The existence of affine isometric actions with unbounded orbits on Lp spaces: dependence on p

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Resumo

Giulia Cardoso Fantato. **A existência de ações isométricas afins com órbitas ilimitadas em espaços Lp: dependência em p**. Dissertação (Mestrado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022.

A direção central de estudo da dissertação é detalhar um teorema e seus corolários do artigo recente [1] de Marrakchi e de la Salle (2020). Esses autores mostram que se um grupo topológico G admite uma ação isométrica afim com órbitas ilimitadas em um espaço L_p, então G admite o mesmo tipo de ação em L_q, para todo $q \geq p$. Para isso, nós exploramos todas as ações de grupo necessárias, como as ações isométricas afins, ações não-singulares e ações de produto torcido, contemplando a teoria dos cociclos. Adicionalmente, investigamos o teorema de Banach-Lamperti, que caracteriza isometrias em L_p, para $p \neq 2$, e analisamos seus aspectos topológicos. O caso $p = 2$ é tratado com outras ferramentas, como as funções condicionalmente de tipo negativo e a construção GNS.

Palavras-chave: Ações de grupos topológicos. Espaços Lp. Isometrias afins.

Abstract

Giulia Cardoso Fantato. **The existence of affine isometric actions with unbounded orbits on Lp spaces: dependence on p**. Thesis (Master's). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2022.

The central direction of the study of this thesis is to detail a theorem and its corollaries from the recent paper [1] by Marrakchi and de la Salle (2020). These authors show that if a topological group G admits an affine isometric action with unbounded orbits on an L_p-space, then G admits the same type of action on L_q, for every $q \geq p$. In order to achieve that, we explore all the group actions needed, such as affine isometric actions, nonsingular actions and skew-product actions, examining the theory of cocycles. Additionally, we investigate the Banach-Lamperti theorem, which characterizes isometries on L_p, for $p \neq 2$, and analyse its topological aspects. The case $p = 2$ is treated with different tools, namely functions conditionally of negative type and the GNS construction.

Keywords: Topological group actions. Lp-spaces. Affine isometries.

List of symbols and abbreviations

$f_*\mu$	the pushforward measure of μ by f
$\mu \ll \nu$	μ is absolutely continuous with respect to ν
$\frac{d\nu}{d\mu}$	the Radon–Nikodym derivative of ν with respect to μ
$\mu \otimes \nu$	the product measure of μ and ν
$\alpha : G \curvearrowright X$	group action of G on X
$N \lhd G$	N is a normal subgroup of G
$G = H \ltimes N$	G is the semidirect product of N and H
C^∞	the class of functions that are differentiable for all degrees of differentiation
$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{T}$	the set of complex numbers with absolute value 1
$\mathbb{R}_+$	the set of non-negative real numbers
$\mathbb{R}_+^*$	the set of positive real numbers
CNT	conditionally of negative type
PT	positive type

Contents

Introduction

The purpose of this thesis is to detail the main theorem and its corollaries from the recent paper [1] by Marrakchi and de la Salle (2020). According to these authors, Affine isometric actions on Hilbert spaces are very well studied — one can see [2]. However, the importance of studying these actions on Banach spaces is due to its relation to other topics such as group cohomology, fixed point properties and geometric group theory.

As reported in [1], studies such as [3–6] led us to expect that for a given topological group G, it should be “easier" to act isometrically on an L_p-space when the value of p increases. This is what the main theorem in [1] states. In order to express this result, mathematical tools needed to understand it will be summarised first.

According to [7], the study of isometries between Banach spaces must be assigned to start at the origins of the theory of Banach spaces, which appeared in Banach's book [8], from 1932. An *isometry* is a transformation between metric spaces that preserves the distance between the elements of the space. Although some authors consider non bijective maps, in this work, we require isometries to be bijective.

Here, the most useful result on isometries is the Banach-Lamperti theorem, which characterizes the linear isometries between L_p-spaces. First, in [8], Banach was able to characterize linear isometries on $L_p([0,1],\lambda)$, for $1 \leq p < \infty$, $p \neq 2$, where λ is the Lebesgue measure. Then, a generalization was given by Lamperti in [9], which we call the Banach-Lamperti theorem. This result includes the case $0 < p < 1$ and characterizes linear isometries on $L_p(X,\mu)$, where (X,μ) is any σ-finite measure space.

For us, however, it is useful to state such theorem for X a Polish space, which defines (X,μ) to be a standard measure space. In this case we can state the Banach-Lamperti theorem in the following way

Theorem 0.0.1. *Let (X,μ) be a σ-finite standard measure space and $0 < p < \infty$, $p \neq 2$. Any linear isometry $S : L_p(X,\mu) \to L_p(X,\mu)$ is of the form*

$$(Sf)(x) = h(x)\left(\frac{d(\mu\circ\varphi)}{d\mu}(x)\right)^{1/p} f(\varphi(x)), \text{ for every } x \in X \text{ and } f \in L_p(X,\mu),$$

where $h : X \to \mathbb{C}$ is a measurable function with $|h(x)| \equiv 1$ a.e. and $\varphi : X \to X$ is a measure class preserving automorphism.

This version of the Banach-Lamperti theorem is crucial to investigate its topological aspects, using the study of isometric actions of topological groups on L_p spaces.

A group action can be seen as a way of extending the notion of group product to be able to multiply an element of a group by an element of a set in such a manner that basic properties are preserved. Precisely, given a group G and a set X, denote by $\mathrm{Bij}(X)$ the group of all bijections $X \to X$, with the composition operation. An *action* of G on X is a group homomorphism $\phi : G \to \mathrm{Bij}(X)$.

Considering a vector space V, denote by $\mathrm{GL}(V)$ the group of all linear bijections $V \to V$. An action π of G on V such that $\phi(g) = \phi_g \in \mathrm{GL}(V)$, for all $g \in G$, is called a *representation* of G on V. In this sense, a representation is a linear action on a vector space.

In this work, we consider actions and representations for a topological group — a group with a topology for which the group operations are continuous. This is relevant for us to be able to define continuity of actions. For the actions described in the next few paragraphs, consider the topology of pointwise convergence in $\mathrm{Bij}(V)$ (or any other subgroup) and contemplate only the actions which are continuous maps.

Given B a Banach space, an affine map $B \to B$ is given by a linear map plus a constant in B, it can be seen as a linear map for which the origin has changed. Although linear actions (representations) are the ones usually studied, here we focus on *affine isometric actions*, for which $\phi_g \in \mathrm{Isom}(B)$, for all $g \in G$, where $\mathrm{Isom}(B)$ is the group of all affine isometries $B \to B$. These types of actions are interesting for two reasons. The first being that, by the Mazur-Ulam theorem, if B is a Banach space over $\mathbb{R}$, then every isometry $B \to B$ is an affine isometry. Therefore, in this context, when we work with affine isometries, we are considering all isometric maps.

The second reason is what we call the cocycle decomposition of an affine isometric action. To understand what this useful concept means, some concepts must be defined first. Consider $\mathcal{O}(B)$ the group of all linear isometries $B \to B$. If $\pi : G \to \mathrm{Isom}(B)$ is an affine isometric action such that $\pi_g \in \mathcal{O}(B)$, for every $g \in G$, then π is called an *isometric representation*. Also, given an isometric representation π we define a *cocycle* with respect to π as a continuous map $c : G \to B$ such that $c(gh) = c(g) + \pi_g(c(h))$, for all $g, h \in G$. Given this, we show that for every affine isometric action α of G on B, there is an isometric representation π of G on B and a cocycle c with respect to π such that

$$\alpha_g(x) = \pi_g(x) + c(g), \qquad \text{for every } g \in G \text{ and } x \in B.$$

Conversely, for any cocycle c with respect to π, the relation above defines an affine isometric action. We have adopted the term *cocycle decomposition* of α to refer to this relation. It is important to note that, for every $g \in G$, $c(g) = \alpha_g(0)$. The set of all cocycles with respect to π is denoted by $Z^1(G, \pi, B)$.

Also, if we denote by $\mathrm{Aut}(X, [\mu])$ the group of all measure class preserving automorphisms of a standard measure space (X, μ) with the topology of pointwise convergence on probability measures, we can define what we call *nonsingular actions*, denoted by $\sigma : G \curvearrowright (X, \mu)$. A map σ is called a *nonsingular action* when it is a continuous homomorphism of the form $\sigma : G \longrightarrow \mathrm{Aut}(X, [\mu])$.

Consider an abelian topological group $(A, *)$ and denote by $L_0(X, \mu, A)$ the group of all measurable maps $X \to A$, with pointwise operation induced by $*$, identifying the maps that are equal μ-a.e.. Similarly to the previous action, we can define a cocycle of σ as a continuous map $c : G \to L_0(X, \mu, A)$ such that, $c(gh) = c(g) * \sigma_g(c(h))$, for all $g, h \in G$. For example, A can be $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle. The set of all cocycles with respect to σ is denoted by $Z^1_\sigma(G, A)$.

Now that we discussed some types of actions that are used in this work, consider the definition of the sets $K^p(G)$, that are crucial for our main theorem.

Definition 0.0.1. Given G a topological group and $p > 0$, let $K^p(G)$ be the set of continuous functions ψ of the form

$$\begin{aligned} \psi : G &\longrightarrow \mathbb{R}^+ \\ g &\longmapsto \psi(g) = \left\|\alpha_g(0)\right\|^p, \end{aligned}$$

for some affine isometric action $\alpha : G \curvearrowright L_p(X, \mu)$.

The set $K^p(G)$ is important for us to be able to show the existence of an affine isometric action of a topological group G on an L_p-space, by showing that it is a nonempty set. With this, the main theorem can be stated as follows,

Theorem 0.0.2. *Let G be a topological group and $0 < p \leq q < \infty$. Then,*

$$K^p(G) \subseteq K^q(G).$$

The proof of this theorem is divided in two cases: the case where $p = 2$ and the one where $p \neq 2$. This has to be done because the Banach-Lamperti theorem 0.0.1 is only valid for $p \neq 2$. In fact, the formula of theorem 0.0.1 still defines a linear isometry in the case $p = 2$, however, there are isometries on L_2 which are not of this form.

In the case $p = 2$, we use the theory of kernels and functions conditionally of negative type and the Gaussian action. We show the following abstract characterization of the set $K^2(G)$: A map belongs to $K^2(G)$ if, and only if, it is conditionally of negative type. Thus, in order for us to show that $K^2(G)$ is nonempty, it is enough to show the existence of a map conditionally of negative type.

The case $p \neq 2$ relies on the Banach-Lamperti theorem 0.0.1, which has the following corollary

Corollary 0.0.1. *For $p \neq 2$, every isometric representation $\pi : G \curvearrowright L_p(X, \mu)$ is of the form*

$$\pi_g = \omega(g)\sigma_g^{p,\mu} \qquad \forall g \in G,$$

for some cocycle $\omega \in Z^1_\sigma(G, \mathbb{T})$ and $\sigma : G \curvearrowright (X, \mu)$ nonsingular action.

Here, $\sigma^{p,\mu} : G \curvearrowright L_p$ is the isometric representation given by

$$\sigma_g^{p,\mu}(f) = \left(\frac{\mathrm{d}\left[(\sigma_g)_*\mu\right]}{\mathrm{d}\mu}\right)^{\frac{1}{p}} \sigma_g(f) \qquad \text{for every } g \in G \text{ and } f \in L_p(X, \mu).$$

Together with this corollary, we use what we call the *skew-product action* and the

Maharam extension to construct the affine isometric action related to $K^q(G)$.

Now, to explain the importance of the main theorem 0.0.2, note that it implies that for a topological group G and $0 < p \leq q < \infty$, we have that for every affine isometric action $\alpha : G \curvearrowright L_p$, there exists an affine isometric action $\beta : G \curvearrowright L_q$ such that $\|\alpha_g(0)\|_{L_p}^p = \|\beta_g(0)\|_{L_q}^q$, for all $g \in G$. This allow us to show the following:

Corollary 0.0.2. *Let G be a topological group. If G admits an affine isometric action on an L_p space with unbounded orbits, then, G admits such an action on L_q, for every $q \geq p$.*

Also, by applying the theorem to $G = \text{Isom}(L_p)$, we obtain the following:

Corollary 0.0.3. *For $0 < p \leq q < \infty$, Isom(L_p) is isomorphic as a topological group to a closed subgroup of Isom(L_q).*

Outline

This thesis is divided into six chapters. Chapter 1 is dedicated to some basic preliminaries, such as nets and continuity, topological groups, group actions and representations, the Haar measure, semidirect product and affine maps.

Since our work relies heavily on measure theory, in Chapter 2 we expose some topics in this subject, such as equivalence of measures, pushforward measure, Radon-Nikodym derivative, product measure, L_p-spaces, total variation of probability measures and Gaussian random variables.

Chapter 3 is a compilation of important results about isometries: the Mazur-Ulam theorem, characterization of isometries on ℓ_p and the Banach-Lamperti theorem. Also, we study isomorphisms between Hilbert spaces and completion of metric spaces.

Chapter 4 is dedicated to all actions used in this work: Affine isometric actions on Banach spaces, nonsingular actions, Affine isometric actions on L_p-spaces, skew-product actions, the Maharam extension and the Gaussian action. Also, we study cocycles such as the Radon-Nikodym cocycle and important results involving the topological aspects of the Banach-Lamperti theorem.

In Chapter 5 we develop the tools and results we need to work on the case $p = 2$ of our main theorem, such as Bernstein functions, kernels and functions conditionally of negative type and of positive type, Schoenberg's theorem and the GNS construction. We finish by demonstrating an abstract characterization of the set $K^2(G)$ using functions conditionally of negative type.

Finally, in Chapter 6 we study the results of paper [1], where we first study the case $p = 2$ and then prove the main theorem and its corollaries.

Chapter 1

Preliminaries

In this chapter, we present classical concepts and statements that an experienced reader might already be familiar with. Since these results are widely known, we are going to omit the proofs and cite references.

1.1 Nets and continuity

While a sequence is indexed by natural numbers, a net is indexed by a directed set. In this section, we define such concepts and present a theorem that provides us with an equivalent definition for continuity of a function between arbitrary topological spaces. A reference for this topic is [10].

1.1.1 Nets and the limit of a net

We begin with the following definition.

Definition 1.1.1. A *directed* set is a set $A \neq \emptyset$ together with a preorder (reflexive and transitive binary relation) $\leq$ such that every pair of elements has an upper bound, which means that for any $a, b \in A$, there is $c \in A$ with $a \leq c$ and $b \leq c$.

The concept of directed set allow us to define what a net is.

Definition 1.1.2. Given a directed set $(A, \leq)$ and a set X, any function $f : A \to X$ is called a *net*. Usually, a net is denoted by $(x_\alpha)_{\alpha \in A}$.

Note that when $A = \mathbb{N}$ with the usual preorder, the last definition gives us the notion of sequence. Also. we can define a notion of convergence in the following way.

Definition 1.1.3. Let $(x_\alpha)_{\alpha \in A}$ be a net in a topological space X and $Y \subseteq X$. We say that $(x_\alpha)_{\alpha \in A}$ is *eventually in* Y if there is $a \in A$ such that

$$\text{for every } b \in A \text{ with } b \geq a, \text{ we have } x_b \in Y.$$

Also, $x \in X$ is said to be the *limit* of the net $(x_\alpha)_{\alpha \in A}$ in X when for every open neighborhood

U of x, the net is eventually in U.

$$\text{Notation: } x_\alpha \longrightarrow x$$

It is important to notice that such limit is not necessarily unique. If X is Hausdorff and a limit of a net in X exists, then it is unique (in fact, the uniqueness of the limit is equivalent to X being Hausdorff). A proof of this can be seen in [10], proposition 4.4.

Remark 1. If (X, d) is a metric space, $(x_\alpha)_{\alpha \in A}$ is a net in X and $x \in X$, then $x_\alpha \longrightarrow x$ if, and only if, $d(x, x_\alpha) \longrightarrow 0$ in $\mathbb{R}$.

1.1.2 Continuity of functions

Let X and Y be topological spaces. A function $f : X \to Y$ is continuous when the inverse image of every open set in Y is open in X.

Another relevant concept is the notion of sequential continuity.

Definition 1.1.4. A function $f : X \to Y$ is said to be *sequentially continuous* at $x \in X$ when for every sequence $(x_n)_n$ in X which converges to x, the sequence $(f(x_n))_n$ converges to $f(x)$ in Y.

As shown in proposition 5.9 of [11], for functions between metric spaces, the notion of sequential continuity and continuity are equivalent; however, this equivalence does not hold for general topological spaces. In fact, the equivalence is true if the domain is a first countable topological space.

Nets are a useful generalization of sequences because the correspondent concept of sequential continuity (but for nets) is equivalent to continuity for functions between any topological spaces.

Theorem 1.1.1. *If X and Y are topological spaces, a function $f : X \to Y$ is continuous at $x \in X$ if, and only if, for every net $(x_\alpha)_{\alpha \in A}$ in X,*

$$x_\alpha \longrightarrow x \text{ in } X \qquad \Longrightarrow \qquad f(x_\alpha) \longrightarrow f(x) \text{ in } Y.$$

Proof. See [10], proposition 4.8. □

1.2 Topological groups, group actions and representations

This section compiles important notions and tools involving groups. Recall that a group is a set together with a binary operation that satisfies associativity, the existence of an identity element and the existence of the inverse element. First we start by allowing a group to have a topology and requiring this topology to be compatible with the binary operation of the group. This gives us the notion of a topological group.

Then, we define the Haar measure, a useful measure for locally compact abelian topological groups. Also, we investigate the concepts of group actions and representations, which are at the core of this work. Finally, we define semidirect product.

1.2.1 Topological groups

We can define a topology on a group that behaves well with the group operation in the following sense

Definition 1.2.1. A group $(G, .)$ is a *topological group* when it is a topological space for which the following functions are continuous

$$\begin{aligned} . \ : G \times G &\longrightarrow G \\ (x, y) &\longmapsto x.y \end{aligned} \qquad\qquad \begin{aligned} ()^{-1} \ : G &\longrightarrow G \\ x &\longmapsto x^{-1} \end{aligned}$$

where $G \times G$ is equipped with the product topology.

Note that every group is a topological group when considering it with the discrete topology.

A specific class of topological groups that are worth attention are the locally compact ones. A topological space X is *locally compact* when every element of X has a compact neighbourhood, meaning that, for every $x \in X$ there is an open set $U \subseteq X$ and a compact set $K \subseteq X$ such that $x \in U \subseteq K$.

Definition 1.2.2. A topological group G is a *locally compact group* when it is a locally compact topological space and Hausdorff (T_2).

1.2.2 Haar measure

Definition 1.2.3. Let G be a locally compact topological group and $\mathcal{B}$ its Borel σ-algebra. A measure μ on $\mathcal{B}$ is called *left-translation-invariant* (resp. *right-translation-invariant*) when

$$\mu(gB) = \mu(B) \qquad \text{(resp. } \mu(Bg) = \mu(B))$$

for all $B \in \mathcal{B}$ and $g \in G$.

Definition 1.2.4. A *left Haar measure* (resp. *right Haar measure*) on G is a measure m on $\mathcal{B}$ that satisfies:

1. m is left-translation-invariant (resp. right-translation-invariant);
2. $m(K) < \infty$, for every $K \subseteq G$ compact;
3. m is outer regular on $B \in \mathcal{B}$, meaning

$$m(B) = \inf\{m(U) \ : \ B \subseteq U,\ U \text{ open}\};$$

4. m is inner regular on open sets $U \subseteq G$, meaning

$$m(U) = \sup\{m(K) \ : \ K \subseteq U,\ K \text{ compact}\}.$$

If G is abelian, both left and right Haar measures coincide and we simply call it the *Haar measure*. Also, Haar's theorem states that there is, up to a positive multiplicative constant, a unique countably additive nontrivial left Haar measure. Moreover, if we specify the measure of a certain set, the Haar measure is unique. In particular, a probability Haar measure is unique.

1.2.3 Actions and representations of a group

Group actions and representations are crucial throughout our work. Here, we define them in a general way. A good reference for this approach is [12].

Definition 1.2.5. Let G be a group with identity e and let X be a set. A map $\phi : G \times X \to X$ is said to be an *action* of G on X when:

1. $\phi(e, x) = x$, for every $x \in X$;
2. $\phi(g, \phi(h, x)) = \phi(g.h, x)$, for every $g, h \in G$ and $x \in X$.

Notation: $\phi : G \curvearrowright X$, action.

Note that if $\phi : G \curvearrowright X$ is an action and $\mathrm{Bij}(X)$ is the group of all bijections $X \to X$ with the composition operation, the following map is a homomorphism.

$$\begin{aligned} \phi : G &\longrightarrow \mathrm{Bij}(X) \\ g &\longmapsto \phi_g : X \longrightarrow X \\ &\qquad\qquad x \longmapsto \phi_g(x) = \phi(g, x) \end{aligned}$$

For a vetor space V, let $\mathrm{GL}(V)$ denote the *general linear group* of V, the group of all automorphisms of V (all bijective linear operators $V \to V$) with the composition operation.

Definition 1.2.6. Let G be a group and V a vector space. A map $\rho : G \to \mathrm{GL}(V)$ is a *representation* of G on V if it is a group homomorphism.
Notation: $\rho : G \curvearrowright V$, representation.

Therefore, a representation is a linear action on a vector space in the sense that the group $\mathrm{Bij}(V)$ is replaced with $\mathrm{GL}(V)$. For this reason, group actions representations are going to be denoted in a similar way.

Below we give the definition for the orbit of an element $v \in V$, with respect to an action $\phi : G \curvearrowright V$.

Definition 1.2.7. Let $\phi : G \curvearrowright V$ be an action. For $v \in V$ fixed, the following set is called the *orbit of v*.

$$G.v := \{\phi_g(v) : g \in G\} \subseteq V$$

If V is also a metric space, when the orbit of v is a bounded subset of V, we call it a *bounded orbit*.

We also have the following.

Definition 1.2.8. Let $\phi : G \curvearrowright V$ be an action. An element $v \in V$ is said to be a *fixed point* when $\phi_g(v) = v$, for all $g \in G$.

Thus, an element $v \in V$ is a fixed point for a group action $\phi : G \curvearrowright V$ if, and only if, $G.v = \{v\}$. Note that, $v \in V$ is a fixed point when the orbit of v can't go anywhere but v.

1.2.4 Semidirect product

The semidirect product is a way of making up a group from two subgroups, one of them being what we call a normal subgroup. A reference for this is [13].

First we need to define what a normal subgroup is.

Definition 1.2.9. A subgroup N of a group G is called a *normal subgroup* when

$$gng^{-1} \in N, \qquad \text{for all } g \in G \text{ and } n \in N$$

This is denoted by $N \lhd G$.

Definition 1.2.10. A group G with identity e is called the *semidirect product* of the subgroups N and H when $N \lhd G$, $G = NH$ and $N \cap H = \{e\}$.

This is denoted by $G = H \ltimes N$.

With the notion of semidirect product, we have the following proposition.

Proposition 1.2.1. *If $G = H \ltimes N$, then for every $g \in G$, there are unique $n \in N$ and $h \in H$ such that $g = nh$.*

Proof. See [14], section 5.5. □

1.3 Affine maps

Definition 1.3.1. Given a vector space V and $\bar{v} \in V$, a *translation* is a map of the form

$$\begin{aligned} t_{\bar{v}} : V &\longrightarrow V \\ v &\longmapsto t_{\bar{v}}(v) = v + \bar{v} \end{aligned}$$

Affine maps are maps between vector spaces that consist of a linear map and a translation. For example, every affine map $f : \mathbb{R} \to \mathbb{R}$ is of the form $f(x) = ax + b$, for each $x \in \mathbb{R}$, where $a, b \in \mathbb{R}$.

Affine maps can be defined more generally between what we call *affine spaces* (vector spaces whose origin we forget about), as it is done in [15]. However, we can restrict ourselves to define them between real or complex vector spaces, since every vector space can be seen as an affine space and these are the types of affine maps used in this work. A reference for this approach is [16].

Definition 1.3.2. Let V, W be two (real or complex) vector spaces. A function $f : V \to W$ is called an *affine map* (or *affine transformation*) when there exists a linear map $T : V \to W$ and a fixed vector $c \in W$ such that

$$f(x) = (t_c \circ T)(x) = T(x) + c \qquad \forall x \in V$$

The function T is usually called the *linear part of* f.

Chapter 2

Topics in Measure Theory

Our work relies on several concepts and results from Measure Theory, for this reason, this Chapter is dedicated to the crucial topics in this subject.

2.1 Equivalence of measures

Given (X, Σ) a measurable space, recall the definition of absolute continuity of measures.

Definition 2.1.1. Given μ and ν measures on (X, Σ), we say that μ is *absolutely continuous* with respect to ν, denoted by $\mu \ll \nu$ when, for every $A \in \Sigma$, $\nu(A) = 0$ implies $\mu(A) = 0$.

Now, for each measure μ on (X, Σ), define the set $\mathcal{N}_\mu = \{A \in \Sigma \,:\, \mu(A) = 0\}$. There is an equivalence relation on the set of all measures on (X, Σ) given by

$$\mu \sim \nu \qquad \iff \qquad \mathcal{N}_\mu = \mathcal{N}_\nu$$

Note that $\mu \sim \nu \iff \mu \ll \nu \;\text{ and }\; \nu \ll \mu$. Also, we denote by $[\mu]$ the equivalence class of μ given by this equivalence relation. We call it the *measure class of* μ.

Theorem 2.1.1. *Any non-zero σ-finite measure μ on a measurable space (X, Σ) is equivalent to a probability measure ν on (X, Σ).*

Proof. Since μ is σ-finite, there are $A_1, A_2, \ldots$ pairwise disjoint measurable sets such that $\mu(A_n) < \infty$, for every $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} A_n = X$.

