The Compactification of the Two Dimensional Monomial Map

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Resumo

Samanta Santos Avelino Silva. **Compactificação do Mapa Monomial de Duas Dimensões**. Tese (Doutorado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2023.

Dada uma matriz

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

em $SL_2(\mathbb{Z})$, podemos definir o mapa monomial associado $f_M : \mathbb{C}^2 \to \mathbb{C}^2$ por:

$$f_M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^a y^b \\ x^c y^d \end{pmatrix}.$$

No aberto $(\mathbb{C}^*)^2$, o mapa f_M é um biholomorfismo e sua dinâmica é bem conhecida Bonnot *et al.*, 2018. No entanto, como discutido por Favre Favre, 2003, essa dinâmica também pode ser estendida para $\mathbb{C}^2$ através da compactificação toroidal. Esse método, apesar de preciso, pode ser bastante técnico.

Nosso objetivo é providenciar uma abordagem alternativa e simplificada ao problema de compactificação, que provê os mesmos resultados de Favre. Usaremos a técnica dos "*blow-ups* de Stern-Brocot", que é similar a proposta por J. Hubbard e P. Papadopol J. Hubbard *et al.*, 2000 e J. H. Hubbard e Papadopol, 2008, para construir um espaço compacto X_M, que contém $(\mathbb{C}^*)^2$ como um subconjunto denso, e tal que f_M se estende a uma aplicação $F_M : X_M \to X_M$ como um sistema dinâmico. Esperamos que esse método ofereça uma perspectiva mais intuitiva e direta para a abordagem do problema.

Palavras-chave: Blow-ups. Mapa monomial. Compactificação. Frações contínuas.

Abstract

Samanta Santos Avelino Silva. **The Compactification of the Two Dimensional Monomial Map**. Thesis (Doctorate). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2023.

Given a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $SL_2(\mathbb{Z})$, we can define its associated monomial map $f_M : \mathbb{C}^2 \to \mathbb{C}^2$ as follows:

$$f_M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^a y^b \\ x^c y^d \end{pmatrix}.$$

In the open set $(\mathbb{C}^*)^2$, f_M is biholomorphic and its dynamics are well known Bonnot *et al.*, 2018. However, as discussed by Favre in Favre, 2003, the dynamics can also be extended to $\mathbb{C}^2$ through toric geometry compactification. This method, while precise, can be somewhat technical.

Our goal is to provide a simpler, alternative approach to the compactification problem that achieves the same results as Favre. We will use the "Stern-Brocot Blow-ups" technique, similar to the one proposed by J. Hubbard and P. Papadopol in J. Hubbard *et al.*, 2000 and J. H. Hubbard and Papadopol, 2008, to construct a compact space X_M, containing $(\mathbb{C}^*)^2$ as a dense subset, such that f_M extends to a map $F_M : X_M \to X_M$ as a dynamic system. We hope this method offers a more intuitive and straightforward perspective on the problem.

Keywords: Blow-ups. Monomial map. Compactification. Continued fractions.

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Introduction

The monomial map $F_M : (\mathbb{C}^*)^2 \longmapsto (\mathbb{C}^*)^2$, defined by

$$F_M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^a y^b \\ x^c y^d \end{pmatrix}$$

is an isomorphism, but there is no obvious way to extend this map to a compact space in a way that clarifies the global dynamics of the map. In order to make the map fully invertible on the entire complex plane, one needs to resolve its indeterminacies; this process is called *compactification* of the map.

For the monomial map, there is a well known compactification via toric geometry by Favre FAVRE, 2003. Another one, that could be simpler and more intuitive is via Farey blow-ups. This technique was invented by J. Hubbard J. HUBBARD *et al.*, 2000, but never fully used in this context. This is our objective.

Before giving more details, recall that a simple blow-up in the context of complex geometry is basically the insertion of a copy of a projective space at a point. Now the whole method of the Farey blow-ups consists in blowing-up points according to a certain order: the one given by the Stern-Brocot tree on the approach of the slopes of the eigenlines of the matrix M, the matrix defining our monomial map. For that reason, and in view of the notation used, we choose to call this chain of blow-ups the *Stern-Brocot-blow-ups* (but essentially they are the same as the *Farey blow-up* defined by Hubbard).

Given the broader range of topics we will need to approach, this thesis will also be an exploration of the relations between algebraic geometry, number theory and holomorphic dynamics. More specifically, it all comes down to how a hyperbolic matrix with positive coefficients can be simultaneously interpreted as: a blow-up, a monomial map and a continued fraction; and later on, how these correlations can be made more flexible.

We organize the text as follows: On Chapter 1 we give basic definitions and results involving blow-ups and continued fractions. On Chapter 2, we characterize matrices of $GL(2, \mathbb{Z})$ and determine the class of matrices we will focus on. This will conclude the introduction on the topics we need to prove our main results.

On Chapter 3, we will make the initial blow-ups necessary to define a birational extension of F_M. On Chapter 4 we construct a compactification for the monomial map, and on Chapter 5 we prove that the resulting space is a infinite chain of blow-ups.

In this work we assume the reader is familiar with basic concepts of Algebraic Geometry as, for instance, the definitions of varieties (affine and projective), rational maps and morphisms. If this is not the case, we advise the reading of sections 1.1 to 1.4 of Hartshorne, 1977 beforehand.

Chapter 1

The monomial map

A monomial map associated with a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is defined as

$$\begin{array}{rccc} f_M : & \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ & \begin{pmatrix} x \\ y \end{pmatrix} & \longmapsto & \begin{pmatrix} x^a y^b \\ x^c y^d \end{pmatrix} \end{array}.$$

If $M \in GL(2, \mathbb{Z})$, the restriction $f_M|_{(\mathbb{C}^*)^2}$ to the set $(\mathbb{C}^*)^2 = (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$ is an isomorphism, with $f_M^{-1} = f_{M^{-1}}$. The description of the dynamics in the hole $\mathbb{C}^2$ is not obvious, since the dynamic system may not be well defined on the lines $x = 0$ and $y = 0$. Nevertheless, the the torus $T = \{(x, y) \in \mathbb{C}^2 : |x| = |y| = 1\}$ is an invariant set and its stable and unstable manifolds are well known (see El Abdalaoui *et al.*, 2016, for instance). Now, if we intend to understand how the dynamics near torus extends to $\mathbb{C}^2$, is desirable to find a compact space where $f_M|_{(\mathbb{C}^*)^2}$ extends a well behaved map.

In this direction, as a fist step, it is natural to ask if f_M can be extended to a projective variety (on which $\mathbb{C}^2$ naturally embedded). If fact, there is more then one way to do that. For us, it will be convenient to consider the Cartesian product of the complex projective line (Segre embedding ensures that $\mathbb{CP}^1 \times \mathbb{CP}^1$ is indeed a projective variety).

First remember that

$$\mathbb{CP}^1 = (\mathbb{C}^2)^* / \sim,$$

where $x_1 \sim x_2$ iff $x_2 = \lambda x_1$ for some $\lambda \in (\mathbb{C}^2)^* = \mathbb{C}^2 \setminus \{0\}$. Writing the equivalence class of (x_1, x_2) in $\mathbb{CP}^1$ as $[x_1 : x_2]$, the map $\mathcal{P} : (x, y) \mapsto \big([x : 1], [y : 1]\big)$ is an embedding of the affine space $\mathbb{C}^2$ into $\mathbb{CP}^1 \times \mathbb{CP}^1$, with inverse $\mathcal{A} : \big([x_1 : x_2], [y_1 : y_2]\big) \mapsto (x_1/x_2, y_1/y_2)$. Moreover, the Definition 1.0.1 gives a natural extension of $f_M|_{(\mathbb{C}^*)^2}$ to $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Definition 1.0.1. Given a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $GL(2, \mathbb{Z})$ with two distinct real eigenvalues, we define the **projective monomial map associated with** M by $F_M = \mathcal{P} \circ f_M \circ \mathcal{A}$, i.e.

$$\begin{array}{rccc} F_M : & \mathbb{CP}^1 \times \mathbb{CP}^1 & \dashrightarrow & \mathbb{CP}^1 \times \mathbb{CP}^1 \\ & \begin{pmatrix} [x_1 : x_2] \\ [y_1 : y_2] \end{pmatrix} & \longmapsto & \begin{pmatrix} [x_1^a y_1^b : x_2^a y_2^b] \\ [x_1^c y_1^d : x_2^c y_2^d] \end{pmatrix} \end{array}.$$

Proposition 1.0.2. *Let M be a matrix in $GL(2,\mathbb{Z})$, then the projective monomial map F_M is a birational map between projective varieties, with inverse given by $F_{M^{-1}}$.*

Proof. For any matrix M, F_M is a rational map since on each chart of $\mathbb{CP}^1 \times \mathbb{CP}^1$, except perhaps at the origin, it is a monomial map (we give more details about this in Section 1.1). The last assertion follows directly from the fact that, for any $M \in GL(2,\mathbb{Z})$,

$$F_{M_1} \circ F_{M_2} = F_{M_1M_2}.$$

$\square$

1.1 Local expressions of the projective monomial map

The space $\mathbb{CP}^1 \times \mathbb{CP}^1$ is covered by the four open subsets $\{([x_1 : x_2],[y_1,y_2]) : x_i, y_j \neq 0\}$, with $i,j \in \{1,2\}$. They are interesting related with the matrices

$$\sigma_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \sigma_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad \sigma_3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \text{and} \quad \sigma_4 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

which we call **signal matrices**. Let $a, b = \pm 1$. Given $x, y \in \mathbb{C}^*$, the choice of a and b will determine the location of the point $([x^a : 1],[y^b : 1])$, in one of those sets. Since all possible combinations are given by $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) = \sigma_i$, $i = 1,2,3,4$, we use the signal matrices to label the covering for $\mathbb{CP}^1 \times \mathbb{CP}^1$ as follows: The open sets are

$$\begin{aligned} \mathcal{O}_{\sigma_1} &= \Big\{ \big([x:1],[y:1]\big) \,:\, x,y \in \mathbb{C} \Big\}, & \mathcal{O}_{\sigma_2} &= \Big\{ \big([x:1],[1:y]\big) \,:\, x,y \in \mathbb{C} \Big\}, \\ \mathcal{O}_{\sigma_3} &= \Big\{ \big([1:x],[y:1]\big) \,:\, x,y \in \mathbb{C} \Big\}, \quad \text{and} & \mathcal{O}_{\sigma_4} &= \Big\{ \big([1:x],[1:y]\big) \,:\, x,y \in \mathbb{C} \Big\}, \end{aligned}$$

and the homeomorphisms, $\varphi_{\sigma_i} : \mathcal{O}_{\sigma_i} \subset \mathbb{CP}^1 \times \mathbb{CP}^1 \longrightarrow \mathbb{C}^2_{\sigma_i}$, are

$$\begin{aligned} \varphi_{\sigma_1} &: \big([x:1],[y:1]\big) \mapsto (x,y)_{\sigma_1}, & \varphi_{\sigma_2} &: \big([x:1],[1:y]\big) \mapsto (x,y)_{\sigma_2}, \\ \varphi_{\sigma_3} &: \big([1:x],[y:1]\big) \mapsto (x,y)_{\sigma_3}, \quad \text{and} & \varphi_{\sigma_i} &: \big([1:x],[1:y]\big) \mapsto (x,y)_{\sigma_4}. \end{aligned}$$

Notice that $\varphi_{\sigma_1} = \mathcal{A}$, and that our original $(\mathbb{C}^*)^2$ is a dense open subset on each copy $\mathbb{C}^2_{\sigma_i}$. Further, the change of coordinates $\varphi_{ij} : \mathbb{C}^2_{\sigma_i} \longrightarrow \mathbb{C}^2_{\sigma_j}$ is the monomial map associated with the product $\sigma_i\sigma_j$, that is $\varphi_{ij} = f_{\sigma_i\sigma_j}$.

Remark 1.1.1. The following algebraic properties are satisfied

1. For any $i \in \{1,2,3,4\}$, we have $(\sigma_i)^{-1} = \sigma_i$ or, in other words, $(\sigma_i)^2 = \sigma_1$;
2. Let $i,j \in \{1,2,3,4\}$, then $\sigma_i\sigma_j = \sigma_j\sigma_i$;
3. The matrices σ_1 and σ_4 commutes with any two-by-two matrix;
4. For $i,j \in \{2,3\}$, with $i \neq j$, $\sigma_i\sigma_j = \sigma_4$.

From now on, we will identify each point $(x,y)_{\sigma_i}$ with its inverse $\varphi^{-1}_{\sigma_i}(x,y)_{\sigma_i}$ in $\mathcal{O}_{\sigma_i}$ and, for each $i = 1,2,3,4$, set $\mathbf{0}_{\sigma_i} = (0,0)_{\sigma_i}$.

Since the dynamic on $(\mathbb{C}^*)^2$ is well known, we would focus our efforts to understand how it extends to the set

$$\big(\{[0:1]\,;[1:0]\}\times\mathbb{CP}^1\big)\cup\big(\mathbb{CP}^1\times\{[0:1]\,;[1:0]\}\big)\,.$$

This is the divisor of $\mathbb{CP}^1\times\mathbb{CP}^1$, it consists of four projective lines intersecting transversely at the origin of the open covering sets. Each of this lines can be labeled such that the sub-index σ_i of their intersections appear as inherited from them. That is, if we define

$$\begin{aligned} \mathsf{L}_{\binom{1}{0}} &= \{[0:1]\}\times\mathbb{CP}^1, &\qquad \mathsf{L}_{\binom{0}{1}} &= \mathbb{CP}^1\times\{[0:1]\},\\ \mathsf{L}_{\binom{-1}{0}} &= \{[1:0]\}\times\mathbb{CP}^1, \quad \text{and} & \mathsf{L}_{\binom{0}{-1}} &= \mathbb{CP}^1\times\{[1:0]\}, \end{aligned}$$

it is easy to check that $\mathbf{0}_{\sigma_1}=\mathsf{L}_{\binom{1}{0}}\cap\mathsf{L}_{\binom{0}{1}}$, $\mathbf{0}_{\sigma_2}=\mathsf{L}_{\binom{1}{0}}\cap\mathsf{L}_{\binom{0}{-1}}$, $\mathbf{0}_{\sigma_3}=\mathsf{L}_{\binom{-1}{0}}\cap\mathsf{L}_{\binom{0}{1}}$, and $\mathbf{0}_{\sigma_4}=\mathsf{L}_{\binom{-1}{0}}\cap\mathsf{L}_{\binom{0}{1}}$, that is, the label of two intersecting lines are the column vectors of the matrix labeling their intersection (see Figure 1.1).

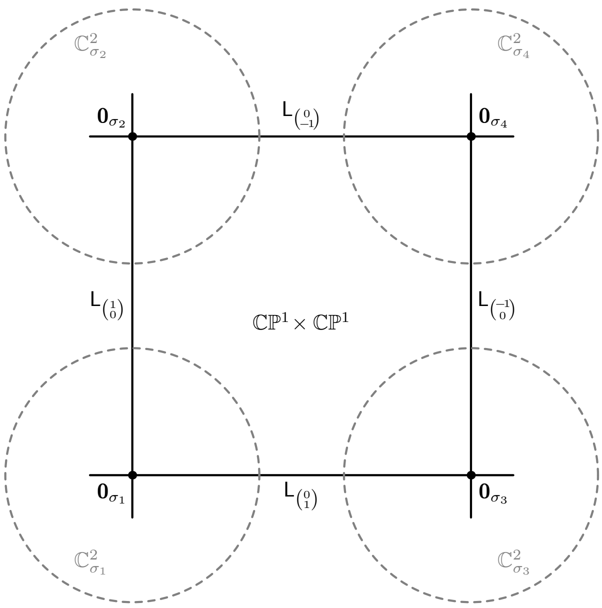


Figure 1.1: *The divisor of* $\mathbb{CP}^1\times\mathbb{CP}^1$.

Finally, the local action of the projective monomial map, that is, the maps $F_M|_{ij}$: $\mathbb{C}^2_{\sigma_i}\cap F_M^{-1}(\mathbb{C}^2_{\sigma_j})\to\mathbb{C}^2_{\sigma_j}$ defined by $F_M|_{ij} := \varphi_{\sigma_j}\circ F_M\circ\varphi_{\sigma_i}^{-1}$ (see Figure 1.2), are also monomial maps. More precisely

$$F_M|_{ij}\equiv f_{\sigma_j M\sigma_i}. \tag{1}$$

At this point, we reach our first problem, the points $\mathbf{0}_{\sigma_1}$, $\mathbf{0}_{\sigma_2}$, $\mathbf{0}_{\sigma_3}$, and $\mathbf{0}_{\sigma_4}$ are indeterminacies either of F_M or F_M^{-1}. Our approach to fix this will be to blow-up the singularities and, after that, fix the range in order to recover a dynamical system.

The issue is, on trying to fix the range we will produce new indeterminacies and, as result, we will end up having to make an infinite chain of blow-ups. This looks problematic, but as we will see, thanks to some nice relations between the continued fractions of

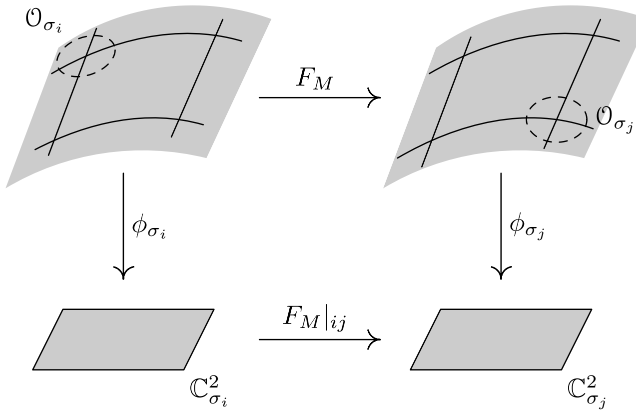


Figure 1.2: *Local expressions of F_M.*

the slope of the eigenlines of the matrix M and the chain of blow-ups that resolve all singularities, we will be able to deal with the final outcome.

1.2 The relation between blow-ups and matrices L and R

We start noticing that the blow-up at the origin of an affine plane is described locally by two monomial maps. More precisely, the blow-up at the origin, $\mathbf{0}_{\mathbb{A}^2}$, of $\mathbb{A}^2$ is defined as the space

$$\widetilde{\mathbb{A}}^2 = \{((x,y),[s:t]) \in \mathbb{A}^2 \times \mathbb{AP}^1 : xt = ys\},$$

together with the projection on the first factor $\pi : \widetilde{\mathbb{A}}^2 \longrightarrow \mathbb{A}^2$. The algebraic variety $\widetilde{\mathbb{A}}^2$ is covered by two affine charts $\mathcal{O}_L := \{((xy,y),[x:1]) : x,y \in \mathbb{A}\}$ and $\mathcal{O}_R := \{((x,xy),[1:y]) : x,y \in \mathbb{A}\}$, where the underscript is motivated by the monomial expression on the first factor which connects the charts to the matrices $L = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. The maps $((xy,y),[x:1]) \mapsto (x,y)$ and $((x,xy),[1:y]) \mapsto (x,y)$ are isomorphisms, and therefore we can identify each chart with a copy of the affine plane, lets say $\mathcal{O}_L \cong \mathbb{A}^2_L$ and $\mathcal{O}_R \cong \mathbb{A}^2_R$. We remind that the blow-up has only a local effect, gluing a copy of the projective line at the blown-up point. In fact the inverse $\pi^{-1} : (x,y) \longmapsto ((x,y),[s:t])$, defines an isomorphism between the blown-up space $\widetilde{\mathbb{A}}^2$ minus the exceptional divisor $\pi^{-1}(\mathbf{0}_{\mathbb{A}^2})$, and the affine chart minus the origin $\mathbb{A}^2 \setminus \{\mathbf{0}_{\mathbb{A}^2}\}$. For more details check Shafarevich, 2013, Chapter II for instance.

Under the identification of charts mentioned above, the restrictions $\pi|_{\mathbb{A}^2_L} : \mathbb{A}^2_L \longrightarrow \mathbb{A}^2$ and $\pi|_{\mathbb{A}^2_R} : \mathbb{A}^2_R \longrightarrow \mathbb{A}^2$ to each chart is given by the monomial maps associated with the matrices L and R from Remark 1.1.1, i.e.,

$$\pi|_{\mathbb{A}^2_L} = f_L \quad \text{and} \quad \pi|_{\mathbb{A}^2_R} = f_R. \tag{2}$$

As we will see, if we properly label the divisor after each blow-up, the identification gave above will allow us to prove that a chain of blow-ups in $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be describe locally by a monomial map associated with a product of the matrices L and R. The labeling

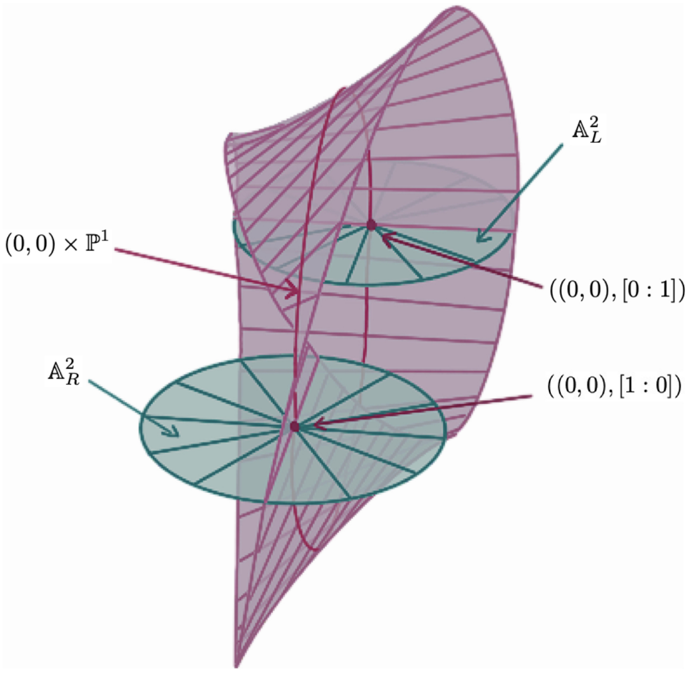


Figure 1.3: *The blow-up of the affine plane at the origin.*

rule that makes all work is the Farey sum used to define Stern-Brocot sequences and the Stern-Brocot tree.

1.3 Stern-Brocot tree and continued fractions

In this section we give a summary on basic definitions and properties regarding Stern-Brocot sequences (SB-sequences, for short) and its tree. More details can be found on Graham *et al.*, 1994, Chapters 4 and 6 or Hatcher, [2022] ©2022, Chapter 2, for instance.

Definition 1.3.1. The **Stern-Brocot sequence of order** n, denoted by s_n, is the ordered sequence of fractions built recursively as follows: Start with $s_0 := \{0/1, 1/0\}$ and then, for $n \geq 1$ natural, define s_n as the sequence obtained by copying s_{n-1} and inserting the *mediant* $\frac{p_1+p_2}{q_1+q_2}$ between all its adjacent fractions $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$. The mediant operation is also known as *Farey addition*. We say that $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are the **parents** of $\frac{p_1+p_2}{q_1+q_2}$, and that $\frac{p_1+p_2}{q_1+q_2}$ is the **child** of $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$.

In any sequence s_n, the fractions appear in ascending order and they are all irreducible. Also, every non-negative rational number is contained in some SB-sequence.

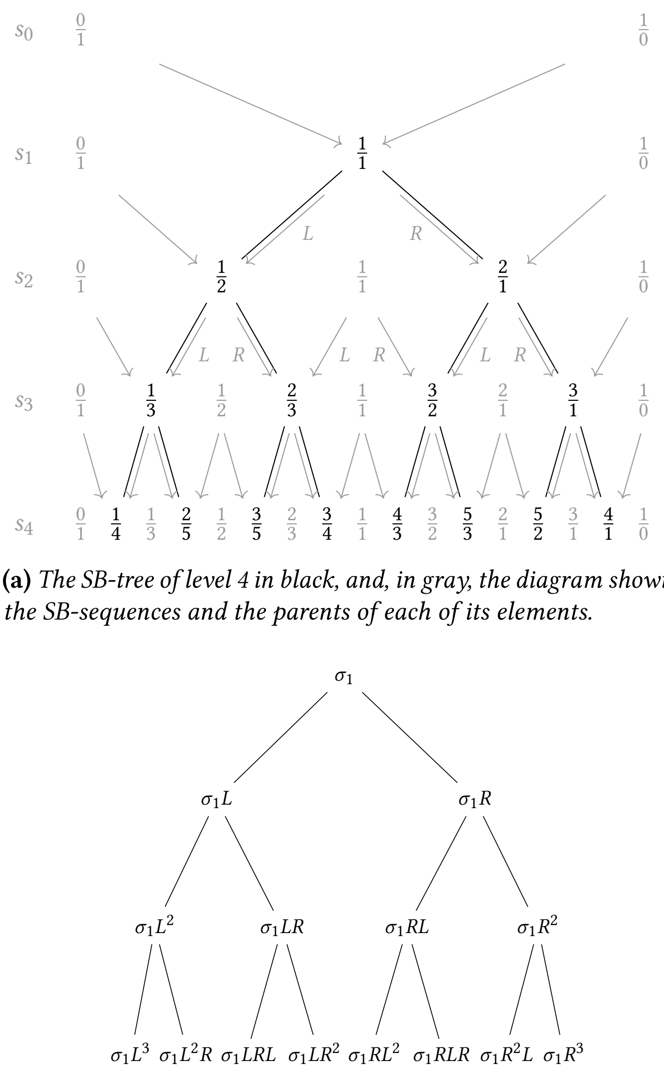
Definition 1.3.2. The tree that take vertices in the set $\bigcup_{n=1}^{\infty} s_n$ and connect each fraction in s_n with its parent created at s_{n-1}, $n \geq 1$, is called the **Stern-Brocot tree**.

The sub-tree having only the vertices in $\bigcup_{n=1}^{N} s_n$ is called **Stern-Brocot tree of level** N.

Let p/q and p'/q' be a parent-child pair of vertices in the SB-tree. If there is an edge between them, we say they are **connected**. Otherwise, they are said **disconnected**.

Remark 1.3.3. Every vertex on the SB-tree has two connected children. The vertex $1/1$ is the only one without a connected parent; any other vertex has exactly one connected parent.

A fraction p/q in the SB-tree can be identified with a vector (q, p) in $\mathbb{Z}^2$, and also with a matrix in $SL(2, \mathbb{Z}_{\geq 0})$ given by $\left(\begin{smallmatrix} q_1 & q_2 \\ p_1 & p_2 \end{smallmatrix}\right)$, where $p_1/q_1, p_2/q_2$ are the ordered parents of p/q. Figure 1.4 exemplifies this correspondence for the SB-tree of level 4.



(a) *The SB-tree of level 4 in black, and, in gray, the diagram showing the SB-sequences and the parents of each of its elements.*

(b) *The analogous SB-tree, using the matrix representation of the vertices.*

Figure 1.4: *Representations of the Stern-Brocot tree of level 4.*

As we will see, the main results presented in the thesis are consequence of this intimate connection between the matrices L, R and continued fractions. So lets elaborate a little more on how the identification mentioned above works. For that, we need to introduce our next definition.

