)|} \arrow[r, "\alpha_A"] \arrow[d, "\widetilde{|N(u)|}"'] & \widehat{A} \arrow[d, "\widehat{u}"] \\
\widetilde{|N(B)|} \arrow[r, "\alpha_B"'] & \widehat{B},
\end{tikzcd}$$

the arrows α_A and α_B are always Artin-Mazur equivalences (by the condition (2)). Finally, applying the property (4) of Artin-Mazur equivalences, we conclude the assertion.

We recall quickly the definition of Artin-Mazur equivalences: a geometric morphism of toposes $f : \mathcal{E} \to \mathcal{F}$ is an Artin-Mazur equivalence if for every (locally constant) sheaf G in $\mathcal{F}$, the morphism

$$H^n(\mathcal{F}; G) \longrightarrow H^n(\mathcal{E}; f^{-1}(G))$$

is an isomorphism for every $n \geq 0$, where, for a topos $\mathcal{X}$:

1. G is a (locally constant) sheaf of sets in $\mathcal{X}$ for $n = 0$, and $H^0(\mathcal{X}; G) = \Gamma(G; \mathcal{X})$, i.e., $H^0(\mathcal{X}; G)$ is the set of global sections of G, defined simply as being the evaluation of G at the terminal object of $\mathcal{X}$.

2. G is a (locally constant) sheaf of groups in $\mathcal{X}$ for $n = 1$, and $H^1(\mathcal{X}; G)$ is the *non-abelian* cohomology of $\mathcal{X}$ with coefficients in G, defined using G-torsors.

3. G is a (locally constant) sheaf of abelian groups for $n > 1$, and $H^n(\mathcal{X}; G)$ is the usual *abelian* cohomology of the topos $\mathcal{X}$ with coefficients in G.

Let's recall briefly the notions of non-abelian and abelian cohomology of a topos $\mathcal{X}$ with coefficients in a locally constant sheaf in $\mathcal{X}$. First we recall that an object G in $\mathcal{X}$ is locally constant if there exists a covering U of $\mathcal{X}$ (i.e., an object $U \in Ob(\mathcal{X})$ such that the unique existent arrow $U \to *$ from U to the terminal object $*$ of $\mathcal{X}$ is an epimorphism) for which the restriction $G|_U$ is a constant sheaf in $\mathcal{X}/U$. Denoting by $\mathcal{X}_{ab}$ the category of locally constant abelian group objects in the topos $\mathcal{X}$, we have that any geometric morphism $f = (f^*, f_*) : \mathcal{X} \to \mathcal{Y}$ implies in the existence of an additive functor of abelian categories

$$f_* : \mathcal{X}_{ab} \longrightarrow \mathcal{Y}_{ab},$$

which, by it's turn, induces n-th derived functors

$$\mathsf{R}^n f_* : \mathcal{X}_{ab} \longrightarrow \mathcal{Y}_{ab}.$$

In particular, taking $\mathcal{Y} = Ens$ and $f = p_{\mathcal{X}} : \mathcal{X} \to Ens$ (with $p_{\mathcal{X}}$ indicating the unique geometric morphism from $\mathcal{X}$ to the terminal topos Ens), then these n-th derived functors are functors of the form

$$\mathsf{R}^n (p_{\mathcal{X}})_* : \mathcal{X}_{ab} \longrightarrow Ab.$$

The n-th abelian cohomology group of $\mathcal{X}$ with coefficients in a object G of $\mathcal{X}_{ab}$, is just the abelian group

$$H^n(\mathcal{X}; G) =_{df} \mathsf{R}^n (p_{\mathcal{X}})_*(G).$$

Now, let G be a locally constant group object in $\mathcal{X}$. A G-torsor is an object X of $\mathcal{X}$ endowed with a formal action

$$G \times X \longrightarrow X,$$

such that the morphism $G \times X \to X \times X$, induced by the action $G \times X \to X$ and the second projection $G \times X \to X$, is an isomorphism, and the canonical arrow $p_X : X \to *$, from X to the terminal object $*$ of $\mathcal{X}$, is an epimorphism. A morphism of G-torsors $f : X \to Y$ is just a morphism in $\mathcal{X}$ compatible with the respective actions of G. We can verify that every morphism of G-torsors is actually an isomorphism. Hence, denoting by $Tors(\mathcal{X}, G)$ the category of G-torsors in $\mathcal{X}$, we have that $Tors(\mathcal{X}, G)$ is a groupoid, and we can define the group:

$$H^1(X; G) =_{df} \pi_0 Tors(\mathcal{X}, G),$$

called the first *non-abelian* cohomology group of $\mathcal{X}$ with coefficients in G [5].

We say that a morphism $u : A \to B$ of small categories is an Artin-Mazur equivalence whenever the geometric morphism $\widehat{u} : \widehat{A} \to \widehat{B}$ is an Artin-Mazur equivalence of toposes.

With the previous definitions, we have the following:

Theorem (Artin-Mazur-Moerdjiik) - *A morphism of small categories $u : A \to B$ is a weak equivalence (i.e., belongs to W_{Cat}) precisely when it is an Artin-Mazur equivalence.*

For a proof of the above theorem, we invite the reader to see [7]. As we showed, this theorem is a consequence of the the formal properties of Artin-Mazur equivalences.

The characterization of the class W_{Cat} of weak equivalences of small categories as Artin-Mazur equivalences is crucial for Grothendieck. Indeed, Grothendieck defines the homotopy category Hot of CW-complexes in [11] as the localization of Cat by the Artin-Mazur equivalences. This leads Grothendieck to an axiomatization of the class of weak equivalence in Cat, that we expose in the sequel.

A class of arrows in W in Cat is a *fundamental localizer* if it satisfies the following axioms:

(LF1). W is weak saturated, i.e.

1. All identity arrow of C belongs to W.

---

[5] Here, $\pi_0 Tors(\mathcal{X}, G)$ denotes the set of isomorphisms classes of objects in $Tors(\mathcal{X}, G)$.

2. If in a commutative triangle of C, two of the morphisms are in W then the same holds for the third morphism.

3. If $r : X' \to X$ and $i : X \to X'$ are two arrows of C, where $r \circ i = Id_{X'}$ and $i \circ r \in W$, then $r \in W$.

(LF2). If A is a small category with a terminal object, then the canonical functor $p_A : A \to e$, from A to the point category e, belongs to W.

(LF3). Given a commutative triangle

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

of small categories, if $A/c \to B/c$ in in W for every $c \in Ob(C)$, then $u \in W$.

Using the formal properties of Artin-Mazur equivalences, Grothendieck verifies that W_{Cat} is a fundamental localizer. The more delicate axiom to verify is (LF3) (the first two axioms are straightforward). This third condition follows from the following local property of cohomology of toposes: given a commutative triangle of geometric morphisms between toposes of the form

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\phi"] \arrow[dr, "\varphi"'] & & \mathcal{Y} \arrow[dl, "\psi"] \\ & \mathcal{E} & \end{tikzcd}$$

if there exists a small family of generators $\mathcal{U}$ of $\mathcal{E}$ such that the induced geometric morphism

$$\phi/U : \mathcal{X}/U \longrightarrow \mathcal{Y}/U$$

is an Artin-Mazur equivalence of toposes for every $U \in \mathcal{U}$, then $\phi : \mathcal{X} \to \mathcal{Y}$ is an Artin-Mazur equivalence. We recall that a small set $\mathcal{U}$ of objects in a topos $\mathcal{E}$ is called a family of generators if for every two parallel arrows $f, g : X \to Y$ in $\mathcal{E}$, such that $fx \neq gx$ for some arrow $x : U \to X$ with $U \in \mathcal{U}$, we have $f \neq g$. Now, the *Lemma of Yoneda* implies that the representable functors $h_a, a \in Ob(A)$, form a small family of generators of the topos $\widehat{A}$. Therefore, using the Artin-Mazur characterization of W_{Cat}, and the canonical equivalences $\widehat{A}/a \simeq \widehat{A/a}$ (for A being a small category and $a \in Ob(A)$), we can verify that W_{Cat} is a fundamental localizer.

On the other hand, the intersection of an arbitrary non-empty class of fundamental localizers is a also a fundamental localizer. Therefore, there exists the *minimal* fundamental localizer W_∞, because $FL(Cat)$ is clearly a fundamental localizer (and the Artin-Mazur equivalences also form a fundamental localizer). The arrows in W_∞ are called ∞-equivalences. Grothendieck conjectured, and Cisinski proved, the following:

Theorem (Cisinski-Grothendieck) - $W_\infty = W_{Cat}$.

For a beautiful and conceptual proof of the the above theorem, we invite the reader to see *Théorème 4.2.15* and *Corollaire 4.2.19* of [15]. A brief sketch of a proof this theorem will also be given in the end of the section (2.3) of this dissertation. A more direct, but completely different proof, can be found in *Théorème 2.2.11* of [18].

In virtue of the characterization of the class W_∞ of ∞-equivalences as the minimal fundamental localizer in Cat, we can now redefine the category of homotopy types as being the category:

$$\mathsf{Hot} =_{df} (W_\infty)^{-1}Cat.$$

This is the staring point of this dissertation. Following Grothendieck, we interpret the category of spaces as being the category Cat of small categories, and the class W_∞ of ∞-equivalences as being the minimal fundamental localizer in Cat. The category Hot of homotopy types will be defined as being the category $(W_\infty)^{-1}Cat$.

The previous definition of the category of homotopy types, combined with the formalism of derivators and the theory of test categories and fundamental localizers, provides an elementary and purely categorical axiomatization of homotopy theory in the spirit of Grothendieck.

The formalism of derivators, test categories and fundamental localizers, which establish together the *Homotopy theory of Grothendieck*, are totally inspired in topos theory. Surprisingly, there is none derivator formalism over toposes in currently mathematics yet. Actually, Grothendieck envisioned to develop a larger unifying theory, called by him *topological algebra*, where homotopical and homological algebra figure just as a part. Another part, which concerns the manuscript *La longue marche à travers la théorie de Galois*, should covers Galois theory. No doubt, the ideas dreamed in this last manuscript, plus the ideas of *Les Dérivateurs* and *Pursuing Stacks*, should going to have an impact for centuries in mathematics.

Remark - As the main references for all the results presented in this section, we indicate [5], [7], [18] and [25].

1

1 Homotopical algebra

1.1 Localizers

Definition 1.1.1. *A localizer is a pair* (C, W) *where* C *is category and* W *is a class of arrows in* C*. The objects of* C *are called models and the arrows in* W *are called weak equivalences (or,* W*-weak equivalence).*

Example 1.1.2. Given any localizer (C, W) and a small category I, we can always form the localizers $(C(I), W_I)$ and (C^I, W_I), where $C(I)$ (resp. C^I) is the category $\underline{Hom}(I^o, C)$ (resp. $\underline{Hom}(I, C)$) and W_I is the class of arrows f in $C(I)$ (resp. in C^I) such that $f_i \in W$ for all $i \in Ob(I)$, called the class of termwise weak equivalences. The localizer $(C(I), W_I)$ is called the *presheaf localizer over* I *at values in* (C, W). We remark that $(C^I, W_I) = (C(I^o), W_{I^o})$.

Example 1.1.3. Given a localizer (C, W), we can always form the *dual localizer* (C^o, W^o), where W^o are the W-equivalences in the dual category C^o of C, i.e., $W^o = \{f^o : f \in W\}$. Since we have a canonical equivalence

$$C^I = \underline{Hom}(I, C) \simeq \underline{Hom}(I^o, C^o)^o = C^o(I)^o,$$

then, from the example (1.1.2), we can redefine the localizer (C^I, W_I) in terms of presheaves localizers as being the pair $(C^o(I)^o, (W^o_I)^o)$, i.e., (C^I, W_I) is the dual localizer of the presheaf localizer $(C^o(I), W^o_I)$.

Definition 1.1.4. *A localizer* (C, W) *is called a category with weak equivalences if the following properties are valid:*

1. W *contain all the identities of* C
2. W *has the property* 2 *out of* 3*: for any commutative triangle*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "s"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

if any two of the arrows f*,* g *or* s *are in* W*, then the same holds for the third.*

Notation: Given a localizer (C, W) and an arbitrary category D, we denote by $\underline{Hom}_W(C, D)$ the full subcategory of $\underline{Hom}(C, D)$ formed by the functors $u : C \to D$ sending W-equivalences to isomorphisms.

Definition 1.1.5. *Let (C, W) be a localizer. A localization of C by W is a category $W^{-1}C$ with a functor*

$$\gamma : C \longrightarrow W^{-1}C$$

such that, if $f \in W$, then $\gamma(f)$ is invertible in $W^{-1}C$, and for every other category D, the functor

$$\underline{Hom}(W^{-1}C, D) \longrightarrow \underline{Hom}_W(C, D), \quad F \mapsto F \circ \gamma, \quad f \mapsto f \star \gamma,$$

where $f \star \gamma$ denotes the usual horizontal composition [6]*, is an isomorphism of categories. Moreover, if $Ob(W^{-1}C) = Ob(C)$ and γ is the identity on the objects, then we say that the pair $(W^{-1}C, \gamma)$ is a Gabriel-Zisman localization.*

Proposition 1.1.6. *Given any localizer (C, W), there exists a category $W^{-1}C$ and a functor $\gamma : C \to W^{-1}C$, such that the following conditions are verified:*

1. *If $f \in W$, then $\gamma(f)$ is invertible in $W^{-1}C$.*
2. *For every functor $u : C \to D$ carrying W-equivalences to isomorphisms, there exists a unique functor $F : W^{-1}C \to D$ such that $u = F \circ \gamma$.*

Proof. Neglecting set theoretic obstructions, there is an easy construction of the category $W^{-1}C$ and the functor $\gamma : C \to W^{-1}C$. For, let Δ_1 be the category $\{0 \xrightarrow{s} 1\}$ and $\overline{\Delta}_1$ be the category $\{0 \xrightarrow{s} 1, 1 \xrightarrow{s^{-1}} 0\}$, where s and s^{-1} are inverses one each other. Then, we have the inclusion functor $\Delta_1 \hookrightarrow \overline{\Delta}_1$, and, for each $f \in W$, a functor $\Delta_1 \to C$, sending s to f. From the existence of the previous functors, we can deduce (by the universal property of the coproducts) the existence of a canonical functor

$$\coprod_W \Delta_1 \longrightarrow \coprod_W \overline{\Delta}_1,$$

and also a canonical functor

$$\coprod_W \Delta_1 \longrightarrow C.$$

---

[6] A precise definition of this composition if given in (1.2.1))

So, we may define $W^{-1}C$ as the inductive limit of the diagram

$$\begin{tikzcd}\coprod_W \Delta_1 \arrow[r] \arrow[d] & C \\ \coprod_W \overline{\Delta}_1 & \end{tikzcd}$$

and γ as the canonical morphism $C \to W^{-1}C$, from where we deduce the existence of a co-cartesian square of the form

$$\begin{tikzcd}\coprod_W \Delta_1 \arrow[r] \arrow[d] & C \arrow[d, "\gamma"] \\ \coprod_W \overline{\Delta}_1 \arrow[r] & W^{-1}C\end{tikzcd}$$

Now, if $u : C \to D$ is a functor which carries W-equivalences to isomorphisms, then, for each $\varphi \in W$, we have a functor

$$\overline{\Delta}_1 \longrightarrow D, \quad s \mapsto u(\varphi), \quad s^{-1} \mapsto u(\varphi)^{-1},$$

from where we deduce (again by the universal property of coproducts), the existence of a unique arrow $\coprod_W \overline{\Delta}_1 \to D$ of categories, making the diagram

$$\begin{tikzcd}\coprod_W \Delta_1 \arrow[r] \arrow[d] & C \arrow[d, "u"] \\ \coprod_W \overline{\Delta}_1 \arrow[r] & D\end{tikzcd}$$

commutative. Therefore, by the universal property of co-cartesian squares, there is a unique functor $F : W^{-1}C \to D$ such that $u = F \circ \gamma$. Hence, it is clear from the previous argument that the only obstruction for the existence of categorical localizations is set theoretical, for the coproducts $\coprod_W \Delta_1$ and $\coprod_W \overline{\Delta}_1$ may not be representable in locally small categories, but possibly only in (very) large categories. In order to have a Gabriel-Zisman localization in the sense of (1.1.5), which furnishes an explicit description of the category $W^{-1}C$, there is an alternative construction sketched by Gabriel-Zisman in the *section (1.1)* of [3]. We expose quickly this last construction. The objects of the category $W^{-1}C$ are the same of C, and the arrows are finite sequences of chains of morphisms of the form

$$X \longleftarrow X_0 \longrightarrow X_1 \longleftarrow ... \longrightarrow X_{n-1} \longleftarrow X_n \longrightarrow Y$$

where the left arrows are W-equivalences and the composition is given by composition of chains, modulo an appropriate equivalence relation, which is the finest equivalence relation such that the evident map $\gamma : C \to W^{-1}C$ is a functor, and for every arrow $f : X \to Y$ in W, the compositions $X \xleftarrow{f} Y \xrightarrow{f} X$ and $Y \xrightarrow{f} X \xleftarrow{f} Y$ are the identities.

$\square$

Corollary 1.1.7. *Every localizer (C, W) admits a Gabriel-Zisman localization in the sense of (1.1.5).*

Terminology: Given a localizer (C, W), we denote by $\mathsf{Ho}_W C$ the category $W^{-1}C$. The category $\mathsf{Ho}_W C$ is called the category of homotopy types of (C, W), and the functor $\gamma : C \to \mathsf{Ho}_W C$ is called the localizing functor. Given a small category I, we denote by $\mathsf{Ho}_W C(I)$ the category $W_I^{-1}C(I)$, and by $\gamma_I : C \to \mathsf{Ho}_W C(I)$ the associated localizing functor.

Example 1.1.8. Any category C is a category of homotopy types, since we have the trivial localizer $(C, Iso(C))$ and $\mathsf{Ho}_{Iso(C)} C \simeq C$.

Remark 1.1.9. The Gabriel-Zisman localization of an arbitrary localizer (C, W) may not be a locally small category, even if the category C is locally small. Yet, if W is a *set* (not a class), of arrows in C, and C is a locally small category, then the Gabriel-Zisman localization $W^{-1}C$ is actually a locally small category (from *Proposition 5.2* of [40]). We going to impose conditions on the localizer (C, W), supposing that C is a locally small category, in order to obtain a locally small category $W^{-1}C$.

Definition 1.1.10. *A localizer (C, W) will be called locally small if for each small category I, the homotopy category $\mathsf{Ho}_W C(I)$ is locally small.*

Definition 1.1.11. *Let C be a locally small category. A reflective localization of C is a functor*

$$\gamma : C \longrightarrow D$$

which admits a right adjoint that is faithful fully. We say that a localization as above is exact when C and D both admit finite projective limits, and the functor γ is exact, i.e., γ commutes with finite projective limits.

The following proposition characterizes reflective localizations and shows that all reflective localization is a case of localization of some localizer.

Proposition 1.1.12. *Let* $\gamma : C \to C'$ *be a functor and* W *be the class of arrows* f *in* C *such that* $\gamma(f)$ *is invertible in* C'*. Equivalent conditions:*

1. $\gamma : C \to C'$ *is a reflective localization ((1.1.11))*
2. *There exists a functor* $\delta : C' \to C$*, right adjoint to* γ*, and the arrow* $\varepsilon : \gamma\delta \to 1_D$*, co-unit of the adjunction* $\gamma \vdash \delta$*, is a natural isomorphism.*
3. *If* D *is a category and* $u : C \to D$ *is a functor where, for every* $f \in W$*,* $u(f)$ *is invertible in* D*, then, there is a unique functor* $F : C' \to D$ *such that* $u = F \circ \gamma$*.*
4. *For any category* D*, the functor*

$$\underline{Hom}(C', D) \longrightarrow \underline{Hom}(C, D), \quad F \mapsto F \circ u$$

is faithful fully.

Proof. See *Proposition (1.3)* of [3].

□

Example 1.1.13. For each category C, we can associate the set $\pi_0(C)$ of the connected components of C, which can be defined as following: first, take the set $Ob(C)$ of objects of C, and define $\sim$ as the minimal equivalence relation which identifies the pairs (x, y) in $Ob(C)$ such that $Hom_C(x, y) \neq \varnothing$, then define $\pi_0(C) = Ob(C)/\sim$. We can verify easily that the function $C \mapsto \pi_0(C)$ is functorial. Hence, if we restrict the construction of π_0 to the locally small category Cat (defined as the category of small categories), then we can verify that the functor

$$\pi_0 : Cat \longrightarrow Ens$$

is left adjoint to the standard inclusion functor

$$i : Ens \longrightarrow Cat,$$

which associates to each set E, the discrete category $i(E)$ (also denoted by E), where the objects are the elements of E and the only arrows are the identities. Therefore, the category Ens of sets is a reflective localization of the category Cat. Moreover, if W_0 denotes the class of arrows $u : A \to B$ in Cat such that $\pi_0(u) : \pi_0(A) \to \pi_0(B)$ is an isomorphism in Ens, then, it follows from the condition (3) of (1.1.12), that $Ens \simeq (W_0)^{-1}Cat$. We going to denote by Hot_0 the homotopy category $\mathsf{Ho}_{W_0}(Cat)$, called the category of 0-homotopy types.

Example 1.1.14. We can also associate to each *small* category C, the localizer $(C, Fl(C))$, which gives us the homotopy category $\pi_1(C) =_{df} \mathsf{Ho}_{Fl(C)}(C)$. Since the category C is small, the category $\mathsf{Ho}_{Fl(C)}(C)$ is always a locally small category ((1.1.9)), and, hence, a small category, for the the localizing functor $\gamma : C \to \pi_1(C)$ is a bijection on the objects. Moreover, we can verify that the category $\pi_1(C)$ is a groupoid. It follows easily from the universal property of localization, that if $u : C \to D$ is any morphism of small categories, then there exists a unique functor $\pi_1(u) : \pi_1(C) \to \pi_1(D)$ such that the diagram

$$\begin{tikzcd}
C \arrow[r, "u"] \arrow[d] & D \arrow[d] \\
\pi_1(C) \arrow[r, "\pi_1(u)"'] & \pi_1(D)
\end{tikzcd}$$

commutes. From the previous fact, the function $C \mapsto \pi_1(C)$ defines a functor

$$\pi_1 : Cat \longrightarrow Gpd,$$

where Gpd denotes the category of small groupoids. Now, let

$$i : Gpd \longrightarrow Cat$$

be the inclusion of Gpd into Cat. From the canonical natural isomorphisms of categories:

$$\underline{Hom}(\pi_1(C), G) \cong \underline{Hom}_{Fl(C)}(C, G) = \underline{Hom}(C, i(G)),$$

induced from the universal properties of localization, with C (resp. G) being a small category (resp. a small groupoid), we have canonical natural isomorphisms

$$Hom_{Cat}(\pi_1(C), G) \cong Hom_{Gpd}(C, i(G)),$$

which means that π_1 is a left adjoint to i. Hence, it follows again from the condition (3) of (1.1.12), that the functor $\pi_1 : Cat \to Gpd$ is a reflective localization. If we denote by W_1 the class of arrows u in Cat such that $\pi_1(u)$ is an equivalence of groupoids, we obtain, by localization, a category Hot_1, called the category of 1-homotopy types, which is the homotopy category $\mathsf{Ho}_{W_1}Cat$. The latter category is not equivalent to the category *Gpd* of groupoids, for, there are isomorphisms in Hot_1 which are not isomorphisms in *Gpd*. Actually, Hot_1 is the category of groupoids up to equivalence.

Example 1.1.15. A locally small category $\mathcal{E}$ is a Grothendieck topos if it admits finite projective limits and it is an exact reflective localization of a category of presheaves, i.e., there exists a *small* category C and an exact reflective localization of the form

$$\gamma : \widehat{C} \longrightarrow \mathcal{E}.$$

In this case, we say that γ is the associated sheaf functor, and the right adjoint $\imath : \mathcal{E} \to \widehat{C}$ of γ is the associated geometric immersion. From an exact reflective localization as above, we can deduce a Grothendieck topology J on C, defining for each object $x \in Ob(C)$, the set $J(x)$ of sieves $i_R : R \hookrightarrow h_x$ (identified as subfunctors of the representable functor h_x), such that $\gamma(i_R)$ is invertible in $\mathcal{E}$. We call these sieves by covering sieves. Moreover, if we deduce from the previous site (C, J) the correspondent category $Sh(C, J)$ of sheaves over (C, J), which is defined as the full subcategory $Sh(C, J)$ of $\widehat{C}$ formed by the presheaves $F : C^o \to Ens$ such that, given any covering sieve $i_R : R \to h_x$ in $J(x)$, the function

$$Hom_{\widehat{C}}(h_x, F) \longrightarrow Hom_{\widehat{C}}(R, F), \quad f \mapsto f|_R = f \circ i_R$$

is a bijection. We recall the reader the construction of the sheafification functor

$$a : \widehat{C} \longrightarrow Sh(C, J)$$

which turns out to be a left exact adjoint of the inclusion functor $Sh(C, J) \hookrightarrow \widehat{C}$. First, we define using the universal property of localization and a local characterization of the isomorphisms in $Sh(C, J)$, we can proof that $\mathcal{E}$ is canonically equivalent to $Sh(C, J)$. Conversely, if J is any Grothendieck topology on C and $\mathcal{X}$ denotes the category of sheaves for the topology J, then we can also proof that $\mathcal{X}$ is an exact reflective localization of $\widehat{C}$, i.e., $\mathcal{X}$ is a Grothendieck topos. From the previous reasons, we can present a Grothendieck topos both as an exact reflective localization of a presheaf category or as a category of sheaves over some site (as it was originally defined in [1]). For more details we indicate the references [1].

Definition 1.1.16. *Let (C, W) be a localizer.*

1. *We say that (C, W) is weak saturated if it satisfies the following conditions:*
   - *(a) All identity arrow of C belongs to W.*
   - *(b) If in a commutative triangle of C, two of the morphisms are in W then the same holds for the third morphism.*
   - *(c) If $r : X' \to X$ and $i : X \to X'$ are two arrows of C, where $r \circ i = Id_{X'}$ and $i \circ r \in W$, then $r \in W$*
2. *We say that (C, W) is strong saturated if the localizing functor*

$$\gamma : C \longrightarrow W^{-1}C$$

is conservative for W, i.e., for every arrow f in C, $f \in W$ if, and only if, $\gamma(f)$ is an isomorphism in $W^{-1}C$.

3. *We say that a localizer (C, W) is reflective, if the localizing functor $\gamma : C \to W^{-1}C$ admits a right adjoint $i : W^{-1}C \to C$ which is faithful fully.*

Lemma 1.1.17. *Every strong saturated localizer is a weak saturated localizer.*

Proof. Indeed, let (C, W) be a strong saturated localizer and $\gamma : C \to W^{-1}C$ be the respective localizing functor. Clearly, all identities are W-equivalences, and it follows easily from the functoriality of γ (and the definition of strong saturated localizers) that W satisfies also the conditions (b) and (c) of (1) in (1.1.16).

□

Lemma 1.1.18. *Every reflective localizer is strong saturated.*

Proof. It is immediate from (1.1.12) and from the definition (1.1.5).

□

Definition 1.1.19. *A morphism of localizers, say, from (C, W) to (C', W'), is a functor $F : C \to C'$ preserving weak equivalences, i.e., if $f \in W$, then $F(f) \in W'$.*

Notation - Given two localizers (C, W) and (C', W') we denote by

$$\underline{Hom}((C, W), (C', W'))$$

the full subcategory of $\underline{Hom}(C, C')$ formed by the objects which are morphisms of localizers.

1.1.20. It follows from the definition (1.1.19) that if $F : (C, W) \to (C', W')$ is a morphism of localizers, then the composed functor

$$C \xrightarrow{\Phi} C' \xrightarrow{\gamma'} \mathsf{Ho}_{W'}(C')$$

sends the arrows in W to isomorphisms in $\mathsf{Ho}_{W'}(C')$. Therefore, by the universal property of the localization $\gamma : C \to \mathsf{Ho}_W(C)$, there exists an unique functor

$$\overline{F} : \mathsf{Ho}_W(C) \longrightarrow \mathsf{Ho}_{W'}(C')$$

such that the diagram

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "\gamma"'] & C' \arrow[d, "\gamma'"] \\
\mathsf{Ho}_W(C) \arrow[r, "\overline{F}"'] & \mathsf{Ho}_{W'}(C')
\end{tikzcd}$$

is commutative. Moreover, we have a functor

$$\underline{Hom}((C, W), (C', W')) \longrightarrow \underline{Hom}_W(C, \mathsf{Ho}_{W'}C'), \quad F \mapsto \gamma' \circ F, \quad f \mapsto \gamma' \star f,$$

given by composition, and, from the universal property of localization, we also have an isomorphism of categories:

$$\underline{Hom}(\mathsf{Ho}_W C, \mathsf{Ho}_{W'}C') \longrightarrow \underline{Hom}_W(C, \mathsf{Ho}_{W'}C'), \quad \Phi \mapsto \Phi \circ \gamma, \quad f \mapsto f \star \gamma.$$

Therefore, given an object F (resp. an arrow f) of $\underline{Hom}((C, W), (C', W'))$, there exists a unique object $\overline{F}$ (resp. a unique arrow $\overline{f}$) in $\underline{Hom}(\mathsf{Ho}_W C, \mathsf{Ho}_{W'}C')$, such that $\overline{F} \circ \gamma = F \circ \gamma$ and $\gamma' \star f = \overline{f} \star \gamma$, which means that we have a functor:

$$\underline{Hom}((C, W), (C', W')) \longrightarrow \underline{Hom}(\mathsf{Ho}_W C, \mathsf{Ho}_{W'}C'), \quad F \mapsto \overline{F}, \quad f \mapsto \overline{f}.$$

1.1.21. If (C, W) is a localizer, then, for each arrow of small categories $u : I \to J$, the inverse image functor

$$u^* : C(J) \longrightarrow C(I), \quad F \mapsto F \circ u^o$$

is a morphism of localizers, for, if $f \in W_J$, then $f_j \in W$ for all $j \in Ob(J)$, which implies that, $u^*(f)_i = f_{u^o(i)} \in W$ for all $i \in Ob(I)$, i.e., $u^*(f) \in W_I$. Then, the inverse image functor u^* induces a canonical functor:

$$u^* : \mathsf{Ho}_W C(J) \longrightarrow \mathsf{Ho}_W C(I),$$

also denoted by u^*, such that the diagram

$$\begin{tikzcd}
C(J) \arrow[r, "u^*"] \arrow[d, "\gamma_J"'] & C(I) \arrow[d, "\gamma_I"] \\
\mathsf{Ho}_W C(J) \arrow[r, "u^*"'] & \mathsf{Ho}_W C(I)
\end{tikzcd}$$

commutes, where γ_I and γ_J denotes the respective localizing functors. Note that, if $\alpha : u \to v$ is a natural transformation between functors from I to J, then we can form the natural transformation $\alpha^* : v^* \to u^*$, which assigns to each object F of $C(J)$, the morphism $\alpha^*(F) : v^*(F) \to u^*(F)$ in $C(I)$, with $\alpha^*(F)_i =_{df} F(\alpha_i) : F(v(i)) \to F(u(i))$.

1.1.22. If (C, W) is a localizer, then the localizing functor $\gamma : C \to \mathsf{Ho}_W C$ induces, for each category D, an isomorphism of categories:

$$\underline{Hom}(\mathsf{Ho}_W C, D) \longrightarrow \underline{Hom}_W(C, D), \quad \Phi \mapsto \Phi \circ \gamma, \quad \alpha \mapsto \alpha \star \gamma$$

where $\alpha \star \gamma$ is the usual horizontal composition (again, a precise definition of this composition is given in (1.2.1)). Now, suppose that $F : C \to C'$ and $G : C' \to C$ are adjoint functors, with $F \vdash G$, where (C, W) and (C', W') are two localizers. Suppose also that $F(W) \subseteq W'$ and $G(W') \subseteq W$, i.e., F and G are both morphisms of localizers. With the above notations we have a canonical commutative squares

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "\gamma"'] & C' \arrow[d, "\gamma'"] \\
\mathsf{Ho}_W C \arrow[r, "\overline{F}"'] & \mathsf{Ho}_W C'
\end{tikzcd}$$

and

$$\begin{tikzcd}
C' \arrow[r, "G"] \arrow[d, "\gamma"'] & C \arrow[d, "\gamma"] \\
\mathsf{Ho}_{W'} C' \arrow[r, "\overline{G}"'] & \mathsf{Ho}_W C,
\end{tikzcd}$$

induced from the universal property of localization. Hence, we have the commutative diagrams

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "\gamma"'] & C' \arrow[r, "G"] \arrow[d, "\gamma'"] & C \arrow[d, "\gamma"] \\
\mathsf{Ho}_W C \arrow[r, "\overline{F}"'] & \mathsf{Ho}_W C' \arrow[r, "\overline{G}"'] & \mathsf{Ho}_W C,
\end{tikzcd}$$

and

$$\begin{tikzcd}
C' \arrow[r, "G"] \arrow[d, "\gamma'"'] & C \arrow[r, "F"] \arrow[d, "\gamma"] & C' \arrow[d, "\gamma'"] \\
\mathsf{Ho}_{W'} C' \arrow[r, "\overline{G}"'] & \mathsf{Ho}_W C \arrow[r, "\overline{F}"'] & \mathsf{Ho}_W C',
\end{tikzcd}$$

and the co-unit $\varepsilon : F \circ G \to 1_{C'}$ and unit $\eta : 1_C \to G \circ F$ arrows of the adjunction $F \vdash G$, inducing, then, canonical arrows $\overline{\varepsilon} : \overline{F} \circ \overline{G} \to 1$ and $\overline{\eta} : 1 \to \overline{G} \circ \overline{F}$. With the previous notations, we can verify that $\overline{F} \vdash \overline{G}$, with $\overline{\varepsilon}$ and $\overline{\eta}$ being respectively the co-unit and unit arrows. We recall from (1.1.20), that $\overline{\varepsilon}$ (resp. $\overline{\eta}$) is the unique arrow in $\underline{Hom}(\mathsf{Ho}_{W'}C', \mathsf{Ho}_{W'}C')$ (resp. in $\underline{Hom}(\mathsf{Ho}_W C, \mathsf{Ho}_W C)$) such that $\gamma' \star \varepsilon = \overline{\varepsilon} \star \gamma'$ (resp. $\gamma \star \eta = \overline{\eta} \star \gamma$). Hence, it follows from the general formalism of localizations that, if $\varepsilon_{c'} \in W'$ and $\eta_c \in W$ for every $c' \in Ob(C')$ and $c \in Ob(C)$, then $\overline{\eta}$ and $\overline{\eta}$ are natural isomorphisms, and the functors $\overline{F}$ and $\overline{G}$ are quasi-inverse one each other, implying in an equivalence of categories:

$$\mathsf{Ho}_W C \simeq \mathsf{Ho}_{W'} C'.$$

We remark that this last statement does not necessarily assume that $F \vdash G$, but only the existence of natural transformations $\varepsilon : F \circ G \to 1$ and $\eta : G \circ F \to 1$ such $\varepsilon_{c'} \in W'$ and $\eta_c \in W$ for all $c' \in Ob(C')$ and $c \in Ob(C)$.

Definition 1.1.23. *Let (C, W) and (C', W) be two localizers. An equivalence $(F, G, \varepsilon, \eta)$ between (C, W) and (C', W') is given by the following data:*

(i). A functor $F : C \to C'$ such that $F(W) \subseteq W'$

(ii). A functor $G : C' \to C$ such that $G(W') \subseteq W$.

(iii). Two natural transformations $\varepsilon : F \circ G \to 1$ and $\eta : 1 \to G \circ F$.

(iv). For every $x \in Ob(C)$, the arrow

$$\varepsilon_x : GF(x) \longrightarrow x$$

is a W-equivalence.

(v). For every $x' \in Ob(C')$, the arrow

$$\eta_{x'} : x' \longrightarrow FG(x')$$

is a a W'-equivalence.

Proposition 1.1.24. *Let (C, W) and (C', W') be two localizers and $(F, G, \varepsilon, \eta)$ be an equivalence between (C, W) and (C', W') according to (1.1.23) . Then, for every small category I, the induced functors*

$$\overline{F}_I : \mathsf{Ho}_W C(I) \longrightarrow \mathsf{Ho}_{W'} C'(I)$$

and

$$\overline{G}_I : \mathsf{Ho}_{W'} C'(I) \longrightarrow \mathsf{Ho}_W C(I)$$

are equivalence of categories, quasi-onverse one each other.

Proof. Under the hypothesis of the proposition, for each small category I, we can form a quadruple $(F_I.G_I, \varepsilon_I, \eta_I)$, where $\varepsilon_I : F_I \circ G_I \to 1$ and $\eta_I : 1 \to G_I \circ F_I$ are natural transformations, defined by the formulas $(\varepsilon_I)_{X',i} =_{df} FG(X'_i) \to X'_i$ and $(\eta_I)_{X,i} =_{df} X_i \to GF(X_i)$, for $i \in Ob(I)$, $X' \in Ob(C'(I))$, and $X \in Ob(C(I))$. With the previous notations, $(\varepsilon_I)_{X'} \in W'_I$ and $(\eta_I)_X \in W_I$ for all $X' \in Ob(C'(I))$ and $X \in Ob(C(I))$, which means that the conditions of (1.1.22) are satisfied, and hence, $\overline{F}_I$ and $\overline{G}_I$ are equivalences of categories, quasi-inverse one each other.

□

Proposition 1.1.25. *Let* (C, W) *and* (C', W') *be two localizers. Then, the canonical functor*

$$\gamma \times \gamma' : C \times C' \longrightarrow \mathsf{Ho}_W C \times \mathsf{Ho}_{W'} C'$$

is a localization of $C \times C'$ *for the class of arrows* $W \times W'$*, and hence,*

$$\mathsf{Ho}_{W \times W'}(C \times C') \simeq \mathsf{Ho}_W C \times \mathsf{Ho}_{W'} C'.$$

Proof. It follows from the following isomorphisms of categories:

$$\begin{aligned}
\underline{Hom}(W^{-1}C \times (W')^{-1}C', D) &\cong \underline{Hom}(W^{-1}C, \underline{Hom}((W')^{-1}C', D)) \\
&\cong \underline{Hom}(W^{-1}C, \underline{Hom}_W(C', D)) \\
&\cong \underline{Hom}_W(C, \underline{Hom}_W(C', D)) \\
&\cong \underline{Hom}_{W \times W'}(C \times C', D) \\
&\cong \underline{Hom}((W \times W')^{-1}(C \times C'), D)
\end{aligned}$$

□

Proposition 1.1.26. *Let* (C, W) *be a localizer such that* C *admits finite products (resp. finite coproducts),* W *contains all the identities and is stable by products of arrows, i.e., if* $f : X \to Y$ *and* $f' : X' \to Y'$ *are two arrows in* W*, then the canonical arrow* $f \times f' : X \times X' \to Y \times Y'$*, induced by the universal property of products, is also in* W*. Then, the localizing functor* $\gamma : C \to \mathsf{Ho}_W C$ *commutes with finite products.*

Proof. The fact that C admits finite products means that the diagonal functor

$$\Delta_C : C \longrightarrow C \times C, \quad X \mapsto (X, X), \quad f \mapsto (f, f)$$

admits a right adjoint

$$\Pi : C \times C \longrightarrow C, \quad (X, Y) \mapsto X \times Y, \quad (f, g) \mapsto f \times g$$

Since we can always verify that $\Delta(W) \subseteq W \times W$, and under the hypothesis that W is stable by finite products, we have $\Pi(W \times W) \subseteq W$, the, it follows from (1.1.22) that the adjunction $\Delta \vdash \Pi$ induces an adjunction $\overline{\Delta} \vdash \overline{\Pi}$ between the categories $W^{-1}C$ and $(W \times W)^{-1}(C \times C)$, and, since $(W \times W)^{-1}(C \times C) \simeq W^{-1}C \times W^{-1}C$, we have that $W^{-1}C$ admits finite products, and it follows from the commutative square

$$\begin{tikzcd}
C \times C \arrow[r, "\Pi"] \arrow[d, "\gamma\times\gamma"'] & C \arrow[d, "\gamma"] \\
W^{-1}C \times W^{-1}C \arrow[r, "\overline{\Pi}"'] & W^{-1}C
\end{tikzcd}$$

that γ commutes with binary products.

□

In the following, we give a brief exposition about model categories in the sense of Quillen, contemplating the main results of this theory, and we indicate [4] as a major reference.

1.1.27. Let C be a category. Given a pair $i : A \to B$ and $p : X \to Y$ of arrows in C, we say that i has the left lift property with respect to p, or, equivalently, that p has the right lift property with respect to i, if for any commutative square of the form

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "b"'] & Y
\end{tikzcd}$$

there exists an arrow $h : B \to X$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "b"'] \arrow[ur, "h"] & Y
\end{tikzcd}$$

commutes. For a class $\mathbf{A}$ of arrows in C, we denote by $l(\mathbf{A})$ (resp. $r(\mathbf{A})$) the class of arrows which has the left (resp. right) lift property with respect to the all arrows in $\mathbf{A}$.

