A functorial approach to Gabriel quiver constructions

Samuel Amador dos Santos Quirino

Thesis presented to the
Institute of Mathematics and Statistics
of the University of São Paulo
in partial fulfillment
of the requirements
for the degree of
Doctor of Science

Program: Matemática
Advisor: Prof. Dr. Kostiantyn Iusenko
Coadvisor: Prof. Dr. John William MacQuarrie

This study was financed in part by the Coordenação de Aperfeiçoamento
de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001

São Paulo
August, 2023

A functorial approach to Gabriel quiver constructions

Samuel Amador dos Santos Quirino

This version of the thesis includes the corrections and modifications suggested by the Examining Committee during the defense of the original version of the work, which took place on August 8, 2023.

A copy of the original version is available at the Institute of Mathematics and Statistics of the University of São Paulo.

Examining Committee:

Prof. Dr. Kostiantyn Iusenko (advisor) – IME-USP
Prof. Dr. Eduardo do Nascimento Marcos – IME-USP
Prof. Dr. Eliezer Batista – UFSC
Prof. Dr. Monique Müller Lopes Rocha – UFSJ
Prof. Dr. Marcelo Muniz Silva Alvez – UFPR

Ficha catalográfica elaborada com dados inseridos pelo(a) autor(a)
Biblioteca Carlos Benjamin de Lyra
Instituto de Matemática e Estatística
Universidade de São Paulo

Quirino, Samuel Amador dos Santos
A functorial approach to Gabriel quiver constructions / Samuel Amador dos Santos Quirino; orientador, Kostiantyn Iusenko; coorientador, John William MacQuarrie. - São Paulo, 2023.
131 p.

Tese (Doutorado) - Programa de Pós-Graduação em Matemática / Instituto de Matemática e Estatística / Universidade de São Paulo.
Bibliografia
Versão corrigida

1. Representation Theory. 2. Rings and Algebras. I. Iusenko, Kostiantyn. II. Título.

Bibliotecárias do Serviço de Informação e Biblioteca
Carlos Benjamin de Lyra do IME-USP, responsáveis pela
estrutura de catalogação da publicação de acordo com a AACR2:
Maria Lúcia Ribeiro CRB-8/2766; Stela do Nascimento Madruga CRB 8/7534.

Agradecimentos

Eu gostaria de agradecer aos meus orientadores, Kostia e John, pela confiança, pelo apoio e pela amizade.

Gostaria de agradecer aos membros da banca, Eduardo, Eliezer, Marcelo e Monique, pelos questionamentos, correções e sugestões.

Gostaria de agradecer aos meus colegas do IME e do ICEX, em particular ao Fernando, ao Lucas e ao Ricardo, por todas as conversas e discussões, matemáticas, filosóficas e políticas.

Gostaria de agradecer à Ana Sofia e à sua família pelo acolhimento, ao Diego pelo sentimento de pertencimento, e ao Carlos por me ajudar a tornar a USP minha segunda casa.

Gostaria de agradecer à minha família por todo o apoio que nunca faltou e à Aline que esteve ao meu lado nessa jornada.

Por fim, gostaria de agradecer à Rose por me ajudar a manter minha cabeça no lugar.

Resumo

Samuel Amador dos Santos Quirino. **Uma abordagem funtorial para as construções da aljava de Gabriel**. Tese (Doutorado). Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2023.

O objetivo deste trabalho é o de estabelecer as construções da aljava de Gabriel de modo funtorial. Por construções da aljava de Gabriel queremos nos referir ao Teorema de Gabriel que estabelece que toda álgebra pontuada de dimensão finita é a álgebra quociente de uma álgebra de caminhos da sua aljava de Gabriel por um ideal admissível. A fim de obtermos tal resultado, consideramos a categoria de coálgebras pontuadas e a categoria de k-aljavas, construímos funtores covariantes entre ambas categorias, que traduzem a coálgebra de caminhos de uma aljava e o quiver de Gabriel de uma coálgebra pontuada, e mostramos que esses funtores induzem um par adjunto quando consideramos a categoria quociente da categoria de coálgebras pontuadas por uma relação de equivalência nos homomorfismos de coálgebras. A unidade da adjunção revela que toda coálgebra pontuada é uma subcoálgebra admissível da coálgebra de caminhos da sua aljava de Gabriel. Por dualidade, obtemos um par de funtores contravariantes entre a categoria de k-aljavas e a categoria quociente da categoria de álgebras pseudocompactas pontuadas por uma relação de equivalência nos homomorfismos de álgebras contínuos, que são adjuntos à esquerda, e concluímos que toda álgebra pseudocompacta pontuada é a álgebra quociente da álgebra de caminhos completa de sua aljava de Gabriel por um ideal admissível. Generalizamos esses resultados para coálgebras básicas com corradical separável e um conceito de k-espécies para coálgebras. Em paralelo, provamos que a álgebra de invariantes de uma álgebra de caminhos completa sob a ação de um grupo homogêneo de automorfismos de álgebras contínuos é uma álgebra de caminhos completa e preserva o tipo de representação finito ou manso da aljava.

Palavras-chave: funtores adjuntos. coálgebras de caminhos. álgebra de caminhos completa. aljava de Gabriel.

Abstract

Samuel Amador dos Santos Quirino. **A functorial approach to Gabriel quiver constructions**. Thesis (Doctorate). Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2023.

The aim of this work is to establish the Gabriel quiver constructions via functors. By Gabriel quiver constructions we mean the Gabriel's theorem which states that every pointed finite dimensional algebra is a quotient of the path algebra of its Gabriel quiver by an admissible ideal. In order to accomplish this, we consider the category of pointed coalgebras and the category of k-quivers, than we construct a pair of covariant functors between both categories, which translates the path coalgebra of a quiver and the Gabriel quiver of a pointed coalgebra, and show that these functors induce an adjoint pair when considering the quotient category of pointed coalgebras by an equivalence relation on coalgebra homomorphisms. The unit of the adjunction shows that every pointed coalgebra is an admissible subcoalgebra of the path coalgebra of its Gabriel quiver. By duality, we obtain a pair of contravariant functors from the category o k-quivers and the quotient category of pointed pseudocompact algebras by an equivalence relation on continuous algebra homomorphisms, which are adjoint on the left, and conclude that every pointed pseudocompact algebra is the quotient of the complete path algebra of its Gabriel quiver by an admissible ideal. We generalize these results for basic coalgebras with separable coradical and the concept of k-species for coalgebras. In parallel, we prove that the algebra of invariants of a complete path algebra under the action of a homogeneous group of continuous algebra automorphisms is a complete path algebra and preserves finite or tame representation type of the quiver.

Keywords: adjoint functors. path coalgebra. complete path algebra. Gabriel quiver.

List of symbols

k	A fixed field
$\mathbb{N}$	Non-negative integers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
id	Identity morphism, wherever it makes sense
$- \otimes -$	Tensor product over k
$- \square_C -$	Cotensor product over the coalgebra C
$- \widehat{\otimes}_A -$	Complete tensor product over the pseudocompact algebra A
$\mathrm{Cot}_C(M)$	Cotensor coalgebra of C and the C-bicomodule M
$\mathrm{T}[[A, U]]$	Complete tensor algebra of A and the psc. A-bimodule U
Cog	Category of coalgebras and coalgebra homomorphisms
$\mathbf{Cog}^{\text{filt}}$	Cat. of cog. with separable coradical and filtered cog. hom.
BCog	Category of basic cog. with separable coradical and cog. hom.
PCog	Category of pointed coalgebras and coalgebra homomorphisms
cog	Category of finite dimensional coalgebras and coalgebra hom.
ALG	Category of algebras and algebra homomorphisms
Alg	Category of pseudocompact algebras and continuous alg. hom.
PAlg	Category of pointed pseudocompact algebras and cont. alg. hom.
alg	Category of finite dimensional algebras and algebra hom.
${}^C\mathcal{M}$	Category of left C-comodules and comodule homomorphisms
$\mathcal{M}^C$	Category of right C-comodules and comodule homomorphisms
${}^C\mathcal{M}^D$	Category of C-D-bicomodules and bicomodule homomorphisms
${}_A\mathcal{M}$	Category of left psc. A-modules and cont. module hom.
$\mathcal{M}_A$	Category of right psc. A-modules and cont. module hom.
${}_A\mathcal{M}_B$	Category of psc. A-B-bimodules and cont. bimodule hom.

Contents

Appendixes

Introduction

Any finite dimensional algebra over an algebraically closed field is Morita equivalent to a quotient of a path algebra by an admissible ideal. Thus, path algebras became a fundamental tool in the study of representation theory of associative algebras.

Every basic algebra over an algebraically closed field is pointed. Furthermore, finite dimensional pointed algebras are isomorphic to a quotient of a path algebra by an admissible ideal and the class of finite dimensional path algebras is precisely the hereditary finite dimensional pointed algebras.

Besides the simplicity for constructing examples (and counter-examples) of path algebras, the finite and tame representation types of finite dimensional path algebras have been classified in a combinatorial way in terms of the underlying graph of its quiver, being the former in correspondence to the simply laced Dynkin diagrams and the latter in correspondence to the Euclidean diagrams.

Moreover, Drozd [Dro80] proved that every finite dimensional algebra over an algebraically closed field that has infinite representation type is either of tame or wild representation type, being the classification of the latter equivalent to classify the representations of any other algebra.

The limiting factor of working with finite dimensional algebras can be surpassed via two dual ways: coalgebras and pseudocompact algebras.

Pseudocompact algebras are topological algebras constructed as the inverse limit of finite dimensional algebras. In this way, they appear naturally as the completed group algebra of profinite algebras, having applications on Galois theory, finite group theory and algebraic geometry.

Coalgebras are defined in the monoidal category of vector spaces by axioms dual to the associative algebra with unit. They compose part of the structure of Hopf algebras, having many applications to group theory and physics. Additionally, the strong finiteness properties of coalgebras make them excellent subjects to generalize the theory from finite dimensional algebras.

The results about finite dimensional algebras mentioned here were dualized for coalgebras. Any coalgebra over an algebraically closed field is Morita-Takeuchi equivalent to a basic coalgebra, which is an admissible pointed subcoalgebra of a path coalgebra. The class of path coalgebras are precisely the hereditary pointed coalgebras.

Basic coalgebras over an algebraically closed field are either of finite, tame, or wild representation type. Path coalgebras of finite representation type are the ones whose underlying graph is a simply laced Dynkin diagram, while the tame ones have either infinite locally Dynkin quivers or its underlying graph is an Euclidean diagram.

In the last decades, the development of category theory allowed the expansion of concepts and results in mathematics and favored the communication between many different areas. With this in mind, one of our objectives was to describe the constructions of path algebra and Gabriel quiver in a functorial way.

When considering finite dimensional pointed algebras as quotient of path algebras, two issues immediately emerge. First, infinite dimensional path algebras appears in the equation as many finite dimensional algebras are quotient of those. Second, the choices involved in the Gabriel quiver of a finite dimensional pointed algebra make the construction impossible to be functorial. In order to solve this, we work with the category of pointed coalgebras and the category of k-quivers, the latter being similar to the category of quivers with the difference that to any two vertices corresponds a vector space instead of a set of arrows.

In Chapter 1, we establish the notations which will be used throughout this text and present general results about category theory, coalgebras and pseudocompact algebras.

We adopt an unusual presentation of the auxiliary results, which are either necessary for other results or important for a better comprehension of the text, writing as “propositions” all the results for which proofs can be found on cited references and writing as “lemmas” the ones we provide a proof because we could not find then on the literature, being either known or unknown. Theorems and corollaries follows as usual, presenting complete proofs for those which we believe no one has done yet.

In Chapter 2 we construct the path coalgebra and the Gabriel quiver as covariant functors between the category of pointed coalgebras and the category of k-quivers. Under an equivalence relation on coalgebra homomorphisms, we obtain the main result of this thesis: the induced path coalgebra functor is a right adjoint for the induced Gabriel k-quiver functor.

Furthermore, we dualize the above result to get a pair of contravariant functors adjoint on the right between the category of k-quivers and a quotient category of pointed pseudocompact algebras. We also show two other covariant functors which form an adjoint pair between the quotient category of pointed pseudocompact algebras and a category of pairs of topologically semisimple pointed pseudocompact algebras and pseudocompact bimodules.

The core content of this chapter is on [IMQ21], which also has an extra section about uniqueness of presentations for (co)algebras in terms of path (co)algebras.

In Chapter 3 we investigate other generalizations of path algebras. First, we take a look into k-species.

In analogy to the relationship of a finite dimensional basic algebra over an algebraically closed field and the path algebra of a quiver (or, equivalently, the tensor algebra of a k-quiver), finite dimensional basic algebras over a perfect field are isomorphic to a quotient of the tensor algebra of a k-species by an admissible ideal. k-species can be thought of as a generalization of k-quivers, having finite dimensional division algebras in place of the vertices and bimodules between two “vertices”.

We generalize the adjunction for pointed coalgebras and k-quivers to an adjunction between a quotient category of coalgebras with separable coradical and filtered coalgebra homomorphisms and a category of pairs consisting of separable coalgebras and bicomodules. Furthermore, this result restricts to an adjunction between the corresponding categories of basic coalgebras, in which case the category of pairs is isomorphic to the category of separable k-cospecies, the analog of k-species in perspective of coalgebras.

In the last section of this chapter we analyze a family of functors which converges to what is known as Peirce decomposition. Making use of the strategy adopted by Radford [Rad82], we consider a category of pairs given by coalgebras with separable coradical and coalgebra projections which are splittings of the canonical inclusion of the coradical into the corresponding coalgebra. Morphisms in this category are filtered coalgebra homomorphisms which are compatible with the chosen projections.

We prove that, under certain equivalence relations on morphisms, the functor $\widetilde{F}_n(C,s) = \left(C_0, \dfrac{C_n}{C_0}\right)$ is a left adjoint for the functor $\widetilde{G}_n(\Sigma, V) = \left(\mathrm{Cot}_\Sigma(V), \pi_0\right)$. The limit functor $\widetilde{F}_\infty(C,s) = \left(C_0, \dfrac{C}{C_0}\right)$ is what we call Peirce decomposition, which corresponds for pseudocompact algebras to $(-)^*\widetilde{F}_\infty(-)^*(A) = \left(\dfrac{A}{J(A)}, J(A)\right)$. As a consequence, this result generalize the adjunction of Radford [Rad82].

In the last chapter we put aside adjunctions and work with the algebra of invariants of a complete path algebra by the action of a homogeneous finite group of continuous algebra automorphisms, $\mathrm{T}[[\Sigma, V]]^G$. First we show that the algebra of invariants of a power series ring, $\mathrm{T}[[k, V]]^G$, is again a power series ring. Then, applying the techniques developed by Cibils and Marcos [CM16], we obtain that $\mathrm{T}[[\Sigma, V]]^G$ is a complete path algebra and conclude that, in this case, the algebra of invariants preserves the finite and tame representation types.

Chapter 1

Preliminaries

The main objective of this chapter is to establish definitions and notations that will be used throughout the text. We also recall results (providing a proof for those not readily found on the literature), which will be summoned when necessary.

In the first section we describe categories, morphisms, functors, natural transformations and adjoint functors. In the second section we introduce coalgebras, comodules, homomorphisms, filtrations and cotensor coalgebras. In the third section we explore the structure of pointed coalgebras. In the forth and last section of this chapter we work with pseudocompact algebras and pseudocompact modules, which are the objects of categories dual to those of coalgebras and comodules, respectively.

1.1 Some category theory

We begin by defining categories, types of morphisms and the structures shared by abelian categories, then we move on to functors and natural transformations, ending the section with two equivalent statements of an adjunction between categories. This section is mainly based on [Mac98] and the appendix of [ASS06].

1.1.1 Categories

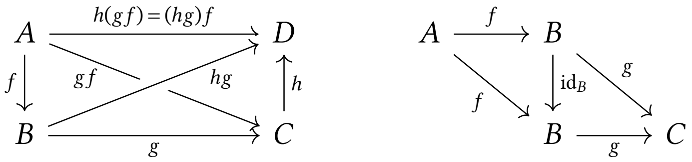
Definition 1.1.1. A *category* $\mathbf{C}$ consists of a class of *objects* and for each pair of objects $A, B \in \mathbf{C}$ a set of *morphisms* from A to B, denoted by $\mathrm{Hom}_{\mathbf{C}}(A, B)$, satisfying the following:

1. for each object $A \in \mathbf{C}$ there exists a morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathbf{C}}(A, A)$ called the *identity morphism* on A;

2. for each triple of objects $A, B, C \in \mathbf{C}$, and each pair of morphisms $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathbf{C}}(B, C)$, there exists a morphism $gf \in \mathrm{Hom}_{\mathbf{C}}(A, C)$ called the *composition* of f and g;

3. for every object $A, B, C, D \in \mathbf{C}$, and every morphism $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$, $g \in$

$\mathrm{Hom}_{\mathbf{C}}(B,C)$ and $h \in \mathrm{Hom}_{\mathbf{C}}(C,D)$, the diagrams:



commute.

A *subcategory* $\mathbf{D}$ of $\mathbf{C}$ is a category consisting of a subclass of objects of $\mathbf{C}$ and, for each pair of objects $A, B \in \mathbf{D}$, a subset $\mathrm{Hom}_{\mathbf{D}}(A,B) \subseteq \mathrm{Hom}_{\mathbf{C}}(A,B)$. The subcategory $\mathbf{D}$ is *full* if $\mathrm{Hom}_{\mathbf{D}}(A,B) = \mathrm{Hom}_{\mathbf{C}}(A,B)$, for every pair of objects $A, B \in \mathbf{D}$.

The *opposite category* $\mathbf{C}^{op}$ of $\mathbf{C}$ is a category with objects the same as $\mathbf{C}$ and for each $A, B \in \mathbf{C}$, $\mathrm{Hom}_{\mathbf{C}^{op}}(A,B) = \mathrm{Hom}_{\mathbf{C}}(B,A)$. Then, if $f \in \mathrm{Hom}_{\mathbf{C}^{op}}(A,B)$ and $g \in \mathrm{Hom}_{\mathbf{C}^{op}}(B,C)$, the composition is given by $g \cdot f = fg$, where fg is the composition in $\mathbf{C}$.

Definition 1.1.2. Let $\mathbf{C}$ be a category and consider $g \in \mathrm{Hom}_{\mathbf{C}}(B,C)$. Then, the morphism g is

1. a *monomorphism* if $gf = gf' \Rightarrow f = f'$, for each $A \in \mathbf{C}$ and each $f, f' \in \mathrm{Hom}_{\mathbf{C}}(A,B)$;
2. an *epimorphism* if $hg = h'g \Rightarrow h = h'$, for each $D \in \mathbf{C}$ and each $h, h' \in \mathrm{Hom}_{\mathbf{C}}(C,D)$;
3. a *split monomorphism* if there exists a $h \in \mathrm{Hom}_{\mathbf{C}}(C,B)$ such that $hg = \mathrm{id}_B$. In this case, h is a *left inverse* for g;
4. a *split epimorphism* if there exists a $f \in \mathrm{Hom}_{\mathbf{C}}(C,B)$ such that $gf = \mathrm{id}_C$. In this case, f is a *right inverse* for g;
5. an *isomorphism* if there exists a $g' \in \mathrm{Hom}_{\mathbf{C}}(C,B)$ which is a left and right inverse for g, i.e. $g'g = \mathrm{id}_B$, and $gg' = \mathrm{id}_C$. In this case, we write $g^{-1} := g'$ the *inverse* of g.

If there exists an isomorphism $g \in \mathrm{Hom}_{\mathbf{C}}(B,C)$, we say that the objects B and C are *isomorphic* and write $B \cong C$.

Denote by $\mathrm{End}_{\mathbf{C}}(A) := \mathrm{Hom}_{\mathbf{C}}(A,A)$ the set of all *endomorphisms* of A in $\mathbf{C}$, and by $\mathrm{Aut}_{\mathbf{C}}(A)$ the subset of $\mathrm{End}_{\mathbf{C}}(A)$ consisting of all isomorphisms, i.e. the *automorphisms* of A in $\mathbf{C}$.

Observe that if the morphism g has a left inverse, then g is a monomorphism, and if g has a right inverse, then it is an epimorphism.

Sometimes it is convenient to work with the following equivalences:

Lemma 1.1.3. *Let* $\mathbf{C}$ *be a category,* $g \in \mathrm{Hom}_{\mathbf{C}}(B,C)$*, and consider the functions*

$$\begin{array}{rccc} g_* : & \mathrm{Hom}_{\mathbf{C}}(A,B) & \longrightarrow & \mathrm{Hom}_{\mathbf{C}}(A,C), \\ & f & \longmapsto & gf \end{array} \qquad \begin{array}{rccc} {}_*g : & \mathrm{Hom}_{\mathbf{C}}(C,D) & \longrightarrow & \mathrm{Hom}_{\mathbf{C}}(B,D). \\ & h & \longmapsto & hg \end{array}$$

Then, the morphism g *is*

1. *a monomorphism if and only if the induced function* g_* *is an injection for every* $A \in \mathbf{C}$*;*

2. *an epimorphism if and only if the induced function ${}_*g$ is an injection for every $D \in \mathbf{C}$;*

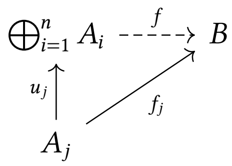
3. *a split monomorphism if and only if the induced function ${}_*g$ is a surjection for every $D \in \mathbf{C}$;*

4. *a split epimorphism if and only if the induced function g_* is a surjection for every $A \in \mathbf{C}$.*

Proof. This result is well known and its proof is straightforward (simple checks). □

1.1.2 Abelian categories

Definition 1.1.4. A *direct sum* of the objects $A_1, \dots, A_n \in \mathbf{C}$ is an object $\bigoplus_{i=1}^n A_i$ together with morphisms $u_j : A_j \to \bigoplus_{i=1}^n A_i$, for $j = 1, \dots, n$ such that for each object $B \in \mathbf{C}$ and morphisms $f_1 : A_1 \to B, \dots, f_n : A_n \to B$, the following diagram commutes:



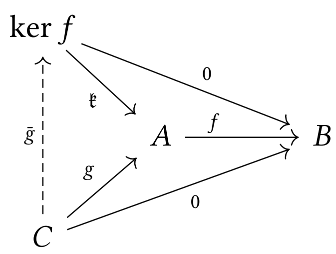
Definition 1.1.5. A *zero object* $0 \in \mathbf{C}$ is such that $|\mathrm{Hom}_{\mathbf{C}}(A, 0)| = |\mathrm{Hom}_{\mathbf{C}}(0, B)| = 1$, for every $A, B \in \mathbf{C}$, where $|-|$ denotes the cardinality of a given set. Thus, for any pair $A, B \in \mathbf{C}$, the composition of the unique morphisms $A \to 0$ and $0 \to B$ establishes the *zero morphism* $0 = 0_B^A : A \to B$, which is unique up to isomorphism.

Definition 1.1.6. A category $\mathbf{C}$ is an *additive category* if it has zero object, any finite set of objects in $\mathbf{C}$ admits a direct sum in $\mathbf{C}$ and each set $\mathrm{Hom}_{\mathbf{C}}(A, B)$ has an abelian group structure such that the composition mappings

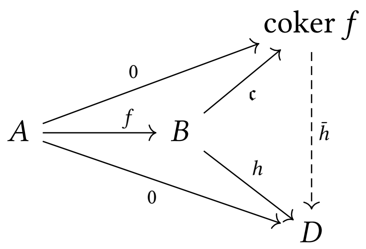
$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{C}}(A, B) \times \mathrm{Hom}_{\mathbf{C}}(B, C) & \longrightarrow & \mathrm{Hom}_{\mathbf{C}}(A, C) \\ (f, g) & \longmapsto & gf \end{array}$$

are group homomorphism in each variable.

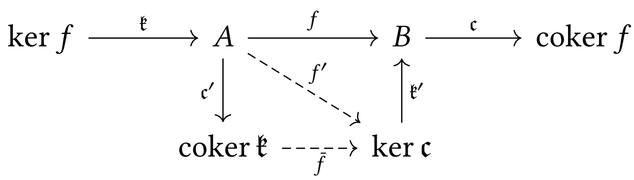
Definition 1.1.7. Let $\mathbf{C}$ be an additive category and $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$. A *kernel* of f is a morphism $\mathfrak{k} : \ker f \to A$ such that $f\mathfrak{k} = 0$, and every morphism $g : C \to A$ with $fg = 0$ factors uniquely through $\mathfrak{k}$ as in the following commutative diagram:



Definition 1.1.8. The dual notion is a *cokernel* of f, i.e. a morphism $\mathfrak{c} : B \to \operatorname{coker} f$ such that $\mathfrak{c}f = 0$, and every morphism $h : B \to D$ with $hf = 0$ factors uniquely through $\mathfrak{c}$ as in the following commutative diagram:



Suppose every morphism in the additive category $\mathbf{C}$ admits a kernel and a cokernel. Then, there exists a unique morphism $\bar{f}$ making the following diagram



(1.1.9)

commutative, where $\mathfrak{c}'$ is a cokernel of $\mathfrak{k}$, $\mathfrak{k}'$ is a kernel of $\mathfrak{c}$, f' is the unique morphism such that $f = \mathfrak{k}'f'$ as $\mathfrak{c}f = 0$, and $\bar{f}$ is the unique morphism such that $f' = \bar{f}\mathfrak{c}'$ as $\mathfrak{k}'f'\mathfrak{k} = f\mathfrak{k} = 0$. Moreover, if $\mathfrak{k}'$ is a monomorphism then $f'\mathfrak{k} = 0$. The object $\ker \mathfrak{c}$ is called the *image* of f and is denoted by $\operatorname{Im} f$.

Definition 1.1.10. An additive category $\mathbf{C}$ is an *abelian category* if for each morphism $f \in \operatorname{Hom}_{\mathbf{C}}(A, B)$ it admits a kernel of f, $\mathfrak{k} : \ker f \to A$, a cokernel of f, $\mathfrak{c} : B \to \operatorname{coker} f$, and the induced morphism $\bar{f} : \operatorname{coker} \mathfrak{k} \to \operatorname{Im} f$ is an isomorphism.

1.1.3 Functors

Definition 1.1.11. A *covariant functor* $F : \mathbf{C} \to \mathbf{D}$ from a category $\mathbf{C}$ to a category $\mathbf{D}$ is an assignment such that each object $A \in \mathbf{C}$ corresponds to an object $F(A) \in \mathbf{D}$ and each morphism $f \in \operatorname{Hom}_{\mathbf{C}}(A, B)$ corresponds to a morphism $F(f) \in \operatorname{Hom}_{\mathbf{D}}\big(F(A), F(B)\big)$ satisfying the equalities

$$F(gf) = F(g)F(f), \qquad F(\mathrm{id}_A) = \mathrm{id}_{F(A)},$$

for any $g \in \operatorname{Hom}_{\mathbf{C}}(B, C)$.

Definition 1.1.12. A *contravariant functor* $G : \mathbf{C} \to \mathbf{D}$ from a category $\mathbf{C}$ to a category $\mathbf{D}$ is an assignment such that each object $A \in \mathbf{C}$ corresponds to an object $G(A) \in \mathbf{D}$ and each morphism $f \in \operatorname{Hom}_{\mathbf{C}}(A, B)$ corresponds to a morphism $G(f) \in \operatorname{Hom}_{\mathbf{D}}\big(G(B), G(A)\big)$ satisfying the equalities

$$G(gf) = G(f)G(g), \qquad G(\mathrm{id}_A) = \mathrm{id}_{G(A)},$$

for any $g \in \operatorname{Hom}_{\mathbf{C}}(B, C)$. Thus, G induces a covariant functor $G' : \mathbf{C}^{op} \to \mathbf{D}$.

Clearly, composition of functors is again a functor.

Definition 1.1.13. A covariant functor $F : \mathbf{C} \to \mathbf{D}$ is an *isomorphism* if there exists a covariant functor $G : \mathbf{D} \to \mathbf{C}$ for which both composites GF and FG are identity functors. In this case, we write $G = F^{-1}$ and $\mathbf{C} \cong \mathbf{D}$.

Definition 1.1.14. Let $F, G : \mathbf{C} \rightrightarrows \mathbf{D}$ be two covariant functors. A *natural transformation* $\tau : F \to G$ is a function which assigns to each object $A \in \mathbf{C}$ a morphism $\tau_A \in \mathrm{Hom}_{\mathbf{D}}\big(F(A), G(A)\big)$ satisfying the following commutative diagram:

$$\begin{array}{ccc} F(A) & \xrightarrow{\tau_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\tau_B]{} & G(B) \end{array} \tag{1.1.15}$$

for every $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$. In this case, we say that τ_A is *natural* in A. The natural transformation τ is a *natural isomorphism* if τ_A is an isomorphism for each $A \in \mathbf{C}$.

Definition 1.1.16. A covariant functor $F : \mathbf{C} \to \mathbf{D}$ is an *equivalence of categories* if there exist a functor $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $\tau : GF \to \mathrm{id}_{\mathbf{C}}$ and $\upsilon : FG \to \mathrm{id}_{\mathbf{D}}$, where $\mathrm{id}_{\mathbf{C}}$ and $\mathrm{id}_{\mathbf{D}}$ are the identity functors on $\mathbf{C}$ and $\mathbf{D}$, respectively.

Definition 1.1.17. A contravariant functor $G : \mathbf{C} \to \mathbf{D}$ is a *duality of categories* if the induced covariant functor $G' : \mathbf{C}^{op} \to \mathbf{D}$ is an equivalence of categories, where $\mathbf{C}^{op}$ is the opposite category of C.

Remark 1.1.18. While an equivalence of categories sends monomorphisms onto monomorphisms and epimorphisms onto epimorphisms, a duality of categories sends monomorphisms onto epimorphisms and epimorphisms onto monomorphisms.

Definition 1.1.19. Let $F : \mathbf{C} \to \mathbf{D}$ be a covariant functor and for each pair $A, B \in \mathbf{C}$ consider the function

$$\begin{aligned} F_{A,B} : \mathrm{Hom}_{\mathbf{C}}(A, B) &\longrightarrow \mathrm{Hom}_{\mathbf{D}}\big(F(A), F(B)\big) \\ f \quad &\longmapsto \quad F(f) \end{aligned}$$

The functor F is *full* if $F_{A,B}$ is a surjection for all $A, B \in \mathbf{C}$, and F is *faithful* if $F_{A,B}$ is an injection for all $A, B \in \mathbf{C}$. If F is both full and faithful, we say that F is *fully faithful*.

A functor $F : \mathbf{C} \to \mathbf{D}$ is *dense* if for any $B \in \mathbf{D}$ there exists $A \in \mathbf{C}$ such that $F(A) \cong B$. A fully faithfull dense functor is an equivalence of categories, see [Mac98, Theorem 4.4.1].

1.1.4 Adjoint functors

Definition 1.1.20. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. An *adjunction* from $\mathbf{C}$ to $\mathbf{D}$ is a triple $\langle F, G, \Psi \rangle : \mathbf{C} \to \mathbf{D}$, where F and G are covariant functors

$$\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D},$$

while Ψ is a function which assigns to each pair of objects $A \in \mathbf{C}$, $B \in \mathbf{D}$ a bijection of sets

$$\Psi = \Psi_{A,B} : \mathrm{Hom}_{\mathbf{C}}\big(A, G(B)\big) \to \mathrm{Hom}_{\mathbf{D}}\big(F(A), B\big)$$

which is natural in A and B. In this case, F is a *left adjoint* for G and G is a *right adjoint* for F.

The natural transformation $\mathcal{H} : \mathrm{id}_{\mathbf{C}} \to GF$ given by

$$\mathcal{H} = \mathcal{H}_A = \Psi^{-1}_{A,F(A)}(\mathrm{id}_{F(A)}) : A \to GF(A) \tag{1.1.21}$$

is the *unit of the adjunction.*

The natural transformation $\mathcal{E} : FG \to \mathrm{id}_{\mathbf{D}}$ given by

$$\mathcal{E} = \mathcal{E}_B = \Psi_{G(B),B}(\mathrm{id}_{G(B)}) : FG(B) \to B \tag{1.1.22}$$

is the *counit of the adjunction.*

Two contravariant functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ are *adjoint on the right* when there exists a bijection $\mathrm{Hom}_{\mathbf{C}}\big(C, G(D)\big) \cong \mathrm{Hom}_{\mathbf{D}}\big(D, F(C)\big)$, natural in C and D.

Similarly, two contravariant functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ are *adjoint on the left* when there exists a bijection $\mathrm{Hom}_{\mathbf{C}}(F(C), D) \cong \mathrm{Hom}_{\mathbf{D}}(G(D), C)$, natural in C and D.

An *adjoint equivalence* of categories is an adjunction $\langle F, G, \Psi\rangle : \mathbf{C} \to \mathbf{D}$ such that F is a left and right adjoint for G.

Proposition 1.1.23. *An adjunction $\langle F, G, \Psi\rangle : \mathbf{C} \to \mathbf{D}$ is completely determined by two covariant functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ and natural transformations $\mathcal{H} : \mathrm{id}_{\mathbf{C}} \to GF$ and $\mathcal{E} : FG \to \mathrm{id}_{\mathbf{D}}$ satisfying the* triangular identities*:*

$$\begin{array}{ccc} F(-) & \xrightarrow{\mathrm{id}_{F(-)}} & F(-) \\ & {\scriptstyle F(\mathcal{H}_-)}\searrow \quad \nearrow{\scriptstyle \mathcal{E}_{F(-)}} & \\ & FGF(-) & \end{array} \quad , \qquad \begin{array}{ccc} G(-) & \xrightarrow{\mathrm{id}_{G(-)}} & G(-) \\ & {\scriptstyle \mathcal{H}_{G(-)}}\searrow \quad \nearrow{\scriptstyle G(\mathcal{E}_-)} & \\ & GFG(-) & \end{array} \tag{1.1.24}$$

In this case, for any morphism $f : A \to G(B)$ and any morphism $f' : F(A) \to B$, the function Ψ is defined by:

$$\Psi_{A,B}(f) = \mathcal{E}_B F(f) : F(A) \to B; \qquad \Psi^{-1}_{A,B}(f') = G(f')\mathcal{H}_A : A \to G(B) \tag{1.1.25}$$

Proof. See [Mac98, Theorem 4.1.2(v)]. □

Proposition 1.1.26. *Consider an adjunction $\langle F, G, \Psi\rangle : \mathbf{C} \to \mathbf{D}$, with unit $\mathcal{H} : \mathrm{id}_{\mathbf{C}} \to GF$ and counit $\mathcal{E} : FG \to \mathrm{id}_{\mathbf{D}}$. Then*

1. *the functor G is faithful if and only if every component $\mathcal{E}_B$ of the counit is an epimorphism;*
2. *the functor G is full if and only if every $\mathcal{E}_B$ is a split monomorphism.*

Hence G is fully faithful if and only if each $\mathcal{E}_B$ is an isomorphism $FG(B) \cong B$.

Proof. See [Mac98, Theorem 4.3.1]. □

Corollary 1.1.27. *Similarly we have:*

1. *the functor F is faithful if and only if every component $\mathcal{H}_A$ of the unit is a monomorphism;*
2. *the functor F is full if and only if every $\mathcal{H}_A$ is a split epimorphism.*

Hence F is fully faithful if and only if each $\mathcal{H}_A$ is an isomorphism $A \cong GF(A)$. Therefore, F and G are equivalences if and only if they are fully faithful.

Proof. The covariant functor F is faithful if and only if the composition function

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbf{C}}(A,B) & \xrightarrow{F} & \mathrm{Hom}_{\mathbf{D}}\big(F(A),F(B)\big) & \xrightarrow{\Psi^{-1}_{A,F(B)}} & \mathrm{Hom}_{\mathbf{D}}\big(A,GF(B)\big) \\ f & \longmapsto & F(f) & \longmapsto & \Psi^{-1}_{A,F(B)}F(f) \end{array}$$

is an injection for every $A, B \in \mathbf{C}$ (see Definition 1.1.19), where the right hand side is a bijection (see Definition 1.1.20). Since

$$\Psi^{-1}_{A,F(B)}F(f) = GF(f)\mathcal{H}_A = \mathcal{H}_B f$$

by naturality of $\mathcal{H}$ (see 1.1.25 and 1.1.15), it follows that $\mathrm{Hom}_{\mathbf{C}}(A,B) \to \mathrm{Hom}_{\mathbf{D}}\big(A,GF(B)\big)$ is an injection if and only if $\mathcal{H}_B$ is a monomorphism for every $B \in \mathbf{C}$ (see Lemma 1.1.3).

If we part from the other way round, the proof follows the same idea, but with interesting and slight different computations, as we show in the sequence:

Each component $\mathcal{H}_B$ of the unit is a monomorphism if and only if the composition function

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbf{C}}(A,B) & \xrightarrow{\mathcal{H}_{B*}} & \mathrm{Hom}_{\mathbf{D}}\big(A,GF(B)\big) & \xrightarrow{\Psi_{A,F(B)}} & \mathrm{Hom}_{\mathbf{D}}\big(F(A),F(B)\big) \\ f & \longmapsto & \mathcal{H}_B f & \longmapsto & \Psi_{A,F(B)}(\mathcal{H}_B f) \end{array}$$

is an injection for every $A, B \in \mathbf{C}$ (see Lemma 1.1.3), where the right hand side is a bijection (see Definition 1.1.20). Since

$$\Psi_{A,F(B)}(\mathcal{H}_B f) = \mathcal{E}_{F(B)}F(\mathcal{H}_B)F(f) = F(f)$$

by the triangular identities (see 1.1.25 and 1.1.24), it follows that

$$\mathrm{Hom}_{\mathbf{C}}(A,B) \to \mathrm{Hom}_{\mathbf{D}}\big(F(A),F(B)\big)$$

is an injection if and only if F is faithful (see Definition 1.1.19).

The proof of part 2 follows in the same way, swapping “injection” for “surjection”. □

1.2 Coalgebras and comodules

Fix once and for all a field k. Algebras, coalgebras, vector spaces, linear maps and tensor products are over k unless specified otherwise. All (co)algebras treated in this text are (co)associative with (co)unit. The symbol $\mathbb{N}$ denotes the set of non-negative integers (including zero).

Given a vector space V, we denote by $V^* = \mathrm{Hom}_k(V,k)$ the space of all linear maps from V to k, namely the *dual space* of V. In the case that V is a topological space, k is treated as a discrete topological ring and $\mathrm{Hom}_k(V,k)$ is the space of all continuous functionals from V to k. Given a map $f : V \to U$, the morphism $f^* : U^* \to V^*$ is defined by $f^*\big(g\big)(v) = g\big(f(v)\big)$, for any $g \in U^*$ and $v \in V$.

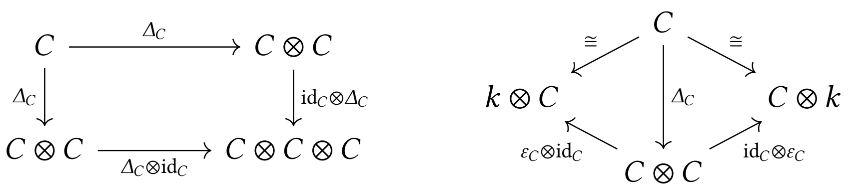
In Section 1.4, we show that the contravariant functor $(-)^* : \mathbf{Cog} \to \mathbf{Alg}$ is a duality of categories between the category of coalgebras (and coalgebra homomorphisms) and the category of pseudocompact algebras (and continuous algebra homomorphisms).

In this section we study coalgebras and comodules, and present important related results as the Dual Wedderburn-Malcev Theorem, the Fundamental Theorem of Coalgebras and Comodules and the Universal Property of Cotensor Coalgebras. For a general introduction to coalgebras and comodules, see, for instance, [Abe80; DNR01; FM20; Mon93; Rad11; Swe69].

1.2.1 Coalgebras

A coalgebra is defined in the monoidal category of vector spaces by axioms dual to those of an algebra, i.e.

Definition 1.2.1. A *coalgebra* $C = (C, \Delta_C, \varepsilon_C)$ is a vector space C together with two linear maps $\Delta_C : C \to C \otimes C$, the *comultiplication* of C, and $\varepsilon_C : C \to k$, the *counit* of C, satisfying the following commutative diagrams:



(1.2.2)

where $\mathrm{id}_C : C \to C$ is the identity map of C, and the maps

$$\begin{array}{rcl} C & \xrightarrow{\cong} & k \otimes C, \\ c & \longmapsto & 1 \otimes c \end{array} \qquad \begin{array}{rcl} C & \xrightarrow{\cong} & C \otimes k, \\ c & \longmapsto & c \otimes 1 \end{array}$$

are the canonical isomorphisms. The first diagram in (1.2.2) is the *coassociativity of the comultiplication* and the second is the *counit property* of C.

A subspace $S \subseteq C$ is a *subcoalgebra* of C if $\Delta_C(S) \subseteq S \otimes S$. In this case, $(S, \Delta_C|_S, \varepsilon_C|_S)$ is a coalgebra.

A coalgebra C is:

1. *simple* if there is no non-zero proper subcoalgebra of C;

2. *cosemisimple* if it is the sum of its simple subcoalgebras;

3. *pointed* if all simple subcoalgebra of C has dimension 1 (as vector space).

Consider a coalgebra C and denote by $C^{cop} = (C, \Delta_C^{cop}, \varepsilon_C)$ the *co-opposite coalgebra* of C with comultiplication $\Delta_C^{cop} = T\Delta_C$, where $T : C \otimes C \to C \otimes C$ is the *twist map* given by $T(a \otimes b) = b \otimes a$.

A coalgebra is *cocomutative* if $C = C^{cop}$.

In particular, if k is an algebraically closed field, then every cocommutative k-coalgebra is pointed, see [Abe80, Theorem 2.3.3].

When it is clear which coalgebra we are referring to, we may drop the subscripts for the comultiplication and counit. The example below give us a family of simple coalgebras, which will be proved simple later on, see Remark 1.4.18, since we need more tools for that:

Example 1.2.3. The *matrix coalgebra* $M^C(n,k)$ is the vector space with basis $\{e_{i,j} \mid 1 \leqslant i, j \leqslant n\}$ given by

$$e_{i,j} = \begin{bmatrix} \delta_{1,i}\delta_{1,j} & \dots & \delta_{1,i}\delta_{n,j} \\ \vdots & \ddots & \vdots \\ \delta_{n,i}\delta_{1,j} & \dots & \delta_{n,i}\delta_{n,j} \end{bmatrix} \tag{1.2.4}$$

(thus is the $n \times n$ matrix), with comultiplication and counit maps given by

$$\Delta(e_{i,j}) = \sum_{l=1}^{n} e_{i,l} \otimes e_{l,j}, \quad \varepsilon(e_{i,j}) = \delta_{i,j}. \tag{1.2.5}$$

The next example is another important family of coalgebras, which are subspaces of the square matrices but not subcoalgebras of $M^C(n,k)$ (which we knew already since the latter is simple).

Example 1.2.6. The *upper triangular matrix coalgebra* $U^C(n,k)$ is the vector space with basis $\{e_{i,j} \mid 1 \leqslant i, j \leqslant n,\ i \geqslant j\}$ given by

$$e_{i,j} = \begin{bmatrix} \delta_{1,i}\delta_{1,j} & \dots & \delta_{1,i}\delta_{n,j} \\ & \ddots & \vdots \\ \text{\huge 0} & & \delta_{n,i}\delta_{n,j} \end{bmatrix} \tag{1.2.7}$$

with comultiplication and counit maps given by

$$\Delta(e_{i,j}) = \sum_{i \leqslant l \leqslant j} e_{i,l} \otimes e_{l,j}, \quad \varepsilon(e_{i,j}) = \delta_{i,j}. \tag{1.2.8}$$

Note that the vector space generated by $e_{i,i}$ is a subcoalgebra of $U^C(n,k)$.

Example 1.2.9. If C and D are coalgebras, then the tensor product of coalgebras $C \otimes D$ is a coalgebra with comultiplication and counit given by:

$$\Delta_{C\otimes D} = (\mathrm{id}_C \otimes T \otimes \mathrm{id}_D)(\Delta_C \otimes \Delta_D), \qquad \varepsilon_{C\otimes D} = \varepsilon_C \otimes \varepsilon_D \tag{1.2.10}$$

Example 1.2.11. The *trigonometric coalgebra* C is a two dimensional coalgebra with basis $\{a, b\}$ and comultiplication and counit given by:

$$\begin{aligned} \Delta(a) &= a \otimes a - b \otimes b, \quad & \varepsilon(a) &= 1; \\ \Delta(b) &= b \otimes a + a \otimes b, \quad & \varepsilon(b) &= 0. \end{aligned} \tag{1.2.12}$$

If there is no root of $\lambda^2 + 1$ in k (for instance, if $k = \mathbb{R}$), then C is simple and cocommutative, but not pointed. Otherwise (for instance, if $k = \mathbb{C}$), let $i = \sqrt{-1}$, then

$$\Delta(a + ib) = a \otimes a - b \otimes b + ib \otimes a + ia \otimes b = (a + ib) \otimes (a + ib)$$

shows that the vector space generated by $(a + ib)$ is a one dimensional subcoalgebra, thus C is cocomutative, pointed and not simple.

The structure of a coalgebra allow us to obtain an algebra from it, called dual algebra.