Definition 1.3.4. The **continuant polynomial** of n parameters, $K_n(x_1, x_2, \ldots, x_n)$, is defined by the following recurrence:

$$\begin{aligned} K_0(\,) &= 1; \\ K_1(x_1) &= x_1; \\ K_n(x_1, \ldots, x_n) &= K_{n-1}(x_1, \ldots, x_{n-1})x_n + K_{n-2}(x_1, \ldots, x_{n-2}). \end{aligned}$$

Using induction, is not hard to show that

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ldots + \cfrac{1}{a_n}}}} = \frac{K_{n+1}(a_0, a_1 \ldots, a_n)}{K_n(a_1, a_2 \ldots, a_n)}, \tag{3}$$

where the first factor is a continued fraction which is denoted by $[a_0; a_1 : a_2 : \ldots : a_n]$, and

$$R^{a_0}L^{a_1}R^{a_2}L^{a_3} \ldots R^{a_{n-2}}L^{a_{n-1}} = \begin{pmatrix} K_{n-2}(a_1, \ldots, a_{n-2}) & K_{n-1}(a_1, \ldots, a_{n-2}, a_{n-1}) \\ K_{n-1}(a_0, \ldots, a_{n-2}) & K_n(a_0, a_1, \ldots, a_{n-2}, a_{n-1}) \end{pmatrix}, \tag{4}$$

where $a_0, a_{n-1} \in \mathbb{Z}_{\geq 0}$, and $a_1, \ldots, a_{n-2} \in \mathbb{Z}_{>0}$. Then, the map $\mathcal{M}_{\mathbb{Q}} : \begin{pmatrix} q_1 & q_2 \\ p_1 & p_2 \end{pmatrix} \mapsto (p_1 + p_2)/(q_1 + q_2)$, applied to the equation (4) yields the “Halphén’s Theorem” (see Halphén, 1877):

$$\mathcal{M}_{\mathbb{Q}}(R^{a_0}L^{a_1}R^{a_2}L^{a_3} \ldots R^{a_{n-2}}L^{a_{n-1}}) = \frac{K_{n+1}(a_0, a_1, \ldots, a_{n-1}, 1)}{K_n(a_1, \ldots, a_{n-1}, 1)}. \tag{5}$$

Finally, combining equations (3) and (5) we arrive at our desired outcome:

Theorem 1.3.5. *Let $a_0, \ldots, a_n$ be non-negative integers and $a_1, \ldots, a_{n-1}$ be positive integers. Then*

$$\mathcal{M}_{\mathbb{Q}}(R^{a_0}L^{a_1} \ldots R^{a_{n-1}}L^{a_n}) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ldots + \cfrac{1}{a_n + \cfrac{1}{1}}}}}.$$

Notice that, given a rational number α on the SB-tree, there exists a unique sequence $(a_0, \ldots, a_n)$, with $a_i \in \mathbb{Z}_{>0}$ such that either $\alpha = [0; a_0 : \ldots : a_n : 1]$ (in case $\alpha < 1$), or

$\alpha = [a_0; a_1 : \dots : a_n : 1]$ (in case $\alpha > 1$). Hence, the matrix

$$M = \left(\prod_{i=0}^{n} \delta_i^{a_i} \right),$$

where either $\delta_{2i} = L$ and $\delta_{2i+1} = R$ (in case $\alpha < 1$), or $\delta_{2i} = R$ and $\delta_{2i+1} = L$ (in case $\alpha > 1$), is the only one such that $\mathcal{M}_{\mathbb{Q}}(M) = \alpha$.

Definition 1.3.6. As just mentioned, a matrix on the SB-tree is given by an alternating product of powers of the matrices L and R. It will be useful to establish a notation that concisely informs the first and last matrix in this product together with the sequence of its (positive) powers. So we define the **SB-matrix** $\Gamma^{\delta_0\delta_n}_{a_0,a_1,\dots,a_n}$ by the word

$$\underbrace{\delta_0 \dots \delta_0}_{a_0 \text{ terms}} \underbrace{\delta_1 \dots \delta_1}_{a_1 \text{ terms}} \dots\dots \underbrace{\delta_{n-1} \dots \delta_{n-1}}_{a_{n-1} \text{ terms}} \underbrace{\delta_n \dots \delta_n}_{a_n \text{ terms}},$$

that is

$$\Gamma^{\delta_0\delta_n}_{a_0,a_1,\dots,a_n} := \left(\prod_{i=0}^{n} \delta_i^{a_i} \right),$$

where either $\delta_0 = L$ or $\delta_0 = R$; and

$$\delta_{i+1} = \begin{cases} R, & \text{if } \delta_i = L \\ L, & \text{if } \delta_i = R \end{cases}.$$

Notice that its enough to inform δ_0 and the sequence $(a_0, a_1, \dots, a_n) \in \left(\mathbb{Z}_{>0}\right)^{n+1}$ to uniquely determine $\Gamma^{\delta_0\delta_n}_{a_0,a_1,\dots,a_n}$ (since the parity of n gives δ_n). Hence, we also define $\Gamma^{\delta_0}_{a_0,a_1,\dots,a_n} := \Gamma^{\delta_0\delta_n}_{a_0,a_1,\dots,a_n}$.

Finally, we denote set of all possible SB-matrices by Γ.

Obviously, each matrix on Γ appear exactly once in the SB-tree. It will also be convenient to define the map

$$\begin{array}{rccc} \mathcal{M}_{\mathbb{Z}^2} : & GL(2,\mathbb{Z}) & \longrightarrow & \mathbb{Z}^2 \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & (a+b, c+d) \end{array}.$$

Before making some examples, we have a nice proposition:

Proposition 1.3.7. *The inverse matrix of a SB-matrix* $M = \Gamma^{\delta_0\delta_n}_{a_0,a_1,\dots,a_{n-1},a_n}$ *is given by*

$$M^{-1} = \sigma_2 \, \Gamma^{\delta_n\delta_0}_{a_n,a_{n-1},\dots,a_1,a_0} \, \sigma_2 = \sigma_3 \, \Gamma^{\delta_n\delta_0}_{a_n,a_{n-1},\dots,a_1,a_0} \, \sigma_3.$$

Proof. This is a direct consequence of the equalities

$$\sigma_2 L \sigma_2 = \sigma_3 L \sigma_3 = L^{-1} \quad \text{and} \quad \sigma_2 R \sigma_2 = \sigma_3 R \sigma_3 = R^{-1},$$

and from the fact that the products $\sigma_2\sigma_2$ and $\sigma_3\sigma_3$ are the identity matrix. □

Example 1.3.8. Lets see some examples for the matrices in definition 1.3.6:

(a) For $\delta_0 = L$ and $(a_0, a_1, a_2, a_4) = (2, 1, 1, 3, 5)$, we have

$$M = \Gamma^{L}_{2,1,1,3,5} = \Gamma^{LL}_{2,1,1,3,5} = L^2R^1L^1R^3L^5 = \begin{pmatrix} 18 & 95 \\ 7 & 37 \end{pmatrix},$$

and

$$M^{-1} = \sigma_2\,\Gamma^{LL}_{5,3,1,1,2}\,\sigma_2 = \begin{pmatrix} 37 & -95 \\ -7 & 18 \end{pmatrix}.$$

The representation of M as an integer pair, is given by

$$\mathcal{M}_{\mathbb{Z}^2}\left(\Gamma^{L}_{2,1,1,3,5}\right) = \mathcal{M}_{\mathbb{Z}^2}\begin{pmatrix} 18 & 95 \\ 7 & 37 \end{pmatrix} = (18 + 95, 7 + 37) = (113, 44),$$

and, as shown by Theorem, 1.3.5, its representation as a rational number, is

$$\mathcal{M}_{\mathbb{Q}}\left(\Gamma^{L}_{2,1,1,3,6}\right) = \frac{18+95}{7+37} = \frac{113}{44} = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{1}}}}}}.$$

(b) For $\delta_0 = R$ and $(a_0, a_1, a_2, a_3) = (4, 2, 3, 1)$, we have

$$M = \Gamma^{R}_{4,2,3,1} = \Gamma^{RL}_{4,2,3,1} = R^4L^2R^3L^1 = \begin{pmatrix} 40 & 31 \\ 9 & 7 \end{pmatrix},$$

and

$$M^{-1} = \sigma_2\,\Gamma^{LR}_{1,3,2,4}\,\sigma_2 = \begin{pmatrix} 40 & -9 \\ -31 & 7 \end{pmatrix}.$$

The representation of M as an integer pair, is

$$\mathcal{M}_{\mathbb{Z}^2}\left(\Gamma^{R}_{4,2,3,1}\right) = \mathcal{M}_{\mathbb{Z}^2}\begin{pmatrix} 7 & 9 \\ 31 & 40 \end{pmatrix} = (16, 71),$$

and its representation as a rational number, is given by

$$\mathcal{M}_{\mathbb{Q}}\left(\Gamma^{R}_{4,2,3,1}\right) = \frac{71}{16} = 4 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1}}}}.$$

We give some basic definitions and results below. They are part of a broader theory, usually referred as Integer Geometry. Nice places to look for detailed demonstrations and to check the bigger context where they are settled in are Graham *et al.*, 1994 and Karpenkov, 2022.

Remark 1.3.9. (i) Any real number (rational or not) has a continued fraction expansion. If the number is irrational then it has infinite continued fraction. The number $[a_0; a_1 : a_2 : \ldots : a_k]$ is called the k-**convergent** to the finite or infinite continued fraction $[a_0; a_1 : a_2 : \ldots]$, and it is such that

$$\lim_{k\to\infty}[a_0; a_1 : a_2 : \ldots : a_k] = [a_0; a_1 : a_2 : \ldots].$$

(ii) For every rational number there exists a unique continued fraction with an odd/even number of entries, such that $\alpha = [a_0 : a_1 : \ldots : a_n] = [a_0 : a_1 : \ldots : a_n - 1 : 1]$. If α is irrational, the continued fraction expansion is unique and infinite, also the k-convergents converge to α.

From Theorem 1.3.5 and Remark 1.3.9, we conclude that the restriction $\mathcal{M}_{\mathbb{Q}}\big|_\Gamma$ is a bijection between SB-matrices and irreducible rational numbers, and $\mathcal{M}_{\mathbb{Z}^2}|_\Gamma$ is a bijection between SB-matrices and points of $\mathbb{Z}^2$ with coprime entries. Further, if α is a positive irrational number with continued fraction given by $[a_0; a_1 : a_2 : a_3 : \ldots]$,

$$\lim_{k\to\infty} \mathcal{M}_{\mathbb{Q}}(R^{a_0}L^{a_1}\ldots R^{a_{n-1}}L^{a_k}) = \alpha.$$

1.3.1 The sail of a positive number

There exists a geometric interpretation for the continued fraction of any number $\alpha > 0$. The associated ray $r_\alpha := \{y = \alpha x : x \geq 0\}$ divides the first quadrant $\{(x, y) : x, y \geq 0\}$ in two angles. The **sail** of such an angle is defined as boundary of the convex hull of all integer points except the origin inside this angle; see Figure 1.5 for an example.

The sails consists of two broken lines, one starting in $A_0 := (1, 0)$ and having vertices $A_1, A_2, \ldots$, denote by $A_0A_1\ldots$, and the other starting in $B_0 := (0, 1)$ and having vertices $B_0, B_1, \ldots$, denoted by $B_0B_1\ldots$. If α is a rational number both sails consists of a ray plus finitely many segments. Further, if they are given by $A_0\ldots A_n$ and $B_0\ldots B_m$, then $m = n - 1$ or $m = n$. In case of an irrational α, each sail is the union of a ray and an infinite broken line. For short, instead of saying that $A_0\ldots A_n$ and $B_0\ldots B_m$ are the sails of the angles defined by the ray r_α, we say simply that they are the sails of α.

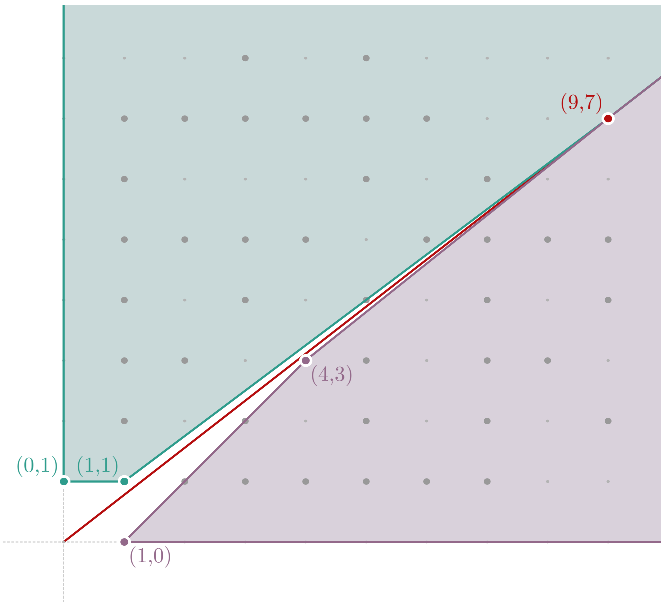


Figure 1.5: *The green and the purple lines are the sails for* $\alpha = 7/9 = [0; 1 : 3 : 2]$*. Points of the SB-sequences are the bigger ones.*

The next proposition, stated in KARPENKOV, 2022, Theorems 3.1 and 3.5, gives a relation between the points of the sails and the continued fraction convergents of a given number α.

Proposition 1.3.10. *Let* $\alpha = [a_0; a_1 : a_2 : \ldots]$ *be a nonnegative real number, and* $A_0A_1A_2\ldots$ *and* $B_0B_1B_2\ldots$ *be the sails (finite or infinite) of* α*, with* $A_0 = (1, 0)$ *and* $B_0 = (0, 1)$*. Then, if* α *is irrational, we have*

(*i*) *for* $\alpha \geq 1$,

$$A_i = (q_{2i-2}, p_{2i-2}) \quad \textit{and} \quad B_i = (q_{2i-1}, p_{2i-1}), \quad i = 1, 2, \ldots$$

(*ii*) *for* $0 < \alpha < 1$,

$$A_i = (q_{2i}, p_{2i}) \quad \textit{and} \quad B_i = (q_{2i-1}, p_{2i-1}), \quad i = 1, 2, \ldots$$

where p_k/q_k *are convergents. If* α *is rational, the same holds except for the last vertices of both sails; they coincide with* (q_n, p_n)*, where* p_n/q_n *is the last convergent, i.e.,* $\alpha = p_n/q_n$*.*

The Corollary 1.3.11 relates SB-matrices with the vertices of the sails of a given positive irrational number. Its proof is a direct consequence of Proposition 1.3.10 and the identification between continued fractions and matrices given by Theorem 1.3.5.

Corollary 1.3.11. *Let* α *be a positive irrational number such that its sails are* $A_0A_1\ldots A_n$ *and* $B_0B_1\ldots B_n$*, where* $A_0 = (1, 0)$ *and* $B_0 = (0, 1)$*. Then, for* $i = 1, 2, 3\ldots$,

(*i*) *if* $\alpha \geq 1$ *and its continued fraction is given by* $[a_0; a_1 : a_2 : a_3 : \ldots]$,

$$A_i = \mathcal{M}_{\mathbb{Z}^2}\left(\Gamma^{RR}_{a_0,\ldots,a_{2i-2}} R^{-1}\right) \quad \textit{and} \quad B_i = \mathcal{M}_{\mathbb{Z}^2}\left(\Gamma^{RL}_{a_0,\ldots,a_{2i-1}} L^{-1}\right),$$

(*ii*) *if* $0 < \alpha < 1$ *and its continued fraction is given by* $[0; a_0 : a_1 : a_2 : \dots]$

$$A_i = \mathcal{M}_{\mathbb{Z}^2}\left(\Gamma^{LR}_{a_0,\dots,a_{2i-1}} R^{-1}\right) \quad B_i = \mathcal{M}_{\mathbb{Z}^2}\left(\Gamma^{LL}_{a_0,\dots,a_{2i}} L^{-1}\right)$$

As shown in Figure 1.5, all the elements of the SB-tree that are on the path which starts at σ_1 and leads to α are in the sails of α. Further, the vertices of these sails occur when we have a change of direction on the path at the SB-three (left to right or vice-versa).

Remark 1.3.12. Let $A_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$. The matrix $\left(\begin{smallmatrix} a_1+b_1 & a_2+b_2 \\ c_1+d_1 & c_2+d_2 \end{smallmatrix}\right)$ has ordered column vectors given by $\mathcal{M}_{\mathbb{Z}^2}(A_1)$ and $\mathcal{M}_{\mathbb{Z}^2}(A_2)$, respectively.

1.3.2 A generalization of the Stern-Brocot tree

For our purpose, it will be necessary to define an extended version of the SB-tree. The idea will be to create three additional branches which will be essentially a copy of the original tree, but with $\sigma_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ replaced by some of the other signal matrices $\sigma_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $\sigma_4 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and gluing then in $\mathbb{Z}^2$ appropriately.

Definition 1.3.13. For $i \in \{1, 2, 3, 4\}$ and $N \neq 0$ natural, let $\mathcal{T}_i^N$ be the tree such that

(i) the vertices are given by all matrices of the form $\sigma_i \Gamma^{\delta_0}_{a_0,a_1,\dots,a_n}$, with $n < N$,

(ii) two vertices A and B are connected by an edge if $B = AL$ or $B = AR$.

The **Stern-Brocot diagram of level** N is the diagram in $\mathbb{Z}^2$ such that, its vertices are matrices M in $\mathcal{T}_i^N$, $i \in \{1, 2, 3, 4\}$, positioned at the point $\mathcal{M}_{\mathbb{Z}^2}(M)$ of $\mathbb{Z}^2$, and its edges are given by the rule (ii) above.

The **Stern-Brocot diagram** is the infinite diagram obtained making $N \to \infty$ in the definition of the Stern-Brocot diagram of level N. That is, it is constituted by four ordered copies of the SB-tree, one for each of the limits $\mathcal{T}_i = \lim_{N\to\infty} \mathcal{T}_i^N$, $i \in \{1, 2, 3, 4\}$. We say that a vertex A of the SB-diagram is at the **level** n_0 if, for some $i \in \{1, 2, 3, 4\}$, $A = \sigma_i \Gamma^{\delta_0}_{a_0,a_1,\dots,a_{n_0}}$. (See Figure 1.6.)

As for the SB-tree, a vertex $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of the SB-diagram is identified with the vector $(a + b, c + d)$ of $\mathbb{Z}^2$. But the identification with $\frac{c+d}{a+b}$ fails since, with the construction gave before, this rational number would represent both the matrices $\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We notice that, the parents of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ still are (a, c) and (b, d) in vector representation.

1.3.3 The sail of a hyperbolic matrix

Given a hyperbolic matrix M in $GL(2, \mathbb{Z})$, that is, with $|tr(M)| > 2$, we can look at the eigenlines of M and, motivated by the definition of sails of a positive number, we also define the following:

Definition 1.3.14. Let M be a hyperbolic matrix in $GL(2, \mathbb{Z})$. Inside each of the four regions delimited by the eigenlines of M, take the convex hull of the set of all integer points, except the origin. The boundary of each convex hull is an infinite broken line called **sail** of M. If the vertices of this sail are $\dots, S_{-2}, S_{-1}, S_0, S_1, S_2, \dots$, we denote the sail

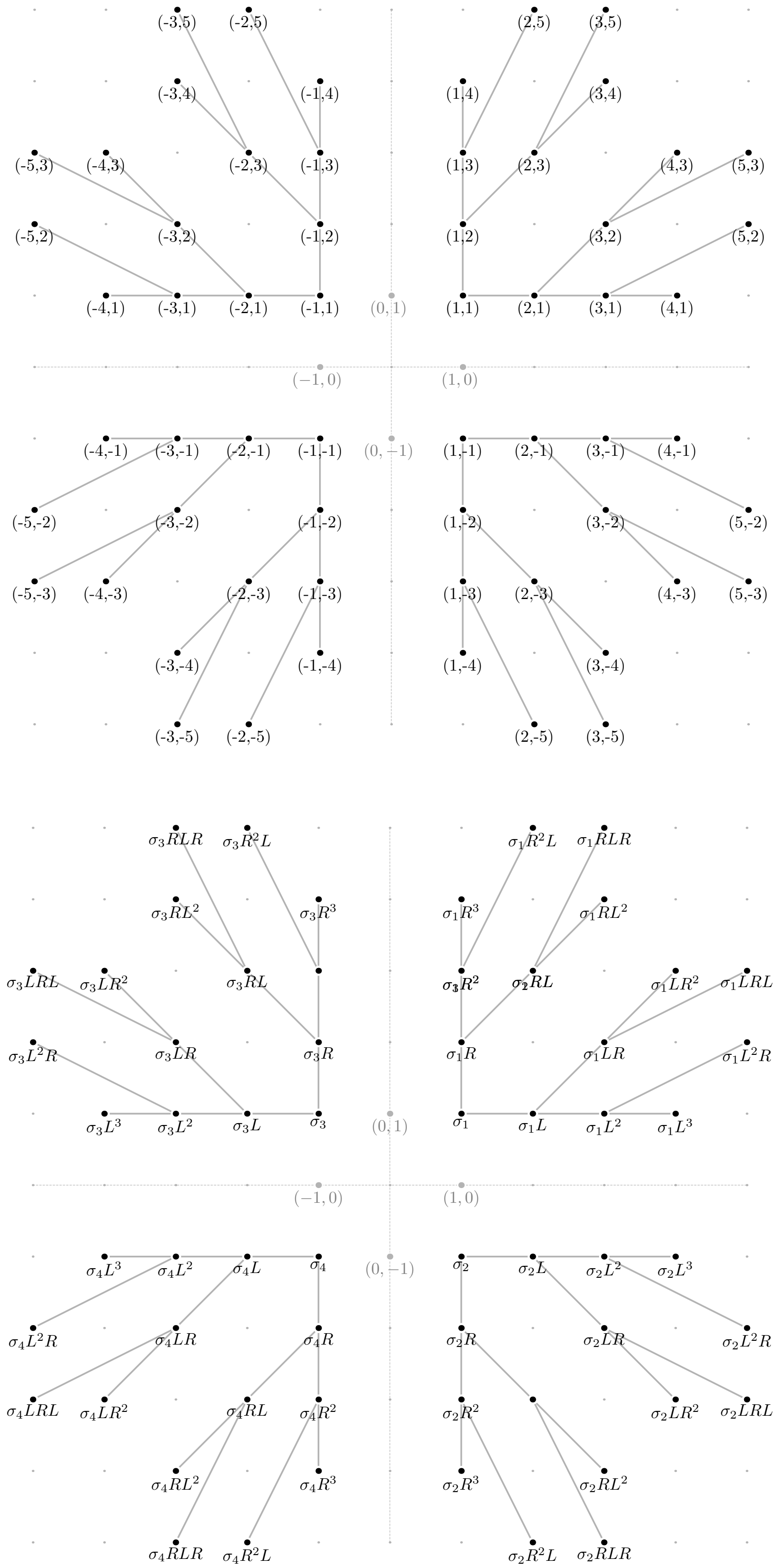


Figure 1.6: *Both diagrams show the SB-diagram of level 4, the left one using vector representation and the right one using matrix representation.*

by $\ldots S_{-2}S_{-1}S_0S_1S_2\ldots$. The union of the four sails of M is called the **geometric continued fraction** of M.

Example 1.3.15. The eigenlines of the matrix $M = \Gamma^R_{1,2,3,1,3,1} = \left(\begin{smallmatrix} 34 & 43 \\ 49 & 52 \end{smallmatrix}\right)$ have slopes

$$\mu_1 = \frac{-7}{2-\sqrt{47}} = [1; 2 : 3 : 1 : 3 : 1 : 1 : \ldots]$$

and

$$\mu_2 = \frac{-7}{2+\sqrt{47}} = -[0; 1 : 3 : 1 : 3 : 2 : 1 : \ldots].$$

Exactly one of the sails of M, lets say S=$\ldots S_{-2}S_{-1}S_0S_1S_2\ldots$, is contained in the region $\{(x,y) \in \mathbb{R}^2 : x \geq 0\}$. From Corollary 1.3.11, the vertices of S can be obtained from the vertices of the rays r_{μ_1} and r_{μ_2}. More precisely, choosing $S_0 = (1,0)$, the vertices with positive y-coordinate are

$$S_i = \mathcal{M}_{\mathbb{Z}^2}\left(\Gamma^{RR}_{a_0,\ldots,a_{2i-2}}R^{-1}\right),$$

where $[a_0; a_1 : a_2 : \ldots]$ is the continued fraction of μ_1; and the vertices with negative y-coordinate are

$$S_{-i} = \sigma_2\mathcal{M}_{\mathbb{Z}^2}\left(\Gamma^{LL}_{b_0,\ldots,b_{2i-2}}L^{-1}\right),$$

where $[b_0; b_1 : b_2 : \ldots]$ is the continued fraction of μ_2. See the geometric continued fraction of M at Figure 1.7. As we will see, is not a coincidence that the underscript sequence of the SB-matrix defining M appears on the continued fraction on the slopes μ_1 and μ_2.

If $M \in SL(2,\mathbb{Z})$ is hyperbolic, that is it has two distinct real eigenvalues, then M has four sails and only one of them is entirely contained in the region $\mathbb{R}^2_{x>0} = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ of the plane. Hence we can order the sails of the geometric continued fraction of M as follows:

Definition 1.3.16. Let M be a hyperbolic matrix in $SL(2,\mathbb{Z})$. The sail of M that is entirely contained on the region $\mathbb{R}^2_{x>0}$ is called the **first sail** of M. From the first sail, the other ones are named **second**, **third** and **forth** sail of M as they appear counterclockwise.

Remark 1.3.17. (i) Since the matrices M, $-M$ and M^{-1} have the same eigenvectors, they all have the same geometric continued fraction.

(ii) A hyperbolic matrix M in $SL(2,\mathbb{Z})$ preserves its eigenlines and takes integers in to integers. Hence, if it also has two positive eigenvalues, each sail is invariant under the action of M. That is, if $\mathbf{q}$ is a point on a sail S of M, then $M(\mathbf{q})$ also belongs to S. Now, if M has negative eigenvalues and $\mathbf{q}$ is in S, then $-M(\mathbf{q})$ also is.

(iii) Let M be an hyperbolic matrix M in $GL(2,\mathbb{Z})$. To determine all vertices of the continued fraction of M, is enough to look at the sails of rays r_{μ_1} and r_{μ_2} which angles are given respectively by the slopes μ_1 and μ_2 of the eigenlines of M. The vertices of r_{μ_1} and r_{μ_2} will give all the vertices in the geometric continued fraction up to multiplication by a signal matrix.