1.1.28. Let C be a category an $f : X \to Y$ be an arrow of C. A retract of f is a second arrow $f' : X' \to Y'$ in C such that we rave a commutative diagram of the form

$$\begin{tikzcd}
X \arrow[r, "i'"] \arrow[d, "f"'] & X' \arrow[r, "r'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y \arrow[r, "i"'] & X' \arrow[r, "r"'] & Y
\end{tikzcd}$$

such that $r'i' = 1_X$ and $ri = 1_Y$. We say that a class of arrows F in C is stable by retracts, if every retract of an arrow in F is also an element of F.

1.1.29. Let C be a category and F be a class of arrows in C. We say that F is stable by direct images, if for every co-cartesian square [7] of the form

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "i'"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

in C, the relation $f \in F$ implies the relation $f' \in F$.

1.1.30. A class of arrows F in a category C is said to be stable by transfinite compositions if for every functor $\Phi : \kappa \to C$ where κ is an ordinal, the following conditions are verified:

1. The inductive limits
$$\varinjlim_{i<\alpha} \Phi_i$$
are representable in C for all $\alpha \in \kappa$.

2. If the canonical arrows
$$\varinjlim_{i<\alpha} \Phi_i \longrightarrow \Phi_\alpha,$$
induced by the universal property of inductive limits, are in F for every $\alpha \in \kappa$, then the canonical morphism
$$\Phi_0 \longmapsto \varinjlim_{\alpha\in\kappa} \Phi_\alpha$$

---

[7] We recall that by the term cartesian (resp. co-cartesian) square, we mean a pullback (resp. pullshout) square.

is also in F.

In particular, if F is stable by transfinite compositions, then F is stable by compositions, taking $\kappa = 2$. If we take $\kappa = \omega$ and a functor $\Phi : \omega \to C$, then we can rewrite the previous conditions as following: if the arrow $\Phi_n \to \Phi_{n+1}$ is an element of F for every $n \in \omega$, then the canonical morphism

$$\Phi_0 \longrightarrow \varinjlim_{n \geq 0} \Phi_n$$

is an element of F.

Definition 1.1.31. *A class of arrows F in a category C is said to be* saturated *if it is stable by retracts, direct images and transfinite compositions.*

Example 1.1.32. Let I be the be the singleton $\{\varnothing \to e\}$ in the category of sets, where e denotes the terminal object in *Ens*. Then, $r(I)$ is the class of all surjections on *Ens*, and $l(r(I))$ is the class of all injections in *Ens*. We say that an injective function is a *cofibration* in the category of sets.

Example 1.1.33. Let *Top* be the category of topological spaces, I be the class of immersions $S^{n-1} \hookrightarrow D^n$, $n \geq 1$, where $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$, and J be the class of arrows $[0,1]^{n-1} \times \{0\} \to [0,1]^n$, $n \geq 1$. Then, $r(J)$ is called the class of Serre fibrations, and $l(r(I))$ is called the class of Serre cofibrations of topological spaces.

Definition 1.1.34. *Let C be a category. A factorization system in C consists in two classes of arrows $(\mathbf{A}, \mathbf{B})$ of C, such that the following conditions are verified:*

(i) $\mathbf{A}$ *and* $\mathbf{B}$ *are both stable by retracts.*

(ii) $\mathbf{A} \subseteq l(\mathbf{B})$ *(or, equivalently,* $\mathbf{B} \subseteq r(\mathbf{A})$*).*

(iii) *Every arrow f of C can be factorized as $f = pi$, where $i \in \mathbf{A}$ and $p \in \mathbf{B}$.*

It follows from the definition (1.1.34) that $\mathbf{A} = l(\mathbf{B})$ and $\mathbf{B} = r(\mathbf{A})$, i.e., $\mathbf{B}$ is necessarily determined by the class $\mathbf{A}$ and vice-versa. Indeed, the previous assertion follows easily from the fact that given any commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

in an arbitrary category C, if $p \in r(f)$ (resp. $p \in l(q)$), then p is a retract of q (resp. of f). Moreover, we can verify that the class $\mathbf{A}$ is necessarily stable by retracts ((1.1.28)), direct images ((1.1.29)) and by transfinite composition ((1.1.30)) [8].

Definition 1.1.35. *A* Quillen model category *is a quadruple* (C, W, Cof, Fib) *where,* Cof *(resp.* Fib*) is a class of arrows of* C*, called the class of* cofibrations *(resp.* fibrations*), and the following axioms are verified:*

(CM1) C *admits finite projective limits and finite inductive limits.*

(CM2) W *is a class of weak equivalences in* C*.*

(CM3) The pairs $(Cof \cap W, Fib)$ *and* $(Cof, Fib \cap W)$ *are weak systems of factorizations in* C*.*

An object X *in* C *is called* fibrating *(resp.* cofibrating*) if the unique arrow* $\varnothing_C \to X$ *(resp.* $X \to *$*) from the initial (resp. terminal) object* $\varnothing_C$ *(resp.* $*$*) of* C *to* X*, is a fibration (resp. cofibration). An object is* cofibrating-fibrating *if it is both fibrating and cofibrating. We denote by* $\mathsf{Ho}_W C$ *the localization* $W^{-1}C$*, and call* $\mathsf{Ho}_W C$ *the* homotopy category *of the model category* (C, W, Cof, Fib)*.*

Definition 1.1.36. *A localizer* (C, W) *is called a Quillen localizer if there exists two classes of arrows* $\mathbf{A}, \mathbf{B} \subset Fl(M)$*, such that* $(C, W, \mathbf{A}, \mathbf{B})$ *is a Quillen model category. A closed Quillen localizer* (C, W) *is a Quillen localizer such that* C *is complete and co-complete. The structure* $(C, W, \mathbf{A}, \mathbf{B})$ *is called a model category structure on the localizer* (C, W)*.*

Lemma 1.1.37. *Every Quillen localizer is strong saturated.*

---

[8] Actually, given any class F of arrows in a category C, we have that $l(F)$ is always stable by retracts, direct images and transfinite compositions.

Proof. See [3], Chap I.5, *Proposition 1.*

□

1.1.38. Let (C, W, Cof, Fib) be a model category. Given an object X of C, we have by the system of factorization $(Cof, Fib \cap W)$ (resp. $(Cof \cap W, Fib)$), that the unique existent arrow $i_X : \varnothing_C \to X$ (resp. $p_X : X \to e_C$) from the initial object $\varnothing_C$ of C to X (resp. from X to the terminal object e_C of C), factors through a cofibration $i_{X_c} : \varnothing_C \to X_c$ and a weak equivalence $\alpha_X : X_c \to X$ (resp. factors through a weak equivalence $\alpha_X : X \to X_f$ and a fibration $p_{X_f} : X_f \to e_C$). With the previous notations, the object X_c (resp. X_f) is cofibrant (resp. fibrant). Now, using the system of factorization $(Cof \cap W, Fib)$, the unique existent arrow $p_{X_c} : X_c \to e_C$, from X_c to the terminal object e_C of C, factors through a weak equivalence $X_c \to X_{cf}$ (which is also a cofibration) and a fibration $X_{cf} \to e_C$. Therefore, we have a cofibration $\varnothing_C \to X_c \to X_{cf}$ and a fibration $X_{cf} \to e_C$, and a diagram

$$\begin{tikzcd}
X_c \arrow[r, "\alpha_X"] \arrow[d, "\beta_{X_c}"'] & X \\
X_{cf} &
\end{tikzcd}$$

where α_X and β_{X_c} are both weak equivalences, i.e., X is isomorphic to a cofibrant-fibrant object X_{cf} in the homotopy category $\mathsf{Ho}_W C$.

Lemma 1.1.39. *Let (C, W, Cof, Fib) be a model category and $F : C \to C'$ be a functor from C to a second category C'. Suppose that there exists a class W' of weak equivalences in C' according to the definition (1.1.4). If F sends trivial cofibrations (resp. trivial fibrations) between cofibrant (resp. fibrant objects) to W'-equivalences, then F sends weak equivalence between cofibrant (resp. fibrant) objects to W'-equivalences.*

Proof. We only proof the case for cofibrant objects, since the case for fibrant objects is analogous. Let $f : X \to Y$ be a weak equivalence in C, with X and Y being both cofibrant. We can form the co-cartesian square

$$\begin{tikzcd}
\varnothing_C \arrow[r, "i_Y"] \arrow[d, "i_X"'] & Y \arrow[d, "j"] \\
X \arrow[r, "i"'] & X \amalg Y
\end{tikzcd}$$

from where we conclude that i and j are both cofibrations, and, in particular, $X \amalg Y$ is cofibrant. Factorizing the arrow $(f, 1_Y) : X \amalg Y \to Y$, induced by the universal property of coproducts, in a cofibration $k : X \amalg Y \to Z$ followed by a trivial fibration $q : Z \to Y$ (since $(Cof, Fib \cap W)$ is a system of factorization in C), we deduce the existence of the commutative triangles:

$$\begin{tikzcd}
X \arrow[rr, "k\circ i"] \arrow[dr, "f"'] & & Z \arrow[dl, "q"] \\
& Y &
\end{tikzcd}$$

and

$$\begin{tikzcd}
Y \arrow[rr, "k\circ j"] \arrow[dr, "1_Y"'] & & Z \arrow[dl, "q"] \\
& Y &
\end{tikzcd}$$

which implies that $k\circ i$ and $k\circ j$ are both trivial cofibrations between cofibrant objects, and hence, $F(k \circ i) \in W'$ and $F(k \circ j) \in W'$, and since $F(1_Y) = 1_{F(Y)} \in W'$, we have that $F(q) \in W'$, from where we conclude that $F(f) \in W'$ (because W' is a class of weak equivalences in C').

□

Theorem 1.1.40. *Let (C, W, Cof, Fib) be a model category. We designate by C_c (resp. C_f, C_{cf}) the full subcategory of C formed by the cofibrant (resp. fibrant, cofibrant-fibrant) objects and $i : C_c \hookrightarrow C$ (resp. $i : C_f \hookrightarrow C$, $i : C_{cf} \hookrightarrow C$) be the inclusion functor. Then, $i : C_c \hookrightarrow C$ induces an eqivalence of categories*

$$\overline{i} : \mathsf{Ho}_{W_c} C_c \longrightarrow \mathsf{Ho}_W C,$$

(resp.

$$\overline{i} : \mathsf{Ho}_{W_f} C_f \longrightarrow \mathsf{Ho}_W C, \qquad \overline{i} : \mathsf{Ho}_{W_{cf}} C_{cf} \longrightarrow \mathsf{Ho}_W C$$

where $W_c = Fl(C_c) \cap W$ (resp. $W_f = Fl(C_f) \cap W$, $W_{cf} = Fl(C_{cf}) \cap W$).

Proof. It follows from (1.1.38) that we can choose, for each object X of C, a cofibrant-replacement $\alpha_X : X' \to X$, where X' is cofibrant and α_X is a weak equivalence, in order to define a functor $F : C \to \mathsf{Ho}_{W_c} C_c$, such that $F(X) = X'$. In particular, F invert the weak equivalences in C, which implies in the existence of a unique functor $\overline{F} : \mathsf{Ho}_W C \to \mathsf{Ho}_{W_c} C_c$ such that the diagram

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "\gamma"'] & C_c \arrow[d, "\gamma_c"] \\
\mathsf{Ho}_W C \arrow[r, "\overline{F}"'] & \mathsf{Ho}_{W_c} C_c
\end{tikzcd}$$

commutes. With the previous notations, we can verify that there exists an isomorphism of functors of the form $i \circ F \to \gamma$, induced from the structural weak equivalences $\alpha_X : X' \to X$, since $X' = i \circ F(X)$ and $X = \gamma(X)$. On the other hand, the inclusion functor $i : C_c \to C$ also induces a canonical functor

$$\bar{i} : \mathsf{Ho}_{W_c} C_c \longrightarrow \mathsf{Ho}_W C$$

such that the diagram

$$\begin{tikzcd}
C_c \arrow[r, "i"] \arrow[d, "\gamma_c"'] & C \arrow[d, "\gamma"] \\
\mathsf{Ho}_{W_c} C_c \arrow[r, "\bar{i}"'] & \mathsf{Ho}_W C
\end{tikzcd}$$

commutes, and we can now verify easily that $\overline{F}$ and $\bar{i}$ are quasi-inverse one each other. For the case of fibrant and cofibrant-fibrant categories C_f and C_{cf}, it's enough to reply the same argument using respectively the fibrant and cofibrant-fibrant replacements presented in (1.1.38).

□

Definition 1.1.41. *A congruence $\sim$ in a category C, is an equivalence relation on the arrows of C which is compatible with the identities and composition law of C, i.e., if f, f', g and g' are four arrows of C, such that $dom(g) = codom(f)$, $dom(g') = codom(f')$, $f \sim f'$ and $g \sim g'$, then $g \circ f \sim g' \circ f'$.*

1.1.42. Given an object X of a model category (C, W, Cof, Fib), the canonical codiagonal arrow $\nabla_X : X \amalg X \to X$, defined from the universal property of coproducts as the unique arrow from $X \amalg X$ to X such that the diagram

$$\begin{tikzcd}
X \arrow[dr] \arrow[drr, "1_X"] & & \\
& X \amalg X \arrow[r, "\nabla_X"] & X \\
X \arrow[ur] \arrow[urr, "1_X"'] & &
\end{tikzcd}$$

commutes, admits a factorization

$$X \amalg X \xrightarrow{(\partial_0,\partial_1)} IX \xrightarrow{\quad\sigma\quad} X$$

where $(\partial_0, \partial_1) : X \amalg X \to IX$ is a cofibration and $\sigma : IX \to X$ is a trivial fibration (in virtue of the axiom (CM3) of the definition (1.1.35)). We say that an obejct IX satisfying the previous condition is a cylinder of X. An object X in C may admit several different cylinders, and they are not isomorphic in general. Yet, they are all weak equivalent to X, and we can define a formal left-homotopy relation $\sim_l$ on the arrows of C as following: we say that a pair of parallel arrows $f, g : X \to Y$ are left-homotopic, and write $f \sim_l g$, if there exists a cylinder IX of X, and an arrow $h : IX \to Y$, such that the diagram

$$\begin{tikzcd}
X \arrow[d, "\partial_0"'] \arrow[dr, "f"] & \\
IX \arrow[r, "h"] & Y \\
X \arrow[u, "\partial_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes. On the other hand, the the diagonal arrow $\triangle_X : X \to X \times X$, which is the unique arrow from X to $X \times X$ such that the diagram

$$\begin{tikzcd}
& & X \\
X \arrow[r, "\triangle_X"] \arrow[urr, "1_X"] \arrow[drr, "1_X"'] & X \times X \arrow[ur] \arrow[dr] & \\
& & X
\end{tikzcd}$$

commutes, admits a factorization

$$X \xrightarrow{\quad s \quad} X^I \xrightarrow{(d^0,d^1)} X \times X$$

where s is a weak equivalence and (d^0, d^1) is a fibration (again by the axiom (CM3) of (1.1.35)). An object X^I satisfying the previous condition is called a path space of X. Again, an object X of C may admit several different path spaces, and they are not isomorphic in general, but they are all weak equivalent to X and we can define a formal right-homotopy relation $\sim_r$ on the arrows of C in the following way: given two parallel arrows $f, g : A \to X$ in C, we say that f is right-homotopic to g, and write $f \sim_r g$, if there exists a path space X^I of X and an arrow $h : A \to X^I$ such that the diagram

$$\begin{tikzcd}
& X \arrow[d, leftrightarrow, "d^0"] \\
A \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & X^I \arrow[d, "d^1"] \\
& X
\end{tikzcd}$$

commutes.

Lemma 1.1.43. *If $f, g : A \to X$ are two parallel arrows in a model category (C, W, Cof, Fib), where A is cofibrant and X is fibrant, then $f \sim_r g$ if, and only if, $f \sim_l g$.*

Proof. Choose a cylinder

$$A \amalg A \xrightarrow{(\partial_0,\partial_1)} IA \xrightarrow{\quad \sigma \quad} A$$

for A and a path space

$$X \xrightarrow{\quad s \quad} X^I \xrightarrow{(d^0,d^1)} X \times X$$

for X, and suppose that $f \sim_l g$ for this choose cylinder. Then, we can form the commutative square

$$\begin{tikzcd}
A \arrow[r, "sg"] \arrow[d, "\partial_1"'] & X^I \arrow[d, "(d^0,d^1)"] \\
IA \arrow[r, "(h,g\sigma)"'] & X \times X,
\end{tikzcd}$$

where $h : IA \to X$ is a left homotopy from f to g according to (1.1.42). Now, it follows from the fact that A is cofibrant, that $\partial_1 : A \to IA$ is a trivial cofibration. In fact, we have, by definition, that $\sigma\partial_1 = 1_A$, and since σ and 1_A are both weak equivalences, then ∂_1 is also a weak equivalence (from the property 2 out of 3 of weak equivalences). Moreover, $\partial_1 = (\partial_0, \partial_1) \circ i_1$, where $i_1 : A \to A \amalg A$ is the canonical inclusion arrow, which is a cofibration from the co-cartesian square:

$$\begin{tikzcd}
\varnothing_C \arrow[r, "i_A"] \arrow[d, "i_A"'] & A \arrow[d, "i_0"] \\
A \arrow[r, "i_1"'] & A \amalg A,
\end{tikzcd}$$

for, i_A is a cofibration. Hence, ∂_1 is a cofibration, because it is the composition of two cofibrations. From the system of factorization $(Cof \cap W, Fib)$, there exists an arrow $l : IA \to X^I$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "sg"] \arrow[d, "\partial_1"'] & X^I \arrow[d, "(d^0,d^1)"] \\
IA \arrow[r, "(h,g\sigma)"'] \arrow[ur, "l"] & X \times X,
\end{tikzcd}$$

commutes. Define $h' = l\partial_0 : A \to X^I$. Then,

$$d^0 h' = d^0 l \partial_0 = h\partial_0 = f$$

and

$$d^1 h' = d^1 l \partial_0 = g\sigma\partial_0 = g 1_A = g$$

which means that $h' : A \to X^I$ is a right-homotopy from f to g. Therefore, if $f \sim_l g$, then $f \sim_r g$. The reciprocal follows by duality considering the fact that X is fibrant.

$\square$

Notation - Given two arrows parallel arrows $f, g : A \to X$ from a cofibrant to a fibrant object in a model category, we say that f is homotopic to g, and write $f \sim g$, if $f \sim_l g$ and/or $f \sim_r g$. The relation $\sim$ is called the homotopy relation.

Lemma 1.1.44. *Let A and X be respectively a cofibrant and a fibrant object in a model category (C, W, Cof, Fib). Then, the homotopy relation $\sim$ is an equivalence relation on the set $Hom_C(A, X)$.*

Proof. The reflexivity is immediate from the definition of the homotopy relation, and the symmetry follows directly from (1.1.43). For the transitivity, supose that $f \sim f'$ and $f' \sim f''$, where f, f' and f'' are arrows from A to X. Choose a cylinder

$$A \amalg A \xrightarrow{(\partial_0, \partial_1)} IA \xrightarrow{\sigma} A$$

for the relation $f \sim f'$, and a second cylinder

$$A \amalg A \xrightarrow{(\partial'_0, \partial'_1)} I'A \xrightarrow{\sigma'} A$$

for the relation $f' \sim f''$, and define $I''A =_{df} I'A \amalg_A IA$ in order to have a co-cartesian square

$$\begin{tikzcd} A \arrow[r, "\partial_0"] \arrow[d, "\partial'_1"'] & IA \arrow[d, "\lambda_0"] \\ I'A \arrow[r, "\lambda_1"'] & I''A. \end{tikzcd}$$

We can verify that $I''A$ is also a cylinder of A. In fact, defining the arrows $\partial''_0 =_{df} \lambda_0 \circ \partial_0 : A \to I''A$ and $\partial''_1 =_{df} \lambda_1 \circ \partial'_1 : A \to I''A$, we deduce (from the universal property of coproducts) the existence of a unique arrow

$$(\partial''_0, \partial''_1) : A \amalg A \longrightarrow I''A$$

satisfying the canonical commutativity conditions of coproducts. Moreover, from the strucutal arrows $\sigma : IA \to A$ and $\sigma' : I'A \to A$, we can also deduce (using the universal property of co-cartesian squares) the existence of a canonical arrow $\sigma'' : I''A \to A$, and using the stability properties of cofibrations and trivial fibrations, we can verify that there is a cylinder:

$$A \amalg A \xrightarrow{(\partial_0'',\partial_1'')} I''A \xrightarrow{\sigma''} A.$$

Now, from the left homotopies $h : IA \to X$ and $h' : I'A \to X$, respectively related to the relations $f \sim f'$ and $f' \sim f''$, we can derive the existence of a third homotopy $h'' : I''A \to X$ (again, using the universal property of co-cartesian squares) which realizes the relation $f \sim f''$.

□

Notation - Given a cofibrant object A and a fibrant object X in a model category (C, W, Cof, Fib), we define the symbol $[A, X] =_{df} Hom_C(A, X)/\sim$ to indicate the quotient of the set $Hom_C(A, X)$ by the homotopy equivalence relation $\sim$.

We remark that the homotopy relation $\sim$ defines a congruence in the full subcategory C_{cf} of cofibrant-fibrant objects of a model category (C, W, Cof, Fib). We denote by $[C_{cf}]$ the quotient category of C_{cf} with respect to the homotopy congruence $\sim$.

Theorem 1.1.45. *(Quillen) Let (C, W, Cof, Fib) be a Quillen model category. For every cofibrant object A, and for every fibrant object X, there exists a natural bijection of the form*

$$[A, X] \longrightarrow Hom_{\mathsf{Ho}_W C}(A, X).$$

In particular, if C_{cf} denotes the full subcategory of C formed by the cofibrat-fibrat objects, then there exists a functor

$$\gamma : C \longrightarrow [C_{cf}],$$

which exhibits $[C_{cf}]$ as the localization of (C, W), i.e., $[C_{cf}] \simeq \mathsf{Ho}_W C$, where $[C_{cf}]$ is the quotient category of C_{cf} with respect to the congruence $\sim$.

Proof. For the first assertion, see Chap I, Th. 1', p. 1.13 of [4]. The second assertion follows from the former by choosing a cofibrant-fibrant replacement functor $\gamma : C \to \mathsf{Ho}_{W_{cf}} C_{cf}$ as in the proof of (1.1.40), since for every pair (A, X) of objects in $\mathsf{Ho}_{W_{cf}} C_{cf}$, we have a natural bijection

$[A, X] \cong Hom_{\mathsf{Ho}_{W_{cf}} C_{cf}}(A, X)$ from the first assertion, which means that $\mathsf{Ho}_{W_{cf}} C_{cf} \simeq [C_{cf}]$, and it follows from (1.1.40) that the inclusion functor $i : C_{cf} \hookrightarrow C$ induces an equivalence of categories $\mathsf{Ho}_{W_{cf}} C_{cf} \simeq \mathsf{Ho}C$, from where we conclude that there exists a functor $\gamma : C \to [C_{cf}]$ which exhibits $[C_{cf}]$ as the localization of the localizer (C, W).

□

Corollary 1.1.46. *Every Quillen localizer is locally small.*

Definition 1.1.47. *Let (C, W) be a localizer and $F : C \to D$ be a functor from C to a second category D. A left (resp. right) derived functor of F (it if exists) is a functor $\mathsf{L}F : \mathsf{Ho}_W C \to D$ (resp. $\mathsf{R}F : \mathsf{Ho}_W C \to D$), endowed with a natural transformation $\eta : \mathsf{L}F \circ \gamma \to F$ (resp. $\varepsilon : F \to \mathsf{R}F \circ \gamma$) satisfying the following universal property: for any other functor $G : \mathsf{Ho}_W C \to D$, endowed with a natural transformation $\eta_G : G \circ \gamma \to F$ (resp. $\varepsilon_G : F \to G \circ \gamma$), there exists a unique natural transformation $\alpha_G : G \to \mathsf{L}F$ (resp. $\beta_G : \mathsf{R}F \to G$), such that $\eta_G = \eta \circ (\alpha_G \star \gamma)$ (resp. $\varepsilon_G = (\beta_G \star \gamma) \circ \varepsilon$).*

Definition 1.1.48. *Let (C, W) and (C', W') be two localizers and $F : C \to C'$ be a functor from C to C'. We say that F admits a total left (resp. right) derived functor, if the functor $C \xrightarrow{F} C' \xrightarrow{\gamma'} \mathsf{Ho}_{W'} C'$ admits a left (resp. right) derived functor, corresponding to the square*

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "\gamma"'] & C' \arrow[d, "\gamma'"] \\
\mathsf{Ho}_W C \arrow[r, "\mathsf{L}F"'] & \mathsf{Ho}_{W'} C'
\end{tikzcd}$$

(resp. to the square

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "\gamma"'] & C' \arrow[d, "\gamma'"] \\
\mathsf{Ho}_W C \arrow[r, "\mathsf{R}F"'] & \mathsf{Ho}_{W'} C'.
\end{tikzcd}$$

In this case, there exists a canonical natural transformation $\eta : \mathsf{L}F \circ \gamma \to \gamma' \circ F$ (resp. $\varepsilon : \gamma' \circ F \to \mathsf{R}F \circ \gamma$).

Lemma 1.1.49. *If $F : (C, W) \to (C', W')$ is a morphism of localizers, then it admits both a left $\mathsf{L}F$ and a right $\mathsf{R}F$ total derived functor, and $\mathsf{L}F \cong \overline{F} \cong$*

$\mathsf{R}F$, where $\overline{F} : \mathsf{Ho}_W C \to \mathsf{Ho}_{W'} C'$ is the canonical functor induced from the universal property of localization.

Proof. It's immediate from the universal property of localization and from the definitions (1.1.47) and (1.1.48), since we have a commutative square:

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "\gamma"'] & C' \arrow[d, "\gamma'"] \\
\mathsf{Ho}_W C \arrow[r, "\overline{F}"'] & \mathsf{Ho}_{W'} C'.
\end{tikzcd}$$

□

Definition 1.1.50. *Let (C, W, Cof, Fib) and (C', W', Cof', Fib') be two model categories. A functor $T : C \to C'$ is called a left (resp. right) Quillen functor if it commutes with inductive (resp. projective) limits and satisfies the relations $T(Cof) \subseteq Cof'$ and $T(Cof \cap W) \subseteq Cof' \cap W'$ (resp. $T(Fib) \subseteq Fib'$ and $T(Fib \cap W) \subseteq Fib' \cap W'$).*

Proposition 1.1.51. *Under the same hypothesis of (1.1.50), let $F : C \to C'$ and $G : C' \to C$ be two functors such that $F \vdash G$. The following conditions are equivalent and define the notion of Quillen adjunction:*

1. *F is a left Quillen functor.*
2. *G is a right Quillen functor.*

Proof. Suppose that $F \vdash G$. If Φ and Ψ are two classes of arrows respectevely in C and C', then $\Psi \subseteq r(F(\Phi))$ if, and only if, $\Phi \subseteq l(G(\Psi))$. The equivalence of the proposition follows immediately from this previous fact and from the definition (1.1.50).

□

Notation - If $F \vdash G$ is an adjunction satisfying any one of the equivalent conditions of (1.1.51), we use the notation $F \vdash_Q G$, to indicate that the pair (F, G) forms a Quillen adjunction.

Theorem 1.1.52. *(Quillen) Let (C, W, Cof, Fib), (C', W', Cof', Fib') and (C'', W'', Cof'', Fib'') be three model categories.*

1. *If $F : C \to C'$ is left (resp. right) Quillen functor, then F admits a total left (resp. right) derived functor $\mathsf{L}F : \mathsf{Ho}_W(C) \to \mathsf{Ho}_{W'}(C')$ (resp. $\mathsf{R}F : \mathsf{Ho}_W(C) \to \mathsf{Ho}_{W'}(C')$.*
2. *If $F : C \to C'$ and $G : C' \to C''$ are two left (resp. right) Quillen functors, then $\mathsf{L}G \circ \mathsf{L}F \cong \mathsf{L}(GF)$ (resp. $\mathsf{R}(GF) \cong \mathsf{R}G \circ \mathsf{R}F$).*
3. *If $F : C \to C'$ and $G : C' \to C$ form a Quillen adjunction, with $F \vdash_Q G$, then $\mathsf{L}F \vdash \mathsf{R}G$.*

Proof. The first assertion is a consequence of (1.1.39). Indeed, the restriction of the functor $F : C \to C'$ to the cofibrant objects of C, defines a functor $F_c : C_c \to C'$ which respect the weak equivalences (since F is a left Quillen functor), and hence, induces a unique functor $\overline{F}_c : \mathsf{Ho}(C_c) \to \mathsf{Ho}(C')$ such that the diagram

$$\begin{tikzcd}
C_c \arrow[r, "F_c"] \arrow[d, "\gamma_c"'] & C' \arrow[d, "\gamma'"] \\
\mathsf{Ho}(C_c) \arrow[r, "\overline{F}_c"'] & \mathsf{Ho}(C')
\end{tikzcd}$$

commutes. Now, choosing for each object X of C, a cofibrant-replacement X_c of X in C, with a weak equivalence $\alpha_X : X_c \to X$, we can define $\mathsf{L}F(X) = F(X_c)$, and since the inclusion functor $i : C_c \to C$ determines an equivalence of categories $\mathsf{Ho}(C_c) \simeq \mathsf{Ho}(C)$ (by (1.1.40)), we can define the left-derived functor $\mathsf{L}F : \mathsf{Ho}(C) \to \mathsf{Ho}(C')$.

The second assertion is a formal consequence of the the definition of the left (resp. right) derived functors and from the trivial fact that left (resp. right) derived functors are stable under composition.

For the third assertion, we have to verify the existence of a natural isomorphism:

$$Hom_{\mathsf{Ho}(C)}(X, \mathsf{R}G(Y)) \cong Hom_{\mathsf{Ho}(C')}(\mathsf{L}F(X), Y),$$

with respect to the variables X and Y. In order to proof the existence of the above natural isomorphism, we can suppose, without any lost of generality (in virtue of (1.1.38) and (1.1.45)), that X and Y are both cofibrant-fibrant, from where we deduce, using the canonical natural isomorphisms

$$Hom_C(X, G(Y)) \cong Hom_{C'}(F(X), Y),$$

relative to the adjunction $F \vdash G$, the natural isomorphisms:

$$[X, G(Y)] \cong [F(X), Y]$$

where $[X, G(Y)]$ and $[F(X), Y]$ are the respective quotients of the sets

$$Hom_C(X, G(Y)), \qquad Hom_{C'}(F(X), Y)$$

by the Quillen congruence of the theorem (1.1.45) [9]. Since $\mathsf{L}F(X) = X$ and $\mathsf{R}G(Y) = Y$, because X is cofibrant and Y is fibrant, then we have a canonical isomorphism:

$$Hom_{\mathsf{Ho}(C)}(X, \mathsf{R}G(Y)) \cong Hom_{\mathsf{Ho}(C')}(\mathsf{L}F(X), Y).$$

□

Definition 1.1.53. *A localizer (C, W) is called a left (resp. right) ideal Quillen localizer, if the following properties are verified:*

1. *C is complete (resp. co-complete).*
2. *For every small category I, the pair $(C(I), W_I)$ is a Quillen localizer.*
3. *For every functor between small categories $u : I \to J$, the functor $u^* : C(J) \to C(I)$ preserves cofibrations (resp. fibrations).*
4. *For every functor between small categories $u : A \to B$ and every $b \in Ob(B)$, the functor $\zeta(u, b) : A/b \to A$ (resp. $\xi(u, b) : b \setminus A \to A$) preserves fibrations (resp. cofibrations).*

We say that (C, W) is an ideal localizer if it is both a left and a right ideal localizer.

Theorem 1.1.54. *Let (C, W) be a left (resp. right) ideal Quillen localizer.*

1. *$\mathsf{Ho}_W C(e) \simeq e$.*

---

[9] We remark that here, the naturality of the isomorphism is a consequence of the fact that the Quillen equivalence relation of the theorem (1.1.45) is in fact a congruence, i.e., it respects the law of composition of the subjacent category.

2. *For every pair* (I, J) *of small categories, the canonical functor*

$$(\imath^*, \jmath^*) : \mathsf{Ho}_W C(I \amalg J) \longrightarrow \mathsf{Ho}_W C(I) \times \mathsf{Ho}_W C(J)$$

   induced from the inclusions $\imath : I \to I \amalg J$ *and* $\jmath : J \to I \amalg J$*, is an equivalence of categories.*

3. *If* A *is a small category and* $f : X \to Y$ *is an arrow in* $\mathsf{Ho}_W(A)$ *such that* $i^*_{A,a}(f)$ *is an isomorphism in* $\mathsf{Ho}_W C$ *for every* $a \in Ob(A)$*, then* f *is an isomorphism in* $\mathsf{Ho}_W C(A)$

4. *For every functor between small categories* $u : I \to J$*, the functor* $u^* : \mathsf{Ho}_W C(J) \to \mathsf{Ho}_W C(I)$ *admits a right (resp. left) adjoint* $\mathsf{R}u_* : \mathsf{Ho}_W(I) \to \mathsf{Ho}_W C(J)$ *(resp.* $\mathsf{L}u_! : \mathsf{Ho}_W C(I) \to \mathsf{Ho}_W C(J)$*).*

5. *For every functor* $u : A \to B$ *between small categories and every* $a \in Ob(B)$*, we have a canonical isomorphism* $i^*_{A,a}\mathsf{R}u_* \cong \mathsf{R}(p_{A/b})_*\zeta(u, b)^*$ *(resp.* $\mathsf{L}(p_{a\backslash B})_!\xi(u, b)^* \cong i^*_{A,a}\mathsf{L}u_!$*).*

Proof. 1. First, note that every arrow in $\mathsf{Ho}_W C(\varnothing)$ is an element of $W_\varnothing$, otherwise, there should be an arrow f in $\mathsf{Ho}_W C(\varnothing)$ which is not a pointwise W-equivalence, which means that f_a is not a W-equivalence for some $a \in \varnothing$, contradicting the fact that there is no elements in $\varnothing$. Now, since $C(\varnothing) \simeq e$, then $\mathsf{Ho}_W C(\varnothing) \simeq e$.

2. It is an easy formal consequence of (1.1.25).

3. Let $f : X \to Y$ be an arrow in $\mathsf{Ho}_W C(A)$. Since (C, W) is a left (or right) ideal Quillen localizer, it follows from (1.1.45) the existence of two full subcategories $C(A)_{cf} \hookrightarrow C(A)$ and $C_{cf} \hookrightarrow C$, of congruences $\sim_A$ and $\sim$, respectively on the arrows of $C(A)$ and C, and functors $\gamma_A : C(A) \to [C(A)_{cf}]$ and $\gamma : C \to [C_{cf}]$, such that γ_A (resp. γ) exhibits $[C(A)_{cf}]$ (resp. C_{cf}) as a localization of the localizer $(C(A), W_A)$ (resp. (C, W)), i.e., $[C(A)_{cf}] \simeq \mathsf{Ho}_W C(A)$ (resp. $[C_{cf}] \simeq \mathsf{Ho}_W C$). Hence, we can suppose that X and Y are both objects of $C(A)_{cf}$ (from (1.1.38)) and that f represents the equivalence class $[\phi]$ of an arrow $\phi : X \to Y$ in $C(A)$, i.e., $f = \gamma_A(\phi)$. Now, given $a \in Ob(A)$, we have a canonical commutative square induced from the universal property of localiza-

tion:

$$\begin{tikzcd}
C(A) \arrow[r, "i^*_{A,a}"] \arrow[d, "\gamma_A"'] & C \arrow[d, "\gamma"] \\
\mathsf{Ho}_W C(A) \arrow[r, "i^*_{A,a}"'] & \mathsf{Ho}_W(C),
\end{tikzcd}$$

which implies that $i^*_{A,a}(f) = i^*_{A,a}\gamma_A(\phi) = \gamma a^*(\phi) = \gamma(\phi_a)$. Then, to say that $i^*_{A,a}(f)$ is an isomorphism is equivalent to say that $\gamma(\phi_a)$ is an isomorphism, which, by it's turn, is equivalent to the assertion that $\phi_a \in W$ (from (1.1.37)), from where we conclude that $i^*_{A,a}(f)$ is an isomorphism if, and only if, $\phi_a \in W$. But, if $i^*_{A,a}(f)$ is an isomorphism for every $a \in Ob(A)$, then $\phi_a \in W$ for every $a \in Ob(A)$, which implies that $\phi \in W_A$, and, hence, $f = \gamma_A(\phi)$ is an isomorphism in $\mathsf{Ho}_W C(A)$.

4. Since (C, W) is a left (resp. right) ideal Quillen localizer, it's immediate from the condition (2) of (1.1.53) that for every arrow $u : I \to J$ of small categories, the functor $u^* : C(J) \to C(I)$ is a left (resp. right) Quillen functor, and the adjunction $u^* \vdash u_*$ (resp. $u_! \vdash u^*$) is actually a Quillen adjunction ((1.1.51)). Moreover, $u^* : C(J) \to C(I)$ is a morphism of localizers, and hence, admits both a left and a right derived functor, given by $\mathsf{L}u^* \cong \mathsf{R}u^* \cong u^* : \mathsf{Ho}_W C(J) \to \mathsf{Ho}_W C(I)$. Then, it follows from (1) of (1.1.52), the existence of a total right (resp. left) Quillen functor $\mathsf{R}u_*$ (resp. $\mathsf{L}u_!$), and from (3) of (1.1.52), it follows that $u^* \cong \mathsf{L}u^* \vdash \mathsf{R}u_*$ (resp. $\mathsf{L}u_! \vdash \mathsf{R}u^* \cong u^*$).

5. Let $u : A \to B$ be an arrow of small categories and $b \in Ob(B)$. Consider the commutative square

$$\begin{tikzcd}
A/b \arrow[r, "\zeta(u,b)"] \arrow[d, "p_{A/b}"'] & A \arrow[d, "u"] \\
e \arrow[r, "i_{B,b}"'] & B
\end{tikzcd}$$

(resp.

$$\begin{tikzcd} b \backslash A \arrow[r, "\xi(u,b)"] \arrow[d, "p_{b\backslash A}"'] & A \arrow[d, "u"] \\ e \arrow[r, "i_{B,b}"'] & B. \end{tikzcd}$$

Since (C, W) is a left (resp. right) ideal Quillen localizer, the category C is complete (resp. co-complete), which implies the canonical isomorphisms $i^*_{B,b}u_* \cong (p_{A/b})_*\zeta(u,b)^*$ (resp. $(p_{b\backslash A})_!\xi(u,b)^* \cong i^*_{B,b}u_!$) [10]. With the previous notations, we proof that we also have a canonical isomorphism

$$(i_{B,b})^*\mathsf{R}u_* \cong \mathsf{R}(p_{A/b})_*\zeta(u,b)^*$$

(resp.

$$\mathsf{L}(p_{b\backslash A})_!\xi(u,b)^* \cong (i_{B,b})^*.$$

First, consider the commutative square

$$\begin{tikzcd} B/b \arrow[r, "\zeta(1_B,b)"] \arrow[d, "p_{B/b}"'] & B \arrow[d, "1_B"] \\ e \arrow[r, "i_{B,b}"'] & B \end{tikzcd}$$

(resp.

$$\begin{tikzcd} b \backslash B \arrow[r, "\xi(1_B,b)"] \arrow[d, "p_{B/b}"'] & B \arrow[d, "1_B"] \\ e \arrow[r, "i_{B,b}"'] & B. \end{tikzcd}$$

It follows respectively from the conditions (3) and (4) of (1.1.53), that $(p_{A/b})_*$ (resp. $(p_{b\backslash B})_!$) and $\zeta(1_B, b)^*$ (resp. $\xi(1_B, b)^*$) are right (resp. left) Quillen functors, and since $(i_{B,b})^* \cong (p_{A/b})_*\zeta(1_B, b)^*$ (resp. $(i_{B,b})^* \cong (p_{b\backslash B})_!\xi(1_B, b)^*$), and right (resp. left) Quillen functors are stable by composition, $(i_{B,b})^*$ is also a right (resp. left) Quilen functor, and it follows from the fact that $i_{B,b}$ is a morphism of localizers, that $(i_{B,b})^* \cong \mathsf{R}(i_{B,b})^* \cong \mathsf{L}(i_{B,b})^*$. Now, in virtue of the condition (4) of

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[10] See Exposé I of [1] or the section ‘Presheaves’ in the ‘Mathematical History of Homotopy Types ’of this dissertation.