Definition 1.2.13. The *dual algebra* of a coalgebra C is the dual vector space $C^* = \mathrm{Hom}_k(C, k)$ endowed with the *multiplication* $\big(\mathbf{m}(f \otimes g)\big)(c) = (f \otimes g)\Delta(c)$, using the canonical isomorphism $k \otimes k \cong k$, and the *unit* $\eta(1)(c) = \varepsilon(c)$, see [Swe69, Proposition 1.1.1].

Example 1.2.14. Consider the matrix coalgebra $M^C(n, k)$ (see Example 1.2.3). Its dual algebra is the matrix algebra $M^C(n, k)^* \cong M(n, k)$.

Example 1.2.15. Let $k = \mathbb{R}$ and consider the trigonometric coalgebra (see Example 1.2.11). Its dual algebra is the algebra of complex numbers.

Definition 1.2.16. The *coradical* of a coalgebra C, denoted by C_0, is the sum of all simple subcoalgebras of C.

The *coradical filtration* of a coalgebra C is the family $\{C_n\}_{n\in\mathbb{N}}$, where C_0 is the coradical of C, and C_n is defined inductively by

$$C_n := \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C) \tag{1.2.17}$$

In this case, each C_n is a subcoalgebra of C, $C_n \subseteq C_{n+1}$ as subcoalgebras, $C = \bigcup_{n=0}^{\infty} C_n$, and $\Delta(C_n) \subseteq \sum_{i=0}^{n} C_{n-i} \otimes C_i$, see [Swe69, §9.1]. See also [Abe80, §2.4.1] and [Mon93, Theorem 5.2.2]. Throughout this text, any numbered subscript on a coalgebra refers to its coradical filtration. We occasionally use the helpful convention $C_{-1} := \{0\}$.

Thus a coalgebra C is cosemisimple if and only if $C = C_0$.

Definition 1.2.18. A coalgebra is *basic* if the dual algebra of each simple subcoalgebra is a division algebra.

A coalgebra C is *separable* if for every extension field L of k, $C \otimes L$ is a cosemisimple coalgebra over L. Equivalently, a coalgebra is separable if, and only if, the dual algebra of each simple subcoalgebra is separable.

Observe that every separable coalgebra is cosemisimple. In particular, every pointed coalgebra is basic and has separable coradical.

Proposition 1.2.19. *Let C be a coalgebra and $D \subseteq C$ be a subcoalgebra. Then, $D_n = C_n \cap D$ for all $n \in \mathbb{N}$.*

Proof. See [HR74, Corollary 2.3.7]. □

1.2.2 Coalgebra homomorphisms

Definition 1.2.20. Consider two coalgebras C and D. A linear map $\rho : C \to D$ is a *coalgebra homomorphism* if it satisfies the commutative diagrams:

$$\begin{array}{ccc} C & \xrightarrow{\ \rho\ } & D \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \Delta_D} \\ C \otimes C & \xrightarrow[\rho\otimes\rho]{} & D \otimes D \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\ \rho\ } & D \\ & {\scriptstyle \varepsilon_C}\searrow \quad \swarrow{\scriptstyle \varepsilon_D} & \\ & k & \end{array} \tag{1.2.21}$$

A coalgebra homomorphism $\rho : C \to D$ is *filtered* if $\rho(C_n) \subseteq D_n$, see [Swe69, p. 229] or [Abe80, p. 92].

Remark 1.2.22. A simple induction on the (coradical) filtration shows that, if $\rho : C \to D$ is a coalgebra homomorphism such that $\rho(C_0) \subseteq D_0$, then ρ is filtered. This happens because $\rho(C_i) \subseteq D_i$, for $0 \leqslant i \leqslant n-1$, implies

$$\Delta_D \rho(C_n) \subseteq (\rho \otimes \rho)(C \otimes C_{n-1} + C_0 \otimes C) \subseteq D \otimes D_{n-1} + D_0 \otimes D,$$

hence $\rho(C_n) \subseteq D_n$.

It is straightforward to check that the image of a coalgebra homomorphism is a subcoalgebra and the composition of coalgebra homomorphisms are coalgebra homomorphisms. Thus, coalgebras and coalgebra homomorphisms form a category denoted by **Cog**.

Denote by **cog** the full subcategory of all finite dimensional coalgebras.

Proposition 1.2.23. *Let C and D be coalgebras and $\rho : C \to D$ be a surjection coalgebra homomorphism, then $D_0 \subseteq \rho(C_0)$. In the case that C is pointed, $\rho(C_0) = D_0$ and D is pointed.*

Proof. See [Mon93, Corollary 5.3.5]. □

In particular, if $\rho : C \to D$ is an injection we have the following:

Lemma 1.2.24. *Let C and D be coalgebras and $\rho : C \to D$ be an injection coalgebra homomorphism, then ρ is filtered. Moreover, if ρ is an isomorphism, then $\rho(C_n) = D_n$.*

Proof. Let $\rho : C \to D$ be a coalgebra isomorphism and $\rho^{-1} : D \to C$ its inverse. Then, $D_0 \subseteq \rho(C_0)$ and $C_0 \subseteq \rho^{-1}(D_0)$ by Proposition 1.2.23. Applying ρ to the last expression gives $\rho(C_0) \subseteq D_0$, which implies $D_0 = \rho(C_0)$. Now, we prove the affirmative by induction on the filtration. Suppose that $\rho(C_i) = D_i$ for $i \leqslant n-1$. Then

$$\Delta_D \rho(C_n) \subseteq (\rho \otimes \rho)(C \otimes C_{n-1} + C_0 \otimes C) = D \otimes D_{n-1} + D_0 \otimes D,$$

shows $\rho(C_n) \subseteq D_n$. Similarly, $\rho^{-1}(D_n) \subseteq C_n$ and, using the same argument as before, $\rho(C_n) = D_n$.

If $\rho : C \to D$ is an injection, then the corestriction to its image is an isomorphism. Then, by what we just proved and Proposition 1.2.19,

$$\rho(C_n) = \rho(C)_n = \rho(C) \cap D_n \subseteq D_n.$$

□

Definition 1.2.25. Consider a coalgebra C. A subspace $I \subseteq C$ is a *coideal* of C if it satisfies:

$$\Delta(I) \subseteq C \otimes I + I \otimes C, \qquad \varepsilon(I) = 0 \tag{1.2.26}$$

Next, we present two fundamental theorems for coalgebras which sustain the important role played by coideals for coalgebras. The theorem below is known as the *Isomorphism Theorem for Coalgebras.*

Theorem 1.2.27. *Let C and D be coalgebras, I a coideal of C, $q : C \to \frac{C}{I}$ be the canonical linear projection, and $\rho : C \to D$ a coalgebra homomorphism. Then:*

1. *the quotient space $\frac{C}{I}$ has a unique coalgebra structure making q into a coalgebra homomorphism;*
2. *the* ker ρ *is a coideal of C, and;*
3. *if $I \subseteq \ker \rho$, then there exists a unique coalgebra homomorphism $\overline{\rho} : \frac{C}{I} \to D$ satisfying the commutative diagram:*

$$\begin{array}{ccc} C & \xrightarrow{\ \rho\ } & D \\ & {\scriptstyle q}\searrow \quad \nearrow{\scriptstyle \overline{\rho}} & \\ & \frac{C}{I} & \end{array} \tag{1.2.28}$$

Proof. See [Swe69, Theorem 1.4.7]. See also [DNR01, Proposition 1.4.9 and Theorem 1.4.10]. □

The next theorem is the *Dual Wedderburn-Malcev Theorem.*

Theorem 1.2.29. *Let C be a coalgebra with separable coradical. Then, there exists a coideal I such that $C = C_0 \oplus I$.*

Proof. See [Abe80, Theorem 2.3.11]. □

In other words, this theorem says that for any coalgebra C with separable coradical, there exists a coalgebra projection (not necessarily unique) $\pi_0 : C \to C_0$, such that $\pi_0 \iota_0 = \mathrm{id}_{C_0}$, where $\iota_0 : C_0 \to C$ is the canonical inclusion. The projection π_0 can be constructed as the composition of the canonical projection (given a coideal $I \subseteq C$, such that $C = C_0 \oplus I$) $q_I : C \to \dfrac{C}{I}$ and the canonical isomorphism $\dfrac{C_0 \oplus I}{I} \cong C_0$.

Theorem 1.2.30. *Let C and D be coalgebras, and $\rho : C \to D$ a coalgebra homomorphism. If C_0 is cocommutative, then ρ is filtered.*

Proof. See [Swe69, Theorem 9.1.4]. □

In particular, if C is pointed every coalgebra homomorphism $\rho : C \to D$ is filtered. The same is true for basic coalgebras, which we will prove next. First we need a well-known result about division algebras, cf. [Pie82, Exercises 2, p. 74], and we will leave a proof because we couldn't find one.

Lemma 1.2.31. *Let A be a division algebra and $B \subseteq A$ be a finite dimensional subalgebra, then B is a division algebra.*

Proof. Let $b \in B$ and consider the map $m_b : B \to B$ given by $x \mapsto bx$. Since B is a subalgebra of a division algebra, it has no nontrivial zero divisor and, because it is finite dimensional, the map m_b is a bijection. Thus b has a right inverse. Similar argument shows that b has also a left inverse (which must be equal the right inverse by associativity) and, since b was arbitrary, B is a division algebra. □

Lemma 1.2.32. *Let C and D be coalgebras and $\rho : C \to D$ be a coalgebra homomorphism. If C is basic, then ρ is filtered.*

Proof. Let $S \subseteq C$ be a simple subcoalgebra of C and consider the dual algebra homomorphism of the restriction $\rho : S \to \rho(S)$, $\rho^* : \big(\rho(S)\big)^* \to S^*$. Since ρ is a surjection onto its image, $\big(\rho(S)\big)^*$ can be seen as a subalgebra of the finite dimensional division algebra S^* via the injection ρ^*. By Lemma 1.2.31, $\big(\rho(S)\big)^*$ is a division algebra and, consequently, $\rho(S)$ is a simple subcoalgebra. The result now follows from Remark 1.2.22. □

Proposition 1.2.33. *Let C and D be coalgebras and $\rho : C \to D$ a coalgebra homomorphism. Then ρ is injective if and only if $\rho|_{C_1} : C_1 \to D$ is injective.*

Proof. See [HR74, Proposition 2.4.2]. See also [Mon93, Theorem 5.3.1]. □

1.2.3 Comodules

Definition 1.2.34. Let C be a coalgebra. A *left C-comodule* $M = (M, \mu_M)$ is a vector space M together with a linear map $\mu_M : M \to C \otimes M$, the *structure map* of M, satisfying the following commutative diagrams:

$$
\begin{array}{ccc}
M & \xrightarrow{\mu_M} & C \otimes M \\
{\scriptstyle \mu_M}\downarrow & & \downarrow{\scriptstyle \Delta_C \otimes \mathrm{id}_C} \\
C \otimes M & \xrightarrow[\mathrm{id}_C \otimes \mu_M]{} & C \otimes C \otimes M
\end{array}
\qquad
\begin{array}{ccc}
 & & M \\
k \otimes M & \xleftarrow{\cong} & \downarrow{\scriptstyle \mu} \\
 & \xleftarrow[\varepsilon \otimes \mathrm{id}]{} & C \otimes M
\end{array}
\tag{1.2.35}
$$

A *right* C*-comodule* $N = (N, \nu_N)$ is defined in a similar fashion, with the *structure map* of N, $\nu_N : N \to N \otimes C$, satisfying the following commutative diagrams:

$$
\begin{array}{ccc}
N & \xrightarrow{\nu_N} & N \otimes C \\
{\scriptstyle \nu_N}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \Delta_C} \\
N \otimes C & \xrightarrow[\nu_N \otimes \mathrm{id}_C]{} & N \otimes C \otimes C
\end{array}
\qquad
\begin{array}{ccc}
N & \xrightarrow{\cong} & \\
{\scriptstyle \nu_N}\downarrow & & N \otimes k \\
N \otimes C & \xrightarrow[\mathrm{id}_N \otimes \varepsilon_C]{} &
\end{array}
\tag{1.2.36}
$$

A subspace $L \subseteq M$ is a *subcomodule* of the left C-comodule M if $\mu_M(L) \subseteq C \otimes L$. In this case $L = (L, \mu_M|_L)$ is a left C-comodule. A subspace $R \subseteq N$ is a *subcomodule* of right C-comodule N if $\nu_N(R) \subseteq R \otimes C$. In this case $R = (R, \nu_N|_R)$ is a right C-comodule.

A left (or right) C-comodule M is *simple* if there is no non-zero proper subcomodule of M.

Now we give some examples of comodules which are obtained from other given structures.

Example 1.2.37.

1. Let C be a coalgebra and M be a left C-comodule. Then M is a right C^{cop}-comodule with structure map $\mu = T\mu_M$, where T is the twist map (see the co-oposite coalgebra at 1.2.1);
2. Every coalgebra C is a left and right C-comodule with structure map $\mu = \nu = \Delta_C$;
3. Any subcoalgebra $S \subseteq C$ of the coalgebra C is a subcomodule of the left (and right) C-comodule C;
4. If $\rho : C \to D$ is a coalgebra homomorphism and M is a left (right) C-comodule, then M is a left (resp. right) D-comodule with structure map $\mu = (\rho \otimes \mathrm{id}_M)\mu_M$ (resp. $\nu = (\mathrm{id}_M \otimes \rho)\nu_M$). In particular, C is a left (and right) D-comodule;
5. Consider a coalgebra C and left (right) C-comodules M and N. Then $M \oplus N$ is a left (resp. right) C-comodule with structure map $\mu = \mu_M + \mu_N$ (resp. $\nu = \nu_M + \nu_N$);
6. Let C be a coalgebra, L a left (right) C-comodule and M and N subcomodules of L. Then $M + N = \{m + n \,|\, m \in M,\ n \in N\}$ and $M \cap N$ are subcomodules.

Remark 1.2.38. In order to simplify computations, we make use of the *sigma notation*, see [Swe69, §1.2 and §2.0], i.e. if C is a coalgebra, M is a right C-comodule, and N is a left

C-comodule, we write

$$\begin{aligned} \Delta_C(c) &= \sum_{i=1}^{r_1} c_{1i} \otimes c_{2i} := \sum_{(c)} c_{(1)} \otimes c_{(2)}; \\ \nu_M(m) &= \sum_{j=1}^{r_2} m_{1j} \otimes m_{2j} := \sum_{(m)} m_{(0)} \otimes m_{(1)}; \\ \mu_N(n) &= \sum_{l=1}^{r_3} n_{1l} \otimes n_{2l} := \sum_{(n)} n_{(-1)} \otimes n_{(0)}; \end{aligned}$$

with $c, c_{1i}, c_{2i}, m_{2j}, n_{1l} \in C$, $m, m_{1j} \in M$, and $n, n_{2l} \in N$, for $i = 1, \ldots, r_1$, $j = 1, \ldots, r_2$, and $l = 1, \ldots, r_3$.

1.2.4 Comodule homomorphisms

Definition 1.2.39. Let C be a coalgebra, L and M be left C-comodules, and R and N be right C-comodules. A linear map $\sigma : L \to M$ is a *comodule homomorphism* of left C-comodules if satisfies the following commutative diagram:

$$\begin{array}{ccc} L & \xrightarrow{\sigma} & M \\ {\scriptstyle \mu_L}\downarrow & & \downarrow{\scriptstyle \mu_M} \\ C \otimes L & \xrightarrow[\mathrm{id}_C \otimes \sigma]{} & C \otimes M \end{array} \tag{1.2.40}$$

A linear map $\sigma' : R \to N$ is a *comodule homomorphism* of right C-comodules if satisfies the following commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{\sigma'} & N \\ {\scriptstyle \nu_R}\downarrow & & \downarrow{\scriptstyle \nu_N} \\ R \otimes C & \xrightarrow[\sigma' \otimes \mathrm{id}_C]{} & N \otimes C \end{array} \tag{1.2.41}$$

It is straightforward to check that composition of comodule homomorphisms are comodule homomorphisms. Thus left C-comodules and comodule homomorphisms form a category denoted by ${}^C\mathcal{M}$. Similarly $\mathcal{M}^C$ denotes the category of right C-comodules and comodule homomorphisms. Denote by ${}^C\mathcal{M}_{\mathrm{f}}$ and by $\mathcal{M}^C_{\mathrm{f}}$ the full subcategories of finite dimensional comodules.

Moreover, the categories ${}^C\mathcal{M}$ and $\mathcal{M}^{C^{cop}}$ are isomorphic, see [DNR01, Proposition 2.1.10]. Hence, any result about left comodules can be translated as a result about right comodules over the cooposite coalgebra, and vice versa.

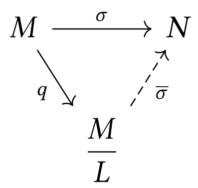
In order to simplify notation, we write $\mathrm{Hom}_{C-}(M, N) := \mathrm{Hom}_{{}^C\mathcal{M}}(M, N)$ if M and N are left C-comodules and $\mathrm{Hom}_{-C}(M, N) := \mathrm{Hom}_{\mathcal{M}^C}(M, N)$ if M and N are right C-comodules.

Similar to the Fundamental Isomorphism Theorem for Coalgebras 1.2.27, we have the

Fundamental Isomorphism Theorem for Comodules:

Theorem 1.2.42. *Let C be a coalgebra, M, N be (right) C-comodules, $L \subseteq M$ be a subcomodule and $\sigma : M \to N$ be a comodule homomorphism. Then:*

1. $\operatorname{Im}\sigma$ *and* $\ker\sigma$ *are subcomodules;*
2. *there exists a unique structure map of $\frac{M}{L}$ making the canonical projection $q : M \to \dfrac{M}{L}$ a comodule homomorphism;*
3. *if $L \subseteq \ker\sigma$, then there exists a unique comodule homomorphism $\overline{\sigma} : \dfrac{M}{L} \to N$ satisfying the following commutative diagram:*



(1.2.43)

Proof. See [Swe69, Theorem 2.0.1]. □

In particular, the canonical projection $q' : N \to \dfrac{N}{\operatorname{Im}\sigma}$ is a cokernel of σ, which leads to the unique isomorphism $\dfrac{M}{\ker\sigma} \cong \operatorname{Im}\sigma$ as in the commutative diagram (1.1.9), see also [DNR01, Theorem 2.1.17]. With some other checks, we get the following:

Proposition 1.2.44. *Let C be a coalgebra. The category of (right) C-comodules $\mathcal{M}^C$ is abelian.*

Proof. See [DNR01, Corollary 2.1.19]. □

Since $\mathcal{M}^C$ is abelian, the second isomorphism theorem applies to comodules, i.e.

Lemma 1.2.45. *Let C be a coalgebra, L be a (right) C-comodule and M and N be subcomodules. Then $\dfrac{N}{M \cap N} \cong \dfrac{M+N}{M}$.*

Proof. First observe that $M + N$ and $M \cap N$ are subcomodules of L (see Example 1.2.37). Moreover, the canonical inclusion $\iota : N \to M+N$ composed with the canonical projection $q : M+N \to \dfrac{M+N}{M}$ is a surjection coalgebra homomorphism. Hence, taking $f = q\iota$ gives: $\ker f = M \cap N$, $\operatorname{coker} f = 0$, $\operatorname{coker}\mathfrak{c} = \dfrac{N}{M \cap N}$ and $\ker\mathfrak{k} = \dfrac{M+N}{M}$, so the isomorphism follows from $\mathcal{M}^C$ being abelian (see 1.1.9 and Definition 1.1.10). □

If C and D are coalgebras and $\rho : C \to D$ is a colalgebra homomorphism, then not only C and D are (right) D-comodules (see 1.2.37), but also $\rho : C \to D$ is a comodule homomorphism. Applying the Fundamental Isomorphism Theorem for Comodules, Theorem 1.2.42, we have the following:

Lemma 1.2.46. *Let C and D be coalgebras and $\rho : C \to D$ be a coalgebra homomorphism. If ρ is filtered, then, for each $n \in \mathbb{N}$, there exists a unique comodule homomorphism such that the following diagram*

$$\begin{array}{ccc} C & \xrightarrow{\rho} & D \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ \dfrac{C}{C_n} & \underset{\overline{\rho}}{\dashrightarrow} & \dfrac{D}{D_n} \end{array} \tag{1.2.47}$$

commute, where q and q' are the canonical projections.

Proof. If ρ is filtered, then

$$\rho(C_n) \subseteq D_n = \ker q' \Longrightarrow C_n \subseteq \ker(q'\rho).$$

The result follows by the Fundamental Isomorphism Theorem for Comodules, Theorem 1.2.42. □

The next example shows that not every coalgebra homomorphism is a comodule homomorphism.

Example 1.2.48. Consider the matrix coalgebra $M^C(n,k)$, with basis $\{e_{i,j} \mid 1 \leqslant i,j \leqslant n\}$ as in Example 1.2.3, and the upper triangular matrix coalgebra $U^C(n,k)$, with basis $\{f_{i,j} \mid 1 \leqslant i,j \leqslant n,\ i \leqslant j\}$ as in Example 1.2.6.

The map $\rho : M^C(n,k) \to U^C(n,k)$ given by

$$\rho(e_{i,j}) = \begin{cases} f_{i,j} & \text{if } i \leqslant j \\ 0 & \text{otherwise} \end{cases}$$

is a coalgebra homomorphism, which is a surjection. In particular, the upper triangular matrix coalgebra $U^C(n,k)$ is isomorphic to a quotient of the triangular matrix coalgebra $M^C(n,k)$.

We have $M^C(n,k)_0 = M^C(n,k)$, but $U^C(n,k)_0$ is generated by the elements of the main diagonal. Hence, the only possible map from $\frac{M^C(n,k)}{M^C(n,k)_0} = 0$ to $\frac{U^C(n,k)}{U^C(n,k)_0}$ would be the zero map, while $q'\rho(e_{1,2}) = [f_{1,2}] \neq 0$.

With some restriction on the coalgebra, the comodules have even more structure.

Theorem 1.2.49. *Let C be a cosemisimple coalgebra and M a right (left) C-comodule. Then, M is injective and projective in the category $\mathcal{M}^C$ (resp. ${}^C\mathcal{M}$).*

Proof. See [DNR01, Theorem 3.1.5]. □

For instance, if $N \subseteq M$ is a subcomodule of M and $\iota : N \to M$ is the canonical inclusion, then, injectivity of N implies that there exists a comodule projection $\pi : M \to N$ such that $\pi\iota = \mathrm{id}_N$. Moreover, any subcomodule of M has a comodule complement in M.

Next, we state the *Fundamental Theorem of Coalgebras and Comodules*, which motivates the study of dualized theorems for finite dimensional algebras in a more general settings for coalgebras.

Theorem 1.2.50. *Let C be a coalgebra and M a (left) C-comodule. Let $X \subseteq C$ and $Y \subseteq M$ be finite subsets. Then, there exists a finite dimensional subcoalgebra $D \subseteq C$ containing X and there exists a finite dimensional subcomodule $N \subseteq M$ containing Y.*

Proof. The book [Swe69, Theorem 2.2.1] deals with the case for coalgebras. For a complete proof of this theorem, see for instance [DNR01, Theorem 1.4.7 and Theorem 2.1.7], [Mon93, Theorem 5.1.1.2] or [Rad11, Theorem 2.2.3 and Theorem 3.2.8]. □

Corollary 1.2.51. *Any simple coalgebra and any simple comodule are finite dimensional.*

Hence, every basic coalgebra (see Definition 1.2.18) over an algebraically closed field is pointed, since its simple subcoalgebras are finite dimensional by the corollary, and the only finite dimensional division algebras over an algebraically closed field k is k itself, e.g. [Coh03, Proposition 5.4.5].

1.2.5 Bicomodules

Definition 1.2.52. Let C and D be coalgebras. A *C-D-bicomodule* $M = (M, \mu_M, \nu_M)$ is a left C-comodule with structure map $\mu_M : M \to C \otimes M$ and a right D-comodule with structure map $\nu_M : M \to M \otimes D$ satisfying the following commutative diagram.

$$\begin{array}{ccc} M & \xrightarrow{\nu_M} & M \otimes D \\ {\scriptstyle \mu_M}\downarrow & & \downarrow{\scriptstyle \mu_M \otimes \mathrm{id}_D} \\ C \otimes M & \xrightarrow[\mathrm{id}_C \otimes \nu_M]{} & C \otimes M \otimes D \end{array} \tag{1.2.53}$$

If M is a C-C-bicomodule, we say that M is a C-bicomodule.

A subspace $N \subseteq M$ is a *subbicomodule* of M if N is a subcomodule of (M, μ_M) and a subcomodule of (M, ν_M).

Remark 1.2.54. Every coalgebra is a bicomodule over itself via comultiplication, and if $\rho : C \to D$ is a coalgebra homomorphism, then C is a D-bicomodule with structure maps $\mu = (\rho \otimes \mathrm{id}_C)\Delta_C$ and $\nu = (\mathrm{id}_C \otimes \rho)\Delta_C$ (see Examples 1.2.37). In this case we say that C is a *D-bicomodule via ρ*.

Definition 1.2.55. Let M and N be C-D-bicomodules. A *bicomodule homomorphism* from M to N is a linear map $\sigma : M \to N$ which is a comodule homomorphism of left C-comodules and a comodule homomorphism of right D-comodules.

Denote by ${}^C\mathcal{M}^D$ the category of all C-D-bicomodules and bicomodule homomorphism. We write $\mathrm{Hom}_{C\text{-}D}(M, N) := \mathrm{Hom}_{{}^C\mathcal{M}^D}(M, N)$ if M and N are C-D-bicomodules.

In analogy with [Rot09, Corollary 2.61], we have the following:

Lemma 1.2.56. *Let C and D be coalgebras. The category ${}^C\mathcal{M}^D$ is isomorphic to the category $\mathcal{M}^{C^{cop}\otimes D}$.*

Proof. Suppose M is a C-D-bicomodule with structure maps μ and ν. We prove that M is a right $C^{cop}\otimes D$-comodule with structure map $\nu' = (T\mu \otimes \mathrm{id}_D)\nu$, where $T : C \to C$ is the twist map, see Definition 1.2.1, and the comultiplication and counit of $C^{cop}\otimes D$ are given by:

$$\Delta' = (\mathrm{id}_C \otimes T \otimes \mathrm{id}_D)(T\Delta_C \otimes \Delta_D), \quad \varepsilon' = \varepsilon_C \otimes \varepsilon_D, \tag{1.2.57}$$

see (1.2.10). Observe that:

$$(\mathrm{id}\otimes T)(T\otimes \mathrm{id})(a\otimes b\otimes c) = (b\otimes c\otimes a) \implies (\mathrm{id}\otimes T)(T\otimes \mathrm{id})(\mathrm{id}\otimes \nu) = (\nu\otimes \mathrm{id})T \tag{1.2.58}$$

Thus

$$\begin{aligned}
(\mathrm{id}\otimes\Delta')\nu' &= \big(\mathrm{id}\otimes(\mathrm{id}_C\otimes T\otimes \mathrm{id}_D)(T\Delta_C\otimes\Delta_D)\big)(T\mu\otimes\mathrm{id}_D)\nu \\
&=\big(\mathrm{id}\otimes(\mathrm{id}_C\otimes T\otimes\mathrm{id}_D)\big)\big((\mathrm{id}\otimes T\Delta_C)T\mu\otimes\mathrm{id}_D\otimes\mathrm{id}_D\big)(\mathrm{id}\otimes\Delta_D)\nu && (1.2.59)\\
&=\big(\mathrm{id}\otimes(\mathrm{id}_C\otimes T\otimes\mathrm{id}_D)\big)\big((T\mu\otimes\mathrm{id}_C)T\mu\otimes\mathrm{id}_D\otimes\mathrm{id}_D\big)(\nu\otimes\mathrm{id}_D)\nu && (1.2.60)\\
&=\big((T\mu\otimes\mathrm{id}_D)\otimes\mathrm{id}_C\otimes\mathrm{id}_D\big)\big((\mathrm{id}\otimes T)(T\otimes\mathrm{id}_D)\otimes\mathrm{id}_D\big)\big((\mu\otimes\mathrm{id}_D)\nu\otimes\mathrm{id}_D\big)\nu && (1.2.61)\\
&=\big((T\mu\otimes\mathrm{id}_D)\otimes\mathrm{id}_C\otimes\mathrm{id}_D\big)\big((\mathrm{id}\otimes T)(T\otimes\mathrm{id}_D)\otimes\mathrm{id}_D\big)\big((\mathrm{id}_C\otimes\nu)\mu\otimes\mathrm{id}_D\big)\nu && (1.2.62)\\
&=\big((T\mu\otimes\mathrm{id}_D)\nu\otimes\mathrm{id}_C\otimes\mathrm{id}_D\big)(T\mu\otimes\mathrm{id}_D)\nu && (1.2.63)\\
&=(\nu'\otimes\mathrm{id}_C\otimes\mathrm{id}_D)\nu' = (\nu'\otimes\mathrm{id}')\nu',
\end{aligned}$$

where (1.2.59) and (1.2.61) are simple rearrangements (using associativity), (1.2.60) is the structure of right comodules (see (1.2.36). See also Example 1.2.37 for the right C^{cop}-comodule structure map), (1.2.62) is the structure of bicomodules (see (1.2.53)), and (1.2.63) follows from (1.2.58). Moreover,

$$(\mathrm{id}\otimes\varepsilon')\nu' = (\mathrm{id}\otimes\varepsilon_C\otimes\varepsilon_D)(T\mu\otimes\mathrm{id}_D)\nu = ((\mathrm{id}\otimes\varepsilon_C)T\mu\otimes\varepsilon_D)\nu \cong (\mathrm{id}\otimes\varepsilon_D)\nu \cong \mathrm{id}.$$

Hence M is a right $C^{cop}\otimes D$-comodule. If $\sigma : M \to N$ is any bicomodule homomorphism, then:

$$\begin{aligned}
(\sigma\otimes\mathrm{id}')\nu'_M &= (\sigma\otimes\mathrm{id}_C\otimes\mathrm{id}_D)(T\mu_M\otimes\mathrm{id}_D)\nu_M\\
&= ((\sigma\otimes\mathrm{id}_C)T\mu_M\otimes\mathrm{id}_D)\nu_M\\
&= (T\mu_N\sigma\otimes\mathrm{id}_D)\nu_M\\
&= (T\mu_N\otimes\mathrm{id}_D)\nu_N\sigma = \nu'_N\sigma,
\end{aligned}$$

shows that σ is a comodule homomorphism of right $C^{cop}\otimes D$-comodules. □

Combining the above Lemma and Proposition 1.2.44, we have

Corollary 1.2.64. *Let C and D be coalgebras. The category of C-D-bicomodules ${}^C\mathcal{M}^D$ is abelian.*

1.2.6 Cotensor coalgebra

Definition 1.2.65. Let C be a coalgebra, M be a right C-comodule and N be a left C-comodule. The *cotensor product* of M and N over C is given by:

$$M \,\square_C\, N \;:= \ker(\nu_M \otimes \mathrm{id}_C - \mathrm{id}_C \otimes \mu_N \;:\; M \otimes N \to M \otimes C \otimes N). \tag{1.2.66}$$

In the case $C = k$, $\square_k \; = \otimes$.

Definition 1.2.67. Let C be a coalgebra and $\{C_{(i)}\}_{i\in\mathbb{N}}$ be a family of vector subspaces of C such that:

$$C = \bigoplus_{i\in\mathbb{N}} C_{(i)}, \qquad \varepsilon_C(C_{(n)}) = 0 \text{ for } n \neq 0, \qquad \Delta_C(C_{(n)}) \subseteq \sum_{i=0}^{n} C_{(i)} \otimes C_{(n-i)}. \tag{1.2.68}$$

Then, C is a *graded coalgebra.*

Example 1.2.69. The *divided power coalgebra* X is the vector space with basis $\{x^n \mid n \in \mathbb{N}\}$ and comultiplication and counit given by:

$$\Delta(x^n) = \sum_{i=0}^{n} x^i \otimes x^{n-i}, \quad \varepsilon(x^n) = \delta_{0,n}. \tag{1.2.70}$$

X is a graded coalgebra.

Remark 1.2.71. The above definition is a special case of *group-graded coalgebras*, see [NT93].

Definition 1.2.72. Let C be a coalgebra and M be a C-bicomodule. The *cotensor coalgebra* of C and M is the graded coalgebra:

$$\mathrm{Cot}_C(M) \;:= \bigoplus_{i\in\mathbb{N}} M^{\,\square_i}, \tag{1.2.73}$$

defined inductively by $M^{\,\square_0} \;:= C$ and $M^{\,\square_i} \;:= M^{\,\square_{i-1}} \,\square_C\, M$, endowed with comultiplication and counit as follows: $\Delta|_C = \Delta_C$, $\varepsilon|_C = \varepsilon_C$ and for any element $m_n \otimes \cdots \otimes m_1 \in M^{\,\square_n}$, with $n \geqslant 1$,

$$\begin{aligned}\Delta(m_n \otimes \cdots \otimes m_1) = \mu_M(m_n) \otimes (m_{n-1} \otimes \cdots \otimes m_1) + (m_n \otimes \cdots \otimes m_2) \otimes \nu_M(m_1)\\ + \sum_{i=1}^{n-1}(m_n \otimes \cdots \otimes m_{i+1}) \otimes (m_i \otimes \cdots \otimes m_1),\end{aligned}$$

and $\varepsilon(m_n \otimes \cdots \otimes m_1) = 0$.

Observe that $I = \bigoplus_{i\geqslant 1} M^{\,\square_i}$ is a coideal of $\mathrm{Cot}_C(M)$ such that the quotient coalgebra $\dfrac{\mathrm{Cot}_C(M)}{I} \cong C$. Hence, the canonical projection $\pi_0 \;:\; \mathrm{Cot}_C(M) \to C$ is a coalgebra homomorphism and, consequently, $\mathrm{Cot}_C(M)$ is a C-bicomodule. One can readily check that the canonical projection $\pi_1 \;:\; \mathrm{Cot}_C(M) \to M$ is a C-bicomodule homomorphism. The next theorem is the *Universal Property of Cotensor Coalgebras*:

Theorem 1.2.74. *Let C and D be coalgebras, and M be a C-bicomodule. Given a coalgebra* *homomorphism $\rho_0 : D \to C$, and a homomorphism of C-bicomodules $\rho_1 : D \to M$ such that $\rho_1(D_0) = 0$, then there exists a unique coalgebra homomorphism $\rho : D \to \mathrm{Cot}_C(M)$ satisfying the following commutative diagrams:*

$$\begin{array}{ccc} & & \mathrm{Cot}_C(M) \\ & \overset{\rho}{\nearrow} & \downarrow \pi_0 \\ D & \xrightarrow[\rho_0]{} & C \end{array} \qquad \begin{array}{ccc} & & \mathrm{Cot}_C(M) \\ & \overset{\rho}{\nearrow} & \downarrow \pi_1 \\ D & \xrightarrow[\rho_1]{} & M \end{array} \tag{1.2.75}$$

where π_0 and π_1 are the canonical projections of $\mathrm{Cot}_C(M)$.

Proof. See [Nic78, Proposition 1.4.2]. □

Remark 1.2.76. For any coalgebra homomorphism $\rho : D \to \mathrm{Cot}_C(M)$, the universal property of cotensor coalgebras gives $\rho = \pi_0\rho + \sum_{i\geqslant 1}(\pi_1\rho)^{\otimes_i}\Delta^{i-1}$.

Definition 1.2.77. A graded coalgebra $C = \bigoplus_{i\in\mathbb{N}} C_{(i)}$ is *coradically graded* if $C_n = \bigoplus_{i\leqslant n} C_{(i)}$, see [Abe80, §2.4.1] and [CM96, Lemma 2.2].

Proposition 1.2.78. *Let C be a cosemisimple coalgebra and M be a C-bicomodule, then* $\mathrm{Cot}_C(M) = \bigoplus_{i=0}^{\infty} M^{\square i}$ *is coradically graded.*

Proof. See [Woo97, Lemma 4.4]. □

The divided power coalgebra X (see Example 1.2.69) is the cotensor coalgebra of $C = \langle x^0 \rangle_k$ and $M = \langle x^1 \rangle_k$, which is coradically graded.

1.3 Pointed coalgebras

The importance of studying finite dimensional basic algebras is due to the Morita Theory, which concludes that for any finite dimensional algebra A, there exists a unique, up to isomorphism, finite dimensional basic algebra B such that ${}_A\mathcal{M} \equiv {}_B\mathcal{M}$, see e.g. [Ben91, §2.2].

Definition 1.3.1. Let A and B be algebras. We say that A is *Morita equivalent* to B if their categories of modules are equivalent.

Takeuchi has dualized these results for coalgebras and Chin and Montgomery proved that, for any coalgebra C, there exists a basic coalgebra D associated to C such that ${}^C\mathcal{M} \equiv {}^D\mathcal{M}$, see [CM97, §1 and §2].

Definition 1.3.2. Let C and D be coalgebras. We say that C is *Morita-Takeuchi equivalent* to D in case that their categories of comodules are equivalent.

Since every pointed coalgebra is basic and every basic coalgebra over an algebraically closed field is pointed, pointed coalgebras form an important class of coalgebras and it is the subject of this section.

Recall that a coalgebra is pointed if every simple subcoalgebra is one dimensional (see Definition 1.2.1). Denote by **PCog** the full subcategory of **Cog** consisting of all pointed coalgebras. It is important to note that any coalgebra $C \in \mathbf{PCog}$ has separable coradical (cf. Definition 1.2.18) and any coalgebra homomorphism $\rho \in \mathrm{Hom}_{\mathbf{PCog}}(C, D)$ is filtered (cf. Theorem 1.2.30).

1.3.1 Group-like elements

Definition 1.3.3. Let C be a coalgebra. An element $g \in C$ is a *group-like element* if satisfies:

$$\Delta_C(g) = g \otimes g, \quad \varepsilon_C(g) = 1. \tag{1.3.4}$$

Denote by $\mathrm{G}(C) := \{g \in C \,|\, \Delta(g) = g \otimes g,\ \varepsilon(g) = 1\}$ the set of all group-like elements of C.

For any set S, the *group-like coalgebra* on S, denoted by kS, is the vector space with basis S and maps given by:

$$\Delta(s) = s \otimes s, \qquad \varepsilon(s) = 1,$$

extended linearly for all $s \in S$. The coalgebra $k\mathrm{G}(C)$ is the *group-like subcoalgebra* on C.

The condition $\varepsilon(g) = 1$ for a group-like element is equivalent to $g \neq 0$, by the counit property (see 1.2.2). The group-like elements play an important role for pointed coalgebras as we will see in the sequence.

Proposition 1.3.5. *Let C be a coalgebra. The elements of $\mathrm{G}(C)$ are linearly independent in C.*

Proof. See [Swe69, Proposition 3.2.1]. □

Proposition 1.3.6. *Let C be a coalgebra and $D \subseteq C$ be a subcoalgebra. If $\dim D = 1$, then $D = k\{g\}$ for some $g \in \mathrm{G}(C)$.*

Proof. See [Swe69, Lemma 8.0.1]. □

Consequently,

Remark 1.3.7. A coalgebra C is pointed if and only if $C_0 = kG(C)$.

Lemma 1.3.8. *Let C and D be coalgebras and $\rho : C \to D$ be a coalgebra homomorphism. Then $\rho\big(\mathrm{G}(C)\big) \subseteq \mathrm{G}(D)$.*

Proof. Consider $g \in \mathrm{G}(C)$. Then,

$$\Delta_D\big(\rho(g)\big) = (\rho \otimes \rho)\Delta_C(g) = \rho(g) \otimes \rho(g), \qquad \varepsilon_D\big(\rho(g)\big) = \varepsilon_C(g) = 1 \tag{1.3.9}$$

(see 1.2.21). Thus $\rho(g) \in \mathrm{G}(D)$. □

1.3.2 Primitive elements

Definition 1.3.10. Let C be a coalgebra and $g, h \in \mathrm{G}(C)$ be group-like elements. An element $p \in C$ is a g, h-*primitive element* if satisfies

$$\Delta_C(p) = p \otimes g + h \otimes p$$

Denote by $\mathrm{P}_{g,h}(C) := \{p \in C \,|\, \Delta(p) = p \otimes g + h \otimes p\}$ the set of all g, h-primitive elements.

A g, h-primitive element that does not belong to the group-like subcoalgebra $k\mathrm{G}(C)$ on C is called *skew primitive*.

Note that if p is a g, h-primitive element of C, then $\varepsilon_C(p) = 0$, by the counit property (see 1.2.2).

Observe that if g and h are group-like elements, then $(h - g)$ is a g, h-primitive element.

Lemma 1.3.11. *Let C be a coalgebra and $g, h, g', h' \in \mathrm{G}(C)$ be group-like elements. Then,*

$$k\mathrm{G}(C) \cap \mathrm{P}_{g,h}(C) = k\{h - g\} \tag{1.3.12}$$

and

$$\mathrm{P}_{g,h}(C) \cap \mathrm{P}_{g',h'}(C) = \begin{cases} \mathrm{P}_{g,h}(C) & \text{if } g' = g,\ h' = h \\ k\{h - g\} & \text{if } g' = h,\ h' = g \\ \{0\} & \text{otherwise} \end{cases} \tag{1.3.13}$$

Proof. Consider $p \in k\mathrm{G}(C) \cap \mathrm{P}_{g,h}(C)$ and write $p = \sum_{e \in \mathrm{G}(C)} \lambda_e e$, for some $\lambda_e \in k$. Then

$$\Big(\sum_{e \in \mathrm{G}(C)} \lambda_e e\Big) \otimes g + h \otimes \Big(\sum_{e \in \mathrm{G}(C)} \lambda_e e\Big) = \Delta\Big(\sum_{e \in \mathrm{G}(C)} \lambda_e e\Big) = \sum_{e \in \mathrm{G}(C)} \lambda_e e \otimes e,$$

implies

$$\sum_{e \in \mathrm{G}(C) \setminus \{g,h\}} \lambda_e (e \otimes g + h \otimes e - e \otimes e) + (\lambda_h + \lambda_g) h \otimes g = 0.$$

Since $\{e \otimes f\}_{e,f \in \mathrm{G}(C)}$ is a linearly independent set in $C \otimes C$ (see Proposition 1.3.5), we must have $\lambda_e = 0$ for $e \in \mathrm{G}(C) \setminus \{g, h\}$, and $\lambda_h = -\lambda_g$. This proves (1.3.12).

Consequently,

$$\mathrm{P}_{g,h}(C) \cap \mathrm{P}_{h,g}(C) \cap k\mathrm{G}(C) = k\{h - g\}, \qquad \mathrm{P}_{g,h}(C) \cap \mathrm{P}_{g',h'}(C) \cap k\mathrm{G}(C) = 0,$$

for any $(g', h') \notin \{(g, h), (h, g)\}$. We claim that if $p \in \mathrm{P}_{g,h}(C)$ is a skew primitive, then $p \notin \mathrm{P}_{g',h'}$ for any $g', h' \in \mathrm{G}(C)$ with $g' \neq g$ or $h' \neq h$.

Suppose not, i.e. let $p \in \mathrm{P}_{g,h}(C)$ be a skew primitive and suppose that $p \in \mathrm{P}_{g',h'}$ for some $g \neq g' \in \mathrm{G}(C)$ (the case $h' \neq h$ is analogous). Then

$$0 = \Delta(p - p) = p \otimes g + h \otimes p - p \otimes g' - h' \otimes p = p \otimes (g - g') + (h - h') \otimes p$$

is a contradiction, since the set $\{(g - g'), p\}$ is linearly independent in C. This completes the proof. □

Remark 1.3.14. Note that the linear maps

$$\begin{array}{rcl} \mu : \mathrm{P}_{g,h}(C) & \longrightarrow & \mathrm{G}(C) \otimes \mathrm{P}_{g,h}(C) \\ p & \longmapsto & h \otimes p \end{array} \qquad \begin{array}{rcl} \nu : \mathrm{P}_{g,h}(C) & \longrightarrow & \mathrm{P}_{g,h}(C) \otimes \mathrm{G}(C) \\ p & \longmapsto & p \otimes g \end{array} \tag{1.3.15}$$

make $\mathrm{P}_{g,h}(C)$ into a $k\mathrm{G}(C)$-bicomodule and $k(h-g)$ is a subbicomodule. Since $k\mathrm{G}(C)$ is a cosemisimple coalgebra, there exists a comodule complement for $k\{h-g\}$ in $\mathrm{P}_{g,h}(C)$, i.e. $\mathrm{P}_{g,h}(C) = k\{h-g\} \oplus \mathrm{P}'_{g,h}(C)$ for some subbicomodule $\mathrm{P}'_{g,h}(C)$ (see Theorem 1.2.49 and comment thereafter). Furthermore, the structure maps of $\mathrm{P}_{g,h}(C)$ are compatible with the structure maps of $k\mathrm{G}(C)$ (which are given by comultiplication) when restricted to $k\{h-g\}$.

