Covering properties, reflections in elementary submodels and partitions on topological spaces

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Resumo

Rodrigo Rey Carvalho. **Propriedades de cobertura, reflexões em submodelos elementares e partições em espaços topológicos**. Tese (Doutorado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022.

Este trabalho trata de dois tópicos distintos. Primeiro tratamos sobre a teoria das partições em espaços topológicos, desenvolvendo os tópicos explorados em [27]. Adaptamos a demonstração do primeiro teorema do artigo previamente citado. Também melhoramos a consistência de um resultado feito com $\diamondsuit$, construindo um exemplo consistente com $\neg CH$. Com relação ao segundo tópico, desenvolvemos sobre os espaços definidos em [25]. Seguimos por um caminho semelhante ao feito na tese [16]. Vemos que, no caso de espaços dispersos, há preservação, com relação a submodelos elementares, para as propriedades de Rothberger, Menger e indestrutivelmente Lindelöf. Ademais continuamos a investigar tais reflexões para espaços mais gerais. Por fim, trabalhamos com espaços da forma $C_p(X)$ e submodelos elementares, estudando a interação entre $C_p(X_M)$ e $C_p(X)_M$.

Palavras-chave: Partição de espaços topológicos. Teoria de Ramsey. Forcing. Propriedades de cobertura. Submodelos elementares. Espaços de funções.

Abstract

Rodrigo Rey Carvalho. **Covering properties, reflections in elementary submodels and partitions on topological spaces**. Thesis (Doctorate). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2022.

This work develops two distinct topics. We first work with partitions on topological spaces, developing some topics found on [27]. We fixed the proof of the first theorem from the previous paper. We also improved the consistency of a result obtained using ◆ by constructing an example consistent with $\neg CH$. In relation with the second topic we studied the spaces developed on [25]. For this we followed the line of work of the thesis [16]. We see that, for scattered spaces the properties Rothberger, Menger and indestructibly Lindelöf are preserved for elementary submodels. Furthermore we continue to investigate these preservations for more general spaces. Finally we worked with C_p spaces and elementary submodels, studying the relation between $C_p(X_M)$ and $C_p(X)_M$.

Keywords: Topological space partitions. Ramsey theory. Forcing. Covering properties. Elementary submodels. Function spaces.

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Introduction

This thesis is split in two central parts. In the first one we present a study of some results obtained working with Ramsey theory and topological spaces. The object of study of this field is known as partitions of topological spaces. It appeared in the literature after the fourth Prague Topological Symposium, dating to 1976 on a work of J. Nešetřil and V. Rödl [34]. One such paper is [27], by P. Komjáth and W. Weiss, which investigates conditions to have monochromatic copies of countable ordinals. Some of the results presented in this thesis are from an accepted paper co-authored by my Ph.D. advisor, L. Junqueira, and G. Fernandes, relating to this previous paper.

The other topic this thesis concerns the use elementary submodels and topology. Many papers started to appear in this area after the systematic study from A. Dow in [13]. We follow a line of work that considers the spaces X_M, introduced in [25]. We will consider two ways of developing this topic. The first is to study the preservation of covering properties as was done by R. Figueiredo in his thesis [16], expanding it to other covering properties such as the Rothberger property. The other is the beginning of a systematic study concerning spaces of the type $C_p(X)_M$ and some variants.

In Chapter 1 we start by setting some of the notations and results that will be used throughout this work. Next in Chapter 2 we present the first topic related to partitions of topological spaces detailing some of the results from [10]. We first start by investigating an alternative version of the Cantor-Bendixson decomposition – the Sequential Cantor-Bendixson decomposition – studying some preliminary results and illustrating this new definition with some examples. After such investigation, we present a revision on a result by W. Weiss and P. Komjáth, which was the motivation for the previous decomposition. Given a coloriong this result describes a condition under which we can find monochromatic copies of each $\alpha < \omega_1$. In the next two sections of Chapter 2 we revisit an example made by P. Komjáth and W. Weiss using $\diamondsuit$ and present a version of the same example without CH. For this, in Section 2.3, we study a new club-like principle $\clubsuit_F$, that extracts the essence of the use of $\diamondsuit$ in the original construction, verifying its consistency with $\neg CH$. In the final section we make use of $\clubsuit_F$ to obtain the desired example in a model without CH.

In Chapter 3 we study spaces of the type X_M as it was done by L. Junqueira, F. Tall, K. Kunen and many others. We proceed in a similar line of work as R. Figueiredo in his thesis [16]. In the first section of this chapter we present some results from the literature to motivate our results. We center this chapter mostly around the works [25], [23], [24] and [16]. In Sections 3.2 and 3.3 we define the Rothberger, Menger and indestructibly Lindelöf properties and study their reflection. Section 3.2 is dedicated to scattered spaces and its effects on the preservation of the previous covering properties. Section 3.3 will investigate the general case for these covering properties. Several reflection results need the assumption of the reflection of the Lindelöf property. Finally, in the last section we verify some reflection results for the weakly Lindelöf property and establish a connection between the reflection of the linearly Lindelöf property with some interesting problems, one of which concerns Dowker spaces.

Finally, in Chapter 4, we study C_p spaces and elementary submodels. The initial motivation behind this investigation was to find concrete examples of topological spaces with the covering properties from the previous chapter and study whether or not we can reflect them. But, we ended up focusing on a study of spaces of the form $C_p(X)_M$ and $C_p(X_M)$. The first section of this chapter presents an overview of some results in the area of function spaces and covering properties. The last chapter presents a result that relates the spaces $C_p(X)_M$ and $C_p(X_M)$ and, assuming some extra conditions, also has a result on the preservation of the tightness for $C_p(X)_M$ and $C_p(X)$.

Chapter 1

Background content

In this chapter we will give some definitions and present some classical results that will be used throughout our work. As it is usual, we refer to [15] for most of our notation regarding topology and [29] to our notation regarding set theory. On a particular note, when we consider regularity and normality we assume that the space is T_1.

1.1 Topology

One of the topics from Chapter 2 is a generalization of the Cantor-Bendixson decomposition for topological spaces. Furthermore, several results from Chapter 3 depend on the structure of scattered spaces. We will present some basic results regarding this topic.

Definition 1.1.1. *Given a topological space $\langle X, \tau \rangle$ we say that:*

- *X is* scattered *if, for every non-empty subspace $Y \subset X$, there is an isolated point $y \in Y$.*
- *X is* perfect *if there are no isolated points in X.*

Definition 1.1.2. *Given any topological space Y, define*

$$I(Y) = \{y \in Y : y \text{ is an isolated point of } Y\}.$$

Definition 1.1.3. *Given a scattered space X, we define recursively:*

- *$X^{(0)} = X$;*
- *$I_\alpha(X) = I(X^{(\alpha)})$;*
- *$X^{(\alpha)} = X \setminus \bigcup_{\beta<\alpha} I_\beta(X)$.*

Note that we obtain a stratified partition of X and we stop this process when we exhaust all of its points. This prompts the following definition.

Definition 1.1.4. *Given a scattered space X and an element $x \in X$,*

- *$ht(x, X) = \alpha$ is the* height *of x, where α is the only ordinal such that $x \in I_\alpha(X)$.*

- $ht(X) = min\{\alpha : I_\alpha(X) = \emptyset\}$.

This stratification of the space also gives us an interesting property relating to the open sets of X.

Proposition 1.1.5. *Given a scattered space X, $x \in X$ and V a neighbourhood of x, there is an open neighbourhood U of x such that $U \subset V \cap \left(\bigcup_{\beta \leq ht(x,X)} I_\beta(X)\right)$ and $U \cap I_{ht(x,X)}(X) = \{x\}$.*

This gives us a basis for the topology formed by such "downward" open sets. Note also that, for scattered spaces, $I_0(X)$ is dense in X, since any open neighbourhood of x must intercept every level of height $< ht(x, X)$.

Now one well-known result is the following:

Proposition 1.1.6. *Given a topological space $\langle X, \tau \rangle$, there is a partition $X_S, X_P \subset X$ of X where X_S is scattered and X_P is perfect.*

The proof of such fact also consists of exhausting the isolated points of X in a iterated manner in such way that what is left must be perfect.

To conclude this section we briefly present some definitions of covering properties and cardinal functions that will be used throughout this thesis.

Definition 1.1.7. *We say that a topological space X is:*

1. Compact *if every open cover of X has a finite subcover;*
2. Countably compact *if every countable open cover of X has a finite subcover;*
3. Lindelöf *if every open cover of X has a countable subcover;*
4. Hereditarily Lindelöf *if for every subspace Y of X, Y is Lindelöf.*

Definition 1.1.8. *Given a topological space X and $x \in X$, we define the following:*

- $L(X) = min\{\kappa : \forall \mathcal{U}$ *open cover of* $X\ \exists \mathcal{U}' \subset \mathcal{U}\ (\mathcal{U}'$ *is a subcover* $\wedge |\mathcal{U}'| \leq \kappa)\}$ *is the* Lindelöf degree of X.
- $\chi(x, X) = min\{\kappa : \exists \mathcal{B}$ *neighbourhood basis for* x *in* $X\ (|\mathcal{B}| \leq \kappa)\}$ *is the* character of x in X.
- $\chi(X) = sup\{\chi(y, X) : y \in X\}$ *is the* character of X.
- $t(x, X) = min\{\kappa : \forall A \subset X\ (x \in \overline{A} \Longrightarrow \exists B \subset A(x \in \overline{B} \wedge |B| \leq \kappa))\}$ *is the* tightness of x in X.
- $t(X) = sup\{t(y, X) : y \in X\}$ *is the* tightness of X.

1.2 Combinatorial properties

In chapter 2 we work with forcing and ♦-like properties. In what follows we give some basic definitions on this topic that can be found in [29] by K. Kunen.

Definition 1.2.1. *Let ρ be an ordinal. We say that $C \subset \rho$ is* unbounded in ρ *if, for all $\alpha \in \rho$, there is $\gamma \in C$ such that $\alpha < \gamma$.*

Definition 1.2.2. *Let κ be a cardinal. We say that $C \subset \kappa$ is a* club in κ *(closed and unbounded) if it is unbounded in κ and, for every $\alpha \in \kappa$, if α is a limit ordinal and $C \cap \alpha$ is unbounded in α, then $\alpha \in C$.*

Definition 1.2.3. *Let κ be a cardinal. We say that $S \subset \kappa$ is a stationary set in κ if, for every club C in κ, we have $C \cap S \neq \emptyset$.*

When there is no ambiguity to which cardinal we are working on we shall just say that C is a club and S is as stationary set. For the rest of this section we will work with ω_1 since it is what we will need, but the definitions and results can be generalized for any regular uncountable cardinal.

Definition 1.2.4. *Let $S \subset \omega_1$ be a stationary set and $f : S \setminus \{\emptyset\} \longrightarrow \omega_1$ a function. f is regressive if, for all $\alpha \in dom(f)$, we have $f(\alpha) < \alpha$.*

The next lemma is also known as the pressing down lemma.

Lemma 1.2.5 (Fodor's lemma). *If $f : S \setminus \{\emptyset\} \longrightarrow \omega_1$ is a regressive function then there are a stationary set $S' \subset S$ and $\alpha \in \omega_1$ such that $f[S] = \{\alpha\}$.*

Now we define the combinatorial principle $\blacklozenge$, that will be used in Chapter 2.

Definition 1.2.6. *A sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ of subsets of ω_1 is called a $\blacklozenge$-sequence if for all $A \subset \omega_1$ the set $\{\alpha : A \cap \alpha = A_\alpha\}$ is stationary.*

Definition 1.2.7. *A sequence $\langle A_\alpha^n : n < \omega \wedge \alpha < \omega_1 \rangle$ is called a $\blacklozenge^*$-sequence if the following holds:*

(1) For every $\alpha \in \omega_1$ and $n \in \omega$, $A_\alpha^n \subset \alpha$;

(2) For every $A \subset \omega_1$, there is a C club in ω_1 such that, for every $\alpha \in C$, there is $n \in \omega$ satisfying $A \cap \alpha = A_\alpha^n$.

Such combinatorial principles are consistent with ZFC and are known to hold, for example, when we consider $V = L$; see e.g. [30].

1.3 Forcing

We follow up with some basic definitions and results on the topic of forcing.

Definition 1.3.1. *A forcing is a triple $\langle \mathbb{P}, \leq, 1 \rangle$ consisting of a partial order $\langle \mathbb{P}, \leq \rangle$ and a maximal element 1. Given $p, q \in \mathbb{P}$ we say that:*

- *q is stronger than p, if $q \leq p$;*
- *q is incompatible with p, denoted by $q \perp p$, if there is no $t \in \mathbb{P}$ such that $t \leq p, q$, otherwise we say that p and q are compatible.*

When there is no ambiguity we shall omit both maximal element and order and refer to the forcing as $\mathbb{P}$.

Definition 1.3.2. *Let $\mathbb{P}$ be a forcing, $t \in \mathbb{P}$ and $D \subset \mathbb{P}$. We say that:*

- *D is dense in $\mathbb{P}$ if, for all $p \in \mathbb{P}$, there is $q \in D$ such that $q \leq p$.*

- *D is dense below t if, for all $p \leq t$, there is $q \in D$ such that $q \leq p$.*

Definition 1.3.3. *Let $\mathbb{P}$ be a forcing and $G \subset \mathbb{P}$. G is said to be a filter over $\mathbb{P}$ if the following conditions hold:*

- *$1 \in G$;*
- *For all $p, q \in G$, they are compatible;*
- *If $p \in G$, then, for all $q \geq p$, we have $q \in G$.*

We say that a filter G is $\mathbb{P}$-generic over M if G intersects all dense sets of $\mathbb{P}$ that are in M.

Definition 1.3.4. *Given a forcing $\mathbb{P}$, we define recursively that τ is a $\mathbb{P}$-name if it is a binary relation and, for all $\langle \sigma, p\rangle \in \tau$, we have that σ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. We refer to $V^{\mathbb{P}}$ as the class of all $\mathbb{P}$-names. Given M a transitive model for $ZF - P$ we set*

$$M^{\mathbb{P}} = V^{\mathbb{P}} \cap M = \{\tau \in M : (\tau \text{ is a } \mathbb{P} - \text{name})^M\}.$$

The following $\mathbb{P}$-names are particularly interesting.

Definition 1.3.5. *Given any set x we define recursively $\check{x} = \{\langle \check{y}, 1\rangle : y \in x\}$.*

We now can consider $\mathcal{FL}_{\mathbb{P}}$ as the language with $\in$ as a binary relation and $\mathbb{P}$-names as constant symbols.

Definition 1.3.6. *Given a forcing $\mathbb{P}$, a $\mathbb{P}$-name τ and $G \subset \mathbb{P}$ we define τ_G recursively as $\tau_G = \{\sigma_G : \langle \sigma, p\rangle \in \tau \wedge p \in G\}$.*

Definition 1.3.7. *As in the conditions of the previous definitions, given $p \in \mathbb{P}$, a formula $\phi(x_1, \cdots, x_n)$ and $\mathbb{P}$-names $\tau_1, \cdots, \tau_n$, we say that p forces $\phi(\tau_1, \cdots, \tau_n)$ and write $p \Vdash \phi(\tau_1, \cdots, \tau_n)$ if $(\phi((\tau_1)_G, \cdots, (\tau_n)_G))^{V[G]}$ holds for every $\mathbb{P}$-generic filter G such that $p \in G$.*

Proposition 1.3.8. *In the conditions of the previous definitions the following are equivalent:*

1. *$p \Vdash \phi(\tau_1, \cdots, \tau_n)$;*
2. *$\forall r < p \; r \Vdash \phi(\tau_1, \cdots, \tau_n)$;*
3. *The set $\{r < p : r \Vdash \phi(\tau_1, \cdots, \tau_n)\}$ is dense below p.*

Lemma 1.3.9 (Truth Lemma)**.** *Let M be a countable transitive model for $ZF - P$ and $\mathbb{P} \in M$ a forcing. If ψ is a sentence of $\mathcal{FL}_{\mathbb{P}} \cap M$ and G is a $\mathbb{P}$-generic filter over M, then $M[G] \models \psi$ if and only if there is a $p \in G$ such that $p \Vdash \psi$.*

Lemma 1.3.10. *For any forcing $\mathbb{P} \in M$ and formula $\phi(x) \in \mathcal{FL}_{\mathbb{P}} \cap M$ with only x as its free variable, we have:*

1. *$p \Vdash \forall x \phi(x)$ if and only if $p \Vdash \phi(\tau)$ for all $\tau \in M^{\mathbb{P}}$;*
2. *$p \Vdash \exists x \phi(x)$ if and only if $\{q \leq p : \exists \tau \in M^{\mathbb{P}}(q \Vdash \phi(\tau))\}$ is dense below p.*

The following is a strengthening of item (2) of the previous lemma.

Theorem 1.3.11 (Maximal Principle)**.** *For any forcing $\mathbb{P} \in M$ and formula $\phi(x) \in \mathcal{FL}_{\mathbb{P}} \cap M$ with only x as its free variable we have: $p \Vdash \exists x \phi(x)$ if and only if $p \Vdash \phi(\tau)$ for some $\tau \in M^{\mathbb{P}}$.*

The following definitions are useful when trying to control what happens to the size of the set of the conditions used to decide a name.

Definition 1.3.12. *Let $\mathbb{P}$ be a forcing. a subset $A \subset \mathbb{P}$ is said to be an antichain if all elements of A are pairwise incompatible, that is for all distinct $p, q \in A$ we have $q \perp p$.*

Definition 1.3.13. *Let $\mathbb{P}$ be a forcing and κ a cardinal. We say that $\mathbb{P}$ has the κ-cc if, for every antichain A in $\mathbb{P}$, $|A| < \kappa$.*

Definition 1.3.14. *Let $\mathbb{P}$ be a forcing and $\tau \in V^{\mathbb{P}}$ a name. A* nice name *for a subset of τ is a name of the form $\bigcup\{\{\sigma\} \times A_\sigma : \sigma \in \operatorname{dom}(\tau) \wedge A_\sigma$ is an antichain in $\mathbb{P}\}$.*

Before considering iterated forcing we define the Cohen forcing.

Definition 1.3.15. *Let $Fn(\omega, 2)$ be the set of the finite functions with domain in ω and image in 2. The triple $\langle Fn(\omega, 2), \emptyset, \supset\rangle$ is the Cohen forcing.*

Now we state the notation used when dealing with iterated forcing. We follow the definition found in K. Kunen [29], but we opt to use the notation $\mathring{x}$ when considering names for the objects in the stages of the iteration.

Definition 1.3.16. *For any ordinal α, the α-stage iterated forcing construction is a pair of sequences*

$$\left\langle \langle(\mathbb{P}_\gamma, \leq_\gamma, 1_\gamma) : \gamma \in \alpha\rangle, \langle(\mathring{\mathbb{Q}}_\gamma, \mathring{\leq}_{\mathring{\mathbb{Q}}_\gamma}, \mathring{1}_{\mathring{\mathbb{Q}}_\gamma}) : \gamma \in \alpha\rangle \right\rangle$$

satisfying the following conditions:

1. *Each $\langle\mathbb{P}_\gamma, \leq_\gamma, 1_\gamma\rangle$ is a forcing;*
2. *Each $\langle\mathring{\mathbb{Q}}_\gamma, \mathring{\leq}_{\mathring{\mathbb{Q}}_\gamma}, \mathring{1}_{\mathring{\mathbb{Q}}_\gamma}\rangle$ is a $\langle\mathbb{P}_\gamma, \leq_\gamma, 1_\gamma\rangle$-name for a forcing;*
3. *Each $p \in \mathbb{P}_\gamma$ is a sequence of the form $\langle\mathring{q}_\xi : \xi < \gamma\rangle$, where, for each $\xi < \gamma$, we have $\mathring{q}_\xi \in dom(\mathring{\mathbb{Q}}_\xi)$;*
4. *If $\xi < \gamma$ and $p \in \mathbb{P}_\gamma$, then $p \restriction_\xi \in \mathbb{P}_\xi$;*
5. *If $\xi < \gamma$, $p \in \mathbb{P}_\xi$ and p' is the sequence given by $p' \restriction_\xi = p$ and $p'(\mu) = \mathring{1}_{\mathring{\mathbb{Q}}_\mu}$ for $\xi \leq \mu < \gamma$, then $p' \in \mathbb{P}_\gamma$ and is denoted by $i_{\xi,\gamma}(p)$;*
6. *1_γ is the sequence $\langle\mathring{1}_{\mathring{\mathbb{Q}}_\xi} : \xi < \gamma\rangle$;*
7. *If $p, p' \in \mathbb{P}_\gamma$, then $p \leq_\gamma p'$ if and only if $p \restriction_\xi \Vdash_{\mathbb{P}_\xi} p(\xi) \leq_\xi p'(\xi)$ for all $\xi < \gamma$;*
8. *If $\gamma + 1 \leq \alpha$, then $\mathbb{P}_{\gamma+1}$ is the set of all $p^\frown \mathring{q}$ such that $p \in \mathbb{P}_\gamma$, $\mathring{q} \in dom(\mathring{\mathbb{Q}}_\gamma)$ and $p \Vdash_{\mathbb{P}_\gamma} \mathring{q} \in \mathring{\mathbb{Q}}_\gamma$.*

Many times it is useful to restrict what happens on the limit stages of an iteration. In this work we will be interested in the iterated forcing with countable support as we state below.

Definition 1.3.17. *An iterated forcing of length α is said to have countable support if, for every limit ordinal $\beta \leq \alpha$, $p \in \mathbb{P}_\beta$ only if the set $\{\gamma < \beta : p(\gamma) \neq \mathring{1}_\gamma\}$ is countable.*

When considering an iterated forcing it is useful to know what is happening in each stage of the iteration in comparison with the final iteration. The next definition helps us with that.

Definition 1.3.18. *In the conditions of the previous definition, if G is a $\mathbb{P}_\alpha$-generic over our ground model M, then, for all $\xi < \alpha$, define $G \restriction \xi = i^{-1}_{\xi,\alpha}(G)$.*

It follows that G_ξ is a $\mathbb{P}_\xi$-generic filter over M and $M[G \restriction \xi] \subset M[G \restriction \gamma]$ if $\xi < \gamma \leq \alpha$. The idea behind this definition is that it is useful to determine when certain (small) objects already appeared in previous steps of the iteration considering a generic G.

1.4 Elementary submodels

We now consider the theory of elementary submodels and take as reference the book [30]. In what follows we state some definitions and results that will be used throughout Chapters 3 and 4.