(1.1.53), the functor $\zeta(u,b)^*$ (resp. $\xi(u,b)^*$) is a right (resp. left) Quillen functor, and being $\zeta(u,b)^*$ (resp. $\xi(u,b)^*$) also a morphism of localizers, we have $\zeta(u,b)^* \cong \mathsf{R}\zeta(u,b)^*$ (resp. $\xi(u,b)^* \cong \mathsf{L}\xi(u,b)^*$). Therefore, as consequence of the assertion (2) of (1.1.52), we can verify the following sequence of canonical isomorphisms:

$$\begin{aligned}(i_{B,b})^*\mathsf{R}u_* &\cong \mathsf{R}(i_{B,b})^*\mathsf{R}u_*\\ &\cong \mathsf{R}(i^*_{B,b}u_*)\\ &\cong \mathsf{R}((p_{A/b})_*\zeta(u,b)^*)\\ &\cong \mathsf{R}(p_{A/b})_*\mathsf{R}\zeta(u,b)^*\\ &\cong \mathsf{R}(p_{A/b})_*\zeta(u,b)^*\end{aligned}$$

(resp.

$$\begin{aligned}\mathsf{L}(p_{b\backslash A})_!\xi(u,b)^* &\cong \mathsf{L}(p_{b\backslash A})_!\mathsf{L}\xi(u,b)^*\\ &\cong \mathsf{L}((p_{b\backslash A})_!\xi(u,b)^*)\\ &\cong \mathsf{L}(i^*_{B,b}u_!)\\ &\cong \mathsf{L}(i_{B,b})^*\mathsf{L}u_!\\ &\cong (i_{B,b})^*\mathsf{L}u_!).\end{aligned}$$

which concludes the proof.

□

Now, we have to investigate sufficient conditions for a localizer to be ideal. Clearly, if we suppose that (C, W, Cof, Fib) is a model category such that C is complete and co-complete, and for each small category I, the quadruple $(C(I), W_I, Cof_I, Fib_I)$ is a model category where Cof_I (resp. Fib_I) is the pointwise cofibrations (resp. pointwise fibrations) and $Fib = r(Cof_I \cap W_I)$ (resp. $Cof_I = l(Fib_I \cap W_I)$), then we can verify that (C, W) is a left (resp. right) ideal Quillen localizer. Yet, it is not true in general that we can endow the pair $(C(I), W_I)$ with the I-injective (resp. I-projective) model category structure. The definition (1.1.53) suggests an investigation for techniques to construct model categories and the study of how certain model categories are presented.

1.1.55. Let I be a (small) set of arrows in a category C. We say that I admits the *small object argument* if for every arrow $f : A \to B$ in I, there exists an infinite regular cardinal κ such that the functor

$$Hom_C(A, ?) : C \longrightarrow Ens$$

commutes with inductive limits indexed by κ-filtered well-ordered sets [11]. We note that if C is a category of finite presentation, then every small set of arrows in C admits the small object argument. Given a (small) set I of arrows in C, we denote by $Sat(I)$ the smallest saturated class of arrows which contains I. Since $l(I)$ (the class of arrows which have the left lift property with respect to all arrows in I) is always saturated, we have $I \subset Sat(I) \subset l(r(I))$. If I admits the small object argument, then $Sat(I) = l(r(I))$, which implies that $(Sat(I), r(I))$ is a system of factorization on C according to (1.1.34). In particular, every arrow f in C can be factorized as $f = pi$, where $p \in r(I)$ and $i \in Sat(I)$.

Definition 1.1.56. *A model category (C, W, Cof, Fib) is called cofibrantly generated if there are two small admissible sets of arrows I and J in C such that $Cof = Sat(I)$ and $Cof \cap W = Sat(J)$.*

Remark 1.1.57. For all cofibrantly generated model category (C, W, Cof, Fib), the pair (C, W) is an ideal Quillen localizer (see p. 218, 219, 220 of [16]). Moreover, the class weak equivalences W is precisely the class of arrows f in C which are of the form $f = pi$, with $p \in r(I)$ and $i \in Sat(J) = l(r(J))$. Therefore, if W' is a class of weak equivalences in C ((1.1.4)) such that $J \subset Cof \cap W'$, $Cof \cap W'$ is saturated, and $r(I) \subset W'$, then $W \subseteq W'$. Now, suppose that every object in C is cofibrating, i.e., the canonical arrow $\varnothing_C \to X$ is a trivial cofibration for every object $X \in Ob(C)$. If $p : X \to Y$ is a trivial fibration in C, then there exists an arrow $s : Y \to X$ such that the diagram

$$\begin{tikzcd}
\varnothing_C \arrow[r, "i_X"] \arrow[d, "i_Y"'] & X \arrow[d, "f"] \\
Y \arrow[r, "Id_Y"'] \arrow[ur, "s"] & Y
\end{tikzcd}$$

commutes, which implies that $p \circ s = Id_Y$. Since $i_X, i_Y \in Cof \cap W \subset Cof \cap W'$, then $s \in W'$, and hence, $p \in W'$, because W' is a class of weak equivalences in C. Therefore, if (C, W, Cof, Fib) is a cofibrantly generated

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[11] A well-ordered set $(S, \leq)$ is κ-filtered if it is non-empty and for every subset E of S, with $\mathfrak{c}(E) < \kappa$, there exists an element $s \in S$ such that $x \leq s$ for all $x \in E$.

model category where every object is cofibrating, with $Cof \cap W = Sat(J)$ for some small set of arrows J in C admitting the small object argument, and W' is a class of weak equivalences in C such that, $Cof \cap W'$ is saturated and $J \subset Cof \cap W'$, then $W \subseteq W'$.

For an appreciation of more technical facts about model categories, including accessibility, cofibrantly generated model categories, combinatorial model categories, techniques to construct model categories, and examples, we indicate the reader to see [21].

We end this section providing a list of several important examples of model categories and Quillen localizers.

Example 1.1.58. If (C, W, Cof, Fib) is a model category, then the dual model category (C^o, W^o, Cof^o, Fib^o) is defined as following: let $?^o : C^o \to C$ be the trivial contravairant functor, which is the identity on the objects and sends an arrow $f : x' \to x$ in C^o to the original arrow $f : x \to x'$ in C. Then, define $W^o = \{f \in Fl(C^o) :?^o(f) \in W\}$, $Cof^o = \{f \in Fl(C) :?^o(f) \in Fib\}$ and $Fib^o = \{f \in Fl(C^o) :?^o(f) \in Cof\}$. With the previous notations, we can verify easily that (C, W, Cof, Fib) is also a model category.

Example 1.1.59. If (C, W, Cof, Fib) is a model category, then, for every $X \in Ob(C)$, we can form the relative model category

$$(C/X, W/X, Cof/X, Fib/X),$$

where...

Example 1.1.60. Let Top be the locally small category of topological spaces and W_{Top} be the class of topological weak equivalences, i.e., morphisms of topological spaces $f : X \to Y$ such that

$$\pi_0(f) : \pi_0(X) \longrightarrow \pi_0(Y)$$

is a bijection, where $\pi_0(X)$ is the set of connected path components of X, and

$$\pi_n(f, x) : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$$

is an isomorphism of groups for every $x \in X$ and $n \geq 1$, where $\pi_n(X, x)$ is the n-th homotopy group of the pointed space (X, x). Then (Top, W_{Top}) is a Quillen localizer (see Chap II, section 3, Th. 1 of [4]). The cofibrations and fibrations of the model category structure over Top are the Serre cofibrations and fibrations. Moreover, the objects which are cofibrant-fibrant for this model category structure on Top are the CW-complexes, and every topological space is fibrant.

In order to present the next example, we recall some terminology and notation about simplicial sets. For $0 \leq i \leq n$, $n \geq 1$, let $\delta_n^i : \Delta_{n-1} \to \Delta_n$ be the unique non-decreasing injective function such that $i \notin Im(\delta_n^i)$ and $\sigma_n^i : \Delta_{n+1} \to \Delta_n$ be the unique non-decreasing surjective function such that $(\sigma_n^i)^{-1}(\{i\})$ has exact two elements. Explicitly,

$$\delta_i^n(k) = k \quad \text{for} \quad k < i, \qquad \delta_i^n(k) = k + 1 \quad \text{for} \quad k \geq i,$$

and

$$\sigma_i^n(k) = k \quad \text{for} \quad k \leq i, \qquad \sigma_i^n(k) = k - 1 \quad \text{for} \quad k > i.$$

For each $n \geq 1$, define the simplicial sets

$$\partial\Delta_n =_{df} \bigcup_{0 \leq i \leq n} Im(\delta_n^i)$$

and

$$\Lambda_n^k =_{df} \bigcup_{0 \leq j \leq n, j \neq k} Im(\delta_n^j),$$

and define $\partial\Delta_0 =_{df} \varnothing_{\widehat{\Delta}}$. Hence, we have the canonical inclusions:

$$i_n : \partial\Delta_n \longrightarrow \Delta_n, \quad j_{n,k} : \Lambda_n^k \longrightarrow \Delta_n.$$

A morphism of simplicial sets $f : X \to Y$ is called a Kan fibration if for every commutative square of the form

$$\begin{tikzcd}
\Lambda_n^k \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta_n \arrow[r] & Y
\end{tikzcd}$$

with $n \geq 1$ and $0 \leq k \leq n$, there exists an arrow $h : \Delta_n \to X$ such that the diagram

$$\begin{tikzcd}
\Lambda_n^k \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta_n \arrow[r] \arrow[ur, "h"] & Y
\end{tikzcd}$$

commutes.

Example 1.1.61. Let Δ be the category of standard simplexes and $\widehat{\Delta}$ be the category of simplicial sets. The category $\widehat{\Delta}$ admits a model category structure, discovered by Quillen, such that the weak equivalences are the arrows $f : X \to Y$ in $\widehat{\Delta}$ such that the topological realization $|f| : |X| \to |Y|$ is a weak equivalence of topological spaces, the cofibrations are the monomorphisms, and the fibrations are the Kan fibrations, i.e., the arrows $f : X \to Y$ which have the right lift property with respect to the canonical inclusions $\Lambda_n^k \to \Delta_n$, for $n \geq 1$ and $0 \leq k \leq n$. For a detailed construction of this model category structure over the simplicial sets, see the *Theorem 11.2* and the *Theorem 11.3* (p. 61-80) of [25]. The fibrant objects in this model category structure over the simplicial sets are called Kan complexes. We remark that in virtue of (1.1.40), we have an equivalence of categories $\mathsf{Ho}(Kan) \simeq \mathsf{Ho}(\widehat{\Delta})$, where Kan denotes the full subcategory of $\widehat{\Delta}$ formed by the Kan complexes. Moreover, $\mathsf{Ho}(\widehat{\Delta}) \simeq \mathsf{Ho}(Top)$. It also follows from the remark (1.1.57) that W_s can be characterized as the minimal class of arrows W in $\widehat{\Delta}$ satisfying the following properties:

1. W is a class of weak equivalences.
2. $Cof \cap W$ is saturated.
3. W contains all the arrows $\Delta_n \to \Delta_0$, $n \geq 0$.

Indeed, we can verify by induction that the last condition above implies that $COf \cap W$ contains all the inclusions $j_{n,k} : \Lambda_n^k \to \Delta_n$, $n \geq 1$, $0 \leq k \leq n$, and $Cof \cap W_s$ is precisely the smallest saturated class of arrows in $\widehat{\Delta}$ containing these inclusions.

Remark 1.1.62. The class of all monomorphisms in $\widehat{\Delta}$ is the class

$$Cof = l(r(\{i_n : \partial\Delta_n \to \Delta_n : n \geq 0\}))$$

and the class of anodyne extensions in $\widehat{\Delta}$ is the class

$$\mathsf{An} =_{df} l(r(\{j_{n,k} : \Lambda_n^k \to \Delta_n : n \geq 1, 0 \leq k \leq n\})).$$

With the notations

$$I = \{i_n : \partial\Delta_n \to \Delta_n : n \geq 0\}$$

and

$$J = \{j_{n,k} : \Lambda_n^k \to \Delta_n : n \geq 1, 0 \leq k \leq n\},$$

we can verify that $Cof = Sat(I) = l(r(I))$ and $\mathsf{An} = Sat(J) = l(r(J))$. Moreover, the pairs $(Sat(I), r(I))$ and $(Sat(J), r(J))$ are systems of factorizations over $\widehat{\Delta}$, and $r(J)$ is precisely the class of Kan fibrations. Then, the Quillen model category $(\widehat{\Delta}, W_s, Cof, Fib)$ is actually the quadruple

$$(\widehat{\Delta}, l(r(J)) \circ r(J), l(r(I)), r(J))$$

Where $Fib \cap W_s = r(Cof) = r(l(r(I)).$

1. W is a class of weak equivalences.
2. $Cof \cap W$ is saturated, i.e., it is stable under direct images, transfinite compositions and retracts.
3. For every $n \in \omega$, the canonical $\Delta_n \to \Delta_0$ is an element of W.

Therefore, we can verify that $W_s \subseteq W$. Note that every arrow $f \in W_s$ can be factorized as $f = pi$, where $p \in Cof \cap W_s = l(r(J)) = Sat(J)$ and $i \in Fib = r(J)$. Since $Cof \cap W$ is saturated and $Cof \cap W_s$ is the smallest saturated class which contains J, it is enough to proof that $J \subset W$.

Example 1.1.63. Let A be a small category. Define the class Cof of cofibrations in $\widehat{A}$ as being the class of all monomophisms of presheaves. A trivial fibration in $\widehat{A}$ is an arrow with the right lift property in relation to all cofibrations. A class W of arrows in $\widehat{A}$ is called an A-localizer if it satisfies the following conditions:

1. W is a class of weak equivalences according to (1.1.4).
2. Every trivial fibration is a W-equivalence (i.e., an element of W).

3. The class $Cof \cap W$ is stable by retracts, direct images and transfinite compositions.

We can verify that the intersection of any non-empty family of A-localizers is also an A-localizer. Then, given a (small) set of arrows S in $\widehat{A}$, we can always form the A-localizer $W(S)$ generated by S, which is the intersection of all localizers containing S. By definition, $W(S)$ is the minimal A-localizer W such that $S \subseteq W$. An A-localizer W is called accessible if it is of the form $W(S)$ for some set of arrows S in $\widehat{A}$. One of the main results in the first chapter of [15] (which is highly non-trivial) affirms that if W is an accessible A-localizer, then $(\widehat{A}, W, Cof, Fib)$ is a model category (see *Thèoréme 1.4.3.* of [15]).

Example 1.1.64. Let $\widehat{\Delta}$ be the category of simplicial sets. Define W as the minimal Δ-localizer containing the arrows $\Delta_n \to \Delta_0$ for $n \in \omega$ and Cof as the class of all monomorphisms in $\widehat{\Delta}$. Then, $(\widehat{\Delta}, W, Cof, Fib)$ is a model category, with $Fib = r(Cof \cap W)$, and it follows from [15] that this model category is precisely the Quillen model category structure on simplicial sets exposed in [4] and (1.1.61). Moreover, the homotopy category $\mathsf{Ho}_W\widehat{\Delta}$ is equivalent to the classical homotopy category of CW-complexes [12].

Example 1.1.65. Let *Ab* be the category of abelian groups, *Comp*(*Ab*) be the category of abelian group complexes (in cohomological notation), and W_{qis} be the class of quasi-isomorphisms in *Comp*(*Ab*), i.e., the arrows which induce an isomorphism on the cohomology groups. Then, the pair $(Comp(Ab), W_{qis})$ is also a Quillen localizer (see Proposition 3.13. of [21]).

Remark 1.1.66. Except for the examples (1.1.58) and (1.1.59), if follows from [21] that all the other examples of Quillen localizers we exposed are ideal Quillen localizers in the sense of the definition (1.1.53).

1.2 *Les Prédérivateurs*

Definition 1.2.1. *A prederivator is a strict* 2*-functor of the form*

$$\mathscr{D} : \mathcal{C}at^o \longrightarrow \mathcal{CAT}.$$

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[12] See Chap II of [15] for a detailed study of this model category structure on the category $\widehat{\Delta}$ of simplicial sets.

It consists of the following data:

(i). For each small category A, a category $\mathscr{D}(A)$;

(ii). For each functor between small categories

$$u : A \to B,$$

a functor

$$u^* = \mathscr{D}(u) : \mathscr{D}(B) \longrightarrow \mathscr{D}(A)$$

(iii). For each natural transformation between morphisms of small categories

$$\begin{tikzcd} A \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow \alpha"] & B \end{tikzcd}$$

a natural transformation

$$\begin{tikzcd} \mathscr{D}(B) \arrow[r, bend left, "v^*"] \arrow[r, bend right, "u^*"'] \arrow[r, phantom, "\Downarrow \alpha^*"] & \mathscr{D}(A) \end{tikzcd}$$

such that the following conditions are satisfied:

(PD1). If $u : A \to B$ and $v : B \to C$ are functors between small categories, then $(vu)^* = u^* v^*$. Moreover, for any small category A, $1_{\mathscr{D}(A)} = (1_A)^*$.

(PD2). If $\alpha, \beta \in Fl(\underline{Hom}(A, B))$, say, $\alpha : u \Rightarrow v$ and $\beta : v \Rightarrow w$, then $(\beta\alpha)^* = \alpha^* \beta^*$. Beyond that, for every functor $u : A \to B$, $1_{u^*} = (1_u)^*$. Diagrammatically, $\mathscr{D}$ sends a 2-diagram of the form

$$\begin{tikzcd} A \arrow[r, bend left=50, "u", ""{name=U, below}] \arrow[r, "v" description, ""{name=V}] \arrow[r, bend right=50, "w"', ""{name=W, above}] \arrow[Rightarrow, from=U, to=V, "\alpha"] \arrow[Rightarrow, from=V, to=W, "\beta"] & B \end{tikzcd}$$

to the 2*-diagram*

$$\begin{tikzcd}
\mathscr{D}(B) \arrow[r, bend left=50, "w^*"] \arrow[r, "v^*"'] \arrow[r, bend right=50, "u^*"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\beta^*"] \arrow[r, phantom, shift right=3ex, "\Downarrow\alpha^*"] & \mathscr{D}(A)
\end{tikzcd}$$

in such a way that $(\beta\alpha)^* = \alpha^*\beta^*$.

(PD3). If $\alpha \in Fl(\underline{Hom}(A,B))$ *and* $\beta \in Fl(\underline{Hom}(B,C))$*, then* $(\beta \star \alpha)^* = \alpha^* \star \beta^*$*. Diagrammatically,* $\mathscr{D}$ *sends each* 2*-diagram of the form*

$$\begin{tikzcd}
A \arrow[r, bend left=50, "u"] \arrow[r, bend right=50, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left=50, "w"] \arrow[r, bend right=50, "t"'] \arrow[r, phantom, "\Downarrow\beta"] & C
\end{tikzcd}$$

in the 2*-diagram*

$$\begin{tikzcd}
\mathscr{D}(C) \arrow[r, bend left=50, "t^*"] \arrow[r, bend right=50, "w^*"'] \arrow[r, phantom, "\Downarrow\beta^*"] & \mathscr{D}(B) \arrow[r, bend left=50, "v^*"] \arrow[r, bend right=50, "u^*"'] \arrow[r, phantom, "\Downarrow\alpha^*"] & \mathscr{D}(A)
\end{tikzcd}$$

in such a way that $(\beta \star \alpha)^* = \alpha^* \star \beta^*$.

We clarify the symbol $\star$ employed in (PD3). Let $u, v : I \to J$ and $w, t : J \to K$ be functors, with $\alpha : u \Rightarrow v$ and $\beta : w \Rightarrow t$ being natural transformations. The morphism $\beta\star\alpha : wu \Rightarrow tv$ may be defined as follows: for each $i \in Ob(I)$, there is an arrow $\alpha_i : u(i) \to v(i)$. Applying w and t respectively, we obtain the arrows $w(\alpha_i) : wu(i) \to wv(i)$ and $t(\alpha_i) : tu(i) \to tv(i)$. Then, by the commutativity of the diagram

$$\begin{tikzcd}
wu(i) \arrow[r, "w(\alpha_i)"] \arrow[d, "\beta_{u(i)}"'] & wv(i) \arrow[d, "\beta_{v(i)}"] \\
tu(i) \arrow[r, "t(\alpha_i)"'] & tv(i)
\end{tikzcd}$$

we design $(\beta \star \alpha)_i := \beta_{v(i)} \circ w(\alpha_i) = t(\alpha_i) \circ \beta_{u(i)}$.

Terminology: Let $\mathscr{D}$ be a prederivator. Given a small category I, the obejcts in $\mathscr{D}(I)$ are called coefficients of type $\mathscr{D}$ over I. If $u : I \to J$ is a functor between small categories, we call $u^* : \mathscr{D}(J) \to \mathscr{D}(I)$ the inverse image induced by $\mathscr{D}$. The objects of $\mathscr{D}(e)$ are the absolute coefficients. For every small category I, there is a unique possible functor $p_I : I \to e$, from which we deduce a functor $p_I^* : \mathscr{D}(e) \to \mathscr{D}(I)$. A coefficient F of type $\mathscr{D}$ over I is constant if there exists at least one absolute coefficient M such that $F \cong p_I^*(M)$. We denote by $\mathscr{D}^c(I)$ the full subcategory of $\mathscr{D}(I)$ formed by constant coefficients.

Example 1.2.2. Let $\mathscr{D}$ be a a prederivator. From $\mathscr{D}$ we can define a prederivator $\mathscr{D}^o$, called the dual prederivator of $\mathscr{D}$, by the formula:

$$\mathscr{D}^o(A) = \mathscr{D}(A^o)^o,$$

for A being a small category. If $u : A \to B$ is a morphism between small categories, then we define the inverse image $u^* : \mathscr{D}^o(B) \to \mathscr{D}^o(A)$ relative to $\mathscr{D}^o$ as being the functor $u^* =_{df} ((u^o)^*)^o : \mathscr{D}(B^o)^o \to \mathscr{D}(A^o)^o$. For a natural transformation $\alpha : u \Rightarrow v$ between functors from A to B, we define α^* as being $\alpha^* =_{df} ((\alpha^o)^*)^o$.

Example 1.2.3. If $\mathscr{D}$ is a prederivator and A is a small category, then we can form a prederivator $\mathscr{D}^{A^o}$ such that, for any object I in Cat,

$$\mathscr{D}^{A^o}(I) = \mathscr{D}(A \times I).$$

In fact, for each arrow $u : I \to J$ in Cat, we have the functor

$$1_A \times u : A \times I \longrightarrow A \times J$$

from which we compute the functor

$$(1_A \times u)^* : \mathscr{D}(A \times J) \longrightarrow \mathscr{D}(A \times I).$$

Given a natural transformation

$$\begin{tikzcd} I \arrow[r, bend left, "u"{name=U}] \arrow[r, bend right, "v"'{name=V}] & J \arrow[from=U, to=V, phantom, "\Downarrow \alpha"] \end{tikzcd}$$

we have the evident morphism of functors:

$$1_A \times \alpha : 1_A \times u \longrightarrow 1_A \times v$$

which induces an arrow

$$(1_A \times \alpha)^* : (1_A \times v)^* \longrightarrow (1_A \times u)^*,$$

of functors from $\mathscr{D}(J)$ to $\mathscr{D}(I)$. With the previous notations, we can verify that $\mathscr{D}^{A^o}$ defines a prederivator.

Example 1.2.4. For every locally small category C, we can associate the prederivator $\widetilde{C} : I \mapsto \underline{Hom}(I^o, C)$.

Example 1.2.5. Let (C, W) be a localizer. Then, we can associate a prederivator $\mathbf{Ho}_W C$, which assigns to each small category I, the category

$$\mathbf{Ho}_W C(I) = (W_I)^{-1} \underline{Hom}(I^o, C).$$

Proposition 1.2.6. *Let $\mathscr{D}$ be a prederivator. If $u : A \to B$ and $v : B \to A$ are adjoint functors, with $u \vdash v$, and $\varepsilon : uv \Rightarrow 1_B$ and $\eta : 1_A \Rightarrow vu$ are respectively the arrows of co-unit and unit of this adjunction, then $u^* \vdash v^*$; and $\eta^* : u^* v^* \Rightarrow 1$ and $\varepsilon^* : 1 \Rightarrow v^* u^*$ are respectively the arrows of co-unit and unit of the adjuntion $u^* \vdash v^*$.*

Proof. By (ii) of (1.2.1), the functors $u : A \to B$ and $v : B \to A$ are assigned to functors $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$ and $v^* : \mathscr{D}(A) \to \mathscr{D}(B)$. From (iii) of (1.2.1) and the natural transformations $\varepsilon : uv \Rightarrow 1_A$ and $\eta : 1_B \Rightarrow vu$, we deduce the natural transformations $\varepsilon^* : 1_A^* \Rightarrow (uv)^*$ and $\eta^* : (vu)^* \Rightarrow 1_B^*$. In virtue of (PD1), we have that $(uv)^* = v^* u^*$, $(vu)^* = u^* v^*$, $1_A^* = 1_{\mathscr{D}(A)}$ and $1_B^* = 1_{\mathscr{D}(B)}$. Now, to affirm that $u \vdash v$ is equivalent to say the equalities

$$1_u = (\varepsilon \star 1_u)(1_u \star \eta)$$

and

$$1_v = (1_v \star \varepsilon)(\eta \star 1_v)$$

are verified. Therefore, by (PD2) and (PD3), we obtain the equalities

$$1_{u^*} = (\eta^* \star 1_{u^\star})(1_{u^*} \star \varepsilon^*)$$

and

$$1_{v^\star} = (1_{v^*} \star \eta^*)(\varepsilon^* \star 1_{v^*}),$$

which means that $u^* \vdash v^*$, being respectivilly $\eta^* : u^*v^* \Rightarrow 1_{\mathscr{D}(B)}$ and $\varepsilon^* : 1_{\mathscr{D}(A)} \Rightarrow v^*u^*$ the arrows of co-unit and unit.

□

Corollary 1.2.7. *Let $\mathscr{D}$ be prederivator. If $u : A \to B$ and $v : B \to A$ are two adjoint functors in Cat, with $u \vdash v$, and u (resp. v) is faithful fully, then $v^* : \mathscr{D}(A) \to \mathscr{D}(B)$ (resp. $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$) is faithful fully.*

Proof. To affirm that u (resp. v) is faithful and full, is the same to say that the co-unit (resp. unit) $\varepsilon : uv \Rightarrow 1_B$ (resp. $\eta : 1_A \Rightarrow vu$), of the adjunction $u \vdash v$, is an isomorphism. By (1.2.6), the prederivator $\mathscr{D}$ induces an adjunction $u^* \vdash v^*$, with co-unit $\eta^* : u^*v^* \Rightarrow 1_{\mathscr{D}(B)}$ and unit $\varepsilon^* : 1_{\mathscr{D}(A)} \Rightarrow v^*u^*$. The 2-strict functionality of $\mathscr{D}$ and the fact that ε (resp. η) is an isomorphism, implies that ε^* (resp. η^*) is also an isomorphism, which by it's turn, is equivalent to say that v^* (resp. u^*) is faithful and full.

□

Corollary 1.2.8. *Let $\mathscr{D}$ be prederivator. If $u : A \to B$ is an equivalence of categories, then $u^* : \mathscr{D}(A) \to \mathscr{D}(B)$ is an equivalence of categories.*

Definition 1.2.9. *A morphism $F : \mathscr{D} \to \mathscr{D}'$ of prederivators is a non-strict morphism of 2-functors, i.e., F consists of the following data:*

(i). For each small category A, a functor $F_A : \mathscr{D}(A) \to \mathscr{D}'(A)$.

(ii). For each functor $u : A \to B$ between small categories, an isomorphism

$$\xi_{F,u} : u^*F_B \longrightarrow F_Au^*$$

which corresponds to the square

$$\begin{tikzcd}
\mathscr{D}(B) \arrow[r, "F_B"] \arrow[d, "u^*"'] & \mathscr{D}'(B) \arrow[d, "u^*"] \\
\mathscr{D}(A) \arrow[r, "F_A"'] & \mathscr{D}'(A)
\end{tikzcd}$$

Satisfying the coherence conditions:

1. *$\xi_{F,1_A} = Id_{F_A}$*
2. *For every commutative triangle*

$$\begin{tikzcd}
A \arrow[rr, "u"] \arrow[dr, "w"'] & & B \arrow[dl, "v"] \\
& C &
\end{tikzcd}$$

in Cat, the triangle

$$\begin{tikzcd}
u^*v^*F_C \arrow[rr, "\xi_{F,vu}"] \arrow[dr, "u^*\xi_{F,v}"'] & & F_Au^*v^* \arrow[dl, "\xi_{F,u}v^*"] \\
& u^*F_Bv^* &
\end{tikzcd}$$

commutes.

3. *Given a natural transformation*

$$\begin{tikzcd}
A \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow \alpha"] & B
\end{tikzcd}$$

in Cat, the square

$$\begin{tikzcd}
v^*F_B \arrow[r, "\xi_{F,v}"] \arrow[d, "\alpha^*\star F_B"'] & F_Av* \arrow[d, "F_A\star\alpha^*"] \\
u^*F_B \arrow[r, "\xi_{F,u}"'] & F_Au^*
\end{tikzcd}$$

commutes.

In the third condition of (1.2.9), the first vertical arrow in the commutative square is associated to the diagram

$$\begin{tikzcd}
\mathscr{D}(B) \arrow[r, "F_B"] & \mathscr{D}'(B) \arrow[r, bend left, "v^*"] \arrow[r, bend right, "u^*"'] \arrow[r, phantom, "\Downarrow \alpha^*"] & \mathscr{D}'(A)
\end{tikzcd}$$

and the second vertical arrow is associated to the diagram

$$\begin{tikzcd}
\mathscr{D}(B) \arrow[r, bend left, "v^*"] \arrow[r, bend right, "u^*"'] \arrow[r, phantom, "\Downarrow \alpha^*"] & \mathscr{D}(A) \arrow[r, "F_A"] & \mathscr{D}'(A).
\end{tikzcd}$$

Definition 1.2.10. *Let $\mathscr{D}$ and $\mathscr{D}'$ be two prederivators and $F, G : \mathscr{D} \to \mathscr{D}'$ be two morphisms of prederivators from $\mathscr{D}$ to $\mathscr{D}'$. A 2-morphism $\theta : F \Rightarrow G$, or, a modification, from F to G, is a an application which associates to each small category A, a natural transformations $\theta_A : F_A \to G_A$, such that, given a morphism of small categories $u : A \to B$, the square*

$$\begin{tikzcd}
u^*F_B \arrow[r, "\xi_{F,u}"] \arrow[d, "u^*\star\theta_B"'] & F_Au^* \arrow[d, "\theta_A\star u^*"] \\
u^*G_B \arrow[r, "\xi_{G,u}"'] & G_Au^*
\end{tikzcd}$$

commutes.

In the definition (1.2.10) the first vertical arrow in the commutative square is associated to the diagram

$$\begin{tikzcd}
\mathscr{D}(B) \arrow[r, bend left, "F_B"] \arrow[r, bend right, "G_B"'] \arrow[r, phantom, "\Downarrow \theta_B"] & \mathscr{D}'(B) \arrow[r, "u^*"] & \mathscr{D}'(A).
\end{tikzcd}$$

and the second vertical arrow in the commutative square is associated to the the diagram

$$\begin{tikzcd}
\mathscr{D}(B) \arrow[r, "u^*"] & \mathscr{D}(A) \arrow[r, bend left, "F_A"] \arrow[r, bend right, "G_A"'] \arrow[r, phantom, "\Downarrow \theta_A"] & \mathscr{D}'(A).
\end{tikzcd}$$

Example 1.2.11. If $\mathcal{F} : \mathscr{D} \to \mathscr{D}'$ is a morphism of prederivators, then we can define the dual morphism $\mathcal{F}^o : \mathscr{D}^o \to (\mathscr{D}')^o$. Indeed, for each small category I, we have a functor

$$\mathcal{F}_{I^o} : \mathscr{D}(I^o) \longrightarrow \mathscr{D}'(I^o).$$

Dualizing $\mathcal{F}_{I^o}$, we derive a functor

$$(\mathcal{F}_{I^o})^o : \mathscr{D}(I^o)^o \longrightarrow \mathscr{D}'(I^o)^o$$

which can be written as

$$\mathcal{F}^o_I : \mathscr{D}^o(I) \longrightarrow (\mathscr{D}')^o(I).$$

If $u : I \to J$ is a morphism of small categories, then we have a natural isomorphism $\xi_{\mathcal{F},u^o} : (u^o)^*\mathcal{F}_{J^o} \to \mathcal{F}_{I^o}(u^o)^*$ corresponding to the square

$$\begin{tikzcd}
\mathscr{D}(J^o) \arrow[r, "\mathcal{F}_{J^o}"] \arrow[d, "(u^o)^*"'] & \mathscr{D}'(J^o) \arrow[d, "(u^o)^*"] \\
\mathscr{D}(I^o) \arrow[r, "\mathcal{F}_{I^o}"'] & \mathscr{D}'(I^o)
\end{tikzcd}$$

and satisfying the compatibility properties of (1.2.9), from where we deduce a natural isomorphism $\xi^o_{\mathcal{F},u^o} : ((u^o)^*)^o\mathcal{F}^o_{J^o} \to \mathcal{F}^o_{I^o}((u^o)^*)^o$, which can be written as $\xi_{\mathcal{F}^o,u} : u^*\mathcal{F}^o_J \to \mathcal{F}^o_I u^*$, since the inverse images of the dual prederivators are defined by the formula $u^* = ((u^o)^*)^o$. With the previous notations, we can verify that $\mathcal{F}^o : \mathscr{D}^o \to (\mathscr{D}')^o$ defines a morphism of prederivators.

Example 1.2.12. Let $\mathscr{D}$ be a prederivator and $u : A \to B$ be a morphism of small categories. Then, u induces a morphism of prederivatros of the form $u^\star : \mathscr{D}^{A^o} \to \mathscr{D}^{B^o}$ such that, for each small category I:

$$u^\star_I =_{df} (u \times 1_I)^* : \mathscr{D}^{A^o}(I) = \mathscr{D}(A \times I) \longrightarrow \mathscr{D}(B \times I) = \mathscr{D}^{B^o}(I).$$

Example 1.2.13. Let $\Phi : C \to C'$ be a functor between locally small categories. Then, for each small category I, we have the functor

$$\Phi_I : C(I) \longrightarrow C'(I), \quad X \mapsto \Phi \circ X$$

defined by composition. Clearly, if $u : I \to J$ is a morphism of small categories, then it defines a commutative diagram

$$\begin{tikzcd}
C(J) \arrow[r, "\Phi_J"] \arrow[d, "u^*"'] & C(J) \arrow[d, "u^*"] \\
C(I) \arrow[r, "\Phi_I"'] & C(J),
\end{tikzcd}$$

which implies that Φ defines actually a morphism of prederivators

$$\widetilde{\Phi} : \widetilde{C} \longrightarrow \widetilde{C}'.$$

In fact, we have a canonical equivalence of categories

$$\mathcal{H}om(\widetilde{C}, \widetilde{C}') \longrightarrow \underline{Hom}(C, C'), \quad \mathcal{F} \mapsto \mathcal{F}_e,$$

with quasi-inverse image being the functor:

$$\underline{Hom}(C, C') \longrightarrow \mathcal{H}om(\widetilde{C}, \widetilde{C}'), \quad \Phi \mapsto \widetilde{\Phi}.$$

Note that $\widetilde{\Phi}_e \cong \Phi$ (because $\widetilde{C}(e) \cong C$ and $\widetilde{C}'(e) \cong C'$). Conversely, $\widetilde{\mathcal{F}_{eA}} \cong \mathcal{F}_A$ for every small category A.

Example 1.2.14. Let $\Phi : (C, W) \to (C', W')$ be a morphism of localizers. Then we can also verify that Φ induces a morphism of prederivators $\overline{\Phi} : \mathbf{Ho}_W C \to \mathbf{Ho}_{W'} C'$. In fact, it already follows from (1.2.13) that Φ induces a morphism of derivators

$$\widetilde{\Phi} : \widetilde{C} \longrightarrow \widetilde{C}',$$

and, hence, for each small category I, we have a functor

$$\Phi_I : \underline{Hom}(I^o, C) \longrightarrow \underline{Hom}(I^o, C'),$$

which sends an arrow $f : F \to G$ of $\underline{Hom}(I^o, C)$ to an arrow $\Phi_I f : \Phi_I F \to \Phi_I G$ in $\underline{Hom}(I^o, C')$, such that, for each $i \in Ob(I)$, $\Phi_I f_i = \Phi(f_i) : \Phi(F_i) \to \Phi(G_i)$. Hence, if $f \in W_I$, i.e., $f_i \in W$ for every $i \in Ob(I)$, then $\Phi(f_i) \in W'$ for every $i \in Ob(I)$ (because Φ is a morphism of localizers), which means that $\Phi_I f \in W'_I$. Therefore, by the universal property of localization, there exists a unique functor

$$\overline{\Phi}_I : \mathbf{Ho}_W C(I) \longrightarrow \mathbf{Ho}_{W'} C'(I)$$

such that the diagram

$$\begin{tikzcd}
C(I) \arrow[r, "\Phi_I"] \arrow[d, "\gamma_I"'] & C'(I) \arrow[d, "\gamma'_I"] \\
\mathbf{Ho}_W C(I) \arrow[r, "\overline{\Phi}_I"'] & \mathbf{Ho}_{W'} C'(I)
\end{tikzcd}$$

commutes. Now, using again the universal property of localization, we can verify easily that for each morphism of small categories $u : I \to J$, we have a commutative square of the form

$$\begin{tikzcd}
\mathbf{Ho}_W C(J) \arrow[r, "\overline{\Phi}_J"] \arrow[d, "u^*"'] & \mathbf{Ho}_{W'} C'(J) \arrow[d, "u^*"] \\
\mathbf{Ho}_W C(I) \arrow[r, "\overline{\Phi}_I"'] & \mathbf{Ho}_{W'} C'(I),
\end{tikzcd}$$

which implies that $\overline{\Phi} : \mathbf{Ho}_W C \to \mathbf{Ho}_{W'} C'$ really defines a morphism of prederivators.

Definition 1.2.15. *We say that a morphism of prederivators $F : \mathscr{D} \to \mathscr{D}'$ is an equivalence of prederivators (resp. faithful, full) if for every small category A, the functor $F_A : \mathscr{D}(A) \to \mathscr{D}'(A)$ is an equivalence of categories (resp. faithful, full).*

Example 1.2.16. If $(F, G, \varepsilon, \delta)$ is an equivalence between the localizers (C, W) and (C', W'), then, in virtue of (1.1.23), the the induced morphism of prederivators $\overline{F} : \mathbf{Ho}_W C \to \mathbf{Ho}_{W'} C'$ (see (1.2.5) and (1.2.14)) is an equivalence of prederivators.

1.2.17. Let $\mathscr{D}$ be a prederivator and A be a small category. If $\gamma : \widetilde{A} \to \mathscr{D}$ is a morphism of prederivators, then, evaluating γ at A^o, we obtain a functor

$$\gamma_{A^o} : \widetilde{A}(A^o) \longrightarrow \mathscr{D}(A^o).$$

Since by definition,

$$\widetilde{A}(A^o) = A(A^o) = \underline{Hom}((A^o)^o, A) = \underline{Hom}(A, A),$$

we can define

$$\gamma_\flat =_{df} \gamma_{A^o}(1_A),$$

which is an object in $\mathscr{D}(A^o)$. Reciprocally, if F is an object of $\mathscr{D}(A^o)$, we can form a morphism of prederivators

$$F^\sharp : \widetilde{A} \longrightarrow \mathscr{D}$$

where, given a small category I and $u \in Ob(A(I))$, which is just a functor $u : I^o \to A$, we have

$$F^\sharp_I(u) =_{df} (u^o)^*(F).$$

Theorem 1.2.18. *The functors $\gamma \mapsto \gamma_\flat$ and $F \mapsto F^\sharp$ are quasi-inverse one each other, and determine an equivalence of categories:*

$$\mathcal{H}om(\widetilde{A}, \mathscr{D}) \simeq \mathscr{D}(A^o).$$

Proof. First, let $\gamma : \widetilde{A} \to \mathscr{D}$ be a morphism or prederivators. Note that $(\gamma^\sharp_\flat)_I(u) = (u^o)^*(\gamma_\flat) = (u^o)^*\gamma_{A^o}(1_A)$ for every small category I and every object $u : I^o \to A$ of $\widetilde{A}(I)$. Now, since γ is a morphism of prederivators from $\widetilde{A}$ to $\mathscr{D}$, the dual morphism $u^o : I \to A^o$ of u induces a square

$$\begin{tikzcd}
\widetilde{A}(A^o) \arrow[r, "\gamma_{A^o}"] \arrow[d, "(u^o)^*"'] & \mathscr{D}(A^o) \arrow[d, "(u^o)^*"] \\
\widetilde{A}(I) \arrow[r, "\gamma_I"'] & \mathscr{D}(I)
\end{tikzcd}$$

which commutes up to a caninonical isomorphism $\xi_{\gamma,u^o} : (u^o)^*\gamma_{A^o} \to \gamma_I(u^o)^*$, and 1_A is an object of $\widetilde{A}(A^o)$, from where we conclude that

$$(\gamma^\sharp_\flat)_I(u) = (u^o)^*\gamma_{A^o}(1_A) \cong \gamma_I(u^o)^*(1_A) = \gamma_I(u),$$

from where we deduce an isomorphism $(\gamma_\flat)^\sharp \cong \gamma$. On the other hand, let F be an object of $\mathscr{D}(A^o)$. Then,

$$(F^\sharp)_\flat = F^\sharp_{A^o}(1_A) = (1^o_A)^*(F) = (1_{A^o})^*(F) = F.$$

From the previous arguments, we have that the functors $\gamma \mapsto \gamma_\flat$ and $F \mapsto F^\sharp$ are quasi-inverse one each other.

