Group cohomology based on partial representations

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À minha avó Azucena,
e às minhas tias Adriana e Gladys.

Resumo

Este trabalho é um estudo completo dos artigos [2] e [5]. Consideraremos a cohomologia parcial $H^n_{par}(G, M)$ de um grupo G com valores num $K_{par}(G)$-módulo M, introduzida em [2], que é definida como o functor derivado à direita do functor de invariantes parciais. Mostrando que o functor de invariantes parciais é representável, poderemos relacionar a cohomologia parcial de grupo com o espaço de derivações parciais e o ideal de aumento parcial; depois, construiremos uma resolução projetiva da álgebra B como $K_{par}(G)$-modulo, onde B é una subálgebra de $K_{par}(G)$. Isto permitirá dar uma outra caracterização da cohomologia parcial de grupo em termos de classes de funções que satisfazem uma certa identidade de n-cociclos. Mostramos a existência de uma sequência espectral de Grothendieck que relaciona a cohomologia do produto smash parcial com a cohomologia parcial do grupo e a cohomologia da álgebra. Dada uma ação parcial unital α de G em uma álgebra $\mathcal{A}$, consideramos a estrutura de $K_{par}(G)$-módulo de $\mathcal{A}$ induzida pela ação α e estudamos o problema de globalização para a cohomologia parcial em $\mathcal{A}$. O problema é reduzido a uma propriedade de extensibilidade de cociclos. Além disso, se $\mathcal{A}$ é um produto de blocos, mostramos que qualquer cociclo é globalizável e que as globalizações de cociclos cohomólogos também são cohomólogas, de onde temos que $H^n_{par}(G, M)$ é isomórfico ao grupo de cohomologia usual $H^n(G, \mathcal{M}(\mathcal{B}))$, onde $\mathcal{B}$ é a álgebra sob a ação envolvente de α e $\mathcal{M}(\mathcal{B})$ é a álgebra de multiplicadores de $\mathcal{B}$.

Abstract

This work is a full study of the papers [2] and [5]. We consider the partial group cohomology $H^n_{par}(G, M)$ of a group G with values in $K_{par}(G)$-module M, introduced in [2], which is defined as the right derived functor of the functor of partial invariants. Showing that the functor of partial invariants is representable, we relate the partial group cohomology with the space of partial derivations and the partial augmentation ideal; next, we construct a projective resolution of the algebra B as a $K_{par}(G)$-module, where B is a commutative subalgebra of $K_{par}(G)$. This allows us to give another characterization of the partial group cohomology in terms of classes of functions that satisfy a certain identity of n-cocycles. We show the existence of a Grothendieck spectral sequence that relates the cohomology of the partial smash product with the partial group cohomology and the algebra cohomology. Given a unital partial action α of G on a algebra $\mathcal{A}$ we consider the $K_{par}(G)$-module structure of $\mathcal{A}$ induced by α and study the globalization problem for the partial cohomology with values in $\mathcal{A}$. The problem is reduced to an extendibility property of cocycles. Moreover, if $\mathcal{A}$ is a product of indecomposable blocks, we show that any cocycle is globalizable, and globalizations of cohomologous cocycles are also cohomologous, whence we have that $H^n_{par}(G, M)$ is isomorphic to the usual cohomology group $H^n(G, \mathcal{M}(\mathcal{B}))$, where $\mathcal{B}$ is the algebra under the enveloping action of α and $\mathcal{M}(\mathcal{B})$ is the multiplier algebra of $\mathcal{B}$.

Index

CHAPTER 1

Introduction

Partial actions, partial representations, the corresponding crossed product and the interaction between them were introduced by R. Exel in [17, 18, 19] as methods of study C^*-algebras. Those works started the development of the theory of partial projective group representations in [10, 11, 12, 13], and the study of group cohomology based on partial actions in [2], [5] and [14]. For an overview of publications on partial actions, related concepts and more details see [16] and [9]. Spectral sequences were invented by J. Leray and R. C. Lyndon in 1940s. In homological algebra and algebraic topology, spectral sequence is a tool that allows us to compute homology groups using approximations of it. In an informal way we can think of a spectral sequence $(E^r, d^r)_{r\geq 1}$ as a book with infinite pages, where the next page is the homology of the previous page, and as we go through the pages we get closer to the homology that we want to compute.

We begin recalling preliminaries in Chapter 2. We give the definition of a partial group representation and an inverse semigroup, next we show some properties of the semigroup $\mathcal{S}(G)$ introduced by R. Exel in [17], with that we define the algebra $K_{par}(G)$ as the K-algebra with base $\mathcal{S}(G)$, and show that the category of partial representations of G and the category of representations of $K_{par}(G)$ are isomorphic. We also recall the definition of a partial action of G on an algebra A, and show that a unital partial action of G on A induces a structure of $K_{par}(G)$-module to A. In Section 2.2 we recall the construction of the partial smash product (also called partial skew group ring) $A \rtimes_\alpha G$, and some kind of universal property for the partial smash product. We also give the definition of a covariant pair, which is a pair in $\operatorname{Rep} A \times \operatorname{ParRep} G$ with certain compatibility property. Then we show that the category of the partial smash representations and the category of covariant pairs are equivalent. Later we show, as is proved in [6], that the partial group algebra $K_{par}(G)$ is isomorphic to a partial smash algebra $B \rtimes_\beta G$, where the algebra B is the commutative algebra generated by the idempotents of $\mathcal{S}(G)$. As a last result of this section we show that the algebra B has a structure of $K_{par}(G)$-module given by conjugation. Finally, in Section 2.3 we recall some definitions and fundamental known results regarding spectral sequences with the final objective of proving Theorem 2.106, which is used to obtain the main result of the Chapter 4.

In Section 3.1 we work with some results obtained in [2]. We define the partial group cohomology $H^n_{par}(G, M)$ of a group G with values in a $K_{par}(G)$-module M as the right derived functor of the functor of partial invariants $(-)^{G_{par}}$, which we prove to be equivalent to functor $\operatorname{Hom}_{K_{par}(G)}(B, -)$, that is, $(-)^{G_{par}}$ is representable. Later we define partial derivations as K-linear maps $\delta : K_{par}(G) \to M$ which satisfy a certain Leibniz rule, and the partial augmentation ideal defined as the kernel of the map $K_{par}(G) \to B$ such that $[g_1][g_2]...[g_n] \mapsto e_{(g_1,g_2,\dots,g_n)}$. After that we relate the partial group cohomology with the vector space of partial derivations $Der_{par}(G, M)$ and the partial augmentation ideal. In Sections 3.2 and 3.3 we study a part of the theory developed in [5]. Using the results obtained in the previous section to give another characterization of the 1-st cohomology group $H^1_{par}(G, M)$ in term of classes of maps $d : G \to M$ of maps that satisfy certain conditions, which form a vector space denoted $D(G, M)$, showing an isomorphism between the vector space of partial derivations $Der_{par}(G, M)$ and $D(G, M)$. Later we define the projective modules P_n as the direct sum of some submodules of $K_{par}(G)$, with the modules P_n we construct a projective resolution of $K_{par}(G)$-module of B in order to obtain another characterization of $H^n_{par}(G, M)$. After that we define the groups $C^n_{par}(G, M)$ in an analogous way to the groups $C^n(G, M)$ used in the construction of the classical cohomology group $H^n(G, M)$. Then we show that $\operatorname{Hom}_{K_{par}(G)}(P_n, M) \cong C^n_{par}(G, M)$ and that $H^n_{par}(G, M)$ is related to some functions of $C^n_{par}(G, M)$.

Chapter 4 corresponds to the final section of [2]. We relate the cohomology of partial smash products with the partial group cohomology and algebra cohomology, showing with Theorem 2.106 that there exists a Grothendieck spectral sequence that relates those cohomologies.

In Section 5.1 we study the final section of [5]. We work with an algebra $\mathcal{A}$ over a commutative ring K and a unital partial action α of G on $\mathcal{A}$. We can define a $K_{par}(G)$-module structure on $\mathcal{A}$ and study the globalization problem for the cohomology with values in this module. We fix an enveloping action $(\mathcal{B}, \beta)$ of α, and since the algebra $\mathcal{B}$ is not necessary unital we work more generally with the multiplier algebra $\mathcal{M}(\mathcal{B})$ of $\mathcal{B}$. First we prove that a cocycle $w \in Z^n_{par}(G, \mathcal{A})$ is globalizable if, and only if, there exists a certain extension $\widetilde{w} : G^n \to \mathcal{A}$ which satisfies some n-cocycle equality. In Section 5.2 we take an arbitrary cocycle $w \in Z^n_{par}(G, \mathcal{A})$ and construct a more manageable n-cocycle $w' \in Z^n_{par}(G, \mathcal{A})$ cohomologus to w. In Section 5.3 using the results obtained in the two previous sections we show that if $\mathcal{A}$ is product of blocks then any cocycle from $Z^n_{par}(G, \mathcal{A})$ is globalizable (Theorem 5.36). We prove Theorem 5.40 which says that globalizations of cohomologous cocycles are also cohomologous. Using Theorems 5.36 and 5.40 we prove that $H^n_{par}(G, M)$ is isomorphic to the usual cohomology group $H^n(G, \mathcal{M}(\mathcal{B}))$.

CHAPTER 2

Preliminaries

In this chapter we recall all the necessary definitions and results that will be used through this work. Let G be a group and K be any field. We denote by 1_G the identity of G.

2.1 Partial representations, inverse semigroups and partial actions

First we will show the definitions and some known results about partial representations, inverse semigroups and partial actions. Most of the results in this part are taken from [16].

Definition 2.1. *A* ***partial representation*** *of G on the K-vector space V is a map $\pi : G \to \mathrm{End}_K(V)$ such that, for any $s, t \in G$, we have:*

(a) $\pi(s)\pi(t)\pi(t^{-1}) = \pi(st)\pi(t^{-1})$,

(b) $\pi(s^{-1})\pi(s)\pi(t) = \pi(s^{-1})\pi(st)$,

(c) $\pi(1_G) = 1$,

where $1 = id_V$. More generally, we recall that a map $\pi : G \to S$ is a partial representation, where S is an unital algebra or just a monoid, if it satisfies items (a), (b) and (c).

In other words, π is a partial representation of G if the equality $\pi(s)\pi(t) = \pi(st)$ holds when the two sides are multiplied either by $\pi(s^{-1})$ on the left or by $\pi(t^{-1})$ on the right.

Example 2.2. *Any representation of G is a partial representation; moreover, if H is any subgroup of G and $\pi : H \to \mathrm{End}_K(V)$ is a partial representation of H, then the map $\tilde{\pi} : H \to \mathrm{End}_K(V)$ given by*

$$\tilde{\pi}(g) = \begin{cases} \pi(g) & \textit{if } g \in H \\ 0 & \textit{otherwise} \end{cases}$$

defines a partial representation of G.

Definition 2.3. *Let $\pi : G \to \mathrm{End}_K(V)$ and $\pi' : G \to \mathrm{End}_K(W)$ be two partial representations of G. A **morphism of partial representations** is a morphism of vector spaces $f : V \to W$, such that the next diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\pi(g)"'] & W \arrow[d, "\pi'(g)"] \\ V \arrow[r, "f"'] & W \end{tikzcd}$$

that is $f \circ \pi(g) = \pi'(g) \circ f \ \forall g \in G$.

The category of partial representations of G, denoted $\mathrm{ParRep}\, G$ is the category whose objects are pairs (V, π), where V is a K-vector space and $\pi : G \to \mathrm{End}_K(V)$ is a partial representation of G on V, and whose morphisms are morphisms of partial representations.

Definition 2.4. *A set S together with a binary operation $\cdot$ is called a **semigroup** if satisfies the associative property, i.e. for all $a, b, c \in S$, the equation $a\cdot(b\cdot c) = (a\cdot b)\cdot c$ holds.*

Definition 2.5. *Given two semigroups S and T, a map $f : S \to T$ is an **homomorphism** between semigroups if satisfies that $f(ab) = f(a)f(b)$ for all $a, b \in S$.*

Definition 2.6. *A semigroup S is said to be **regular**, if for each x in S there exists an element x^* in S such that*

i) $xx^*x = x$,

ii) $x^*xx^* = x^*$.

The element x^* is called an inverse of x. If moreover the idempotent elements of S commute then S is said to be an ***inverse semigroup****.*

Remark 2.7. If S is an inverse semigroup, then for each $x \in S$ the element xx^* is idempotent, since $(xx^*)(xx^*) = x(x^*xx^*) = xx^*$.

The next proposition gives us a definition equivalent to Definition 2.6, showing that the uniqueness of the inverse elements is a necessary and sufficient condition for a regular semigroup to be an inverse semigroup.

Proposition 2.8. *Let S be a regular semigroup. Then any x in S has an unique inverse if, and only if, the idempotents of S commute.*

Proof. Let S be a regular semigroup in which the idempotents of S commute and let u and v be inverses of x. Then

$$u = uxu = u(xvx)u = (ux)(vx)u,$$

where both ux and vx are idempotents. Thus, since idempotents commute, we have

$$u = (ux)(vx)u = (vx)(ux)u = vxu = (vxv)xu = v(xv)(xu),$$

then

$$u = v(xv)(xu) = v(xu)(xv) = v(xux)v = vxv = v.$$

Hence $u = v$.

To prove the converse, let S be a regular semigroup such that each element has a unique inverse. Let e and f arbitrary idempotents in S and let x be the inverse of ef. The element fxe is an idempotent. Indeed,

$$(fxe)^2 = f(xefx)e = fxe.$$

Moreover fxe is the inverse of ef, since

$$(fxe)ef(fxe) = (fxe^2)(f^2xe) = fxe$$

and

$$(ef)(fxe)(ef) = (ef^2)x(e^2f) = (ef)x(ef) = ef.$$

Any idempotent is self-inverse, and by the uniqueness of the inverse we have that $ef = fxe$, so ef is idempotent and it is self-inverse. Notice that we have proved that the product of idempotents is an idempotent, in particular, so is fe. Now observe that

$$ef(fe)(ef) = (ef)(ef) = ef \text{ and } fe(ef)fe = fe,$$

that is, fe is the inverse of ef, but since ef is self-inverse then $fe = ef$. □

Corollary 2.9. *If S is an inverse semigroup, then $(x^*)^* = x$ for any $x \in S$.*

Remark 2.10. Given s and t in an inverse semigroup S, we have that $(st)^* = t^*s^*$. Indeed, since xx^* is idempotent for any x in S, then

$$st(t^*s^*)st = s(tt^*)(s^*s)t = s(s^*s)(tt^*)t = st,$$

and

$$(t^*s^*)st(t^*s^*) = t^*(s^*s)(tt^*)s^* = t^*(tt^*)(s^*s)s^* = t^*s^*.$$

Proposition 2.11. *Let S be an inverse semigroup. Then the relation given by*

$$s \leq t \Leftrightarrow s = te, \text{ for some idempotent } e,$$

is a partial order.

Proof.

- *Reflexive*: For $s \in S$, we have $s \leq s$ since $s = ss^{-1}s$ and $s^{-1}s$ is idempotent.
- *Antisymmetric*: Given $a, b \in S$, if $a \leq b$ and $b \leq a$, then $a = be$ and $b = ai$ where e and i are idempotents, as $ei = ie$ we have:

$$a = be = aie = aiei = bei = ai = b.$$

- *Transitive*: Let a, b and c be in S such that $a \leq b$ and $b \leq c$. Hence $a = be_1$ y $b = ce_2$, Therefore $a = c(e_2e_1)$ since e_2e_1 is idempotent, thus $a \leq c$.

□

The partial order obtained in Proposition 2.11 is called the **natural partial order** on the inverse semigroup.

The next lemma gives equivalent definitions of the natural partial order.

Lemma 2.12. *For an inverse semigroup S the next conditions are equivalent:*

1. $s \leq t$,
2. $s = ft$ *for some idempotent* f,
3. $s^{-1} \leq t^{-1}$,
4. $s = ss^{-1}t$,

5. $s = ts^{-1}s$.

Proof.

- (1) $\Rightarrow$ (2). If $s \leq t$ then $s = te$ for some idempotent e. Taking $f = tet^{-1}$, f is idempotent since

$$f^2 = (tet^{-1})(tet^{-1}) = tt^{-1}tet^{-1} = tet^{-1} = f$$

  and $ft = (tet^{-1})t = tt^{-1}te = te = s$.

- (2) $\Rightarrow$ (3). Since $s = ft$ then $s^{-1} = t^{-1}f$, hence $s^{-1} \leq t^{-1}$.

- (3) $\Rightarrow$ (4). If $s^{-1} \leq t^{-1}$ then $s^{-1} = t^{-1}e$ for some idempotent e, hence $s = et$. Thus $ss^{-1} = ett^{-1}e = ett^{-1}$, Therefore $s = et = (ett^{-1})t = ss^{-1}t$.

- (4) $\Rightarrow$ (5). As $s = ss^{-1}t$ then $s^{-1} = t^{-1}ss^{-1}$ and $s^{-1}s = t^{-1}ss^{-1}t$, hence $s = ss^{-1}(tt^{-1}t) = t(t^{-1}ss^{-1}t) = ts^{-1}s$.

- (5) $\Rightarrow$ (1). It is clear by the definition of the natural partial order of an inverse semigroup.

$\square$

Definition 2.13. *A partial function $f : X \to Y$, is a function $f : X' \to Y'$ where $X' \subseteq X$ and $Y' \subseteq Y$. Given two partial functions $f : X \to Y$ and $g : Y \to W$ the composition $g \circ f : X \to W$ is the partial function with domain $f^{-1}(\operatorname{dom} g \cap \operatorname{im} f)$, such that $g \circ f(x) = g(f(x))$ for any $x \in f^{-1}(\operatorname{dom} g \cap \operatorname{im} f)$.*

Definition 2.14. *Let X be a set. Then*

$$I(X) := \{f : A \to B \mid A \subseteq X,\ B \subseteq X \text{ and } f \text{ is a bijection } \}$$

with the composition of partial functions is an inverse monoid, called ***the symmetric inverse semigroup*** *over X.*

The **Wagner–Preston representation Theorem** says that any inverse semigroup can be embedded in a symmetric inverse semigroup. Then for s in an inverse semigroup S we can understand the elements ss^* as the identity map on the image of s and s^*s as the identity map on the domain of s. Thus we can understand the natural partial order as follows: $s \leq t$ if, and only if, s is a restriction of t.

Definition 2.15. *Let $\sim$ be an equivalence relation of a semigroup S. We say that $\sim$ is a* ***congruence*** *if it is compatible with the semigroup operation, that is, $\sim \subseteq S \times S$ is an equivalence relation and for $x, y, a, b \in S$, if $x \sim y$ and $a \sim b$ then $ab \sim by$.*

It is easy to see that if we have a congruence $\sim$ of a semigroup S, then the set of the equivalence classes $S/\sim$ is a semigroup with the operation induced by S. Moreover, the natural projection $\pi : S \to S/\sim$ is a surjective homomorphism.

Definition 2.16. *Let S be a semigroup, if $R \subseteq S \times S$ we define the congruence generated by R as the intersection of all the congruences that contains R.*

Remark 2.17. The congruence generated by $R \subseteq S \times S$ exists since the arbitrary intersection of congruences is a congruence and the fact that $\sim = S \times S$ is trivially a semigroup congruence of S.

One of the most important inverse monoids in this work is $\mathcal{S}(G)$, defined by R. Exel in [17], which plays an important role in the construction of the *partial group algebra $K_{par}(G)$*.

Definition 2.18. *Let G be a group. Denote by $\mathcal{S}(G)$ the monoid defined by the generators $[t]$, $t \in G$, and relations:*

(1) $[1_G] = 1$;

(2) $[s^{-1}][s][t] = [s^{-1}][st]$;

(3) $[s][t][t^{-1}] = [st][t^{-1}]$;

for any $t, s \in G$. That is, define $\mathcal{S}(G) = [G]/\zeta$, where is $[G]$ the free semigroup generated by the set of symbols $\{[g] \mid g \in G\}$ and ζ is the congruence generated by the set

$$R = \{([s^{-1}][s][t], [s^{-1}][st]),\ ([s][t][t^{-1}], [st][t^{-1}]),\ ([1_G], 1) \mid s, t \in G\} \subseteq [G] \times [G].$$

Remark 2.19. Let $\psi : [G] \to G$ be the semigroup homomorphism given by $[g] \mapsto g$, and let $\pi : [G] \to \mathcal{S}(G)$ be the natural projection . Define

$$\Omega = \{(x, y) \in [G] \times [G] \mid \psi(x) = \psi(y)\}.$$

Clearly Ω is an equivalence relation and it is a congruence since ψ is a homomorphism. Furthermore, notice that $R \subseteq \Omega$, therefore $\zeta \subseteq \Omega$. Thus, for any $z \in \mathcal{S}(G)$ we have that $\psi(\pi^{-1}(z))$ is well-defined. Indeed, if $a, b \in [G]$ are such that $\pi(a) = \pi(b)$

then $(a, b) \in \zeta \subseteq \Omega$, therefore $\psi(a) = \psi(b)$. Then we can define the semigroup homomorphism

$$\begin{aligned} \eta : \mathcal{S}(G) &\to G \\ z &\mapsto \psi(\pi^{-1}(z)) \end{aligned}$$

Therefore, η is a semigroup homomorphism such that $\eta([g]) = g$, for all $g \in G$.

We define for each $g \in G$ the element $e_g = [g][g^{-1}] \in K_{par}(G)$. Notice that for all $g \in G$ the element e_g is idempotent, $e_g e_g = [g][g^{-1}][g][g^{-1}] = [g][g^{-1}] = e_g$. Also these elements satisfy the following relation:

$$[g]e_h = e_{gh}[g].$$

Indeed,

$$\begin{aligned} [g]e_h &= [g][h][h^{-1}] = [gh][h^{-1}] = [gh][h^{-1}g^{-1}g] \\ &= [gh][(gh)^{-1}g] = [gh][(gh)^{-1}][g] \\ &= e_{gh}[g]. \end{aligned}$$

Then we have that the elements e_g commute among themselves,

$$\begin{aligned} e_g e_h &= [g][g^{-1}]e_h = [g]e_{g^{-1}h}[g^{-1}] \\ &= e_{gg^{-1}h}[g][g^{-1}] = e_h e_g. \end{aligned}$$

The next useful results were proved first in [17].

Proposition 2.20. *Any element ω in $\mathcal{S}(G)$ admits a decomposition*

$$\omega = e_{g_1} e_{g_2} ... e_{g_n}[s],$$

where $n \geq 0$ and $g_1, g_2, ..., g_n, s$ are elements of G. In addition, one can assume that

i) $g_i \neq g_j$ for $i \neq j$,

ii) $g_i \neq s$ for any i.

Proof. Let S be the subset of $\mathcal{S}(G)$ consisting of those ω that do admit a decomposition as above. Since $n = 0$ is allowed, we see that each $[s]$ belongs to S. To prove the statement we only have to verify that S is a subsemigroup of $\mathcal{S}(G)$, in view of the fact that the set $\{[s] \in \mathcal{S}(G)\}$ generates $\mathcal{S}(G)$.

Let $\omega = e_{r_1}e_{r_2}...e_{r_n}[s]$. It suffices to demonstrate that $\omega[t]$ belongs to S, since this will prove S to be a right ideal and hence a subsemigroup. Now, note that

$$[s][t] = [s][s^{-1}][s][t] = [s][s^{-1}][st] = e_s[st].$$

So

$$\omega[t] = e_{r_1}...e_{r_n}[s][t] = e_{r_1}...e_{r_n}e_s[s][t] = e_{r_1}...e_{r_n}e_s[st].$$

If $s \neq r_i$ for any $i \in \{1, 2, ..., n\}$ then $e_{r_1}...e_{r_n}e_s[st] \in S$, on the other hand if $s = r_i$ for some $i \in \{1, 2, ..., n\}$ we have $e_{r_1}...e_{r_n}e_s[st] = e_{r_1}...e_{r_n}[st] \in S$ since the elements e_g are central idempotents. $\square$

Remark 2.21. Let $\eta : \mathcal{S}(G) \to G$ be the homomorphism of semigroups defined above. It is clear that $\eta(e_g) = 1_G$ for any $g \in G$. Now let ω be in $\mathcal{S}(G)$, and assume that we have two decomposition $e_{r_1}e_{r_2}...e_{r_n}[s]$ and $e_{h_1}e_{h_2}...e_{h_m}[t]$ of ω. Then

$$s = \eta(e_{r_1}e_{r_2}...e_{r_n}[s]) = \eta(e_{h_1}e_{h_2}...e_{h_m}[t]) = t.$$

Now let e be an idempotent element in $\mathcal{S}(G)$, then by Proposition 2.20 there exists a decomposition $e_{r_1}e_{r_2}...e_{r_n}[g]$ of e. Since e is idempotent then

$$e_{r_1}e_{r_2}...e_{r_n}[g]e_{r_1}e_{r_2}...e_{r_n}[g] = e_{r_1}e_{r_2}...e_{r_n}[g],$$

thus

$$g = \eta(e_{r_1}e_{r_2}...e_{r_n}[g]) = \eta(e_{r_1}e_{r_2}...e_{r_n}[g]e_{r_1}e_{r_2}...e_{r_n}[g]) = g^2.$$

Since G is a group, it follows that $g = 1_G$. Therefore $e = e_{r_1}e_{r_2}...e_{r_n}$, that means that the set $\{e_g \mid g \in G\}$ generates all the idempotents of $\mathcal{S}(G)$.

Finally, if e and f are idempotents in $\mathcal{S}(G)$ we have that $e[s] = f[t]$ if, and only if, $s = t$ and $ee_s = fe_s$. Indeed, by the first part we have $s = t$, then $e[s] = f[s]$ therefore $ee_s = e[s][s^{-1}] = f[s][s^{-1}] = fe_s$.

Remark 2.22. In fact any $\alpha \in \mathcal{S}(G)$ admits a unique standard decomposition

$$\alpha = \varepsilon_{r_1} \dots \varepsilon_{r_n}[s]$$

up to the order of the ε_r's. For more details see [17].

Proposition 2.23. *$\mathcal{S}(G)$ is an inverse semigroup.*

Proof. First observe that by definition of $\mathcal{S}(G)$ for any $g \in G$ we have that $[g^{-1}]$ is an inverse of $[g]$, moreover given $t \in G$ we have that $[t^{-1}][g^{-1}]$ is an inverse of $[g][t]$. Indeed,

$$[g][t][t^{-1}][g^{-1}][g][t] = [g][g^{-1}][g][t][t^{-1}][t] = [g][t],$$

and

$$[t^{-1}][g^{-1}][g][t][t^{-1}][g^{-1}] = [t^{-1}][t][t^{-1}][g^{-1}][g][g^{-1}] = [t^{-1}][g^{-1}].$$

Thus by induction we prove that for $g_1, g_2, ..., g_n \in G$ we have $[g_n^{-1}][g_{n-1}^{-1}] \dots [g_1^{-1}]$ is an inverse of $[g_1][g_2] \dots [g_n]$. Finally by Remark 2.21 idempotents are generated by the set $\{e_g \mid g \in G\}$, and we have that $e_g e_h = e_h e_g$ for any $g, h \in G$. □

Let us denote by s^{-1} the inverse of s in $\mathcal{S}(G)$.

Lemma 2.24. *Let η be the morphism defined in Remark 2.21. Then for each $s \in \mathcal{S}(G)$ we have $s = ss^{-1}[\eta(s)]$.*

Proof. Let s be in $\mathcal{S}(G)$, then by Proposition 2.20 and Remark 2.21 we have $s = e[\eta(s)]$ for some idempotent $e \in E(\mathcal{S}(G))$. Then

$$ss^{-1}[\eta(s)] = s(e[\eta(s)])^{-1}[\eta(s)] = s[\eta(s)^{-1}]e[\eta(s)] = s([\eta(s)^{-1}]e)(e[\eta(s)]) = ss^{-1}s = s.$$

□

Definition 2.25. *Given a group G and a field K, the* ***partial group algebra $K_{par}(G)$*** *is the semigroup algebra of $\mathcal{S}(G)$ over K, i.e. the algebra with K-basis $\mathcal{S}(G)$.*

$K_{par}(G)$ has the following universal property.

Proposition 2.26. *The map*

$$\psi : g \in G \mapsto [g] \in K_{par}(G)$$

is a partial representation, which we will call the ***universal partial representation*** *of G. In addition, for any partial representation π of G in a unital K-algebra B there exists a unique algebra homomorphism*

$$\phi : K_{par}(G) \to B,$$

such that $\pi(g) = \phi([g])$, for any $g \in G$.

Proof. It is clear that by Definition 2.25, ψ is a partial representation.

In order to prove the universal property of $K_{par}(G)$, we define F as the free K-algebra generated by set of symbols $\{[g] \mid g \in G\}$, and I the ideal of F generated by the set

$$\{[s^{-1}][s][t] - [s^{-1}][st], [s][t][t^{-1}] - [st][t^{-1}] \mid s, t \in G\} \subseteq F.$$

Notice that $\overline{[g]} \in F/I \mapsto [g] \in K_{par}(G)$ is an algebra isomorphism. Indeed, by the universal property of free algebras there exists an unique algebra morphism $f : F \to K_{par}(G)$ such that $f([g]) = [g]$. Observe that $\ker f = I$, therefore the map $F/I \to K_{par}(G)$ induced by f is an algebra isomorphism.

By the universal property of free algebras there exists an unique algebra morphism $\phi' : F \to B$ such that $\pi(g) = \phi'([g])$. Notice that $I \subseteq \ker(\phi')$, since

$$\phi'([s^{-1}][s][t] - [s^{-1}][st]) = \pi(s^{-1}st - s^{-1}(st)) = 0.$$

Then there exist an unique morphism $\overline{\phi} : F/I \to B$ such that $\overline{\phi}([g]) = \pi(g)$. Finally, as $K_{par}(G) \cong F/I$, there exist an unique morphism $\phi : K_{par}(G) \to B$ such that $\pi(g) = \phi([g])$.

□

Theorem 2.27. *The categories* $\operatorname{ParRep} G$ *and* $\operatorname{Rep} K_{par}(G)$ *are isomorphic.*

Proof. Let V be a K-vector space and let $\pi_V : G \to \operatorname{End}_K(V)$ be a partial representation of G in V. Then by Proposition 2.26 there is an unique representation $\phi_V : K_{par}(G) \to \operatorname{End}_K(V)$ such that $\phi_V([g]) = \pi_V(g)$. Now given two partial representations $\pi_V : G \to \operatorname{End}_K(V)$ and $\pi_W : G \to \operatorname{End}_K(W)$ of G and a morphism of partial representations $f : V \to W$. Thus from $f \circ \pi_V(g) = \pi_W(g) \circ f$ we have that $f \circ \phi_V([g]) = \phi_W([g]) \circ f$. Therefore f defines a morphism of representation. Conversely, if $\phi_V : K_{par}(G) \to \operatorname{End}_K(V)$ is a representation, then $\pi_V(g) = \phi_V([g])$ gives a partial representation of G in V. □

Another important concept in this work is the notion of a *partial action*.

Definition 2.28. *Let G be a group and A an algebra. A* ***partial action*** *α of G on A is given by a collection $\{D_g\}_{g\in G}$ of ideals of A and a collection $\{\alpha_g : D_{g^{-1}} \to D_g\}_{g\in G}$ of algebra isomorphisms, satisfying the following conditions:*

(1) $D_e = A$ *and* $\alpha_e = id_A$,

(2) $\alpha_h\left(D_{h^{-1}} \cap D_{(gh)^{-1}}\right) = D_h \cap D_{g^{-1}}$,

(3) if $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$ *then* $\alpha_g\alpha_h(x) = \alpha_{gh}(x)$.

Remark 2.29.

1. We can see in [16] that the conditions (2) and (3) can be replaced by the condition:

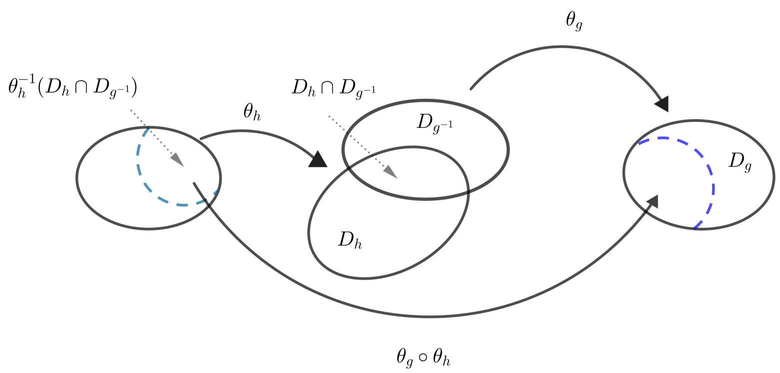
- $\alpha_g \circ \alpha_h \subseteq \alpha_{gh}$ for any g and h in G (where if f and f' are maps then $f \subseteq f'$ means that f is a restriction of f'),

Moreover, condition (2) can be replaced by the "weaker" assumption:

- $D_h \cap D_{g^{-1}} \subseteq \theta_h\left(D_{(gh)^{-1}}\right)$.