Denote by:

$$\bar{\mathrm{P}}_{g,h}(C) := \frac{\mathrm{P}_{g,h}(C)}{k\{h-g\}} \tag{1.3.16}$$

the unique $k\mathrm{G}(C)$-bicomodule such that the canonical projection $q : \mathrm{P}_{g,h}(C) \to \dfrac{\mathrm{P}_{g,h}(C)}{k(h-g)}$ is a bicomodule homomorphism (see the Fundamental Isomorphism Theorem for Comodules, Theorem 1.2.42), and write its elements as $\overline{p} = p + k(h-g)$. It is immediately that $\bar{\mathrm{P}}_{g,h}(C) \cong \mathrm{P}'_{g,h}(C)$, for any such complement.

Lemma 1.3.17. *Let C and D be coalgebras and $\rho : C \to D$ be a coalgebra homomorphism. Then $\rho|_{\mathrm{P}_{g,h}(C)} : \mathrm{P}_{g,h}(C) \to \mathrm{P}_{\rho(g),\rho(h)}(D)$ is a homomorphism of $k\mathrm{G}(D)$-bicomodules for all $g, h \in \mathrm{G}(C)$. Moreover, if ρ is an injection and p is a skew primitive, then $\rho(p)$ is a skew primitive.*

Proof. Consider $g, h \in \mathrm{G}(C)$ and $p \in \mathrm{P}_{g,h}(C)$. Then,

$$\Delta_D\big(\rho(p)\big) = (\rho \otimes \rho)\Delta_C(p) = \rho(p) \otimes \rho(g) + \rho(h) \otimes \rho(p) \tag{1.3.18}$$

implies $\rho(p) \in \mathrm{P}_{\rho(g),\rho(h)}(D)$, since $\rho(g), \rho(h) \in \mathrm{G}(D)$ by Lemma 1.3.8. Hence, the map $\rho|_{\mathrm{P}_{g,h}(C)}$ is well defined. Observe that $\mathrm{P}_{g,h}(C)$ is a $k\mathrm{G}(D)$-bicomodule via ρ (see Example 1.2.37). Thus

$$(\mathrm{id}_D \otimes \rho)\mu(p) = (\mathrm{id}_D \otimes \rho)(\rho \otimes \mathrm{id}_{\mathrm{P}_{g,h}(C)})\mu_{\mathrm{P}_{g,h}(C)}(p) = \rho(h) \otimes \rho(p) = \mu_{\mathrm{P}_{\rho(g),\rho(h)}(D)}\rho(p) \tag{1.3.19}$$

(and the analogous equation for the right D-comodules) show that $\rho|_{\mathrm{P}_{g,h}(C)}$ is a bicomodule homomorphism.

Now, consider ρ an injection and p a skew primitive. If $\rho(p) \in k\big(\rho(h) - \rho(g)\big)$, then $\rho(p) = \lambda\big(\rho(h) - \rho(g)\big)$ for some $\lambda \in k$, and

$$\rho\big(\lambda(h-g)\big) = \lambda\big(\rho(h) - \rho(g)\big) = \rho(p),$$

which contradicts the injectivity of ρ. Therefore, the image of skew primitive elements by injective coalgebra homomorphisms are skew primitive. □

1.3.3 The structure of pointed coalgebras

The next results describe some structure of pointed coalgebras based on their coradical filtration. The description of C_1 is known as *Taft-Wilson Theorem.*

Proposition 1.3.20. *Let C be a pointed coalgebra. Then*

1.
$$C_1 = C_0 \oplus \Big(\bigoplus_{g,h \in \mathrm{G}(C)} \mathrm{P}'_{g,h}(C) \Big), \tag{1.3.21}$$
where $\mathrm{P}'_{g,h}(C)$ is any comodule complement of $k\{h-g\}$ in $\mathrm{P}_{g,h}(C)$;

2. *for any $n \geqslant 1$ and $c \in C_n$,*
$$c = \sum_{g,h \in \mathrm{G}(C)} c_{g,h}, \text{ where } \Delta(c_{g,h}) = c_{g,h} \otimes g + h \otimes c_{g,h} + \omega_{g,h} \tag{1.3.22}$$
for some $\omega_{g,h} \in C_{n-1} \otimes C_{n-1}$.

Proof. See [TW74, Proposition 1 and Proposition 2]. See also [Mon93, Theorem 5.4.1]. □

Note that any coalgebra C with separable coradical is a C_0-bicomodule with structure maps given via a coalgebra projection $\pi_0 : C \to C_0$ (see Theorem 1.2.29 and Remark 1.2.54) and, consequently, the subcoalgebra C_1 is a subbicomodule. In the case that C is pointed, the subcoalgebra C_1, in view of Proposition 1.3.20, has the structure maps given by:

$$\mu = \Delta_{C_0} + \sum_{g,h \in \mathrm{G}(C)} \mu_{\mathrm{P}_{g,h}(C)}, \qquad \nu = \Delta_{C_0} + \sum_{g,h \in \mathrm{G}(C)} \nu_{\mathrm{P}_{g,h}(C)} \tag{1.3.23}$$

where $\mu_{\mathrm{P}_{g,h}(C)}$ and $\nu_{\mathrm{P}_{g,h}(C)}$ are the structure maps of the complements $\mathrm{P}'_{g,h}(C)$ (see Remark 1.3.14 and also Example 1.2.37 for the direct sum of (bi)comodules). Moreover,

$$\frac{C_1}{C_0} = \sum_{g,h \in \mathrm{G}(C)} \frac{\mathrm{P}_{g,h}(C) + C_0}{C_0} \cong \sum_{g,h \in \mathrm{G}(C)} \frac{\mathrm{P}_{g,h}(C)}{C_0 \cap \mathrm{P}_{g,h}(C)} = \bigoplus_{g,h \in \mathrm{G}(C)} \bar{\mathrm{P}}_{g,h}(C).$$

(see Lemma 1.2.45 for the "second isomorphism theorem" and Lemma 1.3.11 for the last term being a direct sum). Thus, the quotient $\dfrac{C_1}{C_0}$ has canonical structure maps of C_0-bicomodules given by the quotient sets of primitive elements $\bar{\mathrm{P}}_{g,h}$.

1.4 Pseudocompact algebras and modules

In Section 1.2.1, we have seen that each coalgebra C corresponds to an algebra, the dual algebra C^* of C. In this section we present the category of pseudocompact algebras and show that the dual algebra of a coalgebra is always a pseudocompact algebra. Moreover, each pseudocompact algebra corresponds to a coalgebra and the functor $(-)^* : \mathbf{Cog} \to \mathbf{Alg}$ is a duality of categories, where $\mathbf{Alg}$ is the category of all pseudocompact algebras and continuous algebra homomorphisms.

Remark 1.4.1. Denote by **ALG** the category of all algebras and algebra homomorphisms. Given any algebra $A \in \mathbf{ALG}$, the *finite dual coalgebra* of A has the vector space:

$$A^{\circ} = \{f \in A^* \mid \ker f \text{ contains a cofinite ideal of } A\}, \tag{1.4.2}$$

see [Swe69, Proposition 6.0.2]. Furthermore, the functors $(-)^{\circ} : \mathbf{ALG} \to \mathbf{Cog}$ and $(-)^* : \mathbf{Cog} \to \mathbf{ALG}$ are adjoint on the right, i.e. $\mathrm{Hom}_{\mathbf{ALG}}(A, C^*) \cong \mathrm{Hom}_{\mathbf{Cog}}(C, A^{\circ})$ (see Definition 1.1.20), see [Swe69, Theorem 6.0.5]. Although this level of generality is very interesting and has many applications, we will rather use the duality between coalgebras and pseudocompact algebras in order to avoid some problems with this adjunction as, for instance, the finite dual coalgebra of any infinite dimensional simple algebra is zero, see [DNR01, Remark 1.5.7] or the Remark right after [Swe69, Proposition 6.0.2].

When dealing with pseudocompact algebras, we treat the field k as a discrete topological ring. For a general introduction to topological rings, topological vector spaces, topological algebras and topological modules, see Appendix A.

1.4.1 Basic definitions

Definition 1.4.3. A *pseudocompact algebra* A is the inverse limit of finite dimensional algebras $\{A_i\}_{i\in I}$, treated as discrete topological algebras, $A = \varprojlim_{i\in I} A_i$.

Denote by **Alg** the category of all pseudocompact algebras and continuous algebra homomorphisms.

For more information about inverse (and direct) limits see, for instance, [RZ10, §1].

The inverse limit inherits a topology, which is complete (like any inverse limit of topological sets) and Hausdorff (since it is the inverse limit of discrete topological sets).

Remark 1.4.4. More often, a pseudocompact algebra is presented by the equivalent definition of a complete Hausdorff topological algebra possessing a fundamental system of neighborhoods of 0 consisting of (two sided) ideals with finite codimension that intersect in 0, see, for instance, [Bru66].

In particular, every finite dimensional algebra is a pseudocompact algebra and the category of all finite dimensional algebras and algebra homomorphisms, **alg**, is a full subcategory of **Alg**.

Definition 1.4.5. Let A be a pseudocompact algebra. A (left) *pseudocompact A-module* is the inverse limit of discrete finite dimensional (left) A-modules $\{U_i\}_{i\in I}$, $U = \varprojlim_{i\in I} U_i$.

Let A and B be pseudocompact algebras. A *pseudocompact A-B-bimodule* is an A-B-bimodule which is a left pseudocompact A-module and a right pseudocompact B-module.

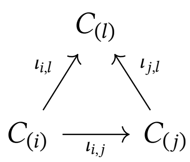
Denote by ${}_A\mathcal{M}$ the category of all left pseudocompact A-modules and continuous module homomorphisms, by $\mathcal{M}_B$ the category of all right pseudocompact B-modules and continuous module homomorphisms and by ${}_A\mathcal{M}_B$ the category of all pseudocompact A-B-bimodules and continuous bimodule homomorphisms. Denote by ${}_A\mathcal{M}^{\mathrm{f}}$ and by $\mathcal{M}_B^{\mathrm{f}}$ the full subcategories of finite dimensional modules.

Definition 1.4.6. A (left) pseudocompact A-module is *simple* if there is no nonzero proper closed submodule. A pseudocompact algebra A is *simple* if it is simple as a pseudocompact A-bimodule, or, equivalently, it has no nonzero proper (two sided) closed ideal.

It follows that every simple pseudocompact A-module (and simple pseucompact algebra) is finite dimensional.

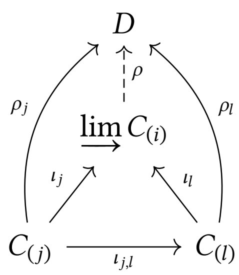
1.4.2 Duality theorems

Since every finitely generated coalgebra is finite dimensional by the Fundamental Theorem of Coalgebras, Theorem 1.2.50, any coalgebra can be realized as the direct limit of its finite dimensional subcoalgebras, as follows: let C be a coalgebra and consider the family $\{C_{(i)}\}_{i\in P}$ of all finite dimensional subcoalgebras of C indexed by the directed poset P with partial order given by $i \leqslant j$ whenever $C_{(i)} \subseteq C_{(j)}$ (P is indeed directed since, for any $i, j \in P$, we take $l \in P$ such that $C_{(l)} = C_{(i)} + C_{(j)}$), which, together with the canonical inclusions $\iota_{i,j} : C_{(i)} \to C_{(j)}$ for $i, j \in P$ with $i \leqslant j$, form a direct system.



(1.4.7)

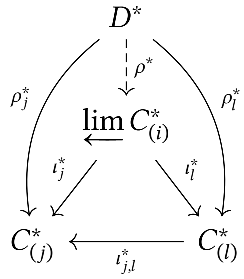
Then, it admits a direct limit, i.e. a coalgebra $\varinjlim C_{(i)}$ and coalgebra maps $\iota_i : C_{(i)} \to \varinjlim C_{(i)}$ such that $\iota_i = \iota_j \iota_{i,j}$ for $i \leqslant j$ and, whenever D is another coalgebra with coalgebra homomorphisms $\rho_i : C_{(i)} \to D$ satisfying $\rho_i = \rho_j \iota_{i,j}$ then there exist a unique coalgebra homomorphism $\rho : \varinjlim C_{(i)} \to D$ satisfying the commutative diagram:



(1.4.8)

Observe that the coalgebra C satisfies the universal property above, since we can define $\rho : C \to D$ as $\rho(c) = \rho_i(c)$ whenever $c \in C_{(i)}$ (such $C_{(i)}$ always exists since the subcoalgebra generated by c is finite dimensional), and the direct system (1.4.7) guarantees that it is well defined. Thus $C \cong \varinjlim C_{(i)}$.

The dual algebra of a finite dimensional coalgebra is finite dimensional and the map $\iota_{i,j}^* : C_{(j)}^* \to C_{(i)}^*$, given by $\iota_{i,j}^*(f)(c) = f\big(\iota_{i,j}(c)\big) = f|_{C_{(i)}}$ is the canonical projection. Hence, applying $(-)^*$ to the direct limit (1.4.8) we obtain an inverse limit



(1.4.9)

Thus

$$C^* \cong \left(\varinjlim C_{(i)}\right)^* \cong \varprojlim C_{(i)}^*$$

is a pseudocompact algebra with an open base of neighborhoods of zero given by the kernels of ι_i^*, which are ideals with finite codimension.

Example 1.4.10. Let X be the divided power coalgebra, see Example 1.2.69. It is clear that each term X_i of the coradical filtration of X is a finite dimensional subcoalgebra and $X = \varinjlim X_i$. The dual algebra of X_i, X_i^*, is isomorphic to the quotient algebra $\dfrac{k[x]}{\langle x^{i+1}\rangle}$ of the polynomial algebra $k[x]$ over the ideal generated by x^{i+1}. Hence, the dual algebra X^* is the inverse limit $\varprojlim \dfrac{k[x]}{\langle x^{i+1}\rangle}$, which is precisely the *power series algebra* $k[[x]]$.

On the other hand, if we start with a pseudocompact algebra and apply $(-)^*$, we obtain a coalgebra. We must first define the dual coalgebra of a finite dimensional algebra.

For any finite dimensional vector space V, the linear injection

$$\theta : V^* \otimes V^* \to (V \otimes V)^*, \tag{1.4.11}$$

given by $\theta(f \times g)(u \times v) = f(u)g(v)$ is actually an isomorphism.

Hence, given any finite dimensional algebra A, its dual space $A^* = \mathrm{Hom}_k(A, k)$ inherits a coalgebra structure with comultiplication $\Delta = \theta^{-1}\mathbf{m}^* : A^* \to A^* \otimes A^*$ and counit $\varepsilon = \eta^* : A^* \to k$, where $\mathbf{m} : A \otimes A \to A$ and $\eta : k \to A$ is the multiplication and unit of A, respectively.

Explicitly, for any $f \in A^*$ and $a, b \in A$,

$$\theta\Delta(f)(a \otimes b) = f\big(\mathbf{m}(a \otimes b)\big), \qquad \varepsilon(f) = f\big(\eta(1)\big), \tag{1.4.12}$$

i.e. writing $\Delta(f) = \sum_{(f)} f_1 \otimes f_2$ means that $f(ab) = \sum_{(f)} f_1(a) f_2(b)$.

It remains to check that applying $(-)^*$ to an inverse limit of finite dimensional algebras, we obtain a direct limit of finite dimensional coalgebras and we are done. Moreover,

Theorem 1.4.13. *The contravariant functor* $(-)^* : \mathbf{Cog} \to \mathbf{Alg}$ *is a duality of categories.*

Proof. See [Sim01, Theorem 3.6]. □

The inverse of the duality $(-)^* : \mathbf{Cog} \to \mathbf{Alg}$ is the contravariant functor $(-)^* : \mathbf{Alg} \to \mathbf{Cog}$ considering $A^* = \mathrm{Hom}_k(A, k)$ the set of all continuous functionals from A to k (it is fundamental that we consider only the continuous functionals, otherwise A^* might not be a coalgebra).

Corollary 1.4.14. *The duality $(-)^* : \mathbf{Cog} \to \mathbf{Alg}$ restricts to a duality $(-)^* : \mathbf{cog} \to \mathbf{alg}$.*

Proof. See [DNR01, Proposition 1.3.14]. □

For any subspace $S \subseteq V$ of V, denote by $S^\perp = \{f \in V^* \mid f(x) = 0, \forall x \in S\}$. Theorem 1.4.13 and Remark 1.1.18 imply the following correspondence:

Corollary 1.4.15. *Consider a coalgebra C. Then:*

$$D \subseteq C \text{ is a subcoalgebra of } C \iff D^\perp \text{ is a closed ideal of } C^*; \tag{1.4.16}$$
$$I \subseteq C \text{ is a coideal of } C \iff I^\perp \text{ is a subalgebra of } C^*. \tag{1.4.17}$$

Remark 1.4.18. Thus, C is a simple coalgebra if, and only if, C^* is a simple pseudocompact algebra. In particular, the matrix coalgebra $M^C(n, k)$ is simple (see Example 1.2.3 and Example 1.2.14).

The categories of comodules and pseudocompact modules are also dual.

Theorem 1.4.19. *The contravariant functor $(-)^* : {}^C\mathcal{M} \to {}_{C^*}\mathcal{M}$ is a duality of categories.*

Proof. See [Sim01, Theorem 4.3]. □

Corollary 1.4.20. *The contravariant functor $(-)^* : {}^C\mathcal{M}^D \to {}_{C^*}\mathcal{M}_{D^*}$ is a duality of categories.*

1.4.3 Jacobson radical

From [Bru66, §1, p. 444]:

Definition 1.4.21. The *Jacobson radical* $J(A)$ of a pseudocompact algebra A is the intersection of all maximal closed ideals of A, see [IM22, Proposition 3.2] for alternative characterizations of $J(A)$.

The *radical* of a pseudocompact A-module M, denoted by $\mathrm{Rad}(M)$, is the intersection of the maximal closed A-submodules of M. For $n \geqslant 1$, define $J^{n+1}(A) = \mathrm{Rad}\big(J^n(A)\big)$.

The Jacobson radical and the coradical are connected as follows:

Proposition 1.4.22. *Let C be a coalgebra. Then:*

$$C_n = \big(J^{n+1}(C^*)\big)^\perp; \tag{1.4.23}$$
$$C_n^\perp = J^{n+1}(C^*). \tag{1.4.24}$$

Proof. The equation (1.4.23) is proved on [Abe80, Corollary 2.3.10] and the equation (1.4.24) follows by applying the first equation and observing that $V^{\perp\perp} = V$, see [Swe69, Appendix I A1], and $V^{*\perp\perp} = \overline{V^*}$, see [Abe80, Theorem 2.2.3], for any vector space V. □

Remark 1.4.25. The above Proposition implies that $C_n^* \cong \dfrac{C^*}{J^{n+1}(C^*)}$.

Definition 1.4.26. A pseudocompact algebra A is:

1. *topologically semisimple* if $J(A) = 0$, or, equivalently, A is the product of square matrix algebras over finite dimensional division algebras, see [IM22, Proposition 3.7] for more characterizations;
2. *basic* if $\dfrac{A}{J(A)}$ is isomorphic to a product of division algebras;
3. *pointed* if $\dfrac{A}{J(A)}$ is isomorphic to a product of copies of k.

Hence, we have the following correspondence:

Lemma 1.4.27. *Let C be a coalgebra. Then:*

$$C \text{ is cosemisimple} \iff C^* \text{ is topologically semisimple;} \tag{1.4.28}$$
$$C \text{ is basic} \iff C^* \text{ is basic;} \tag{1.4.29}$$
$$C \text{ is pointed} \iff C^* \text{ is pointed.} \tag{1.4.30}$$

Proof. Follows from Remark 1.4.25. □

Denote by **PAlg** the full subcategory of **Alg** consisting of all pointed pseudocompact algebras. Thus, Theorem 1.4.13 restricts to a duality $(-)^* : \mathbf{PCog} \to \mathbf{PAlg}$.

Lemma 1.4.31. *Let A and B be pseudocompact algebras and $\alpha : A \to B$ be a continuous algebra homomorphism. If B is basic, then $\alpha\big(J^{n+1}(A)\big) \subseteq J^{n+1}(B)$, for $n \in \mathbb{N}$.*

Proof. Lemma 1.4.27 implies that B^* is a basic coalgebra. Lemma 1.2.31 implies that the coalgebra homomorphism $\alpha^* : B^* \to A^*$ is filtered, i.e. $\alpha^*(B_n^*) \subseteq A_n^*$. By Proposition 1.4.22, $B_n^* = \big(J^{n+1}(B)\big)^\perp$ and $A_n^* = \big(J^{n+1}(A)\big)^\perp$. Thus, for any $f \in \big(J^{n+1}(B)\big)^\perp$, $\alpha^* f \in \big(J^{n+1}(A)\big)^\perp$. Hence, $\alpha^* f(a) = f\big(\alpha(a)\big) = 0$, for every $a \in J^{n+1}(A)$, which implies $\alpha(a) \in J^{n+1}(B)$. □

Definition 1.4.32. A pseudocompact algebra A is a *separable algebra* if A is an inverse limit of separable finite dimensional algebras, see [IM22, Theorem 4.3] for other characterizations.

Thus, a coalgebra C has separable coradical if, and only if, $\dfrac{C^*}{J(C^*)}$ is a separable algebra.

Now, we can state the Wedderburn-Malcev Theorem for pseudocompact algebras.

Theorem 1.4.33. *Let A be a pseudocompact algebra such that $\dfrac{A}{J(A)}$ is a separable algebra. There exists a semisimple subalgebra $S \subseteq A$ such that $A = S \oplus J(A)$. Moreover, if $S' \subseteq A$ is another semisimple subalgebra of A with $A = S' \oplus J(A)$, then there exists an element $\omega \in J(A)$ which satisfies:*

$$S = (1 - \omega)S'(1 - \omega)^{-1}. \tag{1.4.34}$$

Proof. See [IM20, Proposition 2.8]. See also [IM22, Theorem 4.6 and Theorem 4.7]. □

1.4.4 Complete tensor algebra

From [Bru66, §2]:

Definition 1.4.35. Let A be a pseudocompact algebra, U be a right pseudocompact A-module and V be a left pseudocompact A-module. The *complete tensor product* of U and V is a pseudocompact k-module $U \mathbin{\widehat{\otimes}_A} V$ and a A-bihomomorphism $\alpha : U \times V \to U \mathbin{\widehat{\otimes}_A} V$ such that any A-bihomomorphism $f : U \times V \to W$, for some pseudocompact k-module W, factors uniquely by α as in the following commutative diagram:

$$\begin{array}{ccc} U \times V & \xrightarrow{\alpha} & U \mathbin{\widehat{\otimes}_A} V \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \bar{f}} \\ & & W \end{array} \tag{1.4.36}$$

Remark 1.4.37. The complete tensor product is constructed as:

$$U \mathbin{\widehat{\otimes}_A} V = \varprojlim \frac{U}{X} \otimes_A \frac{V}{Y}, \tag{1.4.38}$$

where X and Y runs through the open submodules of U and V, respectively. In this way, $U \mathbin{\widehat{\otimes}_A} V$ is the completion of $U \otimes_A V$ in the topology induced by taking $\varprojlim (U \otimes_A Y + X \otimes_A V)$ as a fundamental system of neighborhoods of 0, see [Bru66, §2].

Example 1.4.39. Consider the power series algebras $A = k[[x]]$ and $B = k[[y]]$. Then, the complete tensor product $A \mathbin{\widehat{\otimes}_k} B$ is the power series algebra in two indeterminates $k[[x, y]]$.

Lemma 1.4.40. *Let C be a coalgebra, M be a right C-comodule and N be a left C-comodule. Then*

$$M \,\square_C\, N \cong (M^* \mathbin{\widehat{\otimes}_{C^*}} N^*)^* \tag{1.4.41}$$

Proof. The required isomorphism is given by

$$x \otimes y \mapsto \big(f \otimes g \mapsto f(x)g(y)\big) \tag{1.4.42}$$

For more details, see Appendix B. □

Definition 1.4.43. Let A be a pseudocompact algebra and U be a pseudocompact A-bimodule. The *complete tensor algebra* $\mathrm{T}[[A,U]]$ is defined to be

$$\mathrm{T}[[A,U]] = \prod_{n=0}^{\infty} U^{\widehat{\otimes}_n}, \tag{1.4.44}$$

where $U^{\widehat{\otimes}_0} = A, U^{\widehat{\otimes}_1} = U$ and $U^{\widehat{\otimes}_n} = \underbrace{U \mathbin{\widehat{\otimes}_A} U \mathbin{\widehat{\otimes}_A} \dots \mathbin{\widehat{\otimes}_A} U}_{n-\text{times}}$ for $n > 1$, see [Gab73, §7.5] for details.

Multiplication is given in the obvious way: the product of the pure tensors $u_1 \mathbin{\widehat{\otimes}_A} \dots \mathbin{\widehat{\otimes}_A} u_m \in U^{\widehat{\otimes}_m}$ and $v_1 \mathbin{\widehat{\otimes}_A} \dots \mathbin{\widehat{\otimes}_A} v_n \in U^{\widehat{\otimes}_n}$ is

$$u_1 \mathbin{\widehat{\otimes}_A} \dots \mathbin{\widehat{\otimes}_A} u_m \mathbin{\widehat{\otimes}_A} v_1 \mathbin{\widehat{\otimes}_A} \dots \mathbin{\widehat{\otimes}_A} v_n \in U^{\widehat{\otimes}_{m+n}}.$$

Which has the following universal property:

Proposition 1.4.45. *Let A and Σ be pseudocompact algebras and U be a pseudocompact Σ-bimodule. Given a continuous algebra homomorphism $\alpha_0 : \Sigma \to A$ and a continuous Σ-bimodule homomorphism $\alpha_1 : U \to A$, with A treated as a Σ-bimodule via α_0, then there exists a unique algebra homomorphism $\alpha : \mathrm{T}[[A,U]] \to A$ such that $\alpha|_\Sigma = \alpha_0$ and $\alpha|_U = \alpha_1$.*

Proof. See [IM20, Lemma 2.11]. □

Remark 1.4.46. Let A and B be pseudocompact algebras and $\alpha : A \to B$ be a continuous algebra homomorphism. Then:

1. A is a A-bimodule with structure maps given by multiplication, i.e. $a' \cdot a \cdot a'' = a'aa''$, for every $a, a', a'' \in A$;
2. B is a *A-bimodule via the algebra homomorphism α*, i.e. $a \cdot b \cdot a' = \alpha(a)b\alpha(a')$, for every $a, a' \in A$ and every $b \in B$.

Lemma 1.4.47. *Let C be a coalgebra, M be a C-bicomodule and consider the cotensor coalgebra $\mathrm{Cot}_C(M)$. Then $\mathrm{Cot}_C(M) \cong \mathrm{T}[[C^*, M^*]]^*$.*

Proof. The poof consists of using the universal properties of the cotensor coalgebra and the complete tensor algebra and applying the duality functors between coalgebras and pseudocompact algebras to construct the desired coalgebra homomorphisms and using these tools again to prove the isomorphism.

Consider $\pi_0 : \mathrm{Cot}_C(M) \to C$ and $\pi_1 : \mathrm{Cot}_C(M) \to M$ the canonical projections and $\iota_0 : C^* \to \mathrm{T}[[C^*, M^*]]$ and $\iota_1 : M^* \to \mathrm{T}[[C^*, M^*]]$ the canonical inclusions. Simple checks show that π_0 is a coalgebra homomorphism, π_1 is a C-bicomodule homomorphism (for $\mathrm{Cot}_C(M)$ treated as a C-bicomodule via π_0), which kills $\mathrm{Cot}_C(M)_0 = C_0$, ι_0 is a continuous algebra homomorphism and ι_1 is a continuous C^*-bimodule homomorphism (for $\mathrm{T}[[C^*, M^*]]$ treated as a pseudocompact C^*-bimodule via ι_0).

Applying $(-)^*$ from Theorem 1.4.13 and Theorem 1.4.19, we obtain a coalgebra homomorphism $\iota_0^* : \mathrm{T}[[C^*, M^*]]^* \to C$ and a C-bicomodule homomorphism $\iota_1^* : \mathrm{T}[[C^*, M^*]]^* \to M$ (observe that, by Proposition 1.4.22, $\mathrm{T}[[C^*, M^*]]_0^* = J(\mathrm{T}[[C^*, M^*]])^{\perp} \subseteq C^*$, implies $\iota_1^*(\mathrm{T}[[C^*, M^*]]_0^*) = 0$), a continuous algebra homomorphism $\pi_0^* : C^* \to \mathrm{Cot}_C(M)^*$ and a continuous C^*-bimodule homomorphism $\pi_1^* : M^* \to \mathrm{Cot}_C(M)^*$.

The universal properties of cotensor coalgebra and complete tensor algebra, Theorem 1.2.74 and Proposition 1.4.45, respectively, give us the unique coalgebra homomorphism $\beta : \mathrm{T}[[C^*, M^*]]^* \to \mathrm{Cot}_C(M)$ satisfying $\pi_0\beta = \iota_0^*$ and $\pi_1\beta = \iota_1^*$, and the unique continuous algebra homomorphism $\alpha : \mathrm{Cot}_C(M)^* \to \mathrm{T}[[C^*, M^*]]$ satisfying $\alpha\iota_0 = \pi_0^*$ and $\alpha\iota_1 = \pi_1^*$.

Duality then give us the equalities $\beta^*\pi_0^* = \iota_0$, $\beta^*\pi_1^* = \iota_1^*$, $\iota_0^*\alpha^* = \pi_0$ and $\iota_1^*\alpha^* = \pi_1^*$. Thus, we have the following commutative diagrams:

$$\begin{array}{ccc} \mathrm{T}[[C^*, M^*]]^* & \xrightarrow{\beta} & \mathrm{Cot}_C(M) \\ {\scriptstyle\alpha^*}\uparrow & \searrow^{\iota_0^*} & \downarrow{\scriptstyle\pi_0} \\ \mathrm{Cot}_C(M) & \xrightarrow[\pi_0]{} & C \end{array} \qquad \begin{array}{ccc} \mathrm{T}[[C^*, M^*]]^* & \xrightarrow{\beta} & \mathrm{Cot}_C(M) \\ {\scriptstyle\alpha^*}\uparrow & \searrow^{\iota_1^*} & \downarrow{\scriptstyle\pi_1} \\ \mathrm{Cot}_C(M) & \xrightarrow[\pi_1]{} & M \end{array} \tag{1.4.48}$$

which implies that $\beta\alpha^* = \mathrm{id}_{\mathrm{Cot}_C(M)}$, by the universal property of the cotensor coalgebra, and

$$\begin{array}{ccc} \mathrm{T}[[C^*, M^*]] & \xrightarrow{\alpha} & \mathrm{Cot}_C(M)^* \\ {\scriptstyle\iota_0}\uparrow & \nearrow^{\pi_0^*} & \downarrow{\scriptstyle\beta^*} \\ C^* & \xrightarrow[\iota_0]{} & \mathrm{T}[[C^*, M^*]] \end{array} \qquad \begin{array}{ccc} \mathrm{T}[[C^*, M^*]] & \xrightarrow{\alpha} & \mathrm{Cot}_C(M)^* \\ {\scriptstyle\iota_1}\uparrow & \nearrow^{\pi_1^*} & \downarrow{\scriptstyle\beta^*} \\ M^* & \xrightarrow[\iota_1]{} & \mathrm{T}[[C^*, M^*]] \end{array} \tag{1.4.49}$$

which implies that $\beta^*\alpha = \mathrm{id}_{\mathrm{T}[[C^*, M^*]]}$, by the universal property of the complete tensor algebra. Therefore, by duality, we have $\alpha^*\beta = \mathrm{id}_{\mathrm{T}[[C^*, M^*]]^*}$. □

1.4.5 Dual correspondences

We finish this chapter with a table which collects the correspondences between coalgebra and pseudocompact algebra structures.

Description	Symbol	Symbol	Description
coalgebra	C	C^*	pseudocompact algebra
simple coalgebra	S	S^*	simple psc. algebra
cosemisimple coalgebra	C_0	C_0^*	topologically semisimple psc. alg.
basic coalgebra	B	B^*	basic psc. algebra
pointed coalgebra	C	C^*	pointed psc. algebra
is a separable coalgebra	C_0	$\frac{C^*}{J(C^*)}$	is a separable psc. algebra
subcoalgebra of C	D	$D^\perp$	ideal of C^*
coideal of C	I	$I^\perp$	subalgebra of C^*
coradical of C	$C_0 = J(C^*)^\perp$	$C_0^\perp = J(C^*)$	Jacobson radical of C^*
n-th term of the coradical of C	$C_n = J^{n+1}(C^*)^\perp$	$C_n^\perp = J^{n+1}(C^*)$	$(n+1)$-th radical of C^*
coalgebra homomorphism	$\rho : C \to D$	$\rho^* : D^* \to C^*$	continuous algebra hom.
(left) C-comodule	M	M^*	(left) psc. C^*-module
comodule homomorphism	$\sigma : M \to N$	$\sigma^* : N^* \to M^*$	continuous module hom.
C-D-bicomodule	M	M^*	pseudocompact C^*-D^*-bimodule
cotensor product	$M \square_C N$	$M^* \widehat{\otimes}_{C^*} N^*$	complete tensor product
cotensor coalgebra	$\mathrm{Cot}_C(M)$	$\mathrm{T}[[C^*, M^*]]$	complete tensor algebra
category of coalgebras	**Cog**	**Alg**	category of psc. algebras
cat. of finite dim. coalgebras	**cog**	**alg**	cat. of finite dim. algebras
cat. of pointed coalgebras	**PCog**	**PAlg**	cat. of pointed psc. algebras
category of left C-comodules	${}^C\mathcal{M}$	${}_{C^*}\mathcal{M}$	category of left psc. C^*-modules
category of C-D-bicomodules	${}^C\mathcal{M}^D$	${}_{C^*}\mathcal{M}_{D^*}$	category of psc. C^*-D^*-bimodules

Table 1.1: *Dual correspondences between coalgebras and pseudocompact algebras structures.*

Chapter 2

A functorial approach to the path coalgebra and Gabriel quiver constructions

Quivers play an important role on the representation theory of associative algebras since every finite dimensional hereditary algebra over an algebraically closed field is Morita equivalent to a path algebra. Moreover, the category of (left) modules over a path algebra is isomorphic to the category of (left) representations of its quiver, which are easier to work with and it is classified for finite and tame representation types.

Furthermore, examples of path algebras are very simple to construct and every finite dimensional basic algebra over an algebraically closed field is a quotient of some path algebra over an admissible ideal. Thus, path algebras are very useful for constructing (counter)examples for what would be general results for such algebras.

This theory has been developed over the years, diversifying what corresponds to finite dimensional algebras and also generalizing the quivers. One way it was done is that every pointed coalgebra (the counterpart of finite dimensional basic algebras over an algebraically closed field) is an admissible subcoalgebra of a path coalgebra.

In this chapter we make this construction in a functorial way, proving that the functor which corresponds to the path coalgebra is right adjoint to the functor corresponding to the Gabriel k-quiver (here we consider k-quivers instead of quivers in order to make it functorial and we consider a quotient by a “good” equivalence relation on coalgebra homomorphisms so that we obtain an adjunction). The unit of the adjunction is the desired inclusion of a pointed coalgebra into the path coalgebra of its Gabriel quiver as an admissible subcoalgebra.

The above is Section 2.3. In Section 2.1 we present the basics of the theory and in Section 2.2 we briefly explain the main result of the article which was our starting point.

We finish this chapter with a parallel for pseudocompact algebras. Applying the duality $(-)^*$ between coalgebras and pseudocompact algebras to the adjunction obtained in Section 2.3 gives us an adjunction on the right between the category of k-quivers and a quotient category of (the category of) pointed pseudocompact algebras by an equivalence relation on continuous algebra homomorphisms. This shows that the relation between pseudocompact algebras and k-quivers are better understood via contravariant functors. Then, we show an adjunction for covariant functors between the quotient category of pointed pseudocompact algebras and a category of pairs, which is related to the category of k-quivers, but loses combinatorial properties. This generalize the main result of Section 2.2.

While working with this thesis, the core content of this chapter was published on [IMQ21].

2.1 Quivers and path (co)algebras

In this section we present the basics of quivers, path algebras and path coalgebras. Whenever k is required to be algebraically closed, we write $k = \bar{k}$.

2.1.1 Quivers

Definition 2.1.1. A *quiver* $Q = (Q_0, Q_1, s, t)$ is a *directed graph*, i.e. a set of *vertices* Q_0, a set of *arrows* Q_1, and two functions $s, t : Q_1 \rightrightarrows Q_0$, where, for any arrow $a \in Q_1$, $s(a)$ represents its *source* and $t(a)$ represents its *target*, see [ARS95, §3.1].

A *map of quivers* $\phi : Q \to R$ consist of a map $\phi_0 : Q_0 \to R_0$ together with a map $\phi_1 : Q_1 \to R_1$ such that $\phi\big(s(a)\big) = s\big(\phi(a)\big)$ and $\phi\big(t(a)\big) = t\big(\phi(a)\big)$ for every $a \in Q_1$.

A *subquiver* $R = (R_0, R_1, s, t)$ of Q is such that $R_0 \subseteq Q_0$, $R_1 \subseteq Q_1$, and for any arrow $a \in R_1$ we have $s(a), t(a) \in R_0$.

A quiver is *finite* if both sets Q_0 and Q_1 are finite.

A quiver is *connected* if it is not the disjoint union of two non-empty subquivers.

The *underlying graph* of a quiver Q, denoted by $\overline{Q}$, is the graph obtained from Q by ignoring its orientation.

Denote by **Quiv** the category of quivers and maps of quivers.

Definition 2.1.2. Let Q be a quiver. A *representation* of Q is a collection $X = ((X_i)_{i \in Q_0}, (X_a)_{a \in Q_1})$ consisting of a vector space X_i for each vertex i and a linear map $X_a : X_{s(a)} \to X_{t(a)}$ for each arrow a.

A *morphism of representations* $\theta : X \to Y$ is a collection $\theta = (\theta_i)_{i \in Q_0}$ for each vertex i satisfying the following commutative diagram

$$\begin{array}{ccc} X_{s(a)} & \xrightarrow{X_a} & X_{t(a)} \\ {\scriptstyle \theta_{s(a)}}\downarrow & & \downarrow{\scriptstyle \theta_{t(a)}} \\ Y_{s(a)} & \xrightarrow[Y_a]{} & Y_{t(a)} \end{array} \tag{2.1.3}$$

If X and Y are two representations of Q, the direct sum $Z = X \oplus Y$ is a representation with $Z_i = X_i \oplus Y_i$, for every $i \in Q_0$, and $Z_a = (X_a, Y_a) : Z_{s(a)} \to Z_{t(a)}$, for every $a \in Q_1$.

A representation is *indecomposable* if it is not isomorphic to the direct sum of two nonzero representations.

Denote by $\mathbf{Rep}_k Q$ such category and by $\mathbf{rep}_k Q$ the full subcategory of all finite dimensional representations.

Definition 2.1.4. A *path* in Q of *length* $l \geqslant 1$ is the formal composition of arrows $a_l a_{l-1} \cdots a_1$ with $s(a_j) = t(a_{j-1})$ and for each vertex $i \in Q_0$ we associate the *stationary path* e_i of length $|e_i| = 0$ with $s(e_i) = t(e_i) = i$.

An *oriented cycle* in a quiver Q is a path $a_l a_{l-1} \cdots a_1$ such that $s(a_1) = t(a_l)$. In case $l = 1$ we say that the oriented cycle a_1 is a *loop*. A quiver is *acyclic* if it contains no oriented cycle.

The *path algebra* $k(Q)$ of the quiver Q is the vector space with basis all finite paths in Q, and multiplication given by concatenation of the paths. If Q_0 is finite, then $k(Q)$ has unit $\sum_{e \in Q_0} e$. The path algebra $k(Q)$ is finite dimensional if, and only if, Q is finite and acyclic.

Definition 2.1.5. Let Q be a quiver and X be a representation of Q. The *support* Q^X of X is the subquiver of Q with vertices $Q_0^X = \{i \in Q_0 \mid X_i \neq 0\}$ and arrows $Q_1^X = \{a \in Q_1 \mid X_a \neq 0\}$.

The representation X is of *finite length* if Q^X is finite and X_i is finite dimensional for every $i \in Q_0^X$.

The representation X is *nilpotent* if there exists an integer $m \geqslant 2$ such that the composition $X_{a_m} X_{a_{m-1}} \cdots X_{a_1} = 0$ for any path $a_m a_{m-1} \cdots a_1$ in Q of length m.

Denote by $\mathbf{nilrep}_k^{\mathrm{fl}} Q$ the full subcategory of $\mathbf{rep}_k Q$ whose objects are nilpotent representations of finite length. Denote by $\mathbf{Rep}_k^{\mathrm{lnfl}} Q$ the full subcategory of $\mathbf{Rep}_k Q$ whose objects are locally nilpotent representations, i.e. direct unions of nilpotent representations of finite length.

The importance of the path algebra construction is due to the following fact.

Proposition 2.1.6. *Let Q be a finite, connected and acyclic quiver. The category $\mathbf{Rep}_k Q$ is equivalent to the category ${}_{k(Q)}\mathcal{M}$. The result holds for the full subcategories $\mathbf{rep}_k Q$ and ${}_{k(Q)}\mathcal{M}^{\mathrm{f}}$.*

Proof. See [ASS06, Corollary 3.1.7]. □

Definition 2.1.7. Let $k(Q)$ be a path algebra and consider the ideal, $R_Q^n = \langle \omega \in Q \mid |\omega| \geqslant n \rangle_k$, generated by the paths of length at least n. An *admissible ideal* of $k(Q)$ is an ideal I such that $R_Q^m \subseteq I \subseteq R_Q^2$, for some $m \in \mathbb{N}$.

Definition 2.1.8. Let A be a finite dimensional basic algebra and $k = \bar{k}$. Denote by Q_A the quiver given as follows:

- The set $(Q_A)_0$ of vertices is in bijective correspondence with a complete set of primitive orthogonal idempotents of A, $\{e_i\}_{i \in I}$, which is in bijection with the unique complete set of primitive orthogonal idempotents of $\frac{A}{J(A)}$;
- The number of arrows from e_i to e_j is the dimension of $e_j \frac{J(A)}{J^2(A)} e_i$.

Q_A is the *Gabriel quiver of the finite dimensional algebra A*, e.g. [Ben91, Definition 4.1.6] or [ASS06, Definition 3.1].

Proposition 2.1.9. *Let A be a finite dimensional basic algebra and $k = \bar{k}$. There exists a surjective algebra homomorphism $\alpha : k(Q_A) \to A$ such that the* $\ker \alpha$ *is an admissible ideal.*

Proof. See [Ben91, Proposition 4.1.7]. □

Definition 2.1.10. An algebra A is a *hereditary algebra* if every submodule of a *projective A-module* is projective, e.g. [Ben91, Definition 4.1.2].

Proposition 2.1.11. *Let A be a finite dimensional basic hereditary algebra and $k = \bar{k}$. Then $A \cong k(Q_A)$.*

Proof. See [Ben91, Proposition 4.2.4]. □

2.1.2 Path coalgebras

In this section, we study the path coalgebra of a quiver.

Definition 2.1.12. The *path coalgebra kQ* of the quiver Q is the vector space with basis all finite paths in Q, and comultiplication and counit maps given by

$$\Delta(w) = \sum_{w=w_2w_1} w_2 \otimes w_1, \qquad \varepsilon(w) = \delta_{|w|,0},$$

where the pairs w_1, w_2 are all paths in Q whose composition gives the path w.

In this way

Proposition 2.1.13. *Let Q be a quiver. Then $kQ \cong \mathrm{Cot}_{kQ_0}(\mathrm{span}\{Q_1\})$.*

Proof. See [Woo97, §4]. □

Hence, kQ is pointed, $\mathrm{G}(kQ)$ consists of stationary paths, $(kQ)_0 = kQ_0$ is the group-like coalgebra on Q_0 (see Definition 1.3.3), and kQ is coradically graded with coradical filtration $\{(kQ)_m\}_{m\in\mathbb{N}}$, in which $(kQ)_m$ is generated by all paths of Q of length strictly less than $m+1$, see e.g. [Sim11, Proposition 7.7].

The above construction defines a covariant functor

$$k- : \mathbf{Quiv} \to \mathbf{PCog}$$

which acts on morphisms as $k\phi(w) = \phi_1(a_l)\phi_1(a_{l-1})\dots\phi_1(a_1)$ for any path $w = a_la_{l-1}\dots a_1$ of length $l \geqslant 1$ and as $k\phi(e_i) = e_{\phi_0(i)}$ for any stationary path e_i.

Proposition 2.1.14. *Let Q be a quiver. The category $\mathbf{Rep}_k^{\mathrm{lnfl}}Q$ is equivalent to the category ${}^{kQ}\mathcal{M}$. The result holds for the full subcategories $\mathbf{nilrep}_k^{\mathrm{fl}}Q$ and ${}^{kQ}\mathcal{M}_{\mathrm{f}}$.*

Proof. See [Sim11, Proposition 7.18]. □

Definition 2.1.15. Let kQ be a path coalgebra and $C \subsetneq kQ$ be a subcoalgebra. C is an *admissible subcoalgebra* if $(kQ)_1 \subseteq C$.

On the other hand,

Definition 2.1.16. Let C be a pointed coalgebra. Denote by ${}_CQ$ the quiver given as follows:

- the set of vertices ${}_CQ_0$ is identified with the set $\mathrm{G}(C)$;
- given two vertices $g, h \in \mathrm{G}(C)$, the arrows from g to h is a basis of the quotient space $\bar{\mathrm{P}}_{g,h}(C)$.