□

Corollary 1.2.19. *If A is a small category and $\mathscr{D}$ is a prederivator, then we have the equivalences of categories*

$$\mathcal{H}om(A^o, \mathscr{D}) \simeq \mathscr{D}(A) \simeq \mathscr{D}(A \times e) = \mathscr{D}^{A^o}(e),$$

where A^o denotes, with an abuse of notation, the prederivator $\widetilde{A}^o$.

Corollary 1.2.20. *If A and B are two small categories, then we have an equivalence of categories:*

$$\mathcal{H}om(\widetilde{A}, \widetilde{B}) \simeq \underline{Hom}(A, B).$$

Proof. It follows immediatily from (1.2.18) since

$$\widetilde{B}(A^o) = \underline{Hom}((A^o)^o, B) = \underline{Hom}(A, B).$$

□

Remark 1.2.21. The corollary (1.2.19) clarifies the notation $\mathscr{D}^{A^o}$ to indicate the prederivator $I \mapsto \mathscr{D}(A \times I)$ defined in (1.2.3). In fact, the absolute coefficients of the prederivator $\mathscr{D}^{A^o}$, which is isomorphic to $\mathscr{D}(A)$, can be thought as the category of *presheaves over A at values in $\mathscr{D}$*. Note that this terminology extends the case when $\mathscr{D}$ is a prederivator represented by a locally small category C. It also follows from (1.2.20) that the category of small categories is a subcategory of the category of prederivators $\mathcal{PD}er$.

1.3 *Les Dérivateurs*

Definition 1.3.1. *Let $\mathscr{D}$ be a prederivator. We say that $\mathscr{D}$ admits left (resp. right) Kan extensions, or, direct cohomological (resp. homological) images, if for any arrow $u : I \to J$ of diagrams, the inverse image u^* admits a right (resp. left) adjoint, which we also denotes by u_* (resp. $u_!$). In this case, we say that u^* (resp. $u_!$) is the right (resp. left) Kan extension, or direct cohomological (resp. homological) image of u.*

Notation - If $\mathscr{D}$ is a prederivator which admits direct cohomological (resp. homological) images, I is a diagram, and F is an object of $\mathscr{D}(I)$, then, we define the symbols

$$\mathsf{H}^*_{\mathscr{D}}(I; F) =_{df} (p_I)_*(F).$$

$$\mathsf{H}^{\mathscr{D}}_*(I; F) =_{df} (p_I)_!(F).$$

We say that $\mathsf{H}^*_{\mathscr{D}}(I;F)$ (resp. $\mathsf{H}^{\mathscr{D}}_*(I;F)$) is the cohomology (resp. homology) of F with coefficients in I for the prederivator $\mathscr{D}$. If F is an object of $\mathscr{D}(I^o)$, we can also define the projective (resp. inductive) homotopy limit of F, as

$$\mathbf{R}\underset{I}{\underrightarrow{\lim}}\, F =_{df} \mathsf{H}^*_{\mathscr{D}}(I^o;F), \qquad \mathbf{L}\underset{I}{\underrightarrow{\lim}}\, F =_{df} \mathsf{H}^{\mathscr{D}}_*(I^o;F).$$

We remark that the previous definition agrees with the categorical notions of inductive and projective limits when $\mathscr{D}$ is the prederivator associated to a co-complete and complete category according to (1.2.4). If $u : A \to B$ is an arrow of small categories and F is an object of $\mathscr{D}(B)$, we also use the notation $F|_A$ to indicate the inverse image $u^*(F)$ of F in $\mathscr{D}(A)$, and call $F|_A$ the restriction of F to A, while $u_*(M)$ (resp. $u_!(M)$) is the right (resp. left) Kan extension to $\mathscr{D}(B)$, of an object M of $\mathscr{D}(A)$.

Proposition 1.3.2. *A prederivator $\mathscr{D}$ admits direct cohomological images if, and only if, $\mathscr{D}^o$ admits direct homological images.*

Proof. Suppose that $\mathscr{D}$ admits direct cohomological images. Then, for each morphism of small categories $u : I \to J$, we have the respective dual functor $u^o : I^o \to J^o$, and, applying $\mathscr{D}$, we have an adjunction $(u^o)^* \vdash (u^o)_*$. Then, dualizing the functors $(u^o)^* : \mathscr{D}(J^o) \to \mathscr{D}(I^o)$ and $(u^o)_* : \mathscr{D}(I^o) \to \mathscr{D}(J^o)$, we have $((u^o)^*)^o : \mathscr{D}(I^o)^o \to \mathscr{D}(J^o)^o$ and $((u^o)_*)^o : \mathscr{D}(I^o)^o \to \mathscr{D}(J^o)^o$. Since $\mathscr{D}^o(A) = \mathscr{D}(A^o)^o$ for each small category A, and $u^* = \mathscr{D}^o(u) = ((u^o)^*)^o$ for each morphism of small categories $u : I \to J$ (see (1.2.2)), then, we can define $u_! =_{df} ((u^o)_*)^o$. With the previous notations, it follows from the adjunction $(u^o)^* \vdash (u^o)_*$, that $((u^o)_*)^o \vdash ((u^o)^*)^o$, which means that $u_! \vdash u^*$. Conversely, if $\mathscr{D}^o$ admits direct homological images, we can apply the same argument to show that $\mathscr{D}$ admits cohomological images, since $\mathscr{D}(A) = \mathscr{D}^o(A^o)^o$ and $u^* = \mathscr{D}(u) = \mathscr{D}^o(u^o)^o$.

□

1.3.3. Consider a square of categories of the form

$$\begin{tikzcd}
X' & X \arrow[l, "g'"'] \\
Y' \arrow[u, "f'"] & Y \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

with a natural transformation $\theta : f'g \to g'f$, and suppose that r (resp. r') is a right adjoint of f (resp. of f'), with the respective unit and co-unit arrows:

$$\varepsilon : fr \longrightarrow 1, \qquad \eta : 1 \longrightarrow rf$$

and

$$\varepsilon' : f'r' \longrightarrow 1, \qquad \eta' : 1 \longrightarrow r'f'.$$

Then, we can obtain a natural transformation $\widetilde{\theta} : gr \to r'g'$, by the composition:

$$gr \xrightarrow{\eta \star gr} r'f'gr \xrightarrow{r' \star \theta \star r} r'g'fr \xrightarrow{r'g' \star \varepsilon} r'g'$$

Not, let $\mathscr{D}$ be a prederivator which admits direct cohomological images and consider a square of small categories of the form

$$\begin{tikzcd}
A' \arrow[r, "v'"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\
B' \arrow[r, "v"'] & B
\end{tikzcd}$$

with a natural transformation $\alpha : uv' \to vu'$. Then, we can derive a square of categories

$$\begin{tikzcd}
\mathscr{D}(A') & \mathscr{D}(A) \arrow[l, "(v')^*"'] \\
\mathscr{D}(B') \arrow[u, "(u')^*"] & \mathscr{D}(B) \arrow[l, "v^*"] \arrow[u, "u^*"']
\end{tikzcd}$$

with a natural transformation $\alpha^* : (u')^* v^* \to (v')^* u^*$. Since u_* (resp. $(u')_*$) is right adjoint to u^* (resp. to $(u')^*$), then, there exists a natural transformation

$$\widetilde{\alpha}^* : v^* u_* \longrightarrow (u')_* (v')^*,$$

called the canonical base change morphism induced from $\mathscr{D}$. Dually, if we suppose that $\mathscr{D}$ admits direct homological images, then we deduce, from the original square

$$\begin{tikzcd}
A' \arrow[r, "v'"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\
B' \arrow[r, "v"'] & B
\end{tikzcd}$$

endowed with the natural transformation $\alpha : uv' \to vu'$, a canonical base change morphism:

$$\widetilde{\alpha}_! : (u')_!(v')^* \longrightarrow v^* u_!.$$

We say that the original square is $\mathscr{D}$-exact if the canonical base change morphism $\widetilde{\alpha}^*$, in the case when $\mathscr{D}$ admits direct cohomological images (resp. $\widetilde{\alpha}_!$, in the case when $\mathscr{D}$ admits direct homological images) is an isomorphism.

Lemma 1.3.4. *Let $\mathscr{D}$ be a prederivator admitting right (resp. left) Kan extensions and consider a digram of the form*

$$\begin{tikzcd}
A'' \arrow[r, "w'"] \arrow[d, "u''"'] & A' \arrow[r, "v'"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\
B'' \arrow[r, "w"'] & B' \arrow[r, "v"'] & B.
\end{tikzcd}$$

Define $p' = v'w'$ and $p = wv$, and suppose that both the squares

$$\begin{tikzcd}
A' \arrow[r, "v'"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\
B' \arrow[r, "v"'] & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
A'' \arrow[r, "w'"] \arrow[d, "u''"'] & A' \arrow[d, "u'"] \\
B'' \arrow[r, "w"'] & B'
\end{tikzcd}$$

are $\mathscr{D}$-exact. Then, the square

$$\begin{tikzcd}
A'' \arrow[r, "p'"] \arrow[d, "u''"'] & A \arrow[d, "u"] \\
B'' \arrow[r, "p"'] & B
\end{tikzcd}$$

is also $\mathscr{D}$-exact.

Proof. It is enough to proof the lemma for the case when $\mathscr{D}$ admits right Kan extensions (the proof for the case when $\mathscr{D}$ admits right Kan extensions is totally analogous, and the reader can reply exact the same arguments).

Under the hypothesis of the lemma, the canonical base change morphisms

$$v^*u_* \longrightarrow (u')_*(v')^* \qquad w^*(u')_* \longrightarrow (u'')_*(w')^*$$

defined in (1.3.3), are both isomorphisms. Now, consider the canonical base change morphism

$$p^*u_* \longrightarrow (u'')_*(p')^*.$$

From the following sequence of *canonical* isomorphisms

$$\begin{aligned} p^*u_* &\cong (vw)^*u_* \\ &\cong w^*v^*u_* \\ &\cong w^*(u')_*(v')^* \\ &\cong (u'')_*(w')^*(v')^* \\ &\cong (u'')_*(v'w')^* \\ &\cong (u'')_*(p')^* \end{aligned}$$

we conclude that $p^*u_* \to (u'')_*(p')^*$ is in fact an isomorphism.

□

1.3.5. Let $\mathscr{D}$ and $\mathscr{D}'$ be two prederivators admitting direct cohomological images and $\mathcal{F} : \mathscr{D} \to \mathscr{D}'$ be a morphism of prederivators from $\mathscr{D}$ to $\mathscr{D}'$. If $u : I \to J$ is a morphism of small categories, then, we have a square of categories

$$\begin{tikzcd} \mathscr{D}'(I) & \mathscr{D}(I) \arrow[l, "\mathcal{F}_I"'] \\ \mathscr{D}'(J) \arrow[u, "u^*"] & \mathscr{D}(J) \arrow[l, "\mathcal{F}_J"] \arrow[u, "u^*"'] \end{tikzcd}$$

with a natural isomorphism $\xi_{\mathcal{F},u} : u^*\mathcal{F}_J \to \mathcal{F}_I u^*$ according to (1.2.9). Since the inverse images u^* admit right adjoints u_*, it follows from the same argument of (1.3.3), the existence of a canonical arrow

$$\widetilde{\xi}_{\mathcal{F},u} : \mathcal{F}_J u_* \longrightarrow u_*\mathcal{F}_I.$$

Dually, if $\mathscr{D}$ and $\mathscr{D}'$ both admit direct homological images, then, there exists a canonical arrow

$$\widetilde{\xi}_{\mathcal{F},u} : u_!\mathcal{F}_I \longrightarrow \mathcal{F}_J u_!.$$

In particular, if we consider a small category A and the canonical functor $p_A : A \to e$, from A to the terminal category e, then, we have for each object F in $\mathscr{D}(A)$, a canonical arrow

$$\widetilde{\xi}_{\mathcal{F},p_A} : \mathcal{F}(p_A)_*(F) \longrightarrow (p_A)_*\mathcal{F}_A(F),$$

which can be written (in cohomological notation) as a morphism

$$\mathcal{F}(\mathsf{H}^*_{\mathscr{D}}(A;F)) \longrightarrow \mathsf{H}^*_{\mathscr{D}'}(A;\mathcal{F}_A(F)).$$

Dually, in the case when $\mathscr{D}$ and $\mathscr{D}'$ both admit direct homological images, we have a morphism

$$\mathsf{H}^{\mathscr{D}}_*(A;\mathcal{F}_A(F)) \longrightarrow \mathcal{F}(\mathsf{H}^{\mathscr{D}}_*(A;F))$$

Definition 1.3.6. *Let $\mathcal{F} : \mathscr{D} \to \mathscr{D}'$ be a morphism of prederivators admitting right (resp. left) Kan extensions.*

1. *We say that $\mathcal{F}$ is a cohomology (resp. homology), if for every small category I, the square*

$$\begin{tikzcd}
\mathscr{D}(I) \arrow[r, "\mathcal{F}_I"] \arrow[d, "(p_I)_*"'] & \mathscr{D}'(I) \arrow[d, "(p_I)_*"] \\
\mathscr{D}(e) \arrow[r, "\mathcal{F}"'] & \mathscr{D}'(e)
\end{tikzcd}$$

   (resp.

$$\begin{tikzcd}
\mathscr{D}(I) \arrow[r, "\mathcal{F}_I"] \arrow[d, "(p_A)_!"'] & \mathscr{D}'(I) \arrow[d, "(p_I)_!"] \\
\mathscr{D}(e) \arrow[r, "\mathcal{F}"'] & \mathscr{D}'(e)
\end{tikzcd}$$

   commutes up to isomorphism through the canonical arrow defined in (1.3.5). In particular

$$\mathcal{F}(\mathsf{H}^*_{\mathscr{D}}(I;X)) \cong \mathsf{H}^*_{\mathscr{D}}(I;\mathcal{F}_I(X))$$

   (resp.

$$\mathsf{H}^{\mathscr{D}}_*(I;\mathcal{F}_I(X)) \cong \mathcal{F}(\mathsf{H}^{\mathscr{D}}_*(I;X))$$

for every object X of $\mathscr{D}(I)$.

2. *We say that $\mathcal{F}$ commutes with right (resp. left) Kan extensions, if for every morphism of small categories $u : I \to J$, the square*

$$\begin{tikzcd}
\mathscr{D}(I) \arrow[r, "\mathcal{F}_I"] \arrow[d, "u_*"'] & \mathscr{D}'(I) \arrow[d, "u_*"] \\
\mathscr{D}(J) \arrow[r, "\mathcal{F}"'] & \mathscr{D}'(J)
\end{tikzcd}$$

(resp.

$$\begin{tikzcd}
\mathscr{D}(I) \arrow[r, "\mathcal{F}_I"] \arrow[d, "u_!"'] & \mathscr{D}'(I) \arrow[d, "u_!"] \\
\mathscr{D}(J) \arrow[r, "\mathcal{F}"'] & \mathscr{D}'(J)
\end{tikzcd}$$

commutes up to isomorphism through the canonical arrow defined in (1.3.5).

3. *We say that $\mathcal{F}$ commutes with homotopy projective (resp. inductive) limits, if for every small category I, the square*

$$\begin{tikzcd}
\mathscr{D}(I^o) \arrow[r, "\mathcal{F}_{I^o}"] \arrow[d, "(p_{I^o})_*"'] & \mathscr{D}'(I^o) \arrow[d, "(p_{I^o})_*"] \\
\mathscr{D}(e) \arrow[r, "\mathcal{F}"'] & \mathscr{D}'(e)
\end{tikzcd}$$

(resp.

$$\begin{tikzcd}
\mathscr{D}(I^o) \arrow[r, "\mathcal{F}_{I^o}"] \arrow[d, "(p_{I^o})_!"'] & \mathscr{D}'(I^o) \arrow[d, "(p_{I^o})_!"] \\
\mathscr{D}(e) \arrow[r, "\mathcal{F}"'] & \mathscr{D}'(e)
\end{tikzcd}$$

commutes up to isomorphism through the canonical arrow defined in (1.3.5). In particular,

$$\mathcal{F}(\mathbf{R}\varprojlim_I X) \cong \mathbf{R}\varprojlim_I \mathcal{F}_I(X)$$

(resp.

$$\mathcal{F}(\mathbf{L}\varinjlim_{I} X) \cong \mathbf{L}\varinjlim_{I} \mathcal{F}_I(X)$$

for every object X *of* $\mathscr{D}(I^o)$*.*

Definition 1.3.7. *Let* $\mathscr{D}$ *and* $\mathscr{D}'$ *be two prederivators and let* $\gamma : \mathscr{D} \to \mathscr{D}'$ *and* $\delta : \mathscr{D}' \to \mathscr{D}$ *be morphisms of prederivators. We say that* γ *is left adjoint to* δ*, and write* $\gamma \vdash_G \delta$ *(we also use the terminology Grothendieck adjunction), if for every small category* I*, the functor* γ_I *is left adjoint to* δ_I*, i.e.,* $\gamma_I \vdash \delta_I$*.*

Proposition 1.3.8. *If* $\gamma : \mathscr{D} \to \mathscr{D}'$ *and* $\delta : \mathscr{D}' \to \mathscr{D}$ *are morphisms between prederivators admitting left (resp. right) Kan extensions, and* $\gamma \vdash_G \delta$*, then* γ *(resp.* δ*) commutes with all left (resp. right) Kan extensions.*

Proof. Let $u : I \to J$ be a morphism of small categories and consider the canonical morphism $u_!\gamma_I \to \gamma_J u_!$, defined in (1.3.5), associated to the diagram

$$\begin{tikzcd}
\mathscr{D}(I) \arrow[r, "\gamma_I"] \arrow[d, "u_!"'] & \mathscr{D}'(I) \arrow[d, "u_!"] \\
\mathscr{D}(J) \arrow[r, "\gamma_J"'] & \mathscr{D}'(J).
\end{tikzcd}$$

From the definition of morphism of prederivators, the square

$$\begin{tikzcd}
\mathscr{D}(I) & \mathscr{D}'(I) \arrow[l, "\delta_I"'] \\
\mathscr{D}(J) \arrow[u, "u^*"] & \mathscr{D}'(J) \arrow[l, "\delta_J"] \arrow[u, "u^*"']
\end{tikzcd}$$

commutes up to isomorphism. Now, since $u_! \vdash u^*$ (for both the prederivators $\mathscr{D}$ and $\mathscr{D}'$), $\gamma_I \vdash \delta_I$ and $\gamma_J \vdash \delta_J$, then $u_!\gamma_I \vdash \delta_I u^*$ and $\gamma_J u_! \vdash u^*\delta_J$, and from the canonical isomorphism $u^*\delta_J \cong \delta_I u^*$, we have $u_!\gamma_I \cong \gamma_J u_!$. Using the same argument now for the square

$$\begin{tikzcd}
\mathscr{D}(I) \arrow[r, "\gamma_I"] & \mathscr{D}'(I) \\
\mathscr{D}(J) \arrow[r, "\gamma_J"'] \arrow[u, "u^*"] & \mathscr{D}'(J), \arrow[u, "u^*"']
\end{tikzcd}$$

which also commutes up to isomorphism because γ is a morphism of prederivators, and using the fact that $u^* \vdash u_*$ (for both the prederivators $\mathscr{D}$ and $\mathscr{D}'$), $\gamma_I \vdash \delta_I$ and $\gamma_J \vdash \delta_J$, we can deduce that δ commutes with right Kan extensions.

□

Definition 1.3.9. *Let $\mathscr{D}$ be a prederivator of domain $\mathcal{D}ia$. Consider the following list of axioms:*

Der 1 *(a). For any two diagrams I and J, the universal functor*

$$(\imath^*, \jmath^*) : \mathscr{D}(I \amalg J) \longrightarrow \mathscr{D}(I) \times \mathscr{D}(J)$$

where $\imath : I \to I \amalg J$ (resp. $\jmath : J \to I \amalg J$) are the canonical functors, is an equivalence of categories.

(b). The category $\mathscr{D}(\varnothing)$ is equivalent to the point category e.

Der 2 *For every small category A, the family of functors*

$$i^*_{A,a} : \mathscr{D}(A) \longrightarrow \mathscr{D}(e), \qquad a \in Ob(A),$$

is conservative.

Der 3g *For any arrow $u : A \to B$ of small categories, the inverse image $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$ has a right adjoint $u_* : \mathscr{D}(A) \to \mathscr{D}(B)$.*

Der 4g *For any arrow $u : A \to B$ of small categories and any object $b \in Ob(B)$, the commutative square*

$$\begin{tikzcd} A/b \arrow[r, "\zeta(u,b)"] \arrow[d, "p_{A/b}"'] & A \arrow[d, "u"] \\ e \arrow[r, "i_{B,b}"'] & B \end{tikzcd}$$

is $\mathscr{D}$-exact, i.e., the canonical morphism

$$i^*_{B,b} u_* \longrightarrow (p_{A/b})_* \zeta(u,b)^*$$

is an isomorphism.

Der 3d *For any arrow of small categories $u : I \to J$, the functor $u^* : \mathscr{D}(J) \to \mathscr{D}(I)$ admits a left adjoint $u_!$.*

Der 4d *For any arrow of small categories $u : I \to J$, and $j \in Ob(J)$, the canonical natural transformation*

$$(p_{j\backslash I})_! \zeta(u,j)^* \longrightarrow j^* u_!$$

induced from the commutative square

$$\begin{tikzcd}
j \backslash I \arrow[r, "\xi(u,j)"] \arrow[d, "p_{j\backslash I}"'] & I \arrow[d, "u"] \\
e \arrow[r, "j"'] & J,
\end{tikzcd}$$

is a natural isomorphism.

We say that $\mathscr{D}$ is a semi-derivator if it satisfies **Der 1** *and* **Der 2**.

We say that $\mathscr{D}$ is a left (resp. right) derivator it is a semi-derivator and satisfies **Der 3g** *and* **Der 4g** *(resp.* **Der 3d** *and* **Der 4d***). Finally, we say that $\mathscr{D}$ is a derivator if it is both a left and a right derivator.*

Example 1.3.10. For every complete and co-complete locally small category C, the prederivatror $\widetilde{C}$ is a derivator. In fact, we have $\widetilde{C}(e) = \underline{Hom}(e, C) \cong C$. Moreover, it's clear that there exists only one presheaf from the category $\varnothing$ to C, which implies $\widetilde{C}(\varnothing) \simeq e$. Now, given two small categories I and J, we can verify easily from the universal properties of products and coproducts of categories, a canonical equivalence:

$$(\imath^*, \jmath^*) : \widetilde{C}(I \amalg J) \longrightarrow \widetilde{C}(I) \times \widetilde{C}(J),$$

where $\imath : I \to I \amalg J$ (resp. $\jmath : J \to I \amalg J$) denotes the canonical inclusion. Hence, $\widetilde{C}$ satisfies **Der 1**. Now, given a small category A and an arrow f in $\widetilde{C}(A) = \underline{Hom}(A^o, C)$, we have that $i^*_{A,a}(f)$ corresponds to the arrow f_a in C for every $a \in Ob(A)$, which implies that $\widetilde{C}$ also satisfies **Der 2**. The axioms **Der 3d**, **Der 3g**, **Der 4d** and **Der 4g** are consequences of the fact that C is complete and co-complete.

Example 1.3.11. If $\mathscr{D}$ is a derivator and A is a small category, then the prederivator $\mathscr{D}^{A^o}$ is also a derivator. Indeed, the condition **Der 1** (a) follows from the distributivity of the binary products with binary coproducts:

$$A \times (I \amalg J) \simeq (A \times I) \amalg (A \times J)$$

for all small categories I and J, and **Der 1** (b) is a consequence of the isomorphism $A \times \varnothing \cong \varnothing$. The axioms **Der 2**, **Der 3g** and **Der 3d** are immediate. The only difficulty is to verify the axioms **Der 4g** and **Der 4d**. We verify **Der 4g** (the argument for **Der 4d** is completely analogous). Let $u : X \to Y$ be a morphism of small categories and $y \in Ob(Y)$. To apply the prederivator $\mathscr{D}^{A^o}$ on the commutative square

$$\begin{tikzcd}
X/y \arrow[r, "\zeta(u,y)"] \arrow[d, "p_{X/y}"'] & X \arrow[d, "u"] \\
e \arrow[r, "i_{Y,y}"'] & Y
\end{tikzcd}$$

is equivalent to apply the derivator $\mathscr{D}$ on the commutative square

$$\begin{tikzcd}
A \times X/y \arrow[r, "1_A \times \zeta(u,y)"] \arrow[d, "1_A \times p_{X/y}"'] & A \times X \arrow[d, "1_A \times u"] \\
A \times e \arrow[r, "1_A \times i_{Y,y}"'] & A \times Y.
\end{tikzcd}$$

Therefore, in order to verify the axiom **Der 4g** for $\mathscr{D}^{A^o}$, we have to proof that the above commutative square in $\mathscr{D}$-exact, i.e., that canonical base change arrow

$$(1_A \times i_{Y,y})^*(1_A \times u)_* \longrightarrow (1_A \times p_{X/y})_*(1_A \times \zeta(u,y))^*$$

is an isomorphism. Applying **Der 2** of $\mathscr{D}$ and the isomorphism of categories $A \times e \cong A$, it's enough to verify that, for each $a \in Ob(A)$, we have an isomorphism

$$i_{A,a}^*(1_A \times i_{Y,y})^*(1_A \times u)_* \longrightarrow i_{A,a}^*(1_A \times p_{X/y})_*(1_A \times \zeta(u,y))^*.$$

Now, from the previous arguments, we have the commutative diagram

$$\begin{tikzcd}
(A \times X/y)/a \arrow[r, "\zeta(q,a)"] \arrow[d, "p"'] & A \times X/y \arrow[r, "1_A \times \zeta(u,y)"] \arrow[d, "q"'] & A \times X \arrow[d, "1_A \times u"] \\
e \arrow[r, "i_{A,a}"'] & A \arrow[r, "\iota_y"'] & A \times Y
\end{tikzcd}$$

where $\iota_y : a \mapsto (a, y)$, with $\iota_y \cong (1_A \times i_{Y,y})$, p is the canonical arrow and q is the canonical projection, with $q \cong 1_A \times p_{X/y}$. Since $\iota_y \circ i_{A,a} = i_{A \times Y,(a,y)}$ and

$$(A \times X/y)/a \cong (A/a \times X/y) \cong (A \times X)/(a, y),$$

we can reformulate the notations and contemplate the commutative diagram

$$\begin{tikzcd}
(A \times X)/(a,y) \arrow[r, "\zeta(q,a)"] \arrow[d, "p"'] & A \times X/y \arrow[r, "1_A \times \zeta(u,y)"] \arrow[d, "q"'] & A \times X \arrow[d, "1_A \times u"] \\
e \arrow[r, "i_{A,a}"'] & A \arrow[r, "\iota_y"'] & A \times Y,
\end{tikzcd}$$

where $i_{A \times Y,(a,y)} = \iota_y i_{A,a}$ and $\zeta(1_A \times u, (a, y)) = (1_A \times \zeta(u, y))\zeta(q, a)$. With the previous notations, we have now to proof that the canonical arrow

$$i^*_{A,a}\iota^*_y(1_A \times u)_* \longrightarrow i^*_{A,a}q_*(1_A \times \zeta(u, y))^*$$

is an isomorphism. In fact, it follows from **Der 4g** of $\mathscr{D}$, that we have the following sequence of canonical isomorphisms:

$$\begin{aligned}
i^*_{A,a}\iota^*_y(1_A \times u)_* &= (\iota_y i_{A,a})^*(1_A \times u)_* \\
&\cong (i_{A \times Y,(a,y)})^*(1_A \times u)_* \\
&\cong p_*\zeta(1_A \times u, (a, y))^* \\
&\cong p_*((1_A \times \zeta(u, y))\zeta(q, a))^* \\
&= p_*\zeta(q, a)^*(1_A \times \zeta(u, y))^* \\
&\cong i^*_{A,a}q_*(1_A \times \zeta(u, y))^*.
\end{aligned}$$

Example 1.3.12. Every prederivator equivalent to a derivator is also a derivator.

Example 1.3.13. If (C, W) is an ideal Quillen localizer, then, it follows from

(1.1.54), that the prederivator $\mathbf{Ho}_W C$, associated to (C, W), is a derivator.

Proposition 1.3.14. *Let $\mathscr{D}$ be a prederivator. Then, $\mathscr{D}$ is a left derivator if, and only if, $\mathscr{D}^o$ is a right derivator. In particular, $\mathscr{D}$ is a derivator if, and only if, $\mathscr{D}^o$ is a derivator.*

Proof. First, we can verify easily that $\mathscr{D}$ is a semi-derivator (according to the definition (1.3.9)) if, and only if, $\mathscr{D}^o$ is also a semi-derivator. The non-trivial part is to proof that $\mathscr{D}$ satisfies **Der 3d** and **Der 4d** if, and only if, $\mathscr{D}^o$ satisfies **Der 3g** and **Der 4g**. Now, since $(\mathscr{D}^o)^o = \mathscr{D}$, it's enough to verify that, if $\mathscr{D}^o$ satisfies **Der 3g** and **Der 4g**, then $\mathscr{D}$ satisfies **Der 3d** and **Der 4d**. Hence, suppose that $\mathscr{D}$ is a prederivator such that $\mathscr{D}^o$ satisfies **Der 3g** and **Der 4g**. Then, for each arrow $u^o : I^o \to J^o$ of small categories, the functor $(u^o)^* : \mathscr{D}^o(I^o) \to \mathscr{D}^o(J^o)$ admits a right adjoint $(u^o)_* : \mathscr{D}^o(I^o) \to \mathscr{D}^o(J^o)$. Since $\mathscr{D}^o(I^o) = \mathscr{D}(I)^o$, $\mathscr{D}^o(J^o) = \mathscr{D}(J)^o$ and $(u^o)^* = \mathscr{D}^o(u^o) = \mathscr{D}(u)^o = (u^*)^o$, we have the equalities:

$$\mathscr{D}(I) = \mathscr{D}^o(I^o)^o, \quad \mathscr{D}(J) = \mathscr{D}^o(J^o)^o, \quad u^* = ((u^o)^*)^o.$$

Therefore, we can define $u_! = ((u^o)_*)^0$, and the adjunction $(u^o)^* \vdash (u^o)_*$ implies (by dualization) in an adjunction $((u^o)_*)^o \vdash ((u^o)^*)^o$, which means that $u_! \vdash u^*$. Moreover, if $u : I \to J$ is an arrow of small categories, then, for each $j \in Ob(J)$, we have, by definition, $j \setminus I = (I^o/j)^o$, which implies that $\mathscr{D}(j \setminus I) = \mathscr{D}^o((j \setminus I)^o)^o = \mathscr{D}^o(I^o/j)^o$. Since the dual of the commutative square

$$\begin{tikzcd}
I^o/j \arrow[r, "\zeta(u^o{,}j)"] \arrow[d, "p_{I^o/j}"'] & I^o \arrow[d, "u^o"] \\
e \arrow[r, "j^o"'] & J^o
\end{tikzcd}$$

corresponds to the commutative square

$$\begin{tikzcd}
j \setminus I \arrow[r, "\xi(u{,}j)"] \arrow[d, "p_{j\setminus I}"'] & I \arrow[d, "u"] \\
e \arrow[r, "j"'] & J,
\end{tikzcd}$$

it follows from the canonical natural isomorphism

$$(j^o)^*(u^o)_* \longrightarrow (p_{I^o/j})_*\zeta(u^o, j)^*$$

that there is a canonical isomorphism of the form

$$(p_{j\backslash I})_!\xi(u, j)^* \longrightarrow j^*u_!.$$

Which means that $\mathscr{D}$ satisfies the axioms **Der 3d** and **Der 4d**.

☐

Proposition 1.3.15. *If $\mathscr{D}$ is a left derivator, then*

1. *$\mathscr{D}(e)$ has a terminal object.*
2. *For every small category I, the category $\mathscr{D}(I)$ admits finite products.*
3. *For every functors $u : I \to J$ and $v : J \to K$, we have a canonical isomorphism $(vu)_* \cong u_*v_*$.*

Proof. 1. The unique existent functor $p_\varnothing : \varnothing \to e$, from the initial (empty) category $\varnothing$ to the terminal category e, induces a morphism $(p_\varnothing)_* : e \simeq \mathscr{D}(\varnothing) \to \mathscr{D}(e)$ (from **??** (b) and **Der 3g**). Since $(p_\varnothing)_*$ is right adjoint to the inverse image functor $p_\varnothing^* : \mathscr{D}(e) \to \mathscr{D}(\varnothing) \simeq e$, it commutes with projective limits, and hence, the image of $(p_\varnothing)_*$ on the objects is given by a single object $*$ in $\mathscr{D}(e)$, which is a terminal object.

2. Since $\mathscr{D}$ is a left derivator, then for every small category I, the prederivator $\mathscr{D}^{I^o}$ is also a left derivator, and $\mathscr{D}^{I^o}(e) \simeq \mathscr{D}(I)$. Hence, it's enough to proof that the category $\mathscr{D}(e)$ admits finite products. From the item (1), the category $\mathscr{D}(e)$ admits a terminal object. In order to verify that $\mathscr{D}(e)$ also admits binary products, we just have to consider the diagram

$$\begin{tikzcd}
e \arrow[dr, "i_0"] & & \\
& e \amalg e \arrow[r, "p_{e\amalg e}"] & e \\
e \arrow[ur, "i_1"'] & &
\end{tikzcd}$$

Applying **Der 1** (a), the functor

$$(i_0^*, i_1^*) : \mathscr{D}(e \amalg e) \longrightarrow \mathscr{D}(e) \times \mathscr{D}(e)$$

is an equivalence of categories. Now, to say that $\mathscr{D}(e)$ admits binary products means to say that the diagonal functor $\mathscr{D}(e) \to \mathscr{D}(e) \times \mathscr{D}(e)$ admits a right adjoint. Using the universal property of products and the previous equivalence of categories between $\mathscr{D}(e \amalg e)$ and $\mathscr{D}(e) \times \mathscr{D}(e)$, we can check that the diagonal functor from $\mathscr{D}(e)$ to $\mathscr{D}(e) \times \mathscr{D}(e)$ is isomorphic to the functor $(p_{e \amalg e})^* : \mathscr{D}(e) \to \mathscr{D}(e \amalg e)$, and since $(p_{e \amalg e})^*$ admits a right adjoint $(p_{e \amalg e})_* : \mathscr{D}(e \amalg e) \to \mathscr{D}(e)$ (by **Der 3g**), the diagonal functor $\mathscr{D}(e) \to \mathscr{D}(e) \times \mathscr{D}(e)$ also admits a right adjoint.

3. It's a general formal consequence of the adjunction.

□

Proposition 1.3.16. *If $\mathscr{D}$ is a right derivator, then*

1. *$\mathscr{D}(e)$ has an initial object.*
2. *For every small category I, the category $\mathscr{D}(e)$ admits finite coproducts.*
3. *For every functors $u : I \to J$ and $v : J \to K$, we have a canonical isomorphism $(vu)_! \cong u_! v_!$.*

Proof. It follows by duality from (1.3.15), for, if $\mathscr{D}$ is a right derivator, then $\mathscr{D}^o$ is a left derivator from (1.3.14), which implies that $\mathscr{D}^o(A)$ admits finite products for every small category A. Since $\mathscr{D}(I) = \mathscr{D}^o(I^o)^o$ and $\mathscr{D}^o(I^o)$ admits finite products, then, by duality, $\mathscr{D}(I)$ admits finite coproducts. The assertion (3) is also immediate from the definition of left adjunction.

□

1.3.17. Let $\mathscr{D}$ be a prederivator and A be a small category. If F (resp. f) is an object (resp. an arrow) of $\mathscr{D}(A)$ and $a \in Ob(A)$, then we denote by F_a (resp. f_a) the object $i_{A,a}^*(F)$ (resp. $i_{A,a}^*(f)$), called the fiber of F at a. A morphism $\varphi : a' \to a$ in A can be regarded as a natural transformation

$$\begin{tikzcd} e \arrow[r, bend left=50, "i_{A,a'}"] \arrow[r, bend right=50, "i_{A,a}"'] \arrow[r, phantom, "\Downarrow \varphi"] & A . \end{tikzcd}$$

Applying the prederivator $\mathscr{D}$ in the above diagram, we have a natural transformation $\varphi^* : i^*_{A,a} \to i^*_{A,a'}$, from where we deduce an arrow $F_\varphi =_{df} \varphi^*(F) : F_a \to F_{a'}$. If $f : F \to F'$ is an arrow in $\mathscr{D}(A)$, then we have a commutative square of the form

$$\begin{tikzcd} F_a \arrow[r, "f_a"] \arrow[d, "F_\varphi"'] & F'_a \arrow[d, "F'_\varphi"] \\ F_{a'} \arrow[r, "f_{a'}"'] & F'_{a'} \end{tikzcd}$$

in $\mathscr{D}(e)$. Therefore, there exists a functor

$$dia_A : \mathscr{D}(A) \longrightarrow \underline{Hom}(A^o, \mathscr{D}(e))$$

such that

$$dia_A(F) : A^o \longrightarrow \mathscr{D}(e), \quad a \mapsto F_a, \quad \varphi \mapsto F_\varphi$$

for every object F of $\mathscr{D}(A)$, and the natural transformation $dia_A(f) : dia_A(F) \to dia_A(F')$, associated to an arrow $f : F \to F'$ in $\mathscr{D}(A)$, is defined by $dia_A(f)_a = f_a$ for $a \in Ob(A)$.

Proposition 1.3.18. *Let $\mathscr{D}$ be a prederivator. Equivalent conditions:*

1. *For every small category A, the functor*

   $$dia_A : \mathscr{D}(A) \longrightarrow \underline{Hom}(A^o, \mathscr{D}(e))$$

   is conservative.

2. *If $\{u : U \to A : u \in \mathcal{U}\}$ is a family of functors which is surjective on the objects of A (i.e., for each $a \in Ob(A)$, there exists $u \in \mathcal{U}$ and $x \in Ob(U)$ such that $u(x) = a$), and f is an arrow of $\mathscr{D}(A)$ such that $u^*(f)$ is an isomorphism in $\mathscr{D}(U)$ for every $u \in \mathcal{U}$, then f is an isomorphism.*

3. *$\mathscr{D}$ is conservative, i.e., $\mathscr{D}$ satisfies* **Der 2**.

Proof. The equivalence between (1) and (3) is just a simple translation of the definitions. We proof the equivalence between (2) and (3). For the implication (2) $\implies$ (3), we just have to apply the condition (2) to the family of arrows $\{i_{A,a} : e \to A : a \in Ob(A)\}$. Conversely, suppose that $\mathscr{D}$ is conservative, i.e., $\mathscr{D}$ satisfies **Der 2**, and let $\{u : U \to A : u \in \mathcal{U}\}$ be a family of functors which is surjective on the objects of A according to (2). If f is an arrow of $\mathscr{D}(A)$ such that $u^*(f)$ is an isomorphism in $\mathscr{D}(U)$ for every morphism $u : U \to A$ in this family, then, given $a \in Ob(A)$, there exist $U \in \mathcal{U}$ and $x \in Ob(U)$ such that $a = u(x)$, which means that $u \circ i_{U,x} = i_{A,a}$. Applying the prederivator $\mathscr{D}$ in the previous equation, we have $i^*_{A,a}(f) = i^*_{U,x}u^*(f)$, and since $u^*(f)$ is an isomorphism in $\mathscr{D}(U)$, then $i^*_{A,a}(f)$ is an isomorphism in $\mathscr{D}(e)$, from where we conclude that $i^*_{A,a}(f)$ is an isomorphism in $\mathscr{D}(e)$ for every $a \in Ob(A)$. Finally, it follows from **Der 2** that f is actually an isomorphism in $\mathscr{D}(A)$.

□

Lemma 1.3.19. *Let $\mathscr{D}$ be a left derivator and $u : A \to B$ be a morphism of small categories. If u is faithful fully, then, for every $a \in Ob(A)$ the morphism*

$$i^*_{B,u(a)}u_* \longrightarrow i^*_{A,a}$$

induced by the co-unit $\varepsilon : u^*u_* \to 1$ of the adjunction $u^* \vdash u_*$, is an isomorphism.

Proof. Under the hypothesis that $u : A \to B$ is faithful fully, the evident functor $A/a \to A/u(a)$ is an isomorphism for every $a \in Ob(A)$. Moreover, $(a, 1_{u(a)})$ is a terminal object of $A/u(a)$. The assertion of the lemma follows easily from this previous fact and from **Der 4g**.

□

Corollary 1.3.20. *If $\mathscr{D}$ is a left (resp. right) derivator and $u : A \to B$ is a faithful fully functor, then the right (resp. left) Kan extension $u_* : \mathscr{D}(A) \to \mathscr{D}(B)$ (resp. $u_! : \mathscr{D}(A) \to \mathscr{D}(B)$) is faithful fully.*

Proof. For the case where $\mathscr{D}$ is a left derivator, the assertion follows from **Der 2** and from the previous lemma. The case where $\mathscr{D}$ is a right derivator follows from the former by duality.