There is a extensive and interesting theory in Integer Geometry that addresses the concept of geometric continued fractions in a broader context. In order to keep a more focused approach on our main problem, we will refrain from addressing most of those

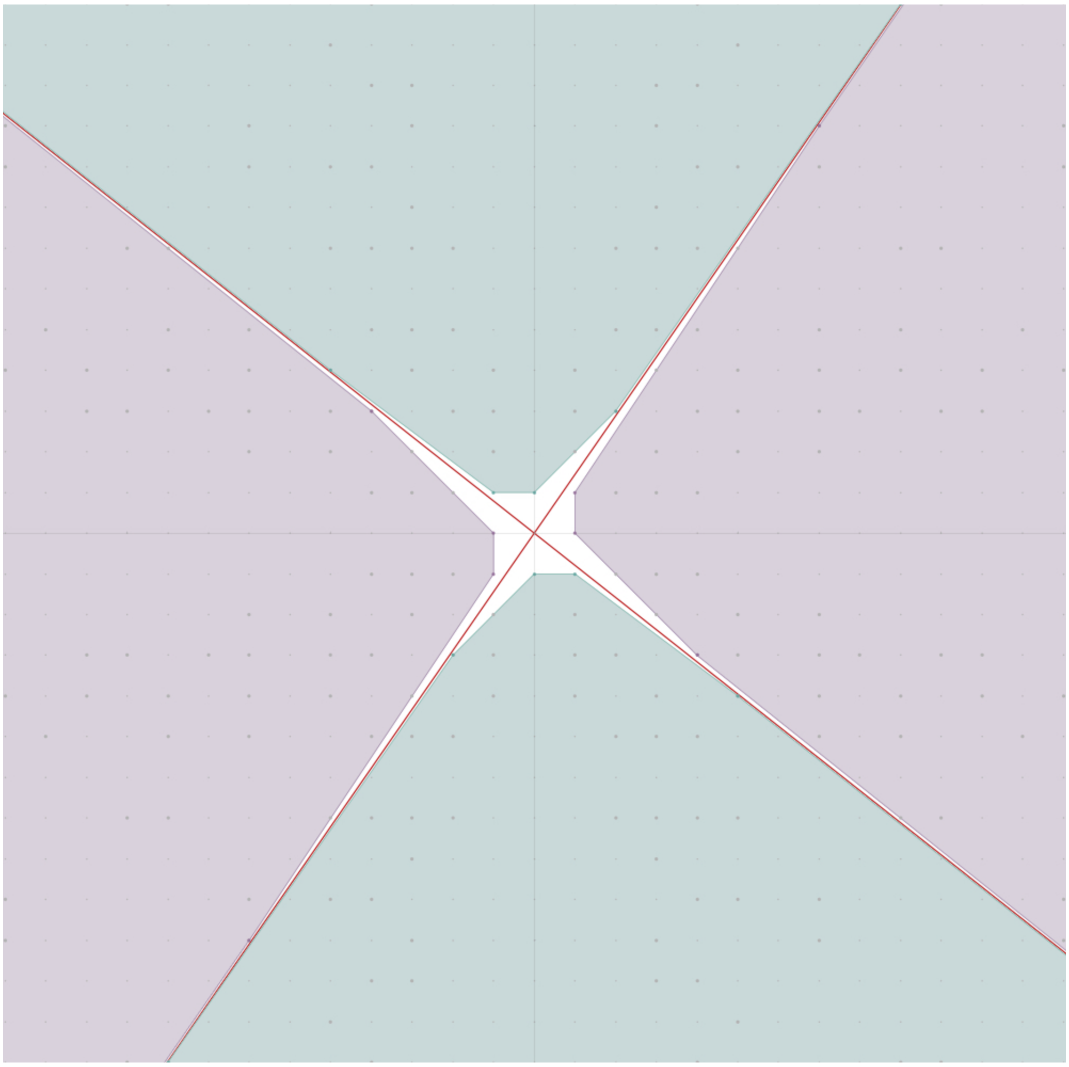


Figure 1.7: *The boundary of the colored regions belong to the geometric continued fraction of* $M = \Gamma^R_{1,2,3,1,3,1}$. *The bigger points, except for* $\pm(1,0)$, $\pm(0,1)$, *belong to the Stern-Brocot diagram.*

topics in this thesis. For interested readers, we highly recommend the book Karpenkov, 2022 as an introductory text.

1.4 SB-blow-ups: A compatible notation for blow-ups chains in $\mathbb{CP}^1 \times \mathbb{CP}^1$

From the explained in the first section of this chapter, a blow-up at the center of a complex affine plane, $\widetilde{\mathbb{C}}^2$, can be described using matrices L and R in a way that the labels $\mathbb{C}^2_L$ and $\mathbb{C}^2_R$ of the covering charts of $\widetilde{\mathbb{C}}^2$ indicate the action on the blow-up in local coordinates. Hence, due to its structure and matrix representation, the Stern-Brocot tree seems a suitable tool for tracking chains of this type of blow-ups. In fact, we will be able to use it to establish a notation that, in itself, gives the description of the blow-up in any of its covering charts.

Definition 1.4.1. Let M be an SB-matrix. The **SB-blow-up** of M, denoted by $\boldsymbol{\pi}_M : \mathbb{B}_M \longrightarrow \mathbb{C}^2$ is a chain of blow-ups performed at $\mathbb{C}^2$ defined recursively as follows:

(i) If $M = \sigma_1$, that is, M is the identity matrix, the SB-blow-up of M is simply the standard blow-up $\pi : \widetilde{\mathbb{C}}^2 \longrightarrow \mathbb{C}^2$ of $\mathbb{C}^2$ at the origin. That is $\boldsymbol{\pi}_{\sigma_1} = \pi$ and $\mathbb{B}_{\sigma_1} = \widetilde{\mathbb{C}}^2$. The blown-up space $\mathbb{B}_{\sigma_1}$ is covered by two copies of the complex plane, $\mathbb{C}^2_L$ and $\mathbb{C}^2_R$, such that $\boldsymbol{\pi}_{\sigma_1}|_{\mathbb{C}^2_L} = f_L$ and $\boldsymbol{\pi}_{\sigma_1}|_{\mathbb{C}^2_R} = f_R$, see Section 1.2. We also label the x and y axis of $\mathbb{C}^2$ by $\mathsf{L}_{\binom{0}{1}}$ and $\mathsf{L}_{\binom{1}{0}}$, respectively.

(ii) Let $M = \delta_1\delta_2 \dots \delta_k$ be a SB-matrix, that is δ_i is either L or R, and set $M' = \delta_1\delta_2 \dots \delta_{k-1}$. The space $\mathbb{B}_M$ is the blow-up of at the origin of the chart $\mathbb{C}^2_M \subset \mathbb{B}_{M'}$, and $\boldsymbol{\pi}_M$ is the composition of the $k+1$ standard blow-ups of complex affine planes at origin

$$\mathbb{B}_{\delta_1\delta_2\dots\delta_k} \xrightarrow{\pi} \mathbb{B}_{\delta_1\delta_2\dots\delta_{k-1}} \xrightarrow{\pi} \quad \dots \quad \xrightarrow{\pi} \mathbb{B}_{\delta_1\delta_2} \xrightarrow{\pi} \mathbb{B}_{\delta_1} \xrightarrow{\pi} \mathbb{B}_{\sigma_1} \xrightarrow{\pi} \mathbb{C}^2.$$

The space $\mathbb{B}_M$ is covered by the charts of $\mathbb{B}_{M'}$ plus two additional ones, $\mathbb{C}^2_{M'L}$ and $\mathbb{C}^2_{M'R}$, where the action of the standard blow-up $\pi : \mathbb{B}_M \longrightarrow \mathbb{B}_{M'}$ is given locally by $\pi|_{\mathbb{C}^2_{M'L}} = f_L$, $\pi|_{\mathbb{C}^2_{M'R}} = f_R$ (outside this charts π act as the identity).

A point on the chart $\mathbb{C}^2_A \subset \mathbb{B}_M$ is denoted by $(x, y)_A$ and for the origin of such chart we write $\mathbf{0}_A$. Finally, we denote the exceptional divisor created on the blow-up of $\mathbf{0}_A$ by $\mathsf{L}_{(\!(A)\!)}$, where $(\!(A)\!) = \mathcal{M}_{\mathbb{Z}^2}(A)$.

Example 1.4.2. The SB-blow-up of $M = RLL$ results in the chain of blow-ups exhibited on Figure 1.8.

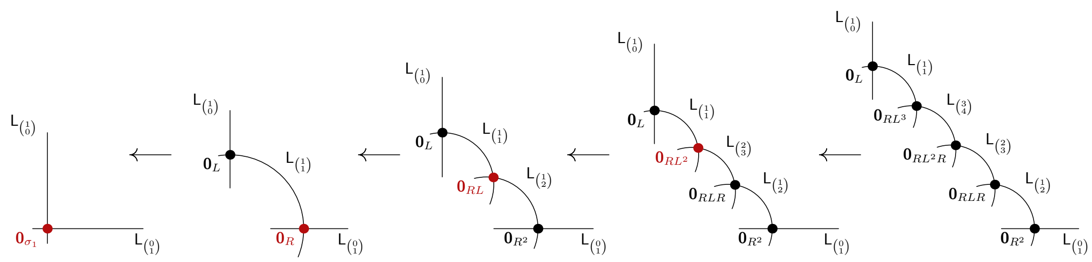


Figure 1.8: *The SB-blow-ups of* $\Gamma^R_{1,2}$.

Remark 1.4.3. A SB-blow-up of $\delta_1\delta_2 \dots \delta_k$ is associated with a branch on the SB-tree: the one contains all the following vertices: σ_1, $(\delta_1\delta_2 \dots \delta_i)$, for $i \in \{1, \dots k\}$, and their children. Starting with the vertex σ_1, after each blow-up, we create two children in the next level of the tree and, if we choose one of them to blow-up again, the other left behind will be associated with one of the charts covering the final blown-up space. Hence, matrices associated with blown-up points will name exceptional divisors, and the ones associated with non blown-up points will name charts.

Example 1.4.4. In Figure 1.9 we show the branch of the SB-tree corresponding to the SB-blow-up of $\Gamma^R_{1,2}$. Notice that, since we blew-up the origin of charts associated with the red matrices, for each of this matrices there is an exceptional labeled by it, in the space $\mathbb{B}_{\Gamma^R_{1,2}}$. Furthermore, the charts covering $\mathbb{B}_{\Gamma^R_{1,2}}$ are the affine complex surfaces associated with the black matrices.

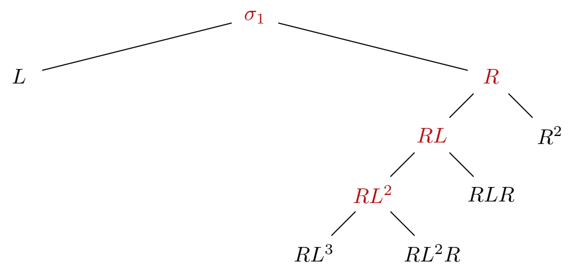


Figure 1.9: *The branch associated with the SB-blow-up of* $\Gamma^R_{1,2}$.

Remark 1.4.5. Let M be a SB-matrix such that $M = \left(\begin{smallmatrix} q_1 & q_2 \\ p_1 & p_2 \end{smallmatrix}\right)$. The chart $\mathbb{C}^2_M$ is centered in the point $\mathbf{0}_M$ and its coordinate axis are given by the divisors $\mathsf{L}_{\binom{q_1}{p_1}}$ and $\mathsf{L}_{\binom{q_2}{p_2}}$. Hence, the charts covering the blow-up of $\mathbb{C}^2_M$ at $\mathbf{0}_M$ are centered at $\mathbf{0}_{ML} = \mathsf{L}_{\binom{q_1}{p_1}} \cap \mathsf{L}_{(\!(M)\!)}$ and $\mathbf{0}_{MR} = \mathsf{L}_{(\!(M)\!)} \cap \mathsf{L}_{\binom{q_2}{p_2}}$.

Since the blow-up in a projective variety is simply the blow-up in one of its affine charts glued with the left ones, we can always consider SB-blow-ups in $\mathbb{CP}^1 \times \mathbb{CP}^1$. Further, we could perform SB-blow-ups in more than one chart of $\mathbb{CP}^1 \times \mathbb{CP}^1$ and then the resulting space would be a succession of blow-ups of $\mathbb{CP}^1 \times \mathbb{CP}^1$ given by the gluing of the SB-blow-ups on each initial chart. In this context, it will be useful to slightly extend the Definition 1.4.1 so one can specify the chart on which an SB-blow-up is performed in $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Definition 1.4.6. Let M be an SB-matrix. The **SB-blow-up** of $\sigma_i M$, denoted $\boldsymbol{\pi}_{\sigma_i M} : \mathbb{B}_{\sigma_i M} \longrightarrow \mathbb{C}^2_{\sigma_i}$, is a chain of blow-ups performed at $\mathbb{C}^2_{\sigma_i}$ defined recursively as follows:

(i) If $M = \sigma_1$, the SB-blow-up of $\sigma_i M$ is simply the standard blow-up $\pi : \widetilde{\mathbb{C}}^2_{\sigma_i} \longrightarrow \mathbb{C}^2_{\sigma_i}$ of the chart $\mathbb{C}^2_{\sigma_i} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ at its origin. That is $\boldsymbol{\pi}_{\sigma_i} = \pi$ and $\mathbb{B}_{\sigma_i} = \widetilde{\mathbb{C}}^2_{\sigma_i}$. The blown-up chart $\mathbb{B}_{\sigma_i}$ is covered by two copies of the complex plane, $\mathbb{C}^2_{\sigma_i L}$ and $\mathbb{C}^2_{\sigma_i R}$, such that $\boldsymbol{\pi}_{\sigma_i}|_{\mathbb{C}^2_{\sigma_i L}} = f_L$ and $\boldsymbol{\pi}_{\sigma_i}|_{\mathbb{C}^2_{\sigma_i R}} = f_R$. We also label the divisors in coordinates of $\mathbb{CP}^1 \times \mathbb{CP}^1$ as in Section **??** (see also Figure 1.1)

(ii) Let $M = \delta_1 \delta_2 \ldots \delta_k$ be a SB-matrix, and set $M' = \delta_1 \delta_2 \ldots \delta_{k-1}$. The space $\mathbb{B}_{\sigma_i M}$ is the blow-up of at the origin of the chart $\mathbb{C}^2_{\sigma_i M} \subset \mathbb{B}_{\sigma_i M'}$, and $\boldsymbol{\pi}_{\sigma_i M}$ is the composition of the $k+1$ standard blow-ups of complex affine planes at origin

$$\mathbb{B}_{\sigma_i \delta_1 \delta_2 \ldots \delta_k} \xrightarrow{\pi} \mathbb{B}_{\sigma_i \delta_1 \delta_2 \ldots \delta_{k-1}} \xrightarrow{\pi} \quad \ldots \quad \xrightarrow{\pi} \mathbb{B}_{\sigma_i \delta_1 \delta_2} \xrightarrow{\pi} \mathbb{B}_{\sigma_i \delta_1} \xrightarrow{\pi} \mathbb{B}_{\sigma_i} \xrightarrow{\pi} \mathbb{C}^2_{\sigma_i}.$$

The space $\mathbb{B}_{\sigma_i M}$ is covered by the charts of $\mathbb{B}_{\sigma_i M'}$ plus two additional ones, $\mathbb{C}^2_{\sigma_i M' L}$ and $\mathbb{C}^2_{\sigma_i M' R}$, where the action of the standard blow-up $\pi : \mathbb{B}_{\sigma_i M} \longrightarrow \mathbb{B}_{\sigma_i M'}$ is given locally by $\pi|_{\mathbb{C}^2_{\sigma_i M' L}} = f_L$, $\pi|_{\mathbb{C}^2_{\sigma_i M' R}} = f_R$.

A point on the chart $\mathbb{C}^2_{\sigma_i A} \subset \mathbb{B}_{\sigma_i M}$ is denoted by $(x, y)_{\sigma_i A}$ and for the origin of such chart we write $\mathbf{0}_{\sigma_i A}$. Finally, we denote the exceptional divisor created on the blow-up of $\mathbf{0}_{\sigma_i A}$ by $\mathsf{L}_{(\!(\sigma_i A)\!)}$.

Notice that, since σ_1 is the identity, the SB-blow-ups $\boldsymbol{\pi}_{\sigma_1 M} : \mathbb{B}_{\sigma_1 M} \longrightarrow \mathbb{C}^2_{\sigma_1}$ and $\boldsymbol{\pi}_M : \mathbb{B}_M \longrightarrow \mathbb{C}^2$ are exactly the same.

The next theorem shows the relation between the SB-blow-ups from Definition 1.4.6 and the SB-diagram.

Theorem 1.4.7. *The label for the divisors of $\mathbb{CP}^1 \times \mathbb{CP}^1$ as established in Section* **??** *is compatible with the notation of the SB-blow-ups. That is, if we make SB-blow-ups in different charts of $\mathbb{CP}^1 \times \mathbb{CP}^1$, their associated branches (as in Remark 1.4.3) appear as ordered branches in the SB-diagram.*

Proof. Since the blow-up is a local construction, to prove the compatibility is sufficient to check that the branches for the SB-blow-ups of σ_i, for $i \in \{1, 2, 3, 4\}$, appear as branches of the SB-diagram.

First notice that, since $\boldsymbol{\pi}_{\sigma_1}|_{\mathbb{C}^2_{\sigma_1 L}} = f_L$,

$$\boldsymbol{\pi}_{\sigma_1}|_{\mathbb{C}^2_{\sigma_1 L}}(x, y)_{\sigma_1 L} = (xy, y)_{\sigma_1} = ([xy : 1], [y : 1])$$

and taking $x = 0$ we see that $\left(\mathsf{L}_{\binom{1}{0}} \setminus \{([0 : 1], [1 : 0])\}\right) \subset \mathbb{C}^2_{\sigma_1 L}$. Hence $\mathbf{0}_{\sigma_1 L} = \mathsf{L}_{\binom{1}{0}} \cap \mathsf{L}_{\binom{1}{1}}$.

From $\boldsymbol{\pi}_{\sigma_1}|_{\mathbb{C}^2_R} = f_R$, we have

$$\boldsymbol{\pi}_{\sigma_1}|_{\mathbb{C}^2_{\sigma_1 R}}(x, y)_{\sigma_1 R} = (x, xy)_{\sigma_1} = ([x : 1], [xy : 1]),$$

which implies $\left(\mathsf{L}_{\binom{0}{1}} \setminus \{([1 : 0], [0 : 1])\}\right) \subset \mathbb{C}^2_{\sigma_1 R}$ and then $\mathbf{0}_{\sigma_1 R} = \mathsf{L}_{\binom{1}{1}} \cap \mathsf{L}_{\binom{0}{1}}$. With similar computations, is possible to show that $\mathbf{0}_{\sigma_2 L} = \mathsf{L}_{\binom{1}{-1}} \cap \mathsf{L}_{\binom{1}{0}}$, $\mathbf{0}_{\sigma_2 R} = \mathsf{L}_{\binom{1}{-1}} \cap \mathsf{L}_{\binom{0}{-1}}$, $\mathbf{0}_{\sigma_3 L} = \mathsf{L}_{\binom{-1}{1}} \cap \mathsf{L}_{\binom{-1}{0}}$, $\mathbf{0}_{\sigma_3 R} = \mathsf{L}_{\binom{-1}{1}} \cap \mathsf{L}_{\binom{0}{1}}$, $\mathbf{0}_{\sigma_4 L} = \mathsf{L}_{\binom{-1}{-1}} \cap \mathsf{L}_{\binom{-1}{0}}$, and $\mathbf{0}_{\sigma_4 R} = \mathsf{L}_{\binom{-1}{-1}} \cap \mathsf{L}_{\binom{0}{-1}}$. See the displayed at Figure 1.10.

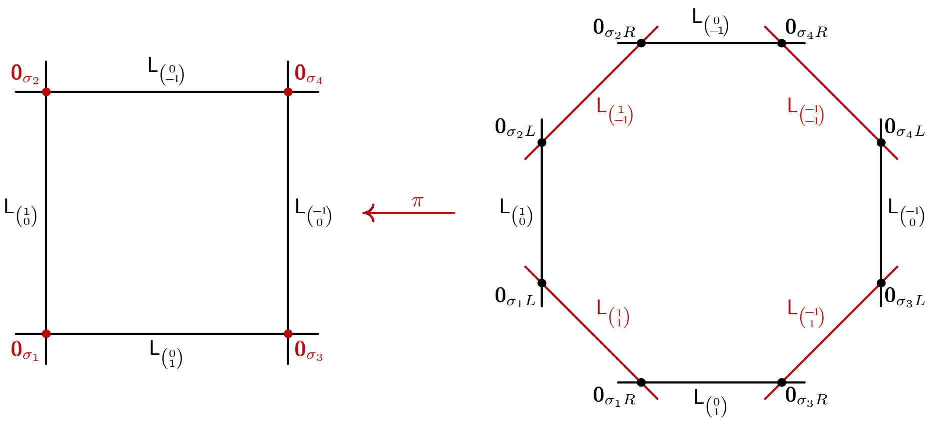


Figure 1.10: *The SB-blow-ups of $\mathbf{0}_{\sigma_1}$, $\mathbf{0}_{\sigma_2}$, $\mathbf{0}_{\sigma_3}$, and $\mathbf{0}_{\sigma_4}$.*

Therefore, the four branches appear as they are ordered on the SB-diagram. In fact, in this case, they are precisely the SB-diagram of level 1, as showed in Figure 1.11. $\square$

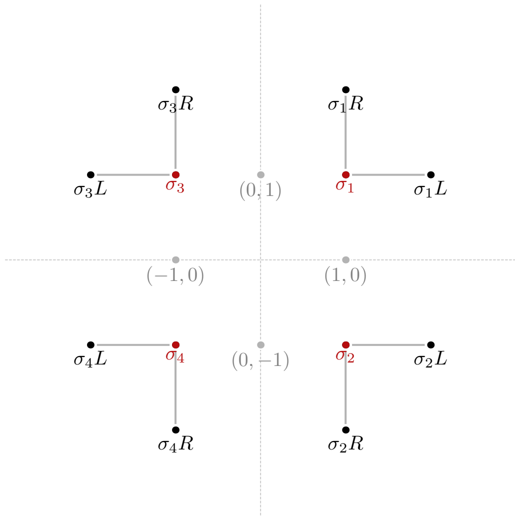


Figure 1.11: *The SB-diagram of level 1.*

Remark 1.4.8. (i) Any cyclic permutation of the labels for the divisors of $\mathbb{CP}^1 \times \mathbb{CP}^1$ is compatible with the notation of SB-blow-ups.

(ii) In $\mathbb{CP}^2$, there is not a notation for divisors that is compatible with the SB-blow-ups notation. This is the reason for choosing $\mathbb{CP}^1 \times \mathbb{CP}^1$ as our initial space.

The propositions below show that the label of charts on an SB-blow-up give the local action of the chain of blow-ups.

Proposition 1.4.9. *Let $\delta_1\delta_2 \dots \delta_n$ be an SB-matrix. Set $M = \sigma_i\delta_1\delta_2 \dots \delta_n$ and $M' = \sigma_i\delta_1\delta_2 \dots \delta_{n-1}$, if $n > 1$, or $M' = \sigma_i$ if $n = 1$. Then the local description of the blow-up $\widetilde{\mathbb{C}}^2_M \subset \mathbb{B}_M \xrightarrow{\pi} \mathbb{C}^2_M \subset \mathbb{B}_{M'}$ at $\mathbf{0}_M$ is given by*

$$\pi|_{\mathbb{C}^2_{ML}} : \begin{array}{ccc} \mathbb{C}^2_{ML} & \longrightarrow & \mathbb{C}^2_M \\ (u,v)_{ML} & \longmapsto & (uv,v)_M \end{array}, \qquad \pi|_{\mathbb{C}^2_{MR}} : \begin{array}{ccc} \mathbb{C}^2_{MR} & \longrightarrow & \mathbb{C}^2_M \\ (u,v)_{MR} & \longmapsto & (u,uv)_M \end{array},$$

where $\mathbb{C}^2_{ML}$ is the affine plane isomorphic to the chart $\{((x,y)_M,[s:1]) \in \mathbb{C}^2_M \times \mathbb{CP} : x = ys\} \subset \widetilde{\mathbb{C}}^2_M$, $\mathbb{C}^2_{MR}$ is isomorphic to $\{((x,y)_M,[1:t]) \in \mathbb{C}^2_M \times \mathbb{P} : y = xt\} \subset \widetilde{\mathbb{C}}^2_M$, and $\mathbb{C}^2_M$ is one of the affine planes covering $\mathbb{B}_{M'}$.

Proof. The isomorphism between $\mathbb{C}^2_{ML}$ and $\{((x,y)_M,[s:1]) \in \mathbb{C}^2_M \times \mathbb{CP} : x = ys\}$ is given by the map

$$(u,v)_{ML} \longmapsto ((uv,v)_M,[u:1]).$$

Then, since π act as the projection on the fist coordinate, we have

$$\pi|_{\mathbb{C}^2_{ML}}(u,v)_{ML} = \pi((uv,v)_M,[u:1]) = (uv,v)_M.$$

Similarly, the isomorphism between $\mathbb{C}^2_{Mr}$ and $\{((x,y)_M,[1:t]) \in \mathbb{C}^2_M \times \mathbb{CP} : y = xt\}$ is

$$(u,v)_{MR} \longmapsto ((u,uv)_M,[1:v]),$$

which gives

$$\pi|_{\mathbb{C}^2_{MR}}(u,v)_{MR} = \pi((u,uv)_M,[1\,:\,v]) = (u,uv)_M,$$

proving the result.

□

Proposition 1.4.10. *Let $M = \prod_{j=1}^{n} \delta_j$, where δ_j is either L or R. Then, $\mathbb{B}_{\sigma_i M}$ is covered by the $n+2$ charts:*

(*i*) *the affine planes $\mathbb{C}^2_{\sigma_i\bar{\delta}_1}$, $\mathbb{C}^2_{\sigma_i ML}$, $\mathbb{C}^2_{\sigma_i MR}$ and*

(*ii*) *the affine planes $\mathbb{C}^2_{\sigma_i\delta_1\ldots\delta_{j-1}\bar{\delta}_j}$, $j = 2,\ldots,n$,*

where

$$\bar{\delta}_j = \begin{cases} L, & \text{if } \delta_j = R \\ R, & \text{if } \delta_j = L \end{cases}, \qquad j = 1,\ldots,n.$$

The restriction $\boldsymbol{\pi}_{\sigma_i M}\big|_{\mathbb{C}^2_{\sigma_i A}}$ is such that $\boldsymbol{\pi}_{\sigma_i M}\big|_{\mathbb{C}^2_{\sigma_i A}} = f_A$, that is

$$\boldsymbol{\pi}_{\sigma_i M}\big|_{\mathbb{C}^2_{\sigma_i A}} : \begin{array}{ccc} \mathbb{C}^2_{\sigma_i A} & \longrightarrow & \mathbb{C}^2_{\sigma_i} \\ (u,v)_{\sigma_i A} & \longmapsto & (u^{q_1}v^{q_2}, u^{p_1}v^{p_2})_{\sigma_i} \end{array}, \tag{6}$$

where $A = \left(\begin{smallmatrix} q_1 & q_2 \\ p_1 & p_2 \end{smallmatrix}\right)$ is the SB-matrix associated with a chart mentioned in (i) or (ii).

