Braid, Knots and Links

Sophia Lopes Ribeiro Fiorotto

Dissertation Submitted
to the
Institute of Mathematics and Statistics
of the
University of São Paulo
for
the degree
of
Master of Science

Program: Mathematics

Supervisor: Prof. Dr. Ivan Struchiner

Co-Supervisor: Drª. Luciana Basualdo Bonatto

During the development of this dissertation the author was financially supported by the CAPES

São Paulo, August 2023

Braids, Knots and Links

This is the original version of the dissertation developed by the candidate Sophia Lopes Ribeiro Fiorotto, as it was submitted to the Judging Committee.

Braids, Knots and Links

This is the original version of the dissertation developed by the candidate Sophia Lopes Ribeiro Fiorotto, as it was submitted to the Judging Committee.

Judging Committee:

- Drª. Luciana Basualdo Bonatto (co-supervisor) - Max Planck Insititute
- Profª. Drª. Lucilia Daruiz Borsari - IME-USP
- Profª. Drª. Carolina de Miranda e Pereiro - UFES

Kabelsalat

From this german words,
'A giant entangled of strings'
We may actually learn
How our journey begins

If every person in our life
creates a new crossing,
which braid do you give rise to?

But after I met you, my love
I keep asking myself,
why don't we close our braids
and become a link already?

Since, no matter the orientation,
I want always to come back to you.

Acknowledgement

Firstly, I would like to thank my co-supervisor Drª. Luciana Basualdo Bonatto for making mathematics more intuitive and joyful and for all the support during the development of this dissertation. I would also link to thank my supervisor Prof. Dr. Ivan Struchiner for all the years of friendship and all the projects we developed together.

I extend my gratitude to the members of the Judging Committee for their thoughtful consideration and valuable insights. I would also like to acknowledge the support provided by CAPES, which which made it possible for me to develop this dissertation.

I would also like to thank the inconditional love and support of my family: my parents Maria da Glória and José Antonio, my brother and sister Otávio e Veridiana and my dearest aunt and second mother Carmen Silvia. Thank you for always believing in me.

In addition, I would like to express my gratitude to all my dearests friends and people in my life that supported me and keep supporting me every day in the challenging and grateful moments of this journey. A special thanks to my dear friend Matheus that shared all great moments with me since undergrad and keeps supporting me everyday. I also feel very grateful for Marize, Camila, Murilo e Duda for all the support and for making my everyday life happier and full of laughs and joy.

Lastly, to all my teachers and students that made part of my journey through education. Being a mathematician makes me a better teacher and reflecting and living education makes me a better mathematician.

Resumo

FIOROTTO, S. L. R. **Tranças, Nós e Links**. Tese (Mestrado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2023.

A teoria das tranças e nós oferece uma via cativante e intuitiva para explorar uma variedade diversificada de ferramentas na topologia algébrica. Nosso objetivo é utilizar o contexto das tranças e links para proporcionar um caminho de estudo na topologia algébrica e explorar resultados importantes na área, como o Teorema de Alexander e o Teorema de Markov.

Esta tese explora os aspectos fundamentais da teoria das tranças, incluindo várias definições de grupos de tranças, noções de equivalência e invariantes. Também fornece noções básicas e resultados da teoria dos nós, como invariantes e superfícies de Seifert. Além disso, investigamos a relação entre tranças e nós. O Teorema de Alexander afirma que todo nó ou link em S^3 pode ser representado como uma trança fechada, enquanto o teorema de Markov oferece insights sobre a relação que as tranças que geram um determinado nó compartilham. Este trabalho oferece um caminho acessível para entender a interação entre tranças, nós e topologia algébrica.

Palavras-chave: Tranças, Nós, Links, Teorema de Alexander, Teorema de Markov.

Abstract

FIOROTTO, S. L. R. **Braids, Knots and Links**. Master Thesis - Mathematics and Statistics Institute, University of Sao Paulo, São Paulo, 2023.

The theory of braids and knots offers a captivating and intuitive avenue for exploring a diverse array of tools in the algebraic topology. We aim to use the context of braids and links to provide a path of study on algebraic topology and exploring important results in the area such as Alexander and Markov's Theorem.

This thesis explores braid theory's fundamental aspects, including various definitions of braid groups, equivalence notions, and invariants. It also provides basic notion and results from knot theory, such as invariants and Seifert Surfaces. Moreover, we investigate the relationship between braids and knots. Alexander's Theorem establishes that every knot or link in S^3 can be represented as a closed braid, while Markov's theorem provides insight into the relationship braids generating a given knot share.

Keywords: Braids, Links, Alexander's Theorem, Markov Theorem

Content

List of Figures

Introduction

The theory of braids is widely studied in the field of algebraic topology. Stemming from an intuitive concept, this theory offers a unique opportunity to explore a multitude of interconnected fields and tools. As a result, delving into the study of braids is not only intriguing but also presents a fascinating avenue for research and exploration.

Essentially, a mathematical braid is a collection of paths living in $\mathbb{R}^3$ that tangles themselves in one another without intersecting and it starts and ends in different points set in parallel planes. Also, equipping the set of braids with the operation of concatenating one another gives us a group structure. Actually, the braid group $\mathcal{B}_n$ was first defined by Emil Artin in [Art25] and further studied in [Art47a] and [Art47b]. However, it was probably first considered by Hurwitz in 1891 when discussing, in modern terminology the fundamental group of configuration spaces. Other definitions were given later and by the time of 1936 it was already known their equivalence [Zar36].

Moreover, Artin introduced the braid group with the idea it could help with knots and links. Knot theory is the theory of studying the embedding of the circle in the space. On the other hand, links are a generalization of knots, namely a link is a finite disjoint union of knots, possibly intertwined. Each knot K_i is called a component of the link. We can also try defining operations on those links such as composition, reflection and reversing the orientation. With this operations we can also define a group structure to the set of knots. A big part of knot theory is trying to define invariants in such a way that given two links that are equivalent, they must have the same invariant. In this text, we will explore some of them, such as the coloring number of a given knot.

In the exploration of braid and knot theories, undeniable intersections emerge, naturally evoking curiosity about potential relationships between braids and links. In fact, we can always obtain a knot from a braid by making the process of closing a braid which, intuitively consists in connecting the initial to the final points of every string without creating any more crossings. Therefore, it's natural to ask if given a knot K, there is a braid that it can be obtained from? If there is, is it unique? Moreover, if the braids that give rise to this knot are not unique, what do they have in common? Actually, the first question was answered positively by Alexander's Theorem that states that every knot or link in S^3 can be represented as a closed braid. In addition, Markov's theorem states that the braids that give rise to a given not are not unique. However, one can be obtained from the other by a braid isotopy or a single stabilization or a destabilization.

In *Chapter 1* we will explore three definitions of braid groups, the notion of equivalence and the braid permutation as an invariant and the pure braid group. We will also investigate some results involving fibration sequence and braids, an the proof of how to obtain Artin's famous presentation of braid groups building upon the configuration space's definition.

In *Chapter 2* we will dive in some definitions on knots and links, how to define the notion of their equivalence and Reidemeister moves. Moreover, we will look closely to operations such as composition, reflection, reverse and defining a group structure under the set of links $\mathcal{L}$. Furthermore, we study some link invariants such as multiplicity, minimum crossing points, the bridge number, the unknotting number and the colouring number of a given knot. Finally, we will look to the process of obtaining a Seifert surface for a given knot.

In *Chapter 3* we will study the relationship between braids and knots. First, we will look closely to the concept of closed braids. Then, we will prove Alexander's Theorem using Yamada-Vogel Algorithm [Yam87] [Vog90]. We will then see a proof of Markov's Theorem, due to o Pawel Traczyk [Tra98] that uses some of the ideas that were developed in the proof of Alexander's theorem.

This text aims to provide a clear and intuitive understanding of the mathematical concepts presented here. Additionally, it offers the option to explore proofs that utilize fundamental tools in algebraic topology, detailed in the Appendices for reference and in-depth study, such as fibration sequences, cell complexes and homotopy. These proofs not only offer a geometric and pleasing interpretation of classical results but also contribute to a comprehensive grasp of the subject matter.

Chapter 1

Braids

In this chapter we will first study the notion of braids and braid groups. We will give three different definitions: geometric definition, configuration spaces and Artin's presentation. Each of these definitions will equip us with essential tools to tackle a variety of results. After giving the geometric definition, we will study the notion of braid equivalence, permutation and pure braid groups. Subsequently, we will introduce the configuration spaces definition for braid groups and pure braid groups. We shall examine outcomes related to the fibration sequence and braid groups. Finally, we will give the Artin's presentation for braid groups and see a prove of how this presentation appears naturally from configuration spaces definition given originally by Fox and Neuwirth [FN62]. In the last section of the chapter we study the Garside Structure and how to obtain a solution for the word problem. The main references used on this chapter are: [MK12], [B$^+$74], [GM11], [Gar69].

1.1 Intuitive approach

In several occasions, in our everyday life, we may have crossed the notion of a braid, mostly in the context of hairstyle. However, mathematically, one doesn't need to be confined only to three strings, or even $\mathbb{R}^3$ (although it will be our initial focus). So, intuitively speaking, a mathematical braid can be understood by the following setup:

For example, imagine two parallel acrylic plates with 4 strings that intertwine (without intersecting) fixed on the plates on both ends. (See Figure 1.1)

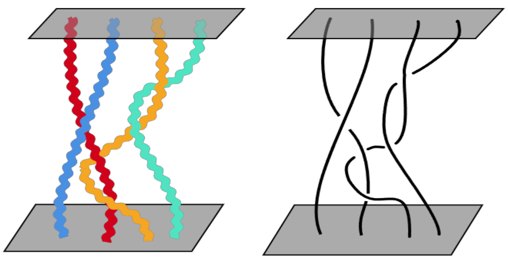


Figure 1.1: *A braid with four strings and a mathematical representation of it.*

The second picture illustrates how we will represent the braids in mathematics. If one string passes behind another, we will represent it by leaving a little blank space.

Another interesting way to think about braids is the following: Imagine three flies standing under an acrylic plate in an instant $t = 0$. When the time starts, they begin going down until they reach another plate in $t = 1$ in such a way that in each time t they are on the same height. When we see the bigger picture, one can imagine how their movement describes a mathematical braid. (See Figure 1.2) However, one may also realize you can make a braid by looking at different configurations of three points on each instant going from $t = 0$ to $t = 1$.

This visualization will help us understanding the configuration spaces definition of a mathematical braid.

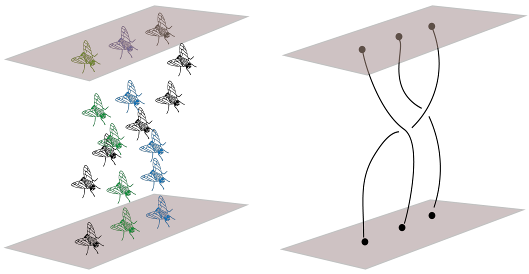


Figure 1.2: *The path done by the flies and the braid they represent it.*

1.2 Geometric definition

Now, we will give our first attempt trying to formalize our notion of a braid:

Definition 1.2.1. Consider $\mathbb{R}^3$ and two parallel planes α, β with the third coordinate being 0 and 1, respectively. Let P_i, Q_i $(1 \leq i \leq n)$ the points with coordinates $(i, 0, 1)$ and $(i, 0, 0)$, such that $\{P_1, \ldots, P_n\} \subset \beta$ in the superior plane and $\{Q_1, \ldots, Q_n\} \subset \alpha$ in the inferior plane.

A braid with n strings is made of n paths, $\gamma_1, \ldots, \gamma_n$, such that γ_i connects P_i from the superior plane to $Q_{\pi(i)}$ in the inferior plane for some permutation $\pi \in \Sigma_n$. Also,

(i) Each γ_i intersects the plane $z = t$ only once for each $t \in [0, 1]$

(ii) The paths $\gamma_1, \ldots, \gamma_n$ intersects the plane $z = t$ always in n distinct points, $\forall t \in [0, 1]$

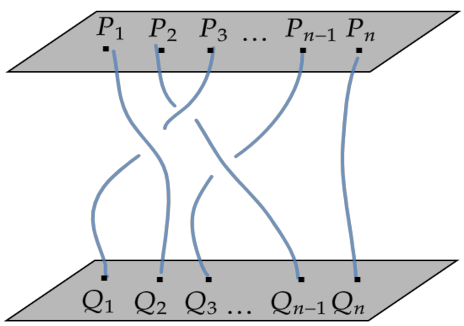


Figure 1.3: *Representation of a generic braid.*

The element π is called the braid permutation and will be detailed later. In other words, the definition says that in a n-string braid, each string crosses each other a finite number of times and they never intersect each other.

Furthermore, one can recognize a group structure under concatenation of braids. We will formalize some details later. Intuitively, the trivial braid (or the group identity) on n-strings is the one with the trivial permutation and no crossings between the strings. Additionally, if one concatenates a given braid with the trivial one, nothing happens. Also, we can try looking into the notion of inverse of a braid by asking: Which braid (if there is any) can we concatenate with a given one to obtain the trivial braid? See the following figures for some examples:

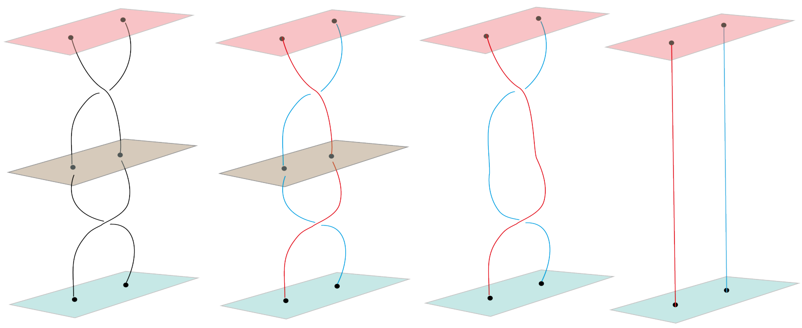


Figure 1.4: *Example of composition of two braids with two strings that gives rise to the trivial braid.*

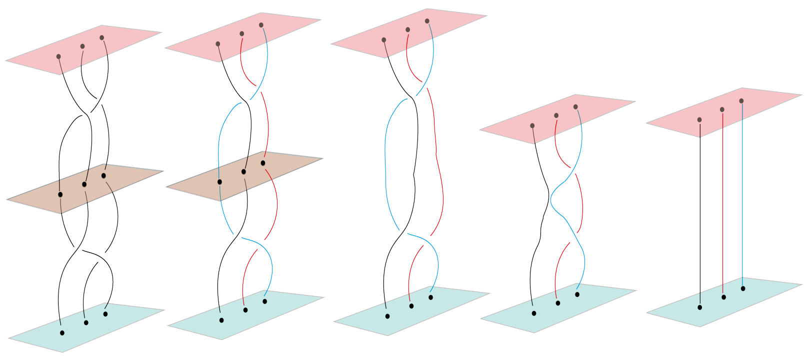


Figure 1.5: *Example of composition of two braids with three strings that gives rise to the trivial braid.*

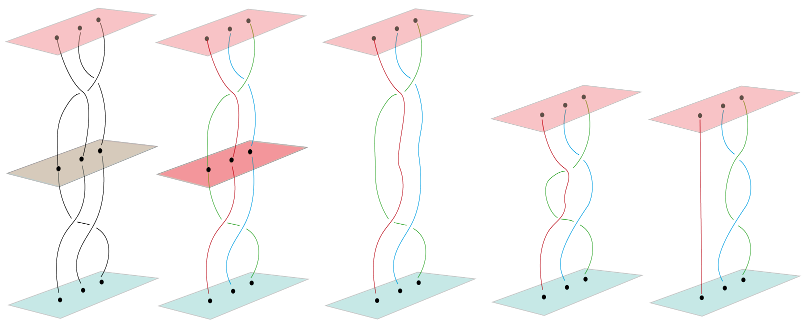


Figure 1.6: *Example of composition of two braids with three strings.*

1.3 Equivalence, Braid Permutation and Pure Braid Group

A big question when studying braid groups is whether given two braids if they are equivalent or not. Thinking about this, we will define an operation that doesn't affect or change the braid we re interested about.

Definition 1.3.1. (Elementary move) Let $\mathcal{C}$ be a unit cube and α_1, α_2 be a pair of parallel faces of the cube that will limit the braid. Let AB be an edge of a string c. (For the purposes of this definition, we need to be strict and work exclusively with the polygonal image of a string.) Let O be a point in $\mathcal{C}$ such that the triangle $\triangle ABO$ (in $\mathcal{C}$) does not intersect any other strings and only meets c along AB. Further, suppose:

(i) $AO \cup OB$ intersects every plane parallel to α and β at most at one point,

(ii) replace AB by the two edges $AO \cup OB$,

(ii') replace $AO \cup OB$ by the edge AB.

If the conditions above holds then the operation, denoted by Φ, or the inverse operation, Φ^{-1} is called an *elementary move* on a braid. Note that the first condition assures the result of operating a given braid β with Φ is also a braid since mathematical braids always flows downward (See Figure 1.7).

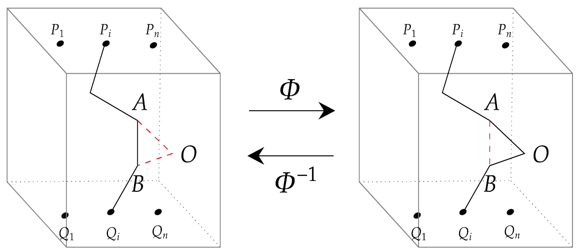


Figure 1.7: *Elementary move performed on a string c.*

Definition 1.3.2 (Equivalence of braids)**.** Two n-braids β and β' is said to be *equivalent* ($\beta \sim \beta'$) if β can be deformed into β' by a finite series of elementary moves, within the cube $\mathcal{C}$. In other words, β is equivalent to β' if there is a finite sequence of β_i, $i \in \{1, \ldots, m\}$ such that β_i is obtained by β_{i-1} by applying Φ or Φ^{-1}.

$$\beta = \beta_0 \xrightarrow{\Phi^{\pm 1}} \beta_1 \xrightarrow{\Phi^{\pm 1}} \cdots \xrightarrow{\Phi^{\pm 1}} \beta_m = \beta'$$

Moreover, it can be shown that this relation is, in fact, an equivalence relation. That means it holds the reflexive, symmetric and transitive properties.

Proposition 1.3.1. Assume we have n-braids $\beta, \beta', \alpha, \alpha'$ such that β is equivalent to β' and α is equivalent to α'. Then, it holds that

$$\beta\alpha \sim \beta'\alpha'.$$

Proof. Considering the definition of braid equivalence (Definition 1.3.2), there exists a finite sequence,

$$\beta = \beta_0 \xrightarrow{\Phi^{\pm 1}} \beta_1 \xrightarrow{\Phi^{\pm 1}} \cdots \xrightarrow{\Phi^{\pm 1}} \beta_m = \beta'.$$

This sequence also induces another sequence:

$$\beta\alpha = \beta_0\alpha \xrightarrow{\Phi^{\pm 1}} \beta_1\alpha \xrightarrow{\Phi^{\pm 1}} \cdots \xrightarrow{\Phi^{\pm 1}} \beta_m\alpha = \beta'\alpha.$$

Thus, we obtain $\beta\alpha \sim \beta'\alpha$.

Similarly, given $\alpha \sim \alpha'$, we have a finite sequence:

$$\alpha = \alpha_0 \xrightarrow{\Phi^{\pm 1}} \alpha_1 \xrightarrow{\Phi^{\pm 1}} \cdots \xrightarrow{\Phi^{\pm 1}} \alpha_k = \alpha',$$

which, in turn, gives rise to another finite sequence:

$$\beta'\alpha = \beta'\alpha_0 \xrightarrow{\Phi^{\pm 1}} \beta'\alpha_1 \xrightarrow{\Phi^{\pm 1}} \cdots \xrightarrow{\Phi^{\pm 1}} \beta'\alpha_k = \beta'\alpha'.$$

Hence, we deduce that $\beta'\alpha \sim \beta'\alpha'$.

The transitivity of braid equivalence allows us to conclude that $\beta\alpha \sim \beta'\alpha'$.

□

Definition 1.3.3. Let β be an n-braid. We will define the mirror of an n-braid (denoted by $\bar{\beta}$ in the following way: if a given string d_i goes from P_i to $Q_{j(i)}$ in the mirror image we will take $\bar{d}_i$ from $P_{j(i)}$ to Q_i. Also, all crossings will be preserved. That means, if d_i goes over d_j then $\bar{d}_i$ will also go over $\bar{d}_j$, for $i \neq j$.

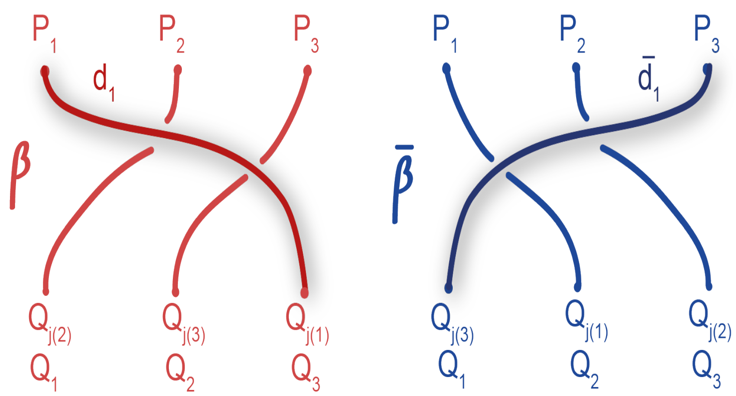


Figure 1.8: *An example of a braid β and it's mirror image $\bar{\beta}$.*

Proposition 1.3.2. For each n-braid β, there exists a n-braid $\bar{\beta}$ such that

$$\beta\bar{\beta} \sim \mathbf{1}_n \text{ and } \bar{\beta}\beta \sim \mathbf{1}_n.$$

Such a n-braid is called the inverse of β and denoted by β^{-1}.

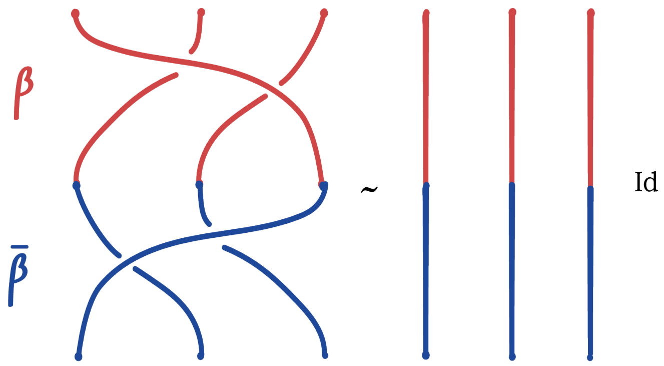


Figure 1.9: *The composite of a braid β and it's mirror image $\bar{\beta}$ gives the identity.*

So far, we have been rather flexible regarding the distinction between an n-braid β (belonging to $\mathcal{B}_n$) and its equivalence class $[\beta]$ (belonging to $\mathcal{B}_n$). However, to establish a group structure, we must now shift our focus to the equivalence classes of n-braids.

A significant consequence of Proposition 1.3.1 is that the product of two elements $[\beta_1]$ and $[\beta_2]$ in $\mathcal{B}_n$ is well-defined, given by $[\beta_1] \cdot [\beta_2] = [\beta_1\beta_2]$ (where the equivalence class $[\beta_1\beta_2]$ remains independent of the choice of representatives for each class). With this result and the previous ones we have now fulfilled all the necessary requirements for $\mathcal{B}_n$ to form a (non-commutative) group.

Theorem 1.3.3. The set of equivalence classes of n-braids, denoted by $\mathbf{B}_n$, indeed constitutes a group. This group is often referred to as the n-braid group or Artin's n-braid group.

Proof. See reference [MK12]. □

It's not difficult to see that if two braids do not have the same amount of strings, they can never be equivalent. However, if we have two n-braids we suspect are not equivalent, how can one prove this is, in fact, the case? What can help on that is to find some sort of *invariant*, that means an algebraic result that can be calculated from a given braid .

Definition 1.3.4 (Braid Invariant)**.** Let f be a mapping from the space of braids $\mathcal{B}$ to some algebraic structure (for instance, number, polynomials, matrices...). If f has the following property:

$$\beta \sim \beta' \implies f(\beta) = f(\beta')$$

then f is said to be a *braid invariant.*

Example 1.3.1. The most intuitive algebraic structure to associate with a braid is the number of strings. In fact, $f : \mathcal{B} \longrightarrow \mathbb{N}$ where $f(\beta)$ equals the number of strings is a braid invariant. Indeed, if $f(\beta) \neq f(\beta')$, then $\beta \nsim \beta'$.

Definition 1.3.5 (Braid Permutation)**.** Let β be an n-braid and consider that the string s_i takes P_i to $Q_{j(i)}$ for $i \in \{1, 2, \ldots, n\}$. Define $\pi : \mathcal{B} \longrightarrow \Sigma_n$ (recall that Σ_n is the set of all permutations of $\{1, 2, \ldots, n\}$) as:

$$\pi(\beta) = \begin{pmatrix} 1 & 2 & \cdots & n \\ j(1) & j(2) & \cdots & j(n) \end{pmatrix}$$

$\pi(\beta)$ is called the *braid permutation* of β.