□

Definition 1.3.21. *A morphism of small categories $u : A \to B$ is a closed (resp. open) immersion, if it is full and the essential image of u is a cosieve (resp. sieve) of B.*

Proposition 1.3.22. *Let $\mathscr{D}$ be a left (resp. right) derivator and $j : Z \to X$ (resp. $i : U \to X$) be a closed (resp. open) immersion of small categories. Then, j_* (resp. $i_!$) is faithful fully and the essential image of j_* (resp. of $i_!$) is formed by the objects F in $\mathscr{D}(X)$ such that F_x is the terminal (resp. initial) object in $\mathscr{D}(e)$ for every $x \in Ob(X) - Ob(Z)$ (resp. $x \in Ob(X) - Ob(U)$).*

Proposition 1.3.23. *Let $\mathcal{F} : \mathscr{D} \to \mathscr{D}'$ be a morphism between left (resp. right) derivators. Equivalent conditions:*

1. *$\mathcal{F}$ is a cohomology (resp. homology).*
2. *$\mathcal{F}$ commutes with homotopy projective (resp. inductive) limits.*
3. *$\mathcal{F}$ commutes with right (resp. left) Kan extensions.*

Proof. The equivalence between (1) and (2) is immediate from the definitions, and it is also clear that (3) implies (1). We proof that (1) implies (3). It is enough to verify only the cohomological case for left derivators, since the homological case for right derivators follows from the former by duality. Hence, suppose that $\mathcal{F} : \mathscr{D} \to \mathscr{D}'$ is a cohomology, which means that, for each small category I, the diagram

$$\begin{tikzcd}
\mathscr{D}(I) \arrow[r, "\mathcal{F}_I"] \arrow[d, "(p_I)_*"'] & \mathscr{D}'(I) \arrow[d, "(p_I)_*"] \\
\mathscr{D}(e) \arrow[r, "\mathcal{F}"'] & \mathscr{D}'(e)
\end{tikzcd}$$

commutes up to isomorphism. Let $u : A \to B$ be a morphism of small categories and consider the canonical arrow $\widetilde{\xi}_{\mathcal{F},u} : \mathcal{F}_B u_* \to u_* \mathcal{F}_A$, corresponding to the square

$$\begin{tikzcd}
\mathscr{D}(A) \arrow[r, "\mathcal{F}_A"] \arrow[d, "u_*"'] & \mathscr{D}'(B) \arrow[d, "u_*"] \\
\mathscr{D}(B) \arrow[r, "\mathcal{F}_B"'] & \mathscr{D}'(B)
\end{tikzcd}$$

defined in (1.3.5). In virtue of **Der 2**, we just have to proof that, for each $b \in Ob(B)$, the morphism $i^*_{B,b}(\widetilde{\xi}_{\mathcal{F},u}) : i^*_{B,b}\mathcal{F}_B u_* \to i^*_{B,b} u_* \mathcal{F}_A$ is an isomorphism in $\mathscr{D}(e)$. But, it follows from **Der 4g**, from (3) of (1.3.15), and from the fact that $\mathcal{F}$ is a morphism of prederivators, the following sequence of isomorphisms:

$$\begin{aligned}
i^*_{B,b}\mathcal{F}_B u_* &\cong \mathcal{F} i^*_{B,b} u_* \\
&\cong \mathcal{F}(p_{A/b})_* \zeta(u,b)^* \\
&\cong (p_{A/b})_* \mathcal{F}_{A/b} \zeta(u,b)^* \\
&\cong p_{A/b})_* \zeta(u,b)^* \mathcal{F}_A \\
&\cong i^*_{B,b} u_* \mathcal{F}_A
\end{aligned}$$

which concludes the proof.

□

Proposition 1.3.24. *Let $\mathscr{D}$ be a left (resp. right) derivator and $u : A \to B$ be a morphism of small categories. Then, for every $b \in Ob(B)$, the canonical arrow*

$$\zeta(1_B, b)^* u_* \longrightarrow (u/b)_* \zeta(u, b)^*$$

(resp.

$$(u/b)_! \xi(u, b)^* \longrightarrow \xi(1_B, b)^* u_!$$

induced from the square

$$\begin{tikzcd}
A/b \arrow[r, "\zeta(u,b)"] \arrow[d, "u/b"'] & A \arrow[d, "u"] \\
B/b \arrow[r, "\zeta(1_B,b)"'] & B
\end{tikzcd}$$

(resp. from the square

$$\begin{tikzcd}
b \setminus A \arrow[r, "\xi(u,b)"] \arrow[d, "b\setminus u"'] & A \arrow[d, "u"] \\
b \setminus B \arrow[r, "\xi(1_B,b)"'] & B
\end{tikzcd}$$

is an isomorphism.

Proof. We only proof the case of left derivators, (the proposition for right derivators follows easily from the last case by duality [13]). The strategy of the proof consists in to apply **Der 2** to the canonical morphism

$$\zeta(1_B, b)^* u_* \longrightarrow (u/b)_* \zeta(u, b)^*$$

i.e., to verify that, for each object (b', φ) of B/b, the arrow

$$(i_{B/b,(b',\varphi)})^* \zeta(1_B, b)^* u_* \longrightarrow (i_{B/b,(b',\varphi)})^* (u/b)_* \zeta(u, b)^*$$

is an isomorphism in $\mathscr{D}(e)$. First, we remark that $(A/b)/(b', \varphi) \simeq A/b'$, which

---

[13] The reader can also follows the same argument exchanging the right Kan extensions by the left Kan extensions.

implies in that we can make the identifications:

$$p_{(A/b)/(b',\varphi)} \cong p_{A/b'}, \qquad \zeta(u,b') \cong \zeta(u,b)\zeta(u/b,(b',\varphi)).$$

Moreover, $i_{B,b'} = \zeta(1_B,b)i_{B/b,(b',\varphi)}$. Now, it follows from **Der 4g** and from the commutative diagram

$$\begin{tikzcd}
(A/b)/(b',\varphi) \arrow[r, "\zeta(u/b,(b',\varphi))"] \arrow[d, "p_{(A/b)/(b',\varphi)}"'] & A/b \arrow[r, "\zeta(u,b)"] \arrow[d, "u/b"'] & A \arrow[d, "u"] \\
e \arrow[r, "i_{B/b,(b',\varphi)}"'] & B/b \arrow[r, "\zeta(1_B,b)"'] & B
\end{tikzcd}$$

that we have the canonical isomorphisms

$$(i_{B/b,(b',\varphi)})^*(u/b)_* \cong (p_{(A/b)/(b',\varphi)})^*\zeta(u/b,(b',\varphi))^*$$

and

$$i^*_{B,b'}u_* \cong (p_{A/b'})_*\zeta(u,b')^* \cong (p_{(A/b)/(b',\varphi)})^*\zeta(u/b,(b',\varphi))^*\zeta(u,b)^*.$$

Therefore,

$$\begin{aligned}
(i_{B/b,(b',\varphi)})^*\zeta(1_B,b)^*u_* &\cong (\zeta(1_B,b)i_{B/b,(b',\varphi)})^*u_* \\
&\cong (i_{B,b'})^*u_* \\
&\cong (p_{A/b'})_*\zeta(u,b')^* \\
&\cong (p_{(A/b)/(b',\varphi)})_*\zeta(u/b,(b',\varphi))^*\zeta(u,b)^* \\
&\cong i^*_{B/b,(b',\varphi)}(u/b)_*\zeta(u,b)^*.
\end{aligned}$$

□

1.3.25. Let $u : A \to B$ be a morphism of small categories. For each object $b \in Ob(B)$, we can form the category A_b, called the fiber of u at B, as following: the objects of A_b are the objects $a \in Ob(A)$ such that $u(a) = b$, and the arrows $\varphi : a' \to a$ of A_b are the arrows φ in A such that $u(\varphi) = 1_b$ in B. The law of composition and identities of A_b are the evident ones induced from A. There is always an inclusion functor $j_b : A_b \to A$ and we also have the evident functors

$$\Theta_b : A_b \longrightarrow A/b, \quad a \mapsto (a, 1_a)$$

and

$$\Xi_b : A_b \longrightarrow b \setminus A, \quad a \mapsto (1_a, a).$$

We say that $u : A \to B$ is a Grothendieck *precofibration* (resp. *prefibration*) if the functor Θ_b admits a left (resp. a right) adjoint Θ'_b (resp. Ξ'_b) for every $b \in Ob(B)$. An arrow $\varphi : a' \to a$ of A is *cartesian* over B if for every morphism $\mu : a'' \to a'$ in A such that $u(\varphi) = u(\mu)$, there exists a unique morphism $\lambda : x \to a$ such that $\varphi \circ \lambda = \mu$. Finally, $u : A \to B$ is a Grothendieck fibration if it is a cartesian prefibration, and a Grohendieck cofibration if $u^o : A^o \to B^o$ is a Grothendieck fibration. We remark that the notions of prefibration and precofibration are dual one each other, i.e., u is a prefibration if, and only if, u^o is a precofibration.

Proposition 1.3.26. *Let $\mathscr{D}$ be a left (resp. right) derivator and consider a cartesian square*

$$\begin{tikzcd}
A_b \arrow[r, "j_b"] \arrow[d, "p_{A_b}"'] & A \arrow[d, "u"] \\
e \arrow[r, "i_{B,b}"'] & B
\end{tikzcd}$$

where u is a Grothendieck cofibration (resp. a Grothendieck fibration). Then, the above square is $\mathscr{D}$-exact, i.e., the canonical base change morphism

$$(i_{B,b})^* u_* \longrightarrow (p_{A_b})_* (j_b)^*$$

(resp.

$$(p_{A_b})_! (j_b)^* \longrightarrow (i_{B,b})^* u_!$$

defined in (1.3.3) is an isomorphism.

Proof. Again, we only proof the proposition for the case where $\mathscr{D}$ is a left derivators (the case when $\mathscr{D}$ is a right derivator follows easily from the last by duality). We remark that we have the commutative diagram

$$\begin{tikzcd}
A_b \arrow[r, "\Theta_b"] \arrow[d, "p_{A_b}"'] & A/b \arrow[r, "1_{A/b}"] \arrow[d, "p_{A/b}"'] & A/b \arrow[r, "\zeta(u,b)"] \arrow[d, "u/b"'] & A \arrow[d, "u"] \\
e \arrow[r, "1"'] & e \arrow[r, "i_{B/b,(b,1_b)}"'] & B/b \arrow[r, "\zeta(1_B,b)"'] & B,
\end{tikzcd}$$

where $i_{B,b} = \zeta(1_B, b)i_{B/b,(b,1_b)}$ and $j_b = \zeta(u,b)1_{A/b}\Theta_b$. Moreover, the square

$$\begin{tikzcd} A/b \arrow[r, "\zeta(u,b)"] \arrow[d, "p_{A/b}"'] & A \arrow[d, "u"] \\ e \arrow[r, "i_{B,b}"'] & B \end{tikzcd}$$

is $\mathscr{D}$-exact (by **Der 4g**), and the fact that u is a Grothendieck cofibration implies that the square

$$\begin{tikzcd} A_b \arrow[r, "\Theta_b"] \arrow[d, "p_{A_b}"'] & A/b \arrow[d, "p_{A/b}"] \\ e \arrow[r, "1"'] & e \end{tikzcd}$$

is also $\mathscr{D}$-exact. Then, it follows from (1.3.4) and (1.3.24) that the square

$$\begin{tikzcd} A_b \arrow[r, "j_b"] \arrow[d, "p_{A_b}"'] & A \arrow[d, "u"] \\ e \arrow[r, "i_{B,b}"'] & B \end{tikzcd}$$

is $\mathscr{D}$-exact.

□

Proposition 1.3.27. *Let $\mathscr{D}$ be a left (resp. right) derivator and consider a cartesian square*

$$\begin{tikzcd} A' \arrow[r, "v'"] \arrow[d, "u'"'] & A \arrow[d, "u"] \\ B' \arrow[r, "v"'] & B \end{tikzcd}$$

where u is a Grothendieck cofibration (resp. a Grothendieck fibration). Then, the above square is $\mathscr{D}$-exact, i.e., the canonical base change morphism

$$v^* u_* \longrightarrow (v')_*(u')^*$$

(resp.

$$(v')_!(u')^* \longrightarrow v^* u_!$$

defined in (1.3.3) is an isomorphism.

Proof. Applying **Der 2**, the statement of the proposition is an easy formal consequence of the definition (1.3.25), and from the propositions (1.3.24) and (1.3.26).

□

Definition 1.3.28. *A localizer (C, W) is called a left (resp. right) Grothednieck localizer if the prederivator $\mathscr{D}_{(C,W)}$, associated to the localizer (C, W), is a left (resp. right) derivator. A localizer is called a Grothendieck localizer if it's both a left and a right Grothendieck localizer.*

Theorem 1.3.29. *Let (C, W) be a localizer. If (C, W) is a Quillen localizer, then (C, W) is a Grothendieck localizer.*

For a detailed proof of the theorem (1.3.29), we indicate [16] (which is actually a paper dedicated to proof the theorem (1.3.29)), or the section *10.3* of [20] (specially, the *Theorem 10.3.3* of [20]). An elegant proof of the theorem (1.3.29) for the special case of combinatorial model categories is exposed in the *Proposition 1.30* of [38]. Indeed, the Theorem (1.1.54) indicates that the unique technical obstruction to proof the theorem (1.3.29) lies on the surprising fact that model categories are not (in general) stable by exponentiation, i.e., given a model category (C, W, Cof, Fib) and an arbitrary small category I, the localizer $(C(I), W_I)$ is not necessarily an ideal Quillen localizer in order to grant the conditions of (1.1.53)). Yet, a definitive solution of this technical obstruction of model categories is given in [19], where Cisinski proposes a more general notion of model category which are stable by exponentiation and includes all Quillen model categories as a special case. The strategy followed in [16] consists in first reduce the assertion of the theorem (1.3.29) to prederivators with domain being only the direct categories in place of all small categories. In this case, we can endow the localizers $(C(I), W_I)$ with a suitable model category structure, called Reedy model category structure, such that the conditions of (1.1.53) are verified, and hence, we can apply the theorem (1.3.13) to this case and proof that the prederivator $\mathbf{Ho}_W C$, associated to the Quillen localizer (C, W), is a derivator at least over the direct categories. Then, we can reduce the general case to the former, and deduce the theorem (1.3.29) in generality.

2 Homotopy Types

2.1 Asphericity

Definition 2.1.1. *Let $\mathscr{D}$ be a prederivator and $u : I \to J$ an arrow of Cat. We say that u is a $\mathscr{D}$-equivalence if for every $M, N \in Ob(\mathscr{D}(e))$, the function*

$$Hom_{\mathscr{D}(J)}(p_J^*(M), p_J^*(N)) \longrightarrow Hom_{\mathscr{D}(I)}(p_I^*(M), p_I^*(N))$$

is bijective.

We denote by $W_{\mathscr{D}}$ the class of $\mathscr{D}$-equivalences in Cat. A small category I is said to be $\mathscr{D}$-aspherical if the unique possible functor $p_I : I \to e$ is a $\mathscr{D}$-equivalence.

Proposition 2.1.2. *For all prederivator $\mathscr{D}$, the localizer $(Cat, W_{\mathscr{D}})$ is strong saturated.*

Proof. Let $W_{\mathscr{D}}$ be the class of $\mathscr{D}$-equivalences in Cat and $\gamma_{\mathscr{D}}$ be localizing functor from Cat to $(W_{\mathscr{D}})^{-1}Cat$. Given $M, N \in Ob(\mathscr{D}(e))$, we can form the evident functor

$$\Phi_{M,N} : Cat \longrightarrow Ens^o, \quad I \mapsto Hom_{\mathscr{D}(I)}((p_I)^*(M), (p_I)^*(N))$$

where, for each small category I, p_I is the canonical functor from I to the terminal category e. Clearly, the functor $\Phi_{M,N}$ sends $\mathscr{D}$-equivalences to isomorphisms, and hence, we have a commutative diagram

$$\begin{tikzcd}
Cat \arrow[rr, "\Phi_{M,N}"] \arrow[dr, "\gamma_{\mathscr{D}}"'] & & Ens^o \\
& (W_{\mathscr{D}})^{-1}Cat \arrow[ur, "\overline{\Phi}_{M,N}"'] &
\end{tikzcd}$$

where $\overline{\Phi}_{M,N}$ is the functor induced by the universal property of localization. Now, let $u : I \to J$ be a functor between small categories. Since $\Phi_{M,N}(u) = \overline{\Phi}_{M,N}\gamma_{\mathscr{D}}(u)$, the fact that $\gamma_{\mathscr{D}}(u)$ is an isomorphism implies that $\Phi_{M,N}(u)$ is an isomorphism, and u is a $\mathscr{D}$-equivalence precisely when $\Phi_{M,N}(u)$ is an isomorphism for every $M, N \in Ob(\mathscr{D}(e))$, from where we conclude that $(Cat, W_{\mathscr{D}})$ is strong saturated.

□

Corollary 2.1.3. *For all prederivator $\mathscr{D}$, the localizer $(Cat, W_{\mathscr{D}})$ is weak saturated.*

Proof. Immediate from (1.1.17) and (2.1.2).

□

Proposition 2.1.4. *Let A be a small category and $\mathscr{D}$ be a prederivator. If A has terminal object, then A is $\mathscr{D}$-aspherical.*

Proof. If e_A is a terminal object of A, then functor $p_A : A \to e$ admits a section $s : e \to A$ which sends $*$ to e_A, which means that $s \vdash p_A$ and s is faithful fully. Then, $(p_A)^* \vdash e_A^*$ and $(p_A)^*$ is faithful fully, from where we conclude that p_A is a $\mathscr{D}$-equivalence.

□

Proposition 2.1.5. *Let $\mathscr{D}$ be a prederivator admitting cohomological direct images and $u : A \to B$ be a morphism of small categories. Equivalent conditions:*

1. *u is a $\mathscr{D}$-equivalence*
2. *The canonical morphism*

$$(p_B)_* p_B^* \longrightarrow (p_B)_* u_* u^* p_B^* \cong (p_A)_* p_A^*$$

induced from the unit $1 \to u_* u^*$ of the adjuntion $u^* \vdash u_*$, is an isomorphism.

3. *For every object F of $\mathscr{D}(e)$, the canonical arrow*

$$\mathsf{H}^*_{\mathscr{D}}(B; F|_B) \longrightarrow \mathsf{H}^*_{\mathscr{D}}(A; F|_A),$$

induced on the cohomologies, is an isomorphism.

Proposition 2.1.6. *Let $\mathscr{D}$ be a prederivator admitting homological direct images and $u : A \to B$ be a morphism of small categories. Equivalent conditions:*

1. *u is a $\mathscr{D}$-equivalence*

2. *The canonical morphism*

$$(p_A)_!p_A^* \cong (p_B)_!u_!u^*p_B^* \longrightarrow (p_B)_!p_B^*$$

induced from the co-unit $u_!u^* \to 1$ *of the adjuntion* $u_! \vdash u^*$*, is an isomorphism.*

3. *For every object* F *of* $\mathscr{D}(e)$*, the canonical arrow*

$$\mathsf{H}_*^{\mathscr{D}}(A; F|_A) \longrightarrow \mathsf{H}_*^{\mathscr{D}}(B; F|_B),$$

induced on the homologies, is an isomorphism.

Definition 2.1.7. *Let* $\mathscr{D}$ *be a prederivator and*

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

be a commutative triangle in Cat*. We say that* $u : A \to B$ *is local* $\mathscr{D}$*-aspherical (resp. local* $\mathscr{D}$*-coaspherical) along* C*, if for every* $c \in Ob(C)$*, the arrow* $u/c : A/c \to B/c$ *(resp.* $c \setminus u : c \setminus A \to c \setminus B$*) is a* $\mathscr{D}$*-equivalence. A morphism of small categories* $u : A \to B$ *is called* $\mathscr{D}$*-aspherical (resp.* $\mathscr{D}$*-coaspherical) if the* u *is local* $\mathscr{D}$*-asphreical (resp. local* $\mathscr{D}$*-coaspherical) along* B *in relation to the commutative triangle*

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "u"'] & & B \arrow[dl, "1_B"] \\ & B & \end{tikzcd}$$

Proposition 2.1.8. *Let* $\mathscr{D}$ *be a left (resp. right) derivator and*

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\ & C & \end{tikzcd}$$

be a commutative triangle in Cat*. If* u *is local* $\mathscr{D}$*-aspherical (resp. local* $\mathscr{D}$*-*

coaspherical) along C, then, the canonical morphism

$$w_*p_B^* \longrightarrow w_*u_*u^*p_B^* \cong v_*p_A^*$$

(resp.

$$v_!p_A^* \cong w_!u_!u^*p_B^* \longrightarrow w_!p_B^*$$

induced from the unit $1 \to u_*u^*$ (resp. co-unit $u_!u^* \to 1$) of the adjunction $u^* \vdash u_*$ (resp. $u_! \vdash u^*$) is an isomorphism in $\mathscr{D}(C)$.

Proof. For the case when $\mathscr{D}$ is a left derivator, consider the canonical morphism

$$w_*(p_B)^* \longrightarrow w_*u_*u^*(p_B)^* \cong v_*(p_A)^*$$

induced from the unit $1 \to u_*u^*$ of the adjunction $u^* \vdash u_*$. In order to proof that the above arrow is a natural isomorphism, it is enough to verify that, for each $c \in Ob(C)$, we have an isomorphism of the form

$$(i_{C,c})^*w_*(p_B)^* \longrightarrow (i_{C,c})^*v_*(p_A)^*$$

in $\mathscr{D}(e)$ (from **Der 2**). Now, in virtue of the axiom **Der 4g**, we have the $\mathscr{D}$-exactness of the commutative squares

$$\begin{tikzcd}
A/c \arrow[r, "\zeta(v,c)"] \arrow[d, "p_{A/c}"'] & A \arrow[d, "v"] \\
e \arrow[r, "i_{C,c}"'] & C
\end{tikzcd}$$

and

$$\begin{tikzcd}
B/c \arrow[r, "\zeta(w,c)"] \arrow[d, "p_{B/c}"'] & B \arrow[d, "w"] \\
e \arrow[r, "i_{C,c}"'] & C
\end{tikzcd}$$

for $c \in Ob(C)$, and hence, canonical natural isomorphisms:

$$(i_{C,c})^*w_* \cong (p_{B/c})_*\zeta(w,c)^*, \qquad (i_{C,c})^*v_* \cong (p_{A/c})_*\zeta(u,c)^*.$$

Moreover, $p_{A/c} = p_A\zeta(v,c)$ and $p_{B/c} = p_B\zeta(w,c)$. On the other hand, the hypothesis that $u : A \to B$ is local $\mathscr{D}$-aspherical along C implies in a (canonical) natural isomorphism

$$(p_{B/c})_*(p_{B/c})^* \longrightarrow (p_{A/c})_*(p_{A/c})^*.$$

for every $c \in Ob(C)$. From the previous facts, we can verify, for all $c \in Ob(C)$, the following sequence of canonical natural isomorphisms:

$$\begin{aligned}
(i_{C,c})^* w_*(p_B)^* &\cong (p_{B/c})_*\zeta(w,c)^*(p_B)^* \\
&\cong (p_{B/c})_*(p_B\zeta(w,c))^* \\
&\cong (p_{B/c})_*(p_{B/c})^* \\
&\cong (p_{A/c})_*(p_{A/c})^* \\
&\cong (p_{A/c})_*(p_A\zeta(v,c))^* \\
&\cong (p_{A/c})_*\zeta(v,c)^*(p_A)^* \\
&\cong (i_{C,c})^* v_*(p_A)^*
\end{aligned}$$

which concludes the proof for left derivators. The assertion of the proposition for right derivators and local $\mathscr{D}$-coasphericity follows by duality.

$\square$

Corollary 2.1.9. *Let $\mathscr{D}$ be a left (resp. right) derivator and*

$$\begin{tikzcd}
A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\
& C &
\end{tikzcd}$$

be a commutative triangle in Cat. If u is local $\mathscr{D}$-aspherical (resp. local $\mathscr{D}$-coaspherical) along C, then u is a $\mathscr{D}$-equivalence. In particular, if $u : A \to B$ is $\mathscr{D}$-aspherical (resp. $\mathscr{D}$-coaspherical), then u is a $\mathscr{D}$-equivalence.

Proof. It follows from (2.1.5) (resp. (2.1.6)) and (2.1.8), since the local $\mathscr{D}$-asphericity (resp. local $\mathscr{D}$-coasphericity) of the arrow $u : A \to B$ along C, implies in the isomorphism $w_*(p_B)^* \to v_*(p_A)^*)$ (resp. $v_!(p_A)^* \to w_!(p_B)^*$), which induces the canonical isomorphism :

$$(p_B)_*(p_B)^* \cong (p_C)_* w_*(p_B)^* \longrightarrow (p_C)_* v_*(p_A)^* \cong (p_A)_*(p_A)^*$$

(resp.

$$(p_A)_!(p_A)^* \cong (p_C)_! v_!(p_A)^* \longrightarrow (p_C)_! w_!(p_B)^* \cong (p_B)_!(p_B)^*).$$

□

Definition 2.1.10. *A fundamental localizer is a class of arrows W of Cat satisfying the following conditions:*

(LF1). W is weak saturated ((1.1.16))

(LF2). If A is a small category with terminal object, then $p_A : A \to e$ is in W

(LF3). For every commutative triangle

$$\begin{tikzcd}
A \arrow[rr, "u"] \arrow[dr, "v"'] & & B \arrow[dl, "w"] \\
& C &
\end{tikzcd}$$

in Cat, if $u/c : A/c \to B/c$ is in W for each $c \in Ob(C)$, then $u \in W$.

Termonilogy: Let W be a fundamental localizer. A small category A is called W-aspherical if the unique existent functor $p_A : A \to e$ is a W-equivalence. A functor $u : A \to B$ of small categories is called W-aspherical if, for every object $b \in Ob(B)$, the category A/b is W-aspherical.

Proposition 2.1.11. *Let W be a fundamental localizer.*

1. *Every isomorphism of small categories is a W-equivalence.*
2. *If an arrow $u : A \to B$ of small categories is W-aspherical, then u is a W-equivalence.*
3. *Every functor of small categories admitting a right adjoint is W-aspherical.*
4. *Every equivalence of categories is a W-equivalence.*
5. *W is auto-dual, i.e. an arrow $u : A \to B$ of small categories is a W-equivalence if, and only if, $u^o : A^o \to B^o$ is a W-equivalence.*

Proof. 1. It is immediate from (LF1).

2. If $u : A \to B$ is W-aspherical, then, for every $b \in Ob(B)$, the category A/b is W-aspherical, and we have the functor $u/b : A/b \to B/b$. Since B/b admits a terminal object, then B/b is W-aspherical (by (LF2)), which implies that u/b is a W-equivalence (by (LF1)). Applying then (LF3) to the commutative triangle

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "u"'] & & B \arrow[dl, "1_B"] \\ & B & \end{tikzcd}$$

we conclude that u is a W-equivalence.

3. Let $u : A \to B$ be a functor which admits a right adjoint $v : B \to A$. From the canonica natural bijections

$$Hom_B(u(a), b) \cong Hom_A(a, v(b)), \qquad a \in Ob(A), b \in Ob(B)$$

we have that A/b is isomorphic to the category $A/v(b)$ for every $b \in Ob(B)$, and since $A/v(b)$ admits a terminal object (because $v(b) \in Ob(A)$), then A/b is W-aspherical (from the item (1)), which implies that $u : A \to B$ is a W-aspherical functor, and hence, a W-equivalence (by the item (2)).

4. It's immediate from the item (3).

5. See *Remarque 1.1.25.* of [13].

□

Proposition 2.1.12. *For every derivator $\mathscr{D}$, the class $W_{\mathscr{D}}$ of $\mathscr{D}$-equivalences is a fundamental localizer.*

Proof. The axiom (LF1) follows from (2.1.3). The axiom (LF2) follows from (2.1.4). The axiom (LF3) follows from (2.1.9).

□

Example 2.1.13. Let $\pi_0 : Cat \to Ens$ be the functor that sends each small category A to the set $\pi_0(A)$ of it's connected components, i.e., $\pi_0 = Ob(A)/\sim$, where $\sim$ is the small equivalence relation in $Ob(A)$ identifying that pairs (a, b) where $Hom_A(a, b) \neq \varnothing$.

2.1.14. For every prederivator $\mathscr{D}$, the class $W_{\mathscr{D}}$ of $\mathscr{D}$-equivalences satisfies the conditions (LF1) and (LF2) of (2.1.10). Hence, the only axiom of fundamental localizers that $W_{\mathscr{D}}$ does not satisfies is (LF3). For this reason, we say that $\mathscr{D}$ is a local prederivator if the class of $\mathscr{D}$-equivalences in Cat satisfies (LF3). In this case, $W_{\mathscr{D}}$ is a fundamental localizer.

2.1.15. Clearly, $Fl(Cat)$ is a fundamental localizer. Hence, the set Λ of all fundamental localizers in Cat is not empty, which implies in the existence of the class

$$W_\infty =_{df} \bigcap \Lambda.$$

It follows easily from the definition of fundamental localizers that W_∞ is a also a fundamental localizer, called *minimal fundamental localizer*. The morphisms in W_∞ are called ∞-*equivalences*.

We end this section by enumerating a list of crucial stability properties of fundamental localizers.

Proposition 2.1.16. *(Maltsiniotis) - If W is a fundamental localizer, then the following conditions are verified:*

1. *An arrow of small categories $u : A \to B$ is a W-equivalence if, and inly if, its opposed arrow $u^o : A^o \to B^o$ is a W-equivalence.*
2. *W is stable by finite products, i.e., if $u : A \to B$ and $v : C \to D$ are two W-equivalences, then the canonical arrow $u \times v : A \times C \to B \times D$ is also a W-equivalence.*
3. *W is stable by small coproducrs, i.e., if $(u_i : A_i \to B_i)_{i \in I}$ is a family of W-equivalences where I is a small set, then the canonical arrow*

$$\coprod_{i \in I} u_i : \coprod_{i \in I} A_i \longrightarrow \coprod_{i \in I} B_i$$

is also a W-equivalence.

4. *W is stable by small filtered inductive limits and by transfinite compositions*[14]*.*
5. *W is stable by retracts.*

---

[14] See (1.1.30)

Proof.
1. See *Proposition 1.1.22.* of [13].
2. See *Proposition 2.1.3.* of [13].
3. See *Proposition 2.1.4.* of [13].
4. See *Proposition 2.4.12, (b)* of [13].
5. It is immediate from the fact that W is stable by small filtered inductive limits.

$\square$

Corollary 2.1.17. *Every fundamental localizer* W *is saturated according to (1.1.31).*

2.2 Hot

Definition 2.2.1. *Let* $\mathscr{S}$ *be a derivator. A second derivator* $\mathscr{D}$ *is called* $\mathscr{S}$*-local if every* $\mathscr{S}$*-equivalence is also a* $\mathscr{D}$*-equivalence.*

Definition 2.2.2. *A right (resp. left) derivator* $\mathscr{S}$ *is called a right (resp. left) free derivator if for every* $\mathscr{S}$*-local right (resp. left) derivator* $\mathscr{D}$*, the functor*

$$\mathcal{H}om_!(\mathscr{S},\mathscr{D}) \longrightarrow \mathscr{D}(e), \quad \mathcal{F} \mapsto \mathcal{F}(e)$$

(resp.

$$\mathcal{H}om^!(\mathscr{S},\mathscr{D}) \longrightarrow \mathscr{D}(e), \quad \mathcal{F} \mapsto \mathcal{F}(e)$$

is an equivalence of categories. In this case, we say that $\mathrm{S} =_{df} \mathscr{S}(e)$ *is the category of spaces induced from* $\mathscr{S}$*. A right (resp. left) derivator* $\mathscr{S}$ *is called absolute if it is a free derivator, and every derivator* $\mathscr{D}$ *is* $\mathscr{S}$*-local.*

Definition 2.2.3. *Let* W_∞ *be the minimal fundamental localizer. We define the derivator* **Hot** *as being the derivator associated to the localizer* (Cat, W_∞)*. The category of homotopy types, denoted by* Hot*, is the category of absolute coefficients of* **Hot***.*

Remark 2.2.4. A theoretical justification of the fact that **Hot** really defines a derivator will be given in the end of the next section, using the theory of Grothendieck test categories. For instance, we assume this result.

Theorem 2.2.5. *If $\mathscr{D}$ is any derivator, then the functor*

$$\mathcal{H}om_!(\mathbf{Hot}, \mathscr{D}) \longrightarrow \mathscr{D}(e), \quad \mathcal{F} \mapsto \mathcal{F}(e)$$

is an equivalence of categories.

Proof - In the following, we give a long and detailed sketch of the proof of the theorem (2.2.5), exhibiting the quasi-inverse image of the functor $\mathcal{F} \mapsto \mathcal{F}(e)$ from $\mathcal{H}om_!(\mathbf{Hot}, \mathscr{D})$ to $\mathscr{D}(e)$. For a complete understanding of this theorem, we indicate the reader to see Chap. III of [11] and Chap. IV of [9], and for a detailed proof, we also indicate [17].

(i). First, given an object F of $\mathscr{D}(e)$, we can form the functor

$$F_! : Cat \longrightarrow \mathscr{D}(e), \quad I \mapsto (p_I)_! p_I^*(F).$$

If $u : I \to J$ is an arrow of small categories, then, we have a canonical morphism

$$(p_I)_! p_I^*(F) \cong (p_J)_! u_! u^* p_J^*(F) \longrightarrow (p_J)_! p_J^*(F),$$

induced from the co-unit $\varepsilon : u_! u^* \to 1$ of the adjunction $u_! \vdash u^*$, which we denote by $F_!(u)$. With the previous notations, we can verify that $F_!$ really defines a functor from Cat to $\mathscr{D}(e)$. On the other hand, if $f : F \to G$ is an arrow of $\mathscr{D}(e)$, then, it follows from the functoriality of $(p_I)_!$ and $(p_I)^*$, that for each arrow $u : I \to J$ in Cat, the square

$$\begin{tikzcd}
(p_I)_! p_I^*(F) \arrow[r, "(p_I)_! p_I^*(f)"] \arrow[d, "F_!(u)"'] & (p_I)_! p_I^*(G) \arrow[d, "G_!(u)"] \\
(p_J)_! p_J^*(F) \arrow[r, "(p_J)_! p_J^*(f)"'] & (p_J)_! p_J^*(G)
\end{tikzcd}$$

is commutative, which means that we can associate to f a natural transformation $f_! : F_! \to G_!$ given by $(f_!)_I = (p_I)_! p_I^*(f)$ in each component I. With the previous notations, we have defined a functor:

$$\mathscr{D}(e) \longrightarrow \underline{Hom}(Cat, \mathscr{D}(e)), \quad F \mapsto F_!, \quad f \mapsto f_!.$$

Moreover, $F_!(e) = (p_e)_! p_e^*(F) \cong F$, since $(p_e)_! \cong 1_{\mathscr{D}(e)} \cong (p_e)^*$. Now, if $u : I \to J$ is an ∞-equivalence, then u is a $\mathscr{D}$-equivalence, because the

$\mathscr{D}$-equivalences form a fundamental localizer and W_∞ is the minimal fundamental localizer, which implies that the canonical arrow $(p_I)_! p_I^* \to (p_J)_! p_J^*$, induced from the adjunction $u_! \vdash u^*$, is a natural isomorphism, and hence, $F_!(u)$ is an isomorphism in $\mathscr{D}(e)$, from where we deduce (in virtue of the universal property of the localization) the existence of a unique functor

$$F_! : \mathsf{Hot} \longrightarrow \mathscr{D}(e)$$

(which we also denote by $F_!$), such that the diagram

$$\begin{tikzcd}
Cat \arrow[r, "F_!"] \arrow[d, "\gamma"'] & \mathscr{D}(e) \\
\mathsf{Hot} \arrow[ur, "F_!"'] &
\end{tikzcd}$$

commutes, where γ denotes the Gabriel-Zisman localization of W_∞. Since the objects of Hot are the same of Cat and γ is the identity map on the objects, we always have $F_!(I) = (p_I)_! p_I^*(F)$ for every object I in Hot. In particular, $F_!(e) \cong F$. In synthesis, there exists a functor

$$\mathscr{D}(e) \longrightarrow \underline{Hom}(\mathsf{Hot}, \mathscr{D}(e)), \quad F \mapsto F_!, \quad f \mapsto f_!$$

such that $F_!(e) \cong F$ (resp. $f_!(e) \cong f$) in $\mathscr{D}(e)$. Denoting by Π the evaluation at the point $F \mapsto F(e)$ from $\underline{Hom}(\mathsf{Hot}, \mathscr{D}(e))$ to $\mathscr{D}(e)$, and by Δ the functor $F \mapsto F_!$ from $\mathscr{D}(e)$ to $\underline{Hom}(\mathsf{Hot}, \mathscr{D}(e))$, we have a natural isomorphism $\Pi\Delta \cong 1_{\mathscr{D}(e)}$.

(ii). In order to extend the functor $F_! : \mathsf{Hot} \to \mathscr{D}(e)$, defined in (i), to a morphism of derivators $F_! : \mathbf{Hot} \to \mathscr{D}$, we can use the Grothendieck cointegration construction, which associates to each functor $X : I^o \to Cat$, with I being a small category, a Grothendieck fibration $p_X : \nabla X \to I$. We recall briefly the construction of ∇X:

- The objects of ∇X are pairs (i, s), where $i \in Ob(I)$ and $s \in Ob(X_i)$;

-The arrows $\varphi : (i, s) \to (i', s')$ of ∇X are pairs (α, φ) such that $\alpha : i \to i'$ is an arrow of I and $\varphi : s \to X_\varphi(s')$ is an arrow of X_i;

-The composition law of ∇X is defined by the formula

$$(\alpha', \varphi') \circ (\alpha, \varphi) = (\alpha' \circ \alpha, X_\varphi(\varphi') \circ \varphi)$$

Then, there exists the evident forgetful functor

$$\zeta_X : \nabla X \longrightarrow I, \quad (i,s) \mapsto i, \quad (\alpha, \varphi) \mapsto \alpha.$$

If $f : X \to Y$ is a natural transformations between functors from I^o to Cat, then, it induces a commutative triangle

$$\begin{tikzcd}
\nabla X \arrow[rr, "\nabla f"] \arrow[dr, "\zeta_X"'] & & \nabla Y \arrow[dl, "\zeta_Y"] \\
& I &
\end{tikzcd}$$

where $\nabla f(i,s) = (i, f_i(s))$ for an object (i,s) of ∇X, and $\nabla f(\alpha, \varphi) = (\alpha, f_i(\varphi))$ for an arrow (α, φ) of ∇X. Moreover, $f_i : X_i \to Y_i$ is an ∞-equivalence for every $i \in Ob(I)$, if, and only if, $i \setminus \nabla f : i \setminus \nabla X \to i \setminus \nabla Y$ is an ∞-equivalence for every $i \in Ob(I)$ (see *Remarque 3.1.13.* of [13]).

(iii). It follows from *Thèoréme 3.1.7.* and *Remarque 3.1.13.* of [13], that for every small category I and object X of $\mathbf{Hot}(I)$ (which is a functor from I^o to Cat), we have $(p_I)_!(X) \cong \nabla X$. Actually, we have the evident functors

$$\Xi_I : Cat/I \longrightarrow \underline{Hom}(I^o, Cat), \quad (A, A \to I) \mapsto (i \mapsto i \setminus A)$$

and

$$\Xi'_I : \underline{Hom}(I^o, Cat) \longrightarrow Cat/I, \quad X \mapsto (\nabla X, \nabla X \xrightarrow{\zeta_X} I).$$

If we define the class W'_I in Cat/I formed by the arrows $u : (A,p) \to (B,q)$, represented as a commutative triangle

$$\begin{tikzcd}
A \arrow[rr, "u"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\
& I &
\end{tikzcd}$$

such that $i \setminus u : i \setminus A \to i \setminus B$ is an ∞-equivalence for every $i \in Ob(I)$, then we can verify the equalities:

$$W'_I = \Xi_I^{-1}(W_I), \qquad W_I = (\Xi'_I)^{-1}(W'_I),$$

and the canonical functors

$$\overline{\Xi}_I : \mathbf{Hot}(I) \longrightarrow (W'_I)^{-1}Cat/I, \qquad \overline{\Xi'}_I : (W'_I)^{-1}Cat/I \longrightarrow \mathbf{Hot}(I)$$

induced from the universal property of localization, are quasi-inverse one each other, establishing, then, an equivalence of categories [15]

$$\mathbf{Hot}(I) = (W_I)^{-1}\underline{Hom}(I^o, Cat) \simeq (W'_I)^{-1}Cat/I.$$

Moreover, any arrow $u : I \to J$ induces, by composition, a functor

$$Cat/u : Cat/I \longrightarrow Cat/J \quad (A, p) \mapsto (A, u \circ p)$$

which sends W'_I-equivalences to W'_J-equivalences, inducing, then, a unique functor

$$\overline{Cat/u} : (W'_I)^{-1}Cat/I \longrightarrow (W'_J)^{-1}Cat/J$$

As a consequence of *Thèoréme 3.1.7.* and *Remarque 3.1.13.* of [13], we can verify that the left adjoint $u_! : \mathbf{Hot}(I) \to \mathbf{Hot}(J)$ of the functor $u^* : \mathbf{Hot}(J) \to \mathbf{Hot}(I)$, is given by

$$u_! = \overline{\Xi}_J \circ \overline{Cat/u} \circ \overline{\Xi'}_I.$$

Applying the previous definition for the unique existent functor $p_I : I \to e$ in Cat, we can verify easily that

$$(p_I)_!(X) = \nabla X$$

for every object X of $\mathbf{Hot}(I)$.