Proof. The set of charts covering $\mathbb{B}_{\sigma_i M}$ are given directly by the branch of the SB-tree $\mathcal{T}_i$ as mentioned on Remark 1.4.3. To find action of $\boldsymbol{\pi}_{\sigma_i M}$ we use induction over j.

Since $\boldsymbol{\pi}_{\sigma_i L}$ and $\boldsymbol{\pi}_{\sigma_i R}$ are respectively the maps f_L and f_R , the first step of the induction is immediate. Now, suppose $A = \prod_{k=1}^{j+1} \delta_i$, and define $A' = \prod_{k=1}^{j} \delta_k = \begin{pmatrix} q_1 & q_2 \\ p_1 & p_2 \end{pmatrix}$. Then,

$$\boldsymbol{\pi}_{\sigma_i M}|_{\mathbb{C}^2_{\sigma_i A'}} : (u,v)_{\sigma_i A'} \mapsto (u^{q_1}v^{q_2}, u^{p_1}v^{p_2})_{\sigma_i},$$

and by Proposition 1.4.9, if $\delta_{j+1} = L$

$$\begin{aligned} \boldsymbol{\pi}_{\sigma_i M}|_{\mathbb{C}^2_{\sigma_i A}}(u,v)_{\sigma_i A} &= \boldsymbol{\pi}_{\sigma_i M}|_{\mathbb{C}^2_{\sigma_i A'}} \circ \pi|_{\mathbb{C}^2_{\sigma_i A'L}}(u,v)_{\sigma_i A'L} \\ &= \boldsymbol{\pi}_{\sigma_i A'}(uv,v)_{\sigma_i A'} \\ &= (u^{q_1}v^{q_1+q_2}, u^{p_1}v^{p_1+p_2})_{\sigma_i} \end{aligned}$$

and, if $\delta_{j+1} = R$

$$\begin{aligned} \boldsymbol{\pi}_{\sigma_i M}|_{\mathbb{C}^2_{\sigma_i A}}(u,v)_{\sigma_i A} &= \boldsymbol{\pi}_{\sigma_i M}|_{\mathbb{C}^2_{\sigma_i A'}} \circ \pi|_{\mathbb{C}^2_{\sigma_i A'R}}(u,v)_{\sigma_i A'R} \\ &= \boldsymbol{\pi}_{\sigma_i M}|_{\mathbb{C}^2_{\sigma_i A}}(u,uv)_{\sigma_i A'} \\ &= (u^{q_1+q_2}v^{q_2}, u^{p_1+p_2}v^{p_2})_{\sigma_i}. \end{aligned}$$

Since $A'L = \begin{pmatrix} q_1 & q_1+q_2 \\ p_1 & p_1+p_2 \end{pmatrix}$ and $A'R = \begin{pmatrix} q_1+q_2 & q_2 \\ p_1+p_2 & p_2 \end{pmatrix}$, we proved the desired.

□

This concludes the description of the SB-blow-ups and we are ready to return to our main problem: the compactification of monomial maps. We will focus our attention on monomial maps associated with matrices in $SL(2,\mathbb{Z})$ having two distinct real eigenvalues. If we hope to use SB-blow-ups to build the desired compact space, we need to understand what exactly is the relation between SB-matrices and $SL(2,\mathbb{Z})$-matrices in general.

Chapter 2

The Stern-Brocot decomposition of matrices in $GL(2, \mathbb{Z})$

In this chapter we show that any matrix M in $GL(2, \mathbb{Z})$ can be decomposed in a product containing a signal matrix, a SB-matrix and perhaps a permutation. We will also establish a relation between this composition, the signs of the eigenvalues and the continued fraction of the slopes of eigenlines of M.

Theorem 2.0.1. *Let M be a matrix in $GL(2, \mathbb{Z})$, then there exists a unique SB-matrix $\Gamma^{\delta_0}_{a_0,\dots,a_n}$ such that*

$$M = \sigma_i \varrho \Gamma^{\delta_0}_{a_0,\dots,a_n} \sigma_j,$$

*where σ_i and σ_j are two signal matrices and ϱ is either the identity or the permutation matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The decomposition above is called the **SB-decomposition** of the matrix M.*

The proof of Theorem 2.0.1 consists from the direct combination of Lemmas 2.1.2 and 2.1.4. The first one is stated and proved in Karpenkov, 2022, Theorem 9.6, and the second one appear for the first time in this thesis.

2.1 Proof of Theorem 2.0.1

For the Lemma 2.1.2 we need to introduce the following definition:

Definition 2.1.1. Given an integer number a, we define

$$M_a = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}.$$

Also, for any sequence of integer numbers $(a_1, \dots, a_n)$ we write

$$M_{a_1,\dots,a_n} = \prod_{i=1}^{n} \begin{pmatrix} 0 & 1 \\ 1 & a_i \end{pmatrix}.$$

Lemma 2.1.2 is a very interesting statement showing that the matrices $M_{a_1,\dots,a_n}$, from

Definition 2.1.1, are related with the continuant polynomials from Definition 1.3.4 and that they are almost all $GL(2,\mathbb{Z})$. It will be useful to consider the usual signal notation: given a number a, its signal is denoted $\operatorname{sgn}(a)$, that is, $\operatorname{sgn}(a) = +1$ if $a > 0$ and $\operatorname{sgn}(a) = -1$ if $a < 0$.

Lemma 2.1.2. *Let*

$$\mathcal{E} = \left\{ \pm \begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix} : k = 1, 2, 3, \ldots \right\}.$$

Then, for any matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $GL(2,\mathbb{Z}) \setminus \mathcal{E}$*, there exists a unique* $n > 0$ *and a unique sequence of numbers* $(a_0, a_1, a_2, \ldots, a_{n-1}, a_n)$*, with* $a_0, a_n \in \mathbb{Z}$ *and* $a_1, a_2, \ldots, a_{n-1} \in \mathbb{Z}_{>0}$*, such that*

(*i*) *If* $a \neq 0$*, then*

$$M = \operatorname{sgn}(a) M_{a_0,a_1,\ldots,a_n} = \operatorname{sgn}(a) \begin{pmatrix} K_{n-2}(a_1, a_2, \ldots, a_{n-2}) & K_{n-1}(a_1, a_2, \ldots, a_n) \\ K_{n-1}(a_1, a_2, \ldots, a_{n-1}) & K_n(a_0, a_1, \ldots, a_n) \end{pmatrix}.$$

The sequence $(a_0, a_1, \ldots, a_n)$ *satisfies*

$$\frac{c}{a} = [a_0; a_1 : \ldots : a_{n-1}],$$

where $[a_0; a_1 : \ldots : a_{n-1}]$ *is the odd regular continued fraction for* c/a *when* $\det(M) = 1$*, and the even regular continued fraction for* c/a *when* $\det(M) = -1$*. The value of* a_n *is the solution of the following linear equation:*

$$K_{n-1}(a_1, \ldots, a_{n-1}, x) = b.$$

(*ii*) *If* $a = 0$*, we have*

$$M = \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} = M_k \quad \text{or} \quad M = \begin{pmatrix} 0 & -1 \\ -1 & -k \end{pmatrix} = -M_k.$$

From now on, we use ϱ exclusively to denote the permutation matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Remark 2.1.3. We have the following relations between the left and right matrices $L = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, the signal matrices $\sigma_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\sigma_3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the permutation matrix $\varrho = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and the matrices M_k:

$$\begin{aligned} \varrho L \varrho &= R, \\ \varrho R \varrho &= L, \\ R\sigma_2\varrho R &= L, \\ L\sigma_2\varrho L &= -R, \\ L\sigma_2\varrho R &= R\sigma_2\varrho L = \sigma_2\varrho, \\ R\sigma_2 R &= L\sigma_2 L = \sigma_2, \end{aligned} \qquad \begin{aligned} R\sigma_2 L &= L\varrho = \varrho R, \\ L\sigma_2 R &= -\varrho L = -R\varrho, \\ R\sigma_3\varrho R &= -L, \\ L\sigma_3\varrho L &= R, \\ L\sigma_3\varrho R &= R\sigma_3\varrho L = \sigma_3\varrho, \\ R\sigma_3 R &= L\sigma_3 L = \sigma_3, \end{aligned} \qquad \begin{aligned} R\sigma_3 L &= -L\varrho = -\varrho R, \\ L\sigma_3 R &= \varrho L = R\varrho, \\ M_k &= \varrho L^k, \\ M_{-k} &= -\sigma_2 R^k \varrho \sigma_2, \\ M_{-k} &= -\sigma_3 R^k \varrho \sigma_3. \end{aligned}$$

They all follow from straightforward calculations and finite induction, in the case of the last three.

Lemma 2.1.4. *The three items below give the SB-decomposition of any matrix* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *in* $GL(2,\mathbb{Z})$.

(i) *If* $a = 0$ *and* $bc > 0$, *we have* $M = \pm \left(\begin{smallmatrix} 0 & 1 \\ 1 & d \end{smallmatrix}\right)$ *and then*

$$M = \begin{cases} \operatorname{sgn}(c)\,\varrho\, L^d, & \text{if } d \geq 0 \\ \operatorname{sgn}(c)\sigma_3\,\varrho\, L^{-d}\sigma_2, & \text{if } d < 0 \end{cases}.$$

(ii) *If* $a = 0$ *and* $bc < 0$, *we have* $M = \pm \left(\begin{smallmatrix} 0 & -1 \\ 1 & d \end{smallmatrix}\right)$ *and then*

$$M = \begin{cases} \operatorname{sgn}(c)\sigma_3\,\varrho\, L^d, & \text{if } d \geq 0 \\ \operatorname{sgn}(c)\,\varrho\, L^{-d}\sigma_2, & \text{if } d < 0 \end{cases}.$$

(iii) *If* $a \neq 0$, *we have the following cases*

(1) $M = \operatorname{sgn}(a)M_{a_1,\dots,a_{2n}} = \operatorname{sgn}(a)\Gamma^{RL}_{a_1,\dots,a_{2n}}$,

(2) $M = \operatorname{sgn}(a)M_{a_1,\dots,a_{2n+1}} = \operatorname{sgn}(a)\,\varrho\,\Gamma^{LL}_{a_1,\dots,a_{2n+1}}$,

(3) $M = \operatorname{sgn}(a)M_{0,a_1,\dots,a_{2n}} = \operatorname{sgn}(a)\,\varrho\,\Gamma^{RL}_{a_1,\dots,a_{2n}}$,

(4) $M = \operatorname{sgn}(a)M_{0,a_1,\dots,a_{2n+1}} = \operatorname{sgn}(a)\Gamma^{LL}_{a_1,\dots,a_{2n+1}}$,

(5) $M = \operatorname{sgn}(a)M_{a_1,\dots,a_{2n},0} = \operatorname{sgn}(a)\,\varrho\,\Gamma^{LR}_{a_1,\dots,a_{2n}}$,

(6) $M = \operatorname{sgn}(a)M_{a_1,\dots,a_{2n+1},0} = \operatorname{sgn}(a)\Gamma^{RR}_{a_1,\dots,a_{2n+1}}$,

(7) $M = \operatorname{sgn}(a)M_{0,a_1,\dots,a_{2n},0} = \operatorname{sgn}(a)\Gamma^{LR}_{a_1,\dots,a_{2n}}$,

(8) $M = \operatorname{sgn}(a)M_{0,a_1,\dots,a_{2n+1},0} = \operatorname{sgn}(a)\,\varrho\,\Gamma^{RR}_{a_1,\dots,a_{2n+1}}$,

(9) $M = \operatorname{sgn}(a)M_{-k,a_1,\dots,a_{2n}} = \operatorname{sgn}(a)\sigma_2\Gamma^{RL}_{k-1,1,a_1-1,a_2\dots,a_{2n}}$,

(10) $M = \operatorname{sgn}(a)M_{-k,a_1,\dots,a_{2n+1}} = \operatorname{sgn}(a)\sigma_2\,\varrho\,\Gamma^{LL}_{k-1,1,a_1-1,a_2\dots,a_{2n+1}}$,

(11) $M = \operatorname{sgn}(a)M_{-k,a_1,\dots,a_{2n},0} = \operatorname{sgn}(a)\sigma_2\,\varrho\,\Gamma^{LR}_{k-1,1,a_1-1,a_2\dots,a_{2n}}$,

(12) $M = \operatorname{sgn}(a)M_{-k,a_1,\dots,a_{2n+1},0} = \operatorname{sgn}(a)\sigma_2\Gamma^{RR}_{k-1,1,a_1-1,a_2\dots,a_{2n+1}}$,

(13) $M = \operatorname{sgn}(a)M_{b,a_1,\dots,a_n,-k} = AL^{a_n-1}RL^{k-1}\sigma_2$, *where* $b \in \mathbb{Z}$ *and* $M_{b,a_1,\dots,a_n} = AL^{a_n}$.

where on each item, the coefficients k *and* a_i, *for any* i, *are positive integer numbers.*

Proof. Since $L^d = \left(\begin{smallmatrix} 1 & d \\ 0 & 1 \end{smallmatrix}\right)$ and $L^{-d} = \sigma_2 L^d \sigma_2 = \left(\begin{smallmatrix} 1 & -d \\ 0 & 1 \end{smallmatrix}\right)$ (see Proposition 1.3.7), items (i) and (ii) are straightforward. Lets prove item (iii). This will require the repeated use of the relations listed in Remark 2.1.3

(1) Supposing that the decomposition of M given by Lemma 2.1.2 is $M = \operatorname{sgn}(a)M_{a_1,\dots,a_{2n}}$, we need to prove that $M = \operatorname{sgn}(a)\Gamma^{RL}_{a_1,\dots,a_{2n}} = \operatorname{sgn}(a)R^{a_1}L^{a_2}\dots R^{a_{2n-1}}L^{a_{2n}}$. We do this by induction over n. If $n = 1$, $M = M_{a_1,a_2} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & a_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & a_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & a_2 \\ a_1 & a_1a_2+1 \end{smallmatrix}\right)$ and $R^{a_1}L^{a_2} = \left(\begin{smallmatrix} 1 & 0 \\ a_1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a_2 \\ 0 & 1 \end{smallmatrix}\right) =$

$\left(\begin{smallmatrix} 1 & a_2 \\ a_1 & a_1a_2+1 \end{smallmatrix}\right)$. Assuming that the induction step $n-1$ is true, let $M = \operatorname{sgn}(a)M_{a_1,\dots,a_{2n}}$. Thus,

$$M = \operatorname{sgn}(a)M_{a_1,\dots,a_{2n-2}}M_{a_{2n-1},a_{2n}} = \operatorname{sgn}(a)\Gamma^{RL}_{a_0,\dots,a_{2n-2}}R^{a_{2n-1}}L^{a_{2n}} = \operatorname{sgn}(a)\Gamma^{RL}_{a_0,\dots,a_{2n}},$$

where in the third equality we used the induction hypothesis.

(*2*) If $M = \operatorname{sgn}(a)M_{a_0,\dots,a_{2n+1}}$, using the relation $M_k = \varrho L^k$ and the item (*1*), we have

$$M = \operatorname{sgn}(a)M_{a_0,\dots,a_{2n+1}} = \operatorname{sgn}(a)M_{a_0,\dots,a_{2n}}M_{a_{2n+1}} = \operatorname{sgn}(a)\Gamma^{RL}_{a_0,\dots,a_{2n}}\varrho L^{a_{2n+1}}.$$

But, since $L = \varrho R\varrho$, it follows that $\Gamma^{RL}_{a_0,\dots,a_{2n}}\varrho = \varrho\Gamma^{LR}_{a_0,\dots,a_{2n}}$ and

$$M = \operatorname{sgn}(a)\varrho\Gamma^{LR}_{a_0,\dots,a_{2n}}L^{a_{2n+1}} = \operatorname{sgn}(a)\varrho\Gamma^{LL}_{a_0,\dots,a_{2n+1}}.$$

Items (*3*) through (*8*) follow directly from $M_0 = \varrho$ together with the fact that L and R and conjugated by ϱ.

(*9*) Suppose that $M = \operatorname{sgn}(a)M_{-k,a_1,\dots,a_{2n}}$. The relation $M_{-k} = -\sigma_2R^k\varrho\sigma_2$ and item (*1*) imply

$$\begin{aligned}\operatorname{sgn}(a)M_{-k}M_{a_1,\dots,a_{2n}} &= -\operatorname{sgn}(a)\sigma_2R^k\varrho\sigma_2\Gamma^{RL}_{a_1,\dots,a_{2n}} = -\operatorname{sgn}(a)\sigma_2R^k\varrho\sigma_2R^{a_1}L^{a_2}\dots R^{a_{2n-1}}L^{a_{2n}} = \\ &= \operatorname{sgn}(a)\sigma_2R^{k-1}LR^{a_1-1}L^{a_2}\dots R^{a_{2n-1}}L^{a_{2n}} = \operatorname{sgn}(a)\sigma_2\Gamma^{RL}_{k-1,1,a_1-1,a_2,\dots,a_{2n}},\end{aligned}$$

where in the third equality we used the relation $R\varrho\sigma_2R = -L$.

(*10*) From $M_{-k} = -\sigma_3R^k\varrho\sigma_3$ and (*2*) it follows

$$\begin{aligned}M &= \operatorname{sgn}(a)M_{-k,a_1,\dots,a_{2n+1}} = -\operatorname{sgn}(a)\sigma_3R^k\varrho\sigma_3\varrho\Gamma^{LL}_{a_1,\dots,a_{2n+1}} = \\ &= -\operatorname{sgn}(a)\sigma_3R^k\varrho\sigma_3\varrho L^{a_1}R^{a_2}\dots R^{a_{2n}}L^{a_{2n+1}} = -\operatorname{sgn}(a)\sigma_3R^k\sigma_2L^{a_1}R^{a_2}\dots R^{a_{2n}}L^{a_{2n+1}} = \\ &= \operatorname{sgn}(a)(-\sigma_3)R^{k-1}\varrho RL^{a_1-1}R^{a_2}\dots R^{a_{2n}}L^{a_{2n+1}} = \operatorname{sgn}(a)\sigma_2\varrho L^{k-1}RL^{a_1-1}R^{a_2}\dots R^{a_{2n}}L^{a_{2n+1}} = \\ &= \operatorname{sgn}(a)\sigma_2\varrho\Gamma^{LL}_{k-1,1,a_1-1,a_2,\dots,a_{2n+1}},\end{aligned}$$

where we used the relations $\varrho\sigma_3\varrho = \sigma_2$, $R\sigma_2L = \varrho R$, $-\sigma_3 = \sigma_2$ and $L = \varrho R\varrho$. Again, items (*11*) and (*12*) follow from (*9*) and (*10*) by using $M_0 = \varrho$.

(*13*) Finally, suppose $M = \operatorname{sgn}(a)M_{b,a_1,\dots,a_n,-k} = \operatorname{sgn}(a)M_{b,a_1,\dots,a_n}M_{-k}$. By the previous items, $M_{b,a_1,\dots,a_n}$ has a decomposition ending in L^{a_n}. Writing, $M_{b,a_1,\dots,a_n} = AL^{a_n}$ we get

$$\operatorname{sgn}(a)M_{b,a_1,\dots,a_n}M_{-k} = -AL^{a_n}\sigma_2R^k\varrho\sigma_2 = AL^{a_n-1}\varrho LR^{k-1}\varrho\sigma_2 = AL^{a_{n-1}}RL^{k-1}\sigma_2,$$

where we used relations $L\sigma_2R = -\varrho L$ and that ϱ conjugates L and R. □

Notice that, since the matrices σ_2, σ_3 and ϱ have determinant equal to -1, the first matrix in (*i*) and the matrices of type (*2*), (*3*), (*5*), (*8*), (*9*) and (*12*) of item (*iii*) are not in $SL(2,\mathbb{Z})$.

Lemma 2.1.4 gives us the following immediate consequence

Corollary 2.1.5. *A matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL(2,\mathbb{Z})$ is a SB-matrix if, and only if, $a,b,c,d > 0$.*

Proof. The forwards is obvious. For the backwards, the only case that is not too obvious is the one for the $SL(2,\mathbb{Z})$ matrices of type (*13*) in Proposition 2.1.4-(*iii*). Lets show that any of such matrices do not have all its coefficients as positive integer numbers.

First, notice that, a matrix $M = AL^{a_n-1}RL^{k-1}\sigma_2$, as in (*13*), belongs to $SL(2,\mathbb{Z})$ if and only if A is of type (*2*), (*3*) or (*9*). If A is of type (*2*) or (*3*), M is the product of a matrix which all entries have the same sign by the matrix σ_2, hence M will have two negative entries. Finally, if A is of type (*9*) then $M = \pm\sigma_2 M'\sigma_2$, where M' is a SB-matrix, hence M also have two negative entries in this case.

□

Also notice that a matrix $M = \sigma_i \Gamma^{\delta_0\delta_n}_{a_0,\dots,a_n}$ is hyperbolic if, and only if, $n > 0$. In fact, if $n = 0$, then the trace of M is equal to ± 2.

2.2 Characterization of sails and continued fractions

From now on, we will work only with matrices in $SL(2,\mathbb{Z})$. Our next goal is to give the expression for the points in the sails of a hyperbolic matrix M, similarly as we stated in Proposition 1.3.11. For SB-matrices this was done in Aicardi, 2009. Since a matrix M, its inverse M^{-1}, and its multiplication by -1, namely $-M$, all have the same geometric continued fraction, it is a good idea to summarize the description of SB-decompositions for matrices in $SL(2,\mathbb{Z})$ grouping their type together with the types of its multiplicative and additive inverses: Let a, b, c, d non negative integer numbers and let $\sigma \in \{\sigma_2, \sigma_3\}$, then we have

(A) the matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is of the form $\Gamma^{\delta_0\delta_n}_{a_0,a_1,\dots,a_n}$, and its inverse $M^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ is given by $\sigma\,\Gamma^{\delta_n\delta_0}_{a_n,a_{n-1},\dots,a_0}\,\sigma$,

(B) the matrix $M = \left(\begin{smallmatrix} a & b \\ -c & -d \end{smallmatrix}\right)$ is of the form $\sigma\varrho\,\Gamma^{\delta_0\delta_n}_{a_0,a_1,\dots,a_n}$, and its inverse $M^{-1} = \left(\begin{smallmatrix} -d & -b \\ c & a \end{smallmatrix}\right)$ is given by $-\sigma\varrho\,\Gamma^{\bar{\delta}_n\bar{\delta}_0}_{a_n,a_{n-1},\dots,a_0}$,

(C) the matrix $M = \left(\begin{smallmatrix} a & -b \\ c & -d \end{smallmatrix}\right)$ is of the form $\varrho\,\Gamma^{\delta_0\delta_n}_{a_0,a_1,\dots,a_n}\sigma$, and its inverse $M^{-1} = \left(\begin{smallmatrix} -d & b \\ -c & a \end{smallmatrix}\right)$ is given by $-\varrho\,\Gamma^{\bar{\delta}_n\bar{\delta}_0}_{a_n,a_{n-1},\dots,a_0}\sigma$.

Now we are ready to characterize the continued fractions of a hyperbolic matrix $M \in SL(2,\mathbb{Z})$ and, as we will see, the points of the sails of M are given by them. As one can imagine, this characterization will involve many cases, so to make the reading more palatable, we split our main results in tree theorems (one for each grouping mentioned above).

On Theorems 2.2.1, 2.2.3, and 2.2.4 we use the notation $[a_0; a_1 : \dots : a_{k-1} : (a_k : a_{k+1} : \dots : a_n)]$ to indicate an infinite continued fraction that is eventually periodic with period $(a_k, a_{k+1}, \dots, a_n)$. If a infinite continued fraction is periodic with period $(a_0, a_1, \dots, a_n)$, we write $[(a_0; a_1 : \dots : a_n)]$.

Characterization Theorem 2.2.1 (type (A)). *Let $M \in SL(2,\mathbb{Z})$ be a hyperbolic matrix, μ_1, μ_2 be the slopes of its eigenlines, and $\sigma \in \{\sigma_2, \sigma_3\}$. Then*

1. *if $M = \Gamma^{RL}_{a_0,a_1\ldots,a_n}$ or $M = \sigma\,\Gamma^{LR}_{a_0,a_1\ldots,a_n}\,\sigma$, the continued fractions of μ_1 and μ_2 are*

$$\mu_1 = [(a_0; a_1 : \ldots : a_n)],$$
$$\mu_2 = -[(a_n; a_{n-1} : \ldots : a_0)];$$

2. *if $M = \Gamma^{LR}_{a_0,a_1\ldots,a_n}$ or $M = \sigma\,\Gamma^{RL}_{a_0,a_1\ldots,a_n}\,\sigma$, we have*

$$\mu_1 = [0; (a_0 : a_1 : \ldots : a_n)],$$
$$\mu_2 = -[0; (a_n : a_{n-1} : \ldots : a_0)];$$

3. *if $M = \Gamma^{RR}_{a_0,a_1\ldots,a_n}$ or $M = \sigma\,\Gamma^{RR}_{a_0,a_1\ldots,a_n}\,\sigma$, we have*

$$\mu_1 = [a_0; (a_1 : \ldots : a_{n-1} : a_n + a_0)],$$
$$\mu_2 = -[a_n; (a_{n-1} : \ldots : a_1 : a_0 + a_n)];$$

4. *if $M = \Gamma^{LL}_{a_0,a_1\ldots,a_n}$ or $M = \sigma\,\Gamma^{LL}_{a_0,a_1\ldots,a_n}\,\sigma$, we have*

$$\mu_1 = [0; a_0 : (a_1 : \ldots : a_{n-1} : a_n + a_0)],$$
$$\mu_2 = -[0; a_n : (a_{n-1} : \ldots : a_1 : a_0 + a_n)].$$

In any of the cases above M has positive eigenvalues.

