Motivic cohomology, Milnor K-theory, and Galois cohomology

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Resumo

Daniel de Almeida Souza. **Cohomologia motívica, K-teoria de Milnor, e cohomologia galoisiana**. Dissertação (Mestrado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022.

Esta dissertação apresenta uma das possíveis fundamentações, baseada em complexos motívicos, para a cohomologia motívica de variedades lisas sobre um dado corpo base k. São discutidas suas propriedades básicas e sua relação com a K-teoria de Milnor e com determinados grupos de cohomologia galoisiana de k. Em particular, é discutida a formulação em termos de cohomologia motívica do homomorfismo do resíduo da norma, que compara os grupos de K-teoria de Milnor módulo um número primo l diferente da característica de k com os grupos de cohomologia galoisiana com coeficientes em potências tensoriais do módulo de raízes l-ésimas da unidade. Por fim, são enunciados alguns resultados preliminares utilizados na caracterização da conjetura de Bloch-Kato em termos de certas afirmações de natureza motívica.

Palavras-chave: Cohomologia motívica. K-teoria de Milnor. Cohomologia galoisiana.

Abstract

Daniel de Almeida Souza. **Motivic cohomology, Milnor K-theory, and Galois cohomology**. Thesis (Master's). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2022.

This dissertation presents one of the possible foundations, based on motivic complexes, for the motivic cohomology of smooth varieties over a given base field k. Its basic properties are discussed, as well as its relation to Milnor K-theory and to certain Galois cohomology groups of k. In particular, we discuss the formulation in terms of motivic cohomology of the norm residue homomorphism, which compares the Milnor K-theory groups modulo a prime number l different from the characteristic of k with the Galois cohomology groups with coefficients in tensor powers of the module of l-th roots of unity. Finally, we list some preliminary results used for characterizing the Bloch-Kato conjecture in terms of certain statements of motivic nature.

Keywords: Motivic cohomology. Milnor K-theory. Galois cohomology.

Contents

Introduction

0.1 Motivic cohomology, Milnor K- theory, and Galois cohomology

In 1970, John Milnor introduced (Milnor, 1970) an algebraic invariant for fields which became known as *Milnor K-theory*. Given a field k and $n \geq 1$, it is defined as the group

$$K_M^n(k) := \frac{\overbrace{k^\times \otimes \ldots \otimes k^\times}^{n \text{ times}}}{(a_1 \otimes \ldots \otimes a_n \,:\, \exists i < j \text{ with } a_i + a_j = 1)}.$$

where $k^\times$ denotes the multiplicative group of units of k; $K_M^0(k)$ is defined as $\mathbb{Z}$. These groups may be organized as a graded ring $K_M^*(k) = \bigoplus_{n\geq 0} K_M^n(k)$; namely, one may consider the quotient of the tensor algebra $T(k^\times)$ by the two-sided homogeneous ideal generated by all tensors of the form $a \otimes (1-a)$ for $a \in k \setminus \{0,1\}$. These are known as the Steinberg relations.

Milnor's notation is due to the fact that these groups serve as an approximation to the then accepted definition of algebraic K-theory groups of a field in degrees 0, 1, and 2; he refers to works by H. Matsumoto, C. Moore, and R. Steinberg. On the other hand, in Milnor, 1970, page 319, Milnor describes his construction of $K_M^n(k)$ for $n \geq 3$ as being *"purely ad hoc"* in the sense that they are defined not in terms of algebraic K-theory in higher degrees, but so that $K_M^*(k)$ corresponds to the quotient of the ring freely generated by $k^\times$ by the two-sided homogeneous ideal generated by certain relations in $k^\times \otimes k^\times$ – the Steinberg relations – which were known to describe the algebraic K-group $K^2(k)$ as a quotient of $k^\times \otimes k^\times$.

One of the reasons for Milnor's interest in $K_M^*(k)$ was its connection, when $char(k) \neq 2$, with the Galois cohomology ring

$$H^*(k, \mathbb{Z}/2) = \bigoplus_{n\geq 0} H^n(k, \mathbb{Z}/2),$$

with $\mathbb{Z}/2 := \mathbb{Z}/2\mathbb{Z}$ is regarded as a discrete $\mathrm{Gal}(k_{sep}/k)$-module – where k_{sep} is a previously chosen separable closure of k – by endowing it with the trivial $\mathrm{Gal}(k_{sep}/k)$-action; note that as $char(k) \neq 2$, it is isomorphic as a $\mathrm{Gal}(k_{sep}/k)$-module to the group μ_2 of square roots of unity in k_{sep}. In order to describe this connection, we may first look at a more general setting. Suppose given a field k and a prime number $l \neq char(k)$. Then we have the Kummer exact sequence

$$1 \longrightarrow \mu_l \longrightarrow k_{sep}^\times \xrightarrow{\wedge l} k_{sep}^\times \longrightarrow 1$$

of discrete $\mathrm{Gal}(k_{sep}/k)$ modules, where $\mu_l \longrightarrow k_{sep}^\times$ is the inclusion of the subgroup of l-th roots of unity, and $\wedge l$ denotes the operation of raising to the l-th power. Then one obtains a long exact sequence of cohomology groups

$$0 \longrightarrow H^0(k, \mu_l) \longrightarrow H^0(k, k_{sep}^\times) \xrightarrow{H^0(k,\wedge l)} H^0(k, k_{sep}^\times) \longrightarrow H^1(k, \mu_l) \longrightarrow H^1(k, k_{sep}^\times) \longrightarrow \cdots,$$

which is isomorphic to

$$0 \longrightarrow \mu_l^{\mathrm{Gal}(k_{sep}/k)} \longrightarrow k^\times \xrightarrow{\wedge l} k^\times \longrightarrow H^1(k, \mu_l) \longrightarrow H^1(k, k_{sep}^\times) \longrightarrow \cdots.$$

A cohomological form of the classical 'theorem 90' by D. Hilbert, also known as 'Hilbert 90', states that the group $H^1(k, k_{sep}^\times)$ is trivial, so one obtains an isomorphism[1]

$$\partial : k^\times/l \xrightarrow{\cong} H^1(k, \mu_l).$$

As the tensor algebra $T(k^\times)$ is generated by degree 1 elements, this map uniquely extends (see Section 1.2) to a graded ring homomorphism

$$T(k^\times)/l \longrightarrow H^*(k, \mu_l^{\otimes *}).$$

On the other hand, it may be proved (in Milnor, 1970, Milnor attributes this result to H. Bass and J. Tate) that the Steinberg relations also hold in the Galois cohomology ring: given $a \neq 0, 1$ in k, one has that $\partial(a)\partial(1-a) \in H^2(k, \mu_l^{\otimes 2})$ is the zero element. Thus one

[1] In the exposition in Section 1.2, the isomorphism ∂ described here is denoted by ∂', as ∂ will then denote $k^\times \longrightarrow H^1(k, \mu_l)$.

obtains a ring homomorphism

$$\nu_* : K_M^*(k)/l \longrightarrow H^*(k, \mu_l^{\otimes *}), \tag{0.1.1}$$

known as the *norm residue homomorphism*. Although Milnor acknowledges in MILNOR, 1970 the existence of this map for arbitrary $l \neq char(k)$, he only studies the case $l = 2 \neq char(k)$. By MILNOR, 1970, Lemma 6.2, $\nu_* : K_M^*(k)/2 \longrightarrow H^*(k, \mu_2^{\otimes *}) \cong H^*(k, \mathbb{Z}/2)$ is an isomorphism whenever k is finite, local, global, real closed, or a direct limit of subfields for which ν_* is bijective. In Theorem 6.3, Milnor proves that if ν_* is an isomorphism for a given k, then it is also an isomorphism for the field of formal power series $k((t))$. He leaves it as an open question, which became known as the *Milnor conjecture*, whether ν_* would be an isomorphism for any field k with $char(k) \neq 2$.

The Milnor conjecture was proved to be true in the mid 1990s by Vladimir Voevodsky (see VOEVODSKY, 1997). Before its solution, however, a more general statement had also been proposed. Although Milnor did not consider the norm residue homomorphism ν_* for $l \neq 2$, it was also not clear whether it would be an isomorphism in general. The claim that $\nu_n : K_M^*(k)/l \longrightarrow H^n(k, \mu_l^{\otimes n})$ for any given field k, $l \neq char(k)$ a prime number, and $n \geq 0$ later became known as the *Bloch-Kato conjecture*, named after Spencer Bloch and Kazuya Kato. In KATO, 1980, Kato states it as Conjecture 1 in page 608. Bloch asked in BLOCH, 2010 (which is the second edition of an exposition based on a series of lectures given in 1979 at Duke University), Lecture 5, whether the cohomology ring $H^*(k, \mu_l^{\otimes *})$ is generated by elements of $H^1(k, \mu_l)$, which is equivalent to asking whether ν_* is surjective as this was already known in degree 1. His question was motivated by his proof that the multiplication map

$$H^1(k, \mu_l)^{\otimes n} \xrightarrow{\cup} H^n(k, \mu_l^{\otimes n})$$

is surjective whenever k is a function field of transcendence degree n over an algebraically closed field.

Particular cases of the Bloch-Kato (and Milnor) conjecture were proved by A. Merkurjev, A. Suslin and M. Rost between the 1980s and early 1990s. Firstly, Merkurjev showed in MERKURJEV, 1981 that the Milnor conjecture holds in degree 2, i.e. that

$$K_M^2(k)/2 \xrightarrow{\nu_2} H^2(k, \mu_2^{\otimes 2}) \cong H^2(k, \mathbb{Z}/2)$$

is an isomorphism for any field k such that $char(k) \neq 2$. Merkurjev and Suslin extended this result in MERKURJEV and A. SUSLIN, 1983 by proving that ν_2 is an isomorphism for any k and l such that $char(k) \neq l$. Rost in ROST, 1986 and Merkurjev and Suslin in MERKURJEV and A. SUSLIN, 1991 showed that ν_3 is always an isomorphism for $l = 2$.

Proofs of the Milnor and Bloch-Kato conjectures were given by Voevodsky in 1996 (see Voevodsky, 1997) and 2008 (see Voevodsky, 2011), respectively, in the latter case with a crucial contribution of M. Rost's work Rost, 2003 on the existence of norm varieties. Despite the original purely algebraic presentation of the conjectures, their proofs rely on a series of algebro-geometric concepts developed between the 1990s and 2000s. We summarize this process below.

The proofs of both conjectures relied on the language and techniques of $\mathbb{A}^1$-homotopy theory, also known as motivic homotopy theory, a theme relating algebraic geometry to algebraic topology that emerged in the mid-1990s from the attempt of a group of mathematicians, particularly F. Morel and V. Voevodsky, to develop a version of homotopy theory for algebraic varieties analogous to the traditional homotopy theory of topological spaces. Within this context, the affine line $\mathbb{A}^1$ over a field k assumes, in the category of varieties over k, a role analogous to that of the unit interval $[0, 1]$ in the category of topological spaces. A reference article is Morel and Voevodsky, 1999. $\mathbb{A}^1$-homotopy theory, in turn, is based on the language of homotopical algebra, introduced by D. Quillen in 1967 (Quillen, 1967), which establishes the use of a certain kind of additional structure on a category – rendering it a *model category* – so that it is endowed with a notion of homotopy theory and thus may be studied by means of several constructions analogous to those of classical homotopy theory, such as homotopies, (co)fibrations, homotopies, cylinders, and path spaces.

Another construction that was important in the proofs of the Milnor and Bloch-Kato conjectures is motivic cohomology. We now make a digression to describe some of the general ideas leading to it.

Motivic cohomology

Motivic cohomology is an invariant of smooth algebraic varieties which in certain ways plays a role analogous to that of singular cohomology in topology. The goals of this introduction will be to make this statement precise, to describe some of the phenomena which originally motivated it and provide a concise timeline for its development during the decades that followed, and also to discuss some of its main connections to phenomena in algebraic geometry, number theory, and topology.

Its origins can be directly traced back to a conjectural framework of mixed motivic sheaves proposed in the 1980s independently by A. Beilinson (see Beilinson, A., 1982) and S. Lichtenbaum (see Lichtenbaum, 1983); their aims were different, but the form and wished-for properties of such theories may be largely identified up to the choice of a Grothendieck topology (on suitable categories of schemes over the given variety) with respect to which one wishes to define sheaves, complexes, derived functors, etc.

One of the expected features of motivic cohomology is a good relation to the algebraic K-theory group of the given variety. In fact, we shall firstly discuss some background motivation from topology (namely, some of the ways in which topological K-theory relates to singular cohomology), state some of the basic structures known to be available for algebraic K-groups of schemes (namely, the γ-filtration), and then, inspired by these, introduce some of the basic requirements for a motivic cohomology theory (e.g. the fact that it should be a bigraded module over any chosen ring of coefficients).

In topology, one remarkable fact about generalized cohomology theories is the existence of a powerful method for approximating (and often actually computing) in such abelian groups by singular cohomology ones, which turn out to be usually both more elementary and more approachable. Indeed, let h be a generalized Eilenberg-Steenrod cohomology theory defined on the category (unpointed) of CW-complexes. Given a CW-complex X, we may consider its skeleta

$$\cdots \hookrightarrow X_{n-1} \hookrightarrow X_n \hookrightarrow \cdots$$

for $n \in \mathbb{Z}$, where we define $X_n = \varnothing$ for $n < 0$. Then this decomposition gives rise to a spectral sequence of abelian groups, known as the *Atiyah-Hirzebruch spectral sequence*, whose E_1-page has terms given by relative (generalized) cohomology groups

$$E_1^{p,q} := h^{p+q}(X_p, X_{p-1})$$

and with E_2-page

$$E_2^{p,q} := H^p(X; h^q(*)),$$

where $*$ denotes a point. It converges to $h^{p+q}(X)$ if, for example, X is a finite CW-complex. The terms $E_\infty^{p,q}$, i.e. the associated graded pieces of the corresponding filtration on the abelian groups $h^{p+q}(X)$, are given explicitly by

$$E_\infty^{p,q} = \frac{F^{p-1}h^{p+q}(X)}{F^p h^{p+q}(X)},$$

where $F^* h^{p+q}(X)$ is the decreasing filtration

$$F^n h^{p+q}(X) := \ker(h^{p+q}(X) \longrightarrow h^{p+q}(X_n))$$

induced from inclusions of skeleta $X_n \hookrightarrow X$.

In particular, by taking h to be (complex, representable) topological K-theory one obtains a spectral sequence

$$E_2^{p,q} = H^p(X; K^q(*)) \Longrightarrow K^{p+q}(X). \tag{0.1.2}$$

Recall that representable topological K-groups of unpointed CW-complexes may be defined as follows: one first defines $K^0(X)$ by endowing with a natural abelian group structure the set $[X, BU \times \mathbb{Z}]$ of homotopy classes of continuous maps to $BU \times \mathbb{Z}$, where BU (usually presented as an infinite Grassmanian) is a delooping of the infinite unitary group $U = \varinjlim_{n \geq 1} U(n)$, and where $\mathbb{Z}$ is endowed with the discrete topology. This is motivated by the fact that if X is a compact (e.g. finite) CW-complex, then $K^0(X)$ is isomorphic to the usual Grothendieck group completion of the monoid of isomorphism classes of $\mathbb{C}$-vector bundles on X; for a *pointed* CW-complex (Y, y) one defines its reduced (representable) K-theory in degree 0 as $\tilde{K}^0(Y, y) := \ker(K^0(Y) \longrightarrow K^0(y))$, so that in particular one has $K^0(X) \cong \tilde{K}^0(X_+)$, where $X_+ = (X_+, *)$ is obtained by taking the disjoint union of X with a point $*$. Topological K-theory in negative degrees is then determined by the suspension axiom for generalized cohomology theories: for each $n \geq 0$ one defines $K^{-n}(Y, y) := K^0(\Sigma^n(Y, y))$ and $\tilde{K}^{-n}(Y, y) := \tilde{K}^0(\Sigma^n(Y, y))$, where Σ^n denotes the n-fold reduced suspension functor; for unpointed X, one takes $K^{-n}(X) := \tilde{K}^0(\Sigma^n(X_+))$. Simple connectedness of BU may be used to prove that for each connected (Y, y) one has (using a $*$ as in $[-, -]_*$ to denote sets of pointed homotopy classes)

$$\tilde{K}^0(Y, y) = \ker([Y, BU \times \mathbb{Z}] \longrightarrow [y, BU \times \mathbb{Z}]) \cong [Y, BU] \overset{(!)}{\cong} [(Y, y), BU]_* \cong [(Y, y), BU \times \mathbb{Z}]_*.$$

It follows that for any (Y, y) and $n > 0$ it holds that

$$\tilde{K}^{-n}(Y, y) =\cong [\Sigma^n(Y, y), BU]_* \cong [\Sigma^n(Y, y), BU \times \mathbb{Z}]_*.$$

In particular, for $n > 0$ we have bijections

$$K^{-n}(*) \cong \tilde{K}^{-n}(S^0, *) \cong \tilde{K}^0(S^n, *) \cong \pi_n(BU) \cong \pi_n(BU \times \mathbb{Z}).$$

For $n = 0$, we directly compute $K^0(*) \cong \pi_0(BU \times \mathbb{Z}) \cong \mathbb{Z}$.

One property which makes topological K-theory greatly more manageable than its algebraic counterpart is the availability of Raoul Bott's *periodicity theorem*, a part of which may be stated as the claim that the loop space functor has period 2 (up to homotopy equivalence) when applied to $BU \times \mathbb{Z}$:

$$\begin{cases} \Omega(BU \times \mathbb{Z}) \simeq U, \\ \Omega^2(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}. \end{cases}$$

Although we shall not discuss real topological K-theory, it is worth mentioning that a role analogous to that of BU and U in the complex case is played by BO and O in the real one (where $O = \varinjlim_{n\geq 1} O(n)$ denotes the infinite orthogonal group). A form of Bott's result is also available for the latter spaces, where we instead have 8-fold periodicity:

$$\Omega^8(BO \times \mathbb{Z}) \simeq BO \times \mathbb{Z}.$$

The homotopy type of each intermediate i-fold loop space of $BO \times \mathbb{Z}$ (i.e. for $1 \leq i \leq 7$) can be explicitly described in terms of O, U, and the infinite symplectic group Sp.

Bott periodicity for $BU \times \mathbb{Z}$ immediately implies 2-fold periodicity for homotopy groups of $BU \times \mathbb{Z}$ and U, and more generally for complex topological K-theory (in non-positive degrees):

(i) Since BU and U are path-connected (equivalently, BU is simply connected), for each $n \geq 0$ we have

$$K^{-n}(*) \cong \pi_n(BU \times \mathbb{Z}) \cong \begin{cases} \pi_0(BU \times \mathbb{Z}) \cong \mathbb{Z}, & n \text{ even}, \\ \pi_0(U) \cong 0, & n \text{ odd}. \end{cases} \tag{0.1.3}$$

(ii) By the usual reduced suspension-loop space adjunction $\Sigma \dashv \Omega$, for each pointed CW-complex (Y, y) and $n > 0$ we have

$$\tilde{K}^{-n}(Y, y) \cong [\Sigma^n(Y, y), BU \times \mathbb{Z}]_* \cong [(Y, y), \Omega^n(BU \times \mathbb{Z})]_* \cong \begin{cases} [(Y, y), BU \times \mathbb{Z}]_*, & n \text{ even}, \\ [(Y, y), U]_*, & n \text{ odd}. \end{cases}$$

For an unpointed CW-complex X, $[X_+, BU \times \mathbb{Z}]_*$ and $[X_+, U]_*$ are further isomorphic to $[X, BU \times \mathbb{Z}]$ and $[X, U]$, respectively, hence complex unpointed representable topological K-theory is given in non-positive degrees by

$$K^{-n}(X) \cong \begin{cases} [X, BU \times \mathbb{Z}], & n \text{ even}, \\ [X, U], & n \text{ odd}. \end{cases} \tag{0.1.4}$$

In order to define K^n for $n > 0$ in such a way that the data $(K^n)_{n\in\mathbb{Z}}$ is part of a generalized cohomology theory, one may resort to the representability of generalized (co)homologies by spectra (in the context of stable homotopy theory), which ultimately follows from Brown's representability theorem. Complex (representable) K-theory is defined to be the generalized cohomology theory represented by the Ω-spectrum KU whose entries are

$$KU^n = \begin{cases} BU \times \mathbb{Z}, & n \text{ even}, \\ U, & n \text{ odd}. \end{cases}$$

and whose structure maps are the weak equivalences

$$KU^{2n} = BU \times \mathbb{Z} \simeq \Omega U = \Omega(KU^{2n+1}),$$

$$KU^{2n-1} = U \simeq \Omega(BU \times \mathbb{Z}) = \Omega(KU^{2n}).$$

From this we obtain a definition of topological complex (unpointed) K-theory as being given in all degrees $n \in \mathbb{Z}$ by the formula 0.1.4, and in particular by 0.1.3 when applied to a point. It follows that the E_2-term of the Atiyah-Hirzebruch spectral sequence for topological complex K-theory, 0.1.2, only depends on ordinary cohomology groups of X with integral coefficients: it assumes the form

$$E_2^{p,q} \cong \begin{cases} H^p(X;\mathbb{Z}), & q \text{ even}, \\ H^p(X;0) \cong 0, & q \text{ odd}. \end{cases} \tag{0.1.5}$$

Moreover, Atiyah and Hirzebruch proved in M.F. Atiyah and Hirzebruch, 1961 that this spectral sequence collapes – i.e. the differentials in the E_2-page are zero – after tensoring with $\mathbb{Q}$. This implies $E_2^{p,q} \cong E_\infty^{p,q}$ for all p, q, so one obtains a decomposition of the form

$$K^n(X) \otimes \mathbb{Q} \cong \bigoplus_{p+q=n} E_\infty^{p,q} \otimes \mathbb{Q} \cong \bigoplus_{p+q=n} E_2^{p,q} \otimes \mathbb{Q} \cong \bigoplus_{q \text{ even}} H^{n-q}(X;\mathbb{Z}) \otimes \mathbb{Q} \cong \bigoplus_{i\in\mathbb{Z}} H^{n-2i}(X;\mathbb{Z}) \otimes \mathbb{Q}. \tag{0.1.6}$$

In the 1980s, A. Beilinson (Beilinson, A., 1982, Beilinson, 1987, Beilinson *et al.*, 1987) and S. Lichtenbaum (Lichtenbaum, 1983) conjectured the existence of a framework of abelian categories of mixed motives which would, in particular, provide the desired invariants of algebraic varieties as Ext groups between certain canonical motives. In what follows, we give a brief and non-exaustive account of the properties initially expected of such categories and invariants.

As for the properties expected of such conjectural cohomological invariants (which Beilinson originally referred to as the "absolute cohomology" groups associated to a given, also conjectural theory of mixed motives), several of them are intrinsic and may be stated independently of the desired categorical presentation, while some are of a structural nature and hence may not. Among the structural ones, some directly concern the expected property of mixed motives to constitute an *abelian* category, while others, quite interestingly, may be entirely formulated in terms of the triangulated structure on the derived category of such a conjectural abelian category. For example, the Ext groups expected to characterize such invariants get replaced by shifted Hom groups. This suggests that it might also be a fruitful task to also look for triangulated categories which, in many respects, would be a satisfactory partial replacement for the abelian counterpart (whose existence would in turn depend on deeper issues).

In the following discussion, π will always denote the morphism of sites $X_{\text{ét}} \longrightarrow X_{\text{Zar}}$ (given by the inclusion functor $X_{\text{Zar}} \hookrightarrow X_{\text{ét}}$) for some scheme X which will be clear from context.

Beilinson's conjectural framework of motivic complexes and motivic cohomology of varieties over a field k would consist, among others, of the following kinds of structure:

(i) For each commutative ring with unit A, a contravariant functor on the category of smooth varieties over k assigning to each X a bigraded A-module $H^{*,*}_{\mathcal{M}}(X, A) \in \mathrm{Mod}_A^{\mathbb{Z}\times\mathbb{Z}}$.

Moreover, there would be homomorphisms

$$H^{m,p}_{\mathcal{M}}(X, A) \otimes H^{n,q}_{\mathcal{M}}(X, A) \longrightarrow H^{m+n,p+q}_{\mathcal{M}}(X, A)$$

for m, n, p, $q \in \mathbb{Z}$ endowing $H^{*,*}_{\mathcal{M}}(X, A)$ with a bigraded ring structure.

The integers m, p as in $H^{m,p}_{\mathcal{M}}(X, A)$ are usually referred to as the *cohomological degree* and *weight*, respectively.

(ii) For each smooth k-variety X, there would exist for $q \geq 0$ chain complexes

$$\mathbb{Z}(q) = (\cdots \longrightarrow \mathbb{Z}(q)_n \xrightarrow{d_{n+1}} \mathbb{Z}(q)_{n-1} \longrightarrow \cdots)$$

(with X implicit) of Zariski sheaves on X, called *motivic complexes*, satisfying:

(ii).**1** Let A be a commutative ring with unit; let us denote by $A(q) = \mathbb{Z}(q) \otimes A$ the complex obtained by degreewise tensoring with the constant abelian sheaf A.

For each $n \in \mathbb{Z}$, Zariski hypercohomology groups $H^n(X, A(q))$ of $A(q)$ are available (i.e. its corresponding hypercohomology spectral sequence with respect to the global section functor converges) and satisfy

$$H^n(X, A(q)) \cong H^{n,q}_{\mathcal{M}}(X, A).$$

(ii).**2** There would be given quasi-isomorphisms $\mathbb{Z}(0) \cong \mathbb{Z}$, where $\mathbb{Z}$ denotes the constant Zariski sheaf, and $\mathbb{Z}(1) \cong \mathcal{O}^\times[-1]$.

(ii).**3** For $q \geq 1$, $\mathbb{Z}(q)$ would be acyclic in all degrees not in $\{1, ..., q\}$. This became known as the Beilinson-Soulé vanishing condition (or "conjecture").

(ii).**4** For each $q \in \mathbb{Z}$ there would be a given a quasi-isomorphism

$$\mathbb{Z}(q) \otimes^L \mathbb{Z}/l \simeq \tau_{\leq q} \mathbf{R}\pi_*(\mathbb{Z}/l)_{\text{ét}}(q).$$

This became known as the Beilinson-Lichtenbaum condition (or "conjecture").

(ii).**5** There would be given maps of complexes

$$\mathbb{Z}(p) \otimes \mathbb{Z}(q) \longrightarrow \mathbb{Z}(p+q)$$

for each p, $q \in \mathbb{Z}$ such that for each commutative ring with unit A, the induced homomorphisms

$$H^n(X, A(p)) \otimes^L H^m(X, A(q)) \longrightarrow H^{n+m}(X, A(p) \otimes A(q)) \longrightarrow H^{n+m}(X, A(p+q))$$

(n, $m \in \mathbb{Z}$) provide, up to the isomorphisms in the previous item, the given bigraded ring structure on $H^{*,*}_{\mathcal{M}}(X, A)$.

(ii).**6** Motivic cohomology with $\mathbb{Q}$-coefficients should be canonically isomorphic, up to re-indexing, to the rationalized associated graded pieces of the γ-filtration on algebraic K-theory groups: for each n, $p \in \mathbb{Z}$ we should have

$$H^{n,p}_{\mathcal{M}}(X, \mathbb{Q}) \cong gr^p_\gamma K^{2p-n}(X) \otimes \mathbb{Q}$$

naturally in X.

Lichtenbaum, on the other hand, proposed a framework (see Lichtenbaum, 1983) closely related to Beilinson's but which would instead rely on the étale topology. His ideas were were incorporated into Beilinson's article Beilinson, 1987. We use the following notation: given a smooth k-variety X as above, for each commutative ring with unit A and $q \in \mathbb{Z}$ we denote the inverse image complex of étale sheaves $\pi^*(A(q))$ by $A(q)_{\text{ét}}$. Then the following would be desired:

1. Given $q \in \mathbb{Z}$ and a prime $l \neq char(k)$, there is an exact triangle of the form

$$\mathbb{Z}(q)_{\acute{e}t} \xrightarrow{l} \mathbb{Z}(q)_{\acute{e}t} \longrightarrow (\mathbb{Z}/l)(q)_{\acute{e}t} \longrightarrow \mathbb{Z}(q)_{\acute{e}t}[1].$$

2. For each integer q, the abelian group $\mathbf{R}^{q+1}\pi_*\mathbb{Z}(q)_{\acute{e}t}$ is trivial.

Higher Chow groups

An important construction in algebraic geometry is that of the Chow groups of a variety[2] over a field k, denoted by $CH^i(X)$ for $i \geq 0$. They are obtained as the quotient of the free abelian group on the set i-codimensional closed subvarieties of X (i.e. its group $\mathcal{Z}_i(X)$ of i-codimensional cycles) by the rational equivalence relation. By using the intersection product homomorphisms

$$CH^i(X) \otimes CH^j(X) \longrightarrow CH^{i+j}(X),$$

these groups may be assembled into a graded ring $CH^*(X)$, the Chow ring of X. In 1986, Spencer Bloch (see Bloch, 1986) applied simplicial techniques to the theory of algebraic cycles to produce a bigraded abelian group-valued invariant of varieties, called *higher Chow groups*; they are denoted by $CH^i(X,q)$ for $i, q \leq 0$, and in particular they recover the classical Chow groups as $CH^i(X) \cong CH^i(X,0)$.

His starting point was the existence for any quasiprojective k-variety X of an isomorphism of $\mathbb{Q}$-vector spaces, due to P. Baum, W. Fulton, and R. MacPherson (see the theorem in Baum *et al.*, 1975, III.1), between $\bigoplus_{i\geq 0}(CH^i(X) \otimes \mathbb{Q})$ and $K^0_{coh}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $K^0_{coh}(X)$ denotes the Grothendieck group of coherent sheaves on X. Moreover, the composite of this isomorphism with the decomposition

$$K^0_{coh}(X) \otimes \mathbb{Q} \cong \bigoplus_{i\geq 0}(\mathrm{Gr}^i_\gamma K^0_{coh}(X) \otimes \mathbb{Q})$$

given by the γ-filtration on $K^0_{coh}(X)$ defines for each $i \geq 0$ an isomorphism

$$CH^i(X) \otimes \mathbb{Q} \cong gr^i_\gamma K^0_{coh}(X) \otimes \mathbb{Q}.$$

Bloch aimed to define higher Chow groups in such a way that for any quasiprojective k-variety they satisfied

$$CH^i(X,q) \otimes \mathbb{Q} \cong gr^i_\gamma K^q_{coh}(X) \otimes \mathbb{Q}$$

[2] By a *variety* we mean an integral, separated, finite type scheme over a given field k.

for each $q, i \leq 0$, hence

$$\bigoplus_{i\geq 0}(CH^i(X,q)\otimes \mathbb{Q}) \cong K^q_{coh}(X)\otimes \mathbb{Q}.$$

This property holds for higher Chow groups as defined in Bloch, 1986 (see below), and is proved in Theorem 9.1 of the same article.

We limit ourselves to giving its definition and stating some of its most important properties; for a longer discussion of these, we refer the reader to Bloch, 1986, Bloch, 2010, and Part 5 of Mazza *et al.*, 2006.

Firstly, for a given field k one considers for each $q \leq 0$ the k-variety $\Delta^q_k := \operatorname{Spec} \frac{k[t_0,\ldots,t_q]}{(t_0+\ldots+t_q-1)}$, which is an algebro-geometric analogue of the standard q-simplex. These may be assembled into a cosimplicial k-variety, i.e. a functor $\Delta^*_k : \Delta \longrightarrow \mathrm{Var}_k$ from the simplex category to the category of k-varieties. We refer to Subsection 2.2.1 for a discussion of this construction. Now, if X is a k-variety, composing Δ^*_k with the product functor $X\times_k - : \mathrm{Var}_k \longrightarrow \mathrm{Var}_k$ yields a cosimplicial k-variety given on objects by $[q]\mapsto X\times_k \Delta^q_k$. The idea is to define, for each $i \geq 0$, a simplicial abelian group $\Delta^{op}\longrightarrow \mathrm{Set}$ that associates to each face inclusion $X\times_k \Delta^r \longrightarrow X\times_k \Delta^q$, $r \leq q$, a pullback map between groups of i-codimensional cycles in the opposite direction. The usual pullback is not available for arbitrary cycles as $X\times_k \Delta^r \longrightarrow X\times_k\Delta^q$ is in general not flat; on the other hand, as remarked in Bloch, 1986, the fact that the image of each face inclusion is a local complete intersection allows one to define the pullback from $X\times_k\Delta^q$ to $X\times_k\Delta^r$ of any cycle which intersects $X\times_k\Delta^r$ properly. Then one defines for each $i, q \leq 0$ a subgroup $\mathcal{Z}^i(X,q)$ of the group $\mathcal{Z}^i(X\times_k\Delta^q)$ of i-codimensional cycles consisting of those cycles that properly intersect all faces $X\times_k\Delta^r\longrightarrow X\times_k\Delta^q$ for $r\leq q$. This defines for each $i, q, p\in\{0,\ldots,q\}$ a map

$$\partial^p_q : \mathcal{Z}^i(X,q)\longrightarrow \mathcal{Z}^i(X,q-1),$$

and by proving that flat pullback under the degeneracy maps $s^p_q : X\times_k\Delta^{q+1}\longrightarrow X\times_k\Delta^q$ for $p\in\{0,\ldots,q\}$ send cycles in $\mathcal{Z}^i(X,q)$ to cycles in $\mathcal{Z}^i(X,q+1)$ one obtains for each $i\leq 0$ a simplicial abelian group

$$\mathcal{Z}^i(X,*).$$

The (i,q)-th *higher Chow group* of X, denoted by $CH^i(X,q)$ is defined as the q-th homology group of the chain complex of abelian groups associated with $\mathcal{Z}^i(X,*)$.

In particular, by identifying the affine line $\mathbb{A}^1_k$ with Δ^1_k via the isomorphism $t\mapsto (t_0,t_1) = (t, 1-t)$ it follows that $CH^i(X,0)$ is the quotient of the group of i-codimensional cycles on X by its subgroup consisting of cycles of the form $\alpha\cap(X\times_k\{0\}) - \alpha\cap(X\times_k\{0\})$

for $\alpha \in \mathcal{Z}(X \times \mathbb{A}^1_k)$ intersecting $X \times \{0\}$ and $X \times \{1\}$ properly. By FULTON, 1984, Prop. 1.6, it is isomorphic to the classical Chow group $CH^i(X)$.

Some of the main features of higher Chow groups (see BLOCH, 1986) are:

(i) They are defined for any variety over k.

(ii) For fixed i, $q \leq 0$, groups $CH^i(-, q)$ can be made functorial in several different ways, all of which recover the usual functoriality of classical Chow groups by taking $i = 0$:

- Covariantly with respect to proper morphisms between quasiprojective k-varieties.
- Contravariantly w.r.t. flat morphisms between quasiprojective k-varieties.
- Contravariantly w.r.t. arbitrary morphisms between smooth quasiprojective k-varieties.

(iii) Higher Chow groups are homotopy invariant: for a (not necessarily quasiprojective) k-variety X, the pullback map $CH^i(X, q) \xrightarrow{\pi^*} CH^i(X \times \mathbb{A}^1_k, q)$ is an isomorphism (BLOCH, 1986, Theorem 2.1).

(iv) There exist localization sequences of the following form: if $Z \hookrightarrow X$ is a closed immersion between quasiprojective k-varieties with open complement $U \hookrightarrow X$, then for each $i \leq 0$, denoting by n the codimension of Y in X, there exists an exact sequence of chain complexes

$$0 \longrightarrow \mathcal{Z}^{i-n}(Z, *) \longrightarrow \mathcal{Z}^i(X, *) \longrightarrow \mathcal{Z}^i(U, *)$$

such that the induced chain map $\mathcal{Z}^i(X, *)/\mathcal{Z}^{i-n}(Z, *) \to \mathcal{Z}^i(U, *)$ is a quasi-isomorphism. Thus there exists a long exact sequence

$$\cdots \to CH^i(U, q+1) \to CH^{i-n}(Z, q) \to CH^i(X, q) \to CH^i(U, q) \to CH^{i-n}(Z, q-1) \to \cdots.$$

This is proved in BLOCH, 1986, 3.1-3.3.

(v) If X is a smooth quasiprojective variety over k, one may define homomorphisms of the form

$$CH^i(X, q) \otimes CH^j(X, r) \longrightarrow CH^{i+j}(X, q + r)$$

that turn $CH^*(-, *)$ into a bigraded ring. This construction is performed in section 5 of BLOCH, 1986.

In the late 1990s, A. Suslin and V. Voevodsky (A. SUSLIN and VOEVODSKY, 1996) used a different approach to produce an invariant for smooth varieties over a field k satisfying many of Beilinson-Lichtenbaum's conditions. Although it is not defined for arbitrary k-schemes, when it does it coincides with Bloch's higher Chow groups. Moreover, it was a significant step towards establishing a cohomological invariant satisfying the structural part of Beilinson-Lichtenbaum's conditions. Firstly, it is indeed computable for each X as the hypercohomology groups (or modules) of certain complexes $\mathbb{Z}(n)$ of Zariski sheaves of abelian groups (or complexes $A(n)$ of sheaves of A-modules) on X; furthermore, such complexes are naturally obtained by restriction of certain complexes of *presheaves with transfers*, which are by definition presheaves on an additive category (of *finite correspondences* over k) in which the category of smooth k-varieties is canonically embedded. Two major consequences of this presentation are:

(i) One is able to produce both Zariski and étale variants of such invariants, as well as canonical change-of-topology maps between them.

(ii) Such invariants arise as (shifted) Hom groups (or modules) in a certain triangulated category $DM(k, \mathbb{Z})$ (or $DM(k, A)$ for a given commutative ring A). As we shall see, the latter are, roughly speaking, obtained by imposing homotopy invariance (with respect to the affine line) to the derived category of sheaves with transfers for the Nisnevich topology. We note that while this procedure (which may be obtained as a Verdier quotient done at the triangulated level, or equivalently as a Bousfield localization on a suitable model structure on the category of complexes of Nisnevich sheaves with transfers) solves the representability issue, it produces a triangulated category which need not arise as the derived category of an abelian category (it remains an open question).

In accordance with current mathematical practice, we will refer to these (and only these) as *(ordinary) motivic cohomology* and *étale motivic cohomology* groups (or modules), respectively. We also refer to the canonical complexes of sheaves computing them as *motivic complexes*, and to categories of the form $DM(k, A)$ as *(triangulated) categories of Voevodsky (mixed) motives*. The Beilinson-Lichtenbaum conditions previously stated are often regarded as axioms for to-be-constructed "motivic cohomology theory" and "motivic complexes", but in order to avoid ambiguity we shall strictly reserve this terminology for Suslin-Voevodsky's construction; this then allows us to regard the former conditions as propositions or conjectures on motivic cohomology/motivic complexes in our sense.

Similarly, the notation $\mathbb{Z}(n)$ (or $A(n)$ for an abelian group or ring A) will also be reserved throughout the text for the (motivic) complexes as constructed by Suslin-Voevodsky.

Motivic cohomology in this sense may be compared with Bloch's higher Chow groups, and this allows one to exchange properties from one construction to the other. By Mazza *et al.*, 2006, Theorem 19.1, if k is a perfect field, then for any smooth, separated, finite type k-scheme X there is an isomorphism

$$CH^q(X, p) \cong H^{p+2q,q}(X, \mathbb{Z})$$

for all $p, q \in \mathbb{Z}$. This is done by constructing (Mazza *et al.*, 2006, 19.8) for perfect k and each $q \in \mathbb{Z}$ a quasi-isomorphism of complexes of Zariski sheaves

$$\mathbb{Z}(q)[2q] \simeq \mathcal{Z}^q(- \times \mathbb{A}^q_k, *)$$

on the category Sm_k of smooth, separated, finite type k-schemes, and by proving (Mazza *et al.*, 2006, 19.12) that for each $p \in \mathbb{Z}$ the Zariski cohomology group $H^p_{\mathrm{Zar}}(X, \mathcal{Z}^q(- \times \mathbb{A}^q_k, *))$ is isomorphic to $CH^q(X, p)$.

0.1.1 Concluding remarks on the use of motivic cohomology for proving the Milnor and Bloch-Kato conjectures

In 2000, Voevodsky published the article Voevodsky, 2000, which presented in a unified way four distinct theories in accordance with the proposal of Beilinson and Lichtenbaum: motivic homology, motivic homology with compact support, motivic cohomology, and motivic cohomology with compact support, with the first one corresponding to the groups defined by Bloch. For that purpose, Voevodsky constructed, for any field k and any abelian group A, a triangulated category known as the *category of Voevodsky motives* over k (with coefficients in A), denoted by $DM(k, A)$. For each k-scheme X, two objects are functorially associated in $DM(k, A)$: the motive of X, denoted $M(X)$, and the motive of X with compact support, denoted by $M^c(X)$. Thus, the four theories mentioned above are given by functors representable in $DM(k, A)$ (after the association $X \longmapsto M(X)$ or $X \longmapsto M^c(X)$). The representing object for $H^{n,q}$ is the n-th shift (in the triangulated category) of the Tate motive $A(q)$. Then one obtains another formulation of the motivic cohomology groups $H^{n,q}(X, A)$, which in this case is also denoted by $H^n(X, A(q))$.

Voevodsky's strategy to prove the Bloch-Kato conjecture consisted in translating the homomorphisms of the form 0.1.1 above as comparison homomorphisms between ordinary and étale motivic cohomology groups.

Firstly, one analyzes the codomain of 0.1.1. A foundational result on étale cohomology shows that when it is computed for étale sheaves on Spec k, it coincides, in a certain sense, with the Galois cohomology groups of k. In the present setting, the Galois cohomology group $H^n(k, \mu_l^{\otimes n})$ is canonically isomorphic to an étale cohomology group $H^n_{\text{ét}}(\mathrm{Spec}\ k, \mathscr{F})$, where $\mathscr{F}$ is a sheaf ("of coefficients") canonically determined by the Gal(k_{sep}/k)-module $\mu_l^{\otimes n}$. It is also worth noting that it is possible to define (cf. Mazza *et al.*, 2006, for example) a variation of the motivic cohomology called étale motivic cohomology. As a result, the group $H^n(k, \mu_l^{\otimes n})$ also becomes canonically isomorphic to a (motivic étale) cohomology group $H^{n,n}_{\text{ét}}(\text{Spec } k, \mathbb{Z}/l)$.

In a less elementary way, the domain, $K^n_M(k)/l$, is also interpreted as a motivic cohomology group. It can be shown that for a d-dimensional variety X over k, for every $n \geq 0$, there exists a canonical homomorphism ∂ such that

$$H^{2d+n,d+n}(X) \cong \text{Coker}\left(\coprod_{x \in X^{(d-1)}} K^{n+1}_M(\kappa(x)) \xrightarrow{\partial} \coprod_{x \in X^{(d)}} K^n_M(\kappa(x))\right),$$

where: $X^{(d-1)}$ (resp. $X^{(d)}$) is the set of $d-1$ (resp. d) codimensional points of the scheme X, i.e. the closed points (resp. irreducible curves) in X; and $\kappa(x)$ is the residue field of X at x. In particular, taking $X = \text{Spec } k$ (which has a single point, of zero dimension) yields $H^{n,n}(\text{Spec } k, \mathbb{Z}) = K^n_M(k)$ and that $H^{n,n}(\text{Spec } k, \mathbb{Z}/l\mathbb{Z}) = K^n_M(k)/l$.

Thus, the Bloch-Kato conjecture claims the existence of certain isomorphisms of the form

$$H^{n,n}(\text{Spec } k, \mathbb{Z}/l) \longrightarrow H^{n,n}_{\text{ét}}(\text{Spec } k, \mathbb{Z}/l).$$

More generally, given a smooth variety X over k and l a prime number different from $char(k)$, there exist canonical homomorphisms

$$H^{n,q}(X, \mathbb{Z}/l\mathbb{Z}) \longrightarrow H^{n,q}_{\text{ét}}(X, \mathbb{Z}/l). \tag{0.1.7}$$

between ordinary and étale motivic cohomology. Thus the Bloch-Kato conjecture may be characterized as a claim about a particular map within a family of comparison homomorphisms. Such an observation suggests, more generally, the search for conditions on X, l, n and q under which 0.1.7 would be an isomorphism.

The so-called Beilinson-Lichtenbaum conjecture, predicted even before Voevodsky's definition of motivic cohomology in Voevodsky, 2000, is the statement that 0.1.7 is an isomorphism whenever $n \leq q$. In 1996, Voevodsky and Suslin showed in A. Suslin and Voevodsky, 1996 that the Bloch-Kato conjecture partially implies the Beilinson-Lichtenbaum conjecture (with the abstract properties of the then conjectural motivic cohomology). Shortly afterwards, T. Geisser and M. Levine proved in Geisser and Levine, 2001 that the implication holds in general.

Another condition turns out to be fundamental for studying the Bloch-Kato conjecture. By taking $X = \operatorname{Spec} k$ and $n = q + 1$ in 0.1.7, one obtains a map from $H^{q+1,q}(\operatorname{Spec} k, \mathbb{Z}/l)$ – which is trivial group – to $H^{q+1,q}_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}/l)$. The claim that $H^{q+1,q}_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}/l)$ is also trivial may be regarded as a generalization of the classical 'theorem 90' by D. Hilbert (also known as 'Hilbert 90'), which states, if framed in terms of Galois cohomology, that the group $H^1(k, k^\times_{sep})$ is trivial. Here, the crucial fact is that this conjectural generalized form of Hilbert 90 implies the Bloch-Kato Conjecture (cf. Voevodsky, 2003, for example), and hence the Beilinson-Lichtenbaum Conjecture.

In Voevodsky, 2000, Voevodsky proved that the generalized Hilbert 90 condition would follow from the existence of certain algebraic varieties, called norm varieties. Its existence has been showed, although not to the full extent originally envisaged, by M. Rost (cf. Rost, 2003, A. Suslin and Joukhovitski, 2006). Based on this, the article Voevodsky, 2011 by Voevodsky, originally published in 2008, makes a series of needed adaptations to the previous text and concludes the proof of the three conjectures.

0.2 Structure of the dissertation; conventions

This text is divided into three chapters.

Chapter 1 discusses Galois cohomology and Milnor K-theory from a classical point of view. In the first section we study the cohomology of discrete groups and of Galois groups. In the second one, the Kummer exact sequence and the 'Hilbert 90' theorem are used to produce, for a given field k, a ring homomorphism from the tensor algebra $T(k^\times)$ of its group of units to a Galois cohomology ring with coefficients in modules of roots of unity. Finally, the vanishing of certain elements of this Galois cohomology ring provides a map from the Milnor K-theory ring to it.

Chapter 2 is an introduction to motivic cohomology in terms of motivic complexes. It is dedicated to presenting general constructions and properties. First we define the additive category of finite correspondences over a given field k, denoted by Cor_k, which can be regarded as an extension of the category of smooth varieties over k. Then we discuss presheaves with transfers, which are additive presheaves of abelian groups (or of modules over a ring) on Cor_k; sheaves with transfers are those presheaves with transfers which satisfy the usual sheaf condition when restricted to the subcategory of Cor_k consisting of usual morphisms of schemes. We define the chain complexes of presheaves with transfers $A(q)$ (which are in fact étale – hence Zariski – sheaves with transfers), where A is an abelian group and $q \leq 0$ is an integer. For an abelian group A and integers p, $q \leq 0$, the motivic cohomology group $H^{p,q}(X, A)$ is defined as the p-th Zariski cohomology of X with respect to the restriction of $A(q)$ to its Zariski site. As $A(q)$ is an étale sheaf, so we also consider étale motivic cohomology groups $H^{p,q}_{\text{ét}}(X, A)$, which is defined analogously in terms of the étale site of X. We discuss the existence of a quasi-isomorphism from $\mathbb{Z}(1)$ to the presheaf with transfers of global units, $\mathscr{O}^{\times}$, placed in cohomological degree 1. Finally, we present some constructions concerning a notion of étale sheafification for presheaves with transfers, and the notion of homotopy invariance of presheaves with transfers.

In Chapter 3 we sketch how motivic cohomology and Voevodsky's theory of mixed motives may be used for providing alternative characterizations of the objects involved in the (former) Bloch-Kato conjecture as well as the conjecture itself. The Milnor K-theory groups of a given k are identified with certain motivic cohomology groups of Spec k; similarly, étale cohomology with coefficients in sheaves of roots of unity are identified with certain étale motivic cohomology groups. The norm residue homomorphism is then characterized as a 'change of topology' map constructed via the adjunction between the category of Zariski sheaves and that of étale sheaves.

We now list some preliminary material and conventions that will be used throughout the text.

We will assume some familiarity with general topology, group theory and ring theory (particularly introductory commutative algebra and Galois theory – see Jacobson, 1964 and M. Atiyah, 1969), as well as with algebraic geometry via schemes. For the latter we refer the reader to Hartshorne, 1977, chapters 1 and 2, and Mumford, 1988. We will also the concepts of smooth and étale morphism of schemes, for which we refer to Mumford, 1988, chapter 3 of Hartshorne, 1977, and Milne, 1980.

Several concepts from category theory will be used throughout the text: categories, functors, natural transformations, adjunctions, representable functors and the Yoneda lemma, limits and colimits; comma categories (especially overcategories); localization of categories; monoidal categories. We will also consider sheaf theory in terms of Grothendieck topologies will be needed: sieves, Grothendieck pretopologies and topologies, sites, (pre)sheaves and categories of (pre)sheaves, direct and inverse image sheaf functors. We refer the reader to Lane, 1971 and Artin *et al.*, 1973.

From homological algebra, we will use additive and abelian categories, categories of (co)chain complexes, (co)homology of complexes and quasi-isomorphisms; additive, left/right exact, and exact functors; derived categories and derived functors; triangulated categories. We refer the reader to C.A. Weibel, 1995, Gelfand and Manin, 2003, Cartan and Eilenberg, 1956, Grothendieck, 1957.

Moreover, the following conventions will be used:

- If $\mathcal{C}$ is a category, we will write $a \in \mathcal{C}$ (instead of $a \in \mathrm{Ob}(\mathcal{C})$, for example) when a is an object of $\mathcal{C}$. We will not use a similar abuse of notation for arrows in a category.
- If $\mathcal{C}$ and $\mathcal{D}$ are categories, the notation $F : \mathcal{C} \longrightarrow \mathcal{D}$ refers to a functor in the covariant sense. By a *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ we will mean a functor $F : \mathcal{C}^{op} \longrightarrow \mathcal{D}$, where $\mathcal{C}^{op}$ denotes the opposite category of $\mathcal{C}$.

  The category of functors and natural transformations from $\mathcal{C}$ to $\mathcal{D}$ will be denoted by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

  We will often refer to a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ (particularly when $\mathcal{D} = \mathrm{Set}$ or Ab) as a $\mathcal{D}$-valued *presheaf* on $\mathcal{C}$. The corresponding *category of presheaves* is defined as

  $$\mathrm{PSh}(\mathcal{C}, D) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D}).$$

- By a *chain complex* in an abelian category $\mathcal{A}$ we will mean a pair $((C_i)_{i\in\mathbb{Z}}, (d_i)_{i\in\mathbb{Z}})$ consisting of a $\mathbb{Z}$-indexed family of objects C_i of $\mathcal{A}$ and a $\mathbb{Z}$-indexed family of arrows in $\mathcal{A}$ of the form $d_i : C_i \longrightarrow C_{i-1}$ with the property that $d_i \circ d_{i+1} = 0$ for every $i \in \mathbb{Z}$. A *cochain complex* is a pair $((C^i)_{i\in\mathbb{Z}}, (d^i)_{i\in\mathbb{Z}})$ consisting of a $\mathbb{Z}$-indexed family of objects C^i of $\mathcal{A}$ and a $\mathbb{Z}$-indexed family of arrows in $\mathcal{A}$ of the form $d^i : C^i \longrightarrow C^{i+1}$ such that $d^{i+1} \circ d^i$ for every $i \in \mathbb{Z}$.

  The term *complex* will be used to refer to a cochain complex.

  In the above notation, a (cochain) complex is said to be *bounded below* (resp. *bounded above*) if there exists $n \in \mathbb{Z}$ such that $C_i \cong 0$ for every $i \leq n$ (resp. for every $i \geq n$). A complex is said to be *bounded* if it is both bounded below and bounded above.

  We will sometimes say that a complex (C^*, d^*) is *concentrated* in a given set of integers to mean that $C^n \cong 0$ whenever n does not belong to that set. For example, being concentrated in degree 0 means that $C^n \cong 0$ for every $n \neq 0$, and being concentrated in non-negative degrees means that $C^n \cong 0$ for every $n < 0$.

  The category of complexes and chain maps in $\mathcal{A}$ will be denoted by $\mathrm{Ch}(\mathcal{A})$. The full subcategories of $\mathrm{Ch}(\mathcal{A})$ whose objects are the bounded below, bounded above, bounded complexes, resp. are denoted by $\mathrm{Ch}^+(\mathcal{A})$, $\mathrm{Ch}^-(\mathcal{A})$, $\mathrm{Ch}^b(\mathcal{A})$. The corresponding derived categories, i.e. the categories obtained from $\mathrm{Ch}^+(\mathcal{A})$, $\mathrm{Ch}^-(\mathcal{A})$, $\mathrm{Ch}^b(\mathcal{A})$ by localization at the quasi-isomorphisms, are denoted by $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$, resp.

- A set X endowed with a left action of a group G will be referred to as a *G-set.* The category of G-sets and functions which preserve the action of G will be denoted by Set_G.

- By a *module* over a group G we will always mean a left module, i.e. an abelian group M endowed with a left group action $\mu : G \times M \longrightarrow M$, say denoted by $(g, m) \mapsto g \cdot m$, such that $g \cdot (a + b) = g \cdot a + g \cdot b$ for every $g \in G$ and $a, b \in M$. We use the notation Mod_G for the category of (left) G-modules and module homomorphisms. Similarly, given a (not necessarily commutative) unital ring R, we use the the *module* to refer to a left module, and the category of (left) R-modules will be denoted by Mod_R.