2. Notice that given a partial action θ of G on A we have that θ_g and θ_h are partial functions, for any $g, h \in G$, therefore $\theta_g \circ \theta_h$ means the composition of partial function. Thus the domain of $\theta_g \circ \theta_h$ is the set

$$\{x \in D_{h^{-1}} : \theta_h(x) \in D_{g^{-1}}\} = \theta_h^{-1}(D_{g^{-1}}) = \theta_h^{-1}(D_h \cap D_{g^{-1}}).$$



Example 2.30. *An action of G on an algebra A is clearly a partial action, defining $D_g = A$ for any $g \in G$ and α_g the map $a \in A \mapsto g(a) \in A$. Moreover, any unital ideal of A carries a partial action: if B is such an ideal, with unit 1_B, then a partial G-action β on B is obtained by defining $D_g = B \cap g(B)$ and β_g to be the restriction of α_g to the ideal $D_{g^{-1}}$. Note that each ideal D_g of B is also unital, with unit $u_g = 1_B g(1_B)$. We can see this example more details in [16, p.15].*

Definition 2.31. *Let $\left(A, \{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}\right)$ and $\left(B, \{E_g\}_{g\in G}, \{\beta_g\}_{g\in G}\right)$ be partial actions. A* ***morphism of partial actions***

$$\varphi : \left(A, \{D_g\}_{g\in G}, \{\theta_g\}_{g\in G}\right) \to \left(B, \{E_g\}_{g\in G}, \{\beta_g\}_{g\in G}\right)$$

is an algebra morphism $\varphi : A \to B$ such that $\varphi(D_g) \subset E_g$ and

$$\begin{tikzcd} D_{g^{-1}} \arrow[r, "\alpha_g"] \arrow[d, "\varphi"'] & D_g \arrow[d, "\varphi"] \\ E_{g^{-1}} \arrow[r, "\beta_g"'] & E_g \end{tikzcd}$$

commutes for each $g \in G$. That is $\varphi(\alpha_g(x)) = \beta_g(\varphi(x))$ for any $g \in G$ and $x \in D_{g^{-1}}$.

Definition 2.32. *A partial action α is called **unital** if each D_g is a unital algebra, that is $D_g = u_g A$ where u_g is a central idempotent in A.*

The next fact is very well known and allows us to relate the concepts of partial action and partial representation. Moreover we can also relate the concept of $K_{par}(G)$-module.

Lemma 2.33. *Let (A, α) be an unital partial action of a group G on an algebra A. Then the map*

$$\begin{aligned} \pi^\alpha : G &\to \mathrm{End}_K(A) \\ g &\mapsto \pi^\alpha_g \end{aligned}$$

where $\pi^\alpha_g(a) = \alpha_g(1_{g^{-1}}a)$ is a partial representation of G.

Proof. We have $\pi^\alpha_{1_G}(a) = \alpha_{1_G}(a) = a$ thus $\pi^\alpha_{1_G} = id_A$. Now observe that

$$\begin{aligned} \pi^\alpha_{g^{-1}}\pi^\alpha_g\pi^\alpha_h(a) &= \alpha_{g^{-1}}(1_g\alpha_g(1_{g^{-1}}\alpha_h(1_{h^{-1}}a))) \\ &= \alpha_{g^{-1}}\alpha_g\alpha_h(1_{h^{-1}}1_{h^{-1}g^{-1}}a) \\ &= \alpha_{g^{-1}}\alpha_{gh}(1_{h^{-1}}1_{(gh)^{-1}}a) \\ &= \alpha_{g^{-1}}(1_g1_{gh}\alpha_{gh}(1_{(gh)^{-1}}a)) \\ &= \alpha_{g^{-1}}(1_g\alpha_{gh}(1_{(gh)^{-1}}a)) = \pi^\alpha_{g^{-1}}\pi^\alpha_{gh}(a) \end{aligned}$$

Finally the last condition of Definition 2.1 is proved in an analogous way.

□

Remark 2.34. By Lemma 2.33 and Theorem 2.27 we have that if (A, α) is an unital partial action of a group G on an algebra A, then A has a structure of $K_{par}(G)$-module given by α.

Lemma 2.35. *If $\pi_M : G \to \mathrm{End}_K(M)$ and $\pi_N : G \to \mathrm{End}_K(N)$ are partial representation, then $\theta : G \to \mathrm{End}_K(\mathrm{Hom}_K(M,N))$, given by $\theta(g)(f) = \pi_N(g) \circ f \circ \pi_M(g^{-1})$, is a partial representation.*

Proof. It is clear that $\theta(1_G) = id_{Hom_K(M,N)}$. Now observe that

$$\begin{aligned}\theta(s^{-1})\theta(s)\theta(t)(f) &= \pi_N(s^{-1})\pi_N(s)\pi_N(t) \circ f \circ \pi_M(t^{-1})\pi_M(s^{-1})\pi_M(s)\\ &= \pi_N(s^{-1})\pi_N(st) \circ f \circ \pi_M(t^{-1}s^{-1})\pi_M(s)\\ &= \theta(s^{-1})\theta(st)(f)\end{aligned}$$

and

$$\begin{aligned}\theta(s)\theta(t)\theta(t^{-1})(f) &= \pi_N(s)\pi_N(t)\pi_N(t^{-1}) \circ f \circ \pi_M(t)\pi_M(t^{-1})\pi_M(s^{-1})\\ &= \pi_N(st)\pi_N(t^{-1}) \circ f \circ \pi_M(t)\pi_M(s^{-1}t^{-1})\\ &= \theta(st)\theta(t^{-1})(f).\end{aligned}$$

□

Lemma 2.36. *Let A be a unital algebra. If $\pi : G \to A$ is a partial representation, then $\theta : G \to A^{op}$, given by $\theta(g) = \pi(g^{-1})$ is a partial representation.*

Proof. First notice that $\theta(1_G) = \pi(1_G) = 1_A$. Furthermore, for any $s, t \in G$ we have that

$$\theta(s^{-1})\theta(s)\theta(t) = \pi(t^{-1})\pi(s^{-1})\pi(s) = \pi(t^{-1}s^{-1})\pi(s) = \theta(s^{-1})\theta(st)$$

and

$$\theta(s)\theta(t)\theta(t^{-1}) = \pi(t)\pi(t^{-1})\pi(s^{-1}) = \pi(t)\pi(s^{-1}t^{-1}) = \theta(st)\theta(t^{-1}).$$

□

Lemma 2.37. *if $\pi : G \to A$, $\theta : G \to B$ are partial representation of G on the unital algebras A and B, then $\Psi : G \to A \otimes_K B$, given by $\Psi(g) = \pi(g) \otimes \theta(g)$, is a partial representation on the algebra $A \otimes_K B$.*

Proof. For any $s, t \in G$, we have that

$$\begin{aligned}\Psi(s^{-1})\Psi(s)\Psi(t) &= (\pi(s^{-1}) \otimes \theta(s^{-1}))(\pi(s) \otimes \theta(s))(\pi(t) \otimes \theta(t))\\ &= \pi(s^{-1})\pi(s)\pi(t) \otimes \theta(s^{-1})\theta(s)\theta(t)\\ &= \pi(s^{-1})\pi(st) \otimes \theta(s^{-1})\theta(st)\\ &= \Psi(s^{-1})\Psi(st).\end{aligned}$$

In an analogous way we have that $\Psi(s)\Psi(t)\Psi(t^{-1}) = \Psi(st)\Psi(t^{-1})$. Finally, since $1_{A\otimes_K B} = 1_A \otimes 1_B$ we have that $\Psi(1_G) = 1_{A\otimes_K B}$. □

2.2 Partial smash product

Now we are able to construct a new algebra called *partial smash product* (also referred to as the "partial skew group ring" or "partial cross product") denoted by $A \rtimes_\alpha G$, where α is a partial action of a group G on an algebra A.

Definition 2.38. *Given a partial action α of G on A, we define the* ***partial smash product*** *$A \rtimes_\alpha G$ to consist of all linear combinations*

$$\sum_{g\in G} a_g\#g,$$

where $a_g \in D_g$ and $a_g = 0$ except for finitely many g's, and the $\#g$ are used as place markers. Therefore

$$A \rtimes_\alpha G = \sum_{g\in G} D_g\#g$$

is a K-module with the product defined as

$$(a_g\#g)(b_h\#h) = \alpha_g(\alpha_{g^{-1}}(a_g)b_h)\#gh.$$

Note that $\alpha_{g^{-1}}(a_g) \in D_{g^{-1}}$, $b_h \in D_h$ and therefore

$$\alpha_g(\alpha_{g^{-1}}(a_g)b_h) \in \alpha_g(D_{g^{-1}}D_h) \subset \alpha_g(D_{g^{-1}} \cap D_h) \subset D_g \cap D_{gh} \subset D_{gh}.$$

Thus the product in Definition 2.38 is well-defined.

Remark 2.39. Notice that $1_A\#1_G$ is the unit element of $A \rtimes_\alpha G$. Indeed,

$$(1_A\#1_G)(b_h\#h) = \alpha_{1_G}\left(\alpha_{1_G^{-1}}(1_A)\, b_h\right)\#1_G h = b_h\#h,$$

$$(a_g\#g)(1_A\#1_G) = \alpha_g\left(\alpha_{g^{-1}}(a_g)\, 1_A\right)\#g1_G = a_g\#g.$$

Example 2.40. *Let A be the commutative algebra $A = k[x,y] / \langle x^2, y^2\rangle$, $G = \langle g : g^2 = 1\rangle$ the cyclic group of order 2 and $I = Ay$ the ideal generated by y (generated by y and xy as vector space). Consider the partial action α of G on A given by $D_g = I$, $\alpha_g(y) = xy$, $\alpha_g(xy) = y$. Then the partial smash product $A \rtimes_\alpha G$ is not associative. In particular for $u = x\#1 + xy\#g$ we have that: $u \cdot u = y\#g$ then $(u \cdot u) \cdot u = 0$ and $u \cdot (u \cdot u) = xy\#g$.*

In what follows we assume that each partial action is unital. In this case the partial smash product is automatically associative and the formula of the product in $A \rtimes_\alpha G$ is simplified as we will see in the next lemma.

Lemma 2.41. *Let* $\left(A, \{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}\right)$ *be a unital partial action, i.e. each domain* D_g *is an ideal of the form* Au_g*, where* u_g *is a central idempotent of* A *for each* $g \in G$*. Then, for* $g, h \in G$*,*

(I) If $Au_g = Ae_g$ *where* e_g *is a central idempotent, then* $u_g = e_g$ *and we have that* $\alpha_g(u_{g^{-1}}) = u_g$*,* $\forall g \in G$*.*

(II) $(au_g\#g)(bu_h\#h) = a\alpha_g(bu_hu_{g^{-1}})u_{gh}\#gh$*, for any* $a, b \in A$*. In particular if* $a = b = 1_A$ *then* $(u_g\#g)(u_h\#h) = u_gu_{gh}\#gh$*.*

(III) The map $\pi_0 : G \to A \rtimes_\alpha G$ *such that* $\pi_0(g) = u_g\#g$ $\forall g \in G$ *is a partial representation of* G *in* $A \rtimes_\alpha G$*.*

(IV) $\alpha_g(u_{g^{-1}}u_h) = u_gu_{gh}$ *and the map* $\alpha_g\big|_{D_{g^{-1}}\cap D_h} : D_{g^{-1}} \cap D_h \to D_g \cap D_{gh}$ *is an isomorphism.*

(V) The smash product $A \rtimes_\alpha G$ *is associative.*

Proof.

(I) If $Au_g = Ae_g$ we have that $u_g = ae_g$ and $e_g = bu_g$, thus

$$u_g = ae_g = ae_ge_g = u_ge_g = e_gu_g = bu_gu_g = bu_g = e_g.$$

(II) Note that by (2) of Definition 2.28 we have $\alpha_g(bu_hu_{g^{-1}}) \in D_g \cap D_{gh}$, thus:

$$\begin{aligned}(au_g\#g)(bu_h\#h) &= \alpha_g\left(\alpha_{g^{-1}}(au_g)\,bu_h\right)\#gh\\ &= \alpha_g\left(\alpha_{g^{-1}}(au_g)\,bu_hu_{g^{-1}}\right)\#gh\\ &= au_g\alpha_g(bu_hu_{g^{-1}})\#gh\\ &= a\alpha_g(bu_hu_{g^{-1}})u_{gh}\#gh.\end{aligned}$$

(III) As $u_{1_G}\#1_G$ is the unit element of $A \rtimes_\alpha G$ then π_0 satisfies (c) of Definition 2.1. Now notice that

$$\begin{aligned}\pi_0(s^{-1})\pi_0(s)\pi_0(t) &= (u_{s^{-1}}\#s^{-1})(u_s\#s)(u_t\#t)\\ &= (u_{s^{-1}}\#1_G)(u_t\#t)\\ &= u_{s^{-1}}u_t\#t\\ &= (u_{s^{-1}}\#s^{-1})(u_{st}\#st)\\ &= \pi_0(s^{-1})\pi_0(st),\end{aligned}$$

which proves part (b) of Definition 2.1, and the item (a) can be proved in an analogous way.

(IV) As the domains D_g are unital ideals for each $g \in G$ then

$$D_{g^{-1}} \cap D_h = Au_{g^{-1}}u_h \text{ and } D_g \cap D_{gh} = Au_g u_{gh}.$$

It is clear that $\alpha_g(D_{g^{-1}} \cap D_h) \subseteq A\alpha_g(u_{g^{-1}}u_h)$, moreover given $a \in A$ we have

$$\alpha_g(\alpha_{g^{-1}}(au_g)u_{g^{-1}}u_h) = \alpha_g(\alpha_{g^{-1}}(au_g))\alpha_g(u_{g^{-1}}u_h) = a\alpha_g(u_{g^{-1}}u_h),$$

hence $A\alpha_g(u_{g^{-1}}u_h) = \alpha_g(D_{g^{-1}} \cap D_h)$. Now by (2) of Definition 2.28

$$\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh},$$

thus

$$D_g \cap D_{gh} = A\alpha_g(u_{g^{-1}}u_h).$$

Then by (I) $\alpha_g(u_{g^{-1}}u_h) = u_g u_{gh}$.

(V) Using (II) we have that

$$\begin{aligned}
&(au_g\#g)((bu_h\#h)(cu_w\#w)) \\
&\quad = (au_g\#g)(b\alpha_h(cu_w u_{h^{-1}})u_{hw}\#hw) \\
&\quad = a\alpha_g(b\alpha_h(cu_w u_{h^{-1}})u_{hw}u_{g^{-1}})u_{ghw}\#ghw \\
&\quad = a\alpha_g(bu_h\alpha_h(cu_w u_{h^{-1}})u_{hw}u_{g^{-1}}u_{g^{-1}})u_{ghw}\#ghw \\
&\quad = a\alpha_g(bu_h u_{g^{-1}})\alpha_g(\alpha_h(cu_w u_{h^{-1}})u_{hw}u_{g^{-1}})u_{ghw}\#ghw \\
&\quad = a\alpha_g(bu_h u_{g^{-1}})\alpha_g(\alpha_h(cu_w u_{h^{-1}})u_{g^{-1}})\alpha_g(u_{hw}u_{g^{-1}})u_{ghw}\#ghw \\
&\quad = a\alpha_g(bu_h u_{g^{-1}})\alpha_g(\alpha_h(cu_w u_{h^{-1}})u_h u_{g^{-1}})u_{ghw}u_g u_{ghw}\#ghw \text{ by (IV)} \\
&\quad = a\alpha_g(bu_h u_{g^{-1}})\alpha_g(\alpha_h(cu_w u_{h^{-1}})\alpha_h(u_{h^{-1}g^{-1}}u_{h^{-1}}))u_{ghw}\#ghw \\
&\quad = a\alpha_g(bu_h u_{g^{-1}})\alpha_g\alpha_h(cu_w u_{h^{-1}g^{-1}}u_{h^{-1}})u_{ghw}\#ghw \\
&\quad = a\alpha_g(bu_h u_{g^{-1}})\alpha_{gh}(cu_w u_{h^{-1}g^{-1}}u_{h^{-1}})u_{ghw}\#ghw \\
&\quad = a\alpha_g(bu_h u_{g^{-1}})\alpha_{gh}(cu_w u_{h^{-1}g^{-1}})\alpha_{gh}(u_{h^{-1}g^{-1}}u_{h^{-1}})u_{ghw}\#ghw \\
&\quad = a\alpha_g(bu_h u_{g^{-1}})\alpha_{gh}(cu_w u_{h^{-1}g^{-1}})u_g u_{gh}u_{ghw}\#ghw \\
&\quad = a\alpha_g(bu_h u_{g^{-1}})u_{gh}\alpha_{gh}(cu_w u_{h^{-1}g^{-1}})u_{ghw}\#ghw \\
&\quad = (a\alpha_g\left(bu_h u_{g^{-1}}\right)u_{gh}\#gh)(cu_w\#w) \\
&\quad = ((au_g\#g)(bu_h\#h))(cu_w\#w).
\end{aligned}$$

□

Remark 2.42. The universal property of $K_{par}(G)$ and the map π_0 given in (III) of Lemma 2.41 endow $A \rtimes_\alpha G$ with a structure of a $K_{par}(G)$-bimodule such that

$$[g] \cdot au_h\#h = (u_g\#g)(au_h\#h) = \alpha_g(au_h u_{g^{-1}})u_{gh}\#gh,$$

and

$$au_h\#h \cdot [g] = (au_h\#h)(u_g\#g) = a\alpha_h(u_g u_{h^{-1}})u_{hg}\#hg = au_h u_{hg}\#hg.$$

Since $A \rtimes_\alpha G$ is associative we have that $([g] \cdot au_h\#h) \cdot [s] = [g] \cdot (au_h\#h \cdot [s])$.

Definition 2.43. *Let A be an algebra on which the group G acts partially. Define the canonical inclusion $\phi_0 : A \to A \rtimes_\alpha G$ by $\phi_0(a) = au_{1_G}\#1_G = a\#1_G$.*

Remark 2.44. Notice that $A = Au_{1_G}$, then $u_{1_G} = 1_A$ the unity of A. It is easy to see that ϕ_0 is a monomorphism of algebras. Indeed,

$$\phi_0(a)\phi_0(b) = (a\#1_G)(b\#1_G) = ab\#1_G = \phi_0(ab),$$

and $\phi_0(a) = au_{1_G}\#1_G = 0\#1_G$ if, and only if, $au_{1_G} = 0$ but $u_{1_G} = 1_A$ then $a = 0$.

Definition 2.45. *Given a K-vector space V and a partial action α of a group G on an algebra A, a pair of maps (ϕ_V, π_V) is said to be a **covariant pair** if $\phi_V : A \to End_K(V)$ is a representation and $\pi_V : G \to End_K(V)$ is a partial representation such that:*

$$\phi_V\left(\alpha_g(au_{g^{-1}})\right) = \pi_V(g)\phi_V(a)\pi_V(g^{-1}).$$

Definition 2.46. *Given two covariant pairs (ϕ_V, π_V) and (ϕ_W, π_W), a **morphism between covariant pairs** $f : (\phi_V, \pi_V) \to (\phi_W, \pi_W)$ is a linear map $f : V \to W$ such that $f \circ \pi_V(g) = \pi_W(g) \circ f$ and $f \circ \phi_V(a) = \phi_W(a) \circ f$ $\forall g \in G$ and $\forall a \in A$.*

We denote by $\mathrm{CovPair}(A, G)$ *the category of covariant pairs (ϕ_V, π_V).*

Remark 2.47. Observe that if (ϕ_V, π_V) a covariant pair then

$$\phi_V(u_g) = \phi_V(\alpha_g(1_A u_{g^{-1}})) = \pi_V(g)\phi_V(1_A)\pi_V(g^{-1}) = \pi_V(g)\pi_V(g^{-1}).$$

The **partial smash product universal property** is given by the following result.

Theorem 2.48. *Let α be an unital partial action of a group G on an algebra A, V a K-vector space and (ϕ_V, π_V) a covariant pair related to these data. Then there exists a unique algebra morphism $\phi : A \rtimes_\alpha G \to End_K(V)$ such that the diagram:*

$$\begin{tikzcd}
 & A \rtimes_\alpha G \arrow[dd, "\phi"] & \\
A \arrow[ur, "\phi_0"] \arrow[dr, "\phi_V"'] & & G \arrow[ul, "\pi_0"'] \arrow[dl, "\pi_V"] \\
 & End_K(V) &
\end{tikzcd}$$

is commutative.

Proof. Define $\phi : A \rtimes_\alpha G \to End_K(V)$ by $\phi(au_g\#g) = \phi_V(a)\pi_V(g)$. Notice that ϕ is well-defined. Indeed, for $a, b \in A$ and $g \in G$ such that $au_g = bu_g$, by Remark 2.47 we have that

$$\begin{aligned}
\phi\left(au_g\#g\right) &= \phi_V(a)\pi_V(g) \\
&= \phi_V(a)\pi_V(g)\pi_V(g^{-1})\pi_V(g) \\
&= \phi_V(a)\phi_V(u_g)\pi_V(g) \\
&= \phi_V(au_g)\pi_V(g) \\
&= \phi_V(bu_g)\pi_V(g) \\
&= \phi_V(b)\phi_V(u_g)\pi_V(g) \\
&= \phi_V(b)\pi_V(g)\pi_V(g^{-1})\pi_V(g) \\
&= \phi_V(b)\pi_V(g) \\
&= \phi\left(bu_g\#g\right).
\end{aligned}$$

Now we will prove that ϕ is an algebra morphism. Let $a, b \in A$ and $g, h \in G$, then by Remark 2.47

$$\begin{aligned}
\phi\left((au_g\#g)(bu_h\#h)\right) &= \phi\left(a\alpha_g\left(bu_hu_{g^{-1}}\right)u_{gh}\#gh\right) \\
&= \phi_V\left(a\alpha_g\left(bu_hu_{g^{-1}}\right)u_{gh}\right)\pi_V(gh) \\
&= \phi_V(a)\phi_V\left(\alpha_g\left(bu_hu_{g^{-1}}\right)\alpha_g(u_hu_{g^{-1}})\right)\pi_V(gh) \\
&= \phi_V(a)\phi_V\left(\alpha_g\left(bu_hu_{g^{-1}}\right)\right)\pi_V(gh) \\
&= \phi_V(a)\pi_V(g)\phi_V\left(bu_h\right)\pi_V(g^{-1})\pi_V(gh) \\
&= \phi_V(a)\pi_V(g)\phi_V\left(bu_h\right)\pi_V(g^{-1})\pi_V(g)\pi_V(h) \\
&= \phi_V(a)\pi_V(g)\phi_V\left(bu_h\right)\phi_V(u_{g^{-1}})\pi_V(h) \\
&= \phi_V(a)\pi_V(g)\phi_V\left(bu_hu_{g^{-1}}\right)\pi_V(h) \\
&= \phi_V(a)\pi_V(g)\phi_V\left(u_{g^{-1}}bu_h\right)\pi_V(h) \\
&= \phi_V(a)\pi_V(g)\phi_V(u_{g^{-1}})\phi_V\left(bu_h\right)\pi_V(h) \\
&= \phi_V(a)\pi_V(g)\phi(bu_h\#h) \\
&= \phi(au_g\#g)\phi(bu_h\#h).
\end{aligned}$$

It only remains to show that the above diagram commutes. For each $g \in G$ we have

$$\begin{aligned}
\phi\pi_0(g) &= \phi\left(u_g\#g\right) \\
&= \phi_V(u_g)\pi_V(g) \\
&= \pi_V(g)\pi_V(g^{-1})\pi_V(g) \\
&= \pi_V(g),
\end{aligned}$$

and for $a \in A$,

$$\begin{aligned}\phi\phi_0(a) &= \phi\left(au_{1_G}\#1_G\right)\\ &= \phi_V(a)\pi_V(1_G)\\ &= \phi_V(a).\end{aligned}$$

Then the diagram commutes. Finally to prove that ϕ is unique, notice that if there exists another ϕ' with the same proprieties, we have that for any $g \in G$ and $a \in A$,

$$\phi(u_g\#g) = \phi\pi_0(g) = \phi'\pi_0(g) = \phi'(u_g\#g)$$

$$\phi(a\#1_G) = \phi\phi_0(a) = \phi'\phi_0(a) = \phi'(a\#1_G)$$

thus $\phi(a\#1_G)\phi(u_g\#g) = \phi'(a\#1_G)\phi'(u_g\#g)$, and then $\phi(au_g\#g) = \phi'(au_g\#g)$.

□

We can generalize the last theorem. Let A be and algebra on which the group G acts partially and S be a monoid, if we have $\phi_S : A \to S$ a representation and $\pi_S : G \to S$ a partial representation such that $\phi_S\left(\alpha_g(au_{g^{-1}})\right) = \pi_S(g)\phi_S(a)\pi_V(g^{-1})$. Then there exist a unique algebra morphism $\phi : A \rtimes_\alpha G \to S$ such that the diagram:

$$\begin{tikzcd} & A \rtimes_\alpha G \arrow[dd, "\phi"] & \\ A \arrow[ur, "\phi_0"] \arrow[dr, "\phi_S"'] & & G \arrow[ul, "\pi_0"'] \arrow[dl, "\pi_S"] \\ & S & \end{tikzcd}$$

is commutative. The proof is analogous to that of the previous theorem.

Proposition 2.49. *Let A be an algebra on which the group G acts partially. Then* $\operatorname{Rep} A \rtimes_\alpha G$ *is isomorphic to* $\operatorname{CovPair}(A, G)$*, the category of covariant pairs.*

Proof. Define the functor $F : \operatorname{CovPair}(A, G) \to \operatorname{Rep} A \rtimes_\alpha G$ as follows: for a covariant pair (ϕ_V, π_V), $F(\phi_V, \pi_V) = \Phi^V$ where Φ^V is the representation obtained using Theorem 2.48. If we have a morphism between covariant pairs $f : (\phi_V, \pi_V) \to (\phi_W, \pi_W)$, where V and W are K-vector spaces, then f defines a morphism between the representations Φ^V and Φ^W obtained using Theorem 2.48 for (ϕ_V, π_V) and (ϕ_W, π_W)

respectively. Indeed, as f is a morphism between covariant pairs we have that, for any $g \in G$ and $a \in A$:

$$f \circ \pi_V(g) = \pi_W(g) \circ f \text{ and } f \circ \phi_V(a) = \phi_W(a) \circ f,$$

then by Theorem 2.48

$$f \circ \Phi^V(\pi_0(g)) = \Phi^W(\pi_0(g)) \circ f \text{ and } f \circ \Phi^V(\phi_0(a)) = \Phi^W(\phi_0(a)) \circ f,$$

thus evaluating $\pi_0(g) = u_g\#g$ and $\phi_0(a) = au_{1_G}\#1_G$

$$f \circ \Phi^V(u_g\#g) = \Phi^W(u_g\#g) \circ f$$

and

$$f \circ \Phi^V(au_{1_G}\#1_G) = \Phi^W(au_{1_G}\#1_G) \circ f,$$

then we have that:

$$\begin{aligned} f \circ \Phi^V(au_g\#g) &= f \circ \Phi^V((u_g\#g)(au_{1_G}\#1_G)) \\ &= f \circ \Phi^V(u_g\#g) \circ \Phi^V(au_{1_G}\#1_G) \\ &= \Phi^W(u_g\#g) \circ f \circ \Phi^V(au_{1_G}\#1_G) \\ &= \Phi^W(u_g\#g) \circ \Phi^W(au_{1_G}\#1_G) \circ f \\ &= \Phi^W(au_g\#g) \circ f. \end{aligned}$$

Thus f is a morphism between the representations Φ^V and Φ^W, then set $F(f) = f$.

Now define the functor $G : \mathrm{Rep}\, A \rtimes_\alpha G \to \mathrm{CovPair}(A, G)$ as follows: for a representation $\Phi : \mathrm{Rep}\, A \rtimes_\alpha G \to End_K(V)$ in $\mathrm{Rep}\, A \rtimes_\alpha G$, define $\phi_{V_\Phi} = \Phi \circ \phi_0$ and $\pi_{V_\Phi} = \Phi \circ \pi_0$, $\phi_{V_\Phi}, \pi_{V_\Phi}$ is a covariant pair, then make $F(\Phi) = (\phi_{V_\Phi}, \pi_{V_\Phi})$. Indeed,

$$\begin{aligned} \phi_{V_\Phi}(\alpha_g(au_g^{-1})) &= \Phi \circ \phi_0\left(\alpha_g(au_g^{-1})\right) \\ &= \Phi\left(\alpha_g(au_g^{-1})u_{1_G}\#1_G\right) \\ &= \Phi\left((u_g\#g)(au_{1_G}u_{g^{-1}}\#g^{-1})\right) \\ &= \Phi(u_g\#g)\,\Phi\left((au_{1_G}\#1_G)(u_{g^{-1}}\#g^{-1})\right) \\ &= \Phi(u_g\#g)\Phi(au_{1_G}\#1_G)\Phi(u_{g^{-1}}\#g^{-1}) \\ &= \left(\Phi \circ \pi_0(g)\right)\left(\Phi \circ \phi_0(a)\right)\left(\Phi \circ \pi_0(g^{-1})\right) \\ &= \pi_{V_\Phi}(g)\phi_{V_\Phi}(a)\pi_{V_\Phi}(g^{-1}). \end{aligned}$$

Let $\Phi : A \rtimes_\alpha G \to V$ and $\Phi' : A \rtimes_\alpha G \to W$ be representations of $A \rtimes_\alpha G$, and a morphism between Φ and Φ' defined by $f : V \to W$. Notice that f defines a morphism of covariant pairs between $(\Phi \circ \phi_0, \Phi \circ \pi_0)$ and $(\Phi' \circ \phi_0, \Phi' \circ \pi_0)$. Indeed, as f defines a morphism between Φ and Φ', we have that $f \circ \Phi(au_g\#g) = \Phi'(au_g\#g) \circ f$ for all $a \in A$ and $g \in G$. Then in particular we have

$$f \circ \Phi(u_g\#g) = \Phi'(u_g\#g) \circ f \Rightarrow f \circ \big(\Phi \circ \pi_0(g)\big) = \big(\Phi' \circ \pi_0(g)\big) \circ f$$

and

$$f \circ \Phi(au_{1_G}\#1_G) = \Phi'(au_{1_G}\#1_G) \circ f \Rightarrow f \circ \big(\Phi \circ \phi_0(a)\big) = \big(\Phi' \circ \phi_0(a)\big) \circ f.$$

Then f defines a morphism between covariant pairs, thus define $G(f) = f$. Now it is easy to see that $FG = I_{\mathrm{Rep}\, A \rtimes_\alpha G}$ and $GF = I_{\mathrm{CovPar}(A,G)}$, where $I_{\mathrm{Rep}\, A \rtimes_\alpha G}$ is the identity functor of $\mathrm{Rep}\, A \rtimes_\alpha G$ and $I_{\mathrm{CovPar}(A,G)}$ is the identity functor of $\mathrm{CovPar}(A, G)$.

□

Notice that the algebra $K_{par}(G)$ has a natural G-grading (see Remark 2.51). This will lead us to show that for any group G the partial group algebra $K_{par}(G)$ is isomorphic to a partial smash product, a fact established in [6, Theorem 6.9]. First let us recall what is a G-graded algebra.

Definition 2.50. *A **G-graded algebra** is an algebra with a decomposition*

$$A = \bigoplus_{g \in G} A_g$$

where each A_g is a subspace of A such that $A_h A_g \subseteq A_{hg}$, for all $g, h \in G$.

Remark 2.51. $K_{par}(G)$ has a natural G-grading:

$$K_{par}(G) = \bigoplus_{g \in G} B_g,$$

where each subspace B_g is generated by elements of the form $[h_1][h_2]...[h_n]$ such that $g = h_1 h_2 ... h_n$, that is:

$$B_g := \langle [h_1][h_2]...[h_n] \mid g = h_1 h_2 ... h_n \rangle .$$

Then for all $x \in B_g$ and $y \in B_h$, $xy \in B_{gh}$, and thus $B_g B_h \subseteq B_{gh}$.

In order to prove that the partial group algebra $K_{par}(G)$ is isomorphic to a partial smash product for any group G we are going to recall that for each $g \in G$ we denote $e_g = [g][g^{-1}] \in K_{par}(G)$. Now define the subalgebra

$$B := \langle e_g \mid g \in G \rangle \subseteq K_{par}(G).$$

Remark 2.52.

- B corresponds to the uniform subalgebra B_{1_G} coming from the natural grading of $K_{par}(G)$. Indeed, if $s = [h_1]...[h_n] \in \mathcal{S}(G)$ is such that $h_1, ..., h_n = 1_G$, then by Proposition 2.20 $s = e_{r_1}...e_{r_m}[t]$, thus using the map η defined in Remark 2.21 we have $\eta(s) = t = 1_G$. Therefore $s = e_{r_1}...e_{r_m}$.
- B is a commutative algebra generated by idempotents.