${}_CQ$ is the *Gabriel quiver of the coalgebra* C, see e.g. [Mon95, Remark 1.2] or [Sim11, Description 4.12].

Remark 2.1.17. The choice of basis makes it impossible to define a functor from the category of pointed coalgebras to the category of quivers which assigns the Gabriel quiver of a coalgebra on objects. In Section 2.3.1 we present a simple solution to bypass this problem.

Proposition 2.1.18. *Let C be a pointed coalgebra. There exists an injective coalgebra homomorphism $\rho : C \to k_CQ$ such that $\rho(C)$ is an admissible subcoalgebra.*

Proof. See [CM97, Theorem 4.3]. □

Definition 2.1.19. A coalgebra C is a *hereditary coalgebra* if homomorphic images of *injective C-comodules* are injective, see [Chi02].

Proposition 2.1.20. *Let C be a pointed hereditary coalgebra. Then $C \cong k_CQ$.*

Proof. See [Chi02, Theorem 1]. □

The dual notion of the path coalgebra is the following

Definition 2.1.21. The *complete path algebra* $k((Q))$ of the quiver Q is the set of sequences in k, $(\lambda_w)_w$, indexed by (oriented) paths in Q, with multiplication defined by:

$$(\lambda_w)_w * (\kappa_v)_v = \Big(\sum_{u=wv} \lambda_w \kappa_v \Big)_u, \tag{2.1.22}$$

see [Iov13, §1].

It follows that $k((Q))$ is a pseudocompact algebra and:

Proposition 2.1.23. *Let Q be a quiver. Then $(kQ)^* \cong k((Q)) \cong \mathrm{T}[[k((Q_0)), k((Q_1))]]$, where*

- *$k((Q_0)) = \prod_{i \in Q_0} ke_i$ is a topologically semisimple pseudocompact algebra;*
- *$kQ_{i,j} = \langle \omega \in Q_1 \mid s(\omega) = i,\ t(\omega) = j \rangle_k$;*
- *$k((Q_1)) = \prod_{i,j \in Q_0} kQ_{i,j}$ is a pseudocompact $k((Q_0))$-bimodule.*

Proof. See [Sim01, Proposition 8.1]. See also [Sim11, Proposition 7.12]. □

2.2 The path algebra as a left adjoint functor

Initially, our aim in this research was to develop for coalgebras a closely related theory of the adjunction obtained by [IM20, Theorem 5.2]. What we got was that the constructions flows even smoother for coalgebras, giving us an insight for how to work with the dual side, which allowed us to generalize the previous results for pseudocompact algebras.

In this section, we present the main results of [IM20], stressing its limitations which we deal with in Section 2.4 using the adjunction obtained in Section 2.3.

First, there is no need for asking the fixed field k (treated in this entire section as a discrete topological ring) to be perfect, since it had only two purposes: for any pseudocompact algebra A, the quotient $\frac{A}{J(A)}$ is a separable algebra, which is always the case when A is pointed, so that the conditions for the Wedderburn-Malcev Theorem for pseudocompact algebras, Theorem 1.4.33, are satisfied; and, considering $A = \Sigma \oplus J(A)$, for $\Sigma \cong \frac{A}{J(A)}$, the projection of Σ-bimodules $J(A) \to \frac{J(A)}{J^2(A)}$ splits, see [IM20, Lemma 2.9], and it does happens because Σ is topologically semisimple and, by [IM22, Proposition 3.7], $J(A)$ is a projective Σ-bimodule.

Definition 2.2.1. A (pointed) *finite Vquiver* $FVQ = (VQ_*, VQ_{e,f})$ consists of a a finite set of vertices $VQ_* = \{*\} \cup VQ_0$ and, for each pair $e, f \in VQ_*$ a finite dimensional vector space $VQ_{e,f}$ such that $VQ_{*,e} = VQ_{e,*} = 0$ for all $e \in VQ_*$.

A *map of Vquivers* $\phi : (VR_*, VR_{e,f}) \to (VQ_*, VQ_{e',f'})$ consists of:

- a surjective map $\phi_0 : VR_* \to VQ_*$ such that $\phi_0(*) =*$, i.e. ϕ_0 is a *pointed map*, and it is injective when restricted to $VR_* \setminus \phi_0^{-1}(\{*\})$;
- and a linear map $\phi_{e,f} : VR_{e,f} \to VQ_{\phi_0(e),\phi_0(f)}$ for each pair of vertices $e, f \in VR_*$.

Denote by **VQuiv** the category whose objects are (pointed) finite Vquivers and morphisms maps of Vquivers.

Definition 2.2.2. Denote by **A** the subcategory of pointed pseudocompact algebras **PAlg** whose objects are (pointed) pseudocompact algebras A such that $\frac{A}{J^2(A)}$ is finite dimensional and morphisms are continuous algebra homomorphisms $\alpha : A \to B$ such that the induced map $\alpha' : \frac{A}{J(A)} \to \frac{B}{J(B)}$ is a surjection.

For $\alpha, \beta \in \mathrm{Hom}_{\mathbf{A}}(A, B)$, define the congruence relation $\alpha \sim \beta$ if

$$(\alpha - \beta)(A) \subseteq J(B), \quad (\alpha - \beta)\big(J(A)\big) \subseteq J^2(B), \tag{2.2.3}$$

see [IM20, Definition 3.11]. Denote by $\mathcal{A}$ the quotient category ${}^{\mathbf{A}}\!/\!_{\sim}$.

Definition 2.2.4. Given a (pointed) finite Vquiver $FVQ = (VQ_*, VQ_{e,f})$, denote by:

$$\Sigma_{VQ} = \prod_{e \in VQ_0} k, \quad VQ_1 = \bigoplus_{e,f \in VQ_0} VQ_{e,f},$$

where Σ_{VQ} is a semisimple pointed algebra and VQ_1 is a Σ_{VQ}-bimodule.

Given a map of Vquivers $\phi : FVR \to FVQ$, denote by $\alpha_0' : \Sigma_{VR} \to \Sigma_{VQ}$ the linear extension of ϕ_0 such that $\alpha_0'(e) = 0$ whenever $\phi(e) =*$, and denote by $\alpha_1' = \sum \phi_{e,f} : \bigoplus VR_{e,f} \to \bigoplus VQ_{\phi_0(e),\phi_0(f)}$ the homomorphism of Σ_{VR}-bimodules. Composing with the corresponding inclusions to $\mathrm{T}[[\Sigma_{VQ}, VQ_1]]$ the universal property of the complete tensor algebra gives a continuous algebra homomorphism $\alpha : \mathrm{T}[[\Sigma_{VR}, VR_1]] \to \mathrm{T}[[\Sigma_{VQ}, VQ_1]]$.

Denote by $\mathcal{T} : \mathbf{VQuiv} \to \mathcal{A}$ the covariant functor given by $\mathcal{T}(FVQ) = \mathrm{T}[[\Sigma_{VQ}, VQ_1]]$ and $\mathcal{T}(\phi) = [\alpha]$, where $[\alpha]$ denotes the congruence class of the continuous algebra homomorphism α, defined above, under the relation $\sim$.

Definition 2.2.5. Let $A \in \mathcal{A}$, P be the complete set of primitive orthogonal idempotents of $\frac{A}{J(A)}$ and $s : \dfrac{A}{J(A)} \to A$ be any splitting of the canonical projection $A \twoheadrightarrow \dfrac{A}{J(A)}$. Let $\mathcal{G}(A) = \{(1+\omega)(-)(1+\omega)^{-1} \mid \omega \in J(A)\}$ be a subgroup of $\mathrm{Aut}(A)$ given by conjugation and, for any $a \in A$, denote by ${}^{\mathcal{G}(A)}a$ the orbit of a under $\mathcal{G}(A)$.

Denote by $\mathcal{F} : \mathcal{A} \to \mathbf{VQuiv}$ the functor given by $\mathcal{F}(A) = (\mathcal{F}(A)_*, \mathcal{F}(A)_{{}^{\mathcal{G}(A)}e, {}^{\mathcal{G}(A)}f})$, where

$$\mathcal{F}(A)_* = \{*\} \cup \{{}^{\mathcal{G}(A)}e \mid e \in s(P)\}, \quad \mathcal{F}(A)_{{}^{\mathcal{G}(A)}e, {}^{\mathcal{G}(A)}f} = f\frac{J(A)}{J^2(A)}e,$$

and, for any continuous algebra homomorphism $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A, B)$, the map of Vquivers $\mathcal{F}(\alpha) = \theta : (\mathcal{F}(A)_*, \mathcal{F}(A)_{{}^{\mathcal{G}(A)}e, {}^{\mathcal{G}(A)}f}) \to (\mathcal{F}(B)_*, \mathcal{F}(B)_{{}^{\mathcal{G}(B)}e, {}^{\mathcal{G}(B)}f})$ is given by:

$$\theta_0({}^{\mathcal{G}(A)}e) = \begin{cases} {}^{\mathcal{G}(B)}\alpha(e) & \text{if } \alpha(e) \neq 0 \\ * & \text{if } \alpha(e) = 0 \end{cases},$$

$$\theta_{{}^{\mathcal{G}(A)}e, {}^{\mathcal{G}(A)}f}\Big(f\big(j + J^2(A)\big)e\Big) = \alpha(f)\big(\alpha(j) + J^2(B)\big)\alpha(e).$$

Theorem 2.2.6. *The covariant functor $\mathcal{T} : \mathbf{VQuiv} \to \mathcal{A}$ is left adjoint to the covariant functor $\mathcal{F} : \mathcal{A} \to \mathbf{VQuiv}$.*

Proof. See [IM20, Theorem 5.2]. □

Remark 2.2.7. The categories treated here are essentially finite: $\frac{A}{J^2(A)}$ is finite dimensional and the continuous algebra homomorphism are subjected to the condition $\alpha' : \frac{A}{J(A)} \to \frac{B}{J(B)}$ is a surjection; Vquivers are finite and their maps have also some restrictions. In Section 2.4, we deal with an adjunction that generalizes the above theorem: any pseudocompact algebra and continuous algebra homomorphism are considered, though the combinatorics of the Vquiver structure is lost.

2.3 The path coalgebra as a right adjoint functor

In this section, we present the main Theorem of this thesis, namely: the path coalgebra as a right adjoint functor. We start by presenting the fundamental blocks of these constructions: the category of k-quivers and a quotient category of the (category of) pointed coalgebras. Then we define the functors between these categories, which corresponds to the classical constructions for the path coalgebra and the Gabriel quiver of coalgebras. Finally, we prove the main theorem.

2.3.1 Category of k-quivers

As briefly discussed on Remark 2.1.17, the category o quivers is not suited for working functorially with the Gabriel quiver for coalgebras because it depends on a choice of basis, which is not canonical. However, there is a simple solution for this problem: the category of k-quivers, see [Gab73, §7.1]. A k-quiver is basically a quiver such that the arrows are a vector space, as we define below.

Definition 2.3.1. A *k-quiver* $VQ = (VQ_0, VQ_{i,j})$ consists of a set of *vertices* VQ_0 together with a vector space $VQ_{i,j}$ for each (ordered) pair $i, j \in VQ_0$, which we call *arrow space*.

A k-subquiver VR of VQ is a k-quiver such that $VR_0 \subseteq VQ_0$ is a subset and, for each $i, j \in VR_0$, the arrow space $VR_{i,j} \subseteq VQ_{i,j}$ is a subspace.

Let VQ and VR be k-quivers. A *map of k-quivers* $\varphi = (\varphi_0, \varphi_{i,j}) : (VQ_0, VQ_{i,j}) \to (VR_0, VR_{i',j'})$ consists of

- a function $\varphi_0 : VQ_0 \to VR_0$.
- a linear map $\varphi_{i,j} : VQ_{i,j} \to VR_{\varphi_0(i),\varphi_0(j)}$ for each pair of vertices $i, j \in VQ_0$.

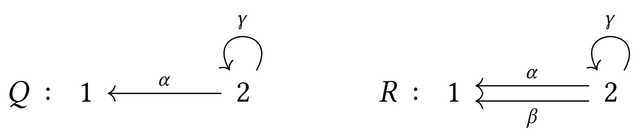
The category k-**Quiv** has objects k-quivers and morphisms maps of k-quivers.

There exists a correspondence between quivers and k-quivers: given a quiver $Q = (Q_0, Q_1)$, for each pair of vertices $i, j \in Q_0$, the vector spaces $Q_{i,j}$ with basis $\{a \in Q_1 \mid s(a) = i,\ t(a) = j\}$ define a k-quiver $VQ = (Q_0, Q_{i,j})$; on the other hand, if we start with a k-quiver $VQ = (VQ_0, VQ_{i,j})$, we obtain a quiver by taking as arrows from i to j a basis of $VQ_{i,j}$. The first correspondence (with the natural assignment for morphisms) defines a functor, which we denote by

$$V(-) : \mathbf{Quiv} \to k\text{-}\mathbf{Quiv} \tag{2.3.2}$$

The second correspondence does not. We observe in passing that the functor $V(-)$ of course does possess a forgetful right adjoint, but we make no use of this functor here.

Example 2.3.3. Let R be a quiver and Q a subquiver as depicted below



(2.3.4)

Consider the canonical inclusion $\iota : Q \to R$. Then, the functor $V(-)$ gives the corresponding k-quivers

(2.3.5)

where $V(R)$ has vertices $V(R)_0 = \{1, 2\}$ and arrow spaces given by

$$V(R)_{i,j} = \begin{cases} \langle \alpha, \beta \rangle \cong k^2 & \text{if } i = 2,\ j = 1 \\ \langle \gamma \rangle \cong k & \text{if } i = j = 2 \\ \{0\} & \text{otherwise} \end{cases}$$

The k-subquiver $V(Q)$ has the same set of vertices and arrow spaces of $V(R)$ except for $V(Q)_{2,1}$, which is $\langle \alpha \rangle$. Thus $V(\iota) : V(Q) \to V(R)$ is the canonical inclusion map of k-quivers.

One of the main advantages of the relationship between quivers and coalgebras is that one obtains a combinatorial description of the comodules for a given coalgebra in terms of representations of quivers — this approach is utilized in the articles [CZ07; CKQ02; Chi10; NS02; KS05; Sim08], among others. We mention that working with k-quivers we maintain this advantage, see Apendix C for more on k-quiver representations of a coalgebra. Representations of k-quivers are defined and their relation to (co)modules discussed, for instance, in [Gab73, §7] and [Sim07, §5].

2.3.2 "Close" coalgebra homomorphisms

Definition 2.3.6. A relation $\sim$ on a set X is a *congruence relation* if, for every $x, y, z \in X$, it satisfies:

1. *reflexivity*, i.e. $x \sim x$;
2. *symmetry*, i.e. $x \sim y \implies y \sim x$;
3. *transitivity*, i.e. $x \sim y,\ y \sim z \implies x \sim z$.

In case X has an operation, say $*$, then a congruence relation on X must be compatible with the operation, i.e. $x \sim y,\ x' \sim y' \implies x * x' \sim y * y'$.

Given two coalgebra homomorphisms $\rho, \gamma : C \to D$, write $\rho \sim \gamma$ if

$$(\rho - \gamma)(C_0) = 0, \quad (\rho - \gamma)(C_1) \subseteq D_0. \tag{2.3.7}$$

Lemma 2.3.8. *The relation $\sim$ defines a congruence relation on* **PCog**.

Proof. That $\rho \sim \rho$ and $\rho \sim \gamma \implies \gamma \sim \rho$ are obvious. Suppose $\rho \sim \gamma$ and $\gamma \sim \sigma$. Then $(\rho - \sigma) = (\rho - \gamma) + (\gamma - \sigma)$ and the result follows since compatibility with composition is again elementary. □

By $\mathbf{PCog}_\sim$ we denote the corresponding quotient category.

Lemma 2.3.9. *Let $\rho, \gamma : C \to D$ be two homomorphisms in* **PCog** *such that $\rho \sim \gamma$. Then $(\rho - \gamma)(C_i) \subseteq D_{i-1}$, for each $i \geqslant 0$.*

Proof. This proof follows a similar philosophy of [TW74, Proposition 4].

We proceed by induction on i. Suppose that $(\rho - \gamma)(C_i) \subseteq D_{i-1}$ for every $i \leqslant n - 1$. Observe that

$$\Delta_D(\rho - \gamma) = (\rho \otimes \rho - \gamma \otimes \gamma)\Delta_C = \big(\rho \otimes (\rho - \gamma) + (\rho - \gamma) \otimes \gamma\big)\Delta_C \tag{2.3.10}$$

since ρ and γ are coalgebra homomorphisms (see Definition 1.2.20). Thus, applying Theorem 1.2.30 to (2.3.10), and provided that $\Delta_C(C_n) \subseteq \sum_{i=0}^n C_i \otimes C_{n-i}$ (see Definition 1.2.16), we get

$$\begin{aligned}\Delta_D(\rho-\gamma)(C_n) &= \big(\rho\otimes(\rho-\gamma)+(\rho-\gamma)\otimes\gamma\big)\Delta_C(C_n)\\ &\subseteq \big(\rho\otimes(\rho-\gamma)+(\rho-\gamma)\otimes\gamma\big)\Big(\sum_{i=0}^{n} C_i\otimes C_{n-i}\Big)\\ &\subseteq \sum_{i=0}^{n} D_i\otimes D_{n-1-i} + \sum_{i=0}^{n} D_{i-1}\otimes D_{n-i}\\ &= \sum_{i=0}^{n-1} D_i\otimes D_{n-1-i} \subseteq D\otimes D_{n-2} + D_0\otimes D.\end{aligned}$$

Hence $(\rho-\gamma)(C_n) \subseteq D_{n-1}$. □

Working with the quotient category $\mathbf{PCog}_\sim := \mathbf{PCog}/\sim$, much of the important information from **PCog** is preserved. For instance:

Lemma 2.3.11. *The projection functor* $\Pi : \mathbf{PCog} \to \mathbf{PCog}_\sim$ *reflects isomorphisms. That is, if* $\rho : C \to D$ *is a coalgebra homomorphism such that* $\Pi(\rho) : C \to D$ *is an isomorphism, then* ρ *is an isomorphism.*

Proof. Note that if $[\rho]$ is an isomorphism, then there exists a coalgebra homomorphism $\gamma : D \to C$ such that $[\gamma][\rho] = [\gamma\rho] = [\mathrm{id}_C]$ and $[\rho][\gamma] = [\rho\gamma] = [\mathrm{id}_D]$. Thus, it is sufficient to show that for any coalgebra endomorphism $\rho : C \to C$, $\rho \sim \mathrm{id}_C$ implies that ρ is an isomorphism.

Since $C = \bigcup_{n\geqslant 0} C_n$, any element $c \in C$ belongs to C_n for some $n \in \mathbb{N}$. In particular, if $c \neq 0$, then there exists a n such that $c \in C_n$, but $c \notin C_{n-1}$.

Let $\rho : C \to C$ be a coalgebra homomorphism such that $\rho \sim \mathrm{id}$, and consider $c \neq 0$ as above.

If $c \in \ker(\rho)$, then, by Lemma 2.3.9,

$$(\mathrm{id}-\rho)(c) = c - \rho(c) = c \in C_{n-1}, \tag{2.3.12}$$

gives a contradiction. Hence $\ker\rho = \{0\}$ and, consequently, ρ is an injection.

Let $c_0 = c$ and, using Lemma 2.3.9 again, define recursively $c_i = -c_{i-1} + \rho(c_{i-1}) \in C_{n-i}$, for $i = 1, \dots, n$. This sequence stops at $c_{n+1} = -c_n + \rho(c_n) = 0$ (it could happens that $\rho(c_i) = c_i$ for $0 \leqslant i < n$, which makes no difference). Writing $c' = \sum_{i=0}^n (-1)^i c_i$ we get:

$$\rho(c') = c_0 \underbrace{\underbrace{-c_0 + \rho(c_0)}_{c_1} - \rho(c_1)}_{-c_2} + \rho(c_2) - \cdots + (-1)^n\rho(c_n) = c_0 + (-1)^n c_{n+1} = c$$

Thus ρ is a surjection and this completes the proof. □

Lemma 2.3.13. *If* $\rho \in \mathrm{Hom}_{\mathbf{PCog}}(C, D)$ *is an injection, then its image* $\Pi(\rho) \in \mathrm{Hom}_{\mathbf{PCog}_\sim}(C, D)$ *is a monomorphism.*

Proof. Suppose $\gamma, \sigma : B \to C$ are two coalgebra homomorphisms such that $\rho\gamma \sim \rho\sigma$. For any $b \in B_0$ we have

$$(\rho\gamma - \rho\sigma)(b) = \rho\big(\gamma(b) - \sigma(b)\big) = 0 \iff \gamma(b) - \sigma(b) = 0, \tag{2.3.14}$$

since ρ is an injection. For $b' \in B_1$, we have

$$(\rho\gamma - \rho\sigma)(b') = \rho\big(\gamma(b') - \sigma(b')\big) \in D_0 \iff \gamma(b') - \sigma(b') \in C_0. \tag{2.3.15}$$

since the image of skew-primitives by injections are skew-primitives (see Lemma 1.2.46. See also Proposition 1.3.20). Thus $\gamma \sim \sigma$ and, therefore, ρ is a monomorphism in $\mathbf{PCog}_\sim$. ☐

2.3.3 Path coalgebra and Gabriel k-quiver functors

We define functors between the categories introduced above.

Definition 2.3.16. Given a k-quiver $VQ = (VQ_0, VQ_{g,h})$, denote by $\Sigma_Q = (kVQ_0, \Delta_0, \varepsilon_0)$ the group-like coalgebra of VQ_0, and by $V_Q = (VQ_1, \mu, \nu)$ the Σ_Q-bicomodule $VQ_1 = \bigoplus_{g,h \in VQ_0} VQ_{g,h}$ with structure maps:

$$\mu(m_{g,h}) = h \otimes m_{g,h}, \quad \nu(m_{g,h}) = m_{g,h} \otimes g, \tag{2.3.17}$$

for each $m_{g,h} \in VQ_{g,h}$.

Definition 2.3.18. Define the *path coalgebra of the* k*-quiver* VQ, $k[VQ]$, as the cotensor coalgebra $\mathrm{Cot}_{\Sigma_Q}(V_Q)$.

For any $\varphi = (\varphi_0, \varphi_{g,h})$ in $\mathrm{Hom}_{k\text{-}\mathbf{Quiv}}(VQ, VR)$, the universal property of the cotensor coalgebra, Theorem 1.2.74, ensures the existence of a unique homomorphism $\rho \in \mathrm{Hom}_{\mathbf{PCog}}(k[VQ], k[VR])$ making the following diagrams commutative:

$$\begin{array}{ccc} \mathrm{Cot}_{\Sigma_Q}(V_Q) & \overset{\rho}{\dashrightarrow} & \mathrm{Cot}_{\Sigma_R}(V_R) \\ {\scriptstyle \pi_0'}\downarrow & \overset{\rho_0}{\searrow} & \downarrow{\scriptstyle \pi_0} \\ \Sigma_Q & \underset{\varphi_0}{\longrightarrow} & \Sigma_R \end{array} \qquad \begin{array}{ccc} \mathrm{Cot}_{\Sigma_Q}(V_Q) & \overset{\rho}{\dashrightarrow} & \mathrm{Cot}_{\Sigma_R}(V_R) \\ {\scriptstyle \pi_1'}\downarrow & \overset{\rho_1}{\searrow} & \downarrow{\scriptstyle \pi_1} \\ V_Q & \underset{\varphi_1}{\longrightarrow} & V_R \end{array} \tag{2.3.19}$$

where π_i', π_i are the canonical projections, φ_i are linear extensions of the maps defined by φ, and $\rho_i := \varphi_i \pi_i'$, for $i = 0, 1$. Set $k[\varphi] := \rho$.

These constructions yield a covariant functor $k[-] : k\text{-}\mathbf{Quiv} \to \mathbf{PCog}$.

Denote by

$$\widetilde{k}[-] : k\text{-}\mathbf{Quiv} \to \mathbf{PCog}_\sim \tag{2.3.20}$$

the covariant functor $\Pi k[-]$, where $\Pi : \mathbf{PCog} \to \mathbf{PCog}_\sim$ is the projection functor.

Example 2.3.21. If $\iota : VQ \hookrightarrow VR$ is an inclusion of k-quivers, then $k[\iota] : k[VQ] \to k[VR]$ is the corresponding inclusion of coalgebras.

Definition 2.3.22. Let C be a pointed coalgebra. Define the *Gabriel k-quiver* of C, $\mathrm{GQ}(C) := \big(\mathrm{GQ}(C)_0, \mathrm{GQ}(C)_{g,h}\big)$, by

$$\mathrm{GQ}(C)_0 := \mathrm{G}(C), \quad \mathrm{GQ}(C)_{g,h} := \bar{\mathrm{P}}_{g,h}(C), \tag{2.3.23}$$

see (1.3.16).

Let $\rho \in \mathrm{Hom}_{\mathbf{PCog}}(C, D)$. By Lemma 1.2.46 (and in view that every coalgebra homomorphism with pointed domain is filtered, see Theorem 1.2.30), there exists a unique coalgebra homomorphism $\bar{\rho} : \frac{C}{C_0} \to \frac{D}{D_0}$ such that the following diagram is commutative:

$$\begin{array}{ccc} C & \xrightarrow{\ \rho\ } & D \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle q'} \\ \frac{C}{C_0} & \xrightarrow[\ \bar{\rho}\ ]{} & \frac{D}{D_0} \end{array} \tag{2.3.24}$$

where q and q' are the canonical quotient maps. The maps

$$\varphi_0 := \rho|_{\mathrm{G}(C)} : \mathrm{G}(C) \to \mathrm{G}(D), \quad \varphi_{g,h} := \bar{\rho}|_{\bar{\mathrm{P}}_{g,h}(C)} : \bar{\mathrm{P}}_{g,h}(C) \to \bar{\mathrm{P}}_{\varphi_0(g),\varphi_0(h)}(D), \tag{2.3.25}$$

define a map of k-quivers $\varphi = (\varphi_0, \varphi_{g,h}) : \mathrm{GQ}(C) \to \mathrm{GQ}(D)$.

This construction yields a covariant functor $\mathrm{GQ}(-) : \mathbf{PCog} \to k\text{-}\mathbf{Quiv}$.

Furthermore,

Lemma 2.3.26. *There is a unique covariant functor*

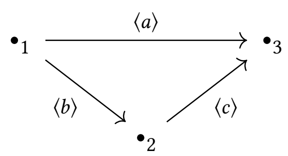
$$\widetilde{\mathrm{GQ}}(-) : \mathbf{PCog}_{\sim} \to k\text{-}\mathbf{Quiv} \tag{2.3.27}$$

such that $\mathrm{GQ}(-) = \widetilde{\mathrm{GQ}}(-)\Pi$, *where* $\Pi : \mathbf{PCog} \to \mathbf{PCog}_{\sim}$ *is the projection functor.*

Proof. Using Remark 1.3.7 and Proposition 1.3.20, one checks that defining $\widetilde{\mathrm{GQ}}(C)$ to be $\mathrm{GQ}(C)$ and $\widetilde{\mathrm{GQ}}([\rho])$ to be $\mathrm{GQ}(\rho)$ (for any representative of $[\rho]$), we obtain a covariant functor satisfying the claim. It is clearly unique. □

Example 2.3.28. A simple example of a path coalgebra is given by the k-quiver

$VQ =$



(2.3.29)

The coalgebra $k[VQ]$ is a 7 dimensional vector space with basis $\{e_1, e_2, e_3, a, b, c, cb\}$, where e_i are group-like elements. The comultiplication of cb, for example, is given by

$$\Delta(cb) = e_3 \otimes cb + c \otimes b + cb \otimes e_1.$$

Let $\rho : k[VQ] \to k[VQ]$ be the linear map that sends a to $a + (e_3 - e_1)$ and fixes all other elements of the given basis. Then ρ is a coalgebra automorphism and $\mathrm{GQ}(\rho) = \mathrm{id}_{VQ}$. Thus $\mathrm{GQ}(-)$ is not faithful.

2.3.4 Adjunction between pointed coalgebras and k-quivers

We prove that the functor $\widetilde{k}[-]$ is right adjoint to $\widetilde{\mathrm{GQ}}(-)$ by presenting a counit $\varepsilon : \widetilde{\mathrm{GQ}}(\widetilde{k}[-]) \to \mathrm{id}_{k\text{-}\mathbf{Quiv}}$ and a unit $\eta : \mathrm{id}_{\mathbf{PCog}_\sim} \to \widetilde{k}[\widetilde{\mathrm{GQ}}(-)]$ of the adjuntion.

First, observe that for any k-quiver VQ, we have the vertex set

$$\mathrm{GQ}(k[VQ])_0 = \mathrm{G}(k[VQ]) = \mathrm{G}\left(\mathrm{Cot}_{kVQ_0}\Big(\bigoplus_{g,h\in VQ_0} VQ_{g,h}\Big)\right) = VQ_0 \tag{2.3.30}$$

and, for each $g, h \in VQ_0$, the arrow space

$$\mathrm{GQ}(k[VQ])_{g,h} = \bar{\mathrm{P}}_{g,h}\left(\mathrm{Cot}_{kVQ_0}\Big(\bigoplus_{g,h\in VQ_0} VQ_{g,h}\Big)\right) = \frac{k\{h-g\} \oplus VQ_{g,h}}{k\{h-g\}} \cong VQ_{g,h} \tag{2.3.31}$$

(see Definition 1.2.72 for the structure of the cotensor coalgebra, and (2.3.17 for the structure maps of the kVQ_0-bicomodule $VQ_{g,h}$). Thus

Definition 2.3.32. Given $VQ \in k$-**Quiv**, define the map of k-quivers

$$\varepsilon_{VQ} : \widetilde{\mathrm{GQ}}(\widetilde{k}[VQ]) \to VQ \tag{2.3.33}$$

by $(\varepsilon_{VQ})_0 = \mathrm{id}_{VQ_0}$ and $(\varepsilon_{VQ})_{g,h} : \mathrm{GQ}(k[VQ])_{g,h} \cong VQ_{g,h}$ is the natural isomorphism, see (2.3.31).

Therefore, the maps ε_{VQ} are isomorphisms and are easily checked to be the components of a natural transformation

$$\varepsilon : \widetilde{\mathrm{GQ}}(\widetilde{k}[-]) \to \mathrm{id}_{k\text{-}\mathbf{Quiv}}. \tag{2.3.34}$$

Given $C \in \mathbf{PCog}$, choose a coalgebra splitting $s : C \to C_0$ of the canonical inclusion $i_0 : C_0 \to C$ (which exists by Theorem 1.2.29, because every pointed coalgebra has separable coradical). We treat C as a C_0-bicomodule via s (see Remark 1.2.54) and choose a splitting $t : C \to C_1$ of the canonical inclusion of C_0-bicomodules $i_1 : C_1 \to C$ (which exists because C_1 is an injective comodule by Theorem 1.2.49). Combining the splitting t with the canonical projection $q : C_1 \to \frac{C_1}{C_0}$ we get a homomorphism of C_0-bicomodules $qt : C \to \frac{C_1}{C_0}$. Since $\frac{C_1}{C_0} \cong \bigoplus_{g,h\in \mathrm{G}(C)} \bar{\mathrm{P}}_{g,h}(C)$ as C_0-bicomodules (see right after Proposition 1.3.20), say θ, the composition $\tilde{t} = \theta qt : C \to \bigoplus_{g,h\in\mathrm{G}(C)} \bar{\mathrm{P}}_{g,h}(C)$ is a homomorphism of C_0-bicomodules which kills C_0 (q guarantee this).

Definition 2.3.35. The maps $s, \tilde{t}$ define, by the universal property of the cotensor coalgebra, Theorem 1.2.74, the coalgebra homomorphism

$$\eta_C^{s,\tilde{t}} : C \to \mathrm{Cot}_{C_0}\Big(\bigoplus_{g,h\in\mathrm{G}(C)} \bar{\mathrm{P}}_{g,h}(C)\Big) = \widetilde{k}[\widetilde{\mathrm{GQ}}(C)]. \tag{2.3.36}$$

Lemma 2.3.37. *The congruence class of $\eta_C^{s,\tilde{t}}$ in* $\mathbf{PCog}_{\sim}$ *does not depend on the choice of splittings s, t.*

Proof. Indeed suppose that s, t and s', t' are two different choices, and $\eta_C^{s,\tilde{t}}$, $\eta_C^{s',\tilde{t}'}$ are the corresponding maps. We must confirm that $\eta_C^{s,\tilde{t}} \sim \eta_C^{s',\tilde{t}'}$. One has

$$(\eta_C^{s,\tilde{t}} - \eta_C^{s',\tilde{t}'})\Big|_{C_0} = \pi_0(\eta_C^{s,\tilde{t}} - \eta_C^{s',\tilde{t}'})\iota_0 = s\iota_0 - s'\iota_0 = 0$$

and

$$(\eta_C^{s,\tilde{t}} - \eta_C^{s',\tilde{t}'})\Big|_{C_1} = (\pi_0 + \pi_1)(\eta_C^{s,\tilde{t}} - \eta_C^{s',\tilde{t}'})\iota_1 = \underbrace{(s - s')}_{\subseteq C_0}\iota_1 + \theta q\underbrace{(t\iota_1 - t'\iota_1)}_{=0} \subseteq C_0 = k[\mathrm{GQ}(C)]_0.$$

□

So we may denote $\eta_C^{s,\tilde{t}}$ simply by η_C.

Remark 2.3.38. The map η_C is the image in $\mathbf{PCog}_{\sim}$ of the coalgebra embedding considered in [Rad82, Corollary 1] and [Woo97, (4.8)], see also [CM97, Theorem 4.3] and [CHZ06, Theorem 3.1]).

Lemma 2.3.39. *The map $\eta_C : C \to \widetilde{k}[\widetilde{\mathrm{GQ}}(C)]$ is the component at C of a natural transformation*

$$\eta : \mathrm{id}_{\mathbf{PCog}_{\sim}} \to \widetilde{k}[\widetilde{\mathrm{GQ}}(-)]. \tag{2.3.40}$$

Proof. Let $\rho : C \to D$ be a morphism in $\mathbf{PCog}$. We must check that the following square commutes in $\mathbf{PCog}_{\sim}$ (see Definition 1.1.14):

$$\begin{array}{ccc} C & \xrightarrow{\quad\rho\quad} & D \\ {\scriptstyle \eta_C}\downarrow & & \downarrow{\scriptstyle \eta_D} \\ \widetilde{k}[\widetilde{\mathrm{GQ}}(C)] & \xrightarrow[\widetilde{k}[\widetilde{\mathrm{GQ}}(\rho)]]{} & \widetilde{k}[\widetilde{\mathrm{GQ}}(D)] \end{array} \tag{2.3.41}$$

Let s, s', t and t' be splittings of the canonical inclusions $\iota_0 : C_0 \to C$, $\iota'_0 : D_0 \to D$, $\iota_1 : C_1 \to C$ and $\iota'_1 : D_1 \to D$ respectively. Denote by $\tilde{\rho}$ the map $\widetilde{k}[\widetilde{\mathrm{GQ}}(\rho)]$. We have that

$$(\eta_D^{s',\tilde{t}'}\rho - \tilde{\rho}\eta_C^{s,\tilde{t}})\Big|_{C_0} = \pi'_0(\eta_D^{s',\tilde{t}'}\rho - \tilde{\rho}\eta_C^{s,\tilde{t}})\iota_0 = (s'\rho - \rho|_{C_0}\,\pi_0\eta_C^{s,\tilde{t}})\iota_0 = \underbrace{s'\rho\iota_0}_{=\rho|_{C_0}} - \rho|_{C_0}\underbrace{s\iota_0}_{=\mathrm{id}_{C_0}} = 0. \tag{2.3.42}$$

and

$$\begin{aligned} \pi'_1(\eta_D^{s',\tilde{t}'}\rho - \tilde{\rho}\eta_C^{s,\tilde{t}})\iota_1 &= (\theta' q' t'\rho - \theta'\,\bar{\rho}|_{\frac{C_1}{C_0}}\,\theta^{-1}\pi_1\eta_C^{s,\tilde{t}})\iota_1 = \theta' q' t'\rho\iota_1 - \theta'\,\bar{\rho}|_{\frac{C_1}{C_0}}\,\theta^{-1}\theta q t\iota_1 \\ &= \theta' q' t'\iota'_1\,\rho|_{C_1} - \theta'\,\bar{\rho}|_{\frac{C_1}{C_0}}\,q = \theta q'(\rho|_{C_1} - \rho|_{C_1}) = 0. \end{aligned} \tag{2.3.43}$$

Hence the classes of $\eta_D^{s',\tilde{t}'}\rho$ and $\tilde{\rho}\eta_C^{s,\tilde{t}}$ are equal in $\mathbf{PCog}_{\sim}$ and η is a natural transformation. □

Theorem 2.3.44. *The functor* $\widetilde{k}[-] : k\text{-}\mathbf{Quiv} \to \mathbf{PCog}_{\sim}$ *is right adjoint to the functor* $\widetilde{\mathrm{GQ}}(-) : \mathbf{PCog}_{\sim} \to k\text{-}\mathbf{Quiv}$.

Proof. We check that the counit-unit equations hold, see Proposition 1.1.23, i.e. for any $C \in \mathbf{PCog}_{\sim}$, and for any $VQ \in k\text{-}\mathbf{Quiv}$, we have the equalities:

$$\mathrm{id}_{\widetilde{\mathrm{GQ}}(C)} = \varepsilon_{\widetilde{\mathrm{GQ}}(C)}\widetilde{\mathrm{GQ}}(\eta_c), \quad \mathrm{id}_{\widetilde{k}[VQ]} = \widetilde{k}[\varepsilon_{VQ}]\eta_{\widetilde{k}[VQ]}. \tag{2.3.45}$$

Observe that

$$\left(\varepsilon_{\widetilde{\mathrm{GQ}}(C)}\widetilde{\mathrm{GQ}}(\eta_C)\right)_0 = (\varepsilon_{\widetilde{\mathrm{GQ}}(C)})_0\ \eta_C|_{\mathrm{G}(C)} = \mathrm{id}_{\mathrm{G}}(C) = (\mathrm{id}_{\widetilde{\mathrm{GQ}}(C)})_0,$$

since $\eta_C|_{C_0} = \mathrm{id}_{C_0}$ (cf. equation 2.3.42). Moreover, the restriction $\eta_C|_{C_1} = s|_{C_1} + \theta\ q|_{C_1}$, where $s : C \to C_0$ is a splitting of the canonical inclusion $\iota_0 : C_0 \to C$, $q : C \to \frac{C}{C_0}$ is the canonical projection and $\theta : \frac{C_1}{C_0} \to \bigoplus_{g,h\in\mathrm{G}(C)} \bar{\mathrm{P}}_{g,h}(C)$ is the natural isomorphism (cf. equation 2.3.43. See also Remark 1.2.76) shows that

$$\eta_C|_{C_1} : C_0 \oplus \Big(\bigoplus_{g,h\in\mathrm{G}(C)} \mathrm{P}'_{g,h}(C)\Big) \to C_0 \oplus \Big(\bigoplus_{g,h\in\mathrm{G}(C)} \bar{\mathrm{P}}_{g,h}(C)\Big)$$

is an isomorphism, with $\mathrm{P}'_{g,h}(C) = \ker(s|_{C_1}) \cap \mathrm{P}_{g,h}(C)$. Thus, for each $g,h \in \mathrm{G}(C)$, the commutative diagram

$$\begin{array}{ccc} \mathrm{P}_{g,h}(C) & \xrightarrow{\eta_C} & k\langle h-g\rangle \oplus \bar{\mathrm{P}}_{g,h}(C) \\ \downarrow q & & \downarrow q' \\ \bar{\mathrm{P}}_{g,h}(C) & \xrightarrow{\bar{\eta}_C} & \frac{k\langle h-g\rangle\oplus\bar{\mathrm{P}}_{g,h}(C)}{k\langle h-g\rangle} \end{array}$$

reveals that $\bar{\eta}_C|_{\bar{\mathrm{P}}_{g,h}(C)}$ is the natural isomorphism $(\varepsilon_{\widetilde{\mathrm{GQ}}(C)})^{-1}_{g,h}$. Hence,

$$\left(\varepsilon_{\widetilde{\mathrm{GQ}}(C)}\widetilde{\mathrm{GQ}}(\eta_C)\right)_{g,h} = (\varepsilon_{\widetilde{\mathrm{GQ}}(C)})_{g,h}\ \bar{\eta}_C|_{\bar{\mathrm{P}}_{g,h}(C)} = \mathrm{id}_{\bar{\mathrm{P}}_{g,h}(C)} = (\mathrm{id}_{\widetilde{\mathrm{GQ}}(C)})_{g,h}.$$

The second equality of (2.3.45) translates as $k[\varepsilon_{VQ}]\eta^{s,\tilde{t}}_{k[VQ]} \sim \mathrm{id}_{k[VQ]}$, where s, t are two splittings constructed as in paragraph just before Definition 2.3.35 and $\eta^{s,\tilde{t}}_{k[VQ]}$ is the corresponding morphism.

We have the following:

$$\begin{aligned}\left(k[\varepsilon_{VQ}]\eta^{s,\tilde{t}}_{k[VQ]} - \mathrm{id}_{k[VQ]}\right)\Big|_{\Sigma_Q} &= \pi_0(k[\varepsilon_{VQ}]\eta^{s,\tilde{t}}_{k[VQ]} - \mathrm{id}_{k[VQ]})\iota_0 = \left((\varepsilon_{VQ})_0\pi'_0\eta^{s,\tilde{t}}_{k[VQ]} - \pi_0\right)\iota_0 \\ &= \mathrm{id}_{\Sigma_Q}s\iota_0 - \pi_0\iota_0 = \mathrm{id}_{\Sigma_Q} - \mathrm{id}_{\Sigma_Q} = 0\end{aligned}$$

as shown on the commutative diagram

$$\begin{array}{ccccc}
k[VQ] & \xrightarrow{\eta^{s,\tilde{t}}_{k[VQ]}} & k[\mathrm{GQ}(k[VQ])] & \xrightarrow{k[\varepsilon_{VQ}]} & k[VQ] \\
{\scriptstyle \iota_0}\uparrow & \searrow{\scriptstyle s} & \downarrow{\scriptstyle \pi_0'} & & \downarrow{\scriptstyle \pi_0} \\
\Sigma_Q & \xrightarrow[\mathrm{id}_{\Sigma_Q}]{} & \Sigma_Q & \xrightarrow[(\varepsilon_{VQ})_0=\mathrm{id}_{\Sigma_Q}]{} & \Sigma_Q
\end{array}$$

and

$$\begin{aligned}
\pi_1(k[\varepsilon_{VQ}]\eta^{s,\tilde{t}}_{k[VQ]} - \mathrm{id}_{k[VQ]})\iota_1 = \big((\varepsilon_{VQ})_1\pi_1'\eta^{s,\tilde{t}}_{k[VQ]} - \pi_1\big)\iota_1 = (\varepsilon_{VQ})_1\theta q t\iota_1 - \pi_1\iota_1 \\
= \big((\varepsilon_{VQ})_1\theta q\big)\Big|_{k[VQ]_1} - \pi_1|_{k[VQ]_1} = 0
\end{aligned}$$

since the projection

$$\Sigma_Q \oplus V_Q \xrightarrow{q} \frac{\Sigma_Q \oplus V_Q}{\Sigma_Q} \xrightarrow{\theta} \bigoplus_{g,h\in VQ_0} \bar{\mathrm{P}}_{g,h}(k[VQ]) \xrightarrow{(\varepsilon_{VQ})_1} \bigoplus_{g,h\in VQ_0} VQ_{g,h} = V_Q$$

is exactly $\pi_1|_{k[VQ]_1}$. Hence,

$$(k[\varepsilon_{VQ}]\eta^{s,\tilde{t}}_{k[VQ]} - \mathrm{id}_{k[VQ]})\Big|_{k[VQ]_1} = (\pi_0 + \pi_1)(k[\varepsilon_{VQ}]\eta^{s,\tilde{t}}_{k[VQ]} - \mathrm{id}_{k[VQ]})\iota_1 \subseteq k[VQ]_0$$

implies $k[\varepsilon_{VQ}]\eta^{s,\tilde{t}}_{k[VQ]} \sim \mathrm{id}_{k[VQ]}$, and the theorem is proved. □

2.3.5 Consequences and examples

Remark 2.3.46. Based on Definition 2.1.15, we call a subcoalgebra H of a path coalgebra $k[VQ]$ *admissible* if H contains $k[VQ]_1$. If C is a pointed coalgebra, any representative in **PCog** of the unit map $\eta_C : C \to \widetilde{k}[\widetilde{\mathrm{GQ}}(C)]$ of Theorem 2.3.44 realizes C as an admissible subcoalgebra of its path coalgebra.

Remark 2.3.47. Using Proposition 1.2.33 and Lemma 1.2.46, one shows that the adjunction of Theorem 2.3.44 restricts to an adjunction between the wide subcategories of k-**Quiv** and $\mathbf{PCog}_\sim$ with morphisms the monomorphisms, cf. [Qui17].

