Pseudocompactness and Ultrafilters

Juliane Trianon Fraga

Thesis presented to the
Institute of Mathematics and Statistics
of the University of São Paulo
in partial fulfillment
of the requirements
for the degree of
Doctor of Science

Program: Mathematics
Advisor: Prof. Dr. Artur Hideyuki Tomita

During the development of this work, the author received
financial support from São Paulo Research Foudation (FAPESP) – grant #2019/12628-0

São Paulo
July, 2023

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This version of the thesis includes the corrections and modifications suggested by the Examining Committee during the defense of the original version of the work, which took place on July 19, 2023.

A copy of the original version is available at the Institute of Mathematics and Statistics of the University of São Paulo.

Examining Committee:

Prof. Dr. Artur Hideyuki Tomita (advisor) - IME-USP

Prof. Dr. Gabriel Zanetti Nunes Fernandes - IME-USP

Prof. Dr. Samuel Gomes da Silva - IME-UFBA

Profª. Drª. Ana Carolina Boero - CMCC-UFABC

Prof. Dr. Renan Maneli Mezabarba - DCEX-UESC

To my beloved parents.

Agradecimentos

Quem me dera ser liberto de mim. Quem me dera perder-me em Ti. Quem me dera não ser mais eu, mas Cristo em mim.
— John Owen

O meu principal agradecimento vai para Aquele que foi ao meu encontro enquanto eu buscava rejeitá-lo. Jesus, que me deu a Vida, ao qual eu pertenço, e para o qual eu dedico tudo o que faço. Com alegria, passarei a eternidade Te agradecendo, mas aqui digo mais uma vez: obrigada!

Agradeço à minha querida mãe, cujas doces palavras têm o poder de acalmar a tempestade, quando ela se forma na minha mente. Agradeço ao meu pai, pela sua sempre presente ajuda, em tudo o que preciso. Este trabalho só existe por causa do amor e da dedicação de vocês para comigo, desde sempre.

Agradeço ao meu orientador, Artur Tomita, pela sua paciência, calma e sabedoria para conduzir este trabalho. Agradeço pela compreensão que ele teve durante os momentos difíceis que passei. Gostaria que todos pudessem ter um orientador assim.

Agradeço à minha amada família, que sempre está ao meu lado, dando conselhos, apoio e leveza. Quero sempre poder retribuir tamanho carinho e dedicação.

Agradeço aos meus amigos, pelas conversas, pela companhia e pela compreensão quando precisei me concentrar no trabalho. Vocês são especiais.

Agradeço aos pesquisadores que gentilmente aceitaram fazer parte da banca examinadora, e que enriqueceram o trabalho com suas valiosas sugestões, correções e comentários.

Agradeço a todos os professores que fizeram parte da minha formação acadêmica. Certamente neste trabalho há um pedaço de cada um de vocês.

Por fim, agradeço à FAPESP, pelo apoio financeiro.

Resumo

Juliane Trianon Fraga. **Pseudocompacidade e Ultrafiltros**. Tese (Doutorado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2023.

Este trabalho apresenta avanços obtidos na teoria de grupos topológicos com propriedades *pseudocompact-like*. Construímos em ZFC um grupo enumeravelmente compacto de cardinalidade $2^{\mathfrak{c}}$ sem sequências convergentes não triviais. Também construímos em ZFC um grupo seletivamente pseudocompacto que não é enumeravelmente pracompacto. Usando a mesma técnica, construímos um grupo que tem todas as potências seletivamente pseudocompactas mas que não é enumeravelmente pracompacto, assumindo a existência de um único ultrafiltro seletivo. Naturalmente, uma pergunta similar à feita por Comfort em 1990 para grupos enumeravelmente compactos também pode ser feita para grupos enumeravelmente pracompactos: para quais cardinais α existe um grupo topológico G tal que G^{γ} é enumeravelmente pracompacto para todos os cardinais $\gamma < \alpha$, mas G^{α} não é enumeravelmente pracompacto? Neste trabalho construímos tal grupo no caso em que $\alpha = \omega$, assumindo a existência de $\mathfrak{c}$ ultrafiltros seletivos incomparáveis, e no caso em que $\alpha = \kappa^{+}$, com $\omega \leq \kappa \leq 2^{\mathfrak{c}}$, assumindo a existência de $2^{\mathfrak{c}}$ ultrafiltros seletivos incomparáveis. Também construímos um grupo topológico Abeliano, não divisível, livre de torção, que é compacto, e mostramos que para qualquer grupo Abeliano G, o grupo $\mathbb{Z} \times G$ não admite topologia de grupo $p-$compacta, para nenhum ultrafiltro livre p. Mostramos que o resultado anterior também é verdadeiro quando substituímos $\mathbb{Z}$ por um subgrupo de $\mathbb{Q}$ que é $r-$divisível para todo primo r, exceto exatamente um deles. Por fim, mostramos que existe uma topologia de grupo $p-$compacta sem sequências convergentes não triviais em $\mathbb{Q}^{(\mathfrak{c})}$ para a qual encontramos um subgrupo fechado $H \subset \mathbb{Q}^{(\mathfrak{c})}$ que contém um elemento não divisível (em H) por nenhum natural.

Palavras-chave: Topologia geral. Pseudocompacidade. Grupo topológico. Compacidade enumerável. Pseudocompacidade seletiva. Pracompacidade enumerável. $p-$compacidade. Ultrafiltro seletivo. Questão de Comfort. Grupo divisível. Sequências convergentes não triviais.

Abstract

Juliane Trianon Fraga. **Pseudocompactness and Ultrafilters**. Thesis (Doctorate). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2023.

This work presents advances obtained in the theory of topological groups with pseudocompact-like properties. We construct in ZFC a countably compact group without non-trivial convergent sequences of size $2^{\mathfrak{c}}$. We also construct in ZFC a selectively pseudocompact group which is not countably pracompact. Using the same technique, we construct a group which has all powers selectively pseudocompact but is not countably pracompact, assuming the existence of a single selective ultrafilter. Naturally, a question similar to that asked by Comfort in 1990 for countably compact groups can also be asked for countably pracompact groups: for which cardinals α is there a topological group G such that G^γ is countably pracompact for all cardinals $\gamma < \alpha$, but G^α is not countably pracompact? In this work we construct such group in the case $\alpha = \omega$, assuming the existence of $\mathfrak{c}$ incomparable selective ultrafilters, and in the case $\alpha = \kappa^+$, with $\omega \leq \kappa \leq 2^{\mathfrak{c}}$, assuming the existence of $2^{\mathfrak{c}}$ incomparable selective ultrafilters. We also construct an Abelian, torsion-free,, non-divisible topological group which is compact, and show that for every Abelian group G, $\mathbb{Z} \times G$ does not admit a $p-$compact group topology for any free ultrafilter p. We show that the previous result is also true when we replace $\mathbb{Z}$ by a subgroup of $\mathbb{Q}$ that is $r-$divisible for every prime r, except exactly one of them. Finally, we show that there exists a $p-$compact group topology on $\mathbb{Q}^{(\mathfrak{c})}$ without non-trivial convergent sequences for which we find a closed subgroup $H \subset \mathbb{Q}^{(\mathfrak{c})}$ which contains an element not divisible (in H) by any natural.

Keywords: General topology. Pseudocompactness. Topological group. Countable compactness. Selective pseudocompactness. Countable pracompactness. p-compactness. Selective ultrafilter. Comfort's Question. Divisible group. Non-trivial convergent sequence.

Contents

Introduction

About this thesis

The objective of this thesis is to present, in the most self-contained way possible, the problems I worked on during my PhD, as well as the original results obtained. Bearing this in mind, in most cases, prerequisites, results from other researchers and additional explanations will be provided if, and only if, they are useful in the motivation, content, or proofs of such results, most of which are published in the following papers that I coauthored:

- A. H. Tomita and J. Trianon-Fraga. Some pseudocompact-like properties in certain topological groups. *Topol. Appl.*, 314:108111, 2022
- A. H. Tomita and J. Trianon-Fraga. On powers of countably pracompact groups. *Topol. Appl.*, 327:108434, 2023

In the last chapter we present the latest topics we have been working on. We intend to submit the results already obtained in the near future.

Following the same philosophy, the proofs of results presented in the text will be given if the proof itself or the ideas contained in it are useful in some way for the presentation of our results, or if they are simple enough not to break the flow of the text. Proofs of well-known results, which are already found in many mathematics textbooks, will generally not be presented, but we will always present a possible source for those interested. There is an effort throughout the thesis to make it clear which results are original, folklore, or from other researchers.

I have always found it challenging to find the sweet spot between overexplaining and underexplaining. Personally, I prefer things that are overexplained rather than overly obscure, and I tried to make the thesis feel that way. I run the risk, therefore, of making the text long-winded and tiring. I hope this is not the case.

The content

In this subsection, we will summarize the topics worked on and the results obtained during the PhD. Everything presented here will be worked on in more detail in the next chapters. Many definitions and important prerequisites are not given yet, thus we refer the reader to the subsequent chapters in case of unfamiliarity with a term presented here. In fact, the idea is just to summarize the background behind the research area and briefly present the results obtained, and thus being familiar with all the terms presented here is not really necessary for now.

We worked with topological groups which have some *pseudocompact-like* property. Throughout the introduction, every topological space will be Tychonoff (Hausdorff and completely regular) and every topological group will be Hausdorff (thus, also Tychonoff). We will explain more about this restriction later.

To introduce the theme, we start from the definition of *pseudocompactness* due to Hewitt [Hew48]: an infinite topological space X is *pseudocompact* if each continuous real-valued function on X is bounded. Although Tychonoff's theorem ensures that the product of any family of compact topological spaces is compact (with respect to the product topology), the same is not true for general pseudocompact topological spaces [Ter52], nor for countably compact topological spaces [Nov53][1]. However, when looking at topological groups, we get an important theorem, proved by Comfort and Ross: the product of any family of pseudocompact topological **groups** is pseudocompact [CR66]. Whether the same holds for countably compact topological groups is an old question, one we will come back to shortly.

In order to establish a weaker notion than compactness, and still hold true certain properties valid for general compact topological spaces (such as, for example, a version of Tychonoff's theorem), Bernstein introduced [Ber70] the notion of *$p-$compactness*. Since then, many related concepts have emerged, which provide new topological spaces, with different properties. Those that are most relevant to the thesis are the following:

- *countable pracompactness*, which we consider folklore;
- *p-pseudocompactness*, introduced in [GS75];
- *ultrapseudocompactness*, introduced in [GS75];
- *selective p-pseudocompactness*, introduced in [AOT14];
- *selective pseudocompactness*, introduced in [GO14].

It is interesting to study how the above notions relate to each other, especially for topological groups, since this class of spaces leads to interesting results, such as the conservation of pseudocompactness by products. For instance, it follows straight from the definitions that, for general topological spaces:

$$\textbf{countable compactness} \Rightarrow \textbf{countable pracompactness} \Rightarrow$$
$$\Rightarrow \textbf{selective pseudocompactness} \Rightarrow \textbf{pseudocompactness}.$$

It is not hard to find an example of a countably pracompact group which is not countably compact. The question whether pseudocompactness implies selective pseudocompactness in topological groups was posed in [AOT14], and solved by Garcia-Ferreira and Tomita, who proved that there exists a pseudocompact group which is not selectively pseudocompact

[1] These papers proved, respectively, that pseudocompactness and countable compactness are not preserved under products for arbitrary topological spaces.

[GT15]. Our first original result, which we state below, answers the remaining question regarding the inversion of the arrows above:

Theorem 1 ([TT22], Theorem 4.1)**.** *There is a selectively pseudocompact group which is not countably pracompact.*

With a similar construction, we also proved that:

Theorem 2 ([TT22], Theorem 5.4)**.** *Assuming the existence of a single selective ultrafilter, there exists a topological group which has all powers selectively pseudocompact and is not countably pracompact.*

Now we will return to a subject that we left pending in a previous paragraph. Comfort, after proving, together with Ross, the mentioned theorem, asked if the same was true for countably compact topological groups. That is, his question was: is the product of countably compact groups countably compact? More generally, he asked the following question in the survey book *Open Problems In topology* [Com90]:

Question 1 ([Com90], Question 477)**.** *Is there, for every (not necessarily infinite) cardinal number $\alpha \leq 2^{\mathfrak{c}}$, a topological group G such that G^{γ} is countably compact for all cardinals $\gamma < \alpha$, but G^{α} is not countably compact?*

The restriction $\alpha \leq 2^{\mathfrak{c}}$ in the question above is due to a result that had already been proved in [GS75]: it is equivalent to a Hausdorff topological space X to have all powers countably compact and be such that $X^{2^{\mathfrak{c}}}$ is countably compact.

Van Douwen gave the first consistent answer to Comfort's original question: he proved under Martin's axiom (MA) that there are two countably compact groups whose product is not countably compact [Dou80]. More specifically, van Douwen proved the two following lemmas:

Lemma 1 ([Dou80])**.** *(ZFC) Every infinite Boolean countably compact group without non-trivial convergent sequences contains two countably compact subgroups whose product is not countably compact.*

Lemma 2 ([Dou80])**.** *(MA) There exists an infinite Boolean countably compact group without non-trivial convergent sequences.*

Using tools outside ZFC, many other examples of countably compact groups without non-trivial convergent sequences were given over the years. For instance:

- in [HJ76], from Continuum Hypothesis (CH);
- in [KTW00], from Martin's Axiom for countable posets ($\mathrm{MA}_{\mathrm{countable}}$);
- in [GTW05], from a single selective ultrafilter.

However, it was left open for a long time whether there exists an example in ZFC. This question was finally solved in 2021, when Hrušák, van Mill, Ramos-García, and Shelah [Hru+21] proved in ZFC the following theorem:

Theorem 3 ([Hru+21])**.** *There exists a Boolean countably compact topological group (of size $\mathfrak{c}$) without non-trivial convergent sequences.*

Due to Lemma 1, this result also solves the original Comfort's question in ZFC.

Then, in [BRT21a], the authors asked whether there exists an (Abelian) countably compact group without non-trivial convergent sequences of size **strictly greater** than $\mathfrak{c}$ in ZFC. With a slight modification to the construction given in [Hru+21], we answered this question:

Theorem 4 ([TT22], Theorem 3.1)**.** *There is a Boolean countably compact group of size* $2^{\mathfrak{c}}$ *without non-trivial convergent sequences.*

We mentioned above the three main results which appear in our first paper ([TT22]). From now on, we will turn to the results of our second paper ([TT23]), which are related to Comfort's more general question (Question 1).

It is natural also to ask productivity questions for countably pracompact and selectively pseudocompact groups. In this regard, Garcia-Ferreira and Tomita proved that if p and q are non-equivalent (according to the Rudin-Keisler order in the set of free ultrafilters on ω) selective ultrafilters on ω, then there are a p-compact group and a q-compact group whose product is not selectively pseudocompact [GT20]. Also, Bardyla, Ravsky and Zdomskyy constructed, under MA, a Boolean countably compact topological group whose square is not countably pracompact [BRZ20]. However, it is still not known whether it is a theorem of ZFC that selective pseudocompactness and countable pracompactness are non-productive in the class of topological groups.

More generally, one can ask Comfort-like questions, such as Question 1, for selectively pseudocompact and countably pracompact groups. In the case of selectively pseudocompact groups, the question is restricted to cardinals $\alpha \leq \omega$, since, as shown in a later chapter, if G is a topological group so that G^ω is selectively pseudocompact, then G^κ is selectively pseudocompact for every cardinal $\kappa \geq \omega$. In the case of countably pracompact groups, it is still not known whether there exists a cardinal κ satisfying that: if a topological group G is such that G^κ countably pracompact, then G^α is countably pracompact, for each $\alpha > \kappa$. Thus, there is no restriction to the cardinals α yet. We write below the non-trivial Comfort-like questions in the case of selectively pseudocompact and countably pracompact groups:

Question 2. *For which cardinals $\alpha \leq \omega$ is there a topological group G such that G^γ is selectively pseudocompact for all cardinals $\gamma < \alpha$, but G^α is not selectively pseudocompact?*

Question 3. *For which cardinals α is there a topological group G such that G^γ is countably pracompact for all cardinals $\gamma < \alpha$, but G^α is not countably pracompact?*

In [GT18], under the assumption of CH, the authors showed that for every positive integer $k > 0$, there exists a topological group G for which G^k is countably compact but G^{k+1} is not selectively pseudocompact. Thus, Question 2 and Question 3 are already solved for finite cardinals under CH. The cardinal $\alpha = \omega$ is the only one for which there are still no consistent answers to Question 2. In [TT23], we answered Question 3 for $\alpha = \omega$, assuming the existence of $\mathfrak{c}$ incomparable selective ultrafilters, and for $\alpha = \kappa^+$, with $\omega \leq \kappa \leq 2^{\mathfrak{c}}$, assuming the existence of $2^{\mathfrak{c}}$ incomparable selective ultrafilters:

Theorem 5 ([TT23], Theorem 3.1)**.** *Suppose that there are $\mathfrak{c}$ incomparable selective ultrafil-*

ters. Then there exists a topological group G which has all finite powers countably pracompact and such that G^ω is not countably pracompact.

Theorem 6 ([TT23], Theorem 4.1)**.** *Suppose that there are $2^{\mathfrak{c}}$ incomparable selective ultrafilters. Let $\kappa \leq 2^{\mathfrak{c}}$ be an infinite cardinal. Then there exists a topological group G such that G^κ is countably pracompact and G^{κ^+} is not countably pracompact.*

As a corollary of the proof of Theorem 5, we also obtained:

Corollary 1 ([TT23])**.** *Suppose that there are $\mathfrak{c}$ incomparable selective ultrafilters. Then, for each $n \in \omega$, $n > 0$, there exists a topological group whose nth power is countably compact and the (n+1)th power is not selectively pseudocompact.*

Since CH implies the existence of $2^{\mathfrak{c}}$ incomparable selective ultrafilters [Bla73], this is a slightly stronger result than what was obtained in [GT18].

The case $\alpha = 2^{\mathfrak{c}}$ of Theorem 6 is particularly interesting since, given a Hausdorff topological space X, if $X^{2^{\mathfrak{c}}}$ is countably compact, then X^α is countably compact for every $\alpha > 2^{\mathfrak{c}}$, as we mentioned.

Finally, we also work a little with the relation between algebraic properties of some Abelian groups and the possibility of endowing them with a $p-$compact topology, for some free ultrafilter p on ω. We have already obtained some results (they will be presented below) but, as mentioned before, they have not yet been submitted.

In this regard, Fuchs showed that a non-trivial free Abelian group does not admit a compact Hausdorff group topology, and Halmos proved that it is possible to topologize the additive group $\mathbb{R}$ so that it becomes a Hausdorff compact topological group [Hal44]. Notice that, algebraically, $\mathbb{R}$ can be considered as the direct sum of $\mathfrak{c}$ copies of $\mathbb{Q}$. Also, Tomita showed the following result:

Theorem 7 ([Tom98])**.** *Let G be an infinite free Abelian group endowed with a group topology. Then, G^ω is not countably compact.*

The proof of the theorem above relies on the fact that the only element of a free Abelian group that is infinitely divisible is 0. This suggests that a good candidate for a torsion-free group that admits a $p-$compact topology might be a divisible group, such as $\mathbb{Q}$. Indeed, Bellini, Rodrigues and Tomita recently showed that, if p is a selective ultrafilter and κ is a cardinal such that $\kappa = \kappa^\omega$, then $\mathbb{Q}^{(\kappa)}$ (the direct sum of κ copies of $\mathbb{Q}$) admits a $p-$compact group topology without non-trivial convergent sequences [BRT21b]. Our first result in this regard is that divisibility can be dropped:

Proposition 1. *There is an Abelian, torsion-free, non-divisible topological group which is compact.*

Since group divisibility is not essential for the existence of p-compact topologies, we tried to change the group $\mathbb{Q}^{(\mathfrak{c})}$ a little so that it loses its divisibility, and study whether we still get such a topology. The most immediate attempt would be to look at the Abelian group $\mathbb{Z} \times \mathbb{Q}^{(\mathfrak{c})}$. We did this, and showed that $\mathbb{Z} \times \mathbb{Q}^{(\mathfrak{c})}$ does not admit a $p-$compact group topology for any $p \in \omega^*$. Actually, we proved a more general result:

Proposition 2. *Let G be an Abelian group. Then, the Abelian group $\mathbb{Z} \times G$ does not admit a $p-$compact group topology for any $p \in \omega^*$.*

In particular, this answers a question asked in [Bel+21].

Interestingly, we also showed that by replacing $\mathbb{Z}$ with a subgroup of $\mathbb{Q}$ which is *very divisible* (in the following sense), the same is true:

Proposition 3. *Let G be an Abelian group, H be a subgroup of $\mathbb{Q}$ and $r > 1$ be a prime number. Suppose that H is t-divisible for each prime $t \neq r$ but is not $r-$divisible. Then, $H \times G$ does not admit a $p-$compact group topology, for any $p \in \omega^*$.*

Finally, using a similar construction to the one done in [BRT21b], we also showed that:

Theorem 8. *Let p be a selective ultrafilter. Then, there exists a p-compact group topology on $\mathbb{Q}^{(\mathfrak{c})}$ without nontrivial convergent sequences and a closed subgroup $H \subset \mathbb{Q}^{(\mathfrak{c})}$ which contains an element not divisible by any $n \in \omega$.*

That is, such H is a $p-$compact subgroup of $\mathbb{Q}^{(\mathfrak{c})}$, without non-trivial convergent sequences, which contains an element not divisible by any $n \in \omega$.

The outline

The thesis is organized as follows.

Chapter 1 deals with the preliminary content necessary for understanding the thesis, which is not directly related to the results that will be presented (such content is in chapter 2). We assume that the reader is already familiar with Mathematics at undergraduate level, but some basic contents were presented in this chapter for the purpose of fixing the notation, or due to its centrality and importance in the theme of the thesis.

In chapter 2, we will introduce our main field of study during the PhD. While making a historical overview, we present preliminary contents that were not covered in chapter 1, as they are directly related to the results that will be presented. We believe it would be better this way, as the topics are already presented in the context in which they will be used, providing greater objectivity and motivation. We also present a general idea of how most of the constructions that will be done in subsequent chapters work, including some facts and results that will be useful to us.

Chapters 3, 4, 5 and 6 are devoted to proving the following:

- **Chapter** 3: Theorem 4;
- **Chapter** 4: Theorem 1 and Theorem 2;
- **Chapter** 5: Theorem 5;
- **Chapter** 6: Theorem 6.

At the end of each of these chapters, we make a brief conclusion, indicating some open questions that we are interested in and intend to study in the future.

The reader will notice that chapter 7 differs somewhat from the other chapters. The reason for this is that the study of the themes presented there is much more recent. Therefore, the historical overview and the insertion of the results obtained in the context of the area are made in the chapter itself, and not in chapter 2. There we also present the proofs of all the original results obtained so far (which are Propositions 1, 2, 3 and Theorem 8), and at the end we present an idea of the next steps that we want to follow in this study.

Chapter 1

Preliminary Content

In this chapter we will group the preliminary content (those that are not directly related to the results that will be presented) necessary for the understanding of the next chapters. By preliminary content, we mean notations, nomenclatures, definitions, as well as results commonly found in undergraduate and graduate mathematics textbooks. We will assume that the reader is already familiar with many basic mathematics concepts and results at the undergraduate level. For this reason, most of the time we will talk about such basic concepts without presenting them (some references for the reader interested in such topics are suggested throughout the text). However, we sometimes find it necessary to recall some basic facts in more detail, as well as to fix the nomenclature we will use. For many readers, much of this chapter can only serve as an occasional reference, and it can be skipped without major problems in understanding the thesis.

1.1 Algebra

In this section we will recall some algebra results and fix some nomenclature we will use in the entire thesis. Much of what is presented here can be found in any basic algebra textbook. For Portuguese speakers, we suggest [Mar10].

Definition 1.1.1. A *group* is a nonempty set G together with a binary operation

$$\cdot : G \times G \to G$$

so that:

G1) *(associativity)* for every $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;

G2) there exists $e \in G$ so that, for each $a \in G$, $e \cdot a = a$;

G3) for every $a \in G$, there is $b \in G$ so that $b \cdot a = e$.

If a group G also satisfies that

G4) for every $a, b \in G$, $a \cdot b = b \cdot a$, we say that G is an *Abelian group*.

As usual, the element e in the property G2 is called the *identity* (or the *neutral element*) of the group. Also, the element b in the property G3 is called the *inverse* of a, and it is denoted by a^{-1}. Notice that we can think of the inverse of elements in a group as a function:

$$\begin{aligned} (\cdot)^{-1} :\ & G \longrightarrow G \\ & a \longmapsto a^{-1}. \end{aligned}$$

Sometimes, when there is no chance of confusion, we may omit the symbol $\cdot$ in the product of elements in a group: for instance, we write ab instead of $a \cdot b$. Recursively we define, for each $n \in \omega$ and $a \in G$, the element $a^n \in G$ in the following way: $a^0 = e$ and $a^{n+1} = a \cdot a^n$.

Given subsets H, K of a group G and $a \in G$, we also define the following:

- $H^{-1} \doteq \{g \in G : g = h^{-1},\ h \in H\}$;
- $HK \doteq \{g \in G : g = h \cdot k,\ h \in H, k \in K\}$;
- for $n \in \omega$, $n > 0$, $H^n \doteq \{g \in G : g = h_0 \cdot \ldots \cdot h_{n-1},\ h_0, \ldots, h_{n-1} \in H\}$;
- H is called *symmetric* if $H = H^{-1}$;
- $aH = \{g \in G : g = a \cdot h,\ h \in H\}$ and $Ha = \{g \in G : g = h \cdot a,\ h \in H\}$.

Note that due to the associativity property of groups, given elements $a, b, c \in G$ and $H \subset G$, one can write, with no chance of confusion, elements and sets as $a(bc)$ or $a(bH)$, for instance, without the parentheses, as abc and abH, respectively.

Next, we define:

Definition 1.1.2. Let G be a group.

1) If G is finite, we define its *order* as the number of its elements. If G is not finite, we say that its order is infinite.
2) The *order of an element $a \in G$* is the smallest positive $m \in \omega$ so that $a^m = e$. If such $m \in \omega$ does not exist, we say that the order of a is infinite.

Definition 1.1.3. A group G is called a *Boolean group* if all its elements, other than identity, have order 2.

Definition 1.1.4. Let G be a group. Then:

1) An element $g \in G$ is a *torsion element* if it has finite order.
2) G is a *non-torsion* group if there exists $g \in G$ which is not a torsion element. Otherwise, G is called a *torsion group.*
3) G is *torsion-free* if the only torsion element of G is the identity.

For Abelian groups, we usually will denote the binary operation as $+$, and the neutral element as 0_G (or simply by 0, if there is no chance of confusion). Also, the inverse of $a \in G$ is denoted by $-a$, and, for each $b, c \in G$, an operation as $b \cdot c^{-1}$ (which in the new notation is $b + (-c)$) will simply be denoted by $b - c$. As done before, we also define, for each $n \in \omega$ and $a \in G$, the element $na \in G$ recursively: $0a = 0$ and $(n+1)a = na + a$. In other words, na will be the element a "added" n times.

Now, we define the following concepts:

Definition 1.1.5. An Abelian group G is *divisible* if, for each $n \in \omega \setminus \{0\}$ and $g \in G$, there is $y \in G$ so that $g = ny$.

Definition 1.1.6. Given a prime number $r \in \omega$, we say that an Abelian group G is $r-$divisible if, for each $g \in G$, there is $y \in G$ so that $g = ry$.

1.2 Topology

In this section we will present the preliminary content of topology.

The meaning of certain basic nomenclatures in topology does not exactly match depending on the source. Thus, we will spend a few pages in this section specifying exactly what we mean by each term that could cause confusion. As usual, a topological space (X, τ) will be denoted simply by X when there is no chance of confusion.

We highlight here that throughout the thesis the set of natural numbers will be identified with the first infinite ordinal, ω. Also, ordinals are identified with cardinals: for instance, ω will be identified with $\aleph_0$. As usual, $\mathfrak{c}$ will denote the cardinality of the continuum.

1.2.1 Basic General Topology

Definition 1.2.1. Let I be a set, $(X_i : i \in I)$ be a family of topological spaces, Y be a nonempty set and, for each $i \in I$, $f_i : Y \to X_i$ be a function. The topology on Y *generated* by the family of functions $(f_i : i \in I)$ is the weakest[1] topology on Y that makes each f_i continuous.

Recall that, given a set I, $(X_i : i \in I)$, Y and $(f_i : i \in I)$ as in the definition above,

$$\mathcal{B} = \Big\{ \bigcap_{i \in F} f_i^{-1}(U_i) \,:\, F \subset I \text{ is finite and } U_i \subset X_i \text{ is an open subset, for each } i \in F \Big\}$$

is a base for the topology generated by the family of functions $(f_i : i \in I)$ on Y.

Let again $(X_i : i \in I)$ be a family of topological spaces. Given $i_0 \in I$, we will usually denote by π_{i_0} the projection from the cartesian product $X \doteq \prod_{i \in I} X_i$ onto X_{i_0}:

$$\begin{aligned} \pi_{i_0} :\ X &\longrightarrow X_{i_0} \\ (x^i)_{i \in I} &\longmapsto x^{i_0}. \end{aligned}$$

Definition 1.2.2. Let $(X_i : i \in I)$ be a family of topological spaces and $X \doteq \prod_{i \in I} X_i$. The *product topology* on X is the topology generated by the family of functions $(\pi_i : i \in I)$.

[1] That is, the intersection of all topologies with the mentioned property.

Remark 1. Given a family of topological spaces $(X_i : i \in I)$, if we somehow say "the topological space $X = \prod_{i \in I} X_i$", without specifying the topology, we mean that X is endowed with the product topology.

Next we define the *separation axioms* for topological spaces. Their precise meanings may vary according to the source. In this thesis, we agree with [Wil04], for example.

Definition 1.2.3. A topological space X is said to be:

1) T_0 if, and only if, given distinct points $x, y \in X$, there is an open subset of X containing one and not the other;
2) T_1 if, and only if, given distinct points $x, y \in X$, there is an open subset of X containing x and not containing y;
3) T_2 (or *Hausdorff*) if, and only if, given distinct points $x, y \in X$, there are disjoint open subsets U and V of X so that $x \in U$ and $y \in V$;
4) *regular* if, and only if, for each closed subset A of X and $x \in X \setminus A$, there are disjoint open subsets U and V of X such that $A \subset U$ and $x \in V$;
5) T_3 if, and only if, X is regular and T_1;
6) *completely regular* if, and only if, for each closed subset A of X and $x \in X \setminus A$, there is a continuous function[2] $f : X \to [0, 1]$ so that $f(x) = 0$ and $f(A) \subset \{1\}$;
7) $T_{3\frac{1}{2}}$ (or *Tychonoff*) if, and only if, X is completely regular and T_1;
8) *normal* if, and only if, given disjoint closed subsets A and B of X, there are disjoint open subsets U and V of X so that $A \subset U$ and $B \subset V$;
9) T_4 if, and only if, X is normal and T_1.

Below we list some important basic facts to remember. Their proofs can be found in many basic topology textbooks (e.g., [Wil04] contains some of them).

Lemma 1.2.4. *A topological space X is T_1 if, and only if, for each $x \in X$, $\{x\}$ is closed.*

Lemma 1.2.5. *Every regular T_0 space is Hausdorff.*

Lemma 1.2.6. *Every completely regular space is regular.*

Corollary 1.2.7. *Every Tychonoff space is Hausdorff.*

Now, let's recall some more definitions and important results.

Definition 1.2.8. Let X be a topological space.

a) A *cover* of X is a collection $\mathcal{U}$ of subsets of X so that $\bigcup \mathcal{U} = X$.
b) An *open cover* of X is a cover of X whose elements are all open subsets.
c) A *subcover* of a cover $\mathcal{U}$ is a subcolletion of $\mathcal{U}$ which is also a cover.

[2] The subset $[0, 1] \subset \mathbb{R}$ is endowed with the subspace topology.

Definition 1.2.9. A topological space X is *compact* if, and only if, every open cover of X has a finite subcover.

Definition 1.2.10. Given a topological space X, we define the following.

1) Let $(x_n)_{n\in\omega}$ be a sequence in X. We say that a point $x \in X$ is an *accumulation point* of $(x_n)_{n\in\omega}$ if, and only if, for each open neighborhood U of x, $\{n \in \omega : x_n \in U\}$ is infinite.

2) Let $A \subset X$. We say that a point $x \in X$ is an *accumulation point* of A if, and only if, for each open neighborhood U of x, $A \cap (U \setminus \{x\}) \neq \emptyset$.

3) Let $A \subset X$. We say that a point $x \in X$ is an $\omega-$*accumulation point* of A if, and only if, for each open neighborhood U of x, $A \cap U$ is infinite.

Definition 1.2.11. A topological space X is *countably compact* if, and only if, it satisfies one of the following equivalent conditions.

1) Every infinite subset of X has an ω-accumulation point.

2) Every sequence on X has an accumulation point.

3) Every countable open cover of X has a finite subcover.

Remark 2. If X is a T_1 topological space, it is equivalent for a point $x \in X$ to be an accumulation point of a subset $A \subset X$ or an $\omega-$accumulation point of A, as we shall see in the next result. For this reason, when dealing with T_1 topological spaces, we will usually use the terminology *accumulation point*, while actually working with the definition of an *$\omega-$accumulation point.*

Lemma 1.2.12. *Let X be a T_1 topological space. Then, $x \in X$ is an accumulation point of a subset $A \subset X$ if, and only if, x is an $\omega-$accumulation point of A.*

Proof. $(\Rightarrow)$ Let $x \in X$ be an accumulation point of a subset $A \subset X$. Suppose that there exists an open neighborhood U of x so that $A \cap (U \setminus \{x\}) = \{x_0, ..., x_m\}$, for $m \in \omega$. Since X is T_1, $V \doteq X \setminus \{x_0, ..., x_m\}$ is an open subset, and therefore $U \cap V$ is an open neighborhood of x which satisfies that $A \cap \big((U \cap V) \setminus \{x\}\big) = \emptyset$, a contradiction, as x is an accumulation point of A. Thus, $A \cap U$ is infinite for every open neighborhood U of x.

$(\Leftarrow)$ It is clear. □

It is also well known that:

Proposition 1.2.13. *Every compact topological space is countably compact.*

Proof. Let X be a compact topological space. Suppose that there is a sequence $(x_n)_{n\in\omega}$ on X which does not have an accumulation point. Then, for each $x \in X$ we may fix a neighborhood V_x of x so that $\{n \in \omega : x_n \in V_x\}$ is finite. Since $(V_x)_{x\in X}$ is an open cover of X, it has a finite subcover, say $\{V_{y_0}, ..., V_{y_m}\}$, $m \in \omega$. Thus, some V_{y_i}, $0 \leq i \leq m$, has to contain infinitely many elements x_n, a contradiction. Therefore, X is countably compact. □

The following definition is not very usual, but we still consider it folklore.

Definition 1.2.14. A topological space X is *countably pracompact* if there exists a dense subset $D \subset X$ such that every sequence on D has an accumulation point in X.

Regarding compact spaces, we have the following important theorem:

Theorem 1.2.15 (Tychonoff's Theorem)**.** *The product of any family of compact topological spaces is compact with respect to the product topology.*

Proof. We refer [Wil04] for a proof. □

1.2.2 Filters and ultrafilters

In this subsection, we will recall the definitions and useful properties of filters and ultrafilters. Much of the content is based on section 12 of [Wil04].

Definition 1.2.16. Let S be a set. A *filter* on S is a collection $\mathcal{F} \subset P(S)$ which satisfies the following properties:

a) $S \in \mathcal{F}$ and $\varnothing \notin \mathcal{F}$;

b) if $F_0, F_1 \in \mathcal{F}$, then $F_0 \cap F_1 \in \mathcal{F}$;

c) if $F_0 \in \mathcal{F}$ and $F_0 \subset F_1$, then $F_1 \in \mathcal{F}$.

Note that, equivalently, we could replace property a) by

$$\text{a)' } \varnothing \subsetneqq \mathcal{F} \subsetneqq P(S).$$

Definition 1.2.17. Let $\mathcal{F}$ be a filter on a set S. A nonempty subcollection $\mathcal{F}_0 \subset \mathcal{F}$ is a *filter base* of $\mathcal{F}$ if, and only if,

$$\mathcal{F} = \{F \in P(S) : F_0 \subset F \text{ for some } F_0 \in \mathcal{F}_0\}.$$

Not every nonempty subcollection of $P(S)$ is a filter base for some filter on a set S. Indeed, the following result holds.

Lemma 1.2.18. *Let S be a nonempty set and $\mathcal{C} \subset P(S)$ be a nonempty collection of nonempty subsets of S. Then, $\mathcal{C}$ is a filter base for some filter on S if, and only if, satisfies the following property:*

$$\textbf{P:} \textit{ for each } C_0, C_1 \in \mathcal{C}, \textit{ there exists } C_2 \in \mathcal{C} \textit{ so that } C_2 \subset C_0 \cap C_1.$$

Proof. ($\Rightarrow$) Suppose that $\mathcal{C}$ is a filter base for a filter $\mathcal{F}$ on S. For each $C_0, C_1 \in \mathcal{C}$, we have that $C_0 \cap C_1 \in \mathcal{F}$. Thus, there exists $C_2 \in \mathcal{C}$ so that $C_2 \subset C_0 \cap C_1$.

($\Leftarrow$) Suppose that $\mathcal{C}$ satisfies the property **P**. Then, we claim that the collection

$$\mathcal{F} \doteq \{F \in P(S) : C \subset F \text{ for some } C \in \mathcal{C}\}$$

is a filter on S (for which, clearly, $\mathcal{C}$ is a filter base). Indeed, as $\mathcal{C}$ is nonempty, we have that $S \in \mathcal{F}$. Also, $\varnothing \notin \mathcal{F}$, otherwise we would have that $\varnothing \in \mathcal{C}$. Thus, the property a) of Definition 1.2.16 is satisfied for $\mathcal{F}$. Property c) is clearly satisfied. Moreover, given

$F_0, F_1 \in \mathcal{F}$, there are $C_0, C_1 \in \mathcal{C}$ so that $C_0 \subset F_0$ and $C_1 \subset F_1$,. Then, $C_0 \cap C_1 \in \mathcal{C}$, and thus $F_0 \cap F_1 \in \mathcal{F}$. Therefore, property b) is also satisfied. □

Next we recall some more useful definitions and results.

Definition 1.2.19. Let $\mathcal{F}$ and $\mathcal{G}$ be filters on a set S. We say that:

a) $\mathcal{F}$ is *finer* than $\mathcal{G}$ if $\mathcal{G} \subset \mathcal{F}$;

b) $\mathcal{F}$ is *strictly finer* than $\mathcal{G}$ if $\mathcal{G} \subsetneq \mathcal{F}$;

c) $\mathcal{F}$ is *fixed* (or *principal*) if $\bigcap \mathcal{F} \neq \emptyset$;

d) $\mathcal{F}$ is *free* if $\bigcap \mathcal{F} = \emptyset$.

Definition 1.2.20. A filter p on S is called an *ultrafilter* if there is no filter on S strictly finer than p[3].

Proposition 1.2.21. *An ultrafilter p on a set S is fixed if, an only if, there is $x \in S$ so that*

$$p = \{F \in P(S) \, : \, x \in F\}.$$

That is, the collection $\{\{x\}\}$ is a filter base for p.

Proof. ($\Rightarrow$) Let p be a fixed ultrafilter on S and $x \in \bigcap p$. It is clear that $p \subset \{F \in P(S) \, : \, x \in F\}$. Let $F \in P(S)$ be so that $x \in F$. If $F \notin p$, then $X \setminus F \in p$, since p is an ultrafilter. However, $x \notin X \setminus F$, a contradiction. Thus, $p = \{F \in P(S) \, : \, x \in F\}$.

($\Leftarrow$) In this case, $\bigcap p = \{x\} \neq \emptyset$, and thus p is fixed. □

Proposition 1.2.22. *An ultrafilter p on a set S is fixed if, and only if, it contains a finite subset.*

Proof. ($\Rightarrow$) By Proposition 1.2.21, if p is a fixed ultrafilter on S, it contains the set $\{x\}$, for some $x \in S$.

($\Leftarrow$) Suppose that p contains the finite subset $\bar{F} \doteq \{x_0, ..., x_n\}$, $n \in \omega$, and that p is free. Then, there are $F_0, ..., F_n \in p$ so that $x_i \notin F_i$, for each $i = 0, ..., n$. This is a contradiction, since $\bar{F} \cap (F_0 \cap ... \cap F_n) = \emptyset$. Thus, p is fixed. □

The following well-known result is very useful when dealing with ultrafilters, as well as the corollary following it.

Proposition 1.2.23. *A filter p on a set S is an ultrafilter if, and only if, for every $X \in P(S)$, either $S \in p$ or $X \setminus S \in p$.*

Proof. We refer [Wil04] for a proof. □

Corollary 1.2.24. *Let p be an ultrafilter on a set S, and suppose that $\bigcup_{i=0}^n A_i = S$, for $A_0, ..., A_n \subset S$ and $n \in \omega$. Then, there is $i \in \{0, ..., n\}$ so that $A_i \in p$.*

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[3] That is, an ultrafilter on S is a maximal filter on S with respect to inclusion.

Proof. In the conditions of the statement, if $A_i \notin p$ for every $i = 0, ..., n$, then, by Proposition 1.2.23, $S \setminus A_i \in p$ for every $i = 0, ..., n$. Thus,

$$\bigcap_{i=0}^{n} (S \setminus A_i) = \emptyset \in p,$$

a contradiction. Therefore, there is $i \in \{0, ..., n\}$ so that $A_i \in p$. □

It is also important to know when a family of subsets can be extended to an ultrafilter. We address this question below.

Definition 1.2.25. Let S be a set and $C \subset P(S)$. We say that C has the

- *finite intersection property* if, and only if, the intersection of any finite subcollection of C is a nonempty subset of S;
- *strong finite intersection property* if, and only if, the intersection of any finite subcollection of C is an infinite subset of S.

Proposition 1.2.26. *Let S be a set and $C \subset P(S)$. Then:*

a) C can be extended to an ultrafilter on S if, and only if, has the finite intersection property;

b) C can be extended to a free ultrafilter on S if, and only if, has the strong finite intersection property.

Proof. a) ($\Rightarrow$) If C can be extended to an ultrafilter p on S, then the intersection of any finite subcollection of C also belongs to p, thus cannot be an empty set.