Definition 1.4.1. *Given a cardinal θ the set $H(\theta)$ is the following:*

$$\{x \in WF : |trcl(x)| < \theta\}$$

Definition 1.4.2. *Let $\phi(x_1, \cdots, x_n)$ be a formula and M and N be two classes such that $M \subset N$. We say that ϕ is absolute between M and N if, for all $a_1, \cdots, a_n \in M$, we have $M \vDash \phi(a_1, \cdots, a_n)$ if and only if $N \vDash \phi(a_1, \cdots, a_n)$. In this case we write $M \prec_\phi N$.*

Definition 1.4.3. *Let M and N be two classes as in the previous definition and Σ be a finite collection of formulas. We write $M \prec_\Sigma N$ if for all $\phi \in \Sigma$ we have $M \prec_\phi N$. In the case that $M \prec_\phi N$ for every formula ϕ we say that M is an elementary submodel of N and write $M \prec N$.*

Theorem 1.4.4 (Reflection Theorem). *If N is a class and, for each ordinal α, $N(\alpha)$ is a set satisfying:*

(1) $\alpha < \beta$ implies $N(\alpha) \subset N(\beta)$;

(2) $N(\gamma) = \bigcup_{\alpha<\gamma} N(\alpha)$ for every limit ordinal γ;

(3) $N = \bigcup_{\alpha \in ON} N(\alpha)$,

then, for any formulas $\phi_1, \cdots, \phi_n$ we have: $\forall\alpha\exists\beta(\phi_1, \cdots, \phi_n$ are absolute for $N(\beta), N)$.

Throughout this work we will be referring to elementary submodels of $H(\theta)$ for a sufficiently large θ. This allows us to have the needed structure to prove what we need and to apply some further results.

Theorem 1.4.5 (Tarski-Vaught criterion). *Let M and N be sets such that $M \subset N$. Then the following are equivalent:*

- *$M \prec N$;*

- *For each $\phi(y, x_1, \cdots, x_n)$ formula and sequence $\{a_1, \cdots, a_n\} \subset M$, if $N \models \exists y \phi(y, a_1, \cdots, a_n)$, then there is $a \in M$ such that $N \models \phi(a, a_1, \cdots, a_n)$.*

This result is a central tool to several results since it guarantees the existence of several objects on the lower model when we work with elementary submodels. Another result that works in the same way considers the idea of definability.

Definition 1.4.6. *A set b is said to be definable by parameters $a_0, \cdots, a_n$ if there exists a formula $\phi(x, y_0, \cdots, y_n)$ such that*

$$\forall x(\phi(x, a_0, \cdots, a_n) \longleftrightarrow x = b)$$

We shall say that a set is definable by parameters in M if the parameter in the definition above are in M.

Proposition 1.4.7. *If b is definable by parameters in M considering the formula $\phi(x, \vec{y})$ and $M \prec_{\{\exists x \phi(x,\vec{y}), \phi(x,\vec{y})\}} H(\theta)$, then $b \in M$.*

The next result is also very important since it allows us to better select elementary submodels, through a closure process, satisfying certain conditions.

Theorem 1.4.8 (Löwenhein-Skolem theorem)**.** *Let A and N be sets such that $A \subset N$. There exists a set M such that $A \subset M$, $M \prec N$, and $|A| + \aleph_0 = |M|$.*

Now the next definition gives us a distinction on elementary submodels.

Definition 1.4.9. *We say that an elementary submodel M is countably closed (ω-closed) if $[M]^\omega \subset M$. We say that an elementary submodel is ω-covering if for every $A \in [M]^\omega$ there exists $B \in M$ such that $A \subset B$.*

The idea of ω-covering elementary submodels first appears in A. Dow's [13]. This is of interest since additional results can be derived using such restrictions on the elementary submodel M. A particularly useful one is the following result by L. Junqueira from [23].

Proposition 1.4.10 (L. Junqueira [23])**.** *If M is an ω-covering elementary submodel, then $\omega_1 \subset M$.*

In regards to the size of such elementary submodels, we know that if M is countably closed then $|M| \geq \mathfrak{c}$. Indeed, since $\omega \subset M$ we must also have $[\omega]^\omega \subset M$. In the same way, the previous result gives that, for and ω-covering M, $|M| \geq \omega_1$. The following result is from [13] and guarantees that it is possible to have M ω-covering of size ω_1.

Proposition 1.4.11 (A. Dow [13])**.** *Let $A \subset H(\theta)$ be such that $|A| \leq \omega$. There exists $M \prec H(\theta)$ such that $A \subset M$, $|M| \leq \omega_1$ and M is ω-covering.*

Proof. We shall define recursively the following $\subset$-increasing sequence of elementary submodels $\{M_\alpha : \alpha \in \omega_1\}$ in the following way:

- M_0 is an elementary submodel containing A of size $\aleph_0$ given by Theorem 1.4.8.
- Assume M_β countable elementary submodel already defined for a $\beta \in \omega_1$. Take $M_{\beta+1}$ countable as in Theorem 1.4.8 containing $M_\beta \cup \{M_\beta\}$.

- If $\alpha \in \omega_1$ is a limit ordinal and M_β is a countable elementary submodel defined for all $\beta < \alpha$ then $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$.

We just need to verify that M_α for the limit step is indeed a countable elementary submodel. Indeed, M_α is countable since it is the countable union of countable sets. Furthermore, by the Tarski-Vaught criterion, to see that $M_\alpha \prec H(\theta)$ we just need to verify that for every formula $\phi(y, x_1, \cdots, x_n)$ and sequence $\{a_1, \cdots, a_n\} \subset M_\alpha$, if $H(\theta) \vDash \exists y \phi(y, a_1, \cdots, a_n)$, then there is $a \in M_\alpha$ such that $H(\theta) \vDash \phi(a, a_1, \cdots, a_n)$. Fix such a formula and a sequence. Since $(M_\beta)_{\beta<\alpha}$ is a $\subset$-increasing sequence we have $\gamma < \alpha$ such that $\{a_1, \cdots, a_n\} \subset M_\gamma$. Now, since $M_\gamma \prec H(\theta)$, there is $a \in M_\gamma \subset M_\alpha$ such that $H(\theta) \vDash \phi(a, a_1, \cdots, a_n)$. Now, in the same way we have $M = M_{\omega_1} = \bigcup_{\alpha<\omega_1} M_\alpha \prec H(\theta)$ and $|M| \leq \omega.\omega_1 = \omega_1$. To finish this proof we will see that M is ω-closed. Indeed, given $B \in [M]^\omega$, by the regularity of ω_1, there is $\gamma \in \omega_1$ such that $B \subset M_\gamma$. But now $B \subset M_\gamma \in M_{\gamma+1} \subset M$, concluding this proof. $\square$

Chapter 2

Partitions of topological spaces

The topic of partitions of topological spaces is a very interesting application of Ramsey theory in topology. It surfaces on the literature by the 70s in works such as [17] by H. Friedman considering mostly ordinals but incorporating some topological elements such as closedness. A more topological approach to this topic first appears in the fourth Prague topological Symposium dating to 1976 on a work of J. Nešetřil and V. Rödl [34]. A more systematic study of this area is done by W. Weiss [42] in the chapter Partitioning Topological Spaces on the book Mathematics of Ramsey Theory.

This chapter will cover in particular some topics of the work [27] of P. Komjáth and W. Weiss, and will present some of the results obtained in [10], a joint work with L. Junqueira and G. Fernandes.

We start by fixing the notation we will use throughout this chapter.

Definition 2.0.1. *Given two topological spaces X, Y and two cardinals λ, κ we say that $X \longrightarrow (Y)^{\lambda}_{\kappa}$ if, for every function $f : [X]^{\lambda} \longrightarrow \kappa$, there exists W a subspace of X homeomorphic to Y such that $f|_{[W]^{\lambda}}$ is constant. We say that W is a monochromatic copy of Y in regards to f.*

With the notation of the previous definition, when there could be ambiguity in regards to Y, as is the case when working with ordinals, we shall make the following distinction $X \longrightarrow (top\, \alpha)^{\lambda}_{\kappa}$.

In the work [27] mentioned above, the authors show the following result:

Theorem 2.0.2 (P. Komjáth, W. Weiss [27])**.** *Let X be a regular topological space with $X \longrightarrow (top\, \omega + 1)^{1}_{\omega}$ and $\chi(X) < \mathfrak{b}$. Then $X \longrightarrow (top\, \alpha)^{1}_{\omega}$ for all $\alpha < \omega_1$.*

In the seminars organised by L. Junqueira a former student of hers R. Rodrigues presented the original proof of this theorem and we found a statement that we could not justify. More specifically, during a step of the construction they needed to guarantee the existence of a certain converging sequence, but we did not think there were sufficient conditions for that in the proof when the character was assumed $< \mathfrak{b}$.

In a private communication, L. Junqueira asked W. Weiss about the proof. He recognized that there was indeed a problem in the way the proof was written and suggested a way to rectify it, changing the kind of decomposition that was used in the proof. The first section of this chapter will cover the decomposition suggested by W. Weiss. In the second section we provide some examples to contextualize Theorem 2.0.2 and show how to fix the proof.

Another question that originates from the same article is whether the bound on the character exhibited in Theorem 2.0.2 is the best possible. In [27] a positive answer for this question is obtained using $\diamondsuit$. In [10] G. Fernandes, L. Junqueira and I show that there is an example that also confirms the bound without the assumption of CH. For this we define a new $\clubsuit$-like principle and explore its properties in the third section. Then we use this principle to construct the example in the final section.

2.1 The sequential decomposition

We start this chapter with the study of the alternative decomposition suggested by W. Weiss. As mentioned before the problem with the use of the Cantor-Bendixson decomposition is that we may not find certain converging sequences when needed, which is not a problem in case the space is first countable. So we do our new decomposition considering converging sequences. That is, using the Cantor-Bendixson construction we remove the isolated points in each step to obtain a stratification of any given topological space, resulting in a scattered subspace and a perfect one by the end of the process. In the new decomposition we remove points that do not have injective ω-sequences converging to them.

Definition 2.1.1. *Given a topological space Y, using recursion on α ordinal define:*

$$SI_0(Y) = \{x \in Y : \text{ there is no injective sequence } s \in Y^\omega \text{ converging to } x\},$$

and $SI_\alpha(Y) = SI_0(Y_S^\alpha)$, *where* $Y_S^\alpha = Y \setminus \bigcup\{SI_\beta(Y) : \beta < \alpha\}$

We set the sequential height of a topological space Y as

$$Sh(Y) = min\{\alpha : SI_\alpha(Y) = \emptyset\}.$$

We note that the sequential height is indeed well defined since at the very least $|Y|^+ \in \{\alpha : SI_\alpha(Y) = \emptyset\}$.

Definition 2.1.2. *A topological space Y is S-scattered if $Y_S^{Sh(Y)} = \emptyset$ or, equivalently, if $Y = \bigcup\{SI_\alpha(Y) : \alpha < Sh(Y)\}$.*

Analogous to the scattered and perfect decomposition of a space, we define:

Definition 2.1.3. *A topological space Y is S-perfect if, for all $x \in Y$, there is an injective sequence converging to x.*

Now, just from this definitions we have:

Corollary 2.1.4. *Given a topological space Y, $Y_S^{Sh(Y)}$ is S-perfect.*

Scattered spaces have some very good properties regarding its structure and neighbourhoods. The following results are an attempt to emulate such properties for the S-scattered spaces.

This next proposition relates to the fact that, given a scattered space and a point x, we may consider open neighbourhoods to be as in Proposition 1.1.5.

Proposition 2.1.5. *Let Y be an S-scattered topological space, $\alpha > 0$ and $x \in SI_\alpha(Y)$. Then, for all $\beta < \alpha$, there is an injective sequence $s : \omega \longrightarrow Y$ converging to x, such that $s[\omega] \subset \bigcup\{SI_\gamma(Y) : \gamma \in [\beta, \alpha)\}$.*

Proof. Suppose that there is a $\beta < \alpha$ such that, for all injective sequences s converging to x, we have

$$s[\omega] \not\subset \bigcup\{SI_\gamma(Y) : \gamma \in [\beta, \alpha)\}.$$

In particular, we must have

$$|s[\omega] \cap (\bigcup\{SI_\gamma(Y) : \gamma \in [\beta, Sh(Y))\})| < \aleph_0,$$

otherwise, we could take a convergent sub-sequence contradicting the hypothesis or contradicting $x \in SI_\alpha(Y)$. This condition gives us that there are no sequences in Y_S^β converging to x and, therefore, in the worst-case scenario, $x \in SI_\beta(Y)$. That is a contradiction, hence, the thesis must hold. □

This proposition is useful in the sense that it gives us a lower estimate for the cardinality of the levels of the sequential decomposition.

Proposition 2.1.6. *Let Y be a topological space and $\alpha < Sh(Y)$. If $\alpha + 1 < Sh(Y)$ then $|SI_\alpha(Y)| \geq \aleph_0$.*

Proof. Note that $\alpha + 1 < Sh(Y)$ yields $SI_{\alpha+1}(Y) \neq \emptyset$. Let $x \in SI_{\alpha+1}$. By Proposition 3, we have an injective sequence $s : \omega \longrightarrow Y$ such that s converges to x and $s[\omega] \subset SI_\alpha(Y)$. Now $\aleph_0 \leq |s[\omega]| \leq |SI_\alpha(Y)|$. □

Note that we may not have the same behaviour as the original Cantor-Bendixson decomposition regarding the open sets as seen in Proposition 1.1.5. But, the previous results guarantee us that any open neighbourhood of x has at least countably many points in the levels lower than the height of x.

We give some examples on how this new decomposition works on some topological spaces.

Example 2.1.7. *The space $\omega_1 + 1$ with the order topology is such that ω_1 is in the 0^{th}-level of the decomposition, together with the successor ordinals that are isolated. The rest of the decomposition is the same as the Cantor-Bendixson for the countable heights. Indeed, the only point with uncountable character was ω_1. So, for all other points, they are isolated if and only if there are no converging injective sequences to it.*

This illustrates the importance of the character when comparing both decompositions, as it was stated in the beginning of this chapter. This motivates the following:

Proposition 2.1.8. *If X is infinite and first countable, then both decompositions are the same on X.*

Proof. Indeed, since being isolated is equivalent to not having an injective converging sequence for such spaces X, each step of the decompositions are identical. □

For an illustration in $\mathbb{R}$ both decompositions are the same since it is a first countable space that is perfect. We will explore this space a little bit more on the next section.

This result concludes this section, but there are some questions that remain relevant. In particular, it would be interesting to know whether the sequential decomposition, or any other decomposition of the same kind, has other applications.

2.2 Rectifying the original theorem

We begin this section by contextualizing the hypothesis in Theorem 2.0.2. We will present some concrete examples of topological spaces satisfying the conditions in its hypothesis.

Example 2.2.1. *The real line $\mathbb{R}$ is regular, has countable character and is such that $\mathbb{R} \longrightarrow (top\, \omega + 1)^1_\omega$.*

Proof. This is due to the fact that $\mathbb{R}$ is hereditarily Lindelöf and first countable. Given any coloring $f : \mathbb{R} \longrightarrow \omega$ there must be $Y \subset \mathbb{R}$ uncountable monochromatic set according to f. Using that Y is Lindelöf, it must have an accumulation point. Therefore, using the countable character, we obtain our monochromatic sequence. □

The example made above is a space in which the original proof of Theorem 2.0.2 can be applied as it is. Indeed, since the space is first countable, the sequence we could not find is obtained by naturally using the countable neighborhood basis. Furthermore, by means of Proposition 2.1.8, the Cantor-Bendixson and the sequential Cantor-Bendixson decompositions are the same. The next example is similar to the previous one but it is not second countable, and since it is separable, it also cannot be metrizable.

Example 2.2.2. *The Sorgenfrey line $\mathbb{R}_S$ is regular, has countable character and is such that $\mathbb{R}_S \longrightarrow (top\, \omega + 1)^1_\omega$.*

As we can see, any hereditarily Lindelöf uncountable space is a space that satisfies part of the hypothesis from Theorem 2.0.2. Furthermore, the classical example $\mathbb{R}$ works with the original proof as seen above. For an example that is not first countable nor hereditarily Lindelöf, we refer to $\omega_1 + 1$.

Example 2.2.3. *The space $\omega_1 + 1$ with the order topology is regular, has character ω_1 and $\omega_1 + 1 \longrightarrow (top\, \omega + 1)^1_\omega$.*

Proof. Indeed, fix a coloring $f : \omega_1 + 1 \longrightarrow \omega$. We will construct a club C such that all colors that are unbounded in ω_1 are also unbounded in the elements of C. Let $A = \{a_n : n \in \omega\} \subset \omega$ be a list of all unbounded colors in ω_1 and $\beta \in \omega_1$ be such that all elements with bounded colors are below β. Our C will be the range of a strictly increasing sequence $\langle \alpha_\gamma : \gamma \in \omega_1 \rangle$ of countable ordinals. For α_0 we take the limit of a strictly increasing ω-sequence of ordinals greater than β satisfying that the colors of A repeat an infinite amount of times in this sequence. We take $\alpha_\gamma = \bigcup_{\beta<\gamma} \alpha_\beta$ if γ is a limit ordinal. Now, $\alpha_{\gamma+1}$ is taken to be the limit of an ω-sequence of ordinals greater than α_γ in the same way we took α_0. Fix α element of C that is above the ones with limited colors. Now $f(\alpha)$ is a colour unlimited in α so we may construct a converging sequence to α of color $f(\alpha)$ using that α has countable cofinality. ☐

The idea behind this example will be the cornerstone of the constructions in Section 2.4, when we find a counterexample for increasing the bound of the character on Theorem 2.0.2. But first let us present a way to fix the problem that appears when we weaken the bound on the character from ω_1 to $\mathfrak{b}$.

After studying the new decomposition and understanding some of its properties we are finally ready to start working on the main result of this section. This next lemma is an auxiliary result which we use to prove Theorem 2.0.2.

Lemma 2.2.4. *Let X be a regular, S-perfect, non empty topological space. If $\chi(X) < \mathfrak{b}$, then, for all $\alpha < \omega_1$, and for each non-empty open set $V \subset X$ there exists $\Phi : \alpha \longrightarrow V$ such that Φ is a homeomorphism on its image.*

Proof. The copy of α will be recursively constructed. Fix α and suppose that, for all $\beta < \alpha$, $y \in X$ and open neighbourhoods A of y, there are $f_\beta : \beta \longrightarrow A$ homeomorphism on its image such that if β is the successor of a limit ordinal γ then $f_\beta(\gamma) = y$. Fix $x \in X$ and V open neighbourhood of x.

Using the Cantor normal decomposition consider $\alpha = \omega^{\beta_1}.n_1 + \cdots + \omega^{\beta_k}.n_k$. We will analyse three cases, $\alpha = \omega^{\beta_1}.1 + 1$, $\alpha = \omega^{\beta_1}.1$, and the case which the Cantor normal decomposition of α is not of the two previous forms. For the latter case, using the S-perfectness, fix an injective sequence s converging to x such that $s[\omega] \subset V$. By the injectivity of s and regularity of X, fix $y \in V$ and open sets W_1, W_2 satisfying $y \in W_1 \subset \overline{W_1} \subset V$, $x \in W_2 \subset \overline{W_2} \subset V$ with $\overline{W_1} \cap \overline{W_2} = \emptyset$. We can easily split α into two smaller ordinals and use the recursion hypothesis to obtain their copies in each of the disjoint open sets above. Using these copies we construct a copy of α as desired.

Now we address the first two cases by considering whether or not β_1 is a limit ordinal. This is the case that was explained in [27] and will be considered in a similar way. The only difference is that we take the sequence using our new decomposition. Suppose $\beta_1 = \gamma + 1$. Now $\langle \omega^\gamma.n : n \in \omega \rangle$ is a sequence converging to ω^{β_1}. By the S-perfectness of X, fix $s : \omega \longrightarrow X$, an injective sequence in $V \setminus \{x\}$ converging to x. The regularity of X yields open sets $\{U_n : n \in \omega\}$, such that $\overline{U_i} \cap \overline{U_j} = \emptyset$, if $i \neq j$, and $s(n) \in U_n \subset \overline{U_n} \subset V$ for all $n \in \omega$. Using the assumption made in the recursion, we take $f_i : \omega^\gamma + 1 \longrightarrow f_i[\omega^\gamma + 1] \subset U_i$ homeomorphism such that $f_i(\omega^\gamma) = s(i)$.

Fix a cofinal function $h : \omega \longrightarrow \omega^\gamma$ and an open basis $\{W_\rho : \rho < \lambda\}$ for x where $\lambda = \chi(x, X) < \mathfrak{b}$. Let $p_\rho : \omega \longrightarrow \omega$ be given by:

$$p_\rho(t) = \begin{cases} min\{i : \forall\delta \geq h(i)\,(f_t(\delta) \in W_\rho)\}, & \text{if } s(t) \in W_\rho \\ 0, & \text{otherwise.} \end{cases}$$

Using the definition of $\mathfrak{b}$, we take $g : \omega \longrightarrow \omega$ satisfying $\forall\rho < \lambda\,(p_\rho \leq^* g)$.

The value $p_\rho(t)$ helps us understand how the open set W_ρ behaves in relation to the homeomorphism f_t using h. The function g will help us translate this behaviour to all the W_ρ's.

With this in mind, the desired copy of $\omega^{\beta_1} + 1$ (or ω^{β_1}) in V will be given by:

$$f(\theta) = \begin{cases} f_t(h(g(t)) + \theta') & \text{if } \theta = \omega^\gamma.t + \theta' \text{ in the normal Cantor form and } 0 < \theta' \leq \omega^\gamma \\ x & \text{if } \theta = \omega^{\gamma+1} \end{cases}$$

Now that f is well-defined and injective, we need only prove that it is open and continuous. For the continuity, fix an open set U in X. Without loss of generality we may assume $U \subset V$. Let $\lambda \in f^{-1}[U]$. If $\lambda \neq \omega^{\beta_1}$, we have that $\lambda = \omega^\gamma.t + \theta'$. Since $f(\lambda) \in U$ if and only if $f_t(h(g(t)) + \theta') \in U$ and f_t is continuous, we have the continuity of f at λ. If $\lambda = \omega^{\beta_1}$, then $f(\lambda) = x$. Consider ρ, such that $x \in W_\rho \subset U$. There must be $k \in \omega$ with the following property: $\forall i \geq k\ (g(i) \geq p_\rho(i))$. Furthermore, since $(\omega^\gamma.n)_{n\in\omega}$ converges to ω^{β_1}, there must be an $l \in \omega$ with

$$\forall i \geq l\ f_i(\omega^\gamma) = f(\omega^\gamma.i) \in W_\rho.$$

Taking $a = max\{k, l\}$ we have that, for all $i \geq a$, $f_i(\omega^\gamma) = f(\omega^\gamma.i) \in W_\rho$ and $g(i) \geq p_\rho$. Therefore, for all $\theta \geq \omega^\gamma.a + 1$, $f(\theta) \in W_\rho \subset U$. Hence, $(\omega^\gamma.a, \omega^{\beta_1}] \subset f^{-1}[U]$, and ω^{β_1} is a point in the interior of $f^{-1}[U]$.