Remark 2.3.48. If C is pointed, then it is hereditary if, and only if, C is isomorphic to $\widetilde{k}[\widetilde{\mathrm{GQ}}(C)]$, cf. Proposition 2.1.20. Therefore, if we restrict $\mathbf{PCog}_\sim$ to the full subcategory of hereditary pointed coalgebras, the adjunction of Theorem 2.3.44 yields an adjoint equivalence of categories.

Remark 2.3.49. Each component of the unit is a monomorphism and each component of the counit is an isomorphism. It follows from Corollary 1.1.27 that the functor $\widetilde{\mathrm{GQ}}(-)$ is faithful and that $\widetilde{k}[-]$ is fully faithful.

Remark 2.3.50. The unit and counit of the adjunction from Theorem 2.3.44 define bijections

$$\Psi = \Psi_{C,VQ} : \mathrm{Hom}_{\mathbf{PCog}_\sim}(C, \widetilde{k}[VQ]) \to \mathrm{Hom}_{k\text{-}\mathbf{Quiv}}(\widetilde{\mathrm{GQ}}(C), VQ), \qquad (2.3.51)$$

with $\Psi([\rho]) = \varepsilon_{VQ}\widetilde{\mathrm{GQ}}([\rho])$ and $\Psi^{-1}(\varphi) = \widetilde{k}[\varphi]\eta_C$, see Proposition 1.1.23.

Remark 2.3.52. The adjunction of Theorem 2.3.44 may be compared with a similar, but different adjunction due to [Rad82]. On the "combinatorial side", Radford's category $(\mathscr{S}\mathscr{V})_k$ is equivalent to k-**Quiv**, but the "algebraic" categories $(\mathscr{C}_p\mathscr{E})_k$ and $\mathbf{PCog}_\sim$ are non-equivalent. While the left adjoint functor $\widetilde{\mathrm{GQ}}(-)$ above corresponds to the Gabriel k-quiver construction, the left adjoint functor in [Rad82] is better thought of as giving a Peirce decomposition of a coalgebra, cf. [HGK10, §2.1] for Peirce decompositions of algebras or [CG02] for a related approach to coalgebras using idempotents. In order to see that the functors are fundamentally different, one may observe that the image of the unit map of Radford's adjunction applied to the coalgebra $k[VQ]$ of Example 2.3.28 does not yield an admissible subcoalgebra.

Example 2.3.53. The adjunction of Theorem 2.3.44 allows us to describe the automorphisms of the path coalgebra $\widetilde{k}[VQ]$ in terms of automorphisms of the corresponding k-quiver VQ. In the following examples we suppress notation: an arrow that should be labelled with a vector space of dimension 1 will be left unlabelled.

1. Consider two k-quivers with underlying graphs:

$$\overline{VQ} = \mathbb{A}_\infty \; : \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

$$\overline{VR} = {}_\infty\mathbb{A}_\infty \; : \qquad \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

   An automorphism of $\widetilde{k}[VQ]$ must fix the vertices. Hence,

$$\mathrm{Aut}_{\mathbf{PCog}_\sim}(\widetilde{k}[VQ]) \cong \prod_{n\in\mathbb{N}} k^\times,$$

   where $k^\times$ is the group of units of k and the product is indexed by the arrow spaces.

   An automorphism of $\widetilde{k}[VR]$ can shift the vertices. Hence,

$$\mathrm{Aut}_{\mathbf{PCog}_\sim}(\widetilde{k}[VR]) \cong \left(\prod_{n\in\mathbb{Z}} k^\times\right) \rtimes \mathbb{Z}.$$

   Note that the automorphism groups of both these coalgebras in **PCog** are quite a bit larger, because for example in $\mathbf{PCog}_\sim$ we don't distinguish between the identity and the automorphism that sends a element x of the arrow space $VQ_{e,f}$ to $x + (f - e)$.

2. If VQ is the k-quiver with one vertex and a loop indexed by the vector space V, then we have $\mathrm{Aut}_{\mathbf{PCog}_\sim}(\widetilde{k}[VQ]) \cong \mathrm{GL}(V) = \mathrm{Aut}_k(V)$. The k-quivers of this form are the only connected k-quivers for which the corresponding automorphism groups in **PCog** and in $\mathbf{PCog}_\sim$ are equal.

3. For the *Kronecker* k-quiver

$$K_V \; : \qquad \bullet \xrightarrow{\;V\;} \bullet$$

with V a vector space, we have

$$\mathrm{Aut}_{\mathbf{PCog}_\sim}(\widetilde{k}[K_V]) \cong \mathrm{GL}(V).$$

2.4 Parallel for pseudocompact algebras

In this section we present two adjunctions for pseudocompact algebras as consequence of the adjunction of Theorem 2.3.44. More precisely, the duality between the category of pseudocompact algebras and coalgebras provide, together with Theorem 2.3.44, a pair of contravariant functors between the quotient category of pointed pseudocompact algebras $\mathbf{PAlg}_\sim$ and the category of k-quivers k-**Quiv**, which are adjoint on the left. Furthermore, the category k-**Quiv** is equivalent to the category **ParPCog**, consisting of pairs of pointed cosemisimple coalgebras and bicomodules; the dual category **ParPAlg** is well defined and consists of topologically semisimple pseudocompact algebras and pseudocompact bimodules; we obtain an adjunction between the categories **PAlg** and **ParPAlg**, which extends Theorem 2.2.6.

2.4.1 Preliminaries and categories

Recall from Section 1.4 that **PAlg** denotes the category of pointed pseudocompact algebras and continuous algebra homomorphisms.

Let $\alpha, \beta : A \to B$ be two homomorphisms in **PAlg**. We write $\alpha \sim \beta$ if

$$(\alpha - \beta)(A) \subseteq J(B), \quad (\alpha - \beta)\big(J(A)\big) \subseteq J^2(B), \tag{2.4.1}$$

see (2.2.3). As with coalgebras, one easily checks that $\sim$ defines a congruence relation on **PAlg**. We denote by $\mathbf{PAlg}_\sim$ the corresponding quotient category. The relation $\sim$ for pseudocompact algebras is dual to the relation $\sim$ for coalgebras in the following sense:

Proposition 2.4.2. *Let $\rho, \gamma : C \to D$ be two homomorphisms in* **PCog**. *Then $\rho \sim \gamma$ if, and only if, $\rho^* \sim \gamma^*$ in* **PAlg**.

Proof. If $\rho', \gamma' : A \to B$ are homomorphisms of pseudocompact algebras, the condition $\rho' \sim \gamma'$ can be interpreted as saying that the compositions

$$A \xrightarrow{\rho' - \gamma'} B \to \frac{B}{J(B)}, \quad J(A) \xrightarrow{\rho' - \gamma'} J(B) \to \frac{J(B)}{J^2(B)} \tag{2.4.3}$$

are the zero map, while if $\rho, \gamma : C \to D$ are homomorphisms of coalgebras, the condition $\rho \sim \gamma$ can be interpreted as saying that the compositions

$$C_0 \to C \xrightarrow{\rho - \gamma} D, \quad \frac{C_1}{C_0} \to \frac{C}{C_0} \xrightarrow{\rho - \gamma} \frac{D}{D_0} \tag{2.4.4}$$

are the zero map. The proposition is thus a formal consequence of duality. □

Proposition 2.4.5. *The duality functors* $(-)^*$ *between* **PCog** *and* **PAlg** *induce a duality between the categories* $\mathbf{PCog}_{\sim}$ *and* $\mathbf{PAlg}_{\sim}$.

Proof. Immediate from Proposition 2.4.2. □

One proves as in [IM20, Lemma 3.8] (or by dualizing Lemma 2.3.9) that given $\alpha, \beta : A \to B$ in **PAlg**, if $\alpha \sim \beta$ then $(\alpha - \beta)\big(J^n(A)\big) \subseteq J^{n+1}(B)$ for every $n \geqslant 0$.

2.4.2 Contravariant adjoint functors

We obtain a new, contravariant adjunction immediately from the adjunction of Theorem 2.3.44 and the duality of categories of Proposition 2.4.5:

Define the contravariant functors

$$\begin{aligned} \widetilde{k}[[-]] : k\text{-}\mathbf{Quiv} &\longrightarrow \mathbf{PAlg}_{\sim}, & \widetilde{\mathrm{GQ}}((-)) : \mathbf{PAlg}_{\sim} &\longrightarrow k\text{-}\mathbf{Quiv}, \\ VQ &\longmapsto \widetilde{k}[VQ]^* & A &\longmapsto \widetilde{\mathrm{GQ}}(A^*) \end{aligned} \tag{2.4.6}$$

with the obvious definition for morphisms. We have

Theorem 2.4.7. *The functors* $\widetilde{\mathrm{GQ}}((-))$ *and* $\widetilde{k}[[-]]$ *are adjoint on the left.*

Proof. This is completely formal. Given $A \in \mathbf{PAlg}$ and $VQ \in k\text{-}\mathbf{Quiv}$ we have

$$\begin{aligned} \mathrm{Hom}_{k\text{-}\mathbf{Quiv}}(\widetilde{\mathrm{GQ}}((A)), VQ) &= \mathrm{Hom}_{k\text{-}\mathbf{Quiv}}(\widetilde{\mathrm{GQ}}(A^*), VQ) \\ &\cong \mathrm{Hom}_{\mathbf{PCog}_{\sim}}(A^*, \widetilde{k}[VQ]) \qquad (2.4.8) \\ &\cong \mathrm{Hom}_{\mathbf{PAlg}_{\sim}}(\widetilde{k}[VQ]^*, A^{**}) \qquad (2.4.9) \\ &\cong \mathrm{Hom}_{\mathbf{PAlg}_{\sim}}(\widetilde{k}[[VQ]], A), \qquad (2.4.10) \end{aligned}$$

where (2.4.8) is due to Theorem 2.3.44 and (2.4.9) (and (2.4.10)) is due to the duality of Proposition 2.4.5, as required (see Definition 1.1.20). □

2.4.3 Covariant adjoint functors

As briefly explained in Section 2.2, Iusenko and MacQuarrie [IM20] define a pair of covariant adjoint functors between a certain category of finite k-quivers and a category whose objects are pseudocompact pointed algebras A such that $\frac{A}{J^2(A)}$ is finite dimensional and whose morphisms are (congruence classes of) those algebra homomorphisms $\alpha : A \to B$ such that the induced map $\frac{A}{J(A)} \to \frac{B}{J(B)}$ is an surjection. The adjunctions from Theorem 2.3.44 and Theorem 2.4.7 are far more general, because there are no finiteness assumptions and there are no conditions on the algebra homomorphisms. We show in this section that if one is willing to leave behind the notion of quiver, one can in fact extend the adjunction of Theorem 2.2.6 to this same level of generality.

The category k-**Quiv** defined in Section 2.3.1 is isomorphic to the "category of pairs", which we define below:

Definition 2.4.11. Denote by **ParPCog** the category whose objects are pairs (Σ, V), where Σ is a pointed cosemisimple coalgebra and V is a Σ-bicomodule. A morphism in $\mathrm{Hom}_{\mathbf{ParPCog}}\big((\Sigma, V), (\Sigma', V')\big)$ is a pair (φ_0, φ_1) consisting of a coalgebra homomorphism $\varphi_0 : \Sigma \to \Sigma'$ and a Σ'-bicomodule homomorphism $\varphi_1 : V \to V'$, with V treated as a Σ'-bicomodule via φ_0 (see Remark 1.2.54).

Lemma 2.4.12. *The categories k-**Quiv** and **ParPCog** are isomorphic.*

Proof. For k-quivers VQ, VR and any map of k-quivers $\varphi : VQ \to VR$, define

$$P(VQ) = (\Sigma_Q, V_Q), \quad P(\varphi) = (\varphi_0, \varphi_1) : (\Sigma_Q, V_Q) \to (\Sigma_R, V_R), \tag{2.4.13}$$

where Σ_Q is the group-like coalgebra of VQ_0 and V_Q is the Σ_Q-bicomodule $\bigoplus_{g,h \in VQ_0} VQ_{g,h}$, as in Definition 2.3.16, and φ_0, $\varphi_1 = \sum_{g,h \in VQ_0} \varphi_{g,h}$ are the respective linear extensions. Hence, $P : k\text{-}\mathbf{Quiv} \to \mathbf{ParPCog}$ is a covariant functor.

In the other direction, we define the covariant functor $Q : \mathbf{ParPCog} \to k\text{-}\mathbf{Quiv}$ by sending (Σ, V) to the k-quiver VQ having vertices $VQ_0 = \mathrm{G}(\Sigma)$, the set of group-like elements of Σ, and, for each pair $g, h \in \mathrm{G}(\Sigma)$,

$$VQ_{g,h} = \{v \in V \mid \mu(v) = h \otimes v \text{ and } \nu(v) = v \otimes g\}. \tag{2.4.14}$$

In order to see that the arrow space $VQ_{g,h}$ is well defined, observe that $\mathrm{G}(\Sigma)$ is a vector space basis for Σ (see Proposition 1.3.5 and Remark 1.3.7) and, using the structure of (bi)comodule of V (see equations 1.2.35 and 1.2.36), any element $v \in V$ can be uniquely written as $v = \sum_{g,h \in \mathrm{G}(\Sigma)} v_{g,h}$, for only finitely many $v_{g,h} \neq 0$ and such that $\mu(v_{g,h}) = h \otimes v_{g,h}$ and $\nu(v_{g,h}) = v_{g,h} \otimes g$. The action on morphisms is simply the restrictions $\varphi_0 = \varphi_0|_{VQ_0}$ and $\varphi_{g,h} = \varphi_1|_{VQ_{g,h}}$. Since $V = \bigoplus VQ_{g,h}$, it is readily seen that the composites QP and PQ are identity functors and, therefore, $Q = P^{-1}$ and P is an isomorphism. □

Definition 2.4.15. Dually, define the category **ParPAlg** to be the category whose objects are pairs (A, U) with A a pointed topologically semisimple pseudocompact algebra and U a pseudocompact A-bimodule. A morphism $(A, U) \to (A', U')$ is a pair (ϕ_0, ϕ_1) consisting of a continuous algebra homomorphism $\phi_0 : A \to A'$ and a continuous A-bimodule homomorphism $\phi_1 : U \to U'$, with U' treated as an A-bimodule via ϕ_0.

Lemma 2.4.16. *The categories **ParPCog** and **ParPAlg** are dual.*

Proof. The assignment $(\Sigma, V) \mapsto (\Sigma^*, V^*)$ clearly defines a duality. □

By composing,

Corollary 2.4.17. *The category k-**Quiv** is dual to the category **ParPAlg**.*

Remark 2.4.18. One could alternatively dualize the category of k-quivers directly, but this is awkward and one loses combinatorial intuition anyway, because the dual of a map of (normal) k-quivers that is not injective on vertices will not be a map of directed graphs between the dual quivers (vertices do not go to vertices).

Consider the covariant functor

$$T[[-]] : \mathbf{ParPAlg} \to \mathbf{PAlg}_{\sim} \tag{2.4.19}$$

given on objects by $T[[(A,U)]] := \mathrm{T}[[A,U]]$ and on morphisms via the universal property of the complete tensor algebra, Proposition 1.4.45, i.e. given $\phi : (A,U) \to (A',U')$, $T[[\phi]] : \mathrm{T}[[A,U]] \to \mathrm{T}[[A',U']]$ is the unique continuous algebra homomorphism such that $T[[\phi]]|_A = \phi_0$ and $T[[\phi]]|_U = \phi_1$.

Consider the covariant functor

$$G[[-]] : \mathbf{PAlg}_{\sim} \to \mathbf{ParPAlg} \tag{2.4.20}$$

given on objects by $G[[A]] := \left(\frac{A}{J(A)}, \frac{J(A)}{J^2(A)}\right)$. Let $\alpha \in \mathrm{Hom}_{\mathbf{PAlg}}(A,B)$. Since B is a pointed, therefore basic, Lemma 1.4.31 implies that $\alpha\big(J(A)\big) \subseteq J(B)$ and $\alpha\big(J^2(A)\big) \subseteq J^2(B)$. Hence, the induced map $\alpha_0 : \frac{A}{J(A)} \to \frac{B}{J(B)}$, given by $\alpha_0\big(a + J(A)\big) = \alpha(a)J(B)$ is a continuous algebra homomorphism, and $\alpha_1 : \frac{J(A)}{J^2(A)} \to \frac{J(B)}{J^2(B)}$, given by $\alpha_1\big(a' + J^2(A)\big) = \alpha(a')J^2(B)$, is a continuous homomorphism of $\frac{A}{J(A)}$-bimodules. Define $G[[\alpha]] = \big([\alpha_0], [\alpha_1]\big)$.

We have the following diagram of categories and functors, wherein arrows marked E are equivalences and arrows marked D are dualities:

$$\begin{array}{ccccc} \mathbf{ParPCog} & \xleftrightarrow{\;E\;} & k\text{-}\mathbf{Quiv} & \underset{\widetilde{\mathrm{GQ}}(-)}{\overset{\widetilde{k}[-]}{\rightleftarrows}} & \mathbf{PCog}_{\sim} \\ {\scriptstyle D}\updownarrow & & & & \updownarrow{\scriptstyle D} \\ \mathbf{ParPAlg} & & \underset{G[[-]]}{\overset{T[[-]]}{\rightleftarrows}} & & \mathbf{PAlg}_{\sim} \end{array} \tag{2.4.21}$$

Lemma 2.4.22. *In the above diagram (2.4.21), the composition*

$$\mathbf{ParPAlg} \to \mathbf{ParPCog} \to k\text{-}\mathbf{Quiv} \to \mathbf{PCog}_{\sim} \to \mathbf{PAlg}_{\sim} \tag{2.4.23}$$

is naturally isomorphic to $T[[-]]$*, and the composition*

$$\mathbf{PAlg}_{\sim} \to \mathbf{PCog}_{\sim} \to k\text{-}\mathbf{Quiv} \to \mathbf{ParPCog} \to \mathbf{ParPAlg} \tag{2.4.24}$$

is naturally isomorphic to $G[[-]]$.

Proof. Simple checks, where (2.4.23) follows from Lemma 1.4.47 and (2.4.24) follows from Proposition 1.4.22 (see also Remark 1.4.25). □

Theorem 2.4.25. *The functor* $T[[-]]$ *is left adjoint to the functor* $G[[-]]$.

Proof. Immediate from Lemma 2.4.22 and Theorem 2.3.44. □

Theorem 2.2.6 can be interpreted as a special case of Theorem 2.4.25: The subcategory $\mathcal{F}$ of **ParPAlg** whose objects are those pairs (A,U) with both A,U finite dimensional and whose morphisms are those (φ_0, φ_1) with φ_0 surjective, is equivalent to the category of finite pointed quivers, see Definition 2.2.1. On the algebra side we restrict $\mathbf{PAlg}_\sim$ to the category $\mathcal{A}$ whose objects are those algebras A in $\mathbf{PAlg}_\sim$ with $\frac{A}{J^2(A)}$ finite dimensional, and whose morphisms are (congruence classes of) those algebra homomorphisms $A \to B$ such that the induced map $\frac{A}{J(A)} \to \frac{B}{J(B)}$ is surjective. Then, Theorem 2.4.25 restrict to adjoint functors

$$\mathcal{F} \underset{G[\![-]\!]}{\overset{T[\![-]\!]}{\rightleftarrows}} \mathcal{A} \tag{2.4.26}$$

and this adjunction is Theorem 2.2.6.

Chapter 3

Related adjunctions

As discussed in the previous chapter, representations of finite dimensional algebras over an algebraically closed field are related to representations of quivers. Moreover, the finite dimensional restriction on the algebras could be surpassed through (pointed) coalgebras (or pseudocompact algebras). Gabriel defined and worked with the concept of k-species, which generalized k-quivers. This field was further developed, mainly by Dlab and Ringel, characterizing the finite and tame representation types of a tensor algebra of a finite k-species in terms of its underlying diagram. Moreover, any finite dimensional basic algebra over a perfect field is a quotient of a tensor algebra of some k-species by an admissible ideal.

In this Chapter we make a brief introduction to k-species and then, with similar constructions from the previous chapter, generalize Theorem 2.3.44, constructing an adjunction between the category of coalgebras with separable coradical and filtered coalgebra homomorphisms and the category of pairs of cosemisimple coalgebras with separable coradical and bicomodules. Moreover, when restricted to basic coalgebras, the category of pairs is isomorphic to an analogous category of k-species for coalgebras.

3.1 k-species

In this section we present the basics of k-species.

Definition 3.1.1. A *k-species* $S = (K_i, E_{i,j})_{i,j\in I}$ consists of a family of finite dimensional division algebras, $\{K_i\}_{i\in I}$, together with K_j-K_i-bimodules, $E_{i,j}$, for each $i, j \in I$, cf. [Gab73, §7.1].

A *morphism of k-species*, $f : (K_i, E_{i,j})_{i,j\in I} \to (K'_{i'}, E'_{i',j'})_{i',j'\in I'}$, consists of an index function $\hat{f} : I \to I'$, a family of algebra homomorphisms $f_i : K_i \to K'_{\hat{f}(i)}$ together with homomorphisms of K_j-K_i-bimodules $f_{i,j} : E_{i,j} \to E'_{\hat{f}(i),\hat{f}(j)}$, where $E'_{\hat{f}(i),\hat{f}(j)}$ is treated as a K_j-K_i-bimodule via f_i and f_j, i.e. the structure of $E'_{\hat{f}(i),\hat{f}(j)}$ is given by $be'a = f_j(b)e' f_i(a)$ for any $a \in K_i$, $b \in K_j$ and $e' \in E'_{\hat{f}(i),\hat{f}(j)}$, cf. [Lem12, Definition 3.3].

A k-species is *finite* if I is a finite set and $\dim_k E_{i,j} < \infty$, for every $i, j \in I$.

The *valued quiver* of a finite k-species S is a quiver Q_S consisting of a set of vertices I, for each $i, j \in I$ there is an arrow from i to j whenever $E_{i,j} \neq 0$, and positive integers $d_i = \dim_k K_i$ and $d_{i,j} = \dim_k E_{i,j}$, whenever $E_{i,j} \neq 0$. A finite k-species is connected if its valued quiver is connected and acyclic if its valued quiver has no oriented cycles.

Denote by k-**Species** the category of k-species and morphisms of k-species.

Definition 3.1.2. Let S be a (connected finite) k-species. A *representation* of S is a collection $X = (X_i, X_{i,j})_{i,j \in I}$ consisting of a K_i-vector space X_i for each $i \in I$ and a K_j-linear map $X_{i,j} : E_{i,j} \otimes_{K_i} X_i \to X_j$ for each $i, j \in I$.

A *morphism of S representations* $\theta : X \to Y$ is a collection $\theta = (\theta_i)_{i \in I}$ of K_i-linear maps $\theta_i : X_i \to Y_i$ satisfying the following commutative diagram

$$\begin{array}{ccc} E_{i,j} \otimes_{K_i} X_i & \xrightarrow{X_{i,j}} & X_j \\ {\scriptstyle \mathrm{id}_{E_{i,j}} \otimes \theta_i} \downarrow & & \downarrow {\scriptstyle \theta_j} \\ E_{i,j} \otimes_{K_i} Y_i & \xrightarrow[Y_{i,j}]{} & Y_j \end{array} \tag{3.1.3}$$

A representation X of a k-species S is *finite dimensional* if X_i is finite dimensional over K_i for every $i \in I$.

Denote by $\mathbf{rep}_k S$ the category of all finite dimensional representations of the connected finite k-species S.

Let S be a connected finite k-species. Denote by $A = \prod_{i \in I} K_i$ and $U = \bigoplus_{i,j \in I} E_{i,j}$. Then U is naturally a A-bimodule. Define

$$\mathrm{T}(S) = \mathrm{T}[A, U] \tag{3.1.4}$$

the *tensor algebra* of S, i.e. $\mathrm{T}[A,U] = \bigoplus_{n=0}^{\infty} U^{\otimes n}$, where $U^{\otimes 0} = A$ and $U^{\otimes n} = U^{\otimes n-1} \otimes_A U$.

Let $\mathrm{T}[A,U]$ be a tensor algebra. An ideal $I \subseteq \mathrm{T}[A,U]$ is called *admissible* if $\bigoplus_{n=m}^{\infty} U^{\otimes n} \subseteq I \subseteq \bigoplus_{n=2}^{\infty} U^{\otimes n}$, for some positive integer m.

Let A be a basic finite dimensional algebra. Consider $\dfrac{A}{J(A)} = \prod\limits_{i \in I} K_i$ the product of extension fields K_i of k and $\dfrac{J(A)}{J^2(A)} = \bigoplus\limits_{i,j \in I} U_{i,j}$ be the decomposition with K_j-K_i-bimodules $U_{i,j}$. The k-species of A is given by $S_A = (K_i, U_{i,j})_{i,j \in I}$.

Proposition 3.1.5. *Let A be a finite dimensional algebra over a perfect field k.*

1. *If A is basic, then $A \cong \dfrac{\mathrm{T}(S_A)}{I}$ for some admissible ideal I of* $\mathrm{T}(S)$*;*
2. *If A is basic and hereditary, then $A \cong \mathrm{T}(S_A)$.*

Proof. See [Lem12, Theorem 4.6]. □

Proposition 3.1.6. *Let S be a connected finite k-species. Then $\mathbf{rep}_k S$ is isomorphic to the category of (left) modules of* $\mathrm{T}(S)$.

Proof. See [DR75, Proposition 10.1]. See also [Lem12, Proposition 7.3]. □

3.2 An adjunction for larger categories

Denote by $\mathbf{Cog}^{\mathrm{filt}}$ the category whose objects are coalgebras with separable coradical and morphisms are filtered coalgebra homomorphisms. Denote by $\mathbf{ParCog}$ the category of pairs (Σ, V), where Σ is a separable coalgebra and V is a Σ-bicomodule. Given two objects (Σ, V) and (Σ', V'), a morphism $\varphi : (\Sigma, V) \to (\Sigma', V')$ is a pair (φ_0, φ_1), such that $\varphi_0 : \Sigma \to \Sigma'$ is a coalgebra homomorphism and $\varphi_1 : V \to V'$ is a Σ'-bicomodule homomorphism regarding V as a Σ'-bicomodule with structure maps $\mu = (\varphi_0 \otimes \mathrm{id})\mu_V$ and $\nu = (\mathrm{id} \otimes \varphi_0)\nu_V$.

In this section, we define a pair of functors $\mathbf{Cog}^{\mathrm{filt}} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{ParCog}$ and show that, under a congruence relation on coalgebra homomorphisms, they form an adjoint pair.

3.2.1 Functors between coalgebras and pairs with separable coradical

Let $C \in \mathbf{Cog}^{\mathrm{filt}}$. Since C has separable coradical, we can apply Theorem 1.2.29 and obtain a projection $s : C \to C_0$, which is a coalgebra homomorphism such that $s\iota_0 = \mathrm{id}_{C_0}$, for the canonical inclusion $\iota_0 : C_0 \to C$. Hence, C can be treated as a C_0-bicomodule with structure maps $\mu_s = (s \otimes \mathrm{id})\Delta$ and $\nu_s = (\mathrm{id} \otimes s)\Delta$. Furthermore, there exists a unique structure of C_0-bicomodule on $\frac{C}{C_0}$ induced by C, $\overline{\mu}_s$, that makes the canonical projection $q : C \to \dfrac{C}{C_0}$ into a C_0-bicomodule homomorphism, i.e.:

$$\begin{array}{ccc} C & \xrightarrow{\quad q \quad} & \dfrac{C}{C_0} \\ {\scriptstyle (s\otimes \mathrm{id})\Delta}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\mu}_s} \\ C_0 \otimes C & \xrightarrow[\mathrm{id}\otimes q]{} & C_0 \otimes \dfrac{C}{C_0} \end{array}$$

commutes (see Theorem 1.2.42).

Let C and D be objects of $\mathbf{Cog}^{\mathrm{filt}}$ and $\rho : C \to D$ be a filtered coalgebra homomorphism. Then C is a D_0-bicomodule with structure maps $\mu_{\rho,s} = (\rho s \otimes \mathrm{id})\Delta$ and $\nu_{\rho,s} = (\mathrm{id} \otimes \rho s)\Delta$. D is also a D_0-bicomodule via a projection of coalgebras $s' : D \to D_0$, see Remark 1.2.54. It is not true, in general, that $\rho : C \to D$ is a D_0-bicomodule homomorphism, for this requires the equality $\rho s = s'\rho$. However, the induced map $\rho_1 : \frac{C_1}{C_0} \to \frac{D_1}{D_0}$ is a D_0-bicomodule homomorphism, as we define and prove in the next Lemma:

Lemma 3.2.1. *Let C and D be coalgebras with separable coradical and $\rho : C \to D$ be a filtered coalgebra homomorphism. Let $s : C \to C_0$ and $s' : D \to D_0$ be coalgebra projections which are splittings of the canonical inclusions $\iota_0 : C_0 \to C$ and $\iota_0' : D_0 \to D$, respectively. Let $q : C \to \dfrac{C}{C_0}$ and $q' : D \to \dfrac{D}{D_0}$ be the canonical projections. Consider C and D as D_0-bicomodules via ρs and s', respectively. Then, the induced map $\rho_1 = \overline{\rho}|_{\frac{C_1}{C_0}} : \frac{C_1}{C_0} \to \frac{D_1}{D_0}$ is a D_0-bicomodule homomorphism.*

Proof. Observe that

$$C_1 \xrightarrow{\Delta} C_1 \otimes C_0 + C_0 \otimes C_1 \xrightarrow{s\otimes q} C_0 \otimes \frac{C_1}{C_0}$$

shows that the induced left C_0-comodule structure on $\frac{C_1}{C_0}$ does not depend on the choice of the projection s, since q kills the left hand side of the sum and s acts as the identity on C_0. The same happens for the induced right C_0-comodule structure on $\frac{C_1}{C_0}$ and the induced left and right D_0-comodule structures on $\frac{C_1}{C_0}$ and $\frac{D_1}{D_0}$. Now, observe the diagram:

$$\begin{array}{ccc} \dfrac{C_1}{C_0} & \xrightarrow{\quad\overline{\rho}\quad} & \dfrac{D_1}{D_0} \\ {\scriptstyle \overline{\mu}_{\rho s}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\mu}_{s'}} \\ D_0 \otimes \dfrac{C_1}{C_0} & \xrightarrow[\mathrm{id}\otimes\overline{\rho}]{} & D_0 \otimes \dfrac{D_1}{D_0} \end{array} \tag{3.2.2}$$

For any $c \in C_1$, the equations (using the sigma notation, see Remark 1.2.38)

$$(\mathrm{id}_{D_0} \otimes \rho_1)\overline{\mu}_{\rho s} q(c) = (\rho s \otimes \overline{\rho} q)\Delta_C(c) = \sum_{(c)} \rho(c_{(1)}) \otimes \overline{\rho} q(c_{(2)}) = \sum_{(c)} \rho(c_{(1)}) \otimes q' \rho(c_{(2)}),$$

$$\overline{\mu}_{s'} \rho_1 q(c) = \overline{\mu}_{s'} q' \rho(c) = (s' \otimes q')\Delta_D\big(\rho(c)\big) = (s'\rho \otimes q'\rho)\Delta_C(c) = \sum_{(c)} s'\rho(c_{(1)}) \otimes q'\rho(c_{(2)}),$$

show that the diagram (3.2.2) commutes, since q is an epimorphism, ρ is filtered and $s'|_{D_0} = \mathrm{id}_{D_0}$. Therefore, $\overline{\rho}|_{\frac{C_1}{C_0}} : \frac{C_1}{C_0} \to \frac{D_1}{D_0}$ is a comodule homomorphism for the left D_0-comodules. Analogous argument works for the right comodule structures. □

Example 3.2.3. Consider the path coalgebra of the quiver:

$$Q : \quad \bullet_3 \xleftarrow{\;b\;} \bullet_2 \xleftarrow{\;a\;} \bullet_1 \tag{3.2.4}$$

and the coalgebra homomorphisms $s, s' : kQ \to kQ_0$ defined on the canonical basis $\{e_1, e_2, e_3, a, b, ba\}$ by:

$$s(x) = \begin{cases} e_i & \text{if } x = e_i,\ \text{for } i = 1, 2 \text{ or } 3 \\ 0 & \text{if } x = a, b \text{ or } ba \end{cases}; \qquad s'(x) = \begin{cases} s(x) & \text{if } x \neq b \\ e_3 - e_2 & \text{if } x = b \end{cases}. \tag{3.2.5}$$

Denote by kQ_s the kQ_0-bicomodule via s and by $kQ_{s'}$ the kQ_0-bicomodule via s'. Then, the induced map $\overline{\mathrm{id}}_{kQ} : \frac{kQ_s}{kQ_0} \to \frac{kQ_{s'}}{kQ_0}$ is not a bicomodule homomorphism, since

$$\left(\overline{\mu}_{s'}\overline{\mathrm{id}} - (\mathrm{id} \otimes \overline{\mathrm{id}})\overline{\mu}_{\mathrm{id}_s}\right)(ba) = (e_3 - e_2) \otimes \overline{a} \neq 0$$

Definition 3.2.6. For each $C \in \mathbf{Cog}^{\mathrm{filt}}$ and $\rho \in \mathrm{Hom}_{\mathbf{Cog}^{\mathrm{filt}}}(C, D)$, define

1. $F(C) = \left(C_0, \dfrac{C_1}{C_0}\right)$;
2. $F(\rho) = (\rho_0, \rho_1)$, where $\rho_0 = \rho|_{C_0} : C_0 \to D_0$ and $\rho_1 = \overline{\rho}|_{\frac{C_1}{C_0}} : \dfrac{C_1}{C_0} \to \dfrac{D_1}{D_0}$ is the unique D_0-bicomodule homomorphism making the diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{\ \rho\ } & D_1 \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ \dfrac{C_1}{C_0} & \underset{\overline{\rho}}{\dashrightarrow} & \dfrac{D_1}{D_0} \end{array} \tag{3.2.7}$$

commute.

Definition 3.2.8. For each $(\Sigma, V) \in \mathbf{ParCog}$ and $\varphi \in \mathrm{Hom}_{\mathbf{ParCog}}\left((\Sigma, V), (\Sigma', V')\right)$, define:

1. $G(\Sigma, V) = \mathrm{Cot}_\Sigma(V)$;
2. $G(\varphi) : \mathrm{Cot}_\Sigma(V) \to \mathrm{Cot}_{\Sigma'}(V')$, given by the universal property of the cotensor coalgebra (see Theorem 1.2.74):

$$\begin{array}{ccc} \mathrm{Cot}_\Sigma(V) & \overset{G(\varphi)}{\dashrightarrow} & \mathrm{Cot}_{\Sigma'}(V') \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_0'} \\ \Sigma & \underset{\varphi_0}{\longrightarrow} & \Sigma' \end{array} \qquad \begin{array}{ccc} \mathrm{Cot}_\Sigma(V) & \overset{G(\varphi)}{\dashrightarrow} & \mathrm{Cot}_{\Sigma'}(V') \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1'} \\ V & \underset{\varphi_1}{\longrightarrow} & V' \end{array} \tag{3.2.9}$$

It is clear that $G(\varphi)$ is filtered (see Remark 1.2.22).

Lemma 3.2.10. *The assignments above define covariant functors $F : \mathbf{Cog}^{\mathrm{filt}} \to \mathbf{ParCog}$ and $G : \mathbf{ParCog} \to \mathbf{Cog}^{\mathrm{filt}}$.*

Proof. Simple checks. □

Now, we define a relation on coalgebra homomorphisms which generalizes the relation defined in Subsection 2.3.2.

Definition 3.2.11. Let $\rho, \gamma \in \mathrm{Hom}_{\mathbf{Cog}^{\mathrm{filt}}}(C, D)$ and $n \in \mathbb{N} \cup \{\infty\}$. Write $C_\infty := C$ and consider $\rho \sim_n \gamma$ if

$$(\rho - \gamma)(C_0) = 0, \quad (\rho - \gamma)(C_n) \subseteq D_0. \tag{3.2.12}$$

Many of the results for $\sim$ can be easily extended to $\sim_n$, since the relation $\sim_n$ implies $\sim_m$ for $m \leqslant n$.

Lemma 3.2.13. *The relation $\sim_n$ is a congruence relation;*

Proof. Follows exactly as in Lemma 2.3.8 □

Lemma 3.2.14. *Let $\rho, \gamma : C \to D$ be filtered coalgebra homomorphisms such that $\rho \sim_n \gamma$. Then $(\rho - \gamma)(C_i) \subseteq D_{i-n}$, for each $i \geqslant n$.*

Proof. Follows by making small changes to proof of Lemma 2.3.9 □

Lemma 3.2.15. *The projection functor $\Pi_n : \mathbf{Cog}^{\text{filt}} \to \mathbf{Cog}^{\text{filt}}/\sim_n$ reflects isomorphisms. That is, if $\rho : C \to D$ is a filtered coalgebra homomorphism such that $\Pi_n(\rho) : C \to D$ is an isomorphism, then ρ is an isomorphism.*

Proof. Since our coalgebra homomorphisms are filtered and $\sim_n$ implies $\sim_1$, this follows as in the proof of Lemma 2.3.11. □

Denote by $\mathbf{Alg}^{\text{filt}}$ the category of pseudocompact algebras dual to $\mathbf{Cog}^{\text{filt}}$, i.e. a pseudocompact algebra A belongs to $\mathbf{Alg}^{\text{filt}}$ if, and only if, $\frac{A}{J(A)}$ is separable, and continuous algebra homomorphisms $\alpha \in \text{Hom}_{\mathbf{Alg}^{\text{filt}}}(A, B)$ satisfies $\alpha\big(J(A)\big) \subseteq J(B)$.

Definition 3.2.16. Let $\alpha, \beta \in \text{Hom}_{\mathbf{Alg}^{\text{filt}}}(A, B)$ and $n \in \mathbb{N} \cup \{\infty\}$. Write $J^{\infty}(A) := \{0\}$ and consider $\alpha \sim_n \beta$ if

$$(\alpha - \beta)(A) \subseteq J(A), \quad (\alpha - \beta)\big(J(A)\big) \subseteq J^n(A). \tag{3.2.17}$$

Following Proposition 2.4.2, the relations $\sim_n$ for coalgebra homomorphisms and continuous algebra homomorphisms are dual, for the corresponding categories.

Proposition 3.2.18. *Let $\rho, \gamma : C \to D$ be two homomorphisms in $\mathbf{Cog}^{\text{filt}}$. Then $\rho \sim_n \gamma$ if, and only if, $\rho^* \sim_n \gamma^*$ in $\mathbf{Alg}^{\text{filt}}$.*

Proof. Follows exactly as in Proposition 2.4.2. □

Denote by $\mathbf{Cog}_n^{\text{filt}} := \mathbf{Cog}^{\text{filt}}/\sim_n$ and by $\mathbf{Alg}_n^{\text{filt}} := \mathbf{Alg}^{\text{filt}}/\sim_n$.

Proposition 3.2.19. *If $\alpha \in \text{Hom}_{\mathbf{Alg}^{\text{filt}}}(A, B)$ is a surjection, then its image $\Pi(\rho) \in \text{Hom}_{\mathbf{Alg}_n^{\text{filt}}}(A, B)$ is an epimorphism.*

Proof. Follows from [IM20, Lemma 3.11] and observing that $\alpha\big(J(A)\big) = J(B)$, see [IM22, Corollary 3.4]. □

Hence, Lemma 2.3.13 follows for this more general context.

Lemma 3.2.20. *The congruence relation $\sim_1$ induces functors $\widetilde{G} = \Pi_1 G : \mathbf{ParCog} \to \mathbf{Cog}_1^{\text{filt}}$ and $\widetilde{F} : \mathbf{Cog}_1^{\text{filt}} \to \mathbf{ParCog}$, such that $\widetilde{F}\Pi_1 = F$.*

Proof. The functor $\widetilde{F}$ is defined by $\widetilde{F}(C) = F(C)$ and $\widetilde{F}\big([\rho]\big) = F(\rho)$ for any representative ρ of the class $[\rho]$. Thus, we must show that $\widetilde{F}$ is well defined (and unique, which is by construction), that is, if $\rho' : C \to D$ is such that $\rho \sim_1 \rho'$, then $F(\rho') = F(\rho)$. But this is obvious, since

$$(\rho - \rho')(C_0) = 0 \iff \rho|_{C_0} = \rho'|_{C_0}$$

and

$$(\rho - \rho')(C_1) \subseteq D_0 \iff q' \, (\rho - \rho')|_{C_1} = 0 \iff \overline{\rho} \; q|_{C_1} = q' \; \rho|_{C_1} = q' \; \rho'|_{C_1} = \overline{\rho'} \; q|_{C_1},$$

which implies $\overline{\rho}|_{\frac{C_1}{C_0}} = \overline{\rho'}|_{\frac{C_1}{C_0}}$ since $q|_{C_1} : C_1 \to \dfrac{C_1}{C_0}$ is an epimorphism. □

3.2.2 The above functors form an adjunction

We prove that the covariant functors $\widetilde{F}$ and $\widetilde{G}$ form an adjunction. The proof consists of presenting a unit and counit of the adjunction, showing that they are natural transformations and satisfy the triangular equalities (see Proposition 1.1.23).

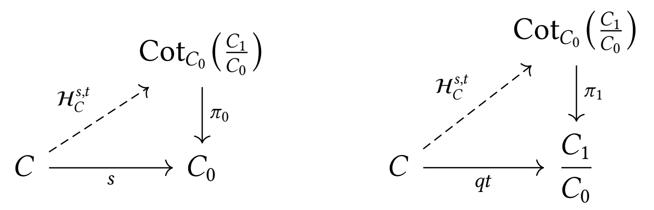
First observe that if $f, f' : C \to \mathrm{Cot}_D(M)$ are filtered coalgebra homomorphisms, then Remark 1.2.76 implies:

$$f \sim_n f' \iff \pi_0 \; f|_{C_0} = \pi_0 \; f'|_{C_0} \text{ and } \pi_1 \; f|_{C_n} = \pi_1 \; f'|_{C_n}. \tag{3.2.21}$$

Moreover, since any C_0-comodule is injective by Theorem 1.2.49, and C_1 is a C_0-subbicomodule of C, there exists a splitting $t : C \to C_1$ of the canonical inclusion $\iota_1 : C_1 \to C$, i.e. t is a C_0-bicomodule homomorphism such that $t\iota_1 = \mathrm{id}_{C_1}$.

Let us define the unit map.

Definition 3.2.22. For each $C \in \mathbf{Cog}^{\mathrm{filt}}$, define $\mathcal{H}_C^{s,t} : C \to \mathrm{Cot}_{C_0}\left(\frac{C_1}{C_0}\right)$ by the universal property of the cotensor coalgebra (see Theorem 1.2.74):



(3.2.23)

where $s : C \to C_0$ is a coalgebra projection, $t : C \to C_1$ is a splitting, $q : C_1 \to \dfrac{C_1}{C_0}$ is the canonical projection, and π_0 and π_1 are the canonical projections of graded coalgebras.

The unit map is independent of the choice of projection and splitting.

Lemma 3.2.24. *If $\hat{s} : C \to C_0$ is any other projection and $\hat{t} : C \to C_1$ any other splitting, then $\mathcal{H}_C^{s,t} \sim_1 \mathcal{H}_C^{\hat{s},\hat{t}}$.*

Proof. This follows immediately from (3.2.21), since any projection $s : C \to C_0$ restricted to C_0 is the identity id_{C_0} and any splitting $t : C \to C_1$ restricted to C_1 is the identity

id_{C_1}. □

Lemma 3.2.25. *The unit $\mathcal{H} : \mathrm{id}_{\mathbf{Cog}_1^{\mathrm{filt}}} \to \widetilde{G}\widetilde{F}$ map is a natural transformation.*

Proof. We must show that for any $[\rho] \in \mathrm{Hom}_{\mathbf{Cog}_1^{\mathrm{filt}}}(C, D)$, the diagram

$$\begin{array}{ccc} C & \xrightarrow{\mathcal{H}_C} & \mathrm{Cot}_{C_0}\left(\frac{C_1}{C_0}\right) \\ {\scriptstyle [\rho]}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{G}\widetilde{F}([\rho])} \\ D & \xrightarrow[\mathcal{H}_D]{} & \mathrm{Cot}_{D_0}\left(\frac{D_1}{D_0}\right) \end{array}$$

commutes (see Definition 1.1.14).