($\Leftarrow$) Suppose that C has the finite intersection property. Let $\mathcal{B}$ the the family of all finite intersections of elements in C. By Lemma 1.2.18, $\mathcal{B}$ is a filter base for some filter $\mathcal{F}_0$ on S, which contains C in particular. Now, let

$$\mathcal{Z} \doteq \{\mathcal{F} \subset P(S) \,:\, C \subset \mathcal{F} \text{ and } \mathcal{F} \text{ is a filter.}\}$$

Since $\mathcal{Z} \neq \emptyset$, an usual application of Zorn's Lemma shows that $\mathcal{Z}$ has a maximal element p, which will therefore be an ultrafilter.

b) ($\Rightarrow$) If C can be extended to a free ultrafilter p on S, then the intersection of any finite subcollection of C also belongs to p. Since p is free, by Proposition 1.2.22, all such intersections are infinite subsets.

($\Leftarrow$) Suppose that C has the strong finite intersection property. Let $\mathcal{B}$ the the family of all finite intersections of elements in C. By Lemma 1.2.18,

$$\mathcal{F}_0 = \{F \in P(S) \,:\, B \subset F, \text{ for some } B \in \mathcal{B}\}$$

is a filter. Also, it is clear that every element of $\mathcal{F}_0$ is an infinite subset and $C \subset \mathcal{F}_0$. Thus, the set

$$\mathcal{Z} \doteq \{\mathcal{F} \subset P(S) \,:\, C \subset \mathcal{F}, \mathcal{F} \text{ is a filter, and every element of } \mathcal{F} \text{ is infinite.}\}$$

is nonempty. Again, we may use Zorn's Lemma to show that there is a maximal element $p \in \mathcal{Z}$. Thus, p is an ultrafiter, and, according to Proposition 1.2.22, it is free. □

Corollary 1.2.27. *Every filter on a set S can be extended to an ultrafilter on S.*

Proof. Follows directly from Proposition 1.2.26, since every filter has the finite intersection property. □

Remark 3. Throughout the thesis, we will denote the set of free ultrafilters on ω by ω^*.

1.2.3 P-points and weak P-points of ω^*

In this subsection, we will define the concepts of P-points, weak P-points, and present some results that we will need.

Definition 1.2.28. Let X be a T_1 topological space. A point $x \in X$ is called *P−point* if, and only if, whenever $(U_n : n \in \omega)$ is a countable family of neighborhoods of x, $x \in \operatorname{int}\big(\bigcap_{n\in\omega} U_n\big)$.

Definition 1.2.29. Let X be a T_1 topological space. A point $x \in X$ is called *weak P−point* if, and only if, x is not an accumulation point of any countable subset of X.

Proposition 1.2.30. *Let X be a T_1 topological space. Then, every P−point of X is a weak P−point.*

Proof. Let $x \in X$ be a P−point of X and suppose that x is an accumulation point of a countable subset A of X. Let $\{y_i : i \in \omega\}$ be an enumeration of $A \setminus \{x\}$. Then, for each $i \in \omega$, we may fix an open neighborhood U_i of x so that $y_i \notin U_i$. Since x is a P−point, $\operatorname{int}\big(\bigcap_{n\in\omega} U_n\big)$ is an open neighborhood of x, thus

$$\operatorname{int}\Big(\bigcap_{n\in\omega} U_n\Big) \cap A$$

is infinite. This is a contradiction, as $y_i \notin \operatorname{int}\big(\bigcap_{n\in\omega} U_n\big)$ for every $i \in \omega$. Therefore, x is a weak P−point. □

Consider a topology in ω^* which has sets of the form

$$A^* \doteq \{p \in \omega^* : A \in p\},$$

for each $A \subset \omega$, as basic open sets. We claim that this topology is T_1. Indeed, given distinct points $p, q \in \omega^*$, there exists $A \subset \omega$ so that $A \in p$ and $A \notin q$. Thus, $p \in A^*$ and $q \notin A^*$.

When ω^* is endowed with the above topology, one can write equivalent definitions for P−points in ω^*:

Proposition 1.2.31. *Let ω^* be endowed with the topology above and $p \in \omega^*$. The following conditions are equivalent.*

(1) p is a P−point of ω^*;

(2) *for every sequence* $(A_n : n \in \omega)$ *of elements of* p*, there exists* $A \in p$ *so that* $A \setminus A_n$ *is finite for each* $n \in \omega$*;*

(3) *for every partition* $(A_n : n \in \omega)$ *of* ω*, either* $A_n \in p$ *for some* $n \in \omega$ *or there is* $A \in p$ *such that* $A \cap A_n$ *is finite for each* $n \in \omega$*.*

Proof. (1) ⇒ (2): Let $p \in \omega^*$ be a P−point, and $(A_n)_{n\in\omega}$ be a sequence of elements of p. Then, $(A_n^* : n \in \omega)$ is a family of open neighborhoods of p, and thus $p \in \operatorname{int}\big(\bigcap_{n\in\omega} A_n^*\big)$. Let $A \subset \omega$ be such that $p \in A^* \subset \operatorname{int}\big(\bigcap_{n\in\omega} A_n^*\big)$. We claim that $A \setminus A_n$ is finite for each $n \in \omega$. Indeed, suppose that $A \setminus A_{n_0}$ is infinite for some $n_0 \in \omega$. Then, there exists a free ultrafilter $q \in \omega^*$ which contains $A \setminus A_{n_0}$. Thus, in particular $A \in q$ and $A_{n_0} \notin q$, and therefore $q \in A^*$ and $q \notin A_{n_0}^*$, which is a contradiction. We conclude that $A \setminus A_n$ is finite for each $n \in \omega$.

(2) ⇒ (1): Suppose that $p \in \omega^*$ satisfies condition (2). We shall prove that p is a P−point. For that, let $(B_n^* : n \in \omega)$ be a countable family of neighborhoods of p, with $B_n \subset \omega$, for each $n \in \omega$. Then, there is an element $B \in p$ so that $B \setminus B_n$ is finite for each $n \in \omega$. We claim that

$$p \in B^* \subset \bigcap_{n\in\omega} B_n^*.$$

Indeed, suppose that $q \in \omega^*$ is such that $B \in q$. Then, if $B_{n_0} \notin q$ for some $n_0 \in \omega$, we would have that $B \cap (\omega \setminus B_{n_0}) \in q$, a contradiction, since $B \setminus B_{n_0}$ is finite. Therefore, the above equation holds, and p is a P−point of ω^*.

(2) ⇒ (3): Suppose that (2) holds, and let $(A_n : n \in \omega)$ be a partition of ω. If $A_n \in p$ for some $n \in \omega$, we are done. Otherwise, $\omega \setminus A_n \in p$ for every $n \in \omega$, thus there exists $A \in p$ so that $A \setminus (\omega \setminus A_n) = A \cap A_n$ is finite, for each $n \in \omega$.

(3) ⇒ (2): Suppose that (3) holds, and let $(A_n : n \in \omega)$ be a sequence of elements of p. Then, $B_n \doteq \omega \setminus A_n \notin p$, for every $n \in \omega$. Define recursively a sequence $(C_n)_{n\in\omega}$, so that $C_0 = B_0 \cup \{0\}$ and for each $n > 0$,

$$C_n \doteq B_n \cup \{n\} \setminus \bigcup_{i=0}^{n-1} C_i.$$

Then, $C_n \notin p$, for every $n \in \omega$. Let $C = \{n \in \omega : C_n \neq \emptyset\}$. Note that $\{C_n : n \in C\}$ is a partition of ω, thus C has to be infinite. Therefore, by hypothesis, there exists $A \in p$ such that $A \cap C_n$ is finite, for each $n \in C$, thus $A \cap B_n (= A \setminus A_n)$ is finite for every $n \in \omega$. □

We highlight that Kunen showed in ZFC that there are $2^{\mathfrak{c}}$ points in ω^* which are weak P−points but not P-points [Kun80]. Also, there exists a model of ZFC in which there are no P−points in ω^* [Wim82].

We end this section with the following useful result.

Proposition 1.2.32. *Let* $(q_i)_{i\in\omega}$ *be a family of distinct weak* P−*points in* ω^**. Then, there exists a family* $(C_i)_{i\in\omega}$ *of pairwise disjoint subsets of* ω *so that* $C_i \in q_i$*, for every* $i \in \omega$*.*

Proof. We will construct such family recursively as follows. Since q_0 is a weak P−point, it cannot be an accumulation point of $\{q_{i+1} : i \in \omega\}$. Thus, there exists $C_0 \in q_0$ so that $C_0 \notin q_i$, for every $i > 0$. Suppose that, for $n \in \omega$, we have constructed a family $(C_i)_{0\leq i\leq n}$ satisfying:

1) $C_i \cap C_j = \emptyset$ for each $i, j \in \{0, ..., n\}$ so that $i \neq j$;
2) $C_i \in q_i$, for each $0 \leq i \leq n$;
3) $C_i \notin q_j$ for every $j \in \omega$, $j \neq i$.

Since q_{n+1} is a weak $P-$point, q_{n+1} cannot be an accumulation point of $\{q_i : i \neq n+1\}$, thus there is $D_{n+1} \in q_{n+1}$ so that $D_{n+1} \notin q_i$ if $i \neq n+1$. Thus, we have that

$$C_{n+1} \doteq D_{n+1} \cap (\omega \setminus C_0) \cap ... \cap (\omega \setminus C_n) \in q_{n+1}.$$

Also, $C_{n+1} \cap C_i = \emptyset$, for every $i \in \{0, ..., n\}$ and $C_{n+1} \notin q_i$ for every $i \in \omega$, $i \neq n+1$. Therefore, the family $(C_i)_{0\leq i\leq n+1}$ also satisfies 1)–3), which ends the proof by recursion. □

1.2.4 Selective ultrafilters

In this brief subsection we will define selective ultrafilters and present some results that will be important throughout the thesis.

Definition 1.2.33. A *selective ultrafilter* on ω is a free ultrafilter p on ω such that for every partition $(A_n : n \in \omega)$ of ω, either there exists $n \in \omega$ such that $A_n \in p$ or there exists $B \in p$ such that $|B \cap A_n| = 1$ for every $n \in \omega$.

It follows straight from item (3) of Proposition 1.2.31 that:

Proposition 1.2.34. *Every selective ultrafilter in ω^* is a $P-$point.*

When handling the combinatorial properties of selective ultrafilters, it is often useful to use some of their equivalent properties, like those given by the well known proposition below.

Proposition 1.2.35. *Let $p \in \omega^*$. The following are equivalent.*

a) p is a selective ultrafilter.

b) For every $f \in \omega^\omega$, there exists $A \in p$ such that $f|_A$ is either constant or one-to-one.

c) For every function $f : [\omega]^2 \to 2$ there exists $A \in p$ such that $f|_{[A]^2}$ is constant.

Proof. See [CN74], for instance. □

The existence of selective ultrafilters is independent of ZFC. In fact, Martin's Axiom implies the existence of $2^{\mathfrak{c}}$ selective ultrafilters [Bla73], while, as we mentioned in the previous subsection, there is a model of ZFC in which there are no $P-$points in ω^* [Wim82].

1.2.5 The Rudin-Keisler order

In this subsection, we will present the definition and some facts regarding the Rudin-Keisler order.

Given an ultrafilter p on ω and a function $f : \omega \to \omega$, we define the set

$$f_*(p) \doteq \{A \subset \omega : f^{-1}(A) \in p\}.$$

Observe that, for each ultrafilter p on ω, $f_*(p)$ is also an ultrafilter on ω.

Definition 1.2.36. Given $p, q \in \omega^*$, we say that $p \leq_{RK} q$ if there exists a function $f : \omega \to \omega$ so that $f_*(q) = p$. Such relation on ω^* is a preorder called the *Rudin-Keisler order.*

We say that $p, q \in \omega^*$ are:

- *incomparable* if neither $p \leq_{RK} q$ or $q \leq_{RK} p$;
- *equivalent* if $p \leq_{RK} q$ and $q \leq_{RK} p$.

For our purposes in this thesis, that is enough about Rudin-Keisler order. For more details on the subject, we suggest [CN74] and [HTT18].

1.2.6 Topological Groups

Let's recall the definition of topological groups:

Definition 1.2.37. A *topological group* is a group G endowed with a topology for which the functions $(\cdot)^{-1} : G \to G$ and $\cdot : G \times G \to G$ are continuous[4].

Notice that, given a topological group G, the continuity of the functions $\cdot$ and $(.)^{-1}$ implies that, for each open subset U and element $g \in G$, the sets U^{-1}, gU and Ug are also open.

When working with the pseudocompact-like properties, we assume that the spaces are at least Tychonoff. Then, in the case of topological groups, it suffices to consider T_0 topological groups since, interestingly, T_0 topological groups are automatically Tychonoff. We present a complete proof of this non-trivial fact in the proposition below because, although it is a folklore basic result about topological groups, it is not often exploited. The proof we present is based on a proof contained in the book [MZ55]. First, we need the following simple lemma.

Lemma 1.2.38 ([MZ55], Section 1.15)**.** *Let G be a topological group and U be an open neighborhood of the identity e. Then, there exists a symmetric open neighborhood W of e such that $W^2 \subset U$.*

Proof. Since G is a topological group, the operation $\cdot : G \times G \to G$ is continuous in (e, e). Therefore, there are open neighborhoods V_0, V_1 of e so that $V_0V_1 \subset U$. We define

$$W \doteq V_0 \cap V_0^{-1} \cap V_1 \cap V_1^{-1},$$

and then W is clearly a symmetric open neighborhood of e. Also, given $w_0, w_1 \in W$, we have that $w_0 \in V_0$, $w_1 \in V_1$, and therefore $w_0 \cdot w_1 \in U$.

[4] Recall that, according to the Remark 1, we are considering that $G \times G$ is endowed with the product topology.

□

Proposition 1.2.39. *The following statements are true.*

a) A T_0 topological group is T_1.

b) Every topological group is completely regular.

Proof. a): Let G be a T_0 topological group. Given distinct elements $g, h \in G$, there is an open subset $U \subset G$ containing one and not the other. Let's say it contains g and not h. Then, $V \doteq hU^{-1}g$ is an open subset containing h and not g. Thus, G is T_1.

b): Let G be a topological group with identity e. We shall prove first that for every closed subset $A \subset G$ so that $e \in G \setminus A$, there exists a continuous function $f : G \to [0,1]$ such that $f(e) = 0$ and $f(A) \subset \{1\}$.

Then let A be a closed subset of G as in the previous paragraph. For each $n \in \omega$, we will construct open neighborhoods $(U_n)_{n\in\omega}$ of e recursively as follows: $U_0 \doteq G \setminus A$ and, for each $n \in \omega$, U_{n+1} is a symmetric open neighborhood of e so that $U_{n+1}^2 \subset U_n \cap U_0$, which exists by Lemma 1.2.38.

Next, for each $n \in \omega$, we also define open neighborhoods $(W_{k,n})_{1\leq k\leq 2^n}$ of e recursively, as follows. First, we put $W_{1,0} \doteq U_0$. Then, for each $n \in \omega$:

$$\begin{cases} W_{1,n+1} \doteq U_{n+1} & \\ W_{2l,n+1} \doteq W_{l,n}, & \text{for each } l \in \omega \text{ so that } 1 \leq l \leq 2^n \\ W_{2l+1,n+1} \doteq W_{1,n+1}W_{l,n}, & \text{for each } l \in \omega \text{ so that } 1 \leq l < 2^n. \end{cases}$$

Now, we will show by induction on $n \in \omega$ that, for each $n \in \omega$, the expression

$$W_{1,n}W_{k,n} \subset W_{k+1,n} \tag{1.1}$$

holds for each $k = 1, ..., 2^n - 1$. If $n = 0$, the claim is a vacuous truth. For $N > 0$, suppose that the claim is true for each $m < N$. We shall prove that it is also true for $n = N$. If $k = 1$, by construction, we have that

$$W_{1,N}W_{1,N} = U_N^2 \subset U_{N-1} = W_{1,N-1} = W_{2,N}.$$

If $k = 2l$ for some $1 \leq l \leq 2^{N-1} - 1$, we have that

$$W_{1,N}W_{2l,N} = W_{1,N}W_{l,N-1} = W_{2l+1,N} = W_{k+1,N},$$

and finally, if $k = 2l + 1$ for some $1 \leq l \leq 2^{N-1} - 1$,

$$W_{1,N}W_{2l+1,N} = (W_{1,N}^2)W_{l,N-1} = (U_N^2)W_{l,N-1} \subset U_{N-1}W_{l,N-1} = W_{1,N-1}W_{l,N-1}.$$

By the induction hypothesis, $W_{1,N-1}W_{l,N-1} \subset W_{l+1,N-1}$, and

$$W_{l+1,N-1} = W_{2l+2,N} = W_{k+1,N},$$

which ends the proof by induction.

Next, we claim that, if $n_0, n_1 \in \omega$, $k_0 \in \{1, ..., 2^{n_0}\}$ and $k_1 \in \{1, ..., 2^{n_1}\}$ satisfy that

$$\frac{k_0}{2^{n_0}} = \frac{k_1}{2^{n_1}},$$

then $W_{k_0,n_0} = W_{k_1,n_1}$. In fact, if $n_0 = n_1$, we have that $k_0 = k_1$ and clearly $W_{k_0,n_0} = W_{k_1,n_1}$. Otherwise, we may suppose, for instance, that $n_0 < n_1$. Thus, $k_1 = 2^{n_1-n_0}k_0$ and, by construction,

$$W_{k_1,n_1} = W_{2^{n_1-n_0-1}k_0,n_1-1} = ... = W_{2^{n_1-n_0-(n_1-n_0)}k_0,n_1-(n_1-n_0)} = W_{k_0,n_0}.$$

Now, let

$$D \doteq \left\{ \frac{k}{2^n} : n \in \omega, k = 1, ..., 2^n \right\},$$

and, for each $d \in D$, say $d = \dfrac{\bar{k}}{2^{\bar{n}}}$ for $\bar{n} \in \omega$ and $\bar{k} \in \{1, ..., 2^{\bar{n}}\}$, we define

$$V_d \doteq W_{\bar{k},\bar{n}}.$$

As shown above, V_d does not depend on the particular representation of d in a fraction of the form $\dfrac{k}{2^n}$, with $n \in \omega$ and $k = 1, ..., 2^n$, thus it is well defined.

Note also that, given $d_0, d_1 \in D$ so that $0 < d_0 < d_1$, we have that $V_{d_0} \subset V_{d_1}$. In fact, one can write $d_0 = \dfrac{k_0}{2^{n_0}}$ and $d_1 = \dfrac{k_1}{2^{n_0}}$, for some $n_0 \in \omega$ and $k_0, k_1 \in \{1, ..., 2^{n_0}\}$. Since W_{1,n_0} is a neighborhood of e, we have that $W_{k_0,n_0} \subset (W_{1,n_0}^{k_1-k_0})W_{k_0,n}$. Using equation (1.1) $k_1 - k_0$ times, we finally obtain that

$$V_{d_0} = W_{k_0,n_0} \subset (W_{1,n_0}^{k_1-k_0-1})W_{k_0+1,n_0} \subset ... \subset W_{k_1,n_0} = V_{d_1}.$$

Define $f : G \to [0, 1]$ by

$$f(g) = \begin{cases} 0, & \text{if } g \in \bigcap_{d \in D} V_d \\ \sup\{d \in D : g \notin V_d\}, & \text{otherwise.} \end{cases}$$

Note that $e \in \bigcap_{d \in D} V_d$, thus $f(e) = 0$, and given $x \in A$, $x \notin V_1 (= U_0)$. Therefore, $f(x) = 1$, and hence $f(A) \subset \{1\}$. We shall show in the next paragraph that f is continuous.

Let $\bar{g} \in G$ and $\epsilon > 0$. Let also $n > 0$ be so that $\dfrac{1}{2^n} < \epsilon$. Suppose first that $f(\bar{g}) = 0$. Then, $V_{\frac{1}{2^n}}$ is an open neighborhood of $\bar{g}$ so that, for each $x \in V_{\frac{1}{2^n}}$,

$$0 \leq f(x) \leq \frac{1}{2^n} < \epsilon.$$

Now, suppose that $0 < f(\bar{g}) < 1$. Since D is dense in $[0,1]$, we may fix $d_0 \in D$ so that

$$f(\bar{g}) < d_0 < 1.$$

Thus, we have that $\bar{g} \in V_{d_0}$. At the same time, we may fix an element $d_1 \in D$ so that $\bar{g} \notin V_{d_1}$. We may then choose $k > n$, $m_0, m_1 \in \{1, ..., 2^k\}$ so that

$$d_0 = \frac{m_0}{2^k},\ d_1 = \frac{m_1}{2^k}.$$

Since $1 \leq m_1 < m_0 < 2^k$, there exists $m \in \{2, ..., 2^k - 1\}$ so that

$$\bar{g} \in V_{\frac{m}{2^k}} \setminus V_{\frac{m-1}{2^k}}.$$

Now, notice that $V_{\frac{1}{2^k}}\bar{g}$ is an open neighborhood of $\bar{g}$ so that, for each $x \in V_{\frac{1}{2^k}}\bar{g}$, $x \in V_{\frac{m+1}{2^k}}$. Indeed, we have already proved that

$$V_{\frac{1}{2^k}}\bar{g} \subset V_{\frac{1}{2^k}}V_{\frac{m}{2^k}} \subset V_{\frac{m+1}{2^k}}.$$

Thus, for each $x \in V_{\frac{m+1}{2^k}}$,

$$f(x) \leq \frac{m+1}{2^k},$$

and we also have that

$$\frac{m-1}{2^k} \leq f(\bar{g}) \leq \frac{m}{2^k}.$$

If $m = 2$,

$$|f(x) - f(\bar{g})| \leq \frac{2}{2^k} \leq \frac{1}{2^n} < \epsilon.$$

If $m > 2$, we claim that $x \notin V_{\frac{m-2}{2^k}}$. In fact, suppose that $x \in V_{\frac{m-2}{2^k}}$. As $V_{\frac{1}{2^k}}$ is a symmetric neighborhood, we have that $\bar{g} \in V_{\frac{1}{2^k}}x$, and hence

$$\bar{g} \in V_{\frac{1}{2^k}}V_{\frac{m-2}{2^k}} \subset V_{\frac{m-1}{2^k}},$$

which is a contradiction. Thus, $x \notin V_{\frac{m-2}{2^k}}$, and therefore we conclude that in this case

$$\frac{m-2}{2^k} \leq f(x) \leq \frac{m+1}{2^k},$$

and again

$$|f(x) - f(\bar{g})| \leq \frac{2}{2^k} \leq \frac{1}{2^n} < \epsilon.$$

Finally, suppose that $f(\bar{g}) = 1$, and let again $k > n$. For each $x \in V_{\frac{1}{2^k}}\bar{g}$, we have that $\bar{g} \in V_{\frac{1}{2^k}}x$. If $x \in V_{\frac{2^k-2}{2^k}}$, we would have

$$\bar{g} \in V_{\frac{1}{2^k}}V_{\frac{2^k-2}{2^k}} \subset V_{\frac{2^k-1}{2^k}},$$

a contradiction. Thus, $x \notin V_{\frac{2^k-2}{2^k}}$, and hence

$$1 - \epsilon \leq 1 - \frac{1}{2^n} \leq \frac{2^k - 2}{2^k} \leq f(x) \leq 1.$$

We conclude then that f is continuous.

Now, let $x \in G$ be an arbitrary element and $C \subset G$ be a closed subset so that $x \in G \setminus C$. Note that $\bar{C} \doteq x^{-1}C$ is a closed subset of G so that $e \in G \setminus \bar{C}$, and thus there exists a continuous function $f : G \to [0,1]$ so that $f(e) = 0$ and $f(\bar{C}) \subset \{1\}$. Consider the function $h : G \to [0,1]$ given by:

$$h(y) \doteq f(x^{-1} \cdot y),$$

for each $y \in G$. Since G is a topological group and f is continuous, it follows that h is continuous, $h(x) = f(x^{-1} \cdot x) = f(e) = 0$ and, for each $y \in A$, $h(y) = f(x^{-1} \cdot y) = 1$. Then, G is completely regular.

□

Corollary 1.2.40. *Every T_0 topological group is Tychonoff.*

We will work again with useful neighborhoods of topological groups in later chapters.

Chapter 2

Pseudocompact-like topologies in groups

In this chapter, we will introduce our main field of study during the PhD. In the first section, we will make a historical overview of the area, presenting at the same time definitions and results that will be relevant to us. Many of these definitions and results require a more detailed discussion, which includes, for example, the presentation of basic properties and non-trivial known facts. For this purpose, the historical timeline will be paused, and such discussions will be done in the form of a digression. There was the possibility of making separate subsections for such topics, prior to the historical presentation, or even including them in chapter 1. However, we judge that the first form captures in a more natural and fluid way the real need for each topic, sticking us with what is really necessary in the proof of the results in the subsequent chapters. We prioritized a natural development of the ideas that led to our results over a possibly more organized presentation. Thus, in the first section, we aim to situate, prepare and motivate the reader in the questions that will be addressed in the future.

In the second section, we present a general idea of how most of the constructions we will make in the next chapters will work, addressing some properties and facts that will serve as a basis for what comes later.

2.1 Introduction

We start with the definition of *pseudocompactness*, by Hewitt, in 1948:

Definition 2.1.1 ([Hew48]). A Tychonoff topological space X is *pseudocompact* if every countinuous function $f : X \to \mathbb{R}$ is bounded.

The following result shows an equivalent definition.

Proposition 2.1.2. *A Tychonoff topological space X is pseudocompact if, and only if, for every sequence $(U_n : n \in \omega)$ of nonempty open sets of X, there exists $x \in X$ such that, for each open neighborhood V of x, $\{n \in \omega : V \cap U_n \neq \emptyset\}$ is infinite.*

Proof. (⇒) Let $\{U_n : n \in \omega\}$ be a family of nonempty open subsets of X. Suppose that, for each $x \in X$, we may choose an open neighborhood V_x of x so that $\{n \in \omega : V_x \cap U_n \neq \emptyset\}$ is finite. For each $n \in \omega$, fix arbitrarily $x_n \in U_n$. Since the space is Tychonoff, for each $n \in \omega$, there exists a continuous function $f_n : X \to \mathbb{R}$ so that $f_n(x_n) = 1$ and $f_n(X \setminus U_n) \subset \{0\}$. Now, notice that, given $y \in X$,

$$\{n \in \omega : f_n(y) \neq 0\} \subset \{n \in \omega : V_y \cap U_n \neq \emptyset\},$$

thus $\{n \in \omega : f_n(y) \neq 0\}$ is finite. Then, the function $f : X \to \mathbb{R}$ given below is well-defined:

$$f(y) = \sum_{\substack{n\in\omega \\ f_n(y)\neq 0}} n f_n(y), \text{ for each } y \in X.$$

Moreover, we claim that f is continuous. Indeed, for each $y \in X$,

$$f|_{V_y} \equiv \sum_{\substack{n\in\omega \\ V_y\cap U_n\neq\emptyset}} n f_n,$$

which is a continuous function.

Finally, f is unbounded, as, for each $n \in \omega$, $f_n(x_n) \geq n$. Thus, X cannot be a pseudocompact space.

(⇐) Assuming that the second condition is true, suppose that X is not pseudocompact, that is, there exists a continuous function $f : X \to \mathbb{R}$ which is unbounded. Then, there exists a strictly increasing family $\{k_n : n \in \omega\} \subset \operatorname{rng}(f)$ so that $k_n > n$ for each $n \in \omega$. Consider, for each $n \geq 1$, the following open subset of $\mathbb{R}$,

$$W_n \doteq \left(\frac{k_{n-1} + k_n}{2}, \frac{k_n + k_{n+1}}{2}\right),$$

and the following open subset of X,

$$U_n \doteq f^{-1}(W_n).$$

Note that $U_n \neq \emptyset$ for every $n \in \omega$, since $k_n \in W_n$ and $k_n \in \operatorname{rng}(f)$, for each $n \in \omega$. By hypothesis, there exists $x \in X$ so that $\{n \in \omega : V \cap U_n \neq \emptyset\}$ is infinite for every open neighborhood V of x. However, letting $V \doteq f^{-1}((f(x) - 1, f(x)))$ and $N \in \omega$ be so that $N > f(x) + 1$, we have that $V \cap U_n = \emptyset$ for every $n \geq N$, a contradiction. Hence, X must be pseudocompact. □

It is well known that:

Corollary 2.1.3. *Every Tychonoff countably compact topological space is pseudocompact.*

Proof. Let X be a Tychonoff countably compact topological space. Let also $(U_n)_{n\in\omega}$ be a family of nonempty open subsets of X, and, for each $n \in \omega$, fix $x_n \in U_n$. Then, there is an element $x \in X$ which is an accumulation point of $(x_n)_{n\in\omega}$. In particular, for each open neighborhood V of x, $\{n \in \omega : V \cap U_n \neq \emptyset\}$ is infinite. Thus, X is pseudocompact. □

Tychonoff's theorem ensures that the product of any collection of compact topological spaces is compact. However, Novák and Terasaka ([Nov53] and [Ter52], respectively) constructed examples of countably compact spaces whose product is not even pseucocompact. Thus, although many properties of compact spaces hold also in countably compact spaces, this is not the case for the useful property of preservation under products. Aiming to construct a property weaker than compactness, but still retaining preservation by products, Bernstein introduced [Ber70] the following notions.

Definition 2.1.4 ([Ber70]). Let $p \in \omega^*$ and $(x_n : n \in \omega)$ be a sequence in a topological space X. We say that $x \in X$ is a *$p-$limit point* of $(x_n : n \in \omega)$ if $\{n \in \omega : x_n \in U\} \in p$ for every open neighborhood U of x.

Definition 2.1.5 ([Ber70]). Let $p \in \omega^*$. A topological space X is *$p-$compact* if every sequence $(x_n : n \in \omega) \subset X$ has a $p-$limit.

Definition 2.1.6 ([Ber70]). A topological space is *ultracompact* if it is $p-$compact for every $p \in \omega^*$.

Remark 4. Actually, Bernstein defined the $p-$limit notion for every ultrafilter on ω, not only for free ultrafilters. However, if p is a fixed ultrafilter on ω, then there is $n_0 \in \omega$ so that $\{n_0\} \in p$. Thus, given a topological space X, for each sequence $\{x_n : n \in \omega\}$, we would have that $\{n \in \omega : x_n \in U\} \in p$ for every open neighborhood U of x_{n_0}. Hence, x_{n_0} would be a so-called $p-$limit point of $\{x_n : n \in \omega\}$, and therefore every topological space would be $p-$compact for every fixed ultrafilter p on ω. That is the reason why we restricted the definition to free ultrafilters.

As desired, the following properties hold. They are proved a little differently in [Ber70], but we also prove them here for completeness.

Proposition 2.1.7. *Every compact topological space is ultracompact.*

Proof. Let X be a compact topological space and $p \in \omega^*$. Suppose that X is not $p-$compact, for some $p \in \omega^*$. Then, there is a sequence $(x_n : n \in \omega)$ in X which does not have a $p-$limit. Hence, for each $x \in X$, we may fix an open neighborhood U_x of x so that $\{n \in \omega : x_n \in U_x\} \notin p$. Since $(U_x)_{x \in X}$ is an open cover of X, it has to contain an open subcover, say $\{U_{y_0}, ..., U_{y_m}\}$, $m \in \omega$. Then, as

$$\bigcup_{i=0}^{m} \{n \in \omega : x_n \in U_{y_i}\} = \omega$$

and p is an ultrafilter, it must exists $i \in \{0, ..., m\}$ such that $\{n \in \omega : x_n \in U_{y_i}\} \in p$, a contradiction. Then, X is ultracompact. □

Proposition 2.1.8. *Let $p \in \omega^*$. The product of any family of $p-$compact topological spaces is $p-$compact.*

Proof. Let I be a set and $(X_i)_{i \in I}$ be a family of $p-$compact topological spaces. Let also $(x_n)_{n \in \omega}$ be a sequence on $\prod_{i \in I} X_i$. Since X_i is $p-$compact for each $i \in I$, $(x_n^i)_{n \in \omega}$ has a $p-$limit $y^i \in X_i$, for each $i \in I$. We claim that $y \doteq (y^i)_{i \in I}$ is a $p-$limit point for $(x_n)_{n \in \omega}$. Indeed, let $U \doteq \prod_{i \in I} U_i$ be a basic open neighborhood of y. Then, there is a finite subset $F \subset I$ so that $U_i = X$ for each $i \in I \setminus F$. Since y^i is a $p-$limit of $(x^i_n)_{n\in\omega}$ for each $i \in I$, we have that $\{n \in \omega : x^i_n \in U_i\} \in p$ for each $i \in F$. Thus, $\bigcap_{i\in F}\{n \in \omega : x^i_n \in U_i\} \in p$, and since

$$\bigcap_{i\in F}\{n \in \omega : x^i_n \in U_i\} = \{n \in \omega : x_n \in U\},$$

we have that $\{n \in \omega : x_n \in U\} \in p$. If V is an arbitrary open neighborhood of y, then there is a basic open neighborhood of y so that $U \subset V$, thus, as shown above, $\{n \in \omega : x_n \in U\} \in p$. Since $\{n \in \omega : x_n \in U\} \subset \{n \in \omega : x_n \in V\}$, we conclude that $\{n \in \omega : x_n \in V\} \in p$. Therefore, $\prod_{i\in I} X_i$ is $p-$compact.

□

Also, it follows straight from the definition that:

Proposition 2.1.9. *If X is a $p-$compact space for some $p \in \omega^*$, then X is countably compact.*

The proof of the previous result is straightforward as, given $p \in \omega^*$, every $p-$limit of a sequence $(x_n)_{n\in\omega}$ is, in particular, an accumulation point of $(x_n)_{n\in\omega}$. But also, the next proposition states that every accumulation point of a sequence $(x_n)_{n\in\omega}$ is a $p-$limit of $(x_n)_{n\in\omega}$ for *some* $p \in \omega^*$.

Proposition 2.1.10. *Let X be a topological space, $(x_n)_{n\in\omega}$ be a sequence in X and x be an accumulation point of $(x_n)_{n\in\omega}$. Then, there is a $p \in \omega^*$ so that x is a $p-$limit of $(x_n)_{n\in\omega}$.*

Proof. By assumption, for each open neighborhood U of x, we have that

$$S_U \doteq \{n \in \omega : x_n \in U\}$$

is infinite. Given $U_0, ..., U_n$ open neighborhoods of x,

$$S_{U_0} \cap ... \cap S_{U_n} = S_{\cap_{i=0}^n U_i},$$

thus $S_{U_0} \cap ... \cap S_{U_n}$ is also infinite. We conclude that the collection

$$S \doteq \{S_U \subset \omega : U \text{ is an open neighborhood of } x\}$$

has the strong finite intersection property. Therefore, S can be extended to a free ultrafilter $p \in \omega^*$. Clearly, in this case x is a $p-$limit of $(x_n)_{n\in\omega}$. □

Next we shall see some basic properties of $p-$limits. Many of them can be found in [Ber70].

We can only guarantee the uniqueness of the $p-$limit of a sequence $(x_n)_{n\in\omega}$, if it exists, when the space in question is Hausdorff:

Proposition 2.1.11. *Let X be a Hausdorff topological space, $(x_n)_{n\in\omega}$ be a sequence in X, and $p \in \omega^*$. If $(x_n)_{n\in\omega}$ has a $p-$limit x in X, then it is unique.*

Proof. In the conditions of the statement, suppose that $y \in X$ is another $p-$limit of $(x_n)_{n\in\omega}$. Since the space is Hausdorff, there are disjoint open neighborhoods U and V of x and y, respectively. Then, we have that $\{n \in \omega : x_n \in U\} \in p$ and $\{n \in \omega : x_n \in V\} \in p$, a contradiction, since $\{n \in \omega : x_n \in U\} \cap \{n \in \omega : x_n \in V\} = \emptyset$. □

Then, if we are working with a Hausdorff topological space X, we will denote

$$x = p - \lim_{n\in\omega} x_n$$

when x is a $p-$limit of a sequence $(x_n)_{n\in\omega}$ in X.

The following properties will be very useful when working with products of topological groups:

Proposition 2.1.12. *Let X, Y be topological spaces, $f : X \to Y$ be a continuous function, and $(x_n)_{n\in\omega}$ be a sequence in X. In these conditions, if $x \in X$ is a $p-$limit of $(x_n)_{n\in\omega}$, then $f(x) \in Y$ is a $p-$limit of $(f(x_n))_{n\in\omega}$.*

Proof. Let U be an open neighborhood of $f(x)$ in Y. Since f is continuous, there exists an open neighborhood V of x so that $f(V) \subset U$. As x is a $p-$limit of $(x_n)_{n\in\omega}$, we have that $\{n \in \omega : x_n \in V\} \in p$ and since $\{n \in \omega : x_n \in V\} \subset \{n \in \omega : f(x_n) \in U\}$, we conclude that $\{n \in \omega : f(x_n) \in U\} \in p$. Hence, $f(x) \in Y$ is a $p-$limit of $(f(x_n))_{n\in\omega}$. □

Proposition 2.1.13. *Let I be a set, $(X^i)_{i\in I}$ be a family of topological spaces, $p \in \omega^*$, and $X \doteq \prod_{i\in I} X^i$. Let also, for each $n \in \omega$, $x_n \doteq (x_n^i)_{i\in I} \in X$. Then, an element $x \doteq (x^i)_{i\in I} \in X$ is a $p-$limit of $(x_n)_{n\in\omega}$ in X if, and only if, for each $i \in I$, x^i is a $p-$limit of the sequence $(x_n^i)_{n\in\omega}$ in X^i.*

Proof. $(\Rightarrow)$ Suppose that $x \doteq (x^i)_{i\in I} \in X$ is a $p-$limit of $(x_n)_{n\in\omega}$ in X. For each $i \in I$, the projection $\pi_i : X \to X^i$ is a continuous function, thus, by Proposition 2.1.12, $\pi_i(x) = x^i \in X^i$ is a p-limit of $(x_n^i)_{n\in\omega}$.

$(\Leftarrow)$ Suppose that, for every $i \in I$, x^i is a $p-$limit of $(x_n^i)_{n\in\omega}$ in X^i. Consider a basic open neighborhood $U \doteq \prod_{i\in I} U_i$ of x. Then, there is a finite subset $F \subset I$ so that $U_i = X$ for each $i \in I \setminus F$. Since x^i is a $p-$limit of $(x_n^i)_{n\in\omega}$ for each $i \in I$, we have that $\{n \in \omega : x_n^i \in U_i\} \in p$, for each $i \in F$. Thus, $\bigcap_{i\in F}\{n \in \omega : x_n^i \in U_i\} \in p$, and since

$$\bigcap_{i\in F}\{n \in \omega : x_n^i \in U_i\} = \{n \in \omega : x_n \in U\},$$

we have that $\{n \in \omega : x_n \in U\} \in p$. If V is an arbitrary open neighborhood of x, then there is a basic open neighborhood of x so that $U \subset V$, thus, as shown above, $\{n \in \omega : (x_n)_{n\in\omega} \in U\} \in p$. Since $\{n \in \omega : x_n \in U\} \subset \{n \in \omega : x_n \in V\}$, we conclude that $\{n \in \omega : x_n \in V\} \in p$. □

As a corollary, we obtain the following:

Corollary 2.1.14. *Let G be a topological group, $(a_n)_{n\in\omega}$, $(b_n)_{n\in\omega}$ be sequences in G, $a, b \in G$, and $p \in \omega^*$. Then, the following properties are true.*

a) If a is a p–limit of $(a_n)_{n\in\omega}$ and b is a p–limit of $(b_n)_{n\in\omega}$, then $a \cdot b$ is a p–limit of the sequence $(a_n \cdot b_n)_{n\in\omega}$ in G.

b) If a is a p–limit of $(a_n)_{n\in\omega}$, then a^{-1} is a p–limit of the sequence $(a_n^{-1})_{n\in\omega}$ in G.

Proof. a) If a is a p–limit of $(a_n)_{n\in\omega}$ and b is a p–limit of $(b_n)_{n\in\omega}$, by Proposition 2.1.13, we obtain that $(a, b) \in G^2$ is a p–limit of the sequence $\big((a_n, b_n)\big)_{n\in\omega}$ in G^2. Also, since the product operation in G is continuous, by Proposition 2.1.12, $a \cdot b$ is a p–limit of the sequence $(a_n \cdot b_n)_{n\in\omega}$ in G.

b) It also follows from Proposition 2.1.12, since the inverse operation in G is continuous. □

Now we shall look at another pseudocompact-like property. As Ginsburg and Saks pointed out in [GS75], there is an useful modification of p–compactness which is suited to the study of pseudocompactness in Tychonoff spaces. We introduce and discuss this new notion below.

Definition 2.1.15 ([GS75])**.** Let X be a Tychonoff topological space, $p \in \omega^*$ and $(S_n : n \in \omega)$ be a sequence of nonempty subsets of X. A point $x \in X$ is called a *p–limit* of $(S_n : n \in \omega)$ if, and only if, for each open neighborhood U of x, $\{n \in \omega : U \cap S_n \neq \emptyset\} \in p$.

Note that there is no chance of confusion when dealing with the different p-limit notions, since one of them refers to a sequence of points, and the other refers to a sequence of subsets.

Definition 2.1.16 ([GS75])**.** Let X be a Tychonoff topological space and $p \in \omega^*$. We say that X is *p–pseudocompact* if, and only if, every sequence $(U_n : n \in \omega)$ of nonempty open subsets of X has a p–limit.

Definition 2.1.17 ([GS75])**.** Let X be a Tychonoff topological space. We say that X is *ultrapseudocompact* if, and only, X is p–pseudocompact for every $p \in \omega^*$.

Remark 5. Whenever we say that a topological space X is pseudocompact, p-pseudocompact, or if we are dealing with p–limits of a sequence of subsets of X, we assume that X is understood to be Tychonoff. This will be emphasized when there is a chance of confusion.