Example 1.3.2. The braid permutation is a braid invariant. In fact, if given two braids such that $\beta \sim \beta'$, then for each $i \in \{1, 2, \dots, n\}$ the ith string must have the same bottom point $Q_{j(i)}$

Example 1.3.3. Consider the braids β and β':

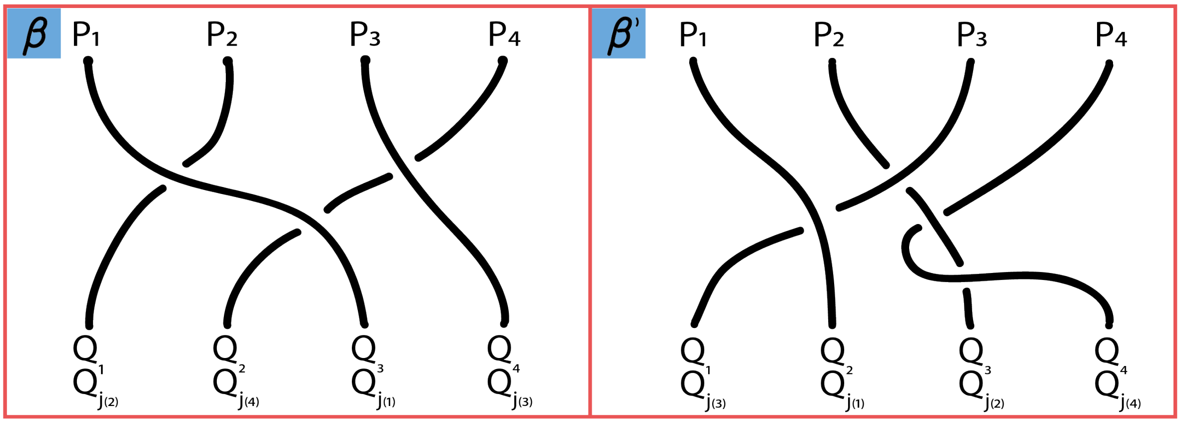


Figure 1.10: *Projection of braids β and β'.*

The permutations are given by:

$$\pi(\beta) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

$$\pi(\beta') = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Since $\pi(\beta) \neq \pi(\beta')$, we may conclude that $\beta \nsim \beta'$.

Even though the pure braid group have already appeared for us, we now present another meaning to it:

Definition 1.3.6 (Pure Braid)**.** Let β be an n-braid. If $\pi(\beta)$ is trivial then $\pi(\beta)$ is called a pure braid. In other words,

$$\pi(\beta) = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}$$

1.4 Configuration Spaces Definition

Artin's definition was generalized by Fox and Neuwirth in [FN62] and defines a braid group not just in the plane, but in any manifold M. Also, that is the definition used by Birman in [B$^+$74].

Definition 1.4.1 (Configuration Spaces of n points on M)**.** Let M be a manifold dim ≥ 2 and let $\prod_{i=1}^{n} M$ the product space of M, n times (i.e. $M \times \cdots \times M$, n times). The configuration space of n points in M, $\tilde{C}_n M$ is:

$$\tilde{C}_n M := \left\{ (z_1, ..., z_n) \in \prod_{i=1}^{n} M \mid z_i \neq z_j \text{ se } i \neq j \right\}$$

Definition 1.4.2. (Pure Braid Group) The fundamental group of $\tilde{C}_n M$, denoted $\pi_1(\tilde{C}_n M)$, is called the Pure Braid Group (or unpermuted) with n strings in a manifold M. More precisely, the Pure Braid Group is $\pi_1(\tilde{C}_n M, z)$, where z is a chosen base point on $\tilde{C}_n M$.

Remark 1.4.1. When we doesn't explicit M, then $M = \mathbb{R}^2$. Normally, when we are in the context of geometrical or Artin's definition, we are always talking about braids in $\mathbb{R}^2$.

Definition 1.4.3 (Unordered Configuration Space)**.** Define an equivalence relation $z, z' \in \tilde{C}_n M$ as it follows: $z \sim z'$ if the coordinates $(z_1, \ldots, z_n)$ of z e $(z'_1, \ldots, z'_n)$ of z' differ only by a permutation. Let C_n be the quotient space $\tilde{C}_n /_{\sim}$ identified with the relation described above.

Definition 1.4.4. (Braid Group) We call the braid group with n strings the set

$$B_n := \pi_1(C_n \mathbb{R}^2)$$

.

Remark 1.4.2. Usually, we will omit $\mathbb{R}^2$ from the braid group notation. Moreover, it can also be studied braid groups in other manifolds, but this won't be the focus in this text.

Note that one have a natural projection $p : \tilde{C}_n \longrightarrow C_n$ that will be useful in many steps ahead. Therefore, more precisely, one may understand B_n as $\pi_1(C_n, \bar{z})$, such that $\bar{z} = p(z)$, where z is the base point chosen on $\tilde{C}_n M$.

Remark 1.4.3. One may understand $\tilde{C}_n M$ as a topological space with the induced topology by the product topology of $M \times \cdots \times M$. Also, since dimM ≥ 2 and M is connected, the homotopy groups $\pi_i(\tilde{C}_n M)$ are independent of the choice of the base point.

1.4.1 Fibration sequence and Braids

Let $Q_m = \{q_1, \ldots, q_m\}$ a fixed set of m distinct points in a manifold M. Define $Q_0 = \emptyset$ and let $\tilde{C}_n^m M := \tilde{C}_n(M - Q_m)$.

Theorem 1.4.4 (Fadell and Neuwirth, 1962)**.** [FVB+62] If $1 \leq r < n$ and

$$\begin{aligned} \pi : & \tilde{C}_n^m M \longrightarrow \tilde{C}_r^m M \\ & (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_r) \end{aligned}$$

Then, $(\pi, \tilde{C}_n^m M)$ is a fibration over $\tilde{C}_r^m M$ with fiber $\tilde{C}_{n-r}^{m+r} M$

Proof. [B+74] We want to get the following fiber bundle:

$$\tilde{C}_{n-r}^{m+r} \overset{i}{\hookrightarrow} \tilde{C}_n^m \overset{\pi}{\to} \tilde{C}_r^m$$

In other words, we need $\pi : \tilde{C}^m_n M \longrightarrow \tilde{C}^m_r M$ such that for each $z \in \tilde{C}^m_r M$ exists a neighborhood U of z such that exists a homeomorphism $h : \pi^{-1}(U) \longrightarrow U \times \tilde{C}^{m+r}_{n-r} M$ that makes the following diagram commute:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "h"] \arrow[dr, "\pi"'] & & U \times \tilde{C}^{m+r}_{n-r} \arrow[dl, "i"] \\ & U & \end{tikzcd}$$

Let $z_1, \ldots, z_r$ be a base point in $\tilde{C}^m_r M$. Consider the fiber

$$\pi^{-1}(z_1, \ldots, z_r) = \{(z_1, \ldots, z_r, y_{r+1}, \ldots, y_n) \in M - Q_m, \text{with distinct coordinates}\}$$

Consider the choice $Q_{m+r} = Q_m \cup \{z_1, \ldots, z_r\}$. Then, we have:

$$\tilde{C}^{m+r}_{n-r} M = \{(y_{r+1}, \ldots, y_n), y_i \in M - Q_{m+r} \forall i \text{ , and } y_i \neq y_j \forall i \neq j\}$$

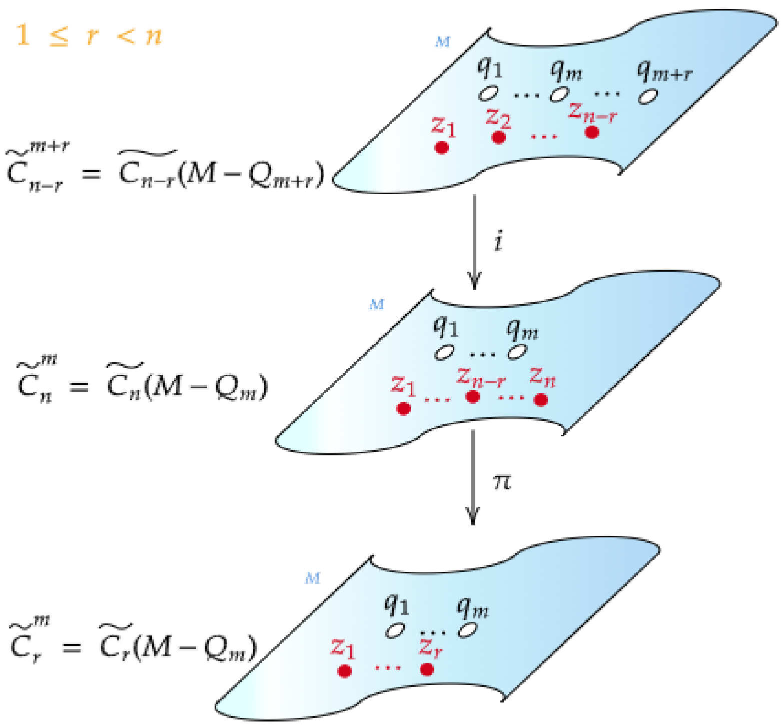


Figure 1.11: *Schematic drawing of fibration $(\pi, \tilde{C}^m_n M)$ over $\tilde{C}^m_r M$ with fiber $\tilde{C}^{m+r}_{n-r} M$.*

And we have the following homeomorphism:

$$h : \tilde{C}^{m+r}_{n-r} M \longrightarrow \pi^{-1}(z_0, \ldots, z_r)$$
$$h(y_{r+1}, \ldots, y_n) = (z_1, \ldots, z_r, y_{r+1}, \ldots, y_n)$$

Now, we prove the local triviality condition for the case r=1. First, fix a point $\bar{x} \in M - Q_m = \tilde{C}^m_1 M = \tilde{C}^m_r M$. Now, consider Q_{m+1} after adding a point q_{m+1} to the set Q_m.

Pick a homeomorphism $\varphi : M \longrightarrow M$ fixing Q_m and such that

$$\begin{aligned} \varphi : M &\longrightarrow M \\ q_{m+1} &\mapsto \bar{x} = \varphi(q_{m+1}) \end{aligned}$$

Let U be an open neighborhood of $\bar{x} \in M - Q_m$ homeomorphic to an open ball. Also, consider $\bar{U}$ the closure of U. Define a map:

$$\begin{aligned} G : & U \times \bar{U} \longrightarrow \bar{U} \\ & (z, y) \mapsto G(z, y) = G_z(y) \end{aligned}$$

such that:

(i) $G_z : \bar{U} \longrightarrow \bar{U}$ is a homeomophism which fixes $\partial\bar{U}$

(ii) $G_z(z) = \bar{x}$.

Now, using (i), let's extend $G : U \times \bar{U} \longrightarrow \bar{U}$ to $G : U \times M \longrightarrow$ by setting $G(z, y) = y$ if $y \notin U$. Note that the first condition guarantees the continuity of G.The required local product representation is given by:

$$\mathrm{U} \times \tilde{C}^{m+1}_{n-1} \underset{F^{-1}}{\overset{F}{\rightleftarrows}} \pi^{-1}(\mathrm{U})$$

$$F(z, z_2, \ldots, z_n) = (z, G_z^{-1}\varphi(z_2), \ldots, G_z^{-1}\varphi(z_n))$$

$$F^{-1}(z, z_2, \ldots, z_n) = (z, \varphi^{-1}G_z(z_2), \ldots, \varphi^{-1}G_z(z_n))$$

The intuition behind F is that if the points of $\tilde{C}^{m+1}_{n-1}$ don't end up inside the neighborhood U, we can just take them as are together with any point in U and we already get a configuration. However, if any of the points end up inside U, then we can perturb it slightly with G which guarantees that the points won't coincide. Note that all coordinates are distinct since $M - \bar{x}$ is fiber and:

$$G_z^{-1}\varphi(z_i) = z \iff \varphi(z_i) = G_z(z) \overset{(ii)}{\iff} \varphi(z_i) = \bar{x}$$

For $r > 1$, the idea is to consider again a neighborhood composed of open balls around the r points:

$$V = \overset{o}{D}(z_1) \times \cdots \times \overset{o}{D^n}(z_{k-1}).$$

and the homeomorfism:

$$h : V \times \tilde{C}^{m+r}_{n-r} \underset{h^{-1}}{\overset{h}{\rightleftarrows}} \pi^{-1}(V)$$

$$h((m_1, \ldots, m_{k-1}), m_k) = \begin{cases} (m_1, \ldots, m_{k-1}, m_k) & \text{if } m_k \notin \cup_{1 \le i \le k-1} D^n(q_i), \text{ and} \\ (m_1, \ldots, m_{k-1}, G(m_i, m_k)) & \text{if } m_k \in D^n(q_i) \end{cases}$$

One can see further details in the computation of G in [Coh10]. □

Corollary 1.4.5. If $\pi_2(M - Q_m) = \pi_3(M - Q_m) = 0$ for each $m \geq 0$, then $\pi_2(\tilde{C}_n M) = 0$

Proof. [B+74] The exact homotopy sequence of the fibration $\pi : \tilde{C}_n^m M \to \tilde{C}_1^m M = M - Q_m$ of the previous theorem gives an exact sequence

$$\cdots \to \pi_3\left(M - Q_m\right) \to \pi_2(\tilde{C}_{n-1}^{m+1} M) \to \pi_2(\tilde{C}_n^m M) \to \pi_2\left(M - Q_m\right) \to \cdots .$$

To understand more about this fact, the reader can consult appendix .1. Note that $\pi_2\left(M - Q_m\right) = \pi_3\left(M - Q_m\right) = 0$. Therefore, follows that $\pi_2(\tilde{C}_{n-1}^{m+1})$ and $\pi_2(\tilde{C}_n^m)$ are isomorphic. One can use an inductive argument to show that

$$\pi_2(\tilde{C}_n M) \cong \pi_2(\tilde{C}_1^{n-1} M) = \pi_2\left(M - Q_{n-1}\right) = 0$$

□

Consider the projection map π from $\tilde{C}_n M$ to $\tilde{C}_{n-1} M$. Let $(z_1, \cdots, z_n)$ be a base point for $\pi_1(\tilde{C}_n M)$, and let $\tilde{C}_1^{n-1} M = M - Q_{n-1} = M - \{z_1, \cdots, z_{n-1}\}$. We define the inclusion map j from $\tilde{C}_1^{n-1} M$ to $\tilde{C}_n M$ as follows: for $z_n \in M - \{z_1, \cdots, z_{n-1}\}$, we have $j\left(z_n\right) = (z_1, \cdots, z_{n-1}, z_n)$.

Corollary 1.4.6. Assume that $\pi_2\left(M - Q_m\right) = \pi_3\left(M - Q_m\right) = \pi_0\left(M - Q_m\right) = 1$ for every $m \geq 0$. Then, the following sequence of groups and homomorphisms is exact:

$$1 \longrightarrow \pi_1\left(\tilde{C}_1^{n-1} M, z\right) \xrightarrow{j_*} \pi_1\left(\tilde{C}_n M, (z_1, \cdots, z_n)\right) \xrightarrow{\pi_*} \pi_1\left(\tilde{C}_{n-1} M, (z_1, \cdots, z_{n-1})\right) \longrightarrow 1,$$

where π_* and j_* are the homomorphisms induced by the mappings π and j, respectively.

Proof. By Theorem 1.4.4, the sequence is part of the exact homotopy sequence of the fibration $\pi : \tilde{C}_n^m M \to \tilde{C}_{n-1}^m M$ for every $m \geq 0$. From the previous corollary, we know that $\pi_2(\tilde{C}_{n-1}^{m+1} M) \cong \pi_2(\tilde{C}_n^m M)$ for every $m \geq 0$. Moreover, we have $\pi_2\left(M - Q_m\right) = \pi_3\left(M - Q_m\right) = \pi_0\left(M - Q_m\right) = 1$, which implies that $\pi_2(\tilde{C}_1^{n-1} M) = \pi_2(\tilde{C}_n M) = \pi_0(\tilde{C}_1^{n-1} M) = 1$. Therefore, the exact sequence reduces to the form stated in the corollary. □

1.5 Artin's presentation

The idea of this definition is to think that any braid in the plane $\mathbb{R}^2$ with n-strings can be seen as formed by a finite number of elementary braids $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that σ_i is the geometric braid that crosses the *ith* string over the $(i+1)th$ string and maintains every other string as the identity.

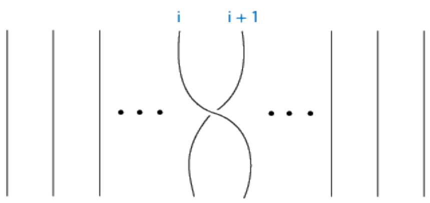


Figure 1.12: *The generator σ_i.*

Also, notice that if one concatenates σ_i with the geometric braid that crosses the $(i+1)th$ string over the *ith* string we will end up with the identity braid, as follows:

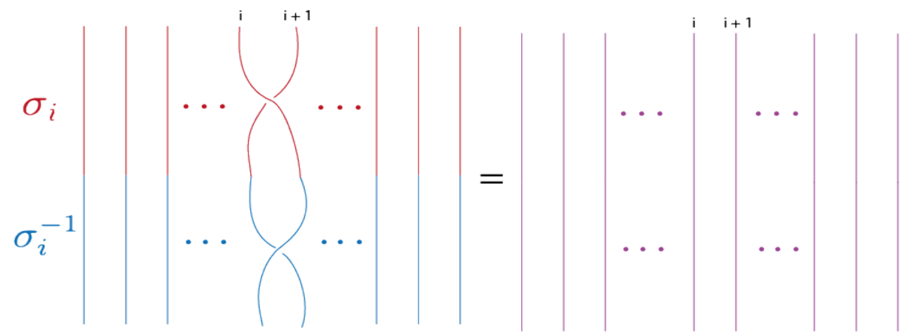


Figure 1.13: *Representation of the composition of σ_i and σ_i^{-1}.*

This classical presentation of braid groups in terms of the generators and relations first appeared in [Art25], as follows:

Definition 1.5.1. The group B_n admits a presentation in terms of generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and defining relations:

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2, i \geq 1, j \leq n-1$

(ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ if $1 \leq i \leq n-2$

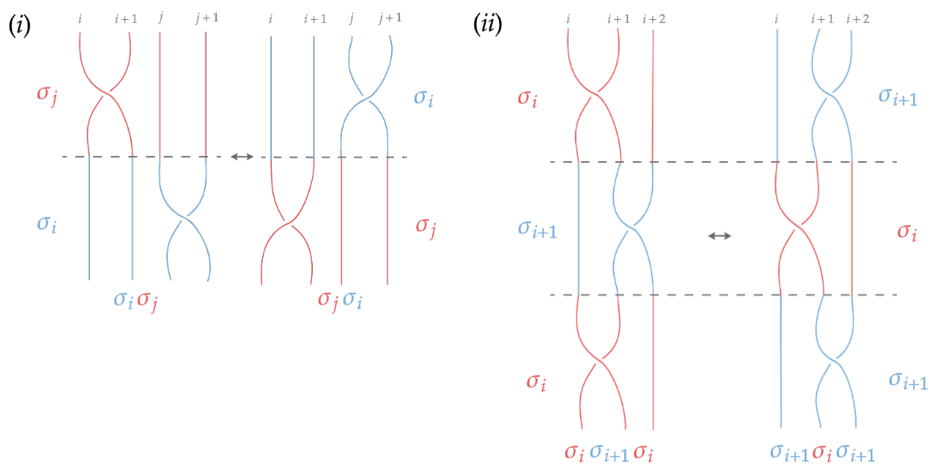


Figure 1.14: *Visual representation of the relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2, i \geq 1, j \leq n-1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ if $1 \leq i \leq n-2$.*

We will prove on the following section how this presentation appears naturally. The first proof given by Artin of the completeness of this presentation is more an indication than a rigorous proof. Other demonstrations have been given by Magnus [Mag34], Bohnenblust [Boh47], Chow [WL48], Fadell and Van Buskirk [FVB+62] and the one we will follow closely given by Fox and Neuwirth [FN62].

Remark 1.5.1. Birman, Ko, and Lee discovered a new presentation for a group, which expanded the set of generators to a more symmetric set. The new generators are denoted by $\sigma_{s,t}$, where $1 \leq s < t \leq n$. We adopt the convention that we will write the smaller subscript first. The defining relations for these generators are as follows:

$$\begin{aligned} \sigma_{s,t}\sigma_{q,r} &= \sigma_{q,r}\sigma_{s,t} \quad \text{if } (t-r)(t-q)(s-r)(s-q) > 0, \\ \sigma_{s,t}\sigma_{r,s} &= \sigma_{r,t}\sigma_{s,t} = \sigma_{r,s}\sigma_{r,t} \text{ if } \quad 1 \leq r < s < t \leq n. \end{aligned}$$

1.5.1 From Configuration Spaces to Artin's presentation

From the configuration spaces definition, we want to show how Artin's presentation appears. The idea of this proof was first given by Fox and Neuwirth in [FN62]. It was also rewritten and appears in [GM11]. We will use cell complexes and the configuration space's definition to prove this fact.

Lemma 1.5.2. Given a cell complex M of dimension n and a subcomplex M' of dimension $n-2$, one can compute a presentation of $\pi_1(M \setminus M')$ in the following way:

(i) Consider a graph Λ having a vertex for each n-cell in $M \setminus M'$ and an edge connecting two vertices for each $(n-1)$-cell next to the corresponding n-cells

(ii) Choose a spanning tree τ in the graph Λ

(iii) Each edge e of $\Lambda \setminus \tau$ (corresponding to a $(n-1)$-cell) will give rise to a generator of $\pi_1(M \setminus M')$. This generator represents a loop that starts at a chosen base vertex, follows τ to one endpoint of e, traverses along e, and then returns to the base vertex along τ.

(iv) For each $(n-2)$-cell in $M \setminus M'$, consider the $(n-1)$-cells adjacent to it. These $(n-1)$-cells form a loop in Λ (up to orientation). Select one of the edges in this loop that does not belong to τ. This selected edge corresponds to one of the generators introduced in step 3. Write down a relation in $\pi_1(M \setminus M')$ by reading the selected edges (generators) in the corresponding order and orientation.

Proof. For further details, consult Example 1.22 in [Hat05]. □

Theorem 1.5.3. The definitions 1.4.4 and 1.5.1 coincide. That means, given the configuration space definition of a braid, one can find and prove a natural presentation given by the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and defining relations:

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2, i \geq 1, j \leq n-1$

(ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ if $1 \leq i \leq n-2$

Proof. We already know that the braid group B_n is the fundamental group of the configuration space, i.e, $B_n = \pi_1(C_n)$. So, the idea here is to construct a n-dimensional cell complex M and M' a subcomplex of M with dimension $n-2$ in the Lemma's condition. In this way, we can compute a presentation of the fundamental group of the cell complex $M \setminus M'$, i.e., $\pi_1(M \setminus M')$ and hence of the corresponding braid group B_n. If we consider a complex number as a pair of real numbers, then $\mathbb{C}^n$ can be ordered by using the lexicographical order. That is, given $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ we will say that $z_1 <_{lex} z_2$ if and only if either:

- $a_1 < a_2$ (and then $z_1 < z_2$)
- $a_1 = a_2$ and $b_1 < b_2$ (then we will say that $z_1 \veebar z_2$)
- $z_1 = z_2$

Now, given a collection of points $\{z_1, \ldots, z_n\}$, they can be totally ordered using the lexicographic order as described above. Hence, we can write $z_1 \square z_2 \square \ldots \square z_n$, where $\square$ can be either $<$, $\veebar$ or $=$. Note, also, that every point in $\mathbb{C}^n/\Sigma_n$ is precisely a family of n unordered points $\{z_1, \ldots, z_n\}$.

Therefore, for each configuration of signs $\theta = (z_1 \square z_2 \square \ldots \square z_n)$ we can associate a cell:

$$C_\theta = \{\{z_1, \ldots, z_n\} \in \mathbb{C}^n/\Sigma_n \text{ such that they are related according to } \theta\}$$

Note that each cell is a configuration of n points that can be geometrically arranged in relation of each other. For example, if we have a cell that is a configuration of four points $\{z_1, z_2, z_3, z_4\}$ such that $z_1 < z_2 < z_3 \veebar z_4$ these are some of the possibilities of how it would look like (See Figure 1.15). Also, intuitively, $z_1 < z_2$ means that z_1 is to the left of z_2 while $z_1 \veebar z_2$ means that z_1 is below z_2.

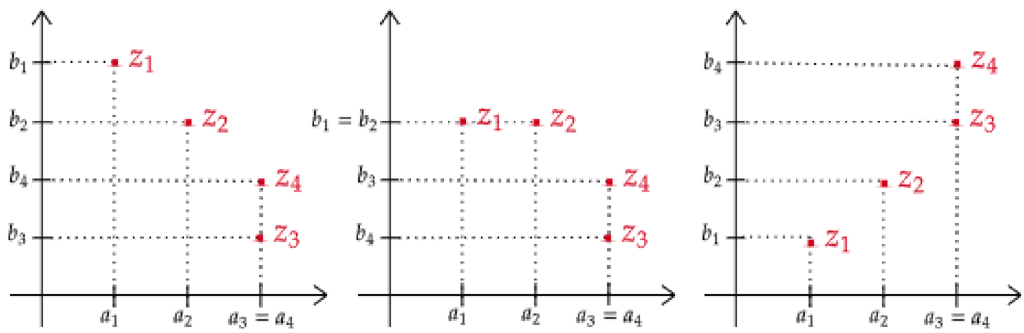


Figure 1.15: *Some possible configurations of points for the cell* $z_1 < z_2 < z_3 \veebar z_4$.