Define the following measure, for each $A \in \Sigma$,

$$\nu(A) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mu(A \cap A_n)}{\mu(A_n)}$$

ν is a probability measure, since

$$\nu(X) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mu(X \cap A_n)}{\mu(A_n)} = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Now let us show that $\nu \sim \mu$. If $A \in \Sigma$ is such that $\mu(A) = 0$, then

$$\nu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mu(A \cap A_n)}{\mu(A_n)} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mu(A)}{\mu(A_n)} = 0$$

Also, if $A \in \Sigma$ is such that $\nu(A) = 0$, then

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mu(A \cap A_n)}{\mu(A_n)} = 0,$$

which implies that $\mu(A \cap A_n) = 0$, for all $n \in \mathbb{N}$. Now, note that

$$\mu(A) = \mu(A \cap X) = \mu(A \cap \bigcup_n A_n) = \mu(\bigcup_n A \cap A_n) = \sum_n \mu(A \cap A_n) = 0$$

Therefore, ν is a probability measure that is equivalent to μ. □

2.2 The pushforward measure

In this subsection we define the pushforward measure, which is going to be important throughout our work, and we discuss some properties. A good reference for this subject is [17]

Definition 2.2.1. Given two measurable spaces (X_1, Σ_1) and (X_2, Σ_2), a function $f : X_1 \to X_2$ is said to be *measurable* when $f^{-1}(A) \in \Sigma_1$, for all $A \in \Sigma_2$.

We can transfer a measure from a measurable space to another by using a measurable function in the following way

Definition 2.2.2. Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces, μ a measure on (X_1, Σ_1) and $f : X_1 \to X_2$ a measurable function. The *pushforward* measure of μ with respect to f is the measure given by

$$\begin{aligned} f_*\mu \, : \, \Sigma_2 &\longrightarrow [0, \infty) \\ A &\longmapsto f_*\mu(A) = \mu\left(f^{-1}(A)\right) \end{aligned}$$

A property of the pushforward measure is that it provides us with a change of variables formula for integration.

Theorem 2.2.1. *In the context of the previous definition, a measurable function g on X_2 is $f_*\mu$-integrable if, and only if, $g \circ f$ is μ-integrable and in this case,*

$$\int_{X_2} g \, d(f_*\mu) = \int_{f^{-1}(X_2)} g \circ f \, d\mu$$

Proof. See [17], theorem 3.6.1. □

For (X, Σ, μ) a measure space, we have the following definitions

Definition 2.2.3. Let $f : X \to X$ be measurable. Then, f is said to be *measure preserving* when

$$f_*\mu(A) = \mu(A) \qquad \forall A \in \Sigma$$

Definition 2.2.4. Let $f : X \to X$ be a measurable function. We say that f *preserves the measure class* $[\mu]$ when $f_*\mu \in [\mu]$.

Definition 2.2.5. A map $f : X \to X$ is said to be an *automorphism* on (X, μ) when f is bijective, measurable and its inverse f^{-1} is also measurable.

2.3 The Radon-Nikodym derivative

The Radon-Nikodym derivative can be defined based on the Radon-Nikodym theorem, written bellow. This derivative describes the rate of change of a measure in relation to another. One can check [18] for this subject.

Theorem 2.3.1. *(Radon-Nikodym) Let μ and ν be σ-finite measures on a measurable space (X, Σ) such that $\nu \ll \mu$. Then, there exists a measurable function $f : X \to [0, \infty)$ such that*

$$\nu(A) = \int_A f \, d\mu, \qquad \forall A \in \Sigma$$

Proof. See [18], theorem 3.8. □

Also, if we have another function g that satisfies the equality above, then $f = g$ μ-a.e. For this reason, we call f the *Radon-Nikodym derivative of ν with respect to μ*, denoted by

$$f =: \frac{\mathrm{d}\nu}{\mathrm{d}\mu}.$$

Important properties of the Radon-Nikodym derivative are written below.

Proposition 2.3.2. *Let ν, μ be σ-finite measures such that $\nu \ll \mu$. Then, if g is a ν-integrable function, we have that $g\frac{d\nu}{d\mu}$ is a μ-integrable function and*

$$\int g \, d\nu = \int g \frac{d\nu}{d\mu} \, d\mu$$

Proof. See [18], proposition 3.9. □

Proposition 2.3.3. *Suppose ν, λ, μ are σ-finite measures such that $\nu \ll \lambda$ and $\lambda \ll \mu$. Then, $\nu \ll \mu$ and*

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda}\frac{d\lambda}{d\mu}, \qquad \mu\text{-a.e}$$

Proof. See [18], proposition 3.9. □

Proposition 2.3.4. *Suppose μ and ν are σ-finite measures such that $\mu \ll \nu$ and $\nu \ll \mu$. Then,*

$$\frac{d\nu}{d\mu} = \left(\frac{d\mu}{d\nu}\right)^{-1}, \qquad \mu\text{-a.e}$$

Proof. See [18], corollary 3.10. □

Next we provide a proof for an useful lemma.

Lemma 2.3.5. *Let μ and ν be two σ-finite measures on a measurable space (X, Σ) such that $\mu \ll \nu$. If $f : X \to X$ is an automorphism, then*

$$\frac{d\mu}{d\nu} \circ f = \frac{d(f_*^{-1}\mu)}{d(f_*^{-1}\nu)} \qquad (f_*^{-1}\nu)\text{-a.e.}$$

Proof. Let $A \in \Sigma$. Using theorem 2.2.1,

$$\begin{aligned}(f_*^{-1}\mu)(A) = \mu(f(A)) &= \int_{f(A)} \mathrm{d}\mu = \int_{f(A)} \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\,\mathrm{d}\nu = \int_{f(A)} \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\,\mathrm{d}(f_*(f_*^{-1}\nu)) \\ &= \int_A \left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu} \circ f\right) \mathrm{d}(f_*^{-1}\nu)\end{aligned}$$

From the uniqueness of the Radon-Nikodym derivative, it follows that $\frac{\mathrm{d}\mu}{\mathrm{d}\nu} \circ f = \frac{\mathrm{d}(f_*^{-1}\mu)}{\mathrm{d}(f_*^{-1}\nu)}$ $(f_*^{-1}\nu)$-a.e. □

2.4 Product measure

A good reference for product measure is also [18].

Given (X, Σ_1, μ) and (Y, Σ_2, ν) measure spaces, we can consider $\Sigma_1 \otimes \Sigma_2$ the σ-algebra on the Cartesian product $X \times Y$ generated by the subsets of the form $A_1 \times A_2$, with $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$.

Definition 2.4.1. A *product measure* on $X \times Y$, denoted by $\mu \otimes \nu$, is a measure on $(X \times Y, \Sigma_1 \otimes \Sigma_2)$ such that

$$(\mu \otimes \nu)(A_1 \times A_2) = \mu(A_1)\nu(A_2) \qquad \forall A_1 \in \Sigma_1 \text{ and } A_2 \in \Sigma_2$$

Remark 2. In the case that μ and ν are σ-finite, $\mu \otimes \nu$ is also σ-finite and is uniquely defined.

Theorem 2.4.1. *(Tonelli's Theorem) Let (X, Σ_1, μ) and (Y, Σ_2, ν) be σ-finite measure spaces.*

If $f : X \times Y \to [0, \infty)$ is a measurable function, then

$$\begin{aligned}\int f\, d(\mu \otimes \nu) &= \int \left[\int f(x,y)\, d\nu(y) \right] d\mu(x) \\ &= \int \left[\int f(x,y)\, d\mu(x) \right] d\nu(y)\end{aligned}$$

Proof. See [18], theorem 2.37. □

Proposition 2.4.2. *Let μ_1, ν_1 be σ-finite measures on (X_1, Σ_1) and μ_2, ν_2 be σ-finite measures on (X_2, Σ_2) such that $\nu_1 \ll \mu_1$ and $\nu_2 \ll \mu_2$. Then, $\nu_1 \otimes \nu_2 \ll \mu_1 \otimes \mu_2$ and*

$$\frac{d(\nu_1 \otimes \nu_2)}{d(\mu_1 \otimes \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2)$$

Proof. Let $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$ and consider the measurable subset $A_1 \times A_2$ in $\Sigma_1 \otimes \Sigma_2$. We have that

$$\begin{aligned}(\nu_1 \otimes \nu_2)(A_1 \times A_2) = \nu_1(A_1)\nu_2(A_2) &= \int_{X_1} \chi_{A_1} \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\, \mathrm{d}\mu_1 \int_{X_2} \chi_{A_2} \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\, \mathrm{d}\mu_2 \\ &= \int_{X_1 \times X_2} \chi_{A_1}\chi_{A_2} \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1} \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\, \mathrm{d}(\mu_1 \otimes \mu_2) \\ &= \int_{A_1 \times A_2} \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1} \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}\, \mathrm{d}(\mu_1 \otimes \mu_2)\end{aligned}$$

By the uniqueness of the Radon-Nikodym derivative, the result follows. □

Now, consider the definition of a radial function.

Definition 2.4.2. A function $\varphi : \mathbb{C} \to \mathbb{R}$ is called *radial* when there exists a function $\varphi_0 : \mathbb{R}_+ \to \mathbb{R}$ such that $\varphi(z) = \varphi_0(|z|)$, for every $z \in \mathbb{C}$.

The following technical lemma is essential for the proof of the main theorem of [1].

Lemma 2.4.3. *Let $0 < p < q < \infty$ and $\varphi : \mathbb{C} \to \mathbb{R}$ be a nonzero, radial, compactly supported, Lipschitz function. Then, there is a constant $C(q) > 0$ such that, for all $w \in \mathbb{C}$, we have*

$$\int_{\mathbb{C}} \int_{\mathbb{R}_+^*} \left| \varphi\left(z + y^{-1/p} w\right) - \varphi(z) \right|^q d\lambda(y)\, d\lambda'(z) = C(q)|w|^p$$

with λ the Lebesgue measure on $\mathbb{R}_+^*$ and λ' the Lebesgue measure on $\mathbb{C}$.

Proof. First, let $y \in \mathbb{R}_+^*$ be fixed. Since φ is Lipschitz, there is a constant $K \geq 0$ such that

$$\left| \varphi\left(z + y^{-1/p}\right) - \varphi(z) \right| \leq K \left| y^{-1/p} \right| = K y^{-1/p}$$

Also, since φ is continuous and compactly supported, it is a bounded function and then, there is $M \geq 0$ such that $|\varphi(z)| \leq M$, for any $z \in \mathbb{C}$. Therefore,

$$\left|\varphi\left(z + y^{-1/p}\right) - \varphi(z)\right| \leq \left|\varphi\left(z + y^{-1/p}\right)\right| + |\varphi(z)| \leq 2M$$

Then, $\left|\varphi\left(z + y^{-1/p}\right) - \varphi(z)\right|^q \leq \min\{K^q y^{-q/p}, 2^q M^q\}$. Note that,

$$\begin{aligned}\int_{\mathbb{C}} \left|\varphi\left(z + y^{-1/p}\right) - \varphi(z)\right|^q \mathrm{d}\lambda'(z) &\leq \int_{z \in \operatorname{supp}\varphi} \left|\varphi\left(z + y^{-1/p}\right) - \varphi(z)\right|^q \mathrm{d}\lambda'(z) \\ &\quad + \int_{z + y^{-1/p} \in \operatorname{supp}\varphi} \left|\varphi\left(z + y^{-1/p}\right) - \varphi(z)\right|^q \mathrm{d}\lambda'(z) \\ &\leq 2\min\{K^q y^{-q/p}, 2^q M^q\} \int_{\operatorname{supp}\varphi} \mathrm{d}\lambda'(z)\end{aligned}$$

Denoting by S the Lebesgue measure λ' of suppφ, we have that

$$\int_{\mathbb{R}_+^*} \int_{\mathbb{C}} \left|\varphi\left(z + y^{-1/p}\right) - \varphi(z)\right|^q \mathrm{d}\lambda'(z)\,\mathrm{d}\lambda(y) \leq 2S \int_{\mathbb{R}_+^*} \min\{K^q y^{-q/p}, 2^q M^q\}\,\mathrm{d}\lambda(y)$$

Note that, since $p < q$,

$$\int_0^\infty \min\{K^q y^{-q/p}, 2^q M^q\}\,\mathrm{d}y < \infty$$

Then, we can define a constant $C(q) > 0$ by

$$C(q) := \int_{\mathbb{C}} \int_{\mathbb{R}_+^*} \left|\varphi\left(z + y^{-1/p}\right) - \varphi(z)\right|^q \mathrm{d}\lambda(y)\,\mathrm{d}\lambda'(z) \tag{2.1}$$

Now, fix $w = |w|e^{i\theta'} \in \mathbb{C}$ and make the substitution $u = |w|^p y$ in the integral 2.1, we then have that

$$C(q)|w|^p = \int_{\mathbb{C}} \int_0^\infty \left|\varphi\left(z + |w|u^{-1/p}\right) - \varphi(z)\right|^q \mathrm{d}u\,\mathrm{d}\lambda'(z)$$

Since φ is radial, there exists a function $\varphi_0 : \mathbb{R}_+ \to \mathbb{R}$ such that $\varphi(z) = \varphi_0(|z|)$, for every $z \in \mathbb{C}$. Because of this, we have that

$$\begin{aligned}\varphi\left(z + |w|u^{-1/p}\right) &= \varphi_0\left(\left|z + |w|u^{-1/p}\right|\right) = \varphi_0\left(\left|z + |w|u^{-1/p}\right| \left|e^{i\theta'}\right|\right) = \varphi_0\left(\left|ze^{i\theta'} + wu^{-1/p}\right|\right) \\ &= \varphi\left(ze^{i\theta'} + wu^{-1/p}\right),\end{aligned}$$

and also, $\varphi(z) = \varphi_0(|z|) = \varphi_0(|ze^{i\theta'}|) = \varphi(ze^{i\theta'})$. Then,

$$C(q)|w|^p = \int_0^\infty \int_{\mathbb{C}} \left|\varphi\left(ze^{i\theta'} + wu^{-1/p}\right) - \varphi(ze^{i\theta'})\right|^q \mathrm{d}\lambda'(z)\,\mathrm{d}u$$

Considering $z = |z|e^{i\theta}$, we have the change of variables

$$\begin{aligned} C(q)|w|^p &= \int_0^\infty \int_{\mathbb{C}} \left|\varphi\left(|z|e^{i(\theta+\theta')} + wu^{-1/p}\right) - \varphi(|z|e^{i(\theta+\theta')})\right|^q |z| \mathrm{d}|z| \,\mathrm{d}\theta \,\mathrm{d}u \\ &= \int_0^\infty \int_{\mathbb{C}} \left|\varphi\left(|z|e^{i\theta} + wu^{-1/p}\right) - \varphi(|z|e^{i\theta})\right|^q |z| \mathrm{d}|z| \,\mathrm{d}\theta \,\mathrm{d}u \\ &= \int_0^\infty \int_{\mathbb{C}} \left|\varphi\left(z + wu^{-1/p}\right) - \varphi(z)\right|^q \mathrm{d}\lambda'(z) \,\mathrm{d}u \end{aligned}$$

□

2.5 L_p spaces

L_p spaces are essential mathematical objects for our work. In this section we define them and discuss some properties. For further reading, one can check [18–20].

Here, (X, μ) is a measure space.

2.5.1 L_p for $1 \leq p \leq \infty$

Let us start with the usual case $1 \leq p < \infty$.

Consider the set $\mathcal{L}_p(X, \mu)$ of p-integrable measurable functions $f : X \to \mathbb{C}$, with the seminorm $\|f\|_p := \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} < \infty$. This forms a seminormed vector space, with the natural operations.

For $p = \infty$, $\mathcal{L}_\infty(X, \mu)$ is the space of measurable functions bounded a.e., with $\|f\|_\infty := \inf\{C \geq 0 : |f(x)| \leq C \text{ for almost every } x \in X\} < \infty$.

Now, for $1 \leq p \leq \infty$ consider the kernel of this seminorm. Since for any measurable function f, we have that $\|f\|_p = 0$ if, and only if, $f = 0$ a.e., this kernel does not depend upon p.

$$\ker(\|\cdot\|_p) = \{f : \|f\|_p = 0\} = \{f : f = 0 \text{ a.e.}\} =: \mathcal{N}$$

Definition 2.5.1. Define $L_p(X, \mu)$ as the quotient space

$$L_p(X, \mu) := \mathcal{L}_p(X, \mu)/\mathcal{N}$$

Proposition 2.5.1. *$\|\cdot\|_p$ is a norm on $L_p(X, \mu)$.*

Proof. See [18], section 6.1. □

That is, $L_p(X, \mu)$ is the normed vector space of p-integrable measurable functions where we identify functions that agree μ-a.e.

Proposition 2.5.2. *$L_p(X, \mu)$ is a Banach space.*

Proof. See [18], theorem 6.6. □

The same can be done for functions $f : X \to \mathbb{R}$, instead of $\mathbb{C}$, achieving the same results. We will sometimes denote this space by $L_p(X, \mu, \mathbb{R})$ to emphasize that the codomain of the functions is $\mathbb{R}$.

2.5.2 L_p for $0 < p < 1$

One can inquire what happens if we define $L_p(X, \mu)$ in the same way we did in the last subsection, but for $0 < p < 1$. In this case, the triangle inequality does not hold and therefore $\|\cdot\|_p$ is not a norm.

Definition 2.5.2. A *quasinorm* on a (real or complex) vector space V is a function $\|\cdot\| : V \to \mathbb{R}_+$ such that

- $\|v\| = 0 \iff v = 0$, for all $v \in V$;
- $\|\alpha v\| = |a|\|v\|$, for all $v \in V$ and $a \in (\mathbb{R}$ or $\mathbb{C})$;
- $\|v + w\| \leq k(\|v\| + \|w\|)$, for every $v, w \in V$ and some $k > 0$.

Proposition 2.5.3. *For $0 < p < 1$, $\|.\|_p$ is a quasinorm on $L_p(X, \mu)$.*

Proof. See [18], section 6.1. □

In this case, $L_p(X, \mu)$ is a complete quasinormed vector space (a quasi-Banach space), where the metric is slightly different from the one induced by the norm (as in the case $1 \leq p < \infty$). The metric is defined by

$$d(f, g) := \|f - g\|_p^p, \qquad f, g \in L_p(X, \mu).$$

2.5.3 L_0

For $(A, *)$ an abelian topological group, we can consider its Borel σ-algebra $\mathcal{B}_A$. Therefore, we can define $L_0(X, \mu, A)$ to be the set of measurable functions $f : X \to A$, where we identify functions that are equal a.e..

We can view $(L_0(X, \mu, A), *)$ as a group by considering the pointwise operation, that is, for $f, g \in L_0(X, \mu, A)$,

$$(f * g)(x) = f(x) * g(x), \qquad \forall x \in X.$$

In this work, we are going to consider $A = \mathbb{R}$ or $\mathbb{C}$ (in this case, $* = +$) and also $A = \mathbb{R}_+^*$ or $\mathbb{T}$ (where $* = \cdot$).

Remark 3. $(\mathbb{T}, \cdot)$ is an abelian group, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle and $\cdot$ is the complex multiplication. Note that since $z.z^* = |z|^2 = 1$, for every $z \in \mathbb{T}$, we have that z^* is the inverse group element of z. Since $\mathbb{T}$ is compact, it is possible to define a Haar measure that is a probability measure on $\mathbb{T}$.

2.5.4 Convergence in measure

We can equip $L_0(X, \mu, \mathbb{R})$ with the topology of convergence in measure. There is a distinction between local and global convergence in measure.

Definition 2.5.3. A sequence $f_n \in L_0(X, \mu, \mathbb{R})$ *converges globally in measure* to $f \in L_0(X, \mu, \mathbb{R})$ when, for every $\epsilon > 0$,

$$\lim_n \mu(\{x \in X : |f(x) - f_n(x)| \geq \epsilon\}) = 0$$

Definition 2.5.4. A sequence $f_n \in L_0(X, \mu, \mathbb{R})$ *converges locally in measure* to $f \in L_0(X, \mu, \mathbb{R})$ when, for every $\epsilon > 0$ and every measurable set $A \subseteq X$ with $\mu(A) < \infty$,

$$\lim_n \mu(\{x \in A : |f(x) - f_n(x)| \geq \epsilon\}) = 0$$

Global convergence implies local convergence, but the converse does not always hold. If $\mu(X) < \infty$, then both notions are equivalent. Because of this we can define convergence in probability in the following way.

Definition 2.5.5. If (X, μ) is a probability space, then we say that a sequence $f_n \in L_0(X, \mu, \mathbb{R})$ *converges in probability* to $f \in L_0(X, \mu, \mathbb{R})$ when it converges locally (or equivalently, globally) in measure.

Proposition 2.5.4. *Let (X, μ) be a probability space. For $1 \leq p < \infty$, convergence in $L_p(X, \mu, \mathbb{R})$ implies convergence in probability in $L_0(X, \mu, \mathbb{R})$.*

Proof. See [21], 245G. □

Corollary 2.5.1. *Let (X, μ) be a probability space. For $1 \leq p < \infty$, the following map is a continuous embedding*

$$\begin{aligned} I_p : L_p(X, \mu, \mathbb{R}) &\longrightarrow L_0(X, \mu, \mathbb{R}) \\ f &\longmapsto I_p(f) = f \end{aligned}$$

Proof. If $(f_n)_n$ is a sequence of functions in $L_p(X, \mu, \mathbb{R})$ that converges to $f \in L_p(X, \mu, \mathbb{R})$, then by proposition 2.5.4 we have that $(f_n)_n$ converges to f in probability as well. Therefore, the map I_p is indeed continuous. □

This corollary is very useful to prove continuity of functions with codomain $L_0(X, \mu, \mathbb{R})$, when (X, μ) is a probability space, because we can consider the same function but with codomain $L_p(X, \mu, \mathbb{R})$ and check its continuity.

We also have the following lemma.

Lemma 2.5.5. *Let (X, μ) be a σ-finite measure space and (X, ν) a probability space such that $\mu \ll \nu$. Then, the following map is continuous*

$$\begin{aligned} I : L_0(X, \nu, \mathbb{R}) &\longrightarrow L_0(X, \mu, \mathbb{R}) \\ f &\longmapsto I(f) = f \end{aligned}$$

when considering the topology of convergence in probability on $L_0(X, \nu, \mathbb{R})$ and topology of local convergence in measure on $L_0(X, \mu, \mathbb{R})$.

Proof. Let $(f_n)_n$ be a sequence of functions in $L_0(X, \nu, \mathbb{R})$ that converges to $f \in L_0(X, \nu, \mathbb{R})$. We need to show that $f_n \to f$ in $L_0(X, \mu, \mathbb{R})$. So, let $\epsilon > 0$ and $A \subseteq X$ measurable with $\mu(A) < \infty$. Define $A_n := \{x \in A : |f(x) - f_n(x)| \geq \epsilon\} \subseteq A$ and fix another $\epsilon' > 0$.

Since the measure μ restricted to $A \subseteq X$ becomes a finite measure and $\mu \ll \nu$, we have that for this $\epsilon' > 0$, there is a $\delta > 0$ such that, if $\nu(A_n) < \delta$, then $\mu(A_n) < \epsilon'$. Since $f_n \to f$ in $L_0(X, \nu, \mathbb{R})$, we have that for this $\delta > 0$, there is an $N \in \mathbb{N}$ such that $\nu(A_n) < \delta$ for all $n \geq N$. Therefore, if $n \geq N$, it follows that $\mu(A_n) < \epsilon'$.

This means that $\lim_n \mu(A_n) = 0$ and therefore $f_n \to f$ in $L_0(X, \mu, \mathbb{R})$. □

2.6 Total variation of probability measures

Let μ be a signed measure on a measurable space (X, Σ). If $|\mu| = \mu^+ + \mu^-$ is the total variation measure of μ, we can define the *total variation norm* of μ by

$$\|\mu\|_1 := |\mu|(X)$$

Now, for ν_1, ν_2 probability measures on (X, Σ), the *total variation distance* between ν_1 and ν_2 is

$$d(\nu_1, \nu_2) := \|\nu_1 - \nu_2\|_1 = \sup_{A \in \Sigma} |\nu_1(A) - \nu_2(A)|$$

There is a relation between the total variation distance of two probability measures and the L_1-norm.