Notice that, according to Proposition 2.1.2, if X is a pseudocompact topological space, given a sequence $(U_n : n \in \omega)$ of nonempty open subsets of X, there is $x \in X$ such that, for each open neighborhood V of x, $\mathcal{B}_V \doteq \{n \in \omega : V \cap U_n \neq \emptyset\}$ is infinite. It is not hard to see that, in this case, the family $(\mathcal{B}_V : V$ is an open neighborhood of $x)$ of subsets of ω has the strong finite intersection property, and thus can be extended to a free ultrafilter $p \in \omega^*$. That is, x is a p–limit point of the family of subsets $(U_n : n \in \omega)$. However, unlike p–pseudocompact spaces, another sequence of nonempty open subsets may not have a p–limit, but a q–limit, for another $q \in \omega^*$. Hence, we may rewrite the definition of pseudocompactness in terms of p–limits:

Proposition 2.1.18. *A Tychonoff space X is pseudocompact if, and only if, every sequence $(U_n : n \in \omega)$ of nonempty open subsets of X has a $p-$limit, for some $p \in \omega^*$.*

Therefore, in a sense, it can be said that the notion of p-compact space is linked to the notion of countably compact space in an analogous way that the notion of p-pseudocompact space is linked to the notion of pseudocompact space (see Proposition 2.1.10). Next, we present some properties related to the new concepts defined above, similarly to what we have done with the $p-$compactness. Many of these properties also appear in [GS75].

First, we point out that, given $p \in \omega^*$, a sequence $(S_n)_{n\in\omega}$ of nonempty open subsets in a topological space X can have infinite, finite or no p-limit points, even X being a Tychonoff space.

Proposition 2.1.19. *For each $p \in \omega^*$, if X is a $p-$pseudocompact topological space, then X is pseudocompact.*

Proof. It follows straight from Proposition 2.1.18. □

Proposition 2.1.20. *For each $p \in \omega^*$, if X is a Tychonoff $p-$compact topological space, then X is $p-$pseudocompact.*

Proof. Let $p \in \omega^*$ and X be a Tychonoff $p-$compact topological space. Given a sequence $(U_n : n \in \omega)$ of nonempty open subsets of X, we may choose arbitrarily $x_n \in U_n$, and then there exists a $p-$limit point $x \in X$ of $(x_n)_{n\in\omega}$. Thus, for each open neighborhood U of x, $\{n \in \omega : x_n \in U\} \in p$. In particular, $\{n \in \omega : U \cap U_n \neq \emptyset\} \in p$. Therefore, X is $p-$pseudocompact. □

Regarding counterexamples, we have that:

- in [GS75], there is an example of a topological space which has all powers pseudocompact, but is not $p-$pseudocompact for any $p \in \omega^*$;
- in [GJ76], there is an example of a countably compact space which is not $p-$pseudocompact for any $p \in \omega^*$;
- in [AOT14], for each $p \in \omega^*$, there is an example of a $p-$pseudocompact space which is not ultrapseudocompact.

We also have the following:

Proposition 2.1.21. *Let I be a set, $(X^i)_{i\in I}$ be a family of topological spaces, $p \in \omega^*$, and $X \doteq \prod_{i\in I} X^i$. Let also, for each $n \in \omega$, $S_n \doteq \prod_{i\in I} S^i_n$, with $S^i_n \in X^i$ for each $n \in \omega$, $i \in I$, be a nonempty subset of X. Then, an element $x \doteq (x^i)_{i\in I} \in X$ is a $p-$limit of $(S_n)_{n\in\omega}$ in X if, and only if, for each $i \in I$, x^i is a $p-$limit of the sequence $(S^i_n)_{n\in\omega}$ in X^i.*

Proof. ($\Rightarrow$) Suppose that $x \doteq (x^i)_{i\in I} \in X$ is a $p-$limit of $(S_n)_{n\in\omega}$ in X. Let $i_0 \in I$, and V be an open neighborhood of x^i in X^i. Consider the open subset $W = \prod_{i\in I} W^i$ of X so that $W^{i_0} = V$ and $W^i = X$ for each $i \in I \setminus \{i_0\}$. Then, $\{n \in \omega : W \cap S_n \neq \emptyset\} \in p$. Since

$\{n \in \omega : W \cap S_n \neq \emptyset\} \subset \{n \in \omega : \pi_{i_0}(W) \cap S_n^{i_0} \neq \emptyset\}$, $\{n \in \omega : V \cap S_n^{i_0} \neq \emptyset\} \in p$. Therefore, x^{i_0} is a p−limit of the sequence $(S_n^{i_0})_{n\in\omega}$ in X^{i_0}.

($\Leftarrow$) Suppose that, for each $i \in I$, x^i is a p−limit of the sequence $(S_n^i)_{n\in\omega}$ in X^i. Let W be an open neighborhood of x in X. Then, there is a basic open neighborhood $V \doteq \prod_{i\in I} V^i$ so that $x \in V \subset W$. Thus, there is a finite subset $F \subset I$ so that $V^i = X^i$ for every $i \in I \setminus F$. Therefore, since

$$\{n \in \omega : W \cap S_n \neq \emptyset\} \supset \{n \in \omega : V \cap S_n \neq \emptyset\} = \bigcap_{i\in F}\{n \in \omega : V^i \cap S_n^i \neq \emptyset\} \in p,$$

x is a p−limit of $(S_n)_{n\in\omega}$ in X.

□

Proposition 2.1.22. *Let $p \in \omega^*$. The product of any family of p-pseudocompact topological spaces is p−pseudocompact.*

Proof. Let I be a set, X^i be a p-pseudocompact topological space, for each $i \in I$, and $X \doteq \prod_{i\in I} X^i$. Let also $(U_n : n \in \omega)$ be a sequence of nonempty open subsets of X. For each $n \in \omega$, we may fix a basic open set $V_n \subset U_n$, say $V_n \doteq \prod_{i\in I} V_n^i$, with V_n^i an open subset of X_n^i, for every $i \in I$ and $n \in \omega$. Then, for each $i \in I$, the sequence $(V_n^i)_{n\in\omega}$ has a p-limit $x^i \in X^i$, and, by Proposition 2.1.21, $x \doteq (x^i)_{i\in I}$ is a p−limit of $(V_n)_{n\in\omega}$ (and thus, also a p−limit of $(U_n)_{n\in\omega}$). Hence, X is p−pseudocompact. □

Then, in 2014, following the ideas in [Ber70], [Gar94] and [GS75], J. Angoa, Y. F. Ortiz-Castillo and Á. Tamariz-Mascarúa introduced the new concept of *selective p−pseudocompactness*[1]:

Definition 2.1.23 ([AOT14])**.** Given $p \in \omega^*$, a Tychonoff topological space X is called *selectively p−pseudocompact* if, and only if, for each sequence $(U_n : n \in \omega)$ of nonempty open subsets of X there are a sequence $(x_n : n \in \omega)$ in X and $x \in X$ such that $x = p - \lim_{n\in\omega} x_n$ and, for each $n \in \omega$, $x_n \in U_n$.

Again, when we say that topological a space X is selectively p−pseudocompact, it is already understood that X is Tychonoff.

Some of the properties listed below are in [AOT14] or [GO14], but we present and prove them here for the sake of completeness.

Proposition 2.1.24. *Let $p \in \omega^*$ and X be a selectively p−pseudocompact space. Then, X is p−pseudocompact.*

Proof. Let $(U_n : n \in \omega)$ be a sequence of nonempty open subsets of X. Then, there is a sequence $(x_n : n \in \omega)$ in X and $x \in X$ such that $x = p - \lim_{n\in\omega} x_n$ and, for each $n \in \omega$,

[1] This concept was originally defined in [AOT14] under the name *strong p−pseudocompactness*, but later the name was changed, since there were already two different properties named in the previous way (in [AG93] and [Dik94]).

$x_n \in U_n$. Thus, for each open neighborhood U of x, $\{n \in \omega : x_n \in U\} \in p$, and hence also $\{n \in \omega : U \cap U_n \neq \emptyset\} \in p$. Therefore, X is $p-$pseudocompact. □

It also follows immediately from the definitions that:

Proposition 2.1.25. *Let $p \in \omega^*$ and X be a Tychonoff $p-$compact topological space. Then, X is selectively $p-$pseudocompact.*

Similar to the previous properties, we have that:

Proposition 2.1.26. *Let $p \in \omega^*$. The product of any family of selectively p-pseudocompact topological spaces is selectively $p-$pseudocompact.*

Proof. Let I be a set and X^i be a selectively $p-$pseudocompact topological space, for each $i \in I$. Let also $X \doteq \prod_{i \in I} X^i$ and $(U_n : n \in \omega)$ be a sequence of nonempty open subsets of X. For each $n \in \omega$, we may fix a basic open set $V_n \subset U_n$, say $V_n \doteq \prod_{i \in I} V_n^i$, with V_n^i and open subset of X_n^i, for every $i \in I$ and $n \in \omega$. Then, for each $i \in I$, there is a sequence $(x_n^i : n \in \omega)$ in X^i and $x^i \in X^i$ such that $x^i = p - \lim_{n \in \omega} x_n^i$ and, for each $n \in \omega$, $x_n^i \in V_n^i$. By Proposition 2.1.13, $x \doteq (x^i)_{i \in I}$ is a $p-$limit of the sequence $((x_n^i)_{i \in I} : n \in \omega)$ in X, and, for each $n \in \omega$, $(x_n^i)_{i \in I} \in V_n$. Therefore, X is selectively $p-$pseudocompact. □

The last pseudocompact-like property we will present in this section is *selective pseudocompactness*[2], introduced by García-Ferreira and Ortiz-Castillo (see [GO14]):

Definition 2.1.27 ([GO14])**.** A Tychonoff topological space X is called *selectively pseudocompact* if, and only if, for each sequence $(U_n : n \in \omega)$ of nonempty open subsets of X there are a sequence $(x_n : n \in \omega)$ in X, $x \in X$ and $p \in \omega^*$ such that $x = p - \lim_{n \in \omega} x_n$ and, for each $n \in \omega$, $x_n \in U_n$.

In other words, X is selectively pseudocompact if, and only if, for each sequence $(U_n : n \in \omega)$ of nonempty open subsets of X we may find a sequence $(x_n : n \in \omega)$ in X which has an accumulation point in X, and so that $x_n \in U_n$, for each $n \in \omega$. Again, when we say that a topological space X is selectively pseudocompact, it is already understood that X is Tychonoff.

It is clear that we have the following:

Proposition 2.1.28. *Every selectively pseudocompact space is pseudocompact.*

Proof. If X is a selectively pseudocompact space, then for each sequence $(U_n : n \in \omega)$ of nonempty open subsets of X, there are a sequence $(x_n : n \in \omega)$ in X, $x \in X$ and $p \in \omega^*$ such that $x = p - \lim_{n \in \omega} x_n$ and, for each $n \in \omega$, $x_n \in U_n$. Thus, x is also a $p-$limit of $(U_n : n \in \omega)$, and therefore X is pseudocompact. □

Recall that we consider the next notion as folklore:

---

[2] This concept was also defined originally under the name *strong-pseudocompactness.*

Definition 2.1.29. A topological space X is *countably pracompact* if there exists a dense subset $D \subset X$ such that every sequence on D has an accumulation point in X.

Proposition 2.1.30. *Every Tychonoff countably pracompact space is selectively pseudocompact.*

Proof. Let X be a Tychonoff countably pracompact space. Thus, there exists a dense subset $D \subset X$ such that every sequence on D has an accumulation point in X. Given a sequence $(U_n : n \in \omega)$ of nonempty open subsets of X, for each $n \in \omega$ we may fix a point $x_n \in U_n \cap D$. Therefore, $(x_n)_{n\in\omega}$ has an accumulation point in X, and hence X is selectively pseudocompact. $\square$

Then, in summary, we have the following diagram of implications for topological spaces in general.

$$
\begin{tikzcd}
\textbf{selective } p\textbf{-pseudocompactness} \arrow[rr, Rightarrow] \arrow[dd, Rightarrow] & & p\textbf{-pseudocompactness} \arrow[dl, Rightarrow] \\
& \textbf{pseudocompactness} & \\
\textbf{selective pseudocompactness} \arrow[ur, Rightarrow] & \textbf{countable compactness} \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \textbf{ultrapseudocompactness} \arrow[uu, Rightarrow] \\
& \textbf{countable pracompactness} \arrow[ul, Rightarrow] &
\end{tikzcd}
$$

Figure 2.1: *Relation between pseudocompact-like properties for general topological spaces. The ultrafilter $p \in \omega^*$ is arbitrary.*

We will now turn our attention more specifically to the properties defined above applied to topological groups. All topological groups mentioned from here will be assumed to be T_0, and therefore, Tychonoff (see Corollary 1.2.40).

In this class of topological spaces, the following result holds. It was proved in [GS97] in a more general setting:

Theorem 2.1.31 ([GS97]). *For a topological group G, the following conditions are equivalent.*

a) G is pseudocompact.

b) There is a $p \in \omega^*$ such that G is p–pseudocompact.

c) G is ultrapseudocompact.

The strength of topological group properties in the proof of topological equivalences, exemplified by the result above, makes it natural to ask whether there are similar equivalences for other pseudocompact-like properties. For instance, the question whether pseudocompactness implies selective pseudocompactness in topological groups was posed in [GO14], and solved by Garcia-Ferreira and Tomita, who proved that there exists a pseudocompact group which is not selectively pseudocompact [GT15]. Hence, the selective pseudocompactness is not another equivalent notion for pseudocompactness in topological groups.

Another related question we might ask is whether countable pracompactness is equivalent to selective pseudocompactness in topological groups. We answered negatively this question in [TT22], constructing a selectively pseudocompact group which is not countably pracompact. The construction will be presented in the next chapter. Therefore, the notion of countable pracompactness is even more strict in topological groups. Assuming the existence of a single selective ultrafilter, we also proved in [TT22] that there exists a topological group which is not countably pracompact and has all powers selectively pseudocompact, a slightly stronger result. The proof will also be presented in the next chapter.

Another important result regarding pseudocompact-like properties in topological groups is the the following theorem due to Comfort and Ross:

Theorem 2.1.32 ([CR66]). *The product of any family of pseudocompact topological groups is pseudocompact.*

As we have seen, a result similar to this one does not apply to topological spaces in general. This result motivated Comfort to question whether the product of countably compact groups is also countably compact. More generally, he asked the following question [Com90][3]:

Question 2.1.33 ([Com90], Question 477). *Is there, for every (not necessarily infinite) cardinal number $\alpha \leq 2^{\mathfrak{c}}$, a topological group G such that G^γ is countably compact for all cardinals $\gamma < \alpha$, but G^α is not countably compact?*

The restriction $\alpha \leq 2^{\mathfrak{c}}$ in the question above is due to the following result:

Theorem 2.1.34 ([GS75], Theorem 2.6). *Let X be a Hausdorff topological space. The following statements are equivalent:*

a) every power of X is countably compact;

b) $X^{2^{\mathfrak{c}}}$ is countably compact;

c) $X^{|X|^\omega}$ is countably compact;

d) there exists $p \in \omega^*$ such that X is $p-$compact.

Van Douwen was the first to prove consistently (under MA) that there are two countably compact groups whose product is not countably compact [Dou80]. More specifically, van Douwen proved the two following lemmas.

Lemma 2.1.35 ([Dou80]). *(ZFC) Every infinite Boolean countably compact group without non-trivial convergent sequences contains two countably compact subgroups whose product is not countably compact.*

[3] This is Question 1 in the Introduction of the thesis.

Tomita proved in ZFC another version of Lemma 2.1.35: the existence of a countably compact Abelian group without non-trivial convergent sequences implies the existence of a countably compact group whose square is not countably compact [Tom05b]. Also, a version for finite and countable powers (for torsion groups) and finite powers (for non-torsion groups) appears in [Tom19].

Lemma 2.1.36 ([Dou80])**.** *(MA) There exists an infinite Boolean countably compact group without non-trivial convergent sequences.*

Together, the two lemmas above show that, under MA, there are two countably compact groups whose product is not countably compact.

Using tools outside ZFC, many other examples of countably compact groups without non-trivial convergent sequences were given over the years. The first one appeared in [HJ76], under CH. In [KTW00], an example was obtained from Martin's Axiom for countable posets, and in [GTW05] from a single selective ultrafilter, improving the technique, since MA implies the existence of selective ultrafilters. Nevertheless, [ST09] showed that the existence of such groups does not imply the existence of selective ultrafilters. It was left open for a long time whether there exists an example in ZFC. Finally, in 2021, Hrušák, van Mill, Ramos-García, and Shelah [Hru+21] proved that:

Theorem 2.1.37 ([Hru+21])**.** *In ZFC, there exists a Hausdorff countably compact topological Boolean group (of size $\mathfrak{c}$) without non-trivial convergent sequences.*

Due to Lemma 2.1.35, this result also solves the original Comfort's question.

Then, in [BRT21a], the authors asked whether there exists an (Abelian) countably compact group without non-trivial convergent sequences of size **strictly greater** than $\mathfrak{c}$ in ZFC. With a slight modification to the construction given in [Hru+21], we answered this question in [TT22], constructing such a group of size $2^{\mathfrak{c}}$.

Bearing in mind the theorem of Comfort and Ross, and also the theorem obtained in [Hru+21], it is also natural to ask productivity questions for countably pracompact and selectively pseudocompact groups. In this regard, Garcia-Ferreira and Tomita proved that if p and q are non-equivalent selective ultrafilters on ω (according to the Rudin-Keisler order in ω^*), then there are a p-compact group and a q-compact group whose product is not selectively pseudocompact [GT20]. Also, Bardyla, Ravsky and Zdomskyy constructed, under MA, a Boolean countably compact topological group whose square is not countably pracompact [BRZ20]. However, it is still not known whether it is a theorem of ZFC that selective pseudocompactness and countable pracompactness are non-productive in the class of topological groups.

More generally, one can ask Comfort-like questions, such as Question 2.1.33, for selectively pseudocompact and countably pracompact groups. In the case of selectively pseudocompact groups, the question is restricted to cardinals $\alpha \leq \omega$, due to the next result.

Lemma 2.1.38. *If G is a topological group such that G^ω is selectively pseudocompact, then G^κ is selectively pseudocompact for every cardinal $\kappa \geq \omega$.*

Proof. Indeed, let $\kappa \geq \omega$ and $(U_n)_{n\in\omega}$ be a family of open subsets of G^κ. For every $n \in \omega$, there are open subsets $U_n^j \subset G$, for each $j < \kappa$, so that $\prod_{j\in\kappa} U_n^j \subset U_n$ and $U_n^j \neq G$ if and only if $j \in F_n$, for a finite subset $F_n \subset \kappa$. Let $F \doteq \bigcup_{n\in\omega} F_n$. For each $n \in \omega$, consider the open subsets $V_n \doteq \prod_{j\in F_n} U_n^j \times \prod_{j\in F\setminus F_n} G \subset G^F$. By assumption, G^F is selectively pseudocompact, thus there is a sequence $\{y_n : n \in \omega\} \subset G^F$ so that $y_n \in V_n$, for every $n \in \omega$, which has an accumulation point y in G^F. Then, given $g \in G$ arbitrarily, the sequence $\{x_n : n \in \omega\} \subset G^\kappa$ defined coordinatewise, for each $n \in \omega$, by

$$x_n^j \doteq \begin{cases} y_n^j, & \text{if } j \in F \\ g, & \text{if } j \in \kappa \setminus F \end{cases}$$

is such that $x_n \in U_n$ for every $n \in \omega$, and has $x \in G^\kappa$ given by

$$x^j \doteq \begin{cases} y^j, & \text{if } j \in F \\ g, & \text{if } j \in \kappa \setminus F \end{cases}$$

as accumulation point. □

Question 2.1.39. *For which cardinals $\alpha \leq \omega$ is there a topological group G such that G^γ is selectively pseudocompact for all cardinals $\gamma < \alpha$, but G^α is not selectively pseudocompact?*

In the case of countably pracompact groups, it is still not known whether there exists a cardinal κ satisfying that: if a topological group G is such that G^κ countably pracompact, then G^α is countably pracompact, for each $\alpha > \kappa$. Thus, there is no restriction to the cardinals α yet:

Question 2.1.40. *For which cardinals α is there a topological group G such that G^γ is countably pracompact for all cardinals $\gamma < \alpha$, but G^α is not countably pracompact?*

It is worth observing that:

Proposition 2.1.41. *Let G be a topological group such that G^ω is* ***countably compact*** *and $\kappa \geq \omega$. Then, G^κ is countably pracompact.*

Proof. Let e be the identity of G. We claim that

$$\Sigma \doteq \{g \in G^\kappa : |\{i \in \kappa : g^i \neq e\}| \leq \omega\}$$

is a dense subset of G^κ for which every sequence has an accumulation point. Indeed, let U be a nonempty basic open subset of G^κ. Then, $U = \prod_{i\in I} U^i$ for some open subsets $U_i \subset G$, and $F \doteq \{i \in \kappa : U^i \neq G\}$ is finite. For each $i \in F$, fix $x^i \in U^i$. Then, $g \doteq (g^i)_{i\in\kappa}$ given by

$$g^i \doteq \begin{cases} x^i, & \text{if } i \in F \\ e, & \text{if } i \notin F \end{cases}$$

belongs to $U \cap \Sigma$. Also, let $(g_n)_{n\in\omega}$ be a sequence in Σ, say $g_n = (g_n^i)_{i\in\kappa}$, for some g_n^i, for every $n \in \omega$ and $i \in \kappa$. Let also $H_n \doteq \{i \in \kappa : g_n^i \neq e\}$, for each $n \in \omega$. Then, $H \doteq \bigcup_{n\in\omega} H_n$ is such that $|H| \leq \omega$. Since G^ω is countably compact, $((g_n^i)_{i\in H} : n \in \omega)$ has an accumulation point in G^H, and thus, $(g_n)_{n\in\omega}$ has an accumulation point in G^κ. Therefore, G^κ is countably pracompact. □

In [GT18], under the assumption of CH, the authors showed that for every positive integer $k > 0$, there exists a topological group G for which G^k is countably compact but G^{k+1} is not selectively pseudocompact. Thus, Question 2.1.39 and Question 2.1.40 are already solved for finite cardinals under CH. The cardinal $\alpha = \omega$ is the only one for which there are still no consistent answers to the Question 2.1.39.

In the paper [TT23]:

(1) assuming the existence of $\mathfrak{c}$ incomparable selective ultrafilters, we answered Question 2.1.40 for $\alpha = \omega$;

(2) assuming the existence of $2^\mathfrak{c}$ incomparable selective ultrafilters, we answered Question 2.1.40 for each successor cardinal $\alpha = \kappa^+$, with $\omega \leq \kappa \leq 2^\mathfrak{c}$.

In particular, the case $\kappa = 2^\mathfrak{c}$ of item (2) shows that there exists a group G so that $G^{2^\mathfrak{c}}$ is countably pracompact but $G^{(2^\mathfrak{c})^+}$ is not countably pracompact. This is particularly interesting, as for Hausdorff topological spaces X, if $X^{2^\mathfrak{c}}$ is countably compact, every power of X is countably compact (see Theorem 2.1.34).

As a corollary of the proof of result (1) above, we also showed in [TT23] that, assuming the existence of $\mathfrak{c}$ incomparable selective ultrafilters, for each $n \in \omega$, $n > 0$, there exists a topological group whose nth power is countably compact and the (n+1)th power is not selectively pseudocompact. Since CH implies the existence of $2^\mathfrak{c}$ incomparable selective ultrafilters [Bla73], this is a slightly stronger result than what was obtained in [GT18].

2.2 The sketch of the constructions and some useful results

In this section we will see in detail the ideas behind the construction of the topologies that will be used in the proof of the main results of the thesis. Each construction has its particularity, but follows a small sketch, which is already frequently used in some way in similar constructions in this field (e.g., in [Hru+21], [GT18], [GT15] and [GT20]).

Let A be a set. As usual, we define, for each cardinal κ:

- $[A]^\kappa \doteq \{B \subset A : |B| = \kappa\}$;
- $[A]^{<\kappa} \doteq \{B \subset A : |B| < \kappa\}$.

Also, we will denote by $\triangle$ the symmetric difference between two sets B, C:

$$B \triangle C \doteq (B \setminus C) \cup (C \setminus B).$$

The following proposition follows straight from the properties of the symmetric difference.

Proposition 2.2.1. *Given a nonempty set A, $[A]^{<\omega}$ becomes a Boolean group when endowed with the symmetric difference as the group operation and $\emptyset$ as the neutral element.*

In most of the constructions presented in this thesis, we will use the technique described below, which is often used in similar constructions in this field[4].

First, a set A is taken as an infinite ordinal, such as ω, $\mathfrak{c}$ or $2^{\mathfrak{c}}$, and the topologies are defined in Boolean groups as $[\omega]^{<\omega}$, $[\mathfrak{c}]^{<\omega}$ or $[2^{\mathfrak{c}}]^{<\omega}$ (with operation and neutral element given by the previous proposition). To establish the topology τ, it is considered a suitable set $\mathcal{A}$ of group homomorphisms $\phi : [A]^{<\omega} \to 2$, where 2 is endowed with the discrete topology, and we let τ be the topology generated by the homomorphisms in $\mathcal{A}$. This makes things a little easier, since Boolean groups have a simpler structure (for instance, recall that they are also vector spaces over the field $\{0, 1\}$), and the only variable to be changed with this choice is the set $\mathcal{A}$ of homomorphisms. That is, the topology of the group is dictated only by the homomorphisms that belong to $\mathcal{A}$, which we can freely choose. The obvious drawback is that this restricts our study to Boolean groups only.

We shall see next some topological properties of the Boolean group $[A]^{<\omega}$, according to the homomorphisms we choose. Such properties are proved in a more general context in the next proposition.

Proposition 2.2.2. *Let G be a group and $\mathcal{H}$ be a set of group homomorphisms $\phi : G \to 2$. Suppose that G is endowed with the topology generated by the homomorphisms in $\mathcal{H}$. Then:*

a) G is a topological group.

b) G is T_0 if, and only if, for each $g, h \in G$, there exists $\phi \in \mathcal{H}$ so that $\phi(g) \neq \phi(h)$ (that is, if $\mathcal{H}$ separates points of G).

c) A point $g \in G$ is an accumulation point of a sequence $(g_n)_{n\in\omega}$ of G if, and only if, there exists $p \in \omega^*$ so that

$$\phi(g) = p - \lim_{n\in\omega} \phi(g_n),$$

for every $\phi \in \mathcal{H}$.

Proof. a) Let

$$\begin{aligned} M : \ G \times G &\longrightarrow G \\ (g, h) &\longmapsto g \cdot h, \end{aligned}$$

$(g_0, h_0) \in G \times G$ and U be an open neighborhood of $g_0 \cdot h_0$. Then, there are $m, n \in \omega$, $\phi_0, ..., \phi_m, \psi_0, ...\psi_n \in \mathcal{H}$ so that

$$g_0 \cdot h_0 \in \bigcap_{i=0}^{m} \phi_i^{-1}(\{0\}) \cap \bigcap_{j=0}^{n} \psi_j^{-1}(\{1\}) \subset U.$$

[4] In fact, the technique described here is a particular case of a more general technique, with which we will work a little in chapter 7.

Letting

$$V \doteq \bigcap_{i=0}^{m} \phi_i^{-1}(\{\phi_i(g_0)\}) \cap \bigcap_{j=0}^{n} \psi_j^{-1}(\{\psi_j(g_0)\})$$

and

$$W \doteq \bigcap_{i=0}^{m} \phi_i^{-1}(\{\phi_i(h_0)\}) \cap \bigcap_{j=0}^{n} \psi_j^{-1}(\{\psi_j(h_0)\}),$$

we have that $(g_0, h_0) \in V \times W$ and, if $(g, h) \in V \times W$, $g \cdot h \in U$. Thus, M is a continuous function.

Now, let

$$\begin{aligned} I :\ & G \longrightarrow G \\ & g \longmapsto g^{-1}. \end{aligned}$$

Analogously, given $g_0 \in G$ and U an open neighborhood of g_0, there are $m, n \in \omega$, $\phi_0, ..., \phi_m, \psi_0, ...\psi_n \in \mathcal{H}$ so that

$$g_0 \in \bigcap_{i=0}^{m} \phi_i^{-1}(\{0\}) \cap \bigcap_{j=0}^{n} \psi_j^{-1}(\{1\}) \subset U.$$

Then, letting

$$V \doteq \bigcap_{i=0}^{m} \phi_i^{-1}(\{1\}) \cap \bigcap_{j=0}^{n} \psi_j^{-1}(\{0\}),$$

we have that $g_0^{-1} \in V$ and, if $g \in V$, $g^{-1} \in U$. Thus, I is also a continuous function.

b) ($\Rightarrow$) If G is T_0, given $g, h \in G$, there is an open subset U that contains one element and not the other. Assume that $g \in U$ and $h \notin U$. Then, there are $n, m \in \omega$ and a finite subset of homomorphisms $\mathcal{J} \doteq \{\phi_0, ..., \phi_n, \psi_0, ..., \psi_m\} \subset \mathcal{H}$ so that

$$g \in \bigcap_{i=0}^{n} \phi_i^{-1}(\{0\}) \cap \bigcap_{j=0}^{m} \psi_j^{-1}(\{1\}) \subset U.$$

Since $h \notin U$, there is some $\sigma \in \mathcal{J}$ so that $\sigma(g) \neq \sigma(h)$.

($\Leftarrow$) Let $g, h \in G$. By assumption, there is $\phi \in \mathcal{H}$ so that $\phi(g) \neq \phi(h)$. Then, defining $U \doteq \phi^{-1}(\{0\})$, we have that either $g \in U$ and $h \notin U$ or $g \in U$ and $h \notin U$.

c) ($\Rightarrow$) Let $g \in G$ be an accumulation point of a sequence $(g_n)_{n\in\omega}$ of G. For each open neighborhood U of g, let $M_U \doteq \{n \in \omega : g_n \in U\}$. Then,

$$\mathcal{M} \doteq \{M_U : U \text{ is an open neighborhood of } g\}$$

has the strong finite intersection property and therefore can be extended to a free ultrafilter

p on ω. We claim that

$$\phi(g) = p - \lim_{n\in\omega} \phi(g_n),$$

for every $\phi \in \mathcal{H}$. Indeed, for each $\phi \in \mathcal{H}$, $M_{\phi^{-1}(\{\phi(g)\})} \in p$.

($\Leftarrow$) Let U be an open neighborhood of g. Then, there are $m, n \in \omega, \phi_0, ..., \phi_m, \psi_0, ...\psi_n \in \mathcal{H}$ so that

$$g \in \bigcap_{i=0}^{m} \phi_i^{-1}(\{0\}) \cap \bigcap_{j=0}^{n} \psi_j^{-1}(\{1\}) \subset U.$$

Since, for each $i = 0, ..., m$ and $j = 0, ...n$, we have that $\{n \in \omega : \phi_i(g) = \phi_i(g_n)\} \in p$ and $\{j \in \omega : \psi_j(g) = \psi_j(g_n)\} \in p$, it follows that $\{n \in \omega : g_n \in U\} \in p$. Therefore, g is an accumulation point of $(g_n)_{n\in\omega}$. $\square$

Note that, due to item b) of the previous proposition and Corollary 1.2.40, for a group G endowed with a topology generated by a set $\mathcal{H}$ of group homomorphisms $\phi : G \to 2$ to be Tychonoff, it is enough that $\mathcal{H}$ separates points of G.

In the next chapters, we will freely use the facts proven in the previous proposition for the case of the Boolean group $[A]^{<\omega}$ endowed with the topology generated by a set $\mathcal{A}$ of group homomorphisms $\phi : [A]^{<\omega} \to 2$, as explained.

The biggest difficulties of such constructions usually are:

1) to determine the properties that homomorphisms should have in order to obtain the desired group;

2) prove that homomorphisms with such characteristics exist.

For item 2), we often need to make use of the assumption of the existence of selective ultrafilters.

Thus, most of the subsequent chapters will have the following structure: the first section will be devoted to auxiliary results, in special the construction of homomorphisms that will be used to form the set $\mathcal{A}$; the second section will contain the construction of the group in question; the third section will bring a brief conclusion of the chapter.

Since we will be dealing with Boolean groups, which are also vector spaces over the field $2 = \{0, 1\}$, we can talk about general linear algebra concepts concerning these groups, such as *linearly independent subsets*. Also, given an infinite set A and $p \in \omega^*$, one may define an equivalence relation on $([A]^{<\omega})^\omega$ by letting, for each $f, g \in ([A]^{<\omega})^\omega$,

$$f \equiv_p g \text{ iff } \{n \in \omega : f(n) = g(n)\} \in p.$$

We also let $[f]_p$ be the equivalence class determined by f, and $([A]^{<\omega})^\omega/p$ will be $([A]^{<\omega})^\omega/\equiv_p$. Notice that, considering $[A]^{<\omega}$ as the Boolean group defined previously, the set $([A]^{<\omega})^\omega/p$ has a natural vector space structure (over the field 2). For each $A_0 \in [A]^{<\omega}$, the constant function in $([A]^{<\omega})^\omega$ which takes only the value A_0 will be denoted by $\vec{A_0}$. If α is an ordinal, $\vec{\{\alpha\}}$ will be denoted simply by $\vec{\alpha}$.

With all that in mind, the following results, which deal with the algebraic aspects of such groups, will be useful to us:

Lemma 2.2.3. *Let A, B and C be subsets of a vector space. Suppose that A is finite and that $A \cup C$, $B \cup C$ are linearly independent. Then there exists $B' \subset B$ such that $|B'| \leq |A|$ and $A \cup C \cup (B \setminus B')$ is linearly independent.*

Proof. We prove the result by induction on $|A|$. First, suppose that $|A| = 1$, that is, A has a single element $a \neq 0$. If $A \cup C \cup B$ is linearly independent, we simply consider $B' = \emptyset$. Otherwise, there is a non-trivial linear combination of elements in $A \cup C \cup B$ that equals zero. Note that a and some element in B must appear in this linear combination, since $A \cup C$ and $B \cup C$ are linearly independent. Thus, we have

$$a = Cl(C)_1 + Cl(B)_1,$$

for some $Cl(B)_1 \neq 0$ and $Cl(C)_1$ linear combinations of elements in B and C, respectively. Choose an element $b \in B$ which appears in $Cl(B)_1$. We claim that $A \cup C \cup (B \setminus \{b\})$ is linearly independent. Indeed, otherwise we would have

$$a = Cl(C)_2 + Cl(B)_2,$$

for some $Cl(B)_2 \neq 0$ and $Cl(C)_2$ linear combinations of elements in $B \setminus \{b\}$ and C, respectively. But this cannot happen, since $B \cup C$ is linearly independent and $Cl(B)_1 \neq Cl(B)_2$. Hence, we have proved the result if A is a set of size 1.

Suppose that the lemma holds for sets of size $n \in \omega$, and that A has a size $n + 1$. In this case, letting $a \in A$, we may apply the hypothesis to the sets $A \setminus \{a\}$, B and C. Thus, we get $B_0 \subset B$ so that $|B_0| \leq n$ and that $(A \setminus \{a\}) \cup C \cup (B \setminus B_0)$ is linearly independent. Now, we apply the result for sets of size 1 to the sets $\{a\}$, $B \setminus B_0$ and $(A \setminus \{a\}) \cup C$, and we are done. $\square$

Corollary 2.2.4. *Let A and B be linearly independent subsets in a vector space with A a finite set. Then there is $B' \subset B$ such that $|B'| \leq |A|$ and $A \cup (B \setminus B')$ is linearly independent.*

The following results, applications of the general facts above, appear in [TT22], and will be used several times in the following chapters.

Lemma 2.2.5. *Let A, B and C be subsets in a Boolean group. Suppose that A is a finite set and that $A \cup C$, $B \cup C$ are linearly independent. Then there exists $B' \subset B$ such that $|B'| \leq |A|$ and $A \cup C \cup (B \setminus B')$ is linearly independent.*

The next result also appears in [Hru+21].

Corollary 2.2.6. *Let A and B be linearly independent subsets in a Boolean group with A a finite set. Then there is $B' \subset B$ such that $|B'| \leq |A|$ and $A \cup (B \setminus B')$ is linearly independent.*

Lemma 2.2.7 ([TT23], Lemma 2.2)**.** *Let X be an infinite set and $\{X_0, ..., X_n\}$ be a partition of X. Let also $(x_k)_{k\in\omega}$ and $(y_k)_{k\in\omega}$ be sequences in the Boolean group $[X]^{<\omega}$ so that:*

- *$\{x_k : k \in \omega\} \cup \{y_k : k \in \omega\}$ is linearly independent;*

- *for every $p \in \{0, ..., n\}$, both $\{x_k \cap X_p : k \in \omega\}$ and $\{y_k \cap X_p : k \in \omega\}$ are linearly independent.*

Then, there exist a subsequence $(k_m : m \in \omega)$ and $n_0 \in \{0, ..., n\}$ so that

$$\{x_{k_m} \cap X_{n_0} : m \in \omega\} \cup \{y_{k_m} \cap X_{n_0} : m \in \omega\}$$

is linearly independent.

Proof. We shall construct inductively a sequence $(A_0^i)_{i\in\omega}$ of subsets of ω as follows. Firstly, if does not exist $k \in \omega$ so that $\{x_k \cap X_0\} \cup \{y_k \cap X_0\}$ is linearly independent, we put $A_0^0 = \emptyset$. Otherwise, we choose the minimum $k_0 \in \omega$ with this property and put $A_0^0 \doteq \{k_0\}$. Suppose that for $l \in \omega$ we have constructed $A_0^0, ..., A_0^l \subset \omega$ such that:

a) $|A_0^i| \leq i + 1$, for each $i = 0, ..., l$;

b) $A_0^i \subset A_0^j$ if $0 \leq i \leq j \leq l$;

c) $\{x_k \cap X_0 : k \in A_0^l\} \cup \{y_k \cap X_0 : k \in A_0^l\}$ is linearly independent.

d) for each $0 \leq i < l$, $A_0^{i+1} \setminus A_0^i = \emptyset$ if, and only if,

$$\{x_k \cap X_0 : k \in A_0^i\} \cup \{y_k \cap X_0 : k \in A_0^i\} \cup \{x_{\tilde{k}} \cap X_0\} \cup \{y_{\tilde{k}} \cap X_0\}$$

is linearly dependent for every $\tilde{k} > \max(A_0^i)$.

In what follows, we will construct A_0^{l+1}. If does not exist $\tilde{k} \in \omega$, $\tilde{k} > \max(A_0^l)$, so that

$$\{x_k \cap X_0 : k \in A_0^l\} \cup \{y_k \cap X_0 : k \in A_0^l\} \cup \{x_{\tilde{k}} \cap X_0\} \cup \{y_{\tilde{k}} \cap X_0\}$$

is linearly independent, we put $A_0^{l+1} = A_0^l$. Otherwise, we choose the minimum $k_{l+1} \in \omega$ with this property, and put $A_0^{l+1} = A_0^l \cup \{k_{l+1}\}$. In any case, $A_0^0, ..., A_0^{l+1}$ satisfy items a)–d), and then, by induction, there exists a sequence $(A_0^i)_{i\in\omega}$ satisfying them. Now, let $A_0 \doteq \bigcup_{i\in\omega} A_0^i$. If A_0 is infinite, then $\{x_k \cap X_0 : k \in A_0\} \cup \{x_k \cap X_0 : k \in A_0\}$ is linearly independent, and we are done.

On the other hand, suppose that A_0 is finite. We may repeat the process above for $X_1, ..., X_n$, constructing analogous subsets $A_1, ..., A_n \subset \omega$. If either of them is infinite, we are done.

Suppose then that $A_0, ..., A_n$ are finite sets. By construction, for each $\tilde{k} > \max(A_0 \cup ... \cup A_n)$ and $j = 0, ..., n$,

$$\{x_k \cap X_j : k \in A_j\} \cup \{y_k \cap X_j : k \in A_j\} \cup \{x_{\tilde{k}} \cap X_j\} \cup \{y_{\tilde{k}} \cap X_j\}$$

is linearly dependent. Also, since, for every $j = 0, ..., n$,

$$C_j \doteq \text{span}(\{x_k \cap X_j : k \in A_j\} \cup \{y_k \cap X_j : k \in A_j\})$$

is finite and both $\{x_k \cap X_j : k \in \omega\}$ and $\{y_k \cap X_j : k \in \omega\}$ are linearly independent, we can fix:

- an infinite subset $A \subset \omega$;
- $c_j \in \mathcal{C}_j$, for each $j = 0, ..., n$,

so that

$$x_{\tilde{k}} \cap X_j = (y_{\tilde{k}} \cap X_j) \vartriangle c_j,$$

for every $\tilde{k} \in A$ and $j = 0, ..., n$. Thus,

$$x_{\tilde{k}} = (x_{\tilde{k}} \cap X_0) \vartriangle ... \vartriangle (x_{\tilde{k}} \cap X_n) = (y_{\tilde{k}} \cap X_0) \vartriangle ... \vartriangle (y_{\tilde{k}} \cap X_n) \vartriangle (c_0 \vartriangle ... \vartriangle c_n) = y_{\tilde{k}} \vartriangle (c_0 \vartriangle ... \vartriangle c_n),$$

for every $\tilde{k} \in A$, which is a contradiction, as $\{x_k : k \in \omega\} \cup \{y_k : k \in \omega\}$ is linearly independent. Hence, $A_0, ..., A_n$ cannot all be finite. □

Lemma 2.2.8 ([TT23], Lemma 2.3). *Let X be an infinite set, $k > 0$ and $\{(x_n^0, ..., x_n^{k-1}) : n \in \omega\} \subset ([X]^{<\omega})^k$ be a sequence. Then, there are:*

- *elements $d_0, ..., d_{k-1} \in [X]^{<\omega}$;*
- *a subsequence $\{(x_{n_l}^0, ..., x_{n_l}^{k-1}) : l \in \omega\}$;*
- *for some[5] $0 \leq t \leq k$, a sequence $\{(y_{n_l}^0, ..., y_{n_l}^{t-1}) : l \in \omega\} \subset ([X]^{<\omega})^t$;*
- *for each $0 \leq s < k$, a function $P_s : t \to 2$,*

satisfying the following:

a) $x_{n_l}^s = \left(\sum_{i=0}^{t-1} P_s(i) y_{n_l}^i \right) \vartriangle d_s$, *for every $l \in \omega$ and $0 \leq s < k$;*

b) $\{y_{n_l}^i : l \in \omega, 0 \leq i < t\}$ *is linearly independent.*

Proof. Fix $q \in \omega^*$, and let

$$\mathcal{M} \doteq \left\{ c \in [X]^{<\omega} : [\vec{c}]_q \in \text{span}(\{[x^0]_q, ..., [x^{k-1}]_q\}) \right\}.$$

We claim that $\mathcal{M}$ is a finite set. Indeed, $\alpha \in \text{span}(\{[x^0]_q, ..., [x^{k-1}]_q\})$ if, and only if, there is a function $F : k \to 2$ so that $\alpha = \sum_{i=0}^{k-1} F(i)[x^i]_q$. Moreover, given $c, \tilde{c} \in [X]^{<\omega}$, $[\vec{c}]_q = [\vec{\tilde{c}}]_q$ if, and only if, $c = \tilde{c}$.