Theorem 2.53. *Given a group G, there is a partial action β of G on the above defined commutative algebra B, such that $K_{par}(G) \cong B \rtimes_\beta G$.*

Proof. We have to define a partial action β of G on B. So define the domains $D_g = e_g B$ and the morphism $\beta_g : D_{g^{-1}} \to D_g$ by:

$$\begin{aligned}\beta_g(e_{g^{-1}}e_{h_1}e_{h_2}...e_{h_n}) &= [g]e_{g^{-1}}e_{h_1}e_{h_2}...e_{h_n}[g^{-1}]\\ &= 1e_{gh_1}e_{gh_2}...e_{gh_n}[g][g^{-1}]\\ &= e_g e_{gh_1}e_{gh_2}...e_{gh_n}.\end{aligned}$$

Then β is a partial action. Indeed, it is clear that $D_{1_G} = B$ and $\beta_{1_G} = id_B$, thus β satisfies the condition (I) of Definition 2.28. Recall that since β_g and β_h are partial functions the domain of $\beta_g\beta_h$ is the set $\beta_{h^{-1}}(D_{g^{-1}} \cap D_h)$. Notice that for any $g \in G$, the ideal $D_g = e_g B$ is unital with unit e_g, so:

$$\begin{aligned}\beta_{h^{-1}}\left(D_{g^{-1}} \cap D_h\right) &= \beta_{h^{-1}}\big(e_{g^{-1}}e_h B\big)\\ &= \beta_{h^{-1}}\big(e_{g^{-1}}e_h\big)\beta_{h^{-1}}\big(e_h B\big)\\ &= e_{(gh)^{-1}}\beta_{h^{-1}}\big(e_h B\big) = e_{(gh)^{-1}}\beta_{h^{-1}}\big(D_h\big)\\ &= e_{(gh)^{-1}}D_{h^{-1}} = D_{(gh)^{-1}} \cap D_{h^{-1}} \subseteq D_{(gh)^{-1}} = \operatorname{dom}\left(\beta_{gh}\right).\end{aligned}$$

Then $\operatorname{dom}(\beta_g\beta_h) \subseteq \operatorname{dom}(\beta_{gh})$, thus:

$$\begin{aligned}\beta_g\beta_h\left(e_{(gh)^{-1}}e_{h^{-1}}e_{w_1}e_{w_2}...e_{w_n}\right) &= \beta_g\left(e_h e_{g^{-1}}e_{hw_1}e_{hw_2}...e_{hw_n}\right)\\ &= e_g e_{gh} e_{ghw_1}e_{ghw_2}...e_{ghw_n}\\ &= e_{gh}e_{(gh)h^{-1}}e_{ghw_1}e_{ghw_2}...e_{ghw_n}\\ &= \beta_{gh}\left(e_{(gh)^{-1}}e_{h^{-1}}e_{w_1}e_{w_2}...e_{w_n}\right).\end{aligned}$$

Then $\beta_g\beta_h \subseteq \beta_{gh}$, and by Remark 2.29 β is a partial action of G on B.

Notice that the map $\pi_0 : G \to B \rtimes_\beta G$ given by $\pi_0(g) = e_g \# g$ is a partial representation of G in $B \rtimes_\beta G$. Indeed, $\pi_0(e) = [1_G]\#1_G$, now observe that for all

$g, h \in G$:

$$\begin{aligned}
\pi_0(g^{-1})\pi_0(g)\pi_0(h) &= (e_{g^{-1}}\#g^{-1})(e_g\#g)(e_h\#h) \\
&= (\beta_{g^{-1}}(e_g)\#1_G)(e_h\#h) = (e_g^{-1}\#1_G)(e_h\#h) \\
&= e_{g^{-1}}e_h\#h \\
&= \beta_{g^{-1}}(e_g)e_{gh})e_h\#h) \\
&= (e_{g^{-1}}\#g^{-1})(e_{gh}\#gh) \\
&= \pi_0(g^{-1})\pi_0(gh).
\end{aligned}$$

Thus by the universal property of $K_{par}(G)$ there exists an unique algebra morphism

$$\hat{\pi} : K_{par}(G) \to B \rtimes_\beta G,$$

such that $\hat{\pi}\big([g]\big) = \pi_0(g) = e_g\#g$.

Observe that $(e_g\#g)(e_h\#h) = \beta_g(e_{g^{-1}}e_h)\#gh = e_g e_{gh}\#gh$. Then

$$\hat{\pi}\big([g_1][g_2]...[g_n]\big) = e_{g_1}e_{g_1g_2}...e_{g_1g_2...g_n}\#g_1g_2...g_n.$$

The canonical inclusion of B into $K_{par}(G)$, $\phi_B(a) = a[1_G]$, and the canonical partial representation $\pi_B(g) = [g]$ form a covariant pair relative to the algebra $K_{par}(G)$. Indeed,

$$\begin{aligned}
\phi_B\left(\beta_g\left(ae_{g^{-1}}\right)\right) &= \phi_B\left(a_g e_g\right) = a_g e_g[1_G] \\
&= a_g[g][g^{-1}][1_G] = [g]a[1_G][g^{-1}] \\
&= \pi_B(g)\phi(a)\pi_B(g^{-1}),
\end{aligned}$$

where $a = e_{h_1}e_{h_2}...e_{h_n}$ and $a_g = e_{gh_1}e_{gh_2}...e_{gh_n}$.

By Theorem 2.48 there is an unique algebra morphism $\varphi : B \rtimes_\beta G \to K_{par}(G)$ such that $\pi_B = \varphi \circ \pi_0$ and $\phi_B = \varphi \circ \phi_0$. Then for $g \in G$ and $a \in B$, $\pi_B(g) = \varphi \circ \pi_0(g)$ and $\phi_B(a) = \varphi \circ \phi_0(a)$, that means $[g] = \varphi(e_g\#g)$ and $a[1_G] = \varphi(a[1_G]\#1_G)$, thus $\varphi(ae_g\#g) = a[g]$.

Finally observe that $\hat{\pi}$ and φ are mutually inverse. Indeed,

$$\begin{aligned}
\varphi \circ \hat{\pi}\big([g_1][g_2]...[g_n]\big) &= \varphi\big((e_{g_1}\#g_1)(e_{g_2}\#g_2)...(e_{g_n}\#g_n)\big) \\
&= [g_1][g_2]...[g_n],
\end{aligned}$$

and for $a = e_{h_1}e_{h_2}...e_{h_n} \in B$

$$\begin{aligned}\hat{\pi} \circ \varphi(ae_g\#g) &= \hat{\pi}\big([g][g^{-1}][h_1][h_1^{-1}][h_2][h_2^{-1}]...[h_n][h_n^{-1}][g]\big)\\ &= (e_g\#g)(e_{g^{-1}}\#g^{-1})(e_{h_1}\#h_1)(e_{h_1^{-1}}\#h_1^{-1})...(e_{h_n}\#h_n)(e_{h_n^{-1}}\#h_n^{-1})(e_g\#g)\\ &= (e_g\#1_G)(e_{h_1}\#1_G)...(e_{h_n}\#1_G)(e_g\#g) = (e_{h_1}e_{h_2}...e_{h_n}\#1_G)(e_g\#g)\\ &= (e_{h_1}e_{h_2}...e_{h_n}e_g\#g)\\ &= ae_g\#g.\end{aligned}$$

Then $\hat{\pi}$ and φ are inverse to each other, thus $K_{par}(G) \cong B \rtimes_\beta G$. □

Theorem 2.54. *Let B be the K-algebra defined above. Then there is a partial representation $\pi : G \to End_K(B)$ defined by $\pi(g)(b) = [g]b[g^{-1}]$ for any $g \in G$ and $b \in B$.*

Proof. First observe that $\pi(1_G)(b) = [1_G]b[1_G] = x$ for all $b \in B$, then $\pi(e) = id_B$, thus π satisfies the first condition of Definition 2.1, for the other two conditions notice that for all $s, t \in G$:

$$\begin{aligned}\pi(s)\pi(t)\pi(t^{-1})(b) &= [s][t][t^{-1}]b[t][t^{-1}][s^{-1}]\\ &= [st][t^{-1}]b[t][(st)^{-1}]\\ &= \pi(st)\big([t^{-1}]b[t]\big)\\ &= \pi(st)\pi(t^{-1})(b),\end{aligned}$$

and analogously $\pi(s^{-1})\pi(s)\pi(t) = \pi(s^{-1})\pi(st)$. □

Corollary 2.55. *B has a structure of a left $K_{par}(G)$-module induced by*

$$\phi_B : K_{par}(G) \to End_K(B),$$

such that $\phi_B\big([g]\big)(x) = [g]x[g^{-1}]$.

Proof. Observe that ϕ_B is the algebra morphism obtained from applying the universal property of $K_{par}(G)$ (Proposition 2.48) to the partial representation $\pi : G \to End_K(B)$ defined in Theorem 2.54. □

2.3 Spectral sequences

In this section we will introduce some definitions and results from spectral sequence theory. The final objective of this sections is to prove Theorem 2.106, which will be used to obtain Theorem 4.4, which shows that there exists a Grothendieck spectral sequence relating cohomology of partial smash products with partial group cohomology and algebra cohomology. The theory in this section is taken from [3], some of the proofs have been given differently.

Definition 2.56. *Let R be any ring. A **complex** $(\mathbf{C}, d)$ for R is an indexed family $\mathbf{C} = \{C_i\}_{i\in\mathbb{Z}}$ of R-modules together with an indexed family of module morphisms $d = \{d_i : C_i \to C_{i-1}\}_{i\in\mathbb{Z}}$ such that $d_{i-1}d_i = 0$.*

$$\cdots \longrightarrow C_{p+2} \xrightarrow{d_{p+2}} C_{p+1} \xrightarrow{d_{p+1}} C_p \xrightarrow{d_p} C_{p-1} \xrightarrow{d_{p-1}} C_{p-2} \longrightarrow \cdots .$$

Given two complexes $(\mathbf{C}, d)$ and $(\mathbf{C}', d')$ a (chain) homomorphism of $\mathbf{C}$ into $\mathbf{C}'$ is an indexed family of module morphisms $\alpha = \{\alpha_i : C_i \to C'_i\}_{i\in\mathbb{Z}}$ such that the next diagram commutes

$$\begin{array}{ccc} C_i & \xrightarrow{d_i} & C_{i-1} \\ {\scriptstyle\alpha_i}\downarrow & & \downarrow{\scriptstyle\alpha_{i-1}} \\ C'_i & \xrightarrow[d'_i]{} & C'_{i-1} \end{array}$$

for any $i \in \mathbb{Z}$.

Definition 2.57. *A complex $(\mathbf{K}, l)$ is a **subcomplex** of a complex $(\mathbf{C}, d)$ if for all $n \in \mathbb{Z}$ we have that:*

(i) $K_n \subseteq C_n$;

(ii) The map l_n is the restriction of d_n to K_n, i.e. $l_n = d_n|_{K_n}$.

Definition 2.58. *Given a subcomplex $(\mathbf{K}, l)$ of a complex $(\mathbf{C}, d)$, the factor complex $\mathbf{C}/\mathbf{K}$ is the family of factor modules $\{C_i/K_i\}_{i\in\mathbb{Z}}$ with the family of boundary maps $\{\overline{d}_i : C_i/K_i \to C_{i-1}/K_{i-1}\}$ induced by d.*

Definition 2.59. *A **graded module** is an indexed family*

$$M = (M_{p\in\mathbb{Z}})$$

of R-modules. Graded modules M are often denoted by $M_\bullet$.

Definition 2.60. *A* ***bigraded module*** *is a doubly indexed family*

$$M = (M_{(p,q)\in\mathbb{Z}\times\mathbb{Z}})$$

of R-modules. Bigraded modules M are often denoted by $M_{\bullet\bullet}$.

Definition 2.61. *Let M and N be bigraded modules, and let $(a, b) \in \mathbb{Z}\times\mathbb{Z}$. A bigraded map of bidegree (a, b), denoted by $f : M \to N$, is a family of module homomorphisms*

$$f = (f_{p,q} : M_{p,q} \to N_{p+a,q+b})_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}\,.$$

The bidegree of f is (a, b), and we denote it by $\deg(f) = (a, b)$.

Given bigraded modules A, B, C and two bigraded maps $f : A \to B$ and $g : B \to C$, the **exactness** of $A \xrightarrow{f} B \xrightarrow{g} C$ means that $\operatorname{im} f = \ker g$; i.e. if $deg(f) = (a, b)$ then $\operatorname{im} f_{p-a,q-b} = \ker g_{p,q}$ for all $p, q \in \mathbb{Z}$.

Definition 2.62. *A* ***bicomplex*** *is an ordered triple (M, d', d''), where $M = (M_{p,q})$ is a bigraded module, d', $d'' : M \to M$ are differentials of bidegree $(-1, 0)$ and $(0, -1)$, respectively (so that d', d'' are morphisms of bigraded modules such that $d'd' = 0$ and $d''d'' = 0$), and*

$$d'_{p,q-1}d''_{p,q} + d''_{p-1,q}d'_{p,q} = 0.$$

Definition 2.63. *If M is a bicomplex, then its* ***total complex*** *, denoted by $\operatorname{Tot}(M)$, is the complex with nth term:*

$$\operatorname{Tot}(M)_n = \bigoplus_{p+q=n} M_{p,q},$$

and with differentials $D_n : \operatorname{Tot}(M)_n \to \operatorname{Tot}(M)_{n-1}$ given by

$$D_n = \sum_{p+q=n} (d'_{p,q} + d''_{p,q}).$$

We can see a bigraded module $M_{\bullet\bullet}$ as the integer pairs in the Cartesian plane where each module $M_{(p,q)} \in M_{\bullet\bullet}$ is represented by the point (p, q), in that sense $\operatorname{Tot}(M)_n$ is the set of the integer pairs in the line defined by $y = -x + n$.

Lemma 2.64. *If M is a bicomplex, then $(\operatorname{Tot}(M), D)$ is a complex.*

Proof. Observe that each direct summand of $\operatorname{Tot}(M)_n$ is a module $M_{p,q}$ such that $p + q = n$, and note that $\operatorname{im} d'_{p,q} \subseteq M_{p-1,q}$ and $\operatorname{im} d''_{p,q} \subseteq M_{p,q-1}$. In both cases $M_{p-1,q}$

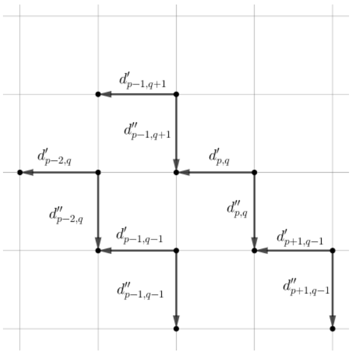


Figure 2.1: Bicomplex

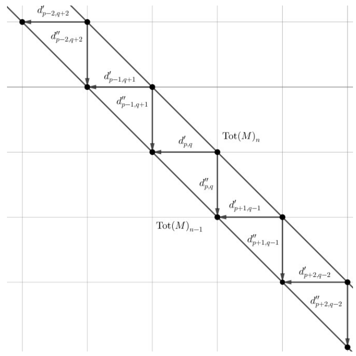


Figure 2.2: Total complex

and $M_{p,q-1}$ are summands of $\mathrm{Tot}(M)_{n-1}$ and therefore $\operatorname{im} D_n \subseteq \mathrm{Tot}(M)_{n-1}$. We show that D is a differential.

$$\begin{aligned} DD &= \sum (d' + d'')(d' + d'') \\ &= \sum d'd' + \sum (d'd'' + d''d') + \sum d''d'' = 0. \end{aligned}$$

□

Definition 2.65. *A **filtration** of a module M is a family $(M_p)_{p\in\mathbb{Z}}$ of submodules of M such that*

$$\cdots \subseteq M_{p-1} \subseteq M_p \subseteq M_{p+1} \subseteq \cdots .$$

*The **factor modules** of this filtration are the modules M_p/M_{p-1} with $p \in \mathbb{Z}$.*

A filtration of a graded module $M_\bullet$ is a family $(F^pM_\bullet)_{p\in\mathbb{Z}}$ of graded modules such that

$$\cdots \subseteq F^{p-1}M_n \subseteq F^pM_n \subseteq F^{p+1}M_n \subseteq \cdots ,$$

for all $n \in \mathbb{Z}$.

Definition 2.66. *A **filtration** of a complex $\mathbf{C}$ is a family of subcomplexes $(F^p\mathbf{C})_{p\in\mathbb{Z}}$ of $\mathbf{C}$ such that*

$$\cdots \subseteq F^{p-1}\mathbf{C} \subseteq F^p\mathbf{C} \subseteq F^{p+1}\mathbf{C} \subseteq \cdots ,$$

where $F^p\mathbf{C} \subseteq F^{p+1}\mathbf{C}$ means that $F^p\mathbf{C}$ is a subcomplex of $F^{p+1}\mathbf{C}$.

Limiting the first or the second index of the direct summands of $\mathrm{Tot}(M)$ we obtain the following filtrations.

Definition 2.67. *The **first filtration** of $\mathrm{Tot}(M)$ is given by*

$$\begin{aligned} \left({}^{\mathrm{I}}F^p\,\mathrm{Tot}(M)\right)_n &= \bigoplus_{i\leq p} M_{i,n-i} \\ &= \cdots \oplus M_{p-2,q+2} \oplus M_{p-1,q+1} \oplus M_{p,q}, \text{ where } q = n - p. \end{aligned}$$

Clearly, varying p for each fixed n, we have a filtration of $\mathrm{Tot}(M)_n$. We denote $\left({}^{\mathrm{I}}F^p \,\mathrm{Tot}(M)\right)$ by ${}^{\mathrm{I}}F^p$. Let us check that $({}^{\mathrm{I}}F^p)_{n\geq 0}$ is a subcomplex of $\mathrm{Tot}(M)$ for each fixed $p \in \mathbb{Z}$:

$$\begin{aligned} D_{i,n-i} M_{i,n-i} = \left(d'_{i,n-i} + d''_{i,n-i}\right) M_{i,n-i} &\subseteq d' M_{i,n-i} + d'' M_{i,n-i} \\ &\subseteq M_{i-1,n-i} \oplus M_{i,n-i-1} \\ &\subseteq \left({}^{\mathrm{I}}F^p \,\mathrm{Tot}(M)\right)_{n-1}. \end{aligned}$$

Definition 2.68. *The* ***second filtration*** *of* $\mathrm{Tot}(M)$ *is given by*

$$\begin{aligned} \left({}^{\mathrm{II}}F^p \,\mathrm{Tot}(M)\right)_n &= \bigoplus_{j\leq p} M_{n-j,j} \\ &= \cdots \oplus M_{q-1,p-2} \oplus M_{q+1,p-1} \oplus M_{q,p}, \ \textit{where } q = n-p. \end{aligned}$$

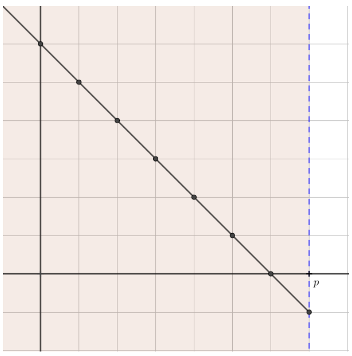


Figure 2.3: First filtration

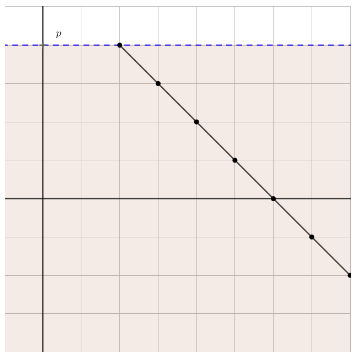


Figure 2.4: Second filtration

We denote $\left({}^{\mathrm{II}}F^p \,\mathrm{Tot}(M)\right)$ by ${}^{\mathrm{II}}F^p$. Analogously to $({}^{\mathrm{I}}F^p)_{n\geq 0}$ we have that $({}^{\mathrm{II}}F^p)_{n\geq 0}$ is a subcomplex of $\mathrm{Tot}(M)$.

Definition 2.69. *An* ***exact couple*** *is a 5-tuple* $(D, E, \alpha, \beta, \gamma)$*, where* D *and* E *are bigraded modules,* $\alpha : D \to D, \beta : D \to E$ *and* $\gamma : E \to D$ *are bigraded maps, and there is exactness at each vertex:* $\ker\alpha = \mathrm{im}\,\gamma$*,* $\ker\beta = \mathrm{im}\,\alpha$*, and* $\ker\gamma = \mathrm{im}\,\beta$*.*

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

Proposition 2.70. *Each filtration $(F^p\mathbf{C})_{p\in\mathbb{Z}}$ of a complex $\mathbf{C}$ determines an exact couple*

$$\begin{tikzcd} D \arrow[rr, "\alpha\,(1,-1)"] & & D \arrow[dl, "\beta\,(0,0)"] \\ & E \arrow[ul, "\gamma\,(-1,0)"] & \end{tikzcd}$$

whose bigraded maps have the displayed bidegrees.

Proof. We write for simplicity $F^p\mathbf{C}$ as F^p. For each fixed p, there is a short exact sequence of complexes,

$$0 \longrightarrow F^{p-1} \xrightarrow{j^{p-1}} F^p \xrightarrow{v^p} F^p/F^{p-1} \longrightarrow 0$$

(where j^{p-1} is the inclusion and v^p is the natural map) that gives rise to the long exact sequence of homology

$$\begin{aligned} \cdots \to H_n(F^{p-1}) &\xrightarrow{\alpha} H_n(F^p) \xrightarrow{\beta} H_n(F^p/F^{p-1}) \xrightarrow{\gamma} \\ & H_{n-1}(F^{p-1}) \xrightarrow{\alpha} H_{n-1}(F^p) \xrightarrow{\beta} H_{n-1}(F^p/F^{p-1}) \to \cdots, \end{aligned}$$

where $\alpha = j_*^{p-1}$, $\beta = v_*^p$, and $\gamma = \partial$ the connecting homomorphism (for more details see [3, p. 333]). We write $q = n - p$, then we have

$$\begin{aligned} \cdots \to H_{p+q}(F^{p-1}) &\xrightarrow{\alpha} H_{p+q}(F^p) \xrightarrow{\beta} H_{p+q}(F^p/F^{p-1}) \xrightarrow{\gamma} \\ & H_{p+q-1}(F^{p-1}) \xrightarrow{\alpha} H_{p+q-1}(F^p) \xrightarrow{\beta} H_{p+q-1}(F^p/F^{p-1}) \to \cdots. \end{aligned}$$

There are two types of homology groups: homology of a subcomplex F^p or F^{p-1} and homology of a quotient complex F^p/F^{p-1}. Define

$$D = (D_{p,q}), \text{ where } D_{p,q} = H_{p+q}(F^p),\ E = (E_{p,q}), \text{ where } E_{p,q} = H_{p+q}(F^p/F^{p-1}).$$

With this notation, the long exact sequence is, for fixed q,

$$\begin{aligned} \cdots \to D_{p-1,q+1} &\xrightarrow[(1,-1)]{\alpha} D_{p,q} \xrightarrow[(0,0)]{\beta} E_{p,q} \xrightarrow[(-1,0)]{\gamma} \\ & D_{p-1,q} \xrightarrow{\alpha} D_{p,q-1} \xrightarrow{\beta} E_{p,q-1} \to \cdots. \end{aligned}$$

Therefore, $(D, E, \alpha, \beta, \gamma)$ is an exact couple with the displayed bidegrees. $\square$

Notation. It is a universal agreement to write $n = p + q$, and we will use this notation from now on.

Each exact couple determines another exact couple, but first we have to introduce another important notion.

Definition 2.71. *A **differential bigraded module** is an ordered pair (M, d), where M is a bigraded module and $d : M \to M$ is a bigraded map with $dd = 0$. If (M, d) is a differential bigraded module, where d has bidegree (a, b), then its **homology** $H(M, d)$ is the bigraded module whose (p, q) term is*

$$H(M, d)_{p,q} = \frac{\ker d_{p,q}}{\operatorname{im} d_{p-a,q-b}}.$$

A bicomplex (M, d', d'') gives rise to two differential bigraded modules, namely, (M, d') and (M, d''). However, $(M, d' + d'')$ is not a differential bigraded module because $d' + d'' : M \to M$ is not a bigraded map.

Remark 2.72. Let (M, d) be a differential bigraded module. Then for any $(p, q) \in \mathbb{Z}\times\mathbb{Z}$ and $z \in \ker d_{p,q}$ we set $\operatorname{cls} z$ as the respective homology class of z.

Proposition 2.73. *If $(D, E, \alpha, \beta, \gamma)$ is an exact couple, then $d^1 = \beta\gamma$ is a differential $d^1 : E \to E$, and there is an exact couple $(D^2, E^2, \alpha^2, \beta^2, \gamma^2)$, called the **derived couple**, with $D^2 = \operatorname{im} \alpha$ and $E^2 = H(E, d^1)$.*

$$\begin{tikzcd}
D^2 \arrow[rr, "\alpha^2"] & & D^2 \arrow[dl, "\beta^2"] \\
& E^2 \arrow[ul, "\gamma^2"] &
\end{tikzcd}$$

Proof. Let α, β, γ have respective bidegrees (a_α, b_α), (a_β, b_β), (a_γ, b_γ). The bigraded map $d^1 : E \to E$, where $d^1 = \beta\gamma$, makes sense since $\beta : D \to E$ and $\gamma : E \to D$. Note that $\gamma\beta = 0$, because the original couple is exact, and so d^1 is a differential: $d^1 d^1 = \beta(\gamma\beta)\gamma = 0$. Since bidegrees add, the bidegree of d^1 is $(a_{d^1}, b_{d^1}) = (a_\beta + a_\gamma, b_\beta + b_\gamma)$. Define $E^2 = H(E, d^1)$. Thus $E^2_{p,q} = \ker d^1_{p,q} / \operatorname{im} d^1_{p-a_{d^1},q-b_{d^1}}$. Define $D^2 = \operatorname{im} \alpha \subseteq D$. Thus, $D^2_{p,q} = \operatorname{im} \alpha_{p-a_\alpha,q-b_\alpha} \subseteq D_{p,q}$. We now define the bigraded maps. Define $\alpha^2 : D^2 \to D^2$ to be the restriction $\alpha|_{D^2}$; that is, $\alpha^2 = \alpha i$, where $i : D^2 \to D$ is the inclusion. Since inclusions have bidegree $(0, 0)$, α^2 has bidegree (a_α, b_α), the same bidegree as that of α. If $x \in D^2_{p,q}$, then $x = \alpha u$ (for $u \in D_{p-a_\alpha,q-b_\alpha}$), and

$$\alpha^2_{p,q} : x = \alpha u \mapsto \alpha x = \alpha\alpha u.$$

Define $\beta^2 : D^2 \to E^2$ as follows. If $y \in D^2_{p,q}$, then $y = \alpha v$ (for $v \in D_{p-a_\alpha, q-b_\alpha}$), and $d^1\beta v = \beta(\gamma\beta)v = 0$ whence we have that βv is a cycle. Since $v = \alpha^{-1}y$, we set

$$\beta^2(y) = \mathrm{cls}(\beta\alpha^{-1}y).$$

We have to prove that β^2 does not depend on the choice v of the preimage $\alpha^{-1}y$, so we must show that if $y = \alpha v'$, then $\mathrm{cls}(\beta v') = \mathrm{cls}(\beta v)$. Now $v' - v \in \ker\alpha = \mathrm{im}\,\gamma$, so that $v' - v = \gamma w$ for some $w \in E$, and hence $\mathrm{cls}(\beta(v' - v)) = \mathrm{cls}(\beta\gamma w) = \mathrm{cls}(d^1 w) = \mathrm{cls}(0)$, then $\beta(v' - v)$ is a boundary. Note that β^2 has bidegree $(a_\beta - a_\alpha, b_\beta - b_\alpha)$. We now define $\gamma^2 : E^2 \to D^2$. Let $\mathrm{cls}(z) \in E^2_{p,q}$, so that $z \in E_{p,q}$ and $d^1 z = \beta\gamma z = 0$. Hence, $\gamma z \in \ker\beta = \mathrm{im}\,\alpha$, thus $\gamma z \in \mathrm{im}\,\alpha = D^2$; displaying subscripts, $\gamma_{p,q} z \in D_{p+a_\gamma, q+b_\gamma}$. Define γ^2 by

$$\gamma^2 : \mathrm{cls}(z) \mapsto \gamma z.$$

We must to show that γ^2 does not depend on the choice of cycle. Indeed, if $w \in \mathrm{im}\, d^1_{p-a_{d^1}, q-b_{d^1}}$ is a boundary, then $w = d^1 x = \beta\gamma x$, for some $x \in E_{p-a_{d^1}, q-b_{d^1}}$, and so $\gamma w = (\gamma\beta)\gamma x = 0$. Observe that γ^2 has the bidegree (a_γ, b_γ), the same bidegree as that of γ. It just remains to prove exactness. Since all the maps are well-defined, there is no reason to display subscripts. First of all, adjacent composites are 0.

$$\begin{aligned}
&\beta^2\alpha^2 : x = \alpha u \mapsto \alpha\alpha u \mapsto \mathrm{cls}(\beta\alpha^{-1}\alpha\alpha u) = \mathrm{cls}(\beta\alpha u) = 0.\\
&\gamma^2\beta^2 : x = \alpha u \mapsto \mathrm{cls}(\beta u) \mapsto \gamma\beta u = 0.\\
&\alpha^2\gamma^2 : \mathrm{cls}(z) \mapsto \gamma z \mapsto \alpha\gamma z = 0.
\end{aligned}$$

We have verified the inclusions of the form im $\subseteq$ ker. Now must proof the reverse inclusions.

$\ker\alpha^2 \subseteq \mathrm{im}\,\gamma^2$. If $x \in \ker\alpha^2$, then $x \in D^2$ and $\alpha x = 0$. Hence, $x \in \ker\alpha = \mathrm{im}\,\gamma$, so that $x = \gamma y$ for some $y \in E$. Now $x \in \mathrm{im}\,\alpha = \ker\beta$, and $0 = \beta x = \beta\gamma y = d^1 y$. Thus, y is a cycle, and $x = \gamma y = \gamma^2\,\mathrm{cls}(y) \in \mathrm{im}\,\gamma^2$.

$\ker\beta^2 \subseteq \mathrm{im}\,\alpha^2$. If $x \in \ker\beta^2$, then $x \in D^2 = \mathrm{im}\,\alpha$ and $\beta^2 x = 0$. Thus, $x = \alpha u$ and $0 = \beta^2 x = \mathrm{cls}(\beta\alpha^{-1}\alpha u) = \mathrm{cls}(\beta u)$. Hence, $\beta u \in \mathrm{im}\, d^1$; that is, $\beta u = d^1 w = \beta\gamma w$ for some $w \in E$. Now $u - \gamma w \in \ker\beta = \mathrm{im}\,\alpha = D^2$, and $\alpha^2(u - \gamma w) = \alpha u - \alpha\gamma w = \alpha u = x$. Therefore, $x \in \mathrm{im}\,\alpha^2$.

$\ker\gamma^2 \subseteq \mathrm{im}\,\beta^2$. If $\mathrm{cls}(z) \in \ker\gamma^2$, then $\gamma^2\,\mathrm{cls}(z) = \gamma z = 0$. Thus, $z \in \ker\gamma = \mathrm{im}\,\beta$, so that $z = \beta v$ for some $v \in D$. Hence, $\beta^2(\alpha v) = \mathrm{cls}(\beta\alpha^{-1}\alpha v) = \mathrm{cls}(\beta v) = \mathrm{cls}(z)$, and $\mathrm{cls}(z) \in \mathrm{im}\,\beta^2$.

□

The next lemma shows a characterization for the map d^1 that will be necessary for some results.

Lemma 2.74. *Let $(F^p\mathbf{C})_{p\in\mathbb{Z}}$ be a filtration of a complex $\mathbf{C}$, and let the corresponding exact couple be $(D, E, \alpha, \beta, \gamma)$. Then the differential $d^1_{p,q} : E_{p,q} \to E_{p-1,q}$ is the connecting homomorphism*

$$H_{p+q}\left(F^p/F^{p-1}\right) \to H_{p+q-1}\left(F^{p-1}/F^{p-2}\right)$$

arising from $0 \to F^{p-1}/F^{p-2} \to F^p/F^{p-2} \to F^p/F^{p-1} \to 0$.

Proof. Let $(\mathbf{C}, d_n)$ be a complex, then we have the next two short exact sequences:

$$0 \longrightarrow F^{p-1} \xrightarrow{i} F^p \xrightarrow{\pi} F^p/F^{p-1} \longrightarrow 0$$

and

$$0 \longrightarrow F^{p-1}/F^{p-2} \xrightarrow{\hat{i}} F^p/F^{p-2} \xrightarrow{\widehat{\pi}} F^p/F^{p-1} \longrightarrow 0,$$

where i is the inclusion, π is the natural map, $\hat{i}$ and $\widehat{\pi}$ are induced by i and π respectively. For x in F^p, let us denote by $\overline{x}$ the class of x in F^p/F^{p-1}, write $\widehat{x}$ for the class of x in F^p/F^{p-2} and we use the notation $\operatorname{cls} z$ to refer to the class of a cycle z in its respective homology group. Define

$$d^p_n : F^pC_n \to F^pC_{n-1}$$

as the restriction of d_n to F^p, and the maps

$$\widehat{d^p_n} : \frac{F^pC_n}{F^{p-2}C_n} \to \frac{F^pC_{n-1}}{F^{p-2}C_{n-1}} \quad \text{and} \quad \overline{d^p_n} : \frac{F^pC_n}{F^{p-1}C_n} \to \frac{F^pC_{n-1}}{F^{p-1}C_{n-1}}$$

as the morphisms induced by d^p_n.