Hence, it is possible to realize that the sets C_θ determines a regular cell decomposition $\mathcal{R}$ of $\mathbb{C}^n/\Sigma_n$, that has real dimension $2n$. (The reader can consult Appendix .2 to see further details about cell complexes). Furthermore, consider the big diagonal $\mathcal{D}/\Sigma_n$. Clearly, it's a subcomplex of $\mathbb{C}^n/\Sigma_n$ and, moreover, $(\mathbb{C}^n/\Sigma_n) \setminus (\mathcal{D}/\Sigma_n) = C_n$, where C_n is the configuration space on n points.

Therefore, since $\pi_1(C_n) = B_n$ we can use the Lemma 1.5.2 to compute a presentation of the braid group. Note that each cell C_θ has real dimension that can be understood as:

$$dim(C_\theta) = 2n - (\#\veebar) - 2(\# =)$$

For example, take $\theta = (z_1 \veebar z_2 < z_3)$. The possible configurations for z_3 are represented in different colors (Figure 1.16). The dimension can be computed as follows:

$$dim(C_\theta) = 2 \cdot 3 - 1 - 2 \cdot 0 = 5$$

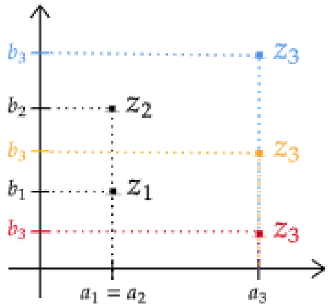


Figure 1.16: *Possible configurations of points for the cell C_θ for $\theta = (z_1 \veebar z_2 < z_3)$.*

Note also those C_θ for which θ involves at least one equality determine a regular cell decomposition of the big diagonal $\mathcal{D}/\Sigma_n$. We can also realize that $dim(C_\theta) = 2n$ if, and only if, $(\#\veebar) = (\# =) = 0$. Hence, the only possible way $dim(C_\theta) = 2n$ is if $\theta = (z_1 < z_2 < \cdots < z_n)$. Therefore, the graph Λ from the Lemma 1.5.2 has only one vertex x, and then $\tau = \{x\}$. We can consider $x = \{1, \ldots, n\}$ the usual base points for braids.

Recall from the lemma that the generators are given by the $(2n-1)$-cells. But note that $dim(C_\theta) = 2n-1$ if, and only if, $(\#\veebar) = 1$ and $(\# =) = 0$. So, the (2n-1)-cells C_{θ_i} will come from $\theta_i = (z_1 < \cdots < z_i \veebar z_{i+1} < \ldots z_n)$ for $i \in \{1, \ldots, n-1\}$. To construct the i-th generator, we will make a loop with base point at x that crosses the C_{θ_i}. This corresponds to moving the points i and $i+1$ in such a way that they swap positions, the i-th one passing bellow the $i+1$-st. (See Figure 1.17) This way we obtain the generator σ_i.

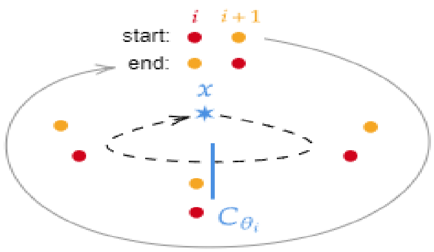


Figure 1.17: *Representation of a loop based at point $x = (1, \ldots, n)$ moving along the $2n$-cell and crossing the $(2n-1)$-cell. This motion represents the generator σ_i.*

Now, to try understanding the relations, following the lemma's result, we will have to look to the $(2n-2)$-cells. Note that $dim(C_\theta) = 2n-2$ if and only if:

- $(\#\veebar) = 0$ and $(\# =) = 1$
- $(\#\veebar) = 2$ and $(\# =) = 0$

However, note that the first case will correspond to a cell decomposition of the big diagonal. So, we will consider only the second one. Also, we will distinguish the second condition into two cases corresponding to the following θ:

(i) $\theta_{i,j} = (z_1 < \cdots < z_i \veebar z_{i+1} < \cdots < z_j \veebar z_{j+1} < \cdots < z_n)$

(ii) $\theta_{i,i+1} = (z_1 < \cdots < z_i \veebar z_{i+1} \veebar z_{i+2} < \cdots < z_n)$

Consider the first scenario. Note that with this conditions, $C_{\theta_{i,j}}$ describes a configuration where two pairs of points, z_i and z_{i+1}, are vertically arranged with z_i below z_{i+1}, and similarly, z_j and z_{j+1} are positioned with z_j below z_{j+1}, while all the other points lie in separate vertical lines. This $(2n-2)$-cell shares an edge with two $(2n-1)$-cells, namely C_{θ_i} and C_{θ_j}, as depicted in Figure 3.2. To make a loop around $C_{\theta_{i,j}}$ one must traverse each adjacent cell twice, once in each direction. Therefore, the corresponding relation is given by $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} = 1$, or equivalently: $\sigma_i\sigma_j = \sigma_j\sigma_i$.

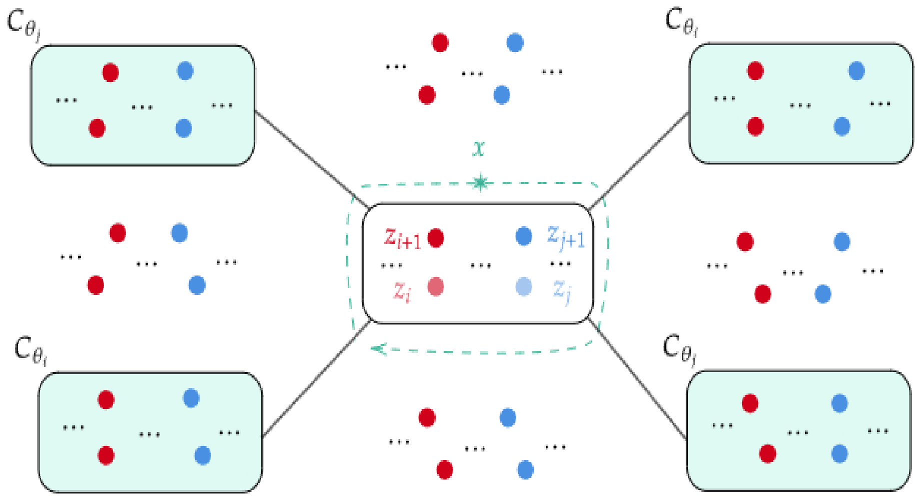


Figure 1.18: *Representation of a loop based at point $x = (1, \ldots, n)$ moving along the $(2n-2)$-cell $C_{\theta_{i,j}}$ and crossing the following $(2n-1)$-cells: C_{θ_i}, C_{θ_j}, C_{θ_i} in the opposite sense and C_{θ_j} in the opposite sense. This motion represents the relation $\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} = 1$.*

The second case corresponds to three points aligned vertically z_i, z_{i+1}, z_{i+2} lying in the $(2n-2)$-cell $C_{\theta_{i,i+1}}$. This cell is adjacent to two $(2n-1)$-cells C_{θ_i} and $C_{\theta_{i+1}}$. In order to do the loop around $C_{\theta_{i,i+1}}$ we must cross each $(2n-1)$-cells three times, with the corresponding orientations, yielding a relation: $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = 1$, or in other words:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

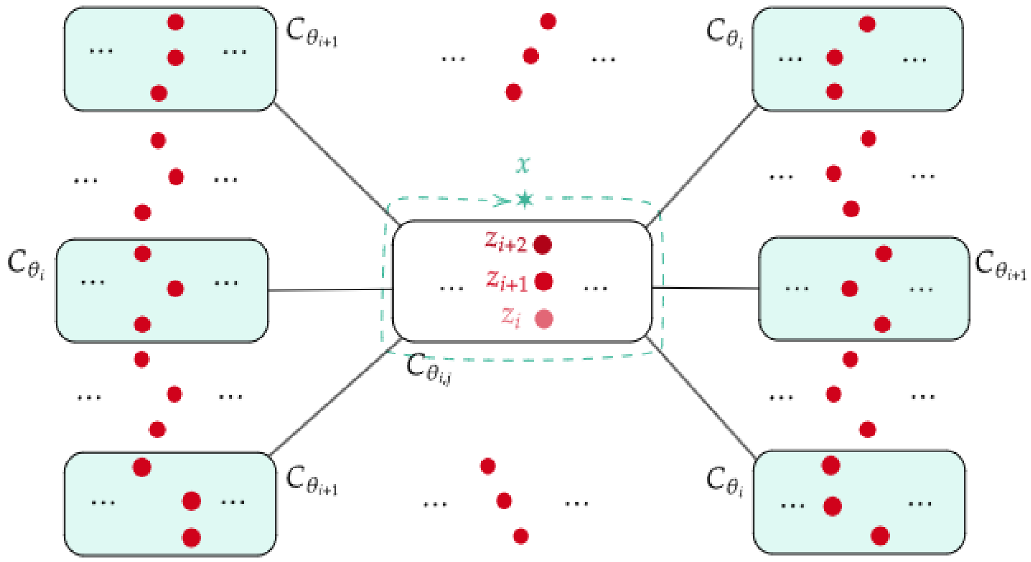


Figure 1.19: *Representation of a loop based at point $x = (1, \ldots, n)$ moving along the $(2n-2)$-cell $C_{\theta_i,i+1}$ and crossing the following $(2n-1)$-cells: C_{θ_i}, $C_{\theta_{i+1}}$, C_{θ_i} again, and now in the opposite sense: $C_{\theta_{i+1}}$, C_{θ_i} and $C_{\theta_{i+1}}$ again. This motion represents the relation $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = 1$.*

This argument given by Fox and Neuwirth gives us not just a presentation for B_n, but also how this relations appears naturally. $\square$

Theorem 1.5.4. The pure n-braid group $\mathbf{P}_n$ has the following presentation,
generators: $A_{j,k}$ $\left(= \sigma_{k-1}\sigma_{k-2}\ldots\sigma_{j+1}\sigma_j^2\sigma_{j+1}^{-1}\ldots\sigma_{k-2}^{-1}\sigma_{k-1}^{-1}\right)$ for $1 \leqslant j < k \leqslant n$
relations:

(A) $A_{r,s} \rightleftharpoons A_{i,j}$ if $1 \leqslant r < s < i < j \leqslant n$ or $1 \leqslant r < i < j < s \leqslant n$

(B) $A_{r,s}A_{r,j}A_{r,s}^{-1} = A_{s,j}^{-1}A_{r,j}A_{s,j}$ if $1 \leqslant r < s < j \leqslant n$

(C) $A_{r,s}A_{s,j}A_{r,s}^{-1} = A_{s,j}^{-1}A_{r,j}^{-1}A_{s,j}A_{r,j}A_{s,j}$ if $1 \leqslant r < s < j \leqslant n$

(D) $A_{i,j}^{-1}A_{s,j}A_{i,j} \rightleftharpoons A_{r,i}$ if $1 \leqslant r < s < i < j \leqslant n$

(D') (Equivalent to (D)) $A_{r,i}A_{s,j}A_{r,i}^{-1} = \left[A_{i,j}^{-1}, A_{r,j}^{-1}\right] A_{s,j} \left[A_{i,j}^{-1}, A_{r,j}^{-1}\right]^{-1}$ if $1 \leqslant r < s < i < j \leqslant n$

Proof. The reader is referred to [MK12] and [B$^+$74] for a complete proof. $\square$

1.6 Garside Structure and the Word Problem

The work of Garside [Gar69] reveals a series of algebraic features of braid groups. This allowed him to solve the word problem in a different way and the conjugacy problem for the first time. We will now introduce the main ideas of his work.

First, notice that the relations in Artin's presentation of the braid group only involves positive powers of the generators σ_i. Hence, one can consider a monoid B_n^+ determined by that presentation. The elements in B_n^+ are words in $\sigma_1, \ldots, \sigma_{n-1}$ (but not their inverses). We will establish the following equivalence:

Definition 1.6.1. Consider two words in B_n^+. We will say that two words are equivalent if and only if one can be obtained from the other by replacing:

- subwords of the form $\sigma_i\sigma_j$ ($|i-j|>1$) by $\sigma_j\sigma_i$
- subwords of the form $\sigma_i\sigma_j\sigma_i$ ($|i-j|=1$) by $\sigma_j\sigma_i\sigma_j$

Moreover, Meneses [GM11] introduced a partial order that helps understanding the results presented by Garside. Let $a,b \in B_n^+$, we say that $a \preccurlyeq b$ if $ac=b$ for some $c \in B_n^+$. We say that a is a *prefix* of b.

One intriguing property of $\preccurlyeq$ is that it is invariant under left multiplication. That is, if $a \preccurlyeq b$, then for any $x \in B_n^+$, we have $xa \preccurlyeq xb$.

A natural question arises: Are there unique greatest common divisors and least common multiples with respect to this partial order? In other words, for $a,b \in B_n^+$, does there exist a unique $d \in B_n^+$ satisfying $d \preccurlyeq a$, $d \preccurlyeq b$, and $d' \preccurlyeq d$ for every d' that is a common prefix of a and b? Similarly, is there a unique $m \in B_n^+$ such that $a \preccurlyeq m$, $b \preccurlyeq m$, and $m \preccurlyeq m'$ for every m' that shares a and b as prefixes? If such unique elements exist, we will denote them by $d = a \wedge b$ (greatest common divisor) and $m = a \vee b$ (least common multiple), respectively. Notably, $xd = xa \wedge xb$ and $xm = xa \vee xb$ for every $x \in B_n^+$.

Additionally, one can define a suffix order $\succcurlyeq$ that is invariant under right multiplication.

Garside's work emphasizes the existence of least common multiples for σ_i and σ_j in B_n^+, which can be expressed as follows:

$$\sigma_i \vee \sigma_j = \begin{cases} \sigma_i\sigma_j & \text{if } |i-j|>1, \\ \sigma_i\sigma_j\sigma_i & \text{if } |i-j|=1. \end{cases}$$

He also demonstrates the cancellativity of B_n^+, meaning that if $xay = xby$, then $a=b$ for all $a,b,x,y \in B_n^+$.

As the relations in B_n's presentation are homogeneous, equivalent words in B_n^+ have the same length, allowing for a well-defined length in B_n^+. Moreover, induction on this length, combined with the cancellativity condition and the above results allows us to show that each pair of elements in B_n^+ possesses unique least common multiples and greatest common divisors.

Garside's investigation centers on a special element Δ, which can be expressed as:

$$\Delta_{n-1} = \sigma_1\,(\sigma_2\sigma_1)\cdots(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)$$

With elementary arguments, he demonstrates that $\Delta = \sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_{n-1}$ in B_n^+ and that $\sigma_i\Delta = \Delta\sigma_{n-i}$ for $i=1,\ldots,n-1$. Consequently, it follows that $\sigma_1,\ldots,\sigma_{n-1}$ are also suffixes of Δ. Furthermore, Δ^2 commutes with every element in B_n^+, and by employing induction on the length, it can be deduced that for every $a \in B_n^+$, there exist non-negative integers m and m' such that $a \preccurlyeq \Delta^m$ and $\Delta^{m'} \succcurlyeq a$.

These findings hold significant implications. Due to the existence of a common multiple (a power of Δ) for any two elements in B_n^+ and the cancellativity of B_n^+, we have that B_n^+ can be embedded in its group of fractions. This group of fractions, as defined by the presentation, precisely corresponds to B_n. Therefore, B_n^+ is not solely an algebraically defined monoid; rather, it constitutes the subset of B_n comprising braids expressible as words involving positive powers of the generators.

Definition 1.6.2 (Positive braids)**.** Braids that can be written as words on positive powers of generators σ_i are referred to as *positive braids*, and B_n^+ is known as the monoid of positive braids.

As a consequence of these properties, the partial order $\preccurlyeq$ (and similarly $\succcurlyeq$) can be extended to encompass B_n. Specifically, $a \preccurlyeq b$ (or $b \succcurlyeq a$) if and only if $ac = b$ (or $b = ca$) for some $c \in B_n^+$. This extension results in a partial order that remains invariant under left-multiplication (or right-multiplication) and admits unique least common multiples and greatest common divisors.

The established structure allows us to unveil numerous desirable properties of braid groups, such as the word problem. In other words, given any two braids $\beta_1, \beta_2 \in B_n$, we want to find a method to decide whether or not $\beta_1 = \beta_2$. We can re-estate the problem by deciding whether or not $\beta = Id$, since if $\beta_1 = \beta_2$ we have $\beta_1\beta_2^{-1} = 1$.

Recall that $\sigma_i \preccurlyeq \Delta$ for every $i = 1, \ldots, n-1$. Put differently, $\Delta = \sigma_i x_i$, for some $x_i \in B_n^+$. Let β be a braid written as a word in $\sigma_1, \ldots, \sigma_{n-1}$ and their inverses. Note that since $\Delta = \sigma_i x_i$, we have:

$$\sigma_i^{-1}\Delta = \sigma_i^{-1}\sigma_i x_i \implies \sigma_i^{-1}\Delta = x_i \implies \sigma_i^{-1}\Delta\Delta^{-1} = x_i\Delta^{-1} \implies \sigma_i^{-1} = x_i\Delta^{-1}$$

Therefore, we can replace every appearance of σ_i^{-1} by $x_i\Delta^{-1}$. Moreover, with more computations, one can show that conjugating every positive braid with Δ^{-1} gives a positive braid. Hence, we can move all appearances of Δ^{-1} to the left and this shows that every braid can be written as $\Delta^p\alpha$ for some $p \in \mathbb{Z}$ and some $\alpha \in B_n^+$. Additionally, if $\Delta \preccurlyeq \alpha$, we can replace Δ^p with Δ^{p+1} and α with $\Delta^{-1}\alpha$. However, this process decreases the length of α, and it can only be done a finite number of times. Consequently, each braid can be uniquely represented as $\Delta^p\alpha$, where $p \in \mathbb{Z}$, $\alpha \in B_n^+$, and $\Delta \not\preccurlyeq \alpha$.

This normal form allows us to solve the word problem by enumerating all positive words representing the positive braid A, applying the braid relations iteratively in every possible way. Although Garside's solution is somewhat inefficient, Elrifai and Morton [EM94] improved it by defining the left normal form of a braid. The left normal form of a braid is given by $\Delta^p a_1 \cdots a_r$, where a_i is a positive proper prefix of Δ (i.e., $1 \prec a_i \prec \Delta$), and $(a_i a_{i+1}) \wedge \Delta = a_i$ for $i = 1, \ldots, r$. This left normal form is unique and provides a decomposition of a braid as a product of a power of Δ and a sequence of proper simple elements (positive prefixes of Δ), known as permutation braids.

Theorem 1.6.1. Braid groups' are torsion-free

Proof. Suppose $x \in B_n$ such that $x^n = 1$ for some $n > 0$. We can consider the element $d = 1 \wedge x \wedge \cdots \wedge x^{n-1}$. Then, $xd = x\left(1 \wedge x \wedge \cdots \wedge x^{n-1}\right) = x \wedge x^2 \cdots \wedge x^{n-1} \wedge 1 = d$. Cancelling d, we get $x = 1$. This demonstrates that braid groups are torsion-free. □

Chapter 2

Knots and Links

In this chapter we will first explore the intuition and first definitions of knots both their polygonal and smooth presentations. Moreover, we will delve into how to define the notion of equivalence obtaining the Ambient Isotopy Equivalence notion and Reidemeister Moves.

Furthermore, we will define operations with knots such as composition, reflection, reverse and see how to obtain a group structure from this operations. Then, we will look closely to the notion of links, that consist of a finite disjoint union of knots K_i, called the components of the link. Since a big part of knot theory consists in studying how to determine whether too given links are or not the same, we will study some knot invariants such as: multiplicity, the minimum crossing points, the bridge number, unknotting number and the colouring number.

Finally, we will end the chapter studying the algorithm of how to obtain a Seifert Surface from a given knot. This notion will help with the proofs from the next chapter. The main references for this chapter are: [Bud12], [MK96], [Cro04].

2.1 Knots definition

Just like the braids can arise from an intuitive point of view, we can do the same for knots. If you simply take a piece of string and do a simple knot (in the most intuitive sense) and glue both ends, you will end up with the first non-trivial knot, the trefoil knot (See Figure 2.1). By non-trivial we mean that even if you pull or stretch all you want, you can never untangle the knot and come back to the unknot (that is, what you obtain from an untangled piece of string if you glue the ends together, as in top left of Figure 2.2).

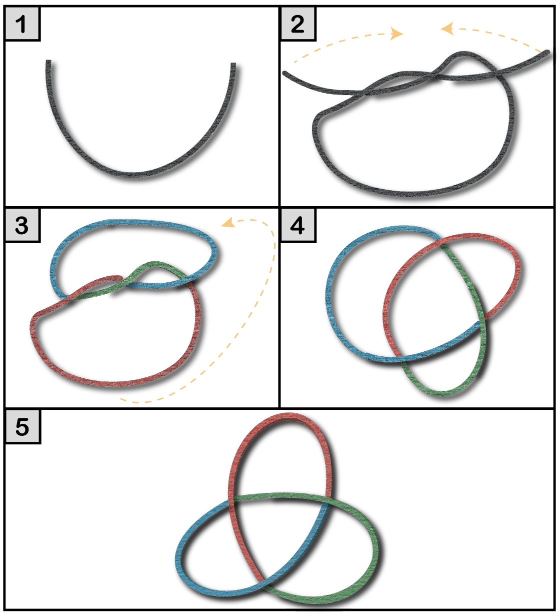


Figure 2.1: *Representation of how to make a trefoil knot with a piece of string.*

It's easy to realize that the same knot can be depicted in different ways, that can come from looking through different perspectives or even changing the rope a little . Each of this "pictures" is called a *projection* of a given knot K (we will formalize this notions ahead in the text). Mathematicians use the projection of a knot to identify them, representing with a tiny space the intersections that go under. These places are called the *crossings* of the knot.

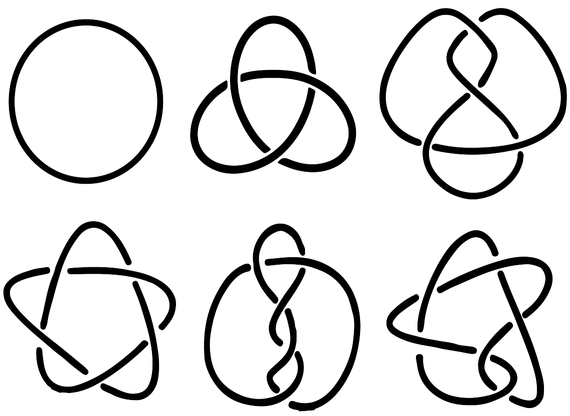


Figure 2.2: *Projections of the trivial knot and the five first non trivial knots.*

Let's try formalizing a little our notion on knots, giving our first definition:

Definition 2.1.1. A *knot* $K \subset \mathbb{R}^3$ is a subset of points homeomorphic to a circle. Alternatively, a knot is an embedding of the circle S^1 into three-dimensional Euclidean space

$\mathbb{R}^3$. We also sometimes consider knots as embeddings of S^1 into S^3, the 3-sphere seen as the one-point compactification of $\mathbb{R}^3$ (since it is equivalent to $\mathbb{R}^3$ with a single point added at ∞).

Note that this definition doesn't specify how the circle should be arranged in space, although until now we were representing as a smooth curve. However, one might represent is as a collection of polygonal curves.

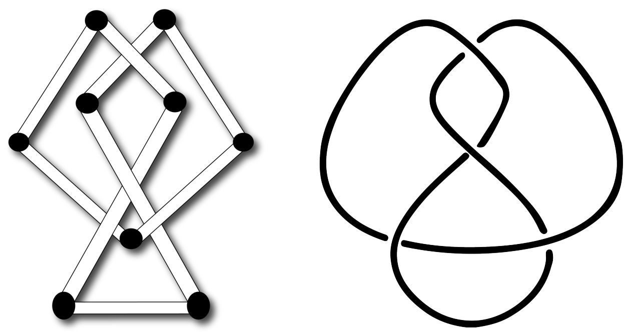


Figure 2.3: *A polygonal and a smooth presentation of the figure eight knot.*

Thinking of a knot with a polygonal representation we can formalize it as a finite set of straight line segments in $\mathbb{R}^3$ which intersect only at their endpoints. We'll call the lines *edges* and their endpoints *vertices*. Note that exactly two edges meet at every vertex. While using this approach, some questions may arise:

- Can every knot be represented by a polygonal arrangement?
- Suppose a knot K can be represented as a polygon. What can one say about the minimum number of edges and vertices?
- Given a number n of edges, what knots can we produce?