Thus, let $j \geq 0$ and $\{c^0, ..., c^{j-1}\} \subset \mathcal{M}$ be so that $\{c^0, ..., c^{j-1}\}$ is a basis for $\text{span}(\mathcal{M}) \subset [X]^{<\omega}$. Then, let also $t \geq 0$ and $y^0, ..., y^{t-1} \in ([X]^{<\omega})^\omega$ be so that $\mathcal{B} \doteq \{[\vec{c^0}]_q, ..., [\vec{c^{j-1}}]_q, [y^0]_q, ..., [y^{t-1}]_q\}$ is a basis for $\text{span}(\{[x^0]_q, ..., [x^{k-1}]_q\})$. Hence, there are $A \in q$, $P_s : t \to 2$ and $C_s : j \to 2$, for each $0 \leq s < k$, so that

$$x_n^s = \sum_{i=0}^{t-1} P_s(i) y_n^i \vartriangle \sum_{i=0}^{j-1} C_s(i) c^i,$$

for every $n \in A$ and $0 \leq s < k$. For each $0 \leq s < k$, let $d_s \doteq \sum_{i=0}^{j-1} C_s(i) c^i$.

---

[5] If $t = 0$, we understand that there is no such sequence and item i) becomes: $x_{n_l}^s = d_s$, for every $l \in \omega$ and $0 \leq s < k$.

We shall prove that there exists an infinite subset $I \subset A$ so that $\{y_n^i : n \in I, 0 \leq i < t\}$ is linearly independent. First, note that for each $c \in [X]^{<\omega}$ and nontrivial function $P : t \to 2$ we have that

$$\sum_{i=0}^{t-1} P(i)[y^i]_q \neq [\vec{c}]_q.$$

Therefore, there exist a subset $A_{P,c} \subset A$, $A_{P,c} \in q$, so that

$$\sum_{i=0}^{t-1} P(i)y_n^i \neq c$$

for each $n \in A_{P,c}$. In particular, we conclude that $\{y_n^i : 0 \leq i < t-1\}$ is linearly independent for every $n \in \bigcap_{\substack{P:t\to 2\\ P\neq 0}} A_{P,\emptyset} \doteq A_0$. We may choose $n_0 \in A_0$.

Now, suppose that, given $p \geq 1$, for each $l = 0, ..., p-1$ we have constructed $A_l \in q$ and $n_l \in A_l$ so that $\{y_{n_l}^i : 0 \leq l < p, 0 \leq i < t\}$ is linearly independent, $(n_l)_{0\leq l<p}$ is strictly increasing and $A_l \subset A$. Let $C_p \doteq \operatorname{span}(\{y_{n_l}^i : 0 \leq l < p, 0 \leq i < t\})$,

$$A_p \doteq \bigcap_{\substack{c\in C_p\\ P:t\to 2\\ P\neq 0}} A_{P,c} \ (\subset A),$$

and fix $n_p \in A_p$, $n_p > n_{p-1}$. It is clear that $A_p \in q$ and also $\{y_{n_l}^i : 0 \leq l \leq p, 0 \leq i < t\}$ is linearly independent, by construction. Then, by induction, there are a sequence $(A_l)_{l\in\omega}$ of elements of q and a strictly increasing sequence $(n_l)_{l\in\omega}$ of naturals so that $n_l \in A_l$ and $\{y_{n_l}^i : l \in \omega, 0 \leq i < t\}$ is linearly independent. Furthermore,

$$x_{n_l}^s = \left(\sum_{i=0}^{t-1} P_s(i)y_{n_l}^i\right) \triangle\, d_s,$$

for every $l \in \omega$ and $0 \leq s < k$. □

The following result, proved in [GT15], claims that every Boolean topological group contains a sequence of open sets satisfying an important algebraic property.

Lemma 2.2.9 ([GT15], Lemma 2.1). *Let G be a non-discrete Boolean topological group. Then, there exists a sequence $(W_n : n \in \omega)$ of nonempty open subsets of G such that, if $x_n \in W_n$ for every $n \in \omega$, then $\{x_n : n \in \omega\}$ is linearly independent.*

The following corollary is immediate from the previous lemma.

Lemma 2.2.10. *Let G be a non-discrete Boolean topological group. Then there exist nonempty open sets $\{U_k^j : k \in \omega, j \in \omega\}$ such that if $u_k^j \in U_k^j$ for each $k, j \in \omega$, then $\{u_k^j : k, j \in \omega\}$ is linearly independent.*

In most group constructions in subsequent chapters, we will also need to enumerate certain sets appropriately. For this, the next results will be important.

Lemma 2.2.11. *Let κ be an infinite cardinal, X be a set so that $|X| \leq \kappa$ and $f : X \to \kappa$ be a* *function so that $f(y) = 0$, for some $y \in X$. Then, there exists a function $g : \kappa \to X$ so that:*

a) for each $\alpha \in \kappa$, $f(g(\alpha)) \leq \alpha$;

b) for each $x \in X$, $|\{\alpha \in \kappa : g(\alpha) = x\}| = \kappa$.

Proof. Let $F : \kappa \to \kappa \times \kappa$ be a bijective function and

$$\begin{aligned} i : \kappa &\longrightarrow X \\ \beta &\longmapsto x_\beta \end{aligned}$$

be a surjective function. We define the function $g : \kappa \to X$ as follows:

$$g(\alpha) \doteq \begin{cases} x_\gamma, & \text{if } F(\alpha) = (\gamma, \delta) \in \kappa \times \kappa \text{ is such that } f(x_\gamma) \leq \alpha \\ y, & \text{otherwise.} \end{cases}$$

It is clear that, for each $\alpha \in \kappa$, $f(g(\alpha)) \leq \alpha$. Now, given $x \in X$, let $\beta \in \kappa$ be so that $x = x_\beta$ and let

$$B \doteq \{\alpha \in \kappa : \exists \delta \in \kappa \text{ so that } F(\alpha) = (\beta, \delta)\}.$$

Since, for each $\eta \in \kappa$, there exists $\alpha \in \kappa$ so that $F(\alpha) = (\beta, \eta)$, we have that $|B| = \kappa$. Also, as $f(x_\beta) < \kappa$, we have that $|B \setminus f(x_\beta)| = \kappa$, and thus

$$|\{\alpha \in \kappa : g(\alpha) = x\}| = \kappa,$$

since $B \setminus f(x_\beta) \subset \{\alpha \in \kappa : g(\alpha) = x\}$. □

Corollary 2.2.12. *Let κ be an infinite cardinal, $J \subset \kappa$ be such that $|J| = \kappa$, X be a set such that $|X| \leq \kappa$ and $f : X \to \kappa$ be a function such that $f(y) = 0$, for some $y \in X$. Then, there exists a function $g : J \to X$ so that:*

a) for each $\alpha \in J$, $f(g(\alpha)) \leq \alpha$;

b) for each $x \in X$, $|\{\alpha \in J : g(\alpha) = x\}| = \kappa$.

Proof. Let $g : \kappa \to X$ be the function given by the previous lemma. As $J \subset \kappa$ and $|J| = \kappa$, there exists an order isomorphism $\phi : J \to \kappa$. Thus, it is clear that $\phi(\alpha) \leq \alpha$, for each $\alpha \in J$, and $\tilde{g} \doteq g \circ \phi$ satisfies the conditions of the statement. □

Chapter 3

A countably compact group without non-trivial convergent sequences of size $2^{\mathfrak{c}}$

This chapter will be devoted to proving the following result, which is in the article [TT22]:

Theorem ([TT22], Theorem 3.1)**.** *There is a Boolean Hausdorff countably compact topological group of size $2^{\mathfrak{c}}$ without non-trivial convergent sequences.*

3.1 Auxiliary Results

We recall that, in 2021, Hrušák, van Mill, Ramos-García, and Shelah [Hru+21] proved in ZFC the following result, which was open for a long time:

Theorem 3.1.1 ([Hru+21], Theorem 4.1)**.** *There exists a Boolean countably compact topological group (of size $\mathfrak{c}$) without non-trivial convergent sequences.*

The main new idea that appears in [Hru+21] when proving Theorem 3.1.1 is the use of a clever filter to generate a suitable family of ultrafilters $\{p_\alpha : \alpha < \mathfrak{c}\} \subset \omega^*$, given by the next result. This family of ultrafilters eliminates the need for selective ultrafilters.

Proposition 3.1.2 ([Hru+21], Claim 4.3)**.** *There is a family $\{p_\alpha : \alpha < \mathfrak{c}\} \subset \omega^*$ such that, for every $D \in [\mathfrak{c}]^\omega$ and $\{f_\alpha : \alpha \in D\}$ such that each f_α is an one-to-one enumeration of linearly independent elements of $[\mathfrak{c}]^{<\omega}$, there is a sequence $(U_\alpha : \alpha \in D)$ that satisfies*

a) $\{U_\alpha : \alpha \in D\}$ is a family of pairwise disjoint subsets of ω;

b) $U_\alpha \in p_\alpha$ for every $\alpha \in D$;

c) $\{f_\alpha(n) : \alpha \in D$ and $n \in U_\alpha\}$ is a linearly independent subset of $[\mathfrak{c}]^{<\omega}$.

Using fundamentally the same idea, with a slight modification, we constructed a similar suitable family of ultrafilters $\{p_\alpha : \alpha < 2^{\mathfrak{c}}\} \subset \omega^*$, which permits, in the same way that was done in [Hru+21], the construction of a group of size $2^{\mathfrak{c}}$ satisfying Theorem 3.1.1. The construction of such family is done in the next result.

Proposition 3.1.3 ([TT22], Proposition 2.4)**.** *There is a family $\{p_\alpha : \alpha < 2^{\mathfrak{c}}\} \subset \omega^*$ such that, for every $D \in [2^{\mathfrak{c}}]^\omega$ and $\{f_\alpha : \alpha \in D\}$ such that each f_α is an one-to-one enumeration of linearly independent elements of $[2^{\mathfrak{c}}]^{<\omega}$, there is a sequence $(U_\alpha : \alpha \in D)$ that satisfies*

a) $\{U_\alpha : \alpha \in D\}$ is a family of pairwise disjoint subsets of ω;

b) $U_\alpha \in p_\alpha$ for every $\alpha \in D$;

c) $\{f_\alpha(n) : \alpha \in D$ and $n \in U_\alpha\}$ is a linearly independent subset of $[2^{\mathfrak{c}}]^{<\omega}$.

Proof. Fix $\{I_n : n \in \omega\}$ a partition of ω into finite sets such that

$$|I_n| > n \cdot \sum_{m<n} |I_m|,$$

and let

$$\mathcal{B} = \{B \subset \omega : \forall n \in \omega, |I_n \setminus B| \leq \sum_{m<n} |I_m|\}.$$

Note that the intersection of every finite subfamily of $\mathcal{B}$ is infinite, thus

$$\mathcal{F} \doteq \{f \subset \omega : C \subset f, \text{ for some } C = B_0 \cap ... \cap B_k, k \in \omega, \text{ and } B_0, ..., B_k \in \mathcal{B}\}$$

is a filter which extends $\mathcal{B}$. If A is an infinite subset of ω, notice that we have, for every $f \in \mathcal{F}$, $|f \cap \bigcup_{n\in A} I_n| = \omega$. Indeed, we shall prove by induction on $k \in \omega$ that, for each $k \in \omega$ and $B_0, ..., B_k \in \mathcal{B}$, we have that $|I_n \cap B_0 \cap ... \cap B_k| \geq (n - k - 1) \cdot \sum_{m<n} |I_m|$, for every $n \in \omega$. For that, suppose first that $k = 0$. Given $B_0 \in \mathcal{B}$, we have that

$$|I_n| = |I_n \setminus B_0| + |I_n \cap B_0|,$$

for every $n \in \omega$. Also, since, for every $n \in \omega$, $|I_n| > n \cdot \sum_{m<n} |I_m|$ and $|I_n \setminus B_0| \leq \sum_{m<n} |I_m|$, we have that $|I_n \cap B_0| \geq (n-1) \cdot \sum_{m<n} |I_m|$, for each $n \in \omega$. Now suppose that the result holds for every $i \leq k_0$, with $k_0 \in \omega$. Then, given $B_0, ..., B_{k_0+1} \in \mathcal{B}$, we may write, for every $n \in \omega$,

$$\begin{aligned}(n - k_0 - 1) \cdot \sum_{m<n} |I_m| \leq |I_n \cap B_0 \cap ... \cap B_{k_0}| &= |I_n \cap B_0 \cap ... \cap B_{k_0+1}| + |I_n \cap B_0 \cap ... \cap B_{k_0} \setminus B_{k_0+1}| \\ &\leq |I_n \cap B_0 \cap ... \cap B_{k_0+1}| + \sum_{m<n} |I_n|,\end{aligned}$$

hence

$$(n - k_0 - 2) \cdot \sum_{m<n} |I_m| \leq |I_n \cap B_0 \cap ... \cap B_{k_0+1}|.$$

Remark 6. In [Hru+21], it was fixed at this point an almost disjoint family[1] $(A_\alpha)_{\alpha<\mathfrak{c}}$ of

[1] We suggest the book [Kun11] for the definition and properties of almost disjoint families.

size $\mathfrak{c}$ of infinite subsets of ω, and each ultrafilter p_α was chosen extending the family

$$\mathcal{F}|_{\bigcup_{n\in A_\alpha} I_n} \doteq \{f \cap \bigcup_{n\in A_\alpha} I_n \,:\, f \in \mathcal{F}\}.$$

As we do not have an almost disjoint family of size greater than $\mathfrak{c}$ of infinite subsets of ω, we need to find another alternative to construct ultrafilters equally suitable for our purposes. The solution found was to use *weak P-points*. Here lies the difference between the two constructions.

Let $\{q_\xi \,:\, \xi < 2^{\mathfrak{c}}\}$ be the set of *weak P-points*[2]. For each $\xi < 2^{\mathfrak{c}}$, we fix a free ultrafilter p_ξ containing the family

$$\left\{ f \cap \bigcup_{n\in A} I_n \,:\, A \in q_\xi,\, f \in \mathcal{F} \right\}.$$

We shall prove that the family of free ultrafilters $(p_\xi)_{\xi<2^{\mathfrak{c}}}$ satisfies the proposition. For this, we fix a set $D = \{\alpha_n \,:\, n \in \omega\} \in [2^{\mathfrak{c}}]^\omega$ and a family $\{f_\alpha \,:\, \alpha \in D\}$ of one-to-one sequences of linearly independent elements of $[2^{\mathfrak{c}}]^{<\omega}$. By Proposition 1.2.32, we may construct a family $(C_{\alpha_n})_{n\in\omega}$ of pairwise disjoints subsets of ω so that $C_{\alpha_n} \in q_{\alpha_n}$, for each $n \in \omega$. Now, let $\{B_n \,:\, n \in \omega\}$ be a partition of ω such that[3] $B_n =^* C_{\alpha_n}$, for every $n \in \omega$ (we could use the sets C_{α_n} directly, but this choice will simplify a bit).

For each $k \in \omega$, let $n_k \in \omega$ be so that $k \in B_{n_k}$. We shall construct a sequence of sets $(R_k)_{k\in\omega}$ satisfying that, for each $k \in \omega$,

1. $R_k \subset I_k$;
2. $|R_k| \leq \sum_{m<k} |I_m|$;
3. $f_{\alpha_{n_0}}[I_0 \setminus R_0] \cup f_{\alpha_{n_1}}[I_1 \setminus R_1] \cup \ldots \cup f_{\alpha_{n_k}}[I_k \setminus R_k]$ is linearly independent.

Let $R_0 = \emptyset$, and, given $N > 0$, assume that we have already constructed sets $(R_k)_{k<N}$ satisfying the conditions above for each $k < N$. Since both $f_{\alpha_N}[I_N]$ and $f_{\alpha_{n_0}}[I_0] \cup f_{\alpha_{n_1}}[I_1 \setminus R_1] \cup \ldots \cup f_{\alpha_{n_{N-1}}}[I_{N-1} \setminus R_{N-1}]$ are linearly independent, Corollary 2.2.6 implies that there exists $R_N \subset I_N$ such that

$$|R_N| \leq \sum_{m<N} |I_m|$$

and

$$f_{\alpha_{n_0}}[I_0] \cup \ldots \cup f_{\alpha_{N_n}}[I_N \setminus R_N] \text{ is linearly independent.}$$

Therefore, there exists a family $(R_k)_{k\in\omega}$ satisfying the three conditions above, for every $k \in \omega$. Hence, the set

$$B \doteq \bigcup_{l\in\omega} (I_l \setminus R_l)$$

satisfies the following:

I1) $I_0 \subset B$;

[2] Recall that Kunen showed that there are $2^{\mathfrak{c}}$ of them in ZFC [Kun80].

[3] Given two sets H, G, we say that $H =^* G$ if, and only if, $H \triangle G$ is finite.

I2) for every $l \in \omega$, $|I_l \setminus B| = |R_l| \leq \sum_{m<l} |I_m|$;

I3) $\{f_{\alpha_n}(m) : n \in \omega, m \in B \cap I_l, l \in B_n\}$ is linearly independent.

It follows from I2) that $B \in \mathcal{B}$, and defining, for each $n \in \omega$,

$$U_n \doteq B \cap \bigcup_{l \in B_n} I_l,$$

it is clear that $(U_n)_{n\in\omega}$ is a family of pairwise disjoint sets. Furthermore, it follows from the definition of p_{α_n} and from the fact that $B_n \in q_{\alpha_n}$, that $U_n \in p_{\alpha_n}$, for every $n \in \omega$. Finally, by construction,

$$\{f_{\alpha_n}(m) : n \in \omega, m \in U_n\}$$

is linearly independent. □

In the next result we will construct the homomorphisms that will be used to define the topology of the required group. Given $E \subset [2^{\mathfrak{c}}]^{<\omega}$ (where $[2^{\mathfrak{c}}]^{<\omega}$ is considered as the Boolean group mentioned in the previous chapter), from now on we will denote as span(E) the vector subspace generated by E.

We fix here the family of ultrafilters $\{p_\alpha : \alpha < 2^{\mathfrak{c}}\}$ constructed in the previous proposition.

Lemma 3.1.4. *Let $\{f_\alpha : \alpha \in I\}$, with $I \subset 2^{\mathfrak{c}}$, be a family of one-to-one enumerations of linearly independent elements of $[2^{\mathfrak{c}}]^{<\omega}$, and $D \in [2^{\mathfrak{c}}]^{\omega}$ be such that, for every $\alpha \in D \cap I$, $\bigcup_{n\in\omega} f_\alpha(n) \subset D$. Consider also $D_0 \in [D]^{<\omega}$ and $F : D_0 \to 2$ a function. Then there exists a homomorphism $\phi : [D]^{<\omega} \to 2$ so that, for every $\alpha \in D \cap I$,*

$$\phi(\{\alpha\}) = p_\alpha - \lim_{n\in\omega} \phi(f_\alpha(n)),$$

and, for each $d \in D_0$, $\phi(\{d\}) = F(d)$.

Proof. Enumerate $(D \cap I)$ as $\{\alpha_n : n \in \omega\}$ so that $\{\alpha_0, ..., \alpha_r\} = D_0 \cap I$, and let $D_0 \setminus I = \{d_0, ..., d_l\}$. According to Proposition 3.1.3, there is a sequence $(U_\alpha : \alpha \in D \cap I)$ that satisfies

a) $\{U_\alpha : \alpha \in D \cap I\}$ is a family of pairwise disjoint subsets of ω;

b) $U_\alpha \in p_\alpha$ for every $\alpha \in D \cap I$;

c) $\{f_\alpha(n) : \alpha \in D \cap I$ and $n \in U_\alpha\}$ is a linearly independent subset of $[2^{\mathfrak{c}}]^{<\omega}$.

Letting

$$E_0 \doteq \{\{d_0\}, ..., \{d_l\}\} \cup \{\{\alpha_0\}, ..., .\{\alpha_r\}\} \cup \{f_{\alpha_m}(n) : m = 0, .., r \text{ and } n \in U_{\alpha_m}\},$$

we shall define a homomorphism $\phi_0 : \text{span}(E_0) \to 2$ so that, for every $j = 0, ..., l$,

$$\phi_0(\{d_j\}) = F(d_j)$$

and, for each $i = 0, ..., r$,

$$\phi_0(\{\alpha_i\}) = F(\alpha_i)$$

and
$$\phi_0(\{\alpha_i\}) = p_{\alpha_i} - \lim_{n \in U_{\alpha_i}} \phi_0(f_{\alpha_i}(n)).$$

This can be done, since $\{\{d_0\}, ..., \{d_l\}\} \cup \{\{\alpha_0\}, ..., \{\alpha_r\}\}$ and $\{f_{\alpha_m}(n) : m = 0, .., r \text{ and } n \in U_{\alpha_m}\}$ are linearly independent, and thus there exists $R_0 \subset \{f_{\alpha_m}(n) : m = 0, .., r \text{ and } n \in U_{\alpha_m}\}$ so that $|R_0| \leq r + l + 2$ and

$$\{\{d_0\}, ..., \{d_l\}\} \cup \{\{\alpha_0\}, ..., \{\alpha_r\}\} \cup (\{f_{\alpha_m}(n) : m = 0, .., r \text{ and } n \in U_{\alpha_m}\} \setminus R_0)$$

is linearly independent.

We shall now define recursively homomorphisms $\phi_k : \mathrm{span}(E_0 \cup \{f_{\alpha_m}(n) : r < m \leq k + r \text{ and } n \in U_{\alpha_m}\} \cup \{\{\alpha_i\} : r < i \leq k + r\}) \to 2$ satisfying that, for each $k \in \omega$,

1) ϕ_0 is the homomorphism defined above;

2) $\phi_k(\{\alpha_{k+r}\}) = p_{\alpha_{k+r}} - \lim_{n \in U_{\alpha_{k+r}}} \phi_k(f_{\alpha_{k+r}}(n))$;

3) ϕ_{k+1} extends ϕ_k.

Suppose that, for $N \in \omega$, we have defined homomorphisms $\phi_0, ..., \phi_N$ satisfying 1), 2) and 3). By Lemma 2.2.5, there is a finite subset R_{N+1} of $\{f_{\alpha_{r+N+1}}(n) : n \in U_{\alpha_{r+N+1}}\}$ such that

$$\mathrm{span}(\{\{d_0\}, ..., \{d_l\}\} \cup \{\{\alpha_0\}, ..., \{\alpha_{r+N}\}\} \cup \{f_{\alpha_m}(n) : 0 \leq m \leq r + N \text{ and } n \in U_{\alpha_m}\}) \bigcap \\ \mathrm{span}(\{f_{\alpha_{r+N+1}}(n) : n \in U_{\alpha_{r+N+1}}\} \setminus R_{N+1}) = \{\emptyset\}.$$

Therefore, we may define the homomorphism ϕ_{N+1} to be equal to ϕ_N in $\mathrm{span}(\{\{d_0\}, ..., \{d_l\}\} \cup \{\{\alpha_0\}, ..., \{\alpha_{r+N}\}\} \cup \{f_{\alpha_m}(n) : 0 \leq m \leq r + N \text{ and } n \in U_{\alpha_m}\})$ and so that

$$\phi_{N+1}(\{\alpha_{r+N+1}\}) = p_{\alpha_{r+N+1}} - \lim_{n \in U_{\alpha_{r+N+1}}} \phi_{N+1}(f_{\alpha_{r+N+1}}(n)).$$

Thus, we have proved that there exists homomorphisms ϕ_k satisfying 1), 2) and 3) for every $k \in \omega$. If ϕ is any homomorphism defined in $[D]^{<\omega}$ extending $\bigcup_{k \in \omega} \phi_k$, then

- $\forall \alpha \in D \cap I$, $\phi(\{\alpha\}) = p_\alpha - \lim_{n \in U_\alpha} \phi(f_\alpha(n))$;
- $\forall d \in D_0$, $\phi(\{d\}) = F(d)$,

as we want. □

Remark 7. Note that the homomorphism ϕ given by the previous lemma may be defined satisfying additional properties, since we have freedom in an infinite subset of $\{f_{\alpha_k}(n) : n \in U_{\alpha_k}\}$, for each $k \in \omega$. For instance, given $\alpha \in D \cap I$, we can choose ϕ satisfying also that

$$\forall i \in 2, |\{n \in \omega : \phi(f_\alpha(n)) = i\}| = \omega.$$

In fact, homomorphisms satisfying this property were constructed in [Hru+21].

3.2 The construction of the group

Theorem 3.2.1 ([TT22], Theorem 3.1)**.** *There is a Boolean Hausdorff countably compact topological group of size $2^{\mathfrak{c}}$ without non-trivial convergent sequences.*

Proof. We shall construct a topology on $[2^{\mathfrak{c}}]^{<\omega}$ as follows.

We begin by stating the following claim, which will be shown after the proof of this theorem.

Claim 1. *There exists a family $\{f_\alpha : \alpha \in [\omega, 2^{\mathfrak{c}})\} \subset ([2^{\mathfrak{c}}]^{<\omega})^\omega$ of one-to-one sequences such that*

1) for every infinite $X \subset [2^{\mathfrak{c}}]^{<\omega}$, there is an $\alpha \in [\omega, 2^{\mathfrak{c}})$ with $rng(f_\alpha) \subset X$;

2) each f_α is a sequence of linearly independent elements;

3) $rng(f_\alpha) \subset [\alpha]^{<\omega}$ for every $\alpha \in [\omega, 2^{\mathfrak{c}})$.

Let $\{p_\alpha : \alpha < 2^{\mathfrak{c}}\} \subset \omega^*$ be the family of free ultrafilters given by Proposition 3.1.3. Define, for each $\phi \in \mathrm{Hom}([\omega]^{<\omega}, 2)$, its extension $\bar{\phi} \in \mathrm{Hom}([2^{\mathfrak{c}}]^{<\omega}, 2)$ recursively, by putting, for every $\alpha \in [\omega, 2^{\mathfrak{c}})$,

$$\bar{\phi}(\{\alpha\}) = p_\alpha - \lim \bar{\phi}(f_\alpha(n)). \tag{3.1}$$

Note that it is enough to define $\bar{\phi}$ in the subset $\{\{\xi\} : \xi \in 2^{\mathfrak{c}}\}$, since this is a basis for $[2^{\mathfrak{c}}]^{<\omega}$, thus we can extend it linearly.

Let τ be the topology on $[2^{\mathfrak{c}}]^{<\omega}$ generated by the homomorphisms in $\{\bar{\phi} : \phi \in \mathrm{Hom}([\omega]^{<\omega}, 2)\}$. For every $\alpha \in [\omega, 2^{\mathfrak{c}})$, it follows that

$$\{\alpha\} = p_\alpha - \lim_{n\in\omega} f_\alpha(n),$$

since the topology is generated by finite intersections of inverse images of $\bar{\phi}$ functions, which satisfy (3.1). Therefore, as the family $\{f_\alpha : \alpha \in [\omega, 2^{\mathfrak{c}})\}$ satisfies 1), the topological space $([2^{\mathfrak{c}}]^{<\omega}, \tau)$ is countably compact.

Next we introduce the notion of *suitably closed* set relative to this construction[4]. Other constructions that we will make later also use an analogous concept, named in the same way, but defined slightly differently depending on the case.

Definition 3.2.2 ([Hru+21])**.** A set $D \in [2^{\mathfrak{c}}]^\omega$ is called *suitably closed* if $\omega \subset D$ and $\bigcup_{n\in\omega} f_\alpha(n) \subset D$, for every $\alpha \in D \setminus \omega$.

The topology also makes the topological group Hausdorff. Indeed, given $x \in [2^{\mathfrak{c}}]^{<\omega} \setminus \{\emptyset\}$, let $D \in [2^{\mathfrak{c}}]^\omega$ be a suitably closed set so that $x \subset D$. We may use Lemma 3.1.4 to construct a homomorphism $\psi : [D]^{<\omega} \to 2$ satisfying (3.1) for each $\alpha \in D \setminus \omega$, and such that $\psi(x) = 1$.

[4] The idea of suitably closed sets already appeared in [KTW00], without using a name. Many subsequent works that used Martin's Axiom for countable posets and selective ultrafilters also used this idea.

Hence, if $\Phi \doteq \psi|_{[\omega]^{<\omega}}$, the homomorphism $\overline{\Phi} \in \mathrm{Hom}([2^{\mathfrak{c}}]^{<\omega}, 2)$ is such that $\overline{\Phi}(x) = 1$, since $\overline{\Phi}|_{[D]^{<\omega}} = \psi$.

To finish, we enunciate the following lemma, which is used to show that the topological space $([2^{\mathfrak{c}}]^{<\omega}, \tau)$ does not contain non-trivial convergent sequences. Versions of this result were already used in previous articles, but we enunciate here the version that appears in [Hru+21], which we also prove for the sake of completeness.

Lemma 3.2.3. *If, for every $D \in [2^{\mathfrak{c}}]^{\omega}$ suitably closed and $\alpha \in D\setminus\omega$, there is $\psi \in Hom([D]^{<\omega}, 2)$ such that*

(1) $\forall \beta \in D \setminus \omega, \psi(\{\beta\}) = p_\beta - \lim_{n\in\omega} \psi(f_\beta(n))$;

(2) $\forall i \in 2, |\{n \in \omega : \psi(f_\alpha(n)) = i\}| = \omega$,

then the topology defined above on $[2^{\mathfrak{c}}]^{<\omega}$ does not contain non-trivial convergent sequences.

Proof. Suppose that $([2^{\mathfrak{c}}]^{<\omega}, \tau)$ contains a non-trivial convergent sequence $(x_n : n \in \omega)$. Let $\alpha \in [\omega, \mathfrak{c})$ be so that $\mathrm{rng}(f_\alpha) \subset \{x_n : n \in \omega\}$. Let D be a suitably closed set containing α. Then, there is $\psi \in \mathrm{Hom}([D]^{<\omega}, 2)$ such that

- $\forall \beta \in D \setminus \omega, \psi(\{\beta\}) = p_\beta - \lim_{n\in\omega} \psi(f_\beta(n))$;
- $\forall i \in 2, |\{n \in \omega : \psi(f_\alpha(n)) = i\}| = \omega$,

Let $\phi \doteq \psi|_{[\omega]^{<\omega}}$. Then, it follows that $\psi = \bar{\phi}|_{[D]^{<\omega}}$, and since the sequence $(\psi(f_\alpha(n)) : n \in \omega)$ takes infinitely many times the values 0 and 1, $(x_n : n \in \omega)$ cannot be a convergent sequence. □

The hypothesis of the lemma above are satisfied, due to Lemma 3.1.4 and Remark 7, following it, thus $([2^{\mathfrak{c}}]^{<\omega}, \tau)$ is a countably compact topological group of size $2^{\mathfrak{c}}$ without non-trivial convergent sequences. □

Now, we shall prove Claim 1:

Proof of Claim 1. Let Y be the set of all sequences of $[2^{\mathfrak{c}}]^{<\omega}$ whose elements are linearly independent. Then, it is not hard to show that $|Y| = 2^{\mathfrak{c}}$. We define $h : X \to 2^{\mathfrak{c}}$ so that, for each $y \in Y$,

$$h(y) \doteq \begin{cases} 0, & \text{if } \sup(\bigcup \mathrm{rng}(y)) \leq \omega \\ \sup(\bigcup \mathrm{rng}(y)) + 1, & \text{otherwise.} \end{cases}$$

Let also $J \doteq [\omega, 2^{\mathfrak{c}})$. By Corollary 2.2.12, there exists a function $f : [\omega, 2^{\mathfrak{c}}) \to Y$ so that, for each $\alpha \in [\omega, 2^{\mathfrak{c}})$, $h(f_\alpha) \leq \alpha$. Then, $\{f_\alpha : \alpha \in [\omega, 2^{\mathfrak{c}})\}$ satisfies 1)–3) of Claim 1.

□

3.3 Conclusion

In this section we will make some additional comments, and present some open problems and natural directions for further studies on the topic addressed in the chapter.

Convergent sequences in topological groups are related to several important concepts concerning these spaces, and hence have also been studied for a long time. We point out that it is possible to find a counterexample to the famous problem posed by Wallace [Wal55] (written in the next question) inside of any non-torsion countably compact topological group without non-trivial convergent sequences (according to [RS96] and [Tom96]).

Question 3.3.1 ([Wal55])**.** *Is every countably compact topological semigroup with two-sided cancellation a topological group?*

A counterexample to Wallace's question has been called a *Wallace semigroup*. Hence, a positive answer to the following question would prove the existence of a Wallace semigroup in ZFC.

Question 3.3.2. *Is there in ZFC a non-torsion countably compact topological group without non-trivial convergent sequences?*

The known examples of Wallace semigroups are under CH [RS96], Martin's Axiom for countable posets [Tom96], $\mathfrak{c}$ incomparable selective ultrafilters (according to the Rudin-Keisler ordering in ω^*) [MT07] and a single selective ultrafilter [BCT19]. With the exception of [Tom96], the articles mentioned obtained the examples as a semigroup of a countably compact free Abelian group without non-trivial convergent sequences.

Chapter 4

A selectively pseudocompact group which is not countably pracompact

This chapter will be mainly devoted to proving the following result, which is in the article [TT22]:

Theorem ([TT22], Theorem 4.1)**.** *There is a Hausdorff selectively pseudocompact group which is not countably pracompact.*

Assuming the existence of a selective ultrafilter p, we will also use a similar construction to show that there exists a selectively $p-$pseudocompact group which is not countably pracompact:

Theorem ([TT22], Theorem 5.4)**.** *If $p \in \omega^*$ is a selective ultrafilter, there exists a Hausdorff selectively $p-$pseudocompact group which is not countably pracompact.*

Since selective $p-$pseudocompactness is productive and implies selective pseudocompactness, we will obtain a group which has all powers selectively pseudocompact and is not countably pracompact.

This chapter will not have a specific section for auxiliary results since the lemmas that we will need in the proof of the first theorem above were already presented in the previous chapter, and the results needed for the proof of the second theorem use notations that will be presented during the proof of the first one.

4.1 The construction of the groups

Theorem 4.1.1 ([TT22], Theorem 4.1)**.** *There is a Hausdorff selectively pseudocompact group which is not countably pracompact.*

Proof. We shall construct a topology on the Boolean group $[\mathfrak{c}]^{<\omega}$ as follows. We start with the following claim.

Claim 2. *There exists a function $F : \mathfrak{c} \times \mathfrak{c} \to 2$ so that:*

- *For every $A \in [\mathfrak{c}]^\omega$, there exists $\beta \in \mathfrak{c}$ such that $F(A \times \{\beta\}) = 1$.*
- *For every $B \in [\mathfrak{c}]^\omega$, there exist $\alpha_0 \in \mathfrak{c}$ and $\alpha_1 \in \mathfrak{c}$ such that $F(\{\alpha_0\} \times B) = 0$ and $F(\{\alpha_1\} \times B) = 1$.*

Proof of the claim. Let $\mathcal{C} = \{C_\beta : \beta < \mathfrak{c}\}$ be an enumeration of $[\mathfrak{c}]^\omega$. We shall recursively construct a family $\{\alpha_\gamma : \gamma < \mathfrak{c}\} \subset \mathfrak{c}$ so that:

- $\{\alpha_\gamma : \gamma < \mathfrak{c}\}$ is a strictly increasing family;
- $\alpha_\gamma \notin \bigcup_{\beta \in C_\gamma} C_\beta$, for each $\gamma < \mathfrak{c}$.

If $\gamma = 0$, we choose arbitrarily α_0 so that $\alpha_0 \in \mathfrak{c} \setminus \bigcup_{\beta \in C_0} C_\beta$, which can be done, since $|\bigcup_{\beta \in C_0} C_\beta| = \omega$. Suppose that, for an ordinal $\kappa < \mathfrak{c}$, we have constructed a family $\{\alpha_\gamma : \gamma < \kappa\}$ satisfying the items above. Then, since

$$|\{\alpha_\gamma : \gamma < \kappa\} \cup \bigcup_{\beta \in C_\kappa} C_\beta| = |\kappa|,$$

we may choose $\alpha_\kappa \in \mathfrak{c} \setminus \bigcup_{\beta \in C_\kappa} C_\beta$ so that $\alpha_\kappa > \alpha_\gamma$ for every $\gamma < \kappa$. Then, there is a family $\{\alpha_\gamma : \gamma < \mathfrak{c}\} \subset \mathfrak{c}$ satisfying the two items above.

Now, let $\{D_0, D_1\}$ be a partition of $\mathfrak{c}$ such that $|D_0| = |D_1| = \mathfrak{c}$, and let $\{\alpha^0_\gamma : \gamma < \mathfrak{c}\}$ and $\{\alpha^1_\gamma : \gamma < \mathfrak{c}\}$ be enumerations of $\{\alpha_\gamma : \gamma \in D_0\}$ and $\{\alpha_\gamma : \gamma \in D_1\}$, respectively. Consider $F : \mathfrak{c} \times \mathfrak{c} \to 2$ defined as follows. Let $(\alpha, \beta) \in \mathfrak{c} \times \mathfrak{c}$. If $\alpha \in C_\beta$, we put $F(\alpha, \beta) = 1$. In the case that $\alpha \notin C_\beta$,

- if $\alpha = \alpha^0_\gamma$ for some $\gamma \in \mathfrak{c}$ and $\beta \in C_\gamma$, we put $F(\alpha, \beta) = 0$;
- if $\alpha = \alpha^1_\gamma$ for some $\gamma \in \mathfrak{c}$ and $\beta \in C_\gamma$, we put $F(\alpha, \beta) = 1$;
- we put $F(\alpha, \beta) = 0$, otherwise.

Now we shall see that F satisfies the properties we want. For that, let $A \in [\mathfrak{c}]^\omega$. Then, there exists $\delta < \mathfrak{c}$ so that $A = C_\delta$. By construction, $F(\alpha, \delta) = 1$, for every $\alpha \in A$, thus $F(A \times \{\delta\}) = 1$. Moreover, for each $\beta \in C_\delta$, we have that $F(\alpha^0_\delta, \beta) = 0$ and $F(\alpha^1_\delta, \beta) = 1$. Therefore, $F(\{\alpha^0_\delta\} \times A) = 0$ and $F(\{\alpha^1_\delta\} \times A) = 1$. □

Now, let $\{p_\xi : \xi < \mathfrak{c}\}$ be the family of free ultrafilters given by Proposition 3.1.2 and $F : \mathfrak{c} \times \mathfrak{c} \to 2$ be the function given by the previous claim. Let also $(J_\beta)_{\beta < \mathfrak{c}}$ be a partition of $\mathfrak{c}$ such that, for each $\beta < \mathfrak{c}$, $|J_\beta| = \mathfrak{c}$. Consider also, for each $\beta < \mathfrak{c}$, a partition $\{J^1_\beta, J^2_\beta\}$ of J_β satisfying that $|J^1_\beta| = |J^2_\beta| = \mathfrak{c}$. We suppose that the initial ω elements of J_β are in J^1_β, for every $\beta < \mathfrak{c}$.

To make the proof clearer, we will enunciate some technical claims, which will be proved at the end of the proof of this theorem.

Claim 3. *Given $\beta < \mathfrak{c}$, there exists a family $\{f^\beta_\xi : \xi \in J^2_\beta\}$ of functions $f^\beta_\xi : \omega \to [J_\beta]^{<\omega}$ such that*

1) each f^β_ξ is an one-to-one enumeration of linearly independent elements of $[J_\beta]^{<\omega}$;

2) *for every infinite* $X \subset [J_\beta]^{<\omega}$*, there exists* $\xi \in J^2_\beta$ *such that* $rng(f^\beta_\xi) \subset X$*;*

3) *for every* $\xi \in J^2_\beta$, $rng(f^\beta_\xi) \subset [\xi]^{<\omega}$.

From now on, we will omit the superscript of f^β_ξ, since for each $\xi \in \bigcup_{\beta<\mathfrak{c}} J^2_\beta$, there is a unique $\beta < \mathfrak{c}$ such that $\xi \in J_\beta$. We also define the sets $J_1 \doteq \bigcup_{\beta<\mathfrak{c}} J^1_\beta$ and $J_2 \doteq \bigcup_{\beta<\mathfrak{c}} J^2_\beta$. Lastly, we fix another partition $(I_\alpha)_{\alpha<\mathfrak{c}}$ of $\mathfrak{c}$ satisfying that $|I_\alpha| = \mathfrak{c}$, for every $\alpha < \mathfrak{c}$.

We shall now define which are the suitably closed sets of this construction.

Definition 4.1.2. A set $A \in [\mathfrak{c}]^\omega$ is *suitably closed* if, for every $\xi \in A \cap J_2$, we have $\bigcup_{n\in\omega} f_\xi(n) \subset A$.

Let $\mathcal{A}$ be the set of all homomorphisms $\sigma : [A]^{<\omega} \to 2$, with $A \in [\mathfrak{c}]^\omega$ suitably closed, satisfying that
$$\sigma(\{\xi\}) = p_\xi - \lim_{n\in\omega} \sigma(f_\xi(n)),$$
for every $\xi \in A \cap J_2$.

Claim 4. *There exists an enumeration* $\{\sigma_\mu : \omega \leq \mu < \mathfrak{c}\}$ *of* $\mathcal{A}$ *so that:*

- $\bigcup dom(\sigma_\mu) \subset \mu$[1], *for each* $\omega \leq \mu < \mathfrak{c}$.
- *For each* $\sigma \in \mathcal{A}$ *and* $\alpha < \mathfrak{c}$*, there exists* $\mu \in I_\alpha$ *so that* $\sigma_\mu = \sigma$.

Consider the enumeration $\{\sigma_\mu : \omega \leq \mu < \mathfrak{c}\}$ of $\mathcal{A}$ given by the previous claim. For each $\mu \in [\omega, \mathfrak{c})$, we shall construct a suitable homomorphism $\overline{\sigma_\mu} : [\mathfrak{c}]^{<\omega} \to 2$. First, we will also need the following result.

Claim 5. *For each* $\beta < \mathfrak{c}$*, there exists an enumeration* $\{g^\beta_\xi : \xi \in J^1_\beta\}$ *of all functions* $g : S \to 2$*, with* $S \in [\mathfrak{c}]^{<\omega}$*, so that* $dom(g^\beta_\xi) \subset \xi$*, for every* $\xi \in J^1_\beta$*, and that for each* $g : S \to 2$ *as above,* $|\{\xi \in J^1_\beta : g^\beta_\xi = g\}| = \mathfrak{c}$.

As done before, from now on we omit the superscript of g^β_ξ, since for each $\xi \in J_1$, there is a unique $\beta < \mathfrak{c}$ such that $\xi \in J^1_\beta$.