Notice that the connecting homomorphism $\gamma_{p,q} : H_n(F^p/F^{p-1}) \to H_{n-1}(F^{p-1})$ arises from the diagram

$$\begin{tikzcd}
 & F^pC_n \arrow[r, "\pi"] \arrow[d, "d^p_n"] & F^pC_n/F^{p-1}C_n \arrow[r] & 0 \\
0 \arrow[r] & F^{p-1}C_{n-1} \arrow[r, "i"] & F^pC_{n-1}. &
\end{tikzcd}$$

Then for $\overline{z} \in \ker \overline{d^p_n}$, $\gamma_{p,q}$ satisfies

$$\gamma_{p,q}(\operatorname{cls}(\overline{z})) = \operatorname{cls}(i^{-1}d^p_n\pi^{-1}(\overline{z})).$$

On the other hand we have the connecting homomorphism $\partial_{p,q} : H_n(F^p/F^{p-1}) \to H_{n-1}(F^{p-1}/F^{p-2})$ arises from

$$\begin{tikzcd} & F^pC_n/F^{p-2}C_n \arrow[r, "\widehat{\pi}"] \arrow[d, "\widehat{d^p_n}"] & F^pC_n/F^{p-1}C_n \\ F^{p-1}C_{n-1}/F^{p-2}C_{n-1} \arrow[r, "\hat{i}"] & F^pC_{n-1}/F^{p-2}C_{n-1}. & \end{tikzcd}$$

Then for $\overline{z} \in \ker \overline{d^p_n}$, $\partial_{p,q}$ satisfies

$$\begin{aligned} \partial_{p,q}(\mathrm{cls}(\overline{z})) &= \mathrm{cls}(\hat{i}^{-1}\widehat{d^p_n}\widehat{\pi}^{-1}(\overline{z})) \\ &= \mathrm{cls}(\hat{i}^{-1}\widehat{d^p_n}(\widehat{\pi^{-1}(\overline{z})})) \\ &= \mathrm{cls}(\hat{i}^{-1}\widehat{d^p_n\pi^{-1}(\overline{z})})) \\ &= \mathrm{cls}\left(\widehat{i^{-1}d^p_n\pi^{-1}(\overline{z})}\right). \end{aligned}$$

Finally recall that $\beta_{p-1,q}$ is the map induced by the natural map $F^{p-1} \to F^{p-1}/F^{p-2}$, that is, $\beta_{p-1,q}(\mathrm{cls}\, w) = \mathrm{cls}(\widehat{w})$. Hence

$$\begin{aligned} \mathrm{cls}\left(\widehat{i^{-1}d^p_n\pi^{-1}(\overline{z})}\right) &= \beta_{p-1,q}\big(\mathrm{cls}(i^{-1}d^p_n\pi^{-1}(\overline{z}))\big) \\ &= \beta_{p-1,q}\gamma_{p,q}(\overline{z}). \end{aligned}$$

Thus $\partial_{p,q} = \beta_{p-1,q}\gamma_{p,q} = d^1_{p,q}$. □

Definition 2.75. *Given an exact couple $(D, E, \alpha, \beta, \gamma)$, we define its r**th derived couple** $(D^r, E^r, \alpha^r, \beta^r, \gamma^r)$ recursively in the next way: the $(r+1)$st derived couple $(D^{r+1}, E^{r+1}, \alpha^{r+1}, \beta^{r+1}, \gamma^{r+1})$ is the derived couple of $(D^r, E^r, \alpha^r, \beta^r, \gamma^r)$.*

We assume that $(D, E, \alpha, \beta, \gamma)$ correspond to its 1st derived couple $(D^1, E^1, \alpha^1, \beta^1, \gamma^1)$.

Corollary 2.76. *Let $(D, E, \alpha, \beta, \gamma)$ be the exact couple arising from a filtration (F^p) of a complex C and let $(D^r, E^r, \alpha^r, \beta^r, \gamma^r)$ be its respective rth derived couple:*

$$\begin{tikzcd} D \arrow[rr, "\alpha\,(1,-1)"] & & D \arrow[dl, "\beta\,(0,0)"] \\ & E \arrow[ul, "\gamma\,(-1,0)"] & \end{tikzcd} \qquad \begin{tikzcd} D^r \arrow[rr, "\alpha^r\,(1,-1)"] & & D^r \arrow[dl, "\beta^r\,(a_r,b_r)"] \\ & E^r \arrow[ul, "\gamma^r\,(-1,0)"] & \end{tikzcd}$$

where $a_r = 1-r$ and $b_r = r-1$. Then $(D^r, E^r, \alpha^r, \beta^r, \gamma^r)$ has the following properties:

(i) the bigraded maps α^r, β^r, γ^r have bidegrees $(1,-1)$, $(1-r, r-1)$, $(-1,0)$, respectively;

(ii) *the differential* d^r *is induced by* $\beta\alpha^{-r+1}\gamma$*, and* d^r *has bidegree* $(-r, r-1)$*;*

(iii) $E^{r+1}_{p,q} = \ker d^r_{p,q} / \operatorname{im} d^r_{p+r,q-r+1}$*;*

(iv) $D^r_{p,q} = im(\alpha_{p-1,q+1})(\alpha_{p-2,q+2})\cdots(\alpha_{p-r+1,q+r-1})$*; in particular, for the exact couple in Proposition 2.70,*

$$D^r_{p,q} = im(j^{p-1}j^{p-2}...j^{p-r+1})_* : H_n(F^{p-r+1}) \to H_n(F^p).$$

Proof.

(i) First observe that the derived couple preserve the bidegrees of α^i and γ^i, i.e. $deg(\alpha) = deg(\alpha^i)$ and $deg(\gamma) = deg(\gamma^i)$, where $i \in \{2, 3, \ldots\}$. Then $deg(\alpha^r) = deg(\alpha) = (1, -1)$ and $deg(\gamma^r) = deg(\gamma) = (-1, 0)$.

Recall that if the maps α^r and β^r have bidegrees $(a_{\alpha^r}, b_{\alpha^r})$ and $(a_{\beta^r}, b_{\beta^r})$, respectively, then β^{r+1} has bidegree $(a_{\beta^r} - a_{\alpha^r}, b_{\beta^r} - b_{\alpha^r})$. Hence, by induction, if

$$deg(\beta^{r-1}) = (1 - (r-1), (r-1) - 1),$$

then

$$deg(\beta^r) = (1 - (r-1) - 1, (r-1) - 1 + 1) = (1 - r, r - 1).$$

Finally to complete the induction observe that for $r = 2$ the map β^2 has bidegree $(-1, 1) = (1 - r, r - 1)$.

(ii) As β^r, γ^r have bidegrees $(1 - r, r - 1), (-1, 0)$ respectively and bidegrees adds, then the bidegree of $d^r = \beta^r\gamma^r$ is $(-r, r - 1)$. Denote by $\mathrm{cls}^2(z) = \mathrm{cls}(\mathrm{cls}(z))$, the class of $\mathrm{cls}(z)$ in E^3, where z is a cycle in the complex (E^2, d^2). Now we define $\mathrm{cls}^r(z)$ recursively by setting $\mathrm{cls}^r(z) = \mathrm{cls}(\mathrm{cls}^{r-1}(z))$, the class of $\mathrm{cls}^{r-1}(z)$ in E^r, where $\mathrm{cls}^{r-1}(z)$ is a cycle in the complex (E^{r-1}, d^{r-1}). Then using the above notation we have

$$\begin{aligned}
d^r(\mathrm{cls}^{r-1}(z)) &= \beta^r\gamma^r(\mathrm{cls}^{r-1}(z)) \\
&= \beta^r\gamma^{r-1}(\mathrm{cls}^{r-2}(z)) \\
&= \beta^r\gamma(z) \\
&= \mathrm{cls}(\beta^{r-1}\alpha^{-1}\gamma(z)) \\
&= \mathrm{cls}(\mathrm{cls}(\beta^{r-2}\alpha^{-2}\gamma(z))) \\
&= \mathrm{cls}^{r-1}(\beta\alpha^{-r+1}\gamma(z)).
\end{aligned}$$

(iii) It is clear since the bidegree of d^r is $(-r, r - 1)$.

(iv) Observe

$$\begin{aligned} D^r_{p,q} &= \alpha_{(p-1,q+1)} D^{r-1}_{(p-1,q+1)} \\ &= \alpha_{(p-1,q+1)}\alpha_{(p-2,q+2)} D^{r-2}_{(p-2,q+2)} \\ &= \alpha_{(p-1,q+1)}\alpha_{(p-2,q+2)}\ldots\alpha_{(p-r+1,q+r-1)} D_{(p-r+1,q+r-1)} \\ &= \operatorname{im}(\alpha_{(p-1,q+1)}\alpha_{(p-2,q+2)}\ldots\alpha_{(p-r+1,q+r-1)}). \end{aligned}$$

For the last statement recall that $\alpha_{p,q} = j^p_* : H_{p+q}(F^p) \to H_{p+q}(F^{p+1})$ and that $j^{p-1}_* j^{p-2}_* \cdots j^{p-r+1}_* = (j^{p-1} j^{p-2} \cdots j^{p-r+1})_*$.

□

Definition 2.77. *A* ***spectral sequence*** *is a sequence $(E^r, d^r)_{r\geq 1}$ of differential bi-graded modules such that $E^{r+1} = H(E^r, d^r)$ for all $r \in \mathbb{Z}^+$.*

Given a spectral sequence $(E^r, d^r)_{r\geq 1}$ and a fixed $r \in \mathbb{Z}^+$, we say that the terms $E^r_{p,q}$ form the rth page of the spectral sequence $(E^r, d^r)_{r\geq 1}$. Thus, it is useful think a spectral sequence as a book where the rth page of the book correspond to the rth page of the spectral sequence, so we have that for each page of the book the next page is its own homology.

Theorem 2.78. *Any filtration of a complex yields a spectral sequence as described Corollary 2.76.*

Proof. A filtration gives an exact couple, as in Proposition 2.70, and the E^r terms of its derived couples define a spectral sequence.

□

Definition 2.79. *If M is a module, then a* ***subquotient*** *of M is a module of the form M'/M'', where $M'' \subseteq M' \subseteq M$.*

If $\{E^r, d^r\}$ is a spectral sequence, then $E^2 = H(E^1, d^1)$ is a subquotient of E^1. Hence, $E^2 = Z^2/B^2 = \ker d^1 / \operatorname{im} d^1$, where

$$B^2 \subseteq Z^2 \subseteq E^1.$$

So any submodule of E^2 is equal to S/B^2 for a unique submodule S of Z^2 with $B^2 \subseteq S$. Hence, in particular, for the relative cycles Z^3 and boundaries B^3 there exist unique submodules B^3_* and Z^3_* of Z^2 such that $B^3 = B^3_*/B^2$ and $Z^3 = Z^3_*/B^2$. Then we can identify Z^3, B^3 with Z^3_*, B^3_* respectively. Therefore $B^3/B^2 \subseteq Z^3/B^2 \subseteq Z^2/B^2 = E^2$, so that

$$B^2 \subseteq B^3 \subseteq Z^3 \subseteq Z^2 \subseteq E^1.$$

More generally, for each r, there is a chain

$$B^2 \subseteq \cdots \subseteq B^r \subseteq Z^r \subseteq \cdots \subseteq Z^2 \subseteq E^1.$$

Definition 2.80. *Given a spectral sequence $\{E^r, d^r\}$ and the above identification of the submodules Z^r and B^r, define $Z^\infty = \bigcap_r Z^r$ and $B^\infty = \bigcup_r B^r$. Then $B^\infty \subseteq Z^\infty$, and the **limit term** of the spectral sequence is the bigraded module E^∞ defined by*

$$E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}.$$

Lemma 2.81. *Let $\{E^r, d^r\}$ be a spectral sequence. Then, for any $p, q \in \mathbb{Z}$,*

(i) $E^{r+1}_{p,q} = E^r_{p,q}$ if and only if $Z^{r+1}_{p,q} = Z^r_{p,q}$ and $B^{r+1}_{p,q} = B^r_{p,q}$,

(ii) If $E^{r+1}_{p,q} = E^r_{p,q}$ for all $r \geq s$, then $E^s_{p,q} = E^\infty_{p,q}$.

Proof. Since it is clear that we are working with the p, q terms of the spectral sequence we will omit (in this proof) the subscripts.

(i) Recall that if X/Y is a subquotient of Z, then $Y \subseteq X \subseteq Z$, and so $X/Y = Z$ if and only if $Y = \{0\}$ and $X = Z$. If $Z^{r+1}/B^{r+1} = E^{r+1} = E^r$, then $B^{r+1} = \{0\}$ in $E^r = Z^r/B^r$; that is, $B^{r+1} \subseteq B^r$, but since $B^r \subseteq B^{r+1}$ we have that $B^{r+1} = B^r$. Hence, $E^{r+1} = Z^{r+1}/B^{r+1} = Z^{r+1}/B^r = E^r = Z^r/B^r$, so that $Z^{r+1} = Z^r$. The converse is obvious.

(ii) If $E^r = E^{r+1}$ for all $r \geq s$, then $Z^s = Z^r$ for all $r \geq s$; hence, $Z^s = \bigcap_{r \geq s} Z^r = Z^\infty$. Also, $B^s = B^r$ for all $r \geq s$; hence, $B^s = \bigcup_{r \geq s} B^r = B^\infty$. Therefore,

$$E^s = Z^s/B^s = Z^\infty/B^\infty = E^\infty.$$

□

Given a filtration (F^p) of a complex $\mathbf{C}$ with inclusions $i^p : F^p \to \mathbf{C}$, we have the map $i^p_* : H_\bullet(F^p) \to H_\bullet(\mathbf{C})$ induced by i^p. Since $F^p \subseteq F^{p+1}$, we have $\operatorname{im} i^p_* \subseteq \operatorname{im} i^{p+1}_*$; that is, $(\operatorname{im} i^p_*)$ is a filtration of $H_\bullet(\mathbf{C})$.

Definition 2.82. *If $(F^P\mathbf{C})$ is a filtration of a complex $\mathbf{C}$ and $i^p : F^p \to \mathbf{C}$ are inclusions, define*

$$\Phi^p H_n(\mathbf{C}) = \operatorname{im} i^p_*.$$

*We call $(\Phi^p H_n(\mathbf{C}))$ the **induced filtration** of $H_n(\mathbf{C})$.*

Definition 2.83. *A filtration (F^pM) of a graded module $M = (M_n)$ is* ***bounded*** *if, for each n, there exist integers $s = s(n)$ and $t = t(n)$ such that*

$$F^sM_n = \{0\} \text{ and } F^tM_n = M_n.$$

Given a bounded filtration $\{F^p\}$ of a complex $\mathbf{C}$, the induced filtration on homology is also bounded, moreover it has the same bounds. Indeed, we have that if $i^p : F^p \to \mathbf{C}$ is the inclusion, then $\Phi^pH_n = \operatorname{im} i^p_*$, where $i^p_* : H_n(F^p) \to H_n(\mathbf{C})$. Since for each n there exist $r, s \in \mathbb{Z}$ such that $F^sC_n = 0$ and $F^tC_n = C_n$, we have $\Phi^sH_n = \{0\}$ and $\Phi^tH_n = H_n$. Hence, for each $n \in \mathbb{Z}$, there is a finite chain,

$$\{0\} = \Phi^sH_n \subseteq \Phi^{s+1}H_n \subseteq \cdots \subseteq \Phi^tH_n = H_n.$$

Of course, it is clear that $\Phi^iH_n = \{0\}$ for all $i \leq s$, and $\Phi^jH_n = H_n$ for all $j \geq t$.

Definition 2.84. *A spectral sequence $(E^r, d^r)_{r\geq 1}$* ***converges*** *to a graded module $H_\bullet$ if there exist some bounded filtration (Φ^pH_n) of $H_\bullet$ such that*

$$E^\infty_{p,q} \cong \Phi^pH_n/\Phi^{p-1}H_n$$

for all n and p, q with $p + q = n$. We denote the convergence by

$$E^r_{p,q} \underset{p}{\Rightarrow} H_n.$$

Since spectral sequences are often referred by its second page, it is common to write the convergence of a spectral sequence as

$$E^2_{p,q} \underset{p}{\Rightarrow} H_n.$$

Theorem 2.85. *Let $(F^p\mathbf{C})_p$ be a bounded filtration of a complex $\mathbf{C}$, and let $(E^r, d^r)_{r\geq 1}$ be the spectral sequence of Theorem 2.78. Then*

(i) for each p, q, we have $E^\infty_{p,q} = E^r_{p,q}$ for large r (depending on p, q),

(ii) $E^2_{p,q} \underset{p}{\Rightarrow} H_n(\mathbf{C})$, by means of the induced filtration of $H_n(\mathbf{C})$.

Proof. Recall that the induced filtration (Φ^pH) is bounded with the same bounds $s(n)$ and $t(n)$ as $(F^p\mathbf{C})$. Then

(i) If p is "large"; that is, $p > t(n)$, then $F^{p-1}C_n = F^pC_n$, and $F^pC_n/F^{p-1}C_n = 0$. By definition, $E_{p,q} = H_{p+q}\left(F^p/F^{p-1}\right)$, and so $E_{p,q} = \{0\}$. Since $E^r_{p,q}$ is a subquotient of $E_{p,q}$, we have $E^r_{p,q} = \{0\}$ for all r. Similarly, if p is "small"; that is, $p < s(n)$, then $F^pC_n = 0$, and $E^r_{p,q} = \{0\}$ for all r. Focus on first subscripts. For any fixed (p,q), $d^r\left(E^r_{p,q}\right) \subseteq E^r_{p-r,\#}$. For large r, the index $p-r$ is small, and so $E^r_{p-r,\#} = \{0\}$. Hence, $\ker d^r_{p,q} = E^r_{p,q}$. Let us compute $E^{r+1}_{p,q} = \ker d^r_{p,q}/\operatorname{im} d^r_{p+r,\#}$. Now $\operatorname{im} d^r_{p+r,\#} = \{0\}$, because the domain of $d^r_{p+r,\#}$ is $E^r_{p+r,\#} = \{0\}$ when r is large. Therefore, $E^{r+1}_{p,q} = \ker d^r_{p,q}/\{0\} = E^r_{p,q}/\{0\} = E^r_{p,q}$ for large r (depending on p,q). Thus, the p,q term of $E^r_{p,q}$ is constant for large r, which says that $E^\infty_{p,q} = E^r_{p,q}$, by Lemma 2.81.

(ii) We continue focusing on the first index in the subscript by writing # for every second index. Consider the exact sequence obtained from the r th derived couple:

$$D^r_{p+r-2,\#} \xrightarrow{\alpha^r} D^r_{p+r-1,\#} \xrightarrow{\beta^r} E^r_{p,q} \xrightarrow{\gamma^r} D^r_{p-1,q}. \quad (1)$$

The indices arise from the bidegrees displayed in Corollary 2.76 (i) : α^r has bidegree $(1,-1)$, β^r has bidegree $(1-r, r-1)$, and γ^r has bidegree $(-1,0)$; as in Corollary 2.76(iv), the module

$$D^r_{p,q} = \operatorname{im}\left(j^{p-1}j^{p-2}\cdots j^{p-r+1}\right)_* : H_n\left(F^{p-r+1}\right) \to H_n\left(F^p\right).$$

Replacing p first by $p+r-1$ and then by $p+r-2$, we have

$$D^r_{p+r-1,\#} = \operatorname{im}\left(j^{p+r-2}\cdots j^p\right)_* \subseteq H_n\left(F^{p+r-1}\right)$$

and

$$D^r_{p+r-2,\#} = \operatorname{im}\left(j^{p+r-3}\cdots j^{p-1}\right)_* \subseteq H_n\left(F^{p+r-2}\right).$$

For large r, $F^{p+r-1}C_n = F^{t(n)}C_n = C_n$, and the composition $j^{p+r-2}\ldots j^p$ of inclusions applied to F^pC_n is just the inclusion $i^p_n : F^pC_n \to C_n$. Therefore, $D^r_{p+r-1,\#} = \operatorname{im} i^p_{*n} = \Phi^pH_n$. Similarly, $D^r_{p+r-2,\#} = \Phi^{p-1}H_n$ for large r. Hence, we may rewrite the exact sequence (1) as

$$\Phi^{p-1}H_n(\mathbf{C}) \xrightarrow{\alpha^r} \Phi^pH_n(\mathbf{C}) \to E^r_{p,q} \to D^r_{p-1,q},$$

where the first arrow is inclusion. Indeed, recall that α^r is just the restriction of $\alpha = j^{p+r-2}_*$ to $D^r_{p+r-2,\#} = \Phi^{p-1}H_n(\mathbf{C})$ and that

$$j^{p+r-2}_* : H_n(F^{p+r-2}) \to H_n(F^{p+r-1}),$$

which for larger r is equal to the identity map

$$j^{p+r-2}_* : H_n(\mathbf{C}) \to H_n(\mathbf{C}).$$

Finally, if r is large

$$D^r_{p-1,q} = \operatorname{im}\left(\left(j^{p-2}\cdots j^{p-r}\right)_* : H_n\left(F^{p-r}\right) \to H_n\left(F^{p-1}\right)\right) = 0,$$

because $F^{p-r}C_n = 0$ for larger r. Hence, we have the next exact sequence

$$0 \to \Phi^{p-1}H_n(\mathbf{C}) \to \Phi^p H_n(\mathbf{C}) \to E^r_{p,q} \to 0,$$

then

$$\Phi^p H_n(\mathbf{C})/\Phi^{p-1}H_n(\mathbf{C}) \cong E^r_{p,q} = E^\infty_{p,q}.$$

□

Definition 2.86. *Let (M, d', d'') be a bicomplex. The transposed bicomplex (M^t, δ', δ'') of (M, d', d'') is the bicomplex such that $M^t_{p,q} = M_{q,p}$, $\delta' = d''$ and $\delta'' = d'$.*

Lemma 2.87. *If $\operatorname{Tot}(M)$ is the total complex of a bicomplex (M, d', d''). Then*

(i) the second filtration of $\operatorname{Tot}(M)$ is equal to the first filtration of $\operatorname{Tot}(M^t)$, i.e.

$$^{\mathrm{II}}F^p \operatorname{Tot}(M)_n = {}^{\mathrm{I}}F^p \operatorname{Tot}(M^t)_n,$$

(ii) $\operatorname{Tot}(M) = \operatorname{Tot}(M^t)$.

Proof. The transpose M^t is defined by $M^t_{p,q} = M_{q,p}$, thus

$$^{\mathrm{II}}F^p \operatorname{Tot}(M)_n = \bigoplus_{j\le p} M_{n-j,j} = \bigoplus_{j\le p} M^t_{j,n-j} = {}^{\mathrm{I}}F^p \operatorname{Tot}(M^t)_n$$

and

$$\operatorname{Tot}(M^t)_n = \bigoplus_{p+q=n} M_{q,p} = \bigoplus_{p+q=n} M_{p,q} = \operatorname{Tot}(M)_n.$$

□

Definition 2.88. *A **first quadrant bicomplex** is a bicomplex $(M_{p,q})$ for which $M_{p,q} = \{0\}$ whenever p or q is negative.*

Theorem 2.89. *Let M be a first quadrant bicomplex, and let $^{\mathrm{I}}E^r$ and $^{\mathrm{II}}E^r$ be the spectral sequences determined by the first and second filtrations of $\operatorname{Tot}(M)$. Then*

(i) The first and second filtrations are bounded, and the bounds for either filtration are $s(n) = -1$ and $t(n) = n$,

(ii) For all p, q, we have ${}^{\mathrm{I}}E_{p,q}^{\infty} = {}^{\mathrm{I}}E_{p,q}^{r}$ and ${}^{\mathrm{II}}E_{p,q}^{\infty} = {}^{\mathrm{II}}E_{p,q}^{r}$ for large r (depending on p, q).

(iii) ${}^{\mathrm{I}}E_{p,q}^{2} \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M))$ and ${}^{\mathrm{II}}E_{p,q}^{2} \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M))$.

Proof. Part (i) is obvious. Statements (ii) and (iii) for ${}^{\mathrm{I}}E$ follow from Theorem 2.85. Since $\mathrm{Tot}\,(M^t) = \mathrm{Tot}\,(M)$, where M^t is the transpose, and since the second filtration of $\mathrm{Tot}(M)$ equals the first filtration of M^t, we have ${}^{\mathrm{II}}E_{p,q}^{\infty} = {}^{\mathrm{II}}E_{p,q}^{r}$ for large r and ${}^{\mathrm{II}}E_{p,q}^{2} \underset{p}{\Rightarrow} H_n\left(\mathrm{Tot}\left(M^t\right)\right) = H_n(\mathrm{Tot}(M))$. □

Let (M, d', d'') be a bicomplex. Define $M_{p,*}$ as the pth column of M (see figure 2.1), hence $(M_{p,*}, d''_{p,*})$ is a complex, where the map $d''_{p,*}$ is the restriction of d'' in $M_{p,*}$. Therefore we can define a new bigraded module $H''(M)$, whose (p, q) term is $H_q\left(M_{p,*}\right)$.

For each fixed q, the qth row $H''(M)_{*,q}$ of $H''(M)$

$$\ldots, H_q\left(M_{p+1,*}\right), H_q\left(M_{p,*}\right), H_q\left(M_{p-1,*}\right), \ldots,$$

can be made into a complex if we define $\overline{d'}_p : H_q\left(M_{p,*}\right) \to H_q\left(M_{p-1,*}\right)$ by

$$\overline{d'}_p : \mathrm{cls}(z) \mapsto \mathrm{cls}\left(d'_{p,q} z\right),$$

where $z \in \ker d''_{p,q}$. There is a new bigraded module whose (p, q) term, denoted by $H'_p H''_q(M)$, is the pth homology of $(H''(M)_{*,q}, \overline{d'})$, i.e. $H'_p H''_q(M) = H_p(H''(M)_{*,q})$.

Definition 2.90. *If (M, d', d'') is a bicomplex, its **first iterated homology**, denoted by $H'H''(M)$, is the bigraded module whose (p, q) term is $H'_p H''_q(M)$.*

Proposition 2.91. *If M is a first quadrant bicomplex, then*

$${}^{\mathrm{I}}E_{p,q}^{1} = H_q\left(M_{p,*}\right) = H''(M)_{p,q}$$

and

$${}^{\mathrm{I}}E_{p,q}^{2} = H'_p H''_q(M) \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M)).$$

Proof. Since it is clear that we are working with ${}^{\mathrm{I}}E_{p,q}^{r}$ let us omit the prescript I for this proof. As in Proposition 2.70 proof, $E_{p,q} = H_n\left(F^p/F^{p-1}\right)$, where

$$\begin{aligned} \left(F^p\right)_n &= \cdots \oplus M_{p-2,q+2} \oplus M_{p-1,q+1} \oplus M_{p,q}, \\ \left(F^{p-1}\right)_n &= \cdots \oplus M_{p-2,q+2} \oplus M_{p-1,q+1}. \end{aligned}$$

Hence, the nth term of F^p/F^{p-1} is $M_{p,q}$. The differential $(F^p/F^{p-1})_n \to (F^p/F^{p-1})_{n-1}$ is

$$\overline{D}_n : a_n + \left(F^{p-1}\right)_n \mapsto D_n a_n + \left(F^{p-1}\right)_{n-1},$$

where $a_n \in (F^p)_n$; we have just seen that we may assume $a_n \in M_{p,q}$. Now $D_n a_n = \left(d'_{p,q} + d''_{p,q}\right) a_n \in M_{p-1,q} \oplus M_{p,q-1}$. But $M_{p-1,q} \subseteq (F^{p-1})_{n-1}$, so that $D_n a_n \equiv d''_{p,q} a_n \bmod (F^{p-1})_{n-1}$. Thus, only d'' survives in F^p/F^{p-1}. More precisely, since $n = p + q$

$$H_n\left(F^p/F^{p-1}\right) = \frac{\ker \overline{D}_n}{\operatorname{im} \overline{D}_{n+1}} = \frac{(\ker d''_{p,q} + (F^{p-1})_n)/(F^{p-1})_n}{(\operatorname{im} d''_{p,q+1} + (F^{p-1})_n)/(F^{p-1})_n} \cong \frac{\ker d''_{p,q}}{\operatorname{im} d''_{p,q+1}} = H_q\left(M_{p,*}\right).$$

Therefore, ${}^{\mathrm{I}}E^1_{p,q} = H_q\left(M_{p,*}\right)$ and the elements of ${}^{\mathrm{I}}E^1_{p,q}$ have the form $\operatorname{cls}(z)$, where $z \in M_{p,q}$ and $d''z = 0$. Now Lemma 2.74 identifies the map d^1 with the connecting homomorphism arising from $0 \to F^{p-1}/F^{p-2} \to F^p/F^{p-2} \to F^p/F^{p-1} \to 0$. So it only remains to prove that $\overline{d'} = d^1$. As $d^1 : H_{p+q}\left(F^p/F^{p-1}\right) \to H_{p+q-1}\left(F^{p-1}/F^{p-2}\right)$ is the connecting homomorphism, it arises from the diagram

$$\begin{tikzcd}
 & & M_{p-1,q+1} \oplus M_{p,q} \arrow[r, "\phi"] \arrow[d, "\overline{D}"] & M_{p,q} \arrow[r] & 0 \\
0 \arrow[r] & M_{p-1,q} \arrow[r, "i"] & M_{p,q-1} \oplus M_{p-1,q}, & &
\end{tikzcd}$$

where $\overline{D} : (a_{p-1,q+1}, a_{p,q}) \mapsto (d''a_{p,q}, d''a_{p-1,q+1} + d'a_{p,q})$, i is the natural inclusion and ϕ is the natural projection. Let $z \in M_{p,q}$ be a cycle; that is, $d''_{p,q} z = 0$. Choose $\phi^{-1} z = (0, z)$, so that $\overline{D}(0, z) = \left(0, d'_{p,q} z\right)$. Then

$$d^1 \operatorname{cls}(z) = \operatorname{cls}\left(i^{-1}\overline{D}\pi^{-1} z\right) = \operatorname{cls}(d'z) = \overline{d'} \operatorname{cls}(z).$$

Hence,

$$E^2_{p,q} = \frac{\ker \overline{d'}_{p,q}}{\operatorname{im} \overline{d'}_{p+1,q}} = H'_p H''_q(M),$$

as required.

$\square$

Now we can construct the analogous of $H'H''(M)$. Let (M, d', d'') a bicomplex, take its transpose bicomplex (M^t, δ', δ''), where $\delta' = d''$ and $\delta'' = d'$. Recall that for each fixed p we have that $(M^t_{p,*}, \delta''_{p,*}) = (M_{*,p}, d'_{*,p})$ is a complex, thus taking homology of pth columns of M^t gives a bigraded module, denoted by $H'(M)$, whose (p,q) term is $H_q(M^t_{p,*})$. For each fixed q, the qth row $H'(M)_{*,q}$ of $H'(M)$

$$\ldots, H_q\left(M^t_{p+1,*}\right), H_q\left(M^t_{p,*}\right), H_q\left(M^t_{p-1,*}\right), \ldots,$$

can be made into a complex if we define $\overline{\delta'}_p : H_q\left(M^t_{p,*}\right) \to H_q\left(M^t_{p-1,*}\right)$ by

$$\overline{\delta'}_p : \mathrm{cls}(z) \mapsto \mathrm{cls}\left(\delta'_{p,q} z\right),$$

where $z \in \ker\delta''_{p,q}$. There is a new bigraded module whose (p,q) term, denoted by $H''_p H'_q(M)$, is the pth homology of $(H''(M)_{*,q}, \overline{\delta'})$, i.e. $H''_p H'_q(M) = H_p(H'(M)_{*,q})$. Notice that $H'(M)$ is just $H''(M^t)$. Therefore

$$H''_p H'_q(M) = H_p(H'(M)_{*,q}) = H_p(H''(M^t)_{*,q}) = H'_p H''_q\left(M^t\right).$$

Definition 2.92. *If (M, d', d'') is a bicomplex, its* ***second iterated homology*** *is the bigraded module whose (p,q) terms is $H''_p H'_q(M)$.*

Proposition 2.93. *If M is a first quadrant bicomplex, then*

$$^{\mathrm{II}}E^1_{p,q} = H_q\left(M_{*,p}\right)$$

and

$$^{\mathrm{II}}E^2_{p,q} = H''_p H'_q(M) \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M)).$$

Proof. By Proposition 2.91 we have

$$^{\mathrm{I}}E^1_{p,q} M^t = H_q\left(M^t_{p,*}\right)$$

and

$$^{\mathrm{I}}E^2_{p,q} M^t = H'_p H''_q(M^t) \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M^t)),$$

where $^{\mathrm{I}}E^r_{p,q} M^t$ refers to the (p,q) term in the r-page of the spectral sequence obtained from the first filtration of $\mathrm{Tot}(M^t)$. By our previous discussion we have that $H'H''\left(M^t\right) = H''H'(M)$, we also know that $H_q\left(M^t_{p,*}\right) = H_q\left(M_{*,p}\right)$ and by Lemma 2.87 the second filtration of $\mathrm{Tot}(M)$ is the first filtration of $\mathrm{Tot}\left(M^t\right)$ and $\mathrm{Tot}(M) = \mathrm{Tot}(M^t)$. Hence,

$$^{\mathrm{II}}E^1_{p,q} = {}^{\mathrm{I}}E^1_{p,q} M^t = H_q\left(M^t_{p,*}\right) = H_q\left(M_{*,p}\right)$$

and

$$^{\mathrm{II}}E^2_{p,q} = {}^{\mathrm{I}}E^2_{p,q} M^t = H'_p H''_q(M^t) = H''_p H'_q(M) \underset{p}{\Rightarrow} H_n(\mathrm{Tot}(M^t)) = H_n(\mathrm{Tot}(M)),$$

as required.