- We will denote the category of schemes by Sch. For a given scheme X, we denote by Sch_X the category of schemes over X: its objects are pairs (Y, f) where Y is a scheme and f is a scheme morphism from Y to X; morphisms from (Y, f) to (Y', f') are scheme morphisms $g : Y \longrightarrow Y'$ such that $f' \circ g = f$. We will usually refer to an object of Sch_X as an *X-scheme*; by abuse of notation, we will often denote (Y, f) by Y.

  The full subcategory of Sch_X whose objects are the finite type X-schemes (i.e. those (Y, f) such that f is a finite type morphism) will be denoted by FTSch_X.

  We will denote by Sm_X the full subcategory of Sch_X whose objects are smooth, separated, finite type schemes over X.

  In case X is an affine scheme Spec A, we denote these categories by Sch_A, FTSch_A, Sm_A (we will mostly deal with the case $X = \operatorname{Spec} k$ for a field k).

- For a given field k: $\mathbb{A}^n_k$ denotes the n-dimensional affine space over k, i.e. $\operatorname{Spec} k[x_1, ..., x_n]$; $\mathbb{P}^n_k$ denotes the n-dimensional projective space over k; following Mazza *et al.*, 2006, the notation $\mathbb{G}_{m,k}$ will be used exclusively for the *pointed* k-scheme $(\mathbb{A}^1_k \setminus \{0\}, s_1)$, where the k-morphism $s_1 : \operatorname{Spec} k \longrightarrow \mathbb{A}^1_k$ is the inclusion of the point $\{1\}$; the unpointed version will be denoted by $\mathbb{A}^1_k \setminus \{0\}$.

  When k is clear from the context, these will be denoted by $\mathbb{A}^n$, $\mathbb{P}^n$, $\mathbb{G}_m$.

Chapter 1

Galois cohomology, Milnor K-theory, and the norm residue homomorphism: the classical point of view

In 1970, when Quillen's general and now widely accepted definition of higher K-theory was still not available, Milnor introduced (see Milnor, 1970) a certain algebraic invariant for fields k which provided an ad-hoc generalization of the algebraic K-theory groups $K^0(k)$, $K^1(k)$, and $K^2(k)$ to higher degrees. More precisely, he defined for each k a graded-commutative ring $K_M^*(k)$ such that $K_M^i(k) \cong K^i(k)$, as abelian groups, for $i = 0, 1, 2$. Although it may be defined in terms of generators and relations, as we shall see, it did not seem to provide an adequate definition of higher K-theory in the sense mathematicians were looking for, since it apparently could not be extended to general rings and it lacked the expected homotopical and homological properties. Despite its external appearance, it turns out that to give a finer description of the internal structure of Milnor's K-theory ring is a problem far from elementary. One particularly striking attempt at (partially) characterizing Milnor's K-groups, already suggested by Milnor himself, was a conjecture claiming that if we reduced Milnor's K-theory ring modulo 2, that is, if we took the quotient $K_M^*(k)/2 \cong \bigoplus_{n\geq 0} K_M^n(k)/2$, denoted simply $k_M^*(k) \cong \bigoplus_{n\geq 0} k_M^n(k)$, then each $k_M^n(k)$ would be given a certain Galois cohomology group of k, and the product operation on the ring $k_M^*(k)$ would even correspond to the usual cup-product in Galois cohomology. As we shall see, this is motivated by the existence, for each $n \geq 0$, of a certain map

$$K_M^n(k) \longrightarrow H^n(k, \mathbb{Z}/2)$$

sending $2a$ to zero for every $a \in K_M^n(k)$, where the group on the right is the n-th Galois cohomology of k with respect to the abelian group $\mathbb{Z}/2$ (endowed with the trivial action of the absolute Galois group of k). The corresponding homomorphisms $k_M^n(k) \longrightarrow H^n(k, \mathbb{Z}/2)$ can be assembled into homomorphism of graded rings $\mu : k_M^*(k) \longrightarrow H^*(k, \mathbb{Z}/2)$. (Note that the ring structure on $H^*(k, \mathbb{Z}/2)$ relies on the canonical isomorphism $\mathbb{Z}/2 \otimes_{\mathbb{Z}}$

$\mathbb{Z}/2 \cong \mathbb{Z}/2$, as the cup product will be given by maps $H^m(k, (\mathbb{Z}/2)^{\otimes m}) \otimes H^n(k, (\mathbb{Z}/2)^{\otimes n}) \longrightarrow H^{m+n}(k, (\mathbb{Z}/2)^{\otimes m+n})$.) No field k was ever discovered for which the map μ was not an isomorphism, and the claim that it is always an isomorphism became known as the *Milnor conjecture.*

1.1 Group cohomology

In this section, we provide an overview of some constructions and results on group cohomology. First we consider a definition which is applicable to groups in general, and then we present a particular notion of cohomology – which is based on the previous one – that is suitable to the study of Galois groups.

1.1.1 Cohomology of abstract groups

Let G be a group. We denote by $\mathbb{Z}[G]$ its corresponding group ring. Recall that it is the (non-commutative, with unit) ring whose underlying additive group is the free abelian group on the set of elements of G, and whose multiplicative structure is given by the unique linear extension of the operation $(1 \cdot g, 1 \cdot h) \longmapsto gh$ on generators. This construction yields a functor Grp $\longrightarrow$ Ring which is left adjoint to the functor Ring $\longrightarrow$ Grp taking a ring R to $R^\times$, its multiplicative group of units. For any such G, there is a canonical isomorphism of categories $\mathrm{Mod}_G \cong \mathrm{Mod}_{\mathbb{Z}[G]}$: given a (left) G-module, the multiplication by elements of G extends uniquely, by linearity, to a multiplication by elements of $\mathbb{Z}[G]$; conversely, for any left $\mathbb{Z}[G]$-module we obtain a G-module by restriction, and one may check that these two constructions are inverse to each other as functors. For this reason, we shall always use the same notation for any G-module and its corresponding left $\mathbb{Z}[G]$-module. We will need the following theorem (see C.A. Weibel, 1995, 2.3):

Proposition 1.1.1. For any (not necessarily commutative) unital ring R, the category Mod_R, of left R-modules, is abelian with enough injectives.

In particular, $\mathrm{Mod}_{\mathbb{Z}[G]}$ and Mod_G are both abelian categories with enough injectives. We recall that given abelian categories $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ has enough injectives, and a left exact functor $F : \mathcal{A} \longrightarrow \mathcal{B}$, then we can define (up to natural isomorphism) its right derived functors $\mathbf{R}^i F : \mathcal{A} \longrightarrow \mathcal{B}$ (where $i \geq 0$). Equivalently, we can define (again, up to natural isomorphism) its so-called total derived functor $\mathbf{R}F : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$, where $D^+(\mathcal{A})$ denotes the full subcategory of $D(\mathcal{A})$, the derived category of $\mathcal{A}$, having as objects the bounded below complexes in $\mathcal{A}$, and analogously for B. In this case, the classical derived functors $\mathbf{R}^i F$ can be recovered as the cohomology objects $\mathbf{R}^i F(A) \cong H^i(RF(A))$ ($i \geq 0$, $A \in \mathcal{A}$), where the argument A in $RF(A)$ denotes the complex consisting of the object A in degree 0, and zero elsewhere. For a full account of these facts, see Gelfand and Manin, 2003.

For any G-module A, we denote by A^G the subgroup of G-invariant elements in A. Now, let $\mathbb{Z}$ denote the additive group of integers with the trivial action of G. Then there is a canonical isomorphism

$$A^G \xrightarrow{\cong} \mathrm{Hom}_G(\mathbb{Z}, A) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A),$$

given by $a \longmapsto \varphi_a$, where $\varphi_a(n) = na$ for $n \in \mathbb{Z}$. The naturality of this isomorphism can be rephrased by saying that the diagram of additive functors

$$\begin{tikzcd} \mathrm{Mod}_G \arrow[dd, "\cong"'] \arrow[dr, "\Gamma_G"] & \\ & \mathrm{Ab} \\ \mathrm{Mod}_{\mathbb{Z}[G]}, \arrow[ur, "\Gamma_{\mathbb{Z}[G]}"'] & \end{tikzcd}$$

where $\Gamma_G : A \longmapsto A^G$ and $\Gamma_{\mathbb{Z}[G]} : A \longmapsto \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A)$, is commutative up to natural isomorphism. Since Γ_G and $\Gamma_{\mathbb{Z}[G]}$ are left exact (as is any functor $\mathrm{Hom}_{\mathcal{A}}(A, -) : \mathcal{A} \longrightarrow \mathrm{Ab}$ for A in an abelian category $\mathcal{A}$), we can define their right derived functors, and the above isomorphism implies that

$$\begin{tikzcd} \mathrm{Mod}_G \arrow[dd, "\cong"'] \arrow[dr, "\mathbf{R}^i\Gamma_G"] & \\ & \mathrm{Ab} \\ \mathrm{Mod}_{\mathbb{Z}[G]} \arrow[ur, "\mathbf{R}^i\Gamma_{\mathbb{Z}[G]}"'] & \end{tikzcd}$$

commutes up to natural isomorphism for each $i \geq 0$, or equivalently that the same holds for

$$\begin{tikzcd} D^+(\mathrm{Mod}_G) \arrow[dd, "\cong"'] \arrow[dr, "R\Gamma_G"] & \\ & D^+(\mathrm{Ab}) \\ D^+(\mathrm{Mod}_{\mathbb{Z}[G]}). \arrow[ur, "R\Gamma_{\mathbb{Z}[G]}"'] & \end{tikzcd}$$

Definition 1.1.2. Let G be an abstract group. For each $i \geq 0$, either of the functors $\mathbf{R}^i\Gamma_G : \mathrm{Mod}_G \longrightarrow \mathrm{Ab}$ and $\mathbf{R}^i\Gamma_{\mathbb{Z}[G]} : \mathrm{Mod}_{\mathbb{Z}[G]} \longrightarrow \mathrm{Ab}$ is called the *i-th (group) cohomology functor* of G and is denoted by $H^i(G, -)$. For each G-module A, $H^i(G, A)$ is called the *i-th* *(group) cohomology group of G with coefficients in A.*

In order to better describe such functors, we remark that they are a particular case of the Ext functor construction from homological algebra. More precisely, suppose $\mathcal{A}$ is an abelian category. Then we have a functor $\mathrm{Hom}_{\mathcal{A}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$ with the properties that

- For any fixed $A \in \mathcal{A}$, the functor $\mathrm{Hom}_{\mathcal{A}}(A,-) : \mathcal{A} \longrightarrow \mathrm{Ab}$ is left exact.
- $\mathcal{A}^{op}$ is also an abelian category, and for any fixed $B \in \mathcal{A}$, the functor $\mathrm{Hom}_{\mathcal{A}}(-,B) : \mathcal{A}^{op} \longrightarrow \mathrm{Ab}$ is left exact.

Assuming that $\mathcal{A}$ has enough injectives, the right derived functors $\mathbf{R}^i\mathrm{Hom}_{\mathcal{A}}(A,-) : \mathcal{A} \longrightarrow \mathrm{Ab}$ are denoted by $\mathrm{Ext}^i_{\mathcal{A}}(A,-)$. Analogously, if $\mathcal{A}^{op}$ has enough injectives, which means that $\mathcal{A}$ has enough *projectives*, then the right derived functors $\mathbf{R}^i\mathrm{Hom}_{\mathcal{A}}(-,B) : \mathcal{A}^{op} \longrightarrow \mathrm{Ab}$ are denoted by $\mathrm{Ext}^i_{\mathcal{A}}(-,B)$. Since this notation is ambiguous, we will (temporarily) denote $\mathrm{Ext}^i_{\mathcal{A}}(A,-)$ by $\text{I-Ext}^i_{\mathcal{A}}(A,-)$ and $\mathrm{Ext}^i_{\mathcal{A}}(-,B)$ by $\text{II-Ext}^i_{\mathcal{A}}(-,B)$, respectively. An astonishing feature of the language of derived categories is that it allows for a clean treatment of Ext functors and leads to the conclusion that in case $\mathcal{A}$ has both enough injectives and enough projectives, we actually have $\text{I-Ext}^i_{\mathcal{A}}(A,B) \cong \text{II-Ext}^i_{\mathcal{A}}(A,B)$ and that this defines a functor $\mathrm{Ext}^i_{\mathcal{A}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Ab}$.

Proposition 1.1.3. Let $\mathcal{A}$ be an abelian category. If $\mathcal{A}$ has enough injectives (resp. enough projectives), then for any $A, B \in \mathcal{A}$, we have

$$\text{I-Ext}^i_{\mathcal{A}}(A,B) \cong \mathrm{Hom}_{D^+(\mathcal{A})}(A, B[n])$$
$$(\text{resp. II-Ext}^i_{\mathcal{A}}(A,B) \cong \mathrm{Hom}_{D^+(\mathcal{A})}(A, B[n]))$$

naturally in A and B. Hence if $\mathcal{A}$ has both enough injectives and enough projectives, we denote by Ext either of the functors I-Ext and II-Ext.

We shall deduce it from the following lemma:

Lemma 1.1.4. Let $\mathcal{A}$ be an abelian category and X^* a complex in $\mathcal{A}$. Then

1. If I^* is a bounded below complex of injective objects in $\mathcal{A}$, then the canonical map $\mathrm{Hom}_{K(\mathcal{A})}(X^*, I^*) \longrightarrow \mathrm{Hom}_{D(\mathcal{A})}(X^*, I^*)$ is an isomorphism.
2. If P^* is a bounded above complex of projective objects in $\mathcal{A}$, then the canonical map $\mathrm{Hom}_{K(\mathcal{A})}(P^*, X^*) \longrightarrow \mathrm{Hom}_{D(\mathcal{A})}(P^*, X^*)$ is an isomorphism.

Proof. See Gelfand and Manin, 2003, p. 183. □

Proof of Proposition 1.1.3. Suppose $\mathcal{A}$ has enough injectives, and let us prove that $\text{I-Ext}^i_{\mathcal{A}}(A,B) \cong \mathrm{Hom}_{D^+(\mathcal{A})}(A, B[n])$. Let I^*_B be an injective resolution of B (recall that this means we have an exact sequence

$$\cdots \longrightarrow 0 \longrightarrow B \longrightarrow I^0_B \longrightarrow I^1_B \longrightarrow \cdots,$$

with B in degree 0, whereas I^*_B is

$$\cdots \longrightarrow 0 \longrightarrow I_B^0 \longrightarrow I_B^1 \longrightarrow \cdots$$

with I_B^0 in degree 0). We have $B \cong I_B^*$ in $D(\mathcal{A})$ (since they are quasi-isomorphic in $Ch(\mathcal{A})$), whence

$$\mathrm{Hom}_{D(\mathcal{A})}(A, B[i]) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, I_B^*[i]) \cong \mathrm{Hom}_{K(\mathcal{A})}(A, I_B^*[i]),$$

where the last isomorphism from Lemma 1.1.4. On the other hand, we have $\text{I-Ext}^i_{\mathcal{A}}(A,B) \cong H^i(\overbrace{\mathrm{Hom}_{\mathcal{A}}(A, I_B^*}^{\in Ch(\mathrm{Ab})})$. Now, define for any complexes X^*, $Y^* \in Ch(\mathcal{A})$ a complex $\mathrm{Hom}^*(X^*, Y^*) \in Ch(\mathrm{Ab})$ in the following way: for each $i \in \mathbb{Z}$, we take $\mathrm{Hom}^i(X^*, Y^*) = \prod_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X^i, Y^{i+j})$, and the differential d is given by

$$\begin{aligned} d^i \,:\, \mathrm{Hom}^i(X^*, Y^*) &\longrightarrow \mathrm{Hom}^{i+1}(X^*, Y^*) \\ f &\longmapsto df = d_Y \circ f - (-1)^n f \circ d_X. \end{aligned}$$

It is immediate to check that this is indeed a complex, and that for each $i \in \mathbb{Z}$, $Z^i(\mathrm{Hom}^*(X^*, Y^*)) \cong \mathrm{Hom}_{Ch(\mathcal{A})}(X^*, Y^*[i])$ with $B^i(\mathrm{Hom}^*(X^*, Y^*))$ corresponding to those chain maps which are homotopic to zero. Hence $H^i(\mathrm{Hom}^*(X^*, Y^*)) \cong \mathrm{Hom}_{K(\mathcal{A})}(X^*, Y^*[i])$, and in particular we conclude that

$$\mathrm{Hom}_{D(\mathcal{A})}(A, B[i]) \cong \mathrm{Hom}_{K(\mathcal{A})}(A, I_B^*[i]) \cong H^i(\mathrm{Hom}^*(A, I_B^*)) \cong \text{I-Ext}^i_{\mathcal{A}}(A, B).$$

It follows analogously that assuming that $\mathcal{A}$ has enough projectives instead of enough injectives, $\text{II-Ext}^i_{\mathcal{A}}(A,B) \cong \mathrm{Hom}_{D(\mathcal{A})}(A[-i], B) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, B[i])$. □

Now we return to group cohomology.

Corollary 1.1.5. For any abstract group G and $i \geq 0$, we have an isomorphism of functors

$$H^i(G, -) \cong \mathrm{Ext}^i_{\mathrm{Mod}_{\mathbb{Z}[G]}}(\mathbb{Z}, -) \,:\, \mathrm{Mod}_{\mathbb{Z}[G]} \longrightarrow \mathrm{Ab}.$$

Thus we can compute cohomology groups $H^i(G, A)$ either by

- Choosing an injective resolution I_A^* of A in $\mathrm{Mod}_{\mathbb{Z}[G]}$ and taking $H^i(G, A) \cong H^i(\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, I_A^*))$, or
- Choosing a projective resolution $P_{\mathbb{Z}}^*$ of $\mathbb{Z}$ in $\mathrm{Mod}_{\mathbb{Z}[G]}$ and taking $H^i(G, A) \cong H^i(\mathrm{Hom}_{\mathbb{Z}[G]}(P_{\mathbb{Z}}^*, A))$.

The second option has the great advantage that we can choose a single $P_{\mathbb{Z}}^*$ once and for all, and it is the one we shall use in practice. In the following we define a particularly convenient projective resolution of $\mathbb{Z}$. For any $i \geq 0$, the underlying abelian group of $\mathbb{Z}[G^{i+1}]$ has a G-module structure induced by $g \cdot (h_0, ..., h_i) = (gh_0, ..., gh_i)$ (equivalently, the group ring construction induces by functoriality a $\mathbb{Z}[G]$-algebra structure on $\mathbb{Z}[G^{i+1}$ from the diagonal group homomorphism $G \longrightarrow G^{i+1}$). Then define G-homomorphisms $d^i : \mathbb{Z}[G^{i+1}] \longrightarrow \mathbb{Z}[G^i]$ for $i \geq 1$ given by $d^i = \sum_{j=0}^{i}(-1)^i r_j^i$, where r_j^i is induced by $r_j^i(h_0, ..., h_i) = (h_0, ..., h_{j-1}, h_{j+1}, ..., h_i)$. Define also $d^0 : \mathbb{Z}[G] \longrightarrow \mathbb{Z}$ induced by $1 \cdot h \longmapsto 1$ for each $h \in G$.

Lemma 1.1.6. For each $i \geq 0$, the $\mathbb{Z}[G]$-module $\mathbb{Z}[G^{i+1}]$ is free (hence projective) with a basis in bijection with G^i. Also,

$$\cdots \xrightarrow{d^3} \mathbb{Z}[G^3] \xrightarrow{d^2} \mathbb{Z}[G^2] \xrightarrow{d^1} \mathbb{Z}[G] \xrightarrow{d^0} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

is an exact sequence of $\mathbb{Z}[G]$-modules, and thus yields a projective resolution of $\mathbb{Z}$, which will be called the *standard resolution* of $\mathbb{Z}$ and it will be denoted by $E^*_{\mathbb{Z}}$.

Proof. See Gille and Szamuely, 2006, p. 56. □

We thus obtain for each G-module A a complex $\mathrm{Hom}_{\mathbb{Z}[G]}(E^*_{\mathbb{Z}}, A)$ given by

$$\cdots \longrightarrow 0 \longrightarrow \overbrace{\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A)}^{\text{degree } 0} \xrightarrow{\mathrm{Hom}(d^1,A)} \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^2], A) \xrightarrow{\mathrm{Hom}(d^2,A)} \cdots .$$

The differentials $\mathrm{Hom}(d^{i+1}, A)$ will be denoted by δ^i. For each $i \geq 0$, the abelian groups $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A)$, $Z^i(\mathrm{Hom}_{\mathbb{Z}[G]}(E^*_{\mathbb{Z}}, A))$ and $B^i(\mathrm{Hom}_{\mathbb{Z}[G]}(E^*_{\mathbb{Z}}, A))$ will be called the (homogeneous) *i-cochains*, *i-cocycles* and *i-coboundaries* of A, respectively, and will be denoted by $C^i(G, A)$, $Z^i(G, A)$ and $B^i(G, A)$. Hence we have $H^i(G, A) \cong H^i(\mathrm{Hom}_{\mathbb{Z}[G]}(E^*_{\mathbb{Z}}, A)) \cong Z^i(G, A)/B^i(G, A)$.

To compute cohomology, we may use the fact that the $\mathbb{Z}[G^{i+1}]$ ($i \geq 0$) are free $\mathbb{Z}[G]$-modules to describe homomorphisms $\mathbb{Z}[G^{i+1}] \longrightarrow A$ as certain maps of sets $G^i \longrightarrow A$. For each $h_1, ..., h_i \in G$, denote by $[h_1, ..., h_n]$ the element $(1, h_1, h_1h_2, ..., h_1h_2 \cdots h_n)$ of $\mathbb{Z}[G^{i+1}]$. It may be proved that such $[h_1, ..., h_n]$ form a basis for $\mathbb{Z}[G^{i+1}]$ as a free $\mathbb{Z}[G]$-module. Also, the differentials are given in this notation by

$$\begin{aligned}
d^i([h_1, ..., h_i]) &= d^i((1, h_1, h_1h_2, ..., h_1h_2 \cdots h_i)) \\
&= (h_1, ..., h_1 \cdots h_i) + \sum_{j=1}^{i-1}(-1)^{j+1}(1, ..., h_1 \cdots h_{j-1}, h_1 \cdots h_{j+1}, ..., h_1 \cdots h_i) \\
&\qquad + (-1)^{i+1}(1, ..., h_1 \cdots h_{i-1}) \\
&= h_1[h_2, ..., h_i] + \sum_{j=1}^{i-1}(-1)^{j+1}[h_1, ..., h_jh_{j+1}, ..., h_i] + (-1)^{i+1}[h_1, ..., h_{i-1}].
\end{aligned}$$

Identifying G^i (as a set) with this basis through $(h_1, ..., h_i) \longleftrightarrow [h_1, ..., h_i]$, we obtain a bijection $C^i(G, A) = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{i+1}], A) \cong \mathrm{Maps}(G^i, A)$. Such maps $a : G^i \longrightarrow A$ are usually called *inhomogeneous cochains*, and we will denote them by $[h_1, ..., h_i] \longmapsto a_{h_1 ... h_i}$. It follows from above formula that the differential $\delta^i : C^i(G, A) \longrightarrow C^{i+1}(G, A)$ associates to a cochain $a : G^i \longrightarrow A$ the cochain given by

$$[h_1, ..., h_{i+1}] \longmapsto h_1 a_{h_2,...,h_{i+1}} + \sum_{j=1}^{i-1} (-1)^{j+1} a_{h_1,...,h_j h_{j+1},...,h_{i+1}} + (-1)^{i+1} a_{h_1,...,h_i}.$$

In particular, elements of $Z^i(G, A)$ correspond precisely to those maps $a : G^i \longrightarrow A$ satisfying $h_1 a_{h_2,...,h_{i+1}} + \sum_{j=1}^{i-1} (-1)^{j+1} a_{h_1,...,h_j h_{j+1},...,h_{i+1}} + (-1)^{i+1} a_{h_1,...,h_i} = 0$ for every $h_1, ..., h_{i+1} \in G$.

Example 1.1.7. Note that elements of $Z^0(G, A)$ correspond to inhomogeneous cocycles of degree 0, i.e. maps $a : \{1\} \longrightarrow A$ with the property that $h_1 a_1 - a_1 = 0$ for every $h_1 \in G$, i.e. such that a_1 is G-invariant. Hence $H^0(G, A) \cong Z^0(G, A)/B^0(G, A) \cong A^G$.

Elements of $Z^1(G, A)$ correspond to inhomogeneous cocycles of degree 1, i.e. maps $a : G \longrightarrow A$ satisfying $h_1 a_{h_2} - a_{h_1 h_2} + a_{h_1} = 0$ for every $h_1, h_2 \in G$ – or equivalently, $a_{gh} = a_g + g a_h$ for every $g, h \in G$. Elements of $B^1(Z, A)$ correspond to maps $a : G \longrightarrow A$ satisfying $a_{h_1} = h_1 a_1 - a_1$ for some $a_1 \in A$ (seen as a map $1 \longrightarrow A$).

Comparison maps in group cohomology

In what follows, we indicate the existence of certain homomorphisms relating the cohomology of a group with that of a given subgroup. Then we finish this subsection by discussing the cup product operation in group cohomology.

Definition 1.1.8. Let G be a group and $H \subset G$ a subgroup. For each H-module A, we define a G-module structure on the abelian group $\mathrm{Hom}_H(\mathbb{Z}[G], A)$ (where we see $\mathbb{Z}[G]$ as an H-module by restricting scalars) with G-action given by $(g \cdot \varphi)(x) = \varphi(xg)$ (where we see $\mathbb{Z}[G]$ as a right G-module). Note that $g \cdot \varphi$ is indeed an H-homomorphism, since $(g \cdot \varphi)(hx) = \varphi(hxg) = h\varphi(xg) = h((g \cdot \varphi)(x))$, and we indeed have a G-action, since $(gg' \cdot \varphi)(x) = \varphi(xgg') = (g' \cdot \varphi)(xg) = (g \cdot (g' \cdot \varphi))(x)$. This G-module is denoted by $M_H^G(A)$ and called the *coinduced module* of A (with G and H implicit). If H is the trivial subgroup, then A is simply an abelian group, and $M_H^G(A)$ is denoted by $M^G(A)$.

It is clear that this construction defines a functor $M_H^G : \mathrm{Mod}_H \longrightarrow \mathrm{Mod}_G$. The reason why it is useful is that it is right adjoint to the functor $\mathrm{Mod}_G \longrightarrow \mathrm{Mod}_H$ given by restricting scalars:

Lemma 1.1.9. Let G be a group and $H \subset G$ a subgroup. Then for any $A \in \mathrm{Mod}_G$ and $B \in \mathrm{Mod}_H$, there is an isomorphism $\mathrm{Hom}_H(A, B) \cong \mathrm{Hom}_G(A, M_H^G(B))$ natural in A and B. The unit of the corresponding adjunction is given by the G-homomorphism

$$A \cong \mathrm{Hom}_G(\mathbb{Z}[G], A) \hookrightarrow \mathrm{Hom}_H(\mathbb{Z}[G], A) = M_H^G(A),$$

where the first isomorphism sends a to the unique G-homomorphism which sends 1 to a.

Proof. See Gille and Szamuely, 2006, p. 60. □

A consequence of this result is:

Lemma 1.1.10 (Shapiro's Lemma). Let G be a group and $H \subset G$ a subgroup. Then for any $A \in \mathrm{Mod}_H$ and $i \geq 0$, there is an isomorphism $H^i(G, M_H^G(A)) \cong H^i(H, A)$, natural in A.

Proof. It suffices to note that any projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module is also a projective $\mathbb{Z}[H]$-resolution, since $\mathbb{Z}[G]$ – and thus any free $\mathbb{Z}[G]$-module – is free as a $\mathbb{Z}[H]$-module, by taking as a basis a set of left coset representatives. The naturality in A follows from that of the adjunction. □

Definition 1.1.11 (Restriction map). Let G be a group and $H \subset G$ a subgroup. For any G-module A, the adjunction unit from Lemma 1.1.9 induces maps in cohomology for each $i \geq 0$, naturally in A:

$$H^i(G, A) \longrightarrow H^i(G, M_H^G(A)) \cong H^i(H, A),$$

where the last isomorphism comes from Lemma 1.1.10. These are called *restriction maps* and are denoted by $Res : H^i(G, A) \longrightarrow H^i(H, A)$.

Definition 1.1.12 (Corestriction map). Let G be a group and $H \subset G$ a subgroup of finite index, say n. For each G-module A, we define a G-homomorphism $\mathrm{Hom}_H(\mathbb{Z}[G], A) = M_H^G(A) \longrightarrow \mathrm{Hom}_G(\mathbb{Z}[G], A) \cong A$ in the following way: for every H-homomorphism $\phi : \mathbb{Z}[G] \longrightarrow A$, it may be proved that

$$\sum_{j=1}^{n} \rho_j \phi(\rho_j^{-1} x),$$

where $\{\rho_j : 1 \leq j \leq n\}$ is a system of left coset representatives for H in G, does not depend on the choice of the ρ_j. Define a G-homomorphism $\phi_H^G : \mathbb{Z}[G] \longrightarrow A$ by

$$\phi_H^G(x) = \sum_{j=1}^{n} \rho_j \phi(\rho_j^{-1} x),$$

where $\{\rho_j : 1 \leq j \leq n\}$ is as above. It may be proved that ϕ_H^G is a G-homomorphism, that $\phi \longmapsto \phi_H^G$ is a G-homomorphism, and that it is natural in A. Applying cohomology to $M_H^G(A) \longrightarrow A$ and using Lemma 1.1.10, we obtain natural morphisms

$$H^i(H, A) \longrightarrow H^i(G, A)$$

for $i \geq 0$. These are called *corestriction maps* and are denoted by $Cor : H^i(H, A) \longrightarrow H^i(G, A)$.

Definition 1.1.13 (Inflation map). Let G be a group and $H \subset G$ a normal subgroup. Note that for each G-module A, the subgroup A^H of A consisting of its H-invariant elements is stable under the action of G. Hence A^H is a G-module on which H acts trivially, so it inherits a G/H-module structure. We now define natural morphisms

$$Inf \,:\, H^i(G/H, A^H) \longrightarrow H^i(G, A)$$

for $i \geq 0$, called *inflation maps*. Observe that if $P^*_{\mathbb{Z}}$ and $Q^*_{\mathbb{Z}}$ are projective resolutions of $\mathbb{Z}$ as a G-module and as a G/H-module, respectively, then the map $G \longrightarrow G/H$ allows us to regard $Q^*_{\mathbb{Z}}$ also as a G-module, so we can extend (by projectivity of $P^*_{\mathbb{Z}}$) the identity $\mathbb{Z} \longrightarrow \mathbb{Z}$ to a map $P^*_{\mathbb{Z}} \longrightarrow Q^*_{\mathbb{Z}}$ of complexes of G-modules which is unique up to chain homotopy. Then for our given G-module A we obtain a chain map $\mathrm{Hom}_{G/H}(Q^*_{\mathbb{Z}}, A^H) \cong \mathrm{Hom}_G(Q^*_{\mathbb{Z}}, A) \longrightarrow \mathrm{Hom}_G(P^*_{\mathbb{Z}}, A)$, where the isomorphism comes from the fact that if Q is a G-module on which H acts trivially, then $\mathrm{Hom}_G(Q, A) \cong \mathrm{Hom}_{G/H}(Q, A^H)$. By applying cohomology, we obtain the desired morphisms $Inf \,:\, H^i(G/H, A^H) \longrightarrow H^i(G, A)$.

Definition 1.1.14 (Conjugation action)**.** Let G be a group and $H \subset G$ a normal subgroup. We shall define for each G-module A a G/H-module structure on $H^i(H, A)$ ($i \geq 0$) which is compatible with long exact sequences in cohomology induced from short exact sequences of G-modules. (One may reframe this as follows: whenever A is an H-module for some group H, then for each way of extending its action to a G-action for some group G such that H is identified with a normal subgroup of G, we obtain an action of G/H on $H^i(H, A)$.)

Note that for any G-modules P and A, the abelian group $\mathrm{Hom}_H(P, A)$ carries a G-action given by $(g \cdot \phi)(x) = g^{-1}\phi(gx)$. Then we have a G/H, since H acts trivially. Now take a projective resolution $P^*_{\mathbb{Z}}$ of $\mathbb{Z}$ as a G-module, which is also an H-projective resolution (as in Lemma 1.1.10), to get a complex $\mathrm{Hom}_H(P^*_{\mathbb{Z}}, A)$ of G/H-modules through the action defined above. By taking cohomology, we obtain the desired action of G/H on $H^i(H, A)$ for $i \geq 0$, which is called the *conjugation action*. Moreover, it follows analogously (using the functoriality of long exact sequences in cohomology with respect to morphisms of short exact sequences of modules) that for any short exact sequence of G-modules $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$, all the maps in the induced long exact sequence are G/H-equivariant.

The cup product operation

We now proceed to define the cup product operation in group cohomology. Recall that given G-modules A and B, in order to describe their cohomologies by using cochains coming from the standard projective resolution $E^*_{\mathbb{Z}}$ of $\mathbb{Z}$, we consider abelian groups of cochains $C^i(G, A) = \mathrm{Hom}_G(\mathbb{Z}[G^{i+1}], A) \cong \mathrm{Maps}(G^i, A)$ ($i \geq 0$) and $C^j(G, B) = \mathrm{Hom}_G(\mathbb{Z}[G^{j+1}], B) \cong \mathrm{Maps}(G^j, B)$ ($j \geq 0$). For fixed i and j, define a bilinear map $C^i(G, A) \times C^j(G, B) \longrightarrow C^{i+j}(G, A \otimes_{\mathbb{Z}} B)$ (note that the tensoring is over $\mathbb{Z}$ and that the G-action is given by $g(a \otimes b) = ga \otimes gb$) by the composite

$$\begin{aligned}\mathrm{Hom}_G(\mathbb{Z}[G^{i+1}], A) \times \mathrm{Hom}_G(\mathbb{Z}[G^{j+1}], B) &\longrightarrow \mathrm{Hom}_G(\mathbb{Z}[G^{i+1}] \otimes_{\mathbb{Z}} \mathbb{Z}[G^{j+1}], A \otimes_{\mathbb{Z}} B) \\ &\longrightarrow \mathrm{Hom}_G(\mathbb{Z}[G^{i+j+1}], A \otimes_{\mathbb{Z}} B),\end{aligned}$$

where the first map comes from the fact that the tensor product is a bifunctor, and the second map is given by precomposition with

$$\begin{aligned} \mathbb{Z}[G^{i+j+1}] &\longrightarrow \mathbb{Z}[G^{i+1}] \otimes_{\mathbb{Z}} \mathbb{Z}[G^{j+1}] \\ (g_0, ..., g_{i+j}) &\longmapsto (g_0, ..., g_i) \otimes (g_i, ..., g_{i+j}). \end{aligned}$$

Hence we have a homomorphism of abelian groups

$$\mathrm{Hom}_G(\mathbb{Z}[G^{i+1}], A) \otimes_{\mathbb{Z}} \mathrm{Hom}_G(\mathbb{Z}[G^{j+1}], B) \longrightarrow \mathrm{Hom}_G(\mathbb{Z}[G^{i+j+1}], A \otimes_{\mathbb{Z}} B),$$

called the *cup product of cochains* and denoted by

$$\cup : C^i(G, A) \otimes_{\mathbb{Z}} C^j(G, B) \longrightarrow C^{i+j}(G, A \otimes_{\mathbb{Z}} B).$$

Explicitly, given cochains $\varphi : \mathbb{Z}[G^{i+1}] \longrightarrow A$ and $\psi : \mathbb{Z}[G^{j+1}] \longrightarrow B$, it sends $\varphi \otimes \psi$ to

$$\begin{aligned} \varphi \cup \psi : \mathbb{Z}[G^{i+j+1}] &\longrightarrow A \otimes_{\mathbb{Z}} B \\ (g_0, ..., g_{i+j}) &\longmapsto \varphi(g_0, ..., g_i) \otimes \psi(g_i, ..., g_{i+j}). \end{aligned}$$

We would like to prove that the cup product operation passes to the level of cohomology. For this to happen, it is necessary and sufficient to check that (i) the cup product of two cocycles is also a cocycle, and (ii) that the cup product of a cocycle with a coboundary – in either order – is a coboundary.

It may be proved that:

Lemma 1.1.15. Let G be a group, and $A, B \in \mathrm{Mod}_G$. Then for any cochains $\varphi \in C^i(G, A)$ and $\psi \in C^j(G, B)$, the following formula holds:

$$\delta^{i+j}(\varphi \cup \psi) = \delta^i(\varphi) \cup \psi + (-1)^i \varphi \cup \delta^j(\psi)$$

Corollary 1.1.16. In the notation of Lemma 1.1.15, if $\delta^i(\varphi) = 0$ and $\delta^j(\psi) = 0$, then $\delta^{i+j}(\varphi \cup \psi) = 0$. Also, if $\varphi = \delta^{i-1}(\varphi')$ for some φ' and $\delta^j(\psi) = 0$, then $\varphi \cup \psi = \delta^{i+j-1}(\theta)$ for some θ (and vice-versa, if φ is a cocycle and ψ a coboundary). □

Definition 1.1.17. Corollary 1.1.16 implies the existence of a homomorphism

$$\cup : H^i(G, A) \otimes_{\mathbb{Z}} H^j(G, B) \longrightarrow H^{i+j}(G, A \otimes_{\mathbb{Z}} B),$$

called the *cup product.*

Proposition 1.1.18. The cup product operation has the following properties:

(i) For fixed G, i, j, the map $\cup : H^i(G, A) \otimes_{\mathbb{Z}} H^j(G, B) \longrightarrow H^{i+j}(G, A \otimes_{\mathbb{Z}} B)$ is natural in A and B.

(ii) It is associative, i.e. for any $A, B, C \in \mathrm{Mod}_G$, $i, j, k \geq 0$, and $\alpha \in H^i(G, A)$, $\beta \in H^j(G, B)$, $\gamma \in H^k(G, C)$, we have $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$.[1]

(iii) It is graded-commutative, in the sense that for any $\alpha \in H^i(G, A)$ and $\beta \in H^j(G, B)$, we have $\alpha \cup \beta = (-1)^{ij}(\beta \cup \alpha)$.[2]

(iv) If $i = j = 0$, then $\cup : H^0(G, A) \otimes_{\mathbb{Z}} H^0(G, B) \longrightarrow H^0(G, A \otimes_{\mathbb{Z}} B)$ is the map $A^G \otimes_{\mathbb{Z}} B^B \longrightarrow (A \otimes_{\mathbb{Z}} B)^G$ given by restricting $id_{A \otimes_{\mathbb{Z}} B}$.

(v) Given a finite index subgroup $H \subset G$, for any $\alpha \in H^i(H, A)$ and $\beta \in H^j(G, B)$ it holds that $Cor(Res(\alpha) \cup \beta) = Cor(\alpha) \cup \beta$. This is known as the *projection formula.*

(vi) Given a subgroup $H \subset G$, for any $\alpha \in H^i(G, A)$ and $\beta \in H^j(G, B)$ it holds that $Res(\alpha \cup \beta) = Res(\alpha) \cup Res(\beta)$.

(vii) Given a normal subgroup $H \subset G$, for any $\alpha \in H^i(G/H, A^H)$ and $\beta \in H^J(G/H, B^H)$ it holds that $Inf(a \cup b) = Inf(a) \cup Inf(b)$.

Proof. See Gille and Szamuely, 2006, 3.4. □

1.1.2 Galois cohomology

Galois cohomology studies the cohomology of modules over Galois groups. Firstly, let us consider a finite Galois extension of fields K/k. Then we may use the above definition of group cohomology and consider for any module M over the Galois group $\mathrm{Gal}(K/k)$ its cohomology groups

$$H^n(\mathrm{Gal}(K, k), M), \quad n \geq 0.$$

We may also be interested in certain systems of modules over Galois groups $\mathrm{Gal}(K/k)$ where K ranges over all finite Galois extensions of k. To study this setting, we fix a separable closure k_{sep} of k, and throughout this subsection we will only deal with finite Galois extensions of k which are contained in k_{sep}.

Let us denote by FinGal_k the direct system of all finite Galois extensions $k \subset K \subset k_{sep}$ ordered by inclusion. Then restriction of k-automorphisms defines for each inclusion $K \subset L$ of two such finite Galois extensions a group homomorphism $\mathrm{Gal}(L/k) \longrightarrow \mathrm{Gal}(K/k)$; this yields a functor

$$\mathrm{Gal}(-/k) : \mathrm{FinGal}^{op} \longrightarrow \mathrm{Grp}.$$

It has the property that the absolute Galois group $\mathrm{Gal}(k_{sep}/k)$ is isomorphic, via the maps $\mathrm{Gal}(k_{sep}/k) \longrightarrow \mathrm{Gal}(K/k)$ also given by restriction of automorphisms, to the limit

$$\varprojlim_{K \in \mathrm{FinGal}^{op}} \mathrm{Gal}(K/k)$$

of $\mathrm{Gal}(-, k)$ in the category of groups.

[1] Note the abuse of notation in this equality, since it only holds up to the isomorphism $H^{i+j+k}(G, (A \otimes_{\mathbb{Z}} B) \otimes_{\mathbb{Z}} C) \cong H^{i+j+k}(G, A \otimes_{\mathbb{Z}} (B \otimes_{\mathbb{Z}} C))$.

[2] It holds up to the isomorphism $H^{i+j}(G, A \otimes_{\mathbb{Z}} B) \cong H^{i+j}(G, B \otimes_{\mathbb{Z}} A)$.

Suppose given a $\mathrm{Gal}(k_{sep}/k)$-module M. For each finite Galois extension K/k contained in k_{sep} we may consider the submodule $M^{\mathrm{Gal}(k_{sep}/K)}$ of elements fixed under the action of the normal subgroup $\mathrm{Gal}(k_{sep}/K) \subset \mathrm{Gal}(k_{sep}/k)$. Then the isomorphism $\mathrm{Gal}(K/k) \cong \mathrm{Gal}(k_{sep}/k)/\mathrm{Gal}(k_{sep}/K)$ endows $M^{\mathrm{Gal}(k_{sep}/K)}$ with a $\mathrm{Gal}(K/k)$-module structure. Moreover, note that if L/k is another finite Galois extension such that $K \subset L \subset k_{sep}$, then by similarly endowing $M^{\mathrm{Gal}(k_{sep}/L)}$ with a $\mathrm{Gal}(L/k)$-module structure we have that

$$(M^{\mathrm{Gal}(k_{sep}/L)})^{\mathrm{Gal}(L/K)} = (M^{\mathrm{Gal}(k_{sep}/L)})^{\mathrm{Gal}(k_{sep}/K)} = M^{\mathrm{Gal}(k_{sep}/K)}.$$

For each finite Galois extension K/k contained in k_{sep}, let us denote by M_K the $\mathrm{Gal}(K/k)$-module $M^{\mathrm{Gal}(k_{sep}/K)}$. Then Definition 1.1.13 provides inflation maps of the following two kinds:

(i) If K and L are finite Galois extensions of k such that $k \subset K \subset L \subset k_{sep}$, we have for each $n \geq 0$ a map

$$Inf_{L/K} : H^n(\mathrm{Gal}(K/k), M_K) = H^n(\mathrm{Gal}(K/k), (M_L)^{\mathrm{Gal}(L/K)}) \longrightarrow H^n(\mathrm{Gal}(L/k), M_L).$$

(ii) If K is a finite Galois extension of k contained in k_{sep}, we have for each $n \geq 0$ a map

$$Inf_K : H^n(\mathrm{Gal}(K/k), M_K) = H^n(\mathrm{Gal}(K/k), M^{\mathrm{Gal}(k_{sep}/K)}) \longrightarrow H^n(\mathrm{Gal}(k_{sep}/k), M).$$

It may be proved that these satisfy $Inf_{S/L} \circ Inf_{L/K} = Inf_{S/K}$ whenever $K \subset L \subset S$, and $Inf_L \circ Inf_{L/K}$ whenever $K \subset L$. Thus we obtain for each $n \geq 0$ a functor

$$H^n(\mathrm{Gal}(-/k), M_-) : \mathrm{FinGal}_k \longrightarrow \mathrm{Ab}$$

and a cocone from the diagram $H^n(\mathrm{Gal}(-/k), M_-)$ to the abelian group $H^n(\mathrm{Gal}(k_{sep}/k), M)$.

This defines for each n an abelian group

$$\varinjlim_{K \in \mathrm{FinGal}} H^n(\mathrm{Gal}(K/k), M_K) \tag{1.1.1}$$

endowed with a homomorphism

$$\varinjlim_{K \in \mathrm{FinGal}} H^n(\mathrm{Gal}(K/k), M_K) \longrightarrow H^n(\mathrm{Gal}(k_{sep}/k), M).$$

Profinite groups

Definition 1.1.19. Recall that a *topological group* is a group G endowed with a topology on its underlying set such that the multiplication map

$$\cdot : G \times G \longrightarrow G$$

and the inversion map

$$(-)^{-1} : G \longrightarrow G$$

are both continuous, where $G \times G$ is given the product topology.

We will use the term *G-set* (resp. *G-module*) to refer to a set endowed with a left action of (resp. to a left module over) the underlying (non-topological) group of G.

A *discrete G-set* is defined to be a G-set X such that the action map

$$G \times X \longrightarrow X$$

is continuous when X is given the discrete topology. The category whose objects are the discrete G-sets and whose morphisms are the functions that preserve the action of G (also known as G-equivariant functions) will be denoted by $\mathscr{C}\mathrm{Set}_G$.

Analogously, a *discrete G-module* is defined to be a G-module M such that the action map

$$G \times M \longrightarrow M$$

is continuous when M is given the discrete topology. The category consisting of discrete G-modules and G-module homomorphisms will be denoted by $\mathscr{C}\mathrm{Mod}_G$.

Remark 1.1.20. Suppose given a topological group G. If X is a G-set (resp. G-module), then it is a discrete G-set (resp. discrete G-module) if and only if the stabilizer $Stab_x$ of every point x of X under the action of G (i.e. the subgroup of G consisting of those g such that $g \cdot x = x$) is open in G. Indeed, if X is a discrete G-set (resp. module), continuity of the action map

$$\mu : G \times X \to X$$

and discreteness of X as a topological space implies that for each $x \in X$, $Stab_x \times \{x\} = \mu^{-1}(\{x\}) \cap (G \times \{x\})$ is an open subset of $G \times \{x\}$, which in turn is homeomorphic to G by projection onto the first coordinate. Hence $Stab_x$ is open in G. Conversely, assume $Stab_x$ is open for every $x \in X$. As the sets $\{x\}$ form a base for the topology on X, it suffices to check that each $\mu^{-1}(\{x\})$ is open in $G \times X$. Since

$$\mu^{-1}(\{x\}) = \bigcup_{y \in X} \{(g, y) \in G \times X | g \cdot y = x\}$$

and X is a discrete topological space, it suffices to prove that $\{g \in G | g \cdot y = x\}$ is open for each $y \in X$. But for fixed y, if $\{g \in G | g \cdot y = x\}$ is nonempty – say it has an element h – then

$$\{g \in G | g \cdot y = x\} = \{g \in G | g \cdot (h \cdot x) = x\} = \{g \in G | gh \cdot x = x\} = Stab_x h^{-1},$$

which is open in G as right multiplication by h^{-1} is a homeomorphism $G \longrightarrow G$. We conclude that the action is continuous.

This is also equivalent to the condition that for each $x \in X$ there exists an open subgroup $H \subset G$ such that H stabilizes x (i.e. $h \cdot x = x$ for every $h \in H$). If Stab_x is open, this condition holds as we can take $H = \mathrm{Stab}_x$. Conversely, if there exists one such H, then for every $g \in \mathrm{Stab}_x$ we have $g \in gH \subset \mathrm{Stab}_x$, so $\mathrm{Stab}_x = \bigcup_{g \in \mathrm{Stab}_x} gH$ is open.

Definition 1.1.21. A topological group G is said to be *profinite* if it can be expressed as a limit of an inverse system of discrete finite groups.

It may be proved (see for example Ribes and Zalesskii, 2013, Th. 2.1.3) that profinite groups may be characterized as those topological groups whose underlying topological space is compact, Hausdorff, and totally disconnected (i.e. its only nonempty connected subspaces are the singletons).

In the above notation, the isomorphism between the Galois group $\mathrm{Gal}(k_{sep}/k)$ and the limit of the inverse system of finite groups $\mathrm{Gal}(-/k) : \mathrm{FinGal}^{op} \to \mathrm{Grp}$ gives $\mathrm{Gal}(k_{sep}/k)$ a profinite group structure by endowing $\mathrm{Gal}(K/k)$ with the discrete topology for each $K \in \mathrm{FinGal}$.

Theorem 1.1.22. Let G be a profinite group. Then the category $\mathscr{C}\mathrm{Mod}_G$ is abelian and has enough injectives.

Proof. See Ribes and Zalesskii, 2013, Ex. 5.3.2 and Prop. 5.4.5. □

Given a profinite group G, let us denote by

$$\Gamma : \mathscr{C}\mathrm{Mod}_G \longrightarrow \mathrm{Ab}$$

the functor given on objects by sending each discrete G-module M to its subgroup M^G of G-invariant elements, and on arrows by restriction. Note that Γ is isomorphic to the functor $\mathrm{Hom}_{\mathscr{C}\mathrm{Mod}_G}(\mathbb{Z}, -)$, where $\mathbb{Z}$ is regarded as a discrete trivial G-module, via the natural transformation whose M-component $\mathrm{Hom}_{\mathscr{C}\mathrm{Mod}_G}(\mathbb{Z}, M) \to M^G$ sends each $f : \mathbb{Z} \to M$ to $f(1)$.

Cohomology of a profinite group G is defined as the right derived functor

$$\mathbf{R}\Gamma : D^+(\mathscr{C}\mathrm{Mod}_G) \longrightarrow D^+(\mathrm{Ab}).$$

Given a bounded below complex M of discrete G-modules (or a discrete G-module, which we then identify with a complex concentrated in degree 0), the n-th (profinite group) cohomology group of G with coefficients in M is defined for each integer n as the n-th cohomology group of $\mathbf{R}\Gamma(M)$,

$$H^n_{disc}(G, M) := H^n(\mathbf{R}\Gamma(M)).$$

By composing $\mathbf{R}\Gamma$ with $H^n : D^+(\mathrm{Ab}) \to \mathrm{Ab}$, these groups may be assembled into functors

$$H^n_{disc}(G, -) : D^+(\mathscr{C}\mathrm{Mod}_G) \longrightarrow \mathrm{Ab}.$$

If k is a field endowed with a separable closure k_{sep}, profinite group cohomology of $\mathrm{Gal}(k_{sep}/k)$ with coefficients in discrete modules is known as *Galois cohomology*. We will use the notation

$$H^n(k, M)$$

for $H^n_{disc}(G, M)$, with the extension k_{sep}/k implicit.

The following proposition establishes that Galois cohomology is given by the abelian group considered in 1.1.1:

Proposition 1.1.23. Let k be a field endowed with a separable closure k_{sep}, and M a discrete $\mathrm{Gal}(k_{sep}/k)$-module. For each $n \geq 0$ there exists an isomorphism

$$H^n(k, M) \cong \varinjlim_{K \in \mathrm{FinGal}} H^n(\mathrm{Gal}(K/k), M_K).$$

Proof. See Ribes and Zalesskii, 2013, 6.5. □

1.2 Galois cohomology applied to the Kummer exact sequence; Milnor K-theory

Throughout this section, k denotes a field and k_{sep} a separable closure of k.

We consider the category Mod_G of discrete modules over G regarded as a topological group via the profinite topology. Firstly, note that the multiplicative group of units $k_{sep}^\times$ has a canonical G-module structure given by the action $G \times k_{sep}^\times \longrightarrow k_{sep}^\times$, $(\varphi, a) \mapsto \varphi(a)$. Now, suppose l is a prime number different from the characteristic of k. Note that the group μ_l of l-th roots of unity admits a (non-canonical) embedding into the multiplicative group of units $k_{sep}^\times$ given by sending any generator of μ_l to a primitive l-th root of unity in $k_{sep}^\times$. Since elements of the image of $\mu_l \hookrightarrow k_{sep}^\times$ are precisely the zeros of $x^l - 1 \in k[x]$, they are permuted under the canonical action of G on $k_{sep}^\times$, so the canonical G-action on $k_{sep}^\times$ restricts to a G-module structure on μ_l.

This defines a short exact sequence

$$1 \longrightarrow \mu_l \longrightarrow k_{sep}^\times \xrightarrow{\wedge l} k_{sep}^\times \longrightarrow 1, \tag{1.2.1}$$

where ^{n} denotes the operation $a \mapsto a^l$. It is known as the *Kummer exact sequence.*

Now, by regarding $\mathbb{Z}$ as a discrete G-module via the trivial action $g \cdot a = a$, we consider the functor

$$\mathrm{Hom}_G(\mathbb{Z}, -) : \mathrm{Mod}_G \longrightarrow \mathrm{Ab},$$

which up to natural isomorphism sends each G-module M to its subgroup M^G consisting of those elements which are fixed under the action of G (i.e. $g \cdot a = a$). By applying the group cohomology functors $H^n(k, -)$ to 1.2.1, we obtain a long exact sequence of cohomology groups

$$0 \to H^0(k, \mu_l) \to H^0(k, k_{sep}^\times) \xrightarrow{H^0(k, \wedge l)} H^0(k, k_{sep}^\times) \to H^1(k, \mu_l) \to H^1(k, k_{sep}^\times) \to \cdots,$$

which is isomorphic to

$$0 \to \mu_l^G \to (k_{sep}^\times)^G \xrightarrow{\wedge l} (k_{sep}^\times)^G \to H^1(k, \mu_l) \to H^1(k, k_{sep}^\times) \to \cdots,$$

and hence to

$$0 \to \mu_l^G \to k^\times \xrightarrow{\wedge l} k^\times \to H^1(k, \mu_l) \to H^1(k, k_{sep}^\times) \to \cdots.$$

Moreover, $H^1(k, k_{sep}^\times)$ may be computed explicitly by means of a classical result in

Galois theory, known as *Hilbert's theorem 90*, or simply *Hilbert 90*.