Given $\mu \in [\omega, \mathfrak{c})$, we start defining an auxiliary homomorphism $\psi_\mu : [\mathfrak{c}]^{<\omega} \to 2$, extending σ_μ. First, if $\xi < \mathfrak{c}$ is such that $\{\xi\} \in \mathrm{dom}(\sigma_\mu)$, we put $\psi_\mu(\{\xi\}) = \sigma_\mu(\{\xi\})$. Otherwise, we have a few cases to consider: firstly, for every $\xi \in J_1$, we put $\psi_\mu(\{\xi\}) = g_\xi(\mu)$ if $\mu \in \mathrm{dom}(g_\xi)$ and $\psi_\mu(\{\xi\}) = 0$ if $\mu \notin \mathrm{dom}(g_\xi)$; for elements $\xi \in J_2$, we define ψ_μ recursively, by putting
$$\psi_\mu(\{\xi\}) = p_\xi - \lim_{n\in\omega} \psi_\mu(f_\xi(n)).$$

Note that since $\{\{\xi\} : \xi < \mathfrak{c}\}$ is a base for $[\mathfrak{c}]^{<\omega}$, the definition above uniquely extends each σ_μ to a homomorphism $\psi_\mu : [\mathfrak{c}]^{<\omega} \to 2$, which satisfy the previous equation for every $\xi \in J_2$, by construction.

[1] In case of a homomorphism $\sigma : [C]^{<\omega} \to 2$, with $C \subset \mathfrak{c}$, note that $\bigcup \mathrm{dom}(\sigma) = C$.

Now, to obtain the homomorphisms $\overline{\sigma_\mu} : [\mathfrak{c}]^{<\omega} \to 2$, we shall make some modifications to the homomorphisms ψ_μ defined before, in the following way. For every $\mu \in [\omega, \mathfrak{c})$ and $\xi < \mathfrak{c}$, we have that $\mu \in I_\alpha$ and $\xi \in J_\beta$ for unique $\alpha, \beta < \mathfrak{c}$, thus we put

$$\begin{cases} \overline{\sigma_\mu}(\{\xi\}) = 0, & \text{if } F(\alpha, \beta) = 0 \\ \overline{\sigma_\mu}(\{\xi\}) = \psi_\mu(\{\xi\}), & \text{if } F(\alpha, \beta) = 1. \end{cases}$$

Again, in this way we define uniquely a homomorphism $\overline{\sigma_\mu} : [\mathfrak{c}]^{<\omega} \to 2$, for each $\mu < \mathfrak{c}$. Note that, for every $\xi \in J_2$ and $\omega \leq \mu < \mathfrak{c}$,

$$\overline{\sigma_\mu}(\{\xi\}) = p_\xi - \lim_{n\in\omega} \overline{\sigma_\mu}(f_\xi(n)).$$

In fact, if $\mu \in I_\alpha$ and $\xi \in J^2_\beta$ are such that $F(\alpha, \beta) = 0$, then $\overline{\sigma_\mu}(\{\xi\}) = \overline{\sigma_\mu}(f_\xi(n)) = 0$ for every $n \in \omega$. Otherwise, $\overline{\sigma_\mu}(\{\xi\}) = \psi_\mu(\{\xi\})$ and $\overline{\sigma_\mu}(f_\xi(n)) = \psi_\mu(f_\xi(n))$, for every $n \in \omega$. Furthermore, note that $\overline{\sigma_\mu}$ is non-trivial for every $\omega \leq \mu < \mathfrak{c}$. Indeed, given $\mu \in I_\alpha$, take $\beta < \mathfrak{c}$ such that $F(\alpha, \beta) = 1$ and let $\xi \in J^1_\beta \setminus \bigcup \mathrm{dom}(\sigma_\mu)$ be such that g_ξ is the function which has $\mathrm{dom}(g_\xi) = \{\mu\}$ and $g_\xi(\mu) = 1$. Hence, we have that $\overline{\sigma_\mu}(\{\xi\}) = g_\xi(\mu) = 1$.

Let now $\overline{\mathcal{A}} \doteq \{\overline{\sigma_\mu} : \omega \leq \mu < \mathfrak{c}\}$ and τ be topology on $[\mathfrak{c}]^{<\omega}$ generated by the homomorphisms in $\overline{\mathcal{A}}$. First, notice that $([\mathfrak{c}]^{<\omega}, \tau)$ is Hausdorff. In fact, given $x \in [\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$, let $D \in [\mathfrak{c}]^\omega$ be a suitably closed set so that $x \subset D$, and let $\alpha < \mathfrak{c}$ be such that $F(\{\alpha\} \times D)$=1. According to Lemma 3.1.4, there exists $\sigma : [D]^{<\omega} \to 2$, $\sigma \in \mathcal{A}$, such that $\sigma(x) = 1$ and, by construction, there exists $\mu_0 \in I_\alpha$ such that $\sigma_{\mu_0} = \sigma$. Hence, we have that $\overline{\sigma_{\mu_0}}(x) = 1$.

Claim 6. *$([\mathfrak{c}]^{<\omega}, \tau)$ is a selectively pseudocompact group.*

Proof of the claim. Let $\{U_n : n \in \omega\}$ be a sequence of nonempty open sets in the group. For each U_n, we fix a function $g_n : S_n \to 2$, with $S_n \in [\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$, so that

$$U_n \supset \bigcap_{\mu\in S_n} \overline{\sigma_\mu}^{-1}(g_n(\mu)).$$

Thus, for each $n \in \omega$, we may define the set $C_n \doteq \{\alpha < \mathfrak{c} : \mu \in I_\alpha \text{ for some } \mu \in S_n\}$, and also $C \doteq \bigcup_{n\in\omega} C_n \in [\mathfrak{c}]^\omega$. By the property of F function, there exists $\beta \in \mathfrak{c}$ such that $F(C \times \{\beta\}) = 1$. For each $n \in \omega$, let $\xi_n \in J^1_\beta \setminus \bigcup_{\mu\in S_n} \left(\cup \mathrm{dom}(\sigma_\mu)\right)$ be such that $g_{\xi_n} = g_n$. We may choose such elements ξ_n pairwise distinct. By the way we have defined the homomorphisms which generate the topology, for every $\mu \in S_n$, $\overline{\sigma_\mu}(\{\xi_n\}) = \psi_\mu(\{\xi_n\}) = g_n(\mu)$, and therefore $\{\xi_n\} \in U_n$, for every $n \in \omega$.

Now, let $\xi \in J^2_\beta$ be such that $\mathrm{rng}(f_\xi) \subset \{\{\xi_n\} : n \in \omega\}$. Since

$$\overline{\sigma_\mu}(\{\xi\}) = p_\xi - \lim_{n\in\omega} \overline{\sigma_\mu}(f_\xi(n)), \tag{4.1}$$

for every $\omega \leq \mu < \mathfrak{c}$, we have that $\{\xi\}$ is an accumulation point of $\{\{\xi_n\} : n \in \omega\}$, ending the proof. □

Claim 7. *$([\mathfrak{c}]^{<\omega}, \tau)$ is not a countably pracompact group.*

Proof of the claim. Suppose that Z is a subset of G that is dense. We shall construct a sequence $\{t_n : n \in \omega\} \subset Z$ that does not have an accumulation point in $([\mathfrak{c}]^{<\omega}, \tau)$. Such sequence shall satisfy that

$$\mathrm{SUPP}(t_n) \setminus \left(\bigcup_{m<n} \mathrm{SUPP}(t_m) \right) \neq \emptyset, \tag{4.2}$$

for every $n \in \omega$, where, for each $D \in [\mathfrak{c}]^{<\omega}$, we define

$$\mathrm{SUPP}(D) \doteq \{\beta < \mathfrak{c} : J_\beta \cap D \neq \emptyset\}.$$

First, fix $t_0 \in Z$ arbitrarily, and suppose that, for $k > 0$, we have defined $\{t_n : n < k\} \subset Z$ satisfying equation (4.2) for every $n < k$. We claim that $B \doteq \bigcup_{z \in Z} \mathrm{SUPP}(z)$ cannot be a countable set. Indeed, if B is countable, by construction of F function, there exists an $\alpha \in \mathfrak{c}$ so that $F(\{\alpha\} \times B) = 0$. Therefore, given $\mu \in I_\alpha$, $\overline{\sigma_\mu}(z) = 0$ for every $z \in Z$, and thus Z cannot be dense in G. Hence, there exists $t_k \in Z$ such that

$$\mathrm{SUPP}(t_k) \setminus \left(\bigcup_{m<k} \mathrm{SUPP}(t_m) \right) \neq \emptyset,$$

which proves the existence of such sequence $\{t_n : n \in \omega\} \subset Z$.

Now we shall show that, for each $x \in [\mathfrak{c}]^{<\omega}$, x is not an accumulation point of $\{t_n : n \in \omega\}$. First, note that there exists $k_0 \in \omega$ such that, for every $n \geq k_0$,

$$\mathrm{SUPP}(t_n) \setminus \left(\bigcup_{m<n} \mathrm{SUPP}(t_m) \cup \mathrm{SUPP}(x) \right) \neq \emptyset.$$

In fact, since $\mathrm{SUPP}(x)$ is finite and (4.2) holds, there cannot be infinite elements t_n such that $\mathrm{SUPP}(t_n) \subset \bigcup_{m<n} \mathrm{SUPP}(t_m) \cup \mathrm{SUPP}(x)$.

Let

$$F_0 \doteq \bigcup_{m<k_0} \mathrm{SUPP}(t_m) \cup \mathrm{SUPP}(x)$$

and, for $i > 0$,

$$F_i \doteq \mathrm{SUPP}(t_{k_0+i-1}) \setminus \left(\bigcup_{m<k_0+i-1} \mathrm{SUPP}(t_m) \cup \mathrm{SUPP}(x) \right).$$

Define also, for each $i \in \omega$,

$$D_i \doteq \{\xi \in \bigcup_{n \in \omega} t_n \cup x : \mathrm{SUPP}(\{\xi\}) \in F_i\},$$

and let A_i be a suitably closed set containing D_i such that $\mathrm{SUPP}(A_i) = \mathrm{SUPP}(D_i)$. Since $(F_i)_{i \in \omega}$ is a family of pairwise disjoint sets, we have that $(A_i)_{i \in \omega}$ is also a family of pairwise disjoint sets.

According to Lemma 3.1.4, we may define a homomorphism $\theta_0 : [A_0]^{<\omega} \to 2$ such that $\theta_0 \in \mathcal{A}$ and $\theta_0(x) = 0$. For $k > 0$, suppose that we have constructed a set of homomorphisms $\{\theta_i : i < k\} \subset \mathcal{A}$ such that

a) $\theta_0(x) = 0$.

b) θ_i is a homomorphism defined in $\left[\bigcup_{j\leq i} A_j\right]^{<\omega}$ taking values in 2, for each $i < k$.

c) θ_i extends θ_{i-1} for each $0 < i < k$.

d) $\theta_i(t_{k_0+j}) = 1$ for each $0 < i < k$ and $j = 0, ..., i-1$.

Let A_k be a suitably closed set containing D_k. Again by Lemma 3.1.4, we may define a homomorphism $\psi_k : [A_k]^{<\omega} \to 2$ so that $\psi_k \in \mathcal{A}$ and

$$\psi_k\Big(t_{k_0+k-1} \setminus \bigcup_{j<k} D_j\Big) + \theta_{k-1}\Big(t_{k_0+k-1} \cap \bigcup_{j<k} D_j\Big) = 1.$$

Now, since $A_k \cap \bigcup_{i<k} A_i = \emptyset$, we may define a homomorphism $\theta_k : \left[\bigcup_{j\leq k} A_j\right]^{<\omega} \to 2$ extending both θ_{k-1} and ψ_k. Then, by construction, we have that $\theta_k(x) = 0$ and $\theta_k(t_{k_0+j}) = 1$ for every $j = 0, ..., k-1$. Also, it follows that $\theta_k \in \mathcal{A}$, since $\psi_k \in \mathcal{A}$ and $\theta_i \in \mathcal{A}$ for every $i < k$. Therefore, there exists a family of homomorphisms $\{\theta_k : k \in \omega\} \subset \mathcal{A}$ satisfying a)-d) for every $k \in \omega$.

Letting $A \doteq \bigcup_{n\in\omega} A_n$ and $\theta \doteq \bigcup_{n\in\omega} \theta_n$, the homomorphism $\theta : [A]^{<\omega} \to 2$ satisfy that $\theta \in \mathcal{A}$, since each $\theta_n \in \mathcal{A}$. Also, $\theta(x) = 0$ and $\theta(t_{k_0+j}) = 1$ for every $j \in \omega$. Let $\alpha < \mathfrak{c}$ be such that $F\Big(\{\alpha\} \times \bigcup_{n\in\omega} \mathrm{SUPP}(t_n) \cup \mathrm{SUPP}(x)\Big) = 1$ and $\mu \in I_\alpha$ be such that $\sigma_\mu = \theta$. Then, $\overline{\sigma_\mu} : [\mathfrak{c}]^{<\omega} \to 2$ satisfies that $\overline{\sigma_\mu}(t_n) = 1$ for each $n \in \omega$, and $\overline{\sigma_\mu}(x) = 0$. Therefore, $x \in [\mathfrak{c}]^{<\omega}$, which was chosen arbitrarily, is not an accumulation point of $\{t_n : n \in \omega\}$, hence $([\mathfrak{c}]^{<\omega}, \tau)$ is not countably pracompact.

□

□

Now we will prove the technical claims made in the proof of the theorem above.

Proof of Claim 3. The proof is analogous to the proof of Claim 1. □

Proof of Claim 4. If $A \in [\mathfrak{c}]^\omega$ is such that $A \subset J_1$, then A is suitably closed, and every homomorphism $\sigma : [A]^{<\omega} \to 2$ belongs to $\mathcal{A}$. Then, $\mathcal{A}$ has at least cardinality $\mathfrak{c}$. Furthermore, since $|[\mathfrak{c}]^\omega| = \mathfrak{c}$, it follows that $|\mathcal{A}| = \mathfrak{c}$.

Now, let $f : \mathcal{A} \to \mathfrak{c}$ be so that, for each $\sigma \in \mathcal{A}$,

$$f(\sigma) \doteq \begin{cases} 0, & \text{if } \sup(\bigcup \mathrm{dom}(\sigma)) \leq \omega \\ \sup(\bigcup \mathrm{dom}(\sigma)) + 1, & \text{otherwise.} \end{cases}$$

Then, by Corollary 2.2.12, there exists $g : [\omega, \mathfrak{c}) \to \mathcal{A}$ so that $f(g(\alpha)) \leq \alpha$ for each $\alpha \in [\omega, \mathfrak{c})$ and that $|\{\alpha \in [\omega, \mathfrak{c}) : g(\alpha) = \sigma\}| = \mathfrak{c}$, for each $\sigma \in \mathcal{A}$.

For each $\alpha < \mathfrak{c}$, let $I_\alpha = \{y^\alpha_\beta : \beta < \mathfrak{c}\}$ be a strictly increasing enumeration (in particular, note that $\beta \leq y^\alpha_\beta$, for every $\alpha, \beta < \mathfrak{c}$. Consider $h : [\omega, \mathfrak{c}) \to \mathcal{A}$ given as follows. For each $\mu \in [\omega, \mathfrak{c})$, let $\alpha < \mathfrak{c}$ be so that $\mu \in I_\alpha$. Then, $\mu = y^\alpha_\beta$, for some $\beta < \mathfrak{c}$. Thus, we put $h(\mu) = g(\beta)$. Then, the enumeration h satisfies the desired properties of Claim 4.

$\square$

Proof of Claim 5. Given $\beta < \mathfrak{c}$, let $X \doteq \bigcup_{S \in [\mathfrak{c}]^{<\omega}} 2^S$. Then, $|X| = \mathfrak{c}$. Let also $f : X \to \mathfrak{c}$ given, for each $x \in X$, by

$$f(x) \doteq \begin{cases} 0, & \text{if } x = \emptyset \\ \max(\mathrm{dom}(x)) + 1, & \text{otherwise.} \end{cases}$$

By Corollary 2.2.12, there exists $h : J^1_\beta \to X$ so that $f(h(\xi)) \leq \xi$ for each $\xi \in J^1_\beta$ and that $|\{\xi \in J^1_\beta : h(\xi) = x\}| = \mathfrak{c}$, for every $x \in X$. Therefore, h is the desired enumeration. $\square$

Assuming the existence of a selective ultrafilter p, we may use the same construction as above to show that there exists a selectively p–pseudocompact group which is not countably pracompact. Since selective p–pseudocompactness is productive and implies selective pseudocompactness, we will obtain a group which has all powers selectively pseudocompact and is not countably pracompact. In order to replace Lemma 3.1.4, we use some versions of results proved in Tomita, Garcia-Ferreira and Watson's paper [GTW05].

Following the proof of Theorem 4.1.1, we consider the same function $F : \mathfrak{c} \times \mathfrak{c} \to 2$, families $(I_\alpha)_{\alpha<\mathfrak{c}}$, $(J_\beta)_{\beta<\mathfrak{c}}$ and partition $\{J^1_\beta, J^2_\beta\}$ of J_β, for each $\beta < \mathfrak{c}$. Using a similar proof to *Lemma 2.1* of [GTW05], one may show the following result.

Lemma 4.1.3 ([GTW05], Lemma 2.1)**.** *If $p \in \omega^*$ is a selective ultrafilter, then, for each $\beta < \mathfrak{c}$, there exists a family of one-to-one functions $\{f_\xi : \xi \in J^2_\beta\} \subset ([J_\beta]^{<\omega})^\omega$ such that*

- *i1) $\bigcup_{n\in\omega} f_\xi(n) \subset max\{\omega, \xi\}$, for every $\xi \in J^2_\beta$.*
- *i2) $\{[f_\xi]_p : \xi \in J^2_\beta\} \cup \{[\vec{\mu}]_p : \mu \in J_\beta\}$ is a base for $([J_\beta]^{<\omega})^\omega / p$.*
- *i3) For every one-to-one function $g \in ([J_\beta]^{<\omega})^\omega$, there are distinct $\xi_0, \xi_1 \in J^2_\beta$ and two increasing sequences of positive integers $(n^0_k)_{k<\omega}$ and $(n^1_k)_{k<\omega}$ such that $f_{\xi_i}(k) = g(n^i_k)$, for every $k < \omega$ and $i \in 2$.*

In what follows, we fix a family $\{f_\xi : \xi \in J_2\}$ so that, for each $\beta < \mathfrak{c}$, $(f_\xi)_{\xi \in J^2_\beta}$ satisfy the three properties stated in the previous result. In this case, it is not hard to show that $\{[f_\xi]_p : \xi \in J_2\} \cup \{[\vec{\mu}]_p : \mu < \mathfrak{c}\}$ is linearly independent in $([\mathfrak{c}]^{<\omega})^\omega / p$ and hence one may repeat the proof of *Lemma 2.3* in [GTW05] to show that[2]:

Lemma 4.1.4 ([GTW05], Lemma 2.3)**.** *Let $p \in \omega^*$ be a selective ultrafilter. For every $E_0 \in [\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$, there are $\{b_i : i < \omega\} \in p$ and $\{E_i : 0 < i < \omega\} \subset [\mathfrak{c}]^{<\omega}$ such that*

[2] To adapt the proof done in [GTW05], we consider $F_0 \doteq E_0$ and, for each $n \geq 0$, $F_{n+1} \doteq F_n \cup \left[\bigcup_{\xi \in F_n \cap J_2} \bigcup_{m \leq n} f_\xi(m)\right]$. The family $\{E_i : 0 < i < \omega\}$ will be a subsequence of $\{F_i : 0 < i < \omega\}$.

1) $E_i \cup \left[\bigcup_{\xi\in E_i\cap J_2} f_\xi(b_i)\right] \subset E_{i+1}$*, for every* $i < \omega$*;*

2) $\{f_\xi(b_i) : \xi \in E_i \cap J_2\} \cup \{\{\mu\} : \mu \in E_i\}$ *is linearly independent, for every* $i < \omega$*.*

Now, we can show the lemma that will replace Lemma 3.1.4. A similar result was also proved in [GTW05] (see *Example 2.4*), but we also show the adapted proof here, for the sake of completeness.

Lemma 4.1.5. *Let* $p \in \omega^*$ *be a selective ultrafilter and* $D \in [\mathfrak{c}]^\omega$ *be such that, for every* $\alpha \in D \cap J_2$, $\bigcup_{n\in\omega} f_\alpha(n) \subset D$*. Then, for each* $E_0 \in [D]^{<\omega} \setminus \{\emptyset\}$*, there exists a homomorphism* $\Phi : [D]^{<\omega} \to 2$ *such that*

(1) $\Phi(\{\xi\}) = p - \lim_{n\in\omega} \Phi(f_\xi(n))$*, for every* $\xi \in D \cap J_2$*.*

(2) $\Phi(E_0) = 1$*.*

Proof. By applying the previous lemma to E_0, we obtain $\{b_i : i < \omega\} \in p$ and $\{E_i : 0 < i < \omega\} \subset [\mathfrak{c}]^{<\omega}$ such that[3]

1) $E_i \cup \left[\bigcup_{\xi\in E_i\cap J_2} f_\xi(b_i)\right] \subset E_{i+1}$, for every $i < \omega$;

2) $\{f_\xi(b_i) : \xi \in E_i \cap J_2\} \cup \{\{\mu\} : \mu \in E_i\}$ is linearly independent, for every $i < \omega$.

Since $\{f_\xi(b_0) : \xi \in E_0 \cap J_2\} \cup \{\{\mu\} : \mu \in E_0\}$ is linearly independent and $E_0 \cup \left[\bigcup_{\xi\in E_0\cap J_2} f_\xi(b_0)\right] \subset E_1$, we may define a homomorphism $\Phi_1 : [E_1]^{<\omega} \to 2$ such that $\Phi_1(E_0) = 1$ and $\Phi_1(f_\xi(b_0)) = \Phi_1(\{\xi\})$ for every $\xi \in J_2 \cap E_0$.

Suppose that, for $0 < i < \omega$, we have defined $\Phi_i : [E_i]^{<\omega} \to 2$ so that $\Phi_i(E_0) = 1$ and $\Phi_i(f_\xi(b_{i-1})) = \Phi_i(\{\xi\})$ for every $\xi \in J_2 \cap E_{i-1}$. Since $\{f_\xi(b_i) : \xi \in E_i \cap J_2\} \cup \{\{\mu\} : \mu \in E_i\}$ is linearly independent and $E_i \cup \left[\bigcup_{\xi\in E_i\cap J_2} f_\xi(b_i)\right] \subset E_{i+1}$, we may define a homomorphism $\Phi_{i+1} : [E_{i+1}]^{<\omega} \to 2$ extending Φ_i so that $\Phi_{i+1}(f_\xi(b_i)) = \Phi_{i+1}(\{\xi\})$ for every $\xi \in E_i \cap J_2$. Thus, such homomorphisms Φ_i exist for every $i > 0$.

Now let $E \doteq \bigcup_{n\in\omega} E_n$ and $\psi \doteq \bigcup_{n>0} \Phi_n : [E]^{<\omega} \to 2$. We may extend ψ to a homomorphism Φ defined on $[D]^{<\omega}$ by putting $\Phi(\{\xi\}) = 0$ if $\xi \in J_1 \cap (D \setminus E)$ and then, recursively,

$$\Phi(\{\xi\}) = p - \lim_{n\in\omega} \Phi(f_\xi(n)),$$

for every $\xi \in J_2 \cap (D \setminus E)$. The homomorphism $\Phi : [D]^{<\omega} \to 2$ satisfy the required properties. Indeed, if $\xi \in J_2 \cap (D \setminus E)$, then (1) follows by the previous equation. On the other hand, if $\xi \in J_2 \cap E$ and $j \in \omega$ is such that $\xi \in E_j$, then $\{b_i : i \geq j\} \subset \{n \in \omega : \Phi(f_\xi(n)) = \Phi(\{\xi\})\}$ and hence

$$\Phi(\{\xi\}) = p - \lim_{n\in\omega} \Phi(f_\xi(n)).$$

□

Next, we will show the mentioned theorem. Since the arguments are analogous to those in the proof of Theorem 4.1.1, we omit some details. As before, a set $A \in [\mathfrak{c}]^\omega$ will be called *suitably closed* if, for every $\xi \in J_2 \cap A$, $\bigcup_{n\in\omega} f_\xi(n) \subset A$.

[3] Note that due to the form of $\{E_i : 0 < i < \omega\}$ sets (see the previous footnote), we have that $\bigcup_{i\in\omega} E_i \subset D$.

Theorem 4.1.6 ([TT22], Theorem 5.4)**.** *If $p \in \omega^*$ is a selective ultrafilter, there exists a Hausdorff selectively p−pseudocompact group which is not countably pracompact.*

Proof. Let $\mathcal{A}$ be the set of all homomorphisms $\sigma : [A]^{<\omega} \to 2$, where A is a suitably closed set, satisfying that

$$\sigma(\{\xi\}) = p - \lim_{n\in\omega} \sigma(f_\xi(n)),$$

for every $\xi \in A \cap J_2$.

Enumerate $\mathcal{A}$ by $\{\sigma_\mu : \omega \leq \mu < \mathfrak{c}\}$, assuming that $\bigcup \mathrm{dom}(\sigma_\mu) \subset \mu$, for each $\omega \leq \mu < \mathfrak{c}$, and also that for each $\sigma \in \mathcal{A}$ and $\alpha < \mathfrak{c}$, there exists $\mu \in I_\alpha$ so that $\sigma_\mu = \sigma$. As before, we shall construct a suitable homomorphism $\overline{\sigma_\mu} : [\mathfrak{c}]^{<\omega} \to 2$, for each $\omega \leq \mu < \mathfrak{c}$.

We consider the same enumeration $\{g_\xi : \xi \in J_1\}$ of all functions $g : S \to 2$ with $S \in [\mathfrak{c}]^{<\omega}$ fixed in Theorem 4.1.1. For each $\omega \leq \mu < \mathfrak{c}$, we define the auxiliary homomorphism $\psi_\mu : [\mathfrak{c}]^{<\omega} \to 2$, extending σ_μ, in the following way. If $\xi < \mathfrak{c}$ is such that $\{\xi\} \in \mathrm{dom}(\sigma_\mu)$, we put $\psi_\mu(\{\xi\}) = \sigma_\mu(\{\xi\})$. Otherwise, we have a few cases to consider: firstly, for every $\xi \in J_1$, we put $\psi_\mu(\{\xi\}) = g_\xi(\mu)$ if $\mu \in \mathrm{dom}(g_\xi)$ and $\psi_\mu(\{\xi\}) = 0$ if $\mu \notin \mathrm{dom}(g_\xi)$; for elements $\xi \in J_2$, we define ψ_μ recursively, by putting

$$\psi_\mu(\{\xi\}) = p - \lim_{n\in\omega} \psi_\mu(f_\xi(n)).$$

It is not hard to see that the homomorphism $\psi_\mu : [\mathfrak{c}]^{<\omega} \to 2$ satisfy the previous equation for every $\xi \in J_2$.

Now, for every $\omega \leq \mu < \mathfrak{c}$ and $\xi < \mathfrak{c}$, we have that $\mu \in I_\alpha$ and $\xi \in J_\beta$ for unique $\alpha, \beta < \mathfrak{c}$, hence we may put

$$\begin{cases} \overline{\sigma_\mu}(\{\xi\}) = 0, & \text{if } F(\alpha,\beta) = 0 \\ \overline{\sigma_\mu}(\{\xi\}) = \psi_\mu(\{\xi\}), & \text{if } F(\alpha,\beta) = 1. \end{cases}$$

As before, in this way we define uniquely a non-trivial homomorphism $\overline{\sigma_\mu} : [\mathfrak{c}]^{<\omega} \to 2$ so that, for every $\xi \in J_2$ and $\omega \leq \mu < \mathfrak{c}$,

$$\overline{\sigma_\mu}(\{\xi\}) = p - \lim_{n\in\omega} \overline{\sigma_\mu}(f_\xi(n)).$$

Let now $\overline{\mathcal{A}} \doteq \{\overline{\sigma_\mu} : \omega \leq \mu < \mathfrak{c}\}$ and τ be the topology on $[\mathfrak{c}]^{<\omega}$ generated by the homomorphisms in $\overline{\mathcal{A}}$. The topological group $([\mathfrak{c}]^{<\omega}, \tau)$ is Hausdorff. Indeed, given $x \in [\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$, let $D \in [\mathfrak{c}]^{\omega}$ be a suitably closed set so that $x \subset D$, and let $\alpha < \mathfrak{c}$ be such that $F(\{\alpha\} \times D)$=1. According to Lemma 4.1.5, there exists $\sigma : [D]^{<\omega} \to 2$, $\sigma \in \mathcal{A}$, such that $\sigma(x) = 1$ and, by construction, there exists $\mu_0 \in I_\alpha$ such that $\sigma_{\mu_0} = \sigma$. Hence, $\overline{\sigma_{\mu_0}}(x) = 1$.

Claim 8. $([\mathfrak{c}]^{<\omega}, \tau)$ *is a selectively p−pseudocompact group.*

Proof of the claim. Let $\{U_n : n \in \omega\}$ be a sequence of nonempty open sets in the group. We proceed the same way as in Claim 6 of Theorem 4.1.1 to construct a sequence of pairwise distinct elements $\{\{\xi_n\} : n \in \omega\}$ such that $\{\xi_n\} \in U_n$ for each $n \in \omega$ and such that, for some fixed $\beta < \mathfrak{c}$, $\xi_n \in J_\beta$ for every $n \in \omega$.

Let $g \in ([J_\beta]^{<\omega})^\omega$ be such that $g(n) = \{\xi_n\}$, for every $n \in \omega$. Since $\{[f_\xi]_p : \xi \in J_\beta^2\} \cup \{[\vec{\mu}]_p : \mu \in J_\beta\}$ is a base for $([J_\beta]^{<\omega})^\omega / p$, there exists $\eta_0, ..., \eta_k \in J_\beta^2$ and $E \in [J_\beta]^{<\omega}$ such that

$$[g]_p = (\triangle_{i \leq k} [f_{\eta_i}]_p) \triangle (\triangle_{\mu \in E} [\vec{\mu}]_p).$$

Hence, there exists $B \in p$ such that, for every $n \in B$,

$$\{\xi_n\} = (\triangle_{i \leq k} f_{\eta_i}(n)) \triangle (\triangle_{\mu \in E} \{\mu\}) = (\triangle_{i \leq k} f_{\eta_i}(n)) \triangle E.$$

Therefore, since for each $i = 0, ..., k$ we have

$$\{\eta_i\} = p - \lim_{n \in \omega} f_{\eta_i}(n)$$

by construction, it follows that

$$(\triangle_{i \leq k} \{\eta_i\}) \triangle E = p - \lim_{n \in \omega} \{\xi_n\},$$

and thus $\{\{\xi_n\} : n \in \omega\}$ has a p–limit. □

Claim 9. *$([\mathfrak{c}]^{<\omega}, \tau)$ is not a countably pracompact group.*

Proof of the claim. It is the same proof done in Claim 7 of Theorem 4.1.1, just changing the use of Lemma 3.1.4 for Lemma 4.1.5. □

□

4.2 Conclusion

In this section we will make some additional comments, and present some open problems and natural directions for further studies on the topic addressed in the chapter.

There are still many open questions regarding the pseudocompact-like properties in topological groups. For instance, [GT15] asks:

Question 4.2.1. *Is there a pseudocompact, non-selectively pseudocompact group which is connected?*

Question 4.2.2. *If an Abelian group admits a pseudocompact group topology, does it admit a selectively pseudocompact group topology?*

Question 4.2.3. *Does every compact group admit a proper dense selectively pseudocompact subgroup?*

In Claim 7 of Theorem 4.1.1, we proved that if $Z \subset ([\mathfrak{c}]^{<\omega}, \tau)$ is a dense subset, then $B \doteq \bigcup_{z \in Z} \mathrm{SUPP}(z)$ is not countable. In particular, this shows that the group we have constructed is not separable. Thus, we ask the following:

Question 4.2.4. *Is there a separable selectively pseudocompact group which is not countably pracompact?*

Chapter 5

A consistent solution to the case $\alpha = \omega$ of the Comfort-like question for countably pracompact groups

This chapter will be devoted to proving the following result, which is in the article [TT23]:

Theorem ([TT23], Theorem 3.1). *Suppose that there are $\mathfrak{c}$ incomparable selective ultrafilters. Then there exists a (Hausdorff) topological group G which has all finite powers countably pracompact and such that G^ω is not countably pracompact.*

5.1 Auxiliary Results

We begin this section by enunciating the **Lemma 3.5** and **Lemma 3.6** of [Tom05a].

Lemma 5.1.1 ([Tom05a], Lemma 3.5). *Let p_0 and p_1 be incomparable selective ultrafilters. Let $\{a^j_k : k \in \omega\} \in p_j$ be a strictly increasing sequence such that $a^j_k > k$ for every $k \in \omega$ and $j \in 2$. Then there exist subsets I_0 and I_1 of ω such that:*

(i1) $\{a^j_k : k \in I_j\} \in p_j$ for each $j \in 2$;

(i2) $\{[k, a^j_k] : j \in 2, k \in I_j\}$ are pairwise disjoint intervals of ω.

As a corollary of the previous lemma, we obtain:

Lemma 5.1.2. *Let $n > 0$ and $\{p_j : j \leq n\}$ be incomparable selective ultrafilters. Let $\{a^j_k : k \in \omega\} \in p_j$ be a strictly increasing sequence such that $a^j_k > k$ for every $k \in \omega$ and $j \leq n$. Then there exists a family $\{I_j : j \leq n\}$ of subsets of ω such that:*

(i1) $\{a^j_k : k \in I_j\} \in p_j$ for each $j \leq n$;

(i2) $\{[k, a^j_k] : j \leq n, k \in I_j\}$ are pairwise disjoint intervals of ω.

Proof. We will show that the lemma is true for each $n > 0$ by induction. The case $n = 1$ is just Lemma 5.1.1.

Suppose that the result is true for a given $n_0 > 0$. We claim that it is also true for $n_0 + 1$. Indeed, let $\{p_j : j \leq n_0 + 1\}$ be incomparable selective ultrafilters and $\{a_k^j : k \in \omega\} \in p_j$ be a strictly increasing sequence such that $a_k^j > k$ for every $k \in \omega$ and $j \leq n_0 + 1$. By hypothesis, there exists a family $\{\tilde{I}_j : j \leq n_0\}$ of subsets of ω so that:

- $\{a_k^j : k \in \tilde{I}_j\} \in p_j$ for each $j \leq n_0$;
- $\{[k, a_k^j] : j \leq n_0, k \in \tilde{I}_j\}$ are pairwise disjoint intervals of ω.

Also, by Lemma 5.1.1, for each $j \leq n_0$ there exist $I_j \subset \tilde{I}_j$ and $K_j \subset \omega$ so that:

- $\{a_k^j : k \in I_j\} \in p_j$ and $\{a_k^{n_0+1} : k \in K_j\} \in p_{n_0+1}$;
- $\{[k, a_k^j] : k \in I_j\} \cup \{[k, a_k^{n_0+1}] : k \in K_j\}$ are pairwise disjoint intervals of ω.

Then, defining $I_{n_0+1} \doteq \bigcap_{j=0}^{n_0} K_j$, we have that $\{I_j : j \leq n_0 + 1\}$ satisfies the hypothesis we want. Therefore, the lemma is true for every $n > 0$. □

The countable version of the previous result is **Lemma 3.6** of [Tom05a]:

Lemma 5.1.3 ([Tom05a], Lemma 3.6)**.** *Let $\{p_j : j \in \omega\}$ be incomparable selective ultrafilters. Let $\{a_k^j : k \in \omega\} \in p_j$ be a strictly increasing sequence such that $a_k^j > k$ for each $k, j \in \omega$. Then there exists a family $\{I_j : j \in \omega\}$ of subsets of ω such that:*

(i1) $\{a_k^j : k \in I_j\} \in p_j$ for each $j \in \omega$;

(i2) $\{[k, a_k^j] : j \in \omega, k \in I_j\}$ are pairwise disjoint intervals of ω.

The following results ensure the existence of the homomorphisms which are necessary to construct the topological groups we want. Their proofs are based on **Lemma 3.7** and **Lemma 4.1** of [Tom05a], and also **Lemma 4.1** of [GT20].

Lemma 5.1.4 ([TT23], Lemma 2.8)**.** *Let:*

- *E be a countable subset of $2^{\mathfrak{c}}$ and $I \subset E$;*
- *$F \subset E$ be a finite subset;*
- *for each $\xi \in I$, $k_\xi \in \omega$;*
- *$\{p_\xi : \xi \in I\}$ be a family of incomparable selective ultrafilters.*
- *for each $\xi \in I$, $g_\xi : \omega \to ([E]^{<\omega})^{k_\xi}$ be a function so that $\{g_\xi^j(m) : j < k_\xi, m \in \omega\}$ is linearly independent;*
- *for each $\xi \in I$, $d_\xi \in ([E]^{<\omega})^{k_\xi}$.*

Then there exist an increasing sequence $\{b_i : i \in \omega\} \subset \omega$, a surjective function $r : \omega \to I$ and a sequence $\{E_i : i \in \omega\}$ of finite subsets of E such that:

a) $F \subset E_0$;

b) $E = \bigcup_{i \in \omega} E_i$;

c) $r(m) \in E_m$ *for each* $m \in \omega$*;*

d) $\bigcup\{d^j_{r(m)} : j < k_{r(m)}\} \subset E_m$*, for each* $m \in \omega$*;*

e) $E_{m+1} \supset \bigcup(\{g^j_\xi(b_m) : \xi \in E_m \cap I, j < k_\xi\}) \cup E_m$*, for each* $m \in \omega$*;*

f) $\{g^j_{r(m)}(b_m) : j < k_{r(m)}\} \cup \{\{\mu\} : \mu \in E_m\}$ *is linearly independent, for each* $m \in \omega$*;*

g) $\{b_i : i \in r^{-1}(\xi)\} \in p_\xi$*, for every* $\xi \in I$*.*

Furthermore, if $\{y_n : n \in \omega\} \subset E$ *is faithfully indexed, then* E_i *can be arranged for each* $i \in \omega$ *so that*

h) $\{n \in \omega : y_n \in E_i\} = 2N_i$*, for some* $N_i \in \omega$*, and* $(N_i)_{i\in\omega}$ *is a strictly increasing sequence.*[1]

Proof. Suppose first that I is infinite. Let $E \doteq \{\xi_n : n \in \omega\}$ be an enumeration and $s : \omega \to \omega$ be a strictly increasing function such that $\{\xi_{s(j)} : j \in \omega\} = I$. We will first define a family $\{F_n : n \in \omega\}$ of finite subsets of E. This family will be used to construct the family $\{E_n : n \in \omega\}$.

Choose $N_0 \in \omega$ so that $\left\{n \in \omega : y_n \in F \cup \{\xi_0\} \cup (\bigcup\{d^j_{\xi_{s(0)}} : j < k_{\xi_{s(0)}}\})\right\} \subset 2N_0$, and define

$$F_0 \doteq \{y_n : n \leq 2N_0\} \cup F \cup \{\xi_0\} \cup (\bigcup\{d^j_{\xi_{s(0)}} : j < k_{\xi_{s(0)}}\}).$$

Suppose that we have defined finite subsets $F_0, ..., F_l \subset E$ so that

1) $\xi_p \in F_p$ for each $0 \leq p \leq l$;
2) $F_{p+1} \supset \bigcup(\{g^j_\beta(m) : m \leq p,\ \beta \in F_p \cap I,\ j < k_\beta\}) \cup F_p$ for each $0 \leq p < l$.
3) $\bigcup\{d^j_{\xi_{s(p)}} : j < k_{\xi_{s(p)}}\} \subset F_p$, for each $0 \leq p \leq l$.
4) $\{n \in \omega : y_n \in F_p\} = 2N_p$, for some $N_p \in \omega$, for each $0 \leq p \leq l$.

Now choose $N_{l+1} > N_l$ so that

$$\left\{n \in \omega : y_n \in \bigcup\left(\{g^j_\beta(m) : m \leq l,\ \beta \in F_l \cap I,\ j < k_\beta\} \cup \{d^j_{\xi_{s(l+1)}} : j < k_{\xi_{s(l+1)}}\}\right) \cup F_l \cup \{\xi_{l+1}\}\right\} \subset 2N_{l+1},$$

and then define

$$F_{l+1} \doteq \{y_n : n \leq 2N_{l+1}\} \cup \bigcup\left(\{g^j_\beta(m) : m \leq l,\ \beta \in F_l \cap I,\ j < k_\beta\} \cup \{d^j_{\xi_{s(l+1)}} : j < k_{\xi_{s(l+1)}}\}\right) \cup F_l \cup \{\xi_{l+1}\}.$$

It is clear that 1), 2), 3) and 4) are also satisfied for $F_0, ..., F_{l+1}$. Then, we may construct recursively a family $\{F_n : n \in \omega\}$ of finite subsets of E satisfying 1)-4) for every $p \in \omega$. We also have that $E = \bigcup_{i\in\omega} F_i$.

For each $\xi \in I$ and $n \in \omega$, let

[1] For every $K \in \omega$, $K \geq 2$, we could also arrange E_i for each $i \in \omega$ so that $\{n \in \omega : y_n \in E_i\} \subset KN_i$, for some $N_i \in \omega$, and $(N_i)_{i\in\omega}$ is strictly increasing. The proof would be analogous.

$$A_n^\xi \doteq \{m \in \omega : \{g_\xi^j(m) : j < k_\xi\} \cup \{\{\mu\} : \mu \in F_n\} \text{ is linearly independent}\}.$$

Since $\{g_\xi^j(m) : j < k_\xi, m \in \omega\}$ is linearly independent and F_n is finite, we have that A_n^ξ is cofinite, and then $A_n^\xi \in p_\xi$, for every $n \in \omega$ and $\xi \in I$. Since selective ultrafilters are $P-$points, for each $\xi \in I$ there exists $A_\xi \in p_\xi$ so that $A_\xi \setminus A_n^\xi$ is finite for every $n \in \omega$.