The first two questions will be explored when we give a more formal definition of a knot ahead in this chapter. The last question can't be answered in general, since the number of cases explodes as n increases. However, we can answer it for small cases. For example, to make the trivial knot we need at least 3 edges. Moreover, it can also be proved that a knotted polygon must have at least six edges.

Another path to specify particular knots is through smooth functions $f : \mathbb{R} \longrightarrow \mathbb{R}^3$. Recall that a smooth function can be differentiated k times and the kth derivative is continuous for some $k > 0$.

Example 2.1.1. We can represent the trivial knot through the image of following function that maps the real line onto the circle with radius 1 in the xy-plane.

$$f : \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$t \mapsto (sin(t), cos(t), 0)$$

Example 2.1.2. Another example is the trefoil knot, that can be given by:

$$f : \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$t \mapsto (sin(t) + 2sin(2t), cos(t) - 2cos(2t), -sin(3t))$$

In general, to represent a knot as the image of a function we need f in which the function coordinates $x(t), y(t), z(t)$ are smooth and f is periodic.

$$f : \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$t \mapsto (x(t), y(t), z(t))$$

Let $p : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the mapping that projects the point P(x, y, z) in $\mathbf{R}^3$ onto the point $\tilde{\mathrm{P}}(x, y, 0)$ in the xy-plane.

For a knot (or link) K, we define $p(\mathrm{K}) = \tilde{\mathrm{K}}$ as the projection of K. If K has an assigned orientation, $\tilde{\mathrm{K}}$ inherits its orientation naturally. However, $\tilde{\mathrm{K}}$ is not a simple closed curve on the plane; it has multiple points of intersection.

Definition 2.1.2 (Regular projection)**.** Given a knot K, a *regular projection* is $\tilde{K} = p(K)$ such that, $p : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is a map that satisfies:

(i) $\tilde{\mathrm{K}}$ has a finite number of intersection points at most.

(ii) If Q is an intersection point of $\tilde{K}$, then the inverse image $p^{-1}(\mathrm{Q}) \cap \mathrm{K}$ of Q in K consists of exactly two points. That is, Q is a double point of $\tilde{\mathrm{K}}$.

(iii) A vertex of K (considered as a polygon) is never mapped onto a double point of $\tilde{\mathrm{K}}$.(See Figures 2.4, 2.5)

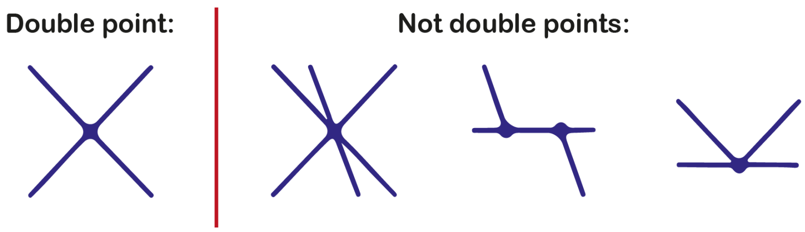


Figure 2.4: *The idea of a double point*

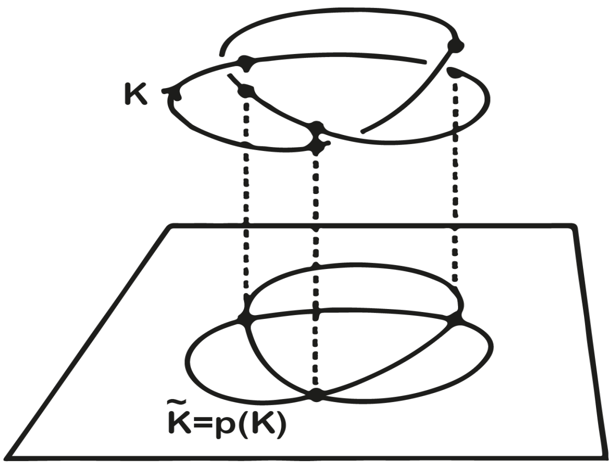


Figure 2.5: *The image of a knot K under p*

Definition 2.1.3 (Regular diagram)**.** A regular diagram D of a knot K is a modification of a regular projection that eliminates the ambiguity at double points. In a regular projection, it's unclear whether the knot passes over or under itself at these double points. To address this ambiguity, we make a slight adjustment to the projection near the double points, effectively creating the illusion of a continuous knot smoothly passing over and under itself. This modified projection is known as a regular diagram.

Definition 2.1.4 (Arcs of a diagram D)**.** Let D be a regular diagram. The arcs of a diagram D are the path components of D.

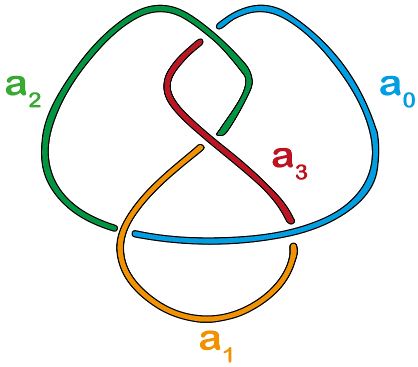


Figure 2.6: *The arcs of the diagram of the figure eight knot*

2.2 Knot Equivalence

A big part of knot theory is to find tools to determine whether two knots are the same or not. In other words, whether they are equivalent.

Intuitively, two knots are equivalent if they can be continuously deformed into the other. So it's natural we would try a definition that explicit this relation. However, we have to be very careful when trying to formalize this definition that involves sliding one knot until it becomes the other. A natural idea would be using the notion of homotopy for doing it (See Appendix .3 for more details on homotopy). That means one could try defining for two knots $K_1, K_2 \subset \mathbb{R}^3$ a continuous map $H : X \times [0,1] \longrightarrow \mathbb{R}^3$ such that $H(x,0) = K_1$ and $H(x,1) = K_2$. In spite of that, doing it would give us a continuous family of knots that can "pull the knot tight" but it would always end up with the trivial knot (See Figure 2.7).

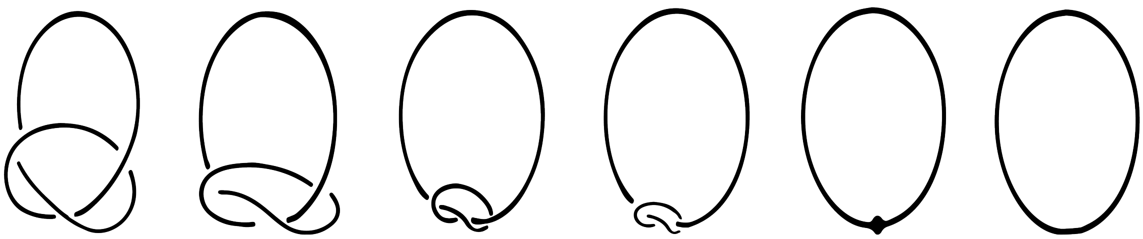


Figure 2.7: *A continuous transformation that makes the knot vanishing.*

Moreover, homotopies allow a curve to pass through itself, which also implies that any knot is homotopic to the trivial one. Therefore, homotopies do not give a good notion of deformation for knots. One way to fix this problem is ensuring that each H_t is one-to-one . If H_t satisfies this property for all $t \in [0,1]$ we will say that H is a *isotopy*. Unfortunately, isotopy also aren't the answer we are looking for. This is because, although they prevent the curves to pass through themselves, another problem arise. Imagine you pull the knot very hard so that the knot becomes very small and tight. Doing this, we would inevitably end up with the trivial knot since mathematical threads have no thickness. The good news, however, is that it's easy to fix this problem. To do it, we only have to ask in our definition that the space containing the knot moves continuously along with the knot and does not become irreparably contorted. Formally, this notion can be translated through this definition:

Definition 2.2.1. Let H be an homeomorphism of $\mathbb{R}^3$. We say that H is *isotopic to the identity* if there is an homotopy $H' : \mathbb{R}^3 \times I \longrightarrow \mathbb{R}^3$ such that each $H'_t : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is an homeomorphism, H'_0 is the identity and $H'_1 = H$.

Now, we can state a definition of equivalence that explicit the deformation of a knot into another:

Definition 2.2.2 (Ambient Isotopy Equivalence)**.** Let $K_1, K_2 \subset R^3$. If we have a homeomorphism H of $\mathbb{R}^3$ which is equivalent to the identity and $H(K_1) = K_2$ we will say that K_1 and K_2 are said to be *ambient isotopic*. Also, the knots $H_t(K_1)$ provides a continuous family which moves progressively from K_1 to K_2 as t increases for 0 to 1.

It can also be shown that ambient isotopy equivalence is an equivalence relation on knots. That said, we can think of each knot actually as the equivalence class of all knots ambient isotopic to it. In practice, this is what we mean that when talking about knots.

Remark 2.2.1. Now, we can also try answering our first question about polygonal knots. Actually, not all knots are equivalent to a polygonal knot. Those that are are called *tame* knots and those that aren't are called *wild* knots. An example of wild knot is tying infinitely knots in a loop one after another (See Figure 2.8) but their study is outside the scope of our work here.

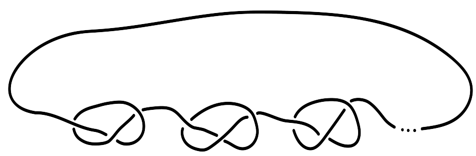


Figure 2.8: *Example of a wild knot.*

Also, a deformation in the knot projection is called **planar isotopy**. This term is due to the fact that we are only deforming the knot within the projection plane. Nonetheless, we can define moves that allows us to go from one projection to another without modifying the knot it represents. Those are called the **Reidemeister Moves**. The first Reidemeister moves allows us to put in or take out a twist from a given string (See Figure 2.9). The second Reidemeister move enable us to add or remove two more crossings to a given string (See Figure 2.10). Finally, the third Reidemeister move allows us to slide a given string under or over some other crossing (See Figure 2.11).

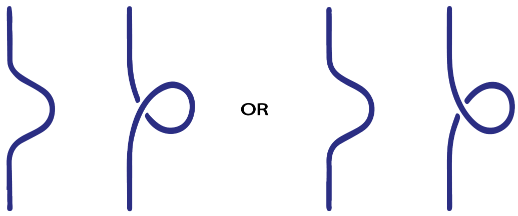


Figure 2.9: *First Reidemeister Move (Type I).*

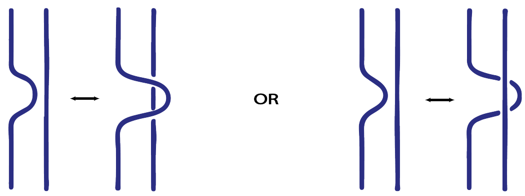


Figure 2.10: *Second Reidemeister Move (Type II).*

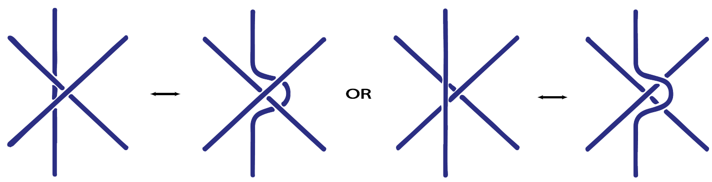


Figure 2.11: *Third Reidemeister Move (Type III).*

Theorem 2.2.2 (Reidemeister's Theorem)**.** Let D, D' be any two diagrams of the same knot or link K. Then there exists a sequence of diagrams $D = D_1 \to D_2 \to \cdots \to D_k = D'$ such that any D_{i+1} in the sequence is obtained from D_i by one of the three Reidemeister moves

Proof. The reader is refereed to [BZK85]. □

2.3 Operating with knots

Composition

Given two projection of knots, K and K' we can remove an arc of each knot (on the outside of the projection to avoid any new crossings) and then connect the four lose ends in a standard way, which does not introduce additional crossings. The resulting knot will be called the **composition** of the other two knots. We will denote the resulting knot by $K \# K'$ (See Figure 2.12).

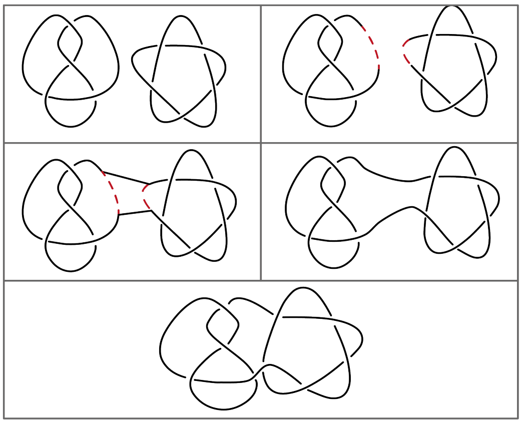


Figure 2.12: *Composition of the figure-8 and the cinquefoil.*

Note that when we composite any knot with the unknot, nothing happens. This observation suggests that the operation of composition admits a neutral element. Also, we call a knot that can be expressed as the composition of two knots a **composite** knot, and the knots that are part of this composition are the **factor** knots. If one knot can't be expressed as the composition of any two nontrivial knots, they will be called **prime** knots. Unfortunately, although all this could suggest an algebraic structure involving knots and their composition, the operation is not very well defined. In fact, given two knots and performing the composition operation in different places can give rise to different knots.

Reflection

A question that may arise at this point is what a knot $K \subset R^3$ and it's image under a homeomorphism $h : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ share. In other words, are K and $h(K)$ ambient isotopic? Indeed, if h is orientation-preserving, the answer is yes.

However, what about the homeomorphisms that are orientation-reversing? To answer this question, let's go back to the construction of the trefoil knot. Note that while constructing it, we made a choice of which side of the string we would put under the other. If we make the other choice, we get the "reflected" image of the knot (we will improve this notion soon). So, one might wonder if it is possible to deform the left-handed trefoil knot into the right-handed trefoil knot.

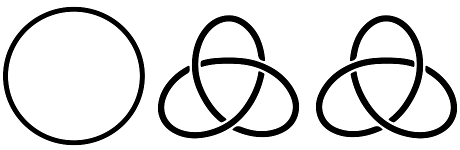


Figure 2.13: *Projection of the trivial knot, the left-handed trefoil knot and the right-handed trefoil knot.*

We can formalize the notion of "reflection" by the following orientation-reversing homeomorphism:

$$\begin{aligned} r : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ (x, y, z) &\mapsto (-x, -y, -z) \end{aligned}$$

Definition 2.3.1. The image of a knot $K \subset \mathbb{R}^3$ under r, $r(K)$ is called the *mirror image* and it's denoted by K^*.

Definition 2.3.2. If K is equivalent to it's mirror image K', we call K *achiral* or *amphichiral*. If this does not happen, we call the knot *chiral* and it occurs in a left-handed or a right handed form

Example 2.3.1. The figure-8 knot is amphichiral since it's equivalent to it's mirror image. One can rotate 90º the mirror image and note you end up with exactly the same knot.

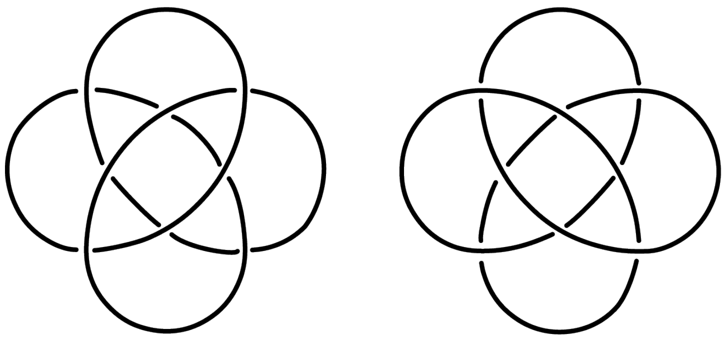


Figure 2.14: *An embedding of the figure-8 knot and it mirror image.*

Remark 2.3.1. Let K be an achiral knot. The reflection of K independs of the choise of r, i.e., K will be ambient isotopic to $r(K)$ under any other orientaion-reversing homeomorphism.

Reverse

Another possibility that knots gives us is orienting them. Given a knot, we have two possible ways of orienting them. Hence, if K is an oriented knot, one can define the *reverse* of K that have the opposite orientation of K and will be denoted by $-K$. (see Figure 2.15 for an example [Cro04])

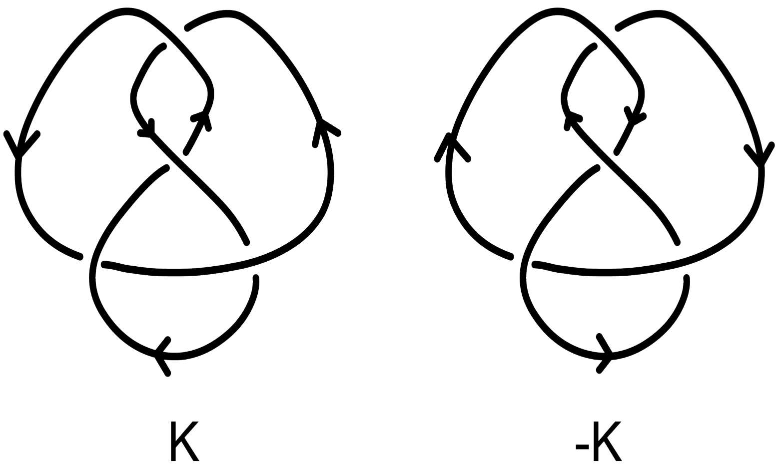


Figure 2.15: *An embedding of the figure-8 knot and it's reverse.*

Definition 2.3.3. At a crossing point, c, of an oriented regular diagram we have two possible configurations considering the orientation of a crossing point. In case (a) we assign sign(c) $= +1$ to the crossing point (and its said to be positive), while in case (b) we assign sign(c) $= -1$ (and its said to be negative).

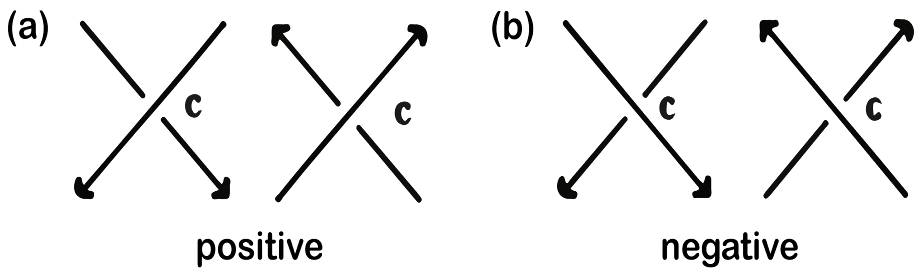


Figure 2.16: *Signs of a given crossing.*

Group Structure

Let r be the operation that transforms K into $-K$, and m be the operation that transforms K into it's mirror image K^*. Moreover, let $i = rm$ be the composite operation that maps K to $-K^*$ (The knot $-K^*$ is known as the inverse of K.). Due to the properties $-(-K) = K$, $(K^*)^* = K$, and $-(K^*) = (-K)^*$, the operations m, r, and i form a group.

	1	m	r	i
1	1	m	r	i
m	m	1	i	r
r	r	i	1	m
i	i	r	m	1

The composition table presented above displays that any oriented knot with two non-trivial symmetries must also possess the third. Consequently, there are five possible topological symmetry types for oriented knots:

$K = -K = K^* = -K^*$	fully symmetric
$K = -K$	reversible
$K = K^*$	+ amphichiral
$K = -K^*$	- amphichiral
none of the above	asymmetric

2.4 Links

Until now, we have focused our attention in studying knots. However, we can generalize the notion of knots by tanking not a single embedding of S^1, but as many as we want.

Definition 2.4.1 (Link)**.** A *link* L is a finite disjoint union of knots K_i, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$, i.e, $L = K_1 \cup \cdots \cup K_n$. Each knot K_i is called *component* of the link and n is called the *multiplicity* of L.

Remark 2.4.1. A *trivial* link of multiplicity n is a collection of n unlinked trivial knots.

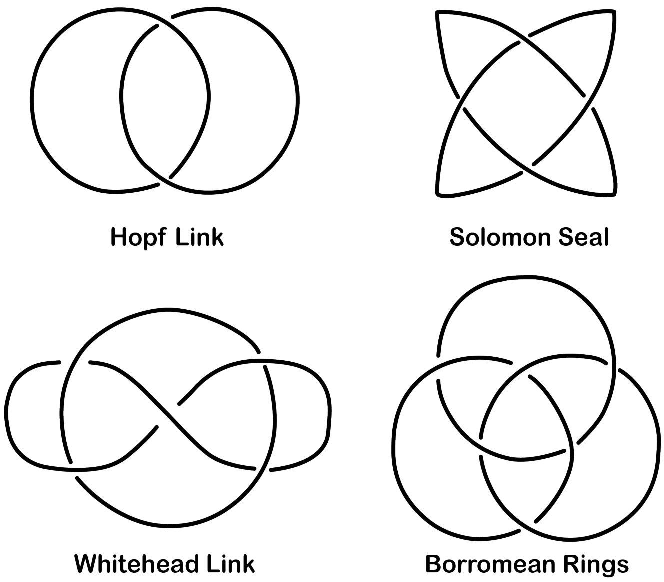


Figure 2.17: *Some examples of simple links.*

Links can also be oriented, which means that each component is assigned an orientation, in the way described for the knots.

Definition 2.4.2 (Weak Link Equivalence)**.** Two links L_1 and L_2 are ambient isotopic if there exists an isotopy $h : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ such that $h(L_1, 0) = h_0(L_1) = L_1$ and $h(L_1, 1) = h_1(L_1) = L_2$.

Note that there is a free choice of how to match up the components of L_1 with those of L_2. Hence, this definition of equivalence is considered weak as it does not impose any restrictions on the isotopy; When we represent a link as $K_1 \cup \cdots \cup K_n$, its components are given arbitrary labels $1, \ldots, n$. The components of a link may also be oriented. A stronger definition of equivalence requires the isotopy to preserve any label or orientations on the links.

All the operations we defined for knots can be generalized for links. When we delve deeper into the symmetries of oriented knots, we found that there were five possible symmetry classes. However, with links, the situation becomes even more intricate. Consider an oriented n-component link $L = K_1 \cup \cdots \cup K_n$. We can represent the link formed from L by reversing the orientations of some components as $\pm K_1 \cup \cdots \cup \pm K_n$, where a - sign indicates a reversal of orientation and a + sign denotes an unchanged orientation. We can also write L followed by the sequence of + and - signs, generating 2^n possible links.

2.5 Invariants

Definition 2.5.1 (Link Invariant)**.** A *link invariant* is a function from the set of links to any other set in such a way that this calculation depends only of the equivalence class the link belongs. In other words, a link invariant is a function from the set of all links $\mathcal{L}$ to another set S, $f : \mathcal{L} \longrightarrow S$ such that:

$$L \sim L' \implies f(L) = f(L')$$

Multiplicity

The simplest link invariant is the multiplicity of the link, i.e., the number of it's components.

The minimum crossing points

Another intuitive association of knots with an integer number would be the amount of crossing points in a given diagram. However, this does not configures an invariant, since the same knot can have diagrams with less or more crossing points. This is because one can perform a Reidemeister move and change the amount of crossing points (See Figure 2.18)

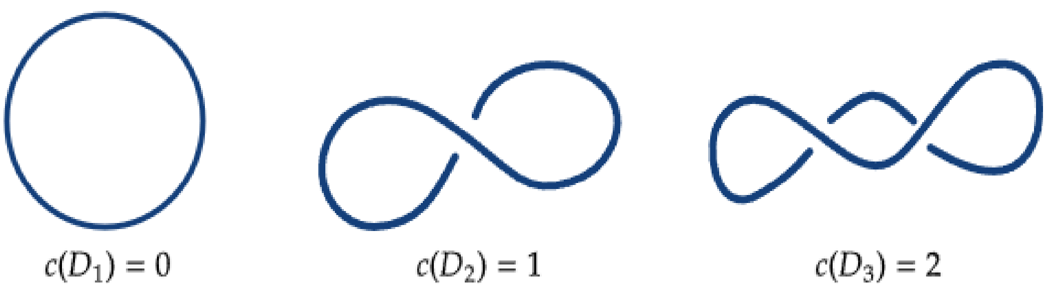


Figure 2.18: *Examples of different diagrams with distinct crossing numbers of the same knot.*

Nevertheless, consider $\mathcal{D}_{\mathcal{L}}$ the set of all diagrams of a given link L. Then, we have:

Proposition 2.5.1. The minimum crossing points of a link L given by

$$\mathrm{c}(L) = \min_{\mathcal{D}_{\mathcal{L}}} \mathrm{c}(\mathrm{D})$$

is a link invariant.

Proof. Let's assume that D stands as the minimum regular diagram of L, i.e, a diagram that haves crossing points equal to the minimum crossing points of a link L. Suppose L' represents a link equivalent to L, and D' is its minimum regular diagram. Note that $\mathrm{c}\,(\mathrm{D}) \leq c\,(D')$ since D′ is the regular diagram for L and L and L′ are equivalent. However, since D qualifies as a regular diagram of L', again by definition, $c\,(D') \leq c\,(D)$. Thus, combining these two inequalities, we conclude $c\,(D) = c\,(D')$, implying $c\,(D)$ remains the minimum number of crossing points for all links equivalent to L. This establishes it as a link invariant. □

In general, there is no method for computing the minimum crossing points of a given link. However, it's been completely determined for alternating knots (i.e. a knot that can be depicted in a knot diagram where crossings consistently alternate between underpasses and overpasses) and for some specific types of non-alternating knots and links. Moreover, if K_1 and K_2 are alternating knots once can prove that:

$$\mathrm{c}\,(\mathrm{K}_1\#\mathrm{K}_2) = \mathrm{c}\,(\mathrm{K}_1) + \mathrm{c}\,(\mathrm{K}_2)$$

The reader is referred to [MK96] for the proof of the facts above mentioned.