(iv). Now, for any small category I, we can form a functor

$$(F_!)_I : Cat(I) = \underline{Hom}(I^o, Cat) \longrightarrow \mathscr{D}(I), \quad X \mapsto (\zeta_X)_!(p_{\nabla X})^*(F).$$

Moreover, if $f : X \to Y$ is an arrow in $\underline{Hom}(I^o, Cat)$ such that $f_i : X_i \to Y_i$ is an ∞-equivalence for every $i \in Ob(I)$, then, the functor $\nabla f : \nabla X \to \nabla Y$

---

[15] See [13] and [14] for all the details.

is ∞-coaspherical along I, i.e., for the commutative triangle

$$\begin{tikzcd}
\nabla X \arrow[rr, "\nabla f"] \arrow[dr, "\zeta_X"'] & & \nabla Y \arrow[dl, "\zeta_Y"] \\
& I &
\end{tikzcd}$$

we have that $i\backslash\nabla f : i\backslash\nabla X \to i\backslash\nabla Y$ is an ∞-equivalence for every $i \in Ob(I)$, which implies that $\nabla f : \nabla X \to \nabla Y$ is local $\mathscr{D}$-coaspherical along I. Then, it follows from (2.1.8), that the canonical arrow

$$(F_!)_I(X) = (\zeta_X)_!(p_{\nabla X})^*(F) \longrightarrow (\zeta_Y)_!(p_{\nabla Y})^*(F) = (F_!)_J(X)$$

is an isomorphism in $\mathscr{D}(I)$. Hence, it follows from the universal property of localization, the existence of a unique arrow

$$(F_!)_I : \mathbf{Hot}(I) \longrightarrow \mathscr{D}(I)$$

also denoted by $(F_!)_I$, such that the diagram

$$\begin{tikzcd}
Cat(I) \arrow[r, "(F_!)_I"] \arrow[d, "\gamma_I"'] & \mathscr{D}(I) \\
\mathbf{Hot}(I) \arrow[ur, "(F_!)_I"'] &
\end{tikzcd}$$

commutes.

(v). In order to verify the pseudo-functoriality of the application

$$(F_!)_I : \mathbf{Hot}(I) \longrightarrow \mathscr{D}(I),$$

for I varying through the small categories i.e., to show that if $u : I \to J$ is a functor between small categories, then the square

$$\begin{tikzcd}
\mathbf{Hot}(J) \arrow[r, "(F_!)_J"] \arrow[d, "u^*"'] & \mathscr{D}(J) \arrow[d, "u^*"] \\
\mathbf{Hot}(I) \arrow[r, "(F_!)_I"'] & \mathscr{D}(I)
\end{tikzcd}$$

commutes up to (a canonical) isomorphism, remark that for each object $X : J^o \to Cat$ of $\mathbf{Hot}(J)$, there is a cartesian square

$$\begin{tikzcd}
\nabla u^*(X) \arrow[r, "\widetilde{u}"] \arrow[d, "\zeta'_X"'] & \nabla X \arrow[d, "\zeta_X"] \\
I \arrow[r, "u"'] & J.
\end{tikzcd}$$

with ζ_X (and, consequently ζ'_X) being a fibration. Now, to say that there exists an isomorphism $(F_!)_I u^*(X) \cong u^*(F_!)_J(X)$, means to say that the canonical base change morphism of the above square

$$(\zeta'_X)_!(\widetilde{u})^* \longrightarrow u^*(\zeta_X)_!$$

is an isomorphism. Indeed, it follows from (1.3.27) that, for any derivator $\mathscr{D}$ and for any cartesian commutative square of small categories of the form:

$$\begin{tikzcd}
A' \arrow[r, "q'"] \arrow[d, "p'"'] & A \arrow[d, "p"] \\
B' \arrow[r, "q"'] & B,
\end{tikzcd}$$

where p (and, consequently, p') is a Grothendieck fibration, the canonical base change morphism

$$(p')_!(q')^* \longrightarrow p^* q_!,$$

presented in (1.3.3), is an isomorphism. Then, since the square

$$\begin{tikzcd}
\nabla u^*(X) \arrow[r, "\widetilde{u}"] \arrow[d, "\zeta'_X"'] & \nabla X \arrow[d, "\zeta_X"] \\
I \arrow[r, "u"'] & J,
\end{tikzcd}$$

is cartesian and $\zeta_X : \nabla X \to J$ is a Grothendieck fibration, we have a canonical isomorphism:

$$(\zeta'_X)_!(\widetilde{u})^* \longrightarrow u^*(\zeta_X)_!,$$

which implies that the square

$$\begin{tikzcd}
\mathbf{Hot}(J) \arrow[r, "(F_!)_J"] \arrow[d, "u^*"'] & \mathscr{D}(J) \arrow[d, "u^*"] \\
\mathbf{Hot}(I) \arrow[r, "(F_!)_I"'] & \mathscr{D}(I)
\end{tikzcd}$$

commutes up to a canonical isomorphism, and we can verify that $F_! : \mathbf{Hot} \to \mathscr{D}$ really defines a morphism of derivators from **Hot** to $\mathscr{D}$ according to (1.2.9).

(vi). Therefore, there exists a morphism of derivators $F_! : \mathbf{Hot} \to \mathscr{D}$, associated to F, and we can verify (extending the arguments of (i)), that if $f : F \to G$ is an arrow of $\mathscr{D}(e)$, then, it induces an arrow $f_! : F_! \to G_!$ in the category $\mathcal{H}om(\mathbf{Hot}, \mathscr{D})$. Moreover, the left Kan extension $(p_I)_!(X)$ of an object X of $\mathbf{Hot}(I)$, for I being a small category, can be defined by the formula:

$$(p_I)_!(X) =_{df} \nabla X.$$

Therefore,

$$\begin{aligned}
(p_I)_!(F_!)_I(X) &\cong (p_I)_!(\zeta_X)_!(p_{\nabla X})^*(F) \\
&\cong (p_I\zeta_X)_!(p_{\nabla X})^*(F) \\
&\cong (p_{\nabla X})_!(p_{\nabla X})^*(F) \\
&\cong F_!(\nabla X) \\
&\cong F_!(p_I)_!(X),
\end{aligned}$$

which means that $F_! : \mathbf{Hot} \to \mathscr{D}$ is a homology from **Hot** to $\mathscr{D}$. Since a morphism of (right) derivators is a homology if, and only if, it commutes with all left Kan extensions (by (1.3.6) and (1.3.23)), $F_!$ commutes with all left Kan extensions, and hence, $F_!$ is actually an object of the category $\mathcal{H}om_!(\mathbf{Hot}, \mathscr{D})$, and we have defined a functor

$$\mathscr{D}(e) \longrightarrow \mathcal{H}om_!(\mathbf{Hot}, \mathscr{D}), \quad F \mapsto F_!$$

such that, for each object F in $\mathscr{D}(e)$, $F_!(e) \cong F$ in $\mathscr{D}(e)$, i.e., the evaluation at the point functor $\mathcal{F} \mapsto \mathcal{F}(e)$ from $\mathcal{H}om_!(\mathbf{Hot}, \mathscr{D})$ to $\mathscr{D}(e)$, is a left quasi-inverse of the the functor $F \mapsto F_!$ from $\mathscr{D}(e)$ to $\mathcal{H}om_!(\mathbf{Hot}, \mathscr{D}(e))$.

(vii). Finally, we can verify that, if $\mathcal{F} : \mathbf{Hot} \to \mathscr{D}$ is an object of $\mathcal{H}om_!(\mathbf{Hot}, \mathscr{D})$, then $\mathcal{F}(e)_! \cong \mathcal{F}$. Therefore, we have an equivalence of categories:

$$\mathcal{H}om_!(\mathbf{Hot}, \mathscr{D}) \longrightarrow \mathscr{D}(e), \quad \mathcal{F} \mapsto \mathcal{F}(e).$$

which ends the proof.

Corollary 2.2.6. *The functor*

$$\mathcal{H}om_!(\mathbf{Hot}, \mathbf{Hot}) \longrightarrow \mathsf{Hot}, \quad \mathcal{F} \mapsto \mathcal{F}(e)$$

is an equivalence of categories.

Corollary 2.2.7. *For every model category* (C, W, Cof, Fib)*, the functor*

$$\mathcal{H}om_!(\mathbf{Hot}, \mathbf{Ho}_W C) \longrightarrow \mathsf{Ho}_W C, \quad \mathcal{F} \mapsto \mathcal{F}(e)$$

is an equivalence of categories.

Proof. Immediate from (1.3.29) and (2.2.5).

□

Definition 2.2.8. *The category* Hot*, of homotopy types, is the category of absolute coefficients of the generic derivator* **Hot***.*

Remark 2.2.9. In the proof of the theorem (2.2.5), which characterizes the derivator **Hot** as a solution of a universal problem, we did not suppose that **Hot** satisfies all the axioms of derivators. We only admitted the existence of direct homological images for **Hot**, which are constructed using only elementary categorical methods in the third chapter of [13]. Moreover, we only need to suppose the following three properties for $\mathscr{D}$:

1. $\mathscr{D}$ admits direct homological images.

2. $W_{\mathscr{D}}$ is a fundamental localizer and $W_\infty \subseteq W_{\mathscr{D}}$

3. Every cartesian square of the form

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "u'"'] & A \arrow[d, "u"] \\ B' \arrow[r] & B \end{tikzcd}$$

in Cat, where u is a Grothendieck fibration, is $\mathscr{D}$-exact.

Clearly, the previous conditions are verified for every right derivator $\mathscr{D}$, but rigorously, the universal property of **Hot** is with respect to the 2-subcategory $\mathcal{P}Der_!$ of $\mathcal{P}Der$ satisfying only these conditions. There exists a dual version concerning the pre-derivator $\mathbf{Hot}^o$ which could be even easier to proof:

1. $\mathscr{D}$ admits direct cohomological images
2. $W_{\mathscr{D}}$ is a fundamental localizer and $W_\infty \subseteq W_{\mathscr{D}}$.
3. Every co-cartesian square of the form

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "u'"'] & A \arrow[d, "u"] \\ B' \arrow[r] & B \end{tikzcd}$$

in Cat, where u' is a Grothendieck cofibration, is $\mathscr{D}$-exact.

Actually, the all the arguments in the proof of (2.2.5) could be dualized and made relative to a fundamental localizer W such that $W \subseteq W_{\mathscr{D}}$, but this last condition means precisely that $\mathscr{D}$ is $\mathbf{Hot}_W$-local. Therefore, if W is a fundamental localizer such that $\mathbf{Hot}_W$ is a right derivator, then $\mathbf{Hot}_W$ is necessarily a right free derivator according to (2.2.2).

In the following, we give a conceptual definition of the singular (co)homology derived from the universal property (2.2.5) of the derivator **Hot**.

2.2.10. In the perspectives adopted by Quillen in [4] and also by Grothendieck in [10], cohomology is the abelianization of homotopy. Following this hypothesis, we going to define a derivator **Hotab**, which will be called the derivator of abelian homotopy types. We end this section with a conceptual definition of singular homology and singular cohomology with coefficients in an abelian group G. This definition can be realized directly in the language of derivators since it is a trivial consequence of (2.2.5). However, in order to illustrate to the reader that this definition really agrees with the traditional definition of singular homology and singular cohomology of topological spaces, we assume all the equivalences exposed in 'A Mathematical History of Homotopy Types'.

2.2.11. We remember that the category of sets Ens is the absolute co-complete category, i.e., given any co-complete category $\mathcal{C}$, the functor

$$\underline{Hom}_!(Ens, \mathcal{C}) \longrightarrow \mathcal{C}, \quad F \mapsto F(pt)$$

is an equivalence of categories, where pt denotes the terminal object in Ens (which is essentially unique up to a canonical isomorphism and can be even defined as the set $1 = \{\varnothing\}$). Now, let Ab be the category of abelian groups. Since Ab is co-complete, there exists a (essentially unique) functor $\mathbb{Z}[?] : Ens \longrightarrow Ab$, commuting with inductive limits, such that $\mathbb{Z}[pt] \cong \mathbb{Z}$ in Ab. Clearly, $\mathbb{Z}[X]$ is the free abelian group of a set X, which means that the functor $\mathbb{Z}[?] : Ens \to Ab$ is the universal abelizanization functor of sets. We can define, then, the homology (resp. the cohomology) of a set X, with coefficients in an abelian group G, by

$$H_*(X; G) =_{df} \mathbb{Z}[X] \otimes G, \qquad H^*(X; G) =_{df} Hom_{Ab}(\mathbb{Z}[X], G).$$

2.2.12. Let $\widehat{\Delta}_{ab}$ be the category of abelian groups in $\widehat{\Delta}$, i.e., the category of functors from Δ^o to the category of abelian groups Ab. The forgetful functor $Ab \to Ens$ induces, by composition, a functor

$$U : \underline{Hom}(\Delta^o, Ab) = \widehat{\Delta}_{ab} \longrightarrow \widehat{\Delta} = \underline{Hom}(\Delta^o, Ens),$$

and the abelianization functor $\mathbb{Z}[?] : Ens \to Ab$ can be extended to a functor

$$\mathbb{Z}[?] : \widehat{\Delta} \longrightarrow \widehat{\Delta}_{ab},$$

which we also denote by $\mathbb{Z}[?]$. We can define the class of arrows W_{ab} in $\widehat{\Delta}_{ab}$ which are carried to an ∞-equivalence in $\widehat{\Delta}$ through the forgetful functor. With the previous notations, we have that $\mathbb{Z}[?] : \widehat{\Delta} \to \widehat{\Delta}_{ab}$ is left adjoint to the forgetful functor $U : \widehat{\Delta}_{ab} \to \widehat{\Delta}$, and from Proposition 2.14. of [25], we have an adjunction between the localizers $(\widehat{\Delta}, W_{\widehat{\Delta}})$ and $(\widehat{\Delta}, W_{ab})$ in the sense of (1.1.22), for $U(W_{ab}) \subseteq W_{\widehat{\Delta}}$ and $\mathbb{Z}[W_{\widehat{\Delta}}] \subseteq W_{ab}$, which implies in a Grothendieck adjunction between the prederivators $\mathbf{Ho}_{W_s}\widehat{\Delta}$ and $\mathbf{Ho}_{W_{ab}}\widehat{\Delta}_{ab}$.

2.2.13. It follows from Section II.4 of [4], or from Theorem 2.6. of [25], that the localizer $(\widehat{\Delta}_{ab}, W_{ab})$, defined in (2.2.12), is actually a Quillen localizer, where the fibrations are the morphisms $f : A \to B$ of simplicial abelian groups which are send to Kan fibrations via the forgetful functor

$U : \widehat{\Delta}_{ab} \to \widehat{\Delta}$, and the cofibrations are the arrows with the left lift property with respect to trivial fibrations (see (1.1.61) for the description of the Quillen model category structure on the category of simplicial sets). Hence, the pair $(\widehat{\Delta}_{ab}, W_{ab})$ is a Grothendieck localizer (from (1.3.29)), and we can define the derivator **Hotab** as being the derivator $\mathbf{Ho}_{W_{ab}}\widehat{\Delta}_{ab}$ associated to the localizer $(\widehat{\Delta}_{ab}, W_{ab})$. The category Hotab, of the absolute coefficients of the derivator **Hotab**, will be called the category of *abelian homotopy types*. Moreover, it follows from the arguments presented in (2.2.12) and from Proposition 2.14. of [25], that there exists a morphism of derivators:

$$\mathbb{Z}[?] : \mathbf{Hot} \simeq \mathbf{Ho}_{W_s}\widehat{\Delta} \longrightarrow \mathbf{Ho}_{W_{Ab}}\widehat{\Delta}_{ab} = \mathbf{Hotab},$$

which admits a right adjunction in each component. Then, $\mathbb{Z}[?]$ commutes with left Kan extensions (in virtue of (1.3.8)), and $\mathbb{Z}[*] \cong \mathbb{Z}$ in Hotab, where $\mathbb{Z}$ denotes (with an abuse of notation) the constant simplicial abelian group with coefficients in $\mathbb{Z}$. Therefore, $\mathbb{Z}[?] : \mathbf{Hot} \to \mathbf{Hotab}$ is essentially the unique morphism of derivators from **Hot** to **Hotab**, commutating with left Kan extensions, and carrying the point to the image of $\mathbb{Z}$ in Hotab (in virtue of the universal property (2.2.5) of the derivator **Hot**).

In the following, we show how the abelianization morphism of derivators $\mathbb{Z}[?] : \mathbf{Hot} \to \mathbf{Hotab}$ is essentially the traditional singular homology of topological spaces. For this, we recall the well known Dold-Kan correspondence, which affirms that the category $\widehat{\Delta}_{ab}$ is equivalent to the category of connective complexes of abelian groups $Comp_{\leq 0}(Ab)$, and that the derived category $D_{\leq 0}(Ab)$ is equivalent to the category Hotab of abelian homotopy types. More generally, we show that the Dold-Kan correspondence allows us to deduce an equivalence between the derivators **Hotab** and $\mathscr{D}_{\leq 0}(Ab)$. The original paper about the Dold-Kan correspondence is [45], and a detailed exposition of the subject (with the proofs of all the results we mention in the sequel) is given in Chapter III, Section 2 of [25].

2.2.14. Let *Ab* be the category of abelian groups and *Comp*(*Ab*) be the category of complexes of abelian groups (in cohomological notation), i.e., the objects K^* of *Comp*(*Ab*) are cohomological complexes of abelian groups:

$$... \to K^{n-1} \xrightarrow{d^{n-1}} K^n \xrightarrow{d^n} K^{n+1} \to ...$$

where $d^{n+1} \circ d^n = 0$. Then, we can form the usual cohomology groups of the complex K^*, defined as $H^n(K) = Ker(d^n)/Im(d^{n-1})$, since, by definition, $Im(d^{n-1}) \subseteq Ker(d^n)$. We oppose the cohomological notation to the homological notation defining, for each complex K^*, the homological complex K_*, such that $K_n = K^{-n}$ and $d_n = d^{-n}$ for $n \in \mathbb{Z}$. We compute the homology groups of K_* from the cohomology groups of K^* by the formula:

$$H_n(K_*) = H^{-n}(K^*).$$

In particular, a connective homological complex K_* of abelian groups, i.e., a homological complex of abelian groups in degree ≥ 0:

$$... \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} ... \to C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

can be thought as a cohomological complex of abelian groups in degree ≤ 0. We recall that the quasi-isomorphisms in $Comp(Ab)$ are the arrows of abelian group complexes which induce an isomorphism in each cohomology group. We denote by W_{qis} the class of quasi-isomorphisms in $Comp(Ab)$. From (1.1.65), the pair $(Comp(Ab), W_{qis})$ is a Quillen localizer, and hence, it is also a Grothendieck localizer (by (1.3.29)), which means that the prederivator $\mathscr{D}(Ab)$, associated to the localizer $(Comp(Ab), W_{qis})$, is acutally a derivator. It is remarkable that the category $D(Ab)$ of absolute coefficients of $\mathscr{D}(Ab)$ is precisely the derived category of abelian groups.

2.2.15. Given an abelian group G, we denote also by G the correspondent abelian group complex concentrated in degree zero, i.e., G denotes the abelian group complex K:

$$... \to K^0 \to ... \to K^{n-1} \to K^n \to K^{n+1} \to ...$$

such that $K^0 = G$ and $K^n = 0$ for $n \neq 0$. For any abelian group G, considered as a complex of abelian groups, there are the functors

$$? \otimes G : Comp(Ab) \longrightarrow Comp(Ab)$$

and

$$Hom(?, G) : Comp(Ab)^o \longrightarrow Comp(Ab)^o.$$

The left (resp. right) derived functor of $? \otimes G$ (resp. $Hom(?, G)$) induces a morphism of derivators

$$? \otimes_{\mathbf{L}} G : \mathscr{D}(Ab) \longrightarrow \mathscr{D}(Ab)$$

(resp.

$$\mathbf{R}Hom(?, G) : \mathscr{D}(Ab)^o \longrightarrow \mathscr{D}(Ab)$$

which commutes with left (resp. right) Kan extensions, because for any model category, the left (resp. right) Kan extensions are defined as left (resp. right) derived functors.

2.2.16. We recall that in the category Δ, there are two special types of arrows: for $n \geq 1$ and $i \in [n]$, we have the i-th face map $\delta_i^n : \Delta_{n-1} \to \Delta_n$, which is the unique injective non-decreasing function from Δ_{n-1} to Δ_n such that $i \notin Im(\delta_i^n)$, and can be explicitly described by the formulas:

$$\delta_i^n(k) = k \quad \text{for} \quad k < i, \qquad \delta_i^n(k) = k + 1 \quad \text{for} \quad k \geq i,$$

and we have the i-th degeneracy maps $\sigma_i^n : \Delta_{n+1} \to \Delta_n$, which is the unique surjective non-decreasing function from $[n+1]$ to $[n]$ such that $(\sigma_i^n)^{-1}(\{i\})$ has precisely two elements. We can also describe the arrows σ_i^n explicitly by the formulas:

$$\sigma_i^n(k) = k \quad \text{for} \quad k \leq i, \qquad \sigma_i^n(k) = k - 1 \quad \text{for} \quad k > i.$$

Given a simplicial set X, we can define the functions $d_i = X(\delta_i^n) : X_n \to X_{n-1}$ and $s_i = X(\sigma_i^n) : X_n \to X_{n+1}$. In particular, if A is a simplicial abelian group, then the arrows $d_i : A_n \to A_{n-1}$ are abelian group homomorphisms and we can form the boundary morphisms

$$\partial_n : A_n \longrightarrow A_{n-1}$$

defined by the formula:

$$\partial_n =_{df} \sum_{i=0}^{n} (-1)^i d_i.$$

An extensive computation proves that $\partial_{n-1} \circ \partial_n = 0$ for every $n \in \omega$. Therefore, we can assign to each simplicial abelian group A, a connective homological abelian group complex:

$$... \to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to ... \to A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} 0 \to 0 \to ...$$

called the *Moore complex* of A. Moreover, if $f : A \to B$ is a morphism between simplical abelian groups, then the morphisms $f_n : A_n \to B_n$ in each component are such that the squares

$$\begin{tikzcd}
A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[d, "f_{n-1}"] \\
B_n \arrow[r, "\partial_n"'] & B_{n-1}
\end{tikzcd}$$

commute, and we can define a functor

$$\mathcal{M} : \widehat{\Delta}_{ab} \longrightarrow Comp_{\leq 0}(Ab),$$

called the *Moore functor*. In order to establish an equivalence between the categories $\widehat{\Delta}_{ab}$ and $Comp_{\leq 0}(Ab)$, we have to normalize the *Moore complex*. The *normalized complex* $\mathcal{N}(A)$ of a simplicial abelian group A is the subcomplex of A such that

$$\mathcal{N}(A)_n =_{df} \bigcap_{i=1}^{n} Ker(d_i : A_n \to A_{n-1}), \quad n > 0,$$

with $\mathcal{N}(A)_0 = A_0$ and $\mathcal{N}(A)_n = 0$ for $n < 0$.

Theorem 2.2.17. (Dold-Kan Correspondece*) The normalized complex functor* $\mathcal{N} : \widehat{\Delta}_{ab} \to Comp_{\leq 0}(Ab)$ *is an equivalence of categories. Moreover, this equivalence provides an equivalence between the derivators* **Hotab** *and* $\mathscr{D}_{\leq 0}(Ab)$*.*

Proof. See Corollary 2.3., Theorem 2.4., and Proposition 2.14. of [25].

□

Remark 2.2.18. An important property of the normilized complex $\mathcal{N}(A)$ of a simplicial abelian group A, is that $\mathcal{N}(A)$ is quasi-isomorphic to the Moore complex $\mathcal{M}(A)$. We remember that the traditional singular homology (resp. singular cohomology) of a topological space X, with coefficients in an abelian group G, is defined as the homology of the complex $\mathcal{M}(Simp(X)) \otimes G$ (resp. $Hom(\mathcal{M}(Simp(X)), G)$), where $Simp(X)$ is the standard simplicial set associated to X, such that

$$Simp(X)_n = Hom_{Top}(|\Delta_n|, X), \qquad n \in \omega.$$

Therefore, the singular homology, with coefficients in an abelian group G, can be defined by the composition of the functors:

$$Top \xrightarrow{Simp} \widehat{\Delta} \xrightarrow{\mathbb{Z}[?]} \widehat{\Delta}_{ab} \xrightarrow{\mathcal{N}} Comp_{\leq 0}(Ab) \hookrightarrow Comp(Ab) \xrightarrow{? \otimes G} Comp(Ab),$$

and the singular cohomology, by the composition of the functors:

$$Top \xrightarrow{Simp} \widehat{\Delta} \xrightarrow{\mathbb{Z}[?]} \widehat{\Delta}_{ab} \xrightarrow{\mathcal{N}} Comp_{\leq 0}(Ab) \hookrightarrow Comp(Ab) \xrightarrow{Hom(?,G)} Comp(Ab)$$

where the last functor $Hom(?, G)$ is contravariant. Since the functors $Simp$ and $\mathcal{N}$ induce respectively the equivalences of derivators $\mathbf{Ho}_{W_{Top}}Top \simeq \mathbf{Ho}_{W_s}\widehat{\Delta}$ and $\mathbf{Hotab} = \mathbf{Ho}_{W_{ab}}\widehat{\Delta}_{ab} \simeq \mathscr{D}_{\leq 0}(Ab)$, and $\mathbb{Z}[?]$ (resp. the inclusion $Comp_{\leq 0}(Ab) \hookrightarrow Comp(Ab)$, $? \otimes G$) induces a morphism of derivators $\mathbf{Ho}_{W_s}\widehat{\Delta} \to \mathbf{Hotab}$ (resp. $\mathscr{D}_{\leq 0}(Ab) \to \mathscr{D}(Ab)$, $? \otimes_{\mathsf{L}} G : \mathscr{D}(Ab) \to \mathscr{D}(Ab)$) commuting with left Kan extensions, then the singular homology defines a morphism of derivators

$$Sing_G : \mathbf{Hot} \simeq \mathbf{Ho}_{W_{Top}}Top \longrightarrow \mathscr{D}(Ab),$$

commuting with left Kan extensions, such that $Sing_G(e) \cong G$ in $D(Ab)$. Analogously, the singular cohomology defines a morphism of derivators

$$Sing_G^* : \mathbf{Hot}^o \simeq (\mathbf{Ho}_{W_{Top}}Top)^o \longrightarrow \mathscr{D}(Ab),$$

commuting with right Kan extensions, such that $Sing_G^*(e) \cong G$ in $D(Ab)^o$.

From the previous digressions, and from (2.2.5), we can now redefine the singular (co)homology as following:

Definition 2.2.19. *Let G be an abelian group. The singular homology (resp. cohomology) with coefficients in G is the (essentially) unique homology of derivators*

$$Sing_G : \mathbf{Hot} \longrightarrow \mathscr{D}(Ab)$$

(resp. cohomology of derivators

$$Sing_G^* : \mathbf{Hot}^o \longrightarrow \mathscr{D}(Ab)$$

such that $Sing_G(e) \cong G$ *(resp.* $Sing_G^*(e) \cong G$*) in* $D(Ab)$*.*

The fact that the previous definition agrees with the traditional singular homology (resp. cohomology) of topological spaces is also a consequence of the results presented in [32].

2.2.20. The definition (2.2.19) allows us to interpret the singular homology with integer coefficients as an abelizanization of the homotopy types. In fact, denoting by $Sing$ the singular homology $Sing_{\mathbb{Z}}$ with coefficients in $\mathbb{Z}$, we have that $Sing$ corresponds to the (essentially unique) morphism of derivators

$$Sing : \mathbf{Hot} \longrightarrow \mathbf{Hotab}$$

commuting with left Kan extensions, such that $Sing(e) \cong \mathbb{Z}$ in Hotab. Analogously, the free abelian group functor $\mathbb{Z}[?] : Ens \to Ab$ is the the unique functor from the category of sets to the category of abelian groups, commuting with left Kan extensions, carrying the point to $\mathbb{Z}$.

Remark 2.2.21. Modelizing the homotopy types with small categories, we have that the homology of homotopy types, with coefficients in an abelian group G, is essentially the specialization of the functor

$$G_! : Cat \longrightarrow D_{\leq 0}(Ab)$$

given by $G_!(X) = (p_X)_! p_X^*(G) = \mathsf{H}_*^{\mathscr{D}_{\leq o}(Ab)}(X; G)$, defined in the proof of (2.2.5). In particular, the class of arrows $u : A \to B$ which induce an isomorphism on the homologies, form a fundamental localizer.

2.3 Test categories

2.3.1. Let A be a small category. Then, we can form the evident functor

$$i_A : \widehat{A} \longrightarrow Cat, \quad X \mapsto A/X$$

which admits a right adjoint

$$i_A^* : Cat \longrightarrow \widehat{A}, \quad C \mapsto (a \mapsto Hom_{Cat}(A/a, C)).$$

We remark that $i_A(h_a) \simeq A/a$ for every $a \in Ob(A)$ (by the Yoneda's Lemma). We denote by $W_{\widehat{A}}$ the class of arrows $f : X \to Y$ in $\widehat{A}$ such that $i_A(f)$ is an ∞-equivalence in Cat. Hence, it follows from the universal property of localization, that there exists a unique morphism $\overline{i}_A : \mathsf{Ho}_{W_{\widehat{A}}}\widehat{A} \to \mathsf{Hot}$, such that the digram

$$\begin{tikzcd}
\widehat{A} \arrow[r, "i_A"] \arrow[d, "\gamma_A"'] & Cat \arrow[d, "\gamma"] \\
\mathsf{Ho}_{W_{\widehat{A}}}\widehat{A} \arrow[r, "\overline{i}_A"'] & \mathsf{Hot}
\end{tikzcd}$$

commutes. If $i_A^*(W_\infty) \subseteq W_{\widehat{A}}$, then the functor i_A^* also induces a canonical functor $\overline{i^*}_A : \mathsf{Hot} \to \mathsf{Ho}_{W_{\widehat{A}}}\widehat{A}$ such that the diagram

$$\begin{tikzcd}
Cat \arrow[r, "i_A^*"] \arrow[d, "\gamma"'] & \widehat{A} \arrow[d, "\gamma_A"] \\
\mathsf{Hot} \arrow[r, "\overline{i^*}_A"'] & \mathsf{Ho}_{W_{\widehat{A}}}\widehat{A}
\end{tikzcd}$$

commutes. Moreover, it follows from (1.1.22) and from the fact that $i_A \vdash i_A^*$, that $\overline{i}_A \vdash \overline{i^*}_A$.

Definition 2.3.2. *Let A be a small category.*

1. *We say that A is a weak test category, if $i_A^*(W_\infty) \subseteq W_{\widehat{A}}$, and the functors $\overline{i^*}_A$ and $\overline{i}_A$ are equivalences of categories, quasi-inverse one each other.*

2. *We say that A is a local test category if for every $a \in Ob(A)$, the category A/a is a weak test category*

3. *We say that A is a test category if it is both a weak and local test category.*

Definition 2.3.3. 1. *A small category A is aspherical if the unique morphism $p_A : A \to e$ is an ∞-equivalence.*

2. *A functor $u : A \to B$ between small categories is aspherical if A/b is aspherical for every $b \in Ob(B)$.*

3. *A presheaf F over a small category A is aspherical if $i_A(F) = A/F$ is an aspherical category.*

4. *A morphism $f : F \to G$ of presheaves over a small category A is aspherical if the functor $i_A(f) : i_A(F) \to i_A(G)$ is an aspherical functor between small categories.*

5. *A morphism $f : F \to G$ of presheaves over a small category A is locally aspherical if the restricted morphism $f|_{A/a} : F|_{A/a} \to G|_{A/a}$ is aspherical over A/a for every $a \in Ob(A)$.*

6. *A presheaf F over A is locally aspherical if $F|_{A/a}$ is an aspherical presheaf over A/a for every $a \in Ob(A)$.*

Proposition 2.3.4. *Let A be a small category. Equivalent conditions:*

1. *A is a weak test category*

2. *A is aspherical and $i_A^*(W_\infty) \subseteq W_{\widehat{A}}$*

3. *For all small categories C, the co-unit morphism $\varepsilon : i_A i_A^*(C) \to C$ of the adjunction $i_A \vdash i_A^*$ is aspherical.*

4. *For all small categories C, the co-unit morphism $\varepsilon : i_A i_A^*(C) \to C$ of the adjunction $i_A \vdash i_A^*$ is an ∞-equivalence.*

5. *For all small categories C admitting a terminal object, the presheaf $i_A^*(C)$ is aspherical.*

Proof. See *Proposition 1.3.9.* of [13].

□

Proposition 2.3.5. *Let F be a presheaf over a small category A. Equivalent conditions:*

1. *F is locally aspherical.*

2. *For every $a \in Ob(A)$, the presheaf $h_a \times F$ is aspherical over A.*

Definition 2.3.6. *Let A be a small category. A separating segment in $\widehat{A}$ is a presheaf $\mathbf{I}$ over A, endowed with two structural arrows $\partial_0, \partial_1 : e_{\widehat{A}} \to \mathbf{I}$, such that the equalizer $Ker(\partial_0, \partial_1)$ is isomorphic to $\varnothing_{\widehat{A}}$.*

Definition 2.3.7. *Let A be a small category. The Lawvere object $\mathbb{L}_{\widehat{A}}$ of $\widehat{A}$ is defined as being the presheaf $i_A^*(\Delta_1)$.*

Lemma 2.3.8. *For all small categories A, the Lawvere object $\mathbb{L}_{\widehat{A}}$ is a strict separating segment.*

Proof. It follows from the fact that, for every object X of $\widehat{A}$, $Hom_{\widehat{A}}(X, \mathbb{L}_{\widehat{A}})$ is naturally isomorphic to the set of sub-objects of X. Since $\varnothing_{\widehat{A}}$ is a strict initial object in $\widehat{A}$, the canonical morphism $\varnothing_{\widehat{A}} \to X$ is a monomorphism for every $X \in Ob(\widehat{A})$, and since the identity arrow $e_{\widehat{A}} \to e_{\widehat{A}}$ and the canonical arrow $\varnothing_{\widehat{A}} \to e_{\widehat{A}}$ are both monomorphisms, they correspond respectively to arrows $\lambda_1 : e_{\widehat{A}} \to \mathbb{L}_{\widehat{A}}$ and $\lambda_0 : e_{\widehat{A}} \to \mathbb{L}_{\widehat{A}}$. With the previous notations, we can verify that $(\mathbb{L}_{\widehat{A}}, \lambda_0, \lambda_1)$ defines a strict separating object in $\widehat{A}$. In fact, consider a commutative square:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & e_{\widehat{A}} \arrow[d, "\lambda_1"] \\
e_{\widehat{A}} \arrow[r, "\lambda_0"'] & \mathbb{L}_{\widehat{A}}.
\end{tikzcd}$$

Now, it follows from the fact that $\varnothing_{\widehat{A}}$ is a strict initial object, that the square:

$$\begin{tikzcd}
\varnothing_{\widehat{A}} \arrow[r] \arrow[d] & \varnothing_{\widehat{A}} \arrow[d] \\
X \arrow[r] & e_{\widehat{A}}
\end{tikzcd}$$

is cartesian. Moreover, the commutative square

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "1_X"'] & e_{\widehat{A}} \arrow[d] \\
X \arrow[r] & e_{\widehat{A}}
\end{tikzcd}$$

is also cartesian. Moreover, the commutativity of the original square implies that $1_X : X \to X$ and $\varnothing_{\widehat{A}} \to X$ represent the same sub-object of X, and hence, the arrow $\varnothing_{\widehat{A}} \to X$ is an isomorphism.

□

Theorem 2.3.9. (Grothendieck-Maltsiniotis) - *Let A be a small category. Equivalent conditions:*

1. *A is a local test category.*
2. *The Lawvere object $\mathbb{L}_{\widehat{A}}$ is locally aspherical.*
3. *There exists a locally aspherical separating segment $(\mathbf{I}, \partial_0, \partial_1)$ in $\widehat{A}$.*

Moreover, if any one of these equivalent conditions are verified, then A is a test category if, and only if, A is aspherical.

Proof. A detailed proof is given in *Thèoréme 1.5.6* of [13]. We give here only a brief sketch. The implication (1) $\implies$ (2) is a consequence of the fact that A/a is a weak test category for every $a \in Ob(A)$ and the restriction $\mathbb{L}_{\widehat{A}}|_{A/a}$, of the Lawvere object $\mathbb{L}_{\widehat{A}}$ of $\widehat{A}$ to A/a, coincides with the Lawvere object $\mathbb{L}_{\widehat{A/a}}$ of the presheaf category $\widehat{A/a}$. Hence, this first implication follows directly from the definition of $\mathbb{L}_{\widehat{A/a}}$ as being the presheaf $i^*_{A/a}(\Delta_1)$, and from the last condition of (2.3.4), since Δ_1 is a category admitting a terminal object. The implication (2) $\implies$ (3) is immediate from (2.3.8). Now, the implication (3) $\implies$ (1) is more delicate. It follows from the fact that, for every $a \in Ob(A)$, the functor $i^*_{A/a}$ preserves projective limits (since it is a right adjoint), and we can verify that for every small category C admitting a terminal object, the presheaf $i^*_{A/a}(C)$ is $\mathbf{I}|_{A/a}$-contractible in $\widehat{A/a}$, which implies that $i^*_{A/a}(C)$ is aspherical over A/a, and hence, A/a is a weak test category (again by the last condition of (2.3.4)). The last assertion in the theorem follows from the condition (2) of (2.3.4) and from the definition of test categories (2.3.2). □

Corollary 2.3.10. (Maltsiniotis criteria) - *If A is a small category, admitting finite products, and there exists a strict separating object $(\mathbf{I}, \partial_0, \partial_1)$ in A, then A is a test category.*

Proof. It follows directly from (2.3.9). In fact, the representable presheaf of the object I defines a locally aspherical strict separating object in $\widehat{A}$ (in virtue of (2.3.5) since A is closed by finite products and any representable presheaf is trivially aspherical). Therefore, A is a local test category by the third condition of (2.3.9), which means that A/a is a weak test category for every $a \in Ob(A)$. Moreover, the hypothesis that A admits finite products implies that, in particular, A admits a terminal object e_A, and hence, A is also a weak test category, for $A \cong A/e_A$. Hence, A is a local test category and a weak test category, which means that A is a test category.

□

Corollary 2.3.11. 1. *The product of a local test category with an arbitrary category is a local test category.*

2. *The product of a test category with an aspherical category is a test category.*

Example 2.3.12. In virtue of the Maltsiniotis criteria, we have the following list of examples of test categories:

1. The category $\widetilde{\Delta}$ of non-empty finite sets admits finite products, and we can verify easily that the triple $(\{0,1\}, \partial_0, \partial_1)$, with ∂_0 (resp. ∂_1) being the function $0 \mapsto 0$ (resp. $0 \mapsto 1$) from $\{0\}$ to $\{0,1\}$, forms a strict separating object in $\widetilde{\Delta}$. Hence, $\widetilde{\Delta}$ is a test category! This means that the category Hot of homotopy types can be studied by the category of presheaves over non-empty finite sets!

2. Fix a locally small category C which admits finite products and an object $\mathbf{I}$ of C. Consider the full subcategory of C formed by the products $\mathbf{I}^n$, for $n \in \omega$, where $\mathbf{I}^0 = *$ (the terminal object of C).

3. The full subcategory *Cart* of the category of topological spaces formed by the cartesian spaces $\mathbb{R}^n$, for $n \in \omega$ (which is closed by products by the formular $\mathbb{R}^m \times \mathbb{R}^n \cong \mathbb{R}^{m+n}$ and admits $\mathbb{R}$ as a strict separating segment).

4. The subcategory $Cart_{diff}$ of *Cart* formed by the same objects, but with only differential maps as morphisms, is again a test category (for the same reasons of *Cart*).

Example 2.3.13. The category Δ is a test category (see *Proposition 1.5.13.* of [13]). It follows from (2.3.9) that, since Δ is an aspherical category (for it admits a terminal object Δ_0) and Δ_1 is a strict separating segment in $\widehat{\Delta}$, with the structural arrows being the unique non-decreasing existent functions

$$\partial_0 : 0 \longmapsto 0, \qquad \partial_1 : 0 \longmapsto 1$$

from Δ_0 to Δ_1 (see *Remargue 1.5.8.* of [13]), the only difficulty lies in to verify that Δ_1 is a locally aspherical presheaf, which, in virtue of the condition (2) of (2.3.5), reduces to verify that, for each $m \geq 0$, the presheaf $\Delta_m \times \Delta_1$ is aspherical. The strategy in [13] is then to compare the presehaf $\Delta_m \times \Delta_1$ with the representable presheaf Δ_{m+1}.