Now, for each $\xi \in I$, let $v_\xi : \omega \to \omega$ be a strictly increasing function so that $A_\xi \setminus A_n^\xi \subset v_\xi(n)$, for each $n \in \omega$. As every p_ξ is a selective ultrafilter, for each $\xi \in I$ there exists $B_\xi \in p_\xi$ such that

$$B_\xi \cap v_\xi(1) = \emptyset, B_\xi \subset A_\xi \text{ and } |[v_\xi(n) + 1, v_\xi(n+1)] \cap B_\xi| \leq 1, \text{ for each } n \in \omega.$$

Let $\{a_n^\xi : n \in \omega\}$ be the strictly increasing enumeration of B_ξ, for each $\xi \in I$. Notice that $a_n^\xi > v_\xi(n) \geq n$ for each $n \in \omega$ and $\xi \in I$. Thus,

$$a_n^\xi \in A_n^\xi, \text{ for each } \xi \in I \text{ and } n \in \omega,$$

and, by Lemma 5.1.3, there exists a family $\{I_\xi : \xi \in I\}$ of subsets of ω such that:

i1) $\{a_i^\xi : i \in I_\xi\} \in p_\xi$ for each $\xi \in I$;

i2) $\{[i, a_i^\xi] : \xi \in I \text{ and } i \in I_\xi\}$ are pairwise disjoint intervals of ω.

By i2), the sets $\{I_\xi : \xi \in I\}$ are pairwise disjoint. We may also assume without loss of generality that $I_{\xi_{s(k)}} \subset \omega \setminus s(k)$ for every $k \in \omega$. Let $\{i_m : m \in \omega\}$ be the strictly increasing enumeration of $\bigcup_{n\in\omega} I_{\xi_{s(n)}}$ and $r : \omega \to I$ be such that $r(m) = \xi_{s(i)}$ if and only if $i_m \in I_{\xi_{s(i)}}$. Define also $b_m \doteq a_{i_m}^{r(m)}$ and $E_m \doteq F_{i_m}$, for each $m \in \omega$.

Conditions a) and b) are trivially satisfied. Moreover, given $m \in \omega$, if $i_m \in I_{\xi_{s(i)}}$, then $i_m \geq s(i)$, and hence $r(m) \in E_m$. Therefore, conditions c) and d) are satisfied. To check condition e), note that $b_m = a_{i_m}^{r(m)} \leq i_{m+1} - 1$ and $E_m = F_{i_m} \subset F_{i_{m+1}-1}$ for each $m \in \omega$, thus

$$\begin{aligned}
&E_m \cup \bigcup(\{g_\xi^j(b_m) : \xi \in E_m \cap I, j < k_\xi\}) \\
&\quad \subset F_{i_{m+1}-1} \cup \bigcup(\{g_\xi^j(p) : p \leq i_{m+1} - 1,\ \xi \in F_{i_{m+1}-1} \cap I,\ j < k_\xi\}) \\
&\quad \subset F_{i_{m+1}} = E_{m+1}.
\end{aligned}$$

Condition f) is also satisfied, since $b_m = a_{i_m}^{r(m)} \in A_{i_m}^{r(m)}$ for each $m \in \omega$, and hence,

$$\{g_{r(m)}^j(b_m) : j < k_{r(m)}\} \cup \{\{\mu\} : \mu \in F_{i_m}\} \text{ is linearly independent.}$$

To check condition g), simply note that, given $\xi \in I$,

$$\{b_m : m \in r^{-1}(\xi)\} = \{a_i^\xi : i \in I_\xi\} \in p_\xi.$$

Condition h) follows by construction.

If I is finite, the proof is basically the same, replacing the use of Lemma 5.1.3 by Lemma

5.1.2. □

Next, we will present all the homomorphism existence lemmas that we will need for the construction done in this chapter, and also for the construction done in the next chapter.

Lemma 5.1.5 ([TT23], Lemma 2.9). *Let:*

- *Z_0 and Z_1 be disjoint countable subsets of $2^{\mathfrak{c}}$, and $E = Z_0 \uplus Z_1$;*
- *$I_0 \subset Z_0$, $I_1 \subset Z_1$, and $I \doteq I_0 \uplus I_1$;*
- *$\mathcal{F} \subset [E]^{<\omega}$ be a finite linearly independent subset and, for each $f \in \mathcal{F}$, let $n_f \in 2$;*
- *for each $\xi \in I$, $k_\xi \in \omega$;*
- *$\{p_\xi : \xi \in I\}$ be a family of incomparable selective ultrafilters;*
- *for each $\xi \in I$, $\delta_\xi = 0$ if $\xi \in I_0$ and $\delta_\xi = 1$ if $\xi \in I_1$;*
- *for every $\xi \in I$, $g_\xi : \omega \to ([Z_{\delta_\xi}]^{<\omega})^{k_\xi}$ be a function so that $\{g_\xi^j(m) : j < k_\xi, m \in \omega\}$ is linearly independent;*
- *for each $\xi \in I$, $d_\xi \in ([Z_{\delta_\xi}]^{<\omega})^{k_\xi}$;*
- *$\{z_n^0 : n \in \omega\} \subset Z_0$ and $\{z_n^1 : n \in \omega\} \subset Z_1$ be sequences of pairwise distinct elements.*

Then, given $(\alpha_0, \alpha_1) \in 2 \times 2$, there exists a homomorphism $\Phi : [E]^{<\omega} \to 2$ such that:

(i1) *$\Phi(f) = n_f$, for every $f \in \mathcal{F}$;*

(i2) *for every $\xi \in I$,*

$$\left\{n \in \omega : \left(\Phi(g_\xi^0(n)), ..., \Phi(g_\xi^{k_\xi-1}(n))\right) = \left(\Phi(d_\xi^0), ...\Phi(d_\xi^{k_\xi-1})\right)\right\} \in p_\xi;$$

(i3) *$\{n \in \omega : \left(\Phi(\{z_n^0\}), \Phi(\{z_n^1\})\right) = (\alpha_0, \alpha_1)\}$ is finite.*

Proof. Firstly we apply Lemma 5.1.4 using the elements given in the hypothesis, $F = \bigcup \mathcal{F}$, and the following sequence $y : \omega \to E$ for item h): for each $n \in \omega$, write $n = 2q + j$ for the unique $q \in \omega$ and $j \in 2$, and put

$$y_{2q+j} = \begin{cases} z_q^0, & \text{if } j = 0 \\ z_q^1, & \text{if } j = 1. \end{cases}$$

Thus we obtain $\{b_i : i \in \omega\} \subset \omega$, $r : \omega \to I$ and $\{E_m : m \in \omega\} \subset [E]^{<\omega}$ satisfying a)–h).

We shall define auxiliary homomorphisms $\Phi_m : [E_m]^{<\omega} \to 2$ inductively. First, we define $\Phi_0 : [E_0]^{<\omega} \to 2$ so that $\Phi_0(f) = n_f$ for each $f \in \mathcal{F}$. Now, suppose that, for $l \in \omega$, we have defined homomorphisms $\Phi_m : [E_m]^{<\omega} \to 2$ for each $m = 0, ..., l$, so that

(1) Φ_{m+1} extends Φ_m for each $0 \leq m < l$;

(2) for every $0 \leq m < l$,

$$\left(\Phi_{m+1}(g^0_{r(m)}(b_m)), ..., \Phi_{m+1}(g^{k_{r(m)}-1}_{r(m)}(b_m))\right) = \left(\Phi_m(d^0_{r(m)}), ..., \Phi_m(d^{k_{r(m)}-1}_{r(m)})\right);$$

(3) $\left(\Phi_m(\{z^0_n\}), \Phi_m(\{z^1_n\})\right) \neq (\alpha_0, \alpha_1)$ for each $0 < m \leq l$ and $n \in \omega$ so that $z^0_n \in E_m \setminus E_{m-1}$.

We shall prove that we may define $\Phi_{l+1} : [E_{l+1}]^{<\omega} \to 2$ so that $\Phi_0, ..., \Phi_{l+1}$ also satisfy (1), (2) and (3). For this, suppose without loss of generality that $r(l) \in I_0$. By item f) of Lemma 5.1.4, $\{g^j_{r(l)}(b_l) : j < k_{r(l)}\} \cup \{\{\mu\} : \mu \in E_l\}$ is linearly independent, and, by item h), for every $n, m \in \omega$, $z^0_n \in E_m$ if, and only if, $z^1_n \in E_m$. Since $g^j_{r(l)}(b_l) \in [Z_0]^{<\omega}$ for every $j < k_{r(l)}$, and $z^1_n \in Z_1$ for every $n \in \omega$, we conclude that

$$\{\{z^1_n\} : z^0_n \in E_{l+1} \setminus E_l\} \cup \{g^j_{r(l)}(b_l) : j < k_{r(l)}\} \cup \{\{\mu\} : \mu \in E_l\} \tag{†}$$

is linearly independent. Therefore, using items d) and e) of Lemma 5.1.4, we may define $\Phi_{l+1} : [E_{l+1}]^{<\omega} \to 2$ extending Φ_l so that

$$(\Phi_{l+1}(g^0_{r(l)}(b_l)), ..., \Phi_{l+1}(g^{k_{r(l)}-1}_{r(l)}(b_l))) = (\Phi_l(d^0_{r(l)}), ..., \Phi_l(d^{k_{r(l)}-1}_{r(l)}))$$

and

$$\Phi_{l+1}(\{z^1_n\}) \neq \alpha_1 \tag{‡}$$

for each $n \in \omega$ such that $z^0_n \in E_{l+1} \setminus E_l$. Thus, we have that $\Phi_0, ..., \Phi_{l+1}$ also satisfy (1), (2) and (3), and therefore there exists a sequence $(\Phi_m)_{m\in\omega}$ of homomorphisms $\Phi_m : [E_m]^{<\omega} \to 2$ satisfying these properties.

We claim that the homomorphism $\Phi \doteq \bigcup_{n\in\omega} \Phi_n : [E]^{<\omega} \to 2$ satisfies the hypothesis we want. In fact, items (i1) and (i3) are clear from the construction and item (i2) follows from the fact that for every $\xi \in I$,

$$(\Phi(g^0_\xi(b_i)), ..., \Phi(g^{k_\xi-1}_\xi(b_i))) = (\Phi(d^0_\xi), ..., \Phi(d^{k_\xi-1}_\xi)),$$

for each $i \in r^{-1}(\xi)$, and that $\{b_i : i \in r^{-1}(\xi)\} \in p_\xi$, by item g) of Lemma 5.1.4. □

The next result is a stronger version of the previous lemma, and uses it in its proof.

Lemma 5.1.6 ([TT23], Lemma 2.10). *Let:*

- *Z_0 and Z_1 be disjoint countable subsets of $2^{\mathfrak{c}}$, and $E = Z_0 \uplus Z_1$;*
- *$I_0 \subset Z_0$, $I_1 \subset Z_1$, and $I \doteq I_0 \uplus I_1$;*
- *$\mathcal{F} \subset [E]^{<\omega}$ be a linearly independent finite subset and, for each $f \in \mathcal{F}$, let $n_f \in 2$;*
- *for each $\xi \in I$, $k_\xi \in \omega$;*
- *$\{p_\xi : \xi \in I\}$ be a family of incomparable selective ultrafilters;*
- *for each $\xi \in I$, $\delta_\xi = 0$ if $\xi \in I_0$ and $\delta_\xi = 1$ if $\xi \in I_1$;*

- *for every $\xi \in I$, $g_\xi : \omega \to ([Z_{\delta_\xi}]^{<\omega})^{k_\xi}$ be a function so that $\{g_\xi^j(m) : j < k_\xi,\ m \in \omega\}$ is linearly independent;*
- *for each $\xi \in I$, $d_\xi \in ([Z_{\delta_\xi}]^{<\omega})^{k_\xi}$;*
- *$\{y_n^0 : n \in \omega\} \subset [Z_0]^{<\omega}$ and $\{y_n^1 : n \in \omega\} \subset [Z_1]^{<\omega}$ be linearly independent subsets.*

Suppose that $|Z_i \setminus \bigcup\{y_n^i : n \in \omega\}| = \omega$, for each $i \in 2$. Then, given $(\alpha_0, \alpha_1) \in 2 \times 2$, there exists a homomorphism $\Phi : [E]^{<\omega} \to 2$ such that:

(i1) $\Phi(f) = n_f$*, for every* $f \in \mathcal{F}$*;*

(i2) for every $\xi \in I$*,*

$$\left\{n \in \omega : \left(\Phi(g_\xi^0(n)), ..., \Phi(g_\xi^{k_\xi-1}(n))\right) = \left(\Phi(d_\xi^0), ..., \Phi(d_\xi^{k_\xi-1})\right)\right\} \in p_\xi;$$

(i3) $\{n \in \omega : (\Phi(y_n^0), \Phi(y_n^1)) = (\alpha_0, \alpha_1)\}$ *is finite.*

Proof. For each $i \in 2$, let $\{z_n^i : n \in \omega\}$ be an enumeration of $\bigcup\{y_n^i : n \in \omega\}$. Next, we extend $\{y_n^i : n \in \omega\}$ to a basis $\mathcal{B}^i$ of $[Z_i]^{<\omega}$ and also $\{\{z_n^i\} : n \in \omega\}$ to a basis $\mathcal{C}^i$ of $[Z_i]^{<\omega}$, for each $i \in 2$. By assumption, $|\mathcal{C}^i \setminus \{\{z_n^i\} : n \in \omega\}| = |\mathcal{B}^i \setminus \{y_n^i : n \in \omega\}| = \omega$, thus we may consider enumerations $\{e_k^i : k \in \omega\}$ of $\mathcal{C}^i \setminus \{\{z_n^i\} : n \in \omega\}\}$ and $\{f_k^i : k \in \omega\}$ of $\mathcal{B}^i \setminus \{y_n^i : n \in \omega\}$. It is clear that both $\mathcal{B}^0 \cup \mathcal{B}^1$ and $\mathcal{C}^0 \cup \mathcal{C}^1$ are basis of $[E]^{<\omega}$.

Let $\theta : [E]^{<\omega} \to [E]^{<\omega}$ be the isomorphism defined by

$$\theta(y_n^i) = \{z_n^i\},$$

and

$$\theta(f_k^i) = e_k^i,$$

for each $i \in 2$ and $n, k \in \omega$. Note that $\theta|_{[Z_i]^{<\omega}} : [Z_i]^{<\omega} \to [Z_i]^{<\omega}$ is also an isomorphism, for each $i \in 2$.

Let, for every $\xi \in I$, $h_\xi : \omega \to ([Z_{\delta_\xi}]^{<\omega})^{k_\xi}$ be given by $h_\xi^j(n) = \theta(g_\xi^j(n))$ for each $n \in \omega$ and $j < k_\xi$, and $\overline{d}_\xi \in ([Z_{\delta_\xi}]^{<\omega})^{k_\xi}$ be given by $\overline{d}_\xi^j = \theta(d_\xi^j)$, for each $j < k_\xi$.

By Lemma 5.1.5, there exists a homomorphism $\overline{\Phi} : [E]^{<\omega} \to 2$ so that:

1) $\overline{\Phi}(\theta(f)) = n_f$, for every $f \in \mathcal{F}$;

2) For every $\xi \in I$, $\left\{n \in \omega : \left(\overline{\Phi}(h_\xi^0(n)), ..., \overline{\Phi}(h_\xi^{k_\xi-1}(n))\right) = \left(\overline{\Phi}(\overline{d}_\xi^0), ...\overline{\Phi}(\overline{d}_\xi^{k_\xi-1})\right)\right\} \in p_\xi$;

3) $\{n \in \omega : (\overline{\Phi}(\{z_n^0\}), \overline{\Phi}(\{z_n^1\})) = (\alpha_0, \alpha_1)\}$ is finite.

Thus, the homomorphism $\Phi \doteq \overline{\Phi} \circ \theta : [E]^{<\omega} \to 2$ satisfies the hypothesis we want. □

Remark 8. Note that in the statement of the previous lemma, item (i3) can be replaced by the following (stronger) condition, for a given $\alpha \in 2$:

(i3) $\{n \in \omega : \Phi(y_n^0 \triangle y_n^1) = \alpha\}$ is finite.

Indeed, we could replace condition (†) in the proof of Lemma 5.1.5 by the fact that

$$\{\{z_n^0, z_n^1\} : z_n^0 \in E_{l+1} \setminus E_l\} \cup \{g_{r(l)}^j(b_l) : j < k_{r(l)}\} \cup \{\{\mu\} : \mu \in E_l\}$$

is linearly independent, thus in equation (‡) we could choose

$$\Phi_{l+1}(\{z_n^0, z_n^1\}) \neq \alpha$$

for each $n \in \omega$ such that $z_n^0 \in E_{l+1} \setminus E_l$. Then, the proof of Lemma 5.1.6 would remain the same, just replacing the old condition with the new one when required.

The next result is an easy corollary of the previous lemma.

Corollary 5.1.7 ([TT23], Corollary 2.11). *Let:*

- *E be a countable subset of $2^{\mathfrak{c}}$;*
- *$I \subset E$;*
- *$\mathcal{F} \subset [E]^{<\omega}$ be a linearly independent finite subset and, for each $f \in \mathcal{F}$, let $n_f \in 2$;*
- *for each $\xi \in I$, $k_\xi \in \omega$.*
- *$\{p_\xi : \xi \in I\}$ be a family of incomparable selective ultrafilters;*
- *for every $\xi \in I$, $g_\xi : \omega \to ([E]^{<\omega})^{k_\xi}$ be a function so that $\{g_\xi^j(m) : j < k_\xi,\ m \in \omega\}$ is linearly independent;*
- *for every $\xi \in I$, $d_\xi \in ([E]^{<\omega})^{k_\xi}$.*

Then there exists a homomorphism $\Phi : [E]^{<\omega} \to 2$ such that:

(i1) $\Phi(f) = n_f$, *for every* $f \in \mathcal{F}$;

(i2) *For every* $\xi \in I$, $\left\{n \in \omega : \left(\Phi(g_\xi^0(n)), ..., \Phi(g_\xi^{k_\xi-1}(n))\right) = \left(\Phi(d_\xi^0), ...\Phi(d_\xi^{k_\xi-1})\right)\right\} \in p_\xi$.

Although the proof of the following result is similar to the proof of Lemma 5.1.5 and Lemma 5.1.6, we present it here for the sake of completeness.

Lemma 5.1.8 ([TT23], Lemma 2.12). *Let:*

- *E be a countable subset of $2^{\mathfrak{c}}$;*
- *$I \subset E$;*
- *$\mathcal{F} \subset [E]^{<\omega}$ be a linearly independent finite subset and, for each $f \in \mathcal{F}$, let $n_f \in 2$;*
- *$n \in \omega$;*
- *$\{p_\xi : \xi \in I\}$ be a family of incomparable selective ultrafilters;*
- *for every $\xi \in I$, $g_\xi : \omega \to ([E]^{<\omega})^n$ be a function so that $\{g_\xi^j(m) : j < n,\ m \in \omega\}$ is linearly independent;*
- *for every $\xi \in I$, $d_\xi \in ([E]^{<\omega})^n$;*

- $\{y_k^j : k \in \omega, \ j \leq n\} \subset [E]^{<\omega}$ *be a linearly independent subset.*

Suppose that $|E \setminus \bigcup\{y_k^j : k \in \omega, \ j \leq n\}| = \omega$. *Then, given* $(\alpha_0, ..., \alpha_n) \in 2^{n+1}$, *there exists a homomorphism* $\Phi : [E]^{<\omega} \to 2$ *such that:*

(i1) $\Phi(f) = n_f$, *for every* $f \in \mathcal{F}$;

(i2) for every $\xi \in I$, $\left\{k \in \omega : \left(\Phi(g_\xi^0(k)), ..., \Phi(g_\xi^{n-1}(k))\right) = \left(\Phi(d_\xi^0), ...\Phi(d_\xi^{n-1})\right)\right\} \in p_\xi$;

(i3) $\{k \in \omega : (\Phi(y_k^0), ..., \Phi(y_k^n)) = (\alpha_0, ..., \alpha_n)\}$ *is finite.*

Proof. We split the proof in two cases.

Case 1: Suppose that each y_k^j is a singleton, that is, $y_k^j = \{z_k^j\}$, for some $z_k^j \in E$, for every $j \leq n$ and $k \in \omega$.

In this case, we apply Lemma 5.1.4 using the elements of the statement, $F \doteq \bigcup \mathcal{F}$, $k_\xi = n$ for each $\xi \in I$, and the following sequence $w : \omega \to E$ in item h): for each $m \in \omega$, write $m = (n+1)q + j$ for the unique $q \in \omega$ and $j \in (n+1)$, and put $w_m = z_q^j$. Thus, we obtain $\{b_i : i \in \omega\}$, $r : \omega \to I$ and $\{E_m : m \in \omega\} \subset [E]^{<\omega}$ satisfying a)-h) of this lemma.

We shall again define auxiliary homomorphisms $\Phi_m : [E_m]^{<\omega} \to 2$, for each $m \in \omega$, inductively. First, define $\Phi_0 : [E_0]^{<\omega} \to 2$ so that $\Phi_0(f) = n_f$, for each $f \in \mathcal{F}$. Suppose that, for $l \in \omega$, we have defined $\Phi_m : [E_m]^{<\omega} \to 2$, for each $m = 0, ..., l$, satisfying that:

(1) Φ_{m+1} extends Φ_m, for each $0 \leq m < l$;

(2) for every $0 \leq m < l$,

$$\left(\Phi_{m+1}(g_{r(m)}^0(b_m)), ..., \Phi_{m+1}(g_{r(m)}^{n-1}(b_m))\right) = \left(\Phi_m(d_{r(m)}^0), ..., \Phi_m(d_{r(m)}^{n-1})\right);$$

(3) $\left(\Phi_m(\{z_k^0\}), ...\Phi_m(\{z_k^n\})\right) \neq \left(\alpha_0, ..., \alpha_n\right)$ for each $0 < m \leq l$ and $k \in \omega$ so that $z_k^0 \in E_m \setminus E_{m-1}$[2] .

Now, since by construction $\{g_{r(l)}^j(b_l) : j < n\} \cup \{\{\mu\} : \mu \in E_l\}$ is linearly independent, we may apply Lemma 2.2.5 with $A \doteq \{g_{r(l)}^j(b_l) : j < n\}$, $B \doteq \{\{z_k^j\} : z_k^0 \in E_{l+1}, \setminus E_l, \ j \leq n\}$ and $C \doteq \{\{\mu\} : \mu \in E_l\}$ to obtain a subset $B' \subset B$ such that $|B'| \leq |A| = n$ and

$$\{g_{r(l)}^j(b_l) : j < n\} \cup \{\{\mu\} : \mu \in E_l\} \cup (\{\{z_k^j\} : z_k^0 \in E_{l+1} \setminus E_l, \ j \leq n\} \setminus B')$$

is linearly independent. Then, for each $k \in \omega$ so that $z_k^0 \in E_{l+1} \setminus E_l$, there exists $0 \leq j^k \leq n$ such that $z_k^{j^k} \in (\{\{z_k^j\} : z_k^0 \in E_{l+1} \setminus E_l, \ j \leq n\} \setminus B')$. Thus, we may define $\Phi_{l+1} : [E_{l+1}]^{<\omega} \to 2$ extending Φ_l so that

$$\left(\Phi_{l+1}(g_{r(l)}^0(b_l)), ..., \Phi_{l+1}(g_{r(l)}^{n-1}(b_l))\right) = \left(\Phi_l(d_{r(l)}^0), ..., \Phi_l(d_{r(l)}^{n-1})\right)$$

and

$$\Phi_{l+1}(\{z_k^{j^k}\}) \neq \alpha_{j^k},$$

[2] Recall that, by construction, given $k, m \in \omega$, $z_k^j \in E_m$ for **some** $0 \leq j \leq n$ if, and only if, $z_k^j \in E_m$ for **every** $0 \leq j \leq n$.

for every $k \in \omega$ so that $z_k^0 \in E_{l+1} \setminus E_l$. Similarly to the proof of Lemma 5.1.5, we have that $\Phi_0, ..., \Phi_{l+1}$ also satisfy (1)-(3), and therefore there exists a sequence $(\Phi_m)_{m\in\omega}$ of homomorphisms $\Phi_m : [E_m]^{<\omega} \to 2$ satisfying such properties. Again, the homomorphism defined by $\Phi \doteq \bigcup_{n\in\omega} \Phi_n : [E]^{<\omega} \to 2$ satisfies the hypothesis we want.

Case 2: The general case. There is no restriction on elements y_k^j.

Let $\{z_k : k \in \omega\}$ be an enumeration of $\bigcup\{y_k^j : k \in \omega, j \leq n\}$ and $\{z_k^0 : k \in \omega\}, ..., \{z_k^n : k \in \omega\}$ be a partition of $\{z_k : k \in \omega\}$. We extend $\{y_k^j : k \in \omega, j \leq n\}$ to a basis $\mathcal{B}$ of $[E]^{<\omega}$ and also $\{\{z_k^j\} : k \in \omega, j \leq n\}$ to a basis $\mathcal{C}$ of $[E]^{<\omega}$. By assumption, $|\mathcal{B} \setminus \{\{z_k^j\} : k \in \omega, j \leq n\}| = |\mathcal{C} \setminus \{y_k^j : k \in \omega, j \leq n\}| = \omega$, thus consider enumerations $\{e_l : l \in \omega\}$ of $\mathcal{B} \setminus \{\{z_k^j\} : k \in \omega, j \leq n\}$ and $\{f_l : l \in \omega\}$ of $\mathcal{C} \setminus \{y_k^j : k \in \omega, j \leq n\}$.

Let $\theta : [E]^{<\omega} \to [E]^{<\omega}$ be the isomorphism defined by:

$$\theta(y_k^j) = \{z_k^j\},$$

for every $k \in \omega$ and $j \leq n$, and

$$\theta(f_l) = e_l,$$

for every $l \in \omega$.

Let also, for each $\xi \in I$, $h_\xi : \omega \to ([E]^{<\omega})^n$ given by $h_\xi^i(m) = \theta(g_\xi^i(m))$, for every $m \in \omega$ and $i < n$, and $\overline{d_\xi} \in ([E]^{<\omega})^n$ given by $\overline{d_\xi}^i = \theta(d_\xi^i)$, for every $i < n$. By the previous case, there exists a homomorphism $\tilde{\Phi} : [E]^{<\omega} \to 2$ so that:

(1) $\tilde{\Phi}(\theta(f)) = n_f$, for each $f \in \mathcal{F}$;

(2) For every $\xi \in I$, $\left\{m \in \omega : \left(\overline{\Phi}(h_\xi^0(m)), ..., \overline{\Phi}(h_\xi^{n-1}(m))\right) = \left(\overline{\Phi}(\overline{d}_\xi^0), ...\overline{\Phi}(\overline{d}_\xi^{n-1})\right)\right\} \in p_\xi$;

(3) $\{k \in \omega : (\overline{\Phi}(\{z_k^0\}), ..., \overline{\Phi}(\{z_k^n\})) = (\alpha_0, ..., \alpha_n)\}$ is finite.

Thus, the homomorphism $\Phi \doteq \overline{\Phi} \circ \theta : [E]^{<\omega} \to 2$ satisfies the hypothesis we want. □

5.2 The construction of the group

Theorem 5.2.1 ([TT23], Theorem 3.1)**.** *Suppose that there are $\mathfrak{c}$ incomparable selective ultrafilters. Then there exists a (Hausdorff) topological group G which has all finite powers countably pracompact and such that G^ω is not countably pracompact.*

Proof. The required group will be constructed giving a suitable topology to the Boolean group $[\mathfrak{c}]^{<\omega}$, as follows.

Let $(X_n)_{n>0}$ be a partition of $\mathfrak{c}$ so that $|X_n| = \mathfrak{c}$ for every $n > 0$. For each $n > 0$, let $(X_n^j)_{j<2}$ be a partition of X_n so that

- $|X_n^0| = |X_n^1| = \mathfrak{c}$;
- X_n^0 contains only limit ordinals and their next ω elements;
- the initial ω elements of X_n are in X_n^1.

For every $n > 0$, let also

$$Y_n^0 \doteq \{\xi \in X_n^0 : \xi \text{ is a limit ordinal}\},$$

and define the sets $X_0 \doteq \dot\bigcup_{n\in\omega} X_n^0$, $X_1 \doteq \dot\bigcup_{n\in\omega} X_n^1$ and $Y_0 \doteq \dot\bigcup_{n\in\omega} Y_n^0$.

As done in a previous chapter, some technical claims will be enunciated. Their proofs are analogous to proofs of claims already made in previous chapters.

Claim 10. *There exists a family of functions $\{f_\xi : \xi \in Y_0\}$ so that:*

1) *for each $n > 0$, $\{f_\xi : \xi \in Y_n^0\}$ is an enumeration of all the sequences $(x_k)_{k\in\omega}$ of elements in $([X_n]^{<\omega})^n$ so that $\{x_k^j : k \in \omega, j < n\}$ is linearly independent;*
2) *given $n > 0$ and $\xi \in Y_n^0$, f_ξ is a function from ω to $([X_n]^{<\omega})^n$ such that $\bigcup_{j<n}\bigcup_{k\in\omega} f_\xi^j(k) \subset \xi$.*

Countable subsets of $\mathfrak{c}$ which have a suitable property of closure related to this construction will also be called *suitably closed*:

Definition 5.2.2. A set $A \in [\mathfrak{c}]^\omega$ is *suitably closed* if, for each $n > 0$ and $\xi \in Y_n^0$ so that $\{\xi + j : j < n\} \cap A \neq \emptyset$, we have that

$$\{\xi + j : j < n\} \cup \bigcup_{j<n}\bigcup_{k\in\omega} f_\xi^j(k) \subset A.$$

Let $\mathcal{A}$ be the set of all homomorphisms $\sigma : [A]^{<\omega} \to 2$, with $A \in [\mathfrak{c}]^\omega$ suitably closed, satisfying that, for every $n > 0$ and $\xi \in A \cap Y_n^0$,

$$\sigma(\{\xi + j\}) = p_\xi - \lim_{k\in\omega} \sigma(f_\xi^j(k)),$$

for each $j < n$.

Claim 11. *There is an enumeration $\{\sigma_\mu : \mu \in [\omega, \mathfrak{c})\}$ so that, for every $\mu \in [\omega, \mathfrak{c})$, $\bigcup dom(\sigma_\mu) \subset \mu$.*

In what follows, we will construct suitable homomorphisms $\overline{\sigma_\mu} : [\mathfrak{c}]^{<\omega} \to 2$, for each $\mu \in [\omega, \mathfrak{c})$. Note that it is enough to define $\overline{\sigma_\mu}$ in the subset $\{\{\xi\} : \xi \in \mathfrak{c}\}$, since this is a basis for $[\mathfrak{c}]^{<\omega}$.

Claim 12. *For each $n > 0$, there is an enumeration $\{g_\xi : \xi \in X_n^1\}$ of all functions $g : S \to 2$, with $S \in [\mathfrak{c}]^{<\omega}$, so that $dom(g_\xi) \subset \xi$, for every $\xi \in X_n^1$, and that for each $g : S \to 2$ as above, $|\{\xi \in X_n^1 : g_\xi = g\}| = \mathfrak{c}$.*

Let $\mu \in [\omega, \mathfrak{c})$. If $\xi < \mathfrak{c}$ is such that $\{\xi\} \in \mathrm{dom}(\sigma_\mu)$, we put $\overline{\sigma}_\mu(\{\xi\}) = \sigma_\mu(\{\xi\})$. Otherwise, we have a few cases to consider:

1) if $\xi \in X_1$ and $\mu \in \mathrm{dom}(g_\xi)$, we put $\overline{\sigma}_\mu(\{\xi\}) = g_\xi(\mu)$;
2) if $\xi \in X_1$ and $\mu \notin \mathrm{dom}(g_\xi)$, we put $\overline{\sigma}_\mu(\{\xi\}) = 0$;

3) for the remaining elements of X_0, $\overline{\sigma}_\mu$ is defined recursively, by putting

$$\begin{cases} \overline{\sigma_\mu}(\{\xi + j\}) = p_\xi - \lim_{k\in\omega} \overline{\sigma_\mu}(f_\xi^j(k)) & \text{if } \xi \in Y_n^0 \text{ and } j < n; \\ \overline{\sigma}_\mu(\{\xi\}) = 0, & \text{if } \xi \notin \{\alpha + j : \alpha \in Y_n^0, j < n\}. \end{cases}$$

The definition above uniquely extends each σ_μ to a homomorphism $\overline{\sigma_\mu} : [\mathfrak{c}]^{<\omega} \to 2$, which satisfies that, for each $n > 0$, $\xi \in Y_n^0$ and $j < n$,

$$\overline{\sigma_\mu}(\{\xi + j\}) = p_\xi - \lim_{k\in\omega} \overline{\sigma_\mu}(f_\xi^j(k)). \tag{*}$$

Let now $\overline{\mathcal{A}} \doteq \{\overline{\sigma_\mu} : \omega \leq \mu < \mathfrak{c}\}$ and τ be the topology on $[\mathfrak{c}]^{<\omega}$ generated by the homomorphisms in $\overline{\mathcal{A}}$. We call G the topological group $([\mathfrak{c}]^{<\omega}, \tau)$. We claim that G is Hausdorff. Indeed, given $x \in [\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$, let A be a suitably closed set containing x. We may use Corollary 5.1.7 with $E = A$, $I = A \cap Y_0$, $\mathcal{F} = \{x\}$ and, for each $n > 0$ and $\xi \in Y_n^0 \cap A$, $d_\xi = (\{\xi\}, ..., \{\xi + n - 1\})$, to fix a homomorphism $\sigma : [A]^{<\omega} \to 2$ so that $\sigma \in \mathcal{A}$ and $\sigma(x) = 1$. By construction, there exists $\mu \in [\omega, \mathfrak{c})$ so that $\sigma_\mu = \sigma$, and hence $\overline{\sigma_\mu}(x) = 1$.

Claim 13. *For every $n > 0$, G^n is countably pracompact.*

Proof of the claim. Fix $n > 0$. We claim that $([X_n]^{<\omega})^n \subset G^n$ is a witness to the countable pracompactness property in G^n. Indeed, if U is a nonempty open subset of G, we may fix a function $g : S \to 2$, with $S \in [\mathfrak{c}]^{<\omega}$, so that

$$U \supset \bigcap_{\mu\in S} \overline{\sigma_\mu}^{-1}(g(\mu)).$$

Then, by construction, we may choose $\xi \in X_n^1 \cap (\mu, \mathfrak{c})$ so that $g_\xi = g$, and thus $\{\xi\} \in U$, which shows that $[X_n]^{<\omega}$ is dense in G, and therefore $([X_n]^{<\omega})^n$ is dense in G^n.

We shall now prove that every infinite sequence $\{x_k : k \in \omega\}$ of elements in $([X_n]^{<\omega})^n$ has an accumulation point in G^n. In fact, by Lemma 2.2.8, there are:

- elements $d_0, ..., d_{n-1} \in [X_n]^{<\omega}$;
- a subsequence $(x_{k_l})_{l\in\omega}$;
- for some $0 \leq t \leq n$, a sequence $(y_l)_{l\in\omega}$ in $([X_n]^{<\omega})^t$
- for each $0 \leq s < n$, a function $P_s : t \to 2$,

satisfying that

i1) $x_{k_l}^s = \left(\sum_{j=0}^{t-1} P_s(j) y_l^j\right) \triangle d_s$, for every $l \in \omega$ and $0 \leq s < n$.

i2) $\{y_l^j : l \in \omega, 0 \leq j < t\}$ is linearly independent.

By construction, there exists $\xi \in Y_n^0$ so that $f_\xi^j(l) = y_l^j$, for every $l \in \omega$ and $0 \leq j < t$. Since

$$\overline{\sigma_\mu}(\{\xi + j\}) = p_\xi - \lim_{l \in \omega} \overline{\sigma_\mu}(f_\xi^j(l)),$$

for each $\mu \in [\omega, \mathfrak{c})$ and $0 \leq j < n$, we conclude that, for each $0 \leq s < n$,

$$\left(\sum_{j=0}^{t-1} P_s(j)\{\xi + j\} \right) \triangle d_s = p_\xi - \lim_{l \in \omega} x_{k_l}^s,$$

and therefore $\{x_k : k \in \omega\}$ has an accumulation point in G^n.[3] □

Claim 14. *G^ω is not countably pracompact.*

Proof of the claim. Let $Y \subset G^\omega$ be a dense subset. Consider the set $\{U_k^j : k \in \omega, j \in \omega\}$ of non-empty open subsets of G given by Lemma 2.2.10. For each $k \in \omega$, we may choose an element $x_k \in Y \cap \prod_{j \leq k} U_k^j \times G^{\omega \setminus k+1}$, and hence

$$\{x_k^j : j \in \omega, k \geq j\}$$

is linearly independent. In what follows, we will show that there exists a subsequence of $\{x_k : k \in \omega\}$ which does not have an accumulation point in G^ω.

For an element $D \in [\mathfrak{c}]^{<\omega}$, we define

$$\mathrm{SUPP}(D) \doteq \{n > 0 : D \cap X_n \neq \emptyset\}.$$

We will split the proof in two cases.

Case 1: There exists $j \in \omega$ so that $\bigcup_{k \in \omega} \mathrm{SUPP}(x_k^j)$ is infinite.

In this case, we may fix a subsequence $\{x_{k_m}^j : m \in \omega\}$ such that

$$\mathrm{SUPP}(x_{k_m}^j) \setminus \left(\bigcup_{p<m} \mathrm{SUPP}(x_{k_p}^j) \right) \neq \emptyset, \tag{5.1}$$

for every $m \in \omega$. We may also assume that $k_0 \geq j$, and hence $\{x_{k_m}^j : m \in \omega\}$ is linearly independent.

Now we shall show that, for each $x \in G$, x is not an accumulation point of $\{x_{k_m}^j : m \in \omega\}$. First, note that, given $x \in G$, there exists $N_0 \in \omega$ such that, for every $m \geq N_0$,

$$\mathrm{SUPP}(x_{k_m}^j) \setminus \left(\bigcup_{p<m} \mathrm{SUPP}(x_{k_p}^j) \cup \mathrm{SUPP}(x) \right) \neq \emptyset.$$

[3] In fact, the accumulation point obtained even belongs to $([X_n]^{<\omega})^n$ itself. This shows that the subgroup $[X_n]^{<\omega}$ has its nth-power countably compact, for each $n > 0$.

In fact, since $\mathrm{SUPP}(x)$ is finite and (5.1) holds, there cannot be infinitely many elements $x^j_{k_m}$ such that $\mathrm{SUPP}(x^j_{k_m}) \subset \bigcup_{p<m} \mathrm{SUPP}(x^j_{k_p}) \cup \mathrm{SUPP}(x)$.

Let

$$F_0 \doteq \bigcup_{p<N_0} \mathrm{SUPP}(x^j_{k_p}) \cup \mathrm{SUPP}(x)$$

and, for $i > 0$,

$$F_i \doteq \mathrm{SUPP}(x^j_{k_{N_0+i-1}}) \setminus \left(\bigcup_{p<N_0+i-1} \mathrm{SUPP}(x^j_{k_p}) \cup \mathrm{SUPP}(x) \right).$$

Define also, for each $i \in \omega$,

$$D_i \doteq \Big(\bigcup_{m\in\omega} x^j_{k_m} \cup x \Big) \cap \Big(\bigcup_{n\in F_i} X_n \Big),$$

and let A_i be a suitably closed set containing D_i such that $A_i \subset \bigcup_{n\in F_i} X_n$. Since $(F_i)_{i\in\omega}$ is a family of pairwise disjoint sets, we have that $(A_i)_{i\in\omega}$ is also a family of pairwise disjoint sets.

Now we may use Corollary 5.1.7 with: $E = A_0$; $I = A_0 \cap Y_0$; $\mathcal{F} = \{x\}$; and, for every $n > 0$ and $\xi \in Y^0_n \cap A_0$, $d_\xi = (\{\xi\}, ..., \{\xi + n - 1\})$, to fix a homomorphism $\theta_0 : [A_0]^{<\omega} \to 2$ such that $\theta_0 \in \mathcal{A}$ and $\theta_0(x) = 0$[4]. For $l > 0$, suppose that we have constructed a set of homomorphisms $\{\theta_i : i < l\} \subset \mathcal{A}$ such that

i1) $\theta_0(x) = 0$.

i2) θ_i is a homomorphism defined in $\left[\bigcup_{p\le i} A_p\right]^{<\omega}$ taking values in 2, for each $i < l$.

i3) θ_i extends θ_{i-1} for each $0 < i < l$.

i4) $\theta_i(x^j_{k_{N_0+p}}) = 1$ for each $0 < i < l$ and $p = 0, ..., i-1$.

Again by Corollary 5.1.7, we may define a homomorphism $\psi_l : [A_l]^{<\omega} \to 2$ so that $\psi_l \in \mathcal{A}$ and

$$\psi_l\Big(x^j_{k_{N_0+l-1}} \setminus \bigcup_{p<l} D_p\Big) + \theta_{l-1}\Big(x^j_{k_{N_0+l-1}} \cap \bigcup_{p<l} D_p\Big) = 1.$$

Now, since $A_l \cap \bigcup_{i<l} A_i = \emptyset$, we may also define a homomorphism $\theta_l : \left[\bigcup_{p\le l} A_p\right]^{<\omega} \to 2$ extending both θ_{l-1} and ψ_l. By construction, we have that $\theta_l(x) = 0$ and $\theta_l(x^j_{k_{N_0+p}}) = 1$ for every $p = 0, ..., l-1$. Also, it follows that $\theta_l \in \mathcal{A}$, since $\psi_l \in \mathcal{A}$ and $\theta_i \in \mathcal{A}$ for every $i < l$. Therefore, there exists a family of homomorphisms $\{\theta_l : l \in \omega\} \subset \mathcal{A}$ satisfying i1)-i4) for every $l \in \omega$.

Letting $A \doteq \bigcup_{i\in\omega} A_i$ and $\theta \doteq \bigcup_{i\in\omega} \theta_i$, the homomorphism $\theta : [A]^{<\omega} \to 2$ satisfies that $\theta \in \mathcal{A}$, since $\theta_i \in \mathcal{A}$ for each $i \in \omega$. Also, $\theta(x) = 0$ and $\theta(x^j_{k_{N_0+p}}) = 1$ for every $p \in \omega$. By construction, there exists $\mu \in [\omega, \mathfrak{c})$ so that $\theta = \sigma_\mu$, thus $\overline{\sigma_\mu} : [\mathfrak{c}]^{<\omega} \to 2$ satisfies that

[4] If $x = \emptyset$, $\mathcal{F}$ is not linearly independent and thus we cannot use Corollary 5.1.7, but it is clear that we can still find such θ_0.

$\overline{\sigma_\mu}(x^j_{k_m}) = 1$ for each $m \geq N_0$, and $\overline{\sigma_\mu}(x) = 0$. Hence, the element $x \in G$, which was chosen arbitrarily, is not an accumulation point of $\{x^j_{k_m} : m \in \omega\}$. In particular, $\{x_{k_m} : m \in \omega\}$ does not have an accumulation point in G^ω.