Before stating it, we recall that if K/k is a finite field extension of degree n, the *norm map*

$$N_{K/k} : K \longrightarrow k$$

is the function sending each element a of K to the determinant of the linear endomorphism $K \longrightarrow K$, $c \mapsto ac$, of K as a finite-dimensional k-vector space. As the determinant of a composite of linear maps equals the product of their determinants, we have that $N_{K/k}(ab) = N_{K/k}(a)N_{K/k}(b)$. If $a \in k$, then $N_{K/k}(a) = a^n$, and in particular $N_{K/k}(1) = 1$. It follows that if $a \neq 0$, we have $N_{K/k}(a)N_{K/k}(a^{-1}) = 1$, so $N_{K/k}(a) \neq 0$. Thus $N_{K/k}$ restricts to a group homomorphism

$$K^\times \longrightarrow k^\times,$$

which will also be denoted by $N_{K/k}$.

It may also be proved (see Jacobson, 1964, I.14) that if K/k is a Galois extension, then for each $a \in K$,

$$\prod_{\varphi \in \mathrm{Gal}(K/k)} \varphi(a),$$

belongs to k and equals $N_{K/k}(a)$. It follows in this case that given $a \in K$ and $\theta \in \mathrm{Gal}(K/k)$, the fact that the function $\mathrm{Gal}(K/k) \longrightarrow \mathrm{Gal}(K/k)$, $\varphi \mapsto \varphi \circ \theta$ is a bijection implies that

$$N_{K/k}(\theta(a)) = \prod_{\varphi \in \mathrm{Gal}(K/k)} \varphi(\theta(a)) = \prod_{\varphi \in \mathrm{Gal}(K/k)} \varphi(a) = N_{K/k}(a).$$

Thus we also have $N_{K/k}(\theta(a)/a) = 1$ whenever $a \in K^\times$.

We will use the following lemma:

Lemma 1.2.1. Let K be a field. Suppose given $n \geq 1$, distinct automorphisms $\varphi_1, \ldots, \varphi_n$, and $a_1, \ldots, a_n \in K$. If $\sum_{i=1}^n a_i\varphi_i = 0$, then $a_1 = \cdots = a_n = 0$.

Proof. Suppose that this is not the case. Let $n \geq 1$ be the smallest number such that there exist distinct automorphisms $\varphi_1, \ldots, \varphi_n$, and $a_1, \ldots, a_n \in K$ not all zero such that $\sum_{i=1}^n a_i\varphi_i = 0$. Then by minimality of n we have that $a_1, \ldots, a_n$ are all nonzero. Moreover, note that $n \geq 2$, as if $a_1 \neq 0$ then $a_1\varphi_1$ cannot be zero since $a_1\varphi_1(1) = a_1$. It follows that $\varphi_1 \neq \varphi_n$, so there exists $b \in K$ such that $\varphi_1(b) \neq \varphi_n(b)$.

Now, for every $c \in K$ we have

$$0 = \sum_{i=1}^n a_i\varphi_i(bc) = \sum_{i=1}^n a_i\varphi_i(b)\varphi_i(c).$$

On the other hand,

$$0 = \varphi_n(b)\sum_{i=1}^n a_i\varphi_i(c) = \sum_{i=1}^n a_i\varphi_n(b)\varphi_i(c).$$

Thus $0 = \sum_{i=1}^n a_i\varphi_i(b)\varphi_i(c) - \sum_{i=1}^n a_i\varphi_n(b)\varphi_i(c) = \sum_{i=1}^n a_i(\varphi_i(b) - \varphi_n(b))\varphi_i(c) = \sum_{i=1}^{n-1} a_i(\varphi_i(b) - \varphi_n(b))\varphi_i(c)$. But this holds for all $c \in K$ and $\varphi_1(b) - \varphi_n(b) \neq 0$, which contradicts the minimality of n. $\square$

Theorem 1.2.2 (Hilbert 90)**.** If K/k is a finite Galois extension, then $H^1(\mathrm{Gal}(K/k), K^\times) \cong 0$.

Proof. Throughout this proof, let G denote $H^1(\mathrm{Gal}(K/k), K^\times)$.

We have to prove that every inhomogeneous cocycle in degree 1 is a coboundary. Suppose given an inhomogeneous cocycle $a : G \longrightarrow K^\times$; it satisfies $a_{gh} = a_g g(a_h)$ for all g, $h \in G$. We need to show that there exists $d \in K^\times$ such that $a_h = h(d)/d$ for every $h \in G$.

By the previous lemma, the function $\sum_{g\in G} a_g g$ is nonzero, so there exists $b \in K^\times$ such that $\sum_{g\in G} a_g g(b) \neq 0$. Denoting $\sum_{g\in G} a_g g(b)$ by c, for each $h \in G$ we have

$$\begin{aligned} h(c) &= h\left(\sum_{g\in G} a_g g(b)\right) \\ &= \sum_{g\in G} h(a_g g(b)) \\ &= \sum_{g\in G} h(a_g) hg(b) \\ &= \sum_{g\in G} a_{hg} a_h^{-1} hg(b) \\ &= a_h^{-1} \sum_{g\in G} a_{hg} hg(b) \\ &= a_h^{-1} \sum_{g\in G} a_g g(b) \\ &= a_h^{-1} c. \end{aligned}$$

Thus we have $a_h = \frac{c}{h(c)} = \frac{h(c^{-1})}{c^{-1}}$ for every $h \in G$, and we may take $d = c^{-1}$. ☐

Theorem 1.2.3 (Hilbert 90, second version)**.** $H^1(k, k_{sep}^\times) \cong 0$.

Proof. Follows from Theorem 1.2.2 by taking the direct limit over all finite Galois extensions of k contained in k_{sep}. ☐

We may use this to prove the following classical theorem:

Theorem 1.2.4 (Hilbert 90, classical version)**.** Suppose K/k is a finite Galois extension such that $\mathrm{Gal}(K/k)$ (which is finite) is a cyclic group, and let θ be any generator of $\mathrm{Gal}(K/k)$. If $a \in K^\times$ satisfies $N_{K/k}(a) = 1$, then there exists $b \in K^\times$ such that $a = \theta(b)/b$.

Proof. Let n be the degree of the extension. We have $a\theta(a)\theta^2(a)\cdots\theta^n(a) = 1$. This assumption allows us to define a function $\alpha : \mathrm{Gal}(K/k) \longrightarrow K^\times$ such that

$$\alpha_{\theta^m} = a\theta(a)\cdots\theta^m(a)$$

for each $m \geq 0$. It is a cocycle since

$$\alpha_{\theta^m\theta^p} = a\theta(a)\cdots\theta^{m+p}(a) = a\theta(a)\cdots\theta^m(a)\theta^m(a\theta(a)\cdots\theta^p(a)) = \alpha_{\theta^m}\theta^m(\alpha_{\theta^p}).$$

By Theorem 1.2.2, α is a coboundary, so there exists $b \in K^\times$ such that $\alpha_\theta = \theta(b)/b$. Hence $a = \theta(b)/b$. □

Now, the initial segment of the long exact sequence of cohomology groups associated to the Kummer exact sequence is isomorphic to

$$0 \longrightarrow \mu_l^G \longrightarrow k^\times \xrightarrow{\wedge l} k^\times \longrightarrow H^1(k, \mu_l) \longrightarrow 0. \tag{1.2.2}$$

Let us denote the connecting homomorphism $k^\times \longrightarrow H^1(k, \mu_l)$ by ∂.

It then follows that we have an isomorphism[3]

$$\partial' \,:\, k^\times/l \xrightarrow{\cong} H^1(k, \mu_l).$$

By using 1.2.2 and the cup product structure on Galois cohomology, we may also obtain comparison maps between tensor powers of $k^\times$ (and $k^\times/l$) and higher Galois cohomology groups with coefficients in tensor powers of $\mu_l^{\otimes n}$. Indeed, recall that the tensor algebra $T(A)$ of an abelian group A is defined as the ring (or $\mathbb{Z}$-algebra) whose underlying abelian group (corresponding to the sum operation) is

$$\bigoplus_{n \geq 0} A^{\otimes n},$$

where $A^{\otimes 0}$ denotes $\mathbb{Z}$, and whose product operation is given on generators by $(a_1 \otimes \cdots \otimes a_m, b_1 \otimes \cdots \otimes b_n) \longmapsto a_1 \otimes \cdots \otimes a_m \otimes b_1 \otimes \cdots \otimes b_n$ and $(1, a_1 \otimes \cdots \otimes a_m) \longmapsto a_1 \otimes \cdots \otimes a_m$ (where $1 \in A^{\otimes 0} = \mathbb{Z}$, and on general elements by linearly extending these rules. It is the free associative ring on the abelian group A, in the sense that if R is an associative ring, then any abelian group morphism $f \,:\, A \longrightarrow R$ extends uniquely to a ring morphism $\overline{f} \,:\, T(A) \longrightarrow R$. Explicitly, $\overline{f}$ is given on each component $A^{\otimes n}$ by the abelian group homomorphism corresponding to the n-linear map $A^n \longrightarrow R$ which sends $(a_1, ..., a_n)$ to $f(a_1) \cdots f(a_n)$, for $n \geq 1$, and $1 \in A^{\otimes 0} = \mathbb{Z}$ to 1_R.

$T(A)$ is a graded ring whose homogeneous elements of degree n are the nonzero elements of $A^{\otimes n}$. Hence if in the above notation R is a graded ring $\bigoplus_{n \geq 0} R_n$ and the image of $A^{\otimes 1} \cong A$ under f is contained in R_1, then for each $n \geq 1$, the fact that every generating tensor in $A^{\otimes n}$ is a product in $T(A)$ of n elements of $A^{\otimes 1}$ implies that $\overline{f}(A^{\otimes n}) \subset R_n$ for each $n \geq 1$; moreover, $\overline{f}(1) = 1_R$ implies that $\overline{f}(A^{\otimes 0}) \subset R_0$. Thus $\overline{f}$ is a graded homomorphism.

In our setting, the map of abelian groups $\partial \,:\, k^\times \longrightarrow H^1(k, \mu_l)$ induces a graded ring homomorphism

$$\partial_* \,:\, T(k^\times) \longrightarrow H^*(k, \mu_l^{\otimes *})$$

from the tensor algebra of $k^\times$ to the Galois cohomology ring $H^*(k, \mu_l^{\otimes *})$. Explicitly, for each $n \geq 0$ its n-th component is the map of abelian groups $\partial_n \,:\, (k^\times)^{\otimes n} \longrightarrow H^n(k, \mu_l^{\otimes n})$ given by

[3] Where $k^\times/l$ denotes the quotient of $k^\times$ by the sub-abelian group, or sub-$\mathbb{Z}$-module, consisting of elements of the form $l \times a$ (in additive notation) for $a \in k^\times$, i.e. of l-th powers (in multiplicative notation) of elements of $k^\times$.

the composite

$$(k^\times)^{\otimes n} \xrightarrow{\partial^{\otimes n}} H^1(k,\mu_l)^{\otimes n} \xrightarrow{\cup} H^n(k,\mu_l^{\otimes n}).$$

Moreover, for any tensor $a_1\otimes a_2\otimes\cdots\otimes a_n \in (k^\times)^{\otimes n}$, we have $l\cdot(a_1\otimes a_2\otimes\cdots\otimes a_n) = a_1^l\otimes a_2\otimes\cdots\otimes a_n$, so $l\cdot(a_1\otimes a_2\otimes\cdots\otimes a_n)$ is sent under $\partial^{\otimes n}$ to

$$\partial(a_1^l)\otimes\partial(a_2)\otimes\cdots\otimes\partial(a_n) = 0\otimes\partial(a_2)\otimes\cdots\otimes\partial(a_n) = 0.$$

Since $(k^\times)^{\otimes n}$ is generated by such tensors, the kernel of ∂_n contains all elements of the form $l\cdot\alpha$ for $\alpha\in(k^\times)^{\otimes n}$. By passing to the quotient, we obtain a map of abelian groups

$$\partial_n' : (k^\times)^{\otimes n}/l \longrightarrow H^n(k,\mu_l^{\otimes n}).$$

As $T(k^\times)/l \cong \bigoplus_{n\geq 0}((k^\times)^{\otimes n}/l)$, these define a ring homomorphism

$$\partial_*' : T(k^\times)/l \longrightarrow H^*(k,\mu_l^{\otimes *}).$$

We now show that certain equalities, called the *Steinberg relations*, hold in $H^2(k,\mu_l^{\otimes 2})$.

Proposition 1.2.5 (Gille and Szamuely, 2006, 4.6.1)**.** For each $a\neq 0,1$ in k, we have $\partial(a)\wedge\partial(1-a) = 0$ in $H^2(k,\mu_l^{\otimes 2})$.

Proof. Let $x^l - a = p_1(x)\cdots p_n(x)$ be a factorization into irreducible monic polynomials in $k[x]$. By definition, a equals the l-th power of each zero in k_{sep} of each of the $p_i(x)$. On the other hand, note that $1-a = p_1(1)\cdots p_n(1)$.

We may describe the factors $p_i(1)$ as follows: for each i, let $a_i\in k_{sep}$ be a zero of p_i. Hence p_i is the minimal polynomial of a_i over k. Moreover, note that (i) $1-a_i$ is a zero of $p_i(1-x)$, so the minimal polynomial of $1-a_i$ over k divides $p_i(1-x)$, and (ii) since $k(a_i) = k(1-a_i)$, the minimal polynomial of $1-a_i$ over k has degree $[k(1-a_i):k] = [k(a_i):k] = \deg(p_i)$. It follows that $(-1)^{\deg(p_i)}p_i(1-x)$ is the minimal polynomial of $1-a_i$ over k, so $N_{k(a_i)/k} = p_i(1-x)(0) = p_i(1)$.

Thus $1-a = N_{k(a_1)/k}(1-a_1)\cdots N_{k(a_n)/k}(1-a_n)$, so

$$\partial(a)\cup\partial(1-a) = \partial(a)\cup\sum_{i=1}^n \partial(N_{k(a_i)/k}(1-a_i)) = \sum_{i=1}^n \partial(a)\cup\partial(N_{k(a_i)/k}(1-a_i)).$$

Now, for each i we have $\partial(N_{k(a_i)/k}(1-a_i)) = Cor(\partial(1-a_i))$, where Cor denotes the corestriction map $H^1(k(a_i),\mu_l)\longrightarrow H^1(k,\mu_l)$ (see Gille and Szamuely, 2006, 4.6.2), hence the projection formula (Proposition 1.1.18(v)) yields

$$\partial(a)\cup\partial(N_{k(a_i)/k}(1-a_i)) = \partial(a)\cup Cor(\partial(1-a_i)) = Cor(Res(\partial(a))\cup\partial(1-a_i)).$$

But denoting by $\partial^i : k(a_i)^\times \longrightarrow H^1(k(a_i),\mu_l)$ the boundary map corresponding to $k(a_i)$, we have $Res(\partial(a)) = \partial^i(a)$ by Gille and Szamuely, 2006, Lemma 4.6.2, which in turn equals 0 as a is an l-th power in $k(a_i)$.

Hence $\partial(a) \cup \partial(N_{k(a_i)/k}(1-a_i)) = 0$ for each i, so $\partial(a) \cup \partial(1-a) = 0$ as desired. □

1.2.1 Milnor K-theory

Definition 1.2.6. Let k be a field. The *Milnor K-theory ring* of k, denoted by $K_M^*(k)$, is the graded associative ring defined as the quotient of the tensor algebra of the abelian group $k^\times$,

$$T(k^\times) = \bigoplus_{n \geq 0} (k^\times)^{\otimes n},$$

by the two-sided homogeneous ideal generated by all tensors of the form $a \otimes (1-a)$ for $a \neq 1$ in $k^\times$.

For each $n \geq 0$, the abelian group $K_M^n(k)$ is called the *n-th Milnor K-theory group* of k. For each $n \geq 1$, the image of a tensor $a_1 \otimes \cdots \otimes a_n$ under the quotient map $T(k^\times) \longrightarrow K_M^*(k)$ will be denoted by $\{a_1, ..., a_n\}$. Elements of this form will be called *n-symbols.*

We remark that we use multiplicative and additive notation for the corresponding operations both between elements of the field and between symbols. For example, given $a, b \in k^\times$, we have $\{a\} + \{b\} = \{ab\}$ and $\{a\}\{b\} = \{a, b\}$, while $\{a + b\}$ is only defined if $a + b \neq 0$ in k.

In the following proposition, we list several properties of Milnor K-theory rings.

Proposition 1.2.7. Let k be a field. Then

(i) For each $n \geq 0$, $K_M^n(k)$ is isomorphic to the abelian group given by generators all the n-symbols $\{a_1\} \otimes \cdots \otimes \{a_n\}$, and by all relations of the form $\{a_1\} \otimes \cdots \otimes \{a_i b_i\} \otimes \cdots \otimes \{a_n\} - \{a_1\} \otimes \cdots \otimes \{a_i\} \otimes \cdots \otimes \{a_n\} - \{a_1\} \otimes \cdots \otimes \{b_i\} \otimes \cdots \otimes \{a_n\}$ and $\{a_1\} \otimes \cdots \otimes \{a_i\} \otimes \{1 - a_i\} \otimes \cdots \otimes \{a_n\}$.

(ii) $K_M^0(k) \cong \mathbb{Z}$, $K_M^1(k) \cong k^\times$, and $K_M^2(k) \cong \frac{k^\times \otimes k^\times}{\langle a \otimes (1-a) \,:\, a \neq 1 \rangle}$.

(iii) For all $a \in k^\times$, $\{a, -a\} = 0$.

(iv) For all $a \in k^\times$, $\{a, -1\} = \{a\}^2 = \{-1, a\}$.

(v) For all $a, b \in k^\times$, $\{a, b\} = -\{b, a\}$.

(vi) $K_M^*(k)$ is a graded-commutative ring, i.e. for all $\alpha \in K_M^m(k)$ and $\beta \in K_M^n(k)$ we have $\alpha\beta = (-1)^{mn}\beta\alpha$.

(vii) If $a_1, ..., a_n \in k^\times$ are such that $a_i + a_j$ is 0 or 1 for some $1 \leq i < j \leq n$, then $\{a_1, ..., a_n\} = 0$.

(viii) If $a_1, ..., a_n \in k^\times$ are such that $a_1 + \cdots + a_n$ is 0 or 1, then $\{a_1, ..., a_n\} = 0$.

Proof.

(i) Follows from a well-known fact about graded rings. Since $K_M^*(k)$ is the quotient of $T(K_M^1(k))$ by the homogeneous ideal, say I, generated by tensors $\{a\} \otimes \{1-a\}$, then its n-th graded component is isomorphic to $K_M^1(k)^{\otimes n}/(I \cap K_M^1(k)^{\otimes n})$.

(ii) Is a particular case of (i).

(iii) By definition, we know that $\{a, a-1\} = \{a^{-1}, a^{-1}-1\} = 0$. But $\{a^{-1}, a^{-1}-1\} = (-1)^2\{a, \frac{1}{a^{-1}-1}\} = \{a, \frac{1}{a^{-1}-1}\}$ by bilinearity, hence

$$0 = \{a, a-1\} + \{a^{-1}, a^{-1}-1\} = \{a, \frac{a-1}{a^{-1}-1}\} = \{a, -a\}.$$

(iv) $\{a, a\} = \{a, (-1)(-a)\} = \{a, -1\} + \{a, -a\} \overset{\text{item }(iii)}{=} \{a, -1\}$, and analogously for the other equality.

(v) Follows by expanding $\{ab, -ab\}$ and using (iii).

(vi) It suffices to prove it for symbols $\alpha = \{a_1, ..., a_m\}$ and $\beta = \{b_1, ..., b_n\}$, and this case is immediate from (e).

(vii) Follows from (f).

(viii) We use induction on $n \geq 2$. The case $n = 2$ is item (iii). Suppose the result is known for some $n-1 \geq 2$ and consider $a_1 + a_2$. If it equals 0, then $\{a_1, ..., a_n\} = 0$ by (g). Otherwise, it follows from $\frac{a_1}{a_1+a_2} + \frac{a_2}{a_1+a_2} = 1$ that

$$0 = \{\frac{a_1}{a_1+a_2}, \frac{a_2}{a_1+a_2}\} = \{a_1, a_2\} - \{a_1, a_1+a_2\} - \{a_1+a_2, a_2\} + \{a_1+a_2, a_1+a_2\}.$$

Multiplying both sides by $\{a_3, ..., a_n\}$, the induction hypothesis for $\{a_1+a_2, a_3, ..., a_n\}$ implies that $\{a_1, ..., a_n\} = 0$.

$\square$

The norm residue homomorphism

By Proposition 1.2.5, the homomorphism

$$\partial_* : T(k^\times) \longrightarrow H^*(k, \mu_l^{\otimes *})$$

factors uniquely through the quotient map $T(k^\times) \longrightarrow K^*_M(k)$, thus defining a ring homomorphism

$$K^*_M(k) \longrightarrow H^*(k, \mu_l^{\otimes *}).$$

The kernel of the latter contains all multiples of l, since ∂_* has this property. Thus one obtains a further ring homomorphism

$$\nu_* : K^*_M(k)/l \longrightarrow H^*(k, \mu_l^{\otimes *}),$$

which is known as the *norm residue homomorphism.*

Definition 1.2.8. Given a field k and an integer $n \geq 0$, we say that the *Bloch-Kato condition* $BK(k,n)$ holds if and only if for every prime number $l \neq char(k)$, the norm residue homomorphism $\nu_n : K^n_M(k)/l \longrightarrow H^n(k, \mu_l^{\otimes n})$ is an isomorphism.

Given $n \geq 0$, we say that $BK(n)$ holds if and only if for every field k, $BK(k,n)$ holds.

The statement that $BK(n)$ holds for every $n \geq 0$ has been known as the *Bloch-Kato conjecture.*

1.3 The Zariski, Nisnevich and étale topologies; Galois cohomology via étale cohomology

1.3.1 The Zariski, Nisnevich and étale topologies

We denote by Ét_X the full subcategory of the category Sch_X of schemes over X whose objects are those (Y, f) (where $f : Y \longrightarrow X$ is a scheme morphism) such that f is étale. In what follows we will usually denote an X-morphism (Y, f) by Y.

An *étale covering* in Ét_X is defined to be a family

$$(f_i : Y_i \longrightarrow Y)_{i \in I}$$

of morphisms in $\text{Ét}/X$, where I is a small set, which are jointly surjective in the sense that $Y = \bigcup_{i \in I} f_i(Y_i)$. It is said to be a *Nisnevich covering* if the following additional condition is satisfied: for every point $y \in Y$, there exist $i \in I$ and a point $z \in Y_i$ such that $f_i(z) = y$ and the homomorphism between residue fields $\kappa(y) \longrightarrow \kappa(z)$ is an isomorphism.

The *étale topology* on X is defined as the Grothendieck topology on $\text{Ét}/X$ generated by the pretopology given by étale coverings. The site thus obtained will be denoted by $X_{\text{ét}}$. Similarly, the *Nisnevich topology*[4] is the Grothendieck topology on $\text{Ét}/X$ induced by the pretopology whose coverings are the Nisnevich coverings. The corresponding site will be denoted by X_{Nis}.

We denote the categories of presheaves of sets and of abelian groups on Ét_X by $\text{PSh}(X)$ and $\text{PSh}_J(X, \text{Ab})$, respectively. For $J = \text{Nis}$ or ét, the categories of sheaves of sets and of abelian groups on X_J will be denoted by $\text{Sh}_J(X)$ and $\text{Sh}_J(X, \text{Ab})$.

[4] It is named after Yevsey Nisnevich, who introduced it in Nisnevich, 1989 and called it the *completely decomposed* (or *cd*) topology.

As every Nisnevich covering is also an étale covering, the identity functor of Ét_X defines a morphism of sites

$$\epsilon\,:\,X_{\text{ét}} \longrightarrow X_{\text{Nis}}.$$

These topologies may be compared with the usual Zariski site X_{Zar} of X, whose underlying category is the category Op_X of open subsets of X and inclusion maps, and whose topology is generated by the pretopology whose coverings of an open subset $U \subset X$ are families

$$(f_i\,:\,U_i \longrightarrow U)_{i\in I}$$

of inclusions such that $U = \bigcup_{i\in I} f_i(U_i)$ $(= \bigcup_{i\in I} U_i)$.

The comparison between X_{Zar}, X_{Nis} and $X_{\text{ét}}$ is obtained by regarding open subsets of X as schemes over X:

An *open subscheme* of X is an open subset $U \subset X$ endowed with the scheme structure whose structure sheaf $\mathscr{O}_U$ is the restriction to U of the structure sheaf of X: for each open subset $V \subset U$ we have $\mathscr{O}_U(V) := \mathscr{O}_X|_U(V) = \mathscr{O}_X(V)$. The inclusion $i\,:\,U \hookrightarrow X$ defines a scheme morphism via the morphism of sheaves of rings $\mathscr{O}_X \longrightarrow i_*\mathscr{O}_U$ whose V-component is the restriction map $\mathscr{O}_X(V) \longrightarrow \mathscr{O}_X(V \cap U) = \mathscr{O}_U(V \cap U) = \mathscr{O}_U(i^{-1}(V)) = i_*\mathscr{O}_U(V)$.

Note that given a further open subset $U' \subset U$, its scheme structure induced from X is equal to that induced from U. Denoting by j the inclusion of U' into U and by $\bar{i}$, $\bar{j}$, $\overline{i\circ j}$ the scheme morphisms corresponding to i, j, $i\circ j$, we have that the composite

$$U' \xrightarrow{\bar{j}} U \xrightarrow{\bar{i}} X$$

is given at the level of structure sheaves by

$$\mathscr{O}_X \xrightarrow{\bar{i}^\#} i_*\mathscr{O}_U \xrightarrow{i_*(\bar{j}^\#)} i_*(j_*\mathscr{O}_{U'}),$$

which is given at each open $V \subset X$ by composing the restriction map $\mathscr{O}_X(V) \longrightarrow \mathscr{O}_X(V\cap U)$ with the $V \cap U$-component of $\bar{j}^\#$. The latter equals the restriction map $\mathscr{O}_U(V \cap U) \longrightarrow \mathscr{O}_U((V\cap U)\cap U')$, hence the restriction map $\mathscr{O}_X(V\cap U) \longrightarrow \mathscr{O}_X(V\cap U')$. Thus (by functoriality of $\mathscr{O}_X$) the above composite equals the restriction map $\mathscr{O}_X(V) \longrightarrow \mathscr{O}_X(V \cap U')$, and we conclude that $\overline{i\circ j} = \bar{i}\circ\bar{j}$.

From this we obtain a functor

$$I\,:\,\text{Op}_X \longrightarrow \text{Sch}_X.$$

It may be proved that it restricts to a functor $\text{Op}_X \longrightarrow \text{Ét}_X$ which sends Zariski coverings to Nisnevich (hence étale) coverings and preserves finite limits (which in Op_X are given by intersections). This induces a morphism of sites

$$\nu\,:\,X_{\text{Nis}} \longrightarrow X_{\text{Zar}},$$

$$\pi = \nu\circ\epsilon\,:\,X_{\text{ét}} \longrightarrow X_{\text{Zar}}.$$

Thus we obtain the following adjunctions between categories of sheaves of abelian groups:

$$\mathrm{Sh}_{\mathrm{Zar}}(X, \mathrm{Ab}) \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathrm{Sh}_{\text{ét}}(X, \mathrm{Ab}), \tag{1.3.1}$$

$$\mathrm{Sh}_{\mathrm{Zar}}(X, \mathrm{Ab}) \underset{\nu_*}{\overset{\nu^*}{\rightleftarrows}} \mathrm{Sh}_{\mathrm{Nis}}(X, \mathrm{Ab}), \tag{1.3.2}$$

$$\mathrm{Sh}_{\mathrm{Nis}}(X, \mathrm{Ab}) \underset{\epsilon_*}{\overset{\epsilon^*}{\rightleftarrows}} \mathrm{Sh}_{\text{ét}}(X, \mathrm{Ab}). \tag{1.3.3}$$

By definition, the direct image functors π_*, ν_* are both given by precomposing sheaves with $I^{op} : \mathrm{Op}_X^{op} \longrightarrow \text{Ét}_X$. On the other hand, ϵ_* is given by precomposition with the identity functor, so it sends an étale sheaf to itself regarded as a Nisnevich sheaf.

$$\mathrm{Sh}_{\mathrm{Zar}}(X, \mathrm{Ab}) \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathrm{Sh}_{\text{ét}}(X, \mathrm{Ab}), \tag{1.3.4}$$

$$\mathrm{Sh}_{\mathrm{Zar}}(X, \mathrm{Ab}) \underset{\nu_*}{\overset{\nu^*}{\rightleftarrows}} \mathrm{Sh}_{\mathrm{Nis}}(X, \mathrm{Ab}), \tag{1.3.5}$$

$$\mathrm{Sh}_{\mathrm{Nis}}(X, \mathrm{Ab}) \underset{\epsilon_*}{\overset{\epsilon^*}{\rightleftarrows}} \mathrm{Sh}_{\text{ét}}(X, \mathrm{Ab}). \tag{1.3.6}$$

Cohomology groups

Definition 1.3.1. Suppose given a scheme X, and let us consider the site X_J where J either be the Zariski, Nisnevich or étale topology. Then we may consider the functor

$$\Gamma : \mathrm{Sh}_J(X, \mathrm{Ab}) \longrightarrow \mathrm{Ab}$$

given by evaluation at X. Cohomology of X with respect to J is defined as the right derived functor

$$\mathbf{R}\Gamma : D^+(\mathrm{Sh}_J(X, \mathrm{Ab})) \longrightarrow D^+(\mathrm{Ab}).$$

If $\mathscr{F}$ is a complex of J-sheaves (or a J-sheaf, which we identify with a complex concentrated in degree 0), the n-th J-cohomology group of X with coefficients in $\mathscr{F}$ is defined for each integer n as the n-th cohomology group of $\mathbf{R}\Gamma(\mathscr{F})$,

$$H^n_J(X, \mathscr{F}) := H^n(\mathbf{R}\Gamma(\mathscr{F})).$$

J-cohomology groups of X may be assembled into functors

$$H^n_J(X, -) : D^+(\mathrm{Sh}_J(X, \mathrm{Ab})) \longrightarrow \mathrm{Ab}.$$

by composing $\mathbf{R}\Gamma$ with each $H^n : D^+(\mathrm{Ab}) \longrightarrow \mathrm{Ab}$.

1.3.2 Étale cohomology and Galois cohomology

We will now state a classical result according to which Galois cohomology of a field k endowed with a separable closure k_{sep}, i.e. profinite group cohomology of $G = \mathrm{Gal}(k_{sep}/k)$

with coefficients in discrete G-modules, is in a certain sense equivalent equivalent to étale cohomology of the scheme Spec k, which is by definition cohomology of sheaves of abelian groups on the étale site Spec $(k)_{\text{ét}}$.

Let F be a presheaf of sets on $\text{Ét}_{\text{Spec } k}$. If K/k is a finite Galois extension contained in k_{sep}, we obtain a left action of $\text{Gal}(K/k)$ on $F(\text{Spec } K)$ given by

$$g \cdot x = F(\text{Spec } g)(x).$$

(Note that $F(\text{Spec } id_K) = F(id_{\text{Spec } K}) = id_{F(\text{Spec } K)}$, and for each $g, h \in \text{Gal}(K/k)$ we have $F(\text{Spec } (g \circ h)) = F(\text{Spec } h \circ \text{Spec } g) = F(\text{Spec } g) \circ F(\text{Spec } h)$ since Spec and F are both contravariant functors.)

If $K \subset L$ are two finite Galois extensions of k contained in k_{sep}, the inclusion $i : K \longrightarrow L$ induces a function $F(i) : F(L) \longrightarrow F(K)$. Denoting by $p : \text{Gal}(L/k) \longrightarrow \text{Gal}(K/k)$ the homomorphism given by restricting automorphisms to K, for each $g \in \text{Gal}(L/k)$ we have $i \circ p(g) = g \circ i$, so

$$F(\text{Spec } i) \circ F(\text{Spec } p(g)) = F(\text{Spec } g) \circ F(\text{Spec } i). \tag{1.3.7}$$

Now, let us consider the set $\varinjlim_{K \in \text{FinGal}} F(\text{Spec } K)$, which can be identified with the quotient of

$$\coprod_{K \in \text{Ob(FinGal)}} F(\text{Spec } K) = \{(K, x) \in \text{Ob(FinGal)} \times \bigcup_{L \in \text{Ob(FinGal)}} F(\text{Spec } L) | x \in F(\text{Spec } K)\}$$

by the equivalence relation $\sim$ given by $(K, x) \sim (K', x')$ if and only if there exists K'' containing both K and K', say with inclusions $i : K \longrightarrow K''$ and $i' : K' \longrightarrow K''$, such that $F(\text{Spec } i(x)) = F(\text{Spec } i'(x'))$. Note that by equation 1.3.7, if $(K, x) \sim (L, x')$ for $K \subset L$, then for each $g \in \text{Gal}(L/k)$ it holds that

$$F(\text{Spec } i)(F(\text{Spec } p(g))(x)) = F(\text{Spec } g)(F(\text{Spec } i)(x)) = F(\text{Spec } g)(x').$$

Then $(K, F(\text{Spec } p(g))(x)) \sim (L, F(\text{Spec } g)(x'))$, which we rewrite as $(K, p(g) \cdot x) \sim (L, g \cdot x')$. This induces an action of $\text{Gal}(k_{sep}/k) \cong \varprojlim_{K \in \text{FinGal}^{op}} \text{Gal}(K/k)$ on $\varinjlim_{K \in \text{FinGal}} F(\text{Spec } K)$ with the property that each class $\overline{(K, x)}$ is stabilized by $\text{Gal}(k_{sep}/K)$, which is an open subgroup. Hence by Remark 1.1.20, $\varinjlim_{K \in \text{FinGal}} F(\text{Spec } K)$ is a discrete $\text{Gal}(k_{sep}/k)$-set.

We have an analogous construction with sheaves of sets replaced by sheaves of abelian groups and actions on sets replaced by modules.

Theorem 1.3.2 (Milne, 1980, Lemma 1.8 and Theorem 1.9). Let us denote the profinite group $\text{Gal}(k_{sep}/k)$ by G. The above construction extends to a functor $\text{PSh}(\text{Ét}_{\text{Spec } k}, \text{Ab}) \longrightarrow \mathscr{C}\text{Mod}_G$ that restricts to an equivalence of categories $\varphi : \text{Sh}_{\text{ét}}(\text{Spec } k, \text{Ab}) \xrightarrow{\simeq} \mathscr{C}\text{Mod}_G$. Moreover, consider the functor

$$P : \text{Ét}_{\text{Spec } k} \longrightarrow \mathscr{C}\text{Set}_G$$

given by sending each $X \in \text{Ét}_{\text{Spec } k}$ to the set $\text{Hom}_{\text{Sch}_k}(\text{Spec } (k_{sep}), X)$ endowed with the

G-action $g \cdot f = f \circ \operatorname{Spec} g$. Then defining

$$\psi : \mathscr{C}\mathrm{Mod}_G \longrightarrow \mathrm{PSh}(\text{Ét}_{\operatorname{Spec} k}, \mathrm{Ab}),$$

by $M \mapsto \mathrm{Hom}_{\mathscr{C}\mathrm{Set}_G}(P(-), M)$, it holds that

(i) $\psi(M)$ is an étale sheaf for every discrete G-module M.

(ii) The restricted functor $\mathscr{C}\mathrm{Mod}_G \longrightarrow \mathrm{Sh}_{\text{ét}}(\operatorname{Spec} k, \mathrm{Ab})$ is an equivalence of categories.

(iii) For each finite K/k, $\psi(M)(\operatorname{Spec} K)$ is isomorphic to $M^{\mathrm{Gal}(k_{sep}/K)}$.

(iv) Products of finite extensions satisfy $\psi(M)(\operatorname{Spec} \prod_{i \in I} K_i) = \prod_{i \in I} \psi(M)(\operatorname{Spec} K_i)$.

As a consequence, denoting by $\Gamma : \mathrm{Sh}_{\text{ét}}(\operatorname{Spec} k, \mathrm{Ab}) \longrightarrow \mathrm{Ab}$ the functor given by evaluation at $\operatorname{Spec} k$ and by $\Gamma' : \mathscr{C}\mathrm{Mod}_G \longrightarrow \mathrm{Ab}$ the functor of G-invariants $M \mapsto M^G$, we have an isomorphism

$$\mathbf{R}\Gamma' \circ \overline{\varphi} \cong \mathbf{R}\Gamma$$

between functors from $D^+(\mathrm{Sh}_{\text{ét}}(\operatorname{Spec} k, \mathrm{Ab}))$ to $D^+(\mathrm{Ab})$, where $\overline{\varphi} : D^+(\mathrm{Sh}_{\text{ét}}(\operatorname{Spec} k, \mathrm{Ab})) \longrightarrow D^+(\mathscr{C}\mathrm{Mod}_G)$ is induced from the universal property of the derived category as a localization (being an equivalence of categories, φ preserves quasi-isomorphisms). By construction, the latter coincides on objects with the extension of φ to a functor $\mathrm{Ch}^+(\mathrm{Sh}_{\text{ét}}(\operatorname{Spec} k, \mathrm{Ab})) \longrightarrow \mathrm{Ch}^+(\mathscr{C}\mathrm{Mod}_G)$, which we also denote by φ by abuse of notation.

It follows that for each bounded below complex $\mathscr{F}$ of sheaves on $(\operatorname{Spec} k)_{\text{ét}}$ there exists for each integer n an isomorphism

$$H^n_{\text{ét}}(\operatorname{Spec} k, \mathscr{F}) \cong H^n_{disc}(G, \varphi(\mathscr{F})) = H^n(k, \varphi(\mathscr{F})).$$

It may be checked via the above description of the equivalence between étale sheaves and discrete G-modules that the module of units $k^\times_{sep}$ corresponds, up to isomorphism, to the étale sheaf of units $\mathscr{O}^\times$ on $\text{Ét}_{\operatorname{Spec} k}$. For a prime number $l \neq char(k)$, the module of l-th roots of unity μ_l corresponds to the étale sheaf μ_l.

Thus the short exact sequence

$$1 \longrightarrow \mu_l \longrightarrow k^\times_{sep} \xrightarrow{\wedge l} k^\times_{sep} \longrightarrow 1$$

of discrete G-modules corresponds, up to isomorphism, to the short exact sequence

$$0 \longrightarrow \mu_l \longrightarrow \mathscr{O}^\times \xrightarrow{\wedge l} \mathscr{O}^\times \longrightarrow 0$$

of étale sheaves.

Chapter 2

Motivic cohomology

In this chapter we discuss motivic cohomology defined in terms of Voevodsky's motivic complexes, having as our main reference the Lecture Notes on Motivic Cohomology by Mazza, Voevodsky and Weibel (Mazza *et al.*, 2006).

Some preliminary constructions will be necessary for discussing motivic complexes and motivic cohomology. The first step will be to define finite correspondences over a given field k. They constitute an additive category whose objects are the smooth, separated, finite type k-schemes and whose morphisms are given by a generalization of the usual morphisms of schemes. It is a cycle-theoretic construction that allows, on the one hand, to consider usual scheme morphisms by identifying them with their graphs, but on the other hand there are additional 'transpose' maps associated to finite surjective morphisms of schemes. Then we discuss presheaves with transfers, which are additive presheaves on the category of finite correspondences, and sheaves with transfers, which are presheaves with transfers whose restriction to the usual category of smooth schemes is a sheaf for a given Grothendieck topology. Then motivic complexes are introduced by using a certain simplicial construction on presheaves with transfers, and motivic cohomology groups of a (smooth, separated, finite type) k-scheme X are defined as hypercohomology groups of the restriction of motivic complexes to either the Zariski or the étale topology on X. As we shall see, the distinction between the invariants provided by these two topologies is an important feature.

Then we will outline a characterization, following Mazza *et al.*, 2006, of the weight 1 motivic complex $\mathbb{Z}(1)$. Namely, it will be quasi-isomorphic as a presheaf with transfers to $\mathscr{O}^{\times}$. This result provides important information on Zariski motivic cohomology and on étale motivic cohomology with torsion coefficients. In the remainder of the chapter, we briefly discuss the existence of a sheafification functor for presheaves with transfers with respect to the étale topology, the construction of comparison ('change of topology') maps between Zariski cohomology and étale cohomology, and also the definition of a homotopy invariant presheaf with transfers.

2.1 Correspondences, (pre)sheaves with transfers

Throughout this section, we let k denote a given field.

Convention 2.1.1. Recall from our conventions that Sm_k denotes the category of smooth, separated, finite type k-schemes. Structure morphisms $X \longrightarrow \operatorname{Spec} k$ will be omitted when this causes no confusion. Note that several properties hold for any such X:

(i) It is regular, as is any smooth scheme over a field. See Raynaud and Grothendieck, 1971, Exposé II, 3.1.

(ii) It is noetherian.

(iii) It is separated, since $\operatorname{Spec} k \longrightarrow \operatorname{Spec} \mathbb{Z}$ is separated. See Grothendieck, 1960, Chap. I, 5.5.1(ii).

(iv) If it is connected, then it is irreducible.

(v) It can be expressed as a finite disjoint union of smooth, separated, integral, finite type k-schemes; these are both its irreducible components and its connected components.

Moreover, by the cancellation property for separated morphisms (i.e. if a composite $g \circ f$ of morphisms of schemes is separated, then so is f; see Grothendieck, 1960, Chap. I, 5.5.1(v)), the underlying morphism of schemes of any morphism in Sm_k is separated.

When it is convenient, we denote finite products in Sm_k (i.e. fibered products over $\operatorname{Spec} k$) by XY, XYZ, etc., and by π_X^{XY}, π_X^{XYZ}, π_{XY}^{XYZ} (or by π_X, π_X, π_{XY}, resp., when the domains cause no confusion), and so on the corresponding canonical projections.

2.1.1 Finite correspondences

In the theory of algebraic cycles, given a field k and k-varieties X and Y, a correspondence, as discussed for example in Fulton, 1984, is defined as a cycle on $X \times_k Z$. When one considers correspondences between smooth k-varieties up to rational equivalence, one may use the intersection product, pullback, and pushforward of cycles to define (Fulton, 1984, Def. 16.6.1) a composition operation on groups of rational equivalence classes of correspondences. We will use a variant of this construction following Mazza *et al.*, 2006, Lecture 1. We will only consider (see Definition) 2.1.2) cycles on a product $X \times_k Y$ in Sm_k whose subvarieties $Z \subset X \times_k Y$ occurring with nonzero coefficient are finite and surjective onto a connected component of X. These are called *finite correspondences*. Composition may be defined for finite correspondences (see Corollary 2.1.6) by following the aforementioned definition of composition of correspondence classes given in Fulton, 1984. However, for finite correspondences, by virtue of Lemma 2.1.5, the cycles between which we need to take the intersection product intersect properly, so in this case composition may be defined at the level of cycles, with no need to identify those which are pairwise rationally equivalent.

Definition 2.1.2. Suppose given $X, Y \in \mathrm{Sm}_k$. A *finite k-correspondence* from X to Y is defined to be a cycle $\alpha = \sum_{i \in I} n_i z_i \in \mathcal{Z}(X \times_k Y)$ such that for each i with $n_i \neq 0$, denoting by $Z_i \hookrightarrow X$ the induced integral closed subscheme structure on $\overline{\{z_i\}}$, the composite

$$Z_i \hookrightarrow X \times_k Y \xrightarrow{\pi_X} X$$

is finite and its image is an irreducible component of X. Note that since finite morphisms are proper, for each z_i this is equivalent to requiring that the following both hold:

(i) The above composite is finite.

(ii) $\pi_X(z_i)$ is the generic point of an irreducible component of X.

Following Cisinski and Déglise, 2019, we denote it by

$$\alpha : X \bullet\!\!\longrightarrow Y.$$

Note that the set of finite k-correspondences from X to Y is a subgroup of $\mathcal{Z}(X \times_k Y)$; we will denote it by $C_k(X, Y)$.

Suppose $f : X \longrightarrow Y$ is a morphism in Sm_k. Since X is separated over k, the graph morphism $\gamma_f : \Gamma_f \longrightarrow X \times_k Y$ is a closed immersion, and the composite $\Gamma_f \xrightarrow{\gamma_f} X \times_k Y \xrightarrow{\pi_X} X$ is an isomorphism. Hence the associated cycle $[\Gamma_f]_{X \times_k Y}$ is a finite correspondence.

Note that f can be recovered from $[\Gamma_f]_{X \times_k Y}$ as

$$X \xrightarrow{\cong} \Gamma_f \xrightarrow{\gamma_f} X \times_k Y \xrightarrow{\pi_Y} Y,$$

where γ_f is the closed immersion associated to $[\Gamma_f]_{X \times_k Y}$, and the first map is inverse to the isomorphism

$$\Gamma_f \xrightarrow{\gamma_f} X \times_k Y \xrightarrow{\pi_X} X.$$

We will often abuse notation and denote Γ_f by f in the context of finite correspondences.

One would like to define composition of finite correspondences by extending the usual composition operation in Sm_k, i.e. one wishes to construct a category of finite correspondences (over k) in such a way that (i) $[\Gamma_g]_{Y \times_k Z} \circ [\Gamma_f]_{X \times_k Y} = [\Gamma_{g \circ f}]_{X \times_k Z}$ for any $X \xrightarrow{f} Y \xrightarrow{g} Z$, and (ii) $[\Gamma_{id_X}]_{X \times_k X} : X \bullet\!\!\longrightarrow X$ is a two-sided identity for any X.

Lemma 2.1.3. Let S be a Noetherian scheme, and $f : X \longrightarrow Y$ a morphism of separated finite type S-schemes. If $Z \subset X$ is an irreducible closed subset which is finite over S, then $f(Z) \subset Y$ is closed, irreducible, and finite over S.

Proof. See Mazza *et al.*, 2006, Lemma 1.4. □

Lemma 2.1.4. Let S be a normal scheme, and X an integral S-scheme with $X \longrightarrow S$ finite and surjective over a connected component of S. Then for any connected S-scheme Y, every connected component of $X \times_S Y$ is finite and surjective over Y.

Proof. See Mazza *et al.*, 2006, Lemma 1.6. □

Lemma 2.1.5. Suppose given $X, Y, Z \in \mathrm{Sm}_k$, and integral closed subschemes $U \hookrightarrow X \times_k Y$, $V \hookrightarrow Y \times_k Z$ which are finite and surjective over some connected component of X, Y, respectively (hence these define finite correspondences $[U] : X \bullet\!\!\longrightarrow Y$ and $[V] : Y \bullet\!\!\longrightarrow Z$). Then any irreducible component of the image of $U \times_k Z \hookrightarrow X \times_k Y \times_k Z$ properly intersects any irreducible component of the image of $X \times_k V \hookrightarrow X \times_k Y \times_k Z$, so the intersection product $[X \times_k V] \cdot [U \times_k Z]$ is defined. Moreover, the pushforward

$$\pi^{XYZ}_{XZ*}([X \times_k V] \cdot [U \times_k Z])$$

is a finite correspondence from X to Z.

Proof. Firstly, note that the scheme-theoretic intersection of $U \times_k Z$ and $X \times_k V$ in $X \times_k Y \times_k Z$ is equivalently given by the pullback $U \times_Y V$ (with respect to the projections $U \xrightarrow{\pi_2} Y$ and $V \xrightarrow{\pi_1} Y$). Hence after taking the respective images in $X \times_k Y \times_k Z$, irreducible components of $U \times_Y V$ are precisely the irreducible components of those subsets given by intersecting an irreducible component of $U \times_k Z$ with one of $X \times_k Z$. Now, we have

(i) Since $V \longrightarrow Y$ is finite and surjective over a connected component, it follows from Lemma 2.1.4 that each irreducible component of $U \times_Y V$ is finite and surjective over some connected component of U, hence also over some connected component of X by the assumption on U. In particular, these have dimension equal to $\dim X$, so the desired intersection is proper; also, the intersection product $[X \times_k V] \cdot [U \times_k Z]$ is defined (by Serre's formula), and the points occurring in it with non-zero coefficient are the same as those in $[U \times_Y V]$.

(ii) By applying Lemma 2.1.3 to the morphism $\pi^{XYZ}_{XZ} : X \times_k Y \times_k Z \longrightarrow X \times_k Z$ of X-schemes, the image in $X \times_k Z$ of each irreducible component of the image of $U \times_Y V$ in $X \times_k Y \times_k Z$ is closed, irreducible, finite, and surjective over a connected component of X.

Since the subspaces of $X \times_k Z$ obtained in (ii) are precisely the supports of points occurring with non-zero coefficient in $\pi^{XYZ}_{XZ*}([U \times_Y V])$, we conclude (by using the remark in (i)) that $\pi^{XYZ}_{XZ*}([X \times_k V] \cdot [U \times_k Z])$ is a finite correspondence from X to Z. □

Corollary 2.1.6. Suppose given $X, Y, Z \in \mathrm{Sm}_k$, and finite correspondences $\alpha : X \bullet\!\!\longrightarrow Y$, $\beta : Y \bullet\!\!\longrightarrow Z$. Then the intersection product $\pi^{XYZ*}_{YZ}\beta \cdot \pi^{XYZ*}_{XY}\alpha$ is defined, and

$$\pi^{XYZ}_{XZ*}(\pi^{XYZ*}_{YZ}\beta \cdot \pi^{XYZ*}_{XY}\alpha)$$

is a finite correspondence from X to Z.

Now, for any such X, Y, Z, α, β, let us define the *composite* $\beta \circ \alpha \in C_k(X, Z)$ of α and β as the above cycle. Then composition is bilinear and associative, and $[\Gamma_{id_X}]_{X \times_k X}$ is a two-sided identity for each $X \in \mathrm{Sm}_k$.

Proof. The first part follows from the linearity of pullbacks and pushforwards, the bilinearity of intersection products, and the fact that α, β are linear combinations of cycles of the form $[U]$, $[V]$, resp., as in Lemma 2.1.5.

For the second part, we refer to Fulton, 1984, Prop. 16.1.1, where the computations are performed more generally for (not necessarily finite) correspondences. □

Definition 2.1.7. Composition of correspondences as in Corollary 2.1.6 defines a (Ab-enriched) category, which we denote by Cor_k and call the *category of finite k-correspondences*, having the same objects as Sm_k and abelian groups of morphisms given by $\mathrm{Hom}_{\mathrm{Cor}_k}(X, Y) := C_k(X, Y)$.

Proposition 2.1.8. Cor_k is an additive category with finite biproducts given by disjoint unions (in particular, the empty scheme is a zero object). Moreover, the data $X \longmapsto X$ and $(g : X \to Y) \longmapsto ([\Gamma_g]_{X\times_k Y} : X \bullet\!\!\longrightarrow Y)$ define a faithful functor $\Gamma : \mathrm{Sm}_k \longrightarrow \mathrm{Cor}_k$.

Proof. It remains to check that for any $f : X \to Y$, $g : Y \to Z$ in Sm_k, it holds that $[\Gamma_g]_{Y\times_k Z} \circ [\Gamma_f] = [\Gamma_{g\circ f}]_{X\times_k Z}$. For this computation, we refer to Fulton, 1984, Prop. 16.1.1(c). □

Proposition 2.1.9. Suppose given finite k-correspondences $\alpha : X \bullet\!\!\longrightarrow Y$ and $\beta : Y \bullet\!\!\longrightarrow Z$. Then $\pi_{YZ}^{XYZ*}(\beta)$ and $\pi_{XY}^{XYZ*}(\alpha)$ intersect properly. Moreover, the support of the cycle $\pi_{XZ*}^{XYZ}(\pi_{YZ}^{XYZ*}(\beta) \cdot \pi_{XY}^{XYZ*}(\alpha))$ is finite and pseudo-dominant over X. It follows that $\pi_{XZ*}^{XYZ}(\pi_{YZ}^{XYZ*}(\beta) \cdot \pi_{XY}^{XYZ*}(\alpha))$ is a finite S-correspondence $X \bullet\!\!\longrightarrow Z$.

Given such f and g, an intersection-theoretic interpretation of (the graph of) $g \circ f$ is provided by the formula

$$[\Gamma_{g\circ f}]_{X\times_k Z} = \pi_{XZ*}^{XYZ}(\pi_{YZ}^{XYZ*}([\Gamma_g]_{Y\times_k Z}) \cdot \pi_{XY}^{XYZ*}([\Gamma_f]_{X\times_k Y}))$$

Transposes and transfers

Note that for any $X, Y \in \mathrm{Sm}_k$ pullback of cycles along the canonical isomorphism $Y \times_k X \xrightarrow{\cong} X \times_k Y$ defines an isomorphism between cycle groups $\mathcal{Z}(X \times_k Y) \xrightarrow{\cong} \mathcal{Z}(Y \times_k X)$; the image of a cycle α under this map will be called its *transpose* and will be denoted by α^t. One of the motivations for considering finite correspondences instead of just ordinary scheme morphisms is the fact that under certain assumptions, the transpose of a correspondence $\alpha : X \bullet\!\!\longrightarrow Y$ defines a correspondence $Y \bullet\!\!\longrightarrow X$. The case we will be most interested in is the following: if $f : X \longrightarrow Y$ in Sm_k is finite and surjective, then the transpose $[\Gamma_f]^t_{X\times_k Y}$ of its graph is a correspondence $Y \bullet\!\!\longrightarrow X$; more explicitly, it is the cycle associated to the composite closed immersion

$$\Gamma_f \xrightarrow{\gamma_f} X \times_k Y \cong Y \times_k X.$$

For any f in Sm_k, we usually denote the cycle $[\Gamma_f]^t_{X\times_k Y} \in \mathcal{Z}(Y \times_k X)$ by f^t (even in case it is not a correspondence).

Monoidal structure

Given $X, Y \in \mathrm{Cor}_k$, let us denote by $X \otimes Y$ the fibered product of schemes $X \times_k Y$ (which is not a cartesian product in Cor_k; see Proposition 2.1.8).