Lemma 2.6.1. *If ν_1, ν_2 and μ are probability measures on (X, Σ) such that $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then*

$$\|\nu_1 - \nu_2\|_1 = \frac{1}{2} \left\| \frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right\|_{L_1}$$

Proof. We have that

$$\left\| \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right\|_{L_1} = \int_X \left| \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right| \mathrm{d}\mu = \int_U \left| \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right| \mathrm{d}\mu + \int_V \left| \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right| \mathrm{d}\mu$$

where

$$\begin{aligned} U &:= \left\{ x \in X \,:\, \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \geq 0 \right\} \\ V &:= \left\{ x \in X \,:\, \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} < 0 \right\} \end{aligned}$$

Also, since for $i = 1, 2$,

$$\int_U \frac{\mathrm{d}\nu_i}{\mathrm{d}\mu}\,\mathrm{d}\mu + \int_V \frac{\mathrm{d}\nu_i}{\mathrm{d}\mu}\,\mathrm{d}\mu = \int_X \frac{\mathrm{d}\nu_i}{\mathrm{d}\mu}\,\mathrm{d}\mu = \nu_i(X) = 1$$

we have that

$$\begin{aligned} \int_U \left| \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right| \mathrm{d}\mu &= \int_U \left(\frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right) \mathrm{d}\mu = \int_U \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu}\,\mathrm{d}\mu - \int_U \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}\,\mathrm{d}\mu \\ &= 1 - \int_V \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu}\,\mathrm{d}\mu - 1 + \int_V \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}\,\mathrm{d}\mu = \int_V \left(\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} \right) \mathrm{d}\mu \\ &= \int_V \left| \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right| \mathrm{d}\mu \end{aligned}$$

Then,

$$\left\| \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right\|_{L_1} = 2 \int_U \left| \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right| \mathrm{d}\mu = 2 \int_V \left| \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right| \mathrm{d}\mu$$

Since the Radon-Nikodym derivative is a measurable function, we have that $U, V \in \Sigma$. Then,

$$\begin{aligned} \|\nu_1 - \nu_2\|_1 &= \sup_{A \in \Sigma} |\nu_1(A) - \nu_2(A)| \geq |\nu_1(U) - \nu_2(U)| \\ &= \int_U \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu}\,\mathrm{d}\mu - \int_U \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}\,\mathrm{d}\mu = \int_U \left| \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right| \mathrm{d}\mu \end{aligned}$$

Therefore,

$$\|\nu_1 - \nu_2\|_1 \geq \frac{1}{2} \left\| \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} - \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu} \right\|_{L_1} \tag{2.2}$$

On the other hand, let $A \in \Sigma$.

$$\begin{aligned}
|\nu_1(A) - \nu_2(A)| &= \left| \int_A \frac{d\nu_1}{d\mu}\, d\mu - \int_A \frac{d\nu_2}{d\mu}\, d\mu \right| = \left| \int_A \left(\frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right) d\mu \right| \\
&= \left| \int_{A \cap U} \left(\frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right) d\mu + \int_{A \cap V} \left(\frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right) d\mu \right| \\
&= \left| \int_{A \cap U} \left| \frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right| d\mu - \int_{A \cap V} \left| \frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right| d\mu \right| \\
&\leq \max \left\{ \int_{A \cap U} \left| \frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right| d\mu, \int_{A \cap V} \left| \frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right| d\mu \right\} \\
&\leq \max \left\{ \int_U \left| \frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right| d\mu, \int_V \left| \frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right| d\mu \right\} = \frac{1}{2} \left\| \frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right\|_{L_1}
\end{aligned}$$

Since $A \in \Sigma$ is arbitrary, it follows that

$$\|\nu_1 - \nu_2\|_1 = \sup_{A \in \Sigma} |\nu_1(A) - \nu_2(A)| \leq \frac{1}{2} \left\| \frac{d\nu_1}{d\mu} - \frac{d\nu_2}{d\mu} \right\|_{L_1} \tag{2.3}$$

From 2.2 and 2.3, the result follows. □

2.7 Gaussian random variables

In this section we define what is a Gaussian random variable and show that it belongs to the L_p-space of its probability space, for every $p > 0$. This will be important for us to define the Gaussian action later on. Let us start with some definitions. A reference for this is [2].

Definition 2.7.1. Given (Ω, μ) a probability space, a real-valued *random variable* X on Ω is a measurable function $X : \Omega \to \mathbb{R}$, where $\mathbb{R}$ is considered with its Borel σ-algebra $\mathcal{B}$. We say that two random variables are *identified* if they are equal a.e.

Definition 2.7.2. The *distribuition* (or the *law*) of X is the pushforward measure:

$$\begin{aligned}
\mu_X : \mathcal{B} &\longrightarrow \mathbb{R} \\
B &\longmapsto \mu_X(B) = \mu(X^{-1}(B))
\end{aligned}$$

Definition 2.7.3. If X is either integrable on Ω or positive-valued, we can define its *expectation* (or *mean*) *value* $E[X] := \int_\Omega X(\omega)\, d\mu(\omega) = \int_\mathbb{R} x\, d\mu_X(x)$.

In the last expression we used the change of variables formula for the pushforward measure, theorem 2.2.1. We also can define the following.

Definition 2.7.4. The *variance* of X is given by $\sigma^2 := E[(X - E[X])^2] = E[X^2] - E[X]^2$. If $E[X] = 0$, X is called *centered*.

If the distribution μ_X is absolutely continuous with respect to the Lebesgue measure ($\mu_X \ll \lambda$), we can apply the Radon-Nikodym theorem 2.3.1 and define

Definition 2.7.5. The *density function* of X, is the map $p : \mathbb{R} \to \mathbb{R}^+$ such that $\mathrm{d}\mu_X(x) = p(x)\mathrm{d}x$ and $\int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = 1$.

Finally, we can define what is a Gaussian random variable.

Definition 2.7.6. A real-valued random variable X on Ω is called *Gaussian* if either X is constant or $\mu_X \ll \lambda$, with a density of the form

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2},$$

for $\sigma > 0$ and $m \in \mathbb{R}$.

Note that if X is a Gaussian random variable, then $E[X] = m$ and the variance of X is σ^2.

Proposition 2.7.1. *For $p > 0$, If X is a centered Gaussian random variable, then $X \in L^p(\Omega, \mu)$.*

Proof. Note that, for $p > 0$, if $g : \mathbb{R} \to \mathbb{R}$ is the function $g(x) = |x|^p$ and X is a centered Gaussian random variable,

$$\begin{aligned}\int_\Omega |X|^p \,\mathrm{d}\mu = \int_\Omega g \circ X \,\mathrm{d}\mu = \int_\mathbb{R} g \,\mathrm{d}\mu_X = \int_{-\infty}^{\infty} |x|^p p(x)\,\mathrm{d}x = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|^p e^{-x^2/2\sigma^2}\,\mathrm{d}x \\ = \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty x^p e^{-x^2/2\sigma^2}\,\mathrm{d}x\end{aligned}$$

the last integral can be solved by calculating the gamma function Γ at $\frac{p+1}{2}$, and is equal to $\frac{\Gamma(\frac{p+1}{2})}{2b^{(p+1)/2}}$, where $b = \frac{1}{2\sigma^2}$.

Hence, $X \in L^p(\Omega, \mu)$. $\square$

Chapter 3

Isometries

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is called an *isometric embedding* when

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2), \qquad \forall x_1, x_2 \in X.$$

Note that such map is automatically injective. We are going to call a function $f : X \to Y$ an *isometry* when it is a surjective isometric embedding.

Therefore, note that all isometries are bijective. It is important to note that, unlike us, some authors use the term isometry even for non-surjective isometric embeddings.

In this chapter, we are going to characterize isometries on Banach spaces and on classical Banach spaces, the most important one for us being the L_p spaces, where we have the Banach-Lamperti theorem.

3.1 Isometries on Banach spaces

Here we present some fundamental results on isometries. Good references for this are [7, 8].

Named after Stanisław Mazur and Stanisław Ulam, the Mazur-Ulam theorem characterizes isometries between normed spaces.

Theorem 3.1.1. *(Mazur-Ulam) If $f : X \to Y$ is an isometry between normed spaces and $f(0) = 0$, then f is real linear.*

Proof. See [7], theorem 1.3.5. □

Note that if $f : X \to Y$ is any isometry between normed spaces, we can define $f' := f - f(0)$. Then, f' is still an isometry and $f'(0) = 0$. Therefore, by the Mazur-Ulam theorem, f' is real linear. This means that $f = f' + f(0)$: any isometry consists of a translation and a real linear isometry. In the case of normed spaces over $\mathbb{R}$, the theorem can be stated as

Theorem 3.1.2. *(Mazur-Ulam) Every isometry between normed spaces over $\mathbb{R}$ is an affine isometry.*

Given K a compact metric space, we denote by $C(K)$ the Banach space of continuous functions $f : K \to \mathbb{R}$ with the supremum norm.

In 1932, Banach characterized linear isometries on $C(K)$.

Theorem 3.1.3. *(Banach) If $T : C(K) \to C(K)$ is a linear isometry, then there is a homeomorphism $\varphi : K \to K$ and $h \in C(K)$ with $|h(x)| \equiv 1$ such that*

$$(Tf)(x) = h(x)f(\varphi(x)) \qquad \forall x \in K, \forall f \in C(K)$$

Proof. See [7], theorem 1.2.2. □

3.1.1 Isometries on ℓ_p

Consider, for $1 \leq p < \infty$,

$$\ell_p = \{(x_n)_n \in \mathbb{R}^{\mathbb{N}} : \sum_n |x_n|^p < \infty\}$$

with the norm $\|x\|_p = \left(\sum_n |x_n|^p\right)^{\frac{1}{p}}$.

There is a characterization for isometries on ℓ_p for $p \neq 2$.

Theorem 3.1.4. *Let $1 \leq p < \infty$ and $p \neq 2$. All linear isometries $T : \ell_p \to \ell_p$ are of the form*

$$T(x_n) = (\epsilon_n x_{\pi(n)})_n \tag{3.1}$$

where $(\epsilon_n)_n$ is a sequence in $\{-1, 1\}$ and $\pi : \mathbb{N} \to \mathbb{N}$ is a permutation of natural numbers. Conversely, for any $(\epsilon_n)_n$ and π as above, the map defined by (3.1) is an isometry.

Proof. See [22], theorem 2.3. □

3.1.2 Isometries on L_p-spaces and the Banach-Lamperti theorem

For $1 \leq p < \infty$, $p \neq 2$, Banach described in [8] the linear isometries on $L_p([0,1], \lambda)$, where λ is the Lebesgue measure. Then, a generalization was given by Lamperti on [9] about linear isometric embeddings on $L_p(X, \mu)$, for (X, μ) any σ-finite measure space and include the case $0 < p < 1$. This generalization is called the Banach-Lamperti theorem.

First, we start stating the theorem described by Banach in [8], theorem I of chapter 11, section 5.

Theorem 3.1.5. *Let $1 \leq p < \infty$, $p \neq 2$. If $S : L_p([0,1], \lambda) \to L_p([0,1], \lambda)$ is a linear isometry, then*

$$(Sf)(t) = h(t)f(\phi(t)) \tag{3.2}$$

where ϕ is a measurable function defined on $[0,1]$ *onto itself and h is a function on* $[0,1]$ *such that*

$$|h|^p = \frac{d(\lambda \circ \phi)}{d\lambda}$$

Conversely, for any h and ϕ as above, the operator S defined by (3.2) is an isometry.

Now, let us discuss the generalization given by Lamperti. Let (X, Σ, μ) be a σ-finite measure space.

Definition 3.1.1. A set map $T : \Sigma \to \Sigma$ defined modulo null sets is called a *regular set isomorphism* when

1. $T(X \setminus A) = T(X) \setminus T(A) \qquad \forall A \in \Sigma$
2. $T(\bigcup A_n) = \bigcup T(A_n) \qquad \forall (A_n)_n$disjoint sequence in Σ
3. $\mu(T(A)) = 0 \qquad \iff \qquad \mu(A) = 0$

Remark 4. If $T : \Sigma \to \Sigma$ is a regular set isomorphism, then T induces a unique linear transformation $U : L_0(X, \mu, \mathbb{C}) \to L_0(X, \mu, \mathbb{C})$ such that

- $U(\chi_A) = \chi_{T(A)} \qquad \forall A \in \Sigma$
- If $(f_n)_n$ is a sequence such that $f_n \to f$, then $U(f_n) \to U(f)$ a.e.
- $(U(f))^{-1}(B) = U(f^{-1}(B))$, for all B Borel set
- $U(fg) = (U(f))(U(g))$, $U(\overline{f}) = \overline{U(f)}$ and $U(|f|) = |U(f)|$, $\forall f, g \in L_0(X, \mu, \mathbb{C})$.

Theorem 3.1.6. *(Banach-Lamperti) Let (X, Σ, μ) be a σ-finite measure space and $0 < p < \infty$, $p \neq 2$. If $S : L_p(X, \mu) \to L_p(X, \mu)$ is a linear isometric embedding, then there exists a regular set isomorphism $T : \Sigma \to \Sigma$ and a function h defined on X such that*

$$(Sf)(x) = h(x)(Uf)(x) \tag{3.3}$$

where U is the linear transformation induced by T and h is such that

$$\int_{T(A)} |h|^p \, d\mu = \int_{T(A)} \frac{d(\mu \circ T^{-1})}{d\mu} \, d\mu = \mu(A) \qquad \forall A \in \Sigma$$

Conversely, for any h and T as above, the operator S defined by (3.3) is an isometry.

Proof. See [7], theorem 3.2.5. □

In subsection 4.3.3, we shall come back to the Banach-Lamperti theorem and investigate its topological aspects, when studying isometric actions of topological groups on L_p-spaces. Also, we are going to adapt the Banach-Lamperti theorem for (X, μ) a σ-finite standard measure space and isometries considered surjective. In this context, we are able to substitute the use of the regular set isomorphism $T : \Sigma \to \Sigma$ for a point mapping $\varphi : X \to X$, in a similar way done in 3.1.5.

One can wonder if the Banach-Lamperti theorem 3.1.6 is valid for $p = 2$, however, this is not the case. One can check that, for $p = 2$, equation 3.3 is an isometry that preserves disjoint supports, yet there are isometries on L_2 that do not preserve disjoint supports and therefore cannot be given by equation 3.3. An example of this is given in [7], section 3.4.

3.2 Isomorphism between Hilbert spaces

In this subsection we present the result that all separable infinite dimensional Hilbert spaces are isomorphic and investigate if there is a similar result for Hilbert spaces that are not necessarily separable.

Definition 3.2.1. Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces. An *isomorphism* between $\mathcal{H}$ and $\mathcal{H}'$ is a linear surjection $I : \mathcal{H} \to \mathcal{H}'$ such that

$$\langle I(x), I(y) \rangle = \langle x, y \rangle \qquad \text{for all } x, y \in \mathcal{H}.$$

In this case, we say that $\mathcal{H}$ and $\mathcal{H}'$ are *isomorphic*.

Proposition 3.2.1. *If $T : \mathcal{H} \to \mathcal{H}'$ is a linear map between Hilbert spaces, then T is an isometric embedding if, and only if,*

$$\langle T(x), T(y) \rangle = \langle x, y \rangle \qquad \textit{for all } x, y \in \mathcal{H}.$$

Proof. See [23], proposition 5.2. □

Therefore, if $I : \mathcal{H} \to \mathcal{H}'$ is an isomorphism, then I is a linear map that preserves the inner product and then it is an isometric embedding. Since I is a surjection, we can say that I is a linear isometry.

Theorem 3.2.2. *All separable infinite dimensional Hilbert spaces are isomorphic.*

Proof. See [23], corollary 5.5. □

One can wonder if there is a result on isomorphisms between Hilbert spaces that are not necessarily separable.

Definition 3.2.2. Let $\mathcal{H}$ be a Hilbert space and A be an index set. We say that $\{u_\alpha : \alpha \in A\} \subseteq \mathcal{H}$ is *orthonormal* when $\|u_\alpha\| = 1$, for all $\alpha \in A$ and $\langle u_\alpha, u_\beta \rangle = 0$, for all $\alpha \neq \beta$. Also, we say that $\{u_\alpha : \alpha \in A\}$ is an *orthonormal basis* when it is orthonormal and a basis for $\mathcal{H}$.

Remark 5. A consequence of Zorn's lemma is that every Hilbert space has an orthonormal basis.

We have defined the sequence spaces ℓ_p on section 3.1.1. But this can be generalized in the following way.

Definition 3.2.3. Let $1 \leq p < \infty$ and A an index set. Define

$$\ell_p(A) = \{(x_\alpha)_{\alpha \in A} \in \mathbb{C}^A \, : \, \sum_\alpha |x_\alpha|^p < \infty\}$$

with the norm $\|x\|_p = \left(\sum_\alpha |x_\alpha|^p\right)^{\frac{1}{p}}$.

Remark 6. $\ell_p(A)$ are Banach spaces and $\ell_2(A)$ is a Hilbert space. Also, the index set A can be turned into a measure space by giving it the discrete σ-algebra and the counting measure ν. One can check that

$$\ell_p(A) = L_p(A, \nu)$$

Theorem 3.2.3. *Let $\mathcal{H}$ be a Hilbert space and $\{u_\alpha \, : \, \alpha \in A\}$ an orthonormal basis for $\mathcal{H}$. Then $\mathcal{H}$ is isomorphic to $\ell_2(A) = L_2(A, \nu)$.*

Proof. See [24], section 4.19. □

3.3 Completion of metric spaces

An important result is that we can uniquely extend an isometry defined on dense subspaces of complete metric spaces. Let us start with the definition of completion.

Definition 3.3.1. A *completion* of a metric space (X, d) is a pair consisting of a complete metric space (X', d') and an isometry $\varphi \, : \, X \to X'$ such that $\varphi(X)$ is dense in X'.

Remark 7. Every metric space has a completion. Also, a completion of an inner product space is a Hilbert space.

Theorem 3.3.1. *Let M, N be complete metric spaces with $X \subseteq M$ and $Y \subseteq N$ dense subspaces and $T' \, : \, X \to Y$ an isometry. Then, T' can be extended to a unique isometry $T \, : \, M \to N$.*

Proof. Proof can be found on [11], proposition 7.10. □

Chapter 4

The Actions

This Chapter is dedicated to exposing the actions and representations used throughout the work. We start by presenting affine isometric actions and then nonsingular actions, introducing for both, the groups and topologies involved in their definition and defining their respective cocycles.

Subsequently, we talk about isometric actions on L_p-spaces, constructing the tools needed to show an important characterization of isometric representations, using a convenient version of the Banach-Lamperti theorem 3.1.6. Then, we define skew-product actions and the Maharam extension, finishing with the Gaussian action.

4.1 Affine isometric actions on Banach spaces

4.1.1 Groups of isometric functions

In section 1.2.3 we defined two groups of functions: $\mathrm{Bij}(X)$ and $\mathrm{GL}(V)$. Now that we know what isometries are, let us define some other groups of functions and fix the notation.

Definition 4.1.1. For V a normed vector space, $\mathcal{O}(V)$ denotes the *orthogonal group* of V: the group of all isometric linear transformations $V \to V$, with the composition operation.

Definition 4.1.2. For V a normed vector space, $\mathrm{Isom}(V)$ denotes the group of all isometric affine transformations $V \to V$, with the composition operation.

Definition 4.1.3. For V a vector space, Denote by $\mathcal{T}(V)$ the group of all translations as in definition 1.3.1, with the composition operation.

Some authors denote the group of translations only by V, but we shall use $\mathcal{T}(V)$ to avoid confusions.

Proposition 4.1.1. *Isom*(V) *is indeed a group.*

Proof. First note that for $f, g \in \mathrm{Isom}(V)$, it follows that for every $x \in V$, $f(x) = L(x) + c$

and $g(x) = T(x) + d$, for $L, T : V \to V$ linear maps and $c, d \in V$. Then,

$$(f \circ g)(x) = f(T(x) + d) = L(T(x) + d) + c = (L \circ T)(x) + L(d) + c$$

with $L \circ T$ linear and $L(d) + c \in V$. Therefore, $f \circ g \in \mathrm{Isom}(V)$.

Also, the composition operation is associative. The identity map $\mathrm{Id}_V : V \to V$ is an affine map because it is a linear map plus the constant $0 \in V$. Finally, since all isometries are bijective, we can consider the inverse of every function in $\mathrm{Isom}(V)$. □

Proposition 4.1.2. *For V a normed vector space, $\mathcal{T}(V)$ and $\mathcal{O}(V)$ are both subgroups of $\mathrm{Isom}(V)$.*

Proof. Every linear isometry is an affine isometry, because we can view the linear map as itself plus 0, with $0 \in V$. Also, every translation is an affine map, since we can view the translation as itself composed with the identity map Id_V. Besides, every translation is an isometry.

Also, it is easy to show that $\mathcal{O}(V)$ and $\mathcal{T}(V)$ are closed under the composition operation and inverse. □

Remark 8. Note that if we consider the normed vector space V to be over $\mathbb{R}$, Mazur-Ulam's Theorem 3.1.2 states that every isometry $V \to V$ is an affine isometry. Therefore, in this case, the group $\mathrm{Isom}(V)$ is precisely the group of all isometries on V.

4.1.2 Strong continuity

Let X be a topological space and $\mathcal{F}(X)$ denote a family of functions $X \to X$. We begin with the definition of strong continuity.

Definition 4.1.4. Let X be a topological space and G be a topological group. A map $\phi : G \to \mathcal{F}(X)$ is said to be *strongly continuous* when the mapping

$$\begin{aligned} G &\longrightarrow X \\ g &\longmapsto \phi_g(x) \end{aligned}$$

is continuous for every $x \in X$.

The aim of this section is to prove that, when considering the topology of pointwise convergence in $\mathcal{F}(X)$, the continuity of $\phi : G \to \mathcal{F}(X)$ is equivalent to strong continuity of this map.

Remember the fact that a net $(f_a)_a$ converges to some $f \in \mathcal{F}(X)$ in the *topology of pointwise convergence* in $\mathcal{F}(X)$ if, and only if, for each $x \in X$, the net $(f_a(x))_a$ converges to $f(x)$ in X.

The equivalence of these continuities will follow easily using theorem 1.1.1.

Theorem 4.1.3. *The map $\phi : G \to \mathcal{F}(X)$ is strongly continuous if, and only if, it is continuous when considering the topology of pointwise convergence in $\mathcal{F}(X)$.*

Proof. ($\Longrightarrow$) Suppose

$$\begin{aligned} G &\longrightarrow X \\ g &\longmapsto \phi_g(x) \end{aligned}$$

is continuous for every $x \in X$. This means that for every $x \in X$, this map is continuous at every $g \in G$.

Let $g \in G$ and $(g_a)_a$ be a net in G such that $g_a \longrightarrow g$ in G. Since for every $x \in X$, the map above is continuous at this g, it follows that $\phi_{g_a}(x) \longrightarrow \phi_g(x)$, for every $x \in X$. Then, by the topology of pointwise convergence, we have that $\phi_{g_a} \longrightarrow \phi_g$ in $\mathcal{F}(X)$. Hence, $\phi : G \to \mathcal{F}(X)$ is continuous.

($\Longleftarrow$) Now, suppose $\phi : G \to \mathcal{F}(X)$ is continuous, where $F(X)$ is equipped with the topology of pointwise convergence.

Let $x \in X$, $g \in G$ and $(g_a)_a$ be a net in G such that $g_a \longrightarrow g$ in G. Then, by hypothesis, $\phi_{g_a} \longrightarrow \phi_g$ in $\mathcal{F}(X)$ and this means that $\phi_{g_a}(x') \longrightarrow \phi_g(x')$, for every $x' \in X$, in particular for $x' = x$. Therefore,

$$\begin{aligned} G &\longrightarrow X \\ g &\longmapsto \phi_g(x) \end{aligned}$$

is continuous. Since $x \in X$ is arbitrary, the result follows. $\square$

4.1.3 Affine isometric actions on Banach spaces

The aim of this subsection is to define affine isometric actions on Banach spaces in a similar way as done in [2], on which they define them on Hilbert spaces.

First let us show that the group of affine isometries is the semidirect product of the group of linear isometries with the translations.

Theorem 4.1.4. *For V a normed vector space, we have that*

$$Isom(V) = \mathcal{O}(V) \ltimes \mathcal{T}(V)$$

Proof. Using definition 1.3.2, we have that if $f : V \to V$ is an affine map, then $f = t_c \circ T$, for $T : V \to V$ a linear map and $c \in V$. Note that, for $v, w \in V$,

$$\|f(v) - f(w)\| = \|T(v) + c - T(w) + c\| = \|T(v) - T(w)\|$$

Therefore, f is an isometry if, and only if, its linear part T is an isometry. Because of this we can say that $\mathrm{Isom}(V) = \mathcal{T}(V)\mathcal{O}(V)$.

Also, if $t_{\bar{v}} \in \mathcal{T}(V) \cap \mathcal{O}(V)$, we have a translation that is also a linear map. Therefore, $t_{\bar{v}}(0) = 0$ and then $\bar{v} = 0$. This means that $t_{\bar{v}} = \mathrm{Id}_V$.

Finally, let $f \in \mathrm{Isom}(V)$ and $t_{\bar{v}} \in \mathcal{T}(V)$. Then, $f = t_c \circ T$, for $c \in V$ and T a linear map.

Now, for any $x \in V$,

$$\begin{aligned}\left(f \circ t_{\bar{v}} \circ f^{-1}\right)(x) &= f(f^{-1}(x) + \bar{v}) = T(f^{-1}(x) + \bar{v}) + c = T(f^{-1}(x)) + T(\bar{v}) + c \\ &= f(f^{-1}(x)) + T(\bar{v}) = x + T(\bar{v}) = t_{T(\bar{v})}(x)\end{aligned}$$

Therefore, $f \circ t_{\bar{v}} \circ f^{-1} = t_{T(\bar{v})} \in \mathcal{T}(V)$. This shows that $\mathcal{T}(V) \triangleleft \mathrm{Isom}(v)$.

Hence, $\mathrm{Isom}(V) = \mathcal{O}(V) \ltimes \mathcal{T}(V)$. □

Definition 4.1.5. Let G be a topological group and B be a Banach space. A map α is called an *affine isometric action* when it is a strongly continuous homomorphism of the form

$$\begin{aligned}\alpha : G &\longrightarrow \mathrm{Isom}(B) \\ g &\longmapsto \alpha_g : B \longrightarrow B \\ &\qquad\qquad x \longmapsto \alpha_g(x)\end{aligned}$$

Notation: $\alpha : G \curvearrowright B$ affine isometric action.

Definition 4.1.6. Let G be a topological group and V be a normed vector space. A map π is called an *isometric representation* when it is a strongly continuous homomorphism of the form

$$\begin{aligned}\pi : G &\longrightarrow \mathcal{O}(V) \\ g &\longmapsto \pi_g : V \longrightarrow V \\ &\qquad\qquad x \longmapsto \pi_g(x)\end{aligned}$$

Notation: $\pi : G \curvearrowright V$ isometric representation.

Note in the previous definitions that, since we are considering a type of continuity, the topological aspect of the action and representation is now relevant.

Also, the strong continuity of $\alpha : G \to \mathrm{Isom}(B)$ and $\pi : G \to \mathcal{O}(V)$ is equivalent to saying that these functions are continuous when considering the topology of pointwise convergence in $\mathrm{Isom}(B)$ and $\mathcal{O}(V)$, respectively, as shown in theorem 4.1.3.