Case 2: For every $j \in \omega$, $M_j \doteq \bigcup_{k\in\omega} \mathrm{SUPP}(x^j_k)$ is finite.

In this case, we claim that for each $j \in \omega$ there exists a subsequence $\{k^j_m : m \in \omega\}$ so that, for every $i \leq j$ and $n \in M_i$, either the family $\{x^i_{k^j_m} \cap X_n : m \in \omega\}$ is linearly independent or constant. Indeed, for $j = 0$ and $n_0 \in M_0$, if there exists an infinite subset of $\{x^0_k \cap X_{n_0} : k \in \omega\}$ which is linearly independent, we may fix a subsequence $\{k^{0,0}_m : m \in \omega\}$ so that $\{x^0_{k^{0,0}_m} \cap X_{n_0} : m \in \omega\}$ is linearly independent; otherwise we may fix a subsequence $\{k^{0,0}_m : m \in \omega\}$ so that $\{x^0_{k^{0,0}_m} \cap X_{n_0} : m \in \omega\}$ is constant. Then, if it exists, we may consider another $n_1 \in M_0$ and repeat the process to obtain a subsequence $\{k^{0,1}_m : m \in \omega\}$ which refines $\{k^{0,0}_m : m \in \omega\}$ and satisfies the desired property for n_0 and n_1. Since M_0 is finite, proceeding inductively we may obtain the required subsequence $\{k^0_m : m \in \omega\}$ in the last step. Then, we repeat the process for the next coordinates, always refining the previous subsequence. Now, fix such subsequences $\{k^j_m : m \in \omega\}$, for each $j \in \omega$. We may also suppose that $k^j_0 \geq j$ for each $j \in \omega$.

For each $j \in \omega$, let

$$\overline{M_j} \doteq \{n \in M_j : \{x^j_{k^j_m} \cap X_n : m \in \omega\} \text{ is linearly independent}\}.$$

Note that $\overline{M_j} \neq \emptyset$ for every $j \in \omega$, since $\{x^j_{k^j_m} \cap X_n : m \in \omega, n \in M_j\}$ generates all the elements in the infinite linearly independent set $\{x^j_{k^j_m} : m \in \omega\}$.

Suppose that there exists $j \in \omega$ so that $|\overline{M_j}| > 1$. Fix then $n_0, n_1 \in \overline{M_j}$ distinct. We shall prove that in this case $\{x^j_{k^j_m} : m \in \omega\}$ does not have an accumulation point in G.

For that, consider:

- $x \in G$ chosen arbitrarily;
- $x^0 \doteq x \cap X_{n_0}$, $x^1 \doteq x \cap X_{n_1}$;
- $Z_0 \subset X_{n_0}$ a suitably closed set containing x^0 and $\bigcup\{x^j_{k^j_m} \cap X_{n_0} : m \in \omega\}$, so that $|Z_0 \setminus \bigcup\{x^j_{k^j_m} \cap X_{n_0} : m \in \omega\}| = \omega$;
- $Z_1 \subset X_{n_1}$ a suitably closed set containing x^1 and $\bigcup\{x^j_{k^j_m} \cap X_{n_1} : m \in \omega\}$, so that $|Z_1 \setminus \bigcup\{x^j_{k^j_m} \cap X_{n_1} : m \in \omega\}| = \omega$;
- $\tilde{E} \doteq Z_0 \uplus Z_1$;
- $I_0 \doteq Z_0 \cap Y_0 (= Z_0 \cap Y^0_{n_0})$, $I_1 \doteq Z_1 \cap Y_0 (= Z_1 \cap Y^0_{n_1})$ and $I \doteq I_0 \uplus I_1$;
- for $\xi \in I$,
$$d_\xi = \begin{cases} (\{\xi\}, ..., \{\xi + n_0 - 1\}), & \text{if } \xi \in I_0 \\ (\{\xi\}, ..., \{\xi + n_1 - 1\}), & \text{if } \xi \in I_1. \end{cases}$$

By Lemma 5.1.6 and Remark 8, there exists a homomorphism $\tilde{\Phi} : [\tilde{E}]^{<\omega} \to 2$ such that:

(i1) for every $s \in x^0 \cup x^1$, $\tilde{\Phi}(\{s\}) = 0$;

(i2) for every $\xi \in I$,

$$\tilde{\Phi}(\{\xi + j\}) = \begin{cases} p_\xi - \lim_{k\in\omega} \tilde{\Phi}(f_\xi^j(k)), \text{ for every } j < n_0, & \text{if } \xi \in I_0 \\ p_\xi - \lim_{k\in\omega} \tilde{\Phi}(f_\xi^j(k)), \text{ for every } j < n_1, & \text{if } \xi \in I_1; \end{cases}$$

(i3) $\left\{m \in \omega : \tilde{\Phi}\left(x^j_{k^j_m} \cap (X_{n_0} \cup X_{n_1})\right) = 0\right\}$ is finite.

Now, fix a suitably closed set E containing $\tilde{E}$, x and $x^j_{k^j_m}$, for each $m \in \omega$, so that $E \cap X_{n_0} = Z_0$ and $E \cap X_{n_1} = Z_1$. Consider the homomorphism $\Phi : [E]^{<\omega} \to 2$ so that, for each $\xi \in E$,

$$\Phi(\{\xi\}) = \begin{cases} \tilde{\Phi}(\{\xi\}), & \text{if } \xi \in \tilde{E} \\ 0, & \text{if } \xi \notin \tilde{E}. \end{cases}$$

In particular, for every $z \in [E]^{<\omega}$ so that $z \cap (X_{n_0} \cup X_{n_1}) = \emptyset$, we have that $\Phi(z) = 0$, and for every $z \in [\tilde{E}]^{<\omega}$, $\Phi(z) = \tilde{\Phi}(z)$.

It follows by construction that $\Phi \in \mathcal{A}$. Furthermore,

$$\begin{aligned} \Phi(x) &= \Phi\Big(\big(x \cap (X_{n_0} \cup X_{n_1})\big) \vartriangle \big(x \setminus (X_{n_0} \cup X_{n_1})\big)\Big) \\ &= \Phi\Big((x \cap X_{n_0}) \vartriangle (x \cap X_{n_1})\Big) + \Phi\Big(x \setminus (X_{n_0} \cup X_{n_1})\Big) = \tilde{\Phi}(x^0) + \tilde{\Phi}(x^1) = 0, \end{aligned}$$

and, for every $m \in \omega$,

$$\begin{aligned} \Phi(x^j_{k^j_m}) &= \Phi\Big(\big(x^j_{k^j_m} \cap (X_{n_0} \cup X_{n_1})\big) \vartriangle \big(x^j_{k^j_m} \setminus (X_{n_0} \cup X_{n_1})\big)\Big) \\ &= \Phi\Big(x^j_{k^j_m} \cap (X_{n_0} \cup X_{n_1})\Big) + \Phi\Big(x^j_{k^j_m} \setminus (X_{n_0} \cup X_{n_1})\Big) = \tilde{\Phi}\Big(x^j_{k^j_m} \cap (X_{n_0} \cup X_{n_1})\Big). \end{aligned}$$

Thus,

$$\left\{m \in \omega : \Phi(x^j_{k^j_m}) = \Phi(x)\right\}$$

is finite. Since, by construction, there exists $\mu \in [\omega, \mathfrak{c})$ so that $\Phi = \sigma_\mu$, we conclude that x cannot be an accumulation point of $\{x^j_{k^j_m} : m \in \omega\}$. As the element $x \in G$ was chosen arbitrarily, the sequence $\{x^j_{k^j_m} : m \in \omega\}$ does not have an accumulation point in G. In particular, $\{x_{k^j_m} : m \in \omega\}$ does not have an accumulation point in G^ω.

Therefore, henceforth we may suppose that $|\overline{M_j}| = 1$ for every $j \in \omega$. We have two subcases to consider.

Case 2.1: There are $j_0, j_1 \in \omega$ distinct so that $\overline{M_{j_0}} \cap \overline{M_{j_1}} = \emptyset$.

Suppose that $j_1 > j_0$, and let $n_0 \in \overline{M_{j_0}}, n_1 \in \overline{M_{j_1}}$. We shall show that the sequence $\{(x^{j_0}_{k^{j_1}_m}, x^{j_1}_{k^{j_1}_m}) : m \in \omega\}$ does not have an accumulation point in G^2. For this, consider:

- $(x^0, x^1) \in G^2$ chosen arbitrarily;
- $y^0 \doteq x^{j_1}_{k^{j_1}_0} \cap X_{n_0}$ and $y^1 \doteq x^{j_0}_{k^{j_1}_0} \cap X_{n_1}$[5];
- $Z_0 \subset X_{n_0}$ a suitably closed set containing $(x^0 \cup x^1) \cap X_{n_0}$, y^0 and $\bigcup\{x^{j_0}_{k^{j_1}_m} \cap X_{n_0} : m \in \omega\}$, so that $|Z_0 \setminus \bigcup\{x^{j_0}_{k^{j_1}_m} \cap X_{n_0} : m \in \omega\}| = \omega$;
- $Z_1 \subset X_{n_1}$ a suitably closed set containing $(x^0 \cup x^1) \cap X_{n_1}$, y^1 and $\bigcup\{x^{j_1}_{k^{j_1}_m} \cap X_{n_1} : m \in \omega\}$, so that $|Z_1 \setminus \bigcup\{x^{j_1}_{k^{j_1}_m} \cap X_{n_1} : m \in \omega\}| = \omega$;
- $\tilde{E} \doteq Z_0 \uplus Z_1$;
- $I_0 \doteq Z_0 \cap Y_0 (= Z_0 \cap Y^0_{n_0})$, $I_1 \doteq Z_1 \cap Y_0 (= Z_1 \cap Y^0_{n_1})$ and $I \doteq I_0 \uplus I_1$;
- for $\xi \in I$,
$$d_\xi = \begin{cases} (\{\xi\}, \dots, \{\xi + n_0 - 1\}), & \text{if } \xi \in I_0 \\ (\{\xi\}, \dots, \{\xi + n_1 - 1\}), & \text{if } \xi \in I_1. \end{cases}$$

By Lemma 5.1.6, there exists a homomorphism $\tilde{\Phi} : [\tilde{E}]^{<\omega} \to 2$ such that:

(i1) for every $s \in (x^0 \cup x^1 \cup y^0 \cup y^1) \cap (X_{n_0} \cup X_{n_1})$, $\tilde{\Phi}(\{s\}) = 0$;

(i2) for every $\xi \in I$,
$$\tilde{\Phi}(\{\xi + j\}) = \begin{cases} p_\xi - \lim_{k \in \omega} \tilde{\Phi}(f^j_\xi(k)), \text{ for every } j < n_0, & \text{if } \xi \in I_0 \\ p_\xi - \lim_{k \in \omega} \tilde{\Phi}(f^j_\xi(k)), \text{ for every } j < n_1, & \text{if } \xi \in I_1; \end{cases}$$

(i3) $\left\{ m \in \omega : \left(\tilde{\Phi}(x^{j_0}_{k^{j_1}_m} \cap X_{n_0}), \tilde{\Phi}(x^{j_1}_{k^{j_1}_m} \cap X_{n_1}) \right) = (0, 0) \right\}$ is finite.

Again, fix a suitably closed set E containing $\tilde{E}$, $x^0 \cup x^1$ and $x^{j_0}_{k^{j_1}_m} \cup x^{j_1}_{k^{j_1}_m}$, for each $m \in \omega$, so that $E \cap X_{n_0} = Z_0$ and $E \cap X_{n_1} = Z_1$. Consider the homomorphism $\Phi : [E]^{<\omega} \to 2$ such that, for each $\xi \in E$,
$$\Phi(\{\xi\}) = \begin{cases} \tilde{\Phi}(\{\xi\}), & \text{if } \xi \in \tilde{E} \\ 0, & \text{if } \xi \notin \tilde{E}. \end{cases}$$

It follows by construction that $\Phi \in \mathcal{A}$ and that, for each $i < 2$,

$$\begin{aligned} \Phi(x^i) &= \Phi\Big(\big(x^i \cap (X_{n_0} \cup X_{n_1})\big) \vartriangle \big(x^i \setminus (X_{n_0} \cup X_{n_1})\big) \Big) \\ &= \Phi\Big((x^i \cap X_{n_0}) \vartriangle (x^i \cap X_{n_1}) \Big) + \Phi\Big(x^i \setminus (X_{n_0} \cup X_{n_1}) \Big) = \tilde{\Phi}(x^i \cap X_{n_0}) + \tilde{\Phi}(x^i \cap X_{n_1}) = 0. \end{aligned}$$

[5] Recall that, by construction, the families $\{x^{j_1}_{k^{j_1}_m} \cap X_{n_0} : m \in \omega\}$ and $\{x^{j_0}_{k^{j_1}_m} \cap X_{n_1} : m \in \omega\}$ are constant.

Furthermore, for every $m \in \omega$ and $i < 2$,

$$\begin{aligned}\Phi(x^{j_i}_{k^{j_1}_m}) &= \Phi\Big(\big(x^{j_i}_{k^{j_1}_m} \cap (X_{n_0} \cup X_{n_1})\big) \vartriangle \big(x^{j_i}_{k^{j_1}_m} \setminus (X_{n_0} \cup X_{n_1})\big)\Big)\\ &= \Phi\Big(x^{j_i}_{k^{j_1}_m} \cap (X_{n_0} \cup X_{n_1})\Big) + \Phi\Big(x^{j_i}_{k^{j_1}_m} \setminus (X_{n_0} \cup X_{n_1})\Big)\\ &= \tilde{\Phi}\Big(x^{j_i}_{k^{j_1}_m} \cap X_{n_0}\Big) + \tilde{\Phi}\Big(x^{j_i}_{k^{j_1}_m} \cap X_{n_1}\Big) = \tilde{\Phi}\Big(x^{j_i}_{k^{j_1}_m} \cap X_{n_i}\Big).\end{aligned}$$

Thus,

$$\Big\{m \in \omega \,:\, \Big(\Phi(x^{j_0}_{k^{j_1}_m}), \Phi(x^{j_1}_{k^{j_1}_m})) = (\Phi(x^0), \Phi(x^1)\Big)\Big\}$$

is finite, and therefore $\{(x^{j_0}_{k^{j_1}_m}, x^{j_1}_{k^{j_1}_m}) \,:\, m \in \omega\}$ does not have an accumulation point in G^2. In particular, $\{x_{k^{j_1}_m} \,:\, m \in \omega\}$ does not have an accumulation point in G^ω.

Case 2.2: For every $j_0, j_1 \in \omega$, $\overline{M_{j_0}} \cap \overline{M_{j_1}} \neq \emptyset$.

In this case, there exists $n_0 > 0$ so that $\overline{M_j} = \{n_0\}$ for every $j \in \omega$. To make the notation simpler, from now on we call $\{k_m \,:\, m \in \omega\}$ the sequence $\{k^{n_0}_m \,:\, m \in \omega\}$. By construction, $\{x^i_{k_m} \,:\, m \in \omega, i \leq n_0\}$ is linearly independent and, for each $i \leq n_0$, there exists $c_i \in [\mathfrak{c}]^{<\omega}$ so that $c_i \cap X_{n_0} = \emptyset$ and $x^i_{k_m} = (x^i_{k_m} \cap X_{n_0}) \vartriangle c_i$, for every $m \in \omega$. Thus, there exists $m_0 \in \omega$ such that

$$\{x^i_{k_m} \cap X_{n_0} \,:\, m \geq m_0,\ i \leq n_0\}$$

is linearly independent.

We shall prove that $\{(x^0_{k_m}, ..., x^{n_0}_{k_m}) \,:\, m \in \omega\}$ does not have an accumulation point in G^{n_0+1}. For that, consider:

- $x = (x^0, ..., x^{n_0}) \in G^{n_0+1}$ chosen arbitrarily;
- for each $i = 0, ..., n_0$, $y^i_m = x^i_{k_m} \cap X_{n_0}$ for every $m \geq m_0$.
- $\tilde{E} \subset X_{n_0}$ a suitably closed set containing $(x^0 \cup ... \cup x^{n_0}) \cap X_{n_0}$ and y^i_m, for every $i \leq n_0$ and $m \geq m_0$, so that $|\tilde{E} \setminus \bigcup\{y^i_m \,:\, m \geq m_0,\ i \leq n_0\}| = \omega$;
- $I = \tilde{E} \cap Y_0$ ($=\tilde{E} \cap Y^0_{n_0}$);
- for each $\xi \in I$, $d_\xi = (\{\xi\}, ..., \{\xi + n_0 - 1\})$.

By Lemma 5.1.8, there exists a homomorphism $\tilde{\Phi} \,:\, [\tilde{E}]^{<\omega} \to 2$ such that

(i1) For every $s \in (x^0 \cup ... \cup x^{n_0}) \cap X_{n_0}$, $\tilde{\Phi}(\{s\}) = 0$.

(i2) For every $\xi \in I$ and $j < n_0$,

$$\tilde{\Phi}(\{\xi + j\}) = p_\xi - \lim_{k \in \omega} \tilde{\Phi}(f^j_\xi(k)).$$

(i3) $\{m \geq m_0 \,:\, (\tilde{\Phi}(y^0_m)..., \tilde{\Phi}(y^{n_0}_m)) = (0, ..., 0)\}$ is finite. Note that by construction $\tilde{\Phi}(x^i \cap X_{n_0}) = 0$ for every $i = 0, ..., n_0$.

Consider E a suitably closed set containing $\tilde{E}$, $x^0 \cup ... \cup x^{n_0}$ and $x^0_{k_m} \cup ... \cup x^{n_0}_{k_m}$, for each $m \geq m_0$, so that $E \cap X_{n_0} = \tilde{E}$. Hence, we may define a homomorphism $\Phi : [E]^{<\omega} \to 2$ such that, for each $\xi \in E$,

$$\Phi(\{\xi\}) = \begin{cases} \tilde{\Phi}(\{\xi\}), & \text{if } \xi \in \tilde{E} \\ 0, & \text{if } \xi \notin \tilde{E}. \end{cases}$$

In particular, for every $z \in [E]^{<\omega}$ so that $z \cap X_{n_0} = \emptyset$, we have that $\Phi(z) = 0$. Moreover, for every $z \in [\tilde{E}]^{<\omega}$, $\Phi(z) = \tilde{\Phi}(z)$. Similarly to **Case 2.1**, it follows by construction that $\Phi \in \mathcal{A}$ and that

$$\{m \geq m_0 : (\Phi(x^0_{k_m}), ..., \Phi(x^{n_0}_{k_m})) = (\Phi(x^0), ..., \Phi(x^{n_0}))\} \text{ is finite.}$$

Again, we conclude that x cannot be an accumulation point of $\{(x^0_{k_m}, ..., x^{n_0}_{k_m}) : m \in \omega\}$.

Therefore, in any case, we showed that there exists a subsequence of $\{x_k : k \in \omega\}$ which does not have an accumulation point in G^ω, and thus the group is not countably pracompact.

□

□

As a corollary of the proof of Theorem 5.2.1, we obtain:

Corollary 5.2.3. *Suppose that there are $\mathfrak{c}$ incomparable selective ultrafilters. Then, for each $n \in \omega$, $n > 0$, there exists a (Hausdorff) topological group whose nth power is countably compact and the (n+1)th power is not selectively pseudocompact.*

Proof. With the same notation of the proof of Theorem 5.2.1, for each $n > 0$, we choose the topological subgroup $H \doteq [X_n]^{<\omega} \subset G$. As already mentioned in a footnote, H^n is countably compact. Also, using Lemma 5.1.8 similarly to what was done in **Case 2.2**, one can show that every sequence $(x^0_k, ..., x^n_k)_{k\in\omega}$ in H^{n+1} so that $\{x^j_k : k \in \omega, j \leq n\}$ is linearly independent does not have an accumulation point in H^{n+1}. Then, it is enough to choose a sequence of nonempty open sets $\{(U^0_k \times ... \times U^n_k) : k \in \omega\} \subset H^{n+1}$, with $\{U^j_k : k \in \omega, j \leq n\}$ as in Lemma 2.2.10, to prove that H^{n+1} is not selectively pseudocompact. □

Recall that in [GT18] the authors proved the same result using CH.

As the next chapter also addresses the theme of productivity of pseudocompact-like properties in topological groups, we will leave a more general conclusion of the subject to be done there.

Chapter 6

Consistent solutions to the Comfort-like question for countably pracompact groups in the case of infinite successor cardinals

This chapter will be devoted to proving the following result, which is in the article [TT23]:

Theorem ([TT23], Theorem 4.1)**.** *Suppose that there are $2^\mathfrak{c}$ incomparable selective ultrafilters. Let $\kappa \leq 2^\mathfrak{c}$ be an infinite cardinal. Then there exists a (Hausdorff) topological group G such that G^κ is countably pracompact and G^{κ^+} is not countably pracompact.*

This chapter will not have a specific section for auxiliary results since the lemmas that we will need in the proof of the theorem above were already presented in the previous chapter.

6.1 The construction of the group

Theorem 6.1.1 ([TT23], Theorem 4.1)**.** *Suppose that there are $2^\mathfrak{c}$ incomparable selective ultrafilters. Let $\kappa \leq 2^\mathfrak{c}$ be an infinite cardinal. Then there exists a (Hausdorff) topological group G such that G^κ is countably pracompact and G^{κ^+} is not countably pracompact.*

Proof. The required group will be constructed giving a suitable topology to the Boolean group $[2^\mathfrak{c}]^{<\omega}$.

Let $\{X_\gamma : \gamma < \kappa\}$ be a partition of $2^\mathfrak{c}$ so that $|X_\gamma| = 2^\mathfrak{c}$ for every $\gamma < \kappa$. For each $\gamma < \kappa$, we enumerate X_γ in strictly increasing order as $\{x^\gamma_\beta : \beta < 2^\mathfrak{c}\}$ (in this case, it is clear that, for every $\gamma < \kappa$ and $\beta < 2^\mathfrak{c}$, $\beta \leq x^\gamma_\beta$). Let also:

- $\{J_0, J_1\}$ be a partition of $2^\mathfrak{c}$ so that $|J_0| = |J_1| = 2^\mathfrak{c}$ and that $\omega \subset J_1$;

- $X_\gamma^0 \doteq \{x_\beta^\gamma : \beta \in J_0\}$ and $X_\gamma^1 \doteq \{x_\beta^\gamma : \beta \in J_1\}$, for each $\gamma < \kappa$;
- $X_0 \doteq \bigcup_{\gamma<\kappa} X_\gamma^0$ and $X_1 \doteq \bigcup_{\gamma<\kappa} X_\gamma^1$;
- $\mathcal{P} \doteq \{p_\xi : \xi \in J_0\}$ be a family of incomparable selective ultrafilters, which exists by hypothesis.

As has been done in previous chapters, we will enunciate some technical claims. Their proofs are analogous to proofs of claims already made in previous chapters

Claim 15. *There is an enumeration $\{I_\alpha : \alpha \in J_0\}$ of all injective sequences of $2^\mathfrak{c}$ so that, for every $\alpha \in J_0$, $rng(I_\alpha) \subset \alpha$.*

Finally, for each $\alpha \in J_0$ and $\gamma < \kappa$, we define the function $f_\alpha^\gamma : \omega \to [X_\gamma]^{<\omega}$ as

$$f_\alpha^\gamma(l) = \{x_{I_\alpha(l)}^\gamma\},$$

for every $l \in \omega$. Note that, for each $\alpha \in J_0$ and $\gamma < \kappa$, $\operatorname{rng}(f_\alpha^\gamma) \subset [x_\alpha^\gamma]^{<\omega}$.

Next we define which are the suitably closed sets of this construction.

Definition 6.1.2. A set $A \in [2^\mathfrak{c}]^\omega$ is suitably closed if, for every $\gamma < \kappa$ and $\beta \in J_0$, if $x_\beta^\gamma \in A$, then $\bigcup_{l\in\omega} f_\beta^\gamma(l) \subset A$.

Let $\mathcal{A}$ be the set of all homomorphisms $\sigma : [A]^{<\omega} \to 2$, with A suitably closed, such that

$$\sigma(\{x_\beta^\gamma\}) = p_\beta - \lim_{l\in\omega} \sigma(f_\beta^\gamma(l)),$$

for every $\gamma < \kappa$ and $\beta \in J_0$ satisfying that $x_\beta^\gamma \in A$.

Claim 16. *There is an enumeration $\{\sigma_\mu : \mu \in [\omega, 2^\mathfrak{c})\}$ of $\mathcal{A}$ so that $\bigcup dom(\sigma_\mu) \subset \mu$, for every $\mu \in [\omega, 2^\mathfrak{c})$.*

We will properly extend each homomorphism σ_μ to a homomorphism $\overline{\sigma_\mu}$ defined in $[2^\mathfrak{c}]^{<\omega}$. For this purpose, we will also need the following claim.

Claim 17. *There is an enumeration $\{b_\beta : \beta \in J_1\}$ of the set*

$$\mathcal{B} \doteq \{\{(\gamma_0, g_0), ..., (\gamma_k, g_k)\} : k \in \omega, |\{\gamma_0, ..., \gamma_k\}| = k + 1, \textit{ and, for each } i = 0, ..., k, \gamma_i < \kappa, \textit{ and } g_i : S_i \to 2, \textit{ for some } S_i \in [2^\mathfrak{c}]^{<\omega}\}$$

so that for every $\beta \in J_1$, $b_\beta = \{(\gamma_0, g_0), ..., (\gamma_k, g_k)\}$ is such that $\bigcup_{i=0}^k dom(g_i) \subset \beta$.

Given $\mu \in [\omega, 2^\mathfrak{c})$, if $\xi < 2^\mathfrak{c}$ is such that $\{\xi\} \in \operatorname{dom}(\sigma_\mu)$, we put $\overline{\sigma_\mu}(\{\xi\}) = \sigma_\mu(\{\xi\})$. Otherwise, we have a few cases to consider. Firstly, we define the homomorphism in the remaining elements of X_γ^1, for each $\gamma < \kappa$, as described in the next paragraph.

Let $\gamma < \kappa$ and $\xi \in X_\gamma^1$ be so that $\{\xi\} \notin \operatorname{dom}(\sigma_\mu)$. Let also $\beta \in J_1$ be the element such that $\xi = x_\beta^\gamma$. Now,

- if there exists a function $g : S \to 2$, $S \in [2^{\mathfrak{c}}]^{<\omega}$, so that $(\gamma, g) \in b_\beta$ and $\mu \in \mathrm{dom}(g)$, we put $\overline{\sigma_\mu}(\{\xi\}) = g(\mu)$;
- otherwise, we put $\overline{\sigma_\mu}(\{\xi\}) = 0$.

Finally, in the remaining elements of X^0_γ, for each $\gamma < \kappa$, we define $\overline{\sigma_\mu}$ recursively, by putting

$$\overline{\sigma_\mu}(\{x^\gamma_\beta\}) = p_\beta - \lim_{l \in \omega} \overline{\sigma_\mu}(f^\gamma_\beta(l)),$$

for each $\beta \in J_0$.

Now we define $\overline{\mathcal{A}} \doteq \{\overline{\sigma_\mu} : \mu \in [\omega, 2^{\mathfrak{c}})\}$. It is clear by the construction that, for each $\mu \in [\omega, 2^{\mathfrak{c}})$,

$$\overline{\sigma_\mu}(\{x^\gamma_\beta\}) = p_\beta - \lim_{l \in \omega} \overline{\sigma_\mu}(f^\gamma_\beta(l)),$$

for every $\gamma < \kappa$ and $\beta \in J_0$. Let G be the group $[2^{\mathfrak{c}}]^{<\omega}$ endowed with the topology generated by the homomorphisms in $\overline{\mathcal{A}}$.

Given $x \in G$, we define, similarly as before,

$$\mathrm{SUPP}(x) = \{\gamma < \kappa : x \cap X_\gamma \neq \emptyset\}.$$

We claim that G is Hausdorff. Indeed, let $x \in [2^{\mathfrak{c}}]^{<\omega} \setminus \{\emptyset\}$ and, given $\gamma \in \mathrm{SUPP}(x)$, $z = x \cap X_\gamma$. Let also $A_0 \subset X_\gamma$ be a suitably closed set containing z. In order to use Corollary 5.1.7, consider:

- $E = A_0$;
- $I = A_0 \cap X^0_\gamma$;
- $\mathcal{F} = \{z\}$;
- $\{q_\xi : \xi \in I\} \subset \mathcal{P}$ so that, for each $\xi \doteq x^\gamma_\beta \in I$, $q_\xi \doteq p_\beta$;

and, for each $\xi \doteq x^\gamma_\beta \in I$,

- $k_\xi = 1$;
- $g_\xi = f^\gamma_\beta$;
- $d_\xi = \{x^\gamma_\beta\}$.

By Corollary 5.1.7, we may fix a homomorphism $\sigma_0 : [A_0]^{<\omega} \to 2$ so that $\sigma_0 \in \mathcal{A}$ and $\sigma_0(z) = 1$. Now, let A be a suitably closed set containing x so that $A \cap X_\gamma = A_0$ and $\sigma : [A]^{<\omega} \to 2$ be a homomorphism so that

$$\sigma(\{\xi\}) = \begin{cases} \sigma_0(\{\xi\}), & \text{if } \xi \in A_0 \\ 0, & \text{if } \xi \notin A_0. \end{cases}$$

Then, $\sigma \in \mathcal{A}$ and $\sigma(x) = \sigma\big((x \cap X_\gamma) \triangle (x \setminus X_\gamma)\big) = 1$. By construction, there exists $\mu \in [\omega, 2^{\mathfrak{c}})$ so that $\sigma_\mu = \sigma$, and hence $\overline{\sigma_\mu}(x) = 1$.

Claim 18. *G^κ is countably pracompact.*

Proof of the claim. We claim that $\{(\{x_\beta^\gamma\})_{\gamma<\kappa} : \beta < 2^{\mathfrak{c}}\} \subset G^\kappa$ is a dense subset for which every sequence has an accumulation point in G^κ.

For $k \in \omega$, let $\{\gamma_0, ..., \gamma_k\} \subset \kappa$ be a finite set of size $k+1$ and, for each $i \in \{0, ..., k\}$, let

- $\mu_0^i, ..., \mu_{j_i}^i \in [\omega, 2^{\mathfrak{c}})$, for some $j_i \in \omega$;
- $g_i : \{\mu_0^i, ..., \mu_{j_i}^i\} \to 2$ be a function.

We shall prove that, if

$$\bigcap_{p=0}^{j_i} (\overline{\sigma_{\mu_p^i}})^{-1}(g_i(\mu_p^i)) \neq \emptyset,$$

for every $i = 0, ..., k$, then there exists $\beta_0 \in J_1$ so that $\{x_{\beta_0}^{\gamma_i}\} \in \bigcap_{p=0}^{j_i} (\overline{\sigma_{\mu_p^i}})^{-1}(g_i(\mu_p^i))$ for each $i = 0, ..., k$. For that, let $\beta_0 \in J_1$ be so that

$$\{(\gamma_0, g_0), ..., (\gamma_k, g_k)\} = b_{\beta_0}.$$

Since, by construction, $\bigcup \mathrm{dom}(\sigma_\mu) \subset \mu$ for every $\mu \in [\omega, 2^{\mathfrak{c}})$, $\bigcup_{i=0}^k \mathrm{dom}(g_i) \subset \beta_0$ and $\beta_0 \leq x_{\beta_0}^\gamma$ for every $\gamma < \kappa$, it follows that $\overline{\sigma_{\mu_p^i}}(\{x_{\beta_0}^{\gamma_i}\}) = g_i(\mu_p^i)$ for each $i = 0, ..., k$ and $p = 0, ..., j_i$, as we wanted.

Furthermore, given an injective sequence $I_\alpha : \omega \to 2^{\mathfrak{c}}$, for some $\alpha \in J_0$, we claim that $\{(\{x_{I_\alpha(l)}^\gamma\})_{\gamma<\kappa} : l \in \omega\}$ has $(\{x_\alpha^\gamma\})_{\gamma<\kappa}$ as accumulation point. Indeed, for every $\mu \in [\omega, 2^{\mathfrak{c}})$, by construction,

$$\overline{\sigma_\mu}(\{x_\alpha^\gamma\}) = p_\alpha - \lim_{l\in\omega} \overline{\sigma_\mu}(\{x_{I_\alpha(l)}^\gamma\}),$$

for each $\gamma < \kappa$. □

Claim 19. *G^{κ^+} is not countably pracompact.*

Proof of the claim. Since the proof of this claim is similar to the proof of Claim 14 of Theorem 5.2.1, we omit the details of some arguments.

Let $Z \subset G^{\kappa^+}$ be a dense subset. We shall show that there exists a sequence in Z that does not have an accumulation point in G^{κ^+}. We will again split the proof of this claim in two cases.

Case 1: There exists $j \in \kappa^+$ so that $\bigcup_{z\in Z} \mathrm{SUPP}(z^j)$ is infinite.

In this case, we may fix a sequence $\{z_m : m \in \omega\} \subset Z$ so that

$$\mathrm{SUPP}(z_m^j) \setminus \bigcup_{p<m} \mathrm{SUPP}(z_p^j) \neq \emptyset,$$

for every $m \in \omega$. We shall show that, for a given $y \in G$, y is not an accumulation point of $\{z_m^j : m \in \omega\}$. In particular, this shows that $\{z_m : m \in \omega\}$ does not have an accumulation point in G^{κ^+}.

Remark 9. Although the arguments are analogous to those in **Case 1** of Claim 14, as we are about to see, there is another technical complication in this case. We can only guarantee the validity of Corollary 5.1.7 for suitably closed sets A so that $A \subset X_\gamma$, for some $\gamma < \kappa$. In fact, while the mapping $\xi \in X_0 \cap A \to q_\xi \in \mathcal{P}$ has to be injective[1], we wish to map x^γ_β, for $\beta \in J_0$ and $\gamma < \kappa$, to p_β.

Let:

1) $M_0 \in \omega$ be such that, for every $m \geq M_0$,

$$\mathrm{SUPP}(z^j_m) \setminus \left(\bigcup_{p<m} \mathrm{SUPP}(z^j_p) \cup \mathrm{SUPP}(y) \right) \neq \emptyset;$$

2) $F_0 \doteq \bigcup_{p<M_0} \mathrm{SUPP}(z^j_p) \cup \mathrm{SUPP}(y)$;

3) for each $i > 0$,

$$F_i \doteq \mathrm{SUPP}(z^j_{M_0+i-1}) \setminus \left(\bigcup_{p<M_0+i-1} \mathrm{SUPP}(z^j_p) \cup \mathrm{SUPP}(y) \right);$$

4) for each $i \in \omega$,

$$D_i \doteq \left(\bigcup_{m\in\omega} z^j_m \cup y \right) \cap \left(\bigcup_{\gamma\in F_i} X_\gamma \right);$$

5) A_i be a suitably closed set containing D_i such that $A_i \subset \bigcup_{\gamma\in F_i} X_\gamma$;

6) for each $i \in \omega$, $\gamma_i \in F_i$ be arbitrarily chosen;

7) for each $i \in \omega$, $A^0_i \subset X_{\gamma_i}$ be a suitably closed set containing $A_i \cap X_{\gamma_i}$.

In order to use Corollary 5.1.7, consider

- $E = A^0_0$;
- $I = A^0_0 \cap X^0_{\gamma_0}$;
- $\{q_\xi : \xi \in I\} \subset \mathcal{P}$ so that, for each $\xi \doteq x^{\gamma_0}_\beta \in I$, $q_\xi \doteq p_\beta$;

and, for each $\xi \doteq x^{\gamma_0}_\beta \in I$,

- $k_\xi = 1$;
- $g_\xi = f^{\gamma_0}_\beta$;
- $d_\xi = \{x^{\gamma_0}_\beta\}$.

[1] Indeed, $X_0 \cap A$ will be the set I, in the notation of Corollary 5.1.7, and then $(q_\xi)_{\xi\in I}$ has to be a family of incomparable selective ultrafilters.

By Corollary 5.1.7, we may ensure the existence of a homomorphism $\tilde{\theta}_0 : [A_0^0]^{<\omega} \to 2$ such that $\tilde{\theta}_0 \in \mathcal{A}$ and $\tilde{\theta}_0(y \cap X_{\gamma_0}) = 0$. Then, we define $\theta_0 : [A_0]^{<\omega} \to 2$ so that, for every $\xi \in A_0$,

$$\theta_0(\{\xi\}) = \begin{cases} \tilde{\theta}_0(\{\xi\}), & \text{if } \xi \in A_0^0 \\ 0, & \text{if } \xi \notin A_0^0. \end{cases}$$

Note that, in this case, we still have $\theta_0 \in \mathcal{A}$, and also

$$\theta_0(y) = \theta_0(y \cap X_{\gamma_0}) + \theta_0(y \setminus X_{\gamma_0}) = \tilde{\theta}_0(y \cap X_{\gamma_0}) = 0.$$

Suppose that we have constructed a set of homomorphisms $\{\theta_i : i < l\} \subset \mathcal{A}$, for $l > 0$, such that:

i1) θ_i is a homomorphism defined in $\left[\bigcup_{p \le i} A_p\right]^{<\omega}$ taking values in 2, for each $i < l$;

i2) $\theta_0(y) = 0$;

i3) θ_i extends θ_{i-1} for each $0 < i < l$;

i4) $\theta_i(z^j_{M_0+p}) = 1$ for each $0 < i < l$ and $p = 0, ..., i-1$.

Again, in order to use Corollary 5.1.7, consider:

- $E = A_l^0$;
- $I = A_l^0 \cap X_{\gamma_l}^0$;
- $\{q_\xi : \xi \in I\} \subset \mathcal{P}$ so that, for each $\xi \doteq x_\beta^{\gamma_l} \in I$, $q_\xi \doteq p_\beta$;

and, for each $\xi \doteq x_\beta^{\gamma_l} \in I$,

- $k_\xi = 1$;
- $g_\xi = f_\beta^{\gamma_l}$;
- $d_\xi = \{x_\beta^{\gamma_l}\}$.

By Corollary 5.1.7, we may ensure the existence of a homomorphism $\tilde{\psi} : [A_l^0]^{<\omega} \to 2$ so that $\tilde{\psi} \in \mathcal{A}$ and

$$\tilde{\psi}\left(z^j_{M_0+l-1} \cap X_{\gamma_l}\right) + \theta_{l-1}\left(z^j_{M_0+l-1} \setminus \bigcup_{\gamma \in F_l} X_\gamma\right) = 1.$$

Then, we define $\psi : [A_l]^{<\omega} \to 2$ so that, for every $\xi \in A_l$,

$$\psi(\{\xi\}) = \begin{cases} \tilde{\psi}(\{\xi\}), & \text{if } \xi \in A_l^0 \\ 0, & \text{if } \xi \notin A_l^0. \end{cases}$$

Let $\theta_l : \left[\bigcup_{p \le l} A_p\right]^{<\omega} \to 2$ be a homomorphism extending both θ_{l-1} and ψ. By construction, we have that $\theta_l(y) = 0$, $\theta_l \in \mathcal{A}$, and also that

$$\begin{aligned}\theta_l(z^j_{M_0+l-1}) &= \theta_l\Big(z^j_{M_0+l-1} \cap \bigcup_{\gamma\in F_l} X_\gamma\Big) + \theta_l\Big(z^j_{M_0+l-1} \setminus \bigcup_{\gamma\in F_l} X_\gamma\Big)\\ &= \tilde{\psi}\Big(z^j_{M_0+l-1} \cap X_{\gamma_l}\Big) + \psi\Big(z^j_{M_0+l-1} \cap \bigcup_{\gamma\in F_l\setminus\{\gamma_l\}} X_\gamma\Big) + \theta_{l-1}\Big(z^j_{M_0+l-1} \setminus \bigcup_{\gamma\in F_l} X_\gamma\Big)\\ &= \tilde{\psi}\Big(z^j_{M_0+l-1} \cap X_{\gamma_l}\Big) + \theta_{l-1}\Big(z^j_{M_0+l-1} \setminus \bigcup_{\gamma\in F_l} X_\gamma\Big) = 1.\end{aligned}$$

Moreover, it follows by construction that $\theta_l(z^j_{M_0+p}) = \theta_{l-1}(z^j_{M_0+p}) = 1$ for each $0 \le p < l-1$. Therefore, there exists a family of homomorphisms $\{\theta_i : i \in \omega\} \subset \mathcal{A}$ satisfying i1)-i4) for every $l \in \omega$.

Letting $A \doteq \bigcup_{i\in\omega} A_i$, the homomorphism $\theta \doteq \bigcup_{i\in\omega} \theta_i : [A]^{<\omega} \to 2$, satisfies that:

- $\theta \in \mathcal{A}$;
- $\theta(y) = 0$;
- $\theta(z^j_{M_0+p}) = 1$ for every $p \in \omega$.

By construction, there exists $\mu \in [\omega, 2^{\mathfrak{c}})$ so that $\theta = \sigma_\mu$, thus $\overline{\sigma_\mu} : [2^{\mathfrak{c}}]^{<\omega} \to 2$ satisfies that $\overline{\sigma_\mu}(z^j_m) = 1$ for each $m \ge M_0$, and $\overline{\sigma_\mu}(y) = 0$. Hence, $y \in G$ is not an accumulation point of $\{z^j_m : m \in \omega\}$.

Case 2: For every $j \in \kappa^+$, $M_j \doteq \bigcup_{z\in Z} \mathrm{SUPP}(z^j)$ is finite.

Since, in this case,

$$\bigcup_{F\in[\kappa]^{<\omega}} \Big\{ j \in \kappa^+ : M_j = F \Big\} = \kappa^+,$$

there exists $F_0 \in [\kappa]^{<\omega}$ so that $N \doteq \Big\{ j \in \kappa^+ : M_j = F_0 \Big\}$ is infinite. Choose $N_0 \subset N$ so that $|N_0| = \omega$, and let $\{j_i : i \in \omega\}$ be an enumeration of N_0.

Now, consider the set $\{U^i_k : k \in \omega, i \in \omega\}$ of nonempty open subsets of G given by Lemma 2.2.10. For each $k \in \omega$, we may choose an element $z_k \in Z \cap \prod_{i\le k} U^i_k \times G^{\kappa^+\setminus\{j_0,\ldots,j_k\}}$. Similarly to what was done in **Case 2** of Claim 14, we can fix a subsequence $\{k^i_m : m \in \omega\}$, for each $i \in \omega$, so that:

- $\{k^{i+1}_m : m \in \omega\}$ refines $\{k^i_m : m \in \omega\}$, for each $i \in \omega$;
- for every $i \in \omega$, $p \le i$ and $\gamma \in F_0$, either the family $\{z^{j_p}_{k^i_m} \cap X_\gamma : m \in \omega\}$ is linearly independent or constant;
- $k^i_0 \ge i$, for each $i \in \omega$.