It is sufficient to show that for any $\rho \in \mathrm{Hom}_{\mathbf{Cog}^{\mathrm{filt}}}(C, D)$ the following equation holds:

$$\pi_i'\big(GF(\rho)\mathcal{H}_C\big)\Big|_{C_i} = \pi_i'(\mathcal{H}_D\rho)\big|_{C_i},$$

for $i = 0, 1$ (see (3.2.21)). Note that composing π_i' with $G(-)$ gives the commutative diagrams in (3.2.9), and composing π_i with $\mathcal{H}$ gives the commutative diagrams in (3.2.23), and also that $F(\rho)_0 = \rho|_{C_0}$ and $F(\rho)_1 = \overline{\rho}|_{\frac{C_1}{C_0}}$. Thus, we can combine these relations in the following commutative diagrams:

$$\begin{array}{ccc} C & & \\ \big\downarrow{\scriptstyle \mathcal{H}_C} & & \\ \mathrm{Cot}_{C_0}\left(\frac{C_1}{C_0}\right) & \xrightarrow{GF(\rho)} & \mathrm{Cot}_{D_0}\left(\frac{D_1}{D_0}\right) \\ \big\downarrow{\scriptstyle \pi_0} & & \big\downarrow{\scriptstyle \pi_0'} \\ C_0 & \xrightarrow[\rho]{} & D_0 \end{array} \quad (s : C \to C_0), \qquad \begin{array}{ccc} C & \xrightarrow{\rho} & D \\ & & \big\downarrow{\scriptstyle \mathcal{H}_D} \\ & & \mathrm{Cot}_{D_0}\left(\frac{D_1}{D_0}\right) \\ & & \big\downarrow{\scriptstyle \pi_0'} \\ & & D_0 \end{array} \quad (s' : D \to D_0),$$

which gives the equality

$$\pi_0'\,\big(GF(\rho)\mathcal{H}_C\big)\Big|_{C_0} = \rho\pi_0\mathcal{H}_C|_{C_0} = \rho\, s|_{C_0} = s'\rho|_{C_0} = \pi_0'\,(\mathcal{H}_D\rho)|_{C_0}$$

and

$$\begin{array}{ccc} C & & \\ \big\downarrow{\scriptstyle \mathcal{H}_C} & & \\ \mathrm{Cot}_{C_0}\left(\frac{C_1}{C_0}\right) & \xrightarrow{GF(\rho)} & \mathrm{Cot}_{D_0}\left(\frac{D_1}{D_0}\right) \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi_1'} \\ \frac{C_1}{C_0} & \xrightarrow[\overline{\rho}]{} & \frac{D_1}{D_0} \end{array} \quad (qt : C \to \tfrac{C_1}{C_0}), \qquad \begin{array}{ccc} C & \xrightarrow{\rho} & D \\ & & \big\downarrow{\scriptstyle \mathcal{H}_D} \\ & & \mathrm{Cot}_{D_0}\left(\frac{D_1}{D_0}\right) \\ & & \big\downarrow{\scriptstyle \pi_1'} \\ & & \frac{D_1}{D_0} \end{array} \quad (q't' : D \to \tfrac{D_1}{D_0}),$$

which gives the equality

$$\pi_1' \left(GF(\rho)\mathcal{H}_C\right)\Big|_{C_1} = \left(\rho\pi_1\mathcal{H}_C\right)\Big|_{C_1} = \overline{\rho}qt|_{C_1} = q't'\rho|_{C_1} = \pi_1' \left(\mathcal{H}_D\rho\right)|_{C_1},$$

because $t|_{C_1} = \mathrm{id}_{C_1}$ and $t'|_{D_1} = \mathrm{id}_{D_1}$ (and ρ is filtered). Thus $GF(\rho)\mathcal{H}_C \sim_1 \mathcal{H}_D\rho$, and the result follows. □

Definition 3.2.26. For each $(\Sigma, V) \in \mathbf{ParCog}$, define the morphism

$$\mathcal{E}_{(\Sigma,V)} : \left(\Sigma, \frac{\mathrm{Cot}_\Sigma(V)_1}{\Sigma}\right) \to (\Sigma, V)$$

by $\mathcal{E}_{(\Sigma,V)_0} = \mathrm{id}_\Sigma : \Sigma \to \Sigma$ and $\mathcal{E}_{(\Sigma,V)_1} : \dfrac{\mathrm{Cot}_\Sigma(V)_1}{\Sigma} = \dfrac{\Sigma \oplus V}{\Sigma} \xrightarrow{\cong} V$ to be the unique isomorphism such that the following diagram commutes (see Lemma 1.2.45):

$$\begin{array}{ccc} \mathrm{Cot}_\Sigma(V)_1 & & \\ {\scriptstyle q}\downarrow & \searrow{\scriptstyle \pi_1} & \\ \dfrac{\mathrm{Cot}_\Sigma(V)_1}{\Sigma} & \xrightarrow[\mathcal{E}_{(\Sigma,V)_1}]{} & V \end{array} \tag{3.2.27}$$

where $q : \mathrm{Cot}_\Sigma(V)_1 \to \dfrac{\mathrm{Cot}_\Sigma(V)_1}{\Sigma}$ is the canonical projection.

Observe that $t = \pi_0 + \pi_1 : \mathrm{Cot}_\Sigma(V) \to \mathrm{Cot}_\Sigma(V)_1 = \Sigma \oplus V$ is a natural splitting for the Σ-bicomodule $\mathrm{Cot}_\Sigma(V)$.

Lemma 3.2.28. *The counit $\mathcal{E} : \widetilde{F}\widetilde{G} \to \mathrm{id}_{\mathbf{ParCog}}$ map is a natural transformation.*

Proof. We must show that for any $\varphi \in \mathrm{Hom}_{\mathbf{ParCog}}\big((\Sigma, V), (\Sigma', V')\big)$, the diagram

$$\begin{array}{ccc} \left(\Sigma, \dfrac{\mathrm{Cot}_\Sigma(V)_1}{\Sigma}\right) & \xrightarrow{\mathcal{E}_{(\Sigma,V)}} & (\Sigma, V) \\ {\scriptstyle \widetilde{F}\widetilde{G}(\varphi)}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \left(\Sigma', \dfrac{\mathrm{Cot}_{\Sigma'}(V')_1}{\Sigma'}\right) & \xrightarrow[\mathcal{E}_{(\Sigma',V')}]{} & (\Sigma', V') \end{array},$$

commutes (see Definition 1.1.14).

Observe that

$$F\big(G(\varphi)\big)_0 = G(\varphi)|_\Sigma = \pi_0' G(\varphi)\iota_0 = \varphi_0\pi_0\iota_0 = \varphi_0$$

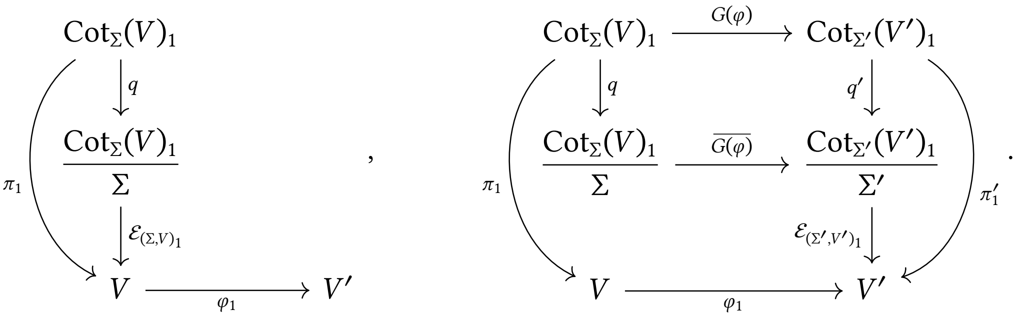
and $\mathcal{E}_{(\Sigma,V)_0} = \mathrm{id}_\Sigma$. Hence

$$\varphi_0\mathcal{E}_{(\Sigma,V)_0} = \varphi_0\mathrm{id}_\Sigma = \mathrm{id}_{\Sigma'}\varphi_0 = \mathcal{E}_{(\Sigma',V')_0}F\big(G(\varphi)\big)_0.$$

Moreover,

$$\mathcal{E}_{(\Sigma',V')}F\big(G(\varphi)\big)_1 q = \mathcal{E}_{(\Sigma',V')}\overline{G(\varphi)}q = \mathcal{E}_{(\Sigma',V')}q'G(\varphi) = \pi_1'G(\varphi) = \varphi_1\pi_1 = \varphi_1\mathcal{E}_{(\Sigma,V)_1}q$$

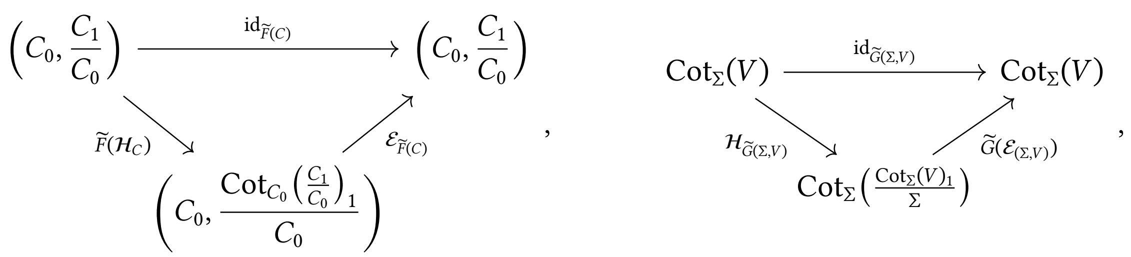
as depicted in the following commutative diagrams:



Since q is an epimorphism, it follows that $\varphi\mathcal{E}_{(\Sigma,V)} = \mathcal{E}_{(\Sigma',V')}F\big(G(\varphi)\big)$, and the result follows.

□

Lemma 3.2.29. *The triangular equalities:*



(3.2.30)

are satisfied.

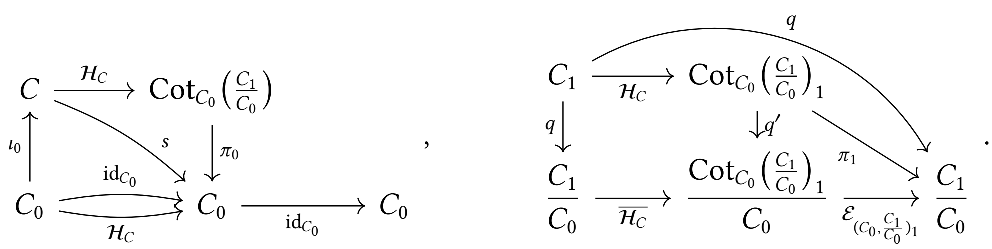
Proof. Observe that

$$\mathcal{E}_{F(C)_0}F(\mathcal{H}_C)_0 = \mathrm{id}_{C_0}\ \mathcal{H}_C|_{C_0} = \mathrm{id}_{C_0}$$

and

$$\mathcal{E}_{F(C)_1}F(\mathcal{H}_C)_1\ q|_{C_1} = \mathcal{E}_{F(C)_1}\overline{\mathcal{H}_C}\ q|_{C_1} = \mathcal{E}_{F(C)_1}q'\ \mathcal{H}_C|_{C_1} = \pi_1'\ \mathcal{H}_C|_{C_1} = qt|_{C_1} = q|_{C_1}$$

as depicted in the following commutative diagrams:



Since q is an epimorphism, it follows that $\mathcal{E}_{F(C)}F\mathcal{H}_C = \mathrm{id}_{(C_0,\frac{C_1}{C_0})}$.

For the second case, note that:

$$\pi_0\left(G\mathcal{E}_{(\Sigma,V)}\mathcal{H}_{G(\Sigma,V)}\right)\big|_{\Sigma} = \left(\mathcal{E}_{(\Sigma,V)_0}\pi_0'\mathcal{H}_{G(\Sigma,V)}\right)\big|_{\Sigma} = \left(\mathcal{E}_{(\Sigma,V)_0}s\right)\big|_{\Sigma} = \mathrm{id}_{\Sigma} = \pi_0|_{\Sigma}$$

and

$$\pi_1\left(G\mathcal{E}_{(\Sigma,V)}\mathcal{H}_{G(\Sigma,V)}\right)\big|_{\mathrm{Cot}_\Sigma(V)_1} = \left(\mathcal{E}_{(\Sigma,V)_1}\pi_1'\mathcal{H}_{G(\Sigma,V)}\right)\big|_{\mathrm{Cot}_\Sigma(V)_1} = \left(\mathcal{E}_{(\Sigma,V)_1}qt\right)\big|_{\mathrm{Cot}_\Sigma(V)_1} = \pi_1|_{\mathrm{Cot}_\Sigma(V)_1}$$

as depicted on the following commutative diagrams:

$$\begin{array}{ccccc}
\mathrm{Cot}_\Sigma(V) & \xrightarrow{\mathcal{H}_{G(\Sigma,V)}} & \mathrm{Cot}_\Sigma\left(\frac{\mathrm{Cot}_\Sigma(V)_1}{\Sigma}\right) & \xrightarrow{G\mathcal{E}_{(\Sigma,V)}} & \mathrm{Cot}_\Sigma(V) \\
& \searrow^{s} & \downarrow{\scriptstyle \pi_0'} & & \downarrow{\scriptstyle \pi_0} \\
& & \Sigma & \xrightarrow[\mathrm{id}_\Sigma]{} & \Sigma
\end{array},$$

$$\begin{array}{ccccc}
\mathrm{Cot}_\Sigma(V) & \xrightarrow{\mathcal{H}_{G(\Sigma,V)}} & \mathrm{Cot}_\Sigma\left(\frac{\mathrm{Cot}_\Sigma(V)_1}{\Sigma}\right) & \xrightarrow{G\mathcal{E}_{(\Sigma,V)}} & \mathrm{Cot}_\Sigma(V) \\
& \searrow^{qt} & \downarrow{\scriptstyle \pi_1'} & & \downarrow{\scriptstyle \pi_1} \\
& & \frac{\mathrm{Cot}_\Sigma(V)_1}{\Sigma} & \xrightarrow{\mathcal{E}_{(\Sigma,V)_1}} & V
\end{array}\quad (\text{with } \pi_1 : \mathrm{Cot}_\Sigma(V) \to V).$$

It follows that $G\mathcal{E}_{(\Sigma,V)}\mathcal{H}_{G(\Sigma,V)} \sim_1 \mathrm{id}_{\mathrm{Cot}_\Sigma(V)}$ (see (3.2.21)).

Therefore, the triangular equalities are satisfied. □

This proves the following Theorem.

Theorem 3.2.31. *The functor* $\widetilde{F} : \mathbf{Cog}_1^{\mathrm{filt}} \to \mathbf{ParCog}$ *is left adjoint to the functor* $\widetilde{G} : \mathbf{ParCog} \to \mathbf{Cog}_1^{\mathrm{filt}}$.

Proof. Follows from all previous results of this subsection. □

Corollary 3.2.32. *The unit of adjunction* $\langle \widetilde{F}, \widetilde{G}, \mathcal{H}, \mathcal{E}\rangle$, $\mathcal{H}_C : C \to \mathrm{Cot}_{C_0}\left(\frac{C_1}{C_0}\right)$, *is an injection of coalgebras.*

Proof. Note that $\mathcal{H}_C|_{C_1} : C_1 \to \mathrm{Cot}_{C_0}\left(\frac{C_1}{C_0}\right)_1$ is an isomorphism. Now the result follows from Proposition 1.2.33. □

The above corollary is [Woo97, Proposition 4.6]. See also [CHZ06, Theorem 3.1].

A subcoalgebra D of a cotensor coalgebra $\mathrm{Cot}_\Sigma(V)$ is said to be *admissible* if $D_1 \subseteq \mathrm{Cot}_\Sigma(V)_1$. Hence, every coalgebra with separable coradical is isomorphic to an admissible subcoalgebra of *its* cotensor coalgebra.

3.2.3 Basic coalgebras and k-species

Let **BCog** denote the full subcategory of $\mathbf{Cog}^{filt}$ which consists of basic coalgebras with separable coradical (and coalgebra homomorphisms) and let **ParBCog** denote the full subcategory of **ParCog** which consists of pairs of separable basic coalgebras and bicomodules. Then, Theorem 3.2.31 restricts to the adjunction:

Corollary 3.2.33. *The functor* $\widetilde{F} : \mathbf{BCog}_1 \to \mathbf{ParBCog}$ *is left adjoint to the functor* $\widetilde{G} : \mathbf{ParBCog} \to \mathbf{BCog}_1$.

We define next a category which resembles the category of k-species described in Section 3.1.

Definition 3.2.34. Denote by k-**CS** the category of separable k-*cospecies*, whose objects are pairs $(S_i, M_{i,j})_{i,j\in I}$, such that:

1. I is an index set;
2. $\{S_i\}_{i\in I}$ is a family of simple separable basic (finite dimensional) coalgebras;
3. $\{M_{i,j}\}_{i,j\in I}$ is a family of S_j-S_i-bicomodules.

A *map of* k*-cospecies* $\psi : (S_i, M_{i,j})_{i,j\in I} \to (S'_{i'}, M'_{i',j'})_{i',j'\in I'}$ consists of:

1. an index function $\hat{\psi} : I \to I'$;
2. a family of coalgebra homomorphisms $\psi_i : S_i \to S'_{\hat{\psi}(i)}$;
3. a family of bicomodule homomorphisms $\psi_{i,j} : M_{i,j} \to M'_{\hat{\psi}(i),\hat{\psi}(j)}$, where the S_j-S_i-bicomodule $M_{i,j} = (M_{i,j}, \mu_{i,j}, \nu_{i,j})$ is treated as a $S'_{\hat{\psi}(j)}$-$S'_{\hat{\psi}(i)}$-bicomodule $M_{i,j} = \big(M_{i,j}, (\psi_j \otimes \mathrm{id})\mu_{i,j}, (\mathrm{id} \otimes \psi_i)\nu_{i,j}\big)$.

In this section we show that the category **ParBCog** is equivalent to k-**CS**.

Clearly, $S := \bigoplus_{i\in I} S_i$ is a separable basic coalgebra and $M := \bigoplus_{i,j\in I} M_{i,j}$ is a S-bicomodule. Hence, the assignments:

$$P\big((S_i, M_{i,j})_{i,j\in I}\big) = \Big(\bigoplus_{i\in I} S_i, \bigoplus_{i,j\in I} M_{i,j}\Big), \qquad P(\psi) = \Big(\sum_{i\in I} \psi_i, \sum_{i,j\in I} \psi_{i,j}\Big)$$

define a covariant functor $P : k\text{-}\mathbf{CS} \to \mathbf{ParBCog}$.

Now, let $\Sigma = \bigoplus_{i\in I} \Sigma_i$ be a separable basic coalgebra, where $\Sigma_i \subseteq \Sigma$ are simple subcoalgebras of Σ, and let V be a Σ-bicomodule (i.e. $(\Sigma, V) \in \mathbf{ParBCog}$). Consider $\{s_{i,j}\}_{j\in I_i}$ a basis of Σ_i, for each $i \in I$. Then, we can describe the structures of comultiplication and left Σ-comodule as follows:

- for each $s_{i,j} \in \Sigma_i$
$$\Delta(s_{i,j}) = \sum_{l\in I_i} s_{i,l} \otimes c_{i,j,l},$$
for only finitely many nonzero $c_{i,j,l} \in \Sigma_i$;

- given a $v \in V$
$$\mu(v) = \sum_{i\in I}\sum_{j\in I_i} s_{i,j} \otimes v_{i,j},$$
for only finitely many nonzero $v_{i,j} \in V$.

Thus, the structure of left comodule, see (1.2.35), give us:

$$(\mathrm{id}\otimes\mu)\mu(v) = \sum_{i\in I}\sum_{j\in I_i} s_{i,j}\otimes\mu(v_{i,j}) = \sum_{i\in I}\sum_{j\in I_i}\sum_{l\in I_i} s_{i,l}\otimes c_{i,j,l}\otimes v_{i,j} = (\Delta\otimes\mathrm{id})\mu(v), \tag{3.2.35}$$
$$v = \sum_{i\in I}\sum_{j\in I_i}\varepsilon(s_{i,j})v_{i,j}, \tag{3.2.36}$$

where (3.2.35) shows that $\mu(v_{i,j}) = \sum_{l\in I_i} c_{i,l,j}\otimes v_{i,l} \subseteq S_i\otimes V$. Hence, the subcomodule generated by the $v_{i,j}$'s (for each fixed $i\in I$), $\langle v_{i,j} \mid j\in I_i\rangle$, is a left S_i-comodule. Moreover, (3.2.36) shows that $v \in \bigoplus_{i\in I}\langle v_{i,j} \mid j \in I_i\rangle$.

Since $v\in V$ was taken arbitrary, we have $V = \bigoplus_{i\in I} V_{i,-}$ as a left Σ-comodule, where, for each $i\in I$, $V_{i,-}$ is a left S_i-comodule. Because similar argument works for right comodules, we have $V = \bigoplus_{i,j\in I} V_{i,j}$, where $V_{i,j} = V_{i,-}\cap V_{-,j}$ is a S_i-S_j-bicomodule.

Let $\varphi \in \mathrm{Hom}_{\mathbf{ParBCog}}\big((\Sigma,V),(\Sigma',V')\big)$, with $\Sigma = \bigoplus_{i\in I} S_i$ and $\Sigma' = \bigoplus_{i'\in I'} S'_{i'}$. Since the simple sucoalgebra S_i is basic, the image $\varphi_0(S_i)$ is a simple subcoalgebra of Σ' (see in the proof of Lemma 1.2.32). Thus, the coalgebra homomorphism $\varphi_0 : \Sigma\to\Sigma'$ defines a unique index function $\hat{Q}(\varphi) : I \to I'$ such that $\varphi_0(S_i) = S'_{\hat{Q}(\varphi)(i)}$. The assignments:

$$Q\big((\Sigma,V)\big) = (S_i, V_{i,j})_{i,j\in I},$$

and,
$$Q(\varphi)_i = \varphi_0|_{S_i} : S_i \to S'_{\hat{Q}(\varphi)(i)}, \quad Q(\varphi)_{i,j} = \varphi_1|_{V_{i,j}} : V_{i,j}\to V'_{\hat{Q}(\varphi)(i),\hat{Q}(\varphi)(j)}$$
define a covariant functor $Q : \mathbf{ParBCog} \to k\text{-}\mathbf{CS}$. The above constructions wield:

Theorem 3.2.37. *The categories* **ParBCog** *and* k-**CS** *are isomorphic.*

In view of Corollary 3.2.33, we have:

Corollary 3.2.38. *The composition of functors* $Q\widetilde{F} : \mathbf{BCog}_1 \to k\text{-}\mathbf{CS}$ *is left adjoint to the composition of functors* $\widetilde{G}P : k\text{-}\mathbf{CS}\to\mathbf{BCog}_1$.

Corollary 3.2.39. *Let C be a basic coalgebra with separable coradical. Then, C is isomorphic to an admissible subcoalgebra of the cotensor coalgebra of the separable k-cospecies $Q\widetilde{F}(C)$.*

3.2.4 The dual case

As in Section 2.4, from Theorem 3.2.31 we obtain two adjunctions for the category $\mathbf{Alg}^{\mathrm{filt}}$.

Corollary 3.2.40. *The functors* $\widetilde{F}(-)^* : \mathbf{Alg}_1^{\mathrm{filt}} \to \mathbf{ParCog}$ *and* $(-)^*\widetilde{G} : \mathbf{ParCog}\to\mathbf{Alg}_1^{\mathrm{filt}}$ *are adjoint on the left.*

Proof. Follows exactly as in Theorem 2.4.7. □

Denote by **BAlg** the full subcategory of $\mathbf{Alg}^{\text{filt}}$ which consists of basic pseudocompact algebras. Combining the above Corollary with Corollary 3.2.38, we get the adjunction:

Corollary 3.2.41. *The contravariant functors* $Q\widetilde{F}(-)^* : \mathbf{BAlg}_1 \to k\text{-}\mathbf{CS}$ *and* $(-)^*\widetilde{G}P : k\text{-}\mathbf{CS} \to \mathbf{BAlg}_1$ *are adjoint on the left.*

Corollary 3.2.42. *Let A be a pseudocompact algebra such that $\frac{A}{J(A)}$ is separable. Then, $\mathcal{H}^*_{A^*} : \mathrm{T}[[\frac{A}{J(A)}, \frac{J(A)}{J^2(A)}]] \to A$ is a continuous algebra homomorphism which is a surjection and satisfies* $\ker \mathcal{H}^*_{A^*} \subseteq J^2\big(\mathrm{T}[[\frac{A}{J(A)}, \frac{J(A)}{J^2(A)}]]\big)$.

Corollary 3.2.43. *Let A be a basic pseudocompact algebra such that $\frac{A}{J(A)}$ is separable. Then, A is isomorphic to a quotient of the complete tensor algebra* $(-)^*\widetilde{G}Q\widetilde{F}(A^*)$.

Let **ParAlg** denote the category of pairs, as in Definition 2.4.15, with objects separable pseudocompact algebras and pseudocompact bimodules.

Define the covariant functors:

- $\widetilde{T} : \mathbf{ParAlg} \to \mathbf{Alg}_1^{\text{filt}}$, with $\widetilde{T}\big((A, U)\big) = \mathrm{T}[[A, U]]$;
- $\widetilde{S} : \mathbf{Alg}_1^{\text{filt}} \to \mathbf{ParAlg}$, with $\widetilde{S}(A) = \left(\frac{A}{J(A)}, \frac{J(A)}{J^2(A)}\right)$.

On morphisms, these functors are defined as $T[[-]]$ of equation 2.4.19 for $\widetilde{T}$, and $G[[-]]$ of equation 2.4.20 for $\widetilde{S}$.

Corollary 3.2.44. *The functor $\widetilde{T}$ is left adjoint to the functor $\widetilde{S}$.*

Proof. Follows as in Theorem 2.4.25. □

3.3 Peirce decomposition and Radford adjunction

One could ask if it is possible to redefine the functor F by a variant functor $F_n : \mathbf{Cog}^{\text{filt}} \to \mathbf{ParCog}$ which assigns for a coalgebra C with separable coradical the pair $F_n(C) = \left(C_0, \dfrac{C_n}{C_0}\right)$. However, as seen in Example 3.2.3, the structure of C_0-bicomodule of $\frac{C_n}{C_0}$ depend on the choice of projection $s : C \to C_0$, and the induced map $\overline{\rho}|_{\frac{C_n}{C_0}} : \frac{C_n}{C_0} \to \frac{D_n}{D_0}$ is not a D_0-bicomodule homomorphism for the filtered coalgebra homomorphism $\rho : C \to D$, in general.

In some cases, given a coalgebra projection $s : C \to C_0$, which is a splitting of the canonical inclusion $\iota_0 : C_0 \to C$, and a coalgebra homomorphism $\rho : C \to D$, one can choose a coalgebra projection $s' : D \to D_0$, which is a splitting of the canonical inclusion $\iota_0' : D_0 \to D$ such that the induced map $\overline{\rho}|_{\frac{C}{C_0}} : \frac{C}{C_0} \to \frac{D}{D_0}$ is a homomorphism of D_0-bicomodules.

Naves [Nav22], in his thesis, proved the following:

Proposition 3.3.1. *Let A be a basic pseudocompact algebra over an algebraically closed field k such that $\dim_k \frac{A}{J^2(A)} < \infty$ and B be a basic finite dimensional algebra. Consider a continuous* *algebra homomorphism* $\alpha : A \to B$ *and a splitting* $s_A : \dfrac{A}{J(A)} \to A$ *of the canonical projection* $\pi_A : A \to \dfrac{A}{J(A)}$*. Then, there exists a splitting* $s_B : \dfrac{B}{J(B)} \to B$ *of the canonical projection* $\pi_B : B \to \dfrac{B}{J(B)}$ *satisfying the commutative diagram:*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle s_A}\uparrow & & \uparrow{\scriptstyle s_B} \\ \dfrac{A}{J(A)} & \xrightarrow{\overline{\alpha}} & \dfrac{B}{J(B)} \end{array} \tag{3.3.2}$$

Moreover, if $\overline{\alpha}$ *is a surjection, then such* s_B *is unique.*

Proof. See [Nav22, Theorem 3.2.4 and Corollary 3.2.7]. □

He also showed that the other way round is not always possible, i.e. given a splitting $s_B : \dfrac{B}{J(B)} \to B$, there is no splitting $s_A : \dfrac{A}{J(A)} \to A$ such that the diagram (3.3.2) commutes, see [Nav22, Remark 3.2.5].

Translating these results to our case, we have the following:

Corollary 3.3.3. *Let* C *be a pointed finite dimensional coalgebra and* D *a pointed coalgebra such that* D_1 *is finite dimensional. Consider* $\rho : C \to D$ *a coalgebra homomorphism and* $s' : D \to D_0$ *a splitting of the canonical inclusion* $\iota' : D_0 \to D$*. Then, there exists a splitting* $s : C \to C_0$ *of the canonical inclusion* $\iota : C_0 \to C$ *such that the induced map* $\overline{\rho}|_{\frac{C}{C_0}} : \frac{C}{C_0} \to \frac{D}{D_0}$ *is a homomorphism of* D_0*-bicomodules.*

Unfortunately, this is very restrictive and not good enough to apply for our intent.

Radford [Rad82] considered a category $\mathcal{C}$ whose objects are pointed coalgebras C together with a coalgebra projection $s : C \to C_0$ (or equivalently a coideal I such that $C = C_0 \oplus I$) and a morphism $\rho : (C, s) \to (D, s')$ is a coalgebra homomorphism $\rho : C \to D$ such that $\rho s = s' \rho$ (or equivalently $\rho(I) \subseteq I'$, for $I' = \ker s'$). He proved that the covariant functor $F' : \mathcal{C} \to k\text{-}\mathbf{Quiv}$, given by $F(C, I) = (\mathrm{G}(C), I)$ and $F(\rho) = (\rho|_{\mathrm{G}(C)}, \rho|_I)$, where $\mathrm{G}(C)$ is the set of group-like elements of C (see Definition 1.3.3), is left adjoint to the covariant functor $G' : k\text{-}\mathbf{Quiv} \to \mathcal{C}$, given by $G'(VQ) = \left(\mathrm{Cot}_{\Sigma_Q}(V_Q), \bigoplus_{n \geqslant 1} V_Q^{\square n}\right)$ (where Σ_Q and V_Q are as in Definition 2.3.16) and on morphism is given by the universal property of the cotensor coalgebra.

Remark 3.3.4. Observe that for $\rho, \gamma \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ such that

$$(\rho - \gamma)(C_0) = 0, \quad (\rho - \gamma)(C) \subseteq D_0,$$

then $\rho = \gamma$, since

$$(\rho - \gamma)(C) = (\rho - \gamma)(C_0) + (\rho - \gamma)(I) \subseteq I' \cap D_0 = \{0\}.$$

Thus, for coalgebra homomorphisms ρ, γ in $\mathcal{C}$, we have $\rho = \gamma \iff \rho \sim_\infty \gamma$ (see Definition 3.2.11).

In this section, following Radford's idea of defining a category with a lot more objects and fewer morphisms (between two objects), we show that the functor F_n can be defined for the right categories and induce an adjunction, which generalize the adjuntion of Radford [Rad82].

Consider the category **Cs** whose objects are pairs (C, s), where C is a coalgebra with separable coradical and $s : C \to C_0$ is a splitting of the canonical inclusion $\iota_0 : C_0 \to C$. A morphism $\rho : (C, s) \to (D, s')$ is a filtered coalgebra homomorphisms $\rho : C \to D$ such that $\rho s = s' \rho$. Denote by $\mathbf{Cs}_n$ the quotient category given by the projection functor $\Pi_n : \mathbf{Cs} \to \mathbf{Cs}/\sim_n$, see Definition 3.2.11. By Remark 3.3.4, $\mathbf{Cs}_\infty = \mathbf{Cs}$. Moreover, $\mathcal{C}$ is the full subcategory of **Cs** restricted to pointed coalgebras.

Define the covariant functor $F_n : \mathbf{Cs} \to \mathbf{ParCog}$ given by $F_n\big((C, s)\big) = \left(C_0, \dfrac{C_n}{C_0}\right)$, where $\frac{C_n}{C_0}$ is a C_0-bicomodule with induced structure via s, and $F_n(\rho) = (\rho|_{C_0}, \overline{\rho}|_{\frac{C_n}{C_0}})$.

Define the covariant functor $G_n : \mathbf{ParCog} \to \mathbf{Cs}$ given by $G_-\big((\Sigma, V)\big) = \big(\mathrm{Cot}_\Sigma(V), \pi_0\big)$, where $\pi_0 : \mathrm{Cot}_\Sigma(V) \to \Sigma$ is the canonical projection of graded algebras, and $G(\varphi) : \mathrm{Cot}_\Sigma(V) \to \mathrm{Cot}_{\Sigma'}(V')$ is given by the universal property as in (3.2.9), which is compatible with the projections by construction.

The projection functor Π_n induces covariant functors $\widetilde{F}_n : \mathbf{Cs}_n \to \mathbf{ParCog}$, such that $F_n = \widetilde{F}_n \Pi_n$, and $\widetilde{G}_n = \Pi_n G_- : \mathbf{ParCog} \to \mathbf{Cs}_n$ (c.f. Lemma 3.2.20).

These lead us to the following:

Theorem 3.3.5. *The covariant functor $\widetilde{F}_n : \mathbf{Cs}_n \to \mathbf{ParCog}$ is left adjoint to the covariant functor $\widetilde{G}_n : \mathbf{ParCog} \to \mathbf{Cs}_n$.*

The proof follows pretty much the same as in Subsection 3.2.2 replacing the level one of the coradical filtration for level n, with other small changes which we describe below.

Definition 3.3.6. For each $(C, s) \in \mathbf{Cs}$, define $\mathcal{H}_C^{s,t} : (C, s) \to \left(\mathrm{Cot}_{C_0}\left(\frac{C_n}{C_0}\right), \pi_0\right)$ by the universal property of the cotensor coalgebra (see Theorem 1.2.74):

$$
\begin{array}{ccc}
 & & \mathrm{Cot}_{C_0}\left(\frac{C_n}{C_0}\right) \\
 & \overset{\mathcal{H}_C^{s,t}}{\nearrow} & \downarrow \pi_0 \\
C & \xrightarrow[s]{} & C_0
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & \mathrm{Cot}_{C_0}\left(\frac{C_n}{C_0}\right) \\
 & \overset{\mathcal{H}_C^{s,t}}{\nearrow} & \downarrow \pi_1 \\
C & \xrightarrow[qt]{} & \dfrac{C_n}{C_0}
\end{array}
\tag{3.3.7}
$$

where $t : C \to C_n$ is a splitting of the canonical inclusion of C_0-bicomodules $\iota_n : C_n \to C$, $q : C_n \to \dfrac{C_n}{C_0}$ is the canonical projection, and π_0 and π_1 are the canonical projections of graded coalgebras.

Lemma 3.3.8. *If $\hat{t} : C \to C_n$ any other splitting, then $\mathcal{H}_C^{s,t} \sim_n \mathcal{H}_C^{s,\hat{t}}$.*

Proof. Follows immediately from (3.2.21) since any splitting $t : C \to C_n$ restricted to C_n is the identity map. □

Lemma 3.3.9. *The unit $\mathcal{H} : \mathrm{id}_{\mathbf{Cs}_n} \to \widetilde{G}_n\widetilde{F}_n$ map is a natural transformation.*

Proof. We must show that for any $[\rho] \in \mathrm{Hom}_{\mathbf{Cs}_n}\big((C,s)(D,s')\big)$, the diagram

$$\begin{array}{ccc} (C,s) & \xrightarrow{\mathcal{H}^s_C} & \left(\mathrm{Cot}_{C_0}\left(\frac{C_n}{C_0}\right),\pi_0\right)\\ {\scriptstyle[\rho]}\Big\downarrow & & \Big\downarrow{\scriptstyle \widetilde{G}_n\widetilde{F}_n([\rho])}\\ (D,s') & \xrightarrow[\mathcal{H}^{s'}_D]{} & \left(\mathrm{Cot}_{D_0}\left(\frac{D_n}{D_0}\right),\pi'_0\right)\end{array}$$

commutes (see Definition 1.1.14).

It is sufficient to show that for any $\rho \in \mathrm{Hom}_{\mathbf{Cs}}\big((C,s)(D,s')\big)$ the following equations hold:

$$\pi'_0\big(G_-(F_n(\rho))\mathcal{H}^s_C\big)\Big|_{C_0} = \pi'_0(\mathcal{H}^{s'}_D\rho)\Big|_{C_0}, \quad \pi'_1\big(G_-(F_n(\rho))\mathcal{H}^s_C\big)\Big|_{C_n} = \pi'_1(\mathcal{H}^{s'}_D\rho)\Big|_{C_n}$$

(see (3.2.21)). Note that

$$\begin{array}{lll} \pi'_0G_-(\varphi) = \varphi_0\pi_0, & \pi_0\mathcal{H}^s_C = s, & F_n(\rho)_0 = \rho|_{C_0}, \\ \pi'_1G_-(\varphi) = \varphi_1\pi_1, & \pi_1\mathcal{H}^s_C = qt, & F_n(\rho)_1 = \overline{\rho}|_{\frac{C_n}{C_0}}. \end{array}$$

Thus, we can combine these relations in the following commutative diagrams:

$$\begin{array}{ccc} C & & \\ \Big\downarrow{\scriptstyle \mathcal{H}^s_C} & & \\ \mathrm{Cot}_{C_0}\left(\frac{C_n}{C_0}\right) & \xrightarrow{G_-(F_n(\rho))} & \mathrm{Cot}_{D_0}\left(\frac{D_n}{D_0}\right) \\ \Big\downarrow{\scriptstyle \pi_0} & & \Big\downarrow{\scriptstyle \pi'_0} \\ C_0 & \xrightarrow[\rho]{} & D_0 \end{array} \quad (s : C \to C_0), \qquad \begin{array}{ccc} C & \xrightarrow{\rho} & D \\ & & \Big\downarrow{\scriptstyle \mathcal{H}^{s'}_D} \\ & & \mathrm{Cot}_{D_0}\left(\frac{D_n}{D_0}\right) \\ & & \Big\downarrow{\scriptstyle \pi'_0} \\ & & D_0 \end{array} \quad (s' : D \to D_0),$$

which gives the equality

$$\pi'_0\left(G_-(F_n(\rho))\mathcal{H}^s_C\right)\Big|_{C_0} = F_n(\rho)_0\pi_0\mathcal{H}^s_C\big|_{C_0} = \rho|_{C_0}\, s|_{C_0} = s'\rho|_{C_0} = \pi'_0\left(\mathcal{H}^{s'}_D\rho\right)\Big|_{C_0}$$

and

$$
\begin{array}{ccc}
C & & \\
\downarrow{\scriptstyle \mathcal{H}^s_C} & & \\
\mathrm{Cot}_{C_0}\left(\frac{C_n}{C_0}\right) & \xrightarrow{G_-(F_n(\rho))} & \mathrm{Cot}_{D_0}\left(\frac{D_n}{D_0}\right) \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1'} \\
\dfrac{C_n}{C_0} & \xrightarrow[\overline{\rho}]{} & \dfrac{D_n}{D_0}
\end{array}
\quad (C \xrightarrow{qt} \tfrac{C_n}{C_0}),
\qquad
\begin{array}{ccc}
C & \xrightarrow{\rho} & D \\
 & & \downarrow{\scriptstyle \mathcal{H}^{s'}_D} \\
 & & \mathrm{Cot}_{D_0}\left(\frac{D_n}{D_0}\right) \\
 & & \downarrow{\scriptstyle \pi_1'} \\
 & & \dfrac{D_n}{D_0}
\end{array}
\quad (D \xrightarrow{q't'} \tfrac{D_n}{D_0}),
$$

which gives the equality

$$
\pi_1' \left(G_-(F_n(\rho))\mathcal{H}^s_C\right)\Big|_{C_n} = \left(F_n(\rho)_1 \pi_1 \mathcal{H}^s_C\right)\Big|_{C_n} = \overline{\rho} qt|_{C_n} = q't'\rho\big|_{C_n} = \pi_1' \left(\mathcal{H}^{s'}_D \rho\right)\Big|_{C_n},
$$

because $t|_{C_n} = \mathrm{id}_{C_n}$ and $t'|_{D_n} = \mathrm{id}_{D_n}$ (and ρ is filtered). Thus $G_-\big(F_n(\rho)\big)\mathcal{H}^s_C \sim_n \mathcal{H}^{s'}_D \rho$, and the result follows. □

Definition 3.3.10. For each $(\Sigma, V) \in \mathbf{ParCog}$, define:

- $\mathcal{E}_{(\Sigma,V)_0} = \mathrm{id}_\Sigma : \Sigma \to \Sigma$;
- $\mathcal{E}_{(\Sigma,V)_1} : \dfrac{\mathrm{Cot}_\Sigma(V)_n}{\Sigma} = \dfrac{\Sigma \oplus \left(\bigoplus_{i=1}^n V^{\square i}\right)}{\Sigma} \cong \displaystyle\bigoplus_{i=1}^n V^{\square i} \twoheadrightarrow V$, is the unique comodule projection such that the following diagram commutes (see (1.1.9)):

$$
\begin{array}{ccc}
\mathrm{Cot}_\Sigma(V)_n & & \\
\downarrow{\scriptstyle q} & \searrow{\scriptstyle \pi_1} & \\
\dfrac{\mathrm{Cot}_\Sigma(V)_n}{\Sigma} & \xrightarrow[\mathcal{E}_{(\Sigma,V)_1}]{} & V
\end{array}
\tag{3.3.11}
$$

where $q : \mathrm{Cot}_\Sigma(V)_n \to \dfrac{\mathrm{Cot}_\Sigma(V)_n}{\Sigma}$ is the canonical projection.

Hence $\mathcal{E}_{(\Sigma,V)} = (\mathcal{E}_{(\Sigma,V)_0}, \mathcal{E}_{(\Sigma,V)_1}) : \left(\Sigma, \dfrac{\mathrm{Cot}_\Sigma(V)_n}{\Sigma}\right) \to (\Sigma, V)$ is a morphism of pairs.

Lemma 3.3.12. *The counit $\mathcal{E} : \widetilde{F}_n \widetilde{G}_n \to \mathrm{id}_{\mathbf{ParCog}}$ map is a natural transformation.*

Proof. We must show that for any $\varphi \in \mathrm{Hom}_{\mathbf{ParCog}}\big((\Sigma, V), (\Sigma', V')\big)$, the diagram

$$
\begin{array}{ccc}
\left(\Sigma, \dfrac{\mathrm{Cot}_\Sigma(V)_n}{\Sigma}\right) & \xrightarrow{\mathcal{E}_{(\Sigma,V)}} & (\Sigma, V) \\
\downarrow{\scriptstyle \widetilde{F}_n \widetilde{G}_n(\varphi)} & & \downarrow{\scriptstyle \varphi} \\
\left(\Sigma', \dfrac{\mathrm{Cot}_{\Sigma'}(V')_n}{\Sigma'}\right) & \xrightarrow[\mathcal{E}_{(\Sigma',V')}]{} & (\Sigma', V')
\end{array},
$$

commutes (see Definition 1.1.14).

Observe that

$$F_n(G_-(\varphi))_0 = G_-(\varphi)|_\Sigma = \pi_0' G_-(\varphi)\iota_0 = \varphi_0\pi_0\iota_0 = \varphi_0$$

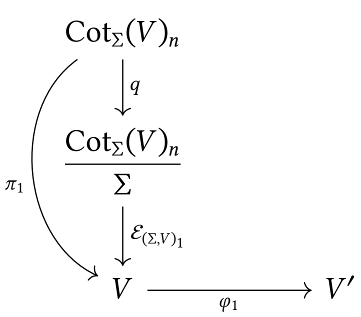
and $\mathcal{E}_{(\Sigma,V)_0} = \mathrm{id}_\Sigma$. Hence

$$\varphi_0\mathcal{E}_{(\Sigma,V)_0} = \varphi_0\mathrm{id}_\Sigma = \mathrm{id}_{\Sigma'}\varphi_0 = \mathcal{E}_{(\Sigma',V')_0}F_n(G_-(\varphi))_0.$$

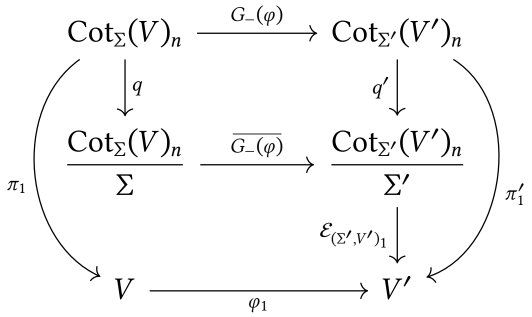
Moreover,

$$\mathcal{E}_{(\Sigma',V')}F_n\big(G_-(\varphi)\big)_1 q = \mathcal{E}_{(\Sigma',V')}\overline{G_-(\varphi)}q = \mathcal{E}_{(\Sigma',V')}q' G_-(\varphi) = \pi_1' G_-(\varphi) = \varphi_1\pi_1 = \varphi_1\mathcal{E}_{(\Sigma,V)_1}q$$

as depicted in the following commutative diagrams:



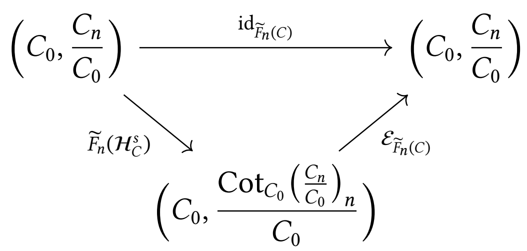
,



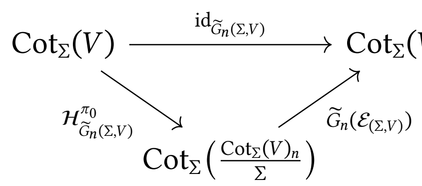
.

Since q is an epimorphism, it follows that $\varphi\mathcal{E}_{(\Sigma,V)} = \mathcal{E}_{(\Sigma',V')}F_n(G_-(\varphi))$, and the result follows. □

Lemma 3.3.13. *The triangular equalities:*



,



,

(3.3.14)

are satisfied.