We now consider for each X, Y, X', $Y' \in \mathrm{Cor}_k$ the bilinear composite

$$\mathcal{Z}(X \times_k X') \times \mathcal{Z}(Y \times_k Y') \longrightarrow \mathcal{Z}((X \times_k X') \times_k (Y' \times_k Y')) \xrightarrow{\cong} \mathcal{Z}((X \times_k Y) \times_k (X' \times_k Y')),$$

where the first map is given by bilinearly extending the function $([U], [V]) \longmapsto [U \times_k V]$ defined on the product of the respective freely generating subsets consisting of cycles associated to integral closed subschemes, i.e. by

$$\left(\sum_{i \in I} [U_i], \sum_{j \in J} [V_j] \right) \longmapsto \sum_{(i,j) \in I \times J} [U_i \times_k V_j],$$

and the second one is given by pulling-back along the canonical isomorphism. Then for any finite correspondences $\alpha \in C_k(X, X')$ and $\beta \in C_k(Y, Y')$, the image of (α, β) under this composite is a finite correspondence from $X \otimes Y$ to $X' \otimes Y'$. Let us denote it by $\alpha \otimes \beta$.

It may be proved that:

Proposition 2.1.10 (Mazza *et al.*, 2006)**.** The above data define a functor $\otimes : \mathrm{Cor}_k \times \mathrm{Cor}_k \longrightarrow \mathrm{Cor}_k$. By also considering the usual associativity and unit isomorphisms for cartesian products of k-schemes, this defines a symmetric monoidal structure on Cor_k.

2.1.2 (Pre)sheaves with transfers

Definition 2.1.11. A *presheaf with transfers* (with respect to k) is defined to be an additive functor

$$F : \mathrm{Cor}_k^{op} \longrightarrow \mathrm{Ab}.$$

If J is a Grothendieck topology on Sm_k (for our purposes, J will be either the Zariski, Nisnevich, or étale topology), such an F is said to be a *J-sheaf with transfers* if its restriction $F \circ \gamma$ to Sm_k^{op} is a J-sheaf.

A *morphism* of presheaves with transfers (resp. J-sheaves with transfers) is defined to be a natural transformation. The Ab-enriched category thus obtained is denoted by $\mathrm{PST}(k)$ (resp. $\mathrm{ST}_J(k)$).

More generally, whenever $\mathcal{A}$ is an abelian category, we define the Ab-enriched category of *$\mathcal{A}$-valued presheaves with transfers* as that consisting of additive functors $\mathrm{Cor}_k^{op} \to \mathcal{A}$ and natural transformations between them. It will be denoted by $\mathrm{PST}(k, \mathcal{A})$.

If $\mathcal{A}$ is a complete abelian category – so we are able to consider the sheaf condition for presheaves $\mathrm{Sm}_k^{op} \longrightarrow \mathcal{A}$ –, the category of *$\mathcal{A}$-valued J-sheaves with transfers*, where J is a Grothendieck topology on Sm_k, is defined as that whose objects are $\mathcal{A}$-valued presheaves with transfers $F : \mathrm{Cor}_k^{op} \to \mathcal{A}$ such that $F \circ \gamma$ is an $\mathcal{A}$-valued J-sheaf on Sm_k, and whose morphisms are natural transformations. We denote it by $\mathrm{ST}_J(k, \mathcal{A})$.

In case $\mathcal{A}$ is the category Mod_A of modules over a commutative ring with unit A, we denote $\mathrm{PST}(k, \mathrm{Mod}_A)$ and $\mathrm{ST}_J(k, \mathrm{Mod}_A)$ by $\mathrm{PST}(k, A)$ and $\mathrm{ST}_J(k, A)$, respectively.

Whenever $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor between abelian categories, composition with φ defines a functor $\mathrm{PST}(k, \mathcal{A}) \longrightarrow \mathrm{PST}(k, \mathcal{B})$. If $f : A \longrightarrow B$ is a homomorphism of commutative rings with unit, then the extension of scalars functor $- \otimes_A B : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_B$

has a right adjoint given by restriction of scalars (the abelian group underlying a B-module M is endowed with an A-module structure $f^*(M)$ with multiplication defined by $a \cdot_{f^*(M)} m := f(a) \cdot_M m$). Thus $- \otimes_A B$ preserves small colimits, and in particular it preserves binary direct sums and the zero object. It follows that $- \otimes_A B$ is additive, so composition of presheaves with transfers with $- \otimes_A B$ defines a functor $\mathrm{PST}(k, A) \longrightarrow \mathrm{PST}(k, B)$; we denote it on objects by $F \longmapsto F \times_A B$, or by $F \otimes B$ when $A = \mathbb{Z}$. Similarly, if M is an A-module, then the tensor product functor $- \otimes_A M : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_A$ has a right adjoint $\mathrm{Hom}_A(M, -)$, so composition with $- \otimes_A M$ defines a functor $\mathrm{PST}(k, A) \longrightarrow \mathrm{PST}(k, A)$; it will be denoted on objects by $F \longmapsto F \otimes_A M$, or by $F \otimes M$ when $A = \mathbb{Z}$.

Definition 2.1.12. Given a scheme $X \in \mathrm{Sm}_k$, the presheaf with transfers $C_k(-, X) : \mathrm{Sm}_k^{op} \longrightarrow \mathrm{Ab}$ will be denoted by $\mathbb{Z}_k^{tr}(X)$. More generally, if A is an abelian group, we denote $\mathbb{Z}_k^{tr}(X) \otimes_{\mathbb{Z}} A$ by $A_k^{tr}(X)$.

By definition, for each $Y \in \mathrm{Sm}_k$ we have $A_k^{tr}(X)(Y) = C_k(Y, X) \otimes_{\mathbb{Z}} A$. Now, since $- \otimes_{\mathbb{Z}} A : \mathrm{Ab} \longrightarrow \mathrm{Ab}$, being a left adjoint, preserves small colimits and in particular small direct sums, we have isomorphisms $\bigoplus_{i \in I} A \cong \bigoplus_{i \in I} (\mathbb{Z} \otimes_{\mathbb{Z}} A) \cong (\bigoplus_{i \in I} \mathbb{Z}) \otimes_{\mathbb{Z}} A$ for any small set I. Hence by definition of $C_k(Y, X)$ it follows that $A_k^{tr}(X)(Y)$ is isomorphic to the free A-module generated by points $z \in Y \times_k X$ such that the integral closed subscheme $Z := \overline{\{z\}} \subset Y \times_k X$ has the property that the composite $Z \hookrightarrow Y \times_k X \xrightarrow{\pi_Y^{YX}} Y$ is finite and its image is a connected component of Y.

It may be proved that:

Proposition 2.1.13 (Mazza *et al.*, 2006, 6.2)**.** Given a scheme $X \in \mathrm{Sm}_k$ and an abelian group A, the functor $\mathrm{Cor}_k(-, X) \otimes_{\mathbb{Z}} A : \mathrm{Sm}_k^{op} \longrightarrow \mathrm{Ab}$ is an étale (hence Nisnevich, Zariski) sheaf. It follows that the representable presheaf with transfers $A_k^{tr}(X) : \mathrm{Cor}_k \longrightarrow \mathrm{Ab}$ is an étale (hence Nisnevich, Zariski) sheaf with transfers.

Thus for each abelian group A and J = étale, Nisnevich, Zariski, we obtain a functor

$$A_k^{tr}(-) : \mathrm{Sm}_k \longrightarrow \mathrm{ST}_J(k).$$

Example 2.1.14. We now sketch a few examples of presheaves with transfers. For a more detailed description we refer to Mazza *et al.*, 2006.

(a) The sheaf of invertible global sections $\mathscr{O}^\times : \mathrm{Sm}_k^{op} \longrightarrow \mathrm{Ab}$ extends to a presheaf with transfers as follows: $p : W \longrightarrow X$ is a (any) finite and surjective morphism of schemes where X is normal, then the extension $K(X) \to K(W)$ of function fields induces a norm map

$$N : K(W)^* \longrightarrow K(X)^*.$$

Since $\mathscr{O}(X)$ is integrally closed in $K(X)$, it follows that for each $f \in \mathscr{O}(W)^*$, both $N(f)$ and $N(f^{-1})$ belong to $\mathscr{O}(X)^*$. We also denote the induced restricted map by

$$N : \mathscr{O}(W)^* \longrightarrow \mathscr{O}(X)^*.$$

Now, given X, $Y \in \mathrm{Sm}_k$ and an integral closed subscheme $W \subset X \times Y$ such that the projection $p : W \longrightarrow X$ is finite and pseudo-dominant (recall that X is regular, hence normal), we obtain a composite map of groups

$$\mathcal{O}^\times(Y) \longrightarrow \mathcal{O}^\times(W) \xrightarrow{N} \mathcal{O}^\times(X).$$

A presheaf with transfers

$$\overline{\mathcal{O}^\times} : \mathrm{Cor}_k^{op} \longrightarrow \mathrm{Ab}$$

may be defined by sending each $X \in \mathrm{Sm}_k$ to $\mathcal{O}(X)$; for each W as above, by sending $1 \cdot [W]_{XY}$ to $\mathcal{O}^\times(Y) \to \mathcal{O}^\times(W) \xrightarrow{N} \mathcal{O}^\times(X)$, and by linearly extending to general finite k-correspondences.

(b) Analogously to the above item (and in the same notation), $K(X) \to K(W)$ induces a trace map (of groups)

$$Tr : K(W) \longrightarrow K(X),$$

which in turn restricts to a map (also denoted by)

$$Tr : \mathcal{O}(W) \longrightarrow \mathcal{O}(X).$$

This provides a map

$$\mathcal{O}(Y) \longrightarrow \mathcal{O}(W) \xrightarrow{Tr} \mathcal{O}(X).$$

By extending linearly as in the above item, we may define a presheaf with transfers

$$\overline{\mathcal{O}} : \mathrm{Cor}_k^{op} \longrightarrow \mathrm{Ab}.$$

(c) Chow groups provide another example of presheaf with transfers. Suppose given $W \subset X \times Y$ as above. Then for each $i \leq 0$, we obtain a map

$$\phi_W : \mathrm{CH}^i(Y) \longrightarrow \mathrm{CH}^i(X)$$

given by sending each cycle class α to $\pi^{XY}_{X*}(W \cdot \pi^{XY*}_{Y}(\alpha))$.

2.2 Motivic complexes

Throughout this section, we let k denote a given field.

2.2.1 A simplicial construction on presheaves with transfers

Recall that the *simplex category*, denoted by Δ, is defined as follows:

1. Objects are the linearly ordered sets of the form $[n] := \{0 < 1 < \cdots < n\}$ for integers $n \geq 0$.

2. Morphisms are order-preserving functions, with composites defined as usual composites of functions.

Given a category $\mathcal{C}$, the presheaf category $\mathrm{PSh}(\Delta, \mathcal{C}) = \mathrm{Fun}(\Delta^{op}, \mathcal{C})$ is called the *category of simplicial objects of* $\mathcal{C}$, and the functor category $\mathrm{Fun}(\Delta, \mathcal{C})$ is called the *category of* *cosimplicial objects of* $\mathcal{C}$. For example, if $\mathcal{C}$ = Set, Ab, Top, or some category of schemes, such a (co)simplicial object is usually referred to as a (co)simplicial set, (co)simplicial abelian group, (co)simplicial topological space, or (co)simplicial scheme.

In topology, one defines for each integer $n \geq 0$ the *standard n-simplex* as the following subspace of $\mathbb{R}^{n+1}$:

$$\Delta^n_{top} = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} | 0 \leq x_i \leq 1 \text{ for each } i, \sum_{i=0}^{n} x_i = 1\}.$$

This defines a cosimplicial topological space $\Delta^*_{top} : \Delta \longrightarrow \text{Top}$ as follows: each $[n] \in \Delta$ is sent to Δ^n_{top}, and each order-preserving function $f : [m] \longrightarrow [n]$ is sent to the map

$$\begin{aligned} \Delta^m_{top} &\longrightarrow \Delta^n_{top} \\ (x_0, ..., x_m) &\longmapsto (\sum_{i \in f^{-1}(0)} x_i, ..., \sum_{i \in f^{-1}(j)} x_i, ..., \sum_{i \in f^{-1}(n)} x_i). \end{aligned}$$

We now consider an analogous construction in the context of algebraic geometry. For each $n \geq 0$, we define the *standard algebraic n-simplex* to be the scheme

$$\Delta^n_{alg} := \text{Spec} \frac{\mathbb{Z}[t_0, ..., t_n]}{(\sum_{i=0}^n t_i - 1)}.$$

These can be organized as a cosimplicial scheme

$$\Delta^*_{alg} : \Delta \longrightarrow \text{Sch}$$

by sending each $[n] \in \Delta$ to Δ^n_{alg}, and each order-preserving function $f : [m] \longrightarrow [n]$ to the morphism of schemes $\Delta^m_{alg} \longrightarrow \Delta^n_{alg}$ corresponding to the ring homomorphism

$$\frac{\mathbb{Z}[t_0, ..., t_m]}{(\sum_{i=0}^m t_i - 1)} \longrightarrow \frac{\mathbb{Z}[t_0, ..., t_n]}{(\sum_{i=0}^n t_i - 1)}$$

given on generators by sending the class of t_j to that of $\sum_{i \in f^{-1}(j)} t_i$.

Given a scheme X and $n \geq 0$, the product of schemes $\Delta^n_{alg} \times X$ will be denoted by Δ^n_X and called the *standard algebraic n-simplex over X*. If $X = \text{Spec } A$ for some ring A, Δ^n_X will be denoted by Δ^n_A. Note that

$$\Delta^n_A = \text{Spec } A \times \Delta^n_X = \text{Spec } (A \otimes \frac{\mathbb{Z}[t_0, ..., t_n]}{(\sum_{i=0}^n t_i - 1)}) \cong \text{Spec } \frac{A[t_0, ..., t_n]}{(\sum_{i=0}^n t_i - 1)}.$$

In particular, $\Delta^n_{alg} \cong \Delta^n_{\mathbb{Z}}$.

We also remark that Δ^n_{alg} is (non-canonically) isomorphic to the n-dimensional affine space over $\mathbb{Z}$, $\mathbb{A}^n_{\mathbb{Z}} = \text{Spec } \mathbb{Z}[x_1, ..., x_n]$. For example, consider the ring homomorphism $\mathbb{Z}[x_1, ..., x_n] \longrightarrow \mathbb{Z}[t_0, ..., t_n]$ given on generators by $x_i \longmapsto t_i$ for each i; by composing with the quotient map, we obtain a homomorphism $\mathbb{Z}[x_1, ..., x_n] \longrightarrow \frac{\mathbb{Z}[t_0, ..., t_n]}{(\sum_{i=0}^n t_i - 1)}$ which can be proved to be an isomorphism.

Thus the corresponding morphism of schemes $\Delta^n_{alg} \longrightarrow \mathbb{A}^n_{\mathbb{Z}}$ is an isomorphism. It follows that for any scheme X there exists an isomorphism $\Delta^n_X = X \times \Delta^n_{alg} \cong X \times \mathbb{A}^n_{\mathbb{Z}} = \mathbb{A}^n_X$. In particular, since any pullback of a smooth, separated, or finite type morphism of schemes also has the corresponding property, it follows that Δ^n_X is an object of Sm_X for each n, so we obtain a cosimplicial object

$$\Delta^*_X : \Delta \longrightarrow \mathrm{Sm}_X \tag{2.2.1}$$

given by composing $\Delta^*_{alg} : \Delta \longrightarrow \mathrm{Sch}$ with the product functor $X \times - : \mathrm{Sch} \longrightarrow \mathrm{Sch}_X$ and then restricting the codomain.

We recall that in the simplex category Δ, one may consider the following two classes of morphisms:

(i) For each $n \geq 1$ and $0 \leq i \leq n$, the *face map* $d^i_n : [n-1] \longrightarrow [n]$ is given by $d^i_n(j) = j$ for $0 \leq j \leq i-1$, and $d^i_n(j) = j+1$ for $i \leq j \leq n-1$. One may show that it is the unique order-preserving injection $[n-1] \longrightarrow [n]$ such that i does not belong to its image.

(ii) For each $n \geq 0$ and $0 \leq i \leq n$, the *degeneracy map* $s^i_n : [n+1] \longrightarrow [n]$ is given by $s^i_n(j) = j$ for $0 \leq j \leq i$, and $d^i_n(j) = j-1$ for $i+1 \leq j \leq n+1$. One may show that it is the unique order-preserving surjection $[n+1] \longrightarrow [n]$ for which i is the image of two elements of $[n+1]$.

Given a category $\mathcal{C}$ and a cosimplicial object $F : \Delta \longrightarrow \mathcal{C}$, the images $F(d^i_n) : F([n-1]) \longrightarrow F([n])$ and $F(s^i_n) : F([n+1]) \longrightarrow F([n])$ of the face and degeneracy maps are usually also denoted by d^i_n and s^i_n, with F implicit, and referred to as face and degeneracy maps, respectively. For a simplicial object $F : \Delta^{op} \longrightarrow \mathcal{C}$, morphisms of the form $F(d^i_n) : F([n]) \longrightarrow F([n-1])$ and $F(s^i_n) : F([n]) \longrightarrow F([n+1])$ are usually also referred to as face and degeneracy maps, but are denoted by ∂^n_i and σ^n_i, respectively. In any of these cases, we often omit n from the notation when it is clear from the context (e.g. d^i denotes d^i_n).

Suppose $\mathcal{A}$ is an abelian category. For a simplicial object $F : \Delta^{op} \longrightarrow \mathcal{A}$, one defines for each $n \geq 0$ the *boundary map*

$$\partial^n : F([n]) \longrightarrow F([n-1])$$

as the alternating sum $\sum_{i=0}^n (-1)^i \partial^n_i$. It may be proved that

$$\cdots \xrightarrow{\partial^{n+1}} F([n]) \xrightarrow{\partial^n} F([n-1]) \xrightarrow{\partial^{n-1}} \cdots \longrightarrow F([0]) \longrightarrow 0 \longrightarrow \cdots$$

is a chain complex in $\mathcal{A}$, which we will regard as a cochain complex concentrated in non-positive degrees, with $F([n])$ placed in degree $-n$ for each $n \geq 0$. We will denote it by $ch(F)$. Moreover, any natural transformation $\eta : F \longrightarrow F'$ in $\mathrm{PSh}(\Delta, \mathcal{A})$ defines a chain map $ch(\eta) : ch(F) \longrightarrow ch(F')$ whose n-th component is $\eta_{[-n]}$ for $n \leq 0$, and 0 for $n \geq 0$. This defines a functor

$$ch : \mathrm{PSh}(\Delta, \mathcal{A}) \longrightarrow \mathrm{Ch}^-(\mathcal{A}). \tag{2.2.2}$$

Construction 2.2.1. Let $\mathcal{C}$ be a category, $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ a functor, $\mathcal{A}$ an abelian category, $F : \mathcal{C}^{op} \longrightarrow \mathcal{A}$ an $\mathcal{A}$-valued presheaf, and $D : \Delta \longrightarrow \mathcal{C}$ a cosimplicial object. Consider the composite functor

$$\Delta^{op} \times \mathcal{C}^{op} \xrightarrow{D^{op}\times id} \mathcal{C}^{op} \times \mathcal{C}^{op} \xrightarrow{\otimes^{op}} \mathcal{C}^{op} \xrightarrow{F} \mathcal{A}. \tag{2.2.3}$$

By considering the isomorphism $\mathrm{Fun}(\Delta^{op}\times\mathcal{C}^{op},\mathcal{A}) \cong \mathrm{Fun}(\mathcal{C}^{op},\mathrm{Fun}(\Delta^{op},\mathcal{A}))$ between functor categories, we obtain a functor

$$S_*^{\otimes,D}(F) : \mathcal{C} \longrightarrow \mathrm{Fun}(\Delta^{op},\mathcal{A})$$

from $\mathcal{C}$ to the category of simplicial objects of $\mathcal{A}$. Explicitly:

(i) Each object $c \in \mathcal{C}$ is sent to the functor $S_*^{\otimes,D}(F)(c) = F(D(*) \otimes c) : \Delta^{op} \longrightarrow \mathcal{A}$.

(ii) Each morphism $f : c \longrightarrow d$ in $\mathcal{C}$ is sent to the natural transformation $S_*^{\otimes,D}(F)(f) : F(D(*) \otimes d) \longrightarrow F(D(*) \otimes c)$ whose $[n]$-component for each $n \geq 0$ is $F(id_{D([n])} \otimes f)$.

We remark that for fixed $\mathcal{C}$, $\mathcal{A}$, and D, this construction is functorial in F as an object of the presheaf category $\mathrm{PSh}(\mathcal{C},\mathcal{A})$: this follows from the fact that the composite 2.2.3 is functorial in F by horizontally composing natural transformations $F \longrightarrow F'$ with the identity natural transformations of $D^{op} \times id$ and $\otimes^{op}$.

Moreover, recall from the above discussion (see 2.2.2) that any simplicial object of an abelian category gives rise, by taking alternating sums of face maps, to a cochain complex concentrated in non-positive degrees. We apply it to our current setting by considering the composite functor

$$\mathrm{PSh}(\mathcal{C},\mathcal{A}) \xrightarrow{S_*^{\otimes,D}(-)} \mathrm{PSh}(\mathcal{C},\mathrm{PSh}(\Delta,\mathcal{A})) \xrightarrow{ch\circ-} \mathrm{PSh}(\mathcal{C},\mathrm{Ch}^-(\mathcal{A})),$$

which we will denote by $C_*^{\otimes,D}(-)$.

Construction 2.2.2. We now apply the above categorical construction to the study of presheaves with transfers over a field k. Following the previous notation, we take $\mathcal{C}$ to be $\mathrm{PST}(k)$, $\otimes$ to be the monoidal product on Cor_k (which restricts to the cartesian product on Sm_k), $\mathcal{A}$ to be the category Ab of abelian groups, and D to be the cosimplicial presheaf with transfers given by the composite

$$\Delta \xrightarrow{\Delta_k^*} \mathrm{Sm}_k \xrightarrow{\Gamma} \mathrm{Cor}_k.$$

Hence we obtain functors

$$S_*^{\otimes,D}(-) : \mathrm{PSh}(\mathrm{Cor}_k,\mathrm{Ab}) \longrightarrow \mathrm{PSh}(\mathrm{Cor}_k,\mathrm{PSh}(\Delta,\mathrm{Ab})),$$

$$C_*^{\otimes,D}(-) : \mathrm{PSh}(\mathrm{Cor}_k,\mathrm{Ab}) \longrightarrow \mathrm{PSh}(\mathrm{Cor}_k,\mathrm{Ch}^-(\mathrm{Ab})).$$

Note that if F is a presheaf with transfers, i.e. an *additive* functor $\mathrm{Cor}_k^{op} \longrightarrow \mathrm{Ab}$, then for each $n \geq 0$, $S_{-n}^{\otimes,D}(F) : \mathrm{Cor}_k^{op} \longrightarrow \mathrm{Ab}$ is given by $F(\Delta_k^n \otimes_k -)$. The latter functor is additive, as F is additive and $\Delta_k^n \otimes_k - : \mathrm{Cor}_k \longrightarrow \mathrm{Cor}_k$ preserves the zero object (the empty scheme) and binary (co)products (given by disjoint unions of schemes). As finite (co)products in $\mathrm{PSh}(\Delta,\mathrm{Ab})$ and $\mathrm{Ch}^-(\mathrm{Ab})$ are computed entrywise, it follows that $S_*^{\otimes,D}(F)$ is a $\mathrm{PSh}(\Delta,\mathrm{Ab})$-valued presheaf with transfers, and $C_*^{\otimes,D}(F)$ is a $\mathrm{Ch}^-(\mathrm{Ab})$-valued presheaf with transfers. Thus by restricting the domain and codomain categories, we obtain functors which will be denoted by

$$S_*(-) : \mathrm{PST}(k) \longrightarrow \mathrm{PST}(k, \mathrm{PSh}(\Delta, \mathrm{Ab})),$$
$$C_*(-) : \mathrm{PST}(k) \longrightarrow \mathrm{PST}(k, \mathrm{Ch}^-(\mathrm{Ab})).$$

We also note that as (i) $\mathrm{PST}(k, \mathcal{A})$ is closed under (co)limits in $\mathrm{PSh}(\mathrm{Cor}_k, \mathcal{A})$ for any (co)complete abelian category $\mathcal{A}$, and (ii) (co)limits in categories of presheaves or chain complexes are computed entrywise, both S_* and C_* preserve (co)limits.[1]

Remark 2.2.3. Suppose $(U_i \xrightarrow{f_i} X)_{i\in I}$ is an étale (resp. Nisnevich, Zariski) covering in Sm_k. Since surjective and étale (resp. Nisnevich, Zariski) morphisms of schemes are stable under base change, we have that $(\Delta^n_k \times_k U_i \xrightarrow{f_i} \Delta^n_k \times X)$ is also an étale (resp. Nisnevich, Zariski) covering.

If F is a presheaf with transfers, by noting that for each $n \geq 0$ and $i, j \in I$ it holds that $\Delta^n_k \times_k (U_i \times_X U_j) \cong (\Delta^n_k \times_k U_i) \times_{\Delta^n_k \times_k X} (\Delta^n_k \times_k U_j)$, it follows that the sheaf condition for $F(\Delta^n_k \times_k -)$ corresponding to $(U_i \xrightarrow{f_i} X)_{i\in I})$ is given by the sheaf condition for F corresponding to $(\Delta^n_k \times U_i \xrightarrow{\Delta^n_k \times_k f_i} \Delta^n_k \times_k X)_{i\in I})$. Thus if $\mathscr{F}$ is an étale (resp. Nisnevich, Zariski) sheaf with transfers, then $C_*(F)$ is a complex of étale (resp. Nisnevich, Zariski) sheaves with transfers.

2.2.2 Motivic complexes

Presheaves with transfers associated to finite families of pointed schemes

Definition 2.2.4. Suppose given a pointed object in Sm_k: a pair (X, x) where X and $x : \mathrm{Spec}\,(k) \longrightarrow X$ are in Sm_k. Then we denote by $\mathbb{Z}^{tr}_k(X, x)$ the cokernel in $\mathrm{PST}(k)$ of $\mathbb{Z}^{tr}_k(\mathrm{Spec}\ k) \xrightarrow{\mathbb{Z}^{tr}_k(x)} \mathbb{Z}^{tr}_k(X)$. Notice that if $t : X \longrightarrow \mathrm{Spec}\ k$ is the structure map of X regarded as a morphism in Sm_k, then $\mathbb{Z}^{tr}_k(x)$ is a section of $\mathbb{Z}^{tr}_k(t) : \mathbb{Z}^{tr}_k(X) \longrightarrow \mathbb{Z}^{tr}_k(\mathrm{Spec}\ k)$, from which the splitting lemma provides a canonical decomposition

$$\mathbb{Z}^{tr}_k(X) \cong \mathbb{Z}^{tr}_k(\mathrm{Spec}\ k) \oplus \mathbb{Z}^{tr}_k(X, x)$$

In particular, the (pointed) multiplicative group scheme $\mathbb{G}_m = (\mathbb{A}^1 \setminus \{0\}, s_1)$, where $s_1 : \mathrm{Spec}\ k \longrightarrow \mathbb{A}^1_k \setminus \{0\}$ is the inclusion of $\{1\}$, defines a presheaf with transfers $\mathbb{Z}^{tr}_k(\mathbb{G}_m)$.

Definition 2.2.5. Suppose given $n \geq 1$ and a family $\{(X_i, x_i) | 1 \leq i \leq n\}$ of pointed objects in Sm_k. Then the smash product $\mathbb{Z}^{tr}_k((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$ is defined as

$$\mathrm{Coker}\left(\bigoplus_{i=1}^{n} \mathbb{Z}^{tr}_k(X_1^{(1)} \times \cdots \times \mathrm{Spec}\ k^{(i)} \times \cdots \times X_n^{(n)}) \xrightarrow{\oplus_i (id, \ldots, x_i, \ldots, id)} \mathbb{Z}^{tr}_k(X_1 \times \cdots \times X_n) \right).$$

We denote $\mathbb{Z}^{tr}_k((X, x)^{(1)} \wedge \cdots \wedge (X, x)^{(n)})$ by $\mathbb{Z}^{tr}_k((X, x)^{\wedge n})$. In particular, note that $\mathbb{Z}^{tr}_k((X, x)^{\wedge 1}) = \mathbb{Z}^{tr}_k(X, x)$. We moreover define $\mathbb{Z}^{tr}_k((X, x)^{\wedge 0}) := \mathbb{Z}^{tr}_k(\mathrm{Spec}\ k)$ and $\mathbb{Z}^{tr}_k((X, x)^{\wedge n}) := 0$ for

---

[1] Although $\mathrm{Ch}^-(\mathrm{Ab})$ is neither complete nor cocomplete, we can work with $\mathrm{Ch}^{\leq 0}(\mathrm{Ab})$: it is (co)complete, and the inclusion $\mathrm{Ch}^{\leq 0}(\mathrm{Ab}) \hookrightarrow \mathrm{Ch}^-(\mathrm{Ab})$ preserves (co)limits; now, as $C_*(F)$ is also an object of $\mathrm{Ch}^{\leq 0}(\mathrm{Ab})$ by construction, the claim follows by expressing $C_*(-)$ as a composite $\mathrm{PST}(k) \longrightarrow \mathrm{PST}(k, \mathrm{Ch}^{\leq 0}(\mathrm{Ab})) \hookrightarrow \mathrm{PST}(k, \mathrm{Ch}^-(\mathrm{Ab}))$.

$n < 0$.[2]

If A is an abelian group, one defines

$$A_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n)) = \mathbb{Z}_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n)) \otimes_{\mathbb{Z}} A.$$

We now show, following Mazza *et al.*, 2006, that $\mathbb{Z}_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$ may be given a convenient presentation by means of a particular description of the image of the morphism of presheaves with transfers

$$\bigoplus_{i=1}^{n} \mathbb{Z}_k^{tr}(X_1^{(1)} \times \cdots \times \operatorname{Spec} k^{(i)} \times \cdots \times X_n^{(n)}) \overset{\oplus_i (id,\ldots,x_i,\ldots,id)}{\longrightarrow} \mathbb{Z}_k^{tr}(X_1 \times \cdots \times X_n).$$

For that purpose, let us introduce the following notation: given $Y \in \mathrm{Sm}_k$ and $i \in \{1, \ldots, n\}$, we say a finite correspondence $\alpha : Y \bullet\!\!\longrightarrow X_1 \times \cdots \times X_n$ is *i-trivial* if it belongs to the image of

$$\mathrm{Cor}_k(Y, X_1^{(1)} \times \cdots \times \operatorname{Spec} k^{(i)} \times \cdots \times X_n^{(n)}) \overset{\mathrm{Cor}_k(Y, id,\ldots,x_i,\ldots,id)}{\longrightarrow} \mathrm{Cor}_k(Y, X_1 \times \cdots \times X_n).$$

Now, let us denote by θ_i the idempotent morphism in Sm_k given by the composite

$$X_1 \times \cdots \times X_n \overset{(id,\ldots,!_i,\ldots,id)}{\longrightarrow} X_1^{(1)} \times \cdots \times \operatorname{Spec} k^{(i)} \times \cdots \times X_n^{(n)} \overset{(id,\ldots,x_i,\ldots,id)}{\longrightarrow} X_1 \times \cdots \times X_n,$$

where $!_i$ denotes the unique morphism $X_i \longrightarrow \operatorname{Spec} k$. Note that if α is an arbitrary finite correspondence from Y to $X_1 \times \cdots \times X_n$, then

$$\theta_i \circ \alpha = (id, \ldots, x_i, \ldots, id) \circ ((id, \ldots, !_i, \ldots, id) \circ \alpha)$$

is i-trivial. On the other, if α is i-trivial then there exists $\beta : Y \bullet\!\!\longrightarrow X_1^{(1)} \times \cdots \times \operatorname{Spec} k^{(i)} \times \cdots \times X_n^{(n)}$ such that

$$\alpha = (id, \ldots, x_i, \ldots, id) \circ \beta,$$

so

$$\theta_i \circ \alpha = (id, \ldots, x_i, \ldots, id)(\circ(id, \ldots, !_i, \ldots, id) \circ (id, \ldots, x_i, \ldots, id)) \circ \beta = (id, \ldots, x_i, \ldots, id) \circ \beta = \alpha.$$

Hence α is i-trivial if and only if $\theta_i \circ \alpha = \alpha$.

If for each i we denote by η_i the composite $X_i \xrightarrow{!_i} \operatorname{Spec} k \xrightarrow{x_i} X_i$, then θ_i is given in coordinates by $(id, \ldots, \eta_i, \ldots, id)$, so α is i-trivial if and only if it satisfies

$$0 = \alpha - \theta_i \circ \alpha = (id - \theta_i) \circ \alpha = (id, \ldots, id - \eta_i, \ldots, id) \circ \alpha,$$

where the differences $id - \theta_i$ and $id - \eta_i$ are taken in $\mathrm{Cor}_k(X_1 \times \cdots \times X_n, X_1 \times \cdots \times X_n)$ and $\mathrm{Cor}_k(X_i, X_i)$, respectively. Defining $\theta_i' = id_{X_1 \times \cdots \times X_n} - \theta_i$, we have that α is i-trivial if and only if $\theta_i' \circ \alpha = 0$.

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[2] From a categorical point of view, the empty cartesian product in Sm_k is the terminal object, which is $\operatorname{Spec} k$; similarly, the empty direct sum of abelian groups is trivial. This is a way of rendering the definition of $\mathbb{Z}_k^{tr}((X, x)^{\wedge 0})$ compatible with that of $\mathbb{Z}_k^{tr}((X, x)^{\wedge n})$ for $n \geq 1$.

Note moreover that by commutativity of the diagram

$$\begin{tikzcd}
X_1^{(1)} \times \cdots \times \operatorname{Spec} k^{(i)} \times \cdots \times X_n^{(n)} \arrow[r, "{(id,\dots,x_i,\dots,id)}"] \arrow[d, "{(id,\dots,!_j,\dots,id)}"'] & X_1 \times \cdots \times X_n \arrow[d, "{(id,\dots,!_j,\dots,id)}"] \\
X_1^{(1)} \times \cdots \times \operatorname{Spec} k^{(i)} \times \cdots \times \operatorname{Spec} k^{(j)} \times \cdots \times X_n^{(n)} \arrow[d, "{(id,\dots,x_j,\dots,id)}"'] & X_1^{(1)} \times \cdots \times \operatorname{Spec} k^{(j)} \times \cdots \times X_n^{(n)} \arrow[d, "{(id,\dots,x_j,\dots,id)}"] \\
X_1^{(1)} \times \cdots \times \operatorname{Spec} k^{(i)} \times \cdots \times X_n^{(n)} \arrow[r, "{(id,\dots,x_i,\dots,id)}"'] & X_1 \times \cdots \times X_n,
\end{tikzcd}$$

where the right vertical composite is θ_j, if $\alpha : Y \bullet\!\!\longrightarrow X_1 \times \cdots \times X_n$ is i-trivial, then $\theta_j \circ \alpha$ is both i-trivial and j-trivial. It follows in this case that

$$\theta_i \circ (\theta_j' \circ \alpha) = \theta_i \circ ((id - \theta_j) \circ \alpha) = \theta_i \circ \alpha - \theta_i \circ (\theta_j \circ \alpha) = \alpha - \theta_j \circ \alpha = (id - \theta_j) \circ \alpha = \theta_j' \circ \alpha,$$

so $\theta_j' \circ \alpha$ is i-trivial.

This allows us to describe when a given α can be expressed as a sum $\sum_i \alpha_i$ where α_i is i-trivial for each i. Define

$$\omega = \theta_n' \circ \cdots \circ \theta_1',$$

which is equal to

$$(id_{X_1 \times \cdots \times X_n} - \theta_n) \circ \cdots \circ (id_{X_1 \times \cdots \times X_n} - \theta_1),$$

hence to

$$(id - \eta_1, \dots, id - \eta_n).$$

Given $\beta : Y \bullet\!\!\longrightarrow X_1 \times \cdots \times X_n$, let us define

$$\beta^j = \theta_j' \circ \cdots \circ \theta_1' \circ \beta$$

for $j = 1, \dots, n$, and $\beta^0 = \beta$.

By the previous remark, α_i^j is i-trivial for any i, j. Thus

$$\omega \circ \alpha_i = \theta_n' \circ \cdots \circ (\theta_i' \circ \alpha_i^{i-1}) = \theta_n' \circ \cdots \circ 0 = 0.$$

Thus $\omega \circ \alpha = \omega \circ (\sum_i \alpha_i) = 0$.

Conversely, suppose given $\alpha : Y \bullet\!\!\longrightarrow X_1 \times \cdots \times X_n$ such that $\omega \circ \alpha = 0$. By induction we obtain that for each $j = 1, \dots, n$, α^j can be written as $\alpha + \beta_j$, where β_j is a sum of i-trivial correspondences for $i = 1, \dots, j$. Indeed, $\alpha^1 = \alpha - \theta_1 \circ \alpha$, so we may take $\beta_1 = -\theta_1 \circ \alpha$; and for $j = 2, \dots, n$ we have

$$\alpha^j = (\alpha + \beta_{j-1}) - \theta_j \circ (\alpha + \beta_{j-1}),$$

so we may take $\beta_j = \beta_{j-1} - \theta_j \circ (\alpha + \beta_{j-1})$. Thus we have

$$0 = \omega \circ \alpha = \alpha^n = \alpha + \beta_n,$$

so $\alpha = -\beta_n$ and there exist i-trivial α_i for $i = 1, \dots, n$ such that $\alpha = \sum_i \alpha_i$.

It follows that the image of

$$\bigoplus_{i=1}^{n} \mathrm{Cor}_k(Y, X_1^{(1)} \times \cdots \times \mathrm{Spec}\ k^{(i)} \times \cdots \times X_n^{(n)}) \xrightarrow{\oplus_i \mathrm{Cor}_k(Y, id, \ldots, x_i, \ldots, id)} \mathrm{Cor}_k(Y, X_1 \times \cdots \times X_n)$$

equals the kernel of $\mathrm{Cor}_k(Y, \omega)$. Since ω is idempotent, we obtain a splitting

$$\mathrm{Cor}_k(Y, X_1 \times \cdots \times X_n) \cong \mathrm{Ker}(\mathrm{Cor}_k(Y, \omega)) \oplus \mathrm{Im}(\mathrm{Cor}_k(Y, \omega)).$$

Thus $\mathrm{Im}(\mathrm{Cor}_k(Y, \omega))$ is the cokernel of $\mathrm{Ker}(\mathrm{Cor}_k(Y, \omega)) \hookrightarrow \mathrm{Cor}_k(Y, X_1 \times \cdots \times X_n)$, which in turn is $\mathbb{Z}_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))(Y)$. In particular, $\mathbb{Z}_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))(Y)$ is a direct summand of $\mathrm{Cor}_k(Y, X_1 \times \cdots \times X_n)$. This defines a functor

$$\mathrm{Im}(\mathrm{Cor}_k(-, \omega)) : \mathrm{Cor}_k \longrightarrow \mathrm{Ab}$$

which is isomorphic to $\mathbb{Z}_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$ and is a direct summand of $\mathbb{Z}_k^{tr}(X_1 \times \cdots \times X_n)$ in $\mathrm{PST}(k)$. Since $\mathbb{Z}_k^{tr}(X_1 \times \cdots \times X_n)$ is an étale (hence Nisnevich, Zariski) sheaf with transfers, it follows that so is $\mathbb{Z}_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$. To see this directly, note that $\mathrm{Im}(\mathrm{Cor}_k(-, \omega))$ is the kernel in $\mathrm{PST}(k)$ of

$$\mathbb{Z}_k^{tr}(X_1 \times \cdots \times X_n) \cong \mathrm{Ker}(\mathrm{Cor}_k(-, \omega)) \oplus \mathrm{Im}(\mathrm{Cor}_k(-, \omega)) \xrightarrow{\text{proj.}} \mathrm{Ker}(\mathrm{Cor}_k(-, \omega)) \hookrightarrow \mathbb{Z}_k^{tr}(X_1 \times \cdots \times X_n).$$

The claim follows from the fact that the kernel of a morphism of sheaves coincides with the kernel of the underlying morphism of presheaves.

This description may be extended to general coefficient groups. Suppose given an abelian group A. Then since tensor products commute with direct sums, $A_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n)) = \mathbb{Z}_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n)) \otimes_{\mathbb{Z}} A$, we have

$$\begin{aligned}
&\mathrm{Ker}(\mathrm{Cor}_k(-, \omega) \otimes_{\mathbb{Z}} A) \oplus A_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n)) \\
&\cong \mathrm{Ker}(\mathrm{Cor}_k(-, \omega) \otimes_{\mathbb{Z}} A) \oplus (\mathbb{Z}_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n)) \otimes_{\mathbb{Z}} A) \\
&\cong \mathrm{Ker}(\mathrm{Cor}_k(-, \omega) \otimes_{\mathbb{Z}} A) \oplus \mathrm{Im}(\mathrm{Cor}_k(-, \omega) \otimes_{\mathbb{Z}} A) \\
&\cong (\mathrm{Ker}(\mathrm{Cor}_k(-, \omega)) \oplus \mathrm{Im}(\mathrm{Cor}_k(-, \omega))) \otimes_{\mathbb{Z}} A \\
&\cong \mathbb{Z}_k^{tr}(X_1 \times \cdots \times X_n) \otimes A \\
&\cong A_k^{tr}(X_1 \times \cdots \times X_n).
\end{aligned}$$

Hence $A_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$ is a direct summand of $A_k^{tr}(X_1 \times \cdots \times X_n)$ and, by the same argument used for $\mathbb{Z}$-coefficients, it is an étale (hence Nisnevich, Zariski) sheaf with transfers.

Remark 2.2.6. We make a remark, which will be needed later, on the projectivity and flatness of certain presheaves with transfers.

For any $X \in \mathrm{Sm}_k$, the presheaf with transfers $\mathbb{Z}_k^{tr}(X) = \mathrm{Cor}_k(-, X) : \mathrm{Cor}_k \longrightarrow \mathrm{Ab}$ has the property, by the Yoneda lemma, that the functor $\mathrm{Hom}_{\mathrm{PST}(k)}(\mathbb{Z}_k^{tr}(X), -)$ from $\mathrm{PST}(k)$ to large abelian groups is naturally isomorphic to the one given by evaluation at X; for each

$F \in \mathrm{PST}(k)$, the F-component of this natural isomorphism is given by

$$\begin{aligned} \mathrm{Hom}_{\mathrm{PST}(k)}(\mathbb{Z}_k^{tr}(X), F) &\longrightarrow F(X) \\ \eta &\longmapsto \eta_X(id_X). \end{aligned}$$

Since a morphism $f : F \longrightarrow G$ in $\mathrm{PST}(k)$ is an epimorphism if and only if $f_Y : F(Y) \longrightarrow G(Y)$ is surjective for all $Y \in \mathrm{PST}(k)$, it follows from the naturality of the above isomorphism that any epimorphism of presheaves with transfers is sent under $\mathrm{Hom}_{\mathrm{PST}(k)}(\mathbb{Z}_k^{tr}(X), F)$ to a surjective map. Thus $\mathbb{Z}_k^{tr}(X)$ is a projective object of $\mathrm{PST}(k)$.

Given pointed schemes $(X_1, x_1), ..., (X_n, x_n)$ in Sm_k, we have that $\mathbb{Z}_k^{tr}((X_1, x_1)\wedge\cdots\wedge(X_n, x_n))$ is a projective object of $\mathrm{PST}(k)$ as it is a direct summand of $\mathbb{Z}_k^{tr}(X_1 \times \cdots \times X_n)$.

If F is a presheaf with transfers such that $F(X)$ is a torsion-free, hence flat abelian group for every $X \in \mathrm{Sm}_k$, then F is flat as a presheaf with transfers, i.e. the functor

$$F \otimes - : \mathrm{PST}(k) \longrightarrow \mathrm{PST}(k)$$

given by taking objectwise tensor products of abelian groups is exact. Conversely, by considering constant presheaves with transfers it follows that any flat presheaf with transfers associates to each object a flat, hence torsion-free abelian group.

In particular, presheaves with transfers of the form $\mathbb{Z}_k^{tr}(X)$ or $\mathbb{Z}_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_n, x_n))$ flat: the former is a sheaf of free, hence torsion-free abelian groups; the latter is torsion-free as it is a sub-presheaf with transfers of $\mathbb{Z}_k^{tr}(X_1 \times \cdots \times X_n)$.

If F is a flat presheaf with transfers, then C_*F is a complex of flat presheaves with transfers.

Motivic complexes

Recall from our conventions that $\mathbb{G}_m$ denotes the pointed k-scheme $(\mathbb{A}^1 \setminus \{0\}, s_1)$, where $s_1 : \mathrm{Spec}\ k \longrightarrow \mathbb{A}^1 \setminus \{0\}$ is the inclusion of $\{1\}$.

Definition 2.2.7. For each integer $q \geq 0$, the (bounded above) cochain complex of presheaves with transfers

$$\mathbb{Z}(q) \in \mathrm{Ch}^-(\mathrm{PST}(k))$$

is defined as $C_*\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q})[-q]$. For $n < 0$ we define $\mathbb{Z}(q) = 0$. Given any abelian group A, we define

$$A(q) := \mathbb{Z}(q) \otimes A.$$

When we need to make the base field k explicit, we will denote $\mathbb{Z}(q)$ by $\mathbb{Z}(q)_k$.

Remark 2.2.8. Unraveling the definition degreewise, for each $q \geq 0$ and $n \in \mathbb{Z}$ we have $\mathbb{Z}(q)^n = C_{q-n}\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q})$ (in particular, $\mathbb{Z}_k^{tr}(q)^n = 0$ for $n > q$), hence for each $X \in \mathrm{Sm}_k$ and $n \leq q$,

$$\mathbb{Z}_k^{tr}(q)^n(X) = \mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge n})(X \times \Delta^{q-n})$$

$$= \left[\mathrm{Coker}\left(\bigoplus_{i=1}^{q} \mathbb{Z}_k^{tr}(\mathbb{G}_m^{(1)}\times\cdots\times\mathrm{Spec}\ k^{(i)}\times\cdots\times\mathbb{G}_m^{(q)}) \xrightarrow{\oplus_i \mathbb{Z}_k^{tr}(id,...,s_1,...,id)} \mathbb{Z}_k^{tr}(\mathbb{G}_m^{(1)}\times\cdots\times\mathbb{G}_m^{(q)}) \right)\right](X\times\Delta^{q-n}),$$

which in turn equals the cokernel of

$$\bigoplus_{i=1}^{q} \mathrm{Cor}_k(X \times \Delta^{q-n}, \mathbb{G}_m^{(1)} \times \cdots \times \mathrm{Spec}\ k^{(i)} \times \cdots \times \mathbb{G}_m^{(q)}) \overset{\oplus_i \mathrm{Cor}_k(X\times\Delta^{q-n}, id, \dots, s_1, \dots, id)}{\longrightarrow} \mathrm{Cor}_k(X \times \Delta^{q-n}, \mathbb{G}_m^{(1)} \times \cdots \times \mathbb{G}_m^{(q)}).$$

Note that $\mathbb{Z}(0) = C_*(\mathbb{Z}_k^{tr}(\mathrm{Spec}\ k))$, i.e. the chain complex associated to the presheaf with transfers $\mathbb{Z}_k^{tr}(\mathrm{Spec}\ k)$ as in Construction 2.2.2.

Moreover, we have $A(q)^n \cong \mathbb{Z}(q)^n \otimes A = C_{q-n}\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q}) \otimes A \cong C_{q-n}A_k^{tr}(\mathbb{G}_m^{\wedge q})$, so

$$A(q) \cong C_* A_k^{tr}(\mathbb{G}_m^{\wedge q})[-q].$$

We have previously seen that: (i) given pointed schemes (X_1, x_1), (X_q, x_q) in Sm_k, $A_k^{tr}((X_1, x_1) \wedge \cdots \wedge (X_q, x_q))$ is an étale (hence Nisnevich, Zariski) sheaf with transfers (2.2.2), and (ii) if F is an étale (resp. Nisnevich, Zariski) sheaf with transfers, then C_*F is a complex of étale (resp. Nisnevich, Zariski) sheaves with transfers (Remark 2.2.3). It follows that $A(q)$ is a complex of étale (hence Nisnevich, Zariski) sheaves with transfers.

Suppose A is a torsion-free abelian group. Then (see Remark 2.2.6) $A_k^{tr}(\mathbb{G}_m^{\wedge q})$, hence $C_*A_k^{tr}(\mathbb{G}_m^{\wedge q})$, hence $A(q)$ is a complex of flat presheaves with transfers.

Definition 2.2.9. Let A be an abelian group. For each $X \in \mathrm{Sm}_k$ and $p, q \in \mathbb{Z}$, the *(ordinary) motivic cohomology* group $H^{p,q}(X, A)$ is defined as the hypercohomology group

$$H^p_{\mathrm{Zar}}(X, A(q)|_{X_{\mathrm{Zar}}}),$$

where $A(q)|_{X_{\mathrm{Zar}}}$ denotes the complex of Zariski sheaves obtained by restricting $A(q)$ to the Zariski site of X. Similarly, the *étale motivic cohomology* group $H^{p,q}(X, A)$ is defined as

$$H^p_{\mathrm{Zar}}(X, A(q)|_{X_{\text{ét}}}),$$

where $A(q)|_{X_{\mathrm{Zar}}}$ is the restriction of $A(q)$ to the étale site of X. We will also denote these groups by $H^p(X, A(q))$ and $H^p_{\text{ét}}(X, A(q)|_{X_{\mathrm{Zar}}})$, respectively.

It may be proved that ordinary motivic cohomology of a given $X \in \mathrm{Sm}_k$ satisfies a boundedness condition depending on the usual dimension of X as a scheme: given an abelian group A and $p, q \in \mathbb{Z}$, it holds that

$$H^{p,q}(X, A) = 0$$

whenever $p > q + \dim(X)$; we refer to Mazza *et al.*, 2006, Theorem 3.6. Moreover, Mazza *et al.*, 2006, Theorem 3.8 shows that motivic cohomology does not depend, in a certain sense, on the choice of a base field: if K is a finite separable extension of k, then any $X \longrightarrow \mathrm{Spec}\ K$ in Sm_K defines an object of Sm_k by composing its structure morphism with $\mathrm{Spec}\ K \longrightarrow \mathrm{Spec}\ k$ (which is smooth, separated, and of finite type), and the complexes $A(q)^K|_{X_{\mathrm{Zar}}}$ and $A(q)^k|_{X_{\mathrm{Zar}}}$ are isomorphic.

Moreover, one may construct (see Mazza *et al.*, 2006, Corollary 3.12) for each $X \in \mathrm{Sm}_k$

and $p, p', q, q' \in \mathbb{Z}$ a pairing

$$H^{p,q}(X,A) \otimes H^{p',q'}(X,A) \longrightarrow H^{p+p',q+q'}(X,A)$$

endowing motivic cohomology with a bigraded ring structure.

2.3 A characterization of $\mathbb{Z}(1)$ up to quasi-isomorphism

This subsection aims to study the proof given in Mazza *et al.*, 2006 that the motivic complex $\mathbb{Z}(1)$ is quasi-isomorphic to the presheaf with transfers $\mathscr{O}^\times$ regarded as a complex concentrated in degree 1.

Outline of the result

Throughout this subsection, k denotes a given field.

Recall that $\mathbb{Z}(1) \in \mathrm{Ch}^-(\mathrm{PST}(k))$ is the cochain complex of presheaves with transfers over k defined as

$$\mathbb{Z}(1) := C_*\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge 1})[-1] \cong C_*\mathbb{Z}_k^{tr}(\mathbb{G}_m)[-1].$$

More explicitly, we have

$$\cdots \xrightarrow{\partial^{n+1}} \mathbb{Z}_k^{tr}(\mathbb{G}_m)(\Delta_k^n \times_k -) \xrightarrow{\partial^n} \mathbb{Z}_k^{tr}(\mathbb{G}_m)(\Delta_k^{n-1} \times_k -) \xrightarrow{\partial^{n-1}} \cdots \longrightarrow \mathbb{Z}_k^{tr}(\mathbb{G}_m)(\Delta_k^0 \times_k -) \longrightarrow 0 \longrightarrow \cdots$$

with $\mathbb{Z}_k^{tr}(\mathbb{G}_m)(\Delta_k^n \times_k -)$ placed in degree $1-n$, where:

(i) ∂^n is the natural transformation whose X-component for each $X \in \mathrm{Cor}_k$ is the alternating sum $\sum_{i=0}^n \mathbb{Z}_k^{tr}(\mathbb{G}_m)(d_n^i \times_k X)$, where $d_n^i : \Delta_k^{n-1} \longrightarrow \Delta_k^n$ is the (n,i)-th face map of the cosimplicial object Δ_k^* defined in Section 2.2.

(ii) $\mathbb{Z}_k^{tr}(\mathbb{G}_m)$ is the cokernel

$$\mathrm{Coker}(\mathbb{Z}_k^{tr}(\mathrm{Spec}\ k) \xrightarrow{\mathbb{Z}_k^{tr}(s_1)} \mathbb{Z}_k^{tr}(\mathbb{A}^1\backslash\{0\})) = \mathrm{Coker}(\mathrm{Cor}_k(-,\mathrm{Spec}\ k) \xrightarrow{s_1\circ -} \mathrm{Cor}_k(-,\mathbb{A}^1\backslash\{0\}))$$

in $\mathrm{PST}(k)$, where $s_1 : \mathrm{Spec}\ k \longrightarrow \mathbb{A}^1 \setminus \{0\}$ is the inclusion of $\{1\}$.