Notice at this point that

$$\Big\{ z^{j_i}_{k^i_m} : i \in \omega, m \in \omega \Big\}$$

is linearly independent. For each $i \in \omega$, let

$$\overline{M_{j_i}} \doteq \left\{\gamma \in F_0 \,:\, \{z^{j_i}_{k^i_m} \cap X_\gamma \,:\, m \in \omega\} \text{ is linearly independent}\right\}.$$

Again, we have that $\overline{M_{j_i}} \neq \emptyset$ for every $i \in \omega$. Then, choose $a, b \in \omega$, $b > a$, so that $M \doteq \overline{M_{j_a}} = \overline{M_{j_b}}$. In this case, there exist $c_a, c_b \in [2^{\mathfrak{c}}]^{<\omega}$ so that

$$z^{j_l}_{k^b_m} = \left(z^{j_l}_{k^b_m} \cap \bigcup_{\gamma \in M} X_\gamma\right) \triangle c_l,$$

for each $l \in \{a, b\}$ and $m \in \omega$. Thus, there exists $m_0 \in \omega$ so that

$$\left\{z^{j_l}_{k^b_m} \cap \bigcup_{\gamma \in M} X_\gamma \,:\, m \geq m_0,\ l \in \{a, b\}\right\}$$

is linearly independent. By Lemma 2.2.7, we may fix a subsequence $\{k_m \,:\, m \in \omega\}$ of $\{k^b_m \,:\, m \in \omega\}$ and $\gamma_0 \in M$ so that

$$\left\{z^{j_l}_{k_m} \cap X_{\gamma_0} \,:\, m \in \omega,\ l \in \{a, b\}\right\}$$

is linearly independent.

We shall show that $\{(z^{j_a}_{k_m}, z^{j_b}_{k_m}) \,:\, m \in \omega\}$ does not have an accumulation point in G^2. For this purpose, consider:

- $x = (x^0, x^1) \in G^2$ chosen arbitrarily;
- for each $l \in \{a, b\}$ and $m \in \omega$, $y^l_m \doteq z^{j_l}_{k_m} \cap X_{\gamma_0}$;
- $\tilde{E} \subset X_{\gamma_0}$ a suitably closed set containing $(x^0 \cup x^1) \cap X_{\gamma_0}$ and y^l_m, for each $l \in \{a, b\}$ and $m \in \omega$, so that $|\tilde{E} \setminus \bigcup\{y^l_m \,:\, l \in \{a, b\},\ m \in \omega\}| = \omega$;
- $I = \tilde{E} \cap X^0_{\gamma_0}$;
- $\{q_\xi \,:\, \xi \in I\} \subset \mathcal{P}$ so that, for each $\xi \doteq x^{\gamma_0}_\beta \in I$, $q_\xi = p_\beta$;
- for each $\xi \doteq x^{\gamma_0}_\beta \in I$, $d_\xi = \{x^{\gamma_0}_\beta\}$;
- for each $\xi \doteq x^{\gamma_0}_\beta \in I$, $g_\xi = f^{\gamma_0}_\beta$.

By Lemma 5.1.8, there exists a homomorphism $\tilde{\Phi} \,:\, [\tilde{E}]^{<\omega} \to 2$ so that:

i1) for every $s \in (x^0 \cup x^1) \cap X_{\gamma_0}$, $\tilde{\Phi}(\{s\}) = 0$;

i2) for every $\xi = x^{\gamma_0}_\beta \in I$,
$$\tilde{\Phi}(\{x^{\gamma_0}_\beta\}) = p_\beta - \lim_{l \in \omega} \tilde{\Phi}(f^{\gamma_0}_\beta(l));$$

i3) $\{m \in \omega \,:\, (\tilde{\Phi}(y^a_m), \tilde{\Phi}(y^b_m)) = (0, 0)\}$ is finite.

Now, we may consider E a suitably closed set containing $\tilde{E}$, $x^0 \cup x^1$, and $z^{j_l}_{k_m}$, for each $l \in \{a, b\}$ and $m \in \omega$, so that $E \cap X_{\gamma_0} = \tilde{E}$. Let $\Phi \,:\, [E]^{<\omega} \to 2$ be the homomorphism such that, for each $\xi \in E$,

$$\Phi(\{\xi\}) = \begin{cases} \tilde{\Phi}(\{\xi\}), & \text{if } \xi \in \tilde{E} \\ 0, & \text{if } \xi \notin \tilde{E}. \end{cases}$$

Then, $\Phi \in \mathcal{A}$,

$$\Phi(x^0) = \Phi(x^1) = 0,$$

and, for each $l \in \{a, b\}$ and $m \in \omega$,

$$\Phi(z^{j_l}_{k_m}) = \tilde{\Phi}(y^l_m) + \Phi(z^{j_l}_{k_m} \setminus X_{\gamma_0}) = \tilde{\Phi}(y^l_m).$$

Thus, we conclude that

$$\left\{m \in \omega \, : \, (\Phi(z^{j_a}_{k_m}), \Phi(z^{j_b}_{k_m})) = (\Phi(x^0), \Phi(x^1))\right\}$$

is finite. Since, by construction, there exists $\mu \in [\omega, 2^{\mathfrak{c}})$ so that $\sigma_\mu = \Phi$, we conclude that x cannot be an accumulation point of $\{(z^{j_a}_{k_m}, z^{j_b}_{k_m}) \, : \, m \in \omega\}$. Since $x \in G^2$ is arbitrary, $\{z_{k_m} \, : \, m \in \omega\} \subset Z$ does not have an accumulation point in G^{κ^+}.

Therefore, G^{κ^+} is not countably pracompact. □

□

6.2 Conclusion

In this section we will make some additional comments, and present some open problems and natural directions for further studies on the topic addressed in this chapter and the previous one.

Productivity of pseudocompact-like properties in topological groups has been widely studied in the last years. As mentioned previously, Comfort and Ross proved that the product of any family of pseudocompact groups is pseudocompact [CR66], and Hrušák, van Mill, Ramos-García, and Shelah proved that there exists two countably compact topological groups whose product is not countably compact [Hru+21]. In ZFC, as mentioned, we do not even know answers to the following questions.

Question 6.2.1 (ZFC). *a) Is there a selectively pseudocompact group whose square is not selectively pseudocompact?*

b) ***(stronger version)*** *Is there a countably compact group whose square is not selectively pseudocompact?*

Question 6.2.2 (ZFC). *a) Is there a countably pracompact group whose square is not countably pracompact?*

b) ***(stronger version)*** *Is there a countably compact group whose square is not countably pracompact?*

As mentioned in chapter 2, the Comfort-like Question 2.1.39 is still not solved consistently only for the case $\alpha = \omega$:

Question 6.2.3. *Is there a topological group G so that G^k is selectively pseudocompact for every $k \in \omega$, but G^ω is not selectively pseudocompact?*

In the case there is a positive consistent answer to the above question, one can also ask:

Question 6.2.4. *Is there a topological group G so that G^k is countably compact for every $k \in \omega$, but G^ω is not selectively pseudocompact?*

Regarding countably pracompact topological groups, Theorem 6.1.1 for $\kappa = 2^\mathfrak{c}$ shows that there exists a group G so that $G^{2^\mathfrak{c}}$ is countably pracompact but $G^{(2^\mathfrak{c})^+}$ is not countably pracompact. Interestingly, for countably compact spaces we know that this is not the case: given a Hausdorff topological space X, if $X^{2^\mathfrak{c}}$ is countably compact, then X^α is countably compact for every $\alpha > 2^\mathfrak{c}$. Thus, it may be interesting to study the following questions further. The first one is a stronger version of Question 2.1.40 for $\alpha = \omega$, which we solved in this paper.

Question 6.2.5. *Is there a topological group G so that G^k is countably compact for every $k \in \omega$ and G^ω is not countably pracompact?*

Question 6.2.6. *For which limit cardinals $\omega < \alpha \leq 2^\mathfrak{c}$ is there a topological group G such that G^γ is countably pracompact for every cardinal $\gamma < \alpha$, but G^α is not countably pracompact?*

Question 6.2.7. *For which cardinals $\alpha > (2^\mathfrak{c})^+$ is there a topological group G such that G^γ is countably pracompact for all cardinals $\gamma < \alpha$, but G^α is not countably pracompact?*

Also, it is natural to ask which the *stopping point* is, if any:

Question 6.2.8. *Is there a cardinal κ such that, for each topological group G, G^κ countably pracompact implies that G^γ is countably pracompact for every $\gamma > \kappa$?*

Chapter 7

On divisibility and p-compact topologies in groups

In this chapter, we will present the most recent topics we are working on and the results already obtained, which we intend to submit in the near future.

7.1 Introduction and Results Obtained

We shall begin with some terminology, definitions and history. We will call $\mathbb{T}$ the Abelian group $\mathbb{R}/\mathbb{Z}$ and, given an infinite cardinal κ, $\mathbb{Q}^{(\kappa)}$ will denote the direct sum of κ copies of $\mathbb{Q}$:

$$\mathbb{Q}^{(\kappa)} \doteq \{g \in \mathbb{Q}^{\kappa} : |\mathrm{supp}(g)| < \omega\},$$

where $\mathrm{supp}(g) \doteq \{\alpha \in \kappa : g^{\alpha} \neq 0\}$. If $C \subset \mathfrak{c}$, we also consider

$$\mathbb{Q}^{(C)} \doteq \{g \in \mathbb{Q}^{(\mathfrak{c})} : \mathrm{supp}(g) \subset C\}$$

as a subgroup of $\mathbb{Q}^{(\mathfrak{c})}$.

Given an ultrafilter q on ω, we define an equivalence relation on $(\mathbb{Q}^{(\mathfrak{c})})^{\omega}$ by letting $f \equiv_q g$ if, and only if, $\{n \in \omega : f(n) = g(n)\} \in q$. We let $[f]_q$ be the equivalence class determined by f and $(\mathbb{Q}^{(\mathfrak{c})})^{\omega}/q$ be $(\mathbb{Q}^{(\mathfrak{c})})^{\omega}/\equiv_q$. This set has a natural $\mathbb{Q}$-vector space structure. We call the group $(\mathbb{Q}^{(\mathfrak{c})})^{\omega}/q$ with this structure the *$q-$ultrapower* of $\mathbb{Q}^{(\mathfrak{c})}$, and it is denoted by $\mathrm{ult}_q(\mathbb{Q}^{(\mathfrak{c})})$.

Given $\mu \in \mathfrak{c}$, we denote χ_{μ} the element of $\mathbb{Q}^{(\mathfrak{c})}$ so that $\mathrm{supp}(\chi_{\mu}) = \{\mu\}$ and $\chi_{\mu}^{\mu} = 1$. Also, given $\mu \in \mathfrak{c}$, we define $\vec{\mu}$ the sequence in $\mathfrak{c}$ such that $\vec{\mu}(n) = \mu$ for every $n \in \omega$. Finally, if $A \subset \omega$ and $\zeta : A \to \mathfrak{c}$, we let $\chi_{\zeta} \in (\mathbb{Q}^{(\mathfrak{c})})^{A}$ be such that $\chi_{\zeta}(n) = \chi_{\zeta(n)}$ for every $n \in A$.

Over the years, the relation between algebraic properties of Abelian groups and the possibility of endowing them with pseudocompact-like topologies has been studied in various ways. In this regard, Fuchs showed that a non-trivial free Abelian group does not admit a compact Hausdorff group topology, and Halmos proved that it is possible to topologize the additive group $\mathbb{R}$ so that it becomes a Hausdorff compact topological group [Hal44]. Notice that, algebraically, $\mathbb{R}$ can be considered as $\mathbb{Q}^{(\mathfrak{c})}$. Also, Tomita showed the following result:

Theorem 7.1.1 ([Tom98], Theorem 17)**.** *Let G be an infinite free Abelian group endowed with a group topology. Then, G^{ω} is not countably compact.*

The proof of the theorem above relies on the fact that the only element of a free Abelian group that is infinitely divisible is 0. This suggests that a good candidate for a torsion-free group that admits a $p-$compact topology might be a divisible group, such as $\mathbb{Q}$. Indeed, Bellini, Rodrigues and Tomita recently showed that, if p is a selective ultrafilter and κ is a cardinal such that $\kappa = \kappa^{\omega}$, then $\mathbb{Q}^{(\kappa)}$ admits a $p-$compact group topology without non-trivial convergent sequences [BRT21b]. Our first result in this regard is that divisibility can be dropped:

Proposition 7.1.2. *There is an Abelian, torsion-free, non-divisible topological group which is compact.*

Proof. Consider the Hausdorff compact group topology in $\mathbb{Q}^{(\mathfrak{c})}$ (as given by Halmos in [Hal44]). Let $\psi : \mathbb{Q}^{(\mathfrak{c})} \to \mathbb{T}$ be a non-trivial continuous group homomorphism[1]. We will prove in the next paragraph that there exists $g \in \mathbb{Q}^{(\mathfrak{c})}$ so that $\psi(g) \neq 0$ has order k, for some prime number $k > 1$.

Suppose that there exists an irrational ζ so that $\zeta + \mathbb{Z} \in \psi[\mathbb{Q}^{(\mathfrak{c})}]$. In this case, $\psi[\mathbb{Q}^{(\mathfrak{c})}]$ is dense in $\mathbb{T}$. Since $\psi[\mathbb{Q}^{(\mathfrak{c})}]$ is compact and therefore closed in $\mathbb{T}$, it follows that $\psi[\mathbb{Q}^{(\mathfrak{c})}] = \mathbb{T}$, and we can take any prime $k > 1$ and find $g \in \mathbb{Q}^{(\mathfrak{c})}$ so that $\psi(g) \neq 0$ and $k\psi(g) = 0$[2]. If there is no such irrational, then let $h \in \mathbb{Q}^{(\mathfrak{c})}$ be so that $\psi(h) \neq 0$. Then, there are $s, t \in \mathbb{Z}$ so that $\gcd(s, t) = 1$, $t > 1$ and $\psi(h) = \frac{s}{t} + \mathbb{Z}$. Suppose that $t = q_0 \cdot \ldots \cdot q_m$, for $m \in \omega$ and prime numbers $q_0, \ldots, q_m > 1$. Then, $g \doteq (q_1 \cdot \ldots \cdot q_m)h \in \mathbb{Q}^{(\mathfrak{c})}$ is such that $\psi(g)$ has order q_0.

Now, fix such prime number k, and let

$$G \doteq \psi^{-1}[\{\mathbb{Z}, \frac{1}{k} + \mathbb{Z}, \ldots, \frac{k-1}{k} + \mathbb{Z}\}].$$

Then, G is a closed subgroup of $\mathbb{Q}^{(\mathfrak{c})}$, and thus is a compact topological group. As $\{mg : m \in \mathbb{Z}\}$ is an infinite subset of G, G is an infinite group. Suppose that there exists $h \in G$ so that $kh = g$. Then,

$$k\psi(h) = \psi(g) \neq 0,$$

and hence $\psi(h) \neq 0$. Since $\psi(h) \in \{0, \frac{1}{k} + \mathbb{Z}, \ldots, \frac{k-1}{k} + \mathbb{Z}\}$, the order of $\psi(h)$ must be k, a contradiction. Therefore, G is not a $k-$divisible group. □

Since group divisibility is not essential for the existence of p-compact topologies, we

[1] More generally, it is well-known that if G is a locally compact Abelian group, the family of continuous homomorphisms $G \to \mathbb{T}$ separates points of G [HR63].

[2] In this case, $\psi(g)$ could be $\frac{1}{k} + \mathbb{Z}$.

can try to change the group $\mathbb{Q}^{(\mathfrak{c})}$ a little so that it loses its divisibility, and study whether we still get such a topology. The most immediate attempt would be to look at the Abelian group $\mathbb{Z} \times \mathbb{Q}^{(\mathfrak{c})}$. We did this, and showed that $\mathbb{Z} \times \mathbb{Q}^{(\mathfrak{c})}$ does not admit a $p-$compact group topology for any $p \in \omega^*$.

Proposition 7.1.3. *The Abelian group $\mathbb{Z} \times \mathbb{Q}^{(\mathfrak{c})}$ does not admit a $p-$compact group topology for any $p \in \omega^*$.*

Proof. Suppose that $\mathbb{Z} \times \mathbb{Q}^{(\mathfrak{c})}$ is a $p-$compact topological group, for some $p \in \omega^*$.

Let $(x_n)_{n\in\omega}$ be a sequence of elements in $\mathbb{Z}$ given by $x_n = 2^n + n!$, for every $n \in \omega$.

We claim that, for each $c \in \mathbb{Z}$, there exists $r \in \mathbb{Z}$ so that $r \nmid 2^n + n! - c$ for all but finitely many $n \in \omega$. Indeed, if $c = 0$, for each odd r, $r \nmid 2^n + n!$ for every $n \in \omega$, $n > 1$. Then, suppose that $c = 2^k l$ for some $k > 0$ and odd $l \in \mathbb{Z}$. If $n > 2^{k+1}$, we have that $2^{k+1}|2^n$ and $2^{k+1}|n!$, but $2^{k+1} \nmid 2^k l$, hence $2^{k+1} \nmid x_n - c$. Hence, $r = 2^{k+1}$ in this case. Finally, if c is an odd number, it is clear that if $r = 2$ and $n > 1$, $r \nmid 2^n + n! - c$.

Let $a \in \mathbb{Q}^{(\mathfrak{c})}$ and $c \in \mathbb{Z}$ be so that

$$p - \lim_{n\in\omega}(x_n, 0) = (c, a).$$

As we showed, there are $r_1, r_2 \in \mathbb{Z}$ so that $r_1 \nmid x_n - c$ and $r_2 \nmid x_n$ for all but finitely many $n \in \omega$. Fix also $r \in \mathbb{Z}$ so that $r > |c|$ and $r_1 r_2 | r$.

Suppose that $c \geq 0$. For each $n \in \omega$, let $e_n \in \mathbb{Z}$, $0 \leq e_n < r$, and $y_n \in \mathbb{Z}$ be so that

$$x_n = r y_n + e_n.$$

Let also $A \in p$ and $e \in \mathbb{Z}$ be so that $e_n = e$ for every $n \in A$. As $\mathbb{Z} \times \mathbb{Q}^{(\mathfrak{c})}$ is a $p-$compact topological group, there are also $b \in \mathbb{Q}^{(\mathfrak{c})}$ and $d \in \mathbb{Z}$ such that

$$p - \lim_{n\in\omega}(y_n, 0) = (d, b).$$

Then,

$$(c, a) = p - \lim_{n\in\omega}(x_n, 0) = p - \lim_{n\in\omega}(r y_n + e_n, 0) = (rd + e, rb),$$

thus $c = rd + e$. This implies that $r|(c - e)$, but $r > c \geq 0$, and $r > e \geq 0$, thus $c = e$. But $r_1 \nmid x_n - c$ for all but finitely many $n \in \omega$, and $r_1 | r$, a contradiction.

Now, suppose that $c < 0$, and analogously, for each $n \in \omega$, let $f_n \in \mathbb{Z}$, $0 \leq f_n < r$ be so that

$$x_n = r y_n + f_n,$$

and $e_n = f_n - r$. Let also $A \in p$ and $e \in \mathbb{Z}$ be so that $e_n = e$ for every $n \in A$. Again, there are $b \in \mathbb{Q}^{(\mathfrak{c})}$ and $d \in \mathbb{Z}$ so that

$$p - \lim_{n\in\omega}(y_n, 0) = (d, b),$$

and then

$$(c, a) = p - \lim_{n\in\omega}(x_n, 0) = p - \lim_{n\in\omega}(r(y_n + 1) + e_n, 0) = (rd + r + e, rb).$$

Analogously, this implies that $r|(c-e)$ and that $c = e$, which again leads to a contradiction. $\square$

In fact, note that we did not use the structure of the group $\mathbb{Q}^{(\mathfrak{c})}$ at any point in the above proof, so a much more general result is valid:

Corollary 7.1.4. *Let G be an Abelian group. Then, the Abelian group $\mathbb{Z} \times G$ does not admit a $p-$compact group topology for any $p \in \omega^*$.*

In particular, this answers the following question of [Bel+21]:

Question 7.1.5 ([Bel+21])**.** *Is there a $p-$compact group topology (without non-trivial convergent sequences) compatible with $\mathbb{Z}^{(\mathfrak{c})} \times \mathbb{Q}^{(\mathfrak{c})}$, for some ultrafilter p? A group topology whose $\omega-$th power is countably compact? What about $\mathbb{Z} \times \mathbb{Q}^{\mathfrak{c}}$?*

Interestingly, if we replace $\mathbb{Z}$ with a subgroup of $\mathbb{Q}$ which is *very divisible*, the same is true:

Proposition 7.1.6. *Let G be an Abelian group, H be a subgroup of $\mathbb{Q}$ and $r > 1$ be a prime number. Suppose that H is t-divisible for each prime $t \neq r$ but is not $r-$divisible. Then, $H \times G$ does not admit a $p-$compact group topology, for any $p \in \omega^*$.*

Proof. Suppose that $H \times G$ is endowed with a $p-$compact group topology, for some $p \in \omega^*$. Let $h \in H$ be so that $h \neq rg$ for each $g \in H$, and $(m_k)_{k\in\omega}$ be an increasing sequence in ω defined inductively, satisfying that $m_0 = 1$ and

$$r^{m_k} > k \sum_{l=0}^{k-1} r^{m_l} + k,$$

for each $k > 0$. Let also $a \in G$ and $c \in \mathbb{Q}$ be such that $ch \in H$ and

$$p - \lim_{n\in\omega} \Big(\sum_{k=0}^{n} r^{m_k} h, 0 \Big) = (ch, a),$$

with $c = \dfrac{u}{v}$, and $u, v \in \mathbb{Z}$ so that $\gcd(u, v) = 1$. Without loss of generality, we may assume that $v > 0$. We have that, for each $n \in \omega$,

$$\sum_{k=0}^{n} r^{m_k} - \frac{u}{v} = \frac{v \sum_{k=0}^{n} r^{m_k} - u}{v}.$$

Let $N_0 \in \omega$ be so that $N_0 > \max(|u|, |v|)$ and that

$$S \doteq v \sum_{k=0}^{N_0-1} r^{m_k} - u > 0.$$

Then,

$$r^{m_{N_0}} > N_0 \sum_{k=0}^{N_0-1} r^{m_k} + N_0 \geq v \sum_{k=0}^{N_0-1} r^{m_k} - u > 0.$$

For each $n \geq N_0$,

$$v \sum_{k=0}^{n} r^{m_k} - u = v \sum_{k=N_0}^{n} r^{m_k} + S.$$

Therefore, $r^{m_{N_0}} \nmid v \sum_{k=0}^{n} r^{m_k} - u$ for any $n \geq N_0$. Let $(y_n)_{n \geq N_0}$ be a sequence in $\mathbb{Z}$ so that, for each $n \geq N_0$,

$$r^{m_{N_0}} y_n = v \sum_{k=N_0}^{n} r^{m_k}.$$

Then, there are $b \in G$ and $d \in \mathbb{Q}$ so that $dg \in H$ and

$$p - \lim_{n \in \omega} \Big(\frac{y_n}{v} h, 0\Big) = (dh, b),$$

with $d = \dfrac{x}{y}$, $\gcd(x, y) = 1$. We claim that $r \nmid y$. In fact, otherwise suppose that $l > 0$ is such that $y = r^l s$, with $s \in \mathbb{Z}$ and $r \nmid s$. In this case, x cannot be divisible by r, and thus there exists $g \in H$ so that $dh = xg$. Therefore, we have that

$$\frac{x}{r^l s} h = xg \Rightarrow h = r^l sg,$$

a contradiction.

Now,

$$p - \lim_{n \in \omega} \Big(\frac{r^{m_{N_0}} y_n}{v} h, 0\Big) = (r^{m_{N_0}} dh, r^{m_{N_0}} b)$$

and therefore,

$$\begin{aligned}
(ch, a) &= p - \lim_{n \in \omega} \Big(\sum_{k=0}^{n} r^{m_k} h, 0\Big) \\
&= p - \lim_{n \in \omega} \Big(\sum_{k=N_0}^{n} r^{m_k} h + \sum_{k=0}^{N_0 - 1} r^{m_k} h, 0\Big) \\
&= \Big(\sum_{k=0}^{N_0 - 1} r^{m_k} h, 0\Big) + p - \lim_{n \in \omega} \Big(\frac{r^{m_{N_0}} y_n}{v} h, 0\Big) \\
&= \Big(\sum_{k=0}^{N_0 - 1} r^{m_k} h + r^{m_{N_0}} dh, r^{m_{N_0}} b\Big).
\end{aligned}$$

Thus,

$$c - \sum_{k=0}^{N_0 - 1} r^{m_k} = -\frac{S}{v} = r^{m_{N_0}} d,$$

and then $-Sy = r^{m_{N_0}} xv$, which is a contradiction, since $r \nmid y$ and $0 < S < r^{m_{N_0}}$. Thus, $H \times G$ does not admit a p−compact group topology, for any $p \in \omega^*$.

□

Finally, we will show that, assuming the existence of a selective ultrafilter $p \in \omega^*$, there exists a p-compact group topology on $\mathbb{Q}^{(\mathfrak{c})}$ without non-trivial convergent sequences and a closed subgroup $H \subset \mathbb{Q}^{(\mathfrak{c})}$ which contains an element not divisible by any $n \in \omega$. That is, H will be a $p-$compact subgroup of $\mathbb{Q}^{(\mathfrak{c})}$, without non-trivial convergent sequences, which contains an element not divisible (in H) by any $n \in \omega$. We will use a construction similar to the one made in [BRT21b].

For that, we let:

- $p \in \omega^*$ be a selective ultrafilter;
- $(J_i)_{i \leq 1}$ be a partition of $[\omega, \mathfrak{c})$ such that $\omega + \omega \in J_1$ and $|J_0| = |J_1| = \mathfrak{c}$;
- $\{f_\alpha : \alpha \in J_0\}$ be an enumeration[3] of $(\mathbb{Q}^{(\mathfrak{c})})^\omega$ such that

$$\bigcup_{n\in\omega} \mathrm{supp}(f_\xi(n)) \subset \xi, \text{ for each } \xi \in J_0.$$

- $I \subset J_0$ be such that $\{[f_\xi]_p : \xi \in I\} \cup \{[\chi_{\vec{\mu}}]_p : \mu \in \mathfrak{c}\}$ is a $\mathbb{Q}-$basis for $\mathrm{ult}_p(\mathbb{Q}^{(\mathfrak{c})})$.

The next result appears in [BRT21b].

Proposition 7.1.7 ([BRT21b], Lemma 3.5)**.** *Let $d \in \mathbb{Q}^{(\mathfrak{c})} \setminus \{0\}$, $r \in \mathbb{Q}^{(I)} \setminus \{0\}$ and $B \in p$. Let C be a countably infinite subset of $\mathfrak{c}$ such that $\omega \cup supp(r) \cup supp(d) \subset C$ and $\bigcup_{n\in\omega} supp(f_\xi(n)) \subset C$ for every $\xi \in C \cap I$. Then there exists a homomorphism $\phi : \mathbb{Q}^{(C)} \to \mathbb{T}$ such that*

a) $\phi(d) \neq 0$;

b) $p - \lim_{n\in\omega} \phi(\frac{1}{N} f_\xi(n)) = \phi(\frac{1}{N}\chi_\xi)$*, for each* $\xi \in I \cap C$ *and* $N \in \omega$*;*

c) $\left(\phi(\sum_{\mu\in supp(r)} r(\mu) f_\mu(n)) : n \in B\right)$ *does not converge.*

Now, we define:

Definition 7.1.8. A set $C \in [\mathfrak{c}]^\omega$ is *suitably closed* if, and only if, for every $\xi \in C \cap I$, we have $\bigcup_{n\in\omega} \mathrm{supp}(f_\xi(n)) \subset C$.

Let $\mathcal{C}$ be the set of all homomorphisms $\phi : \mathbb{Q}^{(C)} \to \mathbb{T}$, with $C \in [\mathfrak{c}]^\omega$ suitably closed, satisfying that

$$p - \lim_{n\in\omega} \phi(\tfrac{1}{N} f_\xi(n)) = \phi(\tfrac{1}{N}\chi_\xi),$$

for each $\xi \in I \cap C$ and $N \in \omega$. We enumerate $\mathcal{C}$ by $\{\phi_\beta : \beta \in [\omega, \mathfrak{c})\}$ assuming, without loss of generality, that given $\beta \in [\omega, \mathfrak{c})$, $\phi_\beta : \mathbb{Q}^{(C)} \to \mathbb{T}$ is such that $C \subset \beta$.

Next we shall extend each homomorphism $\phi \in \mathcal{C}$ to a homomorphism $\overline{\phi}$ defined in $\mathbb{Q}^{(\mathfrak{c})}$ satisfying that

$$p - \lim_{n\in\omega} \overline{\phi}(\tfrac{1}{N} f_\xi(n)) = \overline{\phi}(\tfrac{1}{N}\chi_\xi),$$

for each $\xi \in I$ and $N \in \omega$, in a similar way to what is done in **Lemma 3.6** of [BRT21b]. The difference will be that we wish to control the value of the homomorphisms ϕ_β, when

[3] The construction of such enumeration is analogous to constructions of similar families made in the previous chapters.

$\beta \in [\omega, \omega + \omega)$, in the element $\chi_{\omega+\omega}$. For that, consider $\{\beta_n : n \in \omega\}$ an enumeration of $[\omega, \omega + \omega)$.

Let $\beta \in [\omega, \mathfrak{c})$. Suppose first that $\beta \in [\omega, \omega + \omega)$, say $\beta = \beta_m$, $m \in \omega$. By construction, there is a suitably closed set $C \in [\mathfrak{c}]^\omega$ so that $\phi_\beta : \mathbb{Q}^{(C)} \to \mathbb{T}$, and ϕ_β satisfies that

$$p - \lim_{n \in \omega} \phi_\beta(\tfrac{1}{N} f_\xi(n)) = \phi_\beta(\tfrac{1}{N} \chi_\xi),$$

for each $\xi \in I \cap C$ and $N \in \omega$. Let $\{\xi_\alpha : \alpha < \mathfrak{c}\}$ be a strictly increasing enumeration of $\mathfrak{c} \setminus C$. For each $\alpha < \mathfrak{c}$, let $C_\alpha \doteq C \cup \{\xi_\gamma : \gamma < \alpha\}$. In particular, $C_0 = C$ and $C_\mathfrak{c} = \mathfrak{c}$. Notice that, for each $\alpha < \mathfrak{c}$ and $n \in \omega$, $\mathrm{supp}(f_{\xi_\alpha}(n)) \subset \xi_\alpha$.

We will define recursively homomorphisms $\sigma_\alpha : \mathbb{Q}^{(C_\alpha)} \to \mathbb{T}$, for $\alpha \leq \mathfrak{c}$, satisfying:

a) $\sigma_0 = \phi_\beta$;

b) $\sigma_\delta \subset \sigma_\alpha$ whenever $\delta \leq \alpha \leq \mathfrak{c}$;

c) $p - \lim_{n \in \omega} \sigma_\alpha(\frac{1}{N} f_\xi(n)) = \sigma_\alpha(\frac{1}{N} \chi_\xi)$, for each $\alpha \leq \mathfrak{c}$, $\xi \in C_\alpha \cap I$ and $N \in \omega$.

For that, let $\sigma_0 = \phi_\beta$. Suppose that σ_δ as above is defined for each $\delta < \alpha$, for a given $\alpha < \mathfrak{c}$. First, we define

$$\sigma_\alpha(g) = \Big(\bigcup_{\delta < \alpha} \sigma_\delta\Big)(g),$$

if $g \in \mathbb{Q}^{\left(\bigcup_{\delta<\alpha} C_\delta\right)}$. If $\alpha \in J_1$, we put:

$$\begin{cases} \sigma_\alpha(q\chi_{\xi_\alpha}) = 0, & \text{if } \xi_\alpha \neq \omega + \omega, \text{ for every } q \in \mathbb{Q} \\ \sigma_\alpha(q\chi_{\xi_\alpha}) = q \cdot (\dfrac{1}{m} + \mathbb{Z}), & \text{if } \xi_\alpha = \omega + \omega. \end{cases}$$

Finally, if $\alpha \in J_0$, we put:

$$\sigma_\alpha(q\chi_{\xi_\alpha}) = q \cdot \Big(p - \lim_{n \in \omega} \Big(\bigcup_{\delta < \alpha} \sigma_\delta\Big)(\tfrac{1}{N} f_{\xi_\alpha}(n))\Big),$$

for every $q \in \mathbb{Q}$.

It is not hard to see that there is a unique group homomorphism defined in $\mathbb{Q}^{(C_\alpha)}$ which satisfies all the above definitions. This will be the homomorphism $\sigma_\alpha : \mathbb{Q}^{(C_\alpha)} \to \mathbb{T}$ we wanted, and that ends the definition by recursion. Notice that $\sigma_\mathfrak{c}(\chi_{\omega+\omega}) = \dfrac{1}{m} + \mathbb{Z}$, since $C \subset \beta \subset \omega + \omega$ by construction.

The homomorphism $\sigma_\mathfrak{c} : \mathbb{Q}^{(\mathfrak{c})} \to \mathbb{T}$ will be called $\overline{\phi_\beta}$, and it is the extension of ϕ_β that we were looking for.

Suppose now that $\beta \notin [\omega, \omega + \omega)$. Again, by construction, there is a suitably closed set $C \in [\mathfrak{c}]^\omega$ so that $\phi_\beta : \mathbb{Q}^{(C)} \to \mathbb{T}$, and ϕ_β satisfies that

$$p - \lim_{n \in \omega} \phi_\beta(\tfrac{1}{N} f_\xi(n)) = \phi_\beta(\tfrac{1}{N} \chi_\xi),$$

for each $\xi \in I \cap C$ and $N \in \omega$. Let $\{\xi_\alpha : \alpha < \mathfrak{c}\}$ be a strictly increasing enumeration of $\mathfrak{c} \setminus C$. For each $\alpha < \mathfrak{c}$, let $C_\alpha \doteq C \cup \{\xi_\gamma : \gamma < \alpha\}$.

We will again define recursively homomorphisms $\sigma_\alpha : \mathbb{Q}^{(C_\alpha)} \to \mathbb{T}$, for $\alpha \leq \mathfrak{c}$, satisfying:

a) $\sigma_0 = \phi_\beta$;

b) $\sigma_\delta \subset \sigma_\alpha$ whenever $\delta \leq \alpha \leq \mathfrak{c}$;

c) $p - \lim_{n\in\omega} \sigma_\alpha(\frac{1}{N} f_\xi(n)) = \sigma_\alpha(\frac{1}{N}\chi_\xi)$, for each $\alpha \leq \mathfrak{c}$, $\xi \in C_\alpha \cap I$ and $N \in \omega$.

For that, let $\sigma_0 = \phi_\beta$. Suppose that σ_δ as above is defined for each $\delta < \alpha$, for a given $\alpha < \mathfrak{c}$. First, we define

$$\sigma_\alpha(g) = \big(\bigcup_{\delta<\alpha} \sigma_\delta\big)(g),$$

if $g \in \mathbb{Q}^{\left(\bigcup_{\delta<\alpha} C_\delta\right)}$. If $\alpha \in J_1$, we put $\sigma_\alpha(q\chi_{\xi_\alpha}) = 0$, for every $q \in \mathbb{Q}$. Otherwise, if $\alpha \in J_0$, we put:

$$\sigma_\alpha(q\chi_{\xi_\alpha}) = q \cdot \big(p - \lim_{n\in\omega} \big(\bigcup_{\delta<\alpha} \sigma_\delta\big)(\tfrac{1}{N} f_{\xi_\alpha}(n))\big),$$

for every $q \in \mathbb{Q}$.

There is a unique group homomorphism defined in $\mathbb{Q}^{(C_\alpha)}$ which satisfies all the above definitions. This will be the homomorphism $\sigma_\alpha : \mathbb{Q}^{(C_\alpha)} \to \mathbb{T}$ we wanted, and that ends the definition by recursion.

Again, the homomorphism $\sigma_\mathfrak{c} : \mathbb{Q}^{(\mathfrak{c})} \to \mathbb{T}$ will be called $\overline{\phi_\beta}$, and it is the extension of ϕ_β that we were looking for. Notice that, by construction, for each $\beta \in [\omega, \mathfrak{c})$,

$$p - \lim_{n\in\omega} \overline{\phi_\beta}(\tfrac{1}{N} f_\xi(n)) = \overline{\phi_\beta}(\tfrac{1}{N}\chi_\xi),$$

for each $\xi \in I$ and $N \in \omega$. We define $\overline{C} \doteq \{\overline{\phi_\beta} : \beta \in [\omega, \mathfrak{c})\}$.

Now we may prove the following theorem. Its proof is similar to the proof of **Theorem 3.7** in [BRT21b], but we use the changes made to the homomorphisms to prove that we can find a closed subgroup H of $\mathbb{Q}^{(\mathfrak{c})}$ and an element in H which is not divisible by any $n \in \omega$.

Theorem 7.1.9. *Let p be a selective ultrafilter. Then, there exists a p-compact group topology on $\mathbb{Q}^{(\mathfrak{c})}$ without non-trivial convergent sequences and a closed subgroup $H \subset \mathbb{Q}^{(\mathfrak{c})}$ which contains an element not divisible by any $n \in \omega$.*

Proof. Consider $\mathbb{Q}^{(\mathfrak{c})}$ endowed with the group topology generated by the homomorphisms in $\overline{C}$. We will call this topological group G. Note that, by construction, in this topology we have that

$$\frac{1}{N}\chi_\xi = p - \lim_{n\in\omega} \frac{1}{N} f_\xi(n),$$

for each $\xi \in I$ and $N \in \omega$.

Let $h \doteq (h_i)_{i\in\omega}$ be a sequence of elements in $\mathbb{Q}^{(\mathfrak{c})}$. Then, there are families $(r_\xi : \xi \in I)$

and $(s_\mu : \mu \in \mathfrak{c})$ of rational numbers, where all but finitely many are 0, such that

$$[h]_p = \sum_{i\in I} r_\xi [f_\xi]_p + \sum_{\mu\in\mathfrak{c}} s_\mu [\chi_{\vec{\mu}}]_p.$$

Then,

$$\sum_{i\in I} r_\xi \chi_\xi + \sum_{\mu\in\mathfrak{c}} s_\mu \chi_\mu$$

is a $p-$limit of h, and thus G is $p-$compact.

Now, let g be an injective sequence of elements in G. Again, there are $r \in \mathbb{Q}^{(I)} \setminus \{0\}$ and $s \in \mathbb{Q}^{(\mathfrak{c})}$ such that

$$[g]_p = \sum_{i\in I} r_\xi [f_\xi]_p + \sum_{\mu\in\mathfrak{c}} s_\mu [\chi_{\vec{\mu}}]_p.$$

Then, defining $D \doteq \operatorname{supp}(r)$, there exists $B \in p$ so that

$$g(n) = \sum_{\xi\in D} r_\xi f_\xi(n) + \sum_{\mu\in\mathfrak{c}} s_\mu \chi_\mu,$$

for every $n \in B$. Let $d \in G \setminus \{0\}$ arbitrary, and C be a suitably closed set containing $\omega \cup D \cup \operatorname{supp}(d)$. Applying Proposition 7.1.7, we conclude that there exists a homomorphism $\phi : \mathbb{Q}^{(C)} \to \mathbb{T}$ so that $\left(\phi(\sum_{\mu\in D} r(\mu) f_\mu(n)) : n \in B\right)$ does not converge in $\mathbb{T}$. In particular, $(\sum_{\mu\in D} r(\mu) f_\mu(n) : n \in B)$ does not converge in G, and since $\sum_{\mu\in\mathfrak{c}} s_\mu \chi_\mu$ is constant, $(g(n) : n \in \omega)$ does not converge either.

Finally, we shall see that there is a subgroup $H \subset G$ which contains an element not divisible by any $n \in \omega$. By the construction made previously, $\overline{\phi_{\beta_m}}(\chi_{\omega+\omega}) = \dfrac{1}{m} + \mathbb{Z}$, for every $m \in \omega$. Then,

$$H \doteq \bigcap_{m\in\omega} \overline{\phi_{\beta_m}}^{-1}\left[\{\mathbb{Z}, \frac{1}{m} + \mathbb{Z}, ..., \frac{m-1}{m} + \mathbb{Z}\}\right]$$

is a nontrivial closed subgroup of G, thus it is also $p-$compact. Besides, clearly $\chi_{\omega+\omega} \in H$.

Moreover, given $n \in \omega$, suppose that there exists $v \in H$ so that $\chi_{\omega+\omega} = nv$. Then,

$$\overline{\phi_{\beta_n}}(\chi_{\omega+\omega}) = n\overline{\phi_{\beta}}(v) = 0,$$

a contradiction. Thus, the element $\chi_{\omega+\omega}$ in H is non-divisible in H for every $n \in \omega$. This ends the proof. $\square$

7.2 Conclusion

In conclusion, for now we proved the following:

- There is an Abelian, torsion-free, non-divisible topological group which is compact.
- For every Abelian group G, $\mathbb{Z} \times G$ does not admit a $p-$compact group topology for any $p \in \omega^*$.

- Let G be an Abelian group, H be a subgroup of $\mathbb{Q}$ and $r > 1$ be a prime number. Suppose that H is t-divisible for each prime $t \neq r$ but is not $r-$divisible. Then, $H \times G$ does not admit a $p-$compact group topology, for any $p \in \omega^*$.
- Let p be a selective ultrafilter. Then, there exists a p-compact group topology on $\mathbb{Q}^{(\mathfrak{c})}$ without non-trivial convergent sequences and a closed subgroup $H \subset \mathbb{Q}^{(\mathfrak{c})}$ which contains an element not divisible by any $n \in \omega$.

Next, we intend to continue the study of the relation between divisibility of Abelian groups and pseudocompact-like topologies with the aim of submitting the results in a future article. For now, we highlight the following questions:

Question 7.2.1. *If H is a non-divisible subgroup of $\mathbb{Q}$ and H_0 is a divisible group, does $H \times H_0$ admit a topology whose ω-th power is countably compact?*

Question 7.2.2. *Is there, in ZFC, an Abelian torsion-free topological group which has an element that is not divisible for any $n > 1$ and admits a $p-$compact topology for some $p \in \omega^*$?*

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