Proof. Denote by ℓ_1, ℓ_2 the eigenlines of M. As observed in Remark 1.3.17, the geometric continued fraction of M is determined by the sails of r_{μ_1} and r_{μ_2}, where r_{μ_1}, r_{μ_2} are the rays associated with ℓ_1, ℓ_2, respectively. By Corollary 1.3.11, these sails are given by elements of the form $\sigma_2(\!(\Gamma^{\delta_0\delta_n}_{b_0,\ldots,b_n}\delta_n^{-1})\!)$ (multiplication by σ_2 applies when the slope of the ray is negative). We now use the fact that the sails, seems as the collection of points of this form, are stable by integer powers of M. For instance, suppose that $M = \Gamma^{RL}_{a_0,\ldots,a_n}$. If $(\!(\Gamma^{RL}_{b_0,\ldots,b_m}L^{-1})\!)$ is a point in the sail of M, we have

$$\begin{aligned} M(\!(\Gamma^{RL}_{b_0,\ldots,b_m}L^{-1})\!) &= (\!(M\Gamma^{RL}_{b_0,\ldots,b_m}L^{-1})\!) \\ &= (\!(R^{a_0}\ldots L^{a_n}R^{b_0}\ldots L^{b_m}L^{-1})\!), \end{aligned}$$

which is a point of the sail if and only if $a_0 = b_0, \ldots, a_n = b_n$ (here we are using that $(\!(\cdot)\!)$ is a bijection between SB-matrices and points of $\mathbb{Z}^2$ with coprime entries). Clearly, successive application of M will revel all b_m. Now, if the point is of the form $\sigma_2(\!(\Gamma^{LL}_{b_0,\ldots,b_m}L^{-1})\!)$,

$$\begin{aligned} M\sigma_2(\!(\Gamma^{LL}_{b_0,\ldots,b_m}L^{-1})\!) &= (\!(M\sigma_2\Gamma^{LL}_{b_0,\ldots,b_m}L^{-1})\!) \\ &= (\!(R^{a_0}\ldots L^{a_n}\sigma_2 L^{b_0}\ldots L^{b_m}L^{-1})\!), \end{aligned}$$

where we can use the relation $L\sigma_2 L = \sigma_2$ (see Remark 2.1.3) to conclude that this point is on the sail if and only if $a_n = b_0, \ldots, a_1 = b_{n-1}, a_0 = b_n$ and the resulting point is, assuming for example $m > n$, of the form $\sigma_2(\!(L^{b_{n+1}} \dots L^{b_m} L^{-1})\!)$ and, again, successive application of M reveal all b_m. Applying powers of inverse of M^{-1} lead to the same conclusion and, with minor changes, this considerations prove all the cases. □

Corollary 2.2.2. *Let M be matrix of type (A) as in Theorem 2.2.1 and let $\mu_1 > 0$ and $\mu_2 < 0$ be the slopes of the eigenlines of M. Then, if*

$$\mu_1 = [b_0; b_1 : b_2 \dots] \quad \text{and} \quad \mu_2 = [c_0; c_1 : c_2 \dots]$$

the points of the first sail of M are $A_0 = (1, 0)$ and, for $i = 1, 2, 3 \dots$,

$$A_i = \mathcal{M}_{\mathbb{Z}^2}\left(\Gamma^{LR}_{a_0,\dots,a_{2i-1}} R^{-1}\right) \quad A_{-i} = \mathcal{M}_{\mathbb{Z}^2}\left(\Gamma^{LL}_{a_n,\dots,a_{n-2i}} L^{-1}\right).$$

The proof follows direct from 1.3.10.

Characterization Theorem 2.2.3 (type (B))**.** *Let $M \in SL(2, \mathbb{Z})$ be a hyperbolic matrix and let μ_1, μ_2 be the slopes of its eigenlines. Then*

1. *if $M = \sigma_2\, \varrho\, \Gamma^{RL}_{a_0,a_1\dots,a_n}$ (resp. $M = \sigma_3\, \varrho\, \Gamma^{RL}_{a_0,a_1\dots,a_n}$), where $a_i = a_{n-i}$, for any $i = 1, \dots, k-1$, we have*

   - *if $a_{n-k} > a_k$ (resp. $a_k > a_{n-k}$), then*

     $$\mu_1 = -[0; a_0 : \dots : a_{n-k-1} : (a_{n-k} - a_k - 1 : 1 : a_{k+1} - 1 : a_{k+2} : \dots : a_{n-k-1})],$$
     $$\mu_2 = -[0; a_n : \dots : a_{n-k} : (a_{n-k-1} : \dots : a_{k+2} : a_{k+1} - 1 : 1 : a_{n-k} - a_k - 1)].$$

     Further, for k even, both eigenvalues of M are negative. Otherwise, they are positive.

   - *if $a_k > a_{n-k}$ (resp. $a_{n-k} > a_k$), then*

     $$\mu_1 = -[0; a_0 : \dots : a_{n-k-2} : (a_{n-k-1} - 1 : 1 : a_k - a_{n-k} - 1 : a_{k+1} : \dots : a_{n-k-2})],$$
     $$\mu_2 = -[0; a_n : \dots : a_{n-k-1} : (a_{n-k-2} : \dots : a_{k+1} : a_k - a_{n-k} - 1 : 1 : a_{n-k-1} - 1)].$$

     If k is even, the eigenvalues of M are positive. Otherwise, they are negative.

2. *For $M = \sigma_2\, \varrho\, \Gamma^{LR}_{a_0,a_1\dots,a_n}$ (resp. $M = \sigma_3\, \varrho\, \Gamma^{LR}_{a_0,a_1\dots,a_n}$), where $a_i = a_{n-i}$, for any $i = 1, \dots, k-1$, we have*

   - *if $a_{n-k} > a_k$ (resp. $a_k > a_{n-k}$), then*

     $$\mu_1 = -[a_0; \dots : a_{n-k-1} : (a_{n-k} - a_k - 1 : 1 : a_{k+1} - 1 : a_{k+2} : \dots : a_{n-k-1})],$$
     $$\mu_2 = -[a_n; \dots : a_{n-k-1} : (a_{n-k-2} : \dots : a_{k+1} : a_k - a_{n-k} - 1 : 1 : a_{n-k-1} - 1)].$$

     If k is even, the eigenvalues of M are positive. Otherwise, they are negative.

   - *if $a_k > a_{n-k}$ (resp. $a_{n-k} > a_k$), then*

     $$\mu_1 = -[a_0; \dots : a_{n-k-2} : (a_{n-k-1} - 1 : 1 : a_k - a_{n-k} - 1 : a_{k+1} : \dots : a_{n-k-2})],$$
     $$\mu_2 = -[a_n; \dots : a_{k+1} : (a_k - a_{n-k} - 1 : 1 : a_{n-k-1} - 1 : a_{n-k-2} : a_{n-k-3} : \dots : a_{k+1})]$$

If k is odd, the eigenvalues of M are negative. Otherwise, they are positive.

3. *For $M = \sigma_2\, \varrho\, \Gamma^{LL}_{a_0,a_1\dots,a_n}$ or $M = \sigma_3\, \varrho\, \Gamma^{RR}_{a_0,a_1\dots,a_n}$, we have*

$$\mu_1 = -[a_0;(a_1 : \dots : a_{n-1} : a_n - 1 : 1 : a_0 - 1)],$$
$$\mu_2 = -[0;a_n : (a_{n-1} \dots : a_1 : a_0 - 1 : 1 : a_n - 1)].$$

In this case the eigenvalues of M are negative.

4. *For $M = \sigma_3\, \varrho\, \Gamma^{LL}_{a_0,a_1\dots,a_n}$ or $M = \sigma_2\, \varrho\, \Gamma^{RR}_{a_0,a_1\dots,a_n}$, we have*

$$\mu_1 = -[a_0;(a_1 : \dots : a_{n-1} : a_n - 1 : 1 : a_0 - 1)],$$
$$\mu_2 = -[0;a_n : (a_{n-1} \dots : a_1 : a_0 - 1 : 1 : a_n - 1)].$$

In this case the eigenvalues of M are positive.

Proof. The idea here is similar to the one in the proof of Theorem 2.2.1, but we need to understand what changes when introducing the permutation ϱ. We do this by looking at the powers of M and M^{-1}. Suppose, for instance, that $M = \sigma_2\varrho\Gamma^{RL}_{a_0,\dots a_n}$, $a_i = a_{n-i}$ for every $i = 1,\dots,k-1$, and $a_k > a_{n-k}$ for some even number k. The other cases will follow analogously. Write $M = \sigma_2\Gamma^{LR}_{a_0,\dots,a_{k-1}}\Gamma^{LR}_{a_k,\dots,a_{n-k}}\Gamma^{LR}_{a_{n-k+1},\dots,a_n}\varrho$ (notice that $k < n-k$). Using the relations $L\varrho\sigma_2 R = R\varrho\sigma_2 L = \varrho\sigma_2$ successively together with the hypothesis over the a_i's, we see that $\Gamma^{LR}_{a_{n-k+1},\dots,a_n}\varrho\sigma_2\Gamma^{LR}_{a_0,\dots,a_{k-1}} = \varrho\sigma_2$. This implies

$$M^\ell = \sigma_2\Gamma^{LR}_{a_0,\dots,a_{k-1}}\Gamma^{LR}_{a_k,\dots,a_{n-k}}(\varrho\sigma_2\Gamma^{LR}_{a_k,\dots,a_{n-k}})^{\ell-1}\Gamma^{LR}_{a_{n-k+1},\dots,a_n}\varrho.$$

But, since $a_k > a_{n-k}$,

$$\begin{aligned}\Gamma^{LR}_{a_k,\dots,a_{n-k}}(\varrho\sigma_2\Gamma^{LR}_{a_k,\dots,a_{n-k}}) &= L^{a_k}\dots R^{a_{n-k}}\varrho\sigma_2 L^{a_k}\dots R^{a_{n-k}}\\ &= L^{a_k}\dots L^{a_{n-k-1}}\varrho\sigma_2 L^{a_k-a_{n-k}}R^{a_{k+1}}\dots R^{a_{n-k}}\\ &= L^{a_k}\dots R^{a_{n-k-2}}L^{a_{n-k-1}-1}RL^{a_k-a_{n-k}-1}R^{a_{k+1}}\dots R^{a_{n-k}} \text{ (since } L\varrho\sigma_2 L = R).\end{aligned}$$

Hence

$$M^\ell =$$
$$= \sigma_2\Gamma^{LR}_{a_0,\dots,a_{k-1}}\,L^{a_k}\dots R^{a_{n-k-2}}\left(L^{a_{n-k-1}-1}RL^{a_k-a_{n-k}-1}R^{a_{k+1}}\dots R^{a_{n-k-2}}\right)^\ell L^{a_{n-k-1}}R^{a_{n-k}}\Gamma^{LR}_{a_{n-k+1},\dots,a_n}\varrho.$$

Analogously,

$$M^{-\ell} =$$
$$= \sigma_2\Gamma^{LR}_{a_n,\dots,a_{n-k+1}}\,L^{a_{n-k}}\dots L^{a_{k+1}}\left(R^{a_k-a_{n-k}-1}LR^{a_{n-k-1}-1}L^{a_{n-k-2}}\dots L^{a_{k+1}}\right)^\ell R^{a_k}\Gamma^{LR}_{a_{k-1},\dots,a_n}\varrho.$$

Using these expressions, we obtain the general form of sail points.

Further, since each power of the action of M keeps the sail points in the same quadrant of the plane, follows that the eigenvalues of M are positive. □

Characterization Theorem 2.2.4 (type (C))**.** *Let $M \in SL(2,\mathbb{Z})$ be a hyperbolic matrix and* *let μ_1, μ_2 be the slopes of its eigenlines. Then*

1. *if $M = \varrho\,\Gamma^{RL}_{a_0,a_1\ldots,a_n}\sigma_2$ (resp. $M = \varrho\,\Gamma^{RL}_{a_0,a_1\ldots,a_n}\sigma_3$), where $a_i = a_{n-i}$, for any $i = 1,\ldots,k-1$, we have*

   - *if $a_{n-k} > a_k$ (resp. $a_k > a_{n-k}$), then*

   $$\mu_1 = [0; a_0 \,:\, \ldots \,:\, a_{n-k-1} \,:\, (a_{n-k} - a_k - 1 \,:\, 1 \,:\, a_{k+1} - 1 \,:\, a_{k+2} \,:\, \ldots \,:\, a_{n-k-1})],$$
   $$\mu_2 = [0; a_n \,:\, \ldots \,:\, a_{n-k} \,:\, (a_{n-k-1} \,:\, \ldots \,:\, a_{k+2} \,:\, a_{k+1} - 1 \,:\, 1 \,:\, a_{n-k} - a_k - 1)].$$

   Further, for k even, both eigenvalues of M are negative. Otherwise, they are positive.

   - *if $a_k > a_{n-k}$ (resp. $a_{n-k} > a_k$), then*

   $$\mu_1 = [0; a_0 \,:\, \ldots \,:\, a_{n-k-2} \,:\, (a_{n-k-1} - 1 \,:\, 1 \,:\, a_k - a_{n-k} - 1 \,:\, a_{k+1} \,:\, \ldots \,:\, a_{n-k-2})],$$
   $$\mu_2 = [0; a_n \,:\, \ldots \,:\, a_{n-k-1} \,:\, (a_{n-k-2} \,:\, \ldots \,:\, a_{k+1} \,:\, a_k - a_{n-k} - 1 \,:\, 1 \,:\, a_{n-k-1} - 1)].$$

   If k is even, the eigenvalues of M are positive. Otherwise, they are negative.

2. *For $M = \varrho\,\Gamma^{LR}_{a_0,a_1\ldots,a_n}\sigma_2$ (resp. $M = \varrho\,\Gamma^{LR}_{a_0,a_1\ldots,a_n}\sigma_3$), where $a_i = a_{n-i}$, for any $i = 1,\ldots,k-1$, we have*

   - *if $a_{n-k} > a_k$ (resp. $a_k > a_{n-k}$), then*

   $$\mu_1 = [a_0; \ldots \,:\, a_{n-k-1} \,:\, (a_{n-k} - a_k - 1 \,:\, 1 \,:\, a_{k+1} - 1 \,:\, a_{k+2} \,:\, \ldots \,:\, a_{n-k-1})],$$
   $$\mu_2 = [a_n; \ldots \,:\, a_{n-k-1} \,:\, (a_{n-k-2} \,:\, \ldots \,:\, a_{k+1} \,:\, a_k - a_{n-k} - 1 \,:\, 1 \,:\, a_{n-k-1} - 1)].$$

   If k is even, the eigenvalues of M are positive. Otherwise, they are negative.

   - *if $a_k > a_{n-k}$ (resp. $a_{n-k} > a_k$), then*

   $$\mu_1 = [a_0; \ldots \,:\, a_{n-k-2} \,:\, (a_{n-k-1} - 1 \,:\, 1 \,:\, a_k - a_{n-k} - 1 \,:\, a_{k+1} \,:\, \ldots \,:\, a_{n-k-2})],$$
   $$\mu_2 = [a_n; \ldots \,:\, a_{n-k-1} \,:\, (a_{n-k-2} \,:\, \ldots \,:\, a_{k+1} \,:\, a_k - a_{n-k} - 1 \,:\, 1 \,:\, a_{n-k-1} - 1)].$$

   If k is odd, the eigenvalues of M are negative. Otherwise, they are positive.

3. *For $M = \varrho\,\Gamma^{LL}_{a_0,a_1\ldots,a_n}\sigma_2$ or $M = \varrho\,\Gamma^{RR}_{a_0,a_1\ldots,a_n}\sigma_3$, we have*

   $$\mu_1 = [a_0; (a_1 \,:\, \ldots \,:\, a_{n-1} \,:\, a_n - 1 \,:\, 1 \,:\, a_0 - 1)],$$
   $$\mu_2 = [0; a_n \,:\, (a_{n-1} \ldots \,:\, a_1 \,:\, a_0 - 1 \,:\, 1 \,:\, a_n - 1)].$$

   In this case the eigenvalues of M are negative.

4. *For $M = \varrho\,\Gamma^{LL}_{a_0,a_1\ldots,a_n}\sigma_3$ or $M = \varrho\,\Gamma^{RR}_{a_0,a_1\ldots,a_n}\sigma_2$, we have*

   $$\mu_1 = [a_0; (a_1 \,:\, \ldots \,:\, a_{n-1} \,:\, a_n - 1 \,:\, 1 \,:\, a_0 - 1)],$$
   $$\mu_2 = [0; a_n \,:\, (a_{n-1} \ldots \,:\, a_1 \,:\, a_0 - 1 \,:\, 1 \,:\, a_n - 1)].$$

   In this case the eigenvalues of M are positive.

As the proof of Theorem 2.2.4 follows by arguments that are similar to the ones in the proofs of Theorems 2.2.1 and 2.2.1, it will be omitted.

Chapter 3

Well defined extensions of the projective monomial map

In this chapter, we consider a monomial map f_M, associated with a hyperbolic SB-matrix M and then build an extension of it to a compact space that has $(\mathbb{C}^*)$ as a dense subset.

First, we consider the extension F_M of f_M to a rational map in $\mathbb{CP}^1 \times \mathbb{CP}^1$ as in Definition 1.0.1:

$$\begin{array}{cccc} F_M : & \mathbb{CP}^1 \times \mathbb{CP}^1 & \dashrightarrow & \mathbb{CP}^1 \times \mathbb{CP}^1 \\ & \begin{pmatrix} [x_1 : x_2] \\ [y_1 : y_2] \end{pmatrix} & \longmapsto & \begin{pmatrix} [x_1^a y_1^b : x_2^a y_2^b] \\ [x_1^c y_1^d : x_2^c y_2^d] \end{pmatrix}. \end{array}$$

Since the local expressions of F_M are given by

$$\begin{array}{llll} F_M|_{\sigma_1} : (x,y)_{\sigma_1} & \longmapsto \begin{pmatrix} [x^a y^b : 1] \\ [x^c y^d : 1] \end{pmatrix}, & F_M|_{\sigma_2} : (x,y)_{\sigma_2} & \longmapsto \begin{pmatrix} [x^a : y^b] \\ [x^c : y^d] \end{pmatrix}, \\ F_M|_{\sigma_3} : (x,y)_{\sigma_3} & \longmapsto \begin{pmatrix} [y^b : x^a] \\ [y^d : x^c] \end{pmatrix}, & F_M|_{\sigma_4} : (x,y)_{\sigma_4} & \longmapsto \begin{pmatrix} [1 : x^a y^b] \\ [1 : x^c y^d] \end{pmatrix}. \end{array}$$

where all the coefficients a, b, c and d are positive integer numbers, the map is not defined at $\mathbf{0}_{\sigma_2}$ and $\mathbf{0}_{\sigma_3}$. Similarly, the inverse rational map

$$\begin{array}{cccc} F_M^{-1} : & \mathbb{CP}^1 \times \mathbb{CP}^1 & \dashrightarrow & \mathbb{CP}^1 \times \mathbb{CP}^1 \\ & \begin{pmatrix} [x_1 : x_2] \\ [y_1 : y_2] \end{pmatrix} & \longmapsto & \begin{pmatrix} [x_1^d y_1^{-b} : x_2^d y_2^{-b}] \\ [x_1^{-c} y_1^a : x_2^{-c} y_2^a] \end{pmatrix}. \end{array}$$

is not defined at $\mathbf{0}_{\sigma_1}$ and $\mathbf{0}_{\sigma_4}$.

3.1 Initial blow-ups

Given a rational map from a nonsingular projective surface to a projective space, there is a chain of blow-ups that gives rise to a regular map, which is an extension of the rational map to the blown-up space SHAFAREVICH, 2013, Theorem IV.3.3. This regular map is obtained as the composite of the initial rational map with the blow-ups. In the next theorem we exhibit exactly what are the these chains in terms of the SB-blow-ups. The following notation will come in handy.

Notation. Given an SB-matrix $M = \Gamma^{\delta_0 \delta_n}_{a_0, a_1, \ldots, a_n}$ we define

$$\overleftarrow{M} := \Gamma^{\delta_n \delta_0}_{a_n, a_{n-1}, \ldots, a_0}.$$

Notice that $M^{-1} = \sigma \overleftarrow{M} \sigma$, for $\sigma \in \{\sigma_2, \sigma_3\}$.

Theorem 3.1.1. *Given a hyperbolic SB-matrix $S = \Gamma^{\delta_0 \delta_n}_{a_0, a_1, \ldots, a_n}$. Let*

$$\mathbb{B}_S := \mathbb{B}_{\sigma_1 M} \cup \mathbb{B}_{\sigma_2 \overleftarrow{M}} \cup \mathbb{B}_{\sigma_3 M} \cup \mathbb{B}_{\sigma_4 \overleftarrow{M}},$$

where $M = \Gamma^{\delta_0 \delta_n}_{a_0, a_1, \ldots, a_n} \delta_n^{-1}$, and $\mathbb{B}_{\sigma_i X}$ denotes the SB-blow-up of $\sigma_i X$ as in Definition 1.4.6. Let $\widetilde{F_S} : \mathbb{B}_S \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ be the map defined in the local charts $\mathbb{C}^2_{\sigma_i A} \subset \mathbb{B}_S$ by

$$\widetilde{F_S}\Big|_{\mathbb{C}^2_{\sigma_1 A}} = F_S \circ \boldsymbol{\pi}_{\sigma_1 M}, \qquad \widetilde{F_S}\Big|_{\mathbb{C}^2_{\sigma_4 A}} = F_S \circ \boldsymbol{\pi}_{\sigma_4 M},$$

and

$$\widetilde{F_S}\Big|_{\mathbb{C}^2_{\sigma_2 A}} = F_S \circ \boldsymbol{\pi}_{\sigma_2 \overleftarrow{M}}, \qquad \widetilde{F_S}\Big|_{\mathbb{C}^2_{\sigma_3 A}} = F_S \circ \boldsymbol{\pi}_{\sigma_3 \overleftarrow{M}}.$$

Then $\widetilde{F_S} : \mathbb{B}_S \to \mathbb{CP}^1 \times \mathbb{CP}^1$ is a well defined extension of the projective monomial map F_S to the compact smooth space $\mathbb{B}_S$.

Proof. The space $\mathbb{B}_S$ has the structure of a smooth manifold, since the blow-up of a smooth variety along a smooth subvariety is again a smooth variety, see for instance VAKIL, November 18, 2017, Theorem 22.3.10 or SHAFAREVICH, 2013, Chapter II, 4.1. Also, $\mathbb{B}_S$ is compact, since every nonsingular projective variety is compact SHAFAREVICH, 2013, Chapter II, 2.3.

Since a blow-up is isomorphism outside the exceptional divisor, the local maps $\widetilde{F_S}\Big|_{\mathbb{C}^2_{\sigma_i A}}$ all agree in the overlaps. Hence, to prove that $\widetilde{F_S}$ is well defined, it is enough to look at its local action. From Proposition 1.4.10, we can find the collection of charts covering $\mathbb{B}_S$:

(I) Set $i = 1, 4$; then $\mathbb{B}_{\sigma_i M}$ is covered by $\left(\sum_{j=0}^{n} a_j\right) - n + 1$ charts as follows

   (i) the three affine planes $\mathbb{C}^2_{\sigma_i \bar{\delta}_0}$, $\mathbb{C}^2_{\sigma_i S}$ and $\mathbb{C}^2_{\sigma_i M \bar{\delta}_n}$

(ii) for each $j = 0, \dots, n-1$, the $a_j - 1$ affine planes $\mathbb{C}^2_{\sigma_i A^j_k \bar{\delta}_j}$, where

$$A^j_k = \Gamma^{\delta_0 \delta_j}_{a_0 \dots, a_{j-1}, a_j - k}, \quad k = 1, \dots, a_j - 1. \tag{1}$$

(iii) for $j = n$, the $a_n - 2$ affine planes $\mathbb{C}^2_{\sigma_i A^n_k \bar{\delta}_j}$, where

$$A^n_k = \Gamma^{\delta_0 \delta_n}_{a_0 \dots, a_{n-1}, a_n - k}, \quad k = 2, \dots, a_n - 1. \tag{2}$$

(II) Set $i = 2, 3$; then $\mathbb{B}_{\sigma_i \overleftarrow{M}}$ is covered by $\left(\sum_{j=0}^{n} a_j\right) - n + 3$ charts as follows

(i) the three affine planes $\mathbb{C}^2_{\sigma_i \bar{\delta}_n}$, $\mathbb{C}^2_{\sigma_i \overleftarrow{S}}$ and $\mathbb{C}^2_{\sigma_i \overleftarrow{M} \bar{\delta}_0}$

(i) for each $j = 0, \dots, n-1$, the $a_j - 1$ affine planes $\mathbb{C}^2_{\sigma_i B^j_k \bar{\delta}_{n-j}}$, where

$$B^j_k = \Gamma^{\delta_n \delta_{n-j}}_{a_n \dots, a_{n-j+1}, a_{n-j} - k}, \quad k = 1, \dots, a_j - 1. \tag{3}$$

(iii) for $j = n$, the $a_n - 1$ affine planes $\mathbb{C}^2_{\sigma_i B^j_k \bar{\delta}_{n-j}}$, where

$$B^n_k = \Gamma^{\delta_n \delta_0}_{a_n \dots, a_1, a_0 - k}, \quad k = 2, \dots, a_0 - 1. \tag{4}$$

We will show that the map is well defined on the charts covering $\mathbb{B}_{\sigma_2 \overleftarrow{M}}$ and $\mathbb{B}_{\sigma_1 M}$, the work for the other chats is analogous.