The bridge number

Let D be a diagram of a given link L. We will remove small segments from the diagram that passes over the crossing points such that we end up with a collection of disconnected segments. (See Figure 2.19)

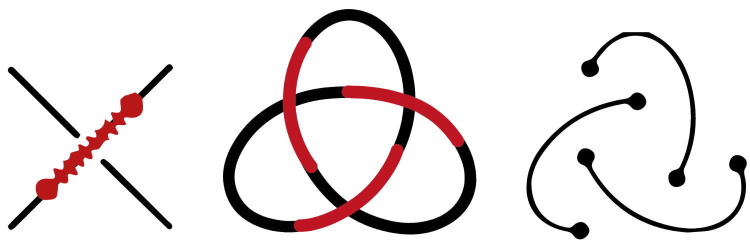


Figure 2.19: *The bridge number of the trefoil knot.*

These segments are called *bridges* and for a given diagram D the number of bridges is called the *bridge number*.

Definition 2.5.2. Let D be a regular diagram of a knot K. If we can partition D into $2n$ polygonal curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ and $\psi_1, \psi_2, \ldots, \psi_n$, such that the following conditions hold, then we say that the bridge number of D, denoted as br(D), is at most n.

(1) The polygonal curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ are mutually disjoint and simple polygonal curves.

(2) The curves $\psi_1, \psi_2, \ldots, \psi_n$ are also mutually disjoint and simple curves.

(3) At the crossing points of D, the segments $\gamma_1, \gamma_2, \ldots, \gamma_n$ pass over the crossing points, while the segments $\psi_1, \psi_2, \ldots, \psi_n$ pass under the crossing points.

In cases where $\mathrm{br(D)} \leq n$ but $\mathrm{br(D)} \nleq n-1$, we define $\mathrm{br(D)} = n$.

Proposition 2.5.2. For a link L, the bridge number

$$\mathrm{br}(L) = \min_{\mathcal{D_L}} \mathrm{br(D)}$$

is an invariant for L and this quantity is called the bridge number (or bridge index) of L

Proof. The proof is analogous of the crossing points proof. □

Unknotting number

Let D be a diagram of a given link L. The unknotting procedure at a crossing consists of exchanging locally the over and under crossing segments on one chosen crossing point until we get a trivial knot (or link).

Proposition 2.5.3. Every diagram D of a given knot (or link) K becomes the unknot after a finite times of performing the unknotting procedure described above (It might also be necessary to perform some Reidemeister moves).

Proof. The proof will proceed by induction on the crossing number of D, $\mathrm{c}(D)$. The initial case, for $\mathrm{c}(D) = 0$ trivially holds, since if $\mathrm{c}(D) = 0$ we have already the trivial knot. Now, suppose the condition holds for all regular diagrams D such that $\mathrm{c}(D) < n$ and we will prove the statement is true for a diagram D with $\mathrm{c}(D) = n$. Let A be an arbitrary point on the diagram that is not on a crossing point. From A we will follow the knot around. If at a crossing point we move along a part that passes over the crossing point, we simply do nothing. However, if we move along a part that passes under the crossing point, we will perform the unknotting operation.

In this manner, we will gradually construct a regular diagram in which, starting from A, we consistently navigate over all knot's crossing points. Continuing our traversal along D and replicating the aforementioned process, we will eventually encounter a crossing point B that we have already traversed (In the case of K being a link, we will return to our initial point A.)

Upon the completion of the procedure, we will have formed a loop that includes the point B. By applying Reidemeister moves, this loop can be eliminated. The resultant new regular diagram, denoted as D′, will possess fewer crossing points than D. Therefore, we can apply the induction hypothesis to D′, thus concluding our proof. □

Similar to our preceding discussions, we define the unknotting number of D as the minimum count of unknotting operations needed to transform D into the regular diagram of the trivial knot (or link). This unknotting number of D will be denoted by u(D). As anticipated, it should be noted that u(D) does not serve as an invariant of K.

As previously discussed, consider all regular diagrams representing K. The minimum number of unknotting operations across all these regular diagrams serves as a knot invariant.

The coloring number of a knot

Let K be a knot and D be it's projection. Suppose P is a crossing point of D and note that locally the diagram is splitted in three arcs a_0, a_1, a_2.

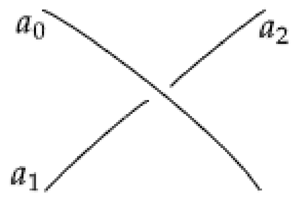


Figure 2.20: *The arcs formed by each crossing on the diagram.*

Definition 2.5.3 (3-colorability)**.** Let K be a knot and D one of its diagrams, with $A = \{a_0, \ldots, a_{n-1}\}$ it's arcs. Then a 3-coloring of K is a function

$$\varphi : A \longrightarrow \{0, 1, 2\}$$

such that in each crossing, the three arcs a_i, a_j, a_k satisfies either:

- $\varphi(a_i) = \varphi(a_j) = \varphi(a_k)$
- $\varphi(a_i), \varphi(a_j)and\varphi(a_k)$ are all distinct.

If $\varphi(a_i) = \varphi(a_j) \;\; \forall i \neq j$ we will call it a trivial coloring. Also, if D admits a non-trivial coloring, we will say that K is $3-$*colorable*.

Each number can be associated with a different color. So, intuitively, for a knot to be 3-colorable we have to color each arc with a color in such a way that as least 2 colors are used and in each crossing all colors have to be equal or all different. (See Figures 2.21 and 2.22)

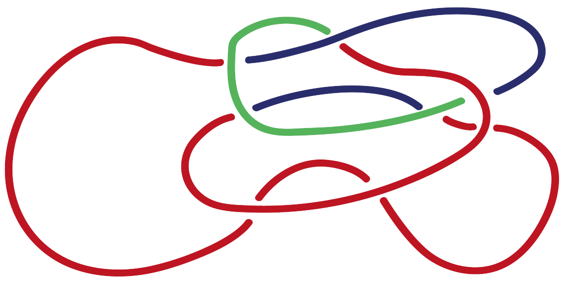


Figure 2.21: *The knot above is 3-colorable*

.

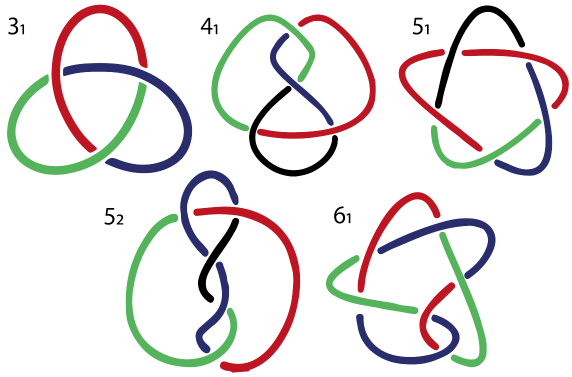


Figure 2.22: *The trefoil (3_1) and 6_1 knots are 3-colorable while 4_1, 5_1, 5_2 are not. Note that in the black strings there are no possible choice of colors we can make so that we still follow the rules*

Proposition 2.5.4. 3-colorability is a knot invariant. In other words, if D is a diagram from a given knot K that is 3-colorable and D' is another diagram of K, then D' is also 3-colorable.

Proof. We want to prove that 3-colorability independs of the choice of diagram representing the knot K. So, because of Reidemeister Theorem, one only have to check that the three types of Reidemeister moves satisfies the 3-colorability conditions. In other words, to prove this proposition it is sufficient to show that each of the regular diagrams obtained after we have performed one of the Reidemeister moves or it's inverses are 3-colorable (See Figure 2.23).

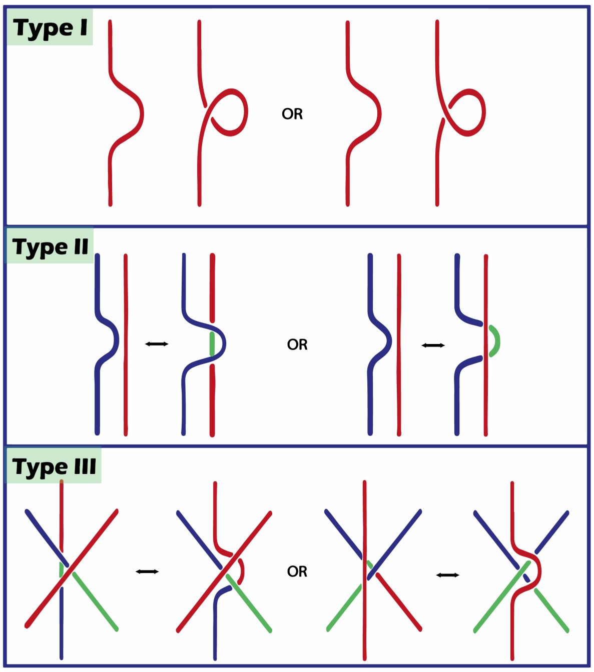


Figure 2.23: *The Reidemeister moves preserves 3-colorability*

□

One might think if we can also define the notion of n-colorability, for an integer n And actually, we can do it, with a slightly difference. One can see further details in [MK96]

Definition 2.5.4 (n-colorability)**.** Let k, n be an integer, K be a knot (or link) and D one of its diagrams, with $A = \{a_0, \ldots, a_{k-1}\}$ it's arcs. Then a n-coloring of K is a function

$$\varphi : A \longrightarrow \{0, 1, \ldots, n-1\}$$

such that at each crossing, if a_r is the label of the strand that is crossing over, and a_s, a_t are the labels of the other two strands, then in each crossing

$$2\varphi(a_r) \equiv \varphi(a_s) + \varphi(a_t) \bmod n$$

If $\varphi(a_i) = 0 \quad \forall i \in \{0, 1, \ldots, n-1\}$ we will call it a trivial coloring. Also, if D admits a non-trivial coloring, we will say that K is $n-$*colorable*.

Note that this definition holds for 3-colorability. If we look closer to Figure 2.22, each crossing satisfies the condition that $2\varphi(a_r) \equiv \varphi(a_s) + \varphi(a_t)$ mod 3.

2.6 Seifert Surfaces

An invariant that plays a big role in the context of knot theory is the Alexander's Polynomial. Although we won't be looking closely to this concept, we will now study part of the initial mechanics one can use to compute it: Seifert Surfaces, since one can define Alexander polynomial via Seifert matrix. This same tool of calculating the Seifert Surface of a given knot will be very important in the proof of Alexander's Theorem in the next chapter.

Theorem 2.6.1. Let K be an arbitrary knot (or link). Then, there exists an orientable and connected surface $S \subset \mathbb{R}^3$ that has as its boundary K.

Proof. Let D be a regular diagram of a given knot (or link) K. We want to decompose the diagram D into several simple closed curves. To do it, we pick an orientation and remove all crossings in the following way:

- draw a circumference centered in every crossing and let A, B, C and D be the intersection with the arcs of the diagram.
- we want to splice the crossing point connecting A to B and C to D (See Figure 2.24)

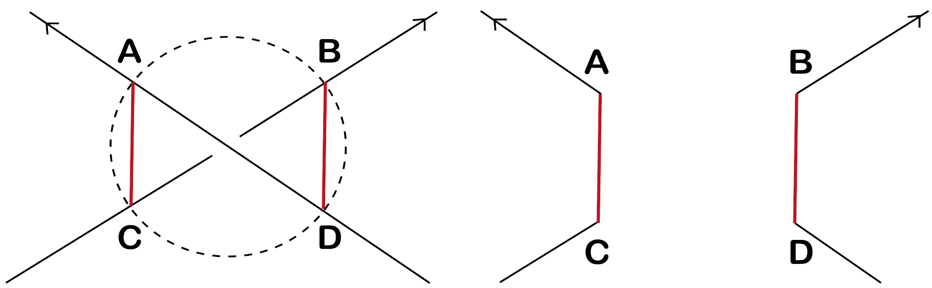


Figure 2.24: *Splicing of a knot K.*

This operation is called splicing of a knot K along its orientation at a crossing point of D. Note that if we perform this operation in every´crossing then we will remove every crossing point of D we will transform the diagram (that will be called Seifert Diagram) into several simple closed curves that may now be spanned by a disk. (See Figure 2.25) This simple closed curves will be called *Seifert Circles*

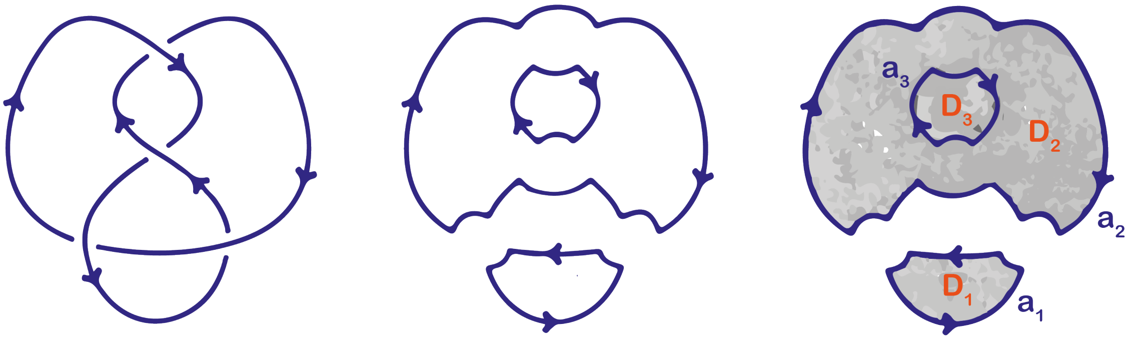


Figure 2.25: *Splicing of the figure-eight knot.*

By performing the splicings we obtain some discs. In order to create the surface, we need to attach a single twisted band between those disks that will lie in different heights. (See Figure 2.26)

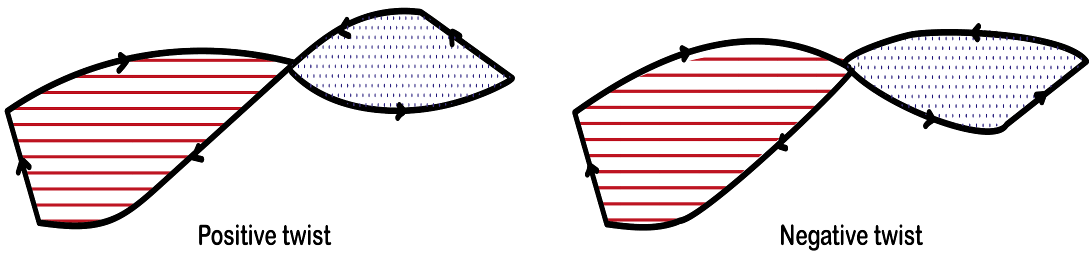


Figure 2.26: *Twisted bands.*

If we attach positive (negative) bands between disks that corresponded to positive (negative) crossing points before they were spliced, then we obtain a connected, orientable surface F. (In the case of a link, K, if we alter K in such a way that the projection of K is connected, then by the above method we can also obtain a connected surface.) The boundary of this surface, F, is plainly the original knot K. Further, it can also be proved that F is also an orientable surface, with orientation induced by the orientation of K. (See Figure 2.27)

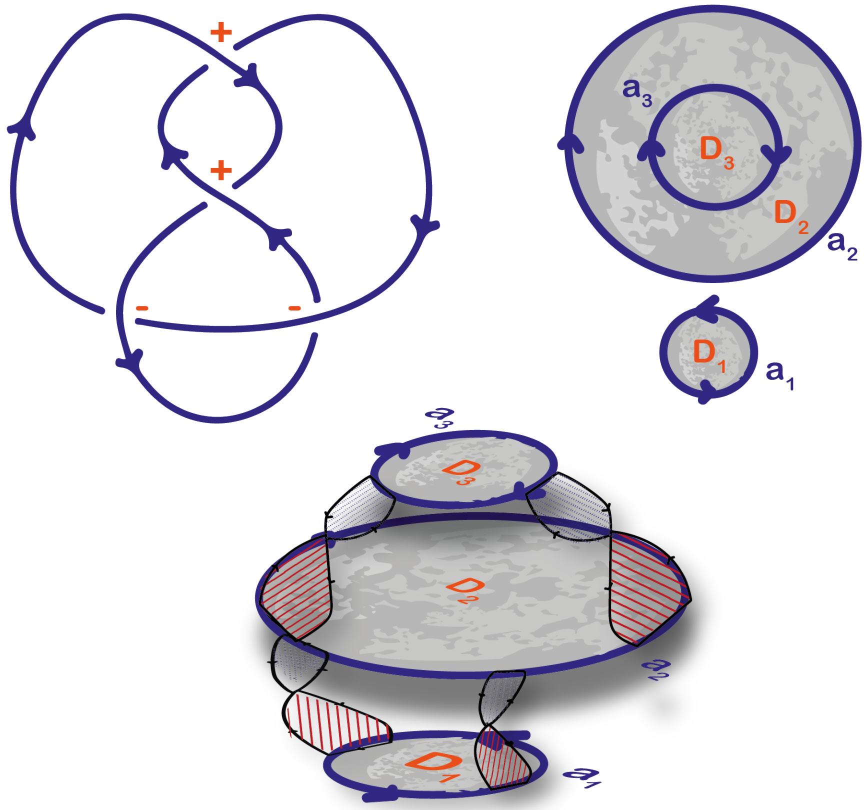


Figure 2.27: *Seifert Surface of the figure-eight knot.*

□

Chapter 3

Relations Between Knots and Braids

In this chapter, we will dive into the fundamental connection between knots and braids. Our exploration begins by elucidating the process of obtaining a knot or a link from a given braid through a process known as 'closing' the braid. This approach leads us to two pivotal questions concerning knots and braids.

One might ask: If is it possible to obtain knots from a given braid, how general is this construction? In other words, is it true that every knot can be obtained from a closed braid? Alexander's theorem answer this question affirmatively. However, it becomes apparent that the correspondence between knots and braids is not one-to-one, as equivalent knots can arise from conjugate braids. This observation naturally leads to the second question: which closed braids represent the same knot type? Markov's theorem answers this question by providing a set of 'moves' that relate any two closed braid representatives of a knot or link while preserving the closed braid structure. The main reference for this chapter is [BB05].

3.1 From Braids to Knots

Let β be a braid on n strings. To obtain a knot or a link, one can simply "close up" the braid as in Figure 3.1. We want to connect P_i to Q_i with paths that doesn't create new crossings. This operation is known as the *closure* of β and will be denoted by $b(\beta)$. The pre-image in $\mathbb{R}^3$ of the 'center point' under the usual projection map is called the *axis* of the braid. Next, we orient the resulting knot or link in a manner such that all the strands of the braid are traveling counterclockwise around the braid axis.

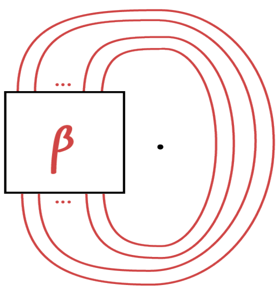


Figure 3.1: *Schematic representation of the operation of closing a braid.*

Remark 3.1.1. If the intention is to examine the knot in S^3, we must incorporate the point at infinity, resulting in the braid axis being an embedded S^1.

Example 3.1.1. The closure of the braid $\beta = \sigma_1\sigma_2$ is the unknot.

Figure 3.2: *The closure of $\beta = \sigma_1\sigma_2$ is trivial.*

Alternatively, let's consider a knot K residing in the three-dimensional sphere S^3. Suppose there exists an embedding $h : S^1 \to S^3$ such that Z is a unknotted within S^3 and lies in the complement of K and $A = h(S^1)$. Additionally, we choose the point at infinity, denoted by $\{\infty\}$, to be included in A. Utilizing standard cylindrical coordinates (ρ, θ, z) on $\mathbb{R}^3$, we identify the resulting subset $\mathbb{R} \cong A - \{\infty\}$ with the z-axis in $\mathbb{R}^3 \cong S^3 - \{\infty\}$. Now, as we travel around the knot K with a suitable cylindrical parametrization, we always ensure that $d\theta/dt > 0$. Under these conditions, we refer to K as a closed braid with respect to the axis A. The closed braid diagram illustrated in Figure 3.1 can then be obtained through projection, parallel to the direction defined by A, onto a plane that is orthogonal to A.

3.2 From Knots to Braids

3.2.1 Alexander's Theorem

While going from braids to knots can be done with a simple procedure, the reverse can be much more complicated. The proof we're going to give, gives an algorithm to find a closed braid that gave rise to a given knot. This proof is originally due to Yamada [Yam87] and was later improved by Vogel [Vog90].

Theorem 3.2.1 (Alexander's Theorem)**.** Every knot or link in S^3 can be represented as a closed braid.

In order to prove Alexander's we will first define some terminology to help in the algorithm and then present the Yamada-Vogel Algorithm to transform a knot into a closed braid.

Definition 3.2.1. Let C_1 and C_2 be two oriented disjoint closed curves in S^2. Then C_1 and C_2 cobound an annulus A. We say that C_1 and C_2 are *coherent* if C_1 and C_2 represent the same element of $H_1(A)$. Otherwise we say that C_1 and C_2 are incoherent.

Remark 3.2.2. Let's interpreter this definition in a more intuitive way. Let C_1, C_2 be two oriented disjoint closed curves in S^2. Then, $[C_1] = [C_2] \in H_1(A)$ if, and only if, $[C_1] - [C_2] = 0 \in H_1(A)$. This means that $[C_1] + [-C_2]$ annihilates in $H_1(A)$. Hence, we have that $[C_1] + [-C_2]$ is boundary of some $2-$cell (or sum of 2-cells). So, consider an arbitrary 2-cell with the boundary oriented compatibly given by $\delta, \alpha, \beta, \gamma$. Note that α and γ cancel each other, so the boundary of this 2-cell is $[\delta] + [\beta]$ (See Figure 3.3). Therefore, in order to $[C_1] + [-C_2] = 0$ we need C_1 and C_2 oriented as δ and β, respectively.

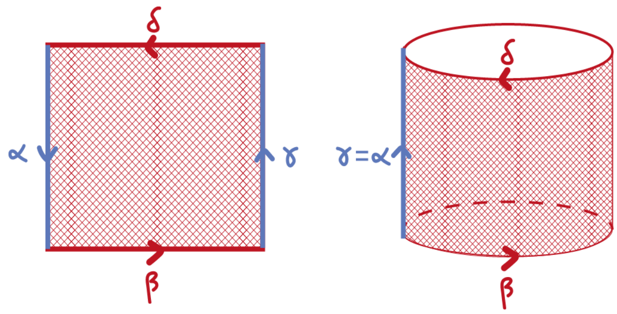


Figure 3.3: *Compatibly oriented 2-cell.*

In other words, $[C_1] + [-C_2] = 0$ if C_1 and $-C_2$ have reversed orientations, and hence, if C_1 and C_2 share the same orientation (or if C_1 and C_2 "point in the same direction" - see Figure 3.4). A visualization that may help determine whether two oriented disjoint curves are coherent is the following: imagine these curves are immersed in a liquid and the orientation of the curve determines the direction of movement in this liquid. If the liquid flows in an organized way, the curves are coherently oriented. Otherwise, they are not coherent.

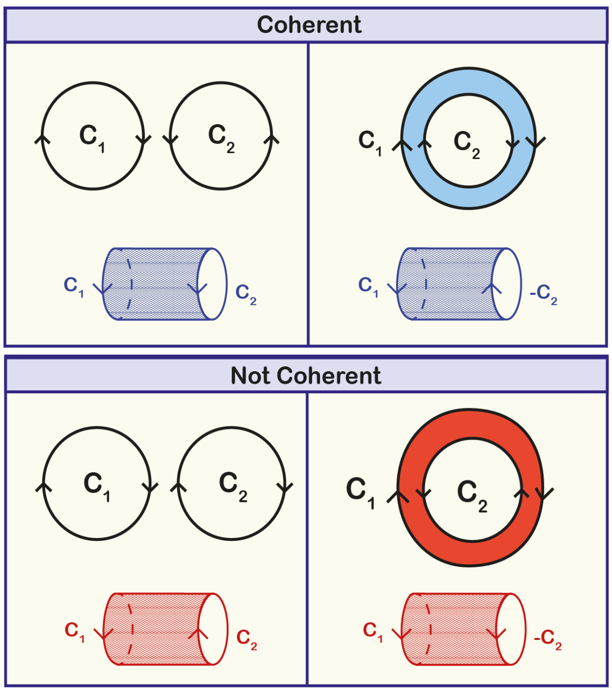


Figure 3.4: *Examples of coherent and not coherent closed curves.*

Definition 3.2.2. The *height* $h(D)$ of a Seifert Diagram D is is the number of incoherent oriented pairs of Seifert circles. If $h(D) = 0$ we will say that the diagram is in the *closed braid form*.

Definition 3.2.3 (Change of infinity)**.** The procedure of *change of infinity* at the knot level, consists of flipping an edge around to the other side of the knot. It can be performed on any edge which borders the outside region of the knot.