In order to study $\mathbb{Z}(1)$, let us first deal with $\mathbb{Z}_k^{tr}(\mathbb{G}_m)$. As described in Definition 2.2.4, s_1 has a retraction $\pi : \mathbb{A}^1 \setminus \{0\} \longrightarrow \mathrm{Spec}\ k$, whence the following is a split exact sequence:

$$0 \longrightarrow \mathbb{Z}_k^{tr}(\mathrm{Spec}\ k) \xrightarrow{\mathbb{Z}_k^{tr}(s_1)} \mathbb{Z}_k^{tr}(\mathbb{A}^1 \setminus \{0\}) \xrightarrow{\mathbb{Z}_k^{tr}(\pi)} \mathbb{Z}_k^{tr}(\mathbb{G}_m) \longrightarrow 0.$$

We will follow the approach used in Mazza *et al.*, 2006. As $\mathbb{A}^1 \setminus \{0\}$ has dimension 1, a finite correspondence from a given scheme $X \in \mathrm{Sm}_k$ to $\mathbb{A}^1 \setminus \{0\}$ is a particular kind of codimension 1 cycle, i.e. of (Weil) divisor (see below), on $X \times (\mathbb{A}^1 \setminus \{0\})$. Such finite correspondences may moreover be identified with divisors on $X \times \mathbb{A}^1$ or on $X \times \mathbb{P}^1$, so we may use a description of a certain quotient – the divisor class group $\mathrm{Cl}(-)$ – of the divisor groups of $X \times \mathbb{A}^1$ and $X \times \mathbb{P}^1$ in terms of that of X: for X connected, one may construct isomorphisms

$$\mathbb{Z} \oplus \mathrm{Cl}(X) \cong \mathrm{Cl}(X \times \mathbb{P}^1),$$
$$\mathrm{Cl}(X) \cong \mathrm{Cl}(X \times \mathbb{A}^1).$$

As a consequence, for connected X it is possible to describe finite correspondences from X to $\mathbb{A}^1 \setminus \{0\}$, regarded as divisors on $X \times \mathbb{P}^1$, as divisors associated to rational functions on $X \times \mathbb{P}^1$ up to sum with (i) a specific multiple of the divisor $X \times \{\infty\}$ and (ii) a divisor of the form $D \times (\mathbb{A}^1 \setminus \{0\})$, where D is a divisor on X which in turn is unique up to sum with a divisor associated to a rational function on X.

This allows to construct a certain split epimorphism

$$P_X : \mathrm{Cor}_k(X, \mathbb{A}^1 \setminus \{0\}) \longrightarrow \mathbb{Z} \oplus \mathscr{O}^\times(X).$$

whose $\mathbb{Z}$-coordinate amounts to the above construction. Then one obtains a decomposition

$$\mathrm{Cor}_k(X, \mathbb{A}^1 \setminus \{0\}) \cong \mathrm{Ker}(P_X) \oplus \mathbb{Z} \oplus \mathscr{O}^\times(X)$$

with the property that the corresponding injection $\mathbb{Z} \longrightarrow \mathrm{Cor}_k(X, \mathbb{A}^1 \setminus \{0\})$ sends 1 to the finite correspondence $X \times \{1\}$; hence up to isomorphism it describes the injection

$$\mathrm{Cor}_k(X, \mathrm{Spec}\ k) \longrightarrow \mathrm{Cor}_k(X, \mathbb{A}^1 \setminus \{0\}).$$

This yields a decomposition

$$\mathbb{Z}_k^{tr}(\mathbb{G}_m)(X) \cong \mathrm{Ker}(P_X) \oplus \mathscr{O}^\times(X),$$

and by varying X it follows that there exists a short exact sequence

$$0 \longrightarrow \mathrm{Ker}(P_-) \longrightarrow \mathbb{Z}_k^{tr}(\mathbb{G}_m) \longrightarrow \mathscr{O}^\times \longrightarrow 0.$$

Moreover, it is possible to give for each X an explicit description of $\mathrm{Ker}(P_X)$ in terms of particular rational functions on $X \times \mathbb{P}^1$. Then one applies the functor $C_* : \mathrm{PST}(k) \longrightarrow \mathrm{PST}(k, \mathrm{Ch}^-(\mathrm{Ab}))$ and checks that (i) $C_*\mathrm{Ker}(P_-)$ is quasi-isomorphic to the zero complex, and (ii) $C_*\mathscr{O}^\times$ is quasi-isomorphic to $\mathscr{O}^\times$. This defines a quasi-isomorphism

$$C_*\mathbb{Z}_k^{tr}(\mathbb{G}_m) \longrightarrow \mathscr{O}^\times,$$

and thus also a quasi-isomorphism

$$\mathbb{Z}(1) \cong C_*\mathbb{Z}_k^{tr}(\mathbb{G}_m)[-1] \longrightarrow \mathscr{O}^\times[-1].$$

Overview of Weil divisors

We recall that any scheme in Sm_k – a smooth, separated, finite type k-scheme – can be expressed as a finite disjoint union of smooth, integral, separated, finite type k-schemes (i.e. of smooth varieties over k). In particular, if $Y \subset X$ is a connected (equivalently, irreducible) component, then for every open subset $U \subset Y$ the ring of regular functions $\mathscr{O}_X(U)$ is an integral k-algebra. For a connected component $Y \subset X$, we denote by $\mathcal{O}_{X,Y}$ the local ring $\mathcal{O}_{X,\eta(Y)}$ of X at the generic point $\eta(Y)$ of Y. It is defined as the colimit of rings

$$\varinjlim_U \mathcal{O}_X(U)$$

where U ranges over the direct system of all nonempty open neighborhoods of $\eta(Y)$ in X (ordered by reverse inclusion). As any such U contains a nonempty neighborhood of $\eta(Y)$ that is contained in Y, e.g. $Y \cap U$, we may (up to isomorphism) compute $\mathcal{O}_{X,Y}$ by ranging U over nonempty open neighborhoods of $\eta(Y)$ contained in Y, i.e. over all nonempty open subsets of Y. By regarding Y as an open subscheme, this is precisely $\mathcal{O}_{Y,Y} = \mathcal{O}_{Y,\eta(Y)}$. A further characterization of $\mathcal{O}_{X,Y}$ may be given as follows: let Spec $A \cong U \subset Y$ be any nonempty affine open subset. As A is an integral domain and $\mathcal{O}_{\text{Spec } A,\eta(\text{Spec } ,A)}$ is isomorphic to the fraction field $\operatorname{Frac}(A)$ of A, the fact that any nonempty open subset of Y has nonempty intersection with U (as Y is irreducible) implies the existence of an isomorphism $\operatorname{Frac}(A) \xrightarrow{\cong} \mathcal{O}_{X,Y}$.

Given $X \in \mathrm{Sm}_k$, let us denote by $c(X)$ the set of connected components of X. We define $\mathfrak{R}(X)$, the *ring of rational functions on* X, as the product

$$\mathfrak{R}(X) = \prod_{Y \in c(X)} \mathcal{O}_{X,Y}$$

of the local rings (which are fields, as noted above) of its connected components.

Now, suppose given a point $x \in X$, say $x \in Y$ where Y is a connected component, and a rational function f, whose Y-component in $\mathcal{O}_{X,Y}$ we denote by $f|_Y$. We say that f is *defined at* x if $f|_Y$ is in the image of the localization map

$$\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}_{X,Y}.$$

In this case, by abuse of notation we also denote by $f|_Y$ the unique element of $\mathcal{O}_{X,x}$ corresponding to $f|_Y$, and we write $f|_Y \in \mathcal{O}_{X,x}$. If this holds, we say that the *value of* f *at* x is the image of $f|_Y$ under the quotient map

$$\mathcal{O}_{X,x} \longrightarrow \kappa(x) = \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x}.$$

We also recall that a *(Weil) divisor on* X is a cycle $\sum_{i \in I} n_i x_i$ such that each irreducible closed subset $Z_i = \overline{\{x_i\}}$ has codimension 1 in X. A point $x \in X$ such that $\overline{\{x\}}$ has codimension 1 in X is said to be a *prime divisor on* X. Under addition, divisors on X form an abelian group, denoted by $\operatorname{Div}(X)$ and called the *(Weil) divisor group of* X.

With x_i and Z_i as above, when it is convenient we will identify a divisor $\sum_{i \in I} n_i x_i$ with the corresponding linear combination $\sum_{i \in I} n_i Z_i$ of irreducible closed subsets of X.

Suppose given a rational function $f \in \mathfrak{R}(X)$, a prime divisor $x \in X$ whose connected component is $Y \subset X$, and suppose that f is defined at x (i.e. $f|_Y \in \mathcal{O}_{X,x}$), and that $f_| Y \neq 0$ in $\mathcal{O}_{X,x}$. The *order of vanishing of* f *at* x, denoted by $\operatorname{ord}_x(f)$, is defined as the length of $\mathcal{O}_{X,x}/(f)$ as an $\mathcal{O}_{X,x}$-module. It may be proved that given another f' subject to the same conditions, it holds that $\operatorname{ord}_x(ff') = \operatorname{ord}_x(f) + \operatorname{ord}_x(f')$. More generally, if f is a rational function and $x \in X$ is a prime divisor with connected component Y such that $f|_Y \neq 0$ in $\mathscr{O}_{X,Y}$ (but not necessarily $f|_Y \in \mathscr{O}_{X,x}$), we may express f as a fraction g/h with $g, h \in \mathscr{O}_{X,x}$. Note that the difference $\mathrm{ord}_x(g) - \mathrm{ord}_x(h)$ does not depend on the choice of g and h: indeed, given g', h' subject to the same conditions, we have $gh' = g'h$, hence $\mathrm{ord}_x(g) + \mathrm{ord}_x(h') = \mathrm{ord}_x(gh') = \mathrm{ord}_x(g'h) = \mathrm{ord}_x(g') + \mathrm{ord}_x(h)$. Thus we define $\mathrm{ord}_x(f) = \mathrm{ord}_x(g) - \mathrm{ord}_x(h)$.

It may be proved that for any rational function $f \in \mathfrak{R}(X)$ such that $f|_Y \neq 0 \in \mathscr{O}_{X,Y}$ for every connected component $Y \subset X$ (equivalently, $\mathrm{ord}_x(f)$ is defined for every $x \in X$), there only exist a finite number of prime divisors $x \in X$ for which $\mathrm{ord}_x(f)$ is nonzero. Let $P(f)$ be the set of such points. The *(Weil) divisor associated to f* is defined as

$$\mathrm{div}(f) = \sum_{x \in P(f)} \mathrm{ord}_x(f)x.$$

Divisors of this form are said to be *principal*. Note that whenever $\mathrm{div}(fg)$ is defined, then $\mathrm{div}(f)$ and $\mathrm{div}(g)$ are both defined and (using that $P(fg) \subset P(f) \cup P(g)$) it holds that

$$\begin{aligned}
\mathrm{div}(fg) &= \sum_{x \in P(fg)} \mathrm{ord}_x(fg)x \\
&= \sum_{x \in P(f)\cup P(g)} \mathrm{ord}_x(fg)x \\
&= \sum_{x \in P(f)\cup P(g)} (\mathrm{ord}_x(f) + \mathrm{ord}_x(g))x \\
&= \sum_{x \in P(f)\cup P(g)} \mathrm{ord}_x(f)x + \sum_{x \in P(f)\cup P(g)} \mathrm{ord}_x(g)x \\
&= \sum_{x \in P(f)} \mathrm{ord}_x(f)x + \sum_{x \in P(g)} \mathrm{ord}_x(g)x \\
&= \mathrm{div}(f) + \mathrm{div}(g).
\end{aligned}$$

Also, the zero divisor equals $\mathrm{div}(1)$. Thus principal divisors form a subgroup of $\mathrm{Div}(X)$. The quotient group, denoted by $\mathrm{Cl}(X)$, is called the *(Weil) divisor class group* of X, and two divisors belonging to the same class in $\mathrm{Cl}(X)$ (equivalently, whose difference is a principal divisor) are said to be *linearly equivalent*. Given a divisor D, its class in $\mathrm{Cl}(X)$ will be denoted by $[D]$.

The following notation will be useful: given $D = \sum_{i \in I} n_i Z_i \in \mathrm{Div}(X)$ and a connected scheme $Y \in \mathrm{Sm}_k$, we write $D \times Y$ for $\sum_{i \in I} n_i (Z_i \times_k Y) \in \mathrm{Div}(X \times_k Y)$.

A decomposition of $\mathbf{Cor}_k(X, \mathbb{A}^1 \setminus \{0\})$

When $X \in \mathrm{Sm}_k$ is connected one may use Hartshorne, 1977, Chapter II, propositions 6.5 and 6.6, to describe the divisor class group of X in terms of that of $X \times \mathbb{P}^1$ and the divisor class of $X \times \{\infty\}$ in $\mathrm{Cl}(X \times \mathbb{P}^1)$.

More precisely, by Hartshorne, 1977, II, Prop. 6.5 there exists an exact sequence

$$\mathbb{Z} \longrightarrow \mathrm{Cl}(X \times \mathbb{P}^1) \longrightarrow \mathrm{Cl}(X \times \mathbb{A}^1) \longrightarrow 0$$

in which the first map sends n to $n \cdot [X \times \{\infty\}]$, and the second one is characterized by the fact that for any codimension 1 irreducible closed subset $Z \subset X \times \mathbb{P}^1$, it sends $[Z]$ to $[Z \cap (X \times \mathbb{A}^1)$, if $Z \cap (X \times \mathbb{A}^1)$ is nonempty, and to 0 otherwise.

On the other hand, by Hartshorne, 1977, II, Prop. 6.6, the map $\mathrm{Div}(X) \longrightarrow \mathrm{Div}(X \times \mathbb{A}^1)$ given by $D \mapsto D \times \mathbb{A}^1$ induces an isomorphism $\mathrm{Cl}(X) \cong \mathrm{Cl}(X \times \mathbb{A}^1)$.

Note that the map $\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(X \times \mathbb{P}^1)$ given by $[D] \mapsto [D \times \mathbb{P}^1]$ is a section of the composite $\mathrm{Cl}(X \times \mathbb{P}^1) \cong \mathrm{Cl}(X \times \mathbb{A}^1) \longrightarrow \mathrm{Cl}(X)$. It may also be proved that $\mathbb{Z} \longrightarrow \mathrm{Cl}(X \times \mathbb{P}^1)$ is injective. This yields an isomorphism $\mathbb{Z} \oplus \mathrm{Cl}(X) \cong \mathrm{Cl}(X \times \mathbb{P}^1)$ given by sending each $(n, [D])$ to $n \cdot [X \times \{\infty\}] + [D \times \mathbb{P}^1]$.

This implies that for every finite correspondence $\alpha \in \mathrm{Cor}_k(X, \mathbb{A}^1)$, identified with an element of $\mathrm{Div}(X \times \mathbb{P}^1)$, the set of

- triples (D, n, D'), where D is a principal divisor on $X \times \mathbb{P}^1$, n is an integer, and D' is a divisor on X such that $\alpha = D + n \cdot (X \times \{\infty\}) + (D' \times \mathbb{P}^1)$ – let us call these *α-triples* –

is nonempty and admits a non-canonical bijective correspondence with the set of principal divisors on X as follows: noting that n does not depend on the given α-triple, for an arbitrary fixed α-triple (D_0, n, D'_0), we send each α-triple (D, n, D') to $D'_0 - D'$ (note that $D - D_0 = (D'_0 - D') \times \mathbb{P}^1$). Thus a principal divisor D on X is such that there exists an α-triple of the form (D, n, D') if and only if there exists a principal divisor E on X such that $D = D_0 + (E \times \mathbb{P}^1)$. By choosing $f_0 \in \mathfrak{R}(X \times \mathbb{P}^1)$ such that $D_0 = \mathrm{div}(f_0)$, we have that principal divisors D with this property are those of the form $D_0 + \mathrm{div}(f) = \mathrm{div}(f_0 f)$ for some invertible regular function f on X, which we identify with a regular function on $X \times \mathbb{P}^1$. It follows that there exists a unique rational function $g\mathfrak{R}(X \times \mathbb{P}^1)$ of the form $f_0 f$ satisfying the normalization condition that g/t^n restricts to the constant function 1 on $X \times \{\infty\}$, namely by taking f as the restriction of t^n/f_0 to $X \times \{\infty\}$. As g only depends on α, let us denote it by g_α.

Then for $X \in \mathrm{Sm}_k$ connected we have (as in Mazza *et al.*, 2006, 4.4) a homomorphism

$$P_X : \mathrm{Cor}_k(X, \mathbb{A}^1 \setminus \{0\}) \longrightarrow \mathbb{Z} \oplus \mathcal{O}^\times(X)$$

which, in the above notation, sends each α to $(n, (-1)^n g_\alpha(0))$, where $g_\alpha(0)$ denotes the restriction of g_α to $X \times \{0\} \cong X$ – it is defined and is invertible in $\mathcal{O}(X)$ as α does not intersect $X \times \{0\}$.

Note that for each $g \in \mathbb{O}^*(X)$, by defining $\alpha = \mathrm{div}(t - g)$, which is the graph of g, we have $g_\alpha = t - g$ and $P_X(\alpha) = (1, (-1)^1(-g)) = (1, g)$. So by taking $\beta = \mathrm{div}(t - 1)$, we have $P_X(\beta) = (1, 1)$, hence $P_X(\alpha - \beta) = (1 - 1, u/1) = (0, u)$. This allows us to construct a section of P_X given by

$$\begin{aligned} S_X : \mathbb{Z} \oplus \mathcal{O}^\times(X) &\longrightarrow \mathrm{Cor}_k(X, \mathbb{A}^1 \setminus \{0\}) \\ (n, g) &\longmapsto n \cdot \mathrm{div}(t - 1) + (\mathrm{div}(t - g) - \mathrm{div}(t - 1)) = \mathrm{div}((t - g)(t - 1)^{n-1}). \end{aligned}$$

Thus P_X is a surjection and $\mathrm{Cor}_k(X, \mathbb{A}^1 \setminus \{0\})$ can be expressed as

$$\mathrm{Cor}_k(X, \mathbb{A}^1 \setminus \{0\}) \cong \mathrm{Ker}(P_X) \oplus \mathbb{Z} \oplus \mathcal{O}^\times(X).$$

Let us consider the corresponding short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Cor}_k(X, \mathbb{A}^1 \setminus \{0\}) \longrightarrow \mathrm{Ker}(P_X) \oplus \mathcal{O}^\times(X) \longrightarrow 0.$$

By construction, $\mathbb{Z} \longrightarrow \mathrm{Cor}_k(X, \mathbb{A}^1 \setminus \{0\})$ sends 1 to $\div(t-1) = X \times \{1\}$, which is equal to the image under 1 of the composite

$$\mathbb{Z} \cong \mathrm{Cor}_k(X, \mathrm{Spec}\ k) \overset{\mathrm{Cor}_k(s_1, \mathbb{A}^1\setminus\{0\})}{\longrightarrow} \mathrm{Cor}_k(X, \mathbb{A}^1 \setminus \{0\}),$$

where the first map sends 1 to $X \times_k \mathrm{Spec}\ k$ regarded as a cycle on itself.

By comparing cokernels, we obtain an isomorphism

$$\mathbb{Z}_k^{tr}(\mathbb{G}_m)(X) \cong \mathrm{Ker}(P_X) \oplus \mathcal{O}^\times(X).$$

It may be checked that the above constructions are natural in $X \in \mathrm{Cor}_k$, so there exists an isomorphism

$$\mathbb{Z}_k^{tr}(\mathbb{G}_m) \cong \mathrm{Ker}(P_-) \oplus \mathcal{O}^\times$$

of presheaves with transfers, hence a short exact sequence

$$0 \longrightarrow \mathrm{Ker}(P_-) \longrightarrow \mathbb{Z}_k^{tr}(\mathbb{G}_m) \longrightarrow \mathcal{O}^\times \longrightarrow 0.$$

The kernel of P_X

In what follows, we will need to consider rational functions on a scheme in Sm_k which are defined at every point of a given subset. We also establish a suitable form of functoriality for this construction.

Construction 2.3.1. Suppose given $X \in \mathrm{Sm}_k$, and $U \subset X$ an arbitrary nonempty subset. For each connected component $Y \subset X$, let us consider the intersection $\bigcap_{x \in U \cap Y} \mathcal{O}_{X,x}$ of subrings of $\mathcal{O}_{X,Y}$. We denote by $\mathcal{O}(X,U)$ the product of these:

$$\mathcal{O}(X,U) := \prod_{Y \in c(X)} \bigcap_{x \in U \cap Y} \mathcal{O}_{X,x}.$$

We may regard this construction as a contravariant functor on pairs (X,U), where morphisms $(X,U) \longrightarrow (X',U')$ are morphisms $f : X \longrightarrow X'$ in Sm_k such that $f(U) \subset U'$. We proceed in the following way: for each connected component $Y \subset X$, there exists a unique connected component $s(Y) \subset X'$ such that $f(Y) \subset s(Y)$. For each $x \in U \cap Y$ it holds that $f(x) \in U' \cap s(Y)$, and moreover we have a homomorphism $\mathcal{O}_{X',f(x)} \longrightarrow \mathcal{O}_{X,x}$ from a subring of $\mathcal{O}_{X',s(Y)}$ to a subring of $\mathcal{O}_{X,Y}$. By ranging x over $U \cap Y$ and taking intersections, we obtain a homomorphism

$$\bigcap_{x' \in U' \cap s(Y)} \mathcal{O}_{X',x'} \longrightarrow \bigcap_{x \in U \cap Y} \mathcal{O}_{X,x}.$$

By composing with the projection $\mathcal{O}(X',U') \longrightarrow \bigcap_{x' \in U' \cap s(Y)} \mathcal{O}_{X',x'}$, this defines a homomorphism $\mathcal{O}(X',U') \longrightarrow \bigcap_{x \in U \cap Y} \mathcal{O}_{X,x}$. Finally, ranging Y over $c(X)$ yields a map

$$\mathcal{O}(X',U') \longrightarrow \mathcal{O}(X',U').$$

Definition 2.3.2. For the purposes of this subsection, for each $X \in \mathrm{Sm}_k$ we denote by $\mathfrak{R}'(X)$ the set of rational functions on $X \times \mathbb{P}^1$ which are defined at every point of $X \times \{0, \infty\}$, and whose value at each point of $X \times \{0, \infty\}$ is 1. It is a submonoid of the underlying multiplicative monoid of the ring $\mathscr{O}(X \times \mathbb{P}^1, X \times \{0, \infty\})$. Note that for any $f : X \longrightarrow Y$ in Sm_k, the morphism $f \times id_{\mathbb{P}^1} : X \times \mathbb{P}^1 \longrightarrow Y \times \mathbb{P}^1$ sends $X \times \{0, \infty\}$ to $Y \times \{0, \infty\}$, so we have a map

$$\mathscr{O}(Y \times \mathbb{P}^1, Y \times \{0, \infty\}) \longrightarrow \mathscr{O}(X \times \mathbb{P}^1, X \times \{0, \infty\})$$

as in the previous construction, which then restricts to a map

$$\mathfrak{R}'(Y) \longrightarrow \mathfrak{R}'(X).$$

Note that $\mathfrak{R}'(X)$ is actually an abelian group, since $X \times \{0, \infty\}$ intersects every connected component of $X \times \mathbb{P}^1$.

In what follows, we identify $\mathbb{A}^1 \setminus \{0\}$ with the open subscheme $\mathbb{P}^1 \setminus \{0, \infty\}$ of $\mathbb{P}^1$.

Remark 2.3.3. As for any $f \in \mathfrak{R}'(X)$ the value of f at every $x \in X \times \{0, \infty\}$ is 1, it holds in particular that for each such f, $\mathrm{ord}_x(f) = 0$ whenever $x \in X \times \{0, \infty\}$ is a prime divisor on $X \times \mathbb{P}^1$. Hence $\mathrm{div}(f)$ may be identified with a divisor on $X \times (\mathbb{A}^1 \setminus \{0\}) \cong X \times (\mathbb{P}^1 \setminus \{0, \infty\}) \hookrightarrow X \times \mathbb{P}^1$, so we obtain a map

$$\mathrm{div} : \mathfrak{R}'(X) \longrightarrow \mathrm{Div}(X \times (\mathbb{A}^1 \setminus \{0\})).$$

The following lemma, which relies on the above remark, allows us to construct finite correspondences from a given $X \in \mathrm{Sm}_k$ to $\mathbb{A}^1 \setminus \{0\}$ in terms of elements of $\mathfrak{R}'(X)$.

Lemma 2.3.4 (Mazza *et al.*, 2006, Lemma 4.3)**.** Given $X \in \mathrm{Sm}_k$ and $f \in \mathfrak{R}'(X)$, it holds that $\mathrm{div}(f) \in \mathrm{Div}(X \times (\mathbb{A}^1 \setminus \{0\}))$ is a finite correspondence from X to $\mathbb{A}^1 \setminus \{0\}$.

Proof. Let us write $\mathrm{div}(f) = \sum_{i \in I} n_i x_i$, where each x_i is a prime divisor on $X \times (\mathbb{A}^1 \setminus \{0\})$. We must prove that each $Z_i := \overline{\{x_i\}}$, where the closure is taken in $X \times \mathbb{A}^1 \setminus \{0\})$, is such that the composite $Z_i \longrightarrow X \times \mathbb{A}^1 \setminus \{0\}) \longrightarrow X$ of the integral closed subscheme inclusion with the canonical projection is finite and its image is a connected component of X.

Firstly, note that since the property of a morphism of schemes being finite and surjective is Zariski-local on the target, we may reduce the problem to the case where X is integral and affine, say $X \cong \mathrm{Spec}\, A$ for an integral domain A. Since $\mathrm{Spec}\, A \times \mathbb{A}^1 \cong \mathrm{Spec}\, (A \otimes_k k[t]) \cong \mathrm{Spec}\, A[t]$, we have $\mathfrak{R}(X \times \mathbb{P}^1) \cong R(X \times \mathbb{A}^1) \cong \mathrm{Frac}(A[t])$. Then any given $f \in \mathfrak{R}(X \times (\mathbb{A}^1 \setminus \{0\}))$ may be expressed as g/h for $g, h \in A[t]$, say $g = c_m t^m + \cdots + c_1 t + c_0$, $h = d_n t^n + \cdots + d_1 t + d_0$. As g and h are both nonzero, we assume c_m and d_n are both nonzero. We also assume c_0 and d_0 are not both 0, as otherwise we reduce to this case by dividing both g and h by some power of t. Then as the restriction of f to $X \times \{0\}$ equals 1, by taking $t = 0$ we obtain that $c_0 = d_0$; hence c_0 and d_0 are both nonzero.

On the other hand, by defining $u = 1/t$ we obtain

$$g(t) = c_m(1/u)^m + \cdots + c_1(1/u) + c_0 = (1/u)^m(c_m + \cdots + c_1 u^{m-1} + c_0 u^m),$$

$$h(t) = d_n(1/u)^n + \cdots + d_1(1/u) + d_0 = (1/u)^n(d_n + \cdots + d_1 u^{n-1} + d_0 u^n),$$

hence
$$\frac{g(t)}{h(t)} = u^{n-m}\frac{c_m + \cdots + c_1 u^{m-1} + c_0 u^m}{d_n + \cdots + d_1 u^{n-1} + d_0 u^n}.$$
By evaluating at $u = 0$, the assumption that f restricts to 1 on $X \times \{\infty\}$ implies that: (i) as in particular its order of vanishing at $X \times \{\infty\}$ is zero, it holds that $m = n$ and (ii) as its value must then be given by c_m/d_n, we have $c_m = d_n$. Then we may reduce to the case where $c_m = d_n = 1$, in which $\operatorname{div}(g)$ and $\operatorname{div}(h)$ are both finite correspondences. Indeed, by restricting g, which is monic by assumption, to the fiber over any point $x \in X$ we obtain a monic, hence nonzero polynomial over the residue field $\kappa(x)$; thus $\operatorname{div}(g) \in \operatorname{Div}(X \times (\mathbb{A}^1 \setminus \{0\}))$ cannot contain any divisor of the form $D \times (\mathbb{A}^1 \setminus \{0\})$ for $D \in \operatorname{Div}(X)$. Similarly for g. $\square$

Thus we obtain a homomorphism of abelian groups
$$Q_X : \mathfrak{R}'(X) \longrightarrow \operatorname{Cor}_k(X, \mathbb{A}^1 \setminus \{0\}).$$
As a rational function $f \in \mathfrak{R}'(X)$ has the property that its restrictions to $X \times \{\infty\}$ and $X \times \{0\}$ are both equal to 1, its image in $\operatorname{Cor}_k(X, \mathbb{A}^1 \setminus \{0\})$ belongs to $\operatorname{Ker}(P_X)$.

This provides a short exact sequence
$$0 \longrightarrow \mathfrak{R}' \longrightarrow \mathbb{Z}_k^{tr}(\mathbb{G}_m) \longrightarrow \mathscr{O}^\times \longrightarrow 0$$
of presheaves with transfers and, as outlined in the beginning of this section, a short exact sequence of complexes
$$0 \longrightarrow C_*\mathfrak{R}' \longrightarrow C_*\mathbb{Z}_k^{tr}(\mathbb{G}_m) \longrightarrow C_*\mathscr{O}^\times \longrightarrow 0.$$
By Mazza *et al.*, 2006, 4.6, $C_*\mathfrak{R}'$ is acyclic, so one obtains:

Proposition 2.3.5. Let k be a field. Then the chain map of complexes of presheaves with transfers
$$\mathbb{Z}(1) \longrightarrow \mathscr{O}^\times[-1]$$
constructed above is a quasi-isomorphism. For each $X \in \mathrm{Sm}_k$ and $n \geq 0$,
$$H^{n,1}(X, \mathbb{Z}) = H^n_{\mathrm{Zar}}(X, \mathbb{Z}(1)|_{X_{\mathrm{Zar}}}) \cong H^n_{\mathrm{Zar}}(X, \mathscr{O}^\times[-1]) \cong H^{n-1}_{\mathrm{Zar}}(X, \mathscr{O}^\times).$$
Hence $H^{1,1}(X, \mathbb{Z}) \cong \mathscr{O}^\times(X)$, and in particular $H^{1,1}(\operatorname{Spec} k, \mathbb{Z}) \cong k^*$.

A description of $\mathbb{Z}/l(1)$ as an étale sheaf

The above result characterizes $\mathbb{Z}(1)$ up to quasi-isomorphism in the category of presheaves with transfers, namely, as $\mathbb{Z}(1) \simeq \mathscr{O}^\times[-1]$. Now, let l be a prime number different from the characteristic of k. Note that the constant presheaf with transfers $\mathbb{Z}/l$ admits a projective resolution
$$\cdots \longrightarrow 0 \longrightarrow \overset{(-1)}{\mathbb{Z}} \overset{l}{\longrightarrow} \overset{(0)}{\mathbb{Z}} \longrightarrow 0 \longrightarrow \cdots$$

in $\mathrm{PST}(k)$. Let us denote it by P. Then we obtain a quasi-isomorphism

$$\mathbb{Z}(1) \otimes P \simeq \mathcal{O}^\times[-1] \otimes P,$$

where the right hand complex is isomorphic to

$$\cdots \longrightarrow 0 \longrightarrow \overset{(0)}{\mathcal{O}^\times} \xrightarrow{l} \overset{(1)}{\mathcal{O}^\times} \longrightarrow 0 \longrightarrow \cdots.$$

On the other hand, since $\mathbb{Z}(1)$ is a complex of flat presheaves with transfers, the quasi-isomorphism $P \longrightarrow \mathbb{Z}/l$ defines a quasi-isomorphism

$$\mathbb{Z}(1) \otimes P \simeq \mathbb{Z}(1) \otimes \mathbb{Z}/l = \mathbb{Z}/l(1).$$

Let us use the subscripts sm and et to regard a bounded above complex of étale sheaves with transfers as an object of $\mathrm{Ch}^-(\mathrm{PST}(k))$ to $\mathrm{Ch}^-(\mathrm{ST}_{\text{ét}}(k))$, respectively. Working in the derived category $\mathrm{D}^-\mathrm{PST}(k)$, we have an isomorphism

$$\mathbb{Z}/l(1)_{sm} \cong (\mathcal{O}^\times[-1] \otimes P)_{sm}.$$

By applying the sheafification functor from presheaves with transfers to étale sheaves with transfers, which is exact, one obtains an isomorphism

$$\mathbb{Z}/l(1)_{\text{ét}} \cong (\mathcal{O}^\times[-1] \otimes P)_{\text{ét}}.$$

in $\mathrm{D}^-\mathrm{ST}_{\text{ét}}(k)$. The complex $(\mathcal{O}^\times[-1] \otimes P)_{\text{ét}}$ may be described by noting that

$$\mathcal{O}^\times_{\text{ét}} \xrightarrow{l} \mathcal{O}^\times_{\text{ét}}$$

is an epimorphism of étale sheaves with transfers (equivalently, the étale sheafification of the presheaf cokernel of $\mathcal{O}^\times_{sm} \xrightarrow{l} \mathcal{O}^\times_{sm}$ is zero). The short exact sequence of étale sheaves with transfers

$$0 \longrightarrow \mu_{l,et} \longrightarrow \mathcal{O}^\times_{\text{ét}} \xrightarrow{l} \mathcal{O}^\times_{\text{ét}} \longrightarrow 0$$

defines a quasi-isomorphism

$$\mu_{l,et} \simeq (\mathcal{O}^\times[-1] \otimes P)_{\text{ét}}.$$

Then one obtains isomorphisms

$$\mu_{l,et} \cong (\mathcal{O}^\times[-1] \otimes P)_{\text{ét}} \cong \mathbb{Z}/l(1)_{\text{ét}}$$

in $\mathrm{D}^-\mathrm{ST}_{\text{ét}}(k)$.

This provides the following result on motivic étale cohomology: for each $X \in \mathrm{Sm}_k$ and $p \geq 0$ we have isomorphisms

$$H^{p,1}_{\text{ét}}(X, \mathbb{Z}/l) = H^p_{\text{ét}}(X, \mathbb{Z}/l(1)) \cong H^p_{\text{ét}}(X, \mu_l).$$

Remark 2.3.6. This may be generalized to the following result (see Mazza *et al.*, 2006): for every $q \leq 1$, there exists a quasi-isomorphism

$$\mu_l^{\otimes q} \simeq \mathbb{Z}/l(q)$$

of complexes of étale sheaves with transfers. Thus for each $X \in \mathrm{Sm}_k$ and $p \geq 0$ one obtains

$$H^{p,q}_{\text{ét}}(X, \mathbb{Z}/l) = H^p_{\text{ét}}(X, \mathbb{Z}/l(q)) \cong H^p_{\text{ét}}(X, \mu_l^{\otimes q}).$$

2.4 Étale sheafification with transfers, change of topology

We let k be a fixed base field throughout this section. The morphism of sites $\pi : (\mathrm{Sm}_k)_{\text{ét}} \longrightarrow (\mathrm{Sm}_k)_{\mathrm{Zar}}$ (given on underlying categories by the identity functor) induces a geometric morphism between the corresponding Grothendieck topoi of sheaves of sets, that is, an adjunction $\pi^* \dashv \pi_*$ where $\pi_* : \mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k) \longrightarrow \mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k)$ is given by restriction (i.e. by precomposition with the identity functor), and $\pi^* : \mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k) \longrightarrow \mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k)$ is the left exact functor given by regarding each Zariski sheaf as a presheaf and then taking its usual étale sheafification. These induce further adjunctions (which we also denote by $\pi^* \dashv \pi_*$) between categories of (cochain complexes of) abelian sheaves:

$$\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}) \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k, \mathrm{Ab}), \tag{2.4.1}$$

$$Ch^?(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab})) \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} Ch^?(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k, \mathrm{Ab})), \tag{2.4.2}$$

where ? in 2.4.2 denotes either +, −, or no boundedness assumption. In each case, π_* is left exact and π^* is exact.

2.4.1 Étale sheafification with transfers

A similar framework is available when considering étale sheaves with transfers. The reference for the following results is Mazza *et al.*, 2006, Lec. 6.

Lemma 2.4.1. Suppose given a finite k-correspondence $\alpha : X \bullet\!\!\longrightarrow Y$ and a surjective étale map $p : Y' \longrightarrow Y$ in Sm_k (identified with its graph in Cor_k). Then there exists in Cor_k a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\alpha'"] \arrow[d, "p'"'] & Y' \arrow[d, "p"] \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

with p' a surjective étale map.

Proof. Note that it suffices to consider the case where $\alpha = [Z]_{X \times_k Y}$ for some integral closed subscheme $i : Z \longrightarrow X \times_k Y$. Indeed, by bilinearity of composition and existence of finite coproducts (given by disjoint unions) in Cor_k, the general case follows by considering the coproduct of the schemes X' obtained for each term of the form $[Z]_{X \times_k Y}$ occurring in α.

The pullback $Z \times_Y Y'$ of $Z \longrightarrow X \times_k Y \longrightarrow Y$ along $p : Y' \longrightarrow Y$ is étale over Z, so $Z \times_Y Y' \longrightarrow Z$ has the property that (by finiteness of $Z \longrightarrow X$) each $x \in X$ has an étale neighborhood $U \longrightarrow X$ such that $(Z \times_Y Y') \times_X U \longrightarrow Z \times_X U$ has a section.

Since X is noetherian, hence quasi-compact, by choosing a finite family of such maps and taking the disjoint union of their domains, which we define to be X', we obtain a surjective étale map $p' : X' \longrightarrow X$ such that $(Z \times_Y Y') \times_X X' \longrightarrow Z \times_X X'$ has a section, say s. See the diagram

$$\begin{tikzcd}
(Z \times_Y Y') \times_X X' \arrow[r] \arrow[d] & Z \times_Y Y' \arrow[r, hook, "i'"] \arrow[d] & X \times_k Y' \arrow[d, "id \times p"] \\
Z \times_X X' \arrow[u, bend left, "s"] \arrow[r, "(\pi \circ i)^*(p')"] \arrow[d, "p'^*(\pi \circ i)"'] & Z \arrow[r, hook, "i"] \arrow[d, "\pi \circ i"] & X \times_k Y \arrow[dl, "\pi"] \\
X' \arrow[r, "p'"'] & X, &
\end{tikzcd}$$

where we have denoted by i' the pullback of i along $id \times p$. Let us also denote by π' the projection $X \times_k Y' \longrightarrow X$.

Now, note that by the pullback lemma, the unique arrow j filling the diagram

$$\begin{tikzcd}
(Z \times_Y Y') \times_X X' \arrow[r, dashed, "j"] \arrow[rr, bend left, "p'^*(\pi' \circ i')"] \arrow[d, "(\pi' \circ i')^*(p')"'] & X' \times_k Y' \arrow[r, "p'^*(\pi')"] \arrow[d, "\pi'^*(p') = p' \times id"] & X' \arrow[d, "p'"] \\
Z \times_Y Y' \arrow[r, "i'"'] & X \times_k Y' \arrow[r, "\pi'"'] & X
\end{tikzcd}$$

is the pullback of i' along $\pi'^*(p')$, hence a closed immersion. Moreover, since the projection $(Z \times_Y Y') \times_X X' \longrightarrow Z \times_X X'$ is separated (as any morphism in Sm_k), s is a closed immersion. We define α' to be the cycle associated to the closed immersion $j \circ s : Z \times_X X' \longrightarrow X' \times_k Y'$. Since any pullback of a finite (resp. surjective) scheme morphism is finite (resp. surjective), α' is a finite correspondence from X' to Y'.

It remains to show that $\alpha \circ p' = p \circ \alpha'$. The unique filler k in

$$\begin{tikzcd}
Z \times_X X' \arrow[r, dashed, "k"] \arrow[rr, bend left, "p'^*(\pi \circ i)"] \arrow[d, "(\pi \circ i)^*(p')"'] & X' \times_k Y \arrow[r, "p'^*(\pi)"] \arrow[d, "\pi^*(p') = p' \times id"] & X' \arrow[d, "p'"] \\
Z \arrow[r, "i"'] & X \times_k Y \arrow[r, "\pi"'] & X
\end{tikzcd}$$

is the pullback of i along $p' \times id$, hence it is a closed immersion whose associated cycle is $\alpha \circ p'$. On the other hand, $p \circ \alpha'$ may be computed as the pushforward of α along

$id \times p : X \times_k Y' \longrightarrow X \times_k Y$; since $\alpha' = (j \circ s)_*[Z \times_X X']_{Z \times_X X'}$, it follows that

$$p \circ \alpha' = ((id \times p) \circ j \circ s)_*[Z \times_X X']_{Z \times_X X'} = k_*[Z \times_X X']_{Z \times_X X'} = \alpha \circ p'.$$

$\square$

Lemma 2.4.2. Suppose given an Ab-enriched category $\mathcal{C}$, a (Set-enriched) subcategory $\mathcal{C}' \hookrightarrow \mathcal{C}$ such that $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{C}')$, a presheaf $F : \mathcal{C}^{op} \longrightarrow \mathrm{Ab}$ whose restriction to $\mathcal{C}'$ we denote by F', a presheaf $G' : \mathcal{C}'^{op} \longrightarrow \mathrm{Ab}$, and a natural transformation $\eta : F' \longrightarrow G'$. For each $X \in \mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{C}')$, we use the notations $h_X := \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \longrightarrow \mathrm{Ab}$, $h'_X := \mathrm{Hom}_{\mathcal{C}'}(-, X) : \mathcal{C}'^{op} \longrightarrow \mathrm{Set}$ for the corresponding represented presheaves. Then it is equivalent to provide the following data:

(i) A presheaf $G : \mathcal{C}^{op} \longrightarrow \mathrm{Ab}$ whose restriction to $\mathcal{C}'^{op}$ equals G' and such that the components of $\eta : F' \longrightarrow G'$ define a natural transformation $\tilde{\eta} : F \longrightarrow G$.

(ii) A choice of abelian group morphisms

$$\varphi_X : G'(X) \longrightarrow \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_X|_{\mathcal{C}'}, G')$$

for each $X \in \mathcal{C}$ subject to the following conditions:

(ii.a) For each $X \in \mathcal{C}$, the diagram

$$\begin{tikzcd} F'(X) \arrow[r, "\psi_X"] \arrow[d, "\eta_X"'] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_X|_{\mathcal{C}'}, F') \arrow[d, "\eta\circ-"] \\ G'(X) \arrow[r, "\varphi_X"'] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_X|_{\mathcal{C}'}, G') \end{tikzcd}$$

commutes, where ψ_X denotes the composite

$$F'(X) = F(X) \overset{\text{Yoneda}}{\cong} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C},\mathrm{Ab})}(h_X, F) \xrightarrow{\text{restriction}} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_X|_{\mathcal{C}'}, F').$$

(ii.b) Given $f : X \longrightarrow Y$ in $\mathcal{C}$, let us denote by

$$ev_f : \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_Y|_{\mathcal{C}'}, G') \longrightarrow G'(X)$$

the map given by evaluation at $f \in h_Y|_{\mathcal{C}'}(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$, and by

$$\mu_f : \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_Y|_{\mathcal{C}'}, G') \longrightarrow \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_X|_{\mathcal{C}'}, G')$$

the one given by precomposition with the restriction of $\mathrm{Hom}_{\mathcal{C}}(-, f)$ to $\mathcal{C}'^{op}$. Then the diagram

$$\begin{tikzcd} G'(Y) \arrow[r, "\varphi_Y"] \arrow[d, "ev_f\circ\varphi_Y"'] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_Y|_{\mathcal{C}'}, G') \arrow[d, "\mu_f"] \\ G'(X) \arrow[r, "\varphi_X"'] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_X|_{\mathcal{C}'}, G') \end{tikzcd}$$

commutes.

(ii.c) For each $X \in \mathcal{C}$, $ev_{id_X} \circ \varphi_X : G'(X) \longrightarrow G'(X)$ is the identity map.

Proof. Suppose given G as in (i). Then for each $X \in \mathcal{C}$ we define φ_X as in the following commutative diagram:

$$\begin{tikzcd}
F'(X) = F(X) \arrow[r, "\cong"] \arrow[d, "\eta_X"'] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C},\mathrm{Ab})}(h_X, F) \arrow[r, "\text{restriction}"] \arrow[d, "\tilde{\eta}\circ-"] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_X|_{\mathcal{C}'}, F') \arrow[d, "\eta\circ-"] \\
G'(X) = G(X) \arrow[r, "\cong"] \arrow[rr, dashed, bend right, "\varphi_X"'] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C},\mathrm{Ab})}(h_X, G) \arrow[r, "\text{restriction}"'] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_X|_{\mathcal{C}'}, G').
\end{tikzcd}$$

Then (ii.a) holds by construction. For (ii.b), note that for any $f : X \longrightarrow Y$ in $\mathcal{C}$, the composite $ev_f \circ \varphi_Y$ equals $G(f)$ by definition of the Yoneda isomorphism $G(Y) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C},\mathrm{Ab})}(h_Y, G)$; the desired equality $\mu_f \circ \varphi_Y = \varphi_X \circ G(f)$ now follows by naturality in X (as an object of $\mathcal{C}$) of both arrows in the definition of φ_X. The equality $ev_{id_X} \circ \varphi_X = G(id_X) = id_{G(X)}$ for each $X \in \mathcal{C}$ yields (ii.c).

Conversely, suppose given $(\varphi_X)_{X \in \mathrm{Ob}(\mathcal{C})}$ as in (ii). We would like to define $G : \mathcal{C}^{op} \longrightarrow \mathrm{Ab}$ on arrows by sending each $f : X \longrightarrow Y$ to $G(f) := ev_f \circ \varphi_Y$; let us then verify the desired properties:

- *Functoriality.* (ii.c) states precisely that G preserves identity morphisms. For compatibility with composition, suppose given $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$; then we use (ii.b) to obtain

$$\begin{aligned}
G(f) \circ G(g) &= (ev_f \circ \varphi_Y) \circ (ev_g \circ \varphi_Z) \\
&= ev_f \circ (\varphi_Y \circ ev_g \circ \varphi_Z) \\
&= ev_f \circ (\mu_g \circ \varphi_Z) \\
&= ev_{g \circ f} \circ \varphi_Z \\
&= G(g \circ f).
\end{aligned}$$

- *The restriction of G to $\mathcal{C}'^{op}$ equals G'.* Suppose given $f : X \to Y$ in $\mathcal{C}'$. Then we may consider the diagram

$$\begin{tikzcd}
G'(Y) \arrow[r, "\varphi_Y"] \arrow[d, "G(f)"'] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_Y|_{\mathcal{C}'}, G') \arrow[r, "\nu_Y"] \arrow[d, "\mu_f"] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h'_Y, G') \arrow[r, "\text{Yoneda} \cong"] \arrow[d, "\mu'_f"] & G'(Y) \arrow[d, "G'(f)"] \\
G'(X) \arrow[r, "\varphi_X"'] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_X|_{\mathcal{C}'}, G') \arrow[r, "\nu_X"'] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h'_X, G') \arrow[r, "\text{Yoneda} \cong"'] & G'(X),
\end{tikzcd}$$

  where μ'_f denotes precomposition with $\mathrm{Hom}_{\mathcal{C}'}(-, f)$, and ν_X (analogously for ν_Y) is given by precomposition with the natural transformation $h'_X \longrightarrow h_X|_{\mathcal{C}'}$ which sends morphisms in $\mathcal{C}'$ to themselves as morphisms in $\mathcal{C}$. Note that it commutes: the left-hand square commutes by definition of $G(f)$ and (ii.b), the middle one by construction, and the right-hand one by naturality of the Yoneda isomorphism. But by (ii.c), the composites of the upper and lower rows are equal to $ev_{id_Y} \circ \varphi_Y = id_{G'(Y)}$ and $ev_{id_X} \circ \varphi_X = id_{G'(X)}$, respectively, so it follows that $G(f) = G'(f)$.

- *η defines a natural transformation $\tilde{\eta} : F \longrightarrow G$.* Given $f : X \longrightarrow Y$ in $\mathcal{C}$, let us consider the diagram

$$\begin{tikzcd} F'(Y) \arrow[r, "\psi_Y"] \arrow[d, "\eta_Y"'] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_Y|_{\mathcal{C}'}, F') \arrow[r, "ev_f"] \arrow[d, "\eta\circ -"] & F'(X) \arrow[d, "\eta_X"] \\ G'(Y) \arrow[r, "\varphi_Y"'] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_Y|_{\mathcal{C}'}, G') \arrow[r, "ev_f"'] & G'(X). \end{tikzcd}$$

The left-hand square commutes by (ii.a), and so does the right-hand one as for each $\varepsilon \in \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}',\mathrm{Ab})}(h_Y|_{\mathcal{C}'}, F')$ we have

$$ev_f(\eta \circ \varepsilon) = (\eta \circ \varepsilon)_X(f) = \eta_X(\varepsilon_X(f)) = \eta_X(ev_f(\varepsilon)).$$

But the outer square is precisely the desired naturality square for f, since $ev_f \circ \psi_Y = F(f)$ and $ev_f \circ \varphi_Y = G(f)$.

It is immediate that the constructions described above are inverse to each other. $\square$

Theorem 2.4.3. Suppose given a presheaf with transfers $F : \mathrm{Cor}_k^{op} \longrightarrow \mathrm{Ab}$. Let $F' := F \circ \gamma$ be its restriction to Sm_k^{op} along the graph functor, and let $F'_{\text{ét}} \in \mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k, \mathrm{Ab})$ be its étale sheafification. Then there exists a unique étale sheaf with transfers $F_{\text{ét}} \in \mathrm{ST}_{\text{ét}}(k)$ satisfying the following properties:

(i) $F'_{\text{ét}} = F_{\text{ét}} \circ \gamma$.

(ii) Let $\eta : F' \longrightarrow F'_{\text{ét}}$ be the usual sheafification morphism. Then η (recall that Sm_k and Cor_k have the same objects) defines a morphism of presheaves with transfers $\eta^{tr} : F \longrightarrow F_{\text{ét}}$. In other words, for any finite correspondence $\alpha \in \mathrm{Cor}_k(X, Y)$, the following diagram commutes:

$$\begin{tikzcd} F(Y) = F'(Y) \arrow[r, "\eta_Y"] \arrow[d, "F(\alpha)"'] & F_{\text{ét}}(Y) = F'_{\text{ét}}(Y) \arrow[d, "F_{\text{ét}}(\alpha)"] \\ F(X) = F'(X) \arrow[r, "\eta_X"'] & F_{\text{ét}}(X) = F'_{\text{ét}}(X). \end{tikzcd}$$

Sketch. We begin by showing that if there exist étale sheaves with transfers F_1, F_2 satisfying the above properties, then $F_1 = F_2$. By (i), F_1 and F_2 coincide on objects and on graphs of scheme morphisms. Let us prove that $F_1(\alpha) = F_2(\alpha)$ for any finite k-correspondence $\alpha : X \bullet\!\!\longrightarrow Y$.

Suppose given $s \in F_1(Y) = F_2(Y)$. Note that there exists a surjective étale map $p : Y' \longrightarrow Y$ in Sm_k such that $F_1(p)(s) = F_2(p)(s)$ belongs to the image of the sheafification map $F'(Y') \longrightarrow F'_{\text{ét}}(Y')$. Indeed, there exists an étale covering $\{p_i : Y_i \longrightarrow Y\}_{i\in I}$ in Sm_k such that for each $i \in I$, $F'_{\text{ét}}(p_i)(s)$ is the image of some s_i along $F'(Y_i) \longrightarrow F'_{\text{ét}}(Y_i)$; since Y is noetherian, hence quasi-compact, and every étale map is open, there exists a finite $J \subset I$ such that $\{p_j : Y_j \longrightarrow Y\}_{j\in J}$ is an étale covering. It follows that the disjoint union of schemes $\coprod_{j\in J} Y_j$ belongs to Sm_k, and since presheaves with transfers send finite coproducts in Cor_k to products (being additive by definition), s restricts via $F'_{\text{ét}}(Y) \longrightarrow F'_{\text{ét}}(\coprod_{j\in J} Y_j)$ to the section corresponding to $(F'_{\text{ét}}(p_j)(s))_{j\in J}$, which is in turn in the image of

$$F'(\coprod_{j\in J} Y_j) \cong \prod_{j\in J} F'(Y_j) \longrightarrow \prod_{j\in J} F'_{\text{ét}}(Y_j) \cong F'_{\text{ét}}(\coprod_{j\in J} Y_j).$$

We then take Y' to be $\coprod_{j\in J} Y_j$.

By Lemma 2.4.1, there exists a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\alpha'"] \arrow[d, "p'"'] & Y' \arrow[d, "p"] \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

with p' surjective étale. Then naturality of $F \longrightarrow F_1$ and $F \longrightarrow F_2$ (which have the same components, namely, those of η) applied to α' plus the assumption on Y' show that $F_1(p \circ \alpha')(s) = F_2(p \circ \alpha')(s)$, hence $F_1(\alpha \circ p')(s) = F_2(\alpha \circ p')(s)$. But $F'_{\text{ét}}(p') = F_1(p') = F_2(p')$ is injective, since $F'_{\text{ét}}$ is an étale sheaf and p' is surjective étale, so $F_1(\alpha)(s) = F_1(\alpha)(s)$. Since α and s were chosen arbitrarily, it follows that $F_1 = F_2$.