In general, for a blow-up $\boldsymbol{\pi}_{\sigma_i X}$ of a SB-matrix X, we can use the local expressions of F, given in (1), to find the local expressions of $\widetilde{F}$:

$$\widetilde{F_S}\Big|_{\mathbb{C}^2_{\sigma_i A}} = F_S\big|_{ji} \circ \boldsymbol{\pi}_{\sigma_i X}\Big|_{\mathbb{C}^2_{\sigma_i A}} = f_{\sigma_j S \sigma_i} \circ \; f_A = f_{\sigma_j S \sigma_i A}.$$

Lets start with the chats of $\mathbb{B}_{\sigma_1 M}$. For the charts of type (ii) or (iii) in (I), we have

$$S \sigma_1 A^j_k \bar{\delta}_j = \Gamma^{\delta_0 \delta_n}_{a_0, \dots, a_n} \, \Gamma^{\delta_0 \delta_j}_{a_0 \dots, a_{j-1}, a_j - k} \bar{\delta}_j, \quad k = 1, \dots a_j - 1,$$

which is a matrix with only positive integer coefficients, since $a_j - k > 0$. Lets say $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, hence we have

$$\widetilde{F_S}\Big|_{\mathbb{C}^2_{\sigma_1 A}} (x, y)_{\sigma_1 A} = (x^a y^b, x^c y^d)_{\sigma_1}, \tag{5}$$

which is well defined for every $(x, y)_{\sigma_1 A} \in \mathbb{C}^2_{\sigma_1 A}$. The three remaining charts of $\mathbb{B}_{\sigma_1 \overleftarrow{M}}$ are the ones associated with $\sigma_1 \bar{\delta}_0$, $\sigma_1 \overleftarrow{S}$, and $\sigma_1 \overleftarrow{M} \bar{\delta}_n$. The fact that all of them have positive coefficients, ensure that $\widetilde{F_S}$ is locally well defined and that its image is contained in the chart $\mathbb{C}^1_{\sigma_1}$ of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

For $\mathbb{B}_{\sigma_2\overleftarrow{M}}$, the charts of type (ii) or (iii) in (II) will be associated with a matrix

$$\begin{aligned} S\sigma_2 B_k^j \bar{\delta}_j &= \Gamma^{\delta_0\delta_n}_{a_0,\dots,a_n}\,\sigma_2\,\Gamma^{\delta_n\delta_{n-j}}_{a_n\dots,a_{n-j+1},a_{n-j}-k}\,\bar{\delta}_{n-j} \\ &= \Gamma^{\delta_0\delta_{n-j-1}}_{a_0,\dots,a_{n-j+1}}\,\delta_{n-j}^{k+a_{n-j}-k}\,\Gamma^{\delta_{n-j+1}\delta_n}_{a_{n-j+1},\dots,a_n}\,\sigma_2\,\Gamma^{\delta_n\delta_{n-j}}_{a_n\dots,a_{n-j+1},a_{n-j}-k}\,\bar{\delta}_{n-j} \\ &= \Gamma^{\delta_0\delta_{n-j}}_{a_0,\dots,a_{n-j+1},k}\,\sigma_2\sigma_2\,\Gamma^{\delta_{n-j}\delta_n}_{a_{n-j}+k,a_{n-j+1},\dots,a_n}\,\sigma_2\,\Gamma^{\delta_n\delta_{n-j}}_{a_n\dots,a_{n-j+1},a_{n-j}-k}\,\bar{\delta}_{n-j} \\ &= \Gamma^{\delta_0\delta_{n-j}}_{a_0,\dots,a_{n-j+1},k}\,\sigma_2\,\left(\Gamma^{\delta_n\delta_{n-j}}_{a_n\dots,a_{n-j+1},a_{n-j}-k}\right)^{-1}\,\Gamma^{\delta_n\delta_{n-j}}_{a_n\dots,a_{n-j+1},a_{n-j}-k}\,\bar{\delta}_{n-j} \\ &= \Gamma^{\delta_0\delta_{n-j}}_{a_0,\dots,a_{n-j+1},k-1}\,\delta_{n-j}\,\sigma_2\,\bar{\delta}_{n-j}. \end{aligned}$$

Now, from Remark 2.1.3,

$$\delta_{n-j}\,\sigma_2\,\bar{\delta}_{n-j} = \begin{cases} \bar{\delta}_{n-j}\varrho, & \text{if} \quad \delta_{n-j} = R \\ -\bar{\delta}_{n-j}\varrho, & \text{if} \quad \delta_{n-j} = L \end{cases}$$

Therefore, $S\sigma_2 B_k^j \bar{\delta}_j$ is a matrix having all coefficients with same sign and $\widetilde{F}_S$ is locally well defined. Further, the action of $\widetilde{F}_S$ is

$$\widetilde{F}_S\Big|_{\mathbb{C}^2_{S\sigma_2 B_k^j \bar{\delta}_{n-j}}}(x,y)_{S\sigma_2 B_k^j \bar{\delta}_{n-j}} = \begin{cases} (x^a y^b, x^c y^d)_{\sigma_1}, & \text{if} \quad \bar{\delta}_{n-j} = R \\ (x^a y^b, x^c y^d)_{\sigma_4}, & \text{if} \quad \bar{\delta}_{n-j} = L \end{cases},$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \Gamma^{\delta_0\delta_{n-j}}_{a_0,\dots,a_{n-j+1}}\delta_{n-j}^{k-1}\bar{\delta}_{n-j}\varrho$.

We still have the tree charts from (i) of (II) to analyze. Let δ be either R or L. Then

$$\begin{aligned} S\sigma_2\overleftarrow{M}\delta &= \Gamma^{\delta_0\delta_n}_{a_0,\dots,a_n}\sigma_2\Gamma^{\delta_n\delta_0}_{a_n,\dots,a_0}\delta_0^{-1}\delta \\ &= \begin{cases} \sigma_2, & \text{if} \quad \delta = \delta_0 \\ \sigma_2\delta_0^{-1}\delta, & \text{if} \quad \delta = \bar{\delta}_0 \end{cases} \\ &= \begin{cases} \sigma_2, & \text{if} \quad \delta = \delta_0 \\ \delta_0\sigma_2\delta, & \text{if} \quad \delta = \bar{\delta}_0 \end{cases} \\ &= \begin{cases} \sigma_2, & \text{if} \quad \delta = \delta_0 \\ L\varrho, & \text{if} \quad \delta = \bar{\delta}_0 \text{ and } \delta_0 = R \\ -R\varrho, & \text{if} \quad \delta = \bar{\delta}_0 \text{ and } \delta_0 = L \end{cases}, \end{aligned}$$

from which we conclude that $\widetilde{F}_S\Big|_{\mathbb{C}^2_{\sigma_2\overleftarrow{M}L}}$ and $\widetilde{F}_S\Big|_{\mathbb{C}^2_{\sigma_2\overleftarrow{M}R}}$ are well defined. In particular, we have

$$\widetilde{F}_S\Big|_{\mathbb{C}^2_{\sigma_2\overleftarrow{S}}}(x,y)_{\sigma_2\overleftarrow{S}} = (x,y)_{\sigma_2} \quad \text{and} \quad \widetilde{F}_S\Big|_{\mathbb{C}^2_{\sigma_2\overleftarrow{M}\delta_0}}(x,y)_{\sigma_2\overleftarrow{M}\bar{\delta}_0} = \begin{cases} (xy, x)_{\sigma_1}, & \text{if} \quad \delta_0 = R \\ (y, xy)_{\sigma_4}, & \text{if} \quad \delta_0 = L \end{cases},$$

Finally, the last chart is $\mathbb{C}^2_{\sigma_2\bar{\delta}_n}$. Since

$$\begin{aligned} S\sigma_2\bar{\delta}_0 &= \Gamma^{\delta_0\delta_n}_{a_0,\dots,a_n}\sigma_2\bar{\delta}_n \\ &= \Gamma^{\delta_0\delta_n}_{a_0,\dots,a_n-1}\delta_n\sigma_2\bar{\delta}_n \\ &= \begin{cases} \Gamma^{\delta_0\delta_n}_{a_0,\dots,a_n-1}L\varrho, & \text{if } \delta_n = R \\ -\Gamma^{\delta_0\delta_n}_{a_0,\dots,a_n-1}L\varrho, & \text{if } \delta_n = L \end{cases}. \end{aligned}$$

is also a matrix with all coefficients with the same sign, hence the map $\left.\widetilde{F}_S\right|_{\mathbb{C}^2_{\sigma_2\bar{\delta}_n}}$ is well defined and

$$\left.\widetilde{F}_S\right|_{\mathbb{C}^2_{\sigma_2\bar{\delta}_0}}(x,y)_{\sigma_2\bar{\delta}_0} = \begin{cases} (xy,x)_{\sigma_1}, & \text{if } \delta_0 = R \\ (xy,x)_{\sigma_4}, & \text{if } \delta_0 = L \end{cases}.$$

□

Theorem 3.1.2. *In the same conditions as in Theorem 3.1.1, let* $\widetilde{F_S^{-1}} : \mathbb{B}_S \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ *be the map defined in the local charts* $\mathbb{C}^2_{\sigma_i A} \subset \mathbb{B}_S$ *by*

$$\left.\widetilde{F_S^{-1}}\right|_{\mathbb{C}^2_{\sigma_1 A}} = F_S^{-1}\circ \boldsymbol{\pi}_{\sigma_1 M}, \qquad \left.\widetilde{F_S^{-1}}\right|_{\mathbb{C}^2_{\sigma_4 A}} = F_S^{-1}\circ \boldsymbol{\pi}_{\sigma_4 M},$$

and

$$\left.\widetilde{F_S^{-1}}\right|_{\mathbb{C}^2_{\sigma_2 A}} = F_S^{-1}\circ \boldsymbol{\pi}_{\sigma_2 \overleftarrow{M}}, \qquad \left.\widetilde{F_S^{-1}}\right|_{\mathbb{C}^2_{\sigma_3 A}} = F_S^{-1}\circ \boldsymbol{\pi}_{\sigma_3 \overleftarrow{M}}.$$

Then $\widetilde{F_S^{-1}} : \mathbb{B}_S \to \mathbb{CP}^1\times\mathbb{CP}^1$ *is a well defined extension of the rational map* F_S^{-1} *to the compact space* $\mathbb{B}_S$.

Proof. The proof of Theorem 3.1.2 is essentially a repetition of the techniques used to show Theorem 3.1.1. But since we will need the local expressions of $(\widetilde{F})^{-1}$, lets prove that this map is well defined in $\mathbb{B}_{\sigma_2\overleftarrow{M}}$ and in $\mathbb{B}_{\sigma_1 M}$. We will use the same notation for the charts as on the proof of Theorem 3.1.1.

Starting with $\mathbb{B}_{\sigma_2\overleftarrow{M}}$; for charts of type (3) and (4), we have

$$\begin{aligned} S^{-1}\sigma_2 B^j_k\bar{\delta}_j &= \sigma_2\,\Gamma^{\delta_n\delta_0}_{a_n,\dots,a_0}\,\sigma_2\sigma_2\,\Gamma^{\delta_n\delta_{n-j}}_{a_n\dots,a_{n-j+1},a_{n-j}-k}\,\bar{\delta}_{n-j} \\ &= \sigma_2\,\Gamma^{\delta_n\delta_0}_{a_n,\dots,a_0}\,\Gamma^{\delta_n\delta_{n-j}}_{a_n\dots,a_{n-j+1},a_{n-j}-k}\,\bar{\delta}_{n-j}, \end{aligned}$$

now, since $a_{n-j}-k \geq 0$, the coefficients of $\Gamma^{\delta_n\delta_0}_{a_n,\dots,a_0}\Gamma^{\delta_n\delta_{n-j}}_{a_n\dots,a_{n-j+1},a_{n-j}-k}\,\bar{\delta}_{n-j}$ are all positive. Hence

$$\left.\widetilde{F_S^{-1}}\right|_{\sigma_2 B^j_k\bar{\delta}_j}(x,y)_{\sigma_2 B^j_k\bar{\delta}_j} = (x^a, y^b, x^c y^d)_{\sigma_2}$$

where $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \Gamma^{\delta_n\delta_0}_{a_n,\dots,a_0}\Gamma^{\delta_n\delta_{n-j}}_{a_n\dots,a_{n-j+1},a_{n-j}-k}$, which is a well defined map. For the charts $\mathbb{C}^2_{\sigma_2\bar{\delta}_n}$, $\mathbb{C}^2_{\sigma_2\overleftarrow{S}}$ and $\mathbb{C}^2_{\sigma_2\overleftarrow{M}\bar{\delta}_0}$, we have a similar situation.

For $\mathbb{B}_{\sigma_1 M}$, the charts of type (1) and (2) will are associated with matrices

$$\begin{aligned}
S^{-1}\sigma_1 A_k^j \bar{\delta}_j &= \sigma_2\,\Gamma^{\delta_n\delta_0}_{a_n,\dots,a_0}\,\sigma_2\sigma_1\,\Gamma^{\delta_0\delta_j}_{a_0\dots,a_{j-1},a_j-k}\bar{\delta}_j \\
&= \sigma_2\,\Gamma^{\delta_n\delta_{j+1}}_{a_n,\dots,a_{j+1}}\,\delta_j^{k+a_j-k}\Gamma^{\delta_{j-1}\delta_0}_{a_{j-1},\dots,a_0}\,\sigma_2\,\Gamma^{\delta_0\delta_j}_{a_0\dots,a_{j-1},a_j-k}\bar{\delta}_j \\
&= \sigma_2\,\Gamma^{\delta_n\delta_j}_{a_n,\dots,a_{j+1},k}\,\sigma_2\sigma_2\,\Gamma^{\delta_j\delta_0}_{a_j-k,a_{j-1},\dots,a_0}\,\sigma_2\,\Gamma^{\delta_0\delta_j}_{a_0\dots,a_{j-1},a_j-k}\bar{\delta}_j \\
&= \sigma_2\,\Gamma^{\delta_n\delta_j}_{a_n,\dots,a_{j+1},k}\,\sigma_2\left(\Gamma^{\delta_0\delta_j}_{a_0\dots,a_{j-1},a_j-k}\right)^{-1}\,\Gamma^{\delta_0\delta_j}_{a_0\dots,a_{j-1},a_j-k}\bar{\delta}_j \\
&= \sigma_2\,\Gamma^{\delta_n\delta_{j+1}}_{a_n,\dots,a_{j+1}}\,\delta_j^k\sigma_2\bar{\delta}_j \\
&= \begin{cases} \sigma_2\,\Gamma^{\delta_n L}_{a_n,\dots,a_{j+1}}\,R^{k-1}L\varrho, & \text{if} \quad \delta_j = R \\ -\sigma_2\,\Gamma^{\delta_n R}_{a_n,\dots,a_{j+1}}\,L^{k-1}R\varrho, & \text{if} \quad \delta_j = L \end{cases}.
\end{aligned}$$

Hence $\left.\widetilde{F}_S^{-1}\right|_{\sigma_1 A_k^j\bar{\delta}_j}$ is well defined, and

$$\left.\widetilde{F}_S^{-1}\right|_{\sigma_1 A_k^j\bar{\delta}_j}(x,y)_{\sigma_1 A_k^j\bar{\delta}_j} = \begin{cases} (x^a y^b, x^c y^d)_{\sigma_2}, & \text{if} \quad \delta_j = R \\ (x^a y^b, x^c y^d)_{\sigma_3}, & \text{if} \quad \delta_j = L \end{cases},$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \Gamma^{\delta_n\delta_{j+1}}_{a_n,\dots,a_{j+1}}\,\delta_j^{k-1}\bar{\delta}_j\varrho$.

The case for the charts $\mathbb{C}^2_{\sigma_1\bar{\delta}_n}$, and $\mathbb{C}^2_{\sigma_1 M\bar{\delta}_0}$ are similar. For $\mathbb{C}^2_{\sigma_1 S}$, we have

$$\begin{aligned}
S^{-1}\sigma_1 S &= \sigma_2\,\Gamma^{\delta_0\delta_n}_{a_0,\dots,a_n}\,\sigma_2\sigma_1\,\Gamma^{\delta_n\delta_0}_{a_n\dots,a_0} \\
&= \sigma_2\,\left(\Gamma^{\delta_0\delta_n}_{a_0\dots,a_n}\right)^{-1}\Gamma^{\delta_0\delta_n}_{a_0\dots,a_n} \\
&= \sigma_2.
\end{aligned}$$

Then

$$\left.\widetilde{F}_S^{-1}\right|_{\sigma_1 S}(x,y)_{\sigma_1 S} = (x,y)_{\sigma_2}.$$

□

Corollary 3.1.3. *The SB-blow-ups of Theorems 3.1.2 and 3.1.1 are the minimal ones that results in a space where both F_S and F_S^{-1} have well defined extensions.*

Proof. In fact if we stop the blow-ups anywhere in the middle of the chains $\boldsymbol{\pi}_{\sigma_2\overleftarrow{M}}$ or $\boldsymbol{\pi}_{\sigma_3\overleftarrow{M}}$, the map $\widetilde{F}_S$ would not be defined on one of the charts covering the final space.

For example, lets say we just make the SB-blow-up of $\sigma_2\Gamma^{\delta_n\delta_{j-1}}_{a_n,\dots,a_{j-1}}$, instead of $\boldsymbol{\pi}_{\sigma_2\overleftarrow{M}}$. Then the covering of the space would have the chart $\mathbb{C}^2_{\sigma_2 A}$ with $A = \Gamma^{\delta_n\delta_{j-1}}_{a_n,\dots,a_j}$ (created in the final blow-up of the chain), and the action of $\widetilde{F}_S$ over there would be

$$\left.\widetilde{F}_S\right|_{\mathbb{C}^2_{\sigma_2 A}}(x,y)_{\mathbb{C}^2_{\sigma_2 A}} = \begin{pmatrix} [x^a : y^b] \\ [\,x^c : \ y^d] \end{pmatrix},$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \Gamma^{\delta_0\delta_{j-2}}_{a_0,\dots,a_{j-2}}$, which is not defined for $x = y = 0$.

□

Notation. In order to write the action of $\widetilde{F_S}$ with a cleaner notation, given a matrix A, we write

$$\widetilde{F_S}\Big|_A := \widetilde{F_S}\Big|_{\mathbb{C}^2_A},$$

and we use $f_S^{\sigma_i A \to \sigma_j A'}$ to indicate the monomial map f_S living from the chart $\mathbb{C}^2_{\sigma_i A}$ and arriving at $\mathbb{C}^2_{\sigma_j A'}$, that is

$$f_S^{\sigma_i A \to \sigma_j A'}(x,y)_{\sigma_i A} = (x^a y^b, x^c y^d)_{\sigma_j A'}, \quad \text{where} \quad S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

Now, from the local expressions of $\widetilde{F_S}$ and $(\widetilde{F_S})^{-1}$ that we derived in the proof of Theorem 3.1.1 and Theorem 3.1.1, we give a summary of their action in any chart of $\mathbb{B}_S$.

Corollary 3.1.4. *Let $S = \Gamma^{\delta_0 \delta_n}_{a_0, a_1, \dots, a_n}$. Then the maps $\widetilde{F_S}$ and $(\widetilde{F_S})^{-1}$ acts on the charts of $\mathbb{B}_S$ as follows*

The local expressions of $\widetilde{F_S}$ are given by monomial maps as follows:

- *For any chart $\mathbb{C}^2_{\sigma_1 A}$ or $\mathbb{C}^2_{\sigma_4 A}$*

$$\widetilde{F_S}\Big|_{\sigma_1 A} = f_{SA}^{\sigma_1 A \to \sigma_1}, \qquad \text{and} \qquad \widetilde{F_S}\Big|_{\sigma_4 A} = f_{SA}^{\sigma_4 A \to \sigma_4}.$$

- *For $j = 0, \dots, n$ and $k = 1, \dots a_j - 1$, set $B^j_k = \Gamma^{\delta_n \delta_{n-j}}_{a_n, \dots, a_{n-j}} \delta^{-k}_{n-j} \bar{\delta}_{n-j}$. Then, we have*

$$\widetilde{F_S}\Big|_{\sigma_2 B^j_k} = \begin{cases} f^{\sigma_2 B^j_k \to \sigma_1}_{\Gamma^{\delta_0 R}_{a_0,\dots,a_{n-j+1}} R^{k-1} L \varrho}, & \text{if} \quad \delta_{n-j} = R \\ f^{\sigma_2 B^j_k \to \sigma_4}_{\Gamma^{\delta_0 L}_{a_0,\dots,a_{n-j+1}} L^{k-1} R \varrho}, & \text{if} \quad \delta_{n-j} = L \end{cases},$$

$$\widetilde{F_S}\Big|_{\sigma_3 B^j_k} = \begin{cases} f^{\sigma_3 B^j_k \to \sigma_4}_{\Gamma^{\delta_0 R}_{a_0,\dots,a_{n-j+1}} R^{k-1} L \varrho}, & \text{if} \quad \delta_{n-j} = R \\ f^{\sigma_3 B^j_k \to \sigma_1}_{\Gamma^{\delta_0 L}_{a_0,\dots,a_{n-j+1}} L^{k-1} R \varrho}, & \text{if} \quad \delta_{n-j} = L \end{cases}.$$

- *In the charts $\mathbb{C}^2_{\sigma_2 \overleftarrow{S}}$ and $\mathbb{C}^2_{\sigma_3 \overleftarrow{S}}$ we have*

$$\widetilde{F_S}\Big|_{\sigma_2 \overleftarrow{S}} = f_{\sigma_1}^{\sigma_2 \overleftarrow{S} \to \sigma_2} \qquad \text{and} \qquad \widetilde{F_S}\Big|_{\sigma_3 \overleftarrow{S}} = f_{\sigma_1}^{\sigma_2 \overleftarrow{S} \to \sigma_3}.$$

The local expressions of $(\widetilde{F_S})^{-1}$ are

- *For any chart $\mathbb{C}^2_{\sigma_2 A}$ or $\mathbb{C}^2_{\sigma_3 A}$*

$$\widetilde{F_S}^{-1}\Big|_{\sigma_2 A} = f_{SA}^{\sigma_2 A \to \sigma_2}, \qquad \text{and} \qquad \widetilde{F_S}^{-1}\Big|_{\sigma_3 A} = f_{SA}^{\sigma_3 A \to \sigma_3}.$$

- *For $j = 0, \dots, n$ and $k = 1, \dots a_j - 1$, set $A_k^j = \Gamma^{\delta_0\delta_{j+1}}_{a_0\dots,a_{j-1},a_j-k}\delta_j^{k-1}\overline{\delta}_j$ we have*

$$\widetilde{F}_S^{-1}\Big|_{\sigma_1 A_k^j} = \begin{cases} f^{\sigma_1 A_k^j \to \sigma_2}_{\Gamma^{\delta_n\delta_{j+1}}_{a_n,\dots,a_{j+1}} R^{k-1}L\varrho}, & if \quad \delta_j = R \\ f^{\sigma_1 A_k^j \to \sigma_3}_{\Gamma^{\delta_n\delta_{j+1}}_{a_n,\dots,a_{j+1}} L^{k-1}R\varrho}, & if \quad \delta_j = L \end{cases},$$

$$\widetilde{F}_S^{-1}\Big|_{\sigma_4 A_k^j} = \begin{cases} f^{\sigma_4 A_k^j \to \sigma_3}_{\Gamma^{\delta_n\delta_{j+1}}_{a_n,\dots,a_{j+1}} R^{k-1}L\varrho}, & if \quad \delta_j = R \\ f^{\sigma_4 A_k^j \to \sigma_2}_{\Gamma^{\delta_n\delta_{j+1}}_{a_n,\dots,a_{j+1}} L^{k-1}R\varrho}, & if \quad \delta_j = L \end{cases}.$$

- *In the charts $\mathbb{C}^2_{\sigma_1 S}$ and $\mathbb{C}^2_{\sigma_4 S}$ we have*

$$\widetilde{F}_S^{-1}\Big|_{\sigma_1 S} = f^{\sigma_1 S \to \sigma_2}_{\sigma_1} \qquad and \qquad \widetilde{F}_S^{-1}\Big|_{\sigma_4 S} = f^{\sigma_4 S \to \sigma_3}_{\sigma_1}.$$

We manage to build a well defined extension of F_S and F_S^{-1} to a compact space. But if we want obtain a dynamical system, there is still a problem: the domain and range on each of those maps do not match.

In the next section, we try to fix this using the inverse of the SB-blow-ups to define a map having $\mathbb{B}_S$ as the range. The price to pay is that, the resulting map will again have points of indeterminacy. But this is a problem we will deal later.

3.2 An extension to a birational map

One way to obtain an extension of the monomial map having the same compact space as domain and range is to define it as $F^\sharp := \pi^{-1} \circ \widetilde{F}$.

Definition 3.2.1. Given a SB-matrix $S = \Gamma^{\delta_0\delta_n}_{a_0,a_1,\dots,a_n}$, let $\mathbb{B}_S$ be the compact space given by Theorem 3.1.1. The **first blow-up** for the monomial map F_S, $\boldsymbol{\pi}^S$, is the map given by the gluing of the SB-blow-ups of Theorem 3.1.1. That is, on each chart $\mathbb{C}^2_{\sigma_i A}$ of $\mathbb{B}_S$, the map $\boldsymbol{\pi}^S : \mathbb{B}_S \longrightarrow \mathbb{B}_S$ is defined by

$$\boldsymbol{\pi}^S\Big|_{\sigma_i A} = \begin{cases} \boldsymbol{\pi}_{\sigma_i S \delta_n^{-1}}\Big|_{\sigma_i A}, & \text{for} \quad i = 2, 3. \\ \boldsymbol{\pi}_{\sigma_i \overleftarrow{S} \delta_0^{-1}}\Big|_{\sigma_i A}, & \text{for} \quad i = 1, 4. \end{cases}$$

We also say that $\mathbb{B}_S$ is the **first blow-up space** for the monomial map F_S.

Theorem 3.2.2. *Let $\boldsymbol{\pi}^S : \mathbb{B}_S \longrightarrow \mathbb{B}_S$ be the first blow-up for the monomial map F_S. Let $F_S^\sharp : \mathbb{B}_S \dashrightarrow \mathbb{B}_S$ and $(F^\sharp)_S^{-1} : \mathbb{B}_S \dashrightarrow \mathbb{B}_S$ be the rational maps defined by*

$$F_S^\sharp = \left(\boldsymbol{\pi}^S\right)^{-1} \circ \widetilde{F}_s \quad and \quad (F^\sharp)_S^{-1} = \left(\boldsymbol{\pi}^S\right)^{-1} \circ \widetilde{F}_S^{-1},$$

with $\widetilde{F}_S$ as in Theorem 3.1.1 and $\widetilde{F}_S^{-1}$ as in Theorem 3.1.2. Then $F_S^\sharp$ is a birational extension of f_S

with inverse $(F^\sharp_S)^{-1}$. *Further, the points where* $F^\sharp_S$ *and* $(F^\sharp_S)^{-1}$ *are not defined are respectively*

$$\mathcal{S}^+ = \{\mathbf{0}_{\sigma_2\overleftarrow{S}}, \mathbf{0}_{\sigma_3\overleftarrow{S}}\} \quad \textit{and} \quad \mathcal{S}^- = \{\mathbf{0}_{\sigma_1 S}, \mathbf{0}_{\sigma_4 S}\}.$$

Proof. In principle, since the inverse of $\boldsymbol{\pi}^S$ is not defined in the points $\mathbf{0}_{\sigma_i}$, $i = i, 2, 3, 4$, we would expect that $F^\sharp$ or $(F^\sharp)^{-1}$ would not be defined in any of the points of the exceptional divisors of $\mathbb{B}_S$. But since the inverse $\boldsymbol{\pi}^S$ is restricted to the image of $\widetilde{F}$, and $(\widetilde{F})^{-1}$, they are defined in a bigger subset of $\mathbb{B}_S$. In fact, from the proofs of Theorems 3.1.1, 3.1.2 and Corollary 3.1.4, we see that any sequence of points in the set $Im(\widetilde{F}) \setminus \{\mathbf{0}_{\sigma_1}, \mathbf{0}_{\sigma_2}, \mathbf{0}_{\sigma_3}, \mathbf{0}_{\sigma_4}\}$, or $Im(\widetilde{F^{-1}}) \setminus \{\mathbf{0}_{\sigma_1}, \mathbf{0}_{\sigma_2}, \mathbf{0}_{\sigma_3}, \mathbf{0}_{\sigma_4}\}$, approaches the point $\mathbf{0}_{\sigma_i}$ through a curve with fixed slope, which allow us to define the locus in the divisor for $(\boldsymbol{\pi}^S)^{-1}(\mathbf{p})$, of any $\mathbf{p}$ in those sets.