To verify that f is open, since f is injective, we only need to check open sets of the forms $[0, \delta)$ and $(\delta, \omega^{\beta_1}]$. Fix $\delta = \omega^\gamma.t + \theta'$. The following holds:

$$f[[0, \delta)] = \left[\left(\bigcup\{U_i : i \in t\}\right) \cup f_t\left[[1, \theta')\right]\right] \cap \operatorname{ran}(f).$$

Since f_t is a homeomorphism, it follows that the set is open in $\operatorname{ran}(f)$. Now, to see that $f\left[(\delta, \omega^{\beta_1}]\right]$ is open, consider $f_t\,[(\theta', \omega^\gamma]]$, the open sets U_i for $i > t$, and a W_ρ, as defined before, disjoint from the sets $\overline{U_0}, \cdots \overline{U_{t+1}}$. The verification is analogous to the ones before.

We must also consider the case where β_1 is a limit ordinal. In this case, we change the convergent sequence $(\omega^\gamma.n : n \in \omega)$ to $(\omega^{\gamma_n} : n \in \omega)$, where $(\gamma_n)_{n\in\omega}$ converges to β_1, which is possible since α is countable. The functions f_i will represent copies of $\omega^{\gamma_i} + 1$ in U_i, and the function h will be changed to an h_n for each corresponding ω^{γ_n}. Changes should also be made to the p_ρ's and f accordingly. The verification is almost identical to the one before.

□

In the original paper by P. Komjáth and W. Weiss, the previous lemma was considered not only for (S-)perfect spaces but also for (S-)scattered with sufficient height. Now we will explain the idea behind this double proof and use Lemma 2.2.4 to conclude the proof of Theorem 2.0.2.

Theorem 2.2.5. *Let X be a regular topological space with $X \longrightarrow (top\,\omega+1)^1_\omega$, and $\chi(X) < \mathfrak{b}$. Then $X \longrightarrow (top\,\alpha)^1_\omega$ for all $\alpha < \omega_1$.*

Proof. Let us consider a partition $X = \bigcup_{n\in\omega} X_n$ given by a coloring $f \in \omega^X$. For each of the subspaces X_n, we take the sequential Cantor-Bendixson decomposition. As in the original proof, we shall consider the following two cases. In the first one, there is an $n \in \omega$ such that $(X_n)_S^{Sh(X_n)}$ is not empty; in the second one, for all $n \in \omega$, X_n is S-scattered.

For the first case, we fix $n \in \omega$ given by the hypothesis. Note that, since $(X_n)_S^{Sh(X_n)}$ is S-perfect, non-empty, and regularity and character are hereditary, here we are in the hypothesis of Lemma 2.2.4; therefore, the monochromatic copies of each $\alpha < \omega_1$ are given by it.

Consider now the second case. First, we note that there must be at least one $j \in \omega$ such that $Sh(X_j) \geq \omega_1$. Otherwise, we would have $Sh(X_n)$ countable for all $n \in \omega$, and then

$$\{SI_\alpha(X_n) \,:\, n \in \omega \text{ and } \alpha < Sh(X_n)\}$$

would be a partition of X that contradicts $X \longrightarrow (top\,\omega+1)^1_\omega$, since from the definition of $SI_\alpha(X_n)$ there is no $s : \omega \longrightarrow SI_\alpha(X_n)$ converging to a point in $SI_\alpha(X_n)$. Fix one such $j \in \omega$. Our goal is to construct the copies of α using the levels of X_j to help us.

In the proof of Lemma 2.2.4, the hypothesis that, for all $\beta < \alpha$, $x \in X$ and open neighbourhoods V of x, there are $f_\beta : \beta \longrightarrow V$ homeomorphisms such that if β is the successor of a limit ordinal γ then $f_\beta(\gamma) = x$, worked for every point x and neighbourhood V. This application, however, is not that simple. It is clear that no point in $SI_0(X_j)$ could represent ω in a homeomorphism, since they cannot have a sequence in X_j converging to them. Therefore, we have to carefully choose the points in each step of the recursive construction. With this in mind, we restate the hypothesis. Suppose that, for all $\beta < \alpha$, there is a $\lambda_\beta < \omega_1$ satisfying that for all $x \in (X_j)_S^{\lambda_\beta}$ and open neighbourhood V of x, there is an homeomorphism $f : \beta \longrightarrow f[\beta] \subset V$ such that if $\beta = \gamma+1$ and γ is a limit ordinal, then $f(\gamma) = x$.

We prove that there is a $\lambda_\alpha < \omega_1$ satisfying the condition in the statement above. Let $\lambda = sup\{\lambda_\beta : \beta < \alpha\}$. For all $\beta < \alpha$, we have $\lambda_\beta < \omega_1$ and $\alpha < \omega_1$; hence, $\lambda < \omega_1$. Let $\lambda_\alpha = \lambda + 1$. Fix $x \in (X_j)_S^{\lambda_\alpha}$ and open neighbourhood V of x. By Proposition 2.1.5, there is a sequence $s : \omega \longrightarrow X_j$ converging to x, with $s[\omega] \subset (X_j)_S^{\lambda} \cap V$. Since $\lambda > \lambda_\beta$, the elements taken for the sequence have the same properties used in Lemma 2.2.4. We can now just repeat the argument used before to obtain the homeomorphism and verify that λ_α is as desired. □

2.3 The principle $\clubsuit_F$ and CS^*-forcing

We will discuss about a possible relaxation on the upper bound on the character of the topological space on the result seen in the previous section. For this we will present, in this section, a study on a variation of the $\clubsuit$ principle that will be used in the next section. In [27] W. Weiss and P. Komjáth create an example using $\blacklozenge$ that guarantee that, at least consistently, $\mathfrak{b}$ is the best bound.

Theorem 2.3.1. *Assuming $\blacklozenge$ there is a topology on ω_1 that refines the order topology such that $X \longrightarrow (top\, \omega + 1)^1_\omega$, $X \not\longrightarrow (top\, \omega^2 + 1)^1_\omega$, and $\chi(X) = \omega_1 = \mathfrak{b}$.*

This construction follows in the same fashion of Example 2.2.3, and will be explored in further detail in the next section. A question that was posed by W. Weiss was if it is possible to construct such a space just using $\clubsuit$. In spite of our efforts we were not able to answer such question. But we were able to obtain an example that works without assuming CH. The original construction uses $\blacklozenge$ to guarantee that $X \longrightarrow (top\, \omega + 1)^1_\omega$. We tried to extract the essence of the $\blacklozenge$ usage in the previous theorem to create a $\clubsuit$-like principle to help us build an example without using CH.

We shall state the principle that we called as $\clubsuit_F$.

Definition 2.3.2. *A sequence $\langle A^n_\alpha : n < \omega \wedge \alpha < \omega_1 \rangle$ is called a $\clubsuit_F$-sequence if the following holds:*

(1) for all $\alpha \in acc(\omega_1)$ and $n \in \omega$ we have that A^n_α is an unbounded subset of α;

(2) for all $f : \omega_1 \longrightarrow \omega$ there are $\alpha \in acc(\omega_1)$ and $n, m \in \omega$ such that $\alpha \in f^{-1}[\{n\}]$ and $A^m_\alpha \subset f^{-1}[\{n\}]$.

The principle $\clubsuit_F$ is the statement: there exists a $\clubsuit_F$-sequence

The idea behind this definition is that we have a matrix of guesses that are relevant at limit ordinals. Furthermore, given any coloring of ω_1 in ω colors we must have a limit ordinal α, a column m of the matrix and a color n such that A^m_α and α are both of color n. This should help us find the monochromatic sequence when constructing the example.

For now, we shall leave the construction of this space aside to verify some properties of $\clubsuit_F$. First we will show the consistency of such principle. Let us first introduce some notation.

Definition 2.3.3. *Let $\vec{B} = \langle B^m_\alpha : \alpha < \omega_1 \wedge n < \omega \rangle$ be a $\blacklozenge^*$-sequence. Consider the sequence $\vec{A}$ given by: $A^n_\alpha = B^n_\alpha$ for all $n \in \omega$ and $\alpha \in \omega_1$ such that B^n_α is unbounded in α, and $A^n_\alpha = \alpha$ otherwise. We say that $\vec{A}$ is the* derived sequence *from $\vec{B}$.*

The next lemma gives us that a model for $\blacklozenge^*$ is also a model for $\clubsuit_F$.

Lemma 2.3.4. *Let $\vec{B}$ be $\blacklozenge^*$-sequence. Then $\vec{A}$, the derived sequence from $\vec{B}$, is a $\clubsuit_F$-sequence.*

Proof. We shall prove that $\langle A^n_\alpha : n \in \omega \wedge \alpha \in \omega_1 \rangle$ is a $\clubsuit_F$-sequence. Indeed, condition (1) is immediate from our definition. To verify that the other condition is also satisfied let $f : \omega_1 \longrightarrow \omega$ be a function. There exists $n \in \omega$ such that $f^{-1}[\{n\}]$ is stationary and therefore $acc(f^{-1}[\{n\}])$ is a club. Applying the $\blacklozenge^*$ property for $f^{-1}[\{n\}]$ we have another club C such that, for all $\beta \in C$, there exists $m \in \omega$ with $f^{-1}[\{n\}] \cap \beta = B^m_\beta$. Finally, for $\beta \in C \cap acc(f^{-1}[\{n\}])$, B^m_β satisfies $B^m_\beta = A^m_\beta$, verifying (2). □

In a communication with A. Rinot it has been brought to our attention that $\blacklozenge^*$ could be weakened to $\blacklozenge$ by considering Lemma 3.5 from his paper with A. Brodsky [9]. Let us state this result and then prove the desired implication.

Lemma 2.3.5 (A. M. Brodsky, A. Rinot[9]). *Suppose that κ, θ are cardinals such that $\kappa^\theta = \kappa$, $S \subset \kappa$ is a stationary set, and that $\clubsuit(S)$ holds. Then there exists a matrix $\langle X^\tau_\delta : \delta \in S \wedge \tau \in \theta \rangle$ such that, for every sequence $\langle X^\tau : \tau \in \theta \rangle$ of cofinal subsets of κ, there exists a stationary $S' \subset S$ satisfying that, for all $\delta \in S'$ and $\tau \in \theta$, $X^\tau_\delta = X^\tau \cap \delta$ and $sup(X^\tau_\delta) = \delta$.*

Now assuming $\blacklozenge$ is enough to secure the validity of the hypothesis from the previous lemma for $\kappa = \omega_1$, $\theta = \omega$ and $S = \omega_1$, the idea to verify that $\clubsuit_F$ holds is to use the sequence that is being guessed by the matrix to translate our monochromatic sets as will be shown below.

Proposition 2.3.6. *Suppose that there exists a matrix $\langle X^\tau_\delta : \delta \in \omega_1 \wedge \tau \in \omega \rangle$ such that, for every sequence $\langle X^\tau : \tau \in \omega \rangle$ of cofinal subsets of ω_1, there exists a stationary $S \subset \omega_1$ such that, for all $\delta \in S$ and $\tau \in \omega$, $X^\tau_\delta = X^\tau \cap \delta$ and $sup(X^\tau_\delta) = \delta$. Then $\clubsuit_F$ holds.*

Proof. Consider $\vec{X}$ the matrix as stated in the hypothesis and consider $\vec{A}$ the matrix derived from $\vec{X}$ as in definition 2.3.3. We argue that $\vec{A}$ verifies $\clubsuit_F$. Indeed condition (1) is satisfied by taking the derived matrix (sequence). Now to verify condition (2) fix $f : \omega_1 \longrightarrow \omega$. Now there must be at least one $n \in \omega$ such that $f^{-1}[\{n\}]$ is uncountable. We list all such $f^{-1}[\{n\}]$, with repetition if necessary, on the sequence $\langle X^i : i \in \omega \rangle$. Consider now a club of limit ordinals on ω_1 such that all colors unbounded in ω_1, given by f, are also unbounded at such elements, and all elements must also have unbounded colors. The stationary subset S given by our hypothesis applied to the sequence mentioned before must intersect this club at an α. Now we just need to select a column i which the corresponding term $f^{-1}[\{n\}]$ of the sequence has the same color as α. Therefore, since $X^i_\alpha = f^{-1}[\{n\}] \cap \alpha$ and $sup(X^i_\alpha) = \delta$, we must have $X^i_\alpha = A^i_\alpha \subset f^{-1}[\{n\}]$ and $f(\alpha) = n$. □

In another communication, A. Rinot told us that he proved that the result from Lemma 2.3.5 is an equivalence, that is, the existence of such a matrix is enough to guarantee $\kappa^\theta = \kappa$ and $\clubsuit(S)$. We give the proof of such fact as it was communicated to us.

Lemma 2.3.7 (A. Rinot). *Suppose that κ, θ are cardinals, S is a stationary set in κ, and there exists a matrix $\langle X^\tau_\delta : \delta \in S \wedge \tau \in \theta \rangle$ such that, for every sequence $\langle X^\tau : \tau \in \theta \rangle$ of cofinal subsets of κ, there exists a stationary $S' \subset S$, such that, for all $\delta \in S'$ and $\tau \in \theta$, $X^\tau_\delta = X^\tau \cap \delta$ and $sup(X^\tau_\delta) = \delta$. Then $\kappa^\theta = \kappa$ and $\clubsuit(S)$ holds.*

Proof. The fact that $\clubsuit(S)$ holds is immediate by just considering one column of the matrix. Now we must show that $\kappa^\theta = \kappa$. Consider $\langle A_\gamma : \gamma < \kappa \rangle$ a partition of κ into κ many cofinal sets. Now for each $\delta \in S$ we consider

$$Z_\delta = \{\alpha < \kappa : A_\alpha \cap \bigcup_{\tau < \theta} X^\tau_\delta \neq \emptyset\}.$$

Given any $Y \in [\kappa]^\theta$ we shall see that there is $\delta \in \kappa$ such that $Y = Z_\delta$, completing the proof. Indeed, for each element β of Y consider A_β. List such A_βs in a sequence of order type θ as in the principle and take the stationary S' given by our hypothesis. Now, since $\langle A_\gamma : \gamma < \kappa \rangle$ is a partition and, for each $\tau < \theta$, X^τ_ρ is contained in a corresponding A_β, the set Z_ρ must only collect all the indexes from our sequence. Therefore $Z_\rho = Y$. □

Note that in both the cases above we verified that $\clubsuit_F$ is valid. But even then, we could not remove it from the presence of CH, or $\blacklozenge$ for that matter. This is troublesome since we cannot make a distinction between $\mathfrak{b}$ and other small cardinals, which is relevant in our original problem.

With this in mind we shall verify that $\clubsuit_F$ is indeed consistent with $\neg CH$, at least relative to the existence of a strongly inaccessible cardinal. For that we will have to work with CS^*-forcing. In [18], Fuchino, Shelah and Soukup developed the concept of CS^*-forcing and used it to show the consistency of the club principle for every stationary subset of limits of ω_1 with $\neg CH$ and MA for countable partial ordered sets. We shall use the same forcing and show that $\clubsuit_F$ holds in its extension.

Definition 2.3.8 (S. Fuchino, S. Shelah, L. Soukup [18])**.** *We say that $\mathbb{P}_\varepsilon = \langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle : \alpha < \varepsilon \rangle$ is a CS^*-iteration if and only if $\langle (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha) : \alpha < \varepsilon \rangle$ is a countable support iteration satisfying the following additional condition: if $\alpha \leq \varepsilon$ and $p < q$ in $\mathbb{P}_\alpha$, then*

$$\operatorname{diff}(p,q) = \{\beta \in dom(p) \cap dom(q) : p \restriction_\beta \not\Vdash p(\beta) = q(\beta)\} \text{ is finite.}$$

Furthermore we say that $p \leq^h_{\mathbb{P}_\alpha} q$ if and only if $p \leq q$ and $p \restriction_{\operatorname{dom}(q)} = q$. We also say that $p \leq^v_{\mathbb{P}_\alpha} q$ if and only if $p \leq q$ and $\operatorname{dom}(p) = \operatorname{dom}(q)$. The inequalities defined above are said to be horizontal and vertical respectively. When there is no risk of ambiguity, we shall omit $\mathbb{P}_\alpha$ from $\leq^h_{\mathbb{P}_\alpha}$ and $\leq^v_{\mathbb{P}_\alpha}$.

This iteration will be done using the Cohen forcing as stated below.

Definition 2.3.9 (Cohen CS^* iteration [18])**.** *If $\mathbb{P}_\varepsilon = \langle (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha) : \alpha < \varepsilon \rangle$ is a CS^*-iteration such that for every $\alpha < \varepsilon$ we have $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha = Fn(\omega, 2)$, then we call $\mathbb{P}_\varepsilon$ a Cohen CS^*-iteration of length ε.*

The following theorem will be central to some of our results. It shows some properties of the Cohen CS^*-iteration when we start with a model where $\blacklozenge^*$ holds.

Theorem 2.3.10. *Let $\mathbb{P}_\kappa$ be a Cohen CS^*-iteration of length κ, where $\kappa \geq \aleph_2$ is a regular cardinal such that for every $\alpha < \kappa$ we have $\alpha^{\aleph_0} < \kappa$. Suppose $\vec{B}$ is a $\blacklozenge^*$-sequence and*

$$\vec{A} = \langle A^m_\alpha : \alpha < \omega_1 \wedge n \in \omega \rangle$$

is the sequence derived from $\vec{B}$. If $p \in \mathbb{P}_\kappa$, σ is a $\mathbb{P}_\kappa$-name and $p \Vdash \sigma : \check{\omega}_1 \longrightarrow \omega$, then there is a condition $p_* < p$, an ordinal $\eta < \omega_1$ and $n, m \in \omega$ such that

$$p_* \Vdash (\check{A}^m_\eta \subset \sigma^{-1}[\{\check{n}\}] \wedge \sigma(\check{\eta}) = \check{n}).$$

In particular $p_* \Vdash \sigma$ is not a bijection.

Before we prove this result we will first prove a corollary.

Corollary 2.3.11. *Assume $\diamondsuit^*$ and that $\kappa \geq \aleph_2$ is a regular cardinal such that for all $\alpha < \kappa$ we have $\alpha^{\aleph_0} < \kappa$. Let $\vec{A}$ be the sequence derived from a $\diamondsuit^*$-sequence $\vec{B}$. Then*

$$\Vdash_{\mathbb{P}_\kappa} \vec{A} \text{ is a } \clubsuit_F\text{-sequence.}$$

Proof. Let σ be a $\mathbb{P}_\kappa$-name and $q \in \mathbb{P}$ such that $q \Vdash \sigma : \omega_1 \longrightarrow \omega$. Fix $p \in \mathbb{P}$ such that $p \leq q$. Applying Theorem 2.3.10 to p and σ, it follows that there exists $p_* \leq p$, $\eta < \omega_1$ and $n, m \in \omega$ such that

$$p_* \Vdash \check{A}^m_\eta \subset \sigma^{-1}[\{\check{n}\}] \wedge \sigma(\check{\eta}) = \check{n}.$$

Therefore

$$\Vdash_{\mathbb{P}_\kappa} \vec{A} \text{ is a } \clubsuit_F\text{-sequence.}$$

□

Now we know that $\clubsuit_F$ holds in this extension, and, as noted in [18], such extension has κ Cohen reals. Therefore we finally have a desired model that shows the consistency of $\clubsuit_F$ with $\neg CH$. Next we focus on some preliminary results that will be used to prove Theorem 2.3.10.

From here until the end of this section the forcing $\mathbb{P}_\alpha$ will denote a Cohen CS^*-iteration of length α for any ordinal α.

Lemma 2.3.12 (S. Fuchino, S. Shelah, L. Soukup[18])**.** *Let γ be an ordinal, and suppose that $\langle p_n : n \in \omega\rangle$ is a sequence of elements of $\mathbb{P}_\gamma$ such that $m < l < \omega$ implies $p_l \leq^h p_n$. Then $r \in \mathbb{P}_\gamma$ for r given by*

$$r(\xi) = \begin{cases} p_n(\xi) & \text{if } \exists n \in \omega\ (\xi \in supp(p_n)) \\ \mathring{1}_\xi & \text{otherwise.} \end{cases}$$

Proof. Indeed, $supp(r)$ is the union of countably many countable supports. To see that $r \in \mathbb{P}_\gamma$ we proceed by induction on γ. Suppose that the lemma is true for all $\beta < \gamma$. If $\gamma = \alpha + 1$ then $p_n \restriction_\alpha$ falls under the condition of our induction hypothesis. Therefore $r \restriction_\alpha \in \mathbb{P}_\alpha$ and $r \restriction_\alpha^\frown r(\alpha) \in \mathbb{P}_\gamma$. If γ is a limit ordinal, then for each $\beta < \gamma$, we have the same argument as above, since $p_n \restriction_\beta \leq^h p_m \restriction_\beta$ for $m \leq n$. Therefore $r \restriction_\beta \in \mathbb{P}_\beta$ for all $\beta < \gamma$. It follows that $r \in \mathbb{P}_\gamma$. □

The previous lemma precedes the following proposition, verifying a countable closedness for decreasing horizontal sequences. From here onward, when we have a sequence as in the hypothesis of Lemma 2.3.12, we shall denote the resulting r as $\bigcup_{\beta<\omega} p_\beta$.

Proposition 2.3.13 (S. Fuchino, S. Shelah, L. Soukup[18])**.** *Let $\langle p_n : n < \omega\rangle$ be a sequence of elements of $\mathbb{P}_\kappa$ such that $n < m$ implies $p_m \leq^h p_m$. Then $\bigcup_{n<\omega} p_n$ is a lower bound of the sequence $\langle p_n : n < \omega\rangle$.*

Proof. We already know that $r \in \mathbb{P}_\kappa$ by the previous lemma. Let us verify that it is a lower bound for the given sequence. Fix $n \in \omega$. To see that $r \leq p_n$ we shall again use induction on κ. Suppose that this result is true for all $\mathbb{P}_\alpha$, $\alpha < \kappa$. For κ limit, for each $\alpha < \kappa$ our induction hypothesis applied to the sequence of restrictions gives us

$$r\restriction_\alpha \Vdash_{\mathbb{P}_\alpha} r(\alpha) \leq_\alpha p_n(\alpha).$$

If $\kappa = \beta + 1$, then $r\restriction_\beta \Vdash_{\mathbb{P}_\beta} r(\beta) \leq_\beta p_n(\beta)$, since $p_n(\beta) = r(\beta)$ or $p_n(\beta) = \mathring{1}_\beta$. We only need to verify that the set $\mathrm{diff}(r, p_n)$ is finite. Indeed,

$$\{\beta \in supp(r) \cap supp(p_n) \,:\, r\restriction_\beta \nVdash_{\mathbb{P}_\beta} r(\beta) = p_n(\beta)\} = \emptyset$$

since, by definition, $r(\beta) = p_n(\beta)$ for all $\beta \in supp(p_n)$. □

Lemma 2.3.14 (S. Fuchino, S. Shelah, L. Soukup[18])**.** *Given an ordinal θ and $p, q \in \mathbb{P}_\theta$, if $p \leq q$, then there is $p' \leq p$ such that, for all $\alpha \in \mathrm{diff}(p', q)$, $p'\restriction_\alpha$ decides the value of $p'(\alpha)$.*

Proof. Fix $p, q \in \mathbb{P}_\theta$, we define $(\alpha_n)_{n\in\omega}$ decreasing sequence of ordinals and $(p_n)_{n\in\omega}$ decreasing sequence of conditions the following way: start by taking

$$\alpha_0 = max\{\alpha \in \mathrm{diff}(p, q) \,:\, p\restriction_\alpha \text{ does not decide } p(\alpha)\}.$$

Now there is $p_0' \in \mathbb{P}_{\alpha_0}$ such that $p_0' \leq p\restriction_{\alpha_0}$ and p_0' decides $p(\alpha_0)$. We take $p_0 = p_0'{}^\frown(p\restriction_{\theta\setminus\alpha_0})$. Suppose that α_n and p_n are defined we consider

$$\alpha_{n+1} = max\{\alpha \in \mathrm{diff}(p_n, q) \,:\, p_n\restriction_\alpha \text{ does not decide } p_n(\alpha)\}$$

if the set is not empty. Again p_n will be given by $p_n = p_n'{}^\frown(p\restriction_{\theta\setminus\alpha_n})$. Note that each p_n taken is a strengthening of its previous conditions. Therefore the values of α_n are also decreasing since each step decides a value for the greatest coordinate of $\mathrm{diff}(p_n, q)$. This process ends in a finite number of steps since α_n is a decreasing sequence of ordinals. Now the condition p_m of this step is the p' we search, since $\{\alpha \in \mathrm{diff}(p_m, q) \,:\, p_n\restriction_\alpha \text{ does not decide } p_m(\alpha)\} = \emptyset$. □

The following lemma allows us to construct the element from the forcing we need to prove Theorem 2.3.10 by replacing a given forcing condition with another one more suitable.