We now sketch the proof of existence of one such $F_{\text{ét}}$; see Mazza *et al.*, 2006. For that purpose we use Lemma 2.4.2 with Cor_k, Sm_k, F, $F_{\text{ét}}$, F', $F'_{\text{ét}}$, η in place of $\mathcal{C}$, $\mathcal{C}'$, F, G, F', G', η, respectively. Let us denote by $P(X)$ the abelian presheaf on Sm_k obtained by restriction of $\mathbb{Z}_k^{tr}(X)$. One needs to define for each $X \in \mathrm{Sm}_k$ a morphism of abelian groups

$$\varphi_X : F'_{\text{ét}}(X) \longrightarrow \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sm}_k,\mathrm{Ab})}(P(X), F'_{\text{ét}})$$

in such a way that they satisfy (ii.a), (ii.b), (ii.c) in 2.4.2. Given $s \in F'_{\text{ét}}(X)$, one chooses a surjective étale map $p : Y \longrightarrow X$ such that there exists a section $t \in F'(Y)$ with the following two properties: (i) $F'_{\text{ét}}(p)(s) \in F'_{\text{ét}}(Y)$ equals $\eta_Y(t)$, and (ii) t belongs to the kernel of $F'(\pi_1) - F(\pi_2) : F'(Y) \longrightarrow F'(Y \times_X Y)$. Then the image of t under

$$F'(Y) = F(Y) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Cor}_k,\mathrm{Ab})}(\mathbb{Z}_k^{tr}(Y), F) \longrightarrow \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sm}_k,\mathrm{Ab})}(P(Y), F') \xrightarrow{\eta_Y \circ -} \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sm}_k,\mathrm{Ab})}(P(Y), F'_{\text{ét}})$$

belongs to the kernel of

$$\mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sm}_k,\mathrm{Ab})}(P(Y), F'_{\text{ét}}) \longrightarrow \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sm}_k,\mathrm{Ab})}(P(Y \times_X Y), F'_{\text{ét}}).$$

So it equals the image of a unique element of $\mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sm}_k,\mathrm{Ab})}(P(X), F'_{\text{ét}})$ under

$$\mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sm}_k,\mathrm{Ab})}(P(X), F'_{\text{ét}}) \longrightarrow \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sm}_k,\mathrm{Ab})}(P(Y), F'_{\text{ét}})$$

by Mazza *et al.*, 2006, 6.12, which in particular states that $\mathbb{Z}_k^{tr}(Y \times_X Y) \longrightarrow \mathbb{Z}_k^{tr}(Y) \longrightarrow \mathbb{Z}_k^{tr}(X) \longrightarrow 0$ is exact in $\mathrm{ST}_{\text{ét}}(k)$, so it is sent to a left exact sequence of abelian groups under $\mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sm}_k,\mathrm{Ab})}(-, F'_{\text{ét}})$. Proving that the element of $\mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sm}_k,\mathrm{Ab})}(P(X), F'_{\text{ét}})$ thus obtained only depends on s yields a map φ_X. The claim follows by checking that the φ_X are homomorphisms and that they satisfy the conditions in 2.4.2. $\square$

By proving that for any morphism of presheaves with transfers $\varphi : F \longrightarrow G$ there exists a unique morphism $\varphi_{\text{ét}} : F_{\text{ét}} \longrightarrow G_{\text{ét}}$ such that the diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\varphi"'] & F_{\text{ét}} \arrow[d, "\varphi_{\text{ét}}"] \\ G \arrow[r] & G_{\text{ét}} \end{tikzcd}$$

commutes, it follows that mapping each F to $F_{\text{ét}}$ and each φ to $\varphi_{\text{ét}}$ defines a functor $\pi^{tr*} : \mathrm{PST}(k) \longrightarrow \mathrm{ST}_{\text{ét}}(k)$. Then it is a left adjoint of the inclusion $\pi_*^{tr} : \mathrm{ST}_{\text{ét}}(k) \longrightarrow \mathrm{PST}(k)$, as in particular for any morphism $\varphi : F \longrightarrow G$ in $\mathrm{PST}(k)$ where G is an étale sheaf with transfers, there exists a unique morphism ψ such that the following diagram commutes:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\varphi"'] & F_{\text{ét}} \arrow[d, "\psi"] \\ G \arrow[r, "id"'] & G. \end{tikzcd}$$

We refer to it as the *sheafification* functor. By restricting π^{tr*} to Zariski sheaves with transfers, one obtains a similar adjunction between $\mathrm{ST}_{\mathrm{Zar}}(k)$ and $\mathrm{ST}_{\text{ét}}(k)$; the latter will also be denoted by $\pi^{tr*} \dashv \pi_*^{tr}$.

2.4.2 Change of topology

By Subsection 2.4.1, there exist pairs of adjoint functors

$$\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}) \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k, \mathrm{Ab}),$$

$$\mathrm{ST}_{\mathrm{Zar}}(k) \underset{\pi_*^{tr}}{\overset{\pi^{tr*}}{\rightleftarrows}} \mathrm{ST}_{\text{ét}}(k),$$

where the right adjoints π_*, π_*^{tr} are given by restriction, π^* is the usual sheafification functor for étale sheaves, and π^{tr*} is characterized by the property that for each $\mathscr{F} \in \mathrm{ST}_{\mathrm{Zar}}(k)$, the $\mathscr{F}$-component of the adjunction unit $1_{\mathrm{ST}_{\mathrm{Zar}}(k)} \Longrightarrow \pi_*^{tr}\pi^{tr*}$ restricts along $\mathrm{Sm}_k^{op} \hookrightarrow \mathrm{Cor}_k^{op}$ to the $\mathscr{F}|_{\mathrm{Sm}_k}$-component of the adjunction unit $1_{\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k)} \Longrightarrow \pi_*\pi^*$ (see Theorem 2.4.3). In particular, for each $\mathscr{F} \in \mathrm{ST}_{\text{ét}}(k)$ we have

$$(\pi_*\mathscr{F})|_{\mathrm{Sm}_k} = \pi_*^{tr}(\mathscr{F}|_{\mathrm{Sm}_k}),$$

and for each $\mathscr{F} \in \mathrm{ST}_{\mathrm{Zar}}(k)$ we have

$$(\pi^*\mathscr{F})|_{\mathrm{Sm}_k} = \pi^{tr*}(\mathscr{F}|_{\mathrm{Sm}_k}).$$

Convention 2.4.4. Due to the above remark, we shall abuse notation and drop the superscript '*tr*' and denote π_*^{tr}, π^{tr*} simply by π_*, π^*, respectively. If $\mathscr{F}$ is an object or complex in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab})$ (resp. $\mathrm{ST}_{\mathrm{Zar}}(k)$), its étale sheafification $\pi^*\mathscr{F}$ (resp. $\pi^*\mathscr{F} := \pi^{tr*}\mathscr{F}$) will be denoted by $\mathscr{F}_{\text{ét}}$.

If $\mathscr{F}$ is an object or complex in $\mathrm{ST}_{\text{ét}}(k)$ or $\mathrm{ST}_{\mathrm{Zar}}(k)$, we will often denote its restriction $\mathscr{F}|_{\mathrm{Sm}_k}$ by $\mathscr{F}$ in case it is clear from the context that we are dealing with objects or complexes in $\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k, \mathrm{Ab})$ or $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab})$. In particular, if $\mathscr{F}$ is an object or complex in $\mathrm{ST}_{\text{ét}}(k)$ such that $\mathbf{R}\pi_*(\mathscr{F}|_{\mathrm{Sm}_k})$ is defined (i.e. such that there exists a quasi-isomorphism $\mathscr{F} \longrightarrow \mathscr{I}$ in $\mathrm{Ch}(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k, \mathrm{Ab}))$ with a complex of injectives $\mathscr{I}$), we will denote $\mathbf{R}\pi_*(\mathscr{F}|_{\mathrm{Sm}_k})$ (which belongs to $D(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$) by $\mathbf{R}\pi_*\mathscr{F}$.

Now we will discuss the fact that given $X \in \mathrm{Sm}_k$ and a complex $\mathscr{F} \in \mathrm{Ch}(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k, \mathrm{Ab}))$ which has an injective resolution, any choice of injective resolution $\mathscr{F} \longrightarrow \mathscr{I}$ induces an isomorphism

$$H^*_{\mathrm{Zar}}(X, \mathbf{R}\pi_*\mathscr{F}) \cong H^*_{\text{ét}}(X, \mathscr{F}).$$

Recall that if an additive functor between abelian categories has an exact left adjoint (which is the case for $\pi_* : \mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k, \mathrm{Ab}) \longrightarrow \mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab})$), then it preserves injective objects. Hence for any complex $\mathscr{F} \in \mathrm{Ch}(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k, \mathrm{Ab}))$ endowed with an injective resolution $\rho : \mathscr{F} \xrightarrow{\simeq} \mathscr{I}$ (regarded as an isomorphism in $D^+(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k, \mathrm{Ab}))$), we have isomorphisms

$$\mathbf{R}\pi_*(\mathscr{F}) \xrightarrow[\mathbf{R}\pi_*(\rho)]{\cong} \mathbf{R}\pi_*(\mathscr{I}) \xrightarrow{\cong} \pi_*(\mathscr{I})$$

in $D^+(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$, i.e. $\pi_*(\mathscr{I})$ is an injective resolution of $\mathbf{R}\pi_*(\mathscr{F})$. Now, fix $X \in \mathrm{Sm}_k$, and let $\Gamma_{\mathrm{Zar}} : \mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}) \longrightarrow \mathrm{Ab}$, $\Gamma_{\text{ét}} : \mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k, \mathrm{Ab}) \longrightarrow \mathrm{Ab}$ be the respective functors given by evaluation at X. Since $\Gamma_{\text{ét}} = \Gamma_{\mathrm{Zar}} \circ \pi_*$, we conclude that étale cohomology of $\mathscr{F}$ is isomorphic to Zariski cohomology of $\mathbf{R}\pi_*(\mathscr{F})$:

$$H^i_{\text{ét}}(X, \mathscr{F}) \cong H^i(\Gamma_{\text{ét}}(\mathscr{I})) \cong H^i(\Gamma_{\mathrm{Zar}}(\pi_*(\mathscr{I})) \cong H^i_{\mathrm{Zar}}(X, \mathbf{R}\pi_*(\mathscr{F})). \tag{2.4.3}$$

Moreover, this allows us to produce canonical comparison maps between Zariski and étale cohomology.

If $\mathscr{F}$ is a complex in $\mathrm{Ch}(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$ such that $\pi^*\mathscr{F}$ has an injective resolution, then we may compose the adjunction unit component $\mathscr{F} \longrightarrow \pi_*\pi^*\mathscr{F}$ with $\pi_*(\pi^*\mathscr{F}) \longrightarrow \mathbf{R}\pi_*(\pi^*\mathscr{F})$ – given by applying π_* to an injective resolution of $\pi^*\mathscr{F}$ – to obtain a chain map $\mathscr{F} \longrightarrow \mathbf{R}\pi_*(\pi^*\mathscr{F})$ of complexes of Zariski sheaves. By taking cohomology with respect to a scheme $X \in \mathrm{Sm}_k$, we obtain a homomorphism

$$H^i_{\mathrm{Zar}}(X, \mathscr{F}) \longrightarrow H^i_{\mathrm{Zar}}(X, \mathbf{R}\pi_*(\pi^*\mathscr{F})) \cong H^i_{\text{ét}}(X, \pi^*\mathscr{F}).$$

We will refer to maps of this form as *change of topology* maps.

If $\mathscr{F}$ is moreover a complex of étale sheaves, then the isomorphism $\mathscr{F} \cong \pi^*\mathscr{F}$ induces an isomorphism $H^i_{\text{ét}}(X, \mathscr{F}) \cong H^i_{\text{ét}}(X, \pi^*\mathscr{F})$, so the change of topology map may be identified up to isomorphism with a homomorphism

$$H^i_{\mathrm{Zar}}(X, \mathscr{F}) \longrightarrow H^i_{\text{ét}}(X, \mathscr{F}).$$

2.5 Homotopy invariant (pre)sheaves with transfers

Throughout this section, we let k be a fixed field. For any $X \in \mathrm{Sm}_k$, we denote by $\pi : X \times \mathbb{A}^1_k \longrightarrow X$ the canonical projection, and by $\iota_0, \iota_1 : X \longrightarrow X \times \mathbb{A}^1_k$ the morphisms given by $x \longmapsto (x, 0)$ and $x \longmapsto (x, 1)$, respectively.

Definition 2.5.1. A presheaf with transfers $F \in \mathrm{PST}(k)$ is said to be *homotopy invariant* if for every $X \in \mathrm{Sm}_k$, $F(\pi) : F(X) \longrightarrow F(X \times \mathbb{A}^1_k)$ is an isomorphism. A complex of presheaves with transfers is said to be *homotopy invariant* if the cohomology presheaf with transfers $H^n F$ is homotopy invariant for every integer n.

Remark 2.5.2. If F is a complex of presheaves with transfers, we shall denote by $F_{\mathbb{A}^1_k}$ the complex of presheaves with transfers given by $F_{\mathbb{A}^1_k}(-) = F(- \times \mathbb{A}^1_k)$. This construction extends to a functor $\mathrm{Ch}(\mathrm{PST}(k)) \longrightarrow \mathrm{Ch}(\mathrm{PST}(k))$ given by precomposition with $- \times \mathbb{A}^1_k :$ $\mathrm{Cor}_k \longrightarrow \mathrm{Cor}_k$. Moreover, for each F we have a chain map $\Pi_F : F \longrightarrow F_{\mathbb{A}^1_k}$ given on each integer n and each $X \in \mathrm{Sm}_k$ by the map $F^n(\pi) : F^n(X) \longrightarrow F^n(X \times \mathbb{A}^1_k) = F^n_{\mathbb{A}^1_k}(X)$ induced by the projection $\pi : X \times \mathbb{A}^1_k \longrightarrow X$. It is natural in F, being given by precomposition with the natural transformation $1_{\mathrm{Cor}_k^{op}} \Rightarrow - \times \mathbb{A}^1_k$ of endofunctors on Cor_k^{op} whose X-component is the arrow dual to the canonical projection $X \times \mathbb{A}^1_k \longrightarrow X$.

By construction, a presheaf with transfers F, which we identify with a complex concentrated in degree 0, is homotopy invariant if and only if $\Pi_F : F \longrightarrow F_{\mathbb{A}^1_k}$ is an isomorphism.

Since small limits and colimits in the category $\mathrm{PSh}(\mathrm{Cor}_k, \mathrm{Ab})$ of (not necessarily additive) abelian presheaves are computed pointwise and finite finite (co)products commute with arbitrary (co)limits in Ab, it follows that any (co)limit in $\mathrm{PSh}(\mathrm{Cor}_k, \mathrm{Ab})$ of a small diagram additive presheaves is additive. Hence small (co)limits in $\mathrm{PST}(k)$ exist and are computed pointwise. This implies that small (co)limits, cohomology objects, shifts, and mapping cones in $\mathrm{Ch}(\mathrm{PST}(k))$ commute with the functor $F \longmapsto F_{\mathbb{A}^1_k}$ defined above.

As a consequence, a complex of presheaves with transfers F is homotopy invariant if and only if $\Pi_F : F \longrightarrow F_{\mathbb{A}^1_k}$ is sent to an isomorphism under the cohomology functor H^n for every integer n, i.e. if Π_F is a quasi-isomorphism, or equivalently, if Π_F becomes an isomorphism in $D(\mathrm{PST}(k))$.

Lemma 2.5.3. Suppose $\varphi : F \longrightarrow G$ is a morphism of complexes of presheaves with transfers (with respect to k) such that:

(i) F and G are homotopy invariant.

(ii) For every field extension K/k such that $\mathrm{Spec}\,(K) \in \mathrm{Sm}_k$, i.e. K/k is finite and separable, the map of complexes of abelian groups $\varphi(\mathrm{Spec}\, K) : F(\mathrm{Spec}\, K) \longrightarrow G(\mathrm{Spec}\, K)$ is a quasi-isomorphism.

Then the Zariski sheafification φ_{Zar} of the restriction of F to Sm_k is a quasi-isomorphism in $\mathrm{Ch}(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$.

Proof. In the notation of Remark 2.5.2, we have a morphism

$$\begin{tikzcd}
F \arrow[r, "\varphi"] \arrow[d, "\Pi_F"'] & G \arrow[r] \arrow[d, "\Pi_G"] & \mathrm{Cone}(\varphi) \arrow[r] \arrow[d, "\Pi_{\mathrm{Cone}(\varphi)}"] & F[1] \arrow[d, "\Pi_{F[1]}=\Pi_F[1]"] \\
F_{\mathbb{A}^1_k} \arrow[r, "\varphi_{\mathbb{A}^1_k}"'] & G_{\mathbb{A}^1_k} \arrow[r] & \mathrm{Cone}(\varphi)_{\mathbb{A}^1_k} = \mathrm{Cone}(\varphi_{\mathbb{A}^1_k}) \arrow[r] & F[1]_{\mathbb{A}^1_k} = F_{\mathbb{A}^1_k}[1].
\end{tikzcd}$$

of distinguished triangles in $D(\mathrm{PST}(k))$. Also by Remark 2.5.2, assumption (i) implies that Π_F and Π_G are isomorphisms in $D(\mathrm{PST}(k))$. By the five lemma, $\Pi_{\mathrm{Cone}(\varphi)}$ is also an isomorphism in $D(\mathrm{PST}(k))$, so $\mathrm{Cone}(\varphi)$ is a homotopy invariant complex of presheaves with transfers. Moreover, assumption (ii) implies that for each integer n, the cohomology presheaf with transfers $H^n\mathrm{Cone}(\varphi)$ vanishes on the spectrum of every finite separable extension of k. By Mazza *et al.*, 2006, Corollary 11.2, $H^n_{\mathrm{Zar}}(\mathrm{Cone}(\varphi)_{\mathrm{Zar}}) = (H^n_{\mathrm{PST}(k)}\mathrm{Cone}(\varphi))_{\mathrm{Zar}}$ is trivial. Hence $\mathrm{Cone}(\varphi_{\mathrm{Zar}}) \cong \mathrm{Cone}(\varphi)_{\mathrm{Zar}}$ is quasi-isomorphic to 0 in $\mathrm{Ch}(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$, and we conclude that φ_{Zar} is a quasi-isomorphism in $\mathrm{Ch}(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$. ☐

Chapter 3

Milnor K-theory via motivic cohomology; variants of the Bloch-Kato conjecture

In this chapter we outline, following mainly Mazza *et al.*, 2006 and Haesemeyer and C. Weibel, 2019, some preliminary results related to the application of motivic cohomology to the study of the classical comparison problem between Milnor K-theory and Galois cohomology discussed in Chapter 1.

We begin by discussing the characterization of Milnor K-theory groups of a field k as certain motivic cohomology groups of Spec k with respect to the Zariski topology. Then we define Voevodsky's triangulated category of mixed motives over a field and state the property of motivic cohomology being representable in it. In the last section we study how the norm residue homomorphism may be described as a change of topology map from Zariski to étale motivic cohomology groups of the given field. We conclude with a succinct exposition, based on the first two chapters of Haesemeyer and C. Weibel, 2019, of some results relating the Bloch-Kato, Beilinson-Lichtenbaum and generalized Hilbert 90 conditions.

3.1 Milnor K-theory as motivic cohomology

In this section, which is based on Mazza *et al.*, 2006, Lecture 5, we aim to compute the motivic cohomology groups $H^{n,n}(\mathrm{Spec}\ k, \mathbb{Z})$ of a given field k in bidegrees of the form (n, n) for $n \geq 0$. More precisely, it will follow from the study of these groups that they are canonically isomorphic to the Milnor K-groups of k; moreover, such isomorphisms will be compatible with the corresponding multiplicative structures (Theorem 3.1.6).

Convention 3.1.1. Throughout this section, S will be used as an abbreviation for the k-scheme $\mathbb{A}^1_k \setminus \{0\}$.

We start by recalling that for $q = 0, 1$, previous results provide a characterization of motivic cohomology groups of the form $H^{p,q}(\mathrm{Spec}\ k, \mathbb{Z})$:

(i) As in Remark 2.2.8, we have $\mathbb{Z}(0) = C_*(\mathbb{Z}_k^{tr}(\operatorname{Spec} k)) \simeq \mathbb{Z}_k^{tr}(\operatorname{Spec} k)$. Hence for each scheme $X \in \mathrm{Sm}_k$ and $p \in \mathbb{Z}$,

$$H^{p,0}(X,\mathbb{Z}) \cong H^p_{\mathrm{Zar}}(X, C_*(\mathbb{Z}_k^{tr}(\operatorname{Spec} k))|_{X_{\mathrm{Zar}}}) \cong H^p_{\mathrm{Zar}}(X, \mathbb{Z}_k^{tr}(\operatorname{Spec} k)|_{X_{\mathrm{Zar}}}),$$

and in particular

$$H^{0,0}(X,\mathbb{Z}) \cong H^0_{\mathrm{Zar}}(X, \mathbb{Z}_k^{tr}(\operatorname{Spec} k)|_{X_{\mathrm{Zar}}}) \cong \mathbb{Z}_k^{tr}(\operatorname{Spec} k)(X) = \mathrm{Cor}_k(X, \operatorname{Spec} k) \cong \mathbb{Z}^{c(X)},$$

where $c(X)$ denotes the (finite) set of connected components of X. Thus for Spec k we obtain

$$H^{0,0}(\operatorname{Spec} k, \mathbb{Z}) \cong \mathbb{Z}.$$

(ii) By Proposition 2.3.5, we have $\mathbb{Z}(1) \simeq \mathcal{O}^\times$. So (as in Proposition 2.3.5) for each $X \in \mathrm{Sm}_k$ and $p \in \mathbb{Z}$,

$$H^{p,1}(X,\mathbb{Z}) = H^p_{\mathrm{Zar}}(X, \mathbb{Z}(1)|_{X_{\mathrm{Zar}}}) \cong H^p_{\mathrm{Zar}}(X, \mathcal{O}^\times[-1]) \cong H^{p-1}_{\mathrm{Zar}}(X, \mathcal{O}^\times),$$

and in particular

$$H^{1,1}(X,\mathbb{Z}) \cong H^0_{\mathrm{Zar}}(X, \mathcal{O}^\times) \cong \mathcal{O}^\times(X).$$

For Spec k we obtain

$$H^{1,1}(\operatorname{Spec} k, \mathbb{Z}) \cong k^\times.$$

Denoting by $\tau : k^\times \xrightarrow{\cong} H^{1,1}(\operatorname{Spec} k, \mathbb{Z})$ the latter isomorphism, we obtain a graded ring homomorphism

$$\tau_* : T(k^\times) \longrightarrow \bigoplus_{n \geq 0} H^{n,n}(\operatorname{Spec} k, \mathbb{Z})$$

which by construction satisfies $\tau_n(a_1 \otimes \cdots \otimes a_n) = \tau(a_1) \cdots \tau(a_n)$ for $a_1, \ldots a_n \in k^\times$. We may then ask whether τ_* fails to be an isomorphism and, if it does, what is its (co)kernel. Following Mazza *et al.*, 2006, we denote $\tau_n(a_1 \otimes \cdots \otimes a_n)$ as above by $[a_1, \ldots, a_n]$. Next we discuss an alternative description of τ_* which allows us to explicitly describe $[a_1, \ldots, a_n]$ in terms of the definition of motivic cohomology for arbitrary n.

One key aspect of the spectrum of a field – which is absent for general schemes – that allows us to study its motivic cohomology (or the Zariski cohomology of other complexes of sheaves) in higher degrees is the fact that its Zariski site $(\operatorname{Spec} k)_{\mathrm{Zar}}$ is equivalent to the lattice $\{\varnothing \subset \operatorname{Spec} k\}$ of open subsets of Spec k, so the global section functor $\Gamma(\operatorname{Spec} k, -) : \mathrm{Sh}_{\mathrm{Zar}}(\operatorname{Spec} k, \mathrm{Ab}) \longrightarrow \mathrm{Ab}$ (given by evaluation at Spec k) is an equivalence of categories.

Thus for each $q \in \mathbb{Z}$, the restriction of the motivic complex $\mathbb{Z}(q)$ to the Zariski site of Spec k corresponds to the complex of abelian groups given by evaluation at Spec k:

$$C_*\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q})[-q](\operatorname{Spec} k).$$

Moreover, its hypercohomology with respect to $\Gamma(\operatorname{Spec} k, -)$ is given by the usual cohomology groups

$$H^p(C_*\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q})[-q](\mathrm{Spec}\ k)) \cong H^{p-q}(C_*\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q})(\mathrm{Spec}\ k)).$$

In other words, for all $p, q \in \mathbb{Z}$ we have an isomorphism

$$H^{p,q}(\mathrm{Spec}\ k, \mathbb{Z}) \cong H^{p-q}(C_*\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q})(\mathrm{Spec}\ k)). \tag{3.1.1}$$

In particular, motivic cohomology in bidegrees (n, n) for $n \geq 0$ acquire the following convenient form:

$$H^{n,n}(\mathrm{Spec}\ k, \mathbb{Z}) \cong H^0(C_*\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge n})(\mathrm{Spec}\ k)) \cong \mathrm{Coker}\left(\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge n})(\mathbb{A}_k^1) \xrightarrow{\partial_0 - \partial_1} \mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge n})(\mathrm{Spec}\ k) \right). \tag{3.1.2}$$

In what follows, it will be necessary to use a more explicit description of the complexes of abelian groups $C_*\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q})(\mathrm{Spec}\ k)$ which compute the (Zariski) motivic cohomology of Spec k. We begin with the following lemma:

Lemma 3.1.2. For any $X \in \mathrm{Sm}_k$, $\mathbb{Z}_k^{tr}(X)(\mathrm{Spec}\ k) = \mathrm{Cor}_k(\mathrm{Spec}\ k, X)$ is the subgroup of $\mathcal{Z}(\mathrm{Spec}\ k \times_k X) \cong \mathcal{Z}(X)$ generated by the closed points of $\mathrm{Spec}\ k \times_k X \cong X$.

Proof. If the integral closed subscheme corresponding to a point $x \in X$ is finite (and surjective) over Spec k, then it is zero-dimensional, so x is closed. Conversely, if x is closed, then its residue field is finitely generated as a k-algebra, hence by Zariski's lemma it is finite as a k-vector space. □

Now, given $q \geq 0$ and an integer $i \leq q$, we obtain that the i-th entry of $\mathbb{Z}(q)(\mathrm{Spec}\ k)$ is

$$\begin{aligned} \mathbb{Z}(q)^i(\mathrm{Spec}\ k) &= \mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q})(\mathrm{Spec}\ k \times \Delta^{q-i}) \\ &= \mathrm{Coker}\Bigg(\bigoplus_{j=1}^{q} \mathrm{Cor}_k(\mathrm{Spec}\ k \times \Delta^{q-i}, S^{(1)} \times \cdots \times \mathrm{Spec}\ k^{(j)} \times \cdots \times S^{(q)}) \longrightarrow \\ &\qquad \xrightarrow{\bigoplus_j \mathrm{Cor}_k(\mathrm{Spec}\ k \times \Delta^{q-i}, id, \ldots, s_1, \ldots, id)} \mathrm{Cor}_k(\mathrm{Spec}\ k \times \Delta^{q-i}, S^{(1)} \times \cdots \times S^{(q)}) \Bigg) \end{aligned}$$

$$\cong \mathrm{Coker}\left(\bigoplus_{j=1}^{q} \mathrm{Cor}_k(\Delta^{q-i}, S^{(1)} \times \cdots \times \mathrm{Spec}\ k^{(j)} \times \cdots \times S^{(q)}) \xrightarrow{\bigoplus_j \mathrm{Cor}_k(\Delta^{q-i}, id, \ldots, s_1, \ldots, id)} \mathrm{Cor}_k(\Delta^{q-i}, S^{(1)} \times \cdots \times S^{(q)}) \right).$$

In order to compute $H^{q,q}(\mathrm{Spec}\ k, \mathbb{Z})$, we need the particular cases $i = q-1, q, q+1$, which are given by

$$\mathbb{Z}(q)^{q-1}(\mathrm{Spec}\ k) \cong \mathrm{Coker}\left(\bigoplus_{j=1}^{q} \mathrm{Cor}_k(\Delta^1, S^{(1)} \times \cdots \times \mathrm{Spec}\ k^{(j)} \times \cdots \times S^{(q)}) \xrightarrow{\bigoplus_j \mathrm{Cor}_k(\Delta^1, id, \ldots, s_1, \ldots, id)} \mathrm{Cor}_k(\Delta^1, S^{(1)} \times \cdots \times S^{(q)}) \right),$$

$$\mathbb{Z}(q)^{q}(\mathrm{Spec}\ k) \cong \mathrm{Coker}\left(\bigoplus_{j=1}^{q} \mathrm{Cor}_k(\Delta^0, S^{(1)} \times \cdots \times \mathrm{Spec}\ k^{(j)} \times \cdots \times S^{(q)}) \xrightarrow{\bigoplus_j \mathrm{Cor}_k(\Delta^0, id, \ldots, s_1, \ldots, id)} \mathrm{Cor}_k(\Delta^0, S^{(1)} \times \cdots \times S^{(q)}) \right),$$

$$\mathbb{Z}(q)^{q+1}(\mathrm{Spec}\ k) \cong 0.$$

Thus $H^{q,q}(\mathrm{Spec}\ k, \mathbb{Z})$ is the cokernel of

$$\partial\ :\ \mathbb{Z}(q)^{q-1}(\mathrm{Spec}\ k) \longrightarrow \mathbb{Z}(q)^{q}(\mathrm{Spec}\ k),$$

where ∂ is induced by the face maps ∂^0 and ∂^1 as in the commutative diagram

$$\begin{tikzcd}
\bigoplus_{j=1}^{q} \mathrm{Cor}_k(\Delta^1, S^{(1)} \times \cdots \times \mathrm{Spec}\ k^{(j)} \times \cdots \times S^{(q)}) \arrow[r] \arrow[d, "\partial^0-\partial^1"'] & \mathrm{Cor}_k(\Delta^1, S^{(1)} \times \cdots \times S^{(q)}) \arrow[r] \arrow[d, "\partial^0-\partial^1"'] & \mathbb{Z}(q)^{q-1}(\mathrm{Spec}\ k) \arrow[d, dashed, "\partial"] \\
\bigoplus_{j=1}^{q} \mathrm{Cor}_k(\Delta^0, S^{(1)} \times \cdots \times \mathrm{Spec}\ k^{(j)} \times \cdots \times S^{(q)}) \arrow[r] & \mathrm{Cor}_k(\Delta^0, S^{(1)} \times \cdots \times S^{(q)}) \arrow[r] & \mathbb{Z}(q)^{q}(\mathrm{Spec}\ k).
\end{tikzcd}$$

As a first step towards comparing Milnor K-groups with motivic cohomology groups of k, note that this provides for each $q \geq 0$ a map of abelian groups

$$(k^\times)^q \longrightarrow H^{q,q}(\mathrm{Spec}\ k, \mathbb{Z}) \tag{3.1.3}$$

given by the composite

$$(k^\times)^q \cong \mathrm{Sm}_k(\mathrm{Spec}\ k, S^q) \longrightarrow \mathrm{Cor}_k(\mathrm{Spec}\ k, S^q) \cong \mathrm{Cor}_k(\Delta^0, S^q)$$

$$\longrightarrow \mathbb{Z}(q)^q(\mathrm{Spec}\ k) \longrightarrow H^{q,q}(\mathrm{Spec}\ k, \mathbb{Z}),$$

where the first map identifies k-tuples of units in k with k-valued points of S^q (which are classified by k-algebra homomorphisms $k[t,t^{-1}]^{\otimes q} \longrightarrow k$), the second one sends morphisms of schemes to finite correspondences via the graph functor, the third one is defined by $\mathrm{Spec}\ k \cong \Delta^0$, and the fourth and fifth ones are given by the above construction.

Let us denote this map by τ'_q. It follows from Remark 2.2.8 and Corollary Mazza *et al.*, 2006 that τ'_0 and τ'_1 are isomorphisms. By definition of the multiplicative structure on motivic cohomology (Mazza *et al.*, 2006, 3.11, 3.12), the map of abelian groups

$$\bigoplus_{q\geq 0} \tau'_q\ :\ \bigoplus_{q\geq 0} (k^\times)^q \longrightarrow \bigoplus_{q\geq 0} H^{q,q}(\mathrm{Spec}\ k, \mathbb{Z})$$

defines a graded ring homomorphism $\tau'_*\ :\ T(k^\times) \longrightarrow \bigoplus_{q\geq 0} H^{q,q}(\mathrm{Spec}\ k, \mathbb{Z})$. As τ' coincides in degree 1 with the homomorphism τ_* defined above, the fact that the tensor algebra is generated by degree 1 elements implies $\tau_* = \tau'_*$. This provides the desired characterization of τ_* in terms of the definition of motivic cohomology groups in arbitrary degrees. In particular, this allows us to use finite correspondences to study certain relations between elements of the form $[a_1, \ldots, a_q] \in H^{q,q}(\mathrm{Spec}\ k, \mathbb{Z})$.

3.1.1 A comparison map in motivic cohomology associated to a finite field extension

In order to study the motivic cohomology groups $H^{q,q}(\operatorname{Spec} k, \mathbb{Z})$, it will be useful to construct certain comparison maps of the form

$$H^{q,q}(\operatorname{Spec} l, \mathbb{Z}) \longrightarrow H^{q,q}(\operatorname{Spec} k, \mathbb{Z}),$$

where l is a finite extension of k. The idea, further explored below, is that while $H^{q,q}(\operatorname{Spec} k, \mathbb{Z})$ may not be generated by elements of the form $[a_1, \ldots, a_q]$ for $a_1, \ldots, a_q \in k^\times$, it is generated by the images under such comparison maps of elements of the form $[b_1, \ldots, b_q] \in H^{q,q}(\operatorname{Spec} l, \mathbb{Z})$ for $b_1, \ldots, b_l \in l^\times$.

Construction 3.1.3. Let $i : k \hookrightarrow l$ be a finite field extension. Then $\operatorname{Spec} i : \operatorname{Spec} l \longrightarrow \operatorname{Spec} k$ is a finite, hence proper, morphism of schemes. It is natural to ask whether this gives rise to a notion of proper pushforward between finite l-correspondences and finite k-correspondences.

For that purpose, suppose given $f : X \longrightarrow \operatorname{Spec} k$, $g : Y \longrightarrow \operatorname{Spec} k$ in Sm_k, and let us consider the commutative diagram (in which we denote $\operatorname{Spec} i$ by i, and canonical projections by π)

$$\begin{tikzcd}
X_l \times_l Y_l \arrow[r, "\pi"] \arrow[d, "f^*(i)\times_i g^*(i)"'] & X_l \arrow[r, "i^*(f)"] \arrow[d, "f^*(i)"] & \operatorname{Spec} l \arrow[d, "i"] \\
X \times_k Y \arrow[r, "\pi"'] & X \arrow[r, "f"'] & \operatorname{Spec} k,
\end{tikzcd}$$

where both inner squares and the outer one are pullbacks. Since $\operatorname{Spec} i$ is finite, hence proper, so is $f^*(i) \times_i g^*(i)$. Let us denote this morphism by p. This allows us to define a pushforward map of abelian groups

$$p_* : \mathcal{Z}(X_l \times_l Y_l) \longrightarrow \mathcal{Z}(X \times_k Y)$$

given on generators as follows: for any irreducible closed subset $Z \subset X_l \times_l Y_l$, its image along p is closed in $X \times_k Y$ (by properness) and irreducible (by continuity); the associated cycle $[Z]_{X_l \times_l Y_l}$ is sent to

$$[\kappa(Z) : \kappa(f(Z))] \cdot [p(Z)]_{X \times_k Y}.$$

Let us now show that p_* sends finite l-correspondences to finite k-correspondences. Suppose given a generating finite l-correspondence of the form $[Z]_{X_l \times_l Y_l}$ as above. This means that by endowing Z with the reduced closed subscheme structure, the composite

$$Z \hookrightarrow X_l \times_l Y_l \xrightarrow{\pi} X_l$$

is a finite morphism and its image is an irreducible component of X_l. Moreover, since $\operatorname{Spec} i : \operatorname{Spec} l \longrightarrow \operatorname{Spec} k$ is flat and flatness is preserved under base change, $f^*(i) : X_l \longrightarrow X$ is also flat. By GROTHENDIECK, 1965, Cor. 2.3.5(ii), it follows that the image, say $W \subset X$, of the composite

$$Z \hookrightarrow X_l \times_l Y_l \xrightarrow{\pi} X_l \xrightarrow{f^*(i)} X$$

is such that $\overline{W}$ is an irreducible component of X. On the other hand, $f^*(i)$ is finite (being a base change of the finite morphism Spec (i)), hence proper, hence in particular closed. Thus $W = \overline{W}$, and W is an irreducible component of X. Furthermore, since finite morphisms are stable under composition, we have that

$$Z \hookrightarrow X_l \times_l Y_l \xrightarrow{\pi} X_l \xrightarrow{f^*(i)} X$$

is finite. Now, since this equals the composite

$$Z \hookrightarrow X_l \times_l Y_l \xrightarrow{p} X \times_k Y \xrightarrow{\pi} X,$$

it follows from Lemma 2.1.3 that the integral closed subscheme $p(Z) \longrightarrow X \times_k Y$ has the property that the composite

$$p(Z) \longrightarrow X \times_k Y \xrightarrow{\pi} X$$

is finite and its image is an irreducible component of X. Thus $[p(Z)]_{X \times_k Y}$ is a generating finite k-correspondence, and the proper pushforward $[\kappa(Z) : \kappa(f(Z)] \cdot [p(Z)]_{X \times_k Y}$ is a finite k-correspondence. This defines a map of abelian groups

$$i_! : \mathrm{Cor}_l(X_l, Y_l) \longrightarrow \mathrm{Cor}_k(X, Y).$$

By construction, it is given by restriction of the proper pushforward map p_* considered above; by abuse of notation, we will sometimes also denote it by p_*.

Let us now consider the following setting, from which we will establish a useful naturality property for maps of the form $i_!$: suppose given $X, Y, Z \in \mathrm{Sm}_k$, and let us denote the corresponding base field change maps by

$$p : X_l \times_l Y_l \longrightarrow X \times_k Y,$$

$$q : Y_l \times_l Z_l \longrightarrow Y \times_k Z,$$

$$r : X_l \times_l Z_l \longrightarrow X \times_k Z,$$

$$s : X_l \times_l Y_l \times_l Z_l \longrightarrow X \times_k Y \times_k Z.$$

We now prove that changing the base field is compatible with composition of correspondences in the sense that the diagram

$$\begin{tikzcd}
\mathrm{Cor}_l(X_l, Y_l) \arrow[r, "p_*"] \arrow[d, "q^*(\beta)\circ -"'] & \mathrm{Cor}_k(X, Y) \arrow[d, "\beta\circ -"] \\
\mathrm{Cor}_l(X_l, Z_l) \arrow[r, "r_*"'] & \mathrm{Cor}_k(X, Z)
\end{tikzcd}$$

commutes for each $\beta \in \mathrm{Cor}_k(Y, Z)$. We proceed by taking any $\alpha \in \mathrm{Cor}_l(X_l, Y_l)$, applying both composite maps, and then comparing the two cycles thus obtained by means of the functoriality of pullbacks, the base change formula, and the projection formula.

The left vertical map sends α to $q^*(\beta) \circ \alpha$, which is given explicitly by

$$\pi^{X_l Y_l Z_l}_{X_l Z_l *}(\pi^{X_l Y_l Z_l *}_{X_l Y_l}(\alpha) \cdot \pi^{X_l Y_l Z_l *}_{Y_l Z_l} q^*(\beta)).$$

It is sent by r_* to

$$r_* \pi^{X_l Y_l Z_l}_{X_l Z_l *}(\pi^{X_l Y_l Z_l *}_{X_l Y_l}(\alpha) \cdot \pi^{X_l Y_l Z_l *}_{Y_l Z_l} q^*(\beta)). \tag{3.1.4}$$

On the other hand, α is sent by the top horizontal map to $p_*(\alpha)$, which is in turn sent by the right vertical map to

$$\pi^{XYZ}_{XZ*}(\pi^{XYZ*}_{XY} p_*(\alpha) \cdot \pi^{XYZ*}_{YZ}(\beta)). \tag{3.1.5}$$

Since $r_* \pi^{X_l Y_l Z_l}_{X_l Z_l *} = (r \circ \pi^{X_l Y_l Z_l}_{X_l Z_l})_* = (\pi^{XYZ}_{XZ} \circ s)_* = \pi^{XYZ}_{XZ*} s_*$, we have that 3.1.4 equals

$$\pi^{XYZ}_{XZ*} s_*(\pi^{X_l Y_l Z_l *}_{X_l Y_l}(\alpha) \cdot \pi^{X_l Y_l Z_l *}_{Y_l Z_l} q^*(\beta)).$$

Thus by comparing it with 3.1.5, it suffices to show that

$$s_*(\pi^{X_l Y_l Z_l *}_{X_l Y_l}(\alpha) \cdot \pi^{X_l Y_l Z_l *}_{Y_l Z_l} q^*(\beta)) = \pi^{XYZ*}_{XY} p_*(\alpha) \cdot \pi^{XYZ*}_{YZ}(\beta).$$

But $\pi^{X_l Y_l Z_l *}_{Y_l Z_l} q^* = (\pi^{X_l Y_l Z_l}_{Y_l Z_l} \circ q)^* = (s \circ \pi^{XYZ}_{YZ})^* = s^* \pi^{XYZ*}_{YZ}$, so the left hand side equals

$$s_*(\pi^{X_l Y_l Z_l *}_{X_l Y_l}(\alpha) \cdot s^* \pi^{XYZ*}_{YZ}(\beta)),$$

which by the projection formula equals

$$s_* \pi^{X_l Y_l Z_l *}_{X_l Y_l}(\alpha) \cdot \pi^{XYZ*}_{YZ}(\beta).$$

The claim then follows by using the base change formula $s_* \pi^{X_l Y_l Z_l *}_{X_l Y_l} = \pi^{XYZ*}_{XY} p_*$.

An analogous computation shows that the diagram

$$\begin{tikzcd}
\mathrm{Cor}_l(Y_l, Z_l) \arrow[r, "q_*"] \arrow[d, "- \circ p^*(\alpha)"'] & \mathrm{Cor}_k(Y, Z) \arrow[d, "- \circ \alpha"] \\
\mathrm{Cor}_l(X_l, Z_l) \arrow[r, "r_*"'] & \mathrm{Cor}_k(X, Z)
\end{tikzcd}$$

commutes for each $\alpha \in \mathrm{Cor}_k(X, Y)$.

Construction 3.1.4. As above, let $i : k \hookrightarrow l$ be a finite field extension. We will now use the previous construction to compare certain chain complexes of abelian groups obtained from the complexes of presheaves with transfers $\mathbb{Z}(q)$ for the two fields. In what follows, we shall denote such complexes by $\mathbb{Z}(q)_k$ and $\mathbb{Z}(q)_l$ when necessary, in order to avoid ambiguity. We recall that the base change functor $(\mathrm{Spec}\, i)^* : \mathrm{Sm}_k \longrightarrow \mathrm{Sm}_l$: preserves finite products; sends $\mathrm{Spec}\, k$ to $\mathrm{Spec}\, l$; sends the algebraic n-simplices over k, Δ^n_k, to the respective ones over l, Δ^n_l; sends the scheme $\mathbb{A}^1_k \setminus \{0\}$ to $\mathbb{A}^1_l \setminus \{0\}$. By using the above naturality property for maps $i_!$, this allows us to define for each $X \in \mathrm{Sm}_k$, $q \geq 0$, a chain map of abelian groups

$$\mathbb{Z}(q)_l(X_l) \longrightarrow \mathbb{Z}(q)_k(X).$$

We will be mainly interested in the case $X = \mathrm{Spec}\, k$, in which we have $X_l \cong \mathrm{Spec}\, l$: the above chain maps then yield, by varying q and taking the p-th cohomology group for each integer p, comparison homomorphisms

$$H^{p,q}(\mathrm{Spec}\ l, \mathbb{Z}) \longrightarrow H^{p,q}(\mathrm{Spec}\ k, \mathbb{Z}).$$

In particular, for each $q \geq 0$ we obtain a map

$$H^{q,q}(\mathrm{Spec}\ l, \mathbb{Z}) \longrightarrow H^{q,q}(\mathrm{Spec}\ k, \mathbb{Z}), \tag{3.1.6}$$

which is called the *norm map* and is denoted by $N_{l/k}$ due to the following properties:

Lemma 3.1.5 (Mazza *et al.*, 2006, 5.3)**.**

(i) $N^0_{l/k}$ equals the composite

$$H^{0,0}(\mathrm{Spec}\ l, \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\cdot[l:k]} \mathbb{Z} \cong H^{0,0}(\mathrm{Spec}\ k, \mathbb{Z}).$$

(ii) $N^1_{l/k}$ equals the composite

$$H^{1,1}(\mathrm{Spec}\ l, \mathbb{Z}) \cong l^\times \xrightarrow{N_{l/k}} \mathbb{k}^\times \cong H^{1,1}(\mathrm{Spec}\ k, \mathbb{Z}),$$

where $N_{l/k}$ denotes the usual norm map.

(iii) For any $x \in H^{*,*}(\mathrm{Spec}\ l, \mathbb{Z})$ and $y \in H^{*,*}(\mathrm{Spec}\ k, \mathbb{Z})$, $N^*_{l/k} : H^{*,*}(\mathrm{Spec}\ l, \mathbb{Z}) \longrightarrow H^{*,*}(\mathrm{Spec}\ k, \mathbb{Z})$ satisfies

$$N^*_{l/k}(y_l \cdot x) = y \cdot N^*_{l/k}(x).$$

(iv) If L is an extension of l which is normal over k, then for any $x \in H^{*,*}(\mathrm{Spec}\ l, \mathbb{Z})$ it holds that

$$N_{l/k}(x)_L = [l : k]_{insep} \sum_{i \in \mathrm{Hom}_k(l,L)} i^*(x).$$

Steinberg relations in $H^{q,q}(\mathrm{Spec}\ k, \mathbb{Z})$, and the comparison with Milnor K-theory

We sketch the proof given in Mazza *et al.*, 2006, 5.9 that motivic cohomology with integral coefficients satisfies a suitable analogue of the Steinberg relations. More precisely, it may be proved that $[a, 1-a]$ is the zero element of $H^{2,2}(\mathrm{Spec}\ k, \mathbb{Z})$ for any $a \in k \backslash \{0, 1\}$.

By using coordinates $\mathbb{A}^1 \cong \mathrm{Spec}\ k[x]$ and $\mathbb{A}^1 \setminus \{0\} \cong \mathrm{Spec}\ k[y, y^{-1}]$, one defines a finite k-correspondence from $\mathbb{A}^1$ to $\mathbb{A}^1 \setminus \{0\}$ as the zero set of

$$y^3 - (a^3 + 1)xy^2 + (a^3 + 1)xy - a^3$$

in $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\})$. Its images under

$$\partial_0, \partial_1 : \mathrm{Cor}_k(\mathbb{A}^1, \mathbb{A}^1 \setminus \{0\}) \longrightarrow \mathrm{Cor}_k(\mathrm{Spec}\ k, \mathbb{A}^1 \setminus \{0\})$$

correspond, by taking $x = 0$ and $x = 1$, respectively, to the zero sets of

$$y^3 - a^3,$$

$$y^3 - (a^3 + 1)y^2 + (a^3 + 1)y - a^3$$

in $\mathbb{A}^1 \setminus \{0\}$. These correspondences are more easily described in a field extension where both polynomials split into linear factors. Let l be obtained by adjoining (if necessary) a cube root of unity α such that $\{1, \alpha, \alpha^2\}$ is the set of all cube roots of unity, or equivalently such that $\alpha^2 + \alpha + 1 = 0$. In particular l/k has degree 1 or 2. Now, the above polynomials split in l as

$$(y - a)(y - \alpha a)(y - \alpha^2 a),$$

$$(y - a^3)(y + \alpha)(y + \alpha^2),$$

respectively. Their associated elements in $\mathrm{Cor}_k(\mathrm{Spec}\ l, \mathbb{A}^1 \setminus \{0\})$ are

$$\{a\} + \{\alpha a\} + \{\alpha^2 a\},$$

$$\{a^3\} + \{-\alpha\} + \{-\alpha^2\}.$$

Now, let us consider the morphism

$$\begin{aligned} s \,:\, \mathbb{A}^1 \setminus \{0, 1\} &\longrightarrow (\mathbb{A}^1 \setminus \{0\})^2 \\ x &\longmapsto (x, 1 - x). \end{aligned}$$

in Sm_k. Composition with s defines a homomorphism

$$\mathrm{Cor}_k(\mathrm{Spec}\ k, s) \,:\, \mathrm{Cor}_k(\mathrm{Spec}\ k, \mathbb{A}^1 \setminus \{0, 1\}) \longrightarrow \mathrm{Cor}_k(\mathrm{Spec}\ k, (\mathbb{A}^1 \setminus \{0\})^2)$$

which for each $a \in k \setminus \{0, 1\}$ sends $\{a\}$ (the graph of the morphism $\mathrm{Spec}\ k \longrightarrow \mathbb{A}^1 \setminus \{0, 1\}$ which classifies a) to $\{(a, 1 - a)\}$ (the graph of the composite $\mathrm{Spec}\ k \longrightarrow \mathbb{A}^1 \setminus \{0, 1\} \xrightarrow{s} (\mathbb{A}^1 \setminus \{0\})^2$).

Moreover, note that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Cor}_k(\mathbb{A}^1_l, \mathbb{A}^1 \setminus \{0, 1\}) \arrow[r, "s\circ-"] \arrow[d, "\partial_0 - \partial_1"'] & \mathrm{Cor}_k(\mathbb{A}^1_l, (\mathbb{A}^1 \setminus \{0\})^2) \arrow[d, "\partial_0 - \partial_1"] \\ \mathrm{Cor}_k(\mathrm{Spec}\ l, \mathbb{A}^1 \setminus \{0, 1\}) \arrow[r, "s\circ-"'] & \mathrm{Cor}_k(\mathrm{Spec}\ l, (\mathbb{A}^1 \setminus \{0\})^2). \end{tikzcd}$$

Hence any element of the image of the composite

$$\mathrm{Cor}_k(\mathbb{A}^1_l, \mathbb{A}^1 \setminus \{0, 1\}) \xrightarrow{\partial_0 - \partial_1} \mathrm{Cor}_k(\mathrm{Spec}\ l, \mathbb{A}^1 \setminus \{0, 1\}) \xrightarrow{s\circ-} \mathrm{Cor}_k(\mathrm{Spec}\ l, (\mathbb{A}^1 \setminus \{0\})^2)$$

is sent to zero in $H^{2,2}(\mathrm{Spec}\ l, \mathbb{Z})$, as it belongs to the image of

$$\mathrm{Cor}_k(\mathrm{Spec}\ l, (\mathbb{A}^1 \setminus \{0\})^2) \xrightarrow{\partial_0 - \partial_1} \mathrm{Cor}_k(\mathrm{Spec}\ l, (\mathbb{A}^1 \setminus \{0\})^2).$$

Thus in our case it holds that

$$\{a, 1-a\} + \{\alpha a, 1-\alpha a\} + \{\alpha^2 a, 1-\alpha^2 a\},$$

$$\{a^3, 1-a^3\} + \{-\alpha, 1+\alpha\} + \{-\alpha^2, 1+\alpha^2\}$$

are elements of $\mathrm{Cor}_k(\mathrm{Spec}\, l, (\mathbb{A}^1 \setminus \{0\})^2)$ whose images in $H^{2,2}(\mathrm{Spec}\, l, \mathbb{Z})$ are equal.

We compute

$$[a, 1-a] + [\alpha a, 1-\alpha a] + [\alpha^2 a, 1-\alpha^2 a] =$$

$$\begin{aligned}
&= [a, 1-a] + ([\alpha, 1-\alpha a] + [a, 1-\alpha a]) + ([\alpha^2, 1-\alpha^2 a] + [a, 1-\alpha^2 a]) \\
&= [a, (1-a)(1-\alpha a)(1-\alpha^2 a)] + [\alpha, 1-\alpha a] + 2[\alpha, 1-\alpha^2 a] \\
&= [a, 1-a^3] + [\alpha, (1-\alpha)(1-\alpha^2 a)^2],
\end{aligned}$$

so it follows that

$$\begin{aligned}
3([a, 1-a] + [\alpha a, 1-\alpha a] + [\alpha^2 a, 1-\alpha^2 a]) &= 3[a, 1-a^3] + 3[\alpha, (1-\alpha)(1-\alpha^2 a)^2] \\
&= [a^3, 1-a^3] + [\alpha^3, (1-\alpha)(1-\alpha^2 a)^2] \\
&= [a^3, 1-a^3] + [1, (1-\alpha)(1-\alpha^2 a)^2] \\
&= [a^3, 1-a^3].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
3([a^3, 1-a^3] + [-\alpha, 1+\alpha] + [-\alpha^2, 1+\alpha^2]) &= 3[a^3, 1-a^3] + [(-\alpha)^3, 1+\alpha] + [(-\alpha^2)^3, 1+\alpha^2]) \\
&= 3[a^3, 1-a^3] + [-1, 1+\alpha] + [-1, 1+\alpha^2]) \\
&= 3[a^3, 1-a^3] + [-1, (1+\alpha)(1+\alpha^2)] \\
&= 3[a^3, 1-a^3] + [-1, 1+(\alpha+\alpha^2)+\alpha^3] \\
&= 3[a^3, 1-a^3] + [-1, 1-1+1] \\
&= 3[a^3, 1-a^3] + [-1, 1] \\
&= 3[a^3, 1-a^3].
\end{aligned}$$

Comparing the above results yields

$$2[a^3, 1-a^3] = 0 \in H^{2,2}(\mathrm{Spec}\, l, \mathbb{Z}).$$

We would like to describe $[a, 1-a] \in H^{2,2}(\mathrm{Spec}\, k, \mathbb{Z})$. For that purpose, first we apply the norm map $N_{l/k}$ and obtain

$$0 = N_{l/k}(2[a^3, 1-a^3]) = 2\deg(l/k)[a^3, 1-a^3],$$

which is either $2[a^3, 1-a^3]$ or $4[a^3, 1-a^3]$. In either case we have $4[a^3, 1-a^3] = 0$ in $H^{2,2}(\mathrm{Spec}\, k, \mathbb{Z})$. This proves that $4[b, 1-b] = 0$ whenever $b = a^3$ for some $a \in k \setminus \{0, 1\}$. For the general case, we may work in a field extension where $b \in k \setminus \{0, 1\}$ is a third power. For example, let L be a field extension of k obtained by adjoining an element a such that

$a^3 = b$. Then we have $4[b, 1-b] = 0$ in $H^{2,2}(\operatorname{Spec} L, \mathbb{Z})$ by the above argument, and

$$0 = N_{L/k}(4[b, 1-b]) = 4\deg(L/k)([b, 1-b]).$$

As $\deg(L/k)$ is either 1, 2, or 3, we have[1]

$$24[b, 1-b] = 0 \in H^{2,2}(\operatorname{Spec} k, \mathbb{Z}).$$

We now use an inductive argument (Mazza *et al.*, 2006, 5.8) to show that if some $n \geq 1$ has the property, say $P(n)$, that $n[a, 1-a] = 0$ in $H^{2,2}(\operatorname{Spec} l, \mathbb{Z})$ for all finite extensions l/k and all $a \in l \setminus \{0, 1\}$, then $[a, 1-a] = 0$ in $H^{2,2}(\operatorname{Spec} l, \mathbb{Z})$ for all such l and a. By taking $n = 24$ it will follow that

$$[a, 1-a] = 0 \in H^{2,2}(\operatorname{Spec} k, \mathbb{Z})$$

for all $a \in k \setminus \{0, 1\}$.