Using the local expressions of $\widetilde{F}$ and and $(\widetilde{F})^{-1}$ and Proposition 1.4.10, we are able to determine the local action of $F^\sharp$ and $(F^\sharp)^{-1}$:

- For any charts $\mathbb{C}^1_{\sigma_1 A}$, $\mathbb{C}^2_{\sigma_2 A}$, $\mathbb{C}^1_{\sigma_3 A}$, $\mathbb{C}^2_{\sigma_4 A}$, we have

$$F^\sharp_S\big|_{\sigma_1 A} = f_S^{\sigma_1 A \to \sigma_1 A}, \qquad F^\sharp_S\big|_{\sigma_4 A} = f_S^{\sigma_4 A \to \sigma_4 A}$$

$$(F^\sharp_S)^{-1}\big|_{\sigma_2 A} = f_{\overleftarrow{S}}^{\sigma_2 A \to \sigma_2 A}, \qquad (F^\sharp_S)^{-1}\big|_{\sigma_3 A} = f_{\overleftarrow{S}}^{\sigma_3 A \to \sigma_3 A}.$$

- For $j = 0, \dots, n$ and $k = 1, \dots a_j - 1$, set $A^j_k = \Gamma^{\delta_0 \delta_{j+1}}_{a_0 \dots, a_{j-1}, a_j - k} \delta_j^{k-1} \bar{\delta}_j$ and $B^j_k = \Gamma^{\delta_n \delta_{n-j}}_{a_n, \dots, a_{n-j}} \delta^{-k}_{n-j} \bar{\delta}_{n-j}$. Then, we have

$$F^\sharp_S\big|_{\sigma_2 \mathbb{B}^j_k} = f_S^{\sigma_2 B^j_k \to \sigma_2 \mathbb{B}^j_k}, \qquad F^\sharp_S\big|_{\sigma_3 \mathbb{B}^j_k} = f_S^{\sigma_3 B^j_k \to \sigma_3 \mathbb{B}^j_k},$$

$$(F^\sharp_S)^{-1}\big|_{\sigma_1\ A^j_k} = f_{\overleftarrow{S}}^{\sigma_1 A^j_k \to \sigma_1 A^j_k}, \qquad (F^\sharp_S)^{-1}\big|_{\sigma_4\ A^j_k} = f_{\overleftarrow{S}}^{\sigma_4 A^j_k \to \sigma_4 A^j_k}.$$

- Finally, for points in $\mathbb{C}^2_{\sigma_1 S} \setminus \{\mathbf{0}_{\sigma_1 S}\}$ or in $\mathbb{C}^2_{\sigma_4 S} \setminus \{\mathbf{0}_{\sigma_4 S}\}$, the map $F^\sharp_S$ acts as a projection, and for any point in $\mathbb{C}^2_{\sigma_2\overleftarrow{S}} \setminus \{\mathbf{0}_{\sigma_2\overleftarrow{S}}\}$ or in $\mathbb{C}^2_{\sigma_3\overleftarrow{S}} \setminus \{\mathbf{0}_{\sigma_3\overleftarrow{S}}\}$ the inverse $(F^\sharp_S)^{-1}$ acts as a projection.

This exhausts all the charts of $\mathbb{B}_S$ and then, we have a complete description of $F^\sharp_S$. We also notice that, by construction $F^\sharp_S \circ (F^\sharp_S)^{-1}\big|_{\sigma_i \mathbb{A}} = (F^\sharp_S)^{-1} \circ F^\sharp_S\big|_{\sigma_i \mathbb{A}}$, for any chart $\mathbb{C}^2_{\sigma_i A}$ of $\mathbb{B}_S$. Further, given the isomorphism between $\mathbb{C}^2_{\sigma_i A} \setminus \{\mathbf{0}_{\sigma_i A}\}$ and $(\mathbb{C}^*)^2$, the restriction $F^\sharp_S\big|_{\mathbb{C}^2_{\sigma_1 A} \setminus \{\mathbf{0}_{\sigma_i A}\}}$ is exactly the monomial map f_S.

□

Definition 3.2.3. Let $(\mathbb{C}^*_{\sigma_i A})^2 = \mathbb{C}^2_{\sigma_i A} \setminus \{\mathbf{0}_{\sigma_i A}\}$. We call the subset $\mathbb{B}_S \setminus (\mathbb{C}^*_{\sigma_i A})^2$ of $\mathbb{B}_S$ the **total divisor** of $\mathbb{B}_S$ and denoted it by $\mathbb{D}_S$.

Theorems 3.2.5 and 3.2.4 are bridges between our study of $SL(2, \mathbb{Z})$ matrices in the previous chapter to dynamical system associated to a monomial map.

Lemma 3.2.4. *The total divisor of $\mathbb{B}_S$ is a cycle of rational curves intersecting transversally. The exceptional divisor labeled $\sigma_i \binom{p}{q}$, where p and q are positive integers, is parameterized by*

$$s \longmapsto \lim_{t\to 0} \begin{pmatrix} s^l t^p \\ s^k t^q \end{pmatrix},$$

where k, l are integers satisfying $lq - kp = 1$.

Proof. The first part of the statement is a direct consequence of the fact that $\mathbb{B}_S$ is obtained after a chain of blow-ups of points.

For the second part, first we notice that a different choice of the pair (k, l) does not change the limit $t \to 0$, in fact it just changes t for some multiple of t. The rest of the proof is made by induction.

For $n = 0$ we have four rational curves intersecting transversaly. They are $(\{0,\infty\} \times \mathbb{P}) \cup (\mathbb{P} \times \{\infty, 0\})$. Lets see the parametrization for $\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$: k must be 1 and l can be anything, then

$$s \mapsto \lim_{t\to 0} \begin{pmatrix} s^l t^p \\ s^k t^q \end{pmatrix} = \lim_{t\to 0} \begin{pmatrix} s^l t^{-1} \\ s t^0 \end{pmatrix} = \begin{pmatrix} \infty \\ s \end{pmatrix}$$

which is a parametrization for $\infty \times \mathbb{P}$ as defined in chapter 1. For the inductive step, let $\begin{pmatrix} p \\ q \end{pmatrix}$ be a label on the level $n + 1$ of the SB-diagram. Then

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + \begin{pmatrix} p_2 \\ q_2 \end{pmatrix},$$

are labels of the neighbors $p_1 q_1$ and $p_2 q_2$ on the level n of the SB-diagram. The two parametrized curves

$$t \mapsto \begin{pmatrix} s^{l_1} t^{p_1} \\ s^{k_1} t^{q_1} \end{pmatrix} \quad \text{and} \quad t \mapsto \begin{pmatrix} s^{l_2} t^{p_2} \\ s^{k_2} t^{q_2} \end{pmatrix}$$

have equations

$$x^{q_1} y^{-p_1} = s_1 \qquad \text{and} \qquad x^{q_2} y^{-p_2} = s_2.$$

Then, since $x = s_1^{-p_2} s_2^{p_1}$ and $y = s_1^{-q_2} s_2^{q_1}$, we can use s_1 and s_2 as coordinates in $(\mathbb{C}^*)^2$. The divisors $\mathsf{L}_{\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}}$ and $\mathsf{L}_{\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}}$ intersect at $s_1 = \infty$ and $s_2 = 0$, and

$$s = s_1 s_2 = x^{q_1+q_2} y^{-(p_1+p_2)},$$

which gives the parametrization of $\mathsf{L}_{\begin{pmatrix} p \\ q \end{pmatrix}}$. $\square$

Theorem 3.2.5. *The map $F_S^\sharp$ sends points of the divisor $\mathsf{L}_{\binom{p}{q}}$ to the divisor $\mathsf{L}_{(\!(S\binom{p}{q})\!)}$, and $(F_S^\sharp)^{-1}$ sends points of the divisor $\mathsf{L}_{\binom{p}{q}}$ to the divisor $\mathsf{L}_{(\!(S^{-1}\binom{p}{q})\!)}$.*

Proof. Firs we remember that for each blown-up point $\mathbf{0}_{\mathbb{C}^2_{\sigma_i A}}$ we create a exceptional divisor labeled by $\mathsf{L}_{(\!(\sigma_i A)\!)}$, see Remark 1.4.3. Also, from the proof of Theorem 3.1.1, we know that the label for the divisors are given by the matrices

$$A^j_k = \Gamma^{\delta_0 \delta_j}_{a_0 \dots, a_{j-1}, a_j - k}, \qquad \text{and} \qquad B^j_k = \Gamma^{\delta_n \delta_{n-j}}_{a_n \dots, a_{n-j+1}, a_{n-j} - k}$$

where $j = 0 \dots n$ and $k = 1, \dots, a_j - 1$. Now, as we saw in Corollary 2.2.2, the matrices A^j_0 and B^j_0 are exactly the matrices associated with the points in the sail of the matrix S. Hence, since a matrix sends points of its sails to points of its sails (including the ones that are not vertices), from the parametrization given in Lemma 3.2.4, follows the desired. $\square$

The correlation between points of of the sails and divisors on the SB-blow-ups also extends to a bigger class of matrices. In fact, we could repeat every step we followed, to find the rational extension $F^\sharp_S$ of any matrix of type (A) from Theorem 2.2.1. For types (B) and (C), the SB-blow-ups would not be exactly the same (since in those cases, the points $\mathbf{0}_{\sigma_2}$ and $\mathbf{0}_{\sigma_2}$ are simultaneously singularities the monomial map and its inverse) but the work is not so different from the one we presented. Further, in any of this cases, the action of the matrix on its sails also gives the description of the action of $F^\sharp_S$ on the divisor.

Proposition 3.2.6. *The subspace $(\mathbb{C}^*_{\sigma_i A})^2$ is dense in $\mathbb{B}_S$.*

Proof. Due to its smooth variety structure, B_S is endowed with a distance d, that can be taken as the one induced by the maximum norm. That is, given $\mathbf{p} = (u_1, v_1)_{\sigma_i A}$ and $\mathbf{q} = (u_2, v_2)_{\sigma_i A}$ in $\mathbb{C}^2_{\sigma_i A} \subset B_S$, $d(\mathbf{p}, \mathbf{q}) = max\{|u_1 - u_2|, |v_2 - v_1|\}$. Then, directly from the definition of the standard blow-up, follows that any point of the divisor can be approached by a sequence in $(\mathbb{C}^*_{\sigma_i A})^2$. $\square$

By construction, is easy to see that the complement of any chart $(\mathbb{C}^*_{\sigma_i A})^2$ is given by the exceptional divisors together with the divisor $\boldsymbol{\pi}_S^{-1}\left(\mathsf{L}_{\binom{1}{0}} \cup \mathsf{L}_{\binom{-1}{0}} \cup \mathsf{L}_{\binom{0}{1}} \cup \mathsf{L}_{\binom{0}{-1}}\right)$.

Final

Chapter 4

The sequence space: a compactification for the monomial map

On the previous chapter, we build an extension for the monomial map having the same compact smooth algebraic variety as its domain and range. But this still does not solve our problem of compactifying $f_S : \mathbb{C} \to \mathbb{C}$ as a dynamical system, since $F_S^\sharp$ still have indeterminacies. We could try to repeat our process and make another sequence of blow-ups on $\mathbb{B}_S$ in order to obtain another well defined rational extension for f_S. But this comes with drawbacks, we would mess with the domain of this extension again.

A way to work around this problem is to consider a generalization for the blow-up definition. The blow-up of $\mathbb{B}_S$ on the singularities of $F^\sharp$ is closure of the graph of $F_S^\sharp$ (see Gathmann, 2019/20, Construction 9.9 and 9.16), and then define a map acting on the space of sequences with all entries on this closure. This idea, replicated from J. Hubbard *et al.*, 2000, is a way to perform infinitely many blow-ups at once and then, use an inductive limit, to get a dynamical system.

4.1 Preliminary results

The following lemma will enable us to start an analysis on the properties of the graph of $F_S^\sharp$.

Lemma 4.1.1 (J. Hubbard *et al.*, 2000)**.** *Let $g : X \dashrightarrow Y$ be a birational transformation between compact smooth algebraic surfaces. Suppose that g is undefined at $\mathcal{S}_g^+ \subset X$ and that g^{-1} is undefined at $\mathcal{S}_g^- \subset Y$. Let*

$$\Gamma_g \subset (X \setminus \mathcal{S}_g^+) \times Y$$

be the graph of g, and let $\bar{\Gamma}_g \subset X \times Y$ be its closure. Then, the space $\bar{\Gamma}_g$ is a smooth manifold, except perhaps at points $(\mathbf{p}, \mathbf{q}) \in \bar{\Gamma}_g$ such that $\mathbf{p} \in \mathcal{S}_g^+$ and $\mathbf{q} \in \mathcal{S}_g^-$.

Proof. Since g is birational, if $\mathbf{p} \notin \mathcal{S}_g^+$, there is a neighborhood of $\mathbf{p}$, $\mathcal{U}_{\mathbf{p}} \subset X$, were g is well defined and induces an isomorphism from $\mathcal{U}_{\mathbf{p}}$ to $g(\mathcal{U}_{\mathbf{p}})$. Hence, the projection on the first coordinate $pr_1 : \bar{\Gamma}_g \to X$ is a local isomorphism with inverse given by $\mathbf{p} \mapsto (\mathbf{p}, g(\mathbf{p}))$. Analogously, if $\mathbf{q} \notin \mathcal{S}_g^-$, the projection on the second coordinate $pr_2 : \bar{\Gamma}_g \to Y$ is a local isomorphism.

$\square$

Applying this to the birational map $F_S^\sharp : \mathbb{B}_S \dashrightarrow \mathbb{B}_S$ defined on the previous chapter, we conclude:

Proposition 4.1.2. *The closure $\bar{\Gamma}_{F_S^\sharp} \subset \mathbb{B}_S \times \mathbb{B}_S$ is a smooth submanifold.*

Proof. As we saw before, the set of the points of indeterminacy of $F_S^\sharp$ is $\mathcal{S}^+ = \{\mathbf{0}_{\sigma_2 \overleftarrow{S}}, \mathbf{0}_{\sigma_3 \overleftarrow{S}}\}$ and the set of the points of indeterminacy of $(F_S^\sharp)^{-1}$ is $\mathcal{S}^- = \{\mathbf{0}_{\sigma_1 S}, \mathbf{0}_{\sigma_4 S}\}$.

Now, any neighborhood of $\mathbf{0}_{\sigma_2 \overleftarrow{S}}$ belongs to $\mathbb{C}^2_{\sigma_2 \overleftarrow{M} \bar{\delta}_0}$. Hence a pair $(\mathbf{0}_{\sigma_1 \overleftarrow{S}}, \mathbf{q})$ is in $\bar{\Gamma}_{F_S^\sharp}$ if and only if, $\mathbf{q}$ is at the union of the exceptional divisors $\mathsf{L}_{⟅\sigma_2\delta_0 L⟆}$ or $\mathsf{L}_{⟅\sigma_2\delta_0 R⟆}$ (see the action of $F_S^\sharp$ in Theorem 3.2.2) and therefore $\mathbf{q} \notin \mathcal{S}^-$. Similarly, if $(\mathbf{0}_{\sigma_3 \overleftarrow{S}}, \mathbf{q})$, then $\mathbf{q} \notin \mathcal{S}^-$. With that we, Lemma 4.1.1 ensure that $\bar{\Gamma}_{F^\sharp}$ is a (compact) smooth manifold.

$\square$

4.2 Characterization of the dynamical system

It is time to present the compactification for the monomial map: the space $\mathbb{B}_S^\infty$ constructed in the following definition is a compact space which, in some sense, contains $\mathbb{C}^2$ as a dense open subset and is such that the monomial map $f_S : \mathbb{C}^2 \to \mathbb{C}^2$ extends to $F_S^\infty : \mathbb{B}_S^\infty \to \mathbb{B}_S^\infty$.

Definition 4.2.1. The **sequence space** $\mathbb{B}_S^\infty$ is the subset of $(\mathbb{B}_S)^{\mathbb{Z}}$ consisting of all indexed sequences $\mathbf{p}_\infty = (\ldots, \mathbf{p}_{-1}, \mathbf{p}_0, \mathbf{p}_1, \ldots)$ such that successive pairs belong to $\bar{\Gamma}_{F^\sharp} \subset \mathbb{B}_S \times \mathbb{B}_S$.

The map $F_S^\infty : \mathbb{B}_S^\infty \to \mathbb{B}_S^\infty$ is defined as the shift

$$F_S^\infty(\ldots, \mathbf{p}_{-1}, \mathbf{p}_0, \mathbf{p}_1, \ldots) := (\ldots, \mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \ldots),$$

and we write $(F_S^\infty(\mathbf{p})_\infty)_k = \mathbf{p}_{k+1}$.

From Tychonoff's theorem follows that $\mathbb{B}_S^\infty$ is a compact topological space, since is a closed subset of a product of compact topological spaces. Furthermore, F_S^∞ is clearly a homeomorphism.

Notation. (i) As mentioned before, we identify the chart $(\mathbb{C}^*)^2_{\sigma_1 A} \subset \mathbb{B}_S$ with $(\mathbb{C}^*)^2$, considering that there is an isomorphism between this two sets and that the map $F_S^\sharp\big|_{(\mathbb{C}^*)^2_{\sigma_1 A}} : \mathbb{C}^2_{\sigma_1 A} \to \mathbb{C}^2_{\sigma_1 A}$ is exactly the monomial map $f_S : \mathbb{C}^2 \to \mathbb{C}^2$. Notice that the set $((\mathbb{C}^*)^2_{\sigma_1 A})^{\mathbb{Z}}$ is obviously contained in $\mathbb{B}_S^\infty$, so we use the notation

$$(\mathbb{C}^*)^2_\infty := ((\mathbb{C}^*)^2_{\sigma_1 A})^{\mathbb{Z}}$$

(ii) We name four particular interesting points of $\mathbb{B}_S^\infty$:

- the one having all entries equal to $\mathbf{0}_{\sigma_2\overleftarrow{S}}$, $(\mathbf{0}_{\sigma_2\overleftarrow{S}})^\infty = (\dots, \mathbf{0}_{\sigma_2\overleftarrow{S}}, \mathbf{0}_{\sigma_2\overleftarrow{S}}, \mathbf{0}_{\sigma_2\overleftarrow{S}}, \dots)$;
- the one with all entries equal to $\mathbf{0}_{\sigma_3\overleftarrow{S}}$, $(\mathbf{0}_{\sigma_3\overleftarrow{S}})^\infty = (\dots, \mathbf{0}_{\sigma_3\overleftarrow{S}}, \mathbf{0}_{\sigma_3\overleftarrow{S}}, \mathbf{0}_{\sigma_3\overleftarrow{S}}, \dots)$;
- the one with all entries equal to $\mathbf{0}_{\sigma_1 S}$, $(\mathbf{0}_{\sigma_1 S})^\infty = (\dots, \mathbf{0}_{\sigma_1 S}, \mathbf{0}_{\sigma_1 S}, \mathbf{0}_{\sigma_1 S}, \dots)$;
- the one having all entries equal to $\mathbf{0}_{\sigma_4 S}$, $(\mathbf{0}_{\sigma_4 S})^\infty = (\dots, \mathbf{0}_{\sigma_4 S}, \mathbf{0}_{\sigma_4 S}, \mathbf{0}_{\sigma_4 S}, \dots)$.

Proposition 4.2.2. *The subspace $(\mathbb{C}^*)^2_\infty$ is dense in $\mathbb{B}_S^\infty$.*

Proof. This is a direct consequence from Proposition 3.2.6 and from the fact that, in product topology, the product of dense subsets is dense in the product space. □

Theorem 4.2.3. *The points of $\mathbb{B}_S^\infty$ lie in one of the following types*

(a) *A sequence with all entries in $(\mathbb{C}^*)^2_{\sigma_i A}$, that is, a point in $(\mathbb{C}^*)^2_\infty$;*

(b) *One of the four sequences $(\mathbf{0}_{\sigma_2\overleftarrow{S}})^\infty, (\mathbf{0}_{\sigma_3\overleftarrow{S}})^\infty, (\mathbf{0}_{\sigma_1 S})^\infty, (\mathbf{0}_{\sigma_4 S})^\infty$;*

(c) *The sequences given by $(\dots, (F_S^\sharp)^{-2}(\mathbf{p}), (F_S^\sharp)^{-1}(\mathbf{p}), \mathbf{p}, F_S^\sharp(\mathbf{p}), (F_S^\sharp)^2(\mathbf{p}), \dots)$, where*

$$\mathbf{p} \in \left(\mathsf{L}_{\binom{1}{0}} \cup \mathsf{L}_{\binom{0}{1}} \cup \mathsf{L}_{\binom{-1}{0}} \cup \mathsf{L}_{\binom{0}{-1}}\right) \setminus \{\mathbf{0}_{\sigma_1 S}, \mathbf{0}_{\sigma_2\overleftarrow{S}}, \mathbf{0}_{\sigma_3\overleftarrow{S}}, \mathbf{0}_{\sigma_4 S}\}$$

Proof. For item (a), notice that a point $(x, y)_{\sigma_i A}$ belongs to $(\mathbb{C}^*)^2_{\sigma_i A}$ if and only if $xy \neq 0$. In this case, since the local expressions of $F_S^\sharp$ and $(F_S^\sharp)^{-1}$ are all ratios of monomials, it follows that their iterates are also in $(\mathbb{C}^*)^2_{\sigma_i A}$. Item ($b$) is immediate from Theorem 3.2.2.

The item (c) is consequence of Lemma 3.2.4 and from the fact that a hyperbolic matrix in $SL(2, \mathbb{Z})$ sends integer segments to integer segments.

□

The statement below, clarifies why F_M^∞ is an extension of F_M.

Proposition 4.2.4. *The space $(\mathbb{B}_S^*)^\infty = \mathbb{B}_S^\infty \setminus \{(\mathbf{0}_{\sigma_2\overleftarrow{S}})^\infty, (\mathbf{0}_{\sigma_3\overleftarrow{S}})^\infty, (\mathbf{0}_{\sigma_1 S})^\infty, (\mathbf{0}_{\sigma_4 S})^\infty\}$ is an algebraic manifold. More precisely, the projection pr_0 onto the 0-th coordinate induces an isomorphism between $\mathbb{B}_S \setminus \{\mathbf{0}_{\sigma_1 S}, \mathbf{0}_{\sigma_2\overleftarrow{S}}, \mathbf{0}_{\sigma_3\overleftarrow{S}}, \mathbf{0}_{\sigma_4 S}\}$ and $(\mathbb{B}_S^\infty)^* = \mathbb{B}_S^\infty \setminus \{(\mathbf{0}_{\sigma_2\overleftarrow{S}})^\infty, (\mathbf{0}_{\sigma_3\overleftarrow{S}})^\infty, (\mathbf{0}_{\sigma_1 S})^\infty, (\mathbf{0}_{\sigma_4 S})^\infty\}$.*

Proof. Given a point $\mathbf{p} \in \mathbb{B}_S^\infty \setminus \{\mathbf{0}_{\sigma_1 S}, \mathbf{0}_{\sigma_2\overleftarrow{S}}, \mathbf{0}_{\sigma_3\overleftarrow{S}}, \mathbf{0}_{\sigma_4 S}\}$, there is a unique $\mathbf{p}_\infty \in (\mathbb{B}_S^\infty)^* = \mathbb{B}_S^\infty \setminus \{(\mathbf{0}_{\sigma_2\overleftarrow{S}})^\infty, (\mathbf{0}_{\sigma_3\overleftarrow{S}})^\infty, (\mathbf{0}_{\sigma_1 S})^\infty, (\mathbf{0}_{\sigma_4 S})^\infty\}$ having $\mathbf{p}$ on the 0-th coordinate position, so the morphism $\mathbf{p} \mapsto (\dots, \mathbf{p}_{-1}, \mathbf{p}, \mathbf{p}_1, \dots)$ is the inverse of pr_0. This concludes the proof. □

Given that $F_S^\sharp = f_S$ in $(\mathbb{C}^*)^2_{\sigma_i A}$, the isomorphism p_0 conjugates F_S^∞ to the monomial map f_S on this subset. That is

$$pr_0 \circ F_S^\infty(\mathbf{p}_\infty) = f_S \circ pr_0(\mathbf{p}_\infty),$$

for any $\mathbf{p}_\infty \in (\mathbb{C}^*)^2_\infty$. Then, using the identification between $(\mathbb{C}^*)^2$ and $(\mathbb{C}^*)^2_\infty$, this shows that F_S^∞ does extend f_S continuously and algebraically in $(\mathbb{B}_S^\infty)^*$.

Chapter 5

The sequence space as a chain of blow-ups

As we saw before, we defined $F_S^\sharp$ on an attempt to make $\widetilde{F}_S$ a well defined dynamical system. The problem is, the map obtained and its inverse are not defined at $\mathbf{0}_{\sigma_2 \overleftarrow{S}}, \mathbf{0}_{\sigma_3 \overleftarrow{S}}, \mathbf{0}_{\sigma_1 S}, \mathbf{0}_{\sigma_4 S} \in \mathbb{B}_S$. Then, the natural next step is to blow-up $\mathbb{B}_S$ until the map defined by the composite of $F_S^\sharp$ with the new blow-ups, which we will call $\widetilde{F}_S^1$, becomes a well defined extension. But, again the range and the domain of this map are not the same, so we should consider the rational map given by the composite of the inverse of the new blow-ups with $\widetilde{F}_S^1$, namely ${F_S^\sharp}^1$. Once more, this map will not be defined at $(\widetilde{F}_S^1)^{-1}(\{\mathbf{0}_{\sigma_3 \overleftarrow{S}}, \mathbf{0}_{\sigma_1 S}, \mathbf{0}_{\sigma_4 S}\})$, hence we should make the blow-ups on the domain of $\widetilde{F}^1$, starting at this inverse image, to get a well defined map. This process will continue through all inverse images of $\{\mathbf{0}_{\sigma_3 \overleftarrow{S}}, \mathbf{0}_{\sigma_1 S}, \mathbf{0}_{\sigma_4 S}\}$, and the strategy to rise a well defined dynamical system will be to take the projective limit of these blow-ups and consider an extension of the monomial map acting in there.

Our next task is to show how we can obtain $(\mathbb{B}_S^\infty)^*$ as an inductive limit on this chain of blow-ups.