Lemma 2.3.15. *Let θ be an ordinal, $p \in \mathbb{P}_\theta$, and $\langle \sigma_\alpha \,:\, \alpha \in \mathrm{dom}\,(p)\rangle$ be a sequence of $\dot{\mathbb{Q}}_\alpha$-names. If for all $\alpha \in \mathrm{dom}\,(p)$ we have*

$$p\restriction_\alpha \Vdash p(\alpha) = \sigma_\alpha,$$

then for any formula $\varphi(x)$ and $\mathbb{P}_\kappa$-name τ we have:

$$p \Vdash \varphi(\tau)$$

if and only if

$$p' \doteq \langle(\alpha, \sigma_\alpha) \,:\, \alpha \in \mathrm{dom}\,(p)\rangle \Vdash \varphi(\tau).$$

Proof. Given $p, p' \in \mathbb{P}_\theta$ as in the hypothesis we prove the following statement by induction on $\gamma \leq \theta$:

$$(\triangle)_\gamma\ \forall r(r \in \mathbb{P}_\gamma \longrightarrow (r \leq p \restriction_\gamma \leftrightarrow r \leq p' \restriction_\gamma))$$

Notice that $(\triangle)_\gamma$ implies that $p \restriction_\gamma$ and $p' \restriction_\gamma$ force the same statements.

Fix $\gamma \leq \theta$ and suppose $r \leq p \restriction_\gamma$, then $r \leq p' \restriction_\gamma$. Indeed, we have $r \restriction_\alpha \Vdash r(\alpha) \leq p(\alpha)$, since $r \restriction_\alpha \leq p \restriction_\alpha$. Therefore $r \restriction_\alpha \Vdash r(\alpha) \leq p(\alpha) = p'(\alpha)$. We need to show now that $\mathrm{diff}(r, p' \restriction_\gamma)$ is finite. If $\alpha \in \mathrm{diff}(r, p' \restriction_\gamma)$ then $r \restriction_\alpha \nVdash r(\alpha) = p'(\alpha)$. Since $r \restriction_\alpha \Vdash p(\alpha) = p'(\alpha)$, $r \restriction_\alpha \nVdash r(\alpha) = p(\alpha)$. Therefore $\alpha \in \mathrm{diff}(r, p \restriction_\gamma)$.

Let $r \leq p' \restriction_\gamma$. If $\zeta < \gamma$, then from our induction hypothesis $(\triangle)_\zeta$ we have that $p' \restriction_\zeta \Vdash p(\zeta) = \sigma_\zeta$. Therefore $r \leq p' \restriction_\gamma$ implies $r \restriction \zeta \Vdash r(\zeta) \leq \sigma_\zeta = p(\zeta)$. Thus $r \leq p \restriction_\gamma$. Now fix $\alpha \in \mathrm{diff}(r, p \restriction_\gamma)$, then $r \restriction_\alpha \nVdash p(\alpha) = r(\alpha)$. Since $\alpha < \gamma$ we have $r \restriction_\alpha \Vdash p(\alpha) = p'(\alpha)$, it follows that $r \restriction_\alpha \nVdash p'(\alpha) = r(\alpha)$. Hence $\alpha \in \mathrm{diff}(r, p' \restriction_\gamma)$ and $(\triangle)_\gamma$ holds.

Note that $(\triangle)_\theta$ implies the lemma. □

We are finally ready to prove Theorem 2.3.10.

Proof of Theorem 2.3.10. Let σ be a name for a given $f : \omega_1 \longrightarrow \omega$ in the extension. We start by taking any $p \in \mathbb{P}$ such that $p \Vdash \sigma : \check{\omega}_1 \longrightarrow \omega$. We will show that there is a condition $p_* < p$, $\eta < \omega_1$ and $n, m \in \omega$ such that

$$p_* \Vdash (\check{A}^m_\eta \subset \sigma^{-1}[\{\check{n}\}] \wedge \sigma(\check{\eta}) = \check{n})$$

which concludes the proof.

Following [11] we shall construct an ω_1-sequence that decides the evaluation of σ on each $\alpha < \omega_1$. We then use a pressing down argument to find a stationary set in V where we can apply a $\clubsuit_F$-sequence derived from a $\blacklozenge^*$-sequence in order to find the condition p_*. The basic idea here is that we cannot decide what happens with the whole coloring but we may decide what happens with a great piece of a stationary subset of ω_1, which, coupled with the $\clubsuit_F$ on the ground model, gives us the desired control over f.

We shall construct inductively $\langle q_\alpha : \alpha < \omega_1 \rangle, \langle n_\alpha : \alpha < \omega_1 \rangle$, together with an auxiliary sequence $\langle p_\alpha : \alpha < \omega_1 \rangle$. These three sequences should satisfy the following conditions for all $\alpha < \omega_1$:

1. $\langle p_\gamma : \gamma < \alpha \rangle$ is a sequence such that $\gamma < \gamma' < \alpha$ implies $p_{\gamma'} \leq^h p_\gamma$;
2. $q_\alpha \leq^v p_\alpha$ and $q_\alpha \Vdash \sigma(\alpha) = \check{n}_\alpha$;
3. $u_\alpha = \mathrm{diff}(q_\alpha, p_\alpha) \subset \mathrm{dom}(p) \cup \bigcup_{\gamma < \alpha} \mathrm{dom}\left(q_\gamma\right)$;
4. $q_\alpha \restriction_{(dom(p_\alpha)\setminus u_\alpha)} = p_\alpha \restriction_{(dom(p_\alpha)\setminus u_\alpha)}$;
5. $q_\alpha \restriction_{u_\alpha} \in Fn(\kappa, \{\check{t} : t \in Fn(\omega, 2)\})$.

Suppose we already constructed $\langle p_\beta : \beta < \alpha \rangle$, $\langle q_\beta : \beta < \alpha \rangle$, and $\langle n_\beta : \beta < \alpha \rangle$.

Let $r_\alpha = p$ if $\alpha = 0$ or $r_\alpha = \bigcup_{\beta \in \alpha} p_\beta$ otherwise. Fix $w_\alpha \leq r_\alpha$ such that there is $n_\alpha \in \omega$ satisfying $w_\alpha \Vdash \sigma(\alpha) = \check{n}_\alpha$. Applying Lemma 2.3.14 on r_α and w_α, we obtain $q^*_\alpha \leq w_\alpha$ such that, for all $\gamma \in \operatorname{diff}(q^*_\alpha, r_\alpha)$, there exists $\check{t}_\gamma$ satisfying $q^*_\alpha \restriction_\gamma \Vdash q^*_\alpha(\gamma) = \check{t}_\gamma$.

Define the following:

- $p_\alpha = r_\alpha \cup (q^*_\alpha \restriction_{dom(q^*_\alpha) \setminus dom(r_\alpha)})$;
- $d_\alpha = \operatorname{diff}(q^*_\alpha, p_\alpha)$;
- $q_\alpha = p_\alpha \restriction_{\operatorname{dom}(p_\alpha) \setminus d_\alpha} \cup \{\check{t}_\gamma : \gamma \in d_\alpha\}$.

Let us verify that p_α and q_α satisfy conditions (1) – (5).

First we note that $p_\alpha \leq^h r_\alpha$, and therefore $p_\alpha \leq^h p_\beta$ for all $\beta < \alpha$. Indeed, $\operatorname{dom}(r_\alpha) \subset \operatorname{dom}(p_\alpha)$ and for all $\gamma \in \operatorname{dom}(r_\alpha)$ we have $p_\alpha \restriction_\gamma \Vdash r_\alpha(\gamma) = p_\alpha(\gamma)$ since they are the same name. Furthermore $\operatorname{diff}(p_\alpha, r_\alpha) = \emptyset$ and, by the definitions given, the inequality holds.

Next, we verify (2). Note that for all $\gamma \in \operatorname{dom}(q^*_\alpha) \setminus d_\alpha$ we have

$$q^*_\alpha \restriction_\gamma \Vdash q^*_\alpha(\gamma) = p_\alpha(\gamma) = q_\alpha(\gamma),$$

and for all $\gamma \in d_\alpha$ we have

$$q^*_\alpha \restriction_\gamma \Vdash q^*_\alpha(\gamma) = \check{t}_\gamma = q_\alpha(\gamma).$$

It follows from Lemma 2.3.15 that q_α forces the same statements that q^*_α forces. In particular, $q_\alpha \Vdash \sigma(\alpha) = \check{\mathrm{n}}_\alpha$.

From the definition of p_α and q_α, it follows that $\operatorname{dom}(q^*_\alpha) = \operatorname{dom}(p_\alpha) = \operatorname{dom}(q_\alpha)$. Hence, in order to verify $q_\alpha \leq^v p_\alpha$, we only have to verify that $q_\alpha \leq p_\alpha$. Consider $\zeta \in dom(p_\alpha)$. If $\zeta \in dom(r_\alpha)$, then $p_\alpha(\zeta) = r_\alpha(\zeta)$ and $q_\alpha \restriction_\zeta \Vdash p_\alpha(\zeta) = r_\alpha(\zeta) \geq q^*_\alpha(\zeta) = q_\alpha(\zeta)$, since $q^*_\alpha \restriction_\zeta$ also forces it.

If $\zeta \in dom(p_\alpha) \setminus dom(r_\alpha)$, since $d_\alpha \subset dom(r_\alpha)$, then $p_\alpha(\zeta) = q^*_\alpha(\zeta)$. Therefore

$$q_\alpha \restriction_\zeta \Vdash q_\alpha(\zeta) = q^*_\alpha(\zeta) = p_\alpha(\zeta).$$

Now we verify condition (3). By Lemma 2.3.15, $\gamma \in \operatorname{diff}(q_\alpha, p_\alpha)$ if and only if $\gamma \in \operatorname{diff}(q^*_\alpha, p_\alpha)$. Therefore $u_\alpha = d_\alpha$. Furthermore, from our induction hypothesis

$$u_\alpha \subset \operatorname{dom}(r_\alpha) \subset dom(p) \cup \bigcup_{\gamma < \alpha} \operatorname{dom}(p_\gamma) = dom(p) \cup \bigcup_{\gamma < \alpha} \operatorname{dom}(q_\gamma).$$

Conditions (4) and (5) follows directly from our definitions of q_α and the fact observed above that $u_\alpha = d_\alpha$.

Consider now a bijection $\Phi : \bigcup_{\beta<\omega_1} dom(q_\beta) \longrightarrow \omega_1$, and let $Y \subset \omega_1$ be a club such that, for all $\alpha \in Y$,

$$\Phi(\bigcup_{\beta<\alpha} dom(q_\beta)) \subset \alpha.$$

Therefore, for all $\alpha \in Y$, we have $a_\alpha := \Phi[u_\alpha] \subset \alpha$.

Let $Y_0 \subset Y$ be a stationary set and $k \in \omega$ be such that for all $\alpha \in Y_0$, $|a_\alpha| = k$. Let $\phi_0 : Y_0 \longrightarrow \omega_1$ be a regressive function given by $\phi_0(\alpha) = min(a_\alpha)$. Applying Fodor's Lemma, we find $Y_1 \subset Y_0$ stationary such that ϕ_0 is constant on Y_1. Recursively, for $n < k$, we construct $\phi_n : Y_n \longrightarrow \omega_1$ such that $\phi_n(\alpha)$ is the n^{th} element of a_α. After k iterations we find Y_{k+1} stationary and $a \in \omega_1^{<\omega}$ such that $\alpha \in Y_{k+1}$ implies $a_\alpha = a$.

For all $\alpha \in Y_{k+1}$ we have $u_\alpha = u$ for a fixed $u \in \kappa^{<\omega}$. The set

$$W = \{r \in Fn(\kappa, Fn(\omega, 2)) : dom(r) = u\}$$

is countable, therefore there is $S \subset Y_{k+1}$ stationary and $r \in W$ such that for all $\alpha \in S$ and $\gamma \in u$ we have $q_\alpha(\gamma) = \check{r}(\gamma)$.

Fix $n^* \in \omega$ such that $T = \{\alpha \in S : n_\alpha = n^*\}$ is stationary. Using that $\vec{B}$ is a $\blacklozenge^*$-sequence, we can find a club C such that, for every $\alpha \in C$, there exists $m \in \omega$ such that $B^m_\alpha = T \cap \alpha$. Finally, let $\eta \in C \cap T \cap acc(T)$, then $sup(B^m_\alpha) = \eta$ and $B^m_\eta = A^m_\eta$.

Next, consider $\xi \in T$ such that $\xi > \eta$. Let $p_* = q_\xi$. We shall verify that that p_* is the condition we are looking for.

First we shall see that if $\alpha \in S \cap \xi$, then $q_\xi \leq^h q_\alpha$. For every $\alpha \in Y$ it holds that $q_\alpha = p_\alpha \restriction_{dom(p_\alpha \setminus d_\alpha)}$. We know that $p_\xi \leq^h p_\alpha$ and $u_\alpha = d_\alpha$, therefore $p_\alpha \restriction_{dom(p_\alpha \setminus d_\alpha)} = p_\xi \restriction_{dom(p_\alpha \setminus d_\alpha)}$. Hence $q_\alpha \restriction_{dom(p_\alpha \setminus d_\alpha)} = q_\xi \restriction_{dom(p_\alpha \setminus d_\alpha)}$. From $\alpha \in S$ we have that $u_\alpha = u = u_\xi$ and $q_\alpha \restriction_{u_\alpha} = q_\xi \restriction_u$. Thus $q_\xi \leq^h q_\alpha$ and we have $q_\xi \Vdash \sigma \restriction_{S\cap\xi} = \langle n_\alpha \mid \alpha \in S \cap \xi \rangle$. □

Next, given a cardinal κ as in the hypothesis of Theorem 2.3.10, we will address what happens with $\mathfrak{d}$ and $2^{\aleph_0}$ in any $\mathbb{P}_\kappa$-generic extension. We will need these results in Section 2.4, where we also obtain that $\mathfrak{b} = \omega_1 < 2^{\aleph_0}$ in a $\mathbb{P}_\kappa$-generic extension as an application of Theorem 2.0.2 (see Corollary 2.4.7).

We first observe that, as in the usual Cohen forcing, under certain assumptions, $\omega_1 < \mathfrak{d}$ holds in our extension. In order to verify that we will use the two following lemmas:

Lemma 2.3.16 (S. Fuchino, S. Shelah, L. Soukup[18]). *Suppose that κ is a strongly inaccessible cardinal $> \omega_1$. Then the forcing $\mathbb{P}_\kappa$ has the κ-cc property.*

Proof. We are in the hypothesis of the Δ-system lemma, that is, $\omega_1 < \kappa$ and for all $\theta < \kappa$ we have $\theta^{<\omega_1} < \kappa$ by the inaccessibility of κ. Let $\{p_\beta : \beta < \kappa\} \subset \mathbb{P}_\kappa$. Without loss of generality we may assume that $\{supp(p_\beta) : \beta < \kappa\}$ has non-empty root x by the Δ-system lemma. Let $\alpha_0 = sup\{\gamma + 1 : \gamma \in x\}$. For each $\beta < \kappa$ we have $p_\beta \restriction_{\alpha_0} \in \mathbb{P}_{\alpha_0}$. Since κ is strongly inaccessible, for each $\alpha < \kappa$, we have $|\mathbb{P}_\alpha| \leq (sup\{|\mathring{\mathbb{Q}}_\beta| : \beta < \alpha\})^\alpha < \kappa$ by induction on $\mathring{\mathbb{Q}}_\beta$. Therefore $|\mathbb{P}_{\alpha_0}| < \kappa$, and we must have $\beta, \beta' < \kappa$ such that $p_\beta \restriction_{\alpha_0} = p_{\beta'} \restriction_{\alpha_0}$. It follows that $q \in \mathbb{P}_\kappa$ given by:

$$q(\gamma) = \begin{cases} p_\beta(\gamma) & \text{if} \gamma \in supp(p_\beta) \\ p_{\beta'}(\gamma) & \text{if} \gamma \in supp(p_{\beta'}) \\ \mathring{1}_\gamma, & \text{otherwise.} \end{cases}$$

verifies the compatibility between p_β and $p_{\beta'}$. □

Lemma 2.3.17. *Let κ be a strongly inaccessible cardinal. Let $\mathbb{P}_\kappa$ be the a CS^*-iteration of Cohen forcing of length κ. Then, in any $\mathbb{P}_\kappa$-generic extension, it holds that $\mathfrak{d} \geq \kappa$.*

Proof. Let $V[G]$ be a $\mathbb{P}_\kappa$-generic extension of V. Consider $\mathcal{F} \in V[G]$ a dominating family of size $\mathfrak{d}$. By contradiction, suppose that $\mathfrak{d} < \kappa$. In $V[G]$, let $\mathcal{F} = \{f_\alpha \mid \alpha < \mathfrak{d}\}$ and let $\Phi : \mathfrak{d} \times \omega \longrightarrow \omega$ such that $\Phi(\alpha, m) = f_\alpha(m)$ for each $\alpha < \mathfrak{d}$ and $m \in \omega$. Then $\Phi \subset \mathfrak{d} \times \omega \times \omega$ codes $\mathcal{F}$. Using that $\mathbb{P}_\kappa$ is κ-cc it follows that $\Phi \in V[G \restriction \xi]$ for some $\xi < \kappa$, and consequently $\mathcal{F} \subset V[G \restriction \xi]$. Indeed, fix a nice name τ for Φ as a subset of $\mathfrak{d} \times \omega \times \omega$. Since P_κ is κ-cc the antichains of $\mathbb{P}_\kappa$ all have size $< \kappa$, therefore all forcing conditions on τ have limited support in κ and there is $\xi < \kappa$ limiting all supports. The ordinal ξ is as we wanted. Let $x_{\xi+1}$ be the Cohen real added at step $\xi + 1$. Consider dense sets in $\mathbb{P}_\xi \in V[G \restriction \xi]$ for all $f \in \omega^\omega \cap V[G \restriction \xi]$ and $n \in \omega$ guaranteeing that there is $m > n$ such that $x_{\xi+1}(m) > f(m)$. It follows that $x_{\xi+1}$ can not be dominated by any real in $V[G \restriction \xi] \supset \mathcal{F}$, contradicting our hypothesis that $\mathcal{F}$ is a dominating family. Thus $\mathfrak{d} \geq \kappa$ in $V[G]$. □

Corollary 2.3.18. *Suppose $\blacklozenge^*$ holds. Let κ be a regular cardinal which is $\aleph_0$-inaccessible. Let $\mathbb{P}_\kappa$ be the a CS^*-iteration of Cohen forcing of length κ. Then $\Vdash_{\mathbb{P}_\alpha} \omega_1 < \mathfrak{d}$.*

Proof. From Theorem 2.3.10 we have that $\Vdash_{\mathbb{P}_\kappa} \omega_1 = \check{\omega}_1$. By Lemma 2.3.17 we have $\Vdash_{\mathbb{P}_\kappa} \kappa \leq \mathfrak{d}$. Thus $\Vdash_{\mathbb{P}_\kappa} \omega_1 < \kappa \leq \mathfrak{d}$. □

We are finally able to conclude that $\clubsuit_F$ is compatible with $\neg CH$ modulo large cardinals.

Corollary 2.3.19. *$Con(ZFC +$ exists a strongly inaccessible cardinal$) \longrightarrow Con(ZFC + \clubsuit_F + \neg CH)$*

2.4 A counterexample for $\neg CH$

In this section we shall present the construction of the regular topological space such that every coloring has a monochromatic copy of $\omega + 1$, has no copy of $\omega^2 + 1$ and has character $\mathfrak{b} < \mathfrak{d}$. In the original paper [27] the authors use ω_1 in their construction, as in the case of the Example 2.2.3. The central idea of the proof is still the same, to use the same club as in the example to find the monochromatic sequence. But we need to carefully refine the neighborhoods of the limit ordinals in order to preserve the monochromatic sequence but still destroying all copies of $\omega^2 + 1$ and increasing the character of the space. For this we use a $\blacklozenge$-sequence of partial functions to guess all ω-colorings of ω_1. The refinement on the limit ordinals is given by such sequence and the full power of the club is used to obtain an element in the stationary set given by the sequence. We shall revisit this construction in a future proposition.

One way of trying to obtain an example without assuming CH is to investigate if there is any preservation of the original example when we add ω_2 Cohen reals to the ground model. In a broader sense, it would be interesting to know "what is left of $\blacklozenge$ after adding Cohen reals", as it is studied in [19] for the Continuum Hypothesis, and whether the resulting property would be strong enough to yield the monochromatic sequence with the same example. One of the problems that such approach faces is the following:

Lemma 2.4.1. *Adding ω_1 Cohen reals to a model of $\blacklozenge$ destroys the previous $\blacklozenge$-sequence.*

Proof. Let $\vec{A} = \langle f_\alpha : \alpha < \omega_1 \rangle$ be a $\blacklozenge$-sequence of partial functions $f_\alpha : \alpha \longrightarrow \omega$. The forcing $Fn(\omega_1, 2)$ is such that, for any generic filter G, the function $g = \bigcup G$ cannot be guessed through $\vec{A}$. Indeed, for $\omega < \alpha < \omega_1$, the following dense sets

$$D_\alpha = \{p \in Fn(\omega_1, 2) : \exists \beta \in dom(p) \cap \alpha (p(\beta) \neq f_\alpha(\beta))\}$$

yield that $g \restriction_\alpha$ is different from all f_α. Therefore, the old sequence cannot guess g in a stationary set. □

The previous lemma points to a possible problem in using the original example after the Cohen forcing to distinguish $\mathfrak{b}$ from $\mathfrak{c}$. Now what about a direct construction using only $\clubsuit$? We could use the elements A_α of the $\clubsuit$-sequence to refine the neighbourhoods of limit ordinals α in a similar way to what is done in the original construction. In this case we have a monochromatic sequence but no control over the color of the limit point. Indeed, given a coloring g, one of its colors must have stationary pre-image S. Now, $\overline{S}$ is a club and $S \cap \overline{S}$ is a stationary set of ω_1. If we try to guess S with the $\clubsuit$-sequence we find a stationary set S'. The problem is that we may have $S' \cap S \cap \overline{S} = \emptyset$.