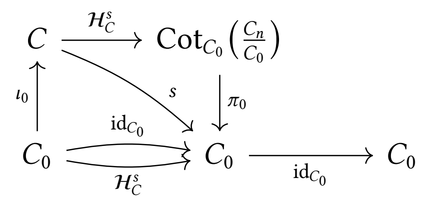
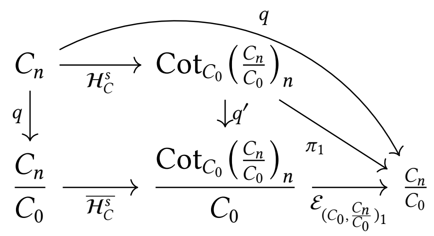
Proof. Observe that

$$\mathcal{E}_{F_n(C)_0}F_n(\mathcal{H}^s_C)_0 = \mathrm{id}_{C_0}\ \mathcal{H}^s_C\big|_{C_0} = \mathrm{id}_{C_0}$$

and

$$\mathcal{E}_{F_n(C)_1}F_n(\mathcal{H}^s_C)_1\ q\big|_{C_n} = \mathcal{E}_{F_n(C)_1}\overline{\mathcal{H}^s_C}\ q\big|_{C_n} = \mathcal{E}_{nF(C)_1}q'\ \mathcal{H}^s_C\big|_{C_n} = \pi_1'\ \mathcal{H}^s_C\big|_{C_n} = qt\big|_{C_n} = q\big|_{C_n}$$

as depicted in the following commutative diagrams:

 ,  .

Since q is an epimorphism, it follows that $\mathcal{E}_{F_n(C)}\, F_n(\mathcal{H}^s_C) = \mathrm{id}_{(C_0, \frac{C_n}{C_0})}$.

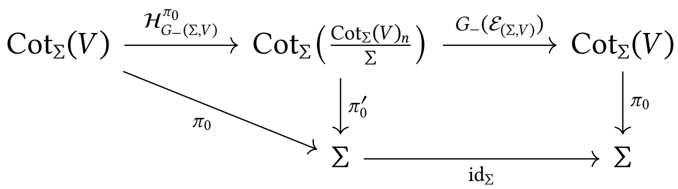
For the second case, note that:

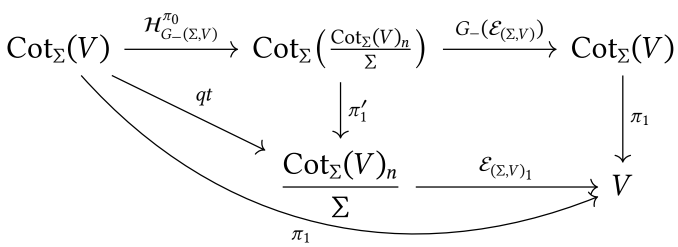
$$\pi_0 \left(G_-(\mathcal{E}_{(\Sigma,V)})\mathcal{H}^{\pi_0}_{G_-(\Sigma,V)}\right)\Big|_\Sigma = \left(\mathcal{E}_{(\Sigma,V)_0}\pi_0'\mathcal{H}^{\pi_0}_{G_-(\Sigma,V)}\right)\Big|_\Sigma = \left(\mathrm{id}_\Sigma \pi_0\right)\Big|_\Sigma = \pi_0|_\Sigma$$

and

$$\pi_1 \left(G_-(\mathcal{E}_{(\Sigma,V)})\mathcal{H}^{\pi_0}_{G_-(\Sigma,V)}\right)\Big|_{\mathrm{Cot}_\Sigma(V)_n} = \left(\mathcal{E}_{(\Sigma,V)_1}\pi_1'\mathcal{H}^{\pi_0}_{G_-(\Sigma,V)}\right)\Big|_{\mathrm{Cot}_\Sigma(V)_n} = \left(\mathcal{E}_{(\Sigma,V)_1} qt\right)\Big|_{\mathrm{Cot}_\Sigma(V)_n} = \pi_1|_{\mathrm{Cot}_\Sigma(V)_n},$$

as depicted on the following commutative diagrams:

 ,

 .

It follows that $G_-(\mathcal{E}_{(\Sigma,V)})\,\mathcal{H}^{\pi_0}_{G_-(\Sigma,V)} \sim_n \mathrm{id}_{\mathrm{Cot}_\Sigma(V)}$ (see (3.2.21)).

Therefore, the triangular equalities are satisfied. □

This proves Theorem 3.3.5.

Corollary 3.3.15. *When restricted to pointed coalgebras, we have* $Q\widetilde{F}_\infty \cong F'$ *and* $\widetilde{G}_\infty P \cong G'$.

Chapter 4

Algebra of invariants

In Section 2.3.5 we describe some group of automorphisms of path coalgebras, but the interest in such objects does not stop there.

In the late 70's, Kharchenko [Kha78] and, independently, Lane [Lan78] proved that the algebra of invariants of a free algebra by the action of a homogeneous group of algebra automorphisms is a free algebra. Few years back, Cibils and Marcos [CM16] proved that the same behavior is true for free linear k-categories, i.e. the category of invariants of a free linear k-category by the action of a finite homogeneous group of automorphisms is again a free linear k-category. Moreover, Cibils and Marcos proved that the category of invariants of a free linear category of finite or tame representation type has finite or tame representation type, respectively, but the category of invariants of a free linear category of wild representation type is not necessarily of wild representation type.

In this chapter we show that the algebra of invariants of a complete path algebra by the action of a homogeneous group of continuous algebra automorphisms is a complete path algebra. In order to do this, in Section 4.2 we prove the the result for power series rings. This extends the Theorem of Kharchenko–Lane (see [Kha78, Proposition 1] and [Lan78, Lemma 1.8]). In Section 4.3, using the techniques developed by Cibils and Marcos (see [CM16, Theorem 3.9]), we prove the main theorem of this chapter.

In the first section we introduce the finite and tame representations of a complete path algebra and prove in the end of this chapter that the algebra of invariants of a complete path algebra by the action of a homogeneous group of continuous algebra automorphisms inherits the representation type of the latter in case it is of finite or tame representation type. We finish this thesis with open questions related to this chapter.

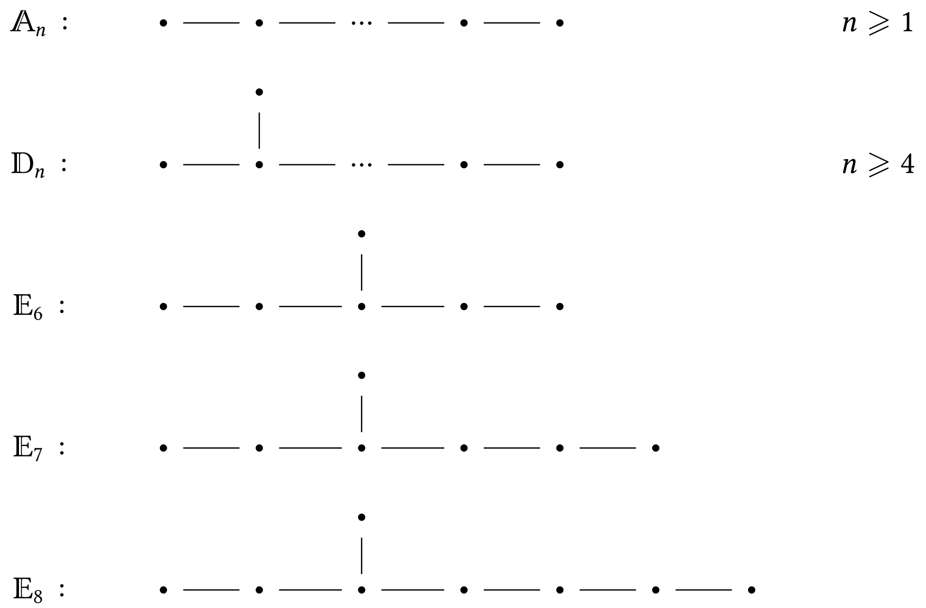
4.1 Representation types

In this section we describe the finite and tame representation types of a path coalgebra. First we present Dynkin diagrams, which are precisely the underlying graphs of finite quivers of finite representation type. Then we present the Euclidian diagrams, which are precisely the underlying graphs of finite quivers of tame representation type. Finally, we describe the finite and tame representation types of path coalgebras.

4.1.1 Dynkin diagrams

Definition 4.1.1. A quiver is said to be of *finite representation type* if it has only finitely many non-isomorphic indecomposable finite dimensional representations. Otherwise it is of *infinite representation type.*

Definition 4.1.2. A simply laced *Dynkin diagram* is one of the following graphs:



(4.1.3)

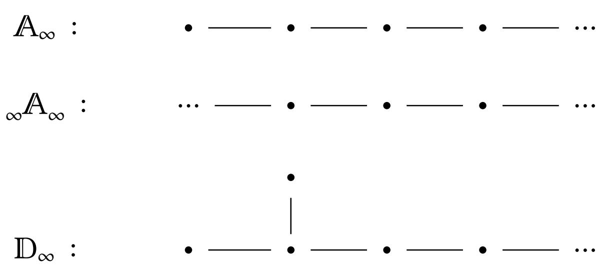
where the numbered subindex indicate the number of vertices.

Definition 4.1.4. A *Dynkin quiver* is a finite quiver whose underlying graph is one of the simply laced Dynkin diagrams.

A quiver is *locally finite* if there exists only finitely many paths between any pair of vertices.

A quiver is a *locally Dynkin quiver* if it is locally finite and any finite subquiver is a Dynkin quiver.

An infinite quiver that is a locally Dynkin quiver has one of the following underlying graphs:



(4.1.5)

The next result is known as Gabriel's Theorem.

Theorem 4.1.6. *A connected finite quiver is of finite representation type if and only if it is a Dynkin quiver.*

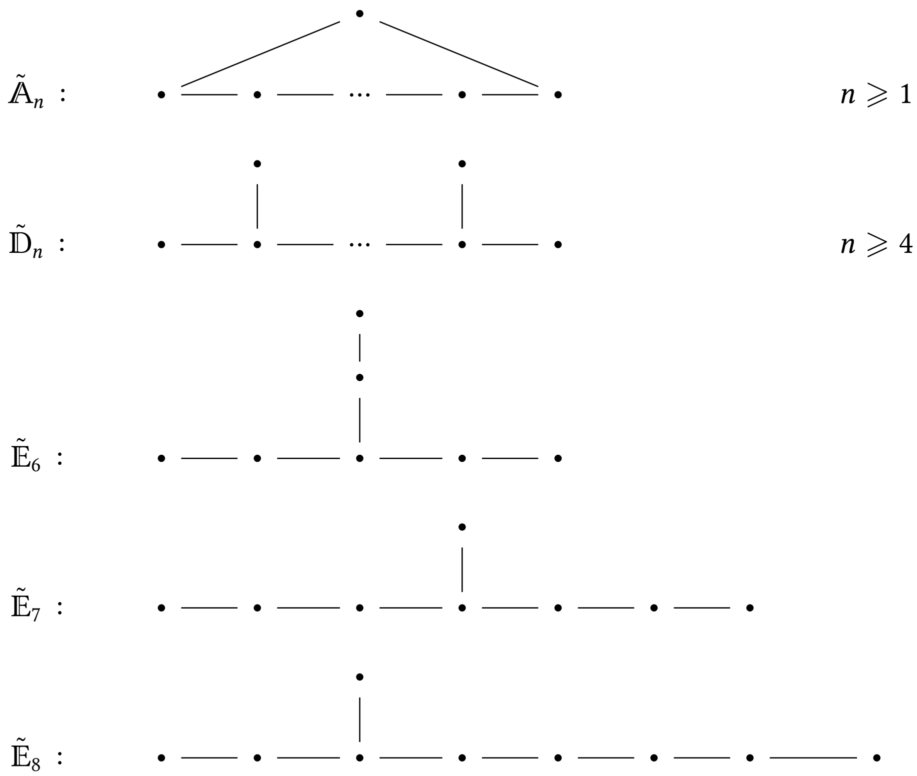
Proof. See [Gab73, §4] and [BGP73]. □

4.1.2 Euclidian diagrams

Nazarova [Naz73], Donovan and Freislich (see for instance [Dok+13]) extended the classification of quivers by means of tame representations type.

Definition 4.1.7. A quiver is said to be of *tame representation type* if it has infinitely many non-isomorphic indecomposable representations such that, for each integer $d \geqslant 1$, all but finitely many non-isomorphic indecomposable representations of dimension d occurs in a finite number of one-parameter families.

Definition 4.1.8. An *Euclidian diagram*, or *extended Dynkin diagram*, is one of the following graphs:



(4.1.9)

where the numbered subindex indicate the number of vertices minus one.

Theorem 4.1.10. *A connected finite quiver is of tame representation type if and only if its underlying graph is one of the Euclidean diagrams.*

4.1.3 Representations of path coalgebras

A comodule is indecomposable if it is not the direct sum of two non-zero subcomodules. One can define finite and tame representation types for coalgebras in the same sense as done for quivers in the previous subsections, regarding left comodules as its representations.

In view of Proposition 2.1.14, we have:

Proposition 4.1.11. *Let Q be a connected quiver and consider the path coalgebra kQ. Then:*

1. *kQ is of finite representation type if, and only if Q is a Dynkin quiver;*
2. *kQ is of tame representation type if, and only if, Q is an infinite locally Dynkin quiver or its underlying graph, $\overline{Q}$, is an Euclidian diagram, including the quiver with one vertex and one loop $\tilde{\mathbb{A}}_0$.*

Proof. See [Sim11, Theorem 7.22]. □

Remark 4.1.12. Since the dual algebra of a path coalgebra is isomorphic to a complete tensor algebra (see Lemma 1.4.47 and propositions 2.1.13 and 2.1.23), and the category of (left) C-comodules is dual to the category of (left) pseudocompact C^*-modules (see Theorem 1.4.19), we conclude that a complete path algebra is of finite or tame representation type if its dual coalgebra is of finite or tame representation type, respectively.

Path coalgebras of finite and tame representations types are the ones which, hopefully, one can completely classify all finite dimensional representations (comodules), up to isomorphism. All other path coalgebras are known as of wild representation type. Initially, Simson defined coalgebras of wild representation type in terms of an embedding of a category of (left) modules into a category of (right) comodules. In order to restrict the definition for comodules, we need the next result about comodules over finite dimensional coalgebras.

Consider a coalgebra C and a left C-comodule (M, μ). Let $\varphi : M \otimes k \to M$ be the canonical isomorphism $\varphi(m \otimes \lambda) = \lambda m$, $\gamma : C \otimes C^* \to k$ be the evaluation of the functional $\gamma(c \otimes f) = f(c)$, and $T : M \otimes C \to C \otimes M$ be the twist map $T(m \otimes c) = c \otimes m$. Define the map $\psi_\mu : M \otimes C^* \to M$ by $\psi_\mu = \varphi(\mathrm{id}_M \otimes \gamma)(T \otimes \mathrm{id}_{C^*})(\mu \otimes \mathrm{id}_{C^*})$, i.e. given $m \in M$ and $\mu(m) = \sum_{(m)} m_{(-1)} \otimes m_{(0)}$, then $\psi_\mu(m \otimes f) = \sum_{(m)} f(m_{-1})m_{(0)}$.

With the above notation, (M, ψ_μ) is a right C^*-module. Moreover,

Proposition 4.1.13. *If C is a finite dimensional coalgebra, then the categories ${}^C\mathcal{M}$ and $\mathcal{M}_{C^*}$ are isomorphic.*

Proof. See [DNR01, Theorem 2.2.5] and [FM20, Remark 3.3.10]. □

Now consider the quiver:

$$Q^3 : \bullet \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; \bullet \tag{4.1.14}$$

Definition 4.1.15. A path coalgebra kQ is of *wild representation type* if there exists a faithful covariant functor $F : {}^{kQ^3}\mathcal{M} \to {}^{kQ}\mathcal{M}$, which preserves indecomposables and short exact sequences and reflects isomorphisms.

Proposition 4.1.16. *Let C be a basic coalgebra and $k = \bar{k}$. Then, C is either of finite representation type, or of tame representation type or of wild representation type.*

Proof. See [Sim11, Corollary 6.8]. □

For the purpose of this chapter, we distinguish tame and finite representations type. Many authors treat finite representation type as a special case of tame representation type. The above result is know as the tame-wild dychotomy for coalgebras.

4.2 Invariants of a power series ring

Definition 4.2.1. Let A be an algebra and G be a group of algebra automorphisms of A. An element $x \in A$ is an *invariant* of G if $g(x) = x$ for all $g \in G$. The set of all invariants of G is a subalgebra of A, denoted by A^G, called the *algebra of invariants* of G.

Definition 4.2.2. Let $A = \bigoplus_{i\in\mathbb{N}} A_i$ be a graded algebra and consider G a group of algebra automorphisms of A. An automorphism $g \in G$ is *homogeneous* if $g(A_i) = A_i$ for every $i \in \mathbb{N}$. We say that G is *homogeneous* if its elements are homogeneous.

The Theorem of Kharchenko–Lane states the following:

Theorem 4.2.3. *Let $R = k\langle X\rangle$ be a free algebra and G a group of automorphisms of R. If G is homogeneous with respect to the grading on R induced by some function $d : X \to \mathbb{N}_{>0}$, then the algebra of invariants of G is free, on a set that is homogeneous with respect to d.*

Proof. See [Coh85, Theorem 6.10.3]. □

The above theorem can be reformulated as: if G is a group of homogeneous automorphisms of the tensor algebra $\mathrm{T}[k, M]$, then $\mathrm{T}[k, M]^G = \mathrm{T}[k, U]$, where $U \subseteq \mathrm{T}[k, M]$ is a subspace generated by homogeneous elements (in $\mathrm{T}[k, M]$). The main tool used to prove this theorem is the fact that an algebra A with a filtration so that $A_0 = k$ is a free algebra if, and only if, A satisfies the weak algorithm (see for instance [Kha78, Proposition 1], [Lan78, Lemma 1.8] or [Coh85, Proposition 2.4.2]).

Using the inverse weak algorithm, we prove at the end of this section that the above theorem can be extended to power series rings.

First, we recall the definitions of the weak algorithm and the inverse weak algorithm, along with associated results. We mainly follow [Coh85, §2.2 and §2.9].

4.2.1 Weak algorithm

Definition 4.2.4. Let R be a ring. A function $\mu : R \to \mathbb{N} \cup \{-\infty\}$ is a *filtration* on R if satisfies:

1. $\mu(x) \geqslant 0$ for $x \neq 0$ and $\mu(0) = -\infty$;
2. $\mu(x - y) \leqslant \max\{\mu(x), \mu(y)\}$;
3. $\mu(xy) \leqslant \mu(x) + \mu(y)$.
4. $\mu(1) = 0$.

In the case (3) is an equality, μ is a *degree function*. In general, we say that $\mu(x)$ is the *degree* of x.

Definition 4.2.5. Let R be a ring. We say that R is *graded* if it can be expressed as the direct sum of abelian groups $R = \bigoplus_{i\in\mathbb{N}} R_i$ such that $R_iR_j \subseteq R_{i+j}$. In this case, R_0 is a subring and each R_i is a R_0-bimodule.

A graded ring $R = \bigoplus_{i\in\mathbb{N}} R_i$ has a natural degree function $\mu : R \to \mathbb{N} \cup \{-\infty\}$ given by:

$$\mu(x) = \begin{cases} \min\{n : x \in \bigcup_{i=0}^{n} R_i\}, & \text{if } x \neq 0; \\ -\infty, & \text{if } x = 0. \end{cases} \tag{4.2.6}$$

In case $x \in R_n$, we say that x is a *homogeneous element* of degree n.

Definition 4.2.7. Let R be a ring with filtration μ. For each $n \in \mathbb{N} \cup \{-\infty\}$, denote by $R_{(n)} = \{x \in R \mid \mu(x) \leqslant n\}$ the set of elements of degree at most n. Then the $R_{(n)}$ are subgroups of the additive group R satisfying:

1. $\{0\} = R_{(-\infty)} \subseteq R_{(0)} \subseteq R_{(1)} \subseteq \cdots$;
2. $\bigcup R_{(n)} = R$;
3. $R_{(i)}R_{(j)} \subseteq R_{(i+j)}$;
4. $1 \in R_{(0)}$.

We can form the *associated graded ring* $\mathrm{gr}(R) = \bigoplus_{n=0}^{\infty} \frac{R_{(i)}}{R_{(i-1)}}$, where $\frac{R_{(0)}}{R_{(-1)}} = \frac{R_{(0)}}{R_{(-\infty)}} = R_{(0)}$.

Definition 4.2.8. Let R be a ring with filtration μ. A family $\{a_i\}_{i\in I}$ of elements of R is *right μ-dependent* if some $a_i = 0$ or there exist $b_i \in R$, almost all 0, such that

$$\mu\Big(\sum a_ib_i\Big) < \max\{\mu(a_i) + \mu(b_i)\}.$$

An element $a \in R$ is right μ-dependent on $\{a_i\}_{i\in I}$ if $a = 0$ or there exist $b_i \in R$, almost all 0, such that

$$\mu\Big(a - \sum a_ib_i\Big) < \mu(a) \quad \text{and} \quad \mu(a_i) + \mu(b_i) \leqslant \mu(a), \ \forall i \in I.$$

The ring R satisfies the n-term *weak algorithm* (with respect to μ) if for any (right) μ-dependent family with at most n members, say $a_1, \dots, a_m$ ($m \leqslant n$), with $\mu(a_1) \leqslant \cdots \leqslant \mu(a_m)$, some a_i is μ-dependent on $a_1, \dots, a_{i-1}$. The ring R satisfies the weak algorithm if it satisfies the n-term weak algorithm for all $n \in \mathbb{N}$.

4.2.2 Inverse weak algorithm

Definition 4.2.9. Let R be a ring. A function $v : R \to \mathbb{N} \cup \{\infty\}$ is an *inverse filtration* on R if satisfies:

1. $v(x) \in \mathbb{N}$ for $x \neq 0$ and $v(0) = \infty$;
2. $v(x - y) \geqslant \min\{v(x), v(y)\}$;
3. $v(xy) \geqslant v(x) + v(y)$.

In the case (3) is an equality, v is an *order function.*

Definition 4.2.10. Let R be a ring with inverse filtration ν. Denote by $R_{[n]} = \{x \in R \mid \nu(x) \geqslant n\}$, which satisfies:

1. $R = R_{[0]} \supseteq R_{[1]} \supseteq \cdots$;
2. $R_{[i]}R_{[j]} \subseteq R_{[i+j]}$;
3. $\bigcap R_{[n]} = 0$.

The *associated graded ring* is $\mathrm{gr}[R] = \bigoplus_{n=0}^{\infty} \frac{R_{[n]}}{R_{[n+1]}}$. If $x \in R$, $x \neq 0$, and $\nu(x) = n$, denote by $\overline{x} = x + R_{[n+1]} \in \frac{R_{[n]}}{R_{[n+1]}}$.

Definition 4.2.11. Let R be a ring with inverse filtration ν. R satisfies the (n-term) *inverse weak algorithm* if the associated graded ring $\mathrm{gr}[R]$ satisfies the (n-term) weak algorithm (with respect to its natural degree function).

Definition 4.2.12. Let R be an inversely filtered ring. Then, R is a topological ring with $R = R_{[0]} \supseteq R_{[1]} \supseteq \cdots$ being its neighborhood base at 0. Denote by $\hat{R}$ its completion. There exists a natural embedding $R \to \hat{R}$, which respects the (inverse) filtration. If this embedding is an isomorphism, we say that R is *complete*.

Proposition 4.2.13. *Let A be a complete inversely filtered algebra such that $\frac{A}{A_{[1]}} = k$. The algebra A is a power series ring if, and only if, A satisfies the inverse weak algorithm.*

Proof. See [Coh85, Proposition 2.9.8]. □

Corollary 4.2.14. *Let $B \subseteq A$ be a closed subalgebra of the power series ring A. If B satisfies the inverse weak algorithm, then B is a power series ring.*

Proof. See [Coh85, Corollary 2.9.9]. □

4.2.3 Power series ring

Let A be an algebra with inverse filtration ν and consider an algebra automorphism g. We say that g preserves the (inverse) filtration of A if $g(A_{[n]}) = A_{[n]}$ for every $n \in \mathbb{N}$, where $A_{[n]} = \{a \in A \mid \nu(a) \geqslant n\}$. In this case, g induce an algebra automorphism in the associated graded algebra $\mathrm{gr}[A]$ given by $g(a + A_{[n+1]}) := g(a) + A_{[n+1]}$ for any representative a of $\overline{a} \in \frac{A_{[n]}}{A_{[n+1]}}$.

Let k be a (discrete) field, A be a pseudocompact k-algebra, M be a pseudocompact A-bimodule, and $\mathrm{T}[[A, M]] = \prod_{i=0}^{\infty} M^{\widehat{\otimes}_i}$ be the complete tensor algebra of A and M (see Definition 1.4.43). The complete tensor algebra $\mathrm{T}[[A, M]]$ has a natural order function

$\nu : \mathrm{T}[[A, M]] \to \mathbb{N} \cup \{\infty\}$ given by:

$$\nu(x) = \begin{cases} \max\{n : x \in \prod_{i=n}^{\infty} M^{\hat{\otimes}_i}\}, & \text{if } x \neq 0; \\ \infty, & \text{if } x = 0. \end{cases}$$

When A is a finite dimensional algebra and M is a finite dimensional A-bimodule, it turns out that $\mathrm{gr}[\mathrm{T}[[A, M]]] \cong \mathrm{T}[A, M]$, considering the above inverse filtration.

An element $x \in \mathrm{T}[[A, M]]$ is called homogeneous if $x \in M^{\hat{\otimes}_n}$ for some $n \in \mathbb{N}$. We say that a algebra automorphisms g of $\mathrm{T}[[A, M]]$ is homogeneous if $g(M^{\hat{\otimes}_n}) = M^{\hat{\otimes}_n}$ for every $n \in \mathbb{N}$. A group of continuous algebra automorphism G of a complete tensor algebra is homogeneous if its elements are homogeneous.

Let G be a homogeneous group of continuous algebra automorphisms of $\mathrm{T}[[A, M]]$. If G is invariant on A, then the pseudocompact A-bimodules $M^{\hat{\otimes}_n}$ are left kG-modules for all $n \in \mathbb{N}$, where kG denotes the *group algebra* of G.

Now, we can prove the main theorem of this subsection.

Theorem 4.2.15. *Let* $\mathrm{T}[[A, M]]$ *be a complete tensor algebra and* G *be a group of continuous algebra automorphisms of* $\mathrm{T}[[A, M]]$*. If* $A = k$*, i.e.* $\mathrm{T}[[k, M]]$ *is a power series ring, and* G *is homogeneous with respect to the natural order function of* $\mathrm{T}[[k, M]]$*, then the algebra of invariants of* G*,* $\mathrm{T}[[k, M]]^G$*, is a power series ring.*

Proof. In view of Corollary 4.2.14 and Proposition 4.2.13, it is sufficient to show that $\mathrm{T}[[k, M]]^G$ is closed and $\mathrm{gr}[\mathrm{T}[[k, M]]^G]$ satisfies the weak algorithm.

It is closed because $\mathrm{T}[[k, M]]^G = \bigcap_{g \in G} \ker(g - \mathrm{id})$.

Since $\mathrm{gr}[\mathrm{T}[[k, M]]]$ satisfies the weak algorithm, $\mathrm{gr}[\mathrm{T}[[k, M]]]^G$ also satisfies the weak algorithm by Theorem 4.2.3. It is clear that $\mathrm{gr}[\mathrm{T}[[k, M]]^G] \subseteq \mathrm{gr}[\mathrm{T}[[k, M]]]^G$. The other way around follows by choosing homogeneous elements.

□

4.3 Invariants of a complete path algebra

With Theorem 4.2.15, we can apply the results of Cibils and Marcos [CM16] to complete path algebras. We adapt the language to the terms developed in this thesis and present full proofs of the results, though they follow exactly or with small changes to those in the article.

We prove that the algebra of invariants of a homogeneous group of continuous algebra automorphisms of a complete path algebra is a complete path algebra. Moreover, if the complete path algebra is of finite or tame representation type, then the algebra of invariants is of finite or tame representation type, respectively.

4.3.1 Composite and irreducible invariants

Let $VQ = (VQ_0, VQ_{e,f})$ be a k-quiver. Consider the complete tensor algebra $\mathrm{T}[[\Sigma, V]]$ of the topologically semisimple pseudocompact algebra $\Sigma = \prod_{e \in VQ_0} k$ and the pseudocompact Σ-bimodule $V = \prod_{e,f \in Q_0} VQ_{e,f}$.

In order to simplify the notation, we shall say that any nonzero vector space $VQ_{e,f}$ is an *arrow space* $V_a = VQ_{e,f}$ from e to f. For any sequence of arrow spaces $V_{a_1}, V_{a_2}, \dots, V_{a_n}$, with $V_{a_i} = VQ_{e_{i-1},e_i}$, the vector space $V_\omega = V_{a_n} \widehat{\otimes}_\Sigma \cdots \widehat{\otimes}_\Sigma V_{a_2} \widehat{\otimes}_\Sigma V_{a_1} \in \mathrm{T}[[\Sigma, V]]$ is the *space of path* ω from e_0 to e_n of length n (in particular, any arrow space is a space of path of length 1). Any subspace of V_ω which is a space of path is called *subpath*.

Let G be a group such that $\{VQ_{e,f}\}_{e,f \in VQ_0}$ is a family of left kG-modules.

A 2-partition of a space of path ω in VQ is any two subpaths ω_1 and ω_2 such that $V_\omega = V_{\omega_2} \widehat{\otimes}_\Sigma V_{\omega_1}$. Write $\omega = \omega_2\omega_1$ for short. Define $\varphi_{\omega_1,\omega_2} : V^G_{\omega_2} \widehat{\otimes}_\Sigma V^G_{\omega_1} \to V^G_\omega$ to be the canonical map. Denote by

$$\varphi_\omega := \sum_{\omega_2\omega_1=\omega} \varphi_{\omega_1,\omega_2} \tag{4.3.1}$$

The image of φ_ω is called *space of composite invariants.* Any complement of $\operatorname{Im}\varphi_\omega$ in the space of invariants is called a *space of irreducible invariants.*

Let us fix a complement and identify it by $V^G_{\omega,\mathrm{irr}}$. If ω has length n and $p = (n_l, \dots, n_1)$ is any *ordered l-partition* of n, denote by

$$V^G_{\omega,p,\mathrm{irr}} := V^G_{\omega_{n_l},\mathrm{irr}} \widehat{\otimes}_\Sigma \cdots \widehat{\otimes}_\Sigma V^G_{\omega_{n_1},\mathrm{irr}},$$

where ω_{n_i} are the unique subpath of ω of length n_i such that $\omega = \omega_{n_l} \cdots \omega_{n_1}$. Consider the canonical map

$$\psi_\omega : \bigoplus_p V^G_{\omega,p,\mathrm{irr}} \to V^G_\omega, \tag{4.3.2}$$

where the direct sum runs through all ordered partitions of length n (the length of ω).

Remark 4.3.3. In [CM16, p. 3124], such partitions are referred to as "non-ordered", but the most commonly used term we know is "ordered" partition.

4.3.2 Irreducible invariants decomposition

The next results follows, with small changes, by [CM16, Lemma 3.7, Proposition 3.8, Theorem 3.9].

Lemma 4.3.4. *ψ_ω is surjective.*

Proof. Follows exactly as [CM16, Lemma 3.7].

It is defined a natural filtration on the invariants and the proof follows by induction.

For each ordered partition $p = (n_l, \dots, n_1)$ of n (the length of ω) consider the map:

$$\varphi_{\omega,p} : V^G_{\omega_{n_l}} \widehat{\otimes}_\Sigma \cdots \widehat{\otimes}_\Sigma V^G_{\omega_{n_1}} \to V^G_\omega.$$

Denote the image of the sum of the maps $\varphi_{\omega,p}$ along all the l-partitions of n by $[U^G_\omega]^l$, called the *space of l-composite invariants*. Thus the 2-composite invariants are the space of composite invariants. The following filtration holds:

$$0 \subseteq [V^G_\omega]^n \subseteq \cdots \subseteq [V^G_\omega]^1 = V^G_\omega.$$

Observe that

$$[V^G_\omega]^n = V^G_{a_n} \widehat{\otimes}_\Sigma \cdots \widehat{\otimes}_\Sigma V^G_{a_1},$$

which is in the image of ψ_ω. Assume that $[V^G_\omega]^l \subseteq \mathrm{Im}(\psi_\omega)$. Let $v \in [V^G_\omega]^{l-1}$ and suppose that v is obtained from a fixed $l-1$ partition. Since $v \in V^G_\omega$, v can be decomposed as a sum of two terms:

1. tensors of irreducible invariants, which belong by definition to $\mathrm{Im}(\psi_\omega)$;
2. $(l-1)$-tensors which contain at least one composite invariant, so belonging to $[V^G_\omega]^l$ which is contained in $\mathrm{Im}(\psi_\omega)$ by hypothesis.

The general term is simply a sum of such terms, completing the proof. □

Proposition 4.3.5. *Consider $V_{a_i} = V$ for all $i \in \{1, \dots, n\}$, i.e. $V_\omega = V^{\widehat{\otimes}_n}$. Then ψ_ω is bijective.*

Proof. Follows as [CM16, Proposition 3.8] after changing the Theorem 4.2.3 for Theorem 4.2.15.

Let $\mathrm{T}[[k,V]]^G = k \times V^G \times (V \widehat{\otimes}_\Sigma V)^G \times \cdots$ be the algebra of invariants of G. By Theorem 4.2.15, there exists a homogeneous k-subbimodule $U \subseteq \mathrm{T}[[k,V]]^G$ such that $\mathrm{T}[[k,U]] = \mathrm{T}[[k,V]]^G$.

Write $U_n = U \cap (V^{\widehat{\otimes}_n})^G$, for each $n \geqslant 1$. Claim: U_n is a vector space of irreducible invariants for every n. For $n = 1$ it is clear since V^G is irreducible. Assume that U_i is a space of irreducible invariants of degree i, for every $i < n$. Because the composites (in degree n) are sums of (complete tensor) products of irreducible of lower degree, they must be obtained from tensors of the U_i's, for $i < n$. Moreover, since $\mathrm{T}[[k,U]]$ is a power series ring, the intersection of the composites with U_n is zero. Hence U_n is a space of irreducible invariants.

Therefore, the isomorphisms $U_i \cong (V^{\widehat{\otimes}_i})^G_{\mathbf{irr}}$ and $\bigoplus_p (U_{n_l} \widehat{\otimes}_\Sigma \cdots \widehat{\otimes}_\Sigma U_{n_1}) \cong (V^{\widehat{\otimes}_n})^G$, where $p = (n_l, \dots, n_1)$ runs through all partitions of n, gives the desired bijection. □

Theorem 4.3.6. *ψ_ω is bijective for any path ω.*

Proof. Follows exactly as [CM16, Theorem 3.9].

Let $V = V_{a_1} \times \cdots \times V_{a_n}$ and $V_\gamma = V^{\widehat{\otimes}_n}$. Then, ψ_γ is the direct sum of the maps $\psi_{a_{i_n} \dots a_{i_1}}$ along all the sequences $(i_n, \dots, i_1)$ of integers belonging to $\{1, \dots, n\}$, which is bijective. Hence, all those maps are invertible, in particular the one corresponding to the path ω (i.e. the sequence $(n, \dots, 1)$). □

4.3.3 Complete path algebra

Theorem 4.3.7. *Let VQ be a k-quiver and G be a finite homogeneous group of continuous algebra automorphism of the complete tensor algebra* $\mathrm{T}[[\Sigma, V]]$*. Then the algebra of invariants* $\mathrm{T}[[\Sigma, V]]^G$ *is isomorphic to a complete tensor algebra.*

Proof. Compare with [CM16, Theorem 4.1].

We construct a family of subbimodules of $\mathrm{T}[[\Sigma, V]]^G$ and show that the complete tensor algebra of the product of such family gives the desired isomorphism.

For each pair of vertices $e, f \in VQ_0$, denote by $\Omega_{e,f}$ the set of all paths from e to f. For each $\omega \in \Omega_{e,f}$ fix a space of irreducible invariants $V^G_{\omega,\mathrm{irr}}$, that is

$$V^G_\omega = V^G_{\omega,\mathrm{irr}} \oplus \mathrm{Im}\,(\varphi_\omega).$$

Let $U_{e,f} = \prod_{\omega \in \Omega_{e,f}} V^G_{\omega,\mathrm{irr}}$ and $U = \prod_{e,f \in VQ_0} U_{e,f}$.

The canonical inclusions $\iota_\Sigma : \Sigma \to \mathrm{T}[[\Sigma, V]]^G$ and $\iota_U : U \to \mathrm{T}[[\Sigma, V]]^G$, and the universal property of the complete tensor algebra, Proposition 1.4.45, gives a continuous algebra homomorphism $\alpha : \mathrm{T}[[\Sigma, U]] \to \mathrm{T}[[\Sigma, V]]^G$.

By Theorem 4.3.6, for any path ω in VQ, V^G_ω is either a space of irreducible invariants or is isomorphic to a direct sum of tensor products of spaces of irreducible invariants (for any fixed choice of such complements).

By continuity of the elements of G, any invariant limit element of a convergent series of elements in $\mathrm{T}[[\Sigma, V]]$ must be the limit of a subseries of invariant elements. Thus, continuity of α implies that such element is the image of a limit element in $\mathrm{T}[[\Sigma, U]]$. Therefore, α is an isomorphism. □

4.3.4 Invariants and representation types

In view of Remark 4.1.12 and Proposition 4.1.11, a complete path algebra of a connected quiver is of finite representation type if, and only if, its quiver is Dynkin and it is of tame representation type if, and only if, its quiver is infinite locally Dynkin or is the underlying graph of an Euclidean graph. Hence, a complete path algebra of finite or tame representation type is not a finite dimensional path algebra only if it is of tame representation type and its quiver is an infinite locally Dynkin or is a cycle.

Theorem 4.3.8. *Let VQ be a k-quiver and G be a homogeneous finite group of continuous automorphism of the complete path algebra* $\mathrm{T}[[\Sigma, V]]$*. If* $\mathrm{T}[[\Sigma, V]]$ *is of finite or tame representation type, then* $\mathrm{T}[[\Sigma, V]]^G$ *is of finite or tame representation type, respectively.*

Proof. Without loss of generality, we may consider VQ connected.

In the case $\mathrm{T}[[\Sigma, V]]$ is finite dimensional, this follows from [CM16, Theorem 5.16]. In general, the proof of [CM16] also follows since the algebra of invariants of a complete path algebra is a complete path algebra, by Theorem 4.3.7. We make it explicit here.

We have two cases:

1. VQ comes from a infinite locally Dynkin quiver;
2. VQ comes from an oriented cycle.

In the first case, any finite subquiver of VQ, say VQ^{f}, is Dynkin and the corresponding tensor algebra $\mathrm{T}[[\Sigma^{\mathrm{f}}, V^{\mathrm{f}}]]$ is a finite dimensional path algebra of finite representation type. Since $\mathrm{T}[[\Sigma, V]]$ is the inverse limit of $\mathrm{T}[[\Sigma^{\mathrm{f}}, V^{\mathrm{f}}]]$, running through all finite subquivers of VQ, it follows that $\mathrm{T}[[\Sigma, V]]^G$ is the inverse limit of $\mathrm{T}[[\Sigma^{\mathrm{f}}, V^{\mathrm{f}}]]^G$, which associated quiver is Dynkin, and, therefore, its associated quiver is locally Dynkin.

If VQ is a cycle, we analyze the corresponding quiver of $\mathrm{T}[[\Sigma, V]]^G$. Given a vertex $e \in VQ_0$ and paths ω_1 and ω_2 in VQ starting at e, then there exists a path ω' such that $\omega_1 = \omega'\omega_2$ or $\omega_2 = \omega'\omega_1$. Thus, for each vertex $e \in VQ_0$ there is at most one irreducible invariant vector space starting at e, which corresponds to an arrow space of $\mathrm{T}[[\Sigma, V]]^G$. Analogous argument shows that there is at most one irreducible invariant vector space ending at f, for each vertex $f \in VQ_0$. Thus, the corresponding quiver of $\mathrm{T}[[\Sigma, V]]^G$ is one or the union of quivers which are a vertex, one directional finite line or a cycle. Therefore, $\mathrm{T}[[\Sigma, V]]^G$ is tame. □

4.4 Further research

We finish this thesis with suggestions for further research and open questions.

In this chapter, we dedicated ourselves in the algebra of invariants of a complete path algebra under the action of a homogeneous group of continuous automorphisms, which extends [Kha78, Proposition 1]. In this same article of Kharchenko [Kha78], the main theorem states that there exists a one-to-one correspondence between the subgroups of a finite group G of homogeneous automorphisms of a free algebra A and the free subalgebras containing A^G. Hence, one could ask if similar correspondence exists for our case, i.e. the subgroups of a homogeneous group of continuous automorphisms G corresponds to the subalgebras of a complete path algebra A, which are complete path algebras and contains A^G?

Ferreira, Murakami, and Paques [FMP04] showed that the above scenario is a special case when G is a group. They proved that the algebra of invariants of a free algebra A under the action of a homogeneous Hopf algebra H is free, [FMP04, Corollary 3.2], and when H is finite dimensional and pointed there exists a correspondence between the right coideal subalgebras of H and the free subalgebras of A containing A^H, [FMP04, Theorem 1.2]. This should work for complete path algebras.

Another question is if it is possible to consider path coalgebras instead and ask what would be a coalgebra of invariants under the action of a (homogeneous) group (or Hopf algebra) of coalgebra automorphisms.

What about the complete tensor product of k-species (or the cotensor product of k-cospecies)?

Dlab and Ringel proved that a connected acyclic finite k-species S is of finite representation type if, and only if, its underlying valued quiver is a Dynkin diagram and S is of tame representation type if, and only if, its underlying valued quiver is an Euclidean diagram, cf. [Lem12, Theorem 7.17]. What happens in the general case, when considering any k-species (or k-cospecies)?

In the opposite direction of the algebra of invariants, Green [Gre83] established a relation between Galois coverings of a finite quiver and finitely generated group-graded algebras, which was generalized to k-categories by Cibils and Marcos [CM06] and dualized to pointed coalgebras by Chin [Chi10]. There should be a correspondence to pointed pseudocompact algebras.

Appendix A

Topological algebras

We define topological algebras. For a general introduction to topological rings, topological vector spaces, topological algebras and topological modules, see [War89].

From [Mun75, §12, p. 76]:

Definition A.1. A *topological space* $X = (X, \tau)$ is a set X together with a *topology* τ, which consists of a collection of subsets of X satisfying:

1. The sets $\emptyset, X \in \tau$;

2. The union of any subcollection of τ belongs to τ;

3. The intersection of the elements of any finite subcollection of τ is in τ.

The elements of τ are called open sets of X.

Example A.2. For any set X, the collection of all subsets of X is a topology on X called the *discrete topology*, while the collection of only X and $\emptyset$ is a topology called the *trivial topology*.

Definition A.3. Let X be a set. A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X, called *basis elements*, satisfying:

1. For any $x \in X$, there exists a $B \in \mathcal{B}$ such that $x \in B$;

2. For any two basis elements B_1, B_2, there exists a basis element B_3 such that $B_3 \subseteq B_1 \cap B_2$.

The topology τ generated by $\mathcal{B}$ is the collection of all unions of basis elements [Mun75, Lemma 13.1].

From [Mun75, §13, p. 82]:

Definition A.4. A *subbasis* $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union is X. The *topology generated by the subbasis* $\mathcal{S}$ is the collection τ of all unions of finite intersections of elements of $\mathcal{S}$.

Definition A.5. Let X and Y be topological spaces. The *product topology* on the cartesian product $X \times Y$ is generated by all basis elements $U \times V$, where U is a basis element of X and V is a basis element of Y [Mun75, Theorem 15.1].

Consider $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ the canonical projections. The collection $\mathcal{S} = \{\pi_1^{-1}(U) \,|\, U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \,|\, V \text{ open in } Y\}$ is a subbasis for the product topology on $X \times Y$ [Mun75, Theorem 15.2]. In general, if $\{X_i\}$ is a family of topological spaces and $\prod X_i$ is the cartesian product of all X_i's, then the product topology on $\prod X_i$ has subbasis $\mathcal{S} = \bigcup \mathcal{S}_{\geqq}$, where $\mathcal{S}_{\geqq} = \{\pi_i^{-1}(U_i) \,|\, U_i \text{ open in } X_i\}$.

From [Mun75, §16, p. 88]:

Definition A.6. Let X be a topological space with topology τ and $Y \subseteq X$. Then $\tau_Y = \{U \cap Y \,|\, U \in \tau\}$ is a topology on Y, called *subspace topology*.

From [Mun75, §17, p. 98]:

Definition A.7. A topological space X is a *Hausdorff space* if for any two elements $x, y \in X$, with $x \neq y$, there exist open sets U and V containing x and y, respectively, such that $U \cap V = \emptyset$.

Definition A.8. Let X be a topological space and $A \subseteq X$ a subset of X. A *neighborhood* V of A is any open set of X that contains A. In case $A = \{x\}$ is a singleton, we simply say that V is a neighborhood of x. A *fundamental system of neighborhoods* $\mathcal{V}$ of a A is a collection of neighborhoods of A such that for any neighborhood U of A there is a $V \in \mathcal{V}$ such that $V \subseteq U$.