Sometimes an isometric representation is called an *orthogonal representation* by some authors.

4.1.4 The cocycle decomposition of an affine isometric action

In this section we are going to define cocycles and coboundaries for an isometric representation. Also, for B a Banach space, we are going to show that any affine isometric action $\alpha : G \curvearrowright B$ can be decomposed in a linear part (consisting of an isometric representation) plus a constant part that consists of a cocycle.

Definition 4.1.7. Let $\pi : G \curvearrowright B$ be an isometric representation. A *cocycle with respect to* π is a continuous map $c : G \to B$ such that, $c(gh) = c(g) + \pi_g(c(h))$, for all $g, h \in G$. This last expression is called the *cocycle relation*. The set of all cocycles with respect to π is denoted by $Z^1(G, \pi, B)$.

Let us prove some cocycle properties.

Proposition 4.1.5. *Let $c \in Z^1(G, \pi, B)$. Then, we have that*

1. $c(e) = 0$;
2. $c(g^{-1}) = -\pi_{g^{-1}}(c(g))$, *for all* $g \in G$;
3. $c(hgh^{-1}) = \pi_h(c(g))$, *for all* $h \in G$ *and* $g \in ker(\pi)$.

Proof. 1. Fix any $g \in G$ and note that

$$c(e) = c(ee) = c(e)+\pi_e(c(e)) = c(e)+\pi_{gg^{-1}}(c(e)) = c(e)+\left(\pi_g \circ \pi_{g^{-1}}\right)(c(e)) = c(e)+c(e)$$

therefore, $c(e) = 0$.

2. Now, notice that, fixing $g \in G$,

$$0 = c(e) = c(g^{-1}g) = c(g^{-1}) + \pi_{g^{-1}}(c(g))$$

hence, $c(g^{-1}) = -\pi_{g^{-1}}(c(g))$.

3. Let $h \in G$ and $g \in \ker(\pi)$. Then, $\pi_g = \mathrm{Id}$. Now, note that

$$\pi_h(c(gh^{-1})) = \pi_h(c(g) + \pi_g(c(h^{-1}))) = \pi_h(c(g)) + \pi_h(c(h^{-1}))$$

and

$$\pi_h(c(h^{-1})) = \pi_h(-\pi^{-1}(c(h))) = -c(h).$$

Therefore,

$$c(hgh^{-1}) = c(h) + \pi_h(c(gh^{-1})) = c(h) + \pi_h(c(g)) - c(h) = \pi_h(c(g)).$$

□

Proposition 4.1.6. *Let $\pi : G \curvearrowright V$ be an isometric representation. If a map $c : G \to V$ is such that there exists $v \in V$ with $c(g) = \pi_g(v) - v$, for all $g \in G$, then c is a cocycle with respect to π.*

Proof. First, let us show that c satisfies the cocycle relation. Let $g, h \in G$

$$c(gh) = \pi_{gh}(v) - v = \pi_g(\pi_h(v)) - v = \pi_g(c(h) + v) + c(g) - \pi_g(v) = c(g) + \pi_g(c(h))$$

From the strong continuity of π, we have that the map

$$\begin{aligned} G &\longrightarrow V \\ g &\longmapsto \pi_g(v) \end{aligned}$$

is continuous. Therefore, $c : G \to V$ is continuous and hence, a cocycle. □

Definition 4.1.8. Let $\pi : G \curvearrowright V$ be an isometric representation. A *coboundary with respect to* π is a cocycle c for which there exists $v \in V$ such that $c(g) = \pi_g(v) - v$, for all $g \in G$. The set of all coboundaries is denoted by $B^1(G, \pi, V) \subseteq Z^1(G, \pi, V)$.

Remark 9. One can check that if π is the trivial representation (meaning, $\pi_g = \mathrm{Id}$, for all $g \in G$), then $B^1(G, \pi, V) = \{0\}$

Since $\mathrm{Isom}(B) = \mathcal{O}(B) \ltimes \mathcal{T}(B)$, every $f \in \mathrm{Isom}(B)$ is of the form $f = t_{\bar{v}} \circ T$ with $t_{\bar{v}} \in \mathcal{T}(B)$ and $T \in \mathcal{O}(B)$. Also note that $f(0) = \bar{v}$, therefore, f is of the form $f(x) = T(x) + f(0)$, for all $x \in B$.

We can consider the following group homomorphism

$$\begin{aligned} p : \mathrm{Isom}(B) &\longrightarrow \mathcal{O}(B) \\ f = t_{\bar{v}} \circ T &\longmapsto p(f) = T \end{aligned}$$

Definition 4.1.9. For an affine isometric action $\alpha : G \curvearrowright B$, we call $\pi := p \circ \alpha : G \to \mathcal{O}(B)$ the *linear part of* α.

Proposition 4.1.7. *The linear part of an affine isometric action $\alpha : G \curvearrowright B$ is an isometric representation.*

Proof. π is a group homomorphism because it is a compostion of two group homomorphisms. Now, note that for each $g \in G$ and $x \in B$,

$$\alpha_g(x) = \pi_g(x) + \alpha_g(0).$$

Then, $\pi_g(x) = \alpha_g(x) - \alpha_g(0)$. Since α is strongly continuous, the map

$$\begin{aligned} G &\longrightarrow B \\ g &\longmapsto \alpha_g(x) \end{aligned}$$

is continuous for each $x \in B$ and in particular for $0 \in B$. This shows that π is strongly continuous. □

Conversely, we would like to know, given $\pi : G \curvearrowright B$ an isometric representation, what the affine isometric actions $G \curvearrowright B$ with linear part π are.

Definition 4.1.10. Let $\pi : G \curvearrowright B$ an isometric representation and $c \in Z^1(G, \pi, B)$. The *affine isometric action associated to the cocycle* c is defined by the map

$$\begin{aligned} \alpha : G &\longrightarrow \mathrm{Isom}(B) \\ g &\longmapsto \alpha_g : B \longrightarrow B \\ &\qquad\qquad x \longmapsto \alpha_g(x) = \pi_g(x) + c(g) \end{aligned}$$

Proposition 4.1.8. *α, the affine isometric action associated to a cocycle $c \in Z^1(G, \pi, B)$, is indeed an affine isometric action.*

Proof. Let $g, h \in G$ and $x \in B$.

$$\begin{aligned}\alpha_{gh}(x) &= \pi_{gh}(x) + c(gh) = \pi_g(\pi_h(x)) + c(g) + \pi_g(c(h)) = \pi_g(\pi_h(x) + c(h)) + c(g) \\ &= \alpha_g(\pi_h(x) + c(h)) = \alpha_g(\alpha_h(x)) = (\alpha_g \circ \alpha_h)(x)\end{aligned}$$

This shows that α is a group homomorphism. Also, for each $g \in G$, $\alpha_g(x) = \pi_g(x) + c(g) \in \mathrm{Isom}(B)$, since $\mathcal{T}(B)$ and $\mathcal{O}(B)$ are subgroups of $\mathrm{Isom}(B)$ (proposition 4.1.2).

Because $c : G \to B$ is continuous and π is strongly continuous, we have that the map

$$\begin{aligned} G &\longrightarrow B \\ g &\longmapsto \alpha_g(x) = \pi_g(x) + c(g)\end{aligned}$$

is continuous for each $x \in B$, meaning that α is strongly continuous. □

Lemma 4.1.9. *Let $\pi : G \curvearrowright B$ be an isometric representation and $\alpha : G \to Isom(B)$ be a map.*

The following are equivalent:

- *α is an affine isometric action of G with linear part π.*
- *There is a cocycle with respect to π, $c : G \to B$, such that α is the affine isometric action associated to c.*

Proof. ($\Longrightarrow$) If α is an affine isometric action of G with linear part π, then for every $g \in G$ and $x \in B$,

$$\alpha_g(x) = \pi_g(x) + \alpha_g(0)$$

Now, we define the following map

$$\begin{aligned} c : G &\longrightarrow B \\ g &\longmapsto c(g) = \alpha_g(0).\end{aligned}$$

This map is continuous since α is strongly continuous. Now, for every $g, h \in G$,

$$c(gh) = \alpha_{gh}(0) = (\alpha_g \circ \alpha_h)(0) = \alpha_g(c(h)) = \pi_g(c(h)) + \alpha_g(0) = \pi_g(c(h)) + c(g)$$

Therefore, c is a cocycle with respect to π and α is the affine isometric action associated to c.

($\Longleftarrow$) Let $c : G \to B$ be the cocycle with respect to π such that α is the affine isometric action associated to c. Proposition 4.1.8 shows that α is an affine isometric action.

Note that, for each $g \in G$, $(p \circ \alpha)(g) = p(\alpha_g) = \pi_g$. Then, $\pi = p \circ \alpha$ is the linear part of α. □

Therefore, given $\alpha : G \curvearrowright B$ an affine isometric action, there is a pair (π, c) consisting of an isometric representation $\pi : G \curvearrowright B$ (the linear part of α) and a cocycle $c \in Z^1(G, \pi, B)$

such that

$$\alpha_g(x) = \pi_g(x) + c(g), \qquad \text{for every } g \in G \text{ and } x \in B.$$

This is the *cocycle decomposition* of α.

4.1.5 Affine isometric actions with unbounded orbits

Having in mind the definitions given in subsection 1.2.3, in order for us to define the notion of an affine isometric action with unbounded orbits, we need the following result.

Proposition 4.1.10. *Let $\alpha : G \curvearrowright B$ be an affine isometric action with $\pi : G \curvearrowright B$ its linear part and $c \in Z^1(G, \pi, B)$ the cocycle associated with α. Then, the following are equivalent:*

1. *c is bounded, meaning that the image of c is bounded on B;*
2. *all orbits of α are bounded;*
3. *some orbit of α is bounded;*

Proof. (1. $\implies$ 2.) Since the image of c is bounded, there is $D > 0$ such that $\|c(g)\| < D$, for all $g \in G$. Let $x \in B$ be fixed. For $g \in G$,

$$\|\alpha_g(x)\| = \|\pi_g(x) + c(g)\| \leq \|\pi_g(x)\| + \|c(g)\| \leq \|x\| + D < \infty$$

Therefore, every orbit of α is bounded.

(2. $\implies$ 3.) Trivial.

(3. $\implies$ 1.) Let $x \in B$ be such that its orbit is bounded, meaning that there exists $C > 0$ such that $\|\alpha_g(x)\| < C$, for all $g \in G$.

Since, for each $g \in G$, $c(g) = \alpha_g(x) - \pi_g(x)$, we have that

$$\|c(g)\| \leq \|\alpha_g(x)\| + \|\pi_g(x)\| \leq C + \|f\| < \infty$$

Thus, c is bounded. □

Note that if an affine isometric action has an unbounded orbit, then by proposition 4.1.10 it cannot have a bounded orbit and then, all its orbits are unbounded. Therefore, we have the following definition.

Definition 4.1.11. Let $\alpha : G \curvearrowright B$ be an affine isometric action. If α has an unbounded orbit (and therefore all orbits unbounded), then we call α an *affine isometric action with unbounded orbits.*

If α is an affine isometric action with unbounded orbits, then it has no fixed points. This is because if α has a fixed point $x \in B$, then its orbit $G.x = \{x\}$ is bounded.

Also note that an isometric representation $\pi : G \curvearrowright B$ has 0 as a fixed point. Therefore, every isometric representation has bounded orbits.

4.2 Nonsingular actions

Another important type of action in our work are the ones given by nonsingular maps, which are given by a weaker requirement than to be measure preserving.

We start by defining the type of measure space used. Recall that a Polish space is a separable and completely metrizable topological space

Definition 4.2.1. We call a measure space (X, Σ, μ) *standard* when X is a Polish space and Σ its Borel σ-algebra.

Let (X, Σ, μ) be a σ-finite standard measure space, that we denote simply by (X, μ).

The structure of a standard measure space gives us the following result.

Lemma 4.2.1. *Let $0 < p < \infty$ and $T : X \to X$ be a measurable map. If $f = f \circ T$, for every $f \in L_p(X, \mu)$, then $T = Id$.*

Proof. For $A \in \Sigma$, let f be the indicator function $f = \chi_A$. Then, for each $x \in X$, $\chi_A(x) = \chi_A(T(x))$. In particular, if $x \in A$, then $T(x) \in A$.

Let $x \in X$ and $\delta > 0$. Consider $B(x, \delta)$ to be the open ball centered in x with radius δ associated to a metric of the Polish space X. Then, we have that $T(x) \in B(x, \delta)$, for every $\delta > 0$. Therefore, $T(x) = x$, for every $x \in X$. □

4.2.1 Groups of automorphisms on a measure space

Now, let us define what we mean by a nonsingular map.

Definition 4.2.2. $f : X \to X$ is said to be *nonsingular* when it is an automorphism on (X, μ) and preserves the measure class $[\mu]$.

Proposition 4.2.2. *If $f : (X, \mu) \to (X, \mu)$ is an automorphism, then f preserves the measure class $[\mu]$ if, and only if, $f_*\mu \ll \mu$ and $f_*^{-1}\mu \ll \mu$.*

Proof. ($\Longrightarrow$) Let f be a measure class preserving automorphism of (X, μ), which means that $f_*\mu \in [\mu]$. Then, $f_*\mu \ll \mu$ and $\mu \ll f_*\mu$. This means that, for $B \in \Sigma$, $\mu(B) = 0 \iff \mu(f^{-1}(B)) = 0$.

Now fix $A \in \Sigma$ such that $\mu(A) = 0$. Then, $\mu(f^{-1}(f(A))) = 0$. Now take $B = f(A) \in \Sigma$ and we have that $\mu(f(A)) = 0$, meaning, $f_*^{-1}\mu(A) = 0$. Therefore, $f_*^{-1}\mu \ll \mu$.

($\Longleftarrow$) If $f_*\mu \ll \mu$ and $f_*^{-1}\mu \ll \mu$, we have that, for $B \in \Sigma$, $\mu(B) = 0 \implies \mu(f(B)) = 0$.

Now fix $A \in \Sigma$ such that $f_*\mu(A) = 0$. This means that $\mu(f^{-1}(A)) = 0$. Choosing $B = f^{-1}(A) \in \Sigma$, we have that $\mu(f(f^{-1}(A))) = 0$ and then $\mu(A) = 0$. Hence, $\mu \ll f_*\mu$.

Therefore, $f_*\mu \ll \mu$ and $\mu \ll f_*\mu$, meaning that $f_*\mu \in [\mu]$. □

Now we define two groups of functions.

Definition 4.2.3. Let $\mathrm{Aut}(X,\mu)$ be the group of all measure preserving automorphisms on (X,μ), in which we identify the automorphisms that are equal a.e., with the composition operation.

Definition 4.2.4. Let $\mathrm{Aut}(X,[\mu])$ be the group of all nonsingular automorphisms on (X,μ), in which we identify the automorphisms that are equal a.e., with the composition operation.

4.2.2 Topology of pointwise convergence on probability measures

The topology defined on the group $\mathrm{Aut}(X,[\mu])$ can be described in terms of convergence in the following way.

Consider $\mathrm{Aut}(X,[\mu])$ with the topology of pointwise convergence on probability measures. In this topology, a net $(\theta_\alpha)_\alpha$ in $\mathrm{Aut}(X,[\mu])$ converges to $\theta \in \mathrm{Aut}(X,[\mu])$ if, and only if,

$$\|(\theta_\alpha)_*\nu - \theta_*\nu\|_1 \longrightarrow 0, \qquad \forall \nu \in [\mu] \text{ probability measure.}$$

Proposition 4.2.3. *$Aut(X,[\mu])$ is a Polish group.*

Proposition 4.2.4. *$Aut(X,\mu)$ is a closed subgroup of $Aut(X,[\mu])$.*

For the last two propositions, see [1].

4.2.3 Nonsingular actions

Definition 4.2.5. Let G be a topological group and (X,μ) a σ-finite standard measure space. A map σ is called a *nonsingular action* when it is a continuous homomorphism of the form

$$\begin{aligned} \sigma \,:\, G &\longrightarrow \mathrm{Aut}(X,[\mu]) \\ g &\longmapsto \sigma_g \end{aligned}$$

Notation: $\sigma \,:\, G \curvearrowright (X,\mu)$ nonsingular action.

A specific type of nonsingular action is the measure preserving action.

Definition 4.2.6. In the case that $\sigma_g \in \mathrm{Aut}(X,\mu)$, for every $g \in G$, the action σ is called a *measure preserving* action. In addition, when (X,μ) is a probability space, σ is called a *probability measure preserving* action, or a *pmp* action.

4.2.4 Cocycles and coboundaries of a nonsingular action

In the same manner as done with affine isometric actions, we can define cocycles for nonsingular actions.

Let us first explain the notation. As explored in section 2.5.3, we have that $L_0(X,\mu,A)$ is a group with the pointwise operation $*$. For $f \in L_0(X,\mu,A)$ we are going to denote by

f^{-1} the inverse group element of f, but be careful not to confuse with the notation for inverse function.

Also, for $\sigma : G \curvearrowright (X, \mu)$ a nonsingular action, if $f \in L_0(X, \mu, A)$, we are going to abuse the notation and write $\sigma_g(f)$ meaning $f \circ \sigma_g^{-1}$.

Definition 4.2.7. Let $\sigma : G \curvearrowright (X, \mu)$ be a nonsingular action and A an abelian topological group. An *A-valued cocycle of* σ is a continuous map $c : G \to L_0(X, \mu, A)$ such that, $c(gh) = c(g) * \sigma_g(c(h))$, for all $g, h \in G$. The set of all A-valued cocycles of σ is denoted by $Z^1_\sigma(G, A)$

Lemma 4.2.5. *Given $\sigma : G \curvearrowright (X, \mu)$ a nonsingular action and A an abelian topological group, for each $f \in L_0(X, \mu, A)$, the mapping*

$$\begin{aligned} G &\longrightarrow L_0(X, \mu, A) \\ g &\longmapsto \sigma_g(f) = f \circ \sigma_g^{-1} \end{aligned}$$

is continuous.

Proof. See [1]. □

Proposition 4.2.6. *Let $\sigma : G \curvearrowright (X, \mu)$ be a nonsingular action. If a map $c : G \to L_0(X, \mu, A)$ is such that there exists an $f \in L_0(X, \mu, A)$ such that $c(g) = \sigma_g(f) * f^{-1}$, for all $g \in G$, then c is a cocycle of σ.*

Proof. First, let $g, h \in G$

$$\begin{aligned} c(gh) &= \sigma_{gh}(f) * f^{-1} = \sigma_g(\sigma_h(f)) * f^{-1} = \sigma_g(f \circ \sigma_h^{-1}) * f^{-1} = (f \circ \sigma_h^{-1} \circ \sigma_g^{-1}) * f^{-1} \\ &= ((c(h) * f) \circ \sigma_g^{-1}) * (f \circ \sigma_g^{-1})^{-1} * c(g) = (c(h) \circ \sigma_g^{-1}) * (f \circ \sigma_g^{-1}) * (f \circ \sigma_g^{-1})^{-1} * c(g) \\ &= c(g) * \sigma_g(c(h)) \end{aligned}$$

From lemma 4.2.5, we get the continuity of c. Hence, c is a cocycle. □

Definition 4.2.8. Let $\sigma : G \curvearrowright (X, \mu)$ be a nonsingular action and A an abelian topological group. An *A-valued coboundary of* σ is a cocycle c for which there exists $f \in L_0(X, \mu, A)$ such that $c(g) = \sigma_g(f) * f^{-1}$, for all $g \in G$. The set of all A-valued coboundaries of σ is denoted by $B^1_\sigma(G, A) \subseteq Z^1_\sigma(G, A)$.

4.3 Isometric actions on L_p-spaces

In this section, we start by defining the β and η mappings and the Radon-Nikodym cocycle, that are used to define the isometric representations $\pi^{p,\mu} : \mathrm{Aut}(X, [\mu]) \ltimes L_0(X, \mu, \mathbb{T}) \curvearrowright L_p(X, \mu)$ and $\sigma^{p,\mu} : G \curvearrowright L_p(X, \mu)$. Finally, we use them to show a characterization of isometric representations, which is a consequence of the Banach-Lamperti theorem 3.1.6.

4.3.1 The maps β and η

First, consider the following lemma that will be used later on.

Lemma 4.3.1. *If $\theta \in Aut(X, [\mu])$, then $\mu\left(\{x \in X : \frac{d(\theta_*\mu)}{d\mu}(x) = 0\}\right) = 0$.*

Proof. From theorem 2.3.1, we have that

$$\theta_*\mu\left(\{x \in X : \frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}(x) = 0\}\right) = \int_{\{x\in X : \frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}(x)=0\}} \frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}\,\mathrm{d}\mu = 0$$

Since $\theta \in \mathrm{Aut}(X, [\mu])$, we have that $\mu\left(\{x \in X : \frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}(x) = 0\}\right) = 0$ as well. $\square$

Consider the following definition.

Definition 4.3.1. Let $p > 0$ and (X, μ) be a σ-finite measure space and define the map

$$\begin{aligned} \beta : \mathrm{Aut}(X, [\mu]) &\longrightarrow \mathcal{O}\left(L_p(X, \mu)\right) \\ \theta &\longmapsto \beta(\theta) =: T_\theta : L_p(X, \mu) \longrightarrow L_p(X, \mu) \\ &\qquad\qquad f \longmapsto T_\theta(f) = \left(\frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}\right)^{1/p} \theta(f) \end{aligned}$$

First, note that since $\theta \in \mathrm{Aut}(X, [\mu])$, $\theta_*\mu \ll \mu$. Then, the Radon-Nikodym derivative $\frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}$ indeed exists.

Proposition 4.3.2. *β is a group homomorphism.*

Proof. To show this, let $\theta_1, \theta_1 \in \mathrm{Aut}(X, [\mu])$ and $f \in L_p(X, \mu)$.

$$T_{\theta_1\circ\theta_2}(f) = \left(\frac{\mathrm{d}\left[(\theta_1 \circ \theta_2)_*\,\mu\right]}{\mathrm{d}\mu}\right)^{1/p} \cdot f \circ (\theta_1 \circ \theta_2)^{-1}$$

and since

$$(\theta_1 \circ \theta_2)_*\,\mu = \mu \circ (\theta_1 \circ \theta_2)^{-1} = \mu \circ \theta_2^{-1} \circ \theta_1^{-1} = [(\theta_2)_*\mu] \circ \theta_1^{-1},$$

we have that

$$\frac{\mathrm{d}\left\{[(\theta_2)_*\mu] \circ \theta_1^{-1}\right\}}{\mathrm{d}\mu} = \frac{\mathrm{d}\left\{[(\theta_2)_*\mu] \circ \theta_1^{-1}\right\}}{\mathrm{d}\left[(\theta_1)_*\mu\right]} \cdot \frac{\mathrm{d}\left[(\theta_1)_*\mu\right]}{\mathrm{d}\mu}$$

Also, from lemma 2.3.5, it follows that

$$\frac{\mathrm{d}\left\{[(\theta_2)_*\mu] \circ \theta_1^{-1}\right\}}{\mathrm{d}\left[(\theta_1)_*\mu\right]} = \frac{\mathrm{d}\left[(\theta_2)_*\mu\right]}{\mathrm{d}\mu} \circ \theta_1^{-1}, \qquad [(\theta_1)_*\mu]\text{-a.e.}$$

and since $\theta_1 \in \mathrm{Aut}(X, [\mu])$, they are equal μ-a.e. as well. Therefore,

$$\begin{aligned}
T_{\theta_1\circ\theta_2}(f) &= \left(\frac{\mathrm{d}\,[(\theta_1)_*\mu]}{\mathrm{d}\mu} \cdot \frac{\mathrm{d}\,[(\theta_2)_*\mu]}{\mathrm{d}\mu} \circ \theta_1^{-1}\right)^{1/p} \cdot f \circ (\theta_1 \circ \theta_2)^{-1} \\
&= \left(\frac{\mathrm{d}\,[(\theta_1)_*\mu]}{\mathrm{d}\mu}\right)^{1/p} \cdot \left[\left(\frac{\mathrm{d}\,[(\theta_2)_*\mu]}{\mathrm{d}\mu}\right)^{1/p} \circ \theta_1^{-1}\right] \cdot f \circ \theta_2^{-1} \circ \theta_1^{-1}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left(T_{\theta_1} \circ T_{\theta_2}\right)(f) = T_{\theta_1}\left(T_{\theta_2}(f)\right) &= T_{\theta_1}\left(\left(\frac{\mathrm{d}\,[(\theta_2)_*\mu]}{\mathrm{d}\mu}\right)^{1/p} \cdot \left(f \circ \theta_2^{-1}\right)\right) \\
&= \left(\frac{\mathrm{d}\,[(\theta_1)_*\mu]}{\mathrm{d}\mu}\right)^{1/p} \cdot \left[\left(\frac{\mathrm{d}\,[(\theta_2)_*\mu]}{\mathrm{d}\mu}\right)^{1/p} \cdot \left(f \circ \theta_2^{-1}\right)\right] \circ \theta_1^{-1} \\
&= \left(\frac{\mathrm{d}\,[(\theta_1)_*\mu]}{\mathrm{d}\mu}\right)^{1/p} \cdot \left[\left(\frac{\mathrm{d}\,[(\theta_2)_*\mu]}{\mathrm{d}\mu}\right)^{1/p} \circ \theta_1^{-1}\right] \cdot f \circ \theta_2^{-1} \circ \theta_1^{-1}
\end{aligned}$$

that means, $T_{\theta_1\circ\theta_2}(f) = \left(T_{\theta_1} \circ T_{\theta_2}\right)(f)$. Therefore, β is a homomorphism. □

The following proposition shows that β is well defined.

Proposition 4.3.3. *$T_\theta \in \mathcal{O}\left(L_p(X, \mu)\right)$, for each $\theta \in Aut(X, [\mu])$.*

Proof. Since $\theta(f) = f \circ \theta^{-1}$, it is straightforward that T_θ is linear.