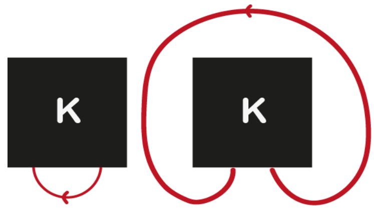


Figure 3.5: *Change of infinity performed at the knot level.*

At the Seifert circles level, we can imagine them lying on the sphere and changing the perspective we are looking at them.

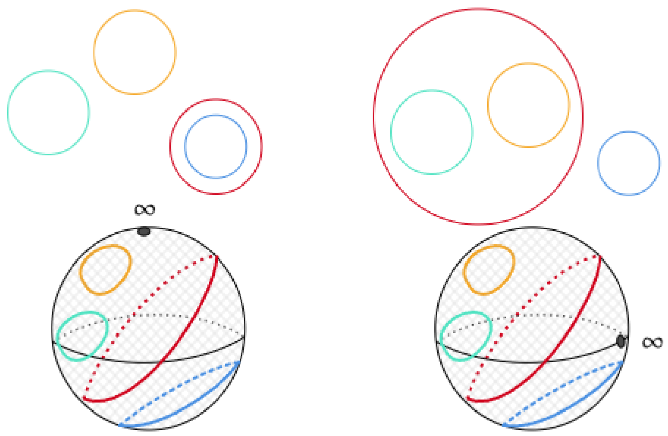


Figure 3.6: *Change of infinity performed at the Seifert circles level.*

Yamada-Vogel Algorithm

1. The algorithm starts in the same way of the Seifert algorithm for constructing a surface from a given knot (See section 2.6). So, given a diagram D, we will obtain a collection of Seifert circles $C_1, \ldots, C_n$. Also, the crossings of the knot will be denoted by signed segments connecting two Seifert circles, in the notion described in Figure 2.16. Note that any two circles joined by a signed arc in any Seifert Diagram are necessarily coherent;

2. If $h(D) = 0$, the knot K is already in the closed braid form, and our task is accomplished. In cases where $h(D) > 0$, we will create a *reducing arc* α. This arc connects an incoherent pair C_1 and C_2, intersecting S only at its endpoints, where S is the Seifert diagram. A region within $S^2 \backslash S$ that contains a reducing arc is will be called

a *defect region*. Employ a reducing move along α, as demonstrated in Figure 3.7, to generate a new Seifert diagram S'. In this revised diagram, a duo of coherent Seifert circles, C_1' and C_2', linked by two arcs with opposite orientations, replaces the former incoherent pair C_1 and C_2. The corresponding alteration in the original diagram D corresponds to a Reidemeister move of type II. In this movement, C_1 is slid over C_2 in a small neighborhood of the arc α, resulting in a fresh diagram D' with two newly introduced crossings. It is important to note that if we instead slide C_1 under C_2, the two new Seifert circles remain the same, although the orientations of the two newly introduced arcs are reversed;

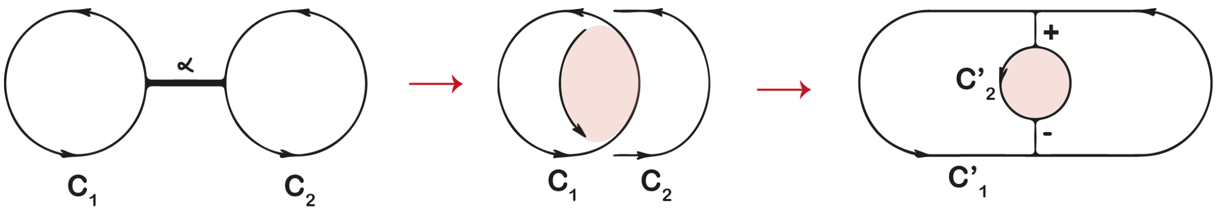


Figure 3.7: *Reducing move operation.*

3. Keep applying reducing moves to incoherent pairs until a diagram with a height of zero is achieved

4. Perform change of infinity procedure, if necessary to help identifying the closed braid.

Example 3.2.1. We apply the Yamada-Vogel algorithm to the diagram of the trefoil knot (3_1). Note that $h(D) = 0$ so no reducing move is needed.

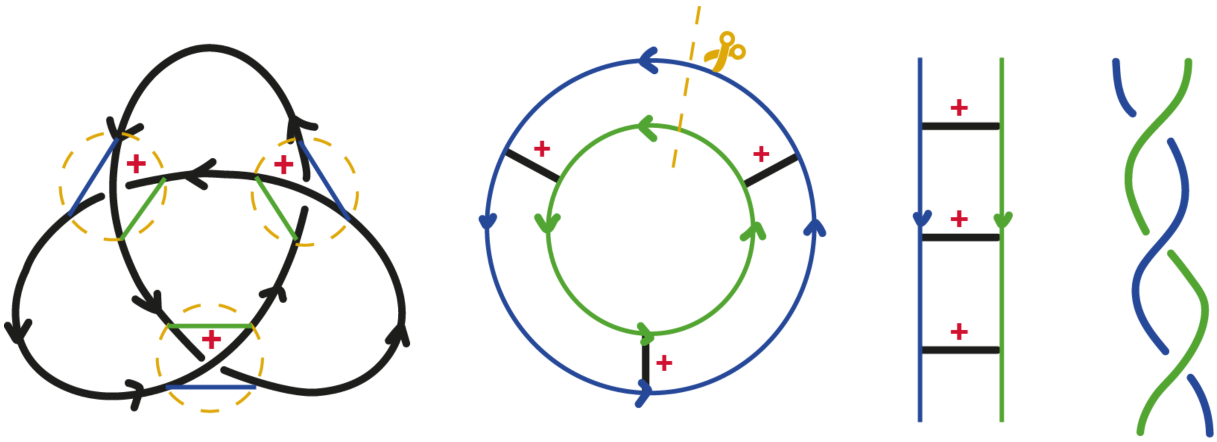


Figure 3.8: *Yamada-Vogel algorithm performed on the trefoil knot.*

Example 3.2.2. We apply the Yamada-Vogel algorithm to the diagram of the trefoil knot (5_2). This example stands out as the first instance in knot tables where the diagram's height is greater than zero. Here, the height of the original knot diagram is 2, since we have two pairs of incoherent Seifert circles. The diagram presents the reducing arcs, α_1, α_2.

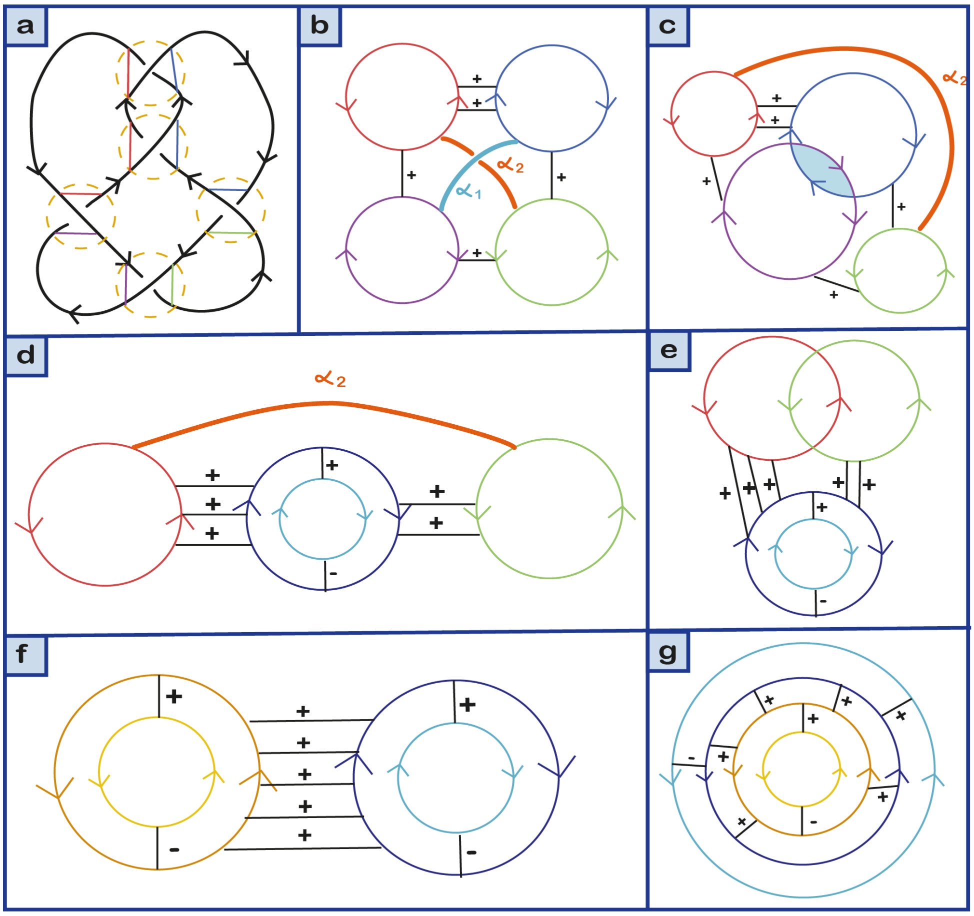


Figure 3.9: *Yamada-Vogel algorithm performed on the 5_2 knot.*

By starting from the positively signed arc positioned at 'eleven o'clock' and proceeding counterclockwise, the equivalence between knot 5_2 and $b(X)$ becomes evident. Here,

$$X = \sigma_2\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2$$

Applying the braid relations to the word defined by X, we see that:

$$X = \sigma_2\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 = \sigma_2\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2 =$$
$$\sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_3^{-1}\sigma_2\sigma_3\sigma_1\sigma_2 = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2^{-1}\sigma_1\sigma_2$$

Since this braid only involves σ_3 once, we may 'delete a trivial loop to get the 8-crossing 3-braid $\sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_2\sigma_2^{-1}\sigma_1\sigma_2$, so the algorithm did not give us minimum braid index. The braid $\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^2\sigma_2$ can indeed be further simplified. This reduction to a 6-crossing 3-braid demonstrates the minimality of 6 crossings. This minimality arises because the crossing number of a 3-braid knot must be even and this knot doesn't possess a diagram with fewer than 5 crossings Consequently, we deduce an additional insight: when utilizing the Yamada-Vogel algorithm to transform a knot into a closed braid form, the crossing number invariably increases.

Proof of Alexander's Theorem

Now that we have Yamada-Vogel Algorithm that allow us to obtain a closed braid from a given knot, there are some things we still need to prove in order to demonstrate Alexander's Theorem. Firstly, we need to see that Yamada-Vogel Algorithm will always lead to a diagram D such that $h(D) = 0$.

Lemma 3.2.3. Performing a reducing move reduces by one the height of the diagram. That means, if D is a Seifert diagram that contains a reducing arc, and D' is the diagram obtained after applying the reducing move with respect to this arc, one have $h(D') = h(D) - 1$.

Proof. Let $C_1, \ldots, C_n$ be the Seifert Circles from a Seifert Diagram S. Also, let C_i, C_j be a pair of non-coherent circles. Note that the union of C_i and C_j divides the 2-sphere S^2 into three sections: an annulus A between C_i and C_j, and two disks, $\mathcal{D}_i$ and $\mathcal{D}_j$, bounded by C_i and C_j respectively. Suppose A can accommodate a reducing arc α. Executing a reducing move along α preserves all other circles C_p for $p \neq i, j$ and replaces C_i and C_j with two new circles. One of these, say C'_j, encloses a disk $\mathcal{D}'_j$ containing no other Seifert circles in the new diagram S'. The other, denoted by C'_i, encloses a disk $\mathcal{D}'_i$ containing all original Seifert circles within annulus A in S. (See Figure 3.10.)

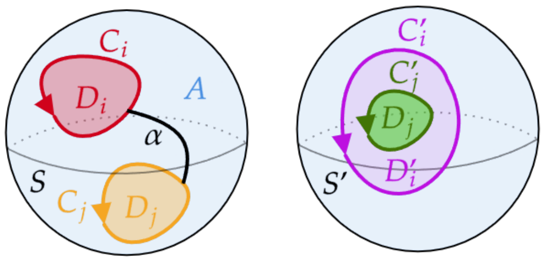


Figure 3.10: *Incoherent Seifert Circles lying on the 2-sphere (on the left) and the circles after the reducing move (on the right).*

For simplicity, we'll use $(C, C') = 1$ if the pair C, C' is coherent, and $(C, C') = -1$ otherwise. If $\{p, q\} \cap \{i, j\} = \emptyset$, then the reducing move on (C_i, C_j) keeps the pair (C_p, C_q) unchanged. Therefore, it suffices to consider how the reducing move affects (C_p, C_x), where $x = i$ or $x = j$ and $p \neq i, j$.

If C_p resides in annulus A in S, then $(C_p, C'_i) = \left(C_p, C'_j\right) = (C_p, C_i) = (C_p, C_j)$. Similarly, if $C_p \subset \mathcal{D}_i$ in S, then $(C_p, C'_i) = (C_p, C_i)$ and $\left(C_p, C'_j\right) = (C_p, C_j)$. Likewise, if $C_p \subset \mathcal{D}_j$ in S, then $(C_p, C'_i) = (C_p, C_j)$ and $\left(C_p, C'_j\right) = (C_p, C_i)$. Consequently, the number of distinct incoherent pairs C_k, C_l in S distinct of C_i, C_j is equivalent to the total number of distinct incoherent pairs in S', denoted as C'_k, C'_l, distinct of C'_i, C'_j. By construction, the substitution of $(C_i, C_j) = -1$ with $\left(C'_i, C'_j\right) = 1$ is achieved. This implies $h\left(D'\right) = h(D) - 1$. □

The proof of Alexander's Theorem will be concluded by the following lemma.

Lemma 3.2.4. [Yam87] If D represents a knot or a link, and $h(D) > 0$, the corresponding Seifert diagram S must contain a defect region.

Proof. We consider the space $S^2 \backslash S$ and let $\mathcal{C}$ denote a connected component of this space. Each boundary component of $\mathcal{C}$ consists of a collection of signed arcs (possibly none) and segments of Seifert circles. Also, each component $\mathcal{C}$ of $S^2 \backslash S$ is a genus-0 surface with at least one boundary component. We will refer to the Seifert circles in S that compose part or all of a boundary component of $\mathcal{C}$ as the *exposed circles* and let m be the number of exposed circles of $\mathcal{C}$. We can analyze the possibilities for $\mathcal{C}$ to not be a defect region.

(i) If $m = 1$ it's not a defect region.

(ii) If $m = 2$ it could be either an annulus or a disk. If it's a disk, the two circles are connected by at least one signed arc, meaning they are coherent, and $\mathcal{C}$ is not a defect region. If $\mathcal{C}$ is an annulus, it's a defect region if the exposed circles are incoherent, otherwise, it's not.

(iii) If $m \geq 3$ there must be an incoherent pair among them, making $\mathcal{C}$ a defect region.

Suppose that no component of $S^2 \backslash S$ is a defect region. Hence, each component is one of the three non-defect region types described above. At least one of these regions must be of the second type, or else $h(D) = 0$. Such a region can be visualized as lying between two nested coherent circles connected by at most one signed arc.

If we start with such a region and try to create a diagram without defect regions, we can't add any circles within the annulus bound by the two nested circles, as this would create a component with three or more exposed circles. We can only add coherent circles that nest with the original two circles (and any number of signed arcs between adjacent pairs). However, such a diagram has height zero. Consequently, $h(D) > 0$ implies the presence of a defect region. □

Definition 3.2.4 (Braid Index)**.** The braid index of a knot or link K is the smallest number n for which a braid $X \in \mathbf{B}_n$ whose closure $b(X)$ represents K exists.

The braid index of K is naturally bounded above by the minimum number of Seifert circles in any of its diagrams. However, the reverse inequality is true according to the Yamada-Vogel algorithm.

Corollary 3.2.5. [Yam87] The minimum number of Seifert circles in any diagram representing a knot or link K is equal to its braid index.

Corollary 3.2.6. [Tra98] [Vog90] Let N denote the length of a sequence of reducing moves required to transform a diagram D into its closed braid form. Then, we have:

$$N = h(D) \leq \frac{(n-1)(n-2)}{2},$$

where n represents the number of Seifert circles associated with D.

The lemma above allow us a way to measure the complexity in the process of transforming a knot into its closed braid form.

3.2.2 Markov's Theorem

Alexander's Theorem, established in the preceding section, provides the assurance that closed braid representations for a knot indeed exist. However, as mentioned earlier, such representations are not unique. Markov's Theorem, gives us a certain amount of control over different closed braid representatives of the same knot. This theorem states that any two of these representatives can be linked by a finite sequence of basic moves, serving as the analogous for closed braids to the Reidemeister Theorem for knots.

Definition 3.2.5 (Markov Moves)**.** The Markov moves consists of three movements that will play a central role in Markov's Theorem: braid isotopy, destabilization and stabilization.

(a) A braid isotopy means isotopy of the closed braid, through braids, in the complement of the braid axis;

(b) A positive stabilization consists in performing a type I like Reidemeister move for the closed braid, around the braid axis, creating a positive crossing. The negative stabilization is the analogous, for a negative crossing. This operation increases the braid index by 1;

(c) A positive (negative) destabilization is undoing the former operation, removing the extra crossing created by a positive (negative) stabilization. This operation decreases the braid index by 1 (See Figure 3.11).

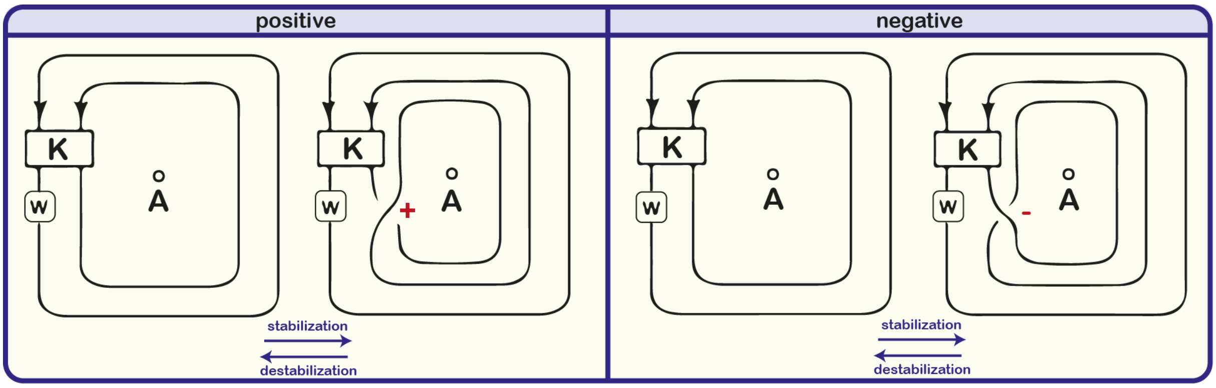


Figure 3.11: *The operations of stabilization and destabilization. The weight w that is attached to one of the braid strands denotes that many 'parallel' strands. The braid inside the box which is labeled K is an arbitrary $(w+1)$-braid.*

Theorem 3.2.7 (Markov's Theorem)**.** Let X and X' be closed braids representations of the same oriented link $K \in \mathbb{R}^3$, then there exists a sequence of closed braid representations of K:

$$X = X_1 \to X_2 \to \cdots \to X_r = X'$$

This sequence is such that each X_{i+1} is derived from X_i through one of the Markov moves. We say that two braid related by a sequence of Markov moves are *Markov equivalent.*

Proof of Markov's Theorem

The following proof is due to Pawel Traczyk [Tra98]. It begins with Reidemeister's theorem, and uses the circle of ideas that were described in Alexander's Theorem proof. Let X and X' be closed braid representatives of the same oriented knot type K. We can assume without loss of generality that X and X' are defined by closed braid diagrams Y and Y' of height $h(Y) = h(Y') = 0$. By Reidemeister's Theorem (Theorem 2.2.2) there is a sequence of knot diagrams $Y = Y_1 \to Y_2 \to \cdots \to Y_k = Y'$, where for $i = 2, \ldots, k-1$, $h(Y_i) \geq 0$. Also, any two diagrams in this sequence are connected by a single Reidemeister move of type I, II, or III. Additionally, Traczyk's proof begins by simplifying the problem to sequences of knot diagrams related by Yamada-Vogel reducing moves:

Lemma 3.2.8. It is sufficient to prove Theorem 3.2.7 for closed braid diagrams Y and Y' that are connected by sequences $Y = Y_1 \to Y_2 \to \cdots \to Y_q = Y'$ satisfying the following conditions:

(i) $h(Y) = h(Y') = 0$,

(ii) $h(Y_i) > 0$ for $i = 2, \ldots, q-1$

(iii) Y_{i+1} is obtained from Y_i by a single Yamada-Vogel reducing move or the inverse of such a move (See Figure 3.7).

Proof. We can always assume that diagrams $Y_2, \ldots, Y_{q-1}$ have a height greater than zero. If not, we can simply replace the given sequences with subsequences that join any two intermediate diagrams of height zero.

A Reidemeister move is considered braid-like if the strands involved in it are locally oriented in a coherent manner, similar to the configuration in a closed braid. Hence, any Reidemeister move of type I is braid-like. To initiate the proof of the lemma, we establish a slightly weaker claim: we contend that we can transition from Y to Y' using a finite sequence of four types of moves and their inverses, all of which are braid-like Reidemeister moves:

- Type I^b: A braid-like Reidemeister move of type I.
- Type II^b: A braid-like Reidemeister move of type II.
- Type III^b: A braid-like Reidemeister move of type III.
- Type $\mathcal{Y}$: A Yamada-Vogel reducing move.

To prove this claim, it suffices to show that non-braid-like Reidemeister moves denoted I^{nb}, $\mathrm{II}^{\mathrm{nb}}$, and $\mathrm{III}^{\mathrm{nb}}$ can be achieved using a finite sequence of moves of types I^b, II^b, II^b, and $\mathcal{Y}$. This can be demonstrated by analyzing the cases of I^{nb}, $\mathrm{II}^{\mathrm{nb}}$, and $\mathrm{II}^{\mathrm{nb}}$ sequentially.

1. Any type I Reidemeister move is equivalent to a braid-like Reidemeister move of type I^b.
2. A move of type $\mathrm{II}^{\mathrm{nb}}$ can be seen as a move of type $\mathcal{Y}^{\pm}$ if the segments involved belong to distinct Seifert circles. Thus, we only need to address the scenario where they are subarcs of the same Seifert circle. The move can be replaced by two moves of type I^b (creating two new Seifert circles), followed by a move of type $\mathcal{Y}$, and finally a move of type $\mathcal{Y}^{-1}$.

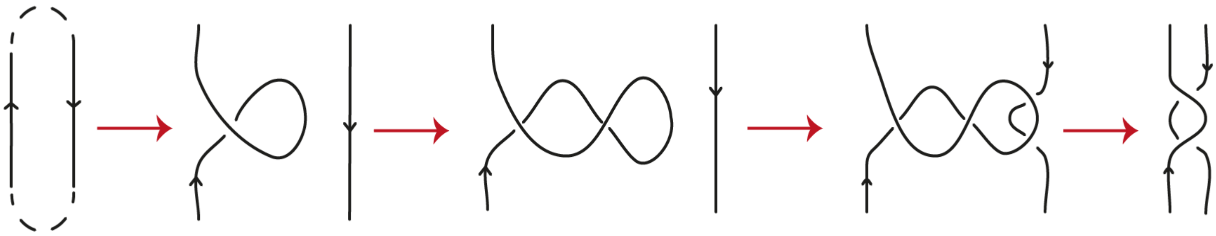


Figure 3.12: *Replacing moves of type II^{nb}*

3. A type III$^{\text{nb}}$ Reidemeister move involves three segments of the knot or braid. There are various scenarios, based on local orientations and the signs of the three crossings, though they all share similarities. We will explore one of those cases with more detail. In this situation, one segment passes under a crossing formed by the other two segments and has an orientation opposite to the other two strands. The sequence of replacements (See Figure 3.13) demonstrates that our type II$^{\text{nb}}$ Reidemeister move can be achieved by a sequence consisting of a type II$^{\text{nb}}$ move, an isotopy, a type III$^{\text{b}}$ move, and finally another type II$^{\text{nb}}$ move. The other cases of type III$^{\text{nb}}$ moves are left for the reader's consideration, reducing the situation to the type II$^{\text{nb}}$ moves.