Let us then understand what is needed to construct a counterexample without the use of CH, and exactly what part $\clubsuit_F$ plays in it. To do this we revisit the original construction of the example made in [27], but modifying it a little to obtain the basic idea behind our desired space.

Proposition 2.4.2. *Suppose $\langle A^m_\alpha : \alpha < \omega_1 \wedge m \in \omega \rangle$ is a $\clubsuit_F$-sequence. Then there is a regular topological space (ω_1, τ) such that $(\omega_1, \tau) \longrightarrow (top\, \omega + 1)^1_\omega$, $(\omega_1, \tau) \not\longrightarrow (top\, \omega^2 + 1)^1_\omega$*

and for every limit ordinal $\alpha \in \omega_1$ and for every $m \in \omega$ there exists an increasing sequence of ordinals in A^m_α converging to α.

Proof. Let $\langle A^n_\alpha : n \in \omega \wedge \alpha \in \omega_1 \rangle$ be a $\clubsuit_F$-sequence and $\alpha \in acc(\omega_1)$. Let γ_n be a strictly increasing sequence converging to α. Define $a_0 = \{t_0(0)\}$ where $t_0(0)$ is the first element of A^0_α. Given $n \in \omega$, suppose that, for all $m < n$, $a_m = \{t_m(0), \cdots, t_m(m)\}$ is already defined. Let $a_n = \{t_n(0), \cdots, t_n(n)\}$ be given by the following: $t_n(j)$ is the least element of A^j_α that is greater than γ_n and all $t_k(l)$ constructed beforehand. Notice that this is possible since all A^n_α are unbounded in α. Order the ordinals $t_k(l)$ as a strictly increasing sequence $s(n)$ converging to α. Now, for each $n \in \omega$, if $s(n)$ is a limit ordinal, let $(\beta^n_i)_{i\in\omega}$ be a strictly increasing sequence converging to $s(n)$. Otherwise consider $\beta^n_i = s(n) - 1$. We shall refine the topology in ω_1 by considering new neighbourhoods of α given by

$$N_\alpha(h, p) = \{\alpha\} \cup (\bigcup\{(\beta^m_{h(m)}, s(m)] : m \geq p\}),$$

where $h \in \omega^\omega$ and $p \in \omega$. As in the construction in [27], our space is regular and it cannot contain any copy of $\omega^2 + 1$ because of the new topology. Then, we just have to verify that $X \longrightarrow (top\,\omega + 1)^1_\omega$. For that we fix any coloring $f : \omega_1 \longrightarrow \omega$. Using (2) of $\clubsuit_F$ there are $m, n \in \omega$ and $\alpha \in \omega_1$ such that $\alpha \in f^{-1}[\{n\}]$ and $A^m_\alpha \subset f^{-1}[\{n\}]$. Now, by construction, the elements $\gamma \in A^m_\alpha$ that are in the new neighbourhood of α together with α constitute the monochromatic copy of $\omega + 1$. □

Notice that we are almost there in terms of the construction, that is, our space is regular, has monochromatic copies of $\omega + 1$ and has no copy of $\omega^2 + 1$. However, since we do not have CH, we cannot control the size of each local basis by ω_1 or $\mathfrak{b}$. In fact we have the following:

Lemma 2.4.3. *Let X be a space as constructed in Proposition 2.4.2. Then $\chi(X) \leq \mathfrak{d}$.*

Proof. Indeed, let $\mathcal{D}$ be a dominating family of size $\mathfrak{d}$. For a limit ordinal α the set $\{N_\alpha(h, p) : h \in \mathcal{D} \wedge p \in \omega\}$ is of size $\mathfrak{d}$ and is a local basis. Given any $N_\alpha(f, n)$, just consider $h \in \mathcal{D}$ such that $f \leq^* h$, and p greater than the maximum among the natural number that verify the previous inequality and n. □

One way to fix this problem would then be to find a model where $\mathfrak{b} = \mathfrak{d}$ and $\clubsuit_F$ holds. We still do not know if such a model is possible. We then have the following question:

Question 2.4.4. $Con(ZFC) \longrightarrow Con(ZFC + \clubsuit_F + 2^{\aleph_0} > \omega_1 + \mathfrak{d} = \mathfrak{b})$*?*

One of the ways to fix this situation would be to see that the example constructed in a ground model, assuming $\blacklozenge^*$ and using Proposition 2.4.2, could be preserved somehow when considering the Cohen CS^*-iteration. That is, we still have monochromatic convergent sequences and no copy of $\omega^2 + 1$, but now, since the basic neighborhoods come from the ground model, we at least have some control on the character of the space. For this we must make sure that our guessing sequence is not destroyed by our forcing iteration.

The next theorem shows us that what we asked above is indeed the case, and finally ends our search for the counterexample without CH.

Theorem 2.4.5. *Assume $\diamondsuit^*$ and that κ is a regular cardinal such that $\kappa \geq \aleph_2$ and, for every $\alpha < \kappa$, we have $\alpha^{\aleph_0} < \kappa$. Let $\mathbb{P}_\kappa$ be the Cohen CS^*-iteration of length κ and G be a $\mathbb{P}_\kappa$-generic filter. Then, in $V[G]$, there exists a topological space (X, τ) such that $X \longrightarrow (top\,\omega + 1)^1_\omega$, $X \not\longrightarrow (top\,\omega^2 + 1)^1_\omega$ and $\chi(X) = \omega_1 < \mathfrak{c}$.*

Proof. Let $\vec{A} = \langle A^m_\alpha : \alpha < \omega_1 \wedge n \in \omega \rangle$ be the $\clubsuit_F$-sequence derived from a $\diamondsuit^*$-sequence $\vec{B}$ in V, as in Lemma 2.3.3. Let $(X, \bar{\tau})$ the space obtained by applying, in V, Theorem 2.4.2 to $\vec{A}$. Let (X, τ) be the topological space generated, in $V[G]$, using $\bar{\tau}$ as a basis. We will prove that (X, τ) is the space we wanted. Let $p \in \mathbb{P}_\kappa$ and let σ be a $\mathbb{P}_\kappa$-name such that

$$p \Vdash \sigma : \omega_1 \longrightarrow \omega.$$

Recall that from Theorem 2.3.10 we have $\Vdash_{\mathbb{P}_\kappa} \check{\omega}_1 = \omega_1$. We will find $q \leq p$, $\dot{S}$ and n such that

$$q \Vdash \sigma[\dot{S}] = \{n\} \wedge \dot{S} \subset \omega_1 \wedge (\dot{S}, \tau) \text{ is homeomorphic to } (\omega + 1, \in).$$

We apply Theorem 2.3.10 to $\vec{A}$, σ and p, to obtain p_*, $\eta < \omega_1$ and $n, m \in \omega$ such that

$$p_* \Vdash (\check{A}^m_\eta \subset \sigma^{-1}[\{\check{n}\}] \wedge \sigma(\check{\eta}) = \check{n})$$

Next, we verify that p_* is the condition q that we are looking for. Let $\vec{w} = \{w_t : t \in \omega\}$ be the sequence given by Theorem 2.4.2 such that $\vec{w} \subset A^m_\eta$ and $\vec{w} \nearrow \eta$. Let $\dot{S} = \check{w} \cup \{\check{\eta}\}$ and notice that $(w \cup \{\eta\}, \bar{\tau})$ is homeomorphic to $(\omega + 1, \in)$. Therefore

$$p_* \Vdash (\dot{S}, \tau) \text{ is homeomorphic to } (\omega + 1, \in).$$

We also have $p_* \Vdash \dot{S} \subset \check{A}^m_\eta \cup \{\eta\} \subset \sigma^{-1}[\{n\}]$. Since CH holds in V we have $\chi(X, \tau) = \omega_1$ in V and therefore $\chi(X, \tau) = \omega_1$ in $V[G]$. From Corollary 2.3.18 we have $\omega_1^{V[G]} < \mathfrak{c}$. Thus (X, τ) is the space we wanted. □

The following corollary just indicates the consistency of the result given by our result above.

Corollary 2.4.6. *Assume the consistency of ZFC. Then ZFC is consistent with the existence of a topological space (X, τ) such that $X \longrightarrow (top\,\omega + 1)^1_\omega$, $X \not\longrightarrow (top\,\omega^2 + 1)^1_\omega$ and $\chi(X) = \omega_1 < \mathfrak{c}$.*

Proof. If $V = L$, then $\diamondsuit^*$ holds. If $\kappa \geq \aleph_2$ is a regular cardinal that is not the successor of a cardinal of countable cofinality, then, for all $\alpha < \kappa$, we have $\alpha^{\aleph_0} < \kappa$. Working in L, let $\kappa \geq \aleph_2$ be such a cardinal. By Theorem 2.3.10, if G is a $\mathbb{P}_\kappa$-generic filter then $V[G]$ is a model of ZFC where, by Theorem 2.4.5, there exists (X, τ) as in the hypothesis of the corollary. □

Another interesting corollary would be to notice that in the extension used in Theorem 2.3.10 we know the value of $\mathfrak{b}$ using the topological results made in this section.

Corollary 2.4.7. *Suppose $\diamondsuit^*$ holds, $\kappa \geq \aleph_2$ is a regular cardinal, and, for all $\alpha < \kappa$, $\alpha^{\aleph_0} < \kappa$. Then in any $\mathbb{P}_\kappa$-generic extension $\mathfrak{b} = \omega_1$.*

Proof. The results above gives us a space of character $\aleph_1$ that satisfies all the hypothesis, except the character, of Theorem 2.0.2 and does not satisfies its thesis. Therefore we must have that the character of such space must be $\geq \mathfrak{b}$. Hence $\omega_1 \geq \mathfrak{b} \geq \omega_1$. □

This is also interesting since it further instigates the question about the existence of a model for $\clubsuit_F + \neg CH + \mathfrak{b} = \mathfrak{d}$.

We now finish this section with the study of what happens when we consider the construction made in 2.4.2 directly in the extension given by the CS^*-iteration, further comparing the two spaces.

Let (Y, τ) be a space obtained by applying Theorem 2.4.2 to a $\clubsuit_F$-sequence $\vec{A}$. We have already seen in 2.4.3 that $\chi(Y) \leq \mathfrak{d}$. In the following lemmas we will present sufficient conditions on $\vec{A}$ which imply that the character of (Y, τ) is $\mathfrak{d}$.

Lemma 2.4.8. *If $\vec{A}$ is a $\clubsuit_F$-sequence such that there exist a limit ordinal $\alpha \in \omega_1$ and $m \in \omega$ satisfying $A^m_\alpha \subset acc(\omega_1)$, then $\chi(Y) = \mathfrak{d}$.*

Proof. We have $\chi(Y) \leq \mathfrak{d}$ since, for any given dominating family $\mathcal{D} \subset \omega^\omega$, the set $\{N_\alpha(h, n) : h \in \mathcal{D} \wedge n \in \omega \wedge \alpha \in acc(\omega_1)\}$ together with the isolated successor ordinals is a basis for Y.

Now, assume that $\chi(Y) < \mathfrak{d}$, and take α and m as in the hypothesis of the lemma. Let $\mathcal{B}$ be a base of open sets for α of size $< \mathfrak{d}$. We write $\mathcal{B} = \{N_\alpha(g_\theta, n_\theta) : \theta < \lambda\}$ for some $\lambda < \mathfrak{d}$. Take $J = \{r \in \omega : s(r) \in A^m_\alpha\}$ which is infinite by construction. Let $\phi : J \longrightarrow \omega$ be the increasing bijection. We will show that $\{g_\theta \restriction_J \circ \phi^{-1} : \theta < \lambda\}$ is a dominating family, which is a contradiction.

Given $f \in \omega^\omega$ consider $h : \omega \longrightarrow \omega$ defined by $h(i) = f \circ \phi(i)$ if $i \in J$ and $h(i) = 0$ otherwise. $N_\alpha(h, 0)$ is a neighbourhood for α so there must be $\gamma < \lambda$ such that $N_\alpha(g_\gamma, n_\gamma) \subset N_\alpha(h, 0)$. If $i \in J \setminus n_\gamma$, since $s(i)$ is a limit ordinal, we must have $\beta^i_{h(i)} < \beta^i_{g_\gamma(i)}$ and therefore $h(i) < g_\gamma(i)$. This implies that $g_\gamma \restriction_J \circ \phi^* \geq f$ since $h \restriction_J = f \circ \phi$. □

Lemma 2.4.9. *If $\vec{A}$ is a $\clubsuit_F$-sequence derived from a $\blacklozenge^*$-sequence $\vec{B}$, then there exist a limit ordinal $\alpha \in \omega_1$ and $m \in \omega$ such that $A^m_\alpha \subset acc(\omega_1)$.*

Proof. Use $\blacklozenge^*$ on the set $acc(\omega_1)$ to find the club C that guesses this set. Now, if $\alpha \in C$ is a limit of limit ordinals, then there is $m \in \omega$ such that $B^m_\alpha = acc(\omega_1) \cap \alpha$. Therefore $B^m_\alpha = A^m_\alpha \subset acc(\omega_1)$. □

The following corollary contrasts with the construction made in Theorem 2.4.5. Here we first consider the extension by a Cohen CS^*-iteration and afterwards construct the topological space using a $\clubsuit_F$-sequence following Theorem 2.4.2.

Corollary 2.4.10. *Let $\mathbb{P}_\kappa$ be a Cohen CS^*-iteration of length κ, where $\kappa \geq \aleph_2$ is a regular cardinal such that for every $\alpha < \kappa$ we have $\alpha^{\aleph_0} < \kappa$. Suppose $\vec{B}$ is a $\blacklozenge^*$-sequence and $\vec{A} = \langle A^m_\alpha : \alpha < \omega_1 \wedge n \in \omega\rangle$ is the sequence derived from $\vec{B}$. If G is a $\mathbb{P}_\kappa$-generic filter, then there exists a topological space (Y, τ) in $V[G]$ such that $(Y, \tau) \longrightarrow (top\, \omega + 1)^1_\omega$, $(Y, \tau) \not\longrightarrow (top\, \omega^2 + 1)^1_\omega$ and $\omega_1 < \chi(Y) = \mathfrak{d}$.*

Proof. By Lemma 2.4.9, there exists $\alpha \in \omega_1$ such that $A^m_\alpha \subset acc(\omega_1)$, which remains true in $V[G]$. By Corollary 2.3.11 we have that $\vec{A}$ is a $\clubsuit_F$-sequence in $V[G]$. If Y is the space obtained by applying Theorem 2.4.2 in $V[G]$ to $\vec{A}$, it follows from Lemma 2.4.8 that $\chi(Y) = \mathfrak{d}$. By Lemma 2.3.17 and Theorem 2.3.10 we have $\mathfrak{d} > \omega_1^{V[G]}$. Therefore (Y, τ) is the space we wanted. □

This is interesting because it highlights that even though the spaces obtained in Theorem 2.4.5 and Corollary 2.4.10 have the same underlying set and are generated considering the same $\clubsuit_F$-sequence, they are different. Indeed, the one from our ground model ends up with a coarser topology than the one generated directly in our extension, to the point where they end up having different characters.

Chapter 3

Elementary submodels and covering properties

In this chapter we will present a study on different covering properties and their interaction with elementary submodels. We shall work similarly to some of the previous papers in the area [23], [24], [25], and R. Figueiredo's thesis [16]. In this introduction we will give a brief historical overview of the relation between elementary submodels and topology. Afterwards, in the first section, we present some results from the previous cited works setting the tone for this chapter. In the two following sections we shall study strengthenings of the Lindelöf property and their preservations, starting with scattered spaces and then proceeding to less restricted spaces. Finally in the last section we study the weakly Lindelöf property in the same way as the other sections.

Elementary submodels had already been introduced as a concept and studied in the field of logic and set theory since the sixties. The interaction between elementary submodels and topology begins a little further from this, in the early eighties, one of the first works being S. Todorčević [41] “Directed sets and cofinal types". But I believe that it was with A. Dow's presentation at the Spring Topology Conference at Gainesville in 1988 and his subsequent paper [13] that the area has come to a flourishment. His work brings to attention some of the advantages of using elementary submodels in topology, such as, insights on the structure of the set-theoretic universe and a way to encompass closing-off arguments, bringing elegance to several proofs. In its wake several other works have been made, such as I. Bandlow's work [6], characterizing Corson compacta, and Z. Balogh's construction of a small Dowker space in ZFC in the paper [5]. With such context in mind L. Junqueira and F. Tall started a systematic study of elementary submodels as means to obtain an operation over a topological space X. This is developed in their work [25] and starts by considering the following definition:

Definition 3.0.1. *Fix a topological space $\langle X, \tau\rangle$ and an elementary submodel M such that $X, \tau \in M$. The space X_M is the topological space $X \cap M$ with the topology τ_M generated by $\{V \cap M : V \in \tau \cap M\}$. When there is no ambiguity about the topology on X we shall only specify the space X.*

The idea behind this definition being: “the way X is perceived through the lenses of M".

Their work focus on several aspects of this operation, such as preservation of properties from X to X_M, when X_M is a subset of X, or even a nice image of a subspace of X, among others. Following this work several others were developed with a focus on this operation, such as F. Tall's [37] and [39], L. Junqueira's [23] and K. Kunen's [28], focusing on several other aspects such as covering properties, preservation of properties from X_M to X and the interaction between compactness, scatteredness and when $X_M = X$. The following will be standard notation throughout this chapter.

Definition 3.0.2. *Given any property P on topological spaces we say that:*

- *The property P is preserved downwards if, for any topological space X with the property P and elementary submodel M, it holds that X_M has the property P.*
- *The property P is preserved upwards if, for any topological space X and elementary submodel M, it holds that if X_M has the property P, then X has the property P.*

Another continuation of the topic of preservation of compactness from these previous works is developed in R. Figueiredo's Ph.D. thesis, investigating whether we have preservation of the Lindelöf property. In this chapter we shall continue this investigation for other covering properties.

3.1 Some prior results

We start this section by stating some results from the aforementioned works, to give some context to this study. In their first work on the topic, [25], L. Junqueira and F. Tall exhibited the following result.

Proposition 3.1.1 (L. Junqueira, F. Tall[25])**.** *For i from 0 up to $3\frac{1}{2}$, the property T_i is preserved downwards.*

The properties T_0 and T_1 are also preserved upwards and, in the paper [37], F. Tall proved the following:

Proposition 3.1.2 (F. Tall[37])**.** *The T_2 and T_3 properties are preserved upwards for any elementary submodel.*

Some properties like $T_{3\frac{1}{2}}$ and T_4 are not preserved, as we can see for Example 1.1 of L. Junqueira's work [23].

Continuing to focus on some properties related to compactness we return to [25] to state the following result:

Proposition 3.1.3 (L. Junqueira, F. Tall[25])**.** *Let X be a countably compact topological space and M be a countably closed elementary submodel. Then X_M also is countably compact.*

The authors also investigate and present several examples regarding preservation. The following one is particularly interesting since it illustrates that the downwards preservation of compactness is not always guaranteed for any elementary submodel.

Example 3.1.4 (L. Junqueira, F. Tall[25])**.** *There are a compact Hausdorff space X and a countably closed elementary submodel M such that $X \in M$ and X_M is not paracompact.*

Proof. Let κ be a cardinal such that $\kappa^\omega = \kappa$ and consider $X = 2^\kappa$ with the product topology. Let $Y \subset X$ be a dense subset of size κ and M an elementary submodel such that $[M]^\omega \subset M$, $|M| = \kappa$ and $Y \cup \kappa \cup \{\kappa, X\} \subset M$. Note that X_M is a subspace of X. Indeed, given $f \in X \cap M$, if $p \subset f$ is finite, then $p \in M$. Since the open set $[p] = \{g \in X : p \subset g\}$ is definable from p and X, we have that $[p] \in M$.

We claim that X_M is not compact: indeed, since $|X \cap M| \leq \kappa < 2^\kappa = |X|$, we have that $X \cap M$ is a proper subset of X. Furthermore, $Y \subset X \cap M$ gives us that $X \cap M$ is dense and, therefore, cannot be closed. Finally, since X is Hausdorff, $X \cap M = X_M$ is not a compact subspace.

Finally we will see that X_M is not paracompact. The Proposition 3.1.3 on X and M gives that X_M is countably compact. Therefore X_M must not be paracompact, otherwise it would be compact, contradicting the previous assertion. □

Note that, as seen in [16], since every regular Lindelöf space is paracompact, the previous example yields the following:

Corollary 3.1.5 (R. Figueiredo[16]). *There is a regular Lindelöf (compact) space X and a countably closed elementary submodel M such that X_M is not Lindelöf.*

It becomes natural to ask in which cases we can guarantee the downwards preservation of compactness. In the paper [24] L. Junqueira and F. Tall develop a study on when $X = X_M$, and the compactness of X_M plays a central role in several of their results. One of the results in this paper, credited to their previous works [25] and [23], is the upwards preservation of compactness for spaces X_M that are Hausdorff. Another result from [24] is the following:

Proposition 3.1.6 (L. Junqueira, F. Tall[24]). *If X_M is compact Hausdorff and X is first countable, then $X = X_M$.*

This is interesting since it implies that if X is a compact Hausdorff first countable space then X_M cannot be compact unless $X = X_M$. Then, in chapter six of [24], the authors ask if the opposite of such result may happen, that is a compact space X such that X_M is compact for every M. Then, in a communication with P. Koszmider relating to the preservation of compactness by generic extensions, both him and the authors proved the following result independently.

Theorem 3.1.7 (P. Kosminder, L. Junqueira, F. Tall[24]). *If X is a compact T_2 scattered space, then X_M is compact for every elementary submodel M in which $X \in M$.*

This means that, for scattered Hausdorff spaces, we have the downwards preservation of compactness for any elementary submodel. In fact, still in chapter six of [24], we have a much stronger result that singles the scatteredness of such spaces as a necessary condition.