If suffices to show that whenever $P(n)$ holds and p is a prime number dividing n, $P(n/p)$ holds. Suppose given a finite extension l/k and $a \in l \setminus \{0, 1\}$. Let L be obtained by adjoining to l an element b such that $b^p = a$. Then

$$\begin{aligned}
(n/p)[a, 1-a] &= (n/p)[b^p, N_{L/l}(1-b)] \\
&= N_{L/l}((n/p)[b^p, 1-b]) \\
&= N_{L/l}((n/p)p[b, 1-b]) \\
&= N_{L/l}(n[b, 1-b]) \\
&= N_{L/l}(0) \\
&= 0,
\end{aligned}$$

as desired.

As $\tau'_* : T(k^\times) \longrightarrow \bigoplus_{q \geq 0} H^{q,q}(\operatorname{Spec} k, \mathbb{Z})$ sends $a \otimes (1-a)$ to $[a, 1-a] = 0$ for all $a \in k \setminus \{0, 1\}$, it defines a graded homomorphism (with notation similar to that in Mazza *et al.*, 2006)

$$\lambda_* : K_M^*(k) \longrightarrow \bigoplus_{q \geq 0} H^{q,q}(\operatorname{Spec} k, \mathbb{Z}).$$

We already know λ_* is an isomorphism in degrees 0 and 1. Let us verify that this is also the case for $q \geq 2$.

This may be done as follows: consider for a given $q \leq 1$ the commutative diagram

$$\begin{tikzcd}
(k^\times)^q \arrow[r, "\omega"] \arrow[d, "\pi'"'] & \operatorname{Cor}_k(\operatorname{Spec} k, S^q) \arrow[d, "\pi"] \\
K_M^q(k) \arrow[r, "\lambda_q"'] & H^{q,q}(\operatorname{Spec} k, \mathbb{Z}),
\end{tikzcd}$$

[1] As in Mazza *et al.*, 2006, one may choose L in such a way that $\deg(L/k)$ is either 1 or 3, as if the minimal polynomial of a over k has degree 2, then k already contains a cube root of b.

where π and π' are the canonical projections, and ω is the composite

$$(k^\times)^q \cong \mathrm{Sm}_k(\mathrm{Spec}\ k, S^q) \longrightarrow \mathrm{Cor}_k(\mathrm{Spec}\ k, S^q).$$

Then one may construct a homomorphism $h : \mathrm{Cor}_k(\mathrm{Spec}\ k, S^q) \longrightarrow K_M^q(k)$ such that:

(1) $\lambda_q \circ h = \pi$. This proves that λ_q is surjective by providing for each element of $H^{q,q}(\mathrm{Spec}\ k, \mathbb{Z})$ one preimage under λ_q for each of its preimages under π.

(2) h factors through π. This provides a unique map $\eta : H^{q,q}(\mathrm{Spec}\ k, \mathbb{Z}) \longrightarrow K_M^q(k)$ such that $h = \eta \circ \pi$. Then it follows from (1) that $\lambda_q \circ \eta \circ \pi = \pi$, hence $\lambda_q \circ \eta = id_{H^{q,q}(\mathrm{Spec}\ k, \mathbb{Z})}$.

(3) For any $a_1, ..., a_q \in k^\times$, it holds that $h(\omega(a_1, ..., a_q))$ equals the Milnor K-theory symbol $\{a_1, ..., a_q\}$. By using η as in (2), this is equivalent to $\eta([a_1, ..., a_q]) = \{a_1, ..., a_q\}$. As $K_M^q(k)$ is generated by symbols of the form $\{a_1, ..., a_q\}$ and λ_q is given by $\{a_1, ..., a_q\} \mapsto [a_1, ..., a_q]$, it follows that $\eta \circ \lambda_q = id_{K_M^q(k)}$.

Thus it would follow from (2) and (3) that λ_q is an isomorphism with inverse η.

We begin by noting that $\mathrm{Cor}_k(\mathrm{Spec}\ k, S^q)$ is freely generated by the finite correspondences from Spec k to $(\mathbb{A}^1 \setminus \{0\})^q$ associated to closed points x of $(\mathbb{A}^1 \setminus \{0\})^q$ (by Zariski's lemma, the residue field of any closed point is a finite extension of k, so it determines a finite correspondence). Note that denoting by l the residue field of x, we may express x as the image of a morphism Spec $l \longrightarrow$ Spec $k \times_k (\mathbb{A}^1 \setminus \{0\})^q \cong (\mathbb{A}^1 \setminus \{0\})^q$ given in coordinates by $(a_1, ..., a_q) \in (l^\times)^q$.

Thus it suffices to define a function from the set of closed points of $(\mathbb{A}^1 \setminus \{0\})^q$ (which we identify with a subset of $\mathrm{Cor}_k(\mathrm{Spec}\ k, S^q)$) to $K_M^q(k)$ such that the induced homomorphism h satisfies (1)-(3) above.

One may proceed by using the above description of closed points of $(\mathbb{A}^1 \setminus \{0\})^q$ and Mazza *et al.*, 2006, 5.11, which states for any finite extension l/k the diagram

$$\begin{tikzcd} K_M^q(l) \arrow[r, "\lambda_q^l"] \arrow[d, "N_{l/k}"'] & H^{q,q}(\mathrm{Spec}\ l, \mathbb{Z}) \arrow[d, "N_{l/k}"] \\ K_M^q(k) \arrow[r, "\lambda_q^k"'] & H^{q,q}(\mathrm{Spec}\ k, \mathbb{Z}) \end{tikzcd}$$

commutes. Hence if we consider the finite correspondence $\{x\}$ from Spec k to $(\mathbb{A}^1 \setminus \{0\})^q$ associated to a closed point $x \in (\mathbb{A}^1 \setminus \{0\})^q$ given in coordinates by $(a_1, ..., a_q) \in (l^\times)^q$, where $l := \kappa(x)$, then $\{x\}$ is the image under the pushforward map

$$\mathrm{Cor}_k(\mathrm{Spec}\ l, (\mathbb{A}^1 \setminus \{0\})^q) \longrightarrow \mathrm{Cor}_k(\mathrm{Spec}\ k, (\mathbb{A}^1 \setminus \{0\})^q)$$

of the finite correspondence $\{y\}$ from Spec l to $(\mathbb{A}^1 \setminus \{0\})^q$ associated to $y = (a_1, ..., a_q) \in$ Spec $l \times_k (\mathbb{A}^1 \setminus \{0\})^q$. But then $\pi(y) \in H^{q,q}(\mathrm{Spec}\ l, \mathbb{Z})$ is the image of $\{a_1, ..., a_q\} \in K_M^q(l)$ under λ_q^l, so commutativity of the above diagram implies that

$$\pi(\{x\}) = \lambda_q^k(N_{l/k}(\{a_1, ..., a_q\})).$$

Let us then define

$$h : \mathrm{Cor}_k(\mathrm{Spec}\, k, S^q) \longrightarrow K_M^q(k)$$

in terms of generators, using the above notation, by

$$\{x\} \longmapsto N_{l/k}(\{a_1, ..., a_q\}).$$

Then (1) holds by construction, and (3) holds since $N_{k/k} = id_{K_M^q(k)}$.

It remains to prove (2). By the description of $H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z})$ as a quotient of $\mathrm{Cor}_k(\mathrm{Spec}\, k, S^q)$, we have that h factors through $\pi : \mathrm{Cor}_k(\mathrm{Spec}\, k, S^q) \longrightarrow H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z})$ if and only if the following two conditions, which from the relations that generate the kernel of π, are satisfied:

(i) The composite

$$\mathrm{Cor}_k(\mathbb{A}^1, S^q) \xrightarrow{\partial_0 - \partial_1} \mathrm{Cor}_k(\mathrm{Spec}\, k, S^q) \xrightarrow{h} K_M^q(k)$$

is the zero map.

(ii) Suppose given an arbitrary generating finite correspondence from Spec k to $(\mathbb{A}^1 \setminus \{0\})^q$, i.e. one associated to a closed point x of $(\mathbb{A}^1 \setminus \{0\})^q$. Denoting by l the residue field of x, let us express x in coordinates by $(a_1, ..., a_q) \in (l^\times)^q$.

If there exists $i \in \{1, ..., q\}$ such that $a_i = 1$, then $h(\{x\}) = 0$.

Item (ii) follows from the fact that if $a_i = 1$ for some i, then $\{a_1, ..., a_q\} = 0 \in K_M^q(l)$, so $h(\{x\}) = N_{l/k}(\{a_1, ..., a_q\}) = N_{l/k}(0) = 0$.

For item (i), we refer to Mazza *et al.*, 2006, where this is proved as a corollary of a theorem due to Suslin (see A. A. Suslin, 1982, 4.4) which establishes a reciprocity law for Milnor K-theory.

Then one obtains the following result:

Theorem 3.1.6. Given a field k and $q \geq 0$, the map

$$\lambda_q : H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z}) \cong K_M^q(k)$$

is an isomorphism of abelian groups. Moreover, the induced map

$$\lambda_* : K_M^*(k) \longrightarrow \bigoplus_{q \geq 0} H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z}).$$

is a ring isomorphism between the Milnor K-theory ring of k and the subring generated by homogeneous elements of bidegree (q, q) of the motivic cohomology ring of k with $\mathbb{Z}$-coefficients.

3.1.2 General coefficients

The characterization of motivic cohomology groups $H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z})$ as in 3.1.2, which uses the fact that the Zariski site of Spec k is trivial, extends to a description of the motivic cohomology groups $H^{q,q}(\mathrm{Spec}\, k, A)$ for an arbitrary abelian group A.

Indeed, we compute

$$\begin{aligned}
H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z}) \otimes_{\mathbb{Z}} A &\cong H^0(C_*\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q})(\mathrm{Spec}\, k)) \otimes_{\mathbb{Z}} A \\
&\cong \mathrm{Coker}\left(\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q})(\mathbb{A}_k^1) \xrightarrow{\partial_0 - \partial_1} \mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q})(\mathrm{Spec}\, k) \right) \otimes_{\mathbb{Z}} A \\
&\cong \mathrm{Coker}\left((\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q})(\mathbb{A}_k^1)) \otimes_{\mathbb{Z}} A \xrightarrow{(\partial_0 - \partial_1)\otimes_{\mathbb{Z}} A} (\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q})(\mathrm{Spec}\, k)) \otimes_{\mathbb{Z}} A \right) \\
&\cong \mathrm{Coker}\left((\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q}) \otimes_{\mathbb{Z}} A)(\mathbb{A}_k^1) \xrightarrow{(\partial_0 \otimes_{\mathbb{Z}} A) - (\partial_1 \otimes_{\mathbb{Z}} A)} (\mathbb{Z}_k^{tr}(\mathbb{G}_m^{\wedge q}) \otimes_{\mathbb{Z}} A)(\mathrm{Spec}\, k) \right) \\
&\cong \mathrm{Coker}\left(A_k^{tr}(\mathbb{G}_m^{\wedge q})(\mathbb{A}_k^1) \xrightarrow{(\partial_0 \otimes_{\mathbb{Z}} A) - (\partial_1 \otimes_{\mathbb{Z}} A)} A_k^{tr}(\mathbb{G}_m^{\wedge q})(\mathrm{Spec}\, k) \right) \\
&\cong H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z}).
\end{aligned}$$

Since these isomorphisms are natural in $A \in \mathrm{Ab}$, we obtain for each $q \leq 0$ an isomorphism of functors

$$H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z}) \otimes - \cong H^{q,q}(\mathrm{Spec}\, k, -)$$

from Ab to Ab.

In particular, suppose given a prime number l different from the characteristic of k. Then we may describe the Milnor K-theory groups of k modulo l, $K_M^q(k)/l$, in terms of motivic cohomology: we have

$$K_M^q(k)/l \cong H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z}) \otimes \mathbb{Z}/l \cong H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z}/l).$$

In fact, this isomorphism is compatible with the projection $\mathbb{Z} \longrightarrow \mathbb{Z}/l$ in the sense that by applying the above isomorphism of functors to $\mathbb{Z} \xrightarrow{l} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/l$, we obtain a commutative diagram

$$\begin{tikzcd}
H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z}) \arrow[r, "l"] \arrow[d, "\cong"'] & H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z}) \arrow[r, "\pi"] \arrow[d, "\cong"] & H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z}/l) \arrow[d, "\cong"] \\
H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[r, "l"] \arrow[d, "\cong"'] & H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "\cong"] & H^{q,q}(\mathrm{Spec}\, k, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/l \arrow[d, "\cong"] \\
K_M^q(k) \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[r, "l"] \arrow[d, "\cong"'] & K_M^q(k) \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[r, "\pi"] \arrow[d, "\cong"] & K_M^q(k) \otimes_{\mathbb{Z}} \mathbb{Z}/l \arrow[d, "\cong"] \\
K_M^q(k) \arrow[r, "l"'] & K_M^q(k) \arrow[r, "\pi"'] & K_M^q(k)/l.
\end{tikzcd}$$

3.2 Voevodsky's mixed motives

Throughout this section, we let k denote a field; the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ will be denoted by $\mathbb{A}^1$, with k implicit, and whenever X, Y are finite type k-schemes, $X \times Y$ will denote the cartesian product of X and Y in Sch_k (rather than in the category of schemes). Moreover, we refer to (pre)sheaves with transfers with respect to k simply as (pre)sheaves with transfers.

Recall that $\mathrm{PST}(k, A)$ denotes the corresponding category of Mod_A-valued presheaves with transfers, and if J is a Grothendieck topology on Sm_k, one denotes by $\mathrm{ST}_J(k, A)$ the category of Mod_A-valued J-sheaves with transfers (see Definition 2.1.11). Given a scheme $X \in \mathrm{Sm}_k$, we have a presheaf with transfers

$$A_k^{tr}(X) = \mathrm{Cor}_k(-, X) \otimes_{\mathbb{Z}} A$$

which is also a J-sheaf with transfers (see Proposition 2.1.13).

$\mathrm{D}^-\mathrm{ST}_J(k, A)$ denotes the derived category of bounded above complexes of sheaves with transfers, i.e. the category obtained by localizing $\mathrm{Ch}^-(\mathrm{ST}_J(k, A))$ of bounded above complexes of J-sheaves with transfers at the (large) set of quasi-isomorphisms.

Convention 3.2.1. Unless otherwise stated, throughout this section J will denote either the Nisnevich or the étale topology on Sm_k.

Definition 3.2.2. Morphisms of (pre)sheaves with transfers of the form $A_k^{tr}(X \times \mathbb{A}^1) \xrightarrow{A^{tr}(\pi)} A^{tr}(X)$, induced by the canonical projection $X \times \mathbb{A}^1 \xrightarrow{\pi} X$ for some $X \in \mathrm{Sm}_k$, will be called *basic* $\mathbb{A}^1$*-weak equivalences* of presheaves with transfers.

A *basic* $\mathbb{A}^1$*-weak equivalence* of J-sheaves with transfers is defined to be a basic $\mathbb{A}^1$-weak equivalence of presheaves with transfers regarded as a morphism of $\mathrm{ST}_J(k, A)$ (where we use the fact that presheaves with transfers of the form $A_k^{tr}(X)$ are J-sheaves with transfers).

Now, let us denote by $\mathcal{E}_J$ the smallest localizing thick subcategory of $\mathrm{D}^-\mathrm{ST}_J(k, A)$ containing a (hence every) cone of

$$A_k^{tr}(X \times \mathbb{A}^1) \xrightarrow{A_k^{tr}(\pi)} A_k^{tr}(X)$$

for each $X \in \mathrm{Sm}_k$. The triangulated category of *(Voevodsky) effective J-motives (over k, with coefficients in A-modules)*, denoted by $\mathrm{DM}_J^{eff,-}(k, A)$, is defined as the Verdier quotient $\mathrm{D}^-\mathrm{ST}_J(k, A)/\mathcal{E}_J$ – as an abstract category, it is a localization of $\mathrm{D}^-\mathrm{ST}_J(k, A)$ at the set, which we denote by W_J, of all morphisms such that one (hence any) of its cones belongs to $\mathcal{E}_J$.

Elements of W_J will be called $\mathbb{A}^1$*-weak equivalences* of complexes of J-sheaves with transfers. Note that by construction, every basic $\mathbb{A}^1$-weak equivalence in the above sense belongs to W_J.

Next we note that any finite type k-scheme gives rise to an object of $\mathrm{DM}_J^{eff,-}(k, A)$. Firstly, we have a functor $\mathrm{Sch}_k \longrightarrow \mathrm{PST}(k, A)$ given on objects by sending each X to the composite

$$\mathrm{Cor}_k^{op} \xrightarrow{C_k(-,X)} \mathrm{Ab} \xrightarrow{-\otimes A} \mathrm{Mod}_A,$$

and on arrows by sending each $f : X \longrightarrow X'$ to the natural transformation whose Y-component for $Y \in \mathrm{Cor}_k$ is $A \otimes C_k(Y, f)$, where $C_k(Y, f)$ is given by composition with the graph of f.

We also have the sheafification functor $\mathrm{PST}(k, A) \longrightarrow \mathrm{ST}_J(k, A)$, and the localization functors $\mathrm{ST}_J(k, A) \longrightarrow \mathrm{D}^-\mathrm{ST}_J(k, A)$ and $\mathrm{D}^-\mathrm{ST}_J(k, A) \longrightarrow \mathrm{DM}_J^{eff,-}(k, A)$. The composite

$$\mathrm{Sch}_k \longrightarrow \mathrm{PST}(k, A) \longrightarrow \mathrm{ST}_J(k, A) \longrightarrow \mathrm{D}^-\mathrm{ST}_J(k, A) \longrightarrow \mathrm{DM}_J^{eff,-}(k, A)$$

will be called the *motive functor (corresponding to k, J, A)* and it will be denoted by $\mathrm{M}_J(-, A)$ (with k implicit), or simply as $\mathrm{M}(-)$ when this causes no ambiguity. For each finite type k-scheme X, $\mathrm{M}_J(X, A)$ will be called the *J-motive of X* (over k, with coefficients in A-modules).

An alternative description of effective motives

Whenever a functor $L : \mathcal{C} \longrightarrow \mathcal{D}$ has a fully faithful right adjoint $R : \mathcal{D} \longrightarrow \mathcal{C}$, then R factors through the full subcategory of $\mathcal{C}$, say $\mathcal{C}'$, consisting of those objects c with the property that for any morphism f in $\mathcal{C}$ such that $L(f)$ is an isomorphism – let us denote by W the set of such arrows –, $\mathrm{Hom}_{\mathcal{C}}(f, c)$ is bijective. The restricted functors $\mathcal{D} \longrightarrow \mathcal{C}'$ and $\mathcal{C}' \longrightarrow \mathcal{D}$ then define an adjoint equivalence between $\mathcal{D}$ and $\mathcal{C}'$. Moreover, L (hence also the composite $\mathcal{C} \xrightarrow{L} \mathcal{D} \xrightarrow{\simeq} \mathcal{C}'$) sends arrows in W to isomorphisms and is actually a localization of $\mathcal{C}$ at W.

For any given category $\mathcal{C}$ endowed with a localization $F : \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$ at some set of morphisms W, it is a natural question whether it arises from a setting as above, i.e. whether F has a fully faithful right adjoint, say R. When this is the case, one is able to identify morphisms between two objects of $\mathcal{C}[W^{-1}]$ with morphisms between their respective images under $R : \mathcal{C}[W^{-1}] \longrightarrow \mathcal{C}$, and the functor $\mathcal{C}[W^{-1}] \longrightarrow R(\mathcal{C}[W^{-1}])$ obtained by restricting the codomain of R is an equivalence of categories.

In our context, one may consider the localization functor

$$L : \mathrm{D}^-\mathrm{ST}_J(k, A) \longrightarrow \mathrm{DM}_J^{eff,-}(k, A)$$

and investigate whether it has a fully faithful right adjoint $R : \mathrm{DM}_J^{eff,-}(k, A) \longrightarrow \mathrm{D}^-\mathrm{ST}_J(k, A)$.

Definition 3.2.3. A complex $\mathscr{F} \in \mathrm{D}^-\mathrm{ST}_J(k, A)$ is said to be $\mathbb{A}^1$-*local* (with respect to J) if the image of any $\mathbb{A}^1$-weak equivalence of complexes of J-sheaves with transfers under $\mathrm{Hom}_{\mathrm{D}^-\mathrm{ST}_J(k,A)}(-, \mathscr{F})$ is an isomorphism of abelian groups.

The full subcategory of $\mathrm{D}^-\mathrm{ST}_J(k, A)$ whose objects are the $\mathbb{A}^1$-local complexes will be denoted by $\mathbb{A}^1\text{-}\mathrm{Loc}_J$.

Suppose given complexes $\mathscr{F}, \mathscr{G} \in \mathrm{D}^-\mathrm{ST}_J(k, A)$ such that $\mathscr{G}$ is $\mathbb{A}^1$-local. By Mazza *et al.*, 2006, 9.19, the map

$$\mathrm{Hom}_{\mathrm{D}^{-}\mathrm{ST}_J(k,A)}(\mathscr{F},\mathscr{G}) \longrightarrow \mathrm{Hom}_{\mathrm{DM}_J^{eff,-}(k,A)}(L(\mathscr{F}),L(\mathscr{G})) = \mathrm{Hom}_{\mathrm{DM}_J^{eff,-}(k,A)}(\mathscr{F},\mathscr{G})$$

given by the localization functor $\mathrm{D}^{-}\mathrm{ST}_J(k,A) \xrightarrow{L} \mathrm{DM}_J^{eff,-}(k,A)$ is bijective. Thus the composite

$$\mathbb{A}^1\text{-Loc}_J \xrightarrow{i} \mathrm{D}^{-}\mathrm{ST}_J(k,A) \xrightarrow{L} \mathrm{DM}_J^{eff,-}(k,A)$$

is fully faithful. By Mazza *et al.*, 2006, 14.4, for any complex $\mathscr{F} \in \mathrm{D}^{-}\mathrm{ST}_J(k,A)$, the morphism $\mathscr{F} \longrightarrow \mathrm{Tot}(C_*(\mathscr{F}))$ is an $\mathbb{A}^1$-weak equivalence, hence an isomorphism in $\mathrm{DM}_J^{eff,-}(k,A)$. As $\mathrm{Tot}(C_*(\mathscr{F}))$ is $\mathbb{A}^1$-local for any such $\mathscr{F}$, it follows that $L \circ i$ is essentially surjective, hence an equivalence of categories.

We now describe an object of $\mathrm{DM}_J^{eff,-}(k,A)$ which will be needed later.

Definition 3.2.4. The motive $\mathbb{Z}(1)[2] \in \mathrm{DM}_J^{eff,-}(k,A)$ is denoted by $\mathbb{L}$ and is called the *Lefschetz motive* (with respect to k, A, J). As a complex of Nisnevich sheaves with transfers, $\mathbb{Z}(1)[2] = C_*A_k^{tr}(\mathbb{G}_m)[1]$ is isomorphic to the cokernel of

$$(C_*A_k^{tr}(\mathrm{Spec}\ k))[1] \xrightarrow{C_*A_k^{tr}(s_1))[1]} (C_*A_k^{tr}(\mathbb{A}^1 \setminus \{0\}))[1].$$

Then we have a distinguished triangle

$$(C_*A_k^{tr}(\mathrm{Spec}\ k))[1] \xrightarrow{C_*A_k^{tr}(s_1))[1]} (C_*A_k^{tr}(\mathbb{A}^1 \setminus \{0\}))[1] \xrightarrow{v} \mathbb{L} \xrightarrow{w} (C_*A_k^{tr}(\mathrm{Spec}\ k))[2]$$

in $\mathrm{DM}_J^{eff,-}(k,A)$. Since $C_*A_k^{tr}(\mathrm{Spec}\ k) \cong A_k^{tr}(\mathrm{Spec}\ k)$ and $C_*A_k^{tr}(\mathbb{A}^1 \setminus \{0\}) \cong A_k^{tr}(\mathbb{A}^1 \setminus \{0\})$ in $\mathrm{DM}_J^{eff,-}(k,A)$, there is a distinguished triangle

$$M(\mathrm{Spec}\ k)[1] \xrightarrow{M(s_1)} M(\mathbb{A}^1 \setminus \{0\})[1] \longrightarrow \mathbb{L} \longrightarrow M(\mathrm{Spec}\ k)[2].$$

For any object D of $\mathrm{DM}_J^{eff,-}(k,A)$, the cohomological functor $\mathrm{Hom}_{\mathrm{DM}_J^{eff,-}(k,A)}(-,D)$ determines the following long exact sequence of abelian groups:

$$\begin{aligned}
\cdots &\xrightarrow{v^*} \mathrm{Hom}_{\mathrm{DM}_J^{eff,-}(k,A)}(M(\mathbb{A}^1 \setminus \{0\}), D[-2]) \xrightarrow{M(s_1)^*} \mathrm{Hom}_{\mathrm{DM}_J^{eff,-}(k,A)}(M(\mathrm{Spec}\ k), D[-2]) \xrightarrow{w^*} \\
&\xrightarrow{w^*} \mathrm{Hom}_{\mathrm{DM}_J^{eff,-}(k,A)}(\mathbb{L}, D) \xrightarrow{v^*} \mathrm{Hom}_{\mathrm{DM}_J^{eff,-}(k,A)}(M(\mathbb{A}^1 \setminus \{0\}), D[-1]) \xrightarrow{M(s_1)^*} \\
&\xrightarrow{M(s_1)^*} \mathrm{Hom}_{\mathrm{DM}_J^{eff,-}(k,A)}(M(\mathrm{Spec}\ k), D[-1]) \xrightarrow{w^*} \cdots .
\end{aligned}$$

Representability of motivic cohomology

Ordinary motivic cohomology groups with A-coefficients are representable in the categories $\mathrm{D}^{-}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k,A))$, $\mathrm{D}^{-}(\mathrm{ST}_{\mathrm{Nis}}(k,A))$, and $\mathrm{DM}_{\mathrm{Nis}}^{eff,-}(k,A)$. More precisely, there exist isomorphisms

$$H^p(X,A(q)) \cong \mathrm{Hom}_{\mathrm{D}^{-}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k,A))}(A_k^{tr}(X)|_{\mathrm{Sm}_k}, A(q)|_{\mathrm{Sm}_k}[p]),$$

$$H^p(X, A(q)) \cong \mathrm{Hom}_{D^-(\mathrm{ST}_{\mathrm{Nis}}(k,A))}(A_k^{tr}(X), A(q)[p]),$$
$$H^p(X, A(q)) \cong \mathrm{Hom}_{\mathrm{DM}_{\mathrm{Nis}}^{eff,-}(k,A)}(M(X), A(q)[p])$$

natural in $X \in \mathrm{Sm}_k$.

These are proved in Mazza *et al.*, 2006, 13.11 and 14.16.

3.3 Motivic characterizations of the Bloch-Kato conjecture

Let us briefly recall the construction, given in Chapter 1, of the norm residue homomorphism. We use the fact that the tensor algebra $T(k^\times)$, whose underlying additive group is $\mathbb{Z} \oplus k^\times \oplus k^{\times\otimes 2} \oplus k^{\times\otimes 3} \oplus \cdots$ and whose multiplication is characterized by $(a_1 \otimes \cdots \otimes a_m, b_1 \otimes \cdots \otimes b_n) \mapsto a_1 \otimes \cdots \otimes a_m \otimes b_1 \otimes \cdots \otimes b_n$, is the free ring generated by the abelian group $k^\times$. Thus the homomorphism $\partial : k^\times \longrightarrow H^1_{\text{ét}}(\mathrm{Spec}\ k, \mu_l)$ defines a graded ring homomorphism

$$\partial_* : T(k^\times) \longrightarrow \bigoplus_{n\geq 0} H^n_{\text{ét}}(\mathrm{Spec}\ k, \mu_l^{\otimes n})$$

to the étale cohomology ring of Spec k with coefficients in μ_l.

By Proposition 1.2.5, each tensor of the form $a \otimes b$ for $a, b \in k^\times$ satisfying $a + b = 1$ is sent under ∂_2 to $0 \in H^2_{\text{ét}}(\mathrm{Spec}\ k, \mu_l^{\otimes 2})$. By passing to the quotient, we obtain a map from Milnor K-theory to étale cohomology:

$$K_M^*(k) = T(k^\times)/(a \times b|\ a, b \in k^\times, a + b = 1) \longrightarrow H^*_{\text{ét}}(\mathrm{Spec}\ k, \mu_l^{\otimes *}).$$

Furthermore, each $a \in k^\times \cong T^1(k^\times)$ satisfies $\partial(a^l) = l \cdot \omega(a) = 0$, so passing once again to the quotient yields a homomorphism – the norm residue homomorphism –

$$\nu_* : K_M^*(k)/l \longrightarrow H^*_{\text{ét}}(\mathrm{Spec}\ k, \mu_l^{\otimes *}).$$

For a field k and an integer $n \geq 0$, the Bloch-Kato condition $BK(k, n)$ is defined to hold if and only if for every prime number $l \neq char(k)$, the map $\nu_n : K_M^n(k)/l \longrightarrow H^n_{\text{ét}}(\mathrm{Spec}\ k, \mu_l^{\otimes n})$ is an isomorphism. The condition $BK(n)$ is defined to hold if and only if $BK(k, n)$ holds for every field k. The Bloch-Kato conjecture states that $BK(n)$ holds for every $n \geq 0$.

In this section, we discuss some constructions which allow one to study the norm residue homomorphism from the point of motivic cohomology. In particular, for each $n \geq 0$ it is possible to characterize the map $\nu_n : K_M^n(k)/l \longrightarrow H^n_{\text{ét}}(\mathrm{Spec}\ k, \mu_l^{\otimes n})$ as a change of topology map from Zariski to étale motivic cohomology; see Corollary 3.3.7.

3.3.1 The Beilinson-Lichtenbaum condition

The following general construction will be needed in what follows: given an abelian category $\mathcal{A}$, a complex $(C, d) \in \mathrm{Ch}(\mathcal{A})$, and an integer n, the truncated complex $\tau^{\leq n}C$ is defined as

$$(\tau^{\leq n}C)^i = \begin{cases} C^i, & i < n, \\ \mathrm{Ker}(d^n : C^n \longrightarrow C^{n+1}) & i = n, \\ 0, & i > n, \end{cases}$$

with differentials given by d^i in degrees $i < n-1$, codomain restriction of $d^{n-1} : C^{n-1} \longrightarrow C^n$ in degree $n-1$, and 0 in degrees $i > n-1$. In particular, $\tau^{\leq n}C$ is acyclic in degrees $i > n$, and $\tau^{\leq n}C \hookrightarrow C$ induces isomorphisms between cohomology objects in degrees $i \leq n$.

Definition 3.3.1. (We follow Haesemeyer and C. Weibel, 2019 up to notation.) As above, we let k denote a fixed field. Suppose given a commutative ring with unit A. Then for each integer $n \geq 0$ we define the *Lichtenbaum motivic complex* $A(n)_{Lich}$ to be the complex

$$A(n)_{Lich} := \tau^{\leq n}\mathbf{R}\pi_*(A(n)|_{\mathrm{Sm}_k})$$

in $D(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$, where $A(n)|_{\mathrm{Sm}_k}$ is the (restriction to Sm_k of the) usual motivic complex as an object of $D(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k, \mathrm{Ab}))$. Let us assume $\mathbf{R}\pi_*(A(n))$ to be computed in terms of the resolution of $A(n)|_{\mathrm{Sm}_k}$ in $\mathrm{Ch}(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_k, \mathrm{Ab}))$ defined in Mazza *et al.*, 2006, 6.20, say G. By construction, each entry of G is endowed with the structure of an étale sheaf with transfers. Hence by applying $\pi_*^{tr} : \mathrm{Ch}(\mathrm{ST}_{\text{ét}}(k)) \longrightarrow \mathrm{Ch}(\mathrm{ST}_{\mathrm{Zar}}(k))$ to G and then truncating $\pi_*^{tr}G$ via $\tau^{\leq n}$ we obtain a complex whose restriction to $\mathrm{Ch}(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$ is equal to $A(n)$. Hence we also denote by $A(n)$, by abuse of notation, the complex of étale sheaves with transfers $\tau^{\leq n}\pi_*^{tr}G$.

Construction 3.3.2. If C is a complex in $\mathrm{Ch}(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$ such that π^*C has an injective resolution, then we may compose the adjunction unit component $C \longrightarrow \pi_*\pi^*C$ with $\pi_*\pi^*C \longrightarrow \mathbf{R}\pi_*\pi^*C$ (obtained by applying π_* to an injective resolution of π^*C) to obtain a chain map $C \longrightarrow \mathbf{R}\pi_*\pi^*C$.

In case C is moreover a complex of étale sheaves on Sm_k which is truncated in degree n (i.e. $\tau^{\leq n}C \cong C$), we have the following morphism in $\mathrm{Ch}(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$:

$$C \cong \tau^{\leq n}C \longrightarrow \tau^{\leq n}\mathbf{R}\pi_*\pi^*C \cong \tau^{\leq n}\mathbf{R}\pi_*C.$$

It will be denoted, with k implicit, by α_n^C.

Now, for any scheme $X \in \mathrm{Sm}_k$ and any integer p we obtain a homomorphism

$$H^p_{\mathrm{Zar}}(X, \alpha_n^C) : H^p_{\mathrm{Zar}}(X, C) \longrightarrow H^p_{\mathrm{Zar}}(X, \tau^{\leq n}\mathbf{R}\pi_*C). \tag{3.3.1}$$

Note that in the particular case $X = \mathrm{Spec}\ k$, triviality of its small Zariski site (so Zariski cohomology is given by usual cohomology of complexes) implies that

$$H^p_{\mathrm{Zar}}(\mathrm{Spec}\ k, \tau^{\leq n}\mathbf{R}\pi_*C) \cong \begin{cases} H^p_{\text{ét}}(\mathrm{Spec}\ k, C), & p \leq n, \\ 0, & p > n, \end{cases}$$

and 3.3.1 becomes isomorphic to

$$\begin{cases} \text{The change of topology map } H^p_{\mathrm{Zar}}(\operatorname{Spec} k, C) \longrightarrow H^p_{\text{ét}}(\operatorname{Spec} k, C), & p \leq n, \\ \text{The trivial map } 0 \longrightarrow 0, & p > n. \end{cases}$$

The Beilinson-Lichtenbaum condition

We now specialize the above construction to the case where $C \in \mathrm{Ch}(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$ is the motivic complex $A(n)$ (restricted to Sm_k, following Convention 2.4.4) for a commutative ring with unit A and an integer $n \geq 0$. Recall that $A(n)$ is an étale sheaf on Sm_k (Proposition 2.1.13) and is truncated in degree n by construction. So we have a morphism

$$\alpha_n^{A(n)} : A(n) \longrightarrow \tau^{\leq n}\mathbf{R}\pi_* A(n) = A(n)_{Lich} \tag{3.3.2}$$

in $\mathrm{Ch}(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$. Whenever it is clear that A denotes a ring, we denote $\alpha_n^{A(n)}$ by α_n^A. Chain maps of this form will be called *Beilinson-Lichtenbaum morphisms.*

We shall be concerned with a few choices of A whose corresponding Beilinson-Lichtenbaum morphisms may be related to each other and to the norm residue homomorphism $\nu_n : K_M^n(k)/l \longrightarrow H^n_{\text{ét}}(\operatorname{Spec} k, \mu_l^{\otimes n})$. As a motivating step, note that

$$\alpha_n^{\mathbb{Z}/l} : \mathbb{Z}/l(n) \longrightarrow \mathbb{Z}/l(n)_{Lich}$$

induces, by applying $H^p_{\mathrm{Zar}}(\operatorname{Spec} k, -)$ to it for a varying integer p as in the previous section, arrows isomorphic to:

$$\begin{cases} \text{The change of topology map } H^p_{\mathrm{Zar}}(\operatorname{Spec} k, \mathbb{Z}/l(n)) \longrightarrow H^p_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}/l(n)), & p \leq n, \\ \text{The trivial map } 0 \longrightarrow 0, & p > n. \end{cases}$$

By taking $p = n$, the quasi-isomorphism $\mathbb{Z}/l(n)_{\text{ét}} \simeq \mu_l^{\otimes n}$ of complexes of étale sheaves shows, by naturality of change of topology maps, that the arrow $H^n_{\mathrm{Zar}}(\operatorname{Spec} k, \alpha_n^{\mathbb{Z}/l})$ is canonically isomorphic to $H^p_{\mathrm{Zar}}(\operatorname{Spec} k, \mathbb{Z}/l(n)) \longrightarrow H^p_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}/l(n))$. By Remark 3.3.5 and Proposition 3.3.6, it is then also canonically isomorphic to the norm residue ν_n.

Definition 3.3.3. Given a field k and an integer $n \geq 0$, we say the *Beilinson-Lichtenbaum condition $BL(k, n)$* holds if and only if for every prime number $l \neq char(k)$, the Beilinson-Lichtenbaum morphism $\alpha_n^{\mathbb{Z}/l} : \mathbb{Z}/l(n) \longrightarrow \mathbb{Z}/l(n)_{Lich}$ is a quasi-isomorphism in $\mathrm{Ch}(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$.

Given $n \geq 0$, we say $BL(n)$ holds if and only if for every field k, $BL(k, n)$ holds.

Lemma 3.3.4 (Haesemeyer and C. Weibel, 2019, 1.29)**.** Suppose given a field k and an integer $n \geq 0$. If $BL(k, n)$ holds, then the Beilinson-Lichtenbaum morphism $\alpha_n^A : A(n) \longrightarrow A(n)_{Lich}$ is also a quasi-isomorphism in $\mathrm{Ch}^-(\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathrm{Ab}))$ for the following rings A:

(i) $\mathbb{Z}/l^i$ for any integer $i \geq 1$.

(ii) $\mathbb{Q}/\mathbb{Z}_{(l)}$.

(iii) $\mathbb{Z}_{(l)}$.

3.3.2 The norm residue homomorphism in terms of motivic cohomology

Stated in terms of Galois cohomology, Hilbert's classical 'theorem 90' (see theorems 1.2.4, 1.2.3) asserts that given a field k endowed with a separable closure k_{sep}, denoting by $G = \mathrm{Gal}(k_{sep}/k)$ the corresponding absolute Galois group, the discrete G-module $k_{sep}^\times$ has trivial (continuous) group cohomology in degree 1:

$$H^1(k, k_{sep}^\times) \cong \varinjlim_{K\in\mathrm{FinGal}} H^n(\mathrm{Gal}(K/k), K_{sep}^\times) \cong 0. \tag{3.3.3}$$

It may then be equivalently stated as the vanishing of the degree 1 étale cohomology of the sheaf $\mathcal{O}^\times$ of global units on (the small étale site of) Spec k:

$$H^1_{\text{ét}}(\mathrm{Spec}\ k, \mathcal{O}^\times) \cong 0. \tag{3.3.4}$$

The quasi-isomorphism of presheaves with transfers $\mathbb{Z}(1) \simeq \mathcal{O}^\times[-1]$ (see 2.3.5) allows us to further restate 3.3.4 in terms of étale motivic cohomology as

$$H^2_{\text{ét}}(\mathrm{Spec}\ k, \mathbb{Z}(1)) \cong H^2_{\text{ét}}(\mathrm{Spec}\ k, \mathcal{O}^\times[-1]) \cong H^1_{\text{ét}}(\mathrm{Spec}\ k, \mathcal{O}^\times) \cong 0. \tag{3.3.5}$$

On the other hand, we have

$$H^1_{\text{ét}}(\mathrm{Spec}\ k, \mathbb{Z}(0)) \cong H^1_{\text{ét}}(\mathrm{Spec}\ k, \mathbb{Z}) \cong 0. \tag{3.3.6}$$

More generally, the discussion below, following Haesemeyer and C. Weibel, 2019, suggests the existence of a pattern relating, for a given field k, the Bloch-Kato condition $BK(n, l)$ for any integer $n \leq 0$ and any prime number $l \neq char(k)$ to a condition on the étale motivic cohomology of Spec k which would extend the above vanishing results to higher degrees and weights.

We start by discussing an alternative description of the norm residue homomorphism. Let us fix a field k and a prime $l \neq char(k)$. Denote by

$$F = \bigoplus_{n\geq 0} F_n \ :\ K^*_M(k) \longrightarrow \bigoplus_{n\geq 0} H^n_{\text{ét}}(\mathrm{Spec}\ k, \mathbb{Z}(n))$$

the composite of the ring isomorphism $\lambda_* : K^*_M(k) \xrightarrow{\cong} \bigoplus_{n\geq 0} H^n(\mathrm{Spec}\ k, \mathbb{Z}(n))$ (see 3.1.6) with the change of topology ring homomorphism $\bigoplus_{n\geq 0} H^n(\mathrm{Spec}\ k, \mathbb{Z}(n)) \longrightarrow \bigoplus_{n\geq 0} H^n_{\text{ét}}(\mathrm{Spec}\ k, \mathbb{Z}(n))$.

Recall from Subsection 3.1.2 that for each $n \geq 0$ there exists a commutative diagram

$$\begin{tikzcd} K^n_M(k) \arrow[r, "l"] \arrow[d, "\cong"'] & K^n_M(k) \arrow[r] \arrow[d, "\cong"] & K^n_M(k)/l \arrow[d, "\cong"] \\ H^{n,n}(\mathrm{Spec}\ k, \mathbb{Z}) \arrow[r, "l"'] & H^{n,n}(\mathrm{Spec}\ k, \mathbb{Z}) \arrow[r] & H^{n,n}(\mathrm{Spec}\ k, \mathbb{Z}/l). \end{tikzcd}$$

Moreover, naturality of change of topology maps yields a commutative diagram

$$\begin{tikzcd}
H^{n,n}(\mathrm{Spec}\ k, \mathbb{Z}) \arrow[r, "l"] \arrow[d] & H^{n,n}(\mathrm{Spec}\ k, \mathbb{Z}) \arrow[r] \arrow[d] & H^{n,n}(\mathrm{Spec}\ k, \mathbb{Z}/l) \arrow[d] \\
H^{n,n}(\mathrm{Spec}\ k, \mathbb{Z})_{\text{ét}} \arrow[r, "l"'] & H^{n,n}(\mathrm{Spec}\ k, \mathbb{Z})_{\text{ét}} \arrow[r] & H^{n,n}(\mathrm{Spec}\ k, \mathbb{Z}/l)_{\text{ét}}.
\end{tikzcd}$$

Then we obtain for each $n \geq 0$ a commutative diagram

$$\begin{tikzcd}
K_M^n(k) \arrow[r, "l"] \arrow[d, "F_n"'] & K_M^n(k) \arrow[r] \arrow[d, "F_n"] & K_M^n(k)/l \arrow[d, "\varphi_n"] \\
H^n_{\text{ét}}(\mathrm{Spec}\ k, \mathbb{Z}(n)) \arrow[r, "l"'] & H^n_{\text{ét}}(\mathrm{Spec}\ k, \mathbb{Z}(n)) \arrow[r] & H^n_{\text{ét}}(\mathrm{Spec}\ k, \mu_l^{\otimes n}),
\end{tikzcd} \tag{3.3.7}$$

where in the bottom right entry we have used the isomorphism $H^n_{\text{ét}}(\mathrm{Spec}\ k, \mu_l^{\otimes n}) \cong H^n_{\text{ét}}(\mathrm{Spec}\ k, \mathbb{Z}/l(q)^n)$ provided by the quasi-isomorphism $\mu_l^{\otimes q} \simeq \mathbb{Z}/l(q)^n$ (see Remark 2.3.6) of complexes of étale sheaves with transfers.

Remark 3.3.5. Note that $H^{n+1}(\mathrm{Spec}\ k, \mathbb{Z}(n)) \cong 0$, since the small Zariski site of Spec k is trivial and $\mathbb{Z}(n)$ is truncated in degree n by construction. Hence the exact sequence

$$H^n(\mathrm{Spec}\ k, \mathbb{Z}(n)) \xrightarrow{l} H^n(\mathrm{Spec}\ k, \mathbb{Z}(n)) \longrightarrow H^n(\mathrm{Spec}\ k, \mu_l^{\otimes n}) \longrightarrow 0$$

combined with the isomorphism $K_M^n(k) \cong H^n(\mathrm{Spec}\ k, \mathbb{Z}(n))$ yields a canonical isomorphism $K_M^n(k)/l \cong H^n(\mathrm{Spec}\ k, \mu_l^{\otimes n})$.

Now, uniqueness of φ_n as a filler in 3.3.9 and naturality of the change of topology maps $H^n(\mathrm{Spec}\ k, -) \longrightarrow H^n_{\text{ét}}(\mathrm{Spec}\ k, -)$ show that φ_n equals the composite

$$K_M^n(k)/l \xrightarrow{\cong} H^n(\mathrm{Spec}\ k, \mathbb{Z}/l^{\otimes n}) \longrightarrow H^n_{\text{ét}}(\mathrm{Spec}\ k, \mathbb{Z}/l^{\otimes n}) \cong H^n_{\text{ét}}(\mathrm{Spec}\ k, \mu_l^{\otimes n}).$$

Proposition 3.3.6. In the above notation, φ_n equals the norm residue homomorphism $\nu_n : K_M^n(k)/l \longrightarrow H^n_{\text{ét}}(\mathrm{Spec}\ k, \mu_l^{\otimes n})$.

Proof. The fact that $F = \bigoplus_{n \geq 0} F_n$ is a ring homomorphism yields a commutative diagram

$$\begin{tikzcd}
K_M^1(k)^{\otimes n} \arrow[r, "\vee"] \arrow[d, "F_1^{\otimes n}"'] & K_M^n(k) \arrow[r] \arrow[d, "F_n"] & K_M^n(k)/l \arrow[d, "\varphi_n"] \\
H^1_{\text{ét}}(\mathrm{Spec}\ k, \mathbb{Z}(1))^{\otimes n} \arrow[r, "\vee"'] & H^n_{\text{ét}}(\mathrm{Spec}\ k, \mathbb{Z}(n)) \arrow[r] & H^n_{\text{ét}}(\mathrm{Spec}\ k, \mu_l^{\otimes n}).
\end{tikzcd}$$

By construction, the ring K_M^* is generated by degree 1 elements, i.e. for each n the multiplication map $\vee : K_M^1(k)^{\otimes n} \longrightarrow K_M^n(k)$ is surjective. Hence so is the composite

$\pi : K^1_M(k)^{\otimes n} \longrightarrow K^n_M(k)/l$ of the upper row. It follows that if we denote by π' the composite

$$K^1_M(k)^{\otimes n} \xrightarrow{F_1^{\otimes n}} H^1_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}(1))^{\otimes n} \xrightarrow{\vee} H^n_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}(n)) \longrightarrow H^n_{\text{ét}}(\operatorname{Spec} k, \mu_l^{\otimes n}),$$

then φ_n is uniquely determined by the condition $\varphi_n \circ \pi = \pi'$. On the other hand, the norm residue $\nu_n : K^n_M(k)/l \longrightarrow H^n_{\text{ét}}(\operatorname{Spec} k, \mu_l^{\otimes n})$ is uniquely characterized by the commutativity of

$$\begin{tikzcd}
K^1_M(k)^{\otimes n} \arrow[r, "\pi"] \arrow[d] & K^n_M(k)/l \arrow[d, "\nu_n"] \\
H^1_{\text{ét}}(\operatorname{Spec} k, \mu_l)^{\otimes n} \arrow[r, "\vee"'] & H^n_{\text{ét}}(\operatorname{Spec} k, \mu_l^{\otimes n}),
\end{tikzcd}$$

so it suffices to show that π' equals the composite $K^1_M(k)^{\otimes n} \longrightarrow H^1_{\text{ét}}(\operatorname{Spec} k, \mu_l)^{\otimes n} \xrightarrow{\vee} H^n_{\text{ét}}(\operatorname{Spec} k, \mu_l^{\otimes n})$. This in turn may be done by proving commutativity of

$$\begin{tikzcd}
 & H^1_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}(1))^{\otimes n} \arrow[r, "\vee"] \arrow[dd, "\psi_1^{\otimes n}"] & H^n_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}(n)) \arrow[dd, "\psi_n"] \\
K^1_M(k)^{\otimes n} \arrow[ur, "\cong"] \arrow[dr] & & \\
 & H^1_{\text{ét}}(\operatorname{Spec} k, \mu_l)^{\otimes n} \arrow[r, "\vee"'] & H^n_{\text{ét}}(\operatorname{Spec} k, \mu_l^{\otimes n}),
\end{tikzcd}$$

where the vertical arrows are induced from morphisms in the derived category of étale sheaves $\mathbb{Z}(q)_{\text{ét}} \longrightarrow \mathbb{Z}/l(q)_{\text{ét}} \cong \mu_l^{\otimes q}$, $q = 1, n$. These give rise to a ring homomorphism $H^*_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}(*)) \longrightarrow H^*_{\text{ét}}(\operatorname{Spec} k, \mu_l^*)$, so the right-hand square commutes. Now, note that commutativity of the left-hand triangle is equivalent, by definition of the maps involved (which rely on the isomorphism $K^1_M(k) \cong H^0_{\text{ét}}(\operatorname{Spec} k, \mathscr{O}^\times)$), to commutativity of

$$\begin{tikzcd}
 & H^1_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}(1)) \arrow[d, "\psi_1"] \\
H^0_{\text{ét}}(\operatorname{Spec} k, \mathscr{O}^\times) \arrow[ur, "\cong"] \arrow[r, "\delta"'] & H^1_{\text{ét}}(\operatorname{Spec} k, \mu_l),
\end{tikzcd} \tag{3.3.8}$$

where $H^0_{\text{ét}}(\operatorname{Spec} k, \mathscr{O}^\times) \cong H^1_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}(1))$ arises from the given quasi-isomorphism $\mathbb{Z}(1)_{\text{ét}} \simeq \mathscr{O}^\times[-1]$ of complexes of étale sheaves, and δ is the connecting homomorphism arising from the Kummer exact sequence of étale sheaves $0 \longrightarrow \mu_l \longrightarrow \mathscr{O}^\times \xrightarrow{l} \mathscr{O}^\times \longrightarrow 0$. Also note that we have distinguished triangles

$$\mathbb{Z}(1)_{\text{ét}} \xrightarrow{-l} \mathbb{Z}(1)_{\text{ét}} \longrightarrow \mathbb{Z}/l(1)_{\text{ét}} \longrightarrow \mathbb{Z}(1)_{\text{ét}}[1],$$

$$\mathscr{O}^\times \xrightarrow{l} \mathscr{O}^\times \longrightarrow \mu_l[1] \longrightarrow \mathscr{O}^\times[1]$$

in the *derived* category of étale sheaves on Spec k, as well as an isomorphism of distinguished triangles (with vertical arrows induced by the quasi-isomorphisms previously described)

$$\begin{tikzcd} \mathbb{Z}(1)_{\text{ét}}[1] \arrow[r, "l"] \arrow[d] & \mathbb{Z}(1)_{\text{ét}}[1] \arrow[r] \arrow[d] & \mathbb{Z}/l(1)_{\text{ét}}[1] \arrow[r] \arrow[d] & \mathbb{Z}(1)_{\text{ét}}[2] \arrow[d] \\ \mathscr{O}^\times \arrow[r, "l"'] & \mathscr{O}^\times \arrow[r] & \mu_l[1] \arrow[r] & \mathscr{O}^\times[1]. \end{tikzcd}$$

We conclude by applying $H^0_{\text{ét}}(\text{Spec } k, -)$ to the middle square and comparing it with 3.3.8. □

Expanding the definition of φ_n yields:

Corollary 3.3.7. The diagram

$$\begin{tikzcd} K^n_M(k)/l \arrow[r, "\cong"] \arrow[d, "\nu_n"'] & H^{n,n}(\text{Spec } k, \mathbb{Z}/l) \arrow[d] \\ H^n(\text{Spec } k, \mu_l^{\otimes n}) \arrow[r, "\cong"'] & H^{n,n}_{\text{ét}}(\text{Spec } k, \mathbb{Z}/l) \end{tikzcd}$$

commutes, where: ν_n is the norm residue homomorphism; the right arrow is the change of topology map; the top arrow is the isomorphism described in Subsection 3.1.2; and the bottom arrow is the isomorphism provided by the quasi-isomorphism $\mu_l^{\otimes n} \simeq \mathbb{Z}/l(n)$ of complexes of étale sheaves with transfers.

3.3.3 The generalized 'Hilbert 90' condition

The short exact sequence

$$0 \longrightarrow \mathbb{Z}(n) \xrightarrow{l} \mathbb{Z}(n) \longrightarrow \mathbb{Z}/l(n) \longrightarrow 0$$

of complexes of étale sheaves with transfers defines a distinguished triangle

$$\mathbb{Z}(n) \xrightarrow{l} \mathbb{Z}(n) \longrightarrow \mathbb{Z}/l(n) \longrightarrow \mathbb{Z}(n)[1]$$

in the derived category. So by using the quasi-isomorphism $\mu_l^{\otimes n} \simeq \mathbb{Z}/l$ one obtains a distinguished triangle

$$\mathbb{Z}(n) \xrightarrow{l} \mathbb{Z}(n) \longrightarrow \mu_l^{\otimes n} \longrightarrow \mathbb{Z}(n)[1].$$

Then let us consider the commutative diagram

$$\begin{tikzcd} K^n_M(k) \arrow[r, "l"] \arrow[d, "F_n"'] & K^n_M(k) \arrow[r] \arrow[d, "F_n"] & K^n_M(k)/l \arrow[r] \arrow[d, "\nu_n"] & 0 \\ H^n_{\text{ét}}(\text{Spec } k, \mathbb{Z}(n)) \arrow[r, "l"'] & H^n_{\text{ét}}(\text{Spec } k, \mathbb{Z}(n)) \arrow[r] & H^n_{\text{ét}}(\text{Spec } k, \mu_l^{\otimes n}) \arrow[r] & H^{n+1}_{\text{ét}}(\text{Spec } k, \mathbb{Z}(n)), \end{tikzcd} \tag{3.3.9}$$

where both rows are exact. If ν_n is an isomorphism, then $H^n_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}(n)) \longrightarrow H^n_{\text{ét}}(\operatorname{Spec} k, \mu_l^{\otimes n})$ is surjective, so by continuing the lower row's long exact sequence, it follows that the map $H^{n+1}_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}(n)) \xrightarrow{l} H^{n+1}_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}(n))$ is injective. This proves:

Lemma 3.3.8. Suppose given a field k, an integer $n \geq 0$, and a prime number $l \neq char(k)$. If the norm residue $\nu_n : K^n_M(k)/l \longrightarrow H^n_{\text{ét}}(\operatorname{Spec} k, \mu_l^{\otimes n})$ is an isomorphism, then the l-torsion subgroup of $H^{n+1}_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}(n))$ is trivial.