□

Definition 2.94. *A spectral sequence* (E^r, d^r) *collapses on the* p*-axis if* $E^2_{p,q} = \{0\}$ *for all* $q \neq 0$*; a spectral sequence* (E^r, d^r) *collapses on the* q*-axis if* $E^2_{p,q} = \{0\}$ *for all* $p \neq 0$*.*

Proposition 2.95. *Let* M *be a first quadrant bicomplex and let* (E^r, d^r) *be the spectral sequence induced by the first or the second filtration of* $\mathrm{Tot}(M)$*.*

(i) If (E^r, d^r) *collapses on either axis, then* $E^\infty_{p,q} = E^2_{p,q}$ *for all* p, q*.*

(ii) If (E^r, d^r) *collapses on the* p*-axis, then* $H_n(\mathrm{Tot}(M)) \cong E^2_{n,0}$*; if* (E^r, d^r) *collapses on the* q*-axis, then* $H_n(\mathrm{Tot}(M)) \cong E^2_{0,n}$*.*

Proof. For the item (i) assume that (E^r, d^r) collapses on the p-axis and choose $r \geq 2$. First of all, $E^r_{p,q} = \{0\}$ for all $r \geq 2$ and $q \neq 0$, because $E^r_{p,q}$ is a subquotient of $E^2_{p,q} = \{0\}$. Now $E^{r+1}_{p,0} = \ker d^r_{p,0} / \operatorname{im} d^r_{p+r,-r+1}$. Now $d^r_{p,0} = 0$, because its target is $E^r_{p-r,r-1}$ which is off the p-axis (see figure 2.5), hence is $\{0\}$; thus, $\ker d^r_{p,0} = E^r_{p,0}$. Also, $d^r_{p+r,-r+1} = 0$, because its domain is off the axis, and so $\operatorname{im} d^r_{p+r,-r+1} = \{0\}$. Therefore, $E^{r+1}_{p,0} = E^r_{p,0}/\{0\} = E^r_{p,0}$ and Lemma 2.81 gives $E^\infty = E^2$. The proof for the case where (E^r, d^r) collapses on the q-axis is analogous.

For the item (ii), observe that since M is a first quadrant bicomplex, by Theorem 2.89 we have that the induced filtration on $H_n = H_n(\mathrm{Tot}(M))$ is

$$\{0\} = \Phi^{-1} H_n \subseteq \Phi^0 H_n \subseteq \cdots \subseteq \Phi^{n-1} H_n \subseteq \Phi^n H_n = H_n.$$

If the spectral sequence collapses on the p-axis, then $\{0\} = E^2_{p,q}$ for all $p \leq n-1$, because $1 \leq q$. By part (i) we have $E^\infty_{p,q} = E^2_{p,q}$. Now by Theorem 2.85 and since the spectral sequence collapses on the p-axis we have that $\{0\} = E^\infty_{p,q} = \Phi^p H_n / \Phi^{p-1} H_n$ for all $p \leq n-1$. Hence, $\{0\} = \Phi^{-1} H_n = \Phi^0 H_n = \cdots = \Phi^{n-1} H_n$ and $H_n = \Phi^n H_n / \Phi^{n-1} H_n \cong E^2_{n,0}$. A similar argument can be given when the spectral sequence collapses on the q-axis. □

Definition 2.96. *A* ***third quadrant bicomplex*** *(or* ***cohomology bicomplex****) is a bicomplex* $(M_{p,q})$ *for which* $M_{p,q} = \{0\}$ *whenever* p *or* q *is positive.*

As it is common, for third quadrant bicomplexes we change the signs of p, q, and n, and we switch its positions.

$$M^{p,q} = M_{-p,-q}.$$

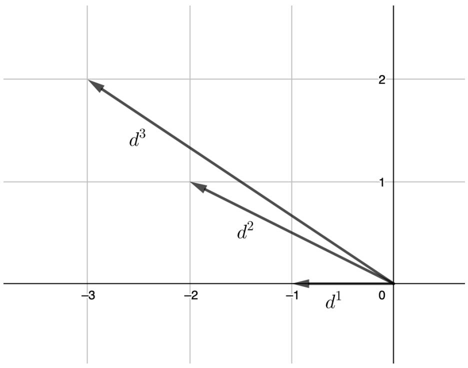


Figure 2.5: Differentials with bidegree $(-r, r-1)$.

Consider the first filtration of $\mathrm{Tot}(M)$ when M is a third quadrant bicomplex.

$$\begin{aligned}\left({}^{\mathrm{I}}F^{-p}\right)_{-n} &= \bigoplus_{i \le -p} M_{i,-n-i} \\ &= M_{-n,0} \oplus \cdots \oplus M_{-p,-n+p}\end{aligned}$$

and

$$\begin{aligned}\left({}^{\mathrm{I}}F^{-p+1}\right)_{-n} &= \bigoplus_{i \le -p+1} M_{i,-n-i} \\ &= M_{-n,0} \oplus \cdots \oplus M_{-p,-n+p} \oplus M_{-p+1,-n+p-1}.\end{aligned}$$

Thus,

$$\{0\} = F^{-n-1} \subseteq F^{-n} \subseteq F^{-n+1} \subseteq \cdots \subseteq F^0 = \mathrm{Tot}(M).$$

If we lower indices and change their sign, we have

$$\{0\} = F_{n+1} \subseteq F_n \subseteq F_{n-1} \subseteq \cdots \subseteq F_0 = \mathrm{Tot}(M),$$

that is, the filtration so labeled is a decreasing filtration. Similarly, lowered indices on the second filtration give another decreasing filtration of $\mathrm{Tot}(M)$. Now think in the induced filtration $(\Phi^p H_n)$ of $H_n(\mathrm{Tot}(M))$, if we define $\Psi_p H^n = \Phi^{-p} H_{-n}$ we obtain an induced filtration $(\Psi_p H^n)$ of $\mathrm{Tot}(M)$ where M is a third quadrant bicomplex.

$$\{0\} = \Psi_{n+1} H^n \subseteq \Psi_n H^n \subseteq \cdots \subseteq \Psi_1 H^n \subseteq \Psi_0 H^n = H^n.$$

A third quadrant spectral sequence will be denoted by (E_r, d_r), and in an analogous way to cohomology we will denote each element of the spectral sequence as $E_r^{p,q}$. Moreover, we write $H_{-n}(\mathrm{Tot}(M)) = H^n(\mathrm{Tot}(M))$, call it the nth cohomology module and, for $1 \le r \le \infty$,

$$d^r_{-p,-q} = d_r^{p,q}, \quad {}^{\mathrm{I}}E^r_{-p,-q} = {}^{\mathrm{I}}E_r^{p,q}, \text{ and } {}^{\mathrm{II}}E^r_{-p,-q} = {}^{\mathrm{II}}E_r^{p,q}.$$

We also have a version of Theorem 2.89 for third quadrant bicomplex.

Theorem 2.97. *Let M be a third quadrant bicomplex, and let ${}^{\mathrm{I}}E_r$ and ${}^{\mathrm{II}}E_r$ be the spectral sequences determined by the first and second filtrations of* $\mathrm{Tot}(M)$*. Then*

(i) The first and second filtrations are bounded.

(ii) For all p, q, we have ${}^{\mathrm{I}}E_\infty^{p,q} = {}^{\mathrm{I}}E_r^{p,q}$ and ${}^{\mathrm{II}}E_\infty^{p,q} = {}^{\mathrm{II}}E_r^{p,q}$ for large r (depending on p, q).

(iii) ${}^{\mathrm{I}}E_2^{p,q} \underset{p}{\Rightarrow} H^n(\mathrm{Tot}(M))$ and ${}^{\mathrm{II}}E_2^{p,q} \underset{p}{\Rightarrow} H^n(\mathrm{Tot}(M))$.

Proof. The bounds are $s(n) = n+1$ and $t(n) = 0$, and so statements (ii) and (iii) for ${}^{\mathrm{I}}E$ follow from Theorem 2.85 . Since Tot $(M^t) =$ Tot (M), where M^t is the transpose, and since the second filtration of $\mathrm{Tot}(M)$ equals the first filtration of M^t, we have ${}^{\mathrm{II}}E_{p,q}^\infty = {}^{\mathrm{II}}E_{p,q}^r$ for large r and ${}^{\mathrm{II}}E_{p,q}^2 \underset{p}{\Rightarrow} H_n\,(\mathrm{Tot}\,(M^t)) = H_n(\mathrm{Tot}(M))$. □

Proposition 2.98. *Let M be a third quadrant bicomplex and let (E_r, d_r) be the spectral sequence induced by the first or the second filtration of* $\mathrm{Tot}(M)$*.*

(i) If (E_r, d_r) collapses on either axis, then $E_\infty^{p,q} = E_2^{p,q}$ for all p, q.

(ii) If (E_r, d_r) collapses on the p-axis, then $H^n(\mathrm{Tot}(M)) \cong E_2^{n,0}$; if (E_r, d_r) collapses on the q-axis, then $H^n(\mathrm{Tot}(M)) \cong E_2^{0,n}$.

Proof. For part (i), assume that (E_r, d_r) collapses on the p-axis, and choose $r \geq 2$. First of all, $E_r^{p,q} = \{0\}$ for all $r \geq 2$ and $q \neq 0$, because $E_r^{p,q}$ is a subquotient of $E_2^{p,q} = \{0\}$. Now $E_{r+1}^{p,0} = \ker d_r^{p,0} / \operatorname{im} d_r^{p+r,-r+1}$. Now $d_r^{p,0} = 0$, because its target is off the axis, hence is $\{0\}$; thus, $\ker d_r^{p,0} = E_r^{p,0}$. Also, $d_r^{p+r,-r+1} = \{0\}$, because its domain is off the axis, and so $\operatorname{im} d_r^{p+r,-r+1} = \{0\}$. Therefore, $E_{r+1}^{p,0} = E_r^{p,0}/\{0\} = E_r^{p,0}$ and Lemma 2.81 gives $E_\infty = E_2$. If (E_r, d_r) collapses on the q-axis the proof is analogous.

For part (ii), we have that the induced filtration on $H^n = H^n(\mathrm{Tot}(M))$ is

$$\{0\} = \Psi_{n+1}H^n \subseteq \Psi_n H^n \subseteq \cdots \subseteq \Psi_1 H^n \subseteq \Psi_0 H^n = H^n.$$

Suppose the spectral sequence collapses on the p-axis; if $p < n$, then $0 < q$. Therefore, $\Psi_p H^n / \Psi_{p+1} H^n = E_\infty^{p,q} = E_2^{p,q} = \{0\}$ because $E_2^{p,q}$ is off the p-axis for all $p < n$. Hence, $\Psi_n H^n = \Psi_{n-1} H^n = \cdots = \Psi_1 H^n = \Psi_0 H^n = H^n$ and $H^n = \Psi_n H^n/\{0\} = \Psi_n H^n / \Psi_{n+1} H^n \cong E_2^{n,0}$. A similar argument can be given when the spectral sequence collapses on the q-axis.

□

Remark 2.99. All concepts defined in this section for the category of modules can be extended to any abelian category thanks to the **Full Imbedding Theorem** [20].

Definition 2.100. *Let* $\mathbf{C}$ *be in* $\mathrm{Ob}(\mathbf{Comp}(\mathcal{A}))$*, where* $\mathcal{A}$ *is an abelian category and* $\mathbf{Comp}(\mathcal{A})$ *denotes the category of the complexes in* $\mathcal{A}$*. A* ***Cartan-Eilenberg projective resolution*** *(or a proper projective resolution) of* $\mathbf{C}$ *is an exact sequence of complexes from* $\mathbf{Comp}(\mathcal{A})$*,*

$$\to M_{\bullet,q} \to \cdots \to M_{\bullet,1} \to M_{\bullet,0} \to 0,$$

such that the following sequences in $\mathcal{A}$ *are projective resolutions for each* p*:*

(i) $\cdots \to M_{p,1} \to M_{p,0} \to C_p \to 0;$

(ii) $\cdots \to Z_{p,1} \to Z_{p,0} \to Z_p(\mathbf{C}) \to 0;$

(iii) $\cdots \to B_{p,1} \to B_{p,0} \to B_p(\mathbf{C}) \to 0;$

(iv) $\cdots \to H_{p,1} \to H_{p,0} \to H_p(\mathbf{C}) \to 0.$

There is a dual notion of **Cartan–Eilenberg injective resolution**.

Definition 2.101. *Let* $\mathbf{C}$ *be a complex in* $\mathbf{Comp}(\mathcal{A})$*, where* $\mathcal{A}$ *is an abelian category. A* ***Cartan-Eilenberg injective resolution*** *(or a proper injective resolution) of* $\mathbf{C}$ *is an exact sequence of complexes from* $\mathbf{Comp}(\mathcal{A})$*,*

$$0 \to M_{\bullet,0} \to M_{\bullet,-1} \to \cdots \to M_{\bullet,-q} \to,$$

such that the following sequences in $\mathcal{A}$ *are injective resolutions for each* p*:*

(i) $0 \to C_p \to M_{p,0} \to M_{p,-1} \to \cdots;$

(ii) $0 \to Z_p(\mathbf{C}) \to Z_{p,0} \to Z_{p,-1} \to \cdots;$

(iii) $0 \to B_p(\mathbf{C}) \to B_{p,0} \to B_{p,-1} \to \cdots;$

(iv) $0 \to H_p(\mathbf{C}) \to H_{p,0} \to H_{p,-1} \to \cdots;$

A **Cartan–Eilenberg projective resolution** can be viewed as a large commutative diagram M in $\mathcal{A}$. For each p, the pth row $M_{p,\bullet}$ is a deleted projective resolution of C_p; for each q, the qth column $M_{\bullet,q}$ is a complex each of whose terms is projective. Also we can see **Cartan–Eilenberg projective resolution** as a bicomplex in the next way: given a Cartan–Eilenberg projective resolution M,

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & M_{n+1,3} \arrow[r, "d''_{n+1,3}"] \arrow[d, "d'_{n+1,3}"] & M_{n+1,2} \arrow[r, "d''_{n+1,2}"] \arrow[d, "d'_{n+1,2}"] & M_{n+1,2} \arrow[r, "d''_{n+1,1}"] \arrow[d, "d'_{n+1,1}"] & M_{n+1,0} \arrow[r] \arrow[d, "d'_{n+1,0}"] & 0 \\
{} \arrow[r] & M_{n,3} \arrow[r, "d''_{n,3}"] \arrow[d, "d'_{n,3}"] & M_{n,2} \arrow[r, "d''_{n,2}"] \arrow[d, "d'_{n,2}"] & M_{n,2} \arrow[r, "d''_{n,1}"] \arrow[d, "d'_{n,1}"] & M_{n,0} \arrow[r] \arrow[d, "d'_{n,0}"] & 0 \\
{} \arrow[r] & M_{n-1,3} \arrow[r, "d''_{n-1,3}"] \arrow[d] & M_{n-1,2} \arrow[r, "d''_{n-1,2}"] \arrow[d] & M_{n-1,2} \arrow[r, "d''_{n-1,1}"] \arrow[d] & M_{n-1,0} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} & {} &
\end{tikzcd}$$

Observe that d' and d'' are differentials with bidegree $(-1,0)$ and $(0,-1)$ respectively, then we can make it into a bicomplex with a **sign change**. Define $\Delta''_{p,q} = (-1)^p d''_{p,q}$. Changing sign does not affect kernels and images, and so $\Delta''\Delta'' = 0$. Finally,

$$\begin{aligned}
d'_{p,q-1}\Delta''_{p,q} + \Delta''_{p-1,q}d'_{p,q} &= (-1)^p d'_{p,q-1}d''_{p,q} + (-1)^{p-1}d''_{p-1,q}d'_{p,q} \\
&= (-1)^p(d'_{p,q-1}d''_{p,q} - d''_{p-1,q}d'_{p,q}) \\
&= 0.
\end{aligned}$$

Therefore, (M, d', Δ'') is a bicomplex. We can do the same construction in the **Cartan–Eilenberg injective resolution** case.

From [1] we have the next well known result.

Proposition 2.102. ***(Horseshoe Lemma).*** *Given a diagram in an abelian category $\mathcal{A}$ with enough projectives,*

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & \\
 & P'_1 \arrow[d] & & P''_1 \arrow[d] & \\
 & P'_0 \arrow[d, "\epsilon'"] & & P''_0 \arrow[d, "\epsilon''"] & \\
0 \arrow[r] & A' \arrow[r, "i"] & A \arrow[r, "q"] & A'' \arrow[r] & 0,
\end{tikzcd}$$

where the columns are projective resolutions and the row is exact, then there exist a projective resolution of A and chain maps so that the three columns form an exact sequence of complexes.

Remark 2.103. The dual of Horseshoe Lemma in which projective resolutions are replaced by injective resolutions is also true.

Theorem 2.104. *If $\mathcal{A}$ is an abelian category with enough projectives (or injectives), then any $\mathbf{C}$ in $\mathrm{Ob}(\mathbf{Comp}(\mathcal{A}))$ has a Cartan–Eilenberg projective (or injective) resolution.*

Proof. Let $\mathbf{C} = \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \to$ be a complex. For each $p \in \mathbb{Z}$, there are exact sequences

$$0 \to B_p \to Z_p \to H_p \to 0 \quad \text{and} \quad 0 \to Z_p \to C_p \to B_{p-1} \to 0.$$

Choose projective resolutions $B_{p,*}$ and $H_{p,*}$ of B_p and H_p, respectively; by Proposition 2.102 (Horseshoe Lemma), there is a projective resolution $Z_{p,*}$ of Z_p so that $0 \to B_{p,*} \to Z_{p,*} \to H_{p,*} \to 0$ is an exact sequence of complexes. Using the Horseshoe Lemma again, there is a projective resolution $M_{p,*}$ of C_p so that $0 \to Z_{p,*} \to M_{p,*} \to B_{p-1,*} \to 0$ is an exact sequence of complexes. Then we have the next two commutative diagrams with exact columns and rows

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ B_{p,1} & \longrightarrow & Z_{p,1} & \longrightarrow & H_{p,1} \\ \downarrow & & \downarrow & & \downarrow \\ B_{p,0} & \longrightarrow & Z_{p,0} & \longrightarrow & H_{p,0} \\ \downarrow & & \downarrow & & \downarrow \\ B_p & \longrightarrow & Z_p & \longrightarrow & H_p \end{array} \qquad \begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ Z_{p,1} & \longrightarrow & M_{p,1} & \longrightarrow & B_{p-1,1} \\ \downarrow & & \downarrow & & \downarrow \\ Z_{p,0} & \longrightarrow & M_{p,0} & \longrightarrow & B_{p-1,0} \\ \downarrow & & \downarrow & & \downarrow \\ Z_p & \longrightarrow & C_p & \longrightarrow & B_{p-1}. \end{array}$$

For each p, define chain maps $d_{p,q} : M_{p,q} \to M_{p-1,q}$ as the composition

$$d_{p,q} : M_{p,q} \to B_{p-1,q} \to Z_{p-1,q} \to M_{p-1,q}.$$

Since the above diagrams are commutative with exact columns and rows, we have that $Z_{p,q} \cong \ker d_{p,q}$, $B_{p,q} \cong \operatorname{im} d_{p+1,q}$ and $H_{p,q} \cong Z_{p,q}/B_{p,q}$. We have a commutative two-dimensional diagram whose columns are the projective resolutions $M_{p,\bullet}$ of C_p.

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & M_{2,1} \arrow[r, "d_{2,1}"] \arrow[d] & M_{1,1} \arrow[r, "d_{1,1}"] \arrow[d] & M_{0,1} \arrow[r, "d_{0,1}"] \arrow[d] & M_{-1,1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & M_{2,0} \arrow[r, "d_{2,0}"] \arrow[d] & M_{1,0} \arrow[r, "d_{1,0}"] \arrow[d] & M_{0,0} \arrow[r, "d_{0,0}"] \arrow[d] & M_{-1,0} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & C_2 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & C_{-1} \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} & {} &
\end{tikzcd}$$

We have constructed a Cartan-Eilenberg projective resolution. □

Definition 2.105. *Let $\mathcal{B}$ be an abelian category with enough projectives (or with enough injectives), and let $F : \mathcal{B} \to \mathrm{Ab}$ be an additive functor of either variance. An object B is called* ***right F-acyclic*** *if $(R^pF)B = \{0\}$ for all $p \geq 1$. An object B is called* ***left F-acyclic*** *if $(L_pF)B = \{0\}$ for all $p \geq 1$.*

Theorem 2.106. *Grothendieck third quadrant spectral sequence.*

Let $\mathcal{A} \overset{G}{\to} \mathcal{B} \overset{F}{\to} \mathcal{C}$ be covariant additive functors, where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are abelian categories with enough injectives. Assume that F is left exact and that GE is right F-acyclic for any injective object E in $\mathcal{A}$. Then for any $A \in \mathrm{Ob}(\mathcal{A})$ there is a third quadrant spectral sequence with

$$E_2^{p,q} = (R^pF)(R^qG)A \underset{p}{\Rightarrow} R^n(FG)A$$

with $n = p + q$.

Proof. For an object A in $\mathcal{A}$ choose an injective resolution $\mathbf{E} = 0 \to A \to E^0 \to E^1 \to$, and apply G to its deletion $\mathbf{E}^A$ to obtain the complex

$$G\mathbf{E}^A = 0 \to GE^0 \to GE^1 \to GE^2 \to \cdots .$$

By Theorem 2.104, there exist a Cartan–Eilenberg injective resolution of $G\mathbf{E}^A$: a third quadrant bicomplex M whose rows are complexes and whose columns are deleted injective resolutions. So the diagram of M with $G\mathbf{E}^A$ after raising indices is

$$\begin{tikzcd}
 & {} & {} & {} & {} & \\
{} \arrow[r] & M^{0,1} \arrow[r] \arrow[u] & M^{1,1} \arrow[r] \arrow[u] & M^{2,1} \arrow[r] \arrow[u] & M^{3,1} \arrow[r] \arrow[u] & {} \\
{} \arrow[r] & M^{0,0} \arrow[r] \arrow[u] & M^{1,0} \arrow[r] \arrow[u] & M^{2,0} \arrow[r] \arrow[u] & M^{3,0} \arrow[r] \arrow[u] & {} \\
{} \arrow[r] & GE^{0} \arrow[r] \arrow[u] & GE^{1} \arrow[r] \arrow[u] & GE^{2} \arrow[r] \arrow[u] & GE^{3} \arrow[r] \arrow[u] & {} \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Consider the bicomplex FM and its total complex $\mathrm{Tot}(FM)$. Let us compute its first iterated homology. For fixed p, the pth column $M^{p,*}$ is a deleted injective resolution of GE^p, and so $FM^{p,*}$ is a complex

$$0 \longrightarrow FM^{p,0} \longrightarrow FM^{p,1} \longrightarrow FM^{p,2} \longrightarrow \cdots$$

whose qth homology is $(R^qF)(GE^p)$:

$$H^q(FM^{p,*}) = (R^qF)(GE^p).$$

Now E^p is injective, so that GE^p is right F-*acyclic*; that is, $(R^qF)(GE^p) = \{0\}$ for all $q \geq 1$. Hence,

$$H^q(FM^{p,*}) = \begin{cases} (R^0F)(GE^p) & \text{if } q = 0 \\ \{0\} & \text{if } q > 0. \end{cases}$$

But F is assumed to be left exact, so that $R^0F = F$. All that survives on the p-axis,

$$0 \to FG(E^0) \to FG(E^1) \to FG(E^2) \to,$$

and this is FG applied to the deleted injective resolution $\mathbf{E}^A$. Hence, its pth homology is $R^p(FG)A$:

$${}^{\mathrm{I}}E_2^{p,q} = \begin{cases} R^p(FG)A & \text{if } q = 0 \\ \{0\} & \text{if } q > 0. \end{cases}$$

Thus, the first spectral sequence of FM collapses on the p-axis, and we have by Proposition 2.98

$$H^n(\mathrm{Tot}(FM)) \cong R^n(FG)A.$$

To compute the second iterated homology of FM we can do it in terms of the first iterated homology of FM^t. We first transpose the indices p, q in the bicomplex FM, noting that

$$H^q(FM^{*,p}) = \frac{\ker Fd^{q,p}}{\operatorname{im} Fd^{q-1,p}}.$$

Apply F to the commutative diagram in which $j : B \to Z$ and $i : Z \to M$ are inclusions, and $\delta : M \to B$ is the surjection arising from d by changing its target; note that $d = ij\delta$:

$$\begin{tikzcd}
 & & & M^{q+1,p} & \\
0 \arrow[r] & Z^{q,p} \arrow[r, "i"] & M^{q,p} \arrow[r, "\delta"] \arrow[ur, "d"] & B^{q+1,p} \arrow[r] \arrow[u, "ij"'] & 0.
\end{tikzcd}$$

We are now going to use the hypothesis that M is a Cartan–Eilenberg injective resolution. Since $Z^{q,p}$ is injective [being a term in the injective resolution of $\mathbf{Z}(GE^p)$], the exact sequence $0 \to Z^{p,q} \xrightarrow{i} M^{p,q} \xrightarrow{\delta} B^{q+1,p} \to 0$ splits. Therefore, the sequence remains exact after applying F, so that Fi is monic, $\ker F\delta = \operatorname{im} Fi$, and $F\delta$ is epic. Similarly, the exact sequence $0 \to B^{q,p} \xrightarrow{j} Z^{q,p} \to H^{q,p} \to 0$ splits, because $B^{q,p}$ is injective, so that it, too, remains exact after applying F. Hence, Fj is monic.

It is clearer to give the next argument in the category of abelian groups **Ab** (this is no loss in generality, thanks to the **Full Imbedding Theorem** [20]). We compute $\ker Fd/\operatorname{im} Fd$. Now $Fd = F(ij\delta)$. Since both Fi and Fj are injections, the numerator

$$\ker Fd = \ker F\delta = \operatorname{im} Fi = (Fi)(FZ).$$

The denominator

$$\operatorname{im} Fd = (Fd)(FM) = (Fi)[(Fj)(F\delta)(FM)] = (Fi)[(Fj)(FB)],$$

because $F\delta : FM \to FB$ is a surjection. Now use the fact that the homomorphism $Fi : FZ \to FM$ and the subgroup $(Fj)(FB) \subseteq FZ$ give a surjection

$$\frac{FZ}{(Fj)(FB)} \longrightarrow \frac{(Fi)(FZ)}{(Fi)[(Fj)(FB)]};$$

moreover, this is an isomorphism because Fi is an injection. Therefore,

$$\frac{\ker Fd}{\operatorname{im} Fd} = \frac{(Fi)(FZ)}{(Fi)[(Fj)(FB)]} \cong \frac{FZ}{(Fj)(FB)}.$$

But $FZ/(Fj)(FB) = \operatorname{coker} Fj \cong FH$, because $0 \to FB \xrightarrow{Fj} FZ \to FH \to 0$ is exact. Restoring indices, we conclude that

$$H^q(FM^{*,p}) = \frac{\ker Fd^{q,p}}{\operatorname{im} Fd^{q-1,p}} \cong FH^{q,p};$$

that is, F commutes with H^q. By hypothesis, each

$$0 \to H^q(G\mathbf{E}^A) \to H^{q,0} \to H^{q,1} \to \cdots \to H^{q,p} \to$$

is an injective resolution of $H^q(G\mathbf{E}^A)$. By definition, $H^q(G\mathbf{E}^A) = (R^qG)A$, so that the modules $H^q(M^{*,p})$ form an injective resolution of $(R^qG)A$. Hence,

$${}^{\mathrm{II}}E_2^{p,q} = H^pH^q(FM) = H^p(FH^q(M)) = (R^pF)(R^qG)A,$$

for F commutes with H^q, and so $(R^pF)(R^qG)A \underset{p}{\Rightarrow} R^n(FG)A$, because both spectral sequences have the same limit by Proposition 2.91 and 2.93 , namely, $R^n(FG)A$. $\square$

CHAPTER 3

Partial group cohomology

Now we will introduce one of our main objects of study, the right derived functor of the functor of partial representations which we will call the *partial group cohomology*.

3.1 Partial group cohomology

The main objective in this section is to relate this cohomology with the vector space of *partial derivations* and the *partial augmentation ideal*. This section corresponds to the study of the first part of [2].

Definition 3.1. *Let G be a group, V a K-vector space and*

$$\phi_V : K_{par}(G) \to End_K(V)$$

an object in $\operatorname{Rep} K_{par}(G)$. The set of ***G-invariants*** *of V is defined as:*

$$V^{G_{par}} := \left\{ v \in V : \phi_V\big([g]\big)(v) = \phi_V(e_g)(v)\ \forall g \in G \right\}.$$

It is easy to see that $V^{G_{par}}$ is a K-vector space. If $f : V \to W$ is a morphism in $\operatorname{Rep} K_{par}(G)$, then:

$$\phi_W\big([g]\big)\left(f(v)\right) = f\left(\phi_V\big([g]\big)(v)\right) = f\left(\phi_V\big(e_g\big)(v)\right) = \phi_W\big(e_g\big)\left(f(v)\right),$$

hence f induces a linear map $f^{G_{par}} : V^{G_{par}} \to W^{G_{par}}$.

Remark 3.2. By Definition 3.1, $(-)^{G_{par}}$ is a functor from $\operatorname{Rep} K_{par}(G)$ to $\operatorname{Rep} K$.

Lemma 3.3. *Let ϕ_V be an object of $\operatorname{Rep} K_{par}(G)$. If $f \in \operatorname{Hom}_{K_{par}(G)}(B, V)$ then f is uniquely defined by the element $f(1)$.*

Proof. Recall the ϕ_B structure of B given by Corollary 2.55. For $e_{g_1}e_{g_2}...e_{g_m} \in B$:

$$\begin{aligned} f\big(e_{g_1}e_{g_2}...e_{g_m}\big) &= f\big([h_1][h_2]...[h_m]1[h_m^{-1}]...[h_2^{-1}][h_1^{-1}]\big) \\ &= f\left(\phi_B\big([h_1][h_2]...[h_m]\big)(1)\right) \\ &= \phi_V\big([h_1][h_2]...[h_m]\big)\left(f(1)\right), \end{aligned}$$

where $h_1 = g_1$ and $h_i = g_{i-1}^{-1}g_i$. Then f only depends of $f(1)$. □

Proposition 3.4. $(-)^{G_{par}} : \operatorname{Rep} K_{par}(G) \to \operatorname{Rep} K$ *is a left exact functor.*

Proof. To see that $(-)^{G_{par}}$ is a left exact functor is enough to see that there exist a natural isomorphism

$$(-)^{G_{par}} \cong \operatorname{Hom}_{K_{par}(G)}(B, -).$$

Define $\eta : (-)^{G_{par}} \to \operatorname{Hom}_{K_{par}(G)}(B, -)$ such that

$$\eta_V : (V)^{G_{par}} \to \operatorname{Hom}_{K_{par}(G)}(B, V)$$

is given by $u \mapsto f_u$ with $f_u(1) = u$. For all $u \in (V)^{G_{par}}$ we have that $\eta_V(u)$ is well-defined by Lemma 3.3. Also $f_{u+v}(1) = u + v = \big(f_u + f_v\big)(1)$, then η_V is a morphism of vector spaces. Moreover $f_v = f_u \Leftrightarrow f_v(1) = f_u(1) \Leftrightarrow v = u$, hence η_V is a monomorphism.

Let $f \in \operatorname{Hom}_{K_{par}(G)}(B, V)$, then we have that:

$$\begin{aligned} \phi_V\big([g]\big)\left(f(1)\right) &= f\left(\phi_B\big([g]\big)(1)\right) \\ &= f\left([g]1[g^{-1}]\right) \\ &= f\left([g][g^{-1}]1[g][g^{-1}]\right) \\ &= f\left(\phi_B\big([g][g^{-1}]\big)(1)\right) \\ &= f\left(\phi_B\big(e_g\big)(1)\right) = \phi_V\big(e_g\big)\left(f(1)\right). \end{aligned}$$

Hence $f(1) \in V^{G_{par}}$. That means that for all $f \in \operatorname{Hom}_{K_{par}(G)}(B, V)$ there exist $u \in V^{G_{par}}$ such that $f_u = f$. Then, η_V is an isomorphism. So η is a natural isomorphism if the next diagram commutes for any morphism $f : V \to W$ in $\operatorname{Rep} K_{par}(G)$

$$\begin{tikzcd}
V^{G_{par}} \arrow[rr, "f^{G_{par}}"] \arrow[d, "\eta_V"'] & & W^{G_{par}} \arrow[d, "\eta_W"] \\
\operatorname{Hom}_{K_{par}(G)}(B,V) \arrow[rr, "\hat{f}"'] & & \operatorname{Hom}_{K_{par}(G)}(B,W)
\end{tikzcd}$$

where $\hat{f} = \operatorname{Hom}_{K_{par}(G)}(B, f)$.

Notice that it is enough to show that $\eta_W \circ f^{G_{par}}(z)\big(1\big) = \operatorname{Hom}_{K_{par}(G)}(B, f) \circ \eta_V(z)\big(1\big)$, because the morphisms in $\operatorname{Hom}_{K_{par}(G)}(B, V)$ are defined only by the image of 1. Let $z \in V$, then:

$$\begin{aligned}
\eta_W \circ f^{G_{par}}(z)\big(1\big) &= \eta_W(f(z))(1) \\
&= f(z) \\
&= f\left(\eta_V(z)\big(1\big)\right) \\
&= \operatorname{Hom}_{K_{par}(G)}(B, f)(\eta_V(z)\big(1\big)).
\end{aligned}$$

This proves that the diagram commutes and hence η is a natural isomorphism.