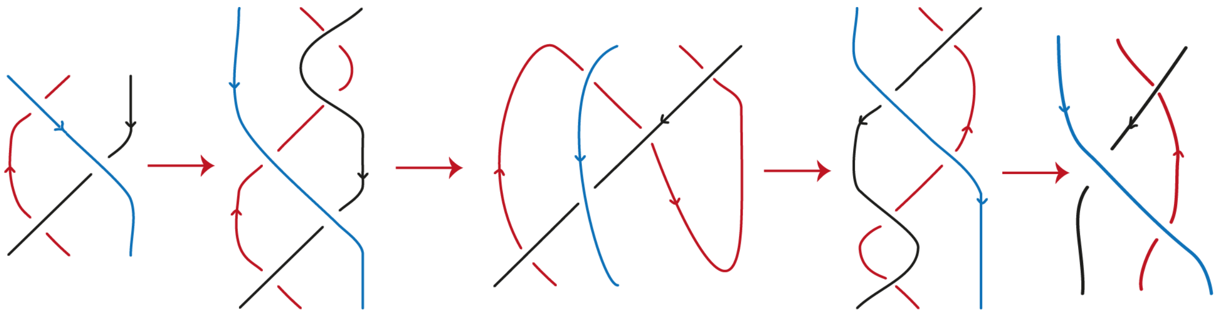


Figure 3.13: *Replacing moves of type II^{nb}*

Consequently, we are left with the case where each diagram in the sequence transitioning from Y to Y' involves only type I^b, II^b, III^b, or $\mathcal{Y}^{\pm}$. To finalize the proof of Lemma 3.2.8, consider a braid-like Reidemeister move t to be performed on diagram Y_i. Suppose $h_i = h(Y_i) > 0$. We can find a sequence of reducing moves $r_1, \ldots, r_{h_i}$ such that applying $r_{h_i} \circ \cdots \circ r_1$ to Y_i results in a braid, and each associated reducing arc $\alpha_1, \ldots, \alpha_{h_i}$ is disjoint from the region where t will be executed. This allows each reducing move r_j to commute with t, enabling us to replace t with its 'conjugate' $r_1^{-1} \circ \cdots \circ r_{h_i}^{-1} \circ t \circ r_{h_i} \circ \cdots \circ r_1$. This positions t to be performed at height 0, meaning on a braid.

In the cases where t is of type II^b or III^b, the proof is complete, as a braid-like move of type II or III executed on a braid corresponds to a braid isotopy. When t is of type I^b, it corresponds to a stabilization (up to isotopy) only if applied to the braid strand closest to the braid axis. However, realizing a type I^b move on any strand in terms of Markov moves is straightforward: merely push the strand beneath the others using type II^t moves, and execute the required stabilization near the braid axis.

Given that we have established that any type II^b or type III^b move can be realized through a finite series of type $\mathcal{Y}^{\pm}$ moves and braid isotopies, this implies that a type I^b

move can also be substituted by a finite sequence of type $\mathcal{Y}^{\pm}$ moves and braid isotopies. The treatment of inverse moves of type I^b follows a similar approach.

By employing this strategy, we have effectively transformed our initial sequence connecting Y to Y' into a new sequence, which, in general, may be lengthier but is exclusively comprised of Markov moves (performed on braid diagrams, i.e., on diagrams with zero height) and type $\mathcal{Y}^{\pm}$ moves. To be more precise, our initial sequence from Y to Y' can be replaced by a sequence of the form $Y = Y_0, \ldots, Y_{a_1}, \ldots, Y_{a_2}, \ldots, Y_{a_n} = Y'$, where $h(Y_{a_i}) = 0$ for all i, and within each subsequence $Y_{a_i}, \ldots, Y_{a_{i+1}}$, either (1) every diagram in the subsequence has zero height, and adjacent diagrams are connected by a single Markov move, or (2) all intermediate diagrams have strictly positive height, and adjacent diagrams are related by a single move of type $\mathcal{Y}$ or $\mathcal{Y}^{-1}$. Therefore, to establish Markov's theorem, it is adequate to examine solely sequences of the second type. $\square$

Let's now consider a sequence of diagrams $Y = Y_1, \ldots, Y_n = Y'$ that satisfy the conditions of Lemma 3.2.8. It's important to note that, similar to the earlier proof, we will not typically distinguish between a diagram and its corresponding Seifert picture. Thus, we'll talk about a Seifert circle in the diagram Y_i, referring to a Seifert circle in the Seifert picture associated with Y_i. Essentially, each circle in a Seifert picture can be considered a part of the corresponding diagram, except in a small neighborhood of signed arcs representing crossings. As reducing arcs avoid signed arcs, there's no ambiguity in referring to a reducing arc in a diagram.

Now, let's delve into the graph of the height function across our sequence. This graph initiates and concludes at height zero, and each intermediate "step" either increments or decrements by 1 since we've reduced our case to sequences of only reducing moves (or their inverses). We're interested in examining the local maxima within this height function. Let $Y(r)$, $\hat{Y}$, and $Y(s)$ be three consecutive diagrams in our sequence, forming a local maximum at $\hat{Y}$. In simpler words, we have two reducing moves, r and s, with corresponding arcs α_r and α_s in $\hat{Y}$ such that performing reducing moves along α_r (resp. α_s) leads to diagrams $Y(r)$ (resp. $Y(s)$) in such a way that makes sense discussing $\alpha_r \cup \alpha_s$.

Definition 3.2.6. The triple $\{Y(r), \hat{Y}, Y(s)\}$ will be called a *peak* in the height function of our sequence. The *height of this peak* is defined as $h(\hat{Y})$, and the *height of the sequence* is the maximum value attained by the height function throughout the sequence. To prove Markov's Theorem, we will use induction based on the height of the sequence.

Lemma 3.2.9. One can assume the reducing arcs associated with any peak in the height function are disjoint. Furthermore, any adjustments required in our sequence of reducing moves to achieve this do not affect the height of the sequence.

Proof. Consider a peak $\{Y(r), \hat{Y}, Y(s)\}$ in the height function with corresponding reducing arcs α_r and α_s. We can assume that these arcs intersect minimally and transversely. If $|\alpha_r \cap \alpha_s| = n \geq 2$, by slightly adjusting some of the intersection points, we can find a new reducing arc $\alpha_{r'}$ with the same endpoints as α_r such that $\alpha_r \cap \alpha_{r'} = \emptyset$ and $|\alpha_{r'} \cap \alpha_s| < n$. This allows us to replace the given peak $\{Y(r), \hat{Y}, Y(s)\}$ with two consecutive peaks $\{Y(r), \hat{Y}, Y_{r'}\}$ and $\{Y_{r'}, \hat{Y}, Y(s)\}$. This process, known as "inserting the reducing operation r' at $\hat{Y}$, effectively replaces one peak with two peaks of the same height. This procedure is continued iteratively until the intersection numbers of adjacent pairs of reducing arcs are at most 1. If the intersection number of the arcs α_r and α_s corresponding to a peak $\{Y(r), \hat{Y}, Y(s)\}$ is 1, and there exists a reducing arc α_t such that $\alpha_t \cap \alpha_r = \alpha_t \cap \alpha_s = \emptyset$, we can insert the reducing operation t at $\hat{Y}$ to generate two peaks, each associated with a disjoint pair of reducing arcs. In case the defect region supporting α_r and α_s contains no third reducing arc disjoint from both α_r and α_s, there is only one possible configuration, as illustrated in Figure 3.13.

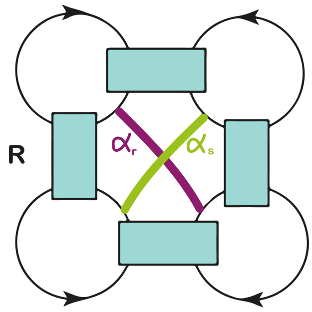


Figure 3.14: *The case where two reducing arcs at a peak intersect once. The green boxes represent the possible signed arcs between the Seifert Circles.*

Now, let's focus on the region labeled as R, which is situated "outside" the circles within our defect region and the signed arcs that connect them. If R contains a Seifert circle, it's evident that there must exist a region somewhere in the diagram with three exposed circles. Such a region, as observed in the previous section, unavoidably represents a defect region containing a reducing arc that is not connected to either α_r or α_s. Conversely, if R doesn't encompass any Seifert circles, then it also cannot include signed arcs. This conclusion is drawn from the fact that all conceivable signed arcs between the four exposed circles of region R are already depicted in Figure 3.14. Consequently, we can introduce a reducing arc in R that connects either pair of diagonally opposed circles.

In summary, when dealing with a peak $\{Y(r), \hat{Y}, Y(s)\}$ that exhibits intersection 1 in its associated arcs, it's always possible to locate a third reducing arc α_t that enables us to insert the reducing operation t at $\hat{Y}$. This operation transforms the initial peak into two distinct peaks, each featuring a pair of reducing arcs that are disjoint. As this process of inserting a reducing operation at a peak maintains the height of the sequence. $\square$

With the preceding lemma, we can now presume that each peak in the graph of our height function corresponds to a separate pair of reducing arcs. Before we introduce the next lemma, which concerns these peaks, we need to familiarize ourselves with a few concepts. Notably, when the reducing arcs engaged in a peak $\{Y(r), \hat{Y}, Y(s)\}$ are disjoint, these reducing moves commute. This means that they can be executed in either order, originating from the diagram $\hat{Y}$ and resulting in the same diagram Y'. Moreover, as long as the reducing arcs α_r and α_s act on three or four distinct Seifert circles, the sequence of the two reducing moves remains interchangeable without altering the outcome. Given that the reducing arcs α_r and α_s are disjoint within the diagram $\hat{Y}$, it's meaningful to discuss the arc α_s (or α_r) in the context of the diagram $Y(r)$ (or $Y(s)$) obtained by reducing $\hat{Y}$ along α_r (or α_s). In this scenario, we have a "commuting pair" of reducing moves associated with the peak. When a peak $\{Y(r), \hat{Y}, Y(s)\}$ exhibits a commuting pair, we can replace it with a "valley" – a subsequence $\{Y(r), Y', Y(s)\}$ where $h(Y') = h(Y) - 2$ and $Y' = Y(s \circ r) = Y(r \circ s)$ results from reducing $Y(r)$ along α_s or vice versa. By this method, we can eliminate any peak corresponding to a commuting pair (such a peak always has a minimum height of 2).

In situations where a peak is characterized by two reducing arcs acting on the same two circles, the second Reidemeister move becomes non-reducing after one move is executed, leading to what is termed a *non-commuting pair* of reducing moves. Consider a peak denoted as $\{Y(r), \hat{Y}, Y(s)\}$, corresponding to a non-commuting pair of reducing arcs. Let C_1 and C_2 be the two Seifert circles involved in this context. If there exists a reducing arc α_t such that $\alpha_t \cap \alpha_r = \alpha_t \cap \alpha_s = \emptyset$ and t involves a circle other than C_1 or C_2, then we can insert t at $\hat{Y}$ to transform the peak $\{Y(r), \hat{Y}, Y(s)\}$ into two new peaks characterized by commuting pairs of reducing moves: $\{Y(r), \hat{Y}, Y(t)\}$ and $\{Y(t), \hat{Y}, Y(s)\}$. Following the earlier pattern, each peak is subsequently replaced by a 'valley': $\{Y(r), Y', Y(t)\}$ and $\{Y(t), Y'', Y(s)\}$, respectively. Here, $Y' = Y(t \circ r) = Y(r \circ t)$ represents the diagram resulting from reducing $Y(r)$ by t (or equivalently, from reducing $Y(t)$ by r), while Y'' corresponds to the diagram resulting from reducing $Y(s)$ by t (or equivalently, from reducing $Y(t)$ by s). This process establishes that the initial non-commuting peak $\{Y(r), \hat{Y}, Y(s)\}$ had a height of at least 2. Consequently, we can substitute such a peak with peaks of strictly lesser height and repeat this procedure until all peaks either possess a height of 1 or do not admit a reducing arc α_t as previously mentioned; such peaks are classified as *irreducible*.

Lemma 3.2.10. We can assume that each peak in the height function of our sequence either has a height of 1 or is irreducible.

Proof. We introduced the concept of irreducible peaks in a way that encompasses all scenarios that prevent height reduction to 1. □

Lemma 3.2.11. We can assume that no peaks in the height function of our sequence possess a height of 1.

Proof. Let $\{Y(r), \hat{Y}, Y(s)\}$ constitute a peak with a height of 1. Recall that a height of 1 indicates that the two reducing arcs are non-commutative and, thus, involve exactly two circles. It can be also be demonstrated that these two circles must be positioned either "inside" or "outside" a band of circles. Additionally, α_r and α_s are equivalent as reducing moves. Consequently, the diagram $Y(r)$ is equivalent to $Y(s)$, allowing for the direct elimination of this peak from our sequence. □

Hence, the last thing we need to address is irreducible peaks within the height function of our sequence. Fortunately, these peaks exhibit a very specific pattern. To elucidate this pattern, we define a weighted Seifert circle similar to the weights associated with closed braid diagrams. A Seifert circle with an attached weight w signifies a collection of w coherently oriented, nested, parallel Seifert circles. The term *band* denotes a Seifert circle with a correspondent weight.

Lemma 3.2.12. If $\{Y(r), \hat{Y}, Y(s)\}$ constitutes an irreducible peak within the height function of our sequence, the diagram $\hat{Y}$ features at most four bands, arranged as depicted in Figure 3.15.

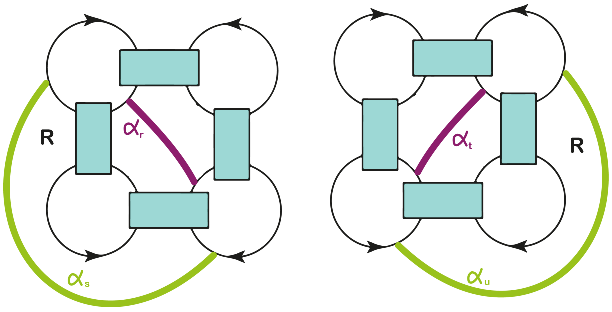


Figure 3.15: *The diagram linked to the irreducible peak $Y(r), \hat{Y}, Y(s)$. Two extra reducing arcs employed for eradicating irreducible peaks. Both pairs, α_r, α_s, and α_t, α_u, represent potential non-commuting pairs associated with the peak.*

Proof. Consider an irreducible peak $\{Y(r), \hat{Y}, Y(s)\}$ and let C_1 and C_2 denote the two circles engaged in the reducing arcs α_r and α_s. For $i = 1, 2$, use D_i to indicate the disk bounded by C_i in S^2, which does not contain the reducing arcs α_r and α_s. Consequently, the region $S^2 \setminus (D_1 \cup D_2 \cup \alpha_r \cup \alpha_s)$ consists of two components. Based on the assumption, neither component can accommodate a defect region (as that would enable the discovery of a reducing arc α_t as described earlier, contradicting the irreducibility of our peak). Consequently, if either component includes Seifert circles, these circles must form a band, positioned opposite to C_1 and C_2. It is admissible that these bands might have a weight of zero. Using similar reasoning, we can deduce that D_i cannot host a defect region for $i = 1, 2$, implying that C_i must serve as the outer circle of a band. It is possible, of course, for some braiding to occur between adjacent coherent circles, as indicated in Figure 3.15. This configuration yields a diagram with each pictured circle representing a band, thus completing the argument. □

The subsequent lemma permits us to replace irreducible peaks in the height function of our sequence with peaks of strictly smaller height.

Lemma 3.2.13. Suppose $\{Y(r), \hat{Y}, Y(s)\}$ is an irreducible peak with a height of $n + 1$ in the height function of our sequence. Then, there exist sequences of diagrams $Y(r) = Y_1^r, \ldots, Y_n^r = Y(p \circ r)$ and $Y(s) = Y_1^s, \ldots, Y_n^s = Y(u \circ s)$ such that Y_{i+1}^r (resp. Y_{i+1}^s) is obtained from Y_i^r (resp. Y_i^s) by a reducing move. Moreover, $h(Y(p \circ r)) = h(Y(u \circ s) = 0$, and $Y(p \circ r)$ and $Y(u \circ s))$ are Markov equivalent.

Before delving into the proof of this lemma, we outline how it contributes to establishing the Markov theorem. Utilizing the preceding lemmas, we have effectively narrowed down the proof of Markov's Theorem to a scenario involving two closed braid diagrams X and X' connected by a sequence of diagrams that involve reducing moves (as well as their inverses). In this sequence, the height of each intermediate diagram is strictly positive, and any peak in the sequence's height function is irreducible, with a height of at least 2. Let $\{Y(r), \bar{Y}, Y(s)\}$ be an irreducible peak with a height of n in the height function of our sequence. Thanks to Lemma 3.2.13, we can substitute this subsequence with a subsequence of strictly smaller height (potentially including a subsequence entirely at height 0 related by Markov moves, which we can disregard as before). In the process, we generate new peaks with heights strictly lower than the peak being replaced. By performing this operation at every irreducible peak, we establish a new sequence linking our closed braid diagrams Y and Y'.

It is important to note that the new peaks may or may not be irreducible, and their corresponding arcs might not even be disjoint. Nevertheless, we find ourselves back in a situation analogous to that before Lemma 3.2.9, albeit with a sequence of reduced height. Hence, by employing induction based on the sequence's height (with the base case provided by Lemma 3.2.11), we can substitute any sequence as described in Lemma 3.2.7 with a sequence entirely composed of diagrams with a height of zero. This successfully concludes the proof of Markov's Theorem.

A brief overview of the proof of Lemma 3.2.13 follows. According to Lemma 3.2.12, the diagram Y has at most four bands, denoted as B_1, B_2, B_3, and B_4, linked by four (potentially trivial) braids X_1, X_2, X_3, X_4, as depicted in Figure 3.15. Let w_i denote the weight of band B_i. Note that, for instance, braid X_1 encompasses $w_1 + w_3$ strands, with similar considerations applicable to the other X_i. If w_i or w_j exceeds 1, a reducing arc connecting B_i and B_j implies a sequence of $w_i w_j$ reducing moves. Multiple approaches are available for constructing such sequences; we adopt the practice of selecting reducing moves in a manner that ensures the strands of one band uniformly slide either below or above the strands of the other band.

Returning to Figure 3.15, we commence by executing the reducing move r along the arc α_r, followed by an additional reduction via arc α_p, involving B_3 and B_4. This leads to a variable number of reducing moves, contingent upon the weights of the bands involved. The resultant diagram $Y(p \circ r)$ assumes the closure of the initial braid, accounting for a choice of arcs passing over or under during the various Type II Reidemeister moves.

To generate the second sequence outlined in the lemma, we initiate the process with the reducing move s along the arc α_s, followed by another reduction involving the arc α_u. The resulting configuration, represented by the diagram $Y(u \circ s)$, takes the form of the second braid presented in Figure 3.16.

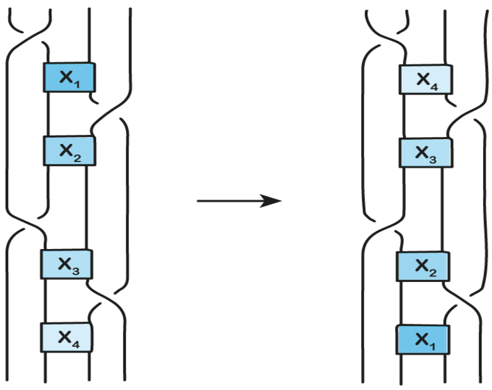


Figure 3.16: *The closures of the two braids are the result of performing the two pairs of reducing moves r,p and s,u indicated in Figure 3.15*

At this point, we have successfully identified our two sequences of reducing moves. As a result, the demonstration of Markov's Theorem has been distilled to a singular computation, specifically, the demonstration of the mutual equivalence (M-equivalence) between the closed braids $Y(p \circ r)$ and $Y(u \circ s)$. Establishing the equivalence between these two braids involves a sequence comprising numerous braid isotopies, along with two stabilizations and two destabilizations. For a comprehensive account of the calculation, readers can refer to [Tra98] and [BB05]. This finalizes the proof Markov's Theorem.

Appendices

.1 Long exact sequence of a fibration

It's well known that in algebraic topology, a "short exact sequence of spaces" $A \hookrightarrow X \to X/A$ gives rise to a long exact sequence of homology groups given some conditions over X and A. However, this is not true when talking about a long exact sequence of homotopy groups due to the failure of excision. Nonetheless, there is a different type of "short exact sequence of spaces" that does give a long exact sequence of homotopy groups. This type of short exact sequence $F \to E \xrightarrow{p} B$, called a fiber bundle, is distinguished from the type $A \hookrightarrow X \to X/A$ in that it has more homogeneity: all the subspaces $p^{-1}(b) \subset E$, which are called fibers, are homeomorphic.

For example, E could be the product $F \times B$ with $p : E \to B$ being the projection. General fiber bundles can be thought of as twisted products. Familiar examples are the Möbius band, which is a twisted annulus with line segments as fibers.

Remark .1.1. One of the most elegant examples of a fiber bundle is the Hopf bundle $S^1 \to S^3 \to S^2$, where S^3 denotes the group of quaternions with unit norm and S^1 is the subgroup of unit complex numbers.

For this particular bundle, the long exact sequence of homotopy groups takes on the following form:

$$\cdots \to \pi_i\left(S^1\right) \to \pi_i\left(S^3\right) \to \pi_i\left(S^2\right) \to \pi_{i-1}\left(S^1\right) \to \pi_{i-1}\left(S^3\right) \to \cdots$$

A remarkable observation is that the exact sequence yields an isomorphism $\pi_2\left(S^2\right) \cong \pi_1\left(S^1\right)$, since the two adjacent terms $\pi_2\left(S^3\right)$ and $\pi_1\left(S^3\right)$ vanish under cellular approximation. This provides a direct proof, from a homotopy-theoretic perspective, that $\pi_2\left(S^2\right) \cong \mathbb{Z}$. Moreover, the proposition that $\pi_i\left(S^1\right) = 0$ for $i > 1$ allows us to deduce that $\pi_i\left(S^3\right) \cong \pi_i\left(S^2\right)$ for all $i \geq 3$, and therefore $\pi_3\left(S^2\right) \cong \pi_3\left(S^3\right) \cong \mathbb{Z}$.

Now, let's introduce the characteristic that leads to a long exact sequence of homotopy groups. We say that a map $p : E \to B$ satisfies the homotopy lifting property concerning a space X, it implies that given a homotopy $g_t : X \to B$ and a map $\tilde{g}_0 : X \to E$ that lifts g_0 that is $p \circ \tilde{g}_0 = g_0$, there exists a homotopy $\tilde{g}_t : X \to E$ that lifts g_t.

A fibration is a map $p : E \to B$ that satisfies the homotopy lifting property for all spaces X. For instance, a projection $B \times F \to B$ is a fibration since we can choose lifts of the form $\tilde{g}_t(x) = (g_t(x), h(x))$, where $\tilde{g}_0(x) = (g_0(x), h(x))$.

Theorem .1.2. [Hat05]Let $p : E \to B$ be a map with the homotopy lifting property with respect to disks D^k for all $k \geq 0$. Let b_0 be a basepoint in B, and $x_0 \in F = p^{-1}(b_0)$. Then, for all $n \geq 1$, the induced map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism. In particular, if B is path-connected, then we have a long exact sequence:

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

To prove this, we will use a relative version of the homotopy lifting property. We say that $p : E \to B$ has the homotopy lifting property for a pair (X, A) if every homotopy $f_t : X \to B$ lifts to a homotopy $\tilde{g}_t : X \to E$ with $\tilde{g}_0$ fixed and $\tilde{g}_t(A) \subset F$. In other words, p has the lift extension property for $(X \times I, X \times \{0\} \cup A \times I)$.

The equivalence between the homotopy lifting property for disks and for all CW pairs (X, A) can be established. The proof relies on a step-by-step construction of the lifting

$\tilde{g}_t$ for each cell of $X - A$, using induction over the skeleta of X. Additionally, the fact that the pairs $\big(D^k \times I, D^k \times \{0\}\big)$ and $\big(D^k \times I, D^k \times \{0\} \cup \partial D^k \times I\big)$ are homeomorphic is used. A map $p : E \to B$ that satisfies the homotopy lifting property for disks is referred to as a Serre fibration.

Proof. Let us first show that p_* is onto. Consider an element $[f] \in \pi_n(B, b_0)$ represented by a map $f : (I^n, \partial I^n) \to (B, b_0)$. Since $p : (E, F) \to (B, b_0)$ satisfies the relative homotopy lifting property for $(I^{n-1}, \partial I^{n-1})$, we can construct a lift $\tilde{f} : I^n \to E$ of f over $J^{n-1} \subset I^n$ by the constant map to x_0. Using the relative homotopy lifting property for $(I^n, \partial I^n)$, we can then extend this lift to a map $\tilde{f} : (I^n, \partial I^n) \to (E, F)$ such that $\tilde{f}(\partial I^n) \subset F$ and $p \circ \tilde{f} = f$. Thus, $p_*([\tilde{f}]) = [f]$ and p_* is onto.

To show that p_* is injective, suppose we have two lifts $\tilde{f}_0, \tilde{f}_1 : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ of maps $f_0, f_1 : (I^n, \partial I^n) \to (B, b_0)$ such that $p_*([\tilde{f}_0]) = p_*([\tilde{f}_1])$. Let $G : (I^n \times I, \partial I^n \times I) \to (B, b_0)$ be a homotopy from $p \circ \tilde{f}_0$ to $p \circ \tilde{f}_1$. We can construct a partial lift $\tilde{G} : (I^n \times I, \partial I^n \times I, J^{n-1} \times I) \to (E, F, x_0)$ of G such that $\tilde{G}(I^n \times \{0\}) = \tilde{f}_0$, $\tilde{G}(I^n \times \{1\}) = \tilde{f}_1$, and $\tilde{G}(J^{n-1} \times I) = x_0$. Using the relative homotopy lifting property for $(I^n, \partial I^n)$, we can then extend this partial lift to a full lift $\tilde{G} : (I^n \times I, \partial I^n \times I) \to (E, F, x_0)$ of G. Thus, $\tilde{G}$ is a homotopy between $\tilde{f}_0$ and $\tilde{f}_1$ and $p_*([\tilde{f}_0]) = p_*([\tilde{f}_1])$ implies that p_* is injective.