Theorem 2.3.14. (Cisinski-Grothendieck) - *Let A be a small category. If A is a local test category, then $\widehat{A}$ admits a model category structure, where $W_{\widehat{A}}$ is the class of weak equivalences, the cofibrations are the monomorphisms, and the fibrations are the arrows which admit the right lift property with respect to the trivial cofibrations.*

Idea of the proof - For a detailed proof of the theorem, see the *Théorème 1.4.3.* and the *Corollaire 4.2.18.* of [15]. We give here a sketch contemplating the crucial points of Cisinski's proof. The idea is to construct a model category structure *ex nihilo* over $(\widehat{A}, W_{\widehat{A}})$, where cofibrations are the monomorphisms and fibrations are the arrows admitting the right lift property with respect to trivial cofibrations. The technique to construct these model category structures is due to Cisinski, and it is developed in details in the first chapter of [15]. The construction of this model category over $\widehat{A}$ is probably inspired in the Theorem 1.7. exposed in [21] and attributed to Smith. Indeed, let κ be a cardinal. An ordered set I is called κ-filtering if for every $J \subseteq I$, such that $\mathfrak{c}(J) \leq \kappa$, there exists $i \in I$ with $j \leq i$ for all $j \in J$. A κ-filtered inductive limit in a locally small category $\mathcal{C}$ is an inductive limit indexed by a κ-filtering ordered set. Suppose that $\mathcal{C}$ is a locally small category which admits κ-filtered inductive limits. We say that an object X of $\mathcal{C}$ is κ-presentable if the functor

$$Hom_{\mathcal{C}}(X, ?) : \mathcal{C} \longrightarrow Ens$$

commutes with κ-filtered inductive limits, and we say that $\mathcal{C}$ is κ-accessible if every object in $\mathcal{C}$ is isomorphic to a κ-filtered inductive limit of κ-presentable objects, and such that the full subcategory $\mathcal{C}_\kappa$ of $\mathcal{C}$, formed by κ-presentable objects, is essentially small. Finally, we say that $\mathcal{C}$ is accessible if it is κ-accessible for some cardinal κ. A functor $F : \mathcal{C} \to \mathcal{C}'$ between accessible categories is accessible if F commutes with κ-filtered inductive limits for some cardinal κ.

Then, we can say that a class W of arrows in a category $\mathcal{C}$ is accessible if, considering W as a full subcategory of the category $FL(\mathcal{C})$ of arrows in $\mathcal{C}$, then W is accessible and the inclusion functor $W \hookrightarrow FL(\mathcal{C})$ is also accessible.

For every small category A, the category $\widehat{A}$ is accessible, and the category Cat of small categories is also accessible. Moreover, the functor $i_A : \widehat{A} \to Cat$ is accessible (because it is a left adjoint). If W is an accessible fundamental localizer (in the above sense), then $i_A^{-1}(W)$ is also accessible in $\widehat{A}$, and the Theorem 1.7. of [21] with the condition (iv) of the *Théorème 4.1.19.* of [15] implies that, in this case, if A is a local test category, then, assuming the accessibility of W_∞, the category $\widehat{A}$ admits a model category structure, where $W_{\widehat{A}}$ is the class of weak equivalences, the cofibrations are the monomorphisms, and the fibrations are the arrows which admit the right lift property with respect to trivial cofibrations. Now, the theory developed by Cisinski in [15] allows us to rewrite the accessibility condition for a fundamental localizer W by the following: W is accessible if there exists a small set S of arrows in Cat such that W is the fundamental localizer generated by S, i.e., W is the minimal fundamental localizer which contains S. Since the minimal fundamental localizer W_∞ is trivially accessible in this previous sense (for, W_∞ is the fundamental localizer generated by the empty set $\varnothing$), the category $\widehat{A}$ admits the alleged model category structure of (2.3.14) whenever A is a local test category.

Corollary 2.3.15. *The prederivator* **Hot** *is a derivator.*

Proof. Choose any test category A, for example, Δ, or any small category satisfying the Maltsiniotis criteria ((2.3.10)), as the small category of non-empty finite sets $\widetilde{\Delta}$. Then, the functors $i_A : \widehat{A} \to Cat$ and $i_A^* : Cat \to \widehat{A}$ establish an equivalence between the localizers $(\widehat{A}, W_{\widehat{A}})$ and (Cat, W_∞), which implies in an equivalence between the prederivators $\mathbf{Ho}_{W_{\widehat{A}}}(\widehat{A})$ and **Hot**. Now, it follows from (2.3.14), that the pair $(\widehat{A}, W_{\widehat{A}})$ is a Quillen localizer, and hence, a Grothendieck localizer (in virtue of the theorem (1.3.29)). Therefore, $\mathbf{Ho}_{W_{\widehat{A}}}\widehat{A}$ is a derivator, from where we conclude that **Hot** is also a derivator.

□

We end this section by giving a brief sketch of the proof that the minimal fundamental localizer W_∞ is precisely the class of arrows $u : A \to B$ in Cat such that $N(u) : N(A) \to N(B)$ is a weak equivalence of simplicial sets. We denote by W_s the class of simplicial weak equivalences in $\widehat{\Delta}$.

We begin introducing the notions of aspherical functors.

Definition 2.3.16. *Let A be a small category and $i : A \to Cat$ be a functor, which, by its turn, induces an evident functor:*

$$i^* : Cat \longrightarrow \widehat{\Delta}, \quad C \mapsto (a \mapsto Hom_{Cat}(i(a), C)).$$

We say that i is aspherical if the following conditions are verified:

1. *For every object $a \in Ob(A)$, the category $i(a)$ admits a terminal object. In particular, $i(a)$ is aspherical.*
2. *For every small category C, we have that C is aspherical precisely when the presheaf $i^*(C)$ is aspherical.*

2.3.17. Let $i : \Delta \to Cat$ be the canonical inclusion from Δ to Cat. Then, the nerve functor $N : Cat \to \widehat{\Delta}$ is precisely the functor $i^* : Cat \to \widehat{\Delta}$ described in the definition (2.3.16). Moreover, we can verify that the nerve functor is actually aspherical (see *Exemple 1.7.18* of [13]).

2.3.18. Let $i : A \to Cat$ be an aspherical functor, and, for each $a \in Ob(A)$, denote by e_a the terminal object of the category $i(a)$. Then, we can form a natural transformation

$$\alpha : i_A i^* \longrightarrow 1_{Cat}$$

given by the formula

$$\alpha_C : i_A i^*(C) = A/i^*(C) \longrightarrow C, \quad (a, i(a) \xrightarrow{v} C) \mapsto v(e_a),$$

in each small category C.

Lemma 2.3.19. *Let $N : Cat \to \widehat{\Delta}$ be the nerve functor. Then, we have the equality:*

$$W_\infty = N^{-1} i_\Delta^{-1}(W_\infty),$$

Proof. Consider the natural transformation

$$\alpha : i_\Delta N \longrightarrow 1_{Cat}$$

described in (2.3.18). Then, we can verify that $\alpha_A : \Delta/N(A) \to A$ is always an ∞-equivalence. Indeed, for every $a \in Ob(A)$, the categories $\Delta/N(A/a)$ and $(\Delta/N(A))/a$ are canonically isomorphic, and since the category A/a admits a terminal object, A/a is aspherical, which implies that $N(A/a)$ is an aspherical presheaf in $\widehat{\Delta}$ (because N is an aspherical functor), and hence, $\Delta/N(A/a)$ and $(\Delta/N(A))/a$ are both aspherical categories. Then, if we consider the functor

$$\alpha_A : \Delta/N(A) = i_\Delta N(A) \longrightarrow A,$$

we have that $(\Delta/N(A))/a$ is aspherical for every $a \in Ob(A)$, which implies that α_A is an ∞-equivalence (by an easy formal application of the axioms of fundamental localizers (2.1.10)). Now, let $u : A \to B$ be an arrow of small categories. From the commutative square

$$\begin{tikzcd}
i_\Delta N(A) \arrow[r, "\alpha_A"] \arrow[d, "i_\Delta N(u)"'] & A \arrow[d, "u"] \\
i_\Delta N(B) \arrow[r, "\alpha_B"'] & B
\end{tikzcd}$$

we have that $u \in W_\infty$ if, and only if, $i_\Delta N(u) \in W_\infty$ (in virtue of the property 2 out of 3 of W_∞, since α_A and α_B are both ∞-equivalences). Therefore, $W_\infty = N^{-1} i_\Delta^{-1}(W_\infty)$.

□

Lemma 2.3.20. *For every fundamental localizer W, we have the inclusion $W_s \subseteq i_\Delta^{-1}(W)$. In particular, $W_s \subseteq i_\Delta^{-1}(W_\infty)$.*

Proof. It follows from the fact that Δ is a test category ((2.3.13)) and for every fundamental localizer W, the class $i_\Delta^{-1}(W)$ is a Δ-localizer (see (1.1.63) and *Proposition 4.2.9.* of [15]). Moreover, $i_\Delta^{-1}(W)$ contains all the arrows of the form $\Delta_n \to \Delta_0$, $n \in \omega$. Hence, it follows from the characterization of W_s given in (1.1.64) as the minimal Δ-localizer containing the arrows $\Delta_n \to \Delta_0$, $n \in \omega$ [16], that $W_s \subseteq i_\Delta^{-1}(W)$ [17].

□

There is an alternative proof for the lemma (2.3.20) using the characterization of the model category structure over $\widehat{\Delta}$ presented in (1.1.61) that is the following: first we proof that for each fundamental localizer W, the class $(i_\Delta)^{-1}(W)$ is a class of weak equivalences in $\widehat{\Delta}$ according to (1.1.4). Then,

---

[16] See *Corollaire 2.1.21* of [15], with the remark that the author of [15] denotes by W_∞ the class W_s.

[17] In [15], Cisinski also characterizes W_s as the minimal Δ-localizer containing all the canonical projections $X \times \Delta_1 \to X$ for X varying trough the objects of $\widehat{\Delta}$.

we verify that the class $Cof \cap (i_\Delta)^{-1}(W)$ is saturated[18] (which follows from (2.1.16)). Clearly, $(i_\Delta)^{-1}(W)$ also contains all the canonical arrows $\Delta_n \to \Delta_0$, $n \in \omega$ (because $i_\Delta(\Delta_0) = e$ and $i_\Delta(\Delta_n) = \Delta/\Delta_n$ admits a terminal object). Using the previous fact, we can proof by induction that $(i_\Delta)^{-1}(W)$ contains all the monomorphisms of the form $j_{n,k} : \Lambda^k_n \to \Delta_n$, $n \geq 1$, $0 \leq j \leq n$. It follows from the fact that $Cof \cap (i_\Delta)^{-1}(W)$ is saturated that $Cof \cap (i_\Delta)^{-1}(W)$ contains $Cof \cap W_s$, because $Cof \cap W_s$ is the smallest saturated class of arrows in $\widehat{\Delta}$ containing the inclusions $j_{n,k} : \Lambda^k_n \to \Delta_n$, $n \geq 1$, $0 \leq k \leq n$. Then, it follows from (1.1.61) that $W_s \subseteq (i_\Delta)^{-1}(W)$.

Theorem 2.3.21. $W_\infty = N^{-1}(W_s)$.

Proof. First, we have that $W_s \subseteq i_\Delta^{-1}(W_\infty)$ (from (2.3.20)), and $W_\infty = N^{-1} i_\Delta^{-1}(W_\infty)$ (from (2.3.20)), which implies $N^{-1}(W_s) \subseteq N^{-1} i_\Delta^{-1}(W_\infty) = W_\infty$. Assuming that $N^{-1}(W_s)$ is a fundamental localizer, we conclude, by the minimality of W_∞, that $W_\infty = N^{-1}(W_s)$. Now, it follows from the Artin-Mazur-Moerdjik characterization of the class $N^{-1}(W_s)$ via topos cohomology (exposed in 'A Mathematical History of Homotopy Types' of this dissertation) that $N^{-1}(W_s)$ is a fundamental localizer, which concludes the proof.

□

We can know conclude the following

Theorem 2.3.22. *Let $u : A \to B$ be a morphism of small categories. Equivalent conditions:*

1. $N(u) : N(A) \to N(B)$ *is a weak equivalence of simplicial sets.*
2. $|N(u)| : |N(A)| \to |N(B)|$ *is a homotopy equivalence of topological spaces.*
3. $u : A \to B$ *is a* W*-equivalence for every fundamental localizer* W.
4. $u : A \to B$ *is an* ∞*-equivalence of small categories.*
5. $u : A \to B$ *is a* **Hot***-equivalence.*
6. $u : A \to B$ *is* $\mathscr{D}$*-equivalence for every derivator* $\mathscr{D}$.
7. $\widehat{u} : \widehat{A} \to \widehat{B}$ *is an Artin-Mazur equivalence of toposes.*

---

[18] We recall that Cof is the class of all monomotphisms in $\widehat{\Delta}$.

8. $\widetilde{|N(u)|} : \widetilde{|N(A)|} \to \widetilde{|N(B)|}$ *is an Artin-Mazur equivalence of toposes.*

Moreover, we have the equalities:

$$W_s = W_{\widehat{\Delta}} = (i_\Delta)^{-1}(W_\infty), \qquad W_\infty = N^{-1} i_\Delta^{-1}(W_\infty)$$

and the functors:

$$N : Cat \longrightarrow \widehat{\Delta}, \qquad i_\Delta : \widehat{\Delta} \longrightarrow Cat$$

are equivalences between the localizers (Cat, W_∞) *and* $(\widehat{\Delta}, W_{\widehat{\Delta}})$*, which implies in an equivalence of prederivators:*

$$\mathbf{Hot} \simeq \mathbf{Ho}_{W_{\widehat{\Delta}}} \widehat{\Delta},$$

and since $(\widehat{\Delta}, W_{\widehat{\Delta}})$ *is a Grothendieck localizer (because* $(\widehat{\Delta}, W_{\widehat{\Delta}})$ *is an ideal Quillen localizer according to (1.1.53)), then* **Hot** *is a derivator (from (1.3.13)), and for every right (resp. left) derivator* $\mathscr{D}$*, the functor*

$$\mathcal{H}om_!(\mathbf{Hot}, \mathscr{D}) \longrightarrow \mathscr{D}(e), \quad \mathcal{F} \mapsto \mathcal{F}(e)$$

(resp.

$$\mathcal{H}om^!(\mathbf{Hot}^o, \mathscr{D}) \longrightarrow \mathscr{D}(e), \quad \mathcal{F} \mapsto \mathcal{F}(e)$$

is an equivalence of categories.

The previous theorem is a synthesis of (2.1.15), (1.3.29), (2.2.5) and from the facts exposed in the last section of ‘A Mathematical History of Homotopy Types’ of this dissertation. We can contemplate the diagram

$$\begin{tikzcd}
 & Ens_\Delta \arrow[dr, "\mathcal{T}"] \arrow[ddl, "|?|"'] & \\
 & Cat \arrow[u, "N"] \arrow[r, "\widehat{?}"] \arrow[dl, "\beta"'] & \mathcal{T}OP \\
Top \arrow[urr, "\widetilde{?}"'] & &
\end{tikzcd}$$

where Ens_Δ is the category $\widehat{\Delta}$ of simplicial sets, $\mathcal{T}$ is the evident functor $X \mapsto \widehat{\Delta}/X$, $|?|$ is the topological realization functor, $\widetilde{?}$ is the functor assigning to each topological space X the usual topos $\widetilde{X}$ of sheaves over the open subsets of X, $\widehat{?}$ is the functor assigning to each small category A the classifying topos $\widehat{A}$ of presheaves over A, and $\beta : Cat \to Top$ is the classifying space functor, defined as the composition:

$$\beta =_{df} |?| \circ N.$$

We remember that there exists a natural transformation

$$\mu : \widetilde{?} \circ \beta \longrightarrow \widehat{?},$$

which we call Moerdjik exchange, such that

$$\gamma_A : \widetilde{\beta A} \longrightarrow \widehat{A}$$

is an Artin-Mazur equivalence for every small category A, and also a natural transformation

$$\vartheta : \mathcal{T} \circ N \longrightarrow \widehat{?}$$

induced from the functors

$$\alpha_A : \Delta / N(A) \longrightarrow A$$

defined in the proof of (2.3.19). It also follows from (2.3.19) and (2.3.22) that for each small category A, the geometric morphism

$$\vartheta_A : \widehat{\Delta} / N(A) \longrightarrow \widehat{A}$$

is an Artin-Mazur equivalence. All the homotopy types of the models Top, Ens_Δ and Cat (and $\widehat{A}$, replacing Δ by any other test category) can be compared in the 2-category $\mathcal{T}OP$ of toposes anf geometric morphisms. Denote by $\mathcal{W}_\infty$ the class of Artin-Mazur equivalences in $\mathcal{T}OP$, called just by ∞-equivalences. Two toposes $\mathcal{X}$ and $\mathcal{Y}$ have the same homotopy type if there is an ∞-equivalence $\mathcal{X} \to \mathcal{Y}$. Following the idea of [1] that toposes are spaces and topology is the study of the 2-category $\mathcal{T}OP$, it becomes clear that all the three notions of spaces

$$Top, \quad Ens_\Delta, \quad Cat$$

are compared in $\mathcal{T}OP$, i.e., if $X \in Ob(\mathscr{C})$, for $\mathscr{C} = Top, Ens_\Delta, Cat$, then we can regard X as a space (it means, a topos) by the respective assignments:

$$\widetilde{X}, \quad \mathcal{T}(X), \quad \widehat{X}.$$

The modelizing story then is actually about subcategories $\mathscr{S}$ of $\mathcal{T}OP$ which parametrize the homotopy types. Then, we can speak about test categories $\widehat{X}$, test toposes $\mathcal{S}$, and test topological spaces $\widetilde{X}$. In all these cases, the homotopy types are modelized by the étale spaces over a base space $\mathcal{S}$:

$$\mathcal{S}/U \longrightarrow \mathcal{S}.$$

Remark 2.3.23. In [11], Grothendieck investigate a complete characterization W_∞, i.e., an 'Eilenberg-Steenrod' list of axioms for W_∞, which yields a full axiomatic picture of homotopy theory via small categories.

3 Homotopy Hypothesis

In this chapter, we give a brief exposition of the homotopy theory of ∞-groupoids in the sense of Grothendieck-Maltsiniotis, which will be called just by ∞-groupoids. Our aim is to provide a precise statement of the *Homotopy Hypothesis* in this context. The main references for this chapter are the papers [14] and the thesis [39].

3.1 Globular language

3.1.1. The category Hot of homotopy types is the category of absolute coefficients of the absolute derivator **Hot**. The homotopy hypothesis can be formulated very vaguely as following:

> *There exists a strict algebraic description of the category* Hot?

The algebraic structure which solves the homotopy hypothesis is the one of ∞-groupoid. As already exposed in the introduction, this is a generalization of the case of 1-homotopy types, for which the ordinary groupoids (which we could name 1-groupoids) is the algebraic structure which described Hot_1 (the category of 1-homotopy types). Hence, the homotopy hypothesis is essentially an algebraic-geometric conjecture [19]. We recall that the category of schemes (in algebraic geometry) can be described as a certain geometric category (pairs $(X, \mathcal{O}_X)$ where X is a topological space endowed with a structural sheaf $\mathcal{O}_X$ of commutative rings), or in purely algebraic terms (functors $X : Comm \to Ens$ where $Comm$ denotes the category of commutative rings) [20]. The theory of schemes is the perfect and paradigmatic situation in mathematics where the principle of algebraic geometry (that geometry is algebra dualized) is fully realized. The idea of Homotopy Hypothesis is to do the same for homotopy types.

3.1.2. From the point of view of the author, a 'category theory' is a formal-

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[19] We could also say that the Homotopy Hypothesis is a Cartesian conjecture, for it was Descartes who introduce for the first time in mathematics the idea that geometry is algebra dualized, which is the foundational hypothesis of every kind of algebraic geometry. The Homotopy Hypothesis is the principle of algebraic geometry applied for the homotopy types.

[20] See the first chapter of [42] for the presentation and equivalence of these two languages for schemes.

ism to produce Kan extensions. In fact, all the concepts which are meaningful from a strict categorical language are automatically derived from Kan extensions. Following this conception, we can retro-act in order to define all the formal concepts necessarily to state a formalism of Kan extensions (functors, natural transformations, adjunctions, equivalences, etc.) in a 'category theory'. Once we formalize the previous concepts, the theory of Kan extensions is naturally presented in the language of derivators, and we derive the existence of the absolute derivator **Hot** again, and, in particular, the category Hot of homotopy types. Hence, if a 'category theory' is proposed, the homotopy types are inevitable. Now, every category is enriched in some other category of structures. For example, every locally small category is trivially enriched in the category of sets Ens, but it could be also enriched in the category of topological spaces (resp. complexes of abelian groups, simplicial sets, etc.). We could then, ask, what is the canonical structure of enrichment for any category? It follows from the theorem (2.2.5) that the homotopy types (and, essentially, only the homotopy types) are the answer for the previous question. We remark that the enrichment of any category in the category of sets may be trivial, but is not the canonical one. Therefore, the homotopy hypothesis has a very deep meaning, and not only for homotopy theory, but for all mathematics, since it potentially provides *the* algebraic structure (which is the structure of ∞-groupoid) in which every other one is canonically enriched.

3.1.3. The theory of $(\infty, 1)$-categories, extensively studied in [26] and [32], is no doubt the most elegant and successful solution for the Homotopy Hypothesis that we have in the actual mathematics. It interprets $(\infty, 1)$-categories as Bordman-Vogt-Joyal quasi-categories, and $(\infty, 1)$-groupoids as Kan complexes (which is a particular case of a quasi-category). Yet, this model is interpreted internally to the category $\widehat{\Delta}$ of simplicial sets, i.e., it makes homotopy theory inside the particular topos $\widehat{\Delta}$. We can adopt this language from a pragmatic perspective, since it is beautiful and works perfectly. Actually, in virtue of the result (2.2.5) and the theory of Quillen model categories, we can even justify conceptually this choice, for, in the Quillen model category on the simplicial sets $\widehat{\Delta}$, which is strictly combinatorial, the fibrating objects are precisely the Kan complexes. Since for any Quillen model category (C, W, Cof, Fib), the homotopy category $\mathsf{Ho}_W(C)$ is equivalent to $\mathsf{Ho}_{W_f}(C_f)$, where C_f is the full subcategory of C formed by the fibrating objects and $W_f = W \cap Fl(C_f)$, then the objects of the category Hot can be described as

Kan complexes, which justifies the modelization of $(\infty, 1)$-groupoids as Kan complexes. But this is still *a* model. Moreover, Kan complexes are not strictly algebraic structures but arguably a certain combinatorial structure. The reader can see in [10] that this is not what Grothendieck had in mind when he thought about ∞-groupoids. Actually, the Grothendieck-Maltsiniotis ∞-groupoids are very closely to Batanin higher groupois ([43] and [44]). The main difference relies only in the orientation: while Grothendieck-Maltsiniotis

In the sequel, we start the exposition of the theory of Grothendieck-Maltsiniotis ∞-groupoids in order to state the Homotopy Hypothesis precisely.

3.1.4. An algebraic structure can be defined by a family of base sets, operations related to certain finite projective limits and equality formulas which can be described as a prescription of commutativity of suitable diagrams. Consider, for example, a very basic algebraic structure, as a magma, i.e., a set E with a binary operation $\mu : E \times E \to E$ (without any equality axioms). In this case, the finite projective limits in question are just the binary products. In order to construct the *universal magma structure*, we start from the universal category $\mathbf{C}_0$ which admits binary products generated by a single object $*$. After Lawvere, we know that the category $\mathbf{C}_0$ is equivalent to the category of non-empty finite sets. The universality of $\mathbf{C}_0$ means that, for any category C admitting binary products, the functor

$$\underline{Hom}_\times(\mathbf{C}_0, C) \longrightarrow C, \quad F \mapsto F(*)$$

is an equivalence of categories, where $\underline{Hom}_\times(\mathbf{C}_0, C)$ denotes the category of functors from $\mathbf{C}_0$ to C preserving binary products. The previous equivalence of categories means that, for any object $x : e \to C$ of C, there exists essentially a unique functor $\mathbf{C}_0 \to C$, commuting with binary products, such that the diagram

$$\begin{tikzcd}
e \arrow[r, "*"] \arrow[dr, "x"'] & \mathbf{C}_0 \arrow[d] \\
& C
\end{tikzcd}$$

commutes. Then, we can construct a canonical extension $h : \mathbf{C}_0 \to \mathbf{C}$ of the category $\mathbf{C}_0$ where $\mathbf{C}$ admits a formal arrow $m : * \times * \to *$. The universality of $\mathbf{C}$ means that, for any category C admitting binary products, for any object x of C and any arrow the form $\mu : x \times x \to x$ in C, there exists essentially a unique functor $F : \mathbf{C} \to C$, commuting with binary products, such that the digram

$$\begin{tikzcd}
e \arrow[r, "*"] \arrow[dr, "x"'] & \mathbf{C}_0 \arrow[r, "h"] \arrow[d] & \mathbf{C} \arrow[dl, "F"] \\
 & C &
\end{tikzcd}$$

commutes. In particular, F sends the arrow $m : * \times * \to *$ to $\mu : x \times x \to x$. The ordinary magmas correspond then to functors of the form $M : \mathbf{C} \to Ens$ commuting with binary products. Now, the forcing of equational axioms on the magma algebraic structure would be co-existent with *coherences* codified by commutativity diagram prescriptions. In this case, these coherences would be conserved by the continuity conditions in the previous universal construction, i.e., by the fact that the universal functor $\mathbf{C} \to C$ induced by an algebraic-structure interpretation in C commutes with binary products.

3.1.5. The definition of Grothendieck-Maltsiniotis ∞-groupoids follows the same strategy of (3.1.4). In fact, any groupoid is a certain type of algebraic-order structure codified by the data

$$(G_0, G_1, s, t, \mu, id, in),$$

where

$$\begin{tikzcd}
G_0 & G_1 \arrow[l, shift right, "t"'] \arrow[l, shift left, "s"]
\end{tikzcd}$$

are two parallel arrows, μ is an arrow of the form

$$G_1 \times_{G_0} G_1 \longrightarrow G_1$$

(the law of composition of the category, with domain being the composable arrows), where $G_1 \times_{G_0} G_1$ is defined in respect to the cartesian square

$$\begin{tikzcd}
G_1 \times_{G_0} G_1 \arrow[r] \arrow[d] & G_1 \arrow[d, "s"] \\
G_1 \arrow[r, "t"'] & G_0,
\end{tikzcd}$$

$id : G_0 \to G_1$ is the identity arrow, such that the diagrams

$$\begin{tikzcd}
G_0 \arrow[r, "id"] \arrow[dr, "Id_{G_0}"'] & G_1 \arrow[d, "s"] \\
& G_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
G_0 \arrow[r, "id"] \arrow[dr, "Id_{G_0}"'] & G_1 \arrow[d, "t"] \\
& G_0
\end{tikzcd}$$

are commutative, and $in : G_1 \to G_1$ is the invert arrow, which assigns to each arrow f of G it's inverse $f^{-1} = in(f)$, with $s(in(f)) = t(f)$ and $t(in(f)) = s(f)$. In the case of an ∞-groupoid G (whatever it means), there should be for each $n \in \omega$ an object G_n, codifying the n-arrows of G, and parallel arrows

$$\begin{tikzcd}
G_{n-1} & G_n \arrow[l, shift right, "t_n"'] \arrow[l, shift left, "s_n"]
\end{tikzcd}$$

for $n \geq 1$, codifying the source and target assignations to an n-arrow, which should satisfy the equalities

$$s_{i-1}s_i = s_{i-1}t_i, \qquad t_{i-1}t_i = t_{i-1}s_i, \qquad i > 1,$$

but also higher compositions $\circ_i$, for $i \geq 1$, higher identities $id_i : G_i \to G_{i+1}$, for $i \geq 0$, and higher invert arrows $in_i : G_i \to G_i$, for $i \geq 1$ (with respect to the composition law $\circ_i$). For the description of $\circ_i$, we consider the cartesian square

$$\begin{tikzcd}
G_i \times_{G_{i-1}} G_i \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & G_i \arrow[d, "t_i"] \\
G_i \arrow[r, "s_i"'] & G_{i-1}.
\end{tikzcd}$$

Then, $\circ_i$ is an arrow from $G_i \times_{G_j} G_i$ to G_i such that $s_i pr_1 \circ_i = t_i pr_2 \circ_i$.

3.1.6. The data

$$(G_i, s_i, t_i, s^i_j, t^i_j, \mu^i_j, id_i, in^i_j)$$

from (3.1.5) is the basic information of an ∞-groupoid structure. The 'axioms' of ∞-groupoids will be coherent conditions associated to *weak* diagram commutativity prescriptions [21], and the models will be functors of the form $G : \mathbf{C} \to Ens$ for a certain *coherator* $\mathbf{C}$ (as in (3.1.4)), preserving a suitable class of iterated fibred products (which are related to the higher compositions). Though the magma structures in a category C with binary products are indexed by the terminal category e (as it's clear from (3.1.4)), the ∞-groupoid structures in a category C, admitting the *iterated fibred products* necessary to define ∞-groupoids, will be indexed by a category $\mathbb{G}$, named the *globular category*. The objects of $\mathbb{G}$ may be thought as n-dimensional versions of the homotopy type of the point, for example, the closed balls

$$D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}, \qquad n \in \omega,$$

but we remark that the definition of $\mathbb{G}$ will be purely formal.

Definition 3.1.7. *The globe category, denoted by $\mathbb{G}$, is the formal category generated by the graph*

$$\begin{tikzcd}
D_0 & D_1 \arrow[l, shift right, "\tau_1"'] \arrow[l, shift left, "\sigma_1"'] & \cdots\cdots \arrow[l, shift right, "\tau_2"'] \arrow[l, shift left, "\sigma_2"'] & D_{i-1} \arrow[l, shift right, "\tau_{i-1}"'] \arrow[l, shift left, "\sigma_{i-1}"'] & D_i \arrow[l, shift right, "\tau_i"'] \arrow[l, shift left, "\sigma_i"'] & \cdots\cdots \arrow[l, shift right, "\tau_{i+1}"'] \arrow[l, shift left, "\sigma_{i+1}"']
\end{tikzcd}$$

with the relations

$$\sigma_{i-1}\sigma_i = \tau_{i-1}\sigma_i, \qquad \tau_{i-1}\tau_i = \sigma_{i-1}\tau_i$$

for $i > 1$. If $i \geq j \geq 0$, we denote respectively by σ^i_j and τ^i_j the following morphisms from D_i to D_j:

$$\sigma^i_j =_{df} \sigma_{j+1} ... \sigma_{i-1}\sigma_i, \qquad \tau^i_j =_{df} \tau_{j+1} ... \tau_{i-1}\tau_i$$

and we have that, for every pair (i, j):

$$Hom_{\mathbb{G}}(D_i, D_j) = \{\sigma^i_j, \tau^i_j\}, \qquad i > j;$$

$$Hom_{\mathbb{G}}(D_i, D_j) = \{Id_{D_i}\}, \qquad i = j;$$

$$Hom_{\mathbb{G}}(D_i, D_j) = \varnothing, \qquad i < j.$$

Notation and Terminology - The objects of the globe category $\mathbb{G}$ will be called globes. For each $n \in \omega$, we denote by $\mathbb{G}_n$ the full sucategory of $\mathbb{G}$

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[21] The term *weak* will be clarified in the next section.

generated by the objects D_i with $0 \leq i \leq n$, called the category of n-globes. We remark that for $0 \leq j \leq i$, σ^i_j (resp. τ^i_j), is the iterated composition of the sequence of arrows

$$\begin{tikzcd} D_i \arrow[r, "\sigma_i"] & D_{i-1} \arrow[r, "\sigma_{i-1}"] & .... \arrow[r] & D_{j+1} \arrow[r, "\sigma_{j+1}"] & D_j \end{tikzcd}$$

(resp.

$$\begin{tikzcd} D_i \arrow[r, "\tau_i"] & D_{i-1} \arrow[r, "\tau_{i-1}"] & .... \arrow[r] & D_{j+1} \arrow[r, "\tau_{j+1}"] & D_j \end{tikzcd}$$

Given a category $\mathbf{C}$ and a functor $\mathbf{D} : \mathbb{G} \to \mathbf{C}$, we going to denote by $\mathbf{D}_i$ the object $\mathbf{D}(D_i)$ in $\mathbf{C}$, and by s_i (resp. t_i) the arrow $\mathbf{D}(\sigma_i)$ (resp. $\mathbf{D}(\tau_i)$). For $0 \leq j \leq i$, we also denote by s^i_j (resp. t^i_j) the arrow $\mathbf{D}(\sigma^i_j)$ (resp. $\mathbf{D}(\tau^i_j)$). Therefore, $\mathbf{D}$ corresponds to a diagram of the form

$$\begin{tikzcd} \mathbf{D}_0 \arrow[r, leftarrow, shift left, "t_1"] \arrow[r, leftarrow, shift right, "s_1"'] & \mathbf{D}_1 \arrow[r, leftarrow, shift left, "t_2"] \arrow[r, leftarrow, shift right, "s_2"'] & ..... \arrow[r, leftarrow, shift left, "t_{i-1}"] \arrow[r, leftarrow, shift right, "s_{i-1}"'] & \mathbf{D}_{i-1} \arrow[r, leftarrow, shift left, "t_i"] \arrow[r, leftarrow, shift right, "s_i"'] & \mathbf{D}_i \arrow[r, leftarrow, shift left, "t_{i+1}"] \arrow[r, leftarrow, shift right, "s_{i+1}"'] & ..... \end{tikzcd}$$

in $\mathbf{C}$. Moreover, given a functor $\mathbf{D} : \mathbb{G} \to \mathbf{C}$ (resp. $\mathbf{B} : \mathbb{G}^o \to \mathbf{C}$) and a pair (i, j) of elements in ω with $0 \leq j \leq i$, then $\mathbf{D}$ (resp. $\mathbf{B}$) carries the diagram

$$\begin{tikzcd} & D_i \arrow[d, "\sigma^i_j"] \\ D_i \arrow[r, "\tau^i_j"'] & D_j \end{tikzcd}$$

to a diagram

$$\begin{tikzcd} & \mathbf{D}_i \arrow[d, "s^i_j"] \\ \mathbf{D}_i \arrow[r, "t^i_j"'] & \mathbf{D}_j \end{tikzcd}$$

(resp.

$$\begin{tikzcd} \mathbf{B}_j \arrow[r, "s^i_j"] \arrow[d, "t^i_j"'] & \mathbf{B}_i \\ \mathbf{B}_i & \end{tikzcd}$$

If the projective (resp. inductive) limit of the above diagram is representable in $\mathbf{C}$, we always by denote by $\mathbf{D}_i \times_{\mathbf{D}_j} \mathbf{D}_j$ (resp. $\mathbf{B}_i \times_{\mathbf{B}_j} \mathbf{B}_i$) it's fibred product (resp. fibred coproduct). The dual category $\mathbb{G}^o$ of the globe category $\mathbb{G}$ will be denoted by $\mathbb{B}$ and called the co-globe category. The objects of $\mathbb{B}$ will be called co-globes..

A *tableau of dimensions* T of width n, for $n \in \omega$ with $n > 1$, consists of two lists :

$$(i_1, ..., i_{n-1}, i_n), \qquad (j_1, ..., j_{n-1})$$

of elements in ω, such that $i_k > j_k$ and $i_{k+1} > j_k$ for $1 \leq k \leq n$. We denote a tableau of dimensions of the previous type as

$$\begin{pmatrix} i_1 & & i_2 & & ..... & i_{n-1} & & i_n \\ & j_1 & & j_2 & ..... & & j_{n-1} & \end{pmatrix}$$

By definition, any tableau of dimensions T as above gives rises to a diagram of the form

$$\begin{tikzcd}
D_{i_1} \arrow[dr, "\sigma^{i_1}_{j_1}"'] & & D_{i_2} \arrow[dl, "\tau^{i_2}_{i_1}"] \arrow[dr, "\sigma^{i_2}_{j_2}"'] & & \cdots & D_{i_{n-1}} \arrow[dr, "\sigma^{i_{n-1}}_{j_{n-1}}"'] & & D_{i_n} \arrow[dl, "\tau^{in}_{j_{n-1}}"] \\
& D_{i_1} & & D_{j_2} & \cdots & & D_{n-1} &
\end{tikzcd}$$

in $\mathbb{G}$, which corresponds, through a functor $\mathbf{D} : \mathbb{G} \to \mathbf{C}$, to a diagram

$$\begin{tikzcd}
\mathbf{D}_{i_1} \arrow[dr, "s^{i_1}_{j_1}"'] & & \mathbf{D}_{i_2} \arrow[dl, "t^{i_2}_{i_1}"] \arrow[dr, "s^{i_2}_{j_2}"'] & & \cdots & \mathbf{D}_{i_{n-1}} \arrow[dr, "s^{i_{n-1}}_{j_{n-1}}"'] & & \mathbf{D}_{i_n} \arrow[dl, "t^{in}_{j_{n-1}}"] \\
& \mathbf{D}_{i_1} & & \mathbf{D}_{j_2} & \cdots & & \mathbf{D}_{n-1} &
\end{tikzcd}$$

in $\mathbf{C}$. When the projective (resp. inductive) limit of the previous diagram is representable in $\mathbf{C}$, we denote it just by

$$(\mathbf{D}_{i_1}, s^{i_1}_{j_1}) \times_{\mathbf{D}_{j_1}} (t^{i_2}_{j_2}, \mathbf{D}_{i_2}, s^{i_2}_{j_2}) \times_{\mathbf{D}_{j_2}} ... \times_{\mathbf{D}_{i_{n-2}}} (t^{i_{n-1}}_{j_{n-1}}, \mathbf{D}_{i_{n-1}}, s^{i_{n-1}}_{j_{i_{n-1}}}) \times_{\mathbf{D}_{j_{n-1}}} (t^{i_n}_{j_{n-1}}, \mathbf{D}_{i_n})$$

(resp.

$$\mathbf{D}_{i_1} \amalg_{\mathbf{D}_{j_1}} \mathbf{D}_{i_2} \amalg_{\mathbf{D}_{j_2}} ... \mathbf{D}_{i_{n-1}} \amalg_{\mathbf{D}_{j_{n-1}}} \mathbf{D}_{i_n},$$

and them the standard iterated fibred products (resp. standard iterated fibred coproducts). We also denote just by T the standard iterated fibred product induced by the tableau of dimensions T represented by a matrix

$$\begin{pmatrix} i_1 & & i_2 & & ..... & i_{n-1} & & i_n \\ & j_1 & & j_2 & ..... & & j_{n-1} & \end{pmatrix}$$

An important case is the tableau of dimension

$$\begin{pmatrix} i & & i \\ & i-1 & \end{pmatrix}$$

which induces the diagram

$$\begin{tikzcd} \mathbf{D}_i \arrow[dr, "s_i"'] & & \mathbf{D}_i \arrow[dl, "t_i"] \\ & \mathbf{D}_{i-1} & \end{tikzcd}$$

with standard iterated fibred product:

$$(\mathbf{D}_i, s_i) \times_{\mathbf{D}_{i-1}} (t_i, \mathbf{D}_i).$$

Definition 3.1.8. *A functor* $\mathbf{D} : \mathbb{G} \to \mathbf{C}$ *will be called a co-globular extension if* $\mathbf{C}$ *admits standard iterated fibred products. We denote just by* $\mathbf{C}$ *the co-globular extension* $\mathbf{D} : \mathbb{G} \to \mathbf{C}$*, omitting the structural functor* $\mathbf{D}$*, which will be always implicit. A morphism* $u : \mathbf{C} \to \mathbf{C}'$ *of co-globular extensions is a functor commuting with standard iterated fibred products, such that the diagram*

$$\begin{tikzcd} & \mathbf{C} \arrow[dd, "u"] \\ \mathbb{G} \arrow[ur, "\mathbf{D}"] \arrow[dr, "\mathbf{D}'"'] & \\ & \mathbf{C}' \end{tikzcd}$$

commutes.

Definition 3.1.9. *A functor* $\mathbf{B} : \mathbb{B} \to \mathbf{C}$ *will be called a globular extension if* $\mathbf{C}$ *admits standard iterated fibred coproducts. We denote just by* $\mathbf{C}$ *the globular extension* $\mathbf{B} : \mathbb{B} \to \mathbf{C}$*, omitting the structural functor* $\mathbf{B}$*, which will be always implicit. A morphism* $u : \mathbf{C} \to \mathbf{C}'$ *of globular extensions is a functor commuting with standard iterated fibred coproducts, such that the* *diagram*

$$\begin{tikzcd} & \mathbf{C} \arrow[dd, "u"] \\ \mathbb{B} \arrow[ur, "\mathbf{B}"] \arrow[dr, "\mathbf{B}'"'] & \\ & \mathbf{C}' \end{tikzcd}$$

commutes.