5.0.1 An infinite chain of blow-ups

We start showing that $\bar{\Gamma}_{F_S^\sharp}$ is the blow-up of $\mathbb{B}_S$ at $\{\mathbf{0}_{\sigma_2 \overleftarrow{S}}, \mathbf{0}_{\sigma_3 \overleftarrow{S}}, \mathbf{0}_{\sigma_1 S}, \mathbf{0}_{\sigma_4 S}\}$ followed by a blow-up of the resulting space at a point of the exceptional divisor. Before that we remember that the total SB-blow-up $\boldsymbol{\pi}_S : \mathbb{B}_S \longrightarrow \mathbb{B}_S$ is constituted by four chains of regular blow-ups. In Lemma 5.0.1 we will need to make reference to each one of them, so we label them as follows

$$\mathbb{B}_{\sigma_i \delta_0 \delta_1 \ldots \delta_k} \xrightarrow{\pi_k} \mathbb{B}_{\sigma_i \delta_0 \delta_1 \ldots \delta_{k-1}} \xrightarrow{\pi_{k-1}} \quad \ldots \quad \xrightarrow{\pi_3} \mathbb{B}_{\sigma_i \delta_0 \delta_1} \xrightarrow{\pi_2} \mathbb{B}_{\sigma_i \delta_0} \xrightarrow{\pi_1} \mathbb{B}_{\sigma_i} \xrightarrow{\pi_0} \mathbb{C}^2_{\sigma_i}.$$

Lemma 5.0.1. *Let* $S = \Gamma^{\delta_0 \delta_n}_{a_0,\ldots,a_n}$ *be SB-matrix and set* $M = \Gamma^{\delta_0 \delta_n}_{a_0,\ldots,a_n - 1}$ *and* $\overleftarrow{M} = \Gamma^{\delta_n \delta_n}_{a_n,\ldots,a_0 - 1}$*. For* *each $j = 1, \dots k$, let $F^j = \pi_j^{-1} \circ F^{j-1}$ and define the maps*

$$y_j : \begin{array}{ccc} \bar{\Gamma}_{F^j} & \longrightarrow & \mathbb{B}_{\sigma_i \delta_0 \dots \delta_{j-1}} \\ (\mathbf{p},\mathbf{q}) & \longmapsto & \mathbf{q} \end{array}, \qquad x_j : \begin{array}{ccc} \bar{\Gamma}_{F^j} & \longrightarrow & \Gamma_{F^{j-1}} \\ (\mathbf{p},\mathbf{q}) & \longmapsto & (\mathbf{p}, \pi_j(\mathbf{q})) \end{array}, \qquad x : \begin{array}{ccc} \bar{\Gamma}_{F^2} & \longrightarrow & \mathbb{B}_{\sigma_i \delta_0} \\ (\mathbf{p},\mathbf{q}) & \longmapsto & \mathbf{p} \end{array}.$$

Then, each one of the squares on the diagram below is a fibered product in the category of analytic spaces.

$$\begin{tikzcd}
\bar{\Gamma}_{F^\sharp} \arrow[r, "y_n"] \arrow[d, "x_k"'] & \mathbb{B}_{\sigma_1 M} \arrow[d, "\pi_{a_0+\dots+a_n+1}"] \\
\bar{\Gamma}_{F^{k-1}} \arrow[r, "y_{k-1}"] \arrow[d, dotted] & \mathbb{B}_{\sigma_1 M \delta_0^{-1}} \arrow[d, dotted] \\
\bar{\Gamma}_{F^3} \arrow[r, "y_3"] \arrow[d, "x_3"'] & \mathbb{B}_{\sigma_1 \delta_0 \delta_1 \delta_2} \arrow[d, "\pi_2"] \\
\bar{\Gamma}_{F^2} \arrow[r, "y_2"] \arrow[d, "x"'] & \mathbb{B}_{\sigma_1 \delta_0 \delta_1} \arrow[d, "\pi_1"] \\
\mathbb{B}_{\sigma_1 M} \arrow[r, "F^1"] & \mathbb{B}_{\sigma_1 \delta_0}
\end{tikzcd}$$

Proof. We will prove the case $\sigma_1 M$, the others are similar. First lets show that the bottom square is a fibered product. Evidently, the diagram commutes and, since all the spaces involved are manifolds, it is enough to prove that the diagram is a fibered product in the category of analytic manifolds, that is, set-theoretically. Now, in the category of sets, the fibered product exists and is given by

$$\mathbb{B}_{\sigma_1 M} * \mathbb{B}_{\sigma_i \delta_0 \delta_1} = \{(\mathbf{p}, \mathbf{q}) \in \mathbb{B}_{\sigma_1 M} \times \mathbb{B}_{\sigma_1 \delta_0 \delta_1} \ : \ \pi_1(\mathbf{q}) = F^1(\mathbf{p})\}.$$

But, if $(\mathbf{p}, \mathbf{q}) \in \mathbb{B}_{\sigma_1 M} * \mathbb{B}_{\sigma_1 \delta_0 \delta_1}$ and $\mathbf{p} \notin \{\mathbf{0}_{\sigma_2 M}, \mathbf{0}_{\sigma_3 M}\}$, then $\mathbf{q} \in \mathsf{L}_{(\!(\delta_0)\!)}$ and, using the expression in charts for F^2, is easy to check that $\{(\mathbf{0}_{\sigma_2 M}, \mathbf{q})(\mathbf{0}_{\sigma_3 M}, \mathbf{q})\} \in \bar{\Gamma}_{F^2}$.

Now, $\mathbb{B}_{\sigma_1 M} * \mathbb{B}_{\sigma_i \delta_0 \delta_1}$ is Zariski closed, because it is the inverse image of the diagonal $\Delta_{\mathbb{B}_{\sigma_i \delta_0 \delta_1}} \subset \mathbb{B}_{\sigma_i \delta_0 \delta_1} \times \mathbb{B}_{\sigma_i \delta_0 \delta_1}$ under the regular map $(\mathbf{p}_1, \mathbf{p}_2) \mapsto (\pi_2(\mathbf{p}_1), F'(\mathbf{p}_2))$, therefore $\bar{\Gamma}_{F^2} \subset \mathbb{B}_{\sigma_1 M} * \mathbb{B}_{\sigma_i \delta_0 \delta_1}$. Then we have $\mathbb{B}_{\sigma_1 M} * \mathbb{B}_{\sigma_i \delta_0 \delta_1} = \bar{\Gamma}_{F^2}$.

To prove that the upper square is a fibered product, we use the following fact: the pullback of $pr_2 : \bar{\Gamma}_{F^2} \to \mathbb{B}_{\sigma_i \delta_0 \delta_1}$ and $\pi_3 : \mathbb{B}_{\sigma_1 M} \to \mathbb{B}_{\sigma_i \delta_0 \delta_1}$ exit and it is given by $(\bar{\Gamma}_{F^2} * \mathbb{B}_{\sigma_1 M}, p_1, p_2)$, where

$$\bar{\Gamma}_{F^2} * \mathbb{B}_{\sigma_1 M} = \{((\mathbf{p}, \mathbf{q}), \mathbf{r}) \in \bar{\Gamma}_{F^2} \times \mathbb{B}_{\sigma_1 M} \ : \ \mathbf{q} = \pi_3(\mathbf{r})\},$$

and $p_1 : \bar{\Gamma}_{F^2} \times \mathbb{B}_{\sigma_1 M} \to \bar{\Gamma}_{F^2}$, $p_2 : \bar{\Gamma}_{F^2} \times \mathbb{B}_{\sigma_1 M} \to \mathbb{B}_{\sigma_i \delta_0 \delta_1}$ are the projections on the first and second factor respectively. We also notice that the map

$$\varphi : \begin{array}{ccc} \bar{\Gamma}_{F^2} * \mathbb{B}_{\sigma_1 M} & \longrightarrow & \bar{\Gamma}_{F_S^\sharp} \\ ((\mathbf{p}, \mathbf{q}), \mathbf{r}) & \longmapsto & (\mathbf{p}, \mathbf{r}) \end{array}$$

is an isomorphism, with inverse given by $(\mathbf{p}, \mathbf{r}) \mapsto ((\mathbf{p}, \pi_2(\mathbf{r})), \mathbf{r})$. The only thing that is not obvious about this statement is that $\varphi(\bar{\Gamma}_{F^2} * \mathbb{B}_{\sigma_1 M}) \subset \bar{\Gamma}_{F_S^\sharp}$, so lets prove it.

Hence π_2 is invertible in a neighborhood of $\mathbf{q}$ and, since $\pi_2(\mathbf{r}) = \mathbf{q} = F^2(\mathbf{p})$, follows that

$$\mathbf{r} = \pi_2^{-1} \circ F^2(\mathbf{p}) = F_S^\sharp(\mathbf{p}).$$

If $(\mathbf{0}_{\sigma_2 M}, \mathbf{q}) \in \bar{\Gamma}_{F^2}$, the expression in local coordinates of F^2 guaranties that $\mathbf{q} \in \mathsf{L}_{\mathbf{q}_2}$, that is, $\mathbf{r} \in \pi_2^{-1}(\mathsf{L}_{(\!(\sigma_i\delta_0\delta_1)\!)}) = \mathsf{L}_{\mathbf{0}_{\sigma_2 M}} \cup \mathsf{L}_{(\!(\sigma_i\delta_0\delta_1)\!)}$. Then, Theorem 4.2.3 implies that $(\mathbf{0}_{\sigma_2 M}, \mathbf{r}) \in \bar{\Gamma}_{F_S^\sharp}$.

$$\begin{tikzcd}
\Gamma \arrow[dr, dashed] \arrow[drrr, bend left, "g"] \arrow[ddrr, bend right, "f"'] & & & \\
& \bar{\Gamma}_{F^2} * \mathbb{B}_{\sigma_1\delta_0\delta_1\delta_2} \arrow[dr, Rightarrow, "\varphi"] \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& & \bar{\Gamma}_{F^\sharp} \arrow[r, "y_3"] \arrow[d, "x_3"] & \mathbb{B}_{\sigma_1\delta_0\delta_1\delta_2} \arrow[d, "\pi_2"] \\
& & \bar{\Gamma}_{F^2} \arrow[r, "y_2"] & \mathbb{B}_{\sigma_1\delta_0\delta_1}
\end{tikzcd}$$

The isomorphism φ and the universal property of the fibered product $(\bar{\Gamma}_{F^2} * \mathbb{B}_{\sigma_1\delta_0\delta_1\delta_2}, p_1, p_2)$ are used to build the diagram below, and its commutativity ensures that the upper square is also a fibered product.

□

We will also use another preliminary Lemma, which statement and proof can be found at Hauser, 2014, Lecture V, Proposition 5.1.

Lemma 5.0.2. *Let $\pi : \widetilde{X} \to X$ be the blow-up of Y along a subvariety Z, and let $\varphi : Y \to X$ be a morphism. Denote by $(Y * \widetilde{X}, pr_1, pr_2)$ the fibered product of φ and π, where $pr_1 : Y * \widetilde{X} \to Y$ and $pr_2 : Y * \widetilde{X} \to \widetilde{X}$ are the the projections onto the first and second factor respectively. Let $S = \varphi^{-1}(Z) \subset Y$ be the inverse image of Z under φ, and let $\widetilde{Y}$ be the Zariski closure of $pr_1^{-1}(Y \backslash S)$ in $Y * \widetilde{X}$. The restriction $\tau : \widetilde{Y} \to Y$ of pr_1 to $\widetilde{Y}$ equals the blow-up of Y along S.*

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, hook] \arrow[dr, "\tau"'] & Y * \widetilde{X} \arrow[r, "p_2"] \arrow[d, "p_1"] & \widetilde{X} \arrow[d, "\pi"] \\
& Y \arrow[r, "\varphi"] & X
\end{tikzcd}$$

Now we are able to show that $\bar{\Gamma}_{F^\sharp}$ is obtained from $\mathbb{B}_{\sigma_1\delta_0\delta_1}$ after two consecutive blow-ups.

Proposition 5.0.3. *$\bar{\Gamma}_{F^2}$ is the blow-up of $\mathbb{B}_{\sigma_1\delta_0\delta_1}$ at the point $\mathbf{0}_{\sigma_1}$ and $\bar{\Gamma}_{F_2^\sharp}$ is the blow-up of $\bar{\Gamma}_{F^2}$ at the point $(\mathbf{0}_{\sigma_1}, \mathbf{0}_{\sigma_1\delta_1})$.*

Proof. Using the fibered product from the bottom square at Lemma 5.0.1 and taking $X = \mathbb{B}_{\sigma_1\delta_0}$, $Y = \mathbb{B}_{\sigma_1\delta_0\delta_1}$, $\pi \equiv \pi_1$ and $\varphi \equiv F^1$ at Lemma 5.0.2, follows that $\overline{\mathcal{A}}_1$, where $\mathcal{A}_1 = (\rho_1^1)^{-1}(\mathbb{B}_{\sigma_1\delta_0\delta_1} \setminus \{\mathbf{0}_{\sigma_1}\})$, is the blow-up of $\mathbb{B}_{\sigma_1\delta_0\delta_1}$ at the point $\mathbf{0}_{\sigma_1} = (F^1)^{-1}(\mathbf{0}_{\sigma_1})$.

Now, we notice that $\overline{\mathcal{A}}_1 = \bar{\Gamma}_{F^2}$: in fact, since $\widetilde{F_S}^{-1}(\mathbf{0}_{\sigma_1})$ is a closed subset of $\mathbb{B}_{\sigma_1\delta_0\delta_1}$, $\mathcal{A}_1$ is a open subset of $\bar{\Gamma}_{F^2}$. Now, evidently $\mathcal{A}_1 \subset \mathbb{B}_{\sigma_1\delta_0\delta_1} * \mathbb{B}_{\sigma_1\delta_0} = \bar{\Gamma}_{F^2}$, hence as every open set inside a variety is a dense subset, $\overline{\mathcal{A}}_1 = \bar{\Gamma}_{F^2}$.

Using the same notation as in Lemma 5.0.1, notice that $pr_2^{-1}(\mathbf{0}_{\sigma_1\delta_0}) = (\mathbf{0}_{\sigma_1}, \mathbf{0}_{\sigma_1\delta_0})$ and therefore $\mathcal{A}_2 := pr_1^{-1}(\bar{\Gamma}_{F^2} \setminus \{(\mathbf{0}_{\sigma_1}, \mathbf{0}_{\sigma_1\delta_0})\})$ is an open subset of $\bar{\Gamma}_{F^2} * \mathbb{B}_{\sigma_1\delta_0\delta_1}$, then $\overline{\mathcal{A}}_2 = \bar{\Gamma}_{F^2} * \mathbb{B}_3$ and Lemma 5.0.2 implies that $\bar{\Gamma}_{F^2} * \mathbb{B}_{\sigma_1\delta_0\delta_1}$ is the blow-up of $\bar{\Gamma}_{F^2}$ at $(\mathbf{0}_{\sigma_1\delta_0}, \mathbf{0}_{\sigma_1\delta_0\delta_1})$.

Finally, by the universal property of the blow-up $pr_1 : \bar{\Gamma}_{F^2} * \mathbb{B}_{\sigma_1\delta_0\delta_1} \to \bar{\Gamma}_{F^2}$, given a morphism $f : \Gamma \to \bar{\Gamma}_{F^2}$ such that $f^{-1}(\mathbf{0}_{\sigma_1\delta_0}, \mathbf{0}_{\sigma_1\delta_0\delta_1})$ is a divisor in Γ, there exists a unique morphism $\tilde{f}$ such that $pr_1 \circ \tilde{f} = f$. Then using the isomorphism φ to build the commutative diagram below, we conclude that ρ_1^2 also satisfies the same universal property.

$$
\begin{tikzcd}
 & \bar{\Gamma}_{F^\sharp} \arrow[d, "\varphi^{-1}"] \arrow[dd, bend left=60, "x"] \\
 & \bar{\Gamma}_{F^\sharp} * \mathbb{B}_{\sigma_i\delta_0\delta_1} \arrow[d, "p_1"] \\
\Gamma \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"'] \arrow[uur, dashed] & \bar{\Gamma}_{F^2}
\end{tikzcd}
$$

Therefore, the second part of the statement is true as well.

∎

With the last two results we can conclude that $\bar{\Gamma}$ is obtained from $\mathbb{B}_S$ from $\boldsymbol{\pi}_S$

Before we can state the generalization on how the next blow-ups and extensions of the monomial map will arise, we need some definitions. The spaces and rational maps below are the building blocks of the chain of blow-ups.

Definition 5.0.4. Let $N, M \in \mathbb{Z}$. For $N < M$, we define the set:

$$\mathbb{B}_{[N,M]} = \left\{ (\mathbf{p}_N, \mathbf{p}_{N+1}, \dots, \mathbf{p}_M) \in \prod_{i=N}^{M} \mathbb{B}_S \ : \ (\mathbf{p}_i, \mathbf{p}_{i+1}) \in \bar{\Gamma}_{F^\sharp}, \text{ for all } N \le i < M \right\},$$

and its subsets

$$\mathbb{D}_{[N,M]} = \left\{ (\mathbf{p}_N, \mathbf{p}_{N+1}, \dots, \mathbf{p}_M) \in \mathbb{B}_{[N,M]} \ : \ \mathbf{p}_i \in \mathbb{D}_S \text{ for all } N \le i \le M \right\},$$

If $M = N$, we set $\mathbb{B}_{[N,N]} = \mathbb{B}_S$, and $\mathbb{D}_{[N,N]} = \mathbb{D}_S$.

Notice that the elements of $\mathbb{B}_{[N,M]}$ and $\mathbb{D}_{[N,M]}$ are finite sequences with $(M-N)+1$ coordinates.

Notation. A point $(\mathbf{p}_N, \mathbf{p}_{N+1}, \dots, \mathbf{p}_M)$ in $\mathbb{B}_{[N,M]}$ will be denoted by $\mathbf{p}_{[N,M]}$, in particular if we want to make clear that $(\mathbf{p}_N, \dots, \mathbf{p}_i, \dots, \mathbf{p}_M)$ is a sequence with i-th entry equals to some $\boldsymbol{\alpha} \in \mathbb{B}_S$, that is $\mathbf{p}_i = \boldsymbol{\alpha}$, we write $(\boldsymbol{\alpha})^i_{[N,M]}$.

Proposition 5.0.5. *The space $\mathbb{B}_{[N,M]}$, is a smooth algebraic surface, and $\mathbb{D}_{[N,M]}$ is a divisor in $\mathbb{B}_{[N,M]}$.*

Proof. In fact the collection of open subsets

$$\mathbb{C}^2_{(\sigma_1 A)^N_{[N,M]}} = \left\{ (\mathbf{p}_N, \mathbf{p}_{N+1}, \dots, \mathbf{p}_{M-1}, \mathbf{p}_M) \,:\, \mathbf{p}_N \in \mathbb{C}^2_{\sigma_1 A} \right\}$$

gives a covering for $\mathbb{B}_{[N,N]}$.

Notice that, by Theorem 4.2.3, the projection on the first factor, $pr_N \,:\, (\mathbf{p}_N, \dots, \mathbf{p}_M) \mapsto \mathbf{p}_N$ is an isomorphism between $\mathbb{C}^2_{(\sigma_i A)^N_{[N,M]}}$ and $\mathbb{C}^2_{\sigma_i A}$.

Then, since the family of open sets given above cover $\mathbb{B}_{[N,M]}$, we can conclude that the collection of these open sets together with their respective projections are system of affine charts for $\mathbb{B}_{[N,M]}$, on which the subscript of each open set indicates the center of the chart.

The fact that $\mathbb{D}_{[N,M]}$ is a divisor in $\mathbb{B}_{[N,M]}$ is a direct consequence of the fact that $\mathbb{D}_S$ is a divisor in $\mathbb{B}_S$ and from the characterization of the system of affine charts sets given above. $\square$

Remark 5.0.6. A consequence of the proof of Proposition 5.0.5 is that, given $\mathbf{p}_{[N,M]} \in \mathbb{B}_{[N,M]}$ there is an unique smaller integer i, $N \le i \le M$, and $\boldsymbol{\alpha} \in \mathbb{B}_S$ such that

$$\mathbf{p}_{[N,M]} = \left((F^\sharp)^{-(i-N)}(\boldsymbol{\alpha}), \dots, (F^\sharp)^{-1}(\boldsymbol{\alpha}), \boldsymbol{\alpha}, F^\sharp(\boldsymbol{\alpha}), \dots, (F^\sharp)^{(M-i)}(\boldsymbol{\alpha}) \right); \tag{1}$$

and, since $F^\sharp$ is birational, this representation is univocal in $\mathbb{B}_{[N,M]}$. Hence, for now on, we are going to use the notation $(\boldsymbol{\alpha})^i_{[N,M]}$ to refer to the unique $\mathbf{p}_{[N,M]}$ in $\mathbb{B}_{[N,M]}$ given by (1).

Theorem 5.0.7. *Let $N, M \in \mathbb{Z}$ with $N+1 < M$, and let $\widetilde{F}_{[N,M]}$, $p_{[N,M]}$ be the projections*

$$\begin{array}{rccc} \widetilde{F}_{[N,M]}: & \mathbb{B}_{[N,M]} & \longrightarrow & \mathbb{B}_{[N+1,M]} \\ & (\mathbf{p}_N, \dots, \mathbf{p}_M) & \longmapsto & (\mathbf{p}_{N+1}, \dots, \mathbf{p}_M) \end{array}, \qquad \begin{array}{rccc} pr_{[N,M]}: & \mathbb{B}''_{[N,M]} & \longrightarrow & \mathbb{B}''_{[N+1,M]} \\ & (\mathbf{p}_N, \dots, \mathbf{p}_M) & \longmapsto & (\mathbf{p}_{N+1}, \dots, \mathbf{p}_M) \end{array}.$$

Then, the diagram below is commutative and each of its columns is a chain of blow-ups, such that blow-ups on a same row are canonically isomorphic.

$$\begin{tikzcd}
\mathbb{B}_{[N,M]} \arrow[r, "\widetilde{F}_{[N,M]}"] \arrow[d, "\rho^2_{[N,M]}"'] & \mathbb{B}_{[N+1,M]} \arrow[d, "\rho^2_{[N+1,M]}"] \\
\mathbb{B}''_{[N,M]} \arrow[r, "pr_{[N,M]}"] \arrow[d, "\rho^1_{[N,M]}"'] & \mathbb{B}''_{[N+1,M]} \arrow[d, "\rho^1_{[N+1,M]}"] \\
\mathbb{B}_{[N,M-1]} \arrow[r, "\widetilde{F}_{[N,M-1]}"] & \mathbb{B}_{[N+1,M-1]}
\end{tikzcd}$$

Proof. First, notice that we may assume that $N = 0$, since the shift on the indices gives an isomorphism $\mathbb{B}_{[N,M]} \to \mathbb{B}_{[0,M-N]}$. Then the rest of the proof follows by induction over M, following the same strategies then on Lemma 5.0.2 5.0.1. $\square$

Finally, we are able to make the connection with the sequence space $\mathbb{B}_\infty$.

Proposition 5.0.8. *The subspace* $\mathbb{B}^*_\infty = \mathbb{B}_\infty \setminus \{(\mathbf{0})_\infty, (\mathbf{a})_\infty, (\mathbf{d}_1)_\infty, (\mathbf{b})_\infty\}$ *is given by the projective limit*

$$\mathbb{B}^*_\infty = \varprojlim_{N \in \mathbb{Z}_{\geq 0}} \mathbb{B}^*_{[-N,N]},$$

where $\mathbb{B}^*_{[-N,N]} = \mathbb{B}_{[-N,N]} \setminus \left\{ (\mathbf{0})^{-N}_{[-N,N]}, (\mathbf{b})^{-N}_{[-N,N]}, (\mathbf{a})^{N}_{[-N,N]}, (\mathbf{d}_1)^{N}_{[-N,N]} \right\}.$

Proof. In fact, the inverse system is given by the pair $\left((\mathbb{B}_{[-i,i]})_{N\in\mathbb{Z}}, (\boldsymbol{\pi}_{ij})_{i\leq j\in\mathbb{Z}}\right)$, where $\boldsymbol{\pi}_{ij} : \mathbb{B}_{[-i,i]} \longrightarrow \mathbb{B}_{[-j,j]}$ acts as the blow-up. $\square$

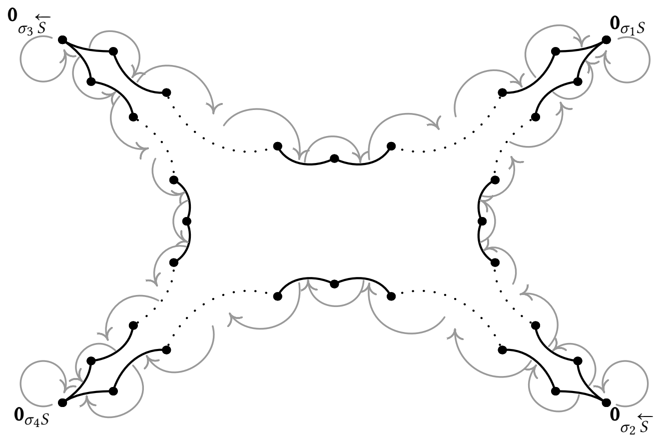


Figure 5.1: *Action of F^∞ on the divisor. It behave as a shift on the space $\mathbb{B}^\infty_S$*

For the sake of our notation, we denote the equivalence class of $\mathbf{p}_{[-N,N]}$ in $\mathbb{B}^*_\infty$ by $\mathbf{p}_\infty$. Then, we define the N^{th} **canonical function** by

$$\begin{aligned} \phi_N \ : \ \mathbb{B}^*_{[-N,N]} &\longrightarrow \mathbb{B}^*_\infty \\ \mathbf{p}_{[-N,N]} &\longmapsto \mathbf{p}_\infty \end{aligned},$$

which is an isomorphism between the subset $(\mathbb{C}^*)^2_{[-N,N]}$, of points in $\mathbb{B}_{[-N,N]}$ having all coordinates in $(\mathbb{C}^*)^2$, and $(\mathbb{C}^*)^2_\infty$. This isomorphism conjugates F_∞ to the map $F^\sharp_{[-N,N]}$, that is,

$$\phi_N \circ F^\sharp_{[-N,N]}(\mathbf{p}_{[-N,N]}) = F_\infty \circ \phi_N(\mathbf{p}_{[-N,N]}),$$

for any $\mathbf{p}_{[-N,N]} \in (\mathbb{C}^*)^2_{[-N,N]}$. This fact gives another way to show that F_∞ is an extension of the monomial map to the compact space $\mathbb{B}_\infty$.

5.0.2 Anosov map and invariant manifolds

Is easy to see that map F_S maps the torus $T = \{(x,y) \in \mathbb{C}^2 \,:\, |x| = |y| = 1\}$ to itself. In fact, the change of coordinates $x = e^{2\pi iu}$, $y = e^{2\pi iv}$ turns the map $F_S|_T \,:\, T \longrightarrow T$ into the Anosov map $A_S \,:\, \mathbb{R}^2/\mathbb{Z}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$ defined by S.

As proved on Lemma 5.2 of El Abdalaoui *et al.*, 2016, the stable and unstable manifolds of the invariant torus are 3-dimensional real analytic manifolds given by the equations $|y| = |x|^{\mu^+}$ and $|y| = |x|^{\mu^-}$, where μ^+ and μ^- are the slopes of the two eigenlines of S. Join this result with the action of F_S^∞ on the divisor, we summarize the dynamic in $\mathbb{B}_S^\infty$ in Theorem 5.0.9.

Theorem 5.0.9. *The torus T has a stable manifold W_S^+ and an unstable manifold W_S^- in $(\mathbb{C}^*)^2$, respectively of equation*

$$|y| = |x|^{\mu^+} \qquad \textit{and } |y| = |x|^{\mu^-}$$

where μ^+ and μ^- are the slopes of the two eigenlines of S. The manifold W_S^+ and W_S^- are real-analytic 3-real submanifolds of $(\mathbb{C}^*)^2$, foliated by Riemann surfaces with dense leaves since $\mu^\pm$ are irrational. Each cuts $(\mathbb{C}^*)^2$ into two, denoted ${}^+(\mathbb{B}_S^\infty)$ and $(\mathbb{B}_S^\infty)^+$ for the complement of W^+ (${}^-(\mathbb{B}_S^\infty)$ and $(\mathbb{B}_S^\infty)^-$ for the complement of W^-). Further, together they cut $\mathbb{B}_S^\infty$ into four pieces, denoted ${}^\pm(\mathbb{B}_S^\infty)^\pm$ in the obvious way, which we will call the M_S-quadrants. These manifolds $W_S^\pm$ accumulate in $\mathbb{B}_S^\infty$ at the non-analytic points.

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