Using Theorem 2.2.1 and Proposition 2.3.2,

$$\begin{aligned}
\|T_\theta(f)\|^p &= \int_X \left|\left(\frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}\right)^{1/p} \theta(f)\right|^p \mathrm{d}\mu = \int_X \left|f \circ \theta^{-1}\right|^p \frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}\,\mathrm{d}\mu \\
&= \int_X \left|f \circ \theta^{-1}\right|^p \mathrm{d}(\theta_*\mu) = \int_X \left|f \circ \theta^{-1}\right|^p \circ \theta\,\mathrm{d}\mu \\
&= \int_X |f|^p\,\mathrm{d}\mu = \|f\|^p
\end{aligned}$$

This means that T_θ is an isometric embedding. The surjectivity is going to follow from Proposition 4.3.2:

Now, for $\theta \in \mathrm{Aut}(X, [\mu])$, if we choose $\theta_1 = \theta$ and $\theta_2 = \theta^{-1}$, it follows that

$$\mathrm{Id}_{L_p} = T_{\mathrm{Id}} = T_{\theta\circ\theta^{-1}} = T_\theta \circ T_{\theta^{-1}}$$

$$\mathrm{Id}_{L_p} = T_{\mathrm{Id}} = T_{\theta^{-1}\circ\theta} = T_{\theta^{-1}} \circ T_\theta$$

Therefore, T_θ is indeed surjective and this finishes the proof that $T_\theta \in \mathcal{O}\left(L_p(X, \mu)\right)$. □

Proposition 4.3.4. *β is injective.*

Proof. Since β is a group homomorphism, it being injective is equivalent to the fact that its kernel is trivial. Therefore, consider $\theta \in \mathrm{Aut}(X, [\mu])$ be such that $\beta(\theta) = \mathrm{Id}_{L_p}$. This means that, for every $f \in L_p(X, \mu)$, $T_\theta(f) = f$.

From the latter, $\left(\frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}\right)^{1/p} f \circ \theta^{-1} = f$. By choosing the constant function $f = 1$, we find out that $\left(\frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}\right)^{1/p} = 1$. Then, we have that $f \circ \theta^{-1} = f$, for every $f \in L_p(X, \mu)$. Thus, from Lemma 4.2.1, $\theta = \mathrm{Id}$. □

Now that we defined the β map and showed some important properties, consider the following.

Definition 4.3.2. Let $p > 0$ and (X, μ) be a σ-finite measure space and define the map

$$\begin{aligned} \eta \,:\, L_0(X, \mu, \mathbb{T}) &\longrightarrow \mathcal{O}\left(L_p(X, \mu)\right) \\ h &\longmapsto \eta_h \,:\, L_p(X, \mu) \longrightarrow L_p(X, \mu) \\ & \qquad\qquad f \longmapsto \eta_h(f) = h.f \end{aligned}$$

It is straightforward that η is a group homomorphism, considering the multiplication operation in $L_0(X, \mu, \mathbb{T})$ and the composition in $\mathcal{O}\left(L_p(X, \mu)\right)$.

Proposition 4.3.5. *η is injective.*

Proof. Again, since η is a group homomorphism, it suffices to show that its kernel is trivial. Let $h \in L_0(X, \mu, \mathbb{T})$ be such that $\eta_h = \mathrm{Id}_{L_p}$. Then, for every $f \in L_p(X, \mu)$, $h.f = f$.

Choose the constant function $f = 1 \in L_p(X, \mu)$ and this finishes the proof. □

Now, let us show that $\mathcal{O}(L_p(X, \mu))$ can be considered as the semidirect product of the ranges of the maps β and η.

Consider

$$\begin{aligned} H_\beta &:= \beta\left[\mathrm{Aut}(X, [\mu])\right] \\ N_\eta &:= \eta\left[L_0(X, \mu, \mathbb{T})\right] \end{aligned}$$

Since β and η are homomorphisms, it is immediate that H_β and N_η are subgroups of $\mathcal{O}(L_p(X, \mu))$.

Proposition 4.3.6. *Define $G' := N_\eta H_\beta$. Then, $G' = H_\beta \ltimes N_\eta$.*

Proof. Let $\eta_h \in N_\eta$ and $\eta_{h'} \circ T_\theta \in G'$. Note that

$$(\eta_{h'} \circ T_\theta) \circ \eta_h \circ (\eta_{h'} \circ T_\theta)^{-1} = \eta_{h'} \circ T_\theta \circ \eta_h \circ T_{\theta^{-1}} \circ \eta_{h'^{-1}}$$

Let $f \in L_p(X, \mu)$.

$$
\begin{aligned}
(T_\theta \circ \eta_h \circ T_{\theta^{-1}})(f) &= T_\theta\left(\eta_h\left(\left(\frac{\mathrm{d}(\theta_*^{-1}\mu)}{\mathrm{d}\mu}\right)^{1/p} \cdot \theta^{-1}(f)\right)\right) \\
&= T_\theta\left(h \cdot \left(\frac{\mathrm{d}(\theta_*^{-1}\mu)}{\mathrm{d}\mu}\right)^{1/p} \cdot (f \circ \theta)\right) \\
&= \left(\frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}\right)^{1/p} \cdot \left[h \cdot \left(\frac{\mathrm{d}(\theta_*^{-1}\mu)}{\mathrm{d}\mu}\right)^{1/p} \cdot (f \circ \theta)\right] \circ \theta^{-1} \\
&= \left(\frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}\right)^{1/p} \cdot \left(h \circ \theta^{-1}\right) \cdot \left[\left(\frac{\mathrm{d}(\theta_*^{-1}\mu)}{\mathrm{d}\mu}\right)^{1/p} \circ \theta^{-1}\right] \cdot \left(f \circ \theta \circ \theta^{-1}\right) \\
&= \left(h \circ \theta^{-1}\right) \cdot (T_\theta \circ T_{\theta^{-1}})(f) = \left(h \circ \theta^{-1}\right) \cdot f = \eta_{h \circ \theta^{-1}}(f)
\end{aligned}
$$

Therefore,

$$(\eta_{h'} \circ T_\theta) \circ \eta_h \circ (\eta_{h'} \circ T_\theta)^{-1} = \eta_{h'} \circ \eta_{h \circ \theta^{-1}} \circ \eta_{h'^{-1}} \in N_\eta$$

Now, take $\eta_h \in N_\eta \cap H_\beta$. Then, $\eta_h = T_\theta$, for some $\theta \in \mathrm{Aut}(X, \mu)$. That means,

$$h \cdot f = \eta_h(f) = T_\theta(f) = \left(\frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}\right)^{1/p} \cdot \left(f \circ \theta^{-1}\right) \qquad \forall f \in L_p(X, \mu)$$

If we take f to be the constant function equal to 1, we obtain $h = \left(\frac{\mathrm{d}(\theta_*\mu)}{\mathrm{d}\mu}\right)^{1/p}$. Then,

$$f = f \circ \theta^{-1} \qquad \forall f \in L_p(X, \mu)$$

From lemma 4.2.1, we have that $\theta = \mathrm{Id}$. This means that $\eta_h = T_\theta = \mathrm{Id}_{L_p}$. □

4.3.2 The Radon-Nikodym cocycle

Let $\sigma : G \curvearrowright (X, \mu)$ be a nonsingular action.

Consider β the map in Definition 4.3.1 and denote by $1 \in L_p(X, \mu)$ the constant function equal to 1. We can define the following map:

$$
\begin{aligned}
D : G &\longrightarrow L_0(X, \mu, \mathbb{R}_+^*) \\
g &\longmapsto D(g) = \left[T_{\sigma_g}(1)\right]^p = \frac{\mathrm{d}\left[(\sigma_g)_*\mu\right]}{\mathrm{d}\mu}
\end{aligned}
$$

This map is called the *Radon-Nikodym cocycle.*

Let us show that indeed $D \in Z^1_\sigma(G, \mathbb{R}^*_+)$. First, let $g, h \in G$ and note that

$$\begin{aligned} D(gh) &= \left[T_{\sigma_{gh}}(1)\right]^p = \left[T_{\sigma_g \circ \sigma_h}(1)\right]^p = \left[\left(T_{\sigma_g} \circ T_{\sigma_h}\right)(1)\right]^p \\ &= \left\{\left(\frac{\mathrm{d}\left[(\sigma_g)_*\mu\right]}{\mathrm{d}\mu}\right)^{1/p}\left[\left(\frac{\mathrm{d}\left[(\sigma_h)_*\mu\right]}{\mathrm{d}\mu}\right)^{1/p} \circ \sigma_g^{-1}\right] 1 \circ \sigma_h^{-1} \circ \sigma_g^{-1}\right\}^p \\ &= \frac{\mathrm{d}\left[(\sigma_g)_*\mu\right]}{\mathrm{d}\mu}\left[\frac{\mathrm{d}\left[(\sigma_h)_*\mu\right]}{\mathrm{d}\mu} \circ \sigma_g^{-1}\right] = D(g).\, \sigma_g\left(D(h)\right) \end{aligned}$$

which is the cocycle relation for D.

Remark 10. If $X = [0, 1]$ and $\mu = \lambda$ is the Lebesgue measure, Note that if $D(g) = \sigma'_{g^{-1}}(t)$ is the derivative, the cocycle relation follows from the chain rule

$$\begin{aligned} D(gh) &= \sigma'_{(gh)^{-1}}(t) = \sigma'_{h^{-1}g^{-1}}(t) = \left(\sigma_{h^{-1}} \circ \sigma_{g^{-1}}\right)'(t) \\ &= \sigma'_{h^{-1}}\left(\sigma_{g^{-1}}(t)\right)\sigma'_{g^{-1}}(t) = \left(\sigma'_{h^{-1}} \circ \sigma_{g^{-1}}\right)(t) D(g) \\ &= \left(D(h) \circ \sigma_{g^{-1}}\right) D(g) = D(g)\, \sigma_g(D(h)) \end{aligned}$$

For the continuity of D, first let us first show the following lemma, considering D when (X, μ) is a probability space.

Lemma 4.3.7. *Let (X, ν) be a probability space. Then,*

$$\begin{aligned} D \,:\, G &\longrightarrow L_0(X, \nu, \mathbb{R}^*_+) \\ g &\longmapsto D(g) = \frac{d\left[(\sigma_g)_*\nu\right]}{d\nu} \end{aligned}$$

is continuous.

Proof. Let ν to be a probability measure on standard measurable space X and $\sigma : G \curvearrowright (X, \nu)$ a nonsingular action. Consider the map

$$\begin{aligned} \mathcal{D} \,:\, G &\longrightarrow L_1(X, \nu, \mathbb{R}) \\ g &\longmapsto \mathcal{D}(g) = \frac{\mathrm{d}\left[(\sigma_g)_*\nu\right]}{\mathrm{d}\nu} \end{aligned}$$

Let $g \in G$ and $(g_\alpha)_\alpha$ be a net in G such that $g_\alpha \longrightarrow g$. Since ν is a probability measure, $\sigma_{g_\alpha *}\nu$ and $\sigma_{g*}\nu$ also are probability measures, for all α. Recall that we are considering $\mathrm{Aut}(X, [\nu])$ with the topology of poinwise convergence on probability measures, discussed in section 4.2.2.

Note that $\sigma_{g_\alpha}, \sigma_g \in \mathrm{Aut}(X, [\nu])$, for every α. Then, $\sigma_{g_\alpha *}\nu \ll \nu$ and $\sigma_{g*}\nu \ll \nu$, for every α. Using Lemma 2.6.1, it follows that

$$\left\| \frac{\mathrm{d}(\sigma_{g_\alpha *}\nu)}{\mathrm{d}\nu} - \frac{\mathrm{d}(\sigma_{g*}\nu)}{\mathrm{d}\nu} \right\|_{L_1} = 2\left\|\sigma_{g_\alpha *}\nu - \sigma_{g*}\nu\right\|_1$$

Since σ is continuous, it follows that $\sigma_{g_\alpha} \longrightarrow \sigma_g$. Because ν is a probability measure, it follows that $\left\|\sigma_{g_\alpha *}\nu - \sigma_{g*}\nu\right\|_1 \longrightarrow 0$ and therefore, $\left\|\frac{\mathrm{d}(\sigma_{g_\alpha *}\nu)}{\mathrm{d}\nu} - \frac{\mathrm{d}(\sigma_{g*}\nu)}{\mathrm{d}\nu}\right\|_{L_1} \longrightarrow 0$.

This means that $\frac{\mathrm{d}(\sigma_{g_\alpha *}\nu)}{\mathrm{d}\nu} \longrightarrow \frac{\mathrm{d}(\sigma_{g*}\nu)}{\mathrm{d}\nu}$ in L_1 and then $\mathcal{D} : G \longrightarrow L_1(X, \nu, \mathbb{R})$ is continuous.

Now, denoting by $\widehat{L_1}(X, \nu, \mathbb{R})$ the subset of $L_1(X, \nu, \mathbb{R})$ consisting of the positive functions of $L_1(X, \nu, \mathbb{R})$. Consider the continuous map $\mathrm{I}_1 : \widehat{L_1}(X, \nu, \mathbb{R}) \to L_0(X, \nu, \mathbb{R}_+^*)$, as in corollary 2.5.1 and consider $D := \mathrm{I}_1 \circ \mathcal{D}$. Then, $D : G \to L_0(X, \nu, \mathbb{R}_+^*)$ is continuous. □

Proposition 4.3.8. *The Radon-Nikodym cocycle D is a continuous map.*

Proof. Let (X, μ) be a σ-finite standard measure space and $\sigma : G \curvearrowright (X, \mu)$ a nonsingular action. Let ν be the probability measure from theorem 2.1.1 such that $[\mu] = [\nu]$.

Note that $\mathrm{Aut}(X, [\mu]) = \mathrm{Aut}(X, [\nu])$.

Using proposition 2.3.3, we have that

$$D : G \longrightarrow L_0(X, \mu, \mathbb{R}_+^*) \tag{4.1}$$
$$g \longmapsto D(g) = \frac{\mathrm{d}\left(\sigma_{g*}\mu\right)}{\mathrm{d}\mu} = \frac{\mathrm{d}\left(\sigma_{g*}\mu\right)}{\mathrm{d}\left(\sigma_{g*}\nu\right)} \frac{\mathrm{d}\left(\sigma_{g*}\nu\right)}{\mathrm{d}\nu} \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \tag{4.2}$$

First, from lemma 4.3.7, the map $D : G \to L_0(X, \nu, \mathbb{R}_+^*)$, given by $g \mapsto \frac{\mathrm{d}\left(\sigma_{g*}\nu\right)}{\mathrm{d}\nu}$ is continuous. Using lemma 2.5.5, we have that the map

$$D : G \longrightarrow L_0(X, \mu, \mathbb{R}_+^*)$$
$$g \longmapsto \frac{\mathrm{d}\left(\sigma_{g*}\nu\right)}{\mathrm{d}\nu}$$

is continuous. Now, let us show that the map $g \to \frac{\mathrm{d}\left(\sigma_{g*}\mu\right)}{\mathrm{d}\left(\sigma_{g*}\nu\right)} \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is a coboundary of σ. Consider $f = \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \in L_0(X, \mu, \mathbb{R}_+^*)$ and note that, from proposition 2.3.4, we have that $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right)^{-1} = f^{-1}$ (μ-a.e.).

Also, from lemma 2.3.5, we have that $\frac{\mathrm{d}\left(\sigma_{g*}\mu\right)}{\mathrm{d}\left(\sigma_{g*}\nu\right)} = \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \circ \sigma_g^{-1} = \sigma_g(f)$, $(\sigma_{g*}\nu)$-a.e. and since $\sigma_g \in \mathrm{Aut}(X, [\mu])$ this equality holds μ-a.e. as well. Hence,

$$\frac{\mathrm{d}\left(\sigma_{g*}\mu\right)}{\mathrm{d}\left(\sigma_{g*}\nu\right)} \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \sigma_g(f)\, f^{-1}$$

Then, from proposition 4.2.6, the map $g \mapsto \frac{\mathrm{d}\left(\sigma_{g*}\mu\right)}{\mathrm{d}\left(\sigma_{g*}\nu\right)} \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is a coboundary of σ, hence continuous. This shows that the map 4.1 is indeed continuous, since it is a product of continuous maps. □

4.3.3 Topological aspects of the Banach-Lamperti theorem

First, let us define another notation.

Definition 4.3.3. Let N, H be groups and $\alpha : N \to N'$ and $\beta : H \to H'$ isomorphisms (bijective group homomorphisms) with $G' = H' \ltimes N'$. Define G to be the *semidirect product relative to α and β* of the groups N and H when $G = N \times H$ is a group with the product

$$(n_1, h_1) \times (n_2, h_2) = (n_3, h_h)$$

when we have that $\alpha(n_1)\beta(h_1)\alpha(n_2)\beta(h_2) = \alpha(n_3)\beta(h_3)$ in G'.

Notation: $G = H \ltimes N$

Proposition 4.3.9. *With the previous definition, the map*

$$\begin{aligned} f : G &\longrightarrow G' \\ (n,h) &\longmapsto f(n,h) = \alpha(n)\beta(h) \end{aligned}$$

is a group isomorphism.

Proof. Let $n_1, n_2 \in N$ and $h_1, h_2 \in H$.

$$f((n_1, h_1) \times (n_2, h_2)) = f(n_3, h_3) = \alpha(n_3)\beta(h_3) = \alpha(n_1)\beta(h_1)\alpha(n_2)\beta(h_2) = f(n_1, h_1)f(n_2, h_2)$$

Then, f is a group homomorphism. Now, let $n'h' \in H' \ltimes N' = G'$. Then, $n' = \alpha(n)$ and $h' = \beta(h)$, for $n \in N$ and $h \in H$. Therefore,

$$n'h' = \alpha(n)\beta(h) = f(n,h)$$

Also, if $(n_1, h_1), (n_2, h_2) \in G$ are such that $f(n_1, h_1) = f(n_2, h_2)$, we have that $\alpha(n_1)\beta(h_1) = \alpha(n_2)\beta(h_2)$. From proposition 1.2.1, it follows that $\alpha(n_1) = \alpha(n_2)$ and $\beta(h_1) = \beta(h_2)$, Then, $n_1 = n_2$ and $h_1 = h_2$, meaning that $(n_1, h_1) = (n_2, h_2)$.

Therefore, f is bijective. □

With this, we can define the following maps

Definition 4.3.4. Let $p > 0$, (X, μ) be a σ-finite measure space and define

$$\begin{aligned} \pi^{p,\mu} : \operatorname{Aut}(X, [\mu]) \ltimes L_0(X, \mu, \mathbb{T}) &\longrightarrow \mathcal{O}\left(L_p(X, \mu)\right) \\ (h, \theta) &\longmapsto \pi^{p,\mu}_{(h,\theta)} := \eta_h \circ T_\theta \end{aligned}$$

Definition 4.3.5. Let $\sigma : G \curvearrowright (X, \mu)$ be a nonsingular action and D be the Radon-Nikodym cocycle. Define

$$\begin{aligned} \sigma^{p,\mu} : G &\longrightarrow \mathcal{O}\left(L_p(X, \mu)\right) \\ g &\longmapsto \sigma^{p,\mu}_g := \pi^{p,\mu}_{(1,\sigma_g)} = \eta_1 \circ T_{\sigma_g} \end{aligned}$$

That is, for $g \in G$ and $f \in L_p(X, \mu)$, we have that

$$\sigma_g^{p,\mu}(f) = \left(\eta_1 \circ T_{\sigma_g}\right)(f) = T_{\sigma_g}(f) = D(g)^{1/p}\, \sigma_g(f)$$

Remark 11. Note that from the way they are defined, the maps $\pi^{p,\mu} : \mathrm{Aut}(X, [\mu]) \ltimes L_0(X, \mu, \mathbb{T}) \curvearrowright L_p(X, \mu)$ and $\sigma^{p,\mu} : G \curvearrowright L_p(X, \mu)$ are isometric representations.

With these means, we are going to show a version of the Banach-Lamperti theorem 3.1.6 adapted to our context; with a σ-finite standard measure space and isometries considered surjective.

Consider $p \neq 2$ and $S : L_p(X, \mu) \to L_p(X, \mu)$ a linear isometric embedding. From the theorem of Banach-Lamperti 3.1.6, it follows that S is of the form

$$(Sf)(x) = h(x)(Uf)(x), \tag{4.3}$$

where U is the transformation induced by a regular set isomorphism T (see remark 4) and h is a function defined on X such that

$$\int_{T(A)} |h|^p \, \mathrm{d}\mu = \int_{T(A)} \frac{\mathrm{d}(\mu \circ T^{-1})}{\mathrm{d}\mu} \, \mathrm{d}\mu = \mu(A) \qquad \forall A \in \Sigma$$

As discussed in chapter 3 of [7], it is natural to want to express 4.3 in the form

$$(Sf)(x) = h(x) f(\varphi(x)), \tag{4.4}$$

where $\varphi : X \to X$ is a point mapping.

Observe that this is the form of the isometries on $L_p([0,1], \lambda)$ expressed in Banach's theorem 3.1.5. In theorem 15.21 of [25] one can find a proof that the set mapping T can be given by a point mapping when X is a complete separable metric space. In fact, the author argues that this is true even when X is topologically complete (has an equivalent metric that makes it complete). For this reason, if (X, μ) is a σ-finite standard measure space, then S is given by 4.4, for $\varphi : X \to X$ a measurable map that preserves the measure class $[\mu]$ and h a function defined on X such that

$$\int_{\varphi^{-1}(A)} |h|^p \, \mathrm{d}\mu = \int_{\varphi^{-1}(A)} \frac{\mathrm{d}(\mu \circ \varphi)}{\mathrm{d}\mu} \, \mathrm{d}\mu \qquad \forall A \in \Sigma \tag{4.5}$$

Now, consider our linear isometric embedding S to be surjective, meaning that $S \in \mathcal{O}(L_p(X, \mu))$. In this case, $S^{-1} \in \mathcal{O}(L_p(X, \mu))$ and then there are $\varphi, \varphi' : X \to X$ measurable maps that preserve the measure class $[\mu]$ and h, h' functions on X, each satisfying the relation 4.5, such that

$$(Sf)(x) = h(x) f(\varphi(x))$$
$$(S^{-1}f)(x) = h'(x) f(\varphi'(x))$$

Therefore,

$$f(x) = ((S \circ S^{-1})f)(x) = (S(S^{-1}f))(x) = h(x)(S^{-1}f)(\varphi(x)) = h(x)h'(\varphi(x)) f(\varphi'(\varphi(x)))$$

Evaluating at the constant function $f \equiv 1 \in L_p(X, \mu)$, we have that

$$h(x)h'(\varphi(x)) \equiv 1$$

Meaning that, for every $f \in L_p(X, \mu)$ and every $x \in X$, $f(x) = f(\varphi'(\varphi(x)))$. From lemma 4.2.1, it follows that $\varphi' \circ \varphi = Id$. Similarly, using $S^{-1} \circ S = Id$, we have that $\varphi \circ \varphi' = Id$, meaning that $\varphi' = \varphi^{-1}$.

Thus, for $S \in \mathcal{O}(L_p(X, \mu))$, the map $\varphi : X \to X$ is a measurable bijection, with φ^{-1} also measurable (an automorphism on (X, μ)). Also, φ preserves the measure class $[\mu]$ and then is nonsingular: $\varphi \in \mathrm{Aut}(X, [\mu])$.

Since φ^{-1} is measurable, $\varphi(A) \in \Sigma$, for every $A \in \Sigma$. Then, 4.5 becomes

$$\int_A |h|^p \, \mathrm{d}\mu = \int_A \frac{\mathrm{d}(\mu \circ \varphi)}{\mathrm{d}\mu} \, \mathrm{d}\mu \qquad \forall A \in \Sigma$$

Therefore, $|h|^p = \frac{\mathrm{d}(\mu \circ \varphi)}{\mathrm{d}\mu}$ μ-a.e. on X. Because of lemma 4.3.1, we can write

$$h = \frac{h}{|h|}|h|$$

Since $\left|\frac{h}{|h|}\right| = 1$, we have a function $\bar{h} := \frac{h}{|h|} \in L_0(X, \mu, \mathbb{T})$ and then we conclude that any isometry $S \in \mathcal{O}(L_p(X, \mu))$ is of the form

$$(Sf)(x) = \bar{h}(x) \left(\frac{\mathrm{d}(\mu \circ \varphi)}{\mathrm{d}\mu}(x) \right)^{1/p} f(\varphi(x)),$$

for $\bar{h} \in L_0(X, \mu, \mathbb{T})$ and $\varphi \in \mathrm{Aut}(X, [\mu])$. This discussion gives rise to the following important result.

Theorem 4.3.10. *For $p \neq 2$, the map $\pi^{p,\mu}$ from definition 4.3.4 is surjective.*

Proof. Since any isometry $S \in \mathcal{O}(L_p(X, \mu))$ is of the form

$$(Sf)(x) = \bar{h}(x) \left(\frac{\mathrm{d}(\mu \circ \varphi)}{\mathrm{d}\mu}(x) \right)^{1/p} f(\varphi(x)),$$

for $\bar{h} \in L_0(X, \mu, \mathbb{T})$ and $\varphi \in \mathrm{Aut}(X, [\mu])$, we can choose $\theta := \varphi^{-1} \in \mathrm{Aut}(X, [\mu])$ and then

$$Sf = \bar{h} \left(\frac{\mathrm{d}(\theta_* \mu)}{\mathrm{d}\mu} \right)^{1/p} (f \circ \theta^{-1}) = (\eta_{\bar{h}} \circ T_\theta)(f)$$

Therefore, $S = \pi^{p,\mu}_{(\bar{h},\theta)}$. ☐

Corollary 4.3.1. *For $p \neq 2$, every isometric representation $\pi : G \curvearrowright L_p(X, \mu)$ is of the form*

$$\pi_g = \omega(g)\sigma^{p,\mu}_g \qquad \forall g \in G,$$

for some cocycle $\omega \in Z^1_\sigma(G,\mathbb{T})$ and $\sigma : G \curvearrowright (X,\mu)$ nonsingular action.