Finally, to show the last statement of the theorem, we plug $\pi_n(B, b_0)$ in for $\pi_n(E, F, x_0)$ in the long exact sequence for the pair (E, F). The map $\pi_n(E, x_0) \to \pi_n(E, F, x_0)$ in the exact sequence then becomes the composition $\pi_n(E, x_0) \to \pi_n(E, F, x_0) \xrightarrow{p_*} \pi_n(B, b_0)$

Thus, we have shown that p_* is onto and injective, which implies that p_* is an isomorphism.

This completes the proof of the theorem. Note that the last statement of the theorem follows from the surjectivity of $\pi_0(F, x_0) \to \pi_0(E, x_0)$, which is a consequence of the fact that B is path-connected. It suffices to note that $\pi_0(B, b_0) = 0$ since B is path connected. □

.2 Cell Complexes

The idea of cellular complexes is based on constructing Hausdorff topological spaces by attaching cells (structures that we will define shortly, but intuitively we can think of 0-cells as points, 1-cells as intervals, and so on). That is, we start with a discrete set of points X_0. In X_0, we attach 1-dimensional disks (intervals) through functions $\chi_\alpha : \partial D^1 = S^0 \to X_0$. In X_1, we attach 2-dimensional disks through $\chi_\alpha : \partial D^2 = S^1 \to X_1$. In this way, by attaching n-dimensional disks in X_{n-1} through $\chi_\alpha : \partial D^n = S^{n-1} \to X_{n-1}$, we obtain X_n.

Definition .2.1 (Open n-cell)**.** Let X be a Hausdorff topological space. An open n-cell in X is a subspace $e \subset X$ together with a homeomorphism

$$\overset{\circ}{h}_e : \overset{\circ}{D}{}^n \to e \subset X$$

Definition .2.2 (n-cell)**.** Furthermore, if the homeomorphism of its open cell extends to the boundary, that is, if it is a continuous map $h : D^n \to X$ such that $h|_{\overset{\circ}{D}{}^n} : \overset{\circ}{D}{}^n \to h(\overset{\circ}{D}{}^n)$ is a homeomorphism, we call $e = h(\overset{\circ}{D}{}^n) \subset X$ together with this homeomorphism an n-cell.

Definition .2.3 (Cellular boundary)**.** We define the cellular boundary as the set given by:

$$\partial_{\text{cell}}(e) = \overline{e} - e$$

Note that it is important to differentiate the boundary of e ($\partial(e)$) and the cellular boundary of e ($\partial_{\text{cell}}(e)$) as they generally do not refer to the same set.

Definition .2.4 (Characteristic function)**.** The characteristic function χ_e is defined as the restriction:

$$\chi_e = h_e|_{S^{n-1}} : S^{n-1} \to X$$

Definition .2.5 (Attaching an n-cell)**.** Let X be a Hausdorff topological space and $A \subset X$ be closed. We say that X is obtained from A by attaching an n-cell if there exists an open n-cell $(e, h_e) \subset X$ such that $X = A \cup e$, $A \cap e = \emptyset$.

Proposition .2.1. Let $X = D^n$ or $X = I^n$. Let $\sim$ be an equivalence relation on ∂X such that $x \sim y \iff x = y$ or $x, y \in \partial X = S^{n-1}$. Suppose $Y = X/\sim$ is Hausdorff and let $B = \pi(\partial X) \subset Y$. Then, Y is obtained from B by attaching an n-cell ($e = \pi(\text{Int}X)$).

Proof. Since I^n is homeomorphic to D^n (via a homeomorphism that preserves their boundaries, i.e., $\partial I^n \cong \partial D^n$), we can assume $X = D^n$. So, we consider $h_e = \pi : D^n \to Y = D^n/\sim$. Thus, we have $e = h_e(\overset{\circ}{D}{}^n)$ and $Y = e \cup B$ (where $B = h_e(\partial D^n)$) such that $e \cap B = \emptyset$ because points in $\overset{\circ}{D}{}^n$ and ∂D^n are not equivalent to each other. □

Lemma .2.2. Let $f : X \to Y$ be a continuous function and $B \subset X$ be open in X. Then, if $f(x) \in f(\overline{B})$, then $f(x) \in \overline{f(B)}$.

Proof. Let V be a neighborhood of $f(x)$. Thus, $f^{-1}(V)$ is an open set containing x. Therefore, it must intersect B at some point y. Thus, V intersects $f(B)$ at the point $f(y)$, which implies that $f(x) \in f(B)$. □

Proposition .2.3. If $e \subset X$ is an n-cell with defining map $h_e : D^n \to X$, then:

(i) $h_e(D^n) = \overline{e}$

(ii) $h_e(S^{n-1}) = \partial_{\text{cell}}(e)$

Proof. (i) We will show that $(h_e(D^n) \subset \overline{e})$ and $(\overline{e} \subset h_e(D^n))$. Since h_e is continuous, by the lemma, we have $h_e(\overline{B}) \subset \overline{h_e(B)}$ for all $B \subset D^n$. Thus, in particular, for $\mathring{D}^n \subset D^n$, we have $h_e(D^n) \subset \overline{e}$, since the closure of the interior of a set is the set itself and $h_e(\mathring{D}^n) = e$. On the other hand, we have $e = h_e(\mathring{D}^n) \subset h_e(D^n)$. Furthermore, $h_e(D^n)$ is compact because it is the image of a compact set under a continuous function, and thus it is closed since X is Hausdorff. Therefore, we conclude that $\overline{e} \subset h_e(D^n)$. From these two inclusions, it follows that $h_e(D^n) = \overline{e}$.

(ii) Now, let's show that $h_e(S^{n-1}) = \overline{e} - e$ by proving both inclusions. Let $y \in \overline{e} - e$. By item (i), we have $y = h_e(x)$ for some $x \in D^n$ such that $x \notin \mathring{D}^n$, since $e = h_e(\mathring{D}^n)$. Therefore, $x \in S^{n-1}$, and we conclude that $\overline{e} - e \subset h_e(S^{n-1})$. Now, let's analyze the other inclusion. Let $y = h_e(x)$ for $x \in S^{n-1}$. We want to show that $y \notin e$. Suppose, by contradiction, that $y \in e$, which means that $y = h_e(x')$ for some $x' \in \mathring{D}^n$. Since D^n is Hausdorff and $x \neq x'$, there exist open sets U and V in D^n such that $x \in U$, $x' \in V$, and $U \cap V = \emptyset$. We can assume that $V \subset \mathring{D}^n$. Since $\mathring{h}_e$ is a homeomorphism, $h_e(V)$ is open in e, and thus it is open in $\overline{e}$. Furthermore, since $h_e : D^n \to \overline{e}$ is continuous, $h_e^{-1}(h_e(V))$ is open in D^n. However, $x \in h_e^{-1}(h_e(V))$ because $y = h'_e(x)$. Therefore, there exists $\epsilon > 0$ such that $(D^n \cap B_\epsilon(x)) \subset h_e^{-1}(h_e(V))$, where $B_\epsilon(x)$ denotes the ball of radius ϵ centered at x with the usual Euclidean metric. Note that we can choose ϵ such that $D^n \cap B_\epsilon(x) \subset U$. Now, let $z \in (\mathring{D}^n \cap B_\epsilon(x))$. From what we have verified above, there exists $z' \in V$ such that $h_e(z) = h_e(z')$. Thus, since z and z' belong to $\mathring{D}^n$, we must have $z = z'$. Therefore, $z \in V$ and $z \in U$, which leads to a contradiction.

□

From the above proposition, we see that the characteristic function of e can be seen as $\chi_e : S^{n-1} \to A$ describing the way A and e interact within X. Alternatively, it describes the relationship between the inclusion $i : A \hookrightarrow X$ and the defining map $h : D^n \to X$. This relationship can be represented by the following diagram:

$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "\text{inclusion}"] \arrow[d, "\chi_e"'] & D^n \arrow[d, "h_e"] \\ A \arrow[r, hook, "i"] & X \end{tikzcd}$$

Furthermore, X can be reconstructed from $A \subset X$ using the characteristic map.

Proposition .2.4 (Universal Property)**.** Suppose X is obtained from A by attaching an n-cell with the defining map $h_e : D^n \to X$. Let Y be another topological space. Then, the function $f : X \to Y$ is continuous if and only if both $f|_A : A \to Y$ and $f \circ h_e : D^n \to Y$ are continuous.

Moreover, $f \mapsto (f|_A, f \circ h_e)$ defines a bijection between:

- continuous functions $f : X \to Y$
- pairs (f_A, f_e) such that $f_A : A \to Y$ and $f_e : D^n \to Y$ are continuous, satisfying $f_A \circ \chi_e = f_e|_{S^{n-1}}$.

$$
\begin{tikzcd}
S^{n-1} \arrow[r, hook, "\text{inclusion}"] \arrow[d, "\chi_e"'] & D^n \arrow[d, "h_e"] \arrow[ddr, bend left, "f_e"] & \\
A \arrow[r, hook, "i"] \arrow[drr, bend right, "f_A"'] & X \arrow[dr, "f"] & \\
 & & Y
\end{tikzcd}
$$

Furthermore, we can also say that the diagram is a pushout in the category of Topological Spaces. This means exactly what we described earlier: If f_A and f_e are such that the outer diagram commutes (i.e., for every $p \in S^{n-1}$, $(f_A \circ \chi_e)(p) = f_e(p)$), then there exists a unique function f such that the inner diagrams commute (i.e., such that $f(a) = f|_A(a)$ for all $a \in A$ and $f_e(p) = f(h_e(p))$ for all $p \in D^n$).

Proof. It is clear that if $f : X \to Y$ is continuous, then $f|_A$ and $f \circ h_e$ are also continuous. Let's prove the converse. Suppose $f|_A$ and $f \circ h_e$ are continuous. We will show that f is continuous by verifying that for every closed subset $F \subset Y$, $f^{-1}(F)$ is closed in X. Note that

$$f^{-1}(F) \cap A = (f|_A)^{-1}(F)$$

Therefore, since $f|_A$ is continuous, $f^{-1}(F) \cap A$ is closed in A. But since A is closed in X, it follows that $f^{-1}(F) \cap A$ is closed in X. Also,

$$f^{-1}(F) \cap \overline{e} = h_e((f \circ h_e)^{-1}(F))$$

Thus, since $(f \circ h_e)^{-1}(F)$ is closed in D^n and therefore compact, $h_e((f \circ h_e)^{-1}(F))$ is compact in a Hausdorff space (since X is Hausdorff) and therefore closed in X. Moreover,

$$f^{-1}(F) = (f^{-1}(F) \cap A) \cup (f^{-1}(F) \cap \overline{e})$$

Therefore, $f^{-1}(F)$ is a union of two closed sets and therefore closed in X.
Furthermore, f exists and is unique since $X = A \cup e$, and:

$$\begin{cases} f(a) = f_A(a) & \forall a \in A \\ f(x) = f_e(h_e^{-1}|_{\mathring{D}^n})(x) & \forall x \in e \end{cases}$$

Moreover, it is well-defined because of $A \cap e = \emptyset$. $\square$

With this, we can prove that:

Proposition .2.5. For any Hausdorff space A and continuous function $\chi : S^{n-1} \to A$, there exists a unique space X, up to homeomorphism, such that X is obtained from A by attaching an n-cell with characteristic map χ.

Definition .2.6. Let X be a Hausdorff space and $A \subset X$ be closed. We say that X is obtained from A by attaching n-cells $e_i \subset X$ $(i \in I)$ if $X = A \cup (\bigcup_{i \in I} e_i)$ and the following conditions are satisfied:

(i) $A \cap e_i = \emptyset$, $\partial_{\text{cell}}(e) \subset A$, and $e_i \cap e_j \neq \emptyset$ for all $i \neq j$.

(ii) $F \subset X$ is closed if and only if $F \cap A$ is closed and $F \cap \overline{e_i}$ is closed for all $i \in I$ (This is also known as the "weak topology").

Definition .2.7 (Cellular Decomposition)**.** Let X be a Hausdorff space. A cellular decomposition of X is a sequence of closed subspaces

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \dots$$

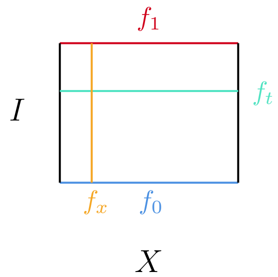
such that:

- $\bigcap_{n \in \mathbf{Z}_{\geq -1}} X_n = X$
- For each $n \geq 0$, X_n is obtained from X_{n-1} by attaching n-cells (the n-cells, along with the defining maps, are part of the structure).
- $F \subset X$ is closed if and only if $F \cap X_n$ is closed for all $n \geq 0$.

Definition .2.8. A CW complex is a Hausdorff topological space X together with a cellular decomposition. Moreover, the subspace X_n is called the n-skeleton of X.

.3 Homotopy

Definition .3.1 (Homotopy)**.** Let X and Y be topological spaces, and let $f, g : X \to Y$ be continuous functions. A homotopy from f to g is a continuous function $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. In this case, we say that f and g are homotopic, denoted as $H : f \simeq g$ or simply $f \simeq g$ when the homotopy is clear from the context.



In the diagram, f_t denotes $H(x, t)$ for a fixed t, and f_x denotes $H(x, t)$ for a fixed x. A homotopy defines a family of functions $H_t(x) = H(x, t)$, where $0 \leq t \leq 1$. We can think of this process as the parameter t representing time, and H as a function that promotes a deformation from f to g.

Proposition .3.1. For any topological spaces X and Y, homotopy is an equivalence relation.

Proof. (i) Every function f is homotopic to itself through the trivial homotopy $H(x, t) = f(x)$. Thus, the homotopy relation is reflexive.

(ii) Suppose $H : f \simeq g$. Then, a homotopy from g to f can be given by $\widetilde{H}(x, t) = H(x, 1 - t)$. Thus, the homotopy relation is symmetric.

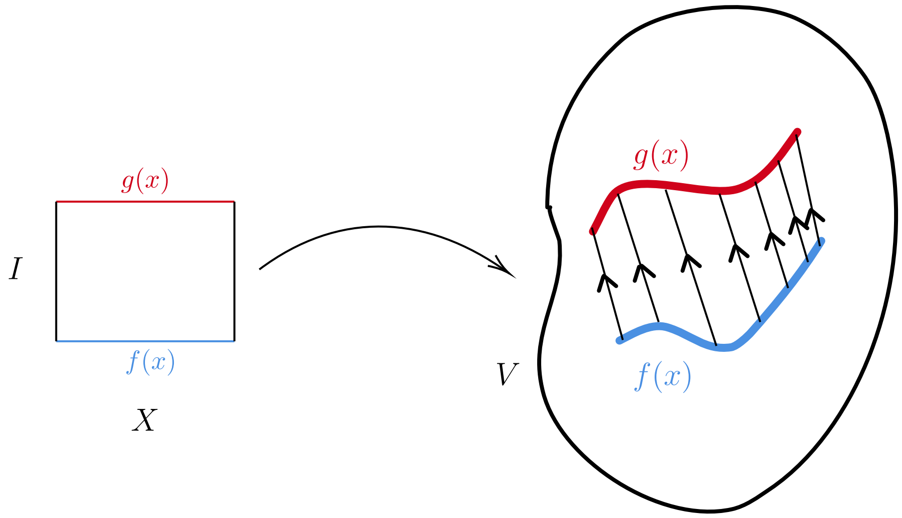
(iii) Suppose $F : f \simeq g$ and $G : g \simeq h$. We can construct $H : X \times I \to Y$ that deforms f into h. The idea is to use F to deform f into g at double speed, and then use G to deform g into h also at double speed.

$$H(x, t) = \begin{cases} F(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ G(x, 2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Note that $F(x, 1) = g(x) = G(x, 0)$, so the function H is well-defined at $t = \frac{1}{2}$. Since H is continuous on the closed sets $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ (as F and G are continuous), H is continuous on $X \times I$ by the Pasting Lemma.

□

Example .3.1. An important example is that of linear homotopy. Let V be a vector space (or $V \subset \mathbb{R}^n$ convex) and $f, g : X \to V$. Then, $f \simeq g$ via the linear homotopy given by $H(x, t) = (1 - t)f(x) + tg(x)$.



Definition .3.2 (Relative Homotopy)**.** Let X and Y be topological spaces, and let $A \subset X$. We say that $f, g : X \to Y$ are homotopic relative to A ($f \simeq g$ rel A) if there exists a homotopy $H : X \times I \to Y$ such that:

- $H(x, 0) = f(x)$
- $H(x, 1) = g(x)$
- $H(a, t) = f(a)$ for all $a \in A$

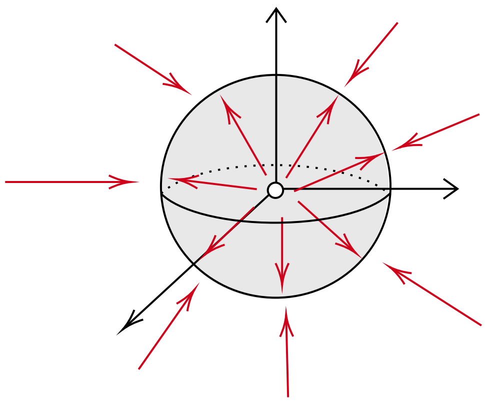
In particular, $f|_A = g|_A$.

Example .3.2. Let's see that $Id_{\mathbb{R}^{n+1}-\{0\}} \simeq r_{S^n}$ rel S^n, where:

$$r_{S^n} : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}$$

$$x \mapsto \frac{x}{\|x\|}$$

Indeed, it suffices to consider the homotopy $H(x,t) = (1-t)x + t\frac{x}{\|x\|}$.



Definition .3.3 (Homotopy Equivalence)**.** We say that the continuous function $f : X \to Y$ is a homotopy equivalence if there exists a continuous function $g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. In this case, X and Y are said to be homotopy equivalent or have the same homotopy type. The function g is called the homotopy inverse of f.

Example .3.3. From the previous example, we have that $\mathbb{R}^{n+1}-\{0\}$ and S^n are homotopy equivalent. We can consider r_{S^n} and $i : S^n \hookrightarrow \mathbb{R}^{n+1} - \{0\}$. Thus, $r_{S^n} \circ i = \mathrm{Id}_{S^n}$ and $i \circ r_{S^n} = \mathrm{Id}_{\mathbb{R}^{n+1}-\{0\}}$.

Definition .3.4 (Contractible)**.** A topological space X is said to be contractible if it is homotopy equivalent to a space consisting of only one point.

Definition .3.5 (Retraction)**.** Let $A \subset X$. A retraction of X onto A is a continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$ (i.e., $r \circ i = \mathrm{Id}_A$).

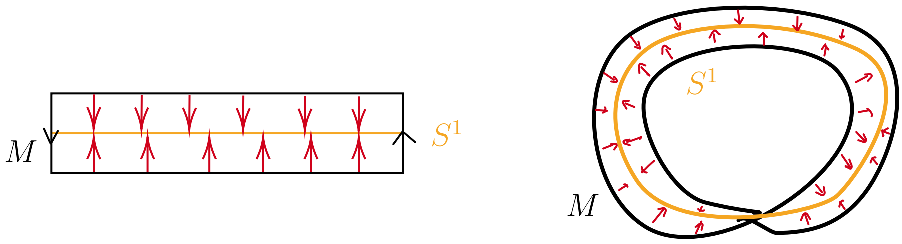
Definition .3.6 (Deformation Retract)**.** We say that a retraction is a deformation retract of X onto A if there exists a homotopy between Id_X and a retraction of X onto A. We can think of a deformation retract as a process of shrinking X onto A during a time interval $0 \leq t \leq 1$, and then defining a family of functions $f_t : X \to X$ such that $f_t(x)$ is the point to which x moves after time t. In other words, a deformation retract of X onto $A \subset X$ is a family of functions $f_t : X \to X$, $t \in I = [0,1]$, such that $f_0 = \mathrm{Id}_X$, $f_1(X) = A$, and $f_t|_A = \mathrm{Id}$ for all $t \in I$.

Example .3.4. Let V be a vector space. Then, V is contractible. In fact, consider $f : V \to \{0\}$ and $g : \{0\} \to V$, where $f \circ g = \mathrm{Id}_{\{0\}}$ and $H : g \circ f \simeq \mathrm{Id}_V$ with $H(v,t) = vt$. Note that $H(v,0) = 0 = (g \circ f)(v)$ and $H(v,1) = v = \mathrm{Id}_V(v)$.

Example .3.5. Let's denote by M the Möbius strip. Then, $M \simeq S^1$. In fact, recall that $M = I \times I/(0,t) \sim (1, 1-t)$. Let's consider S^1 as a subset of M using the homeomorphism $h : S^1 \to S^1 \subset M$ given by $e^{2\pi i s} \mapsto [s, \frac{1}{2}]$. Also, consider $i : S^1 \to M$ such that $[s, \frac{1}{2}] \mapsto [s, \frac{1}{2}]$, and $r : M \to S^1$ such that $[(s,t)] \mapsto [(s, \frac{1}{2})]$. Indeed, this is a deformation retract, as:

(i) $r \circ i = \mathrm{Id}_{S^1}$: $(r \circ i)([s, \frac{1}{2}]) = r(i([s, \frac{1}{2}])) = r([s, \frac{1}{2}]) = [s, \frac{1}{2}]$

(ii) $i \circ r \simeq \mathrm{Id}_M$: Consider the homotopy $H([s,t], \epsilon) = [s, (1-\epsilon)t + \frac{\epsilon}{2}]$



Bibliography

[Art25] Emil Artin. Theorie der zopfe. *Hamburg Abh*, 4:47–72, 1925. 1, 14

[Art47a] Emil Artin. Braids and permutations. *Annals of Mathematics*, páginas 643–649, 1947. 1

[Art47b] Emil Artin. Theory of braids. *Annals of Mathematics*, páginas 101–126, 1947. 1

[B+74] Joan S Birman et al. *Braids, links, and mapping class groups*. Number 82. Princeton University Press, 1974. 3, 9, 10, 13, 19

[BB05] Joan S Birman e Tara E Brendle. Braids: a survey. Em *Handbook of knot theory*, páginas 19–103. Elsevier, 2005. 41, 56

[Boh47] Frederic Bohnenblust. The algebraical braid group. *Annals of Mathematics*, páginas 127–136, 1947. 14

[Bud12] Geoffrey Budworth. *The knot book*. Hachette UK, 2012. 22

[BZK85] G Burde, H Zieschang e De Knots. Gruyter studies in math. 5. *Berlin, New York*, 1985. 28

[Coh10] Frederick R Cohen. Introduction to configuration spaces and their applications. Em *Braids*, páginas 183–261. World Scientific Publisher Hackensack, NJ, 2010. 12

[Cro04] Peter R Cromwell. *Knots and links*. Cambridge university press, 2004. 22, 30

[EM94] Elsayed A Elrifai e Hugh R Morton. Algorithms for positive braids. *Quarterly Journal of Mathematics*, 45(180):479–497, 1994. 21

[FN62] Ralph Fox e Lee Neuwirth. The braid groups. *Mathematica Scandinavica*, 10:119–126, 1962. 3, 9, 14, 15

[FVB+62] Edward Fadell, James Van Buskirk et al. The braid groups of e2 and s2. *Duke Math. J*, 29(243-257):7, 1962. 10, 14

[Gar69] Frank A Garside. The braid group and other groups. *The Quarterly Journal of Mathematics*, 20(1):235–254, 1969. 3, 19

[GM11] Juan González-Meneses. Basic results on braid groups. Em *Annales Mathématiques Blaise Pascal*, volume 18, páginas 15–59, 2011. 3, 15, 20

[Hat05] Allen Hatcher. *Algebraic topology.* 2005. 15, 59

[Mag34] Wilhelm Magnus. Über automorphismen von fundamentalgruppen berandeter flächen. *Mathematische Annalen*, 109(1):617–646, 1934. 14

[MK96] Kunio Murasugi e Bohdan Kurpita. *Knot theory and its applications.* Springer, 1996. 22, 34, 38

[MK12] Kunio Murasugi e Bohdan Kurpita. *A study of braids*, volume 484. Springer Science & Business Media, 2012. 3, 8, 19

[Tra98] Paweł Traczyk. A new proof of markov's braid theorem. *Banach Center Publications*, 42(1):409–419, 1998. 2, 48, 50, 56

[Vog90] Pierre Vogel. Representation of links by braids: A new algorithm. *Comment. Math. Helv*, 65(1):104–113, 1990. 2, 42, 48

[WL48] Chow Wei-Liang. On the algebraical braid group. *Ann. Math*, 49:654–658, 1948. 14

[Yam87] Shuji Yamada. The minimal number of seifert circles equals the braid index of a link. *Inventiones mathematicae*, 89(2):347–356, 1987. 2, 42, 47, 48

[Zar36] Oscar Zariski. On the poincaré group of rational plane curves. *American Journal of Mathematics*, 58(3):607–619, 1936. 1