Theorem 3.1.8 (L. Junqueira, F. Tall[24]). *If there is a countable elementary submodel M such that $X \in M$ and X_M is compact Hausdorff, then X is scattered.*

Following this line of work K. Kunen in his paper [28] has developed the concept of squashable spaces, that is, a compact Hausdorff space X such that there exist a regular cardinal θ and an elementary submodel M satisfying $X \in M \prec H(\theta)$, X_M is compact and $X \not\subset M$. In this paper K. Kunen investigates the relation between squashability and large cardinals.

Another continuation for the line of work mentioned previously is in R. Figueiredo's Ph.D. thesis. The author studied the Lindelöf property and its interaction with elementary submodels. One of the results he obtained is the following:

Theorem 3.1.9 (R. Figueiredo [16])**.** *Let X be a regular scattered space and κ a cardinal. If $L(X) \leq \kappa$ then, for any elementary submodel M with $X \in M$ and $\kappa \in M$, $L(X_M) \leq \kappa$.*

This result is analogous to the one before relating to the downwards preservation of compactness to the Lindelöf property. Since scatteredness plays a central role in these results, also in his thesis R. Figueiredo explores the preservation of scatteredness.

Proposition 3.1.10 (R. Figueiredo[16])**.** *Let X be a topological space and M be an elementary submodel. Then X is scattered if and only if X_M is scattered.*

Now, in relation to the upwards preservation for the Lindelöf property, analogous to the preservation of compactness, we have the following result from [23] by L. Junqueira.

Proposition 3.1.11 (L. Junqueira[23])**.** *The Lindelöf property is preserved upwards if we assume M to be ω-covering.*

Note that in contrast to the compactness result it was necessary to assume that M is ω-covering. Indeed, any non-Lindelöf space X verifies that X_M has the Lindelöf property if we assume M countable. Finally a result from I. Juhász and W. Weiss's paper [21] guarantees that the Lindelöf property is preserved through forcing for scattered regular spaces.

As a direct corollary of the previously mentioned results we have the following:

Corollary 3.1.12. *If X is a regular scattered space then the following are equivalent:*

(a) *X is Lindelöf;*

(b) *X_M is Lindelöf for every elementary submodel M with $X \in M$;*

(c) *X_M is Lindelöf for every ω-covering elementary submodel with $X \in M$;*

(d) *X_M is Lindelöf for some ω-covering elementary submodel with $X \in M$;*

(e) *X is Lindelöf in every forcing extension;*

(f) *X is Lindelöf in every ω-closed forcing extension;*

(g) *X is Lindelöf for some ω-closed forcing extension.*

In his thesis R. Figueiredo also studies preservations on being a P-space. This is interesting since it is a natural consideration when working with properties of both compact and Lindelöf spaces. Indeed, a usual line of research is to investigate when Lindelöf P-spaces behave like compact spaces. This is also interesting since many covering properties stronger than Lindelöf are implied when we consider a Lindelöf P-space, for example the Rothberger and Alster properties.

Definition 3.1.13. *Given any topological space $\langle X, \tau \rangle$, the set $\{G \subset X : G \text{ is a } G_\delta \text{ set}\}$ forms a base to a topology τ_δ on X, which we call the G_δ topology. In this case we denote X_δ the topological space given by the G_δ topology on X.*

Notice that if X is a P-space then $X = X_\delta$. The next lemma by R. Figueiredo guarantees us that P-spaces are preserved for a specific class of elementary submodels.

Lemma 3.1.14 (R. Figueiredo [16]). *Let $\langle X, \tau \rangle$ be a topological space and M be an ω-covering elementary submodel. Then $(X_\delta)_M = (X_M)_\delta$.*

3.2 Stronger covering properties: scattered spaces

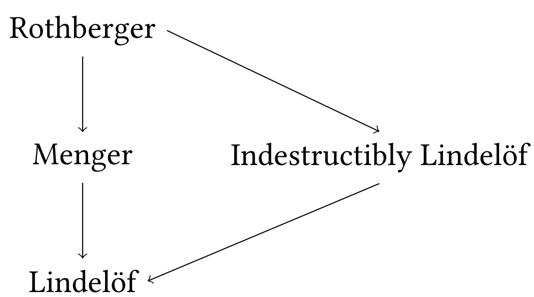
Given the results in the previous section it is natural to ask what happens to stronger covering properties. In [26] a study of Rothberger spaces, Menger spaces and their selection principles is made. Also, in [38], the notion of an indestructibly Lindelöf space is defined. In this section we are interested in exploring these properties and verifying their preservation considering elementary submodels and scattered spaces. In what follows we shall give the definitions and relations between these covering properties.

Definition 3.2.1. *Let $\langle X, \tau \rangle$ be a topological space. We say that X is Rothberger if, for every sequence of open covers $(\mathcal{U}_n)_{n\in\omega}$, there is a selection $f : \omega \longrightarrow \tau$ such that $f(n) \in \mathcal{U}_n$ and $f[\omega]$ is a cover of X.*

Definition 3.2.2. *Let $\langle X, \tau \rangle$ be a topological space. We say that X is Menger if, for every sequence of open covers $(\mathcal{U}_n)_{n\in\omega}$, there is a selection $f : \omega \longrightarrow [\tau]^{<\omega}$ such that $f(n) \subset \mathcal{U}_n$ and $\bigcup f[\omega]$ is a cover of X.*

Definition 3.2.3. *Let $\langle X, \tau \rangle$ be a topological space. We say that X is indestructibly Lindelöf if and only if X is Lindelöf and, for all countably closed forcings, the space X remains Lindelöf in the extension.*

Now the following diagram illustrates the relationship between the properties mentioned above. A more complete diagram considering several other properties and relations can be found for instance on L. Aurichi and F. Tall's paper [4].



Most of the relations on the above diagram are a direct consequence of the definitions. The fact that Rothberger implies indestructibly Lindelöf is proved in [36] as a consequence of a characterization of indestructibility from the authors and a result of Pawlikowski and can be found as Corollary 10 from [36].

It is known that these properties are indeed distinct. One example that shows this distinction for the Rothberger and Lindelöf properties is the real line $\mathbb{R}$. This space is Lindelöf but cannot be Rothberger. For a quick verification of this fact just consider the sequence of covers given by $\mathcal{V}_n = \{]x - \frac{1}{2^n}, x + \frac{1}{2^n}[\ :\ x \in \mathbb{R}\}$. Regardless of the choice made, the union of any selection using this sequence will have finite measure, therefore cannot contain $\mathbb{R}$. This will be used in future results from this chapter.

It is natural to ask, considering Corolary 3.1.12, if, in the case of regular scattered spaces, we have the preservation, either downwards or upwards, of the previous covering properties. In what follows we will present some known results from this area that will help us better understand the relation between some of these properties and the Lindelöf property.

Theorem 3.2.4 (R. Levy, M. D. Rice[31])**.** *Let X be a regular scattered Lindelöf space. Then X_δ is Lindelöf.*

The next result is proved in [36] in the chain of implications in Theorem 47 from this article. The authors attributed this result to F. Galvin, and in what follows we shall present a more direct proof of the implication we are interested in.

Theorem 3.2.5 (M. Scheepers, F. Tall[36])**.** *Let X be a Lindelöf P-space. Then X is a Rothberger space.*

Proof. Fix a sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X. For each $x \in X$ let $U_n^x \in \mathcal{U}_n$ be an open set containing x. Since X is a P-space we have that $V_x = \bigcap_{n\in\omega} U_n^x$ is an open set of X containing x. It follows that $\mathcal{V} = \{V_x : x \in X\}$ is an open cover for X. By Lindelöfness we take a countable subcover $\{V_{x_n} : n \in \omega\}$. The selection of elements from $(\mathcal{U}_n)_{n\in\omega}$ given by $(U_n^{x_n})_{n\in\omega}$ verifies the Rothberger property, since $X = \bigcup_{n\in\omega} V_{x_n} \subset \bigcup_{n\in\omega} U_n^{x_n}$. □

This next result verifies that the diagram above collapses to one property when considering regular scattered spaces.

Theorem 3.2.6. *Let X be a regular scattered space. The following are equivalent:*

(a) X is Lindelöf;

(b) X is Rothberger;

(c) X is Menger;

(d) X is indestructibly Lindelöf.

Proof. Most of the implications we need to verify follow from the diagram above. We only need to concern ourselves with (a) implies (b) to finish this proof. Due to Theorem 3.2.4, since X is regular scattered and Lindelöf, we have that X_δ is Lindelöf. Then X_δ is a Lindelöf P-space, which implies that X_δ is a Rothberger space by Theorem 3.2.5. It follows, from the fact that the topology in X_δ is finer than that of X, that X must be Rothberger. □

Now Corollary 3.1.12 holds for all the previous properties:

Corollary 3.2.7. *If X is a regular scattered space and T is any of the properties Rothberger, Menger or indestructibly Lindelöf then the following are equivalent:*

(a) X is T;

(b) X_M is T for every elementary submodel M with $X \in M$;

(c) X_M is T for every ω-covering elementary submodel with $X \in M$;

(d) X_M is T for some ω-covering elementary submodel with $X \in M$;

(e) X is T in every forcing extension;

(f) X is T in every ω-closed forcing extension;

(g) X is T for some ω-closed forcing extension.

Proof. Fix a regular scattered topological space X. To see that (a) implies (b) note that T implies Lindelöf and, by Theorem 3.1.9, for every elementary submodel M we have that X_M is also Lindelöf. Now, by Propositions 3.1.1 and 3.1.10, X_M also is regular and scattered. Therefore, by Theorem 3.2.6, X_M is T. The implications from (b) to (c) and (c) to (d) are immediate; we will see that (d) implies (a). Since X_M is Lindelöf and M is ω-covering, by Theorem 3.1.11, so is X. By the results 3.1.10 and 3.1.2 X also is regular and scattered, therefore X is T. Now (a) implies (e) follows from the fact that X is Lindelöf on the ground model and by Corollary 3.1.12 X also is Lindelöf in the extension. Now, since scatteredness and regularity are preserved in the extension we have that X is also T. The implications from (e) to (f) and (f) to (g) are immediate; we are only left with (g) implies (a). Indeed, since X is Lindelöf in the extension X must also be Lindelöf in the ground model. This is given by the fact that the forcing is countably closed, therefore, any countable subcover of elements from the topology of X in the ground model must also be in the ground model. Since X is scattered and regular in the ground model we have that X also is T. □

3.3 Stronger covering properties: general case

In this section we shall continue the results from the previous section by looking at the general case. A more detailed study of the given covering properties will be done without assuming our space is regular and scattered. We start by checking some upward preservation results.

Theorem 3.3.1. *Let X be a topological space and M be a countably closed elementary submodel such that $X \in M$. If X_M is Rothberger, then X must also be Rothberger.*

Proof. Suppose that X is not Rothberger. Then we have the following:

$$\exists f(f \text{ is a function} \wedge dom(f) = \omega \wedge \forall n \in \omega(f(n) \subset \tau \wedge \bigcup f(n) = X) \wedge$$

$$\wedge(\forall g : \omega \longrightarrow \tau((\forall n \in \omega(g(n) \in f(n))) \Rightarrow \bigcup_{n\in\omega} g(n) \neq X))).$$

Using the Tarski-Vaught criterion, we may take $f \in M$. Let $f = (\mathcal{U}_n)_{n\in\omega}$, no selection $g = (U_n)_{n\in\omega}$ such that $U_n \in \mathcal{U}_n$ can be an open cover of X. Now consider

$$\mathcal{U}'_n = \{U \cap M : U \in \mathcal{U}_n \cap M\}.$$

We will see that $\mathcal{U}'_n$ is an open cover of X_M. Note that, for all $n \in \omega$, $\mathcal{U}_n = f(n) \in M$. Therefore, using absoluteness, we reflect the fact that $\bigcup \mathcal{U}_n = X$ to M obtaining

$$\forall x \in X \cap M \exists U \in \mathcal{U}_n \cap M(x \in U).$$

This is enough to see that $\bigcup \mathcal{U}'_n = X_M$ and, since the open sets U from the cover were taken from M, we have that each $U \cap M \in \tau_M$. Using the Rothberger property of X_M there is a selection $(U_n \cap M)_{n\in\omega}$ from the sequence $(\mathcal{U}'_n)_{n\in\omega}$ that is an open cover of X_M. Therefore, since M is countably closed and each $U_n \in M$, we have $(U_n)_{n\in\omega} \in M$. Now we reflect the fact that $M \models \bigcup_{n\in\omega} U_n = X$ and obtain that $(U_n)_{n\in\omega}$ is a selection from f that is an open cover of X, contradicting our assumption. □

The same argument can be made with the Menger property, yielding the following.

Theorem 3.3.2. *Let X be a topological space and M a countably closed elementary submodel such that $X \in M$. If X_M is Menger, then X must also be Menger.*

Proof. Again assume that X is not Menger and use the Tarski-Vaught criterion to take $(\mathcal{U}_n)_{n\in\omega}$ sequence of open covers in M such that no selection $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ gives that $\bigcup_{n\in\omega} \mathcal{V}_n$ is an open cover of X. Now consider

$$\mathcal{U}'_n = \{U \cap M : U \in \mathcal{U}_n \cap M\}.$$

Again, this set is an open cover of X_M since, for all $n \in \omega$, $\mathcal{U}_n \in M$. Using the Menger property of X_M there is a selection $(\mathcal{V}'_n)_{n\in\omega}$ of finite subsets such that $\bigcup_{n\in\omega} \mathcal{V}'_n$ is an open cover of X_M. Let

$$\mathcal{V}_n = \{U \in \mathcal{U}_n : U \cap M \in \mathcal{V}'_n\}.$$

For all n, $\mathcal{V}_n \in M$. Since M is countably closed, we have $(\mathcal{V}_n)_{n\in\omega} \in M$, and we can reflect that it is a selection that contradicts the hypothesis. □

Working with the indestructibly Lindelöf property is more complex in a sense. A result by F. Tall in [38] characterizes indestructibility by means of covering trees. In a way this can be seen as a weaker version of the Rothberger property since we replace the sequence of open covers by a tree structure of height ω_1. To verify the upwards preservation we first need to define covering trees and verify a result from [38].

Definition 3.3.3. *Given a topological space $\langle X, \tau\rangle$, a covering tree T for X is a function $T : \omega^{<\omega_1} \longrightarrow \tau$ such that, for all $f \in \omega^{<\omega_1}$ the set $\{T(f \cup \{\langle domf, n\rangle\}) : n \in \omega\}$ is a cover of X.*

In this way each branch of this tree can be seen as a selection of open sets in each open cover. The next theorem is F. Tall's result mentioned before that characterizes indestructibility. The basic idea behind this is that for any given covering tree we can have a "countable selection" that gives an open cover for the topological space.

Theorem 3.3.4 (F. Tall[38])**.** *For a Lindelöf space X the following are equivalent:*

(a) X is indestructible;

(b) *X cannot be destroyed by the forcing $Fn(\omega_1, \omega, \omega_1)$;*

(c) *For each covering tree T for the space X, the set $\{f \in \omega^{<\omega_1} : ranT|_{pred(f)}$ is a cover of $X\}$ is dense in $\omega^{<\omega_1}$;*

(d) *For each covering tree T for the space X and $f \in \omega^{<\omega_1}$, there is $\overline{f} \in \omega^{\omega_1}$ such that $\overline{f}|_{domf} = f$ and $\{T(\overline{f}|_\alpha) : \alpha < \omega_1\}$ is a cover of X;*

(e) *For each covering tree T for the space X, there is $f \in \omega^{<\omega_1}$ such that the set $ranT|_{pred(f)}$ is a cover of X.*

Now we are ready to show the preservation result.

Theorem 3.3.5. *Let X be a topological space and M be a countably closed elementary submodel such that $X, \omega^{<\omega_1} \in M$. If X_M is indestructibly Lindelöf, then X must also be indestructibly Lindelöf.*

Proof. First we note that, by Proposition 1.4.10, $\omega_1 \subset M$ since M is countably closed and, therefore, ω-covering. This implies, also by countably closedness, that $\omega^{<\omega_1} \subset M$. Indeed, each $f \in \omega^{<\omega_1}$ is a countable set of ordered pairs of the form $\langle \alpha, n \rangle$ for $\alpha \in \omega_1 \subset M$ and $n \in \omega \subset M$. Now, if we assume that X is not indestructibly Lindelöf, by virtue of Theorem 3.3.4, there is a covering tree contradicting (e). Fix such a tree T. By the Tarski-Vaught criterion, we may take $T \in M$. Hence, for each $f \in \omega^{<\omega_1} \subset M$, $T(f) \in M$. Defining $\tilde{T} : \omega^{<\omega_1} \longrightarrow \tau_M$ by $\tilde{T}(f) = T(f) \cap M$, we have that $\tilde{T}$ is a covering tree for X_M following Definition 3.3.3. Indeed, for all $f \in \omega^{<\omega_1}$, since

$$\{T(f \cup \{\langle domf, n \rangle\}) : n \in \omega\} \text{ is a cover of } X \text{ and a subset of } M,$$

it follows that

$$\{\tilde{T}(f \cup \{\langle domf, n \rangle\}) : n \in \omega\} \text{ is a cover of } X_M.$$

Now, applying our hypothesis that X_M is indestructibly Lindelöf and Theorem 3.3.4, fix $g \in \omega^{<\omega_1}$ such that the set $ran\tilde{T}|_{pred(g)}$ is a cover of X_M, following (e) again. By countable closedness, $ranT|_{pred(g)} \in M$ since it is a countable set of elements from M. Therefore, by our choice of g and elementarity, we have $M \vDash \bigcup ranT|_{pred(g)} = X$, contradicting our choice of T. □

Now we will verify the downwards preservation for these stronger properties. It would be interesting to see if the strengthenings of the Lindelöf property are enough to give us any kind of preservation. It would be especially interesting if at least we had that X_M is Lindelöf. More so if we are starting from an X with the indestructibly Lindelöf property, seeing that its definition is based in another kind of preservation for the Lindelöf property. The following results show us that in fact the downwards preservation of the Lindelöf property is related to the preservation of all the other properties. That is, for certain elementary submodels M, if the space X has a stronger covering property and X_M is Lindelöf, then X_M also has the same covering property.

We start by verifying the case for the Rothberger property.

Theorem 3.3.6. *Let X be a Rothberger space and M be a countably closed elementary submodel such that $X \in M$. If X_M is Lindelöf, then X_M is Rothberger.*

Proof. To see that X_M is Rothberger fix a sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X_M. Using the Lindelöfness of X_M we can take each $\mathcal{U}_n$ to be a countable cover by open sets from $\tau \cap M$, taking a refinement if necessary. Now each of these covers $\mathcal{U}_n$ gives an open cover of X. Indeed, for each $U \in \mathcal{U}_n$ take the open set $W_U \in \tau \cap M$ such that $W_U \cap M = U$. Let $\mathcal{W}_n = \{W_U : U \in \mathcal{U}_n\}$; by countable closedness of M, $\mathcal{W}_n \in M$. Now we have

$$M \models \mathcal{W}_n \text{ is a cover of } X$$

so we reflect this and obtain that $\mathcal{W}_n$ is a cover of X. Now apply the Rothberger property on $(\mathcal{W}_n)_{n\in\omega}$ to find the selection $(W_{U_n})_{n\in\omega}$ that is an open cover. We have

$$X_M = \bigcup_{n\in\omega}(W_{U_n} \cap M) = \bigcup_{n\in\omega} U_n.$$

Therefore the selection $(U_n)_{n\in\omega}$ proves that X_M is Rothberger. □

Again, there is an analogous proof for the Menger property, requiring only a few adjustments on the previous one:

Theorem 3.3.7. *Let X be a Menger space and M be a countably closed elementary submodel. If X_M is Lindelöf, then X_M is Menger.*

Proof. Take $(\mathcal{U}_n)_{n\in\omega}$ and $\mathcal{W}_n$ as in the proof of Theorem 3.3.6. Apply the Menger property to find the selection $(\mathcal{V}_n)_{n\in\omega}$ such that $\mathcal{V}_n \in [\mathcal{W}_n]^{<\omega}$ and $\bigcup_{n\in\omega} \mathcal{V}_n$ is a cover of X. Restricting $\mathcal{V}_n$ to M, as in the proof before, verifies that X_M is Menger. □

To verify the downward preservation of the indestructibly Lindelöf property we will need a lemma, which complements the result in Theorem 3.3.4 by adding another condition to the list of equivalences. That is, given a base for the topology on X, the new condition allows us to consider only covering trees made by basic open sets.

Lemma 3.3.8. *For a Lindelöf space X the following are equivalent:*

(a) X is indestructible;

(b) For any covering tree T for the space X the set $\{f \in \omega^{<\omega_1} : ranT|_{pred(f)} \text{ is a cover of } X\}$ is dense in $\omega^{<\omega_1}$;

(c) For any base of open sets $\mathcal{B}$ of X, and any covering tree T for the space X composed by elements of $\mathcal{B}$, the set $\{f \in \omega^{<\omega_1} : ranT|_{pred(f)} \text{ is a cover of } X\}$ is dense in $\omega^{<\omega_1}$.

Proof. The equivalence between (a) and (b) is in [38] and (b) implies (c) is trivial. We need to concern ourselves with the implication from (c) to (a). For this, we will have to redo the proof of (b) implies (a) incorporating the desired base. Suppose that X is destructible and fix $\mathbb{P}$ a countably closed forcing and G a $\mathbb{P}$-generic filter witnessing this destructibility. We fix a base $\mathcal{B}$ for X in V and a cover of elements of $\mathcal{B}$ in $V[G]$ such that $\mathcal{B}$ does not have countable subcovers. Using the truth lemma, we can fix $p \in \mathbb{P}$ and σ a $\mathbb{P}$-name such that:

$$p \Vdash \sigma \subset \check{\mathcal{B}} \wedge \bigcup \sigma = \check{X} \wedge \forall s((s \subset \sigma \wedge |s| = \check{\omega}) \Longrightarrow \bigcup s \neq \check{X})$$

We now define recursively p_η for all $\eta \in \omega^{<\omega_1}$. We first take $p_\emptyset = p$. Now, for limit $\alpha < \omega_1$ and $\eta \in \omega^\alpha$, define p_η as a $q \in \mathbb{P}$ satisfying $q \leq p_{\eta|_\beta}$ for all $\beta < \alpha$. For $\alpha = \gamma + 1$ and $\eta \in \omega^\gamma$, consider

$$\mathcal{U}_\eta = \{U \in \mathcal{B} : \exists q \leq p_\eta (q \Vdash U \in \sigma)\} \in V.$$

$\mathcal{U}_\eta$ is a cover of X and, using Lindelöfness of X in V we select $\{U_\eta^n : n \in \omega\}$ a subcover of $\mathcal{U}_{\eta|_\gamma}$. We also fix $q_n \in \mathbb{P}$ witnessing that $U_\eta^n \in \mathcal{U}_\eta$. We put $p_{\eta ^\frown \{\langle \gamma, n\rangle\}} = q_n$. Now, the tree formed by the open sets selected on the successor steps and the empty set otherwise, contradicts (c). □

Theorem 3.3.9. *Let $\langle X, \tau\rangle$ be an indestructibly Lindelöf space and M a countably closed elementary submodel. If X_M is Lindelöf, then X_M is indestructibly Lindelöf.*

Proof. Suppose X_M is not indestructibly Lindelöf. By item (c) of Lemma 3.3.8 there exist a basic covering tree T for X_M, using $\mathcal{B} = \{V \cap M : V \in \tau \cap M\}$, and $g \in \omega^{<\omega_1}$ such that no extension f of g can yield an open cover of X_M through $ranT|_{pred(f)}$. For each $f \in \omega^{<\omega_1}$ let $U_f \in \tau \cap M$ be such that $T(f) = U_f \cap M$. We can consider $\tilde{T}$ given by $\tilde{T}(f) = U_f$. Note that $\tilde{T}$ is a covering tree for X. Indeed, for each $f \in \omega^{<\omega_1}$, the set

$$\{\tilde{T}(f \cup \{\langle domf, n\rangle\}) : n \in \omega\}$$

is in M by countable closedness, and T being a covering tree guarantees that

$$M \models \{\tilde{T}(f \cup \{\langle domf, n\rangle\}) : n \in \omega\} \text{ is a cover of } X.$$

Using the indestructibility of X and Theorem 3.3.4 we find $f \in \omega^{<\omega_1}$ such that $g \subset f$ and $ran\tilde{T}|_{pred(f)}$ is a cover of X. Therefore $ranT|_{pred(f)}$ is a cover for X_M, contradicting our previous assumption. □

Now we present some examples to put in perspective the results mentioned above. One natural question that comes to mind is whether there is any possibility to obtain the same results considering a less restricted M, maybe ω-covering? Other possible question is to verify whether we could abandon the hypothesis that X_M is Lindelöf.