Proof. If $\pi : G \curvearrowright L_p(X,\mu)$ is an isometric representation with $p \neq 2$, we have that $\pi_g \in \mathcal{O}(L_p(X,\mu))$, for every $g \in G$.

From theorem 4.3.10, it follows that for every $g \in G$, $\pi_g = \pi^{p,\mu}_{(h_g,\theta_g)} = \eta_{h_g} \circ T_{\theta_g}$, where $h_g \in L_0(X,\mu,\mathbb{T})$ and $\theta_g \in \mathrm{Aut}(X,[\mu])$. Define:

$$\begin{aligned} \omega : G &\longrightarrow L_0(X,\mu,\mathbb{T}) \\ g &\longmapsto \omega(g) := h_g \end{aligned}$$

$$\begin{aligned} \sigma : G &\longrightarrow \mathrm{Aut}(X,[\mu]) \\ g &\longmapsto \sigma(g) := \theta_g \end{aligned}$$

Let $g, \bar{g} \in G$. Let us prove that

$$\eta_{h_g} \circ T_{\theta_g} \circ \eta_{h_{g'}} \circ T_{\theta_{g'}} = \eta_{h_g} \circ \eta_{\theta_g(h_{g'})} \circ T_{\theta_g} \circ T_{\theta_{g'}}$$

To do this, let $f \in L_p(X,\mu)$

$$\begin{aligned} \eta_{h_g}\left(T_{\theta_g}\left(\eta_{h_{g'}}\left(T_{\theta_{g'}}(f)\right)\right)\right) &= h_g.T_{\theta_g}\left(h_{g'}.T_{\theta_{g'}}(f)\right) = h_g.\left(\frac{\mathrm{d}(\theta_{g*}\mu)}{\mathrm{d}\mu}\right)^{1/p}.\left(h_{g'}.T_{\theta_{g'}}(f)\right)\circ\theta_g^{-1} \\ &= h_g.\left(\frac{\mathrm{d}(\theta_{g*}\mu)}{\mathrm{d}\mu}\right)^{1/p}.\left(h_{g'}\circ\theta_g^{-1}\right).\left(T_{\theta_{g'}}(f)\circ\theta_g^{-1}\right) \\ &= h_g.\theta_g\left(h_{g'}\right).\left(\frac{\mathrm{d}(\theta_{g*}\mu)}{\mathrm{d}\mu}\right)^{1/p}.\theta_g\left(T_{\theta_{g'}}(f)\right) \\ &= h_g.\theta_g\left(h_{g'}\right).T_{\theta_g}\left(T_{\theta_{g'}}(f)\right) = \left(\eta_{h_g}\circ\eta_{\theta_g(h_{g'})}\circ T_{\theta_g}\circ T_{\theta_{g'}}\right)(f) \end{aligned}$$

Now, using that π, η and β are homomorphisms,

$$\begin{aligned} \eta_{hgg'} \circ T_{\theta_{gg'}} &= \pi_{gg'} = \pi_g \circ \pi_{g'} = \eta_{h_g} \circ T_{\theta_g} \circ \eta_{h_{g'}} \circ T_{\theta_{g'}} = \eta_{h_g} \circ \eta_{\theta_g(h_{g'})} \circ T_{\theta_g} \circ T_{\theta_{g'}} \\ &= \eta_{h_g.\theta_g(h_{g'})} \circ T_{\theta_g \circ \theta_{g'}} \end{aligned}$$

In proposition 4.3.6, we proved that $H_\beta \ltimes N_\eta$. Now, using proposition 1.2.1, it follows that $\eta_{hgg'} = \eta_{h_g.\theta_g(h_{g'})}$ and $T_{\theta_{gg'}} = T_{\theta_g \circ \theta_{g'}}$. Using that η and β are injective (Propositions 4.3.5 and 4.3.4), it follows that σ is a homomorphism and that ω satisfies the cocycle relation. $\square$

4.4 Skew-product actions

Definition 4.4.1. Let $\sigma : G \curvearrowright (X,\mu)$ be a nonsingular action and $(A,*)$ be a locally compact abelian group with m its Haar measure. For every $c \in Z^1_\sigma(G,A)$ we can define the following map

$$\begin{aligned}\sigma \rtimes c : G &\longrightarrow \mathrm{Aut}(X \times A, [\mu \otimes m]) \\ g &\longmapsto (\sigma \rtimes c)_g : X \times A \longrightarrow X \times A \\ &\qquad\qquad (x, a) \longmapsto (\sigma_g(x), a * c(g^{-1})(x))\end{aligned}$$

this map is what we call the *skew-product action of σ by c.*

Notation: $\sigma \rtimes c : G \curvearrowright (X \times A, \mu \otimes m)$.

Proposition 4.4.1. *The skew-product action $\sigma \rtimes c : G \curvearrowright (X \times A, \mu \otimes m)$ is a nonsingular action.*

Proof. See [1]. □

Proposition 4.4.2. *The following map is a coboundary*

$$\begin{aligned}c' : G &\longrightarrow L_0(X \times A, \mu \otimes m, A) \\ g &\longmapsto c'(g) : X \times A \longrightarrow A \\ &\qquad (x, a) \longmapsto c'(g)(x, a) = c(g)(x)\end{aligned}$$

Proof. Consider the projection map

$$\begin{aligned}p : X \times A &\longrightarrow A \\ (x, a) &\longmapsto p(x, a) = a\end{aligned}$$

and note that, for $(x, a) \in X \times A$,

$$\begin{aligned}\left[(\sigma \rtimes c)_g(p) * p^{-1}\right](x, a) &= \left[p \circ (\sigma \rtimes c)_g^{-1} * p^{-1}\right](x, a) \\ &= p(\sigma_{g^{-1}}(x), a * c(g)(x)) * p^{-1}(x, a) \\ &= a * c(g)(x) * a^{-1} = c(g)(x) = c'(g)(x, a)\end{aligned}$$

therefore, c' is a coboundary. □

Lemma 4.4.3. *Let $\sigma \rtimes c : G \curvearrowright (X \times A, \mu \otimes m)$ be a skew-product action. Then,*

$$(\sigma \rtimes c)_{g*}(\mu \otimes m) = (\sigma_{g*}\mu) \otimes m,$$

for every $g \in G$.

Proof. Fix $g \in G$ and let $f : X \times A \to \mathbb{R}^+$ be a measurable function. Denote $\sigma \rtimes c$ by $\tilde{\sigma}$.

Using theorems 2.2.1 and 2.4.1, we have that

$$\begin{aligned}I &= \int f \,\mathrm{d}(\tilde{\sigma}_{g*}(\mu \otimes m)) = \int f \circ \tilde{\sigma}_g \,\mathrm{d}(\mu \otimes m) = \int \int (f \circ \tilde{\sigma}_g)(x, a) \,\mathrm{d}m(a) \,\mathrm{d}\mu(x) \\ &= \int \int f(\sigma_g(x), a * c(g^{-1})(x)) \,\mathrm{d}m(a) \,\mathrm{d}\mu(x)\end{aligned}$$

Now, note that for each $x \in X$,

$$\int f(\sigma_g(x), a * c(g^{-1})(x))\,\mathrm{d}m(a) = \int f(\sigma_g(x), a)\,\mathrm{d}m(a)$$

where we changed the variable $a * c(g^{-1})(x) \to a$ and used that the Haar measure m is translation invariant with respect to the fixed element $c(g^{-1})(x) \in A$. Then,

$$I = \int\int f(\sigma_g(x), a)\,\mathrm{d}m(a)\,\mathrm{d}\mu(x) = \int\int f(\sigma_g(x), a)\,\mathrm{d}\mu(x)\,\mathrm{d}m(a)$$

For each $a \in A$, $\int f(\sigma_g(x), a)\,\mathrm{d}\mu(x) = \int f(x, a)\,\mathrm{d}(\sigma_{g*}\mu)(x)$. Then,

$$I = \int\int f(x, a)\,\mathrm{d}(\sigma_{g*}\mu)(x)\,\mathrm{d}m(a) = \int f\,\mathrm{d}((\sigma_{g*}\mu) \otimes m)$$

Therefore, for every measurable function $f : X \times A \to \mathbb{R}^+$, we have that

$$I = \int f\,\mathrm{d}(\tilde{\sigma}_{g*}(\mu \otimes m)) = \int f\,\mathrm{d}((\sigma_{g*}\mu) \otimes m)$$

For each B measurable set in $X \times A$, choose the function f to be χ_B, the indicator function of B. Then, we have that $(\tilde{\sigma}_{g*}(\mu \otimes m))(B) = ((\sigma_{g*}\mu) \otimes m)(B)$, for every measurable set B of $X \times A$. Therefore,

$$\tilde{\sigma}_{g*}(\mu \otimes m) = (\sigma_{g*}\mu) \otimes m$$

□

4.5 The Maharam extension

We can define a measure preserving action in a similar way as done for the skew-product action. This action is called the Maharam extension, named after D. Maharam.

Definition 4.5.1. Consider the abelian group $(\mathbb{R}^*_+, .)$ with the Lebesgue measure λ. Let $\sigma : G \curvearrowright (X, \mu)$ be a nonsingular action and $D \in Z^1_\sigma(G, \mathbb{R}^*_+)$ be the Radon-Nikodym cocycle. The following map is called the *Maharam extension of* σ

$$\begin{aligned} \tilde{\sigma} : G &\longrightarrow \mathrm{Aut}(X \times \mathbb{R}^*_+, [\mu \otimes \lambda]) \\ g &\longmapsto \tilde{\sigma}_g : X \times \mathbb{R}^*_+ \longrightarrow X \times \mathbb{R}^*_+ \\ &\qquad\qquad (x, y) \longmapsto (\sigma_g(x), y\,.\,\left[D(g^{-1})(x)\right]^{-1}) \end{aligned}$$

Proposition 4.5.1. *$\tilde{\sigma}$ is a measure preserving action.*

Proof. See [26], Theorem 1 of Section 3.2. □

4.6 The Gaussian action

For this section, have in mind the definitions and results from section 2.7. The *Gaussian process* is a type of random process (a collection of random variables). According to [27], section 2.3, we have the following.

For H a Hilbert space, there exists a standard probability space (Ω, μ) and a collection of random variables $(\widehat{v})_{v \in H}$ on Ω (called the Gaussian process) such that

1. $\widehat{v}$ is a centered Gaussian random variable of variance $\|v\|^2$, for every $v \in H$;
2. The map
$$\begin{aligned} \wedge : H &\longrightarrow L_2(\Omega, \mu, \mathbb{R}) \\ v &\longmapsto \widehat{v} \end{aligned}$$
is linear;
3. The random variables $(\widehat{v})_{v \in H}$ generate the σ-algebra of measurable subsets of (Ω, ν).

Also, we have that for every $T \in \mathcal{O}(H)$, there exists a measure preserving automorphism $\widehat{T}$ of (Ω, μ) such that, for every $v \in H$,

$$\widehat{T(v)} \circ \widehat{T} = \widehat{v} \tag{4.6}$$

Because of this, we have that given G a topological group, H a Hilbert space and $\pi : G \curvearrowright H$ an isometric representation, we can construct a probability measure preserving action on a standard probability space (Ω, ν) . This action is called the *Gaussian action associated with* π, denoted by $\sigma^\pi : G \curvearrowright (\Omega, \mu)$. Also, equation 4.6 gives us

$$\widehat{v} \circ (\sigma^\pi_g)^{-1} = \widehat{\pi_g(v)} \qquad \text{for every } g \in G \text{ and } v \in H.$$

Chapter 5

Characterizing the set $K^2(G)$

In this Chapter we are going to develop some of the tools needed to prove the case $p = 2$ of our main theorem. The objective is to provide an abstract characterization of the set $K^2(G)$ using what we call functions conditionally of negative type. We start by defining Bernstein functions, then we define kernels conditionally of negative type and kernels of positive type and enunciate the Schoenberg's theorem, that relates both of them.

Next, we demonstrate an important theorem called the GNS construction, which shows that for every kernel conditionally of negative type, there exists a Hilbert space and a map with specific properties that allow us to finally show that a function belongs to $K^2(G)$ if, and only if, it is conditionally of negative type. This is the characterization that we need.

5.1 Bernstein functions

Definition 5.1.1. A function $f : (0, \infty) \longrightarrow \mathbb{R}$ is called a *Bernstein function* when

1. f is of class C^∞;
2. $f(x) \geq 0$, for all $x \in (0, \infty)$;
3. $(-1)^{n-1} f^{(n)}(x) \geq 0$, for all $n \in \mathbb{N}$ and $x \in (0, \infty)$.

As an example, we have that the function $x \longmapsto x^\alpha$, for $0 < \alpha \leq 1$, is a Bernstein function.

Theorem 5.1.1. *(Lévy-Khintchine representation) A function $f : (0, \infty) \longrightarrow \mathbb{R}$ is a* Bernstein function if, and only if, it admits the representation

$$f(x) = a + bx + \int_{(0,\infty)} (1 - e^{-xt}) \,\mathrm{d}\mu(t),$$

where $a, b \geq 0$ and μ is a Radon measure on $(0, \infty)$ such that

$$\int_{(0,\infty)} \frac{t}{1+t} \,\mathrm{d}\mu(t) < \infty$$

Proof. See theorem 3.2 of [28]. □

5.2 Kernels CNT and PT

First we start by defining kernels conditionally of negative type and kernels of positive type. Not to be confused with the classical use of the word kernel in algebra, in this context, kernel is a type of function. This usage of the word kernel is usual in the context of integral operator equations.

Definition 5.2.1. Let X be a topological space. A kernel *conditionally of negative type* on X is a continuous function $\Psi : X \times X \to \mathbb{R}$ such that:

1. $\Psi(x, x) = 0$, for all $x \in X$;
2. $\Psi(y, x) = \Psi(x, y)$, for all $x, y \in X$;
3. Given $n \in \mathbb{N}$,for any $x_1, \cdots, x_n \in X$ and $c_1, \cdots, c_n \in \mathbb{R}$ with $\sum_i c_i = 0$, we have that $\sum_i \sum_j c_i c_j \Psi(x_i, x_j) \leq 0$.

Definition 5.2.2. A kernel of *positive type* on a topological space X is a continuous function $\Phi : X \times X \to \mathbb{C}$ such that, given $n \in \mathbb{N}$, for any $x_1, \cdots, x_n \in X$ and $c_1, \cdots, c_n \in \mathbb{C}$, we have that $\sum_i \sum_j c_i \overline{c_j} \Phi(x_i, x_j) \geq 0$.

5.2.1 Schoenberg's theorem

Schoenberg's theorem relates the concept of kernels of positive type with kernels conditionally of negative type.

Theorem 5.2.1. *(Schoenberg) Let X be a topological space and $\Psi : X \times X \longrightarrow \mathbb{R}$ be a continuous function on X such that $\Psi(x, x) = 0$ and $\Psi(x, y) = \Psi(y, x)$, for all $x, y \in X$. Then, the following are equivalent*

1. *Ψ is a kernel conditionally of negative type;*
2. *$e^{-t\Psi}$ is a kernel of positive type, for all $t \geq 0$.*

Proof. See [2], theorem C.3.2. □

5.3 The GNS construction

According to [2], the GNS construction is named after Israel Gelfand, Mark Naimark, and Irving Segal and it shows how a kernel conditionally of negative type on a topological space X is related to specific mapping from X to a Hilbert space.

Theorem 5.3.1. *(GNS construction) If Ψ is a kernel conditionally of negative type on a topological space X and if $x_0 \in X$, then there exists a real Hilbert space $\mathcal{H}$ and a continuous mapping $f : X \to \mathcal{H}$ with the following properties:*

1. *$\Psi(x, y) = \|f(x) - f(y)\|^2, \forall x, y \in X$*

2. *$\{f(x)-f(x_0) : x \in X\} \subseteq \mathcal{H}$ is a total set, meaning that the linear span of $\{f(x)-f(x_0) : x \in X\}$ is dense in $\mathcal{H}$.*

Also, $\mathcal{H}$ and f are unique up to canonical isomorphism, that means, if $\mathcal{H}'$ and f' also satisfy the above two conditions, then there exists a unique affine isometry $T : \mathcal{H} \to \mathcal{H}'$ such that $f' = T \circ f$.

Proof. Fix Ψ is a kernel conditionally of negative type on X and $x_0 \in X$ and let us construct the Hilbert space and the function desired.

Considering $\mathcal{B}$ the Borel σ-algebra on X, let δ_x denote the Dirac measure at the point $x \in X$: For $A \in \mathcal{B}$,

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Note that we can define $\delta_x : X \to X$ by $\delta_x(y) = \delta_x(\{y\})$, for every $y \in X$.

Consider the real vector space

$$V = \{\sum_{i=1}^{n} c_i \delta_{x_i} \ : \ n \in \mathbb{N},\ x_i \in X,\ c_i \in \mathbb{R} \text{ with } \sum_{i=1}^{n} c_i = 0\}$$

Define, for $v = \sum_{i=1}^{n} c_i \delta_{x_i}$ and $w = \sum_{j=1}^{m} a_j \delta_{x'_j}$,

$$\langle v, w \rangle = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} c_i a_j \Psi(x_i, x'_j)$$

This is not quite an inner product for V, but it can be shown that it is a positive semi-definite symmetric form: it fails only the positive-definiteness condition for inner products (however, $\langle v, v \rangle \geq 0$, for all $v \in V$). The good thing is that we can still use Cauchy-Schwarz inequality on positive semi-definite symmetric forms. Define

$$N = \{v \in V \ : \ \langle v, v \rangle = 0\}$$

and note that, for $\lambda_1, \lambda_2 \in \mathbb{R}$ and $v_1, v_2 \in N$,

$$\langle \lambda_1 v_1 + \lambda_2 v_2, \lambda_1 v_1 + \lambda_2 v_2 \rangle = 2\lambda_1 \lambda_2 \langle v_1, v_2 \rangle$$

and $|\langle v_1, v_2 \rangle|^2 \leq \langle v_1, v_1 \rangle \langle v_2, v_2 \rangle = 0$, which shows us that $\langle v_1, v_2 \rangle = 0$. Therefore, $\lambda_1 v_1 + \lambda_2 v_2 \in N$, which proves that $N \subseteq V$ is a subspace. Now, considering the quotient space V/N we can define, for $[v], [w] \in V/N$,

$$\langle [v], [w] \rangle := \langle v, w \rangle.$$

It can be shown that this is an inner product for V/N. Now, we consider the Hilbert space $\mathcal{H}$ given by the completion of the inner product space V/N. We abuse the notation by omitting the isometry between V/N and $\mathcal{H}$ and just considering V/N as a dense subspace of

$\mathcal{H}$. Consider the map

$$\begin{aligned} f : X &\longrightarrow \mathcal{H} \\ x &\longmapsto f(x) = \left[\delta_x - \delta_{x_0}\right] \end{aligned}$$

Note that, for $x, y \in X$,

$$\begin{aligned} \|f(x) - f(y)\|^2 &= \langle f(x) - f(y), f(x) - f(y)\rangle = \langle \left[\delta_x - \delta_y\right], \left[\delta_x - \delta_y\right]\rangle \\ &= -\frac{1}{2}\left(\Psi(x,x) + (1)(-1)\Psi(x,y) + (-1)(1)\Psi(y,x) + \Psi(y,y)\right) \\ &= \Psi(x,y) \end{aligned}$$

then, f satisfies property 1. Also, the fact that Ψ is continuous tells us that $\lim_{x\to y}\|f(x) - f(y)\|^2 = \Psi(y,y) = 0$. Hence, f is continuous. Now for property 2., let D denote the linear span of $\{f(x) - f(x_0) : x \in X\}$. Take $[v] \in {}^{V}\!/\!{}_{N}$.

$$\begin{aligned} [v] = \left[\sum_{i=1}^{n} c_i\delta_{x_i}\right] &= \left[\sum_{i=1}^{n} c_i(\delta_{x_i} - \delta_{x_0})\right] = \sum_{i=1}^{n} c_i\left[\delta_{x_i} - \delta_{x_0}\right] \\ &= \sum_{i=1}^{n} c_i f(x_i) = \sum_{i=1}^{n} c_i\left(f(x_i) - f(x_0)\right) \in D, \end{aligned}$$

since $\sum_{i=1}^{n} c_i = 0$. Therefore, $D = {}^{V}\!/\!{}_{N}$. From the definition of completion, ${}^{V}\!/\!{}_{N}$ is dense in $\mathcal{H}$, hence D is dense in $\mathcal{H}$.

Note that, fixing any $x_0' \in X$ and defining D' as the linear span of $\{f(x) - f(x_0') : x \in X\}$, if $[v] \in {}^{V}\!/\!{}_{N}$, again we have that

$$[v] = \sum_{i=1}^{n} c_i f(x_i) = \sum_{i=1}^{n} c_i\left(f(x_i) - f(x_0')\right) \in D'$$

and therefore, D' is also dense in $\mathcal{H}$. This observation is important because we can say that the linear span of $\{f(x) - f(x_0') : x \in X\}$ is dense in $\mathcal{H}$, for any $x_0' \in X$, not only x_0 that was used to define f.

Now, for the uniqueness, let $\mathcal{H}'$ and f' also satisfy both properties and define the linear map

$$\begin{aligned} T' : {}^{V}\!/\!{}_{N} &\longrightarrow \mathcal{H}' \\ \left[\sum_{i=1}^{n} c_i\delta_{x_i}\right] &\longmapsto \sum_{i=1}^{n} c_i f'(x_i) \end{aligned}$$

T' is well defined, but before we show that, one needs to see that for any $[v] \in {}^{V}\!/\!{}_{N}$,

$\|[v]\| = \|T'([v])\|$:

$$\begin{aligned}
\|[v]\|^2 &= \left\| \left[\sum_{i=1}^{n} c_i \delta_{x_i} \right] \right\|^2 = \left\| \left[\sum_{i=1}^{n} c_i (\delta_{x_i} - \delta_{x_0}) \right] \right\|^2 \\
&= \langle \sum_{i=1}^{n} c_i (\delta_{x_i} - \delta_{x_0}), \sum_{j=1}^{n} c_j (\delta_{x_j} - \delta_{x_0}) \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \langle \delta_{x_i} - \delta_{x_0}, \delta_{x_j} - \delta_{x_0} \rangle \\
&= -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \left(\Psi(x_i, x_j) - \Psi(x_i, x_0) - \Psi(x_0, x_j) \right) \\
&= -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \left(\left\| f'(x_i) - f'(x_j) \right\|^2 - \left\| f'(x_i) - f'(x_0) \right\|^2 - \left\| f'(x_0) - f'(x_j) \right\|^2 \right)
\end{aligned}$$

Note that

$$\begin{aligned}
\left\| f'(x_i) - f'(x_j) \right\|^2 &= \langle f'(x_i) - f'(x_j), f'(x_i) - f'(x_j) \rangle \\
&= \langle f'(x_i), f'(x_i) \rangle - \langle f'(x_j), f'(x_i) \rangle - \langle f'(x_i), f'(x_j) \rangle + \langle f'(x_j), f'(x_j) \rangle.
\end{aligned}$$

In the same way,

$$\begin{aligned}
\left\| f'(x_i) - f'(x_0) \right\|^2 &= \langle f'(x_i), f'(x_0) \rangle - \langle f'(x_0), f'(x_i) \rangle - \langle f'(x_i), f'(x_0) \rangle + \langle f'(x_0), f'(x_0) \rangle \\
\left\| f'(x_0) - f'(x_j) \right\|^2 &= \langle f'(x_0), f'(x_0) \rangle - \langle f'(x_j), f'(x_0) \rangle - \langle f'(x_0), f'(x_j) \rangle + \langle f'(x_j), f'(x_j) \rangle
\end{aligned}$$

Hence,

$$\begin{aligned}
\|[v]\|^2 &= -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j (-2\langle f'(x_i), f'(x_j) \rangle + 2\langle f'(x_i), f'(x_0) \rangle - 2\langle f'(x_0), f'(x_0) \rangle \\
&\qquad + 2\langle f'(x_j), f'(x_0) \rangle) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \langle f'(x_i) - f'(x_0), f'(x_j) - f'(x_0) \rangle \\
&= \left\langle \sum_{i=1}^{n} c_i (f'(x_i) - f'(x_0)), \sum_{j=1}^{n} c_j (f'(x_j) - f'(x_0)) \right\rangle \\
&= \left\| \sum_{i=1}^{n} c_i (f'(x_i) - f'(x_0)) \right\|^2 = \left\| \sum_{i=1}^{n} c_i f'(x_i) - f'(x_0) \sum_{i=1}^{n} c_i \right\|^2 \\
&= \left\| \sum_{i=1}^{n} c_i f'(x_i) \right\|^2 = \left\| T'([v]) \right\|^2
\end{aligned}$$

Since T' is linear, this shows that T' is an isometric embedding.

This equality also shows that if $v \in N$,

$$\left\| T'([v]) \right\|^2 = \|[v]\|^2 = \|[0]\|^2 = 0,$$

then $T'([v]) = 0$. Therefore, if we have $[v] = [w]$, then $v - w \in N$ and $0 = T'([v - w]) = T'([v]) - T'([w])$. This shows that T' is indeed well defined.