From [War89, p. 2]:

Definition A.9. Let G be a group. A topology τ on G is a group topology if the operation of the group is continuous from $G \times G$, furnished with the cartesian product topology defined by τ, to G and if the inverse map $g \mapsto g^{-1}$ is continuous from G to G. In this case we say that $G = (G, \tau)$ is a *topological group*.

A set F of subsets of E is a *filter* on E if $E \in F$, $\emptyset \notin F$ and $A \cap B \in F$ for any $A, B \in F$. If E is a topological space, a *neighborhood* of a point $x \in E$ is any subset of E containing an open set U such that $x \in U$ (in particular, every open set is a neighborhood of some point). The set of all neighborhoods of x is a filter on E.

A set B of subsets of E is a *filter base* on E if the set of all subsets F of E for which there exists $b \in B$ such that $b \subseteq F$ is a filter, called the *filter generated* by B. Thus B is a filter base if and only if $B \neq \emptyset$, $\emptyset \notin B$ and the intersection of two members of B contains a member of B. In a topological space E, a *fundamental system of neighborhoods* of $x \in E$ is a filter base generating the filter of neighborhoods of x.

Proposition A.10. *Let G be a group. If B is a fundamental system of neighborhoods of* 0 *for a group topology on G, then the following conditions hold:*

1. *For each $V \in B$ there exists $U \in B$ such that $UU \subseteq V$, where $UU = \{x, y \in G \,|\, x \in U,\ y \in U\}$;*
2. *if $V \in B$, then there exists $U \in B$ such that $U \subseteq V^{-1}$, where $V^{-1} = \{x^{-1} \in G \,|\, x \in V\}$;*

3. *if $V \in B$, then for each $a \in G$ there exists $U \in B$ such that $U \subseteq aVa^{-1}$.*

Conversely, if B is a filter base on G satisfying the previous conditions, then there is a unique group topology on G for which B is a fundamental system of neighborhoods of 0.

Proof. See [War89, Corollary 1.5]. □

If X is a commutative group (denoted additively), then a filter base B on X is a fundamental system of neighborhoods of zero for a group topology on X if and only if for each $V \in B$ there exist $U \in X$ such that $U + U \subseteq V$ and $U \subseteq -V$.

Proposition A.11. *Let G be a topological group, let B be a fundamental system of neighborhoods of 0, and let $A \subseteq G$. If O is an open subset of G, then:*

1. *AO and OA are open;*
2. *hence for any neighborhood V of 0, AV and VA are neighborhoods of A;*
3. *the symmetric open neighborhoods of 0 form a fundamental system of neighborhoods of 0;*
4. *$\overline{A} = \cap\{AV \mid V \in B\} = \cap\{VA \mid V \in B\}$;*
5. *in particular, $\overline{\{0\}} = \cap\{V \mid V \in B\}$;*
6. *the closed symmetric neighborhoods of 0 form a fundamental system of neighborhoods of 0.*

Proof. See [War89, Theorem 1.6]. □

Proposition A.12. *Let G be a topological group. The following statements are equivalent:*

1. *$\{0\}$ is closed;*
2. *$\{0\}$ is the intersection of all neighborhoods of 0;*
3. *G is Hausdorff;*
4. *G is Hausdorff and for each $b \in G$ the closed neighborhoods of b form a fundamental system of neighborhoods of b.*

Proof. See [War89, Theorem 1.7]. □

From [War89, p. 24]:

Let X be a topological space and B be a filter base on X. The filter base B *converges* to $x \in X$ if every neighborhood of x contains a member of B. In the case X is Hausdorff, B converges to at most one point of X.

If $f : X \to Y$ is a continuous function and B is a filter base on X that converges to $x \in X$, then $f(B)$ is a filter base on Y that converges to $f(x) \in Y$.

Definition A.13. Let G be a topological group and $e \in G$ its identity element. A filter base B on G is *Cauchy* if every neighborhood V of e contains an element $U \in B$ such that $U^{-1}U \subseteq V$.

Definition A.14. A topological group G is *complete* if every Cauchy filter base on G converges to a point on G.

Definition A.15. Let G be a Hausdorff group. A complete Hausdorff group $\hat{G}$ is a *completion* of G if G is a dense topological subgroup of $\hat{G}$.

From [War89, p. 85]:

Definition A.16. Let X be a vector space. A topology τ on X is a vector topology if τ is an additive group topology on X such that the mapping $(\lambda, x) \mapsto \lambda x$ is continuous from $k \times X$ to X where $k \times X$ is furnished with the cartesian product topology. In this case we say that $X = (X, \tau)$ is a *topological vector space.*

From [War89, p. 77]:

Definition A.17. Let R be a ring. A topology τ on R is a ring topology if τ is an additive group topology on R such that the multiplication of R is continuous where $R \times R$ is furnished with the cartesian product topology. In this case we say that $R = (R, \tau)$ is a *topological ring.*

From [War89, p. 85]:

Definition A.18. Let R be a topological ring and M be a (left) R-module. A topology τ on M is a *module topology* if τ is an additive group topology on M and the mapping $(r, m) \mapsto rm$ is continuous from $R \times M$ to M where $R \times M$ is furnished with the cartesian product topology. In this case we say that $M = (M, \tau)$ is a *topological (left) R-module.*

From [War89, p. 90]:

Definition A.19. Let R be a commutative topological ring and A be an R-algebra. A topology τ on A is an *algebra topology* if τ is both a ring topology and a R-module topology on A. In this case we say that $A = (A, \tau)$ is a *topological algebra.*

Definition A.20. A topological algebra A is *complete* if the underlying topological additive group is complete.

Appendix B

Cotensor product and complete tensor product are dual

We present the details of the proof of Lemma 1.4.40.

Proof of Lemma 1.4.40. By the Fundamental Theorem of Comodules, Theorem 1.2.50, $N = \varinjlim_{i\in I} N_i$ with N_i finite dimensional left C-comodules. Then

$$\begin{aligned}
(M \square_C N)^* &= \mathrm{Hom}_k(M \square_C N, k) = \mathrm{Hom}_k\big(M \square_C \varinjlim_{i\in I} N_i, k\big) \\
&\cong \mathrm{Hom}_k\big(\varinjlim_{i\in I}(M \square_C N_i), k\big) \\
&\cong \varprojlim_{i\in I} \mathrm{Hom}_k(M \square_C N_i, k) && \text{(B.1)} \\
&\cong \varprojlim_{i\in I} \mathrm{Hom}_k\big(\mathrm{Hom}_{-C}(N_i^*, M), k\big) && \text{(B.2)} \\
&\cong \varprojlim_{i\in I} \mathrm{Hom}_k\big(\mathrm{Hom}_{-C^*}(M^*, N_i^{**}), k\big) && \text{(B.3)} \\
&= \varprojlim_{i\in I} \mathrm{Hom}_k\big(\mathrm{Hom}_{-C^*}(M^*, \mathrm{Hom}_k(N_i^*, k)), k\big) \\
&\cong \varprojlim_{i\in I} \mathrm{Hom}_k\big(\mathrm{Hom}_k(M^* \widehat{\otimes}_{C^*} N_i^*, k), k\big) && \text{(B.4)} \\
&= \varprojlim_{i\in I} (M^* \widehat{\otimes}_{C^*} N_i^*)^{**} \cong \varprojlim_{i\in I} (M^* \widehat{\otimes}_{C^*} N_i^*) \\
&\cong M^* \widehat{\otimes}_{C^*} \varprojlim_{i\in I} N_i^* = M^* \widehat{\otimes}_{C^*} N^*,
\end{aligned}$$

where (B.1) follows from an analogous proof of [Rot09, Theorem 2.31], (B.2) is due to [DNR01, Proposition 2.3.7], (B.3) is due to Theorem 1.4.19, and (B.4) follows from [CE99, Proposition 5.2'], see [Bru66, Lemma 2.4]. □

Appendix C

Equivalence of k-quiver representations and comodules

We define a k-quiver representation which is equivalent to comodules of the corresponding path coalgebra. For convenience we define k-quiver representations on the right and make the computations for right comodules. Similar calculations applies for the left side.

Definition C.1. Let VQ be a k-quiver. A *local representation* of VQ, $X = (X_g, X_{g,h})_{g,h \in VQ_0}$, is a family of k-vector spaces X_g and linear transformations $X_{g,h} : X_g \to X_h \otimes_k VQ_{g,h}$ satisfying, for each $g \in VQ_0$ and each $\iota : k \to X_g$,

1. there exists an integer $m > 0$ such that $X_{g,g_1,\ldots,g_m}\, \iota \equiv 0$, where $X_{g,g_1,\ldots,g_m}$ is the composition

$$(X_{g_{m-1},g_m} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \ldots (X_{g_1,g_2} \otimes \mathrm{id}) X_{g,g_1} : X_g \to X_{g_m} \otimes VQ_{g_{m-1},g_m} \otimes \cdots \otimes VQ_{g,g_1};$$

2. and $\sum_{h \in VQ_0} \dim_k(\mathrm{Im}\,(X_{g,h}\, \iota)) < \infty$.

A *morphism of local representations* of VQ, $f : X \to Y$, is a family $(f_g)_{g \in VQ_0}$ of linear maps $f_g : X_g \to Y_g$ satisfying the commutative diagram:

$$\begin{array}{ccc} X_g & \xrightarrow{X_{g,h}} & X_h \otimes VQ_{g,h} \\ {\scriptstyle f_g}\downarrow & & \downarrow{\scriptstyle f_h \otimes \mathrm{id}} \\ Y_g & \xrightarrow{Y_{g,h}} & Y_h \otimes VQ_{g,h} \end{array}$$

Denote by $\mathbf{Rep}^l_k(VQ)$ the category of all local representations of VQ.

Given a right $k[VQ]$-comodule M, we obtain a local representation of VQ in the following way:

- Write the counity of kVQ_0 as $\varepsilon_0 = \sum_{g \in VQ_0} \varepsilon_g$, where $\varepsilon_g(h) = \delta_{g,h}$ for any $h \in VQ_0$. The comodule structure, $(\mathrm{id} \otimes \varepsilon)\nu(m) \cong m$, together with the construction of the counity of the path coalgebra $k[VQ]$ give us $M = \bigoplus_{g \in VQ_0} M_g$, for $M_g := (\mathrm{id} \otimes \varepsilon_g)\nu(M)$;
- Define the linear map $X_{g,h} : M_g \to M_h \otimes VQ_{g,h}$ by $X_{g,h} = (\mathrm{id} \otimes \varepsilon_h \otimes \mathrm{id})(\nu \otimes \mathrm{id})(\mathrm{id} \otimes \pi_1)\, \nu|_{M_g}$, where $\pi_1 : k[VQ] \to VQ_1$ is the canonical projection. Observe that $(\nu \otimes \mathrm{id})\nu = (\mathrm{id} \otimes \Delta)\nu$ implies

$$\begin{aligned}((\mathrm{id} \otimes \varepsilon_h)\nu \otimes \pi_1)\, \nu|_{M_g} =& (\mathrm{id} \otimes \varepsilon_h \otimes \mathrm{id})(\nu \otimes \mathrm{id})(\mathrm{id} \otimes \pi_1)\, \nu|_{M_g} \\ =& ((\mathrm{id} \otimes \varepsilon_h \otimes \pi_1)(\mathrm{id} \otimes \Delta)\nu \otimes \mathrm{id})(\mathrm{id} \otimes \varepsilon_g)\nu \\ =& (\mathrm{id} \otimes (\varepsilon_h \otimes \pi_1)\Delta \otimes \varepsilon_g)(\nu \otimes \mathrm{id})\nu \\ =& (\mathrm{id} \otimes ((\varepsilon_h \otimes \pi_1)\Delta \otimes \varepsilon_g)\Delta)\nu \\ =& (\mathrm{id} \otimes (\varepsilon_h \otimes \pi_1 \otimes \varepsilon_g)\Delta^2)\nu.\end{aligned}$$

  Thus $X_{g,h}(M_g) \subset M_h \otimes VQ_1 \cap M \otimes VQ_{g,h} = M_h \otimes VQ_{g,h}$.

For any $g \in VQ_0$ and any $x \in M_g$, the subcomodule generated by x, $\langle x \rangle \subset \bigoplus_{g \in VQ_0} M_g$, is finite dimensional by the Fundamental Theorem for Comodules, Theorem 1.2.50. Hence $\sum_{h \in VQ_0} \dim_k(X_{g,h}(k\{x\})) < \infty$.

The composition X_{g,g_1,g_2} is given by

$$\begin{aligned}(X_{g_1,g_2} \otimes \mathrm{id})X_{g,g_1} =& ((\mathrm{id} \otimes \varepsilon_{g_2} \otimes \pi_1)(\mathrm{id} \otimes \Delta)\nu \otimes \mathrm{id})(\mathrm{id} \otimes (\varepsilon_{g_1} \otimes \pi_1 \otimes \varepsilon_g)\Delta^2)\nu \\ =& ((\mathrm{id} \otimes \varepsilon_{g_2} \otimes \pi_1)(\mathrm{id} \otimes \Delta) \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes (\varepsilon_{g_1} \otimes \pi_1 \otimes \varepsilon_g)\Delta^2)(\nu \otimes \mathrm{id})\nu \\ =& (\mathrm{id} \otimes \varepsilon_{g_2} \otimes \pi_1 \otimes \varepsilon_{g_1} \otimes \pi_1 \otimes \varepsilon_g)(\mathrm{id} \otimes (\Delta \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta^2))(\mathrm{id} \otimes \Delta)\nu \\ =& (\mathrm{id} \otimes (\varepsilon_{g_2} \otimes \pi_1 \otimes \varepsilon_{g_1} \otimes \pi_1 \otimes \varepsilon_g)\Delta^4)\nu.\end{aligned}$$

And in general,

$$X_{g,g_1,\ldots,g_m} = (\mathrm{id} \otimes (\varepsilon_{g_m} \otimes \pi_1 \otimes \cdots \otimes \varepsilon_{g_2} \otimes \pi_1 \otimes \varepsilon_{g_1} \otimes \pi_1 \otimes \varepsilon_g)\Delta^{2m})\nu.$$

Since

$$\Delta^{2m} C_n \subset \sum_{i_1+i_2+\cdots+i_{2m} \leqslant n} C_{n-i_1-i_2-\cdots-i_{2m}} \otimes C_{i_1} \otimes C_{i_2} \otimes \cdots \otimes C_{i_{2m}},$$

we get $X_{g,g_1,\ldots,g_m}(c) = 0$ for any $c \in C_n$ and $m > n$.

Therefore, $X = (M_g, X_{g,h})_{g,h \in VQ_0}$ is a local representation.

We show the other direction. Given a local representation $X = (X_g, X_{g,h})_{g,h \in VQ_0}$, we obtain a right $k[VQ]$-comodule $M = \bigoplus_{g \in VQ_0} X_g$ with comodule structure given as follows: for any $x \in X_g$, with $X_{g,\ldots,g_{m+1}}(x) = 0$,

$$\nu(x) = x \otimes g + \sum_{g_1 \in VQ_0} X_{g,g_1}(x) + \sum_{g_1,g_2 \in VQ_0} X_{g,g_1,g_2}(x) + \cdots + \sum_{g_1,\ldots,g_m \in VQ_0} X_{g,g_1,\ldots,g_m}(x),$$

where $X_{g,g_1,g_2,\ldots,g_m}(x) \in M \otimes VQ_{g_{m-1},g_m} \otimes \cdots \otimes VQ_{g,g_1} \subset M \otimes VQ^{\square m}$, since for any $u \otimes v \in VQ_{g_1,g_2} \otimes VQ_{g,g_1}$,

$$(\nu \otimes \mathrm{id})(u \otimes v) = u \otimes g_1 \otimes v = (\mathrm{id} \otimes \mu)(u \otimes v),$$

implies $VQ_{g_1,g_2} \otimes VQ_{g,g_1} = VQ_{g_1,g_2} \square_{kVQ_0} VQ_{g,g_1}$. Note that $(\mathrm{id} \otimes \varepsilon)\nu = \mathrm{id}$. For any $x \in X_g$, write $X_{g,g_1}(x) = X^0_{g,g_1}(x) \otimes X^1_{g,g_1}(x)$ and $X_{g,g_1,\ldots g_m}(x) = X^0_{g,g_1,\ldots g_m}(x) \otimes X^m_{g,g_1,\ldots g_m}(x) \otimes \cdots \otimes X^1_{g,g_1}(x)$. Thus,

$$\begin{aligned}
(\nu \otimes \mathrm{id})\nu(x) =& x \otimes g \otimes g + \sum_{g_1 \in VQ_0} X^0_{g,g_1}(x) \otimes g_1 \otimes X^1_{g,g_1}(x) \\
&+ \sum_{g_1,g_2 \in VQ_0} X^0_{g,g_1,g_2}(x) \otimes g_2 \otimes \left(X^2_{g,g_1,g_2}(x) \otimes X^1_{g,g_1}(x)\right) + \ldots \\
&+ \sum_{g_1,\ldots,g_m \in VQ_0} X^0_{g,g_1,\ldots,g_m}(x) \otimes g_m \otimes \left(X^m_{g,g_1,\ldots g_m}(x) \otimes \cdots \otimes X^1_{g,g_1}(x)\right) \\
&+ \sum_{g_1 \in VQ_0} X^0_{g,g_1}(x) \otimes X^1_{g,g_1}(x) \otimes g + \sum_{g_1,g_2 \in VQ_0} X^0_{g,g_1,g_2}(x) \otimes X^2_{g,g_1,g_2}(x) \otimes X^1_{g,g_1}(x) + \ldots \\
&+ \sum_{g_1,\ldots,g_m \in VQ_0} X^0_{g,g_1,\ldots,g_m}(x) \otimes X^m_{g,g_1,\ldots,g_m}(x) \otimes \left(X^{m-1}_{g,g_1,\ldots,g_{m-1}}(x) \otimes \cdots \otimes X^1_{g,g_1}(x)\right) + \ldots \\
&+ \sum_{g_1,\ldots,g_m \in VQ_0} X^0_{g,g_1,\ldots,g_m}(x) \otimes \left(X^m_{g,g_1,\ldots,g_m}(x) \otimes \cdots \otimes X^1_{g,g_1}(x)\right) \otimes g \\
=& x \otimes g \otimes g \\
&+ \sum_{g_1 \in VQ_0} X^0_{g,g_1}(x) \otimes \left(g_1 \otimes X^1_{g,g_1}(x) + X^1_{g,g_1}(x) \otimes g\right) \\
&+ \sum_{g_1,g_2 \in VQ_0} X^0_{g,g_1,g_2}(x) \otimes \Big(g_2 \otimes \left(X^2_{g,g_1,g_2}(x) \otimes X^1_{g,g_1}(x)\right) + X^2_{g,g_1,g_2}(x) \otimes X^1_{g,g_1}(x) + \\
&\left(X^2_{g,g_1,g_2}(x) \otimes X^1_{g,g_1}(x)\right) \otimes g\Big) + \ldots \\
&+ \sum_{g_1,\ldots,g_m \in VQ_0} X^0_{g,g_1,\ldots,g_m}(x) \otimes \Big(g_m \otimes \left(X^m_{g,g_1,\ldots,g_m}(x) \otimes \cdots \otimes X^1_{g,g_1}(x)\right) + \\
&\sum_{i=1}^{m-1} \left(X^m_{g,g_1,\ldots,g_m}(x) \otimes \cdots \otimes X^{i+1}_{g,g_1,\ldots,g_{i+1}}(x)\right) \otimes \left(X^i_{g,g_1,\ldots,g_i}(x) \otimes \cdots \otimes X^1_{g,g_1}(x)\right) + \\
&\left(X^m_{g,g_1,\ldots,g_m}(x) \otimes \cdots \otimes X^1_{g,g_1}(x)\right) \otimes g\Big) \\
=& (\mathrm{id} \otimes \Delta)\nu(x),
\end{aligned}$$

shows that ν is indeed a (right) comodule structure for M.

Let $\sigma : M \to N$ be a comodule homomorphism. The equality $\nu_N \sigma = (\sigma \otimes \mathrm{id})\nu_M$, implies $\sigma(M_g) \subset N_g$, since

$$\sigma(\mathrm{id} \otimes \varepsilon_g)\nu_M = (\mathrm{id} \otimes \varepsilon_g)(\sigma \otimes \mathrm{id})\nu_M = (\mathrm{id} \otimes \varepsilon_g)\nu_N \sigma.$$

Set $\sigma_g = \sigma|_{M_g} : M_g \to N_g$, for each $g \in VQ_0$. Then

$$
\begin{aligned}
Y_{g,h}\sigma_g =&((\mathrm{id}\otimes\varepsilon_h)\nu_N\otimes\pi_1)\ \nu_N|_{N_g}\ \sigma|_{M_g} = ((\mathrm{id}\otimes\varepsilon_h)\nu_N\otimes\pi_1)(\sigma\otimes\mathrm{id})\ \nu_M|_{M_g}\\
=&((\mathrm{id}\otimes\varepsilon_h)\nu_N\sigma\otimes\pi_1)\ \nu_M|_{M_g} = ((\mathrm{id}\otimes\varepsilon_h)(\sigma\otimes\mathrm{id})\nu_M\otimes\pi_1)\ \nu_M|_{M_g}\\
=&(\sigma|_{M_h}(\mathrm{id}\otimes\varepsilon_h)\nu_M\otimes\pi_1)\ \nu_M|_{M_g} = (\sigma|_{M_h}\otimes\mathrm{id})((\mathrm{id}\otimes\varepsilon_h)\nu_M\otimes\pi_1)\ \nu_M|_{M_g}\\
=&(\sigma_h\otimes\mathrm{id})X_{g,h}
\end{aligned}
$$

shows that $(\sigma_g)_{g\in VQ_0}$ is a morphism of local representations.

Extending the commutative diagram for the morphism $f : X\to Y$ of local representations

$$
\begin{array}{ccccccccc}
X_g & \xrightarrow{X_{g,g_1}} & X_{g_1}\otimes VQ_{g,g_1} & \xrightarrow{X_{g_1,g_2}\otimes\mathrm{id}} & X_{g_2}\otimes VQ_{g_1,g_2}\otimes VQ_{g,g_1} & \longrightarrow & \dots & \xrightarrow{X_{g_{m-1},g_m}\otimes\mathrm{id}^{m-1}} & X_{g_m}\otimes VQ_{g_{m-1},g_m}\otimes\dots\otimes VQ_{g,g_1}\\
{\scriptstyle f_g}\downarrow & & \downarrow{\scriptstyle f_{g_1}\otimes\mathrm{id}} & & \downarrow{\scriptstyle f_{g_2}\otimes\mathrm{id}^2} & & & & \downarrow{\scriptstyle f_{g_m}\otimes\mathrm{id}^m}\\
Y_g & \xrightarrow{Y_{g,g_1}} & Y_{g_1}\otimes VQ_{g,g_1} & \xrightarrow{Y_{g_1,g_2}\otimes\mathrm{id}} & Y_{g_2}\otimes VQ_{g_1,g_2}\otimes VQ_{g,g_1} & \longrightarrow & \dots & \xrightarrow{Y_{g_{m-1},g_m}\otimes\mathrm{id}^{m-1}} & Y_{g_m}\otimes VQ_{g_{m-1},g_m}\otimes\dots\otimes VQ_{g,g_1}
\end{array}
$$

is easy to see that, for any $x\in X_g$,

$$
\begin{aligned}
(f\otimes\mathrm{id})\nu_X(x) =& f_g(x)\otimes g + \sum_{g_1\in VQ_0}(f_{g_1}\otimes\mathrm{id})X_{g,g_1}(x) + \sum_{g_1,g_2\in VQ_0}(f_{g_2}\otimes\mathrm{id}^2)X_{g,g_1,g_2}(x)+\dots\\
&+\sum_{g_1,\dots,g_m\in VQ_0}(f_{g_m}\otimes\mathrm{id}^m)X_{g,g_1,\dots,g_m}(x)\\
=& f_g(x)\otimes g + \sum_{g_1\in VQ_0}Y_{g,g_1}(f_g(x)) + \sum_{g_1,g_2\in VQ_0}Y_{g,g_1,g_2}(f_g(x))+\dots\\
&+\sum_{g_1,\dots,g_m\in VQ_0}Y_{g,g_1,\dots,g_m}(f_g(x)) = \nu_Y f(x),
\end{aligned}
$$

showing that $f : \bigoplus_{g\in VQ_0}X_g \to \bigoplus_{g\in VQ_0}Y_g$ is a comodule homomorphism.

Therefore, the correspondence between comodules and local representations is functorial. Moreover, if M is a (right) ($k[VQ]$-)comodule and X is the local representation obtained from M, then,

$$
\nu_X = \sum_{i=0}^{m}\left(\sum_{g_0,\dots g_i\in VQ_0}\big(\mathrm{id}\otimes(\varepsilon_{g_i}\otimes\pi_1\otimes\dots\otimes\pi_1\otimes\varepsilon_{g_0})\Delta^{2i}\big)\nu_M\right) = \nu_M.
$$

If X is a local representation and M its associated comodule, then

$$
\begin{aligned}
M_{g,h}(x) =& \big(\mathrm{id}\otimes(\varepsilon_h\otimes\pi_1\otimes\varepsilon_g)\Delta^2\big)\nu_X(x)\\
=& \big(\mathrm{id}\otimes(\varepsilon_h\otimes\pi_1\otimes\varepsilon_g)\Delta^2\big)\left(x\otimes g + \sum_{g_1\in VQ_0}X_{g,g_1}(x)+\dots+\sum_{g_1,\dots,g_m\in VQ_0}X_{g,g_1,\dots,g_m}(x)\right)\\
=& X_{g,h}(x).
\end{aligned}
$$

Hence the categories $\mathcal{M}^{k[VQ]}$ and $\mathbf{Rep}_k^l(VQ)$ are isomorphic.

References

[Abe80] Eiichi Abe. *Hopf algebras.* Vol. 74. Cambridge Tracts in Mathematics. Translated from the Japanese by Hisae Kinoshita and Hiroko Tanaka. Cambridge University Press, Cambridge-New York, 1980, pp. xii+284. ISBN: 0-521-22240-0 (cit. on pp. 12–15, 17, 25, 34).

[ASS06] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1.* Vol. 65. London Mathematical Society Student Texts. Techniques of representation theory. Cambridge University Press, Cambridge, 2006, pp. x+458. ISBN: 9780511614309. DOI: 10.1017/CBO9780511614309. URL: https://doi.org/10.1017/CBO9780511614309 (cit. on pp. 5, 41, 42).

[ARS95] Maurice Auslander, Idun Reiten, and Sverre O. Smalø. *Representation theory of Artin algebras.* Vol. 36. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995, pp. xiv+423. ISBN: 0-521-41134-3. DOI: 10.1017/CBO9780511623608. URL: http://dx.doi.org/10.1017/CBO9780511623608 (cit. on p. 40).

[Ben91] David J. Benson. *Representations and cohomology. I.* Vol. 30. Cambridge Studies in Advanced Mathematics. Basic representation theory of finite groups and associative algebras. Cambridge University Press, Cambridge, 1991, pp. xii+224. ISBN: 0-521-36134-6 (cit. on pp. 25, 42).

[BGP73] Joseph N. Bernstein, Izrail' M. Gel' fand, and V. A. Ponomarev. "Coxeter functors and Gabriel's theorem". *Uspehi Mat. Nauk* 28.2(170) (1973), pp. 19–33. ISSN: 0042-1316. DOI: 10.1070/RM1973v028n02ABEH001526. URL: https://dx.doi.org/10.1070/RM1973v028n02ABEH001526 (cit. on p. 83).

[Bru66] Armand Brumer. "Pseudocompact algebras, profinite groups and class formations". *J. Algebra* 4 (1966), pp. 442–470. ISSN: 0021-8693. DOI: 10.1016/0021-8693(66)90034-2. URL: http://dx.doi.org/10.1016/0021-8693(66)90034-2 (cit. on pp. 30, 33, 35, 99).

[CE99] Henri Cartan and Samuel Eilenberg. *Homological algebra.* Princeton Landmarks in Mathematics. With an appendix by David A. Buchsbaum, Reprint of the 1956 original. Princeton University Press, Princeton, NJ, 1999, pp. xvi+390. ISBN: 0-691-04991-2 (cit. on p. 99).

[CHZ06] Xiao-Wu Chen, Hualin Huang, and Pu Zhang. "Dual Gabriel theorem with applications". *Sci. China Ser. A* 49.1 (2006), pp. 9–26. ISSN: 1006-9283. DOI: 10.1007/s11425-004-5235-4. URL: https://doi.org/10.1007/s11425-004-5235-4 (cit. on pp. 52, 71).

[CZ07] Xiao-Wu Chen and Pu Zhang. “Comodules of $U_q(\mathrm{sl}_2)$ and modules of $\mathrm{SL}_q(2)$ via quiver methods”. *J. Pure Appl. Algebra* 211.3 (2007), pp. 862–876. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2007.03.010. URL: https://doi.org/10.1016/j.jpaa.2007.03.010 (cit. on p. 47).

[Chi02] William Chin. “Hereditary and path coalgebras”. *Comm. Algebra* 30.4 (2002), pp. 1829–1831. ISSN: 0092-7872. DOI: 10.1081/AGB-120013219. URL: http://dx.doi.org/10.1081/AGB-120013219 (cit. on p. 43).

[Chi10] William Chin. “Galois coverings of pointed coalgebras”. *J. Algebra* 324.11 (2010), pp. 3169–3182. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2010.06.020. URL: https://doi.org/10.1016/j.jalgebra.2010.06.020 (cit. on pp. 47, 93).

[CKQ02] William Chin, Mark Kleiner, and Declan Quinn. “Almost split sequences for comodules”. *J. Algebra* 249.1 (2002), pp. 1–19. ISSN: 0021-8693. DOI: 10.1006/jabr.2001.9086. URL: https://doi.org/10.1006/jabr.2001.9086 (cit. on p. 47).

[CM97] William Chin and Susan Montgomery. “Basic coalgebras”. In: *Modular interfaces (Riverside, CA, 1995)*. Vol. 4. AMS/IP Stud. Adv. Math. Amer. Math. Soc., Providence, RI, 1997, pp. 41–47. DOI: 10.1090/amsip/004/03. URL: https://doi.org/10.1090/amsip/004/03 (cit. on pp. 25, 43, 52).

[CM96] William Chin and Ian M. Musson. “The coradical filtration for quantized enveloping algebras”. *J. London Math. Soc. (2)* 53.1 (1996), pp. 50–62. ISSN: 0024-6107. DOI: 10.1112/S0024610799008248. URL: https://doi.org/10.1112/S0024610799008248 (cit. on p. 25).

[CM06] Claude Cibils and Eduardo N. Marcos. “Skew category, Galois covering and smash product of a k-category”. *Proc. Amer. Math. Soc.* 134.1 (2006), pp. 39–50. ISSN: 0002-9939. DOI: 10.1090/S0002-9939-05-07955-4. URL: https://doi.org/10.1090/S0002-9939-05-07955-4 (cit. on p. 93).

[CM16] Claude Cibils and Eduardo N. Marcos. “Invariants of a free linear category and representation type”. *J. Pure Appl. Algebra* 220.9 (2016), pp. 3119–3132. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2016.02.007. URL: https://doi.org/10.1016/j.jpaa.2016.02.007 (cit. on pp. 3, 81, 88–91).

[Coh85] Paul M. Cohn. *Free rings and their relations*. Second. Vol. 19. London Mathematical Society Monographs. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1985, pp. xxii+588. ISBN: 0-12-179152-1 (cit. on pp. 85, 87).

[Coh03] Paul M. Cohn. *Basic algebra*. Groups, rings and fields. Springer-Verlag London, Ltd., London, 2003, pp. xii+465. ISBN: 1-85233-587-4. DOI: 10.1007/978-0-85729-428-9. URL: https://doi.org/10.1007/978-0-85729-428-9 (cit. on p. 22).

[CG02] Juan Cuadra and José Gómez-Torrecillas. “Idempotents and Morita-Takeuchi theory”. *Comm. Algebra* 30.5 (2002), pp. 2405–2426. ISSN: 0092-7872. DOI: 10.1081/AGB-120003476. URL: https://doi.org/10.1081/AGB-120003476 (cit. on p. 55).

[DNR01] Sorin Dăscălescu, Constantin Năstăsescu, and Şerban Raianu. *Hopf algebras*. Vol. 235. Monographs and Textbooks in Pure and Applied Mathematics. An introduction. Marcel Dekker, Inc., New York, 2001, pp. x+401. ISBN: 0-8247-0481-9 (cit. on pp. 12, 16, 19–22, 30, 33, 84, 99).

[DR75] Vlastimil Dlab and Claus Michael Ringel. "On algebras of finite representation type". *J. Algebra* 33 (1975), pp. 306–394. ISSN: 0021-8693. DOI: 10.1016/0021-8693(75)90125-8. URL: https://doi.org/10.1016/0021-8693(75)90125-8 (cit. on p. 63).

[Dok+13] Michael A. Dokuchaev et al. "Dynkin diagrams and spectra of graphs". *São Paulo J. Math. Sci.* 7.1 (2013), pp. 83–104. ISSN: 1982-6907. DOI: 10.11606/issn.2316-9028.v7i1p83-104. URL: https://doi.org/10.11606/issn.2316-9028.v7i1p83-104 (cit. on p. 83).

[Dro80] Ju. A. Drozd. "Tame and wild matrix problems". In: *Representation Theory II.* Ed. by Vlastimil Dlab and Peter Gabriel. Berlin, Heidelberg: Springer Berlin Heidelberg, 1980, pp. 242–258. DOI: 10.1007/BFb0088467. URL: https://doi.org/10.1007/BFb0088467 (cit. on p. 1).

[FM20] Vitor de Oliveira Ferreira and Lucia Satie Ikemoto Murakami. *Uma introdução às álgebras de Hopf.* Vol. 5. Textuniversitários. Editora Livraria da Física, São Paulo, 2020, pp. xiv+302. ISBN: 978-85-7861-646-5 (cit. on pp. 12, 84).

[FMP04] Vitor O. Ferreira, Lúcia S. I. Murakami, and Antonio Paques. "A Hopf–Galois correspondence for free algebras". *Journal of Algebra* 276.1 (2004), pp. 407–416. ISSN: 0021-8693. DOI: https://doi.org/10.1016/S0021-8693(03)00502-7. URL: https://www.sciencedirect.com/science/article/pii/S0021869303005027 (cit. on p. 92).

[Gab73] Peter Gabriel. "Indecomposable representations. II". In: *Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971).* Academic Press, London, 1973, pp. 81–104 (cit. on pp. 36, 46, 47, 61, 83).

[Gre83] Edward L. Green. "Graphs with relations, coverings and group-graded algebras". *Trans. Amer. Math. Soc.* 279.1 (1983), pp. 297–310. ISSN: 0002-9947. DOI: 10.2307/1999386. URL: https://doi.org/10.2307/1999386 (cit. on p. 93).

[HGK10] Michiel Hazewinkel, Nadiya Gubareni, and V. V. Kirichenko. *Algebras, rings and modules.* Vol. 168. Mathematical Surveys and Monographs. Lie algebras and Hopf algebras. American Mathematical Society, Providence, RI, 2010, pp. xii+411. ISBN: 978-0-8218-5262-0. DOI: 10.1090/surv/168. URL: https://doi.org/10.1090/surv/168 (cit. on p. 55).

[HR74] Robert G. Heyneman and David E. Radford. "Reflexivity and coalgebras of finite type". *J. Algebra* 28 (1974), pp. 215–246. ISSN: 0021-8693. DOI: 10.1016/0021-8693(74)90035-0. URL: https://doi.org/10.1016/0021-8693(74)90035-0 (cit. on pp. 15, 17).

[Iov13] M. C. Iovanov. "Complete path algebras and rational modules". *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* 56(104).3 (2013), pp. 349–364. ISSN: 1220-3874 (cit. on p. 43).

[IM20] Kostiantyn Iusenko and John William MacQuarrie. "The path algebra as a left adjoint functor". *Algebr. Represent. Theory* 23.1 (2020), pp. 33–52. ISSN: 1386-923X. DOI: 10.1007/s10468-018-9836-y. URL: https://doi.org/10.1007/s10468-018-9836-y (cit. on pp. 35, 36, 44, 45, 57, 66).

[IM22] Kostiantyn Iusenko and John William MacQuarrie. "Semisimplicity and separability for pseudocompact algebras". *arXiv e-prints*, arXiv:2207.06281 (July 2022), arXiv:2207.06281. DOI: 10.48550/arXiv.2207.06281. arXiv: 2207.06281 [math.RA] (cit. on pp. 33–35, 44, 66).

[IMQ21] Kostiantyn Iusenko, John William MacQuarrie, and Samuel Quirino. “A functorial approach to Gabriel k-quiver constructions for coalgebras and pseudocompact algebras”. *Bull. Braz. Math. Soc. (N.S.)* 52.3 (2021), pp. 697–719. ISSN: 1678-7544. DOI: 10.1007/s00574-020-00227-4. URL: https://doi.org/10.1007/s00574-020-00227-4 (cit. on pp. 2, 40).

[Kha78] Vladislav K. Kharchenko. “Algebras of invariants of free algebras”. *Algebra i Logika* 17.4 (1978), pp. 478–487, 491. ISSN: 0373-9252 (cit. on pp. 81, 85, 92).

[KS05] Justyna Kosakowska and Daniel Simson. “Hereditary coalgebras and representations of species”. *J. Algebra* 293.2 (2005), pp. 457–505. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2005.04.019. URL: https://doi.org/10.1016/j.jalgebra.2005.04.019 (cit. on p. 47).

[Lan78] David Richard Lane. “Free algebras of rank two and their automorphisms”. PhD thesis. Bedford College, University of London, 1978. URL: https://ethos.bl.uk/OrderDetails.do?uin=uk.bl.ethos.462675 (cit. on pp. 81, 85).

[Lem12] Joel Lemay. “Valued Graphs and the Representation Theory of Lie Algebras”. *Axioms* 1.2 (2012), pp. 111–148. ISSN: 2075-1680. DOI: 10.3390/axioms1020111. URL: https://www.mdpi.com/2075-1680/1/2/111 (cit. on pp. 61–63, 93).

[Mac98] Saunders MacLane. *Categories for the working mathematician*. Second. Vol. 5. Graduate Texts in Mathematics. Springer-Verlag, New York, 1998, pp. xii+314. ISBN: 0-387-98403-8 (cit. on pp. 5, 9–11).

[Mon93] Susan Montgomery. *Hopf algebras and their actions on rings*. Vol. 82. CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993, pp. xiv+238. ISBN: 0-8218-0738-2. DOI: 10.1090/cbms/082. URL: https://doi.org/10.1090/cbms/082 (cit. on pp. 12, 14, 15, 17, 22, 29).

[Mon95] Susan Montgomery. “Indecomposable coalgebras, simple comodules, and pointed Hopf algebras”. *Proc. Amer. Math. Soc.* 123.8 (1995), pp. 2343–2351. ISSN: 0002-9939. DOI: 10.2307/2161257. URL: https://doi.org/10.2307/2161257 (cit. on p. 43).

[Mun75] James R. Munkres. *Topology: a first course*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975, pp. xvi+413 (cit. on pp. 95, 96).

[NT93] C. Năstăsescu and B. Torrecillas. “Graded coalgebras”. *Tsukuba J. Math.* 17.2 (1993), pp. 461–479. ISSN: 0387-4982. DOI: 10.21099/tkbjm/1496162273. URL: https://doi.org/10.21099/tkbjm/1496162273 (cit. on p. 24).

[Nav22] Fernando R. Naves. “Adjunctions between categories of algebras and bifinite quotient extensions”. PhD thesis. Belo Horizonte, Brasil: Instituto de Ciências Exatas, Universidade Federal de Minas Gerais, Aug. 2022. URL: http://hdl.handle.net/1843/47367 (cit. on pp. 74, 75).

[Naz73] L. A. Nazarova. “Representations of quivers of infinite type”. *Izv. Akad. Nauk SSSR Ser. Mat.* 37 (1973), pp. 752–791. ISSN: 0373-2436 (cit. on p. 83).

[Nic78] Warren D. Nichols. “Bialgebras of type one”. *Comm. Algebra* 6.15 (1978), pp. 1521–1552. ISSN: 0092-7872. DOI: 10.1080/00927877808822306. URL: https://doi.org/10.1080/00927877808822306 (cit. on p. 25).

[NS02] Sebastian Nowak and Daniel Simson. "Locally Dynkin quivers and hereditary coalgebras whose left comodules are direct sums of finite dimensional comodules". *Comm. Algebra* 30.1 (2002), pp. 455–476. ISSN: 0092-7872. DOI: 10.1081/AGB-120006503. URL: https://doi.org/10.1081/AGB-120006503 (cit. on p. 47).

[Pie82] Richard S. Pierce. *Associative algebras.* Vol. 88. Graduate Texts in Mathematics. Studies in the History of Modern Science, 9. Springer-Verlag, New York-Berlin, 1982, pp. xii+436. ISBN: 0-387-90693-2 (cit. on p. 17).

[Qui17] Samuel Quirino. "Path coalgebra as a right adjoint functor". MA thesis. Departamento de Matemática, Universidade Federal de Minas Gerais, Brasil, Feb. 2017. URL: http://hdl.handle.net/1843/EABA-AQXL4B (cit. on p. 54).

[Rad82] David E. Radford. "On the structure of pointed coalgebras". *J. Algebra* 77.1 (1982), pp. 1–14. ISSN: 0021-8693. DOI: 10.1016/0021-8693(82)90274-5. URL: https://doi.org/10.1016/0021-8693(82)90274-5 (cit. on pp. 3, 52, 55, 75, 76).

[Rad11] David E. Radford. *Hopf algebras.* Vol. 49. Series on Knots and Everything. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011, pp. xxii+559. ISBN: 9789814335997. DOI: 10.1142/8055. URL: https://www.worldscientific.com/doi/abs/10.1142/8055 (cit. on pp. 12, 22).

[RZ10] Luis Ribes and Pavel Zalesskii. *Profinite groups.* Second. Vol. 40. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2010, pp. xvi+464. ISBN: 978-3-642-01641-7. DOI: 10.1007/978-3-642-01642-4. URL: https://doi.org/10.1007/978-3-642-01642-4 (cit. on p. 30).

[Rot09] Joseph J. Rotman. *An introduction to homological algebra.* Second. Universitext. Springer, New York, 2009, pp. xiv+709. ISBN: 978-0-387-24527-0. DOI: 10.1007/b98977. URL: https://doi.org/10.1007/b98977 (cit. on pp. 22, 99).

[Sim01] Daniel Simson. "Coalgebras, comodules, pseudocompact algebras and tame comodule type". *Colloq. Math.* 90.1 (2001), pp. 101–150. ISSN: 0010-1354. DOI: 10.4064/cm90-1-9. URL: http://dx.doi.org/10.4064/cm90-1-9 (cit. on pp. 32, 33, 43).

[Sim07] Daniel Simson. "Path coalgebras of profinite bound quivers, cotensor coalgebras of bound species and locally nilpotent representations". *Colloq. Math.* 109.2 (2007), pp. 307–343. ISSN: 0010-1354. DOI: 10.4064/cm109-2-12. URL: https://doi.org/10.4064/cm109-2-12 (cit. on p. 47).

[Sim08] Daniel Simson. "Tame-wild dichotomy for coalgebras". *J. Lond. Math. Soc. (2)* 78.3 (2008), pp. 783–797. ISSN: 0024-6107. DOI: 10.1112/jlms/jdn047. URL: https://doi.org/10.1112/jlms/jdn047 (cit. on p. 47).

[Sim11] Daniel Simson. "Coalgebras of tame comodule type, comodule categories, and a tame-wild dichotomy problem". In: *Representations of algebras and related topics.* EMS Ser. Congr. Rep. Eur. Math. Soc., Zürich, 2011, pp. 561–660. DOI: 10.4171/101-1/12. URL: https://doi.org/10.4171/101-1/12 (cit. on pp. 42, 43, 84).

[Swe69] Moss E. Sweedler. *Hopf algebras.* Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969, pp. vii+336 (cit. on pp. 12, 14–18, 20, 22, 26, 30, 34).

[TW74] Earl J. Taft and Robert Lee Wilson. "On antipodes in pointed Hopf algebras". *J. Algebra* 29 (1974), pp. 27–32. ISSN: 0021-8693. DOI: 10.1016/0021-8693(74)90107-0. URL: https://doi.org/10.1016/0021-8693(74)90107-0 (cit. on pp. 29, 47).

[War89] Seth Warner. *Topological fields*. Vol. 157. North-Holland Mathematics Studies. Notas de Matemática [Mathematical Notes], 126. North-Holland Publishing Co., Amsterdam, 1989, pp. xiv+563. ISBN: 0-444-87429-1 (cit. on pp. 95–98).

[Woo97] David Woodcock. "Some categorical remarks on the representation theory of coalgebras". *Comm. Algebra* 25.9 (1997), pp. 2775–2794. ISSN: 0092-7872. DOI: 10.1080/00927879708826022. URL: https://doi.org/10.1080/00927879708826022 (cit. on pp. 25, 42, 52, 71).

Index