We first consider a result of L. Junqueira from the paper [23].

Theorem 3.3.10 (L. Junqueira[23])**.** *Under the continuum hypothesis an elementary submodel M is ω-covering if and only if M is countably closed.*

Next we state some results due to A. Miller and D. Fremlin that consider the size of Rothberger and Menger subspaces of the real line.

Proposition 3.3.11 (D. Fremlin, A. Miller[32])**.** *The minimal cardinality of a subspace of $\mathbb{R}$ that is not Menger is $\mathfrak{d}$.*

Proposition 3.3.12 (D. Fremlin, A. Miller[32])**.** *Any Lindelöf space with cardinality strictly less than $cov(\mathcal{M})$ must also be Rothberger.*

The examples that follow show that in ZFC we cannot weaken our hypothesis for the Rothberger property.

Example 3.3.13. *Assume the negation of the Continuum Hypothesis and $cov(\mathcal{M}) = \mathfrak{c}$, for example in a model with $\aleph_2$ Cohen reals. There is a topological space X such that, for every ω-covering elementary submodel M such that $X \in M$ and $|M| \leq \aleph_1$, X_M is Rothberger but X is not.*

Proof. Just consider $\mathbb{R}$ and M as stated. Since $\mathbb{R}$ is first countable, by a result from [25], we have that $\mathbb{R}_M$ is a subspace of $\mathbb{R}$. Now, given that $|M| \leq \aleph_1$, we have $|\mathbb{R}_M| \leq \aleph_1 < cov(\mathcal{M})$. Hence, by Proposition 3.3.12, $\mathbb{R}_M$ is Rothberger and $\mathbb{R}$ is not. □

The example above shows that the upwards preservation can fail for the Rothberger property. The next example is from [22], and has already been used by R. Figueiredo in [16]. It is related with the downwards preservation of the Rothberger property.

Example 3.3.14 (I. Juhász, W. Weiss[22])**.** *Suppose there is a Kurepa tree with no Aronszajn subtrees, such as in a model of $V = L$, as seen in [12]. Then there is a regular Lindelöf P-space X, of weight $\aleph_1$, such that, for every ω-covering elementary submodel M of size ω_1 with $X \in M$, we have that X_M is not Lindelöf.*

Proof. The space X constructed in [22] under such conditions is a linearly ordered P-space, with the Lindelöf property, size $> \omega_1$ and weight ω_1 having the basis as a Kurepa line. If M is an ω-covering elementary submodel, by Proposition 1.4.10 we have $w(X) = \omega_1 \subset M$. It follows that X_M is a dense subspace of X. Indeed, since $w(X) \subset M$, by the Tarski-Vaught criterion, we have a base $\mathcal{B} \subset M$ for the topology of size ω_1. This means that $X_M = X \cap M$ with the subspace topology. Now, by the Tarski-Vaught criterion, for each $B \in \mathcal{B}$, since B is not empty there must be $x \in B \cap M$. Furthermore $|X_M| \leq |M| = \omega_1 < |X|$, therefore $X \cap M$ cannot be closed, in X, otherwise by density $|X_M| = |X|$. Finally, since X is a P-space X_M cannot be Lindelöf, otherwise it would be closed. □

Note that, in the Example 3.3.14, since X is a regular P-space and Lindelöf, it must also be Rothberger. This example, as it stands, addresses the weakening of our hypothesis on the Lindelöf property. Since we considered $V = L$ we have CH and therefore, by Theorem 3.3.10, the ω-covering elementary submodel is countably closed and, at the same time, X_M is not Lindelöf. On the other hand, if it is possible to obtain such a Kurepa tree in tandem with $\neg CH$ we would have the weakening of the submodel hypothesis. But in this case we still would lose the fact that X_M is Lindelöf. It is still unknown to us whether there is such a counterexample weakening The constraint on M with X_M Lindelöf. This motivates the following corollary:

Corollary 3.3.15. *It is consistent with ZFC that there exists a Rothberger space X of weight $\aleph_1$ such that, for every ω-covering elementary submodel M, X_M is not Rothberger.*

Now we can also ask these questions about the Menger property. The last example is also the one that shows downwards preservation does not hold for Menger spaces, since a Rothberger space is also a Menger space. For an upwards counterexample we return our focus to $\mathbb{R}$.

Example 3.3.16. *Assuming the negation of the Continuum Hypothesis and $\mathfrak{d} > \aleph_1$, there is a topological space X such that, for every ω-covering elementary submodel M of size $\aleph_1$, X_M is Menger, but X is not.*

Proof. First we note that, by Proposition 3.3.11, there is a non-Menger subspace $X \subset \mathbb{R}$ such that $|X| = \mathfrak{d} > \aleph_1$. Let M be a ω-covering elementary submodel of size $\aleph_1$ with $X \in M$. Since X is a subspace of $\mathbb{R}$ it is first countable and, therefore, X_M is a subspace of $\mathbb{R}$. Since $|X_M| \leq |M| = \aleph_1 < \mathfrak{d} = |X|$, we have that X_M is Menger by Proposition 3.3.11. □

Now, we are better equipped to understand the downward preservation for the Lindelöf Property and its aforementioned strengthenings. The results 3.3.7, 3.3.6 and 3.3.9 guarantee us that for the preservation of the Menger, Rothberger and indestructibly Lindelöf properties it is only necessary to verify the preservation of the Lindelöf property. It adds to the question of what conditions are enough to guarantee the preservation of the Lindelöf property. But, in contrast with these strengthenings, Example 3.3.14 gives us that it is consistent with ZFC that not even Lindelöf P-spaces, which implies Rothberger, are enough to guarantee the downwards preservation of the Lindelöf property. Furthermore, Corollary 3.1.5 states that even compactness is not enough for this preservation. This whole situation motivates the following definition:

Definition 3.3.17. *We say that a topological space X is elementary Lindelöf if, for every countably closed elementary submodel M, the space X_M is Lindelöf.*

Now, even though indestuctibly Lindelöf is not enough to guarantee the downwards preservation of the Lindelöf property, we turn our attention to [38]. Since the definition of indestructibility is based in the preservation of the Lindelöf properties for countably closed forcings, we hoped to find a direction for the parallel study involving elementary submodels. And such research bore some fruits. A result by F. Tall in his paper lists some properties which guarantee that a space is indestructibly Lindelöf.

Proposition 3.3.18 (F. Tall[38])**.** *Given a Lindelöf space X, each of the following conditions imply that X is indestructibly Lindelöf.*

(a) X is hereditarily Lindelöf;

(b) $|X| \leq \aleph_1$;

(c) $|X| < 2^{\aleph_1}$ and some form of generalized Martin's axiom holds so that one can meet $< 2^{\aleph_1}$ dense subsets of $Fn(\omega_1, \omega, \omega_1)$;

(d) X is T_1 and has a point countable base;

(e) X has at most $\aleph_1$ non-isolated points;

(f) CH holds, and every subset of X of cardinality $\leq \aleph_1$ has closure of cardinality $\leq \aleph_1$;

(g) X is scattered and regular.

We shall see that some of the properties mentioned by Tall are enough to guarantee downwards preservation of Lindelöfness for some types of elementary submodels.

Theorem 3.3.19. *Given a Lindelöf space X and an elementary submodel M, each of the following conditions on X and M imply that X_M is Lindelöf.*

(a) $X \cap M$ *is Lindelöf;*

(b) X *is hereditarily Lindelöf;*

(c) $|X| \leq \mathfrak{c}$ *and* M *is countably closed with* $\mathfrak{c} \in M$*;*

(d) $|X| \leq \aleph_1$ *and* M *is* ω*-covering with* $\aleph_1 \in M$*;*

(e) X *has at most* $\aleph_1$ *non-isolated points and* M *is* ω*-covering with* $\aleph_1 \in M$*;*

(f) X *has at most* $\mathfrak{c}$ *non-isolated points and* M *is countably closed with* $\mathfrak{c} \in M$*;*

(g) X *is scattered and regular.*

Proof. (a) It follows from the definitions that the topology on $X \cap M$ as a subspace of X is finer than that of X_M. Now, from the fact that $X \cap M$ is Lindelöf, so is X_M.

(b) Since X is hereditarily Lindelöf the space $X \cap M$ is Lindelöf, and, by (a), so is X_M

(c) Since M is countably closed we have $\mathfrak{c} \subset M$ by virtue of the Tarski-Vaught criterion and the fact that $\mathcal{P}(\omega) \subset M$. Now, since $|X| \leq \mathfrak{c}$ and $\mathfrak{c} \in M$, again by the Tarski-Vaught criterion, we have $X \subset M$. Now $X \cap M = X$ is a Lindelöf space, and, by (a), we have that X_M is Lindelöf.

(d) The proof is the same as in (c) replacing $\mathfrak{c}$ by $\aleph_1$ and using the fact that $\omega_1 \subset M$ for M ω-covering.

(e) Let X' be the set of all non-isolated points of X. Since

$$X' = \{x \in X : \forall U \in \tau(x \in U \Longrightarrow \exists y \in U(y \neq x))\}$$

is definable by parameters in M, we have $X' \in M$. Using that $|X'| \leq \aleph_1$ we have a surjective function $f : \omega_1 \longrightarrow X'$. Using the Tarski-Vaught criterion we assume $f \in M$. Now, since M is ω-covering $\omega_1 \subset M$ and therefore $ran(f) = X' \subset M$. Let $\mathcal{C}$ be any cover for X_M by open sets from $\{U \cap M : U \in \tau \cap M\}$. Take $\tilde{\mathcal{C}} = \{U \in \tau \cap M : U \cap M \in \mathcal{C}\}$. The fact that $X' \subset X \cap M$ guarantees that

$$\mathcal{W} = \tilde{\mathcal{C}} \cup \{\{x\} : x \in X \setminus X'\}$$

is an open cover of X. Using the fact that X is Lindelöf we extract a countable subcover $\mathcal{W}'$ from $\mathcal{W}$. Now,

$$\{U \cap M : U \in \mathcal{W} \setminus \{\{x\} : x \in X \setminus X'\}\} \subset \mathcal{C}$$

is the countable subcover desired.

(f) The proof is the same as in (d) replacing $\aleph_1$ by $\mathfrak{c}$.

(g) The proof is given by Corollary 3.1.12.

□

By the Theorem 3.3.19 we have several classes of topological spaces that are elementary Lindelöf.

Now, we can ask how the property from Definition 3.3.17 relates with other covering properties, not only those presented throughout this chapter. In particular the following question is interesting by the similarity of the definitions involved and the nature of the examples we obtained.

Question 3.3.20. *Is there an elementary Lindelöf space that is not indestructibly Lindelöf?*

Another interesting and possibly more complex question would be the following:

Question 3.3.21. *Is there a combinatorial characterization for the preservation of Lindelöfness by elementary submodels?*

3.4 Preservation for weakly Lindelöf spaces

In this final section we will explore the weakly Lindelöf property and show some preservation results. We also bring to attention an existing problem, also pointed by R. Figueiredo, related to the linearly Lindelöf property and its preservation by elementary submodels.

Definition 3.4.1. *A topological space X is said to be weakly Lindelöf if, for any open cover $\mathcal{U}$ of X, there is a countable subset $\mathcal{U}' \subset \mathcal{U}$ satisfying that $\bigcup \mathcal{U}'$ is dense in X. We also define*

$$wL(X) = min\{\kappa : \forall \mathcal{U} \text{ open cover of } X\ \exists \mathcal{U}' \subset \mathcal{U}(\bigcup \mathcal{U}' \text{ is dense in } X \text{ and } |\mathcal{U}'| \leq \kappa)\}$$

The next result shows a sufficient condition for the upwards preservation of the weakly Lindelöf property.

Lemma 3.4.2. *Let X be a topological space and M be an ω-covering elementary submodel with $X \in M$. If X_M is weakly Lindelöf then X is also weakly Lindelöf.*

Proof. Suppose that X is not weakly Lindelöf. Then we have the following:

$\exists \mathcal{U}$ open cover of X such that, for all countable $\mathcal{U}' \subset \mathcal{U}$, the set $\bigcup \mathcal{U}'$ is not dense in X.

Using the Tarski-Vaught criterion we can take such $\mathcal{U} \in M$. Since $\mathcal{U} \in M$ is an open cover of X we may reflect this to M to obtain that, $\{U \cap M : U \in \mathcal{U} \cap M\}$ is a cover of X_M by open sets of X_M. Since X_M is weakly Lindelöf, there is a countable $\mathcal{V} \subset \mathcal{U} \cap M$ such that

$$\{V \cap M : V \in \mathcal{V})\} \subset \{U \cap M : U \in \mathcal{U} \cap M\}$$

and

$$\bigcup\{V \cap M : V \in \mathcal{V})\} \text{ is dense in } X_M.$$

Using that M is ω-covering we may take a countable $\mathcal{V}' \in M$ such that $\mathcal{V} \subset \mathcal{V}'$. Furthermore, since $\mathcal{U} \in M$, we have $\mathcal{V}'' = \mathcal{V}' \cap \mathcal{U} \in M$. We now show that $\bigcup \mathcal{V}''$ is dense in X, which contradicts our initial assumption. Indeed, since $\mathcal{V}'' \in M$ is a collection of open sets and $\mathcal{V} \subset \mathcal{V}''$ we have the following;

$$M \models \forall W \in \tau \setminus \emptyset \exists V \in \mathcal{V}''(W \cap V \neq \emptyset).$$

Reflecting this affirmation we verify the denseness of $\bigcup \mathcal{V}''$. □

Again, as in the other covering properties cases, the scatteredness of the space allows the downward preservation in a similar way.

Theorem 3.4.3. *Let X be a regular scattered space and κ be a cardinal. If $wL(X) \leq \kappa$ then, for any elementary submodel M with $X \in M$ and $\kappa \in M$, $wL(X_M) \leq \kappa$.*

Proof. We shall prove this theorem by induction on the height of our scattered space X. The case where the topological space has height zero is trivial because the space is empty. Now assume that δ is an ordinal and, for every regular scattered space Y of height less than δ, if $wL(Y) \leq \kappa$, then $wL(X_M) \leq \kappa$. Fix $\langle X, \tau \rangle$, as in the conditions before, such that $ht(X) = \delta$ and fix an elementary submodel M. We must show that $wL(X_M) \leq \kappa$.

By Proposition 1.1.5 and regularity, consider $\langle U_x : x \in X \rangle$ such that $x \in U_x \in \tau$ and, if $x \neq y$ and $ht(x, X) \leq ht(y, X)$, then $y \notin cl_\tau(U_x)$. By the Tarski-Vaught criterion we can assume that $\langle U_x : x \in X \rangle \in M$.

Claim: If $x \in X \cap M$, then $cl_\tau(U_x)_M$ has a weak Lindelöf degree $\leq \kappa$.

Proof of Claim: We divide this proof in two cases: $ht(cl_\tau(U_x)) < \delta$ and $ht(cl_\tau(U_x)) = \delta$. In the first one our claim is clear by virtue of the induction hypothesis. Now, in the second case, we must have that $\delta = \gamma + 1$. Indeed, by the way we selected U_x, $\{x\}$ must be the last level of $cl_\tau(U_x)$, therefore $ht(x, cl_\tau(U_x)) = \gamma$ and $ht(cl_\tau(U_x)) = \gamma + 1$. Fix

$$\mathcal{U} \text{ open cover of } cl_\tau(U_x)_M \text{ by elements of } \{U \cap cl_\tau(U_x)_M : U \in \tau \cap M\}.$$

Since $x \in X \cap M$, there is an $W \in \mathcal{U}$ such that $x \in W$. Now $cl_\tau(U_x) \setminus W$ is such that $ht(cl_\tau(U_x) \setminus W) < \kappa$. By the induction hypothesis, $(cl_\tau(U_x) \setminus W)_M$ has weak Lindelöf degree $\leq \kappa$. Since

$$\{U \cap (cl_\tau(U_x) \setminus W)_M : U \in \mathcal{U}\} \text{ is a cover of } (cl_\tau(U_x) \setminus W)_M \text{ by open sets,}$$

there must be $\mathcal{V} \subset \mathcal{U}$ of size $\leq \kappa$ such that

$$\bigcup \{V \cap (cl_\tau(U_x) \setminus W)_M : V \in \mathcal{V}\} \text{ is dense in } (cl_\tau(U_x) \setminus W)_M.$$

It follows that $\mathcal{V} \cup \{W\}$ is a subcover of $\mathcal{U}$ of size $\leq \kappa$ and $\bigcup(\mathcal{V} \cup \{W\})$ is dense in $cl_\tau(U_x)$, concluding the proof of the claim. ■

Now, by the Tarski-Vaught criterion, we can find $\mathcal{H} \in M$ such that $\mathcal{H}$ is a subset of $ran(\langle U_x : x \in X \rangle)$ of size $\leq \kappa$ with dense union in X. Therefore, reflecting the previous assertion to M we have that

$$\bigcup\{(U_x)_M : U_x \in \mathcal{H} \cap M\} \text{ is dense in } X_M.$$

Indeed, given W non-empty open set in X_M there is $\tilde{W} \in \tau \cap M$ and $U_y \in \mathcal{H} \cap M$ such that $\tilde{W} \cap M = W$ and $\tilde{W} \cap U_y \cap M \neq \emptyset$. It follows that

$$\bigcup\{cl_\tau(U_x)_M : U_x \in \mathcal{H} \cap M\} \text{ is also dense in } X_M.$$

Fix $\mathcal{U}$ an open cover of X_M. We have that $\mathcal{U}$ is also a cover for $cl_\tau(U_x)_M$, for each $U_x \in \mathcal{H}$. Now we apply the claim to obtain a family of size $\leq \kappa$ of open sets of $\mathcal{U}$ whose union is dense in $cl_\tau(U_x)_M$. The union of those families verifies that $wL(X_M) \leq \kappa$. □

The two results above guarantee that an analogous of Corollary 3.2.7 holds for the weakly Lindelöf property.

Corollary 3.4.4. *If X is a regular scattered space then the following are equivalent:*

(a) X is weakly Lindelöf;

(b) X_M is weakly Lindelöf for every elementary submodel M with $X \in M$;

(c) X_M is weakly Lindelöf for every ω-covering elementary submodel with $X \in M$;

(d) X_M is weakly Lindelöf for some ω-covering elementary submodel with $X \in M$.

Another important weakening of Lindelöfness is linear Lindelöfness, which has been extensively studied. We first present some definitions and results from the literature related to this property, to then finally study its relations with elementary submodels. The central idea behind its definition is that, for compact spaces, we have the following classic characterization:

Proposition 3.4.5 ([15])**.** *A topological space X is compact if and only if every infinite subset of X admits a complete accumulation point.*

The natural extrapolation for this result, using uncountable subsets of regular cardinality, turns out to not be sufficient to characterize the Lindelöf property, although Lindelöfness implies this generalization. One example that verifies the non equivalence can be found in Mischenko's work [33]. So the following definition makes sense:

Definition 3.4.6. *A topological space is said to be linearly Lindelöf if every uncountable subset of regular cardinality admits a complete accumulation point.*

The name linearly Lindelöf was, in fact, given in a later date, in view of the following equivalence:

Proposition 3.4.7. *A topological space X is linearly Lindelöf if and only if, for every open cover γ of X such that, for $A, B \in \gamma$, $A \subset B$ or $B \subset A$, we have a countable subcover $\gamma' \subset \gamma$.*

In the paper [3] from A. Arhangel'skii and R. Buzyakova the relation between those two properties is systematically studied and several questions are posed.

It follows that it should be interesting to verify an analogous to Theorem 3.1.9 for the linearly Lindelöf property, as was also asked in [16].

Question 3.4.8. *If X is a scattered, regular, linearly Lindelöf P-space, must X_M be linearly Lindelöf for every elementary submodel M?*

This seems to be a perfunctory question at first, but in what follows we show that it is actually of interest in helping to answer some unknown questions from this area's literature. But first we give one more definition in the same style of the linearly Lindelöf.

Definition 3.4.9. *Given a cardinal κ and a topological space X, we say that X is κ-compact if every subset of X of size κ admits a complete accumulation point*

From the paper [20] from I. Juhász and Z. Szentmiklóssy, relating to κ-compactness and Shelah's PCF theory, we have the following result:

Theorem 3.4.10 (I. Juhász, Z. Szentmiklóssy[20])**.** *Every linearly Lindelöf and $\aleph_\omega$-compact space is Lindelöf.*

This result implies, using the definition of $\aleph_\omega$-compact, the following immediate corollary:

Corollary 3.4.11. *Every linearly Lindelöf space of size $< \aleph_\omega$ is Lindelöf.*

The following result is a consequence of Theorem 3.1.12 and Corollary 3.4.11 and was also made in [16].