Now, if D' denotes the linear span of $\{f'(x) - f'(x_0) : x \in X\}$, we are going to show that the range $T'(V/N)$ is equal to D' . First, if $\sum_{i=1}^n c_i f'(x_i) \in T'(V/N)$, then $\sum_{i=1}^n c_i = 0$ and therefore

$$\sum_{i=1}^{n} c_i f'(x_i) = \sum_{i=1}^{n} c_i (f'(x_i) - f'(x_0)) \in D'.$$

Now, if $\sum_{i=1}^n \gamma_i(f'(x_i) - f'(x_0)) \in D'$, define new coefficients by $c_i := \gamma_i$, for $i = 1, \cdots, n$ and $c_{n+1} := -\sum_{i=1}^n \gamma_i$. Note that $\sum_{i=1}^{n+1} c_i = 0$ and if we denote $x_{n+1} := c_0$, we have that

$$\begin{aligned}\sum_{i=1}^{n} \gamma_i(f'(x_i) - f'(x_0)) &= \sum_{i=1}^{n} \gamma_i f'(x_i) - \sum_{i=1}^{n} \gamma_i f'(x_0) = \sum_{i=1}^{n} c_i f'(x_i) + c_{n+1} f'(x_0) \\ &= \sum_{i=1}^{n+1} c_i f'(x_i) \in T'(V/N).\end{aligned}$$

Hence, $T'(V/N) = D'$ and since D' is dense in $\mathcal{H}'$, we have that the range $T'(V/N)$ is dense in $\mathcal{H}'$. Now, if we consider $T' : V/N \to T'(V/N)$, we have an isometry between dense subspaces of complete spaces $\mathcal{H}$ and $\mathcal{H}'$. By Theorem 3.3.1, there is a unique isometry $T : \mathcal{H} \to \mathcal{H}'$ that extends T'.

Using sequences, one can check that T is also linear. Now, for $x \in X$,

$$(T \circ f)(x) = T\left(\left[\delta_x - \delta_{x_0}\right]\right) = T'\left(\left[\delta_x - \delta_{x_0}\right]\right) = f'(x) - f'(x_0)$$

therefore, we can define an affine function A by using T as its linear part and adding a constant $f'(x_0)$:

$$\begin{aligned} A : \mathcal{H} &\longrightarrow \mathcal{H}' \\ x &\longmapsto A(x) = T(x) + f'(x_0) \end{aligned}$$

and this is the affine isometric map such that $A \circ f = f'$. The only thing left to check is that A is unique. Suppose there is another affine isometry $B : \mathcal{H} \to \mathcal{H}'$ such that $B \circ f = f'$. Then, for each $x \in X$,

$$A(f(x)) = B(f(x)), \tag{5.1}$$

replacing $x = x_0$, we have that $A(f(x_0)) = B(f(x_0))$ and since $f(x_0) = [0]$, we know that the constant part of both affine maps is equal, that is, $A([0]) = B([0]) =: c$. Therefore, for every $h \in \mathcal{H}$,

$$\begin{aligned} A(h) &= T(h) + c \\ B(h) &= L(h) + c \end{aligned}$$

Then from 5.1, $T(f(x)) = L(f(x))$, for all $x \in X$. Since T and L are linear, they will also coincide in D, which is dense in $\mathcal{H}$. Therefore, $T = L$ in $\mathcal{H}$ and this concludes the uniqueness part. □

5.4 Functions conditionally of negative type

We are interested in looking into maps of the form $\psi(g) = ||\alpha_g(0)||^2$ for some continuous affine isometric action $\alpha : G \curvearrowright L_2(X, \mu)$. We shall prove that there is an abstract characterization of such maps, using kernels conditionally of negative type.

Definition 5.4.1. Given G a topological group, let $K^2(G)$ be the set of continuous functions ψ of the form

$$\begin{aligned} \psi : G &\longrightarrow \mathbb{R}^+ \\ g &\longmapsto \psi(g) = \left\|\alpha_g(0)\right\|^2, \end{aligned}$$

for some affine isometric action $\alpha : G \curvearrowright L_2(X, \mu)$.

Definition 5.4.2. Let G be a topological group. A continuous function $\psi : G \to \mathbb{R}$ is *conditionally of negative type* when the kernel Ψ on G given by $\Psi(g, h) = \psi(h^{-1}g)$, for $g, h \in G$, is conditionally of negative type.

The aim of this section is to prove the following theorem.

Theorem 5.4.1. *A map belongs to $K^2(G)$ if, and only if, it is conditionally of negative type.*

We will first prove one direction.

Proposition 5.4.2. *The functions on $K^2(G)$ are conditionally of negative type.*

Proof. $K^2(G)$ is the set of all continuous functions $\psi : G \to \mathbb{R}^+$ of the form $\psi(g) = \left\|\alpha_g(0)\right\|^2$, for some continuous affine isometric action $\alpha : G \curvearrowright L_2(X, \mu)$. Let us check that such a function ψ is conditionally of negative type.

We shall consider the kernel $\Psi : G \times G \to \mathbb{R}$ given by $\Psi(g, h) = \psi(h^{-1}g)$, for $g, h \in G$. Let (π, b) be the pair associated with the cocycle decomposition of α.

Now, for any $g, h \in G$, using proposition 4.1.5,

$$\begin{aligned} \Psi(g, h) = \psi(h^{-1}g) &= \left\|\alpha_{h^{-1}g}(0)\right\|^2 = \left\|b(h^{-1}g)\right\|^2 = \left\|b(h^{-1}) + \pi_{h^{-1}}(b(g))\right\|^2 \\ &= \left\|-\pi_{h^{-1}}(b(h)) + \pi_{h^{-1}}(b(g))\right\|^2 = \left\|b(g) - b(h)\right\|^2, \end{aligned}$$

since $\pi_{h^{-1}}$ is an isometry. From this,

$$\Psi(g, g) = \left\|b(g) - b(g)\right\|^2 = 0$$

Also,

$$\Psi(g, h) = \left\|b(g) - b(h)\right\|^2 = \left\|b(h) - b(g)\right\|^2 = \Psi(h, g)$$

Now, for the third condition, fix $n \in \mathbb{N}$, $g_1, ..., g_n \in G$ and $c_1, \cdots, c_n \in \mathbb{R}$ with $\sum_i c_i = 0$.

$$\begin{aligned}
\sum_i \sum_j c_i c_j \Psi(g_i, g_j) &= \sum_i \sum_j c_i c_j \left\| b(g_i) - b(g_j) \right\|^2 = \sum_i \sum_j c_i c_j \left\langle b(g_i) - b(g_j), b(g_i) - b(g_j) \right\rangle \\
&= \sum_i \sum_j c_i c_j (\|b(g_i)\|^2 + \left\|b(g_j)\right\|^2 - \left\langle b(g_i), b(g_j) \right\rangle - \left\langle b(g_j), b(g_i) \right\rangle) \\
&= 2 \sum_i c_i \|b(g_i)\|^2 \sum_j c_j - 2 \sum_i \sum_j \left\langle c_i b(g_i), c_j b(g_j) \right\rangle = -2 \left\| \sum_i c_i b(g_i) \right\|^2 \leq 0
\end{aligned}$$

□

Conversely, we need the following proposition.

Proposition 5.4.3. *If ψ is a function conditionally of negative type on G, a topological group, then there exists $\mathcal{H}_\psi$ a real Hilbert space and $\alpha^\psi : G \curvearrowright \mathcal{H}_\psi$ an affine isometric action such that $\psi(g) = \left\| \alpha_g^\psi(0) \right\|^2, \forall g \in G$.*

Proof. Applying Theorem 5.3.1 on the kernel of ψ and $x_0 \in G$, we obtain a real Hilbert space $\mathcal{H}_\psi$ and a continuous mapping $f : G \to \mathcal{H}_\psi$ such that

1. $\psi(y^{-1}x) = \|f(x) - f(y)\|^2, \forall x, y \in X$
2. $\{f(x) - f(x_0) : x \in X\} \subseteq \mathcal{H}_\psi$ is a total set.

Observe that we can consider $f(e) = 0$, because if we replace f by $f - f(e)$, such function continues to satisfy these two conditions. Define, for each $g \in G$

$$\begin{aligned}
f_g : G &\longrightarrow \mathcal{H}_\psi \\
h &\longmapsto f_g(h) = f(gh)
\end{aligned}$$

For $x, y \in G$, we have that

$$\left\| f_g(x) - f_g(y) \right\|^2 = \|f(gx) - f(gy)\|^2 = \psi((gy)^{-1}gx) = \psi(y^{-1}g^{-1}gx) = \psi(y^{-1}x)$$

which shows that each f_g satisfies condition (1). Now, it is easy to check that $\{f(gx) - f(gx_0) : x \in G\} = \{f(x) - f(gx_0) : x \in G\}$. Also, remember from the proof of 5.3.1, that the set $\{f(x) - f(x_0') : x \in G\}$ is total in $\mathcal{H}$, for any $x_0 \in G$. Therefore, condition (2) is also satisfied by each f_g.

Hence, by the uniqueness part of the theorem, for each $g \in G$, there exists a unique affine isometry $T_g : \mathcal{H}_\psi \to \mathcal{H}_\psi$ such that $f_g = T_g \circ f$. Finally, let us show that the following is the affine isometric action that we want.

$$\begin{aligned}
\alpha^\psi : G &\longrightarrow \mathrm{Isom}(\mathcal{H}_\psi) \\
g &\longmapsto \alpha_g^\psi = T_g
\end{aligned}$$

For $g, h \in G$,

$$f_{gh}(x) = f_g(hx) = T_g(f(hx)) = T_g(f_h(x)) = T_g(T_h(f(x))) = [(T_g \circ T_h) \circ f](x),$$

for each $x \in G$. Hence, $T_g \circ T_h$ is an affine isometry such that $f_{gh} = (T_g \circ T_h) \circ f$. By the uniqueness of such functions, it follows that $T_{gh} = (T_g \circ T_h)$ and then, α^ψ is a homomorphism.

Now, for the strong continuity, first let us show that the map $g \mapsto \alpha_g^\psi(v)$ is continuous for $v = f(x)$, for all $x \in G$.

For a fixed $x \in G$, if $g, h \in G$,

$$\left\|\alpha_g^\psi(f(x)) - \alpha_h^\psi(f(x))\right\|^2 = \|f(gx) - f(hx)\|^2 = \psi(x^{-1}h^{-1}gx).$$

If we take the limit as $h \to g$, since ψ is continuous, this results in $\psi(e) = 0$. This means that $\lim_{g\to h}\left\|\alpha_g^\psi(f(x)) - \alpha_h^\psi(f(x))\right\| = 0$ and then, it can be shown (by working the definitions) that this implies that the map

$$\begin{aligned} G &\longrightarrow \mathcal{H}_\psi \\ g &\longmapsto \alpha_g^\psi(f(x)) \end{aligned}$$

is continuous for each $x \in G$.

Now, we need to show that this implies that all maps of the form $g \mapsto \alpha_g^\psi(v)$ are continuous for any $v \in \mathcal{H}_\psi$.

Let $v \in \mathcal{H}_\psi$, $g \in G$ and $(g_a)_a$ be a net in G such that $g_a \longrightarrow g$. Also, fix $\epsilon > 0$.

Note that $\{f(x) - f(e) : x \in G\} = \{f(x) : x \in G\}$ is a total subset of $\mathcal{H}_\psi$. Therefore, there are $k \in \mathbb{N}$, $x_i \in G$ and constants λ_i such that

$$\left\|v - \sum_{i=1}^{k} \lambda_i f(x_i)\right\| < \epsilon. \tag{5.2}$$

Now,

$$\begin{aligned} \left\|\alpha_{g_a}^\psi(v) - \alpha_g^\psi(v)\right\| \leq &\left\|\alpha_{g_a}^\psi\left(\sum \lambda_i f(x_i)\right) - \alpha_g^\psi\left(\sum \lambda_i f(x_i)\right)\right\| + \\ &\left\|\alpha_{g_a}^\psi(v) - \alpha_{g_a}^\psi\left(\sum \lambda_i f(x_i)\right)\right\| + \left\|\alpha_g^\psi\left(\sum \lambda_i f(x_i)\right) - \alpha_g^\psi(v)\right\| \end{aligned} \tag{5.3}$$

Since α_g^ψ is an isometry for any $g \in G$, each of the last two terms are equal to $\|v - \sum \lambda_i f(x_i)\|$ and then are each $< \epsilon$, by (5.2).

For the first term, we need to consider that, for any $w \in \mathcal{H}_\psi$,

$$\begin{aligned} \alpha_{g_a}^\psi(w) &= T_a(w) + c_a, \qquad \forall a \\ \alpha_g^\psi(w) &= T(w) + c \end{aligned}$$

where T_a and T are linear maps on $\mathcal{H}_\psi$ and $c_a, c \in \mathcal{H}_\psi$. This can be done because α^ψ_g is an affine isometry for any $g \in G$.

With this,

$$\begin{aligned}\left\|\alpha^\psi_{g_a}\left(\sum \lambda_i f(x_i)\right) - \alpha^\psi_g\left(\sum \lambda_i f(x_i)\right)\right\| &= \left\|\sum \lambda_i \left[T_a(f(x_i)) - T(f(x_i))\right] + c_a - c\right\| \\ &\leq \sum |\lambda_i| \, \|T_a(f(x_i)) - T(f(x_i))\| + \|c_a - c\|\end{aligned}$$

Note that $c_a = \alpha^\psi_{g_a}(0) = \alpha^\psi_{g_a}(f(e))$ and $c = \alpha^\psi_g(0) = \alpha^\psi_g(f(e))$.

Since $g \mapsto \alpha^\psi_g(f(x))$ is continuous particularly for $e \in G$, we have that $c_a \longrightarrow c$.

Now, since all $g \mapsto \alpha^\psi_g(f(x_i))$ are continuous and $c_a \longrightarrow c$, it can be shown that $\|T_a(f(x_i)) - T(f(x_i))\| < \epsilon$, for all $i = 1, \cdots, k$. Because of this we can finally say that the first term of (5.3) also gets arbitrarily small. Therefore, $g \mapsto \alpha^\psi_g(v)$ is continuous for all $v \in \mathcal{H}_\psi$.

Hence, α^ψ is an affine isometric action such that

$$\psi(g) = \|f(g) - f(e)\|^2 = \left\|f_g(e)\right\|^2 = \left\|T_g(f(e))\right\|^2 = \left\|\alpha^\psi_g(0)\right\|^2.$$

□

Proof. (of theorem 5.4.1) The first direction was shown in proposition 5.4.2. For the converse, we need to construct an affine isometric action $\alpha : G \curvearrowright L_2(X, \mu)$, for some $L_2(X, \mu)$, such that $\left\|\alpha_g(0)\right\|^2 = \left\|\alpha^\psi_g(0)\right\|^2$, for each $g \in G$, where $\alpha^\psi : G \curvearrowright \mathcal{H}_\psi$ is the affine isometric action of proposition 5.4.3.

By theorem 3.2.3, we can say that there is an isomorphism between $\mathcal{H}_\psi$ and some $L_2(X, \mu)$ space, $I : \mathcal{H}_\psi \to L_2(X, \mu)$. As discussed in section 3.2, I is a linear isometry.

Now we can define

$$\begin{aligned}\alpha : G &\longrightarrow \operatorname{Isom}(L_2(X, \mu)) \\ g &\longmapsto \alpha_g := I \circ \alpha^\psi_g \circ I^{-1}\end{aligned}$$

First, let us check that α is an affine isometric action. In order to do this, we are going to show that there is an isometric representation $\pi : G \curvearrowright L_2(X, \mu)$ and a cocycle $c' \in Z^1(G, \pi, L_2(X, \mu))$ such that α is the affine isometric action associated to c'.

Let (π^ψ, c) be the pair associated to the cocycle decomposition of α^ψ and define

$$\begin{aligned}\pi : G &\longrightarrow \mathcal{O}(L_2(X, \mu)) \\ g &\longmapsto \pi_g := I \circ \pi^\psi_g \circ I^{-1}\end{aligned}$$

First, let us show that π is an isometric representation. Let $g, h \in G$. Then,

$$\pi_{gh} = I \circ \pi^\psi_{gh} \circ I^{-1} = I \circ \pi^\psi_g \circ \pi^\psi_h \circ I^{-1} = I \circ \pi^\psi_g \circ I^{-1} \circ I \circ \pi^\psi_h \circ I^{-1} = \pi_g \circ \pi_h$$

Then, π is a homomorphism and, for each $g \in G$, π_g is a linear isometry, since it is the composition of linear isometries. For the strong continuity, let $x \in L_2(X, \mu)$ and note that the map

$$\begin{aligned} f : G &\longrightarrow \mathcal{H}^\psi \\ g &\longmapsto f(g) = \pi_g^\psi(I^{-1}(x)) \end{aligned}$$

is continuous, since π^ψ is strongly continuous. Now, the map

$$\begin{aligned} G &\longrightarrow L_2(X, \mu) \\ g &\longmapsto \pi_g(x) = I(\pi_g^\psi(I^{-1}(x))) = (I \circ f)(g) \end{aligned}$$

is continuous since it is the composition of two continuous maps and this shows that π is strongly continuous, hence, an isometric representation.

Let $c' := I \circ c$ and note that it is continuous since it is a composition of continuous maps. Also, for $g, h \in G$,

$$\begin{aligned} c'(gh) &= I(c(gh)) = I(c(g) + \pi_g^\psi(c(h))) = c'(g) + (I \circ \pi_g^\psi)(c(h)) \\ &= c'(g) + (I \circ \pi_g^\psi \circ I^{-1} \circ I)(c(h)) = c'(g) + (I \circ \pi_g^\psi \circ I^{-1})(c'(h)) = c'(g) + \pi_g(c'(h)) \end{aligned}$$

Then, $c' \in Z^1(G, \pi, L_2(X, \mu))$. Note that, for $g \in G$ and $x \in L_2(X.\mu)$,

$$\begin{aligned} \alpha_g(x) &= (I \circ \alpha_g^\psi \circ I^{-1})(x) = I(\alpha_g^\psi(I^{-1}(x))) = I(\pi_g^\psi(I^{-1}(x)) + c(g)) \\ &= (I \circ \pi_g^\psi \circ I^{-1})(x) + I(c(g)) = \pi_g(x) + c'(g) \end{aligned}$$

Now, from proposition 4.1.8, we have that α is an affine isometric action. Note that α is the one we want, since

$$\left\|\alpha_g(0)\right\|^2 = \left\|(I \circ \alpha_g^\psi \circ I^{-1})(0)\right\|^2 = \left\|I(\alpha_g^\psi(I^{-1}(0)))\right\|^2 = \left\|I(\alpha_g^\psi(0))\right\|^2 = \left\|\alpha_g^\psi(0)\right\|^2$$

For every $g \in G$. $\square$

Chapter 6

Main results

In this last chapter, we are going to study the main theorem of [1] and its corollaries, starting by working on the case $p = 2$ and then giving a proof of the main theorem, which shows that $K^p(G) \subseteq K^q(G)$, for $0 < p \leq q < \infty$. Finally, we analyse two corollaries of this main theorem. The first one says that if a topological group G admits an affine isometric action with unbounded orbits on an L_p-space, then G admits the same type of action on L_q, for every $q \geq p$. The second corollary shows that for $0 < p \leq q < \infty$, $\mathrm{Isom}(L_p)$ is isomorphic as a topological group to a closed subgroup of $\mathrm{Isom}(L_q)$.

6.1 The case $p = 2$

Definition 6.1.1. Given G a topological group and $p > 0$, let $K^p(G)$ be the set of continuous functions ψ of the form

$$\begin{aligned} \psi \,:\, G &\longrightarrow \mathbb{R}^+ \\ g &\longmapsto \psi(g) = \left\|\alpha_g(0)\right\|^p, \end{aligned}$$

for some affine isometric action $\alpha \,:\, G \curvearrowright L_p(X, \mu)$.

Note that this generalizes definition 5.4.1.

Proposition 6.1.1. *Let G be a topological group, $\psi \in K^2(G)$ and $p > 0$. Then, there exists a continuous probability measure preserving action $\sigma \,:\, G \curvearrowright (\Omega, \nu)$ and a $\mathbb{R}$-valued cocycle of σ $c \in Z^1_\sigma(G, \mathbb{R})$ such that $\psi(g)^{\frac{p}{2}} = \|c(g)\|^p_{L_p}$, for all $g \in G$. In particular, $\psi^{\frac{p}{2}} \in K^p(G)$, for all $p > 0$.*

Proof. By definition, $\psi \,:\, G \to \mathbb{R}^+$ is a continuous map of the form $\psi(g) = \left\|\alpha_g(0)\right\|^2$, for some affine isometric action $\alpha \,:\, G \curvearrowright L_2(X, \mu)$. Consider the pair (π, c) associated to the cocycle decomposition of α.

This means that, for any $g \in G$ and $x \in L_2(X, \mu)$, $\alpha_g(x) = \pi_g(x) + c(g)$ and then, $\psi(g) = \|c(g)\|^2$.

Let $\sigma^\pi : G \curvearrowright (\Omega, \nu)$ be the Gaussian action associated to π on some standard probability space (Ω, ν), as discussed in section 4.6. Then, we have a linear map

$$\begin{aligned} \wedge : L_2(X, \mu) &\longrightarrow L_0(\Omega, \nu, \mathbb{R}) \\ v &\longmapsto \widehat{v} \end{aligned}$$

where each $\widehat{v}$ is a centered Gaussian random variable of variance $\|v\|^2$. Also,

$$\sigma_g^\pi(\widehat{v}) = \widehat{\pi_g(v)}$$

where $\sigma_g^\pi(\widehat{v}) := \widehat{v} \circ (\sigma_g^\pi)^{-1}$, as defined before. Now, consider

$$\begin{aligned} \widehat{c} : G &\longrightarrow L_0(\Omega, \nu, \mathbb{R}) \\ g &\longmapsto \widehat{c}(g) = \widehat{c(g)} \end{aligned}$$

First note that, for each $g \in G$, $\widehat{c(g)} \in L_p(\Omega, \nu, \mathbb{R})$, for every $p > 0$. This is true because each $\widehat{c(g)}$ is a centered Gaussian random variable.

Now, let X be any centered Gaussian random variable of variance σ^2. If we consider the Gaussian variable given by $Y := \frac{X}{\sigma}$, a simple calculation shows that Y is also centered and its variance is equal to 1. Also,

$$\|Y\|_{L_p}^p = \int_\Omega |Y|^p \, \mathrm{d}\nu = \int_{-\infty}^{\infty} |x|^p \, p(x) \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|^p \, e^{-x^2/2} \, \mathrm{d}x =: C_p$$

On the other hand, $\|Y\|_{L_p}^p = \left\|\frac{X}{\sigma}\right\|_{L_p}^p = \frac{1}{\sigma^p} \|X\|_{L_p}^p$. Therefore, $\|X\|_{L_p}^p = C_p \, \sigma^p$.

Then, given $v \in L_2(X, \mu)$, if $X = \widehat{v}$, the variance of X is $\|v\|^2$. Hence, $\left\|\widehat{v}\right\|_{L_p}^p = C_p \, \|v\|^p$. This shows that the function

$$\begin{aligned} \wedge_p : L_2(X, \mu) &\longrightarrow L_p(\Omega, \nu, \mathbb{R}) \\ v &\longmapsto \widehat{v} \end{aligned}$$

is continuous, since it is linear and bounded. Now, from corollary 2.5.1, the map $\mathrm{I}_p : L_p(\Omega, \nu, \mathbb{R}) \longrightarrow L_0(\Omega, \nu, \mathbb{R})$ is continuous and then so is $\wedge = \mathrm{I}_p \circ \wedge_p$. Now, since $\widehat{c} = \wedge \circ c$, we obtain the continuity of $\widehat{c}$.

Now, let $g, h \in G$.

$$\widehat{c}(gh) = \widehat{c(gh)} = \widehat{c(g)} + \widehat{\pi_g(c(h))} = \widehat{c}(g) + \sigma_g^\pi(\widehat{c}(h)).$$

Therefore, $\widehat{c}$ is a $\mathbb{R}$-valued cocycle of σ^π.

Now, we shall show that $\left\|\widehat{c}(g)\right\|_{L_p}^p = C_p \, \|c(g)\|^p$, for all $g \in G$ and some constant $C_p > 0$.

For $X = \widehat{c}(g)$, its variance is $\|c(g)\|^2$, then, $\left\|\widehat{c}(g)\right\|_{L_p}^p = C_p \, \|c(g)\|^p$. Hence, for any $g \in G$,

$\psi(g)^{\frac{p}{2}} = \|c(g)\|^p = \frac{1}{C_p}\left\|\widehat{c}(g)\right\|_{L_p}^p$, then, if we simply define

$$\tilde{c} := \frac{\widehat{c}}{(C_p)^{1/p}}$$

this is the cocycle that we want. □

Corollary 6.1.1. *For every topological group G, $K^2(G) \subseteq K^p(G)$, for all $p \geq 2$.*

Proof. Let $\psi \in K^2(G)$. From Theorem 5.4.1, we have that $K^2(G)$ is the set of all continuous functions on G that are conditionally of negative type. Therefore, ψ is a function conditionally of negative type. That means that the kernel Ψ on G given by $\Psi(g,h) = \psi(h^{-1}g)$ is conditionally of negative type. Then, by Theorem 5.2.1, it follows that $e^{-t\Psi}$ is a kernel of positive type, for all $t \geq 0$.

If $0 < \alpha \leq 1$, we know that

$$\begin{aligned} f \,:\, (0,\infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = x^\alpha \end{aligned}$$

is a Bernstein function. Hence, it admits the following representation (Theorem 5.1.1)

$$f(x) = a + bx + \int_{(0,\infty)} (1 - e^{-xt})\, \mathrm{d}\mu(t),$$

where $a, b \geq 0$ and μ is a Radon measure on $(0,\infty)$ such that $\int_{(0,\infty)} \frac{t}{1+t}\, d\mu(t) < \infty$.