□

Now we are able to define the *partial cohomology groups* of a group G and a $K_{par}(G)$-module M.

Definition 3.5. *If G is a group and M an object in $\operatorname{Rep} K_{par}(G)$, then the partial cohomology groups of G with coefficients in M are defined as:*

$$H^n_{par}\big(G, M\big) = \operatorname{Ext}^n_{K_{par}(G)}\big(B, M\big),$$

that is, $H^n_{par}\big(G, M\big)$ is the right derived functor of $(-)^{G_{par}} \cong \operatorname{Hom}_{K_{par}(G)}(B, -)$.

Remark 3.6. Recall that $\operatorname{Ext}^n_{K_{par}(G)}(-, M) = R^n \operatorname{Hom}_{\{} KparG\}(-, M)$, so:

$$\begin{aligned}
H^n_{par}(G, M) &= \operatorname{Ext}^n_{K_{par}(G)}\big(B, M\big) \\
&= R^n \operatorname{Hom}_{K_{par}(G)}(B, M) = H^n\left(\operatorname{Hom}_{K_{par}(G)}(B, M)\right).
\end{aligned}$$

If (C, ϵ) is a protective resolution of B of $K_{par}(G)$-modules,

$$0 \leftarrow B \overset{\epsilon}{\leftarrow} C_0 \overset{d_1}{\leftarrow} C_1 \overset{d_2}{\leftarrow} C_2 \overset{d_3}{\leftarrow} C_3 \overset{d_4}{\leftarrow} \cdots,$$

then we have the cochain complex

$$0 \to \mathrm{Hom}_{K_{par}(G)}(B,M) \xrightarrow{\gamma_\epsilon} \mathrm{Hom}_{K_{par}(G)}(C_0,M) \xrightarrow{\gamma_{d_1}} \cdots ,$$

where $\gamma_\epsilon = \mathrm{Hom}_{K_{par}(G)}(\epsilon, M)$ and $\gamma_{d_i} = \mathrm{Hom}_{K_{par}(G)}(d_i, M)$, the functor $\mathrm{Hom}_{K_{par}(G)}(-,M)$ applied to the morphisms ϵ and d_i respectively, for $i \geq 1$. Thus

$$H^n_{par}(G,M) = \frac{\ker \mathrm{Hom}(d_{n+1},M)}{\mathrm{im}\,\mathrm{Hom}\,(d_n,M)}.$$

Let ϵ be the following morphism

$$\epsilon : K_{par}(G) \to B,$$

given by $\epsilon\left([g_1][g_2]...[g_n]\right) = [g_1][g_2]...[g_n][g_n^{-1}]...[g_2^{-1}][g_1^{-1}] = e_{g_1}e_{g_1g_2}...e_{g_1g_2...g_n}$.

Notice that ϵ is a $K_{par}(G)$-module morphism. Indeed, for all $a \in K_{par}(G)$, we have that $\epsilon(a) = \phi_B(a)(1)$.

Remark 3.7. Observe that for all $x \in \mathcal{S}(G)$ we have that:

$$\epsilon\left([g_1][g_2]...[g_n]x\right) = [g_1][g_2]...[g_n]\epsilon(x)[g_n^{-1}]...[g_2^{-1}][g_1^{-1}].$$

Lemma 3.8. *The morphism $\epsilon : K_{par}(G) \to B$ verifies the following properties.*

(a) $\epsilon(xy)x = x\epsilon(y)$ for all $x,y \in \mathcal{S}(G)$;

(b) $\epsilon(xy) = \epsilon(xy)\epsilon(x)$; for all $x,y \in \mathcal{S}(G)$.

Proof. Take $x = [g_1][g_2]...[g_r]$ and $y = [h_1][h_2]...[h_s]$, recall that B is commutative and $[g]e_h = e_{gh}[g] = e_g e_{gh}[g]$. Then we have that:

$$\begin{aligned}
x\epsilon(y) &= [g_1][g_2]...[g_r]e_{h_1}e_{h_1h_2}...e_{h_1h_2...h_s} \\
&= [g_1][g_2]...[g_{r-1}]e_{g_rh_1}[g_r]e_{h_1h_2}...e_{h_1h_2...h_s} \\
&= e_{g_1g_2...g_rh_1}[g_1][g_2]...[g_r]e_{h_1h_2}...e_{h_1h_2...h_s} \\
&= e_{g_1g_2...g_rh_1}e_{g_1g_2...g_rh_1h_2}...e_{g_1g_2...g_rh_1h_2...h_s}[g_1][g_2]...[g_r] \\
&= e_{g_1g_2...g_rh_1}e_{g_1g_2...g_rh_1h_2}...e_{g_1g_2...g_rh_1h_2...h_s}e_{g_1}[g_1]e_{g_2}[g_2]...e_{g_r}[g_r] \\
&= e_{g_1}e_{g_1g_2}...e_{g_1g_2...g_r}e_{g_1g_2...g_rh_1}e_{g_1g_2...g_rh_1h_2}...e_{g_1g_2...g_rh_1h_2...h_s}[g_1][g_2]...[g_r] \\
&= \epsilon(xy)x,
\end{aligned}$$

and

$$\begin{aligned}\epsilon(xy)\epsilon(x) &= e_{g_1}e_{g_1g_2}\ldots e_{g_1g_2\ldots g_r}e_{g_1g_2\ldots g_rh_1}e_{g_1g_2\ldots g_rh_1h_2}\ldots e_{g_1g_2\ldots g_rh_1h_2\ldots h_s}e_{g_1}e_{g_1g_2}\ldots e_{g_1g_2\ldots g_r}\\ &= e_{g_1}e_{g_1g_2}\ldots e_{g_1g_2\ldots g_r}e_{g_1g_2\ldots g_rh_1}e_{g_1g_2\ldots g_rh_1h_2}\ldots e_{g_1g_2\ldots g_rh_1h_2\ldots h_s}\\ &= \epsilon(xy).\end{aligned}$$

□

Definition 3.9. *The set $IG = \ker\epsilon$ is called the* ***partial augmentation ideal****.*

Remark 3.10. Since ϵ is a $K_{par}(G)$-module morphism then $IG = \ker\epsilon$ is a left ideal of $K_{par}(G)$, hence is a left $K_{par}(G)$-module.

Definition 3.11. *Let G be a group and M a left K-module. Define the* ***vector space of partial derivations*** *as follows:*

$$\mathrm{Der}_{par}(G,M) = \{\delta \in \mathrm{Hom}_K(K_{par}(G),M) : \delta(ab) = a\delta(b) + \epsilon(ab)\delta(a)\ \forall a,b \in \mathcal{S}(G)\}.$$

In particular, we say that $\delta \in \mathrm{Der}_{par}(G,M)$ is ***inner*** *if $\delta\big([g]\big) = [g]m - e_g m$ for some $m \in M$. We denote by $\mathrm{Int}_{par}(G,M)$ the space of inner partial derivations.*

Remark 3.12. Notice that if w is an idempotent of $\mathcal{S}(G)$ then

$$\begin{aligned}\delta(w) = \delta(ww) &= w\delta(w) + \epsilon(w)\delta(w)\\ &= w\delta(w) + w\delta(w)\\ &= 2w\delta(w).\end{aligned}$$

Thus $\delta(w) = 2w\delta(w) = 4w\delta(w) = 2\delta(w)$, hence $\delta(w) = 0$.

In order to prove some relation between the groups $H^n_{par}(G,M)$ and $\mathrm{Der}_{par}(G,M)$ we start with the following exact sequence in $\mathrm{Rep}\,K_{par}(G)$

$$0 \to IG \xrightarrow{i} K_{par}(G) \xrightarrow{\epsilon} B \to 0.$$

Observe that $\epsilon\big(\epsilon(x)\big) = \epsilon(x)$ for all $x \in K_{par}(G)$. Then $x - \epsilon(x) \in IG$ for all $x \in K_{par}(G)$.

Proposition 3.13. *There is a natural isomorphism*

$$\mathrm{Hom}_{K_{par}(G)}(IG,-) \cong \mathrm{Der}_{par}(G,-).$$

Proof. Define
$$\eta : \mathrm{Hom}_{K_{par}(G)}(IG, -) \to \mathrm{Der}_{par}(G, -),$$
given by
$$\begin{aligned}\eta_M\colon\ \mathrm{Hom}_{K_{par}(G)}(IG, M) &\to \mathrm{Der}_{par}(G, M)\\ f\quad &\mapsto \widehat{f}\end{aligned}$$
such that
$$\begin{aligned}\widehat{f}\colon K_{par}(G) &\to M\\ x\ &\mapsto f(x - \epsilon(x)).\end{aligned}$$
Notice that $\widehat{f}$ is a partial derivation. Indeed, for $x, y \in \mathcal{S}(G)$ by Lemma 3.8 we have that:
$$\begin{aligned}\widehat{f}(xy) = f(xy - \epsilon(xy)) &= f(xy - x\epsilon(y) + \epsilon(xy)x - \epsilon(xy)\epsilon(x))\\ &= xf(y - \epsilon(y)) + \epsilon(xy)f(x - \epsilon(x))\\ &= x\widehat{f}(y) + \epsilon(xy)\widehat{f}(x).\end{aligned}$$
Then $\widehat{f} \in \mathrm{Der}_{par}(G, M)$.

Now we want to check that for any $\alpha : M \to M'$ morphism of $K_{par}(G)$-modules we have that:
$$\eta_{M'} \circ \mathrm{Hom}_{K_{par}(G)}(IG, \alpha) = \mathrm{Der}_{par}(G, \alpha) \circ \eta_M.$$
Let $f \in \mathrm{Hom}_{K_{par}(G)}(IG, M)$ then
$$\begin{aligned}\eta_{M'} \circ \mathrm{Hom}_{K_{par}(G)}(IG, \alpha)(f) &= \eta_{M'}(\alpha \circ f)\\ &= \widehat{\alpha \circ f} : x \in K_{par}(G) \mapsto \alpha f\big(x - \epsilon(x)\big) \in M'\end{aligned}$$
and
$$\mathrm{Der}_{par}(G, \alpha) \circ \eta_M(f) = \alpha \circ \widehat{f} : x \in K_{par}(G) \mapsto \alpha f\big(x - \epsilon(x)\big) \in M'.$$
Then η is a natural transformation.

Finally observe that η_M is an isomorphism for all M. Let $f \in \mathrm{Hom}_{K_{par}(G)}(IG, M)$ such that $\eta_M(f) = \widehat{f} = 0$, thus
$$\begin{aligned}\widehat{f}(x) = 0\ \forall x \in K_{par}(G) &\Leftrightarrow f(x - \epsilon(x)) = 0\ \forall x \in K_{par}(G)\\ &\Rightarrow 0 = f(x - \epsilon(x)) = f(x)\ \forall x \in IG\\ &\Rightarrow f(x) = 0\ \forall x \in IG\\ &\Rightarrow f = 0.\end{aligned}$$

Hence η_M is a monomorphism.

Let $\delta \in \mathrm{Der}_{par}(G, M)$, define

$$\begin{aligned}\delta' \colon IG &\to M\\ x &\mapsto \delta(x).\end{aligned}$$

Then $\delta' \in \mathrm{Hom}_{K_{par}(G)}(IG, M)$. Indeed, let $x \in IG$ and $a \in K_{par}(G)$, thus:

$$x = \sum_{j \in J} x_j \text{ and } a = \sum_{i \in I} a_i,$$

where $x_j, a_i \in K_{par}(G)$ and recall that IG is a left ideal of $K_{par}(G)$. Then:

$$\begin{aligned}\delta'(ax) &= \sum_{i,j} \delta(a_i x_j) = \sum_{i,j} a_i \delta(x_j) + \epsilon(a_i x_j)\delta(a_i)\\ &= a\delta'(x) + \sum_{i \in I} \epsilon(a_i x)\delta(a_i)\\ &= a\delta'(x).\end{aligned}$$

Hence $\delta' \in \mathrm{Hom}_{K_{par}(G)}(IG, M)$ and by Remark 3.12,

$$\eta_M(\delta')(x) = \delta'\big(x - \epsilon(x)\big) = \delta\big(x - \epsilon(x)\big) = \delta(x) - \delta\big(\epsilon(x)\big) = \delta(x),$$

for all $x \in K_{par}(G)$. Thus $\eta_M(\delta') = \delta$ hence $\mathrm{Hom}_{K_{par}(G)}(IG, M) \cong \mathrm{Der}_{par}(G, M)$. □

Lemma 3.14. *If $\delta \in \mathrm{Int}_{par}(G, M)$, then $\delta(x) = (x - \epsilon(x)) \cdot m$ for all $x \in K_{par}(G)$, for some $m \in M$.*

Proof. Recall that for $g, h \in G$, $\delta\big([h][g]\big) = [h]\delta\big([g]\big) + \epsilon\big([h][g]\big)\delta\big([h]\big)$. Then

$$\begin{aligned}\delta\big([h][g]\big) &= [h]\big([g] - e_g\big) \cdot m + e_h e_{hg}\big([h] - e_h\big) \cdot m\\ &= ([h][g] - [h]e_g + e_h e_{hg}[h] - e_h e_{hg} e_h) \cdot m\\ &= ([h][g] - [h]e_g + e_h[h]e_g - e_h e_{hg}) \cdot m\\ &= \big([h][g] - [h]e_g + [h]e_g - \epsilon\big([h][g]\big)\big) \cdot m\\ &= \big([h][g] - \epsilon\big([h][g]\big)\big) \cdot m.\end{aligned}$$

Then take $x_n = [g_1][g_2]...[g_n]$, by induction over n, if our lemma is valid for each x_k with $k \leq n$. Then

$$\begin{aligned}\delta(x_n[h]) &= x_n\delta\big([h]\big) + \epsilon\big(x_n[h]\big)\delta\big(x_n\big)\\ &= x_n\big([h]-e_h\big)\cdot m + \epsilon\big(x_n[h]\big)\big(x_n - \epsilon(x_n)\big)\cdot m\\ &= \big(x_n[h] - x_ne_h + \epsilon\big(x_n[h]\big)x_n - \epsilon\big(x_n[h]\big)\epsilon\big(x_n\big)\big)\cdot m\\ &= \big(x_n[h] - x_ne_h + x_n\epsilon\big([h]\big) - \epsilon\big(x_n[h]\big)\big)\\ &= \big(x_n[h] - \epsilon\big(x_n[h]\big)\big)\cdot m.\end{aligned}$$

Hence $\delta(\lambda) = \big(\lambda - \epsilon\big(\lambda\big)\cdot m\big)$ for all $\lambda \in \mathcal{S}(G)$. As $\mathcal{S}(G)$ is a basis of $K_{par}(G)$ as an algebra, then $\delta(x) = (x - \epsilon(x))\cdot m$ for all $x \in K_{par}(G)$. $\square$

The following theorem is one of the main results of this work, it gives a characterization of partial cohomology groups via the vector space of partial invariants, the vector space of partial derivations and the partial augmentation ideal.

Theorem 3.15. *Let G be a group and M an object in* $\operatorname{Rep} K_{par}(G)$. *Then*

$$\begin{aligned}&H^0_{par}(G,M) = M^{G_{par}} = \operatorname{Hom}_{K_{par}(G)}(B,M);\\ &H^1_{par}(G,M) = \operatorname{Der}_{par}(G,M)\big/\operatorname{Int}_{par}(G,M);\\ &H^n_{par}(G,M) = \operatorname{Ext}^{n-1}_{K_{par}(G)}(IG,M),\ n\geq 2.\end{aligned}$$

Proof. Associated to the short exact sequence

$$0 \to IG \xrightarrow{i} K_{par}(G) \xrightarrow{\epsilon} B \to 0$$

there is a long exact sequence

$$\begin{aligned}0 &\longrightarrow \operatorname{Hom}_{K_{par}(G)}(B,M) \xrightarrow{\epsilon^*} \operatorname{Hom}_{K_{par}(G)}(K_{par}(G),M) \xrightarrow{i^*} \operatorname{Hom}_{K_{par}(G)}(IG,M)\\ &\longrightarrow \operatorname{Ext}^1_{K_{par}(G)}(B,M) \longrightarrow \operatorname{Ext}^1_{K_{par}(G)}(K_{par}(G),M) \longrightarrow \operatorname{Ext}^1_{K_{par}(G)}(IG,M)\\ &\longrightarrow \operatorname{Ext}^2_{K_{par}(G)}(B,M) \longrightarrow \operatorname{Ext}^2_{K_{par}(G)}(K_{par}(G),M) \longrightarrow ...\end{aligned}$$

where $i^* = \operatorname{Hom}_{K_{par}(G)}(i,M)$ and $\epsilon^* = \operatorname{Hom}_{K_{par}(G)}(\epsilon,M)$. Then we have that:

$$H^0_{par}(G,M) = \operatorname{Ext}^0_{K_{par}(G)}(B,M) \cong \operatorname{Hom}_{K_{par}(G)}(B,M) \cong M^{G_{par}}.$$

Besides, since $K_{par}(G)$ is projective, we have that $\operatorname{Ext}^n_{K_{par}(G)}(K_{par}(G),M) = 0$ for any $n \in \mathbb{N}$, then for $n \geq 2$ we have the next exact sequence:

$$0 \to \operatorname{Ext}^{n-1}_{K_{par}(G)}(IG,M) \to \operatorname{Ext}^n_{K_{par}(G)}(B,M) \to 0.$$

Hence $H^n_{par}(G,M) = \operatorname{Ext}^{n-1}_{K_{par}(G)}(IG,M), \ \forall n \geq 2.$

We also have the next exact sequence:

$$\begin{aligned} 0 \to \operatorname{Hom}_{K_{par}(G)}(B,M) &\xrightarrow{\epsilon^*} \operatorname{Hom}_{K_{par}(G)}(K_{par}(G),M) \\ &\xrightarrow{i^*} \operatorname{Hom}_{K_{par}(G)}(IG,M) \longrightarrow \operatorname{Ext}^1_{K_{par}(G)}(B,M) \to 0. \end{aligned}$$

Then $H^1_{par}(G,M) = \operatorname{Ext}^1_{K_{par}(G)}(B,M) \cong \operatorname{Hom}_{K_{par}(G)}(IG,M)\big/ \operatorname{im} i^*$. Using the isomorphism η_M defined in Proposition 3.13 we have that

$$\operatorname{Hom}_{K_{par}(G)}(IG,M) \cong \operatorname{Der}_{par}(G,M),$$

thus if we check that $\eta_M\big(\operatorname{im} i^*\big) = \operatorname{Int}_{par}(G,M)$ then:

$$H^1_{par}(G,M) = \operatorname{Der}_{par}(G,M)\big/ \operatorname{Int}_{par}(G,M).$$

Indeed, let $f' \in \operatorname{im} i^*$, then $f' = f \circ i$ for some $f \in \operatorname{Hom}_{K_{par}(G)}(K_{par}(G),M)$, hence we have that $\eta_M(f') = \widehat{f \circ i}$. Thus for all $x \in \mathcal{S}(G)$

$$\eta_M\big(f'\big)(x) = f \circ i(x - \epsilon(x)) = f(x - \epsilon(x)) = (x - \epsilon(x))f(1).$$

Hence $\eta_M\big(\operatorname{im} i^*\big) \subseteq \operatorname{Int}_{par}(G,M)$.

Finally let $\delta \in \operatorname{Int}_{par}(G,M)$, then by Lemma 3.14 $\delta(x) = \big(x - \epsilon(x)\big) \cdot m$ for all $x \in K_{par}(G)$ and some $m \in M$, then take $f \in Hom_{K_{par}(G)}(K_{par}(G),M)$ defined by $f(1) = m$. Thus

$$\begin{aligned} \widehat{f \circ i}(x) &= f \circ i(x - \epsilon(x)) = f(x - \epsilon(x)) \\ &= (x - \epsilon(x)) \cdot m = \delta(x). \end{aligned}$$

Hence $\eta_M(f \circ i) = \delta$, then $\eta_M\big(\operatorname{im} i^*\big) = \operatorname{Int}_{par}(G,M)$.

$\square$

3.2 The 1-st cohomology group

In this section we are going to use Theorem 3.15 to give another characterization of the *1-st cohomology group* $H^1_{par}(G, M)$ using maps $d : G \to M$ satisfying a certain property. This section correspond to the study of [5].

Recall that B is the subalgebra of $K_{par}(G)$ generated by the set of the idempotent elements of $\mathcal{S}(G)$ and the action of $K_{par}(G)$ on B is given by conjugation

$$s \cdot e = ses^{-1}.$$

First we will show some small results in order to have the necessary tools to reach our goal.

Lemma 3.16. *For arbitrary* $\delta \in \mathrm{Der}_{par}(G, M)$ *and* $s \in \mathcal{S}(G)$*,* e *an idempotent element of* $\mathcal{S}(G)$ *one has*

(i) $\delta(es) = e \cdot \delta(s)$,

(ii) $\delta(se) = ses^{-1} \cdot \delta(s)$.

Proof. By Remark 3.12 we have $\delta(e) = 0$ for any idempotent $e \in \mathcal{S}(G)$, then

(i) $\delta(es) = e\delta(s) + \epsilon(es)\delta(e) = e\delta(s)$,

(ii) $\delta(se) = s\delta(e) + \epsilon(se)\delta(s) = s\epsilon(e)s^{-1}\delta(s) = ses^{-1}\delta(s)$.

□

We will use the isomorphism

$$H^1_{par}(G, M) \cong \mathrm{Der}_{par}(G, M) \big/ \mathrm{Int}_{par}(G, M)$$

given in Theorem 3.15 to obtain another interpretation of $H^1_{par}(G, M)$ in terms of certain maps $f \colon G \to M$. We introduce those maps in the following lemma.

Lemma 3.17. *Let* M *be a* $K_{par}(G)$*-module and* $d : G \to M$ *a map such that for all* $g, h \in G$

$$e_g \cdot d(gh) = [g] \cdot d(h) + e_{gh} \cdot d(g).$$

Then the K*-linear map* $\delta : K_{par}(G) \to M$ *defined by*

$$\delta(e[g]) = e \cdot d(g),$$

where e *is an idempotent of* $\mathcal{S}(G)$*, is a partial derivation.*

Proof. First of all, we show that δ is well-defined. Since

$$e_{1_G} \cdot d(1_G) = [1_G] \cdot d(1_G) + e_{1_G} \cdot d(1_G),$$

then $d(1_G) = 0$. Hence

$$0 = e_g \cdot d(gg^{-1}) = [g] \cdot d(g^{-1}) + d(g)$$

thus $d(g) = -[g]d(g^{-1})$. Therefore

$$d(g) = -[g]d(g^{-1}) = -[g](-[g^{-1}]d(g)) = e_g d(g).$$

By Remark 2.21 we have that $e[g] = f[h]$ in $\mathcal{S}(G)$ if, and only if, $g = h$ and $e_g e = e_g f$, where e and f are idempotents of $\mathcal{S}(G)$. Then

$$\delta(e[g]) = e \cdot d(g) = ee_g \cdot d(g) = fe_g \cdot d(g) = f \cdot d(g) = \delta(f[g]) = \delta(f[h]).$$

Now given two arbitrary elements $e[g]$ and $f[h]$ of $\mathcal{S}(G)$ their product is

$$\begin{aligned} e[g]f[h] &= e[g]e_{g^{-1}}f[h] \\ &= e[g]f[g^{-1}][g][h] \\ &= e[g]f[g^{-1}][gh]. \end{aligned}$$

Recall that $e[g]f[g^{-1}]$ is an idempotent, then

$$\delta(e[g]f[h]) = \delta(e[g]f[g^{-1}][gh]) = e[g]f[g^{-1}] \cdot d(gh).$$

On the other hand we have that

$$\begin{aligned} e[g] \cdot \delta(f[h]) + &\epsilon(e[g]f[h]) \cdot \delta(e[g]) \\ &= e[g]f \cdot d(h) + \epsilon(e[g]f[h])e \cdot d(g) \\ &= e[g]fe_{g^{-1}} \cdot d(h) + e[g]fe_h f[g^{-1}]e \cdot d(g) \\ &= e[g]f[g^{-1}][g] \cdot d(h) + e[g]f[g^{-1}]e_{gh} \cdot d(g) \\ &= e[g]f[g^{-1}]\left([g] \cdot d(h) + e_{gh} \cdot d(g)\right) \\ &= e[g]f[g^{-1}]\left(e_g \cdot d(gh)\right) \\ &= e[g]f[g^{-1}] \cdot d(gh) = \delta(e[g]f[h]). \end{aligned}$$

Thus δ is a partial derivation.

□

Definition 3.18. *Let us denote by* $\mathrm{D}(G, M)$ *the* K*-vector space of maps* $d : G \to M$ *which satisfy*

$$e_g \cdot d(gh) = [g] \cdot d(h) + e_{gh} \cdot d(g).$$

Proposition 3.19. *There is a bijective correspondence between the partial derivations of $K_{par}(G)$ with values in M and the elements of* $\mathrm{D}(G,M)$.

Proof. Let $\delta \in \mathrm{Der}_{par}(G,M)$, then we have that

$$\begin{aligned} e_g \cdot \delta([gh]) = \delta(e_g[gh]) = \delta([g][h]) &= [g]\cdot\delta([h]) + \epsilon([g][h])\cdot\delta([g]) \\ &= [g]\cdot\delta([h]) + e_g e_{gh}\cdot\delta([g]) \\ &= [g]\cdot\delta([h]) + e_{gh}\cdot\delta(e_g[g]) \\ &= [g]\cdot\delta([h]) + e_{gh}\cdot\delta([g]). \end{aligned}$$

Hence if we define $d : G \to M$ by $d(g) = \delta([g])$ then $d \in \mathrm{D}(G,M)$. Conversely, if $d \in \mathrm{D}(G,M)$, then δ given by $\delta(e[g]) = e \cdot d(g)$ is in $\mathrm{Der}_{par}(G,M)$ by Lemma 3.17. Now observe that if $d \mapsto \delta \mapsto d'$ then

$$d'(g) = \delta([g]) = \delta(1_{\mathcal{S}(G)}[g]) = 1_{\mathcal{S}(G)}\cdot d(g) = d(g),$$

and if $\delta \mapsto d \mapsto \delta'$

$$\delta'(e[g]) = e\cdot d(g) = e\cdot\delta([g]) = \delta(e[g]).$$

$\square$

Definition 3.20. *Define the set*

$$\mathrm{PD}(G,M) := \{d : G \to M \mid \exists m \in M\ \forall g \in G : d(g) = [g]\cdot m - e_g \cdot m\}$$

Remark 3.21. Observe that $\mathrm{PD}(G,M)$ is a K-subspace of $\mathrm{D}(G,M)$. To see that it is enough to notice that given $d \in \mathrm{PD}(G,M)$ we have

$$\begin{aligned} [g]\cdot d(h) + e_{gh}\cdot d(g) &= [g]\cdot([h]\cdot m - e_h\cdot m) + e_{gh}\cdot([g]\cdot m - e_g\cdot m) \\ &= [g][h]\cdot m - [g]e_h\cdot m + e_{gh}[g]\cdot m - e_{gh}e_g\cdot m \\ &= e_g[gh]\cdot m - e_{gh}[g]\cdot m + e_{gh}[g]\cdot m - e_g e_{gh}\cdot m \\ &= e_g\cdot([gh]\cdot m - e_{gh}\cdot m) = e_g\cdot d(gh). \end{aligned}$$

Then $\mathrm{PD}(G,M) \subseteq \mathrm{D}(G,M)$.

Theorem 3.22. *Let M be a $K_{par}(G)$-module. Then $H^1_{par}(G,M)$ is isomorphic to the quotient of the additive group* $\mathrm{D}(G,M)$ *modulo the subgroup* $\mathrm{PD}(G,M)$.

Proof. We know by Theorem 3.15 that $H^1_{par}(G,M) \cong \mathrm{Der}_{par}(G,M)/\,\mathrm{Int}_{par}(G,M)$ and by Proposition 3.19 we have that $\mathrm{Der}_{par}(G,M) \cong \mathrm{D}(G,M)$ via

$$\varphi : \mathrm{Der}_{par}(G,M) \to D(G,M),$$

given by $d(g) = \varphi(\delta)(g) = \delta([g])$.

Then is enough show that $\varphi(\mathrm{Int}_{par}(G, M)) = \mathrm{PD}(G, M)$. Let $\delta \in \mathrm{Int}_{par}(G, M)$, then $\delta([g]) = [g] \cdot m - e_g \cdot m$ and $d = \varphi(\delta)$ satisfies $d(g) = [g] \cdot m - e_g \cdot m$ for all g in G, thus $\varphi(\delta) \in \mathrm{PD}(G, M)$. On the other hand, given $d \in \mathrm{PD}(G, M)$, then $d(g) = [g] \cdot m - e_g \cdot m$. Now define δ by $\delta(e[g]) = e \cdot d(g)$, thus the map δ is in $\mathrm{Int}_{par}(G, M)$ since

$$\delta([g]) = \delta(1_{\mathcal{S}(G)}[g]) = 1_{\mathcal{S}(G)} \cdot d(g) = [g] \cdot m - e_g \cdot m.$$

Hence

$$\varphi(\delta)(g) = \delta([g]) = \delta(1_{\mathcal{S}(G)}[g]) = 1_{\mathcal{S}(G)} \cdot d(g) = d(g).$$

Thus $\varphi(\delta) = d$. Therefore $\mathrm{PD}(G, M) \cong \mathrm{Int}_{par}(G, M)$ via φ.

□

Corollary 3.23. *Let $(\mathcal{A}, \alpha)$ be a unital partial G-module. Consider the corresponding $K_{par}(G)$-module structure on $\mathcal{A}$ given by Lemma 2.33. Then $H^1_{par}(G, \mathcal{A})$ is isomorphic to the quotient of the additive group of functions*

$$\{f : G \to \mathcal{A} \mid \forall g \in G : 1_g f(gh) = \alpha_g(1_{g^{-1}} f(h)) + 1_{gh} f(g)\}$$

by the subgroup

$$\{f : G \to \mathcal{A} \mid \exists a \in \mathcal{A}\ \forall g \in G : f(g) = \alpha_g(1_{g^{-1}} a) - 1_g a\}.$$

3.3 A projective resolution of B

Theorem 3.15 gives us a characterization of $H^n_{par}(G, M)$ without explicitly showing an exact resolution of B. We will construct an exact resolution of B that will allow us to calculate the partial cohomology groups. As we did in Theorem 3.22 we will use the classes of certain maps $f : G \to M$ to characterize the elements of $H^n_{par}(G, M)$.

Lemma 3.24. *Let R be a unital ring and $\{e_i\}_{i \in I} \subseteq E(R)$ a set of idempotents of R. Then the left R-module $\bigoplus_{i \in I} Re_i$ is projective.*

Proof. Indeed, each Re_i is a projective left R-module, since $Re_i \oplus R(1_R - e_i)$ is isomorphic to R, a free module of rank 1. Finally, a direct sum of projective modules is projective (see [15, Proposition 3.10]). □

Notation. Denote by $e_{(g_1,\dots,g_n)}$ the product of idempotent elements $e_{g_1} e_{g_1 g_2} \dots e_{g_1 g_2 \dots g_n} \in E(\mathcal{S}(G))$.

Remark 3.25. Notice that

$$e_{(g_1,\ldots,g_n)} = e_{g_1}e_{g_1g_2}\ldots e_{g_1g_2\ldots g_n} = e_{g_1}(e_{g_1g_2}\ldots e_{g_1g_2\ldots g_n}) = e_{g_1}e_{(g_1g_2,g_3,\ldots,g_n)}.$$

More generally

$$\begin{aligned} e_{(g_1,\ldots,g_n)} &= e_{g_1}e_{g_1g_2}\ldots e_{g_1g_2\ldots g_i}e_{g_1g_2\ldots g_ig_{i+1}}\ldots e_{g_1g_2\ldots g_n} \\ &= e_{g_1g_2\ldots g_i}(e_{g_1}e_{g_1g_2}\ldots e_{g_1g_2\ldots g_ig_{i+1}}\ldots e_{g_1g_2\ldots g_n}) \\ &= e_{g_1g_2\ldots g_i}e_{(g_1,\ldots,g_ig_{i+1},\ldots,g_n)}. \end{aligned}$$

Now we will define the family of projective $K_{par}(G)$-modules $\{P_i\}_{i\in\mathbb{N}\cup\{0\}}$, which are the modules that will constitute the projective resolution of B.

Definition 3.26. *Define*

$$\begin{aligned} P_0 &= K_{par}(G) \\ P_n &= \bigoplus_{g_1,g_2,\ldots,g_n\in G} K_{par}(G)\cdot e_{(g_1,g_2\ldots,g_n)},\ n\in\mathbb{N}. \end{aligned}$$

By Lemma 3.24 each P_n is a projective $K_{par}(G)$-module.

Given $\omega \in P_n$ we have that $\omega(g_1, g_2, \ldots, g_n)$ is generated by the elements $se_{(g_1,g_2,\ldots g_n)}$ with $s \in \mathcal{S}(G)$, with that idea in mind it would be convenient to us to have an equivalent description of the modules P_n such that each generator $s = se_{(g_1,g_2,\ldots g_n)}$ is identified with an element $s(g_1, ..., g_n)$ that satisfies certain conditions.