Corollary 3.4.12 (R. Figueiredo[16])**.** *If X is a scattered, regular, linearly Lindelöf space, then X is Lindelöf if, and only if, there exists an ω-covering elementary submodel M of cardinality $< \aleph_\omega$ such that X_M is linearly Lindelöf.*

Proof. For the first implication suppose that X is a Lindelöf space and fix an ω-covering elementary submodel M of size $\aleph_1$ that exists by Proposition 1.4.11. Using Theorem 3.1.12 we have that X_M is Lindelöf and therefore, X_M is linearly Lindelöf as well. For the other implication fix M as in the hypothesis of the corollary such that X_M is linearly Lindelöf. Since $|M| < \aleph_\omega$ we have $|X_M| < \aleph_\omega$. By Corollary 3.4.11 X_M is also Lindelöf, and by Theorem 3.1.12 X must also be Lindelöf. □

This is interesting since it gives a condition for the equivalence of the Lindelöf and linearly Lindelöf properties for the class of scattered regular spaces. Then, since an affirmative answer to the Question 3.4.8 gives the second statement in the equivalence of Corollary 3.4.12, it is equivalent to a negative answer for the following:

Question 3.4.13. *Is there a topological space that is linearly Lindelöf, regular and scattered but not Lindelöf?*

This could be interesting for example when considering the following question from [1] by L. Junqueira, O. Alas and R. Wilson

Question 3.4.14. *Is there a linearly Lindelöf regular P-space which is not Lindelöf?*

In this case we might need to consider that the example cannot be scattered if an affirmative answer for Question 3.4.12. Another use of such restriction is when considering the following question:

Question 3.4.15. *Is a normal linearly Lindelöf space Lindelöf?*

It was noted by Miščenko, as is shown on [35], that such a counterexample for this question must be a Dowker space. In the case Question 3.4.12 has an affirmative answer, this counterexample, if it exists, cannot be scattered.

Chapter 4

Reflection and function spaces

In this chapter we start the study of reflections on function spaces. This was motivated by the previous studies relating to the Rothberger and indestructibly Lindelöf properties. We wanted to find possible counterexamples related to the reflection of such properties. This constituted part of the search for concrete examples of spaces with these covering properties. The idea behind the investigation of such spaces is that their particular structure might help us decide the reflection results regarding the Lindelöf property. This is particularly interesting when we study function spaces considering some historical focus on the interplay between the Lindelöf property and function spaces.

We note that, beyond our motivation considering covering properties, this study is interesting in and of itself. Several studies that have been made considering the relation between elementary submodels and function spaces, for example I. Bandlow's works [6], that characterizes Corson compacta by means of elementary submodels and retractions, and [7], that further develops the previous paper also investigating Corson compact subspaces of $C_p(X)$ using elementary submodels. Another one of such studies is T. Eisworth's work [14], that systematically investigates a quotient space obtained by analysing $C_p(X) \cap M$, that already appeared in the previous works from I. Bandlow and, independently, from A. Dow. In his paper T. Eisworth incorporates the definition of the space X_M and works with monotonically normal compacta. But, to the best of our knowledge, in spite of the extensive works on this field, no study had been made considering the spaces $C_p(X)_M$ and $C_p(X_M)$.

This chapter is divided in two sections. In the first one we state some previous results from the literature setting the groundwork for the rest of the chapter. In the second section we study the relation between $C_p(X)_M$ and $C_p(X_M)$ passing through the space X/M. Furthermore we verify a result concerning the downward preservation of the tightness of $C_p(X)_M$.

4.1 Background content for function spaces

In this section we will state some results that will be useful to have in mind when investigating spaces of the type $C_p(X)$ and covering properties. Furthermore we will introduce some notation and results from T. Eisworth [14] that will be used in this chapter. Throughout this chapter we will assume that our topological spaces are Tychonoff as is usual when considering continuous functions. Let us then start with the basic definitions.

Definition 4.1.1. *Given topological spaces X and Y the* function space $C_p(X, Y)$ *is the set* $\{f \in Y^X : f$ *is a continuous function*$\}$ *seen as a subspace of* Y^X *with the product topology.*

When $Y = \mathbb{R}$ we will shorten $C_p(X, \mathbb{R})$ to $C_p(X)$. One particular aspect we shall demand from all our elementary submodels from now on is that $\mathbb{R} \in M$, so that we may better express the topology from $C_p(X)$. The next result will be used throughout this chapter, and it is a direct application of a property from [25].

Proposition 4.1.2 (L. Junqueira, F. Tall[25])**.** *If M is an elementary submodel and $\mathbb{R} \in M$ then there is a basis of open sets $\mathcal{B} \in M$ of $\mathbb{R}$ such that $\mathcal{B} \subset M$ is countable.*

Proof. Indeed, since $\mathbb{R}$ is second countable we may use the Tarski-Vaught criterion to select such base in M. Since $\omega \subset M$ and M has a function that enumerates the elements of $\mathcal{B}$ we must also have $\mathcal{B} \subset M$. □

Throughout the rest of this chapter we will need to use this result. Therefore we preemptively fix such $\mathcal{B} \subset M$ to use in the next results as the *standard basis for* $\mathbb{R}$.

Now we see the relationship between a function space $C_p(X)$ and σ-compactness. A result in this direction can be found in [40] and guarantees that there are severe restrictions on X if we have that $C_p(X)$ is σ-compact.

Proposition 4.1.3. *Let X be a topological space. Then $C_p(X)$ is σ-compact if and only if X is finite.*

This is particularly interesting since many of the strengthenings of Lindelöfness can be derived from σ-compactness, such as the Menger, Alster and Hurewicz properties. The Rothberger property in particular is not derived from it. The following folklore result is very useful to set a restriction on Y when a function space $C_p(X, Y)$ can be Rothberger.

Proposition 4.1.4. *Let X and Y be non-empty topological spaces. If Y is Hausdorff, then there is a closed copy of Y in $C_p(X, Y)$. In particular there is a closed copy of $\mathbb{R}$ in $C_p(X)$.*

Proof. For each $y \in Y$ consider the following function $f_y : X \longrightarrow Y$ as being the constant function assuming value y. We shall argue that $\Phi : Y \longrightarrow C_p(X, Y)$ given by $\Phi(y) = f_y$ is a homeomorphism on its image and $\Phi[Y]$ is closed. Indeed, let $g \in C_p(X, Y) \setminus \Phi[Y]$. There must be $x, z \in X$ such that $g(z) \neq g(x)$. Using Hausdorffness we take disjoint open sets W_x and W_z from Y separating the images. Now

$$pr_x^{-1}[W_x] \cap pr_z^{-1}[W_z] \cap C_p(X, Y)$$

is an open set containing g that does not have any constant functions. This function clearly is 1-1; we just need to see that it is continuous and open. Given any open set $V \subset Y$ and $x \in X$ we have

$$\Phi[V] = pr_x^{-1}[V] \cap \Phi[Y].$$

In an analogous manner, given any basic open set of $\Phi[Y]$,

$$W = \bigcap_{i \in n} pr_{x_i}^{-1}[V_i] \cap \Phi[Y],$$

we have $\Phi^{-1}[W] = \bigcap_{i \in n} V_i$. □

This result gives us the following two corollaries regarding the Rothberger property.

Corollary 4.1.5. *A function space $C_p(X, Y)$ can be Rothberger only if Y is Rothberger.*

Proof. Indeed, the Rothberger property is preserved through taking closed subspaces and homeomorphisms. Therefore, if $C_p(X, Y)$ is Rothberger, by Proposition 4.1.4, so must be Y. □

Corollary 4.1.6. *No function spaces of the form $C_p(X)$ can be Rothberger.*

Proof. This is immediate from the corollary above and fact that $\mathbb{R}$ is not Rothberger. □

It follows that we must choose the space Y to be at least Rothberger if we want a chance of $C_p(X, Y)$ being Rothberger. For a recent example on such investigation, in a paper from 2016 [8], D. Bernal-Santos studied necessary and sufficient conditions on a Isbell-Mrowka space $\psi(\mathcal{A})$ in order to $C_p(\psi(\mathcal{A}), 2)^n$ have the Rothberger property.

For now, we will state some basic definitions and results from [14] that will be used in the next section.

Definition 4.1.7. *Given a topological space X, a subset A of X is said to be a* cozero set *if there is $f \in C_p(X)$ such that $A = f^{-1}[\mathbb{R} \setminus \{0\}]$. Analogously A is said to be a* zero set *if there is $f \in C_p(X)$ such that $A = f^{-1}[\{0\}]$.*

Definition 4.1.8 (T. Eisworth [14])**.** *Given a topological space X, an elementary submodel M and $x, y \in X$, we define the relation $x \sim_M y$ if and only if, for all $f \in C_p(X) \cap M$, $f(x) = f(y)$.*

This definition above is an equivalence relation and we may consider the following quotient space:

Definition 4.1.9 (T. Eisworth [14])**.** *Let X/M be the topological space given by $\{[x] : x \in X\}$ where $[x]$ represents the equivalence class of x considering $\sim_M$. We can also consider the projection $\pi_M : X \longrightarrow X/M$ given by $\pi_M(x) = [x]$. The topology on X/M will be the one generated by the set $\{\pi_M[U] : U \in M \wedge U$ is a cozero set$\}$.*

The following results will help us understand better some properties of X/M and its relation with X_M.

Proposition 4.1.10 (T. Eisworth [14])**.** *Given X and M as in the definition above, the following hold:*

- *the projection $\pi_M : X \longrightarrow X/M$ is continuous;*
- *$x \sim_M y$ if and only if, for every cozero set $U \in M$, $x \in U \iff y \in U$;*

- $[x] = \bigcap\{Z \in M : x \in Z \wedge Z$ *is a zero set* $\}$;
- $[x] = \bigcap\{\overline{U} : x \in U \in M \wedge U$ *is a cozero set* $\}$;
- X/M *is a Hausdorff space.*

Proposition 4.1.11 (T. Eisworth [14])**.** *X_M is homeomorphic to a dense subspace of X/M. That is, there is a homeomorphism $I : X_M \longrightarrow I[X_M] \subset X/M$ such that the image is dense in X/M.*

4.2 $C_p(X)_M$, $C_p(X/M)$ and $C_p(X_M)$

In this section we will focus on the relation between the function spaces $C_p(X_M)$, $C_p(X/M)$ and the reflected function space $C_p(X)_M$. Using the results from the previous section we will prove a series of new results comparing these spaces.

Proposition 4.2.1. *The function $R : C_p(X/M) \longrightarrow C_p(X_M)$, given by $R(g) = g \circ I$, is continuous and 1-1.*

Proof. First we note that R is well defined since I is a homeomorphism. Fix $f, g \in C_p(X/M)$ distinct. If $R(f) = R(g)$, then we must have $f \restriction_{I(X_M)} = g \restriction_{I(X_M)}$. By density of $I(X_M)$ and continuity of f and g we must have $f = g$. Now we must see that R is continuous. Fix a basic open set V in $C_p(X_M)$ given by $V = \bigcap_{i \in n} \pi_{x_i}^{-1}[U_i]$ where $x_i \in X_M$ and $U_i \in \mathcal{B}$. Now

$$R^{-1}[V] = \{f \in C_p(X/M) : R(f) \in V\} = \{f \in C_p(X/M) : \forall i \in n\, (f \circ I(x_i) \in U_i)\} =$$

$$= \bigcap_{i \in n} \{f \in C_p(X/M) : f([x_i]) \in U_i\},$$

which is open. □

Now that we found a continuous function relating $C_p(X/M)$ to $C_p(X_M)$ we shall take a look at $C_p(X) \cap M$. Since this space has two possible topologies, the subspace one and $C_p(X)_M$, the next two propositions will cover both cases, illustrating some differences between them.

Definition 4.2.2. *Given a function $g \in C_p(X)_M$, define $\tilde{g}$ by $\tilde{g}([x]) = g(x)$.*

We note that the definition above is sound. Given $g \in C_p(X)_M$ we have $\tilde{g} \in \mathbb{R}^{X/M}$. Indeed, because $g \in C_p(X) \cap M$, it does not matter what element of $[x]$ we take to compute g on. Now we give one condition to see that $\tilde{g}$ is continuous.

Proposition 4.2.3. *Given a function $g \in C_p(X)_M$ and $\tilde{g}$ as in Definition 4.2.2, if, for every $[x] \in X/M$, there is $y \in X \cap M$ such that $y \in [x]$, then $\tilde{g}$ is continuous.*

Proof. Fix $U \in \mathcal{B}$. Taking $[x] \in \tilde{g}^{-1}[U]$, we have $g(x) \in U$. Using Tychonoffness and $x, g, U \in M$ we can use the Tarski-Vaught criterion to take a co-zero set Z in M such that $x \in Z$ and $g[z] \subset U$. Now, $[x] \in \pi_M[Z]$, $\pi_M[Z]$ is an open set, and $\tilde{g}[\pi_M[Z]] \subset U$. □

For the next results we shall assume that, for all $g \in C_p(X)_M$, $\tilde{g}$, as in Definition 4.2.2, is continuous.

Proposition 4.2.4. *The function $L_0 : C_p(X)_M \longrightarrow C_p(X/M)$, given by $L_0(g) = \tilde{g}$, is open in its image and 1-1.*

Proof. Let $f, g \in C_p(X)_M$. If $f \neq g$, then there is $x \in X$ such that $f(x) \neq g(x)$. Therefore $\tilde{f}([x]) \neq \tilde{g}([x])$. To see that L_0 is open we consider an open subset V of $C_p(X)_M$ and take $W = L_0[V]$. Let $g \in L_0[V]$, that is, there is $f \in V$ such that $g = L_0(f)$. Since $V \in M$ and $f \in M$, there must be $n \in \omega$, $x_i \in M$ and $U_i \in \mathcal{B}$ for each $i \in n$ such that $f \in \bigcap_{i \in n} pr_{x_i}^{-1}[U_i] \subset V$. Therefore $g \in \bigcap_{i \in n} pr_{[x_i]}^{-1}[U_i] \cap L_0[C_p(X)_M] \subset L_0[V]$ □

Proposition 4.2.5. *The function $L_1 : C_p(X) \cap M \longrightarrow C_p(X/M)$, given by $L_1(g) = \tilde{g}$, is continuous and 1-1.*

Proof. Since L_1 has the same base function as L_0 the fact that L_1 is 1-1 is proved in same way as L_0. Therefore we only need to worry about the topological property relating to this function. Let V be a basic open set of $C_p(X/M)$ given by $V = \bigcap_{i \in n} pr_{[x_i]}^{-1}[U_i]$. We have the following:

$$L_1^{-1}[V] = \{f \in C_p(X) \cap M : L_1(f) \in V\} = \{f \in C_p(X) \cap M : f(x_i) \in U_i\} = \bigcap_{i \in n} pr_{x_i}^{-1}[U_i] \cap M,$$

which is open. □

Notice that, since our topology on $C_p(X) \cap M$ is finer that that of $C_p(X)_M$, it makes sense that L_0 is open and L_1 is continuous. The results from before motivate us to study the composition of the previous functions. We then are able to obtain several ways to relate $C_p(X_M)$ and $C_p(X)_M$. This is interesting since we might be able to derive some properties of X_M from $C_p(X)_M$ and vice versa.

Corollary 4.2.6. *The function $R \circ L_1$ is 1-1 and continuous.*

Corollary 4.2.7. *The function $R \circ L_0$ is 1-1 and open on its image.*

Proof. The 1-1 part is immediate from the previous propositions. To verify the openness we repeat the argument used in Proposition 4.2.4 exchanging L_0 by $R \circ L_0$. □

We shall return to these results later. The next two results give us insights on two other interesting properties that these compositions have.

Proposition 4.2.8. *For $i \in 2$, $R \circ L_i$ is a ring-isomorphism on its image.*

Proof. Let $f, g \in C_p(X) \cap M$. For all $x \in X_M$ we have:

$$\begin{aligned} R \circ L_i(f+g)(x) &= L_i(f+g) \circ I(x) = L_i(f+g)([x]) \\ &= (f+g)(x) = f(x) + g(x) = (L_i(f) + L_i(g))([x]) \\ &= (L_i(f) + L_i(g)) \circ I(x) = (R \circ L_i(f) + R \circ L_i(g))(x). \end{aligned} \tag{4.1}$$

In the same way we verify that $R \circ L_i(f.g) = R \circ L_i(f).R \circ L_i(g)$.

□

Proposition 4.2.9. *For $i \in 2$, $\operatorname{ran}(R \circ L_i)$ is a dense subset of $C_p(X_M)$.*

Proof. Let V be a non-empty basic open set from $C_p(X_M)$. There are $x_1, \cdots, x_n \in X_M$ and $V_0, \cdots, V_n$ basic open sets of $\mathbb{R}$ such that

$$V = C_p(X_M) \cap \left(\bigcap_{i \in n+1} pr^{-1}_{X_M, x_i}[V_i] \right).$$

By the Tarski-Vaught criterion, reflecting the fact that X is Tychonoff and, for all $j \in n+1$, $x_j, V_j \in M$, we can take $f \in C_p(X) \cap M$ such that $f(x_j) \in V_j$ for all $j \in n+1$. Now $R \circ L_i(f)$ is such that $R \circ L_i(f)(x_j) \in V_j$. This guarantees that $R \circ L_i(f) \in V$. □

The previous results give us a foundation and what to expect when dealing with such function spaces. For example, Propositions 4.2.4 and 4.2.5 illustrate the difference between $C_p(X) \cap M$ and $C_p(X)_M$. It would be interesting to see what conditions would be necessary for one of the functions $R \circ L_i$ to be a homeomorphism. One such condition that we will explore later is the case where $X \subset M$. Note that the following holds:

Proposition 4.2.10. *If $X \subset M$ then, for every $[x] \in X/M$, there is $y \in X \cap M$ such that $y \in [x]$.*

This guarantees that we are in the condition of the previous results.

Proposition 4.2.11. *If $X \subset M$ then $R \circ L_0$ is also continuous.*

Proof. When we argue that L_1 is continuous in Proposition 4.2.5 the main argument is that the sets $pr^{-1}_{x_i}[U_i] \cap M$ must be open in the domain. Now, for all $i \in n$, since $x_i \in X \subset M$ and $U_i \in M$ we must have $pr^{-1}_{x_i}[U_i] \in M$ and therefore $pr^{-1}_{x_i}[U_i] \cap M$ is an open set of $C_p(X)_M$ when we make the same arguments for L_0, concluding the proof. □

Corollary 4.2.12. *There is a dense copy of $C_p(X)_M$ inside $C_p(X_M)$.*

Proof. This is immediate since $R \circ L_0$ is an homeomorphism with dense image by the previous results. □

A consequence of these results is a downward preservation of the countable tightness of the function space $C_p(X)$. This relies on some classical results from the theory of function spaces and results from Chapter 3.

Let us start with one such result due to A. Arhangel'skii [2].

Theorem 4.2.13 (A. Arhangel'skii [2])**.** *For a space X it holds that $t(C_p(X)) = sup\{l(X^n) : n \in \omega\}$. In particular, $C_p(X)$ is countably tight if and only if X^n is Lindelöf for each $n \in \omega$.*

Since we will work with product spaces it becomes relevant to present the following result.

Proposition 4.2.14. *Let X, Y be topological spaces and M be an elementary submodel such that $X, Y \in M$. Then $(X \times Y)_M = X_M \times Y_M$.*

Proof. First we note that $(X \cap M) \times (Y \cap M) = (X \times Y) \cap M$, that is, both topological spaces have the same underlying set. We only need to see that their topologies coincide. Indeed, take a basic open set $U = (U_X \cap M) \times (U_Y \cap M)$ from $(X_M) \times (Y_M)$. We have $U = (U_X \cap M) \times (U_Y \cap M) = (U_X \times U_Y) \cap M$, which is a basic open set of $(X \times Y)_M$ since $U_X \times U_Y \in \tau_{X \times Y} \cap M$. The other inclusion is analogous. □

Now we can prove the preservation of the tightness.

Theorem 4.2.15. *Let X be a Tychonoff space and M an elementary submodel such that $X \subset M$. If $C_p(X)$ has countable tightness, then so does $C_p(X)_M$.*

Proof. Notice that, since $C_p(X)$ is countably tight, we must have X^n Lindelöf for all n by Theorem 4.2.13. Now since $X \subset M$ we also have $X^n \subset M$. That implies, by Proposition 4.2.14 and Theorem 3.3.19 item (a), $(X_M)^n$ is Lindelöf for all $n \in \omega$. This means that $C_p(X_M)$ also has countable tightness. By our Corollary 4.2.12 and the fact that the tightness preserves the inequality for subspaces and is preserved by homeomorphisms, we have that $C_p(X)_M$ also has countable tightness. □

One interesting question is whether $X \subset M$ is enough to guarantee that the subspace topology on $C_p(X) \cap M$ is equal to the topology on $C_p(X)_M$. This is in fact true, which means that $X \subset M$ is a stronger condition on the reflected C_p-space.

Proposition 4.2.16. *If X is a topological space and M is an elementary submodel such that $X \subset M$, then $C_p(X)_M$ is a subspace of $C_p(X)$.*

Proof. We only need to show that the topology from $C_p(X)_M$ is a basis for the subspace topology. Indeed, let A be a basic open set of $C_p(X)$. We have then $n \in \omega$, $x_i \in X$, for $i \in n$, and $V_i \in \mathcal{B}$ such that $A = \bigcap_{i \in n} \{f \in C_p(X) : f(x_i) \in V_i\}$. Now, since $X, \mathcal{B} \subset M$ we have that A is definable by elements of M. Therefore $A \in M$. □

Now the previous results implies the following corollary for cardinal functions that are hereditary.

Corollary 4.2.17. *If there is $i \in 2$ such that $R \circ L_i$ is a homeomorphism, then for any cardinal function $f \in \{\chi, \psi, s, w\}$ we have $f(C_p(X)_M) \leq f(C_p(X_M))$.*

One possible way of exploring this topic would be to verify the converse result. That is, what does $C_p(X)_M$ being a subspace of $C_p(X)$ imply? Another possibility would be to analyse whether one of the functions $R \circ L_i$ could be an homeomorphism without demanding that $C_p(X)_M$ has the subspace topology.

In what follows we will show a corollary obtained by adjusting previous results from the literature to the language of function spaces. Theorem 2.12 from [25] gives us a way to analyse whether X_M is a subspace of X by considering X with pointwise countable type:

Theorem 4.2.18 (L. Junqueira, F. Tall[25])**.** *If Y is a Hausdorff space with countable tightness and pointwise countable type, and M is a countably closed elementary submodel, then Y_M is a subspace of Y.*

Now the following result can be found in [40].

Proposition 4.2.19. *If X is any space and there exists $K \subset C_p(X)$ compact such that $\chi(K, C_p(X)) \leq \omega$, then X is countable.*

From these results we can conclude:

Corollary 4.2.20. *Let X be such that $C_p(X)$ has pointwise countable type and M is countably closed. Then $C_p(X)_M$ is a subspace of $C_p(X)$.*

Proof. We already know that $C_p(X)$ is Hausdorff and has pointwise countable type. The only thing that is left to check in order to apply Theorem 4.2.18 is the countable tightness. But now, since $C_p(X)$ is non-empty and has pointwise countable type, there is a compact set as in Proposition 4.2.19. It follows that X is countable and, therefore, $C_p(X)$ is first countable, hence countably tight. □

Most of the results in this section have been made considering $\mathbb{R}$. But they can be verified replacing $\mathbb{R}$ by any Y second countable when it makes sense. We conclude this section by stating that this study has just begun and we believe that it is a nice topic for further developments.

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