Now, we shall prove that $f \circ \Psi$ is a kernel conditionally of negative type as well. Note that $(f \circ \Psi)(x,x) = f(\Psi(x,x)) = f(0) = 0^\alpha = 0$ and $(f \circ \Psi)(y,x) = f(\Psi(y,x)) = f(\Psi(x,y)) = (f \circ \Psi)(x,y)$, for all $x, y \in G$. Now, let $n \in \mathbb{N}$, $x_1, \cdots, x_n \in G$ and $c_1, \cdots, c_n \in \mathbb{R}$ with $\sum_i c_i = 0$. Then,

$$\begin{aligned} \sum_i \sum_j c_i c_j (f \circ \Psi)(x_i, x_j) &= \sum_i \sum_j c_i c_j \left[a + b\Psi(x_i,x_j) + \int_{(0,\infty)} (1 - e^{-\Psi(x_i,x_j)t})\, \mathrm{d}\mu\,(t) \right] \\ &= a \sum_i \sum_j c_i c_j + b \sum_i \sum_j c_i c_j \Psi(x_i,x_j) + \\ &\quad + \int_{(0,\infty)} \sum_i \sum_j c_i c_j (1 - e^{-t\Psi(x_i,x_j)})\, \mathrm{d}\mu(t) \end{aligned}$$

Note that the first term is zero and the second is ≤ 0, since Ψ is conditionally of negative type and $b \geq 0$. Now, the last term:

$$\begin{aligned} \int_{(0,\infty)} \sum_i \sum_j (1 - e^{-t\Psi(x_i,x_j)})\, \mathrm{d}\mu(t) &= \int_{(0,\infty)} (\sum_i \sum_j c_i c_j - \sum_i \sum_j c_i c_j e^{-t\Psi(x_i,x_j)})\, \mathrm{d}\mu(t) \\ &= -\int_{(0,\infty)} \sum_i \sum_j c_i c_j e^{-t\Psi(x_i,x_j)}\, \mathrm{d}\mu(t) \end{aligned}$$

is also ≤ 0, since $e^{-t\Psi}$ is a kernel of positive type, for all $t \geq 0$. This completes the proof that $f \circ \Psi$ is a kernel conditionally of negative type. Hence, $f \circ \psi$ is a function conditionally of negative type, which means that $f \circ \psi = \psi^{\alpha} \in K^2(G)$.

Now, let $p \geq 2$. From Proposition 6.1.1 we have that $(\psi^{\alpha})^{\frac{p}{2}} \in K^p(G)$. Taking $\alpha = \frac{2}{p}$, we end up with $\psi \in K^p(G)$. $\square$

6.2 The main theorem

To prove the main theorem, the following proposition is needed.

Proposition 6.2.1. *Let G be a topological group, $p > 0$ and $\psi \in K^p(G)$. Then, there exists a nonsingular action $\sigma : G \curvearrowright (X, \mu)$ and a cocycle $c \in Z^1(G, \sigma^{p,\mu}, L_p(X, \mu))$ such that $\psi(g) = \|c(g)\|^p_{L_p}$, for all $g \in G$.*

Proof. If $p = 2$, from Proposition 6.1.1 there is a continuous probability measure preserving action $\sigma : G \curvearrowright (X, \mu)$ and a cocycle $c \in Z^1_\sigma(G, \mathbb{R})$ such that $\psi(g) = \|c(g)\|^p_{L_p}$, for all $g \in G$.

Since σ is measure preserving, in particular it is a nonsingular action. Also, for each $g \in G$, since $(\sigma_g)_*\mu = \mu$, we have that $D(g) = 1$.

Note that from the construction of the cocycle c in Proposition 6.1.1, we have that $c(g) \in L_2(X, \mu)$, for all $g \in G$. Also, for each $g, h \in G$,

$$\begin{aligned} c(gh) &= c(g) + \sigma_g(c(h)) = c(g) + D(g)^{1/2}\,\sigma_g(c(h)) \\ &= c(g) + T_{\sigma_g}(c(h)) = c(g) + \sigma_g^{2,\mu}(c(h)) \end{aligned}$$

Therefore, $c \in Z^1(G, \sigma^{2,\mu}, L_2(X, \mu))$ and the Proposition is verified for $p = 2$.

Now, let $p \neq 2$ and $\psi \in K^p(G)$, meaning that $\psi(g) = \left\|\alpha_g(0)\right\|^p$ for some affine isometric action $\alpha : G \curvearrowright L_p(X, \mu)$. Consider the pair (π, c) associated with the cocycle decomposition of α. Since $\pi : G \curvearrowright L_p(X, \mu)$ is an isometric representation, from corollary 4.3.1 it follows that, for every $g \in G$, $\pi_g = \omega(g)\sigma_g^{p,\mu}$, with $\omega \in Z^1_\sigma(G, \mathbb{T})$ and $\sigma : G \curvearrowright (X, \mu)$ nonsingular action.

Let us consider the skew-product action $\tilde{\sigma} := \sigma \rtimes \omega : G \curvearrowright (X \times \mathbb{T}, \mu \otimes m)$, meaning that $\tilde{\sigma}_g(x, z) = (\sigma_g(x), z.\omega(g^{-1})(x))$. Also, consider the projection $u : X \times \mathbb{T} \to \mathbb{T}$ with $u(x, z) = z$. Define

$$\begin{aligned} \tilde{c} : G &\longrightarrow L_p(X \times \mathbb{T}, \mu \otimes m) \\ g &\longmapsto \tilde{c}(g) : X \times \mathbb{T} \longrightarrow \mathbb{C} \\ &\qquad (x, z) \longmapsto \tilde{c}(g)(x, z) = u(x, z).c(g)(x) = z.c(g)(x) \end{aligned}$$

First, let us show that $\tilde{c} \in Z^1(G, \tilde{\sigma}^{p,\mu\otimes m}, L_p(X \times \mathbb{T}, \mu \otimes m))$

Let $g, h \in G$, $z \in \mathbb{T}$ and $x \in X$.

$$\tilde{c}(gh)(x, z) = z.c(gh)(x) = z.\left(c(g)(x) + \pi_g(c(h))(x)\right) = z.c(g)(x) + z.\pi_g(c(h))(x)$$

with

$$\pi_g(c(h)) = \left(\omega(g)\sigma_g^{p,\mu}\right)(c(h)) = \omega(g).\sigma_g^{p,\mu}(c(h)) = \omega(g).D(g)^{1/p}.\sigma_g(c(h))$$

On the other hand,

$$\begin{aligned}\tilde{\sigma}_g^{p,\mu\otimes m}(\tilde{c}(h))(x,z) &= \left(D'(g)^{1/p}\tilde{\sigma}_g(\tilde{c}(h))\right)(x,z) = D'(g)^{1/p}(x,z).\tilde{c}(h)\left(\tilde{\sigma}_g^{-1}(x,z)\right)\\ &= D'(g)^{1/p}(x,z).\tilde{c}(h)\left(\sigma_g^{-1}(x), z.\omega(g)(x)\right)\\ &= D'(g)^{1/p}(x,z).z.\omega(g)(x).c(h)\left(\sigma_g^{-1}(x)\right)\end{aligned}$$

From lemma 4.4.3 and proposition 2.4.2, we have that

$$\begin{aligned}D'(g)^{1/p}(x,z) &= \left(\frac{\mathrm{d}\left(\tilde{\sigma}_{g*}(\mu\otimes m)\right)}{\mathrm{d}(\mu\otimes m)}\right)^{1/p}(x,z) = \left(\frac{\mathrm{d}\left((\sigma_{g*}\mu)\otimes m)\right)}{\mathrm{d}(\mu\otimes m)}\right)^{1/p}(x,z)\\ &= \left(\frac{\mathrm{d}(\sigma_{g*}\mu)}{\mathrm{d}\mu}(x)\frac{\mathrm{d}m}{\mathrm{d}m}(z)\right)^{1/p} = \left(\frac{\mathrm{d}(\sigma_{g*}\mu)}{\mathrm{d}\mu}(x)\right)^{1/p} = (D(g)(x))^{1/p}\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{c}(gh)(x,z) &= z.c(g)(x) + z.\omega(g)(x).\left(D(g)(x)\right)^{1/p}.\sigma_g(c(h))(x) =\\ &= z.c(g)(x) + \tilde{\sigma}_g^{p,\mu\otimes m}(\tilde{c}(h))(x,z)\\ &= \left(\tilde{c}(g) + \tilde{\sigma}_g^{p,\mu\otimes m}(\tilde{c}(h))\right)(x,z)\end{aligned}$$

which is the cocycle relation for $\tilde{c}$.

Also, note that for every $g \in G$,

$$\begin{aligned}\|\tilde{c}(g)\|_{L_p}^p &= \int_{X\times\mathbf{T}} |\tilde{c}(g)|^p \,\mathrm{d}(\mu\otimes m) = \int_{\mathbf{T}}\int_X |\tilde{c}(g)(x,z)|^p \,\mathrm{d}\mu(x)\,\mathrm{d}m(z)\\ &= \int_{\mathbf{T}}\int_X |z|^p|c(g)(x)|^p \,\mathrm{d}\mu(x)\,\mathrm{d}m(z) = \int_{\mathbf{T}}\int_X |c(g)(x)|^p \,\mathrm{d}\mu(x)\,\mathrm{d}m(z)\\ &= \int_X |c(g)(x)|^p \,\mathrm{d}\mu(x) = \|c(g)\|_{L_p}^p = \|\alpha_g(0)\|_{L_p}^p = \psi(g)\end{aligned}$$

□

In order to construct the cocycle needed in the following main theorem, we have to go to a bigger space, using both skew-product action and the Maharam extension.

Theorem 6.2.2. *Let G be a topological group and $0 < p < q < \infty$. Then, for every $\psi \in K^p(G)$, there exists a measure preserving action $\sigma : G \curvearrowright (Y,\nu)$ and a function $h \in L_\infty(Y,\nu)$ such that $b(g) = \sigma_g(h) - h \in L_q(Y,\nu)$, with $\psi(g) = \|b(g)\|_{L_q}^q$, for all $g \in G$. In particular, $\psi \in K^q(G)$.*

Proof. By proposition 6.2.1, there exists a nonsingular action $\sigma : G \curvearrowright (X,\mu)$ and a cocycle $c \in Z^1(G, \sigma^{p,\mu}, L_p(X,\mu))$ such that $\psi(g) = \|c(g)\|_{L_p}^p$, for all $g \in G$.

Consider $\tilde{\sigma} : G \curvearrowright (X \times \mathbb{R}_+^*, \mu\otimes\lambda)$ to be the Maharam extension of σ, which means that, for every $g \in G$ and $(x, y) \in X \times \mathbb{R}_+^*$,

$$\tilde{\sigma}_g(x, y) = (\sigma_g(x), y \,.\, \left[D(g^{-1})(x)\right]^{-1}),$$

where $D \in Z_\sigma^1(G, \mathbb{R}_+^*)$ is the Radon-Nikodym cocycle of σ. Define the following map

$$\begin{aligned} \tilde{c} \,:\, G &\longrightarrow L_0(X \times \mathbb{R}_+^*, \mu \otimes \lambda, \mathbb{C}) \\ g &\longmapsto \tilde{c}(g) \,:\, X \times \mathbb{R}_+^* \longrightarrow \mathbb{C} \\ &\qquad\qquad (x, y) \longmapsto \tilde{c}(g)(x, y) = y^{-1/p} c(g)(x) \end{aligned}$$

Let us show that $\tilde{c}$ is a cocycle with respect to $\tilde{\sigma}$, meaning, $\tilde{c} \in Z_{\tilde{\sigma}}^1(G, \mathbb{C})$. Let $g, h \in G$ and $(x, y) \in X \times \mathbb{R}_+^*$.

$$\begin{aligned} \tilde{c}(gh)(x, y) &= y^{-1/p} c(gh)(x) = y^{-1/p} \left[c(g) + \sigma_g^{p,\mu}(c(h))\right](x) \\ &= y^{-1/p} c(g)(x) + y^{-1/p} D(g)^{1/p}(x) c(h)\left(\sigma_g^{-1}(x)\right) \\ &= \tilde{c}(g)(x, y) + \tilde{c}(h)\left(\sigma_g^{-1}(x), y.D(g)^{-1}(x)\right) \\ &= \tilde{c}(g)(x, y) + \tilde{c}(h)\left(\tilde{\sigma}_{g^{-1}}(x, y)\right) \\ &= \left(\tilde{c}(g) + \tilde{\sigma}_g(\tilde{c}(h))\right)(x, y) \end{aligned}$$

Now, consider the skew product action $\rho = \tilde{\sigma} \rtimes \tilde{c} \,:\, G \curvearrowright (Y, \nu)$, where $(Y, \nu) = (\tilde{X} \times \mathbb{C}, \tilde{\mu} \otimes \lambda')$, with $\tilde{X} = X \times \mathbb{R}_+^*$, $\tilde{\mu} = \mu \otimes \lambda$ and λ' is the Lebesgue measure of $\mathbb{C}$. This means that, for every $(\tilde{x}, z) \in Y$,

$$\rho_g(\tilde{x}, z) = \left(\tilde{\sigma}_g(\tilde{x})\,,\; z + \tilde{c}(g^{-1})(\tilde{x})\right)$$

Now, let $\varphi \,:\, \mathbb{C} \to \mathbb{R}$ be a nonzero, radial, compactly supported, Lipschitz function and define

$$\begin{aligned} h \,:\, Y &\longrightarrow \mathbb{R} \\ (\tilde{x}, z) &\longmapsto h(\tilde{x}, z) = \varphi(z) \end{aligned}$$

Note that $h \in L_\infty(Y, \nu)$, since continuous functions with compact support are bounded. Also, for every $g \in G$, let

$$b(g) = \rho_g(h) - h = g \circ \rho_g^{-1} - h$$

Let us show that $b(g) \in L_q(Y,\nu)$, for every $g \in G$.

$$\begin{aligned}
\|b(g)\|_{L_q}^q = \int_Y |b(g)|^q \,\mathrm{d}\nu &= \int_Y \left| h\left(\rho_g^{-1}(\tilde{x},z)\right) - h(\tilde{x},z)\right|^q \,\mathrm{d}\nu(\tilde{x},z) \\
&= \int_Y \left| h\left(\tilde{\sigma}_{g^{-1}}(\tilde{x})\,,\, z + \tilde{c}(g)(\tilde{x})\right) - \varphi(z)\right|^q \,\mathrm{d}\nu(\tilde{x},z) \\
&= \int_Y |\varphi(z+\tilde{c}(g)(\tilde{x})) - \varphi(z)|^q \,\mathrm{d}(\tilde{\mu}\otimes\lambda')(\tilde{x},z) \\
&= \int_{\mathbb{C}}\int_{\tilde{X}} |\varphi(z+\tilde{c}(g)(\tilde{x})) - \varphi(z)|^q \,\mathrm{d}\tilde{\mu}(\tilde{x})\,\mathrm{d}\lambda'(z) \\
&= \int_{\mathbb{C}}\int_X\int_{\mathbb{R}_+^*} |\varphi(z+\tilde{c}(g)(\tilde{x})) - \varphi(z)|^q \,\mathrm{d}\lambda(y)\,\mathrm{d}\mu(x)\,\mathrm{d}\lambda'(z) \\
&= \int_X\int_{\mathbb{C}}\int_{\mathbb{R}_+^*} \left|\varphi(z+y^{1/p}c(g)(x)) - \varphi(z)\right|^q \,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda'(z)\,\mathrm{d}\mu(x)
\end{aligned}$$

Note that, for each $x \in X$, we have $c(g)(x) \in \mathbb{C}$. Now, using lemma 2.4.3, we get a constant $C(q) > 0$ such that,

$$\|b(g)\|_{L_q}^q = \int_X C(q)|c(g)(x)|^p \,\mathrm{d}\mu(x) = C(q)\int_X |c(g)|^p \,\mathrm{d}\mu(x) = C(q)\,\|c(g)\|_{L_p}^p < \infty$$

Note that we also got the result that we want, since we can easily redefine the function b to absorb the constant $C(q)$. □

For any topological group G and any $0 < p \le q < \infty$, we therefore have the inclusion $K^p(G) \subseteq K^q(G)$. This means that, for every affine isometric action $\alpha : G \curvearrowright L_p$, there exists an affine isometric action $\beta : G \curvearrowright L_q$ such that $\|\alpha_g(0)\|_{L_p}^p = \|\beta_g(0)\|_{L_q}^q$, for all $g \in G$.

6.2.1 Existence of affine isometric actions with unbounded orbits on L_p

The main theorem 6.2.2 gives us the following corollary.

Corollary 6.2.1. *Let G be a topological group. If G admits an affine isometric action on an L_p space with unbounded orbits, then, G admits such an action on L_q, for every $q \ge p$.*

Proof. Let $0 < p < \infty$ and $\alpha : G \curvearrowright L_p$ be an affine isometric action with unbounded orbits. Consider its linear part $\pi : G \curvearrowright L_p$ and $c \in Z^1(G,\pi,L_p)$ associated with the cocycle decomposition of α.

It follows from 6.2.2 that, for any q such that $p \le q < \infty$, there is an affine isometric action $\beta : G \curvearrowright L_q$ such that $\|\alpha_g(0)\|_{L_p}^p = \|\beta_g(0)\|_{L_q}^q$, for all $g \in G$.

Considering $\pi' : G \to L_p$ the linear part of β and $b \in Z^1(G,\pi',L_q)$ associated with the cocycle decomposition of β, we have that $\|c(g)\|_{L_p}^p = \|b(g)\|_{L_q}^q$, for all $g \in G$.

Since α has unbounded orbits, the cocycle c is unbounded. Then, since

$$\|b(g)\|_{L_q} = \|c(g)\|_{L_p}^{p/q},$$

the cocycle b is also unbounded. Hence, β is also an affine isometric action with unbounded orbits. $\square$

Also, if we apply the main theorem 6.2.2 on the group $\mathrm{Isom}(L_p)$, we get the following result.

Corollary 6.2.2. *For $0 < p \leq q < \infty$, Isom(L_p) is isomorphic as a topological group to a closed subgroup of Isom(L_q).*

Proof. Consider $G = \mathrm{Isom}(L_p)$ and $\alpha : \mathrm{Isom}(L_p) \curvearrowright L_p$ the canonical affine isometric action

$$\begin{aligned} \alpha : \mathrm{Isom}(L_p) &\longrightarrow \mathrm{Isom}(L_p) \\ g &\longmapsto \alpha_g = g \end{aligned}$$

Then, by theorem 6.2.2, we have another affine isometric action $\beta : \mathrm{Isom}(L_p) \curvearrowright L_q$, such that $\|g(0)\|_{L_p}^p = \|\beta_g(0)\|_{L_q}^q$, for all $g \in \mathrm{Isom}(L_p)$. Therefore, we have a continuous homomorphism

$$\beta : \mathrm{Isom}(L_p) \longrightarrow \mathrm{Isom}(L_q)$$

Denote the image of β by $B \subseteq \mathrm{Isom}(L_q)$. Since β is a homomorphism, we have that B is a subgroup of $\mathrm{Isom}(L_q)$.

Now, let $(\beta_{g_n})_n$ be a sequence in B such that $\beta_{g_n} \to \mathrm{Id}$. Take $f \in L_p$ and let t_f be the translation by f. Note that $t_f \in \mathrm{Isom}(L_p)$. Also, observe that

$$g_n(f) - f = t_f^{-1}(g_n(f)) = t_f^{-1}(g_n(0+f)) = t_f^{-1}(g_n(t_f(0))) = \left(t_f^{-1} \circ g_n \circ t_f\right)(0)$$

We then have that

$$\|g_n(f) - f\|_{L_p}^p = \|\left(t_f^{-1} \circ g_n \circ t_f\right)(0)\|_{L_p}^p = \|\beta_{t_f^{-1} \circ g_n \circ t_f}(0)\|_{L_q}^q = \|(\beta_{t_f}^{-1} \circ \beta_{g_n} \circ \beta_{t_f})(0)\|_{L_q}^q$$

Since $\beta_{g_n} \to \mathrm{Id}$, we have that this last expression converges to 0. Therefore, $g_n(f) \to f$ in L_p, for every $f \in L_p$, hence, $g_n \to \mathrm{Id}$ in $\mathrm{Isom}(L_p)$.

Therefore, we showed that for every sequence $(\beta_{g_n})_n$ in B such that $\beta_{g_n} \to \mathrm{Id}$, we have that $g_n \to \mathrm{Id}$ in $\mathrm{Isom}(L_p)$. From this we get injectivity of β, since $\ker(\beta) = \{\mathrm{Id}\}$.

Now, consider β^{-1} the inverse map of β on its image B. If $(\beta_{g_n})_n$ ia a sequence in B such that $\beta_{g_n} \to \mathrm{Id}$, then $g_n \to \mathrm{Id}$ and this is exactly the same as $\beta^{-1}(\beta(g_n)) \to \beta^{-1}(\mathrm{Id})$. Therefore, β^{-1} is continuous on Id and hence it is continuous.

Also, since B is the preimage by β^{-1} of $\mathrm{Isom}(L_p)$, we have that it is closed. Hence, B is a closed subgroup of $\mathrm{Isom}(L_q)$ and the homomorphism β is a homeomorphism on its image B. $\square$

Bibliography

[1] Amine Marrakchi and Mikael de la Salle. *Isometric actions on Lp-spaces: dependence on the value of p.* 2020. DOI: 10.48550/ARXIV.2001.02490. URL: https://arxiv.org/abs/2001.02490 (cit. on pp. iii, v, 1, 4, 15, 40, 41, 52, 67).

[2] Alain Valette Bachir Bekka Pierre de la de la Harpe. *Kazhdan's property.* New mathematical monographs 11. Cambridge University Press, 2008 (cit. on pp. 1, 22, 33, 56).

[3] Pierre Pansu. "Cohomologie Lp des variétés à courbure négative, cas du degré 1". In: *Partial Differential equations and Geometry - Torino 1988 - Rend. Sem. Mat. Torino, fasc. spez.* (1988), pp. 95–120 (cit. on p. 1).

[4] Marc Bourdon and Hervé Pajot. "Cohomologie lp et espaces de Besov". In: *Journal Fur Die Reine Und Angewandte Mathematik - J REINE ANGEW MATH* 2003 (Jan. 2003), pp. 85–108 (cit. on p. 1).

[5] Guoliang Yu. *Hyperbolic groups admit proper affine isometric actions on lp-spaces.* 2004 (cit. on p. 1).

[6] Bogdan Nica. "Proper isometric actions of hyperbolic groups on Lp-spaces". In: *Compositio Mathematica* 149.5 (Feb. 2013), pp. 773–792 (cit. on p. 1).

[7] James E. Jamison Richard J. Fleming. *Isometries on Banach spaces: function spaces.* 1st ed. Monographs and Surveys in Pure and Applied Math. Chapman and Hall/CRC, 2002 (cit. on pp. 1, 25–28, 49).

[8] S. Banach. *Theory of Linear Operations.* North-Holland Mathematical Library 38. North-Holland, 1994 (cit. on pp. 1, 25, 26).

[9] John Lamperti. "On the isometries of certain function-spaces." In: *Pacific Journal of Mathematics* 8.3 (1958), pp. 459–466 (cit. on pp. 1, 26).

[10] Norman R. Howes. *Modern Analysis and Topology.* 1st ed. Universitext. Springer, 1995 (cit. on pp. 5, 6).

[11] Elon Lages Lima. *Espaços Métricos.* Projeto Euclides. 2014 (cit. on pp. 6, 29).

[12] Michael Kapovich Cornelia Drutu. *Geometric Group Theory.* Colloquium Publications 63. American Mathematical Society, 2018 (cit. on p. 8).

[13] George McCarty. *Topology: An Introduction with Application to Topological Groups.* Dover Books on Mathematics. Dover Publications, 2011 (cit. on p. 9).

[14] Richard M. Foote David S. Dummit. *Abstract Algebra.* 3rd ed. John Wiley & Sons, Inc, 2004 (cit. on p. 9).

[15] Marcel Berger. *Geometry I.* Corrected. Universitext. Springer, 1987 (cit. on p. 9).

[16] Quaintance J Gallier J. *Fundamentals of linear algebra and optimization.* 2017 (cit. on p. 9).

[17] Vladimir I. Bogachev. *Measure Theory Volume 1*. 1st ed. Springer, 2007 (cit. on pp. 12, 13).
[18] Gerald B. Folland. *Real analysis: modern techniques and their applications*. 2nd ed. PAM. Wiley, 1999 (cit. on pp. 13–15, 17, 18).
[19] M. P. Wolff H. H. Schaefer. *Topological Vector Spaces*. 2nd ed. Graduate Texts in Mathematics №3. Springer, 1999 (cit. on p. 17).
[20] J. W. Roberts N. J. Kalton N. T. Peck. *An F-space Sampler*. 1st ed. London Mathematical Society Lecture Note Series. Cambridge University Press, 1985 (cit. on p. 17).
[21] D. H Fremlin. *Measure Theory: Broad Foundations (Vol. 2)*. 2001 (cit. on p. 19).
[22] Leandro Antunes and Kevin Beanland. *Surjective isometries on Banach sequence spaces: a survey*. 2021 (cit. on p. 26).
[23] John B Conway. *A Course in Functional Analysis*. Springer, 2007 (cit. on p. 28).
[24] Walter Rudin. *Real and complex analysis*. 3rd ed. McGraw-Hill, 1987 (cit. on p. 29).
[25] Halsey Royden. *Real Analysis*. 3rd ed. 1988 (cit. on p. 49).
[26] D. Maharam. “Incompressible transformations”. In: *Fundamenta Mathematicae* 56.1 (1964), pp. 35–50 (cit. on p. 53).
[27] Amine Marrakchi and Stefaan Vaes. “Nonsingular Gaussian actions: Beyond the mixing case”. In: *Advances in Mathematics* 397 (Mar. 2022), p. 108190 (cit. on p. 54).
[28] Rene L Schilling; Renming Song; Zoran Vondracek. *Bernstein functions : theory and applications*. 2nd ed. De Gruyter studies in mathematics, 37. De Gruyter, 2012 (cit. on p. 56).