Remark 3.27. For each $n \in \mathbb{N}$ the module P_n is isomorphic, as a K-vector space to the vector space over K with basis

$$R_n = \{s(g_1, ..., g_n) \mid s \in \mathcal{S}(G),\ g_1, ..., g_n \in G, s^{-1}s \le e_{(g_1,...,g_n)}\}$$

where

$$s(g_1, ..., g_n) = t(h_1, ..., h_n) \Leftrightarrow \begin{cases} (g_1, ..., g_n) = (h_1, ..., h_n), \\ s = t. \end{cases}$$

Proof. We understand the element $se_{(g_1,...,g_n)} \in P_n$ such that it is specifically in the coordinate $(g_1, \ldots, g_n)$ of P_n. The set $\{se_{(g_1,...,g_n)} \mid s \in \mathcal{S}(G)\}$ of elements of P_n form a basis of the K-vector space P_n. Now define the map $\psi : R_n \to P_n$ given by

$$\psi(s(g_1, \ldots, g_n)) = se_{(g_1,...,g_n)}.$$

It is clear that ψ is an epimorphism, since for all $s \in \mathcal{S}(G)$ we have that

$$s^{-1}se_{(g_1,\dots,g_n)} \leq e_{(g_1,\dots,g_n)},$$

thus $se_{(g_1,\dots,g_n)}(g_1,\dots,g_n) \in R_n$, hence $\psi(se_{(g_1,\dots,g_n)}(g_1,\dots,g_n)) = se_{(g_1,\dots,g_n)}$. Besides if $\psi(t(g_1,...,g_n)) = \psi(s(g_1,...,g_n))$, then $se_{(g_1,\dots,g_n)} = te_{(g_1,\dots,g_n)}$. Therefore

$$t(g_1,...,g_n) = s(g_1,...,g_n).$$

Thus ψ is an isomorphism. $\square$

Let us identify each element $t \in K_{par}(G) \cdot e_{(g_1,\dots,g_n)} \subseteq P_n$ with $t(g_1,\dots,g_n)$. We extend the characterization of P_n from Remark 3.3 to $n = 0$ by identifying P_0 with the K-vector space with basis

$$\{s(\ ) \mid s \in \mathcal{S}(G)\}.$$

Notice that for all $t \in \mathcal{S}(G)$ and $s(g_1,...,g_n)$ we have

$$t \cdot s(g_1,...,g_n) = tse_{(g_1,\dots,g_n)} = ts(g_1,...,g_n).$$

Now observe

$$(ts)^{-1}ts = s^{-1}t^{-1}ts \leq s^{-1}s \leq e_{(g_1,\dots,g_n)}.$$

Therefore $tse_{(g_1,\dots,g_n)} = ts(g_1,...,g_n)$ is well-defined.

In order to show that the projective modules P_n form a projective resolution of B we have to define the morphisms $P_n \to P_{n-1}$. But first we need the following definition.

Definition 3.28. *Define the K-linear maps $\partial_n : P_0 \to B$, and $\partial_0 : P_n \to P_{n-1}$, $n \in \mathbb{N}$, as follows*

$$\begin{aligned}
\partial_0(s(\ )) &= ss^{-1} \\
\partial_1(s(g)) &= s([g](\ ) - (\ )) \\
\partial_n\left(s\left(g_1,\dots,g_n\right)\right) &= s\Big([g_1]\left(g_2,\dots,g_n\right) \\
&\quad + \sum_{i=1}^{n-1}(-1)^i \left(g_1,\dots,g_ig_{i+1},\dots,g_n\right) \\
&\quad + (-1)^n \left(g_1,\dots,g_{n-1}\right) \Big).
\end{aligned}$$

Each ∂_n is a $K_{par}(G)$-module morphism. Since the $K_{par}(G)$-module structure on B is given by the conjugation, we have that for ∂_0

$$\partial_0(ts(\,)) = ts(ts)^{-1} = tss^{-1}t^{-1} = t \cdot ss^{-1} = t \cdot \partial_0(s(\,)).$$

Now, for $n \geq 0$ observe

$$\begin{aligned}\partial_n(ts(g_1, ..., g_n)) &= ts\left([g_1](g_2, ..., g_n) + \ldots + (-1)^n(g_1, ..., g_{n-1})\right)\\ &= t \cdot \partial_n(s(g_1, ..., g_n)).\end{aligned}$$

Observe that if $s(g_1, ..., g_n) \in P_n$ then $s[g_1](g_2, ..., g_n) \in P_{n-1}$. Indeed, since $s(g_1, ..., g_n) \in P_n$ we have $s^{-1}s \leq e_{(g_1,...,g_n)}$, thus $s = se_{(g_1,...,g_n)}$, then

$$\begin{aligned}s[g_1] &= se_{(g_1,...,g_n)}[g_1]\\ &= se_{g_1}e_{g_1g_2}...e_{g_1g_2...g_n}[g_1]\\ &= se_{g_1}[g_1]e_{g_2}...e_{g_2...g_n}\\ &= s[g_1]e_{(g_2,...,g_n)},\end{aligned}$$

thus $(s[g_1])^{-1}s[g_1] \leq e_{(g_2,...,g_n)}$ and $s[g_1](g_2, ..., g_n) \in P_{n-1}$.

Let us denote B by P_{-1} and consider the morphism $\eta : \mathcal{S}(G) \to G$ from Remark 2.21.

Definition 3.29. *Define K-linear maps $\sigma_n : P_n \to P_{n+1}$, $n \in \mathbb{N} \cup \{-1, 0\}$, as follows*

$$\begin{aligned}\sigma_{-1}(e) &= e(\,),\\ \sigma_0(s(\,)) &= ss^{-1}(\eta(s)),\\ \sigma_n(s(g_1, g_2..., g_n)) &= ss^{-1}(\eta(s), g_1, g_2, ..., g_n),\ n \in \mathbb{N}.\end{aligned}$$

By Lemma 2.24 we have $s \leq [\eta(s)]$ and $s^{-1} \leq [\eta(s)^{-1}]$, then $ss^{-1} \leq e_{\eta(s)}$ for all $s \in \mathcal{S}(G)$, and if moreover $s^{-1}s \leq e_{(g_1,...,g_n)}$, then

$$\begin{aligned}ss^{-1} = s(s^{-1}s)s^{-1} &\leq [\eta(s)]s^{-1}s[\eta(s)^{-1}]\\ &\leq [\eta(s)]e_{(g_1,...,g_n)}[\eta(s)^{-1}] = e_{(\eta(s),g_1,...,g_n)}.\end{aligned}$$

Thus $ss^{-1} \leq e_{(\eta(s),g_1,...,g_n)}$ and σ_n is well-defined.

Lemma 3.30. *We have that*

$$\begin{aligned}\partial_0 \circ \sigma_{-1} &= \mathrm{id}_B,\\ \partial_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \partial_n &= \mathrm{id}_{P_n},\ n \in \mathbb{N} \cup \{0\}.\end{aligned}$$

Proof. It suffices to verify on the K-basis of P_n, $n \in \mathbb{N} \cup \{-1, 0\}$. The first case

$$\sigma_0 \circ \sigma_{-1}(e) = \partial_0(e(\,)) = e.$$

Now for $n = 0$

$$\begin{aligned}(\partial_1 \circ \sigma_0 + \sigma_{-1} \circ \partial_0)\,(s(\,)) &= \partial_1(ss^{-1}(\eta(s))) + \sigma_{-1}(ss^{-1}) \\ &= ss^{-1}[\eta(s)](\,) - ss^{-1}(\,) + ss^{-1}(\,) = s(\,).\end{aligned}$$

For $n \geq 1$ observe

$$\begin{aligned}(\partial_{n+1} \circ \sigma_n)(s(g_1, ..., g_n)) =& \,\partial_{n+1}(ss^{-1}(\eta(s), g_1, ..., g_n)) \\ =& \,ss^{-1}\big([\eta(s)](g_1, g_2, ..., g_n) - (\eta(s)g_1, ..., g_n) \\ &+ \sum_{i=1}^{n-1}(-1)^{i+1}(\eta(s), g_1, ..., g_i g_{i+1}, ..., g_n) \\ &+ (-1)^{n+1}(\eta(s), g_1, g_2, ..., g_{n-1})\big),\end{aligned}$$

and

$$\begin{aligned}(\sigma_{n-1} \circ \partial_n)(s(g_1, ..., g_n)) =& \,\sigma_{n-1}\big(s([g_1](g_2, ..., g_n) \\ &+ \sum_{i=1}^{n-1}(-1)^i(g_1, ..., g_i g_{i+1}, ..., g_n) \\ &+ (-1)^n(g_1, ..., g_{n-1})\big) \\ =& \,s[g_1](s[g_1])^{-1}(\eta(s[g_1]), g_2, ..., g_n)) \\ &+ \sum_{i=1}^{n-1}(-1)^i ss^{-1}(\eta(s), g_1, .., g_i g_{i+1}, ..., g_n) \\ &+ (-1)^n ss^{-1}(\eta(s), g_1, g_2, ..., g_{n-1}).\end{aligned}$$

Finally, notice that $\eta(s[g_1]) = \eta(s)g_1$, and

$$\begin{aligned}s[g_1](s[g_1])^{-1} = s[g_1](ss^{-1}[\eta(s)][g_1])^{-1} &= se_{g_1}[\eta(s)^{-1}]ss^{-1} \\ &= s[\eta(s)^{-1}]ss^{-1}e_{\eta(s)g_1} = ss^{-1}e_{\eta(s)g_1}.\end{aligned}$$

Then

$$s[g_1](s[g_1])^{-1}(\eta(s[g_1]), g_2, ..., g_n) = ss^{-1}([\eta(s)]g_1, g_2, ..., g_n).$$

Therefore

$$(\partial_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \partial_n)(s(g_1, g_2, ..., g_n)) = s(g_1, g_2, ..., g_n).$$

□

Lemma 3.31. *For $n \in \mathbb{N} \cup \{0\}$ the set $\sigma_n(P_n)$ generates P_{n+1} as a $K_{par}(G)$-module.*

Proof. Let $s(g_1, ..., g_{n+1}) \in P_{n+1}$. Since $s^{-1}s \leq e_{(g_1,g_2,...,g_{n+1})}$ we have

$$se_{g_1} = se_{(g_1,g_2,...,g_n)}e_{g_1} = se_{(g_1,g_2,...,g_n)} = s,$$

consider $t = [\eta(s)^{-1}]s[g_1] \in \mathcal{S}(G)$, then we obtain

$$tt^{-1} = [\eta(s)^{-1}]se_{g_1}s^{-1}[\eta(s)] = [\eta(s)^{-1}]ss^{-1}[\eta(s)] = s^{-1}s$$

Furthermore, as $ss^{-1} \leq e_{\eta(s)}$, then $ss^{-1} = e_{\eta(s)}ss^{-1}$, thus $s = e_{\eta(s)}s$. Whence

$$\begin{aligned} t^{-1}t = [g_1^{-1}]s^{-1}e_{\eta(s)}s[g_1] = [g_1^{-1}]s^{-1}s[g_1] &\leq [g^{-1}]e_{(g_1,...,g_{n+1})}[g_1] \\ &= e_{(g_2,...,g_{n+1})}e_{g_1^{-1}} \leq e_{(g_2,...,g_{n+1})}. \end{aligned}$$

Which means that $t(g_2, ..., g_{n+1})$ is well-defined. Finally,

$$\eta(t) = \eta([\eta(s)^{-1}]s[g_1]) = \eta(s)^{-1}\eta(s)g_1 = g_1.$$

Therefore

$$s \cdot \sigma_n(t(g_2, ..., g_{n+1})) = s \cdot tt^{-1}(\eta(t), g_2, ..., g_{n+1}) = ss^{-1}s(g_1, ..., g_{n+1}) = s(g_1, ..., g_n).$$

Then $s(g_1, g_2, ..., g_{n+1})$ is in the $K_{par}(G)$-module generated by $\sigma_n(P_n)$.

□

Proposition 3.32. *The sequence*

$$\cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} B \to 0$$

is a projective resolution of B.

Proof. The exactness in B is clear since ∂_0 is an epimorphism by the first item of Lemma 3.30. By Lemma 3.24 each P_n is projective then we only have to prove that the sequence is exact. The inclusion $\ker \partial_n \subseteq \operatorname{im} \partial_{n+1}$, is follows from the second item of Lemma 3.30. Indeed, given $z \in \ker \partial_n$ we have

$$z = (\partial_{n+1} \circ \sigma_n + \sigma_{n-1} \circ \partial_n)(z) = \partial_{n+1} \circ \sigma_n(z).$$

Therefore $\ker \partial_n \subseteq \operatorname{im} \partial_{n+1}$. For the converse inclusion first observe

$$\begin{aligned} \partial_0 \circ \partial_1 \circ \sigma_0(s(\,)) &= \partial_0 \circ \partial_1(ss^{-1}(\eta(s))) \\ &= \partial_0(ss^{-1}([\eta(s)](\,) - (\,))) \\ &= \partial_0(s(\,) - ss^{-1}(\,)) = ss^{-1} - ss^{-1}ss^{-1} = 0. \end{aligned}$$

By Lemma 3.31 we have that $\sigma_0(P_0)$ generates P_1, then $\partial_0 \circ \partial_1 = 0$. By Lemma 3.30 we have $\partial_{n+1} \circ \sigma_n = \mathrm{id}_{P_n} - \sigma_{n-1} \circ \partial_n$ and $\partial_n \circ \sigma_{n-1} = \mathrm{id}_{P_{n-1}} - \sigma_{n-2} \circ \partial_{n-1}$. Thus using an inductive argument, if $\partial_{n-1} \circ \partial_n = 0$ then

$$\begin{aligned}\partial_n \circ \partial_{n+1} \circ \sigma_n &= \partial_n \circ (\mathrm{id}_{P_n} - \sigma_{n-1} \circ \partial_n)\\ &= \partial_n - \partial_n \circ \sigma_{n-1} \circ \partial_n\\ &= \partial_n - (\mathrm{id}_{P_{n-1}} - \sigma_{n-2} \circ \partial_{n-1}) \circ \partial_n\\ &= \partial_n - \partial_n + \sigma_{n-2} \circ \partial_{n-1} \circ \partial_n\\ &= 0.\end{aligned}$$

By Lemma 3.31 $\sigma_n(P_n)$ generates P_{n+1} then $\partial_n \circ \partial_{n+1} = 0$. □

Definition 3.33. *Let M be a $K_{par}(G)$-module. Define the following additive groups*

$$\begin{aligned}&C^0_{par}(G, M) = M,\\ &C^n_{par}(G, M) = \{f : G^n \to M \mid f(g_1, g_2, ..., g_n) \in e_{(g_1,g_2,...,g_n)} \cdot M\},\ n \in \mathbb{N}.\end{aligned}$$

Lemma 3.34. *Let M be a $K_{par}(G)$-module. Then*

$$\mathrm{Hom}_{K_{par}(G)}(P_n, M) \cong C^n_{par}(G, M).$$

Proof. For $n = 0$ we have that $P_0 = K_{par}(G)$, then

$$\mathrm{Hom}_{K_{par}(G)}(P_0, M) \cong M = C^0_{par}.$$

Recall that $\mathrm{Hom}\,(\bigoplus A_i, -) \cong \prod \mathrm{Hom}\,(A_i, -)$ (for any family of modules $\{A_i\}$). Then, for $n \in \mathbb{N}$,

$$\begin{aligned}\mathrm{Hom}_{K_{par}(G)}(P_n, M) &\cong \prod_{g_1,...,g_n \in G} \mathrm{Hom}_{K_{par}(G)}(K_{par}(G)e_{(g_1,..,g_n)}, M)\\ &\cong \prod_{g_1,...,g_n \in G} e_{(g_1,..,g_n)} \cdot M\\ &= C^n_{par}(G, M).\end{aligned}$$

□

Remark 3.35. The two isomorphisms used in Lemma 3.34 are

$$\mathrm{Hom}_{K_{par}(G)}(P_n, M) \to \prod_{g_1,...,g_n \in G} \mathrm{Hom}_{K_{par}(G)}(K_{par}(G)e_{(g_1,...,g_n)}, M)$$

given by $\varphi \mapsto \hat{\varphi}$ such that $\hat{\varphi}_{(g_1,\dots,g_n)}(se_{(g_1,\dots,g_n)}) = \varphi(se_{(g_1,\dots,g_n)}(g_1,...,g_n))$. And

$$\mathrm{Hom}_{K_{par}(G)}(K_{par}(G)e_{(g_1,\dots,g_n)}, M) \to C^n_{par}(G,M)$$

given by $\psi \mapsto \Phi_\psi$ such that $\Phi_\psi(g_1,...,g_n) = \psi(e_{(g_1,\dots,g_n)})$. Then the isomorphism

$$\begin{aligned}\mathrm{Hom}_{K_{par}(G)}(P_n, M) &\longrightarrow C^n_{par}(G,M)\\ \varphi \quad &\longmapsto \quad f_\varphi,\end{aligned}$$

is such that

$$f_\varphi(g_1,...,g_n) = \Phi_{\hat{\varphi}}(g_1,...,g_n) = \hat{\varphi}(e_{(g_1,\dots,g_n)}) = \varphi(e_{(g_1,\dots,g_n)}(g_1,...,g_n))$$

Conversely, each $f' \in C^n_{par}(G,M)$ corresponds to $\varphi \in \mathrm{Hom}_{K_{par}(G)}(P_n, M)$ defined by

$$\varphi(s(g_1,...,g_n)) = s \cdot f'(g_1,...,g_n).$$

Definition 3.36. *Let M be a $K_{par}(G)$-module and $n \in \mathbb{N} \cup \{0\}$. Define the K-linear map $\delta^n : C^n_{par}(G,M) \to C^{n+1}_{par}(G,M)$ as follows:*

$$\begin{aligned}\left(\delta^0 m\right)(g) &= [g]\cdot m - e_g \cdot m, m \in C^0_{par}(G,M)\\ \left(\delta^n f\right)(g_1,\dots,g_{n+1}) &= [g_1]\cdot f(g_2,\dots,g_{n+1})\\ &\quad + \sum_{i=1}^{n}(-1)^i e_{g_1\dots g_i}\cdot f(g_1,\dots,g_ig_{i+1},\dots,g_{n+1})\\ &\quad + (-1)^{n+1}e_{g_1\dots g_{n+1}}\cdot f(g_1,\dots,g_n), n\in\mathbb{N}, f \in C^n_{par}(G,M).\end{aligned}$$

Lemma 3.37. *For all $n \in \mathbb{N}\cup\{0\}$ and $f \in C^n_{par}(G,M)$ we have*

$$\delta^n f = f \circ \partial_{n+1},$$

where f and $\delta^n f$ are identified with the morphisms from $\mathrm{Hom}_{K_{par}(G)}(P_n, M)$ *as in Remark 3.35. In particular,*

$$C^0_{par}(G,M) \xrightarrow{\delta^0} C^1_{par}(G,M) \xrightarrow{\delta^1} \cdots$$

is a cochain complex of abelian groups.

Proof. Remark 3.35 allows us to understand $f \in C^n_{par}(G,M)$ as a map in $\mathrm{Hom}_{K_{par}(G)}(P_n, M)$ such that

$$f(s(g_1,...,g_n)) = s \cdot f(g_1,...,g_n),$$

and let us understand $\delta^n f \in \mathrm{Hom}_{K_{par}(G)}(P_n, M)$ as a map in $C^n_{par}(G,M)$ such that

$$\delta^n f(g_1, ..., g_n) = \delta^n f(e_{(g_1,...,g_n)}(g_1, ..., g_n)).$$

Thus it suffices to verify $\delta^n f = f \circ \partial_{n+1}$ on the generators

$$\{e_{(g_1,...,g_{n+1})} \mid g_1, g_2, ..., g_{n+1} \in G\}$$

of P_{n+1}. We first consider the case $n = 0$. Let $m \in C^0_{par}(G,M)$. Then as an element of $\mathrm{Hom}_{K_{par}(G)}(K_{par}(G), M)$, m sends $s(\ )$ to $s \cdot m$. Then

$$\begin{aligned} m \circ \partial_1(e_g(g)) &= m(e_g([g](\ ) - (\ )) \\ &= [g] \cdot m - e_g \cdot m = (\delta^0 m)(g) \\ &= (\delta^0 m)(e_g(g)). \end{aligned}$$

Now let $n \in \mathbb{N}$ and f a function from $C^n_{par}(G,M)$. Thus

$$\begin{aligned} f \circ \partial_{n+1}\left(e_{(g_1,...,g_{n+1})}(g_1, \dots, g_{n+1})\right) =& e_{(g_1,...,g_{n+1})} \cdot \Big([g_1] \cdot f(g_2, \dots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \dots, g_n)\Big) \\ =& e_{(g_1,...,g_{n+1})} \cdot \Big([g_1] \cdot f(g_2, \dots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i e_{g_1 \dots g_i} \cdot f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ &+ (-1)^{n+1} e_{g_1 \dots g_{n+1}} \cdot f(g_1, \dots, g_n)\Big) \\ =& e_{(g_1,...,g_{n+1})} \cdot (\delta^n f)(g_1, \dots, g_{n+1}) \\ =& (\delta^n f)\left(e_{(g_1,...,g_{n+1})}(g_1, \dots, g_{n+1})\right) \end{aligned}$$

□

Definition 3.38. *Denote* $\ker \delta^n$ *by* $Z^n_{par}(G,M), n \in \mathbb{N} \cup \{0\}$, *and* $\mathrm{im}\, \delta^{n-1}$ *by* $B^n_{par}(G,M)$ $n \in \mathbb{N}$, *where* δ^n *is given by Definition 3.36.*

Theorem 3.39. *Let G be a group and M a $K_{par}(G)$-module. Then $H^0_{par}(G,M) \cong Z^0_{par}(G,M)$ and $H^n_{par}(G,M) \cong Z^n_{par}(G,M)/B^n_{par}(G,M)$.*

Proof. This follows from Proposition 3.32, Lemma 3.34 and Lemma 3.37. □

Remark 3.40. For a $K_{par}(G)$-module $\mathcal{A}$ coming from an unital partial G-module $(\mathcal{A},\alpha)$ and $n\in\mathbb{N}$ we have

$$C^n_{par}(G,\mathcal{A}) = \left\{f : G^n \to \mathcal{A} \mid f\left(g_1,\dots,g_n\right) \in 1_{(g_1,\dots,g_n)}\mathcal{A}\right\},$$

where

$$1_{(g_1,\dots,g_n)} = 1_{g_1}1_{g_1g_2}\dots 1_{g_1\dots g_n} \in \mathcal{A}.$$

Then formulas of Definition 3.36 take the following form

$$\begin{aligned}\left(\delta^0 a\right)(g) &= \alpha_g\left(1_{g^{-1}}a\right) - 1_g a, a \in C^0_{par}(G,\mathcal{A}) = \mathcal{A},\\ \left(\delta^n f\right)\left(g_1,\dots,g_{n+1}\right) &= \alpha_{g_1}\left(1_{g_1^{-1}}f\left(g_2,\dots,g_{n+1}\right)\right)\\ &\quad + \sum_{i=1}^{n}(-1)^i 1_{g_1\dots g_i} f\left(g_1,\dots,g_ig_{i+1},\dots,g_{n+1}\right)\\ &\quad + (-1)^{n+1}1_{g_1\dots g_{n+1}} f\left(g_1,\dots,g_n\right)\end{aligned}$$

$n\in\mathbb{N}, f\in C^n_{par}(G,M)$.

Example 3.41. *Let G be the cyclic group $C_2 = \{1, x \mid x^2 = 1\}$. Observe that $\mathcal{S}(G) = \{e, [x], e_x\}$ where $e = [1]$. Take $M = K_{par}(C_2)$, therefore*

$$C^0_{par}(C_2, K_{par}(C_2)) = K_{par}(C_2),$$

and, for $n \geq 1$,

$$C^n_{par}(C_2, K_{par}(C_2)) = \{f : (C_2)^n \to K_{par}(C_2) \mid f(g_1, g_2, \dots, g_n) \in e_{(g_1,g_2,\dots,g_n)} \cdot K_{par}(C_2)\}.$$

For the case $n = 1$ we have

$$C^1_{par}(C_2, K_{par}(C_2)) = \{f : C_2 \to K_{par}(C_2) \mid f(g) \in e_g \cdot K_{par}(C_2)\},$$

notice that $f(1) \in e\cdot K_{par}(C_2) = K_{par}(C_2)$ and $f(x) \in e_x \cdot K_{par}(C_2) \cong KC_2$. Indeed, it is easy to see that $e_x \cdot \mathcal{S}(C_2) = \{[x], e_x\} \cong C_2$ and since $[x]$ generates $e_x \cdot K_{par}(C_2)$ as an algebra, we have that $e_x \cdot K_{par}(C_2) \cong KC_2$. Thus,

$$C^1_{par}(C_2, K_{par}(C_2)) \cong K_{par}(C_2) \oplus KC_2.$$

Now, take any $e_{(g_1,g_2,\dots,g_n)} \in \mathcal{S}(C_2)$, if there exist $j \in \{1,\dots,n\}$ such that $g_j = x$ then $e_{(g_1,g_2,\dots,g_n)} = e_x$. Indeed, let us take $j \in \{1,\dots,n\}$ such that is the smallest element that satisfies $g_j = x$, therefore $g_i = 1$ for all $i < j$. Thus,

$$e_{(g_1,g_2,\dots,g_n)} = e_{g_1}e_{g_1g_2}\dots e_{g_1g_2\dots g_j}e_{g_1g_2\dots g_j}\dots e_{g_1g_2\dots g_n} = e_xe_{xg_{j+1}}\dots e_{x\dots g_n} = e_x.$$

Hence, for $f \in C^n_{par}(C_2, K_{par}(C_2))$ *we have that* $f(g_1, \ldots, g_n) \in e \cdot K_{par}(C_2) = K_{par}(C_2)$ *if* $g_i = 1$ *for all* $i \in \{1, \ldots, n\}$ *and* $f(g_1, \ldots, g_n) \in e_x \cdot K_{par}(C_2)$ *otherwise. Therefore*

$$C^n_{par}(C_2, K_{par}(C_2)) \cong K_{par}(C_2) \oplus (KC_2)^{n-1}.$$

If we check δ^0 *in the basis* $\{e, [x], e_x\}$ *of* $C^0_{par}(C_2, K_{par}(C_2))$ *we have that*

$$\begin{aligned} \delta^0(e) &= (0, [x] - e_x), \\ \delta^0([x]) &= (0, e_x - [x]), \\ \delta^0(e_x) &= (0, [x] - e_x), \end{aligned}$$

where the first coordinate correspond to $e \cdot K_{par}(C_2)$ *and the second coordinate correspond to* $e_x \cdot K_{par}(C_2)$. *Therefore* $\ker \delta^0 = H^0_{par}(C_2, K_{par}(C_2))$ *is the vector space generated by* $\{e + [x], e_x + [x]\}$, *which is* $(K_{par}(C_2))^{G_{par}}$ *as Theorem 3.15 says.*

CHAPTER 4

Grothendieck spectral sequence

We will use spectral sequence theory to relate cohomology of partial smash product with partial group cohomology and algebra cohomology. In order to do that we will show that there exists a pair of functors which satisfies the conditions of Theorem 2.106, thus we will obtain a Grothendieck spectral sequence relating the desired cohomologies. This section corresponds to the study of the final part of [2].

Definition 4.1. *If A is a k-algebra, where k is a commutative ring, then its* ***enveloping algebra*** *is*

$$A^e = A \otimes_k A^{op}.$$

Proposition 4.2. *Let R and S be k-algebras, where k is a commutative ring. Then any (R, S)-bimodule M is a left $R \otimes_k S^{op}$-module, where*

$$(r \otimes_k s)m = rms.$$

In particular if A is a k-algebra, then A is a A^e-bimodule.

Definition 4.3. *(Cohomology modules of an algebra).* [1] *Let A be an algebra and let M be an A-bimodule (A^e-module), we define the nth cohomology module $H^n(A, M)$ of A with coefficients in M as $\mathrm{Ext}^n_{A^e}(A, M)$.*

[1]For more information about cohomology of algebras see [15, Section 6.11].

Theorem 4.4. *For any $A \rtimes_\alpha G$-bimodule M there is a third quadrant cohomology spectral sequence starting with E_2 and converging to $H^\bullet(A \rtimes_\alpha G, M)$:*

$$E_2^{p,q} = H^p_{par}(G, H^q(A, M)) \underset{p}{\Rightarrow} H^{p+q}(A \rtimes_\alpha G, M).$$

We need some preparation in order to construct the functors that will be used in the proof of Theorem 4.4. Recall that we are working with an unital partial action $\left(A, \{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}\right)$, and that we denote the unity of D_g by u_g.