Example 3.1.10. Every finitely complete (resp. co-complete) category $\mathbf{C}$ which admits a functor from $\mathbb{G}$ to $\mathbf{C}$ (resp. from $\mathbb{B}$ to $\mathbf{C}$) is a co-globular (resp. globular) extension.

Example 3.1.11. Given a globular extension $\mathbf{B} : \mathbb{B} \to \mathbf{C}$, we can always form a co-globular extension $\mathbf{D} : \mathbb{G} \to \mathbf{C}^o$, defining $\mathbf{D} = \mathbf{B}^o$. Reciprocally, given a co-globular extension $\mathbf{D} : \mathbb{G} \to \mathbf{C}$, we can form the co-globular extension $\mathbf{B} : \mathbb{B} \to \mathbf{C}^o$, defining $\mathbf{B} = \mathbf{D}^o$. The previous correspondence is actually functorial, and defines an equivalence between the category of globular extensions and co-globular extensions.

Remark 3.1.12. Some authors define the category $\mathbb{B}$ as the globe category, formalizing the theory in terms of globular extensions rather than co-globular extensions. This has future advantages in the definition of ∞-groupoids of topological spaces. The reader may think that our choice is cohomologically oriented, while it's dual language, in terms of globular extensions, is homologically oriented. The advantage of our orientation is that the definition of ∞-groupoids are more closely related to an algebraic structure

Example 3.1.13. The dual category Top^o of the category of topological spaces is a co-globular extension (and hence, Top is a globular extension). Indeed, for each $n \in \omega$, we define the topological space

$$\mathbf{D}_n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

with the topology induced from the usual topology of $\mathbb{R}^n$, and the arrows

$$s_n, t_n : \mathbf{D}_n \to \mathbf{D}_{n-1}$$

in Top^o, correspond respectively to the continuous functions

$$x \mapsto (x, -\sqrt{1 - \|x\|^2}), \qquad x \mapsto (x, \sqrt{1 - \|x\|^2})$$

from $\mathbf{D}_{n-1}$ to $\mathbf{D}_n$ in Top. The standard iterated fibred products in Top^o correspond to the standard iterated fibred coproducts in Top.

Definition 3.1.14. *A co-globular set is a presheaf over the globe category* $\mathbb{G}$*. The category of co-globular sets is just the category* $\widehat{\mathbb{G}}$*.*

3.1.15. It is immediate from the definition of co-globular extensions that the category of co-globular sets is a co-globular extension (since it is complete), where the structural functor is given by the Yoneda immersion

$$h : \mathbb{G} \longrightarrow \widehat{\mathbb{G}}.$$

3.1.16. From the definition (3.1.8), given a morphism of co-globular extensions $u : \mathbf{C} \to \mathbf{C}'$, we have that $\mathbf{D}'_i = u(\mathbf{D}_i)$ for every $i \in \omega$, and all standard iterated fibred products are preserved. Moreover, there exists a canonical co-globular extension $\mathbf{C}_0$, called the *co-globular envelope*, which can be constructed as the strict full subcategory of $\widehat{\mathbb{G}}$ generated by the globe category $\mathbb{G}$ and, for each *tableau of dimensions* T, a choice of a standard iterated fibred product of T in $\widehat{\mathbb{G}}$. From the previous description, we can verify that $\mathbf{C}_0$ is the initial object in the category of co-globular extensions. There exists yet a completely combinatorial description of $\mathbf{C}_0$ (see (*2.3*) of [39] for more details).

Definition 3.1.17. *Let* $\mathbf{C}$ *be a co-globular extension. We say that a pair of parallel arrows of the form*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbf{D}_i, \end{tikzcd}$$

in $\mathbf{C}$*, for* $i \in \omega$*, is admissible, if* $i = 0$*, or* $i > 0$ *and we have the equalities:*

$$s_i f = s_i g, \qquad t_i f = t_i g.$$

A lift of the pair (f, g) *is a morphism* $h : X \to \mathbf{D}_{i+1}$ *in* $\mathbf{C}$ *such that*

$$f = s_{i+1} h, \qquad g = t_{i+1} h.$$

3.1.18. Let $\mathbf{C}$ be a co-globular extension. We remark that if a pair (f, g) of parallel arrows of the form

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbf{D}_i \end{tikzcd}$$

in $\mathbf{C}$ admits a lift, then it is necessarily admissible. In fact, if $i = 0$, then (f, g) is admissible by definition. Now, if $i > 0$, let $h : X \to \mathbf{D}_{i+1}$ be a lift of (f, g). Since $f = s_{i+1}h$ and $g = t_{i+1}h$, we have

$$s_i f = s_i s_{i+1} h = s_i t_{i+1} h = s_i g$$

and

$$t_i g = t_i t_{i+1} h = t_i s_{i+1} h = t_i f,$$

which means that (f, g) is admissible. From now though, we say that a pair (f, g) of parallel arrows of the form

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbf{D}_i \end{tikzcd}$$

in $\mathbf{C}$, for $i \in \omega$, is an i-pair. Therefore, from the previous argument, we have that every i-pair which admits a lift is admissible. We say that the co-globular extension $\mathbf{C}$ is *contractible* when the reciprocal is also verified, i.e., when every admissible i-pair admits a lift.

Definition 3.1.19. *Let $\mathbf{C}$ be a co-globular extension. Given an object X of $\mathbf{C}$, we say that a formal i-arrow of X, for $i \in \omega$, is a morphism of the form $f : X \to \mathbf{D}_i$ in $\mathbf{C}$. We say that X is contractible if every pair of admissible i-arrows:*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbf{D}_i \end{tikzcd}$$

admits a lift.

Remark 3.1.20. It is immediate from definitions (3.1.18) and (3.1.19) that a co-globular extension is contractible precisely when all its objects are contractible. Moreover, it is also an easy consequence of the definitions (3.1.8) and (3.1.19) that any morphism of co-globular extensions preserves contractible objects. In particular, if a morphism of co-globular extensions $\mathbf{C} \to \mathbf{C}'$ induces a bijection on the objects and $\mathbf{C}$ is contractible, then $\mathbf{C}'$ is also contractible.

3.1.21. Let $\mathbf{C}$ be a co-globular extension and E be a set of admissible parallel arrows in $\mathbf{C}$. Even if the elements of E does not admit a lift in $\mathbf{C}$, we can canonically extend $\mathbf{C}$ to a second co-globular extension $\mathbf{C}'$ in order to grant the existence of these lifts in $\mathbf{C}'$. More precisely, there exits always a morphism of co-globular extensions $u : \mathbf{C} \to \mathbf{C}'$ such that

1. For every pair (f, g) in E, there exists a lift h of the pair $(u(f), u(g))$ in $\mathbf{C}$.
2. If $u' : \mathbf{C} \to \mathbf{C}''$ is another morphism of co-globular extensions such that the previous condition is verified, then, there exists a unique morphism of co-globular extension $v : \mathbf{C}' \to \mathbf{C}''$ such that the diagram

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "u"] \arrow[dr, "u'"'] & \mathbf{C}' \arrow[d, "v"] \\
 & \mathbf{C}''
\end{tikzcd}$$

   commutes.

We call $\mathbf{C}'$ the universal co-globular extension generated from $\mathbf{C}$ by the set of admissible pairs E. Conversely, we say that a morphism of co-globular extensions $\mathbf{C} \to \mathbf{C}'$ is *admissible* if $\mathbf{C}'$ is the universal co-globular extension generated from $\mathbf{C}$ by the set of *all* admissible pairs in $\mathbf{C}$. We remark that, by construction, if $\mathbf{C} \to \mathbf{C}'$ is an admissible morphism of co-globular extensions, then it induces a bijection on the objects. Therefore, if the original co-globular extension $\mathbf{C}$ is contractible, then $\mathbf{C}'$ is also contractible (see (3.1.20)).

Construction 3.1.22. We can formally construct the *free contractible co-globular extension* $\mathbf{C}_\infty$, called the *canonical coherator*, proceeding in the following way:

1. Starting from the universal co-globular extension $\mathbb{G} \to \mathbf{C}_0$ (the co-globular envelope), we consider the set E_0 of all admissible pairs in $\mathbf{C}_0$, and then, we construct the universal admissible co-globular morphism $\mathbf{C}_0 \to \mathbf{C}_1$.
2. For $n \geq 1$, we denote by E_n the set of all admissible pairs in $\mathbf{C}_n$, and then, we construct the universal admissible co-globular morphism $\mathbf{C}_n \to \mathbf{C}_{n+1}$.
3. The previous construction gives rises to a diagram of the form

$$\mathbb{G} \to \mathbf{C}_0 \to \mathbf{C}_1 \to ..... \to \mathbf{C}_n \to \mathbf{C}_{n+1} \to ....$$

in the category of small categories, such that, for each $n \in \omega$ and each object X of $\mathbf{C}_n$, the image of X by the morphism $\mathbf{C}_n \to \mathbf{C}_{n+1}$ is contractible in $\mathbf{C}_{n+1}$, and the functor $\mathbf{C}_n \to \mathbf{C}_{n+1}$ induces a bijection on the objects.

4. We can compute the inductive limit of the diagram

$$\mathbb{G} \to \mathbf{C}_0 \to \mathbf{C}_1 \to ..... \to \mathbf{C}_n \to \mathbf{C}_{n+1} \to ....$$

in the category of small categories, which is representable by a small category $\mathbf{C}_\infty$.

5. Finally, it follows from the properties of direct inductive limit that $\mathbf{C}_\infty$ is a contractible co-globular extension, and hence, a coherator, which can be considered as a canonical coherator.

To prove the last statement, it is enough to verify that for every $n \in \omega$, the canonical functor $\mathbf{C}_n \to \mathbf{C}_\infty$ is a bijection on the objects. We sketch the complete reasoning. In fact, if that is the case, then $\mathbf{C}_\infty$ would be a co-globular extension and each functor $\mathbf{C}_n \to \mathbf{C}_\infty$ would be a morphism of co-globular extensions. Now, since every object X of $\mathbf{C}_\infty$ is isomorphic to the image of an object X_n of $\mathbf{C}_n$ by the canonical functor $\mathbf{C}_n \to \mathbf{C}_\infty$ for some $n \in \omega$, and the projection X_{n+1} of X_n in $\mathbf{C}_{n+1}$ is contractible, then X would be contractible in $\mathbf{C}_\infty$, because X is also isomorphic to the image of X_{n+1} trough the morphism of co-globular extensions $\mathbf{C}_{n+1} \to \mathbf{C}_\infty$, and morphisms of co-globular extensions preserve contractible objects ((3.1.20)). Therefore, consider the diagram

$$\mathbb{G} \to \mathbf{C}_0 \to \mathbf{C}_1 \to ..... \to \mathbf{C}_n \to \mathbf{C}_{n+1} \to .... \to \mathbf{C}_\infty = \varinjlim \mathbf{C}_n$$

of small categories. For every object X of $\mathbf{C}$, there exists $n \in \omega$ such that X is isomorphic to the image of an object of $\mathbf{C}_n$ by the canonical morphism $\mathbf{C}_n \to \mathbf{C}$. Then, we just have to verify (by induction) that for each $n \in \omega$, the morphism of co-globular extensions $\mathbf{C}_n \to \mathbf{C}_{n+1}$ is a bijection on the objects, which is a consequence of (3.1.21) since $\mathbf{C}_n \to \mathbf{C}_{n+1}$ is always an admissible morphism of co-globular extensions.

Definition 3.1.23. *A co-globular extension* $\mathbf{C}$ *is called a weak coherator if it satisfy the following conditions:*

1. $\mathbf{C}$ *is a contractible co-globular extension.*

2. *For every conctractible co-globular extension $\mathbf{C}'$, there exists at least one (not necessarily unique) morphism of co-globular extensions from $\mathbf{C}$ to $\mathbf{C}'$.*

3. *The canonical co-globular morphism $\mathbf{C}_0 \to \mathbf{C}$ induces a bijection on the objects.*

Remark 3.1.24. In order to proof that a co-globular extension $\mathbf{C}$ satisfying the conditions (1) and (3) of (3.1.23) is a weak coherator, i.e., $\mathbf{C}$ satisfies also (2) of (3.1.23), it is enough to verify that for every co-globular extension $\mathbf{C}'$, there exists a functor $u : \mathbf{C} \to \mathbf{C}'$, for, the conditions (1) and (3) imply that all the objects of $\mathbf{C}$ are isomorphic to a standard iterated fibred product.

Proposition 3.1.25. *The free contractible co-globular extension $\mathbf{C}_\infty$ is a weak coherator. Moreover, if $\mathbf{C}'$ is any contractible co-globular extension, then, there exists a (essentially unique) morphism of co-globular extension from $\mathbf{C}_\infty$ to $\mathbf{C}'$.*

Proof. The conditions (1) and (3) of the definition (3.1.23) follow from the proper construction (3.1.22) of $\mathbf{C}_\infty$. For both the condition (2) of (3.1.23) and the second assertion of the proposition, let $\mathbf{D} : \mathbb{G} \to \mathbf{C}'$ be a contractible co-globular extension. Then, by the universal definition of the co-globular envelope $\mathbf{C}_0$ ((3.1.16)), $\mathbb{G} \to \mathbf{C}_0$ is the universal completation of $\mathbb{G}$ with respect to the standard iterated fibred products, which means that there exists a (essentially unique) morphism of co-globular extensions $u_0 : \mathbf{C}_0 \to \mathbf{C}'$, such that the diagram

$$\begin{tikzcd}
\mathbb{G} \arrow[r] \arrow[dr, "\mathbf{D}"'] & \mathbf{C}_0 \arrow[d, "u_0"] \\
 & \mathbf{C}'
\end{tikzcd}$$

commutes. Now, using the hypothesis that $\mathbf{C}$ is *contractible*, we can verify by finite induction that for each $n \in \omega$, a morphism of co-globular extensions $u_n : \mathbf{C}_n \to \mathbf{C}'$ can be canonically extended to a morphism of co-globular extensions $u_{n+1} : \mathbf{C}_{n+1} \to \mathbf{C}'$, and hence, $\mathbf{C}'$ is a directed inductive co-cone of the diagram

$$\mathbb{G} \to \mathbf{C}_0 \to \mathbf{C}_1 \to ..... \to \mathbf{C}_n \to \mathbf{C}_{n+1} \to ....$$

of the construction (3.1.22), which implies in the exitence of a unique functor

$u_\infty : \mathbf{C}_\infty \to \mathbf{C}'$ such that the diagram

$$\begin{tikzcd}
\mathbb{G} \arrow[r] \arrow[dr, "\mathbf{D}"'] & \mathbf{C}_n \arrow[r] \arrow[d, "u_n"] & \mathbf{C}_\infty \arrow[dl, "u_\infty"] \\
& \mathbf{C}' &
\end{tikzcd}$$

commutes for every $n \in \omega$. Moreover, it follows from (3.1.24) that u_∞ is a morphism of co-globular extensions, which concludes the proof.

□

Remark 3.1.26. It follows from the third condition of (3.1.23) that every weak coherator is a small category. In particular, the canonical coherator $\mathbf{C}_\infty$ is also a small category.

3.2 ∞-groupoids

Definition 3.2.1. *Let* $\mathbf{C}$ *be a small co-globular extension. An* ∞*-model is a functor* $G : \mathbf{C} \to Ens$ *commuting with standard iterated fibred products. We denote by* $\infty\text{-}\mathcal{M}od_{\mathbf{C}}$ *the full subcategory of* $\underline{Hom}(C, Ens)$ *formed by the* ∞*-models. When* $\mathbf{C}$ *is a weak coherator (resp. the canonical coherator* $\mathbf{C}_\infty$*), we denote by* $\infty\text{-}\mathcal{G}pd_{\mathbf{C}}$ *(resp. by* $\infty\text{-}\mathcal{G}pd$*) the category* $\infty\text{-}Mod_{\mathbf{C}}$*, called the category of* ∞*-groupoids of type* $\mathbf{C}$ *(resp. the category of* ∞*-groupoids).*

3.2.2. Let $\mathbf{C}$ be a co-globular extension. By definition, every ∞-model G over $\mathbf{C}$ also induces a co-globular extension $G \circ \mathbf{D} : \mathbb{G} \to Ens$. For each $i \in \omega$, we denote by G_i the set $G(\mathbf{D}_i)$, called the set of i-arrows of G, and we maintain the notation s_i (resp. t_i) to indicate the function $G(s_i)$ (resp. $G(t_i)$). We also denote by s^i_j (resp. t^i_j) the function $G(s^i_j)$ (resp. $G(t^i_j)$), for $0 \leq j \leq i$. For $i = 0$ (resp. $i = 1$) we say just that G_0 (resp. G_1) is the set of objects (resp. arrows) of G. Since for each $i \in \omega$, with $i \geq 1$, we have the parallel arrows

$$\begin{tikzcd}
G_{i-1} & G_i, \arrow[l, shift right, "t_i"'] \arrow[l, shift left, "s_i"]
\end{tikzcd}$$

then, given an i-arrow f of G, we say that $s_i(f)$ (resp. $t_i(f)$) is the domain (resp. codomain) of f.

Remark 3.2.3. By definition, an ∞-groupoid G is just an ∞-model over a weak coherator $\mathbf{C}$. We remark that different weak coherators can induce non-equivalent categories of ∞-groupoids. Yet, as we going to see in the last section of this chapter, the homotopy hypothesis implies that at least the *homotopy category* of these ∞-groupoids must be equivalent, and hence, at least homotopically, the category of ∞-groupoids should be independent of the choice of a weak coherator type. Yet, we can consider only the case of ∞-models over the *canonical coherator* $\mathbf{C}_\infty$ as ∞-groupoids.

Construction 3.2.4. We construct the ∞-groupoid associated to a topolicial space X. First, we recall that the dual category Top^o of the category of topological space is a co-globular extension ((3.1.13)). Suppose that the structural functor $\mathbf{D} : \mathbb{G} \to Top^o$ of co-globular extensions can be extended to a co-globular extension $F : \mathbf{C}_\infty \to Top^o$, i.e., F is a functor commuting with the standard iterated fibred products. Then, we can associate to each topological space X, a functor

$$\Pi_\infty X : \mathbf{C}_\infty \longrightarrow Ens, \quad c \mapsto Hom_{Top}(F_c, X),$$

which is the composition of the functor $F : \mathbf{C}_\infty \to Top^o$ with the functor

$$Hom_{Top}(?, X) : Top^o \longrightarrow Ens.$$

Since F commutes with standard iterated fibred products (which is a special type of projective limit) and $Hom_{Top}(?, X)$ commutes with projective limits, then $\Pi_\infty X$ commutes with the standard iterated fibred products, which means that $\Pi_\infty X$ is an ∞-groupoid according to the definition (3.2.1). Then, in order to construct the ∞-groupoid of a topological space X, we just have to proof that $\mathbf{D} : \mathbb{G} \to Top^o$ is a *contractible co-globular extension* ((3.1.18)), for, in this case, it follows from (3.1.25) that there exists a (essentially unique) co-globular extension $F : \mathbf{C}_\infty \to Top^o$ extending $\mathbf{D} : \mathbb{G} \to Top^o$, and hence, we can apply the previous construction to define the ∞-groupoid $\Pi_\infty X$ of a topological space X. Indeed, it is a consequence of (3.1.13) by duality arguments, that we just have to verify the following: given two parallel arrows of topological spaces of the form

$$\begin{tikzcd} \mathbf{D}_n \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X, \end{tikzcd}$$

where $\mathbf{D}_n = D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, such that $fs_n = gs_n$ and $ft_n = gt_n$, there exists an arrow $h : D^{n+1} \to X$ satisfying the relations $f = hs_{n+1}$ and

$g = ht_{n+1}$, where $s_n : D^{n-1} \to D^n$ and $t_n : D^{n-1} \to D^n$ denote respectively the maps:

$$x \mapsto (x, -\sqrt{1 - \|x\|^2}), \qquad x \mapsto (x, \sqrt{1 - \|x\|^2}),$$

for $n > 0$ (see (3.1.13) in order to remember the definition of the co-globular extension $\mathbf{D} : \mathbb{G} \to Top^o$). With the previous notations, we have $s_{n+1}(D^n) \subseteq S^n$ and $t_{n+1}(D^n) \subseteq S^n$, which means that the arrow $s_{n+1} : D^n \to D^{n+1}$ (resp. $t_{n+1} : D^n \to D^{n+1}$) factors through the inclusion $i_n : S^n \hookrightarrow D^{n+1}$, i.e., there is an arrow $s'_{n+1} : D^n \to S^n$ (resp. $t'_{n+1} : D^n \to S^n$) such that the triangle

$$\begin{tikzcd}
D^n \arrow[d, "s'_{n+1}"'] \arrow[dr, "s_{n+1}"] & \\
S^n \arrow[r, "i_n"'] & D^{n+1}
\end{tikzcd}$$

(resp.

$$\begin{tikzcd}
D^n \arrow[d, "t'_{n+1}"'] \arrow[dr, "t_{n+1}"] & \\
S^n \arrow[r, "i_n"'] & D^{n+1})
\end{tikzcd}$$

commutes. Beyond that, $s_{n+1}|_{S^{n-1}} = t_{n+1}|_{S^{n-1}}$, which means that the square

$$\begin{tikzcd}
S^{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "i_{n-1}"'] & D^n \arrow[d, "s_{n+1}"] \\
D^n \arrow[r, "t_{n+1}"'] & D^{n+1}
\end{tikzcd}$$

commutes. Then, by the universal property of co-cartesian squares, there exists a unique map of topological spaces

$$(s_{n+1}, t_{n+1}) : S^n \cong D^n \amalg_{S^{n-1}} D^n \longrightarrow D^{n+1}$$

satisfying the relations:

$$s_{n+1} = (s_{n+1}, t_{n+1}) \circ s'_{n+1}, \qquad t_{n+1} = (s_{n+1}, t_{n+1}) \circ t'_{n+1},$$

from where we conclude that (s_{n+1}, t_{n+1}) coincides with the inclusion i_n :

$S^n \hookrightarrow D^{n+1}$. Now, let $f, g : D^n \to X$ be two parallel arrows of topological spaces such that $fs_n = gs_n$ and $ft_n = gt_n$. Since $s_n|_{S^{n-1}} = t_n|_{S^{n-1}}$, we have a commutative square

$$\begin{tikzcd}
S^{n-1} \arrow[r, "i_n"] \arrow[d, "i_n"'] & D^n \arrow[d, "f"] \\
D^n \arrow[r, "g"'] & X
\end{tikzcd}$$

Therefore, by the universal property of co-cartesian squares again, there exists a unique arrow of topological spaces $(f, g) : S^n \cong D^n \amalg_{S^{n-1}} D^n \to X$ satisfying the relations:

$$f = (f, g) \circ s_n, \qquad g = (f, g) \circ t_n.$$

In particular, we have a commutative square:

$$\begin{tikzcd}
S^n \arrow[r, "{(f,g)}"] \arrow[d, "{(s_{n+1},t_{n+1})}"'] & X \arrow[d] \\
D^{n+1} \arrow[r] & pt.
\end{tikzcd}$$

In the usual model category structure on topological spaces (1.1.60), every object is fibrant, and since the inclusion $S^n \hookrightarrow D^{n+1}$ is a Serre cofibration, there exists an arrow $h : D^{n+1} \to X$ such that the diagram

$$\begin{tikzcd}
S^n \arrow[r, "{(f,g)}"] \arrow[d, "{(s_{n+1},t_{n+1})}"'] & X \arrow[d] \\
D^{n+1} \arrow[r] \arrow[ur, "h"] & pt.
\end{tikzcd}$$

commutes, implying in the relations

$$f = hs_{n+1}, \qquad g = ht_{n+1}.$$

Therefore, *Top* is in fact a contractible co-globular extension, and hence, we can construct the ∞-groupoid $\Pi_\infty X$ associated to a given topological space X.

The application $X \mapsto \Pi_\infty X$, from *Top* to ∞-$\mathcal{G}pd$, is functorial by definition, and determines a functor:

$$\Pi_\infty : Top \longrightarrow \infty\text{-}\mathcal{G}pd.$$

3.3 The homotopy category of ∞-groupoids

In this section, we define homotopy groups for ∞-groupoids and also the homotopy category of ∞-groupoids. For more details, we indicate the reader to see [14] and [39].

Using only the formal properties of contractible co-globular extensions and the definition of ∞-groupoids (3.2.1), Maltsiniotis shows in [14] that every ∞-groupoid G over a weak coherator $\mathbf{C}$ admits a basic structure:

$$G_i, \quad s_i, \quad t_i, \quad k_i, \quad \star_1^i, \quad \star_2^i, \quad in_1^i.$$

where

$$s_i, t_i : G_i \longrightarrow G_{i-1}$$

$$k_i : G_i \longrightarrow G_{i+1}$$

$$\star_1^i : (G_i, s_i) \times_{G_{i-1}} (t_i, G_i) \longrightarrow G_i$$

$$\star_2^i : (G_i, s_2^i) \times_{G_{i-2}} (t_2^i, G_i) \longrightarrow G_i$$

$$in_1^i : G_i \longrightarrow G_i.$$

Moreover, this basic structure of morphisms are defined by the following conditions:

1. $s_{i-1}s_i = t_{i-1}s_i$ and $t_{i-1}t_i = s_{i-1}t_i$, for $i \geq 1$;
2. $s_{i+1}k_i = 1_{G_i} = t_{i+1}k_i$, for $i \geq 0$;
3. $s_i\star_1^i = s_i pr_2$ and $t_i\star_1^i = t_i pr_1$, for $i \geq 1$.
4. $s_i\star_2^i = \star_1^{i-1}(s_i \times_{G_{i-1}} s_i)$ and $t_i\star_2^i = \star_1^{i-1}(t_i \times_{G_{i-1}} t_i)$, for $i \geq 1$.
5. $s_i in_1^i = t_i$ and $t_i in_1^i = s_i$, for $i \geq 1$.

Definition 3.3.1. *Let* $\mathbf{C}$ *be a coherator and* G *be an* ∞*-groupoid over* $\mathbf{C}$*. For* $i \in \omega$ *and* $f, g \in G_i$*, we say that* f *is homotopic to* g*, and write* $f \sim_i g$*, if there exists* $h \in G_{i+1}$ *such that* $s_{i+1}h = f$ *and* $t_{i+1}h = g$*.*

Proposition 3.3.2. *Let* $\mathbf{C}$ *be a coherator. For every* ∞*-groupoid* G *over* $\mathbf{C}$*, the homotopy relation on* i*-arrows, for* $i \in \omega$*, is an equivelence relation on the set* G_i*.*

Proof. (*Reflexivity*) - If $f \in G_i$, then $s_{i+1}(k_i f)$ and $t_i(k_i f) = f$, which implies that $f \sim_i f$.

(*Symmetry*) - If $f \sim_i g$, then there exists $h \in G_{i+1}$ such that $s_{i+1}h = f$ and $t_{i+1}h = g$. Therefore, $s_{i+1}(in_1^{i+1}h) = t_{i+1}h = g$ and $t_{i+1}(in_1^{i+1}) = s_{i+1}h = f$, which implies that $g \sim_i f$.

(*Transitivity*) - If $f, f', f'' \in G_i$ are such that $f \sim_i f'$ and $f' \sim_i f''$, then there are $h, h' \in G_{i+1}$ such that the following equalities are verified:

$$s_{i+1}h = f, \quad t_{i+1}h = f', \quad s_{i+1}h' = f', \quad t_{i+1}h' = f'',$$

which implies that (h', h) is an element of $(G_{i+1}, s_{i+1}) \times_{G_i} (t_{i+1}, G_{i+1})$, and hence, we can compute $h'' = h' \star_1^{i+1} h$, which is an element of G_{i+1}. Moreover, $s_{i+1}h'' = s_{i+1}h = f$ and $t_{i+1}h'' = t_{i+1}h' = f''$, which implies that $f \sim_i f''$. □

3.3.3. Given an ∞-groupoid G over a coherator $\mathbf{C}$, we denote by $[G_i]$ the quotient of the set G_i by the homotopy equivalence relation on i-arrows. The set of connected components of G, denoted by $\pi_0(G)$, is defined by

$$\pi_0(G) =_{df} [G_0].$$

Hence, two objects x and y of G are connected precisely when exists an arrow $h \in G_1$ such that $s_1(h) = x$ and $t_1(h) = y$.

Proposition 3.3.4. *For a* ∞*-groupoid* G*, the applications* $s_i, t_i : G_i \to G_{i-1}$ *and the compositions* $\star_1^i : G_i \to G_{i-1}$ *are compatible with the homotopy relation* $\sim_i$ *(3.3.1) for every* $i > 0$*.*

Proof. The fact that the functions s_i and t_i are compatible with the homotopy relation is immediate from the the definition (3.3.1). For the application $\star_1^i$, suppose that (g, f) and (g', f) are two elements of $(G_i, s_i) \times_{G_{i-1}} (t_i, G_i)$, and $g \sim_i g'$. Then, there exists $h \in G_{i+1}$ such that $s_{i+1}h = g$ and $t_{i+1}h = g'$, and we can define $h' =_{df} \star_2^{i+1}(h, k_i f)$. Then

$$s_{i+1}h' = s_{i+1} \star_2^{i+1} (h, k_i f) = \star_1^i(s_{i+1}h, s_{i+1}k_i f) = \star_1^i(g, f) = g \star_1^i f$$

and

$$t_{i+1}h' = t_{i+1} \star_1^{i+2} (h, k_i f) = \star_1^i (t_{i+1}h, t_{i+1}k_i f) = \star_1^i (g', f) = g' \star_1^i f,$$

which concludes the proof.

$\square$

3.3.5. Using the homotopy relation on i-arrows of an ∞-groupoid G over a weak coherator $\mathbf{C}$, we can form, to each $i \geq 1$, a small category $\varpi_i(G)$, described in the following way:

1. $Ob(\varpi_i(G)) = G_{i-1}$
2. Given two objects x and y in G_{i-1}, we define

   $$Hom_{\varpi_i(G)}(x, y) = Map_{G_i}(x, y)/ \sim_i$$

   where
   $$Map_{G_i}(x, y) = \{f \in G_i : s_i(f) = x, \quad t_i(f) = y\}.$$

   We indicate by $\varphi : x \to y$ an element of $Hom_{\varpi_i(G)}(x, y)$, which represents the homotopy equivalence class of an element $f \in G_i$ such that $s_i(f) = x$ and $t_i(f) = y$. An element of $Map_{G_i}(x, y)$ will be called a *map* from x to y, and we also indicate by $f : x \to y$ an element of $Map_{G_i}(x, y)$.
3. The law of composition $\circ_i$ of $\varpi_i((G)$, is given by the formula

   $$\varphi \circ_i \psi =_{df} g \star_1^i f$$

   where ψ (resp. φ) represents the homotopy equivalence class of a map $f : x \to y$ (resp. $g : y \to z$). From (3.3.4), this previous definition does not depend of the choice of f and g.
4. The identity 1_x of an object $x \in G_{i-1}$ is the homotopy equivalence class of the element $k_{i-1}(x)$.

With the previous definition, we can verify that the category $\varpi_i(G)$ is a groupoid (using the structural arrows $in_1^i : G_i \to G_i$ of G). In particular, given an object x of $\varpi_i(G)$ (which is just an element of G_{i-1}), the set $Hom_{\varpi_i(G)}(x, x)$ is a group with the law of composition of $\varpi_i(G)$.

Definition 3.3.6. *Given an ∞-groupoid G over a coherator* $\mathbf{G}$ *and an object x of G, we define, for each $i \geq 1$, the group*

$$\pi_i(G,x) =_{df} Hom_{\varpi_i(G)}(k^0_{i-1}(x), k^0_{i-1}(x))$$

called the i-homotopy group of the pair (G,x), where k^j_i denotes the arrow $k_{j+i-1}...k_{j+1}k_j$.

By functoriality, if $f : G \to G'$ is a morphism of ∞-groupoids and $x \in G_{i-1}$, with $i \geq 1$, then f is compatible with the homotopy equivalence relation in each component, and we can verify the existence of a canonical group homomorphism

$$\pi_i(f,x) : \pi_i(G,x) \longrightarrow \pi_i(G', f(x))$$

induced from f. Moreover, the function $f_0 : G_0 \to G'_0$ also induces a function

$$\pi_0(f) : \pi_0(G) \longrightarrow \pi_0(G')$$

on the quotients.

Definition 3.3.7. *A morphism $f : G \to G'$ of ∞-groupoids over a coherator* $\mathbf{C}$ *is called a homotopy weak equivalence if*

1. *The induced function $\pi_0(f) : \pi_0(G) \to \pi_0(G')$ is a bijection*
2. *For every object x of G, the induced group morphisms*

   $$\pi_i(f,x) : \pi_i(G,x) \longrightarrow \pi_i(G', f(x)), \qquad i \geq 1$$

   are all isomorphisms.

We denote by $\mathcal{W}_{\mathbf{C},\infty}$ the class of homotopy weak equivalences in ∞-$\mathcal{G}pd_{\mathbf{C}}$, and by $\mathsf{Ho}(\infty\text{-}\mathcal{G}pd_{\mathbf{C}})$ the localised category $(\mathcal{W}_{\infty,\mathbf{C}})^{-1}\mathcal{G}pd_{\mathbf{C}}$, called the homotopy category of ∞-groupoids over $\mathbf{C}$.

Conjecture 3.3.8. *(Homotopy Hypothesis)- For every weak coherator* $\mathbf{C}$*, we have an equivalence of categories:*

$$\mathsf{Ho}(\infty\text{-}\mathcal{G}pd_{\mathbf{C}}) \simeq \mathsf{Hot}.$$

Following Maltsiniotis, the conjecture (3.3.8) is precisely stated via the theory of test categories, and it is actually a consequence of four other conjectures. We expose a variation of these conjectures in the sequel. For the original Maltsiniotis conjectures we indicate [14]. First, fix a weak coherator $\mathbf{C}$ and let A be the opposed category $\mathbf{C}^o$. By the condition (3) of (3.1.23), the category A is a small category. We denote by ∞-$\mathcal{G}pd_{\mathbf{C}}$ the category of ∞-groupoids of type $\mathbf{C}$. Then, ∞-$\mathcal{G}pd_{\mathbf{C}}$ is a full subcategory of $\widehat{A}$, and there exists an inclusion functor

$$i : \infty\text{-}\mathcal{G}pd_{\mathbf{C}} \hookrightarrow \widehat{A}.$$

Conjecture 3.3.9. 1. *The category A is test category.*

2. *A morphism of ∞-$\mathcal{G}pd_{\mathbf{C}}$ is a homotopy weak equivalence according to (3.3.7)) if, and only if, its image under the inclusion functor $i : \infty$-$\mathcal{G}pd_{\mathbf{C}} \hookrightarrow \widehat{A}$ is a $W_{\widehat{A}}$-equivalence. In particular, i is a morphism of localizers from $(\infty\text{-}\mathcal{G}pd_{\mathbf{C}}, \mathcal{W}_{\infty,\mathbf{C}})$ to $(\widehat{A}, W_{\widehat{A}})$, and hence, induces a canonical functor:*

$$\bar{i} : \mathsf{Ho}(\infty\text{-}\mathcal{G}pd_{\mathbf{C}}) \longrightarrow \mathsf{Ho}_{W_{\widehat{A}}}\widehat{A}.$$

3. *The functor $i : \infty$-$\mathcal{G}pd_{\mathbf{C}} \hookrightarrow \widehat{A}$ admits a right adjoint $r : \widehat{A} \to \infty$-$\mathcal{G}pd_{\mathbf{C}}$ which is also a morphism of localizer, i.e., $r(W_{\widehat{A}}) \subseteq \mathcal{W}_{\infty,\mathbf{C}}$, and hence, the functor r induces a canonical functor:*

$$\bar{r} : \mathsf{Ho}_{W_{\widehat{A}}}\widehat{A} \longrightarrow \mathsf{Ho}(\infty\text{-}\mathcal{G}pd_{\mathbf{C}}).$$

4. *The functors $\bar{i} : \mathsf{Ho}(\infty\text{-}\mathcal{G}pd_{\mathbf{C}}) \to \mathsf{Ho}_{W_{\widehat{A}}}\widehat{A}$ and $\bar{r} : \mathsf{Ho}_{W_{\widehat{A}}}\widehat{A} \to \mathsf{Ho}(\infty\text{-}\mathcal{G}pd_{\mathbf{C}})$ are equivalence of categories, quasi-inverse one each other.*

Assuming the conjectures (3.3.9), it follows from the definition of test categories (2.3.2) that $\mathsf{Ho}(\infty\text{-}\mathcal{G}pd_{\mathbf{C}})$ is equivalent to the category Hot of homotopy types.

3.3.10. We defined in (3.2.4) the ∞-groupoid functor

$$\Pi_\infty : Top \longrightarrow \infty\text{-}\mathcal{G}pd,$$

which assigns to each topological space X an ∞-groupoid $\Pi_\infty(X)$. We also defined respectively in (3.3.6) and (3.3.7) the notions of homotopy groups and homotopy weak equivalences of ∞-groupois. We note that from the construction (3.2.4), we have (essentially) $Pt(X) = (\Pi_\infty X)_0$, where $Pt(X)$ is the set of points of the topological space X. Then, it would be expected the following isomorphisms:

$$\pi_0(X) \cong \pi_0(\Pi_\infty(X)), \quad \text{and} \quad \pi_i(X,x) \cong \pi_i(\Pi_\infty X, x) \quad \text{for} \quad i > 0.$$

In this case, it would be a canonical functor

$$\overline{\Pi}_\infty : \mathsf{Hot} \longrightarrow \mathsf{Ho}(\infty\text{-}\mathcal{G}pd).$$

such that the diagram

$$\begin{tikzcd}
Top \arrow[r, "\Pi_\infty"] \arrow[d] & \infty\text{-}\mathcal{G}pd \arrow[d] \\
\mathsf{Hot} \arrow[r, "\overline{\Pi}_\infty"'] & \mathsf{Ho}(\infty\text{-}\mathcal{G}pd)
\end{tikzcd}$$

commutes. It would be expected that $\overline{\Pi}_\infty$ is an equivalence of categories.

3.3.11. We could associate to a weak coherator $\mathbf{C}$ a pre-derivator $\mathbf{Ho}_{\mathcal{W}_\infty}\infty\text{-}\mathcal{G}pd_{\mathbf{C}}$ induced from the localizer $(\infty\text{-}\mathcal{G}pd_{\mathbf{C}}, \mathcal{W}_{\infty,\mathbf{C}})$. We denote by $\infty\text{-}\mathscr{G}pd_{\mathbf{C}}$ the pre-derivator $\mathbf{Ho}_{\mathcal{W}_\infty}\infty\text{-}\mathcal{G}pd_{\mathbf{C}}$. Then, it would be expected that $\infty\text{-}\mathscr{G}pd_{\mathbf{C}}$ is a right derivator and for any right derivator $\mathscr{D}$, the functor

$$\mathcal{H}om_!(\infty\text{-}\mathscr{G}pd_{\mathbf{C}}, \mathscr{D}) \longrightarrow \mathscr{D}(e), \quad \mathcal{F} \mapsto \mathcal{F}(e)$$

is an equivalence of categories. In this case, it would follow from (2.2.5) an equivalence of derivators

$$\mathbf{Hot} \simeq \infty\text{-}\mathscr{G}pd_{\mathbf{C}}$$

and, in particular, an equivalence of categories

$$\mathsf{Hot} \simeq \mathsf{Ho}(\infty\text{-}\mathcal{G}pd_{\mathbf{C}}),$$

since

$$\mathsf{Ho}(\mathcal{G}pd_{\mathbf{C}}) = \infty\text{-}\mathscr{G}pd_{\mathbf{C}}(e).$$

The previous reasoning is actually an alternative strategy to proof (3.3.8) via the theory of derivators. Yet, it may be very difficult to construct the direct homological images of the pre-derivator $\infty\text{-}\mathscr{G}pd_{\mathbf{C}}$ (supposing that they exist).

3

List of Symbols

$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integer numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set o real numbers
$\mathbb{C}$	set of complex numbers
$\varnothing$	empty set, or, according to the context, the initial object of a category
pt	unitary set, the point
$*$	the terminal object of a category, the point
ω	set of finite ordinals
e	the point category, i.e., a category with an unique object and it's identity arrow
e_C	the terminal object of a category C
$\varnothing_C$	the initial object of a category C
$\mathcal{P}(x)$	power set of the set x
$X \cong Y$	the object X is isomorphic to Y
$A \simeq B$	the category A is equivalent the category B
Cat	the category of small categories
$\mathcal{C}at$	the 2-category of small categories
$\mathcal{C}AT$	the 2-category of locally small categories
Ens	the category of sets (see the general conventions in the introduction)
Top	the category of topological spaces (see the general conventions in the introduction)
Grp	the category of groups (see the general conventions in the introduction)
Ab	the category of abelian groups (see the general conventions in the introduction)
Gpd	the category of small groupoids
∞-$\mathcal{G}pd$	the category of ∞-groupoids

3

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