On the other hand, it may be proved (see Haesemeyer and C. Weibel, 2019, 1.6) that $H^{n+1}_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}(n))$ is a torsion group whose l-primary subgroup is $H^{n+1}_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}_{(l)}(n))$. Hence if ν_n is an isomorphism, then $H^{n+1}_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}_{(l)}(n)) \cong 0$.

Definition 3.3.9. Given a field k and an integer $n \geq 0$, we say the (*generalized 'Hilbert 90'*) condition $H90(k, n)$ holds if and only if for every prime number $l \neq char(k)$, $H^{n+1}_{\text{ét}}(\operatorname{Spec} k, \mathbb{Z}_{(l)}(n)) \cong 0$.

Given $n \geq 0$, we say $H90(n)$ holds if and only if for every field k, $H90(k, n)$ holds.

Proposition 3.3.10 (Haesemeyer and C. Weibel, 2019, 2.10)**.** Suppose U is a nonempty open subscheme of $\mathbb{A}^1_k \setminus \{0\}$. Let us consider the following distinguished triangle in $\mathrm{DM}(k)$:

$$\bigoplus_{x \in \mathbb{A}^1_k \setminus U} M(x)(1)[1] \longrightarrow M(U) \longrightarrow M(\operatorname{Spec} k) \longrightarrow \bigoplus_{x \in (\mathbb{A}^1_k \setminus \{0\}) \setminus U} M(x)(1)[2].$$

Suppose $\mathscr{F}$ is a complex of Zariski sheaves with transfers whose underlying complex of presheaves with transfers is homotopy invariant (as in Definition 2.5.1). Then for each integer $n \geq 1$ there exists a split exact sequence

$$0 \longrightarrow H^n(k, \mathscr{F}) \longrightarrow H^n(k(t), \mathscr{F}) \xrightarrow{\partial} \bigoplus_{x \in \mathbb{A}^1_k} H^{n-1}(\{x\}, \mathscr{F}(-1)) \longrightarrow 0.$$

Proof. Suppose U is a dense open subset of $\mathbb{A}^1_k \setminus \{0\}$. Since U is dense in $\mathbb{A}^1_k$, it contains a zero-cycle of degree 1, yielding a map $p : \operatorname{Spec} k \longrightarrow U$ of degree 1 in Cor_k. Now, let us consider the sequence

$$\operatorname{Spec} k \xrightarrow{p} U \hookrightarrow \mathbb{A}^1_k.$$

By applying the motive functor $M : \mathrm{Cor}_k \to \mathrm{DM}(k)$, we obtain a sequence

$$M(\operatorname{Spec} k) \xrightarrow{M(p)} M(U) \longrightarrow M(\mathbb{A}^1_k)$$

whose composite is an isomorphism inverse to the morphism of motives induced by the terminal map $\mathbb{A}^1_k \to \operatorname{Spec} k$. By the splitting lemma, the short exact sequence

$$0 \longrightarrow H^n(\operatorname{Spec} k, \mathscr{F}) \longrightarrow H^n(U, \mathscr{F}) \longrightarrow \bigoplus_{x \in \mathbb{A}^1_k \setminus U} H^{n-1}(\{x\}, \mathscr{F}(-1)) \longrightarrow 0$$

splits. $\square$

Corollary 3.3.11. Adequately choosing $\mathscr{F}$ and $n \geq 1$ in Proposition 3.3.10 yields the following split short exact sequences, where l denotes a prime number different from $char(k)$:

(i) For $\mathscr{F} = \mathbb{Z}(n)$,

$$0 \longrightarrow K_n^M(k) \longrightarrow K_n^M(k(t)) \xrightarrow{\partial} \bigoplus_{x \in \mathbb{A}_k^1} K_{n-1}^M(k(x)) \longrightarrow 0.$$

(ii) For $\mathscr{F} = \mathbb{Z}_{(l)}(n)$,

$$0 \longrightarrow H^n_{\mathrm{Zar}}(k, \mathbb{Z}_{(l)}(n)) \longrightarrow H^n_{\mathrm{Zar}}(k(t), \mathbb{Z}_{(l)}(n)) \xrightarrow{\partial} \bigoplus_{x \in \mathbb{A}_k^1} H^{n-1}_{\mathrm{Zar}}(k(x), \mathbb{Z}_{(l)}(n-1)) \longrightarrow 0.$$

(iii) For $\mathscr{F} = R\pi_* \mu_l^{\otimes n}$,

$$0 \longrightarrow H^n_{\mathrm{\acute{e}t}}(k, \mu_l^{\otimes n}) \longrightarrow H^n_{\mathrm{\acute{e}t}}(k(t), \mu_l^{\otimes n}) \xrightarrow{\partial} \bigoplus_{x \in \mathbb{A}_k^1} H^{n-1}_{\mathrm{\acute{e}t}}(k(x), \mu_l^{\otimes n-1}) \longrightarrow 0.$$

(iv) For $\mathscr{F} = R\pi_* \mathbb{Z}(n)_{\mathrm{\acute{e}t}}$,

$$0 \longrightarrow H^{n+1}_{\mathrm{\acute{e}t}}(k, \mathbb{Z}(n)) \longrightarrow H^{n+1}_{\mathrm{\acute{e}t}}(k(t), \mathbb{Z}(n)) \xrightarrow{\partial} \bigoplus_{x \in \mathbb{A}_k^1} H^{n}_{\mathrm{\acute{e}t}}(k(x), \mathbb{Z}(n-1)) \longrightarrow 0.$$

□

Corollary 3.3.12. The following hold:

(i) Suppose given a field k, an integer $n \geq 1$, and a prime number $l \neq char(k)$. If $H^{n+1}_{\mathrm{\acute{e}t}}(k(t), \mathbb{Z}_{(l)}(n)) \cong 0$, then $H^{n}_{\mathrm{\acute{e}t}}(k, \mathbb{Z}_{(l)}(n-1)) \cong 0$.

By varying l among all primes different from $char(k) = char(k(t))$, it follows that $H90(k(t), n+1)$ implies $H90(k, n)$. By also varying k, it follows that for every $n \geq 1$, if $H90(n+1)$ holds, then $H90(n)$ holds.

(ii) Suppose given a field k, an integer $n \geq 1$, and a prime number $l \neq char(k)$. If $H^n_{\mathrm{Zar}}(k(t), \mathbb{Z}/l(n)) \longrightarrow H^n_{\mathrm{Zar}}(k(t), \mathbb{Z}/l(n)_{Lich}) \cong H^n_{\mathrm{\acute{e}t}}(k(t), \mathbb{Z}/l(n))$ is an isomorphism, then so is $H^{n-1}_{\mathrm{Zar}}(k, \mathbb{Z}/l(n-1)) \longrightarrow H^{n-1}_{\mathrm{Zar}}(k, \mathbb{Z}/l(n-1)_{Lich}) \cong H^{n-1}_{\mathrm{\acute{e}t}}(k, \mathbb{Z}/l(n-1))$.

By varying l among all primes different from $char(k) = char(k(t))$, we have that $BK(k(t), n+1)$ implies $BK(k, n)$. And by varying k, it follows that for every $n \geq 1$, if $BK(n+1)$ holds, then $BK(n)$ holds.

(iii) Suppose given a field k, an integer $n \geq 1$, and a prime number $l \neq char(k)$. If $H^n_{\mathrm{Zar}}(k(t), \mathbb{Z}/l(n)) \longrightarrow H^n_{\mathrm{Zar}}(k(t), \mathbb{Z}/l(n)_{Lich}) \cong H^n_{\mathrm{\acute{e}t}}(k(t), \mathbb{Z}/l(n))$ is surjective, then so is $H^{n-1}_{\mathrm{Zar}}(k, \mathbb{Z}/l(n-1)) \longrightarrow H^{n-1}_{\mathrm{Zar}}(k, \mathbb{Z}/l(n-1)_{Lich}) \cong H^{n-1}_{\mathrm{\acute{e}t}}(k, \mathbb{Z}/l(n-1))$.

Proof.

(i) Note that $H^n_{\text{ét}}(k, \mathbb{Z}_{(l)}(n-1)) \cong H^n_{\text{ét}}(k(0), \mathbb{Z}_{(l)}(n-1))$ (where 0 denotes the origin in $\mathbb{A}^1_k$) is a direct summand of $\bigoplus_{x \in \mathbb{A}^1_k} H^n_{\text{ét}}(k(x), \mathbb{Z}_{(l)}(n-1))$, which is in turn a direct summand of $H^{n+1}_{\text{ét}}(k(t), \mathbb{Z}_{(l)}(n))$.

(ii) Let us consider the canonical morphism $\mathbb{Z}/l(n) \longrightarrow R\pi_* \mathbb{Z}/l(n)_{\text{ét}}$ of Zariski sheaves on Sm_k. Naturality of the split exact sequence from Proposition 3.3.10 yields the left-hand square in the following commutative diagram where all horizontal arrows are split surjections:

$$\begin{tikzcd}
H^n_{\text{Zar}}(k(t), \mathbb{Z}/l(n)) \arrow[r, two heads, "\partial"] \arrow[d] & \bigoplus_{x \in \mathbb{A}^1_k} H^{n-1}_{\text{Zar}}(k(x), \mathbb{Z}/l(n-1)) \arrow[r, two heads] \arrow[d] & H^{n-1}_{\text{Zar}}(k, \mathbb{Z}/l(n-1)) \arrow[d] \\
H^n_{\text{ét}}(k(t), \mathbb{Z}/l(n)) \arrow[r, two heads, "\partial"'] & \bigoplus_{x \in \mathbb{A}^1_k} H^{n-1}_{\text{ét}}(k(x), \mathbb{Z}/l(n-1)) \arrow[r, two heads] & H^{n-1}_{\text{ét}}(k, \mathbb{Z}/l(n-1)).
\end{tikzcd}$$

Since the splitting is natural, if $H^n_{\text{Zar}}(k(t), \mathbb{Z}/l(n)) \longrightarrow H^n_{\text{ét}}(k(t), \mathbb{Z}/l(n))$ is an isomorphism, then so is $H^{n-1}_{\text{Zar}}(k, \mathbb{Z}/l(n-1)) \longrightarrow H^{n-1}_{\text{ét}}(k, \mathbb{Z}/l(n-1))$.

(iii) The same diagram as in the proof of (ii) shows that if $H^n_{\text{Zar}}(k(t), \mathbb{Z}/l(n)) \longrightarrow H^n_{\text{ét}}(k(t), \mathbb{Z}/l(n))$ is onto, then so is $H^{n-1}_{\text{Zar}}(k, \mathbb{Z}/l(n-1)) \longrightarrow H^{n-1}_{\text{ét}}(k, \mathbb{Z}/l(n-1))$.

□

3.3.4 Consequences of the existence of the contraction functor

Following Definition 3.2.3, the full subcategory of $\mathrm{D}^-\mathrm{ST}_{\text{Nis}}(k, \mathbb{Z})$ consisting of $\mathbb{A}^1$-local complexes will be denoted by $\mathbb{A}^1\text{-Loc}_{\text{Nis}}(k, \mathbb{Z})$. As stated in the previous section, the composite of the inclusion functor $i : \mathbb{A}^1\text{-Loc}_{\text{Nis}}(k, \mathbb{Z}) \hookrightarrow \mathrm{D}^-\mathrm{ST}_{\text{Nis}}(k, \mathbb{Z})$ with the localization $L : \mathrm{D}^-\mathrm{ST}_{\text{Nis}}(k, \mathbb{Z}) \longrightarrow \mathrm{DM}^{eff,-}_{\text{Nis}}(k, \mathbb{Z})$ defines an equivalence of categories from $\mathbb{A}^1\text{-Loc}_{\text{Nis}}(k, \mathbb{Z})$ to the category of Voevodsky (Nisnevich) motives over k with integral coefficients. The following discussion uses the existence of a tensor triangulated structure $\otimes$ on $\mathbb{A}^1\text{-Loc}_{\text{Nis}}(k, \mathbb{Z})$ and a corresponding internal hom functor RHom, as in Mazza *et al.*, 2006, 14.11.

Definition 3.3.13. Let F be a homotopy invariant presheaf with transfers. Note that for any $X \in \mathrm{Cor}_k$ the projection $X \times (\mathbb{A}^1 \setminus \{0\}) \xrightarrow{\pi} X$ has a section $X \xrightarrow{\iota_1} X \times (\mathbb{A}^1 \setminus \{0\})$, and that both π and ι_1 are natural in X. Hence we have maps $F(X) \xrightarrow{F(\pi)} F(X \times (\mathbb{A}^1 \setminus \{0\}))$ with retractions $F(X \times (\mathbb{A}^1 \setminus \{0\})) \xrightarrow{F(\iota_1)} F(X)$, both natural in $X \in \mathrm{Cor}_k$.

Thus we obtain a split monomorphism $F \longrightarrow F(- \times (\mathbb{A}^1 \setminus \{0\}))$ which, denoting its cokernel by G, fits into a split exact sequence

$$0 \longrightarrow F \longrightarrow F(- \times (\mathbb{A}^1 \setminus \{0\})) \longrightarrow G \longrightarrow 0.$$

The splitting lemma yields an isomorphism

$$F(- \times (\mathbb{A}^1 \setminus \{0\})) \xrightarrow{\cong} F \oplus G.$$

In this setting, G will be denoted by F_{-1} and called the *contraction* of F. As (co)kernels of presheaves with transfers are computed objectwise, we have isomorphisms $F_{-1}(X) \cong F(X \times (\mathbb{A}^1 \setminus \{0\}))/F(X)$ natural in $X \in \mathrm{Cor}_k$.

One recursively defines $F_{-(n+1)} = (F_{-n})_{-1}$ for $n \geq 1$.

In what follows, we will denote by Σ the shift endofunctor $\mathscr{F} \mapsto \mathscr{F}[1]$ on $\mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k, \mathbb{Z})$, and Σ^{-1} denotes its inverse $\mathscr{F} \mapsto \mathscr{F}[-1]$.

Recall from Definition 3.2.4 that the Lefschetz motive $\mathbb{L}$ is defined as the complex $\mathbb{Z}(1)[2] \in \mathrm{D}^{-}\mathrm{ST}_{\mathrm{Nis}}(k)$.

The following proposition consists in a summary of properties of a particular construction on $\mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k, \mathbb{Z}) \simeq \mathrm{D}^{-}\mathrm{ST}_{\mathrm{Nis}}(k, \mathbb{Z})$ which is intended to extend the above definition of contractions of sheaves to more general complexes. Such properties allow one to compare motivic cohomology groups of a given $X \in \mathrm{Sm}_k$ in different degrees. Explicitly, one considers the derived hom functor $\mathrm{RHom} : \mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k, \mathbb{Z}) \times \mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k, \mathbb{Z}) \longrightarrow \mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k, \mathbb{Z})$ and the Lefschetz motive $\mathbb{L}$ (see Definition 3.2.4) and defines the endofunctor

$$\mathrm{Cont} := \mathrm{RHom}(\mathbb{L}, -)[1]$$

on $\mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k, \mathbb{Z})$. It is denoted in Haesemeyer and C. Weibel, 2019 also by $\mathscr{F} \mapsto \mathscr{F}_{-1}$.

The properties below also concern a natural transformation $\delta : \Sigma^{-1} \Longrightarrow \mathrm{Cont}(- \otimes \mathbb{Z}(1))$ which is essentially given by the unit natural transformation η corresponding to the adjunction $- \otimes \mathbb{L} \dashv \mathrm{RHom}(\mathbb{L}, -)$ between endofunctors on $\mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k, \mathbb{Z})$. Namely, by horizontally composing $\eta : 1_{\mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k,\mathbb{Z})} \Rightarrow \mathrm{RHom}(\mathbb{L}, - \otimes \mathbb{L})$ with the identity natural transformation $1_{\Sigma^{-1}}$ one obtains a natural transformation

$$\Sigma^{-1} \Longrightarrow \mathrm{RHom}(\mathbb{L}, - \otimes \mathbb{L})[-1].$$

Then one defines δ by vertically composing it with the natural isomorphisms

$$\mathrm{RHom}(\mathbb{L}, - \otimes \mathbb{L})[-1] \cong \mathrm{RHom}(\mathbb{L}, - \otimes \mathbb{Z}(1)[2])[-1] \cong \mathrm{RHom}(\mathbb{L}, - \otimes \mathbb{Z}(1))[1] = \mathrm{Cont}(- \otimes \mathbb{Z}(1)).$$

Proposition 3.3.14. $\mathrm{Cont} : \mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k, \mathbb{Z}) \longrightarrow \mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k, \mathbb{Z})$ and $\delta : \Sigma^{-1} \Longrightarrow \mathrm{Cont}(- \otimes \mathbb{Z}(1))$ have the following properties:

(i) Cont is compatible with cohomology in the sense that for each $p \in \mathbb{Z}$ there exist isomorphisms $\mathrm{Cont}(H^p(-, \mathscr{F})) \cong H^p(-, \mathrm{Cont}(\mathscr{F}))$ in $\mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k, \mathbb{Z})$ which are natural in $\mathscr{F} \in \mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k, \mathbb{Z})$.

(ii) Cont is compatible with truncation in the sense that for each $n \in \mathbb{Z}$ there exist isomorphisms $\mathrm{Cont}(\tau^{\leq n}(\mathscr{F})) \cong \tau^{\leq n}(\mathrm{Cont}(\mathscr{F}))$ in $\mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k, \mathbb{Z})$ which are natural in $\mathscr{F} \in \mathbb{A}^1\text{-Loc}_{\mathrm{Nis}}(k, \mathbb{Z})$.

(iii) Given $q \geq 0$ and l a prime number different from $char(k)$, the components of δ associated with each of the motivic complexes $\mathbb{Z}/l(q)$ and $\mathbf{R}\pi_*(\mathbb{Z}/l(q))$ are isomorphisms.

Suppose given $q \geq 0$ and l a prime number different from $char(k)$. Then item (iii) of

Proposition 3.3.14 yields isomorphisms

$$\delta_{\mathbb{Z}/l(q)} : \mathbb{Z}/l(q)[-1] \longrightarrow \mathrm{Cont}(\mathbb{Z}/l(q) \otimes \mathbb{Z}(1)) \cong \mathrm{Cont}(\mathbb{Z}/l(q+1)),$$

$$\delta_{\mathbb{Z}/l(q)_{Lich}} : \mathbb{Z}/l(q)_{Lich}[-1] \longrightarrow \mathrm{Cont}(\mathbb{Z}/l(q)_{Lich} \otimes \mathbb{Z}(1)) \cong \mathrm{Cont}(\mathbb{Z}/l(q+1)_{Lich})$$

in $\mathbb{A}^1$-$\mathrm{Loc}_{\mathrm{Nis}}(k,\mathbb{Z})$. Then by naturality we have the following commutative square in $\mathbb{A}^1$-$\mathrm{Loc}_{\mathrm{Nis}}(k,\mathbb{Z})$ associated to $\alpha_q^{\mathbb{Z}/l} : \mathbb{Z}/l(q) \longrightarrow \mathbb{Z}/l(q)_{Lich}$:

$$\begin{tikzcd}
\mathbb{Z}/l(q)[-1] \arrow[r, "\cong"] \arrow[d, "\alpha_q^{\mathbb{Z}/l}[-1]"'] & \mathrm{Cont}(\mathbb{Z}/l(q+1)) \arrow[d, "\alpha_{q+1}^{\mathbb{Z}/l} = \mathrm{Cont}(\alpha_q^{\mathbb{Z}/l} \otimes \mathbb{Z}(1))"] \\
\mathbb{Z}/l(q)[-1] \arrow[r, "\cong"'] & \mathrm{Cont}(\mathbb{Z}/l(q+1)_{Lich}).
\end{tikzcd}$$

As a consequence, if $\alpha_{q+1}^{\mathbb{Z}/l}$ is an isomorphism, then $\alpha_q^{\mathbb{Z}/l}[-1]$, hence $\alpha_q^{\mathbb{Z}/l}$, is an isomorphism. Thus if $BK(k, q+1)$ holds, then $BL(k, q)$ holds. This is Theorem 2.9 in Mazza *et al.*, 2006.

3.3.5 Some comparison results

Definition 3.3.15. Let X be any finite type k-scheme. We define the presheaf of abelian groups $\mathbb{Z}^{Sch}(X) : \mathrm{Sch}_k^{op} \to \mathrm{Ab}$ as the composite of the presheaf $\mathrm{Hom}_{\mathrm{Sch}_k}(-, X) : \mathrm{Sch}_k^{op} \to \mathrm{Set}$ with the free abelian group functor $\mathbb{Z}[-] : \mathrm{Set} \to \mathrm{Ab}$. Explicitly, we have

$$\begin{aligned}
\mathbb{Z}^{Sch}(X) : \mathrm{Sch}_k^{op} &\longrightarrow \mathrm{Ab} \\
Y &\longmapsto \mathbb{Z}[\mathrm{Hom}_{\mathrm{Sch}_k}(Y, X)] \\
(f : Y' \to Y) &\longmapsto (\sum_{i=1}^{m} n_i g_i \longmapsto \sum_{i=1}^{m} n_i (g_i \circ f)).
\end{aligned}$$

We will denote by:

(i) $\mathbb{Z}^{Sm}(X) : \mathrm{Sm}_k^{op} \to \mathrm{Ab}$ the restriction of $\mathbb{Z}^{Sch}(X)$ along $\mathrm{Sm}_k^{op} \hookrightarrow \mathrm{Sch}_k^{op}$.[2]

(ii) $\mathbb{Z}^{Sm}_{Nis}(X) : \mathrm{Sm}_k^{op} \to \mathrm{Ab}$ the Nisnevich sheafification of $\mathbb{Z}^{Sm}(X)$.

(iii) $\mathbb{Z}^{Sm}_{et}(X) : \mathrm{Sm}_k^{op} \to \mathrm{Ab}$ the étale sheafification of $\mathbb{Z}^{Sm}(X)$.

For any complex $\mathscr{F}$ of Nisnevich (resp. étale) sheaves on Sm_k, we define the *smooth-type Nisnevich (resp. étale) complex* of X with coefficients in $\mathscr{F}$ as

$$\mathrm{RHom}_{D(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Ab}))}(\mathbb{Z}^{Sm}_{Nis}(X), \mathscr{F}),$$
$$\mathrm{RHom}_{D(\mathrm{Sh}_{\mathrm{\acute{e}t}}(\mathrm{Sm}_k, \mathrm{Ab}))}(\mathbb{Z}^{Sm}_{et}(X), \mathscr{F}).$$

The *smooth-type Nisnevich (resp. étale) cohomology groups* of X with coefficients in $\mathscr{F}$ are

[2] Note that since Sm_k is a full subcategory of Sch_k, if $X \in \mathrm{Sm}_k$, then $\mathbb{Z}^{Sm}(X)$ is equal to the composite of the presheaf on Sm_k represented by X with the free abelian group functor.

defined as

$$\begin{aligned} H^n_{Sm,Nis}(X,\mathscr{F}) &:= H^n(\mathrm{RHom}_{D(\mathrm{Sh_{Nis}}(\mathrm{Sm}_k,\mathrm{Ab}))}(\mathbb{Z}^{Sm}_{Nis}(X),\mathscr{F})),\\ H^n_{Sm,et}(X,\mathscr{F}) &:= H^n(\mathrm{RHom}_{D(\mathrm{Sh_{\acute{e}t}}(\mathrm{Sm}_k,\mathrm{Ab}))}(\mathbb{Z}^{Sm}_{et}(X),\mathscr{F})).\end{aligned}$$

We then have

$$\begin{aligned} H^n_{Sm,Nis}(X,\mathscr{F}) &\cong \mathrm{Hom}_{D(\mathrm{Sh_{Nis}}(\mathrm{Sm}_k,\mathrm{Ab}))}(\mathbb{Z}^{Sm}_{Nis}(X),\mathscr{F}[n]),\\ H^n_{Sm,et}(X,\mathscr{F}) &\cong \mathrm{Hom}_{D(\mathrm{Sh_{\acute{e}t}}(\mathrm{Sm}_k,\mathrm{Ab}))}(\mathbb{Z}^{Sm}_{et}(X),\mathscr{F}[n]).\end{aligned}$$

Definition 3.3.16. Suppose given a finite type k-scheme X endowed with a presentation as a finite union of closed subsets, say $X = \bigcup_{i\in I} X_i$, where $I = \{1,\dots,N\}$ for some $n \geq 1$. We denote these data by $(X,\{X_i\}_{i\in I})$. In what follows, we assume each intersection $\bigcap_{i\in J} X_i$ for nonempty $J \subset I$ to be endowed with its reduced structure as a closed subscheme of X.

We define the *Čech complex of presheaves* of $(X,\{X_i\}_{i\in I})$, denoted by $\check{\mathbb{Z}}^{Sch}(X,\{X_i\}_{i\in I})$, as the following complex of presheaves of abelian groups on Sch_k:

$$\overset{-N}{0} \to \mathbb{Z}^{Sch}(\bigcap_{i\in I} X_i) \to \cdots \to \bigoplus_{\substack{J\subset I\\ |J|=m+1}} \mathbb{Z}^{Sch}(\bigcap_{i\in J} X_i) \to \bigoplus_{\substack{J\subset I\\ |J|=m}} \mathbb{Z}^{Sch}(\bigcap_{i\in J} X_i) \to \cdots \to \overset{0}{\bigoplus_{i\in I}} \mathbb{Z}^{Sch}(X_i) \to \overset{1}{0},$$

where for each m the differential $\bigoplus_{\substack{J\subset I\\ |J|=m+1}} \mathbb{Z}^{Sch}(\bigcap_{i\in J} X_i) \to \bigoplus_{\substack{J\subset I\\ |J|=m}} \mathbb{Z}^{Sch}(\bigcap_{i\in J} X_i)$ is defined as having components

$$\mathbb{Z}^{Sch}(\bigcap_{i\in J} X_i) \longrightarrow \mathbb{Z}^{Sch}(\bigcap_{i\in K} X_i)$$

for each $J, K \subset I$ with $|J| = m+1$, $|K| = m$ given by

$$\begin{cases} (-1)^{m+j}\mathbb{Z}^{Sch}(\bigcap_{i\in J} X_i \hookrightarrow \bigcap_{i\in K} X_i), & \text{if } K \subset J \text{ and } J\setminus K = \{j\},\\ 0, & \text{if } K \not\subset J.\end{cases}$$

This allows us to define several related complexes of (pre)sheaves which will be useful in what follows:

1. $\check{\mathbb{Z}}^{Sm}(X,\{X_i\}_{i\in I})$ will denote the complex of presheaves of abelian groups on Sm_k obtained by restriction of $\check{\mathbb{Z}}^{Sch}(X,\{X_i\}_{i\in I})$ along $\mathrm{Sm}_k^{op} \hookrightarrow \mathrm{Sch}_k^{op}$.
2. $\check{\mathbb{Z}}^{Sm}_{Nis}(X,\{X_i\}_{i\in I})$ will denote the Nisnevich sheafification of $\check{\mathbb{Z}}^{Sm}(X,\{X_i\}_{i\in I})$.
3. $\check{\mathbb{Z}}^{Sm}_{et}(X,\{X_i\}_{i\in I})$ will denote the étale sheafification of $\check{\mathbb{Z}}^{Sm}(X,\{X_i\}_{i\in I})$.

Note that by (either left or right) exactness of sheafification functors, entries of $\check{\mathbb{Z}}^{Sm}_{Nis}$ (resp. $\check{\mathbb{Z}}^{Sm}_{et}$) are of the form $\bigoplus_{\substack{J\subset I\\ |J|=m}} \mathbb{Z}^{Sm}_{Nis}(\bigcap_{i\in J} X_i)$ (resp. $\bigoplus_{\substack{J\subset I\\ |J|=m}} \mathbb{Z}^{Sm}_{et}(\bigcap_{i\in J} X_i)$), with differentials computed as in $\check{\mathbb{Z}}^{Sm}_{Nis}$ with $\mathbb{Z}^{Sm}_{Nis}$ (resp. $\mathbb{Z}^{Sm}_{et}$) applied to inclusions $\bigcap_{i\in J} X_i \hookrightarrow \bigcap_{i\in K} X_i$ (where $K \subset J \subset \{1,\dots,N\}$ and $J \setminus K$ is a singleton) instead of $\mathbb{Z}^{Sch}$.

For each $i \in I$, the closed immersion $X_i \hookrightarrow X$ induces a morphism of sheaves

$\mathbb{Z}^{Sch}(X_i) \longrightarrow \mathbb{Z}^{Sch}(X)$. Passing to the coproduct, we obtain a morphism $\bigoplus_{i\in I} \mathbb{Z}^{Sm}_{Nis}(X_i) \longrightarrow \mathbb{Z}^{Sm}_{Nis}(X)$ which, if placed in degree 0, defines a chain map

$$\check{\mathbb{Z}}^{Sch}(X, \{X_i\}_{i\in I}) \longrightarrow \mathbb{Z}^{Sch}(X)$$

with the zero map placed in each nonzero degree.

By restricting to Sm_k^{op} and applying the Nisnevich and étale sheafification functors, we obtain chain maps

$$\begin{aligned}
\check{\mathbb{Z}}^{Sm}(X, \{X_i\}_{i\in I}) &\longrightarrow \mathbb{Z}^{Sm}(X),\\
\check{\mathbb{Z}}^{Sm}_{Nis}(X, \{X_i\}_{i\in I}) &\longrightarrow \mathbb{Z}^{Sm}_{Nis}(X),\\
\check{\mathbb{Z}}^{Sm}_{et}(X, \{X_i\}_{i\in I}) &\longrightarrow \mathbb{Z}^{Sm}_{et}(X).
\end{aligned}$$

If F is a presheaf of abelian groups on Sch_k, we will denote by $[\check{\mathbb{Z}}^{Sch}(X, \{X_i\}_{i\in I}), F]$ the complex of abelian groups given by $[\check{\mathbb{Z}}^{Sch}(X, \{X_i\}_{i\in I}), F]^m = \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sch}_k,\mathrm{Ab})}(\check{\mathbb{Z}}^{Sch}(X, \{X_i\}_{i\in I})^{-m}, F)$ for each integer m, and whose m-th differential is given by precomposition with the $(-m-1)$-th differential in $\check{\mathbb{Z}}^{Sch}(X, \{X_i\}_{i\in I})$:

$$\begin{gathered}
\mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sch}_k,\mathrm{Ab})}(\check{\mathbb{Z}}^{Sch}(X, \{X_i\}_{i\in I})^{-m}, F) \longrightarrow \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sch}_k,\mathrm{Ab})}(\check{\mathbb{Z}}^{Sch}(X, \{X_i\}_{i\in I})^{-m-1}, F)\\
\varphi \longmapsto \varphi \circ d^{-m-1}.
\end{gathered}$$

Note that since $\check{\mathbb{Z}}^{Sch}(X, \{X_i\}_{i\in I})$ is concentrated in nonpositive degrees, $[\check{\mathbb{Z}}^{Sch}(X, \{X_i\}_{i\in I}), F]$ is concentrated in nonnegative degrees; and for each $m \geq 0$, the Yoneda lemma defines an isomorphism

$$\begin{aligned}
[\check{\mathbb{Z}}^{Sch}(X, \{X_i\}_{i\in I}), F]^m &= \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sch}_k,\mathrm{Ab})}(\check{\mathbb{Z}}^{Sch}(X, \{X_i\}_{i\in I})^{-m}, F)\\
&= \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sch}_k,\mathrm{Ab})}\Big(\bigoplus_{\substack{J\subset I\\ |J|=m+1}} \mathbb{Z}^{Sch}\big(\bigcap_{i\in J} X_i\big), F\Big)\\
&\cong \prod_{\substack{J\subset I\\ |J|=m+1}} \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sch}_k,\mathrm{Ab})}\big(\mathbb{Z}^{Sch}\big(\bigcap_{i\in J} X_i\big), F\big)\\
&\cong \prod_{\substack{J\subset I\\ |J|=m+1}} F\big(\bigcap_{i\in J} X_i\big).
\end{aligned}$$

The following variant of this construction will also be useful: suppose X and $\{X_i\}_{i\in I}$ as above are such that $X \in \mathrm{Sm}_k$ and for each nonempty $J \subset I$, $\bigcap_{i\in J} X_i \in \mathrm{Sm}_k$. Then if F is a presheaf of abelian groups on Sm_k, we similarly define $[\check{\mathbb{Z}}^{Sm}(X, \{X_i\}_{i\in I}), F]$ as the complex of abelian groups given by $[\check{\mathbb{Z}}^{Sm}(X, \{X_i\}_{i\in I}), F]^m = \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Sm}_k,\mathrm{Ab})}(\check{\mathbb{Z}}^{Sm}(X, \{X_i\}_{i\in I})^{-m}, F)$ for each integer m, and whose m-th differential is given by precomposition with the $-m-1$-th differential in $\check{\mathbb{Z}}^{Sm}(X, \{X_i\}_{i\in I})$. Analogously, $[\check{\mathbb{Z}}^{Sm}(X, \{X_i\}_{i\in I}), F]$ is concentrated in nonnegative degrees, and for each $m \geq 0$ the Yoneda lemma (for presheaves on Sm_k

instead of Sch_k) yields

$$[\check{\mathbb{Z}}^{Sm}(X, \{X_i\}_{i\in I}), F]^m \cong \prod_{\substack{J\subset I \\ |J|=m+1}} F(\bigcap_{i\in J} X_i).$$

Recall that if $\mathscr{F}$ is a Nisnevich sheaf of abelian groups on Sm_k, the n-th smooth-type Nisnevich cohomology group of X, denoted by $H^n_{Sm,Nis}(X, \mathscr{F})$, was defined as the n-th cohomology group of the derived hom complex

$$\mathrm{RHom}_{D(\mathrm{Sh_{Nis}}(\mathrm{Sm}_k,\mathrm{Ab}))}(\mathbb{Z}^{Sm}_{Nis}(X), \mathscr{F})).$$

Since $\mathrm{RHom}_{D(\mathrm{Sh_{Nis}}(\mathrm{Sm}_k,\mathrm{Ab}))}$ preserves quasi-isomorphisms, this complex is quasi-isomorphic to

$$\mathrm{RHom}_{D(\mathrm{Sh_{Nis}}(\mathrm{Sm}_k,\mathrm{Ab}))}(\check{\mathbb{Z}}^{Sm}_{Nis}(X, \{X_i\}_{i\in I}), \mathscr{F})),$$

so it suffices to compute cohomology groups of the latter complex.

Let us denote by Δ^n the standard algebraic n-simplex over k, i.e. $\Delta^n_k = \Delta^n_{alg} \times \mathrm{Spec}\ k \cong \mathrm{Spec}\ \frac{k[t_0,...,t_n]}{(\sum_{i=0}^n t_i - 1)}$. The *boundary* of Δ^n is defined as the closed sub-k-scheme obtained by imposing the vanishing of some coordinate among $t_0, ..., t_n$, namely,

$$\partial\Delta^n := \mathrm{Spec}\ \frac{k[t_0, ..., t_n]}{(\sum_{i=0}^n t_i - 1, \prod_{i=0}^n t_i)}.$$

Proposition 3.3.17 (Haesemeyer and C. Weibel, 2019, 2.15)**.** Suppose given $n \geq 0$, $q \geq 0$, and a homotopy invariant complex $\mathscr{F}$ of Nisnevich sheaves of abelian groups on Sm_k, there exists an isomorphism

$$H^q_{Sm,Nis}(\partial\Delta^n, \mathscr{F}) \cong H^q_{Sm,Nis}(\mathrm{Spec}\ k, \mathscr{F}) \oplus H^{q+1-m}_{Sm,Nis}(\mathrm{Spec}\ k, \mathscr{F}).$$

Proposition 3.3.18 (Haesemeyer and C. Weibel, 2019, 2.19)**.** Let $X \in \mathrm{Sm}_k$ be semilocal and presented as a finite union $X = \bigcup_{i=1}^N X_i$ of closed subsets such that for each nonempty $J \subset \{1, ..., N\}$, $\bigcap_{i\in J} X_i$ is smooth. Suppose $\mathscr{F}$ is a complex of Nisnevich sheaves with transfers whose cohomology sheaves are homotopy invariant. Then there is an isomorphism

$$H^n_{Sm,Nis}(X, \mathscr{F}) \xrightarrow{\cong} H^n([\check{\mathbb{Z}}^{Sm}(X, \{X_i\}_{i\in I}), F])$$

Recall from Definition 3.3.15 the functorial construction of the presheaf $\mathbb{Z}^{Sm}(X) : \mathrm{Sm}_k^{op} \longrightarrow \mathrm{Ab}$ associated to a finite type k-scheme X, and its Nisnevich sheafification $\mathbb{Z}^{Sm}_{Nis}(X) : \mathrm{Sm}_k^{op} \longrightarrow \mathrm{Ab}$.

In what follows, we refer to a pair (X, Z) consisting of a finite type k-scheme X and a closed subset $Z \subset X$ as a *closed pair* over k.

Definition 3.3.19. Suppose (X, Z) is a closed pair over k. We denote by $\mathbb{Z}^{Sm}(X, Z)$ the cokernel of $\mathbb{Z}^{Sm}(X \setminus Z) \longrightarrow \mathbb{Z}^{Sm}(X)$ in the category of presheaves of abelian groups on Sm_k, where $X \setminus Z \subset X$ is endowed with the open subscheme structure.

We denote by $\mathbb{Z}^{Sm}_{Nis}(X,Z)$ the Nisnevich sheafification of $\mathbb{Z}^{Sm}(X,Z)$, which by (right) exactness of the sheafification functor is isomorphic to the cokernel of $\mathbb{Z}^{Sm}_{Nis}(X \setminus Z) \longrightarrow \mathbb{Z}^{Sm}_{Nis}(X)$ in $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Ab})$ via the canonical map $\mathrm{Coker}(\mathbb{Z}^{Sm}_{Nis}(X \setminus Z) \longrightarrow \mathbb{Z}^{Sm}_{Nis}(X)) \longrightarrow \mathbb{Z}^{Sm}_{Nis}(X,Z)$.

Remark 3.3.20.

(i) Note that this extends the construction of $\mathbb{Z}^{Sm}_{Nis}(X)$ for a finite type k-scheme X given in Definition 3.3.15 in the following sense: by taking $Z \subset X$ to be X itself, we have $X \setminus Z = \emptyset$ as a finite type k-scheme. Now, $\mathbb{Z}^{Sm}(\emptyset)$ is not isomorphic to the zero presheaf $0 : \mathrm{Sm}_k^{op} \longrightarrow \mathrm{Ab}$, since although $\mathbb{Z}[\mathrm{Hom}_{\mathrm{Sch}_k}(Y,\emptyset)] = \mathbb{Z}[\emptyset] \cong 0$ (where $\emptyset$ in $\mathbb{Z}[\emptyset]$ denotes just the empty set) whenever Y is nonempty, we still have $\mathbb{Z}[\mathrm{Hom}_{\mathrm{Sch}_k}(\emptyset,\emptyset)] \cong \mathbb{Z}[\{*\}] \cong \mathbb{Z}$. However, the Nisnevich sheafification of $\mathbb{Z}^{Sm}(\emptyset)$ is isomorphic to 0, since (i) 0 is a Nisnevich sheaf, and (ii) any Nisnevich sheaf $\mathscr{F} : \mathrm{Sm}_k^{op} \longrightarrow \mathrm{Ab}$ satisfies $\mathscr{F}(\emptyset)$ (by applying the sheaf condition to the Nisnevich covering $\{\emptyset \longrightarrow \emptyset\}$), so the only morphism of abelian presheaves $\mathbb{Z}^{Sm}(\emptyset) \longrightarrow \mathscr{F}$ is the zero morphism, which then factors uniquely through $\mathbb{Z}^{Sm}(\emptyset) \longrightarrow 0$.

We conclude that the cokernel map $\mathbb{Z}^{Sm}_{Nis}(X) \longrightarrow \mathrm{Coker}(\mathbb{Z}^{Sm}_{Nis}(\emptyset) \longrightarrow \mathbb{Z}^{Sm}_{Nis}(X)) \cong \mathbb{Z}^{Sm}_{Nis}(X,X)$ is an isomorphism.

(ii) If $U \longrightarrow X$ is any open immersion (or indeed any monomorphism) of finite type k-schemes, then $\mathrm{Hom}_{\mathrm{Sch}_k}(Y,U) \longrightarrow \mathrm{Hom}_{\mathrm{Sch}_k}(Y,X)$ is injective for any $Y \in \mathrm{Sm}_k$ (or indeed any $Y \in \mathrm{Sch}_k$), so by taking free abelian groups it follows that $\mathbb{Z}^{Sm}(U) \longrightarrow \mathbb{Z}^{Sm}(X)$ is a monomorphism in $\mathrm{PSh}(\mathrm{Sm}_k, \mathrm{Ab})$. By applying sheafification, which is (left) exact, to the exact sequence $0 \longrightarrow \mathbb{Z}^{Sm}(U) \longrightarrow \mathbb{Z}^{Sm}(X)$, we obtain an exact sequence $0 \longrightarrow \mathbb{Z}^{Sm}_{Nis}(U) \longrightarrow \mathbb{Z}^{Sm}(X)$ in $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Ab})$, so $\mathbb{Z}^{Sm}_{Nis}(U) \longrightarrow \mathbb{Z}^{Sm}(X)$ is a monomorphism in the latter category.

In our context, given X and Z as in Definition 3.3.19, we have that $\mathbb{Z}^{Sm}(X \setminus Z) \longrightarrow \mathbb{Z}^{Sm}(X)$ and $\mathbb{Z}^{Sm}_{Nis}(X \setminus Z) \longrightarrow \mathbb{Z}^{Sm}_{Nis}(X)$ are monomorphisms in $\mathrm{PSh}(\mathrm{Sm}_k, \mathrm{Ab})$ and $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Ab})$, resp. Hence we shall also use the quotient notation $\mathbb{Z}^{Sm}(X)/\mathbb{Z}^{Sm}(X \setminus Z)$ and $\mathbb{Z}^{Sm}_{Nis}(X)/\mathbb{Z}^{Sm}_{Nis}(X \setminus Z)$ for $\mathbb{Z}^{Sm}(X,Z)$ and $\mathbb{Z}^{Sm}_{Nis}(X,Z)$, resp.

(iii) Suppose given a finite type k-scheme X and two inclusions of closed subsets $W \subset Z \subset X$. Then since $(X \setminus W) \setminus (Z \setminus W) = X \setminus Z$ and (co)kernels of abelian presheaves are computed objectwise, we have the following short exact sequence in $\mathrm{PSh}(\mathrm{Sm}_k, \mathrm{Ab})$:

$$0 \longrightarrow \frac{\mathbb{Z}^{Sm}(X \setminus W)}{\mathbb{Z}^{Sm}((X \setminus W) \setminus (Z \setminus W))} \longrightarrow \frac{\mathbb{Z}^{Sm}(X)}{\mathbb{Z}^{Sm}(X \setminus Z)} \longrightarrow \frac{\mathbb{Z}^{Sm}(X)}{\mathbb{Z}^{Sm}(X \setminus W)} \longrightarrow 0,$$

which is

$$0 \longrightarrow \mathbb{Z}^{Sm}(X \setminus W, Z \setminus W) \longrightarrow \mathbb{Z}^{Sm}(X,Z) \longrightarrow \mathbb{Z}^{Sm}(X,W) \longrightarrow 0.$$

By exactness of the sheafification functor, we obtain a short exact sequence

$$0 \longrightarrow \mathbb{Z}^{Sm}_{Nis}(X \setminus W, Z \setminus W) \longrightarrow \mathbb{Z}^{Sm}_{Nis}(X,Z) \longrightarrow \mathbb{Z}^{Sm}_{Nis}(X,W) \longrightarrow 0$$

in $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Ab})$.

By using this construction, cohomology of k-schemes in the sense of Definition 3.3.15

may be extended to a relative setting. If $\mathscr{F}$ is a complex of Nisnevich (resp. étale) sheaves on Sm_k and (X, Z) is a closed pair over k, we define the *smooth-type Nisnevich complex* of (X, Z) with coefficients in $\mathscr{F}$ as

$$\mathrm{RHom}_{D(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k,\mathrm{Ab}))}(\mathbb{Z}^{Sm}_{Nis}(X, Z), \mathscr{F}).$$

The corresponding cohomology groups will be denoted by

$$H^n_{Sm,Nis}(X, Z, \mathscr{F}) := H^n(\mathrm{RHom}_{D(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k,\mathrm{Ab}))}(\mathbb{Z}^{Sm}_{Nis}(X, Z), \mathscr{F})).$$

Then it holds that

$$H^n_{Sm,Nis}(X, Z, \mathscr{F}) \cong \mathrm{Hom}_{D(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k,\mathrm{Ab}))}(\mathbb{Z}^{Sm}_{Nis}(X, Z), \mathscr{F}[n]).$$

Proposition 3.3.21 (Haesemeyer and C. Weibel, 2019, 2.27)**.** Suppose given $q \geq 0$ and a closed pair over k of the form $(\partial\Delta^n, Z)$ such that Z does not contain any vertex of $\partial\Delta^n$. Then $BL(q-1)$ implies that the map between cohomology groups

$$H^q_{Sm,Nis}(\partial\Delta^n, Z, \mathbb{Z}/l(q)) \longrightarrow H^q_{Sm,Nis}(\partial\Delta^n, Z, \mathbb{Z}/l(q)_{Lich})$$

associated to the morphism $\alpha^{\mathbb{Z}/l}_q : \mathbb{Z}/l(q) \longrightarrow \mathbb{Z}/l(q)_{Lich}$ is an isomorphism.

In what follows, we consider the semilocalization of $\partial\Delta^n$ at its set of vertices $\{v_0, ..., v_n\}$, where v_i is the closed point corresponding to the maximal ideal

$$(t_0, ..., t_{i-1}, t_i - 1, t_{i+1}, ..., t_n) \supset (\sum_{i=0}^n t_i - 1, \prod_{i=0}^n t_i)$$

in $k[t_0, ..., t_n]$. Let us denote it by $\partial_{loc}\Delta^n$.

Proposition 3.3.22 (Haesemeyer and C. Weibel, 2019, 2.35)**.** Suppose given $n \geq 0$. Then for each $q \leq 0$ and $p > q$, the k-scheme $\partial_{loc}\Delta^n$ satisfies

$$H^p_{Sm,Nis}(\partial_{loc}\Delta^n, \mathbb{Z}(q)) \cong 0,$$
$$H^p_{Sm,Nis}(\partial_{loc}\Delta^n, \mathbb{Z}(q)/l) \cong 0.$$

Lemma 3.3.23 (Haesemeyer and C. Weibel, 2019, 2.36)**.** Suppose that the following condition holds: for every field k and every prime number $l \neq char(k)$, the comparison homomorphism

$$H^n_{\mathrm{Zar}}(\mathrm{Spec}\ k, \alpha^{\mathbb{Z}/l}_n) : H^n_{\mathrm{Zar}}(\mathrm{Spec}\ k, \mathbb{Z}/l(n)) \longrightarrow H^n_{\mathrm{Zar}}(\mathrm{Spec}\ k, \mathbb{Z}/l(n)_{Lich})$$

is surjective. Suppose given a scheme $X \in \mathrm{Sch}_k$ with the following properties:

(i) X is semilocal.

(ii) X is a finite union of smooth semilocal closed subschemes, say $X = \bigcup_{i=1}^N X_i$, such that for every nonempty $I \subset \{1, ..., N\}$, $\bigcap_{i\in I} X_i$ is smooth.

Then
$$H^n_{\mathrm{Zar}}(X, \alpha_n^{\mathbb{Z}/l}) \,:\, H^n_{\mathrm{Zar}}(X, \mathbb{Z}/l(n)) \longrightarrow H^n_{\mathrm{Zar}}(X, \mathbb{Z}/l(n)_{Lich})$$
is surjective.

Theorem 3.3.24. Let l be a prime number and n a non-negative integer. Suppose the comparison homomorphism

$$H^n_{\mathrm{Zar}}(\mathrm{Spec}\ k, \alpha_n^{\mathbb{Z}/l}) \,:\, H^n_{\mathrm{Zar}}(\mathrm{Spec}\ k, \mathbb{Z}/l(n)) \longrightarrow H^n_{\mathrm{Zar}}(\mathrm{Spec}\ k, \mathbb{Z}/l(n)_{Lich})$$

is surjective for every field k such that $char(k) \neq l$. Then for every field k,

$$\alpha_n^{\mathbb{Z}/l} \,:\, \mathbb{Z}/l(n) \longrightarrow \mathbb{Z}/l(n)_{Lich}$$

is a quasi-isomorphism of complexes of Zariski sheaves on Sm_k.

Proof. For fixed l, we proceed by induction on n. The case $n = 0$ follows from the fact that the Bloch-Kato conjecture holds in dimension 0.

Suppose that the result is known for $0, \ldots, n-1$, and that the statement's assumption holds for n. This assumption implies in particular that for any field k, the homomorphism $H^n_{\mathrm{Zar}}(\mathrm{Spec}\ k(t), \alpha_n^{\mathbb{Z}/l}) \,:\, H^n_{\mathrm{Zar}}(\mathrm{Spec}\ k(t), \mathbb{Z}/l(n)) \longrightarrow H^n_{\mathrm{Zar}}(\mathrm{Spec}\ k(t), \mathbb{Z}/l(n)_{Lich})$ associated to its function field is surjective. Then by Corollary 3.3.12, $H^{n-1}_{\mathrm{Zar}}(\mathrm{Spec}\ k, \alpha_n^{\mathbb{Z}/l}) \,:\, H^{n-1}_{\mathrm{Zar}}(\mathrm{Spec}\ k, \mathbb{Z}/l(n)) \longrightarrow H^{n-1}_{\mathrm{Zar}}(\mathrm{Spec}\ k, \mathbb{Z}/l(n)_{Lich})$ is surjective. Since this holds for every field k, it follows from the induction hypothesis that for every field k,

$$\alpha_{n-1}^{\mathbb{Z}/l} \,:\, \mathbb{Z}/l(n-1) \longrightarrow \mathbb{Z}/l(n-1)_{Lich}$$

is a quasi-isomorphism of complexes of Zariski sheaves on Sm_k.

□

Theorem 3.3.25 (Haesemeyer and C. Weibel, 2019, 2.38)**.** For each integer $n \geq 0$, if $H90(n)$ holds, then $BL(n)$ holds.

Proof. $\mathbb{Z}/l(n)$ and $\mathbb{Z}/l(n)_{Lich}$ are homotopy invariant complexes of presheaves with transfers. Hence by Lemma 2.5.3, it suffices to prove that for every prime number l, any field k such that $char(k) \neq l$, and any $p \leq n$, the comparison homomorphism

$$H^p_{\mathrm{Zar}}(\mathrm{Spec}\ k, \mathbb{Z}/l(n)) \xrightarrow{H^p_{\mathrm{Zar}}(\mathrm{Spec}\ k, \alpha_n^{\mathbb{Z}/l})} H^p_{\mathrm{Zar}}(\mathrm{Spec}\ k, \mathbb{Z}/l(n)_{Lich}) \cong H^p_{\text{ét}}(\mathrm{Spec}\ k, \mathbb{Z}/l(n))$$

is an isomorphism.

Now, let k be a fixed field and l a prime different from $char(k)$. By 3.3.23, if X is a finite disjoint union of semilocal schemes in Sm_k, then

$$H^n_{\mathrm{Zar}}(X, \alpha_n^{\mathbb{Q}/\mathbb{Z}_{(l)}}) \,:\, H^n_{\mathrm{Zar}}(X, \mathbb{Q}/\mathbb{Z}_{(l)}(n)) \longrightarrow H^n_{\mathrm{Zar}}(X, \mathbb{Q}/\mathbb{Z}_{(l)}(n)_{Lich})$$

is surjective for every $n \geq 0$. By 3.3.24,

$$H^p_{\mathrm{Zar}}(\mathrm{Spec}\ k, \alpha_n^{\mathbb{Q}/\mathbb{Z}_{(l)}}) \,:\, H^p_{\mathrm{Zar}}(\mathrm{Spec}\ k, \mathbb{Q}/\mathbb{Z}_{(l)}(n)) \longrightarrow H^p_{\mathrm{Zar}}(\mathrm{Spec}\ k, \mathbb{Q}/\mathbb{Z}_{(l)}(n)_{Lich})$$

is an isomorphism for every $p \leq n$. By considering long exact sequences associated to the short exact sequence of coefficient complexes $0 \longrightarrow \mathbb{Z}/l(n) \longrightarrow \mathbb{Q}/\mathbb{Z}_{(l)}(n) \xrightarrow{l} \mathbb{Q}/\mathbb{Z}_{(l)}(n)$, we obtain for each $p \leq n$ a commutative diagram (where we write k for Spec k):

$$\begin{array}{ccccccccc}
H^{p-1}_{\mathrm{Zar}}(k, \mathbb{Q}/\mathbb{Z}_{(l)}(n)) & \to & H^{p-1}_{\mathrm{Zar}}(k, \mathbb{Q}/\mathbb{Z}_{(l)}(n)) & \to & H^{p}_{\mathrm{Zar}}(k, \mathbb{Z}/l(n)) & \to & H^{p}_{\mathrm{Zar}}(k, \mathbb{Q}/\mathbb{Z}_{(l)}(n)) & \to & H^{p}_{\mathrm{Zar}}(k, \mathbb{Q}/\mathbb{Z}_{(l)}(n)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{p-1}_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}_{(l)}(n)) & \to & H^{p-1}_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}_{(l)}(n)) & \to & H^{p}_{\text{ét}}(k, \mathbb{Z}/l(n)) & \to & H^{p}_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}_{(l)}(n)) & \to & H^{p}_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}_{(l)}(n)).
\end{array}$$

By the five lemma, the middle vertical arrow is an isomorphism, as desired. $\square$

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