Take a pair of representations

$$\phi_X : K_{par}(G) \to \mathrm{End}_K(X) \in \mathrm{Ob}(\mathrm{Rep}\, K_{par}(G))$$

and

$$\Phi_M : (A \rtimes_\alpha G)^e \to \mathrm{End}_K(M) \in \mathrm{Ob}(\mathrm{Rep}(A \rtimes_\alpha G)^e).$$

Taking $h = 1$ in Remark 2.42 we obtain the homomorphism $K_{par}(G) \to A \rtimes_\alpha G$ of algebras given by $[g] \mapsto u_g\#g$, which induce the algebra homomorphism $B \to (A \rtimes_\alpha G)^e$ defined by $e_s \mapsto u_s\#1_G \otimes 1_{(A\rtimes_\alpha G)^{op}}$. It follows that M is a B-module. Moreover, X is a bimodule over B, because B is a commutative subalgebra of $K_{par}(G)$. Thus we can consider the representation

$$\Delta : (A \rtimes_\alpha G)^e \to \mathrm{End}_K(X \otimes_B M) \in \mathrm{Rep}(A \rtimes_\alpha G)^e,$$

given by

$$\Delta(au_g\#g \otimes bu_h\#h)(x \otimes m) = \phi_X([g])(x) \otimes \Phi_M(au_g\#g \otimes bu_h\#h)(m).$$

It follows from the definition of the module structures that for $x \otimes m$ in $X \otimes_B M$ we have

$$\phi_X(e_s)(x) \otimes m = x \otimes \Phi_M(u_s\#1_G \otimes 1)(m).$$

Thus in order to verify that Δ is well defined we have to show that

$$\Delta(au_g\#g \otimes bu_h\#h)(\phi_X(e_s)(x) \otimes m) = \Delta(au_g\#g \otimes bu_h\#h)(x \otimes \Phi_M(u_s\#1_G \otimes 1)(m)).$$

Indeed,

$$\begin{aligned}
&\Delta(au_g\#g \otimes bu_h\#h)(\phi_X(e_s)(x) \otimes m)\\
&\qquad = \phi_X([g])(\phi_X(e_s)(x)) \otimes \Phi_M(au_g\#g \otimes bu_h\#h)(m)\\
&\qquad = \phi_X([g]e_s)(x) \otimes \Phi_M(au_g\#g \otimes bu_h\#h)(m)\\
&\qquad = \phi_X(e_{gs}[g])(x) \otimes \Phi_M(au_g\#g \otimes bu_h\#h)(m)\\
&\qquad = \phi_X(e_{gs})\phi_X([g])(x) \otimes \Phi_M(au_g\#g \otimes bu_h\#h)(m)\\
&\qquad = \phi_X([g])(x) \otimes \Phi_M(u_{gs}\#1_G \otimes 1)\Phi_M(au_g\#g \otimes bu_h\#h)(m)\\
&\qquad = \phi_X([g])(x) \otimes \Phi_M((u_{gs}\#1_G)(au_g\#g) \otimes bu_h\#h)(m) \qquad (1)\\
&\qquad = \phi_X([g])(x) \otimes \Phi_M((au_g\#g)(u_s\#1_G) \otimes bu_h\#h)(m) \qquad (2)\\
&\qquad = \phi_X([g])(x) \otimes \Phi_M(au_g\#g \otimes bu_h\#h)\Phi_M(u_s\#1_G \otimes 1)(m)\\
&\qquad = \Delta(au_g\#g \otimes bu_h\#h)(x \otimes \Phi_M(u_s\#1_G \otimes 1)(m)),
\end{aligned}$$

where from (1) to (2) we use that by Lemma 2.41

$$\begin{aligned}
(u_{gs}\#1_G)(au_g\#g) &= au_gu_{gs}\#g\\
&= au_g\alpha_g(u_su_{g^{-1}})\#g\\
&= (au_g\#g)(u_s\#1_G).
\end{aligned}$$

In particular, if we take $M = A \rtimes_\alpha G$ we have that $X \otimes_B (A \rtimes_\alpha G)$ is an object in $\mathrm{Rep}(A \rtimes_\alpha G)^e$, where

$$(au_g\#g \otimes bu_h\#h) \cdot (x \otimes cu_s\#s) = \phi_X([g])(x) \otimes (au_g\#g)(cu_s\#s)(bu_h\#h). \qquad (4.1)$$

Observe that M can be viewed as an object in $\mathrm{Rep}\, A^e$, where the morphism $\phi_M : A^e \to \mathrm{End}_K(M)$ is the composition

$$A^e \xrightarrow{\phi_0 \otimes \phi_0} (A \rtimes_\alpha G)^e \xrightarrow{\Phi_M} \mathrm{End}_K(M),$$

given by

$$a \otimes b \mapsto au_{1_G}\#1_G \otimes bu_{1_G}\#1_G \mapsto \Phi_M(au_{1_G}\#1_G \otimes bu_{1_G}\#1_G).$$

Recall that by (III) of Lemma 2.41 we have that $G \to A \rtimes_\alpha G$ given by $g \mapsto u_g\#g$ is a partial representation. Furthermore, by Lemmas 2.36 and 2.37 we have that M is an object in $\mathrm{ParRep}\, G$ with the map $G \to \mathrm{End}_K(M)$ given by

$$g \in G \mapsto (u_g\#g \otimes u_{g^{-1}}\#g^{-1}) \in (A \rtimes_\alpha G)^e \mapsto \Phi_M(u_g\#g \otimes u_{g^{-1}}\#g^{-1}) \in \mathrm{End}_K(M).$$

Therefore, by Lemma 2.35 there exists a partial representation

$$\mu : G \to \mathrm{End}_K(\mathrm{Hom}_K(A, M)),$$

given by

$$\mu(g)(f)(x) = \Phi_M(u_g\#g \otimes u_{g^{-1}}\#g^{-1})f(\alpha_{g^{-1}}(u_g x)),$$

where $f \in \mathrm{Hom}_K(A, M)$ and $x \in A$. Notice that μ induces a partial representation

$$\pi : G \to \mathrm{End}_K(\mathrm{Hom}_{A^e}(A, M)),$$

given by

$$\pi(g)(f)(x) = \mu(g)(f)(x) = \Phi_M(u_g\#g \otimes u_{g^{-1}}\#g^{-1})f(\alpha_{g^{-1}}(u_g x)),$$

where $f \in \mathrm{Hom}_{A^e}(A, M)$ and $x \in A$. So we only have to verify that $\pi(g)(f) \in \mathrm{Hom}_{A^e}(A, M)$ for all $f \in \mathrm{Hom}_{A^e}(A, M)$

$$\begin{aligned}
&\pi(g)(f)(axb)\\
&= \Phi_M(u_g\#g \otimes u_{g^{-1}}\#g^{-1})f(\alpha_{g^{-1}}(u_g axb))\\
&= \Phi_M(u_g\#g \otimes u_{g^{-1}}\#g^{-1})f(\alpha_{g^{-1}}(u_g a)\alpha_{g^{-1}}(u_g x)\alpha_{g^{-1}}(u_g b)) && (3)\\
&= \Phi_M(u_g\#g \otimes u_{g^{-1}}\#g^{-1})\Phi_M(\alpha_{g^{-1}}(u_g a)u_{1_G}\#1_G \otimes \alpha_{g^{-1}}(u_g b)u_{1_G}\#1_G)f(\alpha_{g^{-1}}(u_g x)) && (4)\\
&= \Phi_M((u_g\#g)(\alpha_{g^{-1}}(u_g a)u_{1_G}\#1_G) \otimes (\alpha_{g^{-1}}(u_g b)u_{1_G}\#1_G)(u_{g^{-1}}\#g^{-1}))f(\alpha_{g^{-1}}(u_g x)) && (5)\\
&= \Phi_M((au_{1_G}\#1_G)(u_g\#g) \otimes (u_{g^{-1}}\#g^{-1})(bu_{1_G}\#1_G))f(\alpha_{g^{-1}}(u_g x)) && (6)\\
&= \Phi_M(au_{1_G}\#1_G \otimes bu_{1_G}\#1_G)\Phi_M(u_g\#g \otimes u_{g^{-1}}\#g^{-1})f(\alpha_{g^{-1}}(u_g x))\\
&= \Phi_M(au_{1_G}\#1_G \otimes bu_{1_G}\#1_G)\pi(g)(f)(x).
\end{aligned}$$

From (3) to (4) recall that $f \in \mathrm{Hom}_{A^e}(A, M)$, from (5) to (6) we use that

$$\begin{aligned}
(u_g\#g)(\alpha_{g^{-1}}(u_g a)u_{1_G}\#1_G) &= u_g\alpha_g(\alpha_{g^{-1}}(u_g a)u_{g^{-1}})u_g\#g\\
&= u_g\alpha_g(\alpha_{g^{-1}}(u_g a))\#g\\
&= u_g a\#g = (au_{1_G}\#1_G)(u_g\#g)
\end{aligned}$$

and analogously,

$$\begin{aligned}
(\alpha_{g^{-1}}(u_g b)u_{1_G}\#1_G)(u_{g^{-1}}\#g^{-1}) &= \alpha_{g^{-1}}(u_g b)u_{g^{-1}}\#g^{-1}\\
&= u_{g^{-1}}\alpha_{g^{-1}}(bu_{1_G}u_g)u_{g^{-1}}\#g^{-1}\\
&= (u_{g^{-1}}\#g^{-1})(bu_{1_G}\#1_G).
\end{aligned}$$

Then $\pi(g)(f) \in \mathrm{Hom}_{A^e}(A, M)$. Therefore $\pi \in \mathrm{RepPar}\, K$, and take $\tilde{\pi} \in \mathrm{Rep}\, K_{par}(G)$ such that $\pi(g) = \tilde{\pi}([g])$.

Now we consider the natural transformations

$$\mathrm{Hom}_{K_{par}(G)}(-, \mathrm{Hom}_{A^e}(A, M)) \xrightarrow{\Gamma} \mathrm{Hom}_{(A\rtimes_\alpha G)^e}(- \otimes_B (A \rtimes_\alpha G), M)$$

and

$$\mathrm{Hom}_{(A\rtimes_\alpha G)^e}(-\otimes_B (A\rtimes_\alpha G), M) \xrightarrow{\Lambda} \mathrm{Hom}_{K_{par}(G)}(-, \mathrm{Hom}_{A^e}(A, M))$$

defined as follows: given a $K_{par}(G)$-module X and $H \in \mathrm{Hom}_{K_{par}(G)}(X, \mathrm{Hom}_{A^e}(A, M))$, the map $\Gamma_X(H)$ is defined by

$$\Gamma_X(H)(x \otimes au_g\#g) := \Phi_M(1 \otimes au_g\#g)H(x)(1_A),$$

and given $T \in \mathrm{Hom}_{(A\rtimes_\alpha G)^e}(X \otimes_B (A \rtimes_\alpha G), M)$, the map $\Lambda_X(T)$ is defined by

$$\Lambda_X(T)(x)(a) := T(x \otimes (au_{1_G}\#1_G)).$$

The map $\Gamma_X(H)$ is well defined since

$$\begin{aligned}
&\Gamma_X(H)\,(e_h \cdot x \otimes au_g\#g) \\
&\quad = \Phi_M\,(1_{A\rtimes_\alpha G} \otimes au_g\#g)\,H\,(e_h \cdot x)\,(1_A) \\
&\quad = \Phi_M\,(1 \otimes au_g\#g)\,(\tilde{\pi}\,(e_h))\,H(x)(1) \\
&\quad = \Phi_M\,(1 \otimes au_g\#g)\,\pi(h)\pi(h^{-1})H(x)(1) \\
&\quad = \Phi_M(1 \otimes au_g\#g)\Phi_M(u_h\#h \otimes u_{h^{-1}}\#h^{-1})\pi(h^{-1})H(x)(\alpha_{h^{-1}}\,(u_h 1)) \\
&\quad = \Phi_M\,(1 \otimes au_g\#g)\,\Phi_M\left(u_h\#h \otimes u_{h^{-1}}\#h^{-1}\right)\Phi_M\left(u_{h^{-1}}\#h^{-1} \otimes u_h\#h\right) \\
&\qquad H(x)\,(\alpha_h\,(u_{h^{-1}}\alpha_{h^{-1}}\,(u_h 1))) \\
&\quad = \Phi_M\,(1 \otimes au_g\#g)\,\Phi_M\left((u_h\#h)\left(u_{h^{-1}}\#h^{-1}\right) \otimes (u_h\#h)\left(u_{h^{-1}}\#h^{-1}\right)\right) \\
&\qquad H(x)\,(u_h) \\
&\quad = \Phi_M\,(1 \otimes au_g\#g)\,\Phi_M\,((u_h\#1_G) \otimes (u_h\#1_G))\,H(x)\,(u_h) \\
&\quad = \Phi_M\,(u_h\#1_G \otimes (u_h\#1_G)\,(au_g\#g))\,H(x)\,(u_h) \\
&\quad = \Phi_M\,(1 \otimes (u_h\#1_G)\,(au_g\#g))\,\Phi_M(u_h\#1_G \otimes u_{1_G}\#1_G)H(x)\,(u_h) \\
&\quad = \Phi_M\,(1 \otimes (u_h\#1_G)\,(au_g\#g))\,\phi_M(u_h \otimes u_{1_G})(H(x)\,(u_h 1)) \\
&\quad = \Phi_M\,(1 \otimes (u_h\#1_G)\,(au_g\#g))\,H(x)\,(u_h\,(u_h 1)) \\
&\quad = \Phi_M\,(1 \otimes (u_h\#1_G)\,(au_g\#g))\,H(x)\,(1u_h) \\
&\quad = \Phi_M\,(1 \otimes (u_h\#1_G)\,(au_g\#g))\,\phi_M(u_{1_G} \otimes u_h)H(x)\,(1) \\
&\quad = \Phi_M\,(1 \otimes (u_h\#1_G)\,(au_g\#g))\,\Phi_M(u_{1_G}\#1_G \otimes u_h\#1_G)H(x)\,(1) \\
&\quad = \Phi_M\,(1 \otimes (u_h\#1_G)\,(u_h\#1_G)\,(au_g\#g))\,H(x)(1) \\
&\quad = \Phi_M\,(1 \otimes (u_h\#1_G)\,(au_g\#g))\,H(x)(1) \\
&\quad = \Gamma_X(H)\,(x \otimes (u_h\#1_G)\,(au_g\#g)) \\
&\quad = \Gamma_X(H)\,(x \otimes e_h \cdot (au_g\#g))\,.
\end{aligned}$$

Next we show that $\Gamma_X(H)$ is a homomorphism of $(A \rtimes_\alpha G)^e$-modules:

$$\begin{aligned}
&\Gamma_X(H)\left(\Delta\left(cu_h\#h \otimes du_s\#s\right)\left(x \otimes au_g\#g\right)\right)\\
&\qquad \overset{4.1}{=} \Gamma_X(H)\big(\phi_X([h])(x) \otimes ((cu_h\#h) \otimes (du_s\#s)) \cdot (au_g\#g)\big)\\
&\qquad = \Gamma_X(H)\left(\phi_X([h])(x) \otimes (cu_h\#h)\left(au_g\#g\right)\left(du_s\#s\right)\right)\\
&\qquad = \Phi_M\left(1_{A\rtimes_\alpha G} \otimes (cu_h\#h)\left(au_g\#g\right)\left(du_s\#s\right)\right) H\left(\phi_X([h])(x)\right)(1_A)\\
&\qquad = \Phi_M\left(1 \otimes (cu_h\#h)\left(au_g\#g\right)\left(du_s\#s\right)\right) \pi(h)(H(x))(1)\\
&\qquad = \Phi_M\left(1 \otimes (cu_h\#h)\left(au_g\#g\right)\left(du_s\#s\right)\right) \Phi_M\left(u_h\#h \otimes u_{h^{-1}}\#h^{-1}\right) H(x)(u_{h^{-1}})\\
&\qquad = \Phi_M\left(1 \otimes (cu_h\#h)\left(au_g\#g\right)\left(du_s\#s\right)\right) \Phi_M\left(u_h\#h \otimes u_{h^{-1}}\#h^{-1}\right)\\
&\qquad\qquad \cdot \phi_M(u_{1_G} \otimes u_{h^{-1}})H(x)(1)\\
&\qquad = \Phi_M\left(1 \otimes (cu_h\#h)\left(au_g\#g\right)\left(du_s\#s\right)\right) \Phi_M\left(u_h\#h \otimes u_{h^{-1}}\#h^{-1}\right)\\
&\qquad\qquad \cdot \Phi_M(u_{1_G}\#1_G \otimes u_{h^{-1}}\#1_G)H(x)(1)\\
&\qquad = \Phi_M\left(1 \otimes (cu_h\#h)\left(au_g\#g\right)\left(du_s\#s\right)\right) \Phi_M\left(u_h\#h \otimes u_{h^{-1}}\#h^{-1}\right) H(x)(1)\\
&\qquad = \Phi_M\left(u_h\#h \otimes \left(\alpha_{h^{-1}}\left(cu_h\right)\#1_G\right)\left(au_g\#g\right)\left(du_s\#s\right)\right) H(x)(1)\\
&\qquad = \Phi_M\left(u_h\#h \otimes \left(au_g\#g\right)\left(du_s\#s\right)\right) \Phi_M((u_{1_G}\#1_G) \otimes \alpha_{h^{-1}}(cu_h)\#1_G)H(x)(1)\\
&\qquad = \Phi_M\left(u_h\#h \otimes \left(au_g\#g\right)\left(du_s\#s\right)\right) \phi_M(u_{1_G} \otimes \alpha_{h^{-1}}(cu_h))H(x)(1)\\
&\qquad = \Phi_M\left(u_h\#h \otimes \left(au_g\#g\right)\left(du_s\#s\right)\right) H(x)\left(\alpha_{h^{-1}}\left(cu_h\right)\right)\\
&\qquad = \Phi_M\left(u_h\#h \otimes \left(au_g\#g\right)\left(du_s\#s\right)\right) \phi_M(\alpha_{h^{-1}}(cu_h) \otimes u_{1_G})H(x)(1)\\
&\qquad = \Phi_M\left(u_h\#h \otimes \left(au_g\#g\right)\left(du_s\#s\right)\right) \Phi_M\left(\alpha_{h^{-1}}\left(cu_h\right)\#1_G \otimes (u_{1_G}\#1_G)\right) H(x)(1)\\
&\qquad = \Phi_M\left(\left(u_h\#h\right)\left(\alpha_{h^{-1}}\left(cu_h\right)\#1_G\right) \otimes \left(au_g\#g\right)\left(du_s\#s\right)\right) H(x)(1)\\
&\qquad = \Phi_M\left(\left(u_h\alpha_h\left(u_{h^{-1}}\alpha_{h^{-1}}\left(cu_h\right)\right)\#h\right) \otimes \left(au_g\#g\right)\left(du_s\#s\right)\right) H(x)(1)\\
&\qquad = \Phi_M\left(cu_h\#h \otimes \left(au_g\#g\right)\left(du_s\#s\right)\right) H(x)(1)\\
&\qquad = \Phi_M\left(cu_h\#h \otimes du_s\#s\right) \Phi_M\left(1 \otimes au_g\#g\right) H(x)(1)\\
&\qquad = \Phi_M\left(cu_h\#h \otimes (du_s\#s)\right) \Gamma_X(H)\left(x \otimes au_g\#g\right).
\end{aligned}$$

On the other hand, $\Lambda_X(T) \in \mathrm{Hom}_{K_{par}(G)}(X, \mathrm{Hom}_{A^e}(A, M))$ because

$$\begin{aligned}
\Lambda_X(T)(x)((c \otimes d) \cdot a) &= T\left(x \otimes (cu_{1_G}\#1_G)\left(au_{1_G}\#1_G\right)\left(du_{1_G}\#1_G\right)\right)\\
&= T\left(x \otimes ((cu_{1_G}\#1_G) \otimes (du_{1_G}\#1_G)) \cdot (au_{1_G}\#1_G)\right)\\
&= T\left(\Delta((cu_{1_G}\#1_G) \otimes (du_{1_G}\#1_G))(x \otimes (au_{1_G}\#1_G))\right)\\
&= \Phi_M\left(cu_{1_G}\#1_G \otimes du_{1_G}\#1_G\right) T\left(x \otimes au_{1_G}\#1_G\right)\\
&= \Phi_M\left(cu_{1_G}\#1_G \otimes du_{1_G}\#1_G\right) \Lambda_X(T)(x)(a),
\end{aligned}$$

which shows that $\Lambda_X(T)(x)$ is a homomorphism of A^e-modules. Furthermore, $\Lambda_X(T)$

is a homomorphism of $K_{par}(G)$-modules, since

$$\begin{aligned}
&\Lambda_X(T)\left(\phi_X([g])(x)\right)(a)\\
&\qquad= T\left(\phi_X([g])(x)\otimes au_{1_G}\#1_G\right) = T\left(\phi_X(e_g[g])(x)\otimes au_{1_G}\#1_G\right)\\
&\qquad= T\left(\left(\phi_X\left(e_g\right)\phi_X([g])\right)(x)\otimes au_{1_G}\#1_G\right)\\
&\qquad= T\left(\phi_X([g])(x)\otimes\left(u_g\#1_G\right)\left(au_{1_G}\#1_G\right)\right) = T\left(\phi_X([g])(x)\otimes au_g\#1_G\right)\\
&\qquad= T\left(\phi_X([g])(x)\otimes\left(u_g\#g\right)\left(\alpha_{g^{-1}}\left(au_g\right)\#1_G\right)\left(u_{g^{-1}}\#g^{-1}\right)\right)\\
&\qquad\overset{4.1}{=} T\left(\Delta(u_g\#g\otimes u_{g^{-1}}\#g^{-1})(x\otimes\left(\alpha_{g^{-1}}\left(au_g\right)\#1_G\right))\right)\\
&\qquad= \Phi_M\left(u_g\#g\otimes u_{g^{-1}}\#g^{-1}\right)T\left(x\otimes\alpha_{g^{-1}}\left(au_g\right)\#1_G\right)\\
&\qquad= \Phi_M\left(u_g\#g\otimes u_{g^{-1}}\#g^{-1}\right)\Lambda_X(T)(x)\left(\alpha_{g^{-1}}\left(au_g\right)\right) = \pi(g)\Lambda_X(T)(x)(a).
\end{aligned}$$

Moreover, $\Lambda\circ\Gamma = id$, since

$$\begin{aligned}
\Lambda_X(\Gamma_X(H))(x)(a) &= \Gamma_X(H)(x\otimes au_{1_G}\#1_G)\\
&= \Phi_M(u_{1_G}\#1_G\otimes au_{1_G}\#1_G)H(x)(1)\\
&= \phi_M(u_{1_G}\otimes au_{1_G})H(x)(1)\\
&= H(x)(a),
\end{aligned}$$

and $\Gamma\circ\Lambda = id$, because

$$\begin{aligned}
\Gamma_X(\Lambda_X(T))(x\otimes au_g\#g) &= \Phi_M(1_{A\rtimes_\alpha G}\otimes au_g\#g)\Lambda_X(T)(x)(1_A)\\
&= \Phi_M(1\otimes au_g\#g)T(x\otimes u_{1_G}\#1_G) = T(x\otimes au_g\#g).
\end{aligned}$$

Finally, observe that for any morphism $f : X \to Y$ in $\operatorname{Rep} K_{par}(G)$ and $H \in \operatorname{Hom}_{K_{par}(G)}(Y, \operatorname{Hom}_{A^e}(A, M))$ we have

$$\begin{aligned}
&\Gamma_X\big(\operatorname{Hom}_{K_{par}(G)}(f, \operatorname{Hom}_{A^e}(A, M))(H)\big)(x\otimes au_g\#g)\\
&\quad= \Gamma_X(H\circ f)(x\otimes au_g\#g)\\
&\quad= \Phi_M(1_{A\rtimes_\alpha G}\otimes au_g\#g)\big((H\circ f)(x)(1_A)\big)\\
&\quad= \Gamma_Y(H)(f(x)\otimes au_g\#g)\\
&\quad= \Gamma_Y(H)\circ(f\otimes id)(x\otimes au_g\#g)\\
&\quad= \operatorname{Hom}_{(A\rtimes_\alpha G)^e}(f\otimes_B(A\rtimes_\alpha G), M)\big(\Gamma_Y(H)\big)(x\otimes au_g\#g).
\end{aligned}$$

Thus the next diagram commutes

$$\begin{tikzcd}
\operatorname{Hom}_{K_{par}(G)}(Y, \operatorname{Hom}_{A^e}(A, M)) \arrow[r, "f^*"] \arrow[d, "\Gamma_Y"'] & \operatorname{Hom}_{K_{par}(G)}(X, \operatorname{Hom}_{A^e}(A, M)) \arrow[d, "\Gamma_X"] \\
\operatorname{Hom}_{(A\rtimes_\alpha G)^e}(Y\otimes_B(A\rtimes_\alpha G), M) \arrow[r, "f_*"'] & \operatorname{Hom}_{(A\rtimes_\alpha G)^e}(X\otimes_B(A\rtimes_\alpha G), M)
\end{tikzcd}$$

where $f^* = \mathrm{Hom}_{K_{par}(G)}(f, \mathrm{Hom}_{A^e}(A, M))$ and $f_* = \mathrm{Hom}_{(A\rtimes_\alpha G)^e}(f \otimes_B (A \rtimes_\alpha G), M)$.

The previous facts lead us to the following two propositions.

Proposition 4.5. *The functors*

$$\mathrm{Hom}_{K_{par}(G)}(-, \mathrm{Hom}_{A^e}(A, M)) \text{ and } \mathrm{Hom}_{(A\rtimes_\alpha G)^e}(- \otimes_B (A \rtimes_\alpha G), M)$$

are naturally isomorphic by means of Γ.

Proposition 4.6. *Up to the natural isomorphism* Γ*, we have the commutative diagram of functors*

$$\begin{tikzcd}
\mathrm{Rep}(A \rtimes_\alpha G)^e \arrow[rr, "F"] \arrow[dr, "F_1"'] & & \mathrm{Rep}\, K \\
& \mathrm{Rep}\, K_{par}(G) \arrow[ur, "F_2"'] &
\end{tikzcd}$$

where

$$F_1(M) = \mathrm{Hom}_{A^e}(A, M),\ F_2(X) = \mathrm{Hom}_{K_{par}(G)}(B, X)$$

and

$$F(M) = \mathrm{Hom}_{(A\rtimes_\alpha G)^e}(A \rtimes_\alpha G, M).$$

Proof. Recall that $F_1(M) = \mathrm{Hom}_{A^e}(A, M) \in \mathrm{Ob}(\mathrm{Rep}\, K_{par}(G))$ due the map π defined above, and B is a $K_{par}(G)$-module by Corollary 2.55. Now, if we take $X = B$ and $M = A \rtimes_\alpha G$, the map Δ defines a $(A \rtimes_\alpha G)^e$-module structure for $B \otimes_B (A \rtimes_\alpha G)$ given by

$$\begin{aligned}(au_g\#g \otimes bu_h\#h) \cdot (w \otimes cu_s\#s) &= [g] \cdot w \otimes (au_g\#g \otimes bu_h\#h) \cdot (cu_s\#s) \\ &= [g]w[g^{-1}] \otimes (au_g\#g)(cu_s\#s)(bu_h\#h).\end{aligned}$$

Furthermore, observe that for any $x \in A \rtimes_\alpha G$ we have

$$\begin{aligned}(au_g\#g \otimes bu_h\#h) \cdot (1_B \otimes x) &= e_g \otimes (au_g\#g)x(bu_h\#h) \\ &= 1_B \otimes (u_g\#1_G)(au_g\#g)x(bu_h\#h) \\ &= 1_B \otimes (au_g\#g)x(bu_h\#h).\end{aligned}$$

Whence we get that $B \otimes_B (A \rtimes_\alpha G)$ and $A \rtimes_\alpha G$ are isomorphic as $(A \rtimes_\alpha G)^e$-modules, because $B \otimes_B (A \rtimes_\alpha G) \cong A \rtimes_\alpha G$ as abelian groups by means of the map $b \otimes x \mapsto b \cdot x$, where $b \in B$ and $x \in A \rtimes_\alpha G$. Then using Proposition 4.5 to the particular case $X = B$ we have for any $M \in \mathrm{Ob}(\mathrm{Rep}(A \rtimes_\alpha G)^e)$ that

$$\begin{aligned} F_2 F_1 M &= \mathrm{Hom}_{K_{par}(G)}(B, \mathrm{Hom}_{A^e}(A, M)) \\ &\simeq \mathrm{Hom}_{(A \rtimes_\alpha G)^e}(B \otimes_B (A \rtimes_\alpha G), M) \\ &\simeq \mathrm{Hom}_{(A \rtimes_\alpha G)^e}(A \rtimes_\alpha G, M). \end{aligned}$$

Finally, if $f : X \to Y$ is a morphism in $\mathrm{Rep}(A \rtimes_\alpha G)^e$ we want the next diagram to commute

$$\begin{tikzcd} \mathrm{Hom}_{K_{par}(G)}(B, \mathrm{Hom}_{A^e}(A, X)) \arrow[r, "F_2F_1(f)"] \arrow[d, "\gamma_X"'] & \mathrm{Hom}_{K_{par}(G)}(B, \mathrm{Hom}_{A^e}(A, Y)) \arrow[d, "\gamma_Y"] \\ \mathrm{Hom}_{(A \rtimes_\alpha G)^e}(B \otimes_B (A \rtimes_\alpha G), X) \arrow[r, "F(f)"'] & \mathrm{Hom}_{(A \rtimes_\alpha G)^e}(B \otimes_B (A \rtimes_\alpha G)Y) \end{tikzcd}$$

where γ_X and γ_Y are the isomorphisms given by Γ_B, when $M = X$ and $M = Y$ respectively, i.e.

$$\gamma_X(H)(b \otimes au_g\#g) = (1_{A \rtimes_\alpha G} \otimes au_g\#g) \cdot H(b)(1_A)$$

and

$$\gamma_Y(H')(b \otimes au_g\#g) = (1_{A \rtimes_\alpha G} \otimes au_g\#g) \cdot H'(b)(1_A),$$

for $H \in \mathrm{Hom}_{K_{par}(G)}(B, \mathrm{Hom}_{A^e}(A, X))$, $H' \in \mathrm{Hom}_{K_{par}(G)}(B, \mathrm{Hom}_{A^e}(A, Y))$, $a \in A$ and $b \in B$. Now, observe that

$$\big(F_2F_1(f)(H)\big)(b)(a) = f(H(b)(a))$$

and

$$\big(F(f)(W)\big)(b \otimes au_g\#g) = f\big(W(b \otimes au_g\#g)\big),$$

where $W \in \mathrm{Hom}_{(A \rtimes_\alpha G)^e}(B \otimes_B (A \rtimes_\alpha G), X)$. Therefore,

$$\begin{aligned} \gamma_Y\big(F_2F_1(f)H\big)(b \otimes au_g\#g) &= (1_{A \rtimes_\alpha G} \otimes au_g\#g) \cdot \big(F_2F_1(f)H\big)(b)(1_A) \\ &= (1_{A \rtimes_\alpha G} \otimes au_g\#g) \cdot f\big(H(b)(1_A)\big) \\ &= f\big((1_{A \rtimes_\alpha G} \otimes au_g\#g) \cdot H(b)(1_A)\big) \\ &= f\big(\gamma_X(H)(b \otimes au_g\#g)\big) \\ &= F(f)\big(\gamma_X(H)\big)(b \otimes au_g\#g). \end{aligned}$$

Thus, the above diagram commutes. $\square$

Observe that if the functor F_2 is left exact and $F_1(N)$ is right F_2-*acyclic* for any injective object N in $\operatorname{Rep}(A \rtimes_\alpha G)^e$, then by Theorem 2.106 for any object M in $\operatorname{Rep}(A \rtimes_\alpha G)^e$ there exists a third quadrant spectral sequence with

$$E_2^{p,q} = (R^p F_2)(R^q F_1)M \underset{p}{\Rightarrow} R^{p+q}(F_2 F_1)M.$$

Now notice that

$$\begin{aligned}(R^p F_2)(R^q F_1)M &= (R^p F_2)(R^q \operatorname{Hom}_{A^e}(A, M)) \\ &= (R^p F_2)H^q(A, M) \\ &= R^p \operatorname{Hom}_{K_{par}(G)}(B, H^q(A, M)) \\ &= H^p_{par}(G, H^q(A, M))\end{aligned}$$

and

$$\begin{aligned}R^n(F_2 F_1)M &= (R^n F)M \\ &= R^n(\operatorname{Hom}_{(A \rtimes_\alpha G)^e}(A \rtimes_\alpha G, M)) \\ &= H^n(A \rtimes_\alpha G, M).\end{aligned}$$

Thus

$$E_2^{p,q} = H^p_{par}(G, H^q(A, M)) \underset{p}{\Rightarrow} H^{p+q}(A \rtimes_\alpha G, M),$$

which proves Theorem 4.4. So it only remains to check that the functors F_1 and F_2 have the desired properties. But first we have to show some necessary results.

The next fact is a corollary of [4, Theorem 1].

Lemma 4.7. *Let S be a finite commutative semigroup, in which all the elements are idempotents. Then $\mathbb{Z}S$ has a basis of orthogonal idempotents. Consequently the same basis works for KS.*

Proof. First define the map $\zeta : S \times S \to \mathbb{Z}$ by $\zeta(a, b) = 1$ if $a \leq b$ and $\zeta(a, b) = 0$ otherwise. Now define for each $c \in S$ the $\mathbb{Z}$-linear map $\zeta_c : \mathbb{Z}S \to \mathbb{Z}$ given by $\zeta_c(a) = \zeta(c, a)$, where $a \in S$. Then for $a, b \in S$ we have that:

- if $c \leq a$ and $c \leq b$, then $c \leq ab$, whence $\zeta_c(ab) = 1 = \zeta_c(a)\zeta_c(b)$,
- if $c \nleq a$ or $c \nleq b$, then $c \nleq ab$, whence $\zeta_c(ab) = 0 = \zeta_c(a)\zeta_c(b)$.

Therefore, ζ_c is a homomorphism of $\mathbb{Z}S$ into $\mathbb{Z}$. If there exists $x \in \mathbb{Z}S$ such that $\zeta_c(x) = 0$ for all $c \in S$, then $x = \sum_{a \in S} \psi(a)a$ for some set $\{\psi(a) \in \mathbb{Z} \mid a \in S\}$, and applying ζ_c we have that $0 = \sum_{a \geq c} \psi(a)$ for all $c \in S$. Since S is finite it has maximal elements, thus for each maximal element m we have that $0 = \sum_{a \geq m} \psi(a) = \psi(m)$. Hence, by descending induction we have that $\psi(a) = 0$ for all $a \in S$, therefore $x = 0$. Whence we have that if $x, y \in \mathbb{Z}S$ and $\zeta_c(x) = \zeta_c(y)$ for all $c \in S$, then $x = y$. Now define $\mu : S \times S \to \mathbb{Z}$ recursively as follows:

- $\mu(a, b) = 0$ if $a \nleq b$,
- $\mu(a, a) = 1$,
- if $a < b$, suppose that $\mu(a, z)$ has been defined on the set $\{z \mid a \leq z < b\}$. Then define
$$\mu(a, b) = -\sum_{a \leq z < b} \mu(a, z),$$
for $a, b \in S$.

Notice that
$$\sum_{a \leq z \leq b} \mu(a, z) = \delta_{a,b},$$
where $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ otherwise. For each $a \in S$ define
$$w_a = \sum_{b \in S} \mu(b, a) b \in \mathbb{Z}S.$$
Observe that, since $\mu(b, a) = 0$ if $b \nleq a$ and $\zeta_c(a) = 0$ if $c \nleq b$, then
$$\zeta_c(w_a) = \sum_{b \in S} \mu(b, a)\zeta_c(b) = \sum_{c \leq b \leq a} \mu(b, a) = \delta_{a,c}.$$
Then for $a, c \in S$ and $x \in \mathbb{Z}S$
$$\zeta_c(xw_a) = \zeta_c(x)\zeta_c(w_a) = \zeta_c(x)\delta_{a,c} = \zeta_a(x)\delta_{a,c} = \zeta_a(x)\zeta_c(w_a) = \zeta_c(\zeta_a(x)w_a).$$
Therefore
$$xw_a = \zeta_a(x)w_a, \text{ for all } a \in S \text{ and } x \in \mathbb{Z}S.$$
Hence, for any pair $a, b \in S$, $w_b w_a = \zeta_a(w_b)w_a = \delta_{a,b}w_a$, so that the w_a are pairwise orthogonal idempotents of $\mathbb{Z}S$. Finally, let $w = \sum_{a \in S} w_a$, observe that $\zeta_c(w) = 1$ for all $c \in S$. Then $\zeta_c(bw) = \zeta_c(b)\zeta_c(w) = \zeta_c(b)$ for all $c \in S$ so that $bw = w$ and w is thus an identity element of $\mathbb{Z}S$. Therefore given $x \in \mathbb{Z}S$ we may write
$$x = xe = \sum_{a \in S} xw_a = \sum_{a \in S} \zeta_c(x)w_a.$$
□

Lemma 4.8. *Let S be a commutative semigroup consisting of idempotent. Let, furthermore, K be a field and KS be the semigroup algebra of S. If I is a finitely generated ideal of KS then I is generated by an idempotent of KS.*

Proof. Let I be a finitely generated ideal of KS and let $r_1, r_2, ..., r_m$ be generators of I. Choose idempotents $u_1, ..., u_n$ of S such that each r_i is a combination of these idempotents, and let T be the subsemigroup of S generated by $u_1, ..., u_n$. Then T is a finite commutative semigroup consisting only of idempotents, and by Lemma 4.7 the space KT has a basis of orthogonal idempotents $w_1, ..., w_N$. Each generator r_i lies in KT and therefore we may write $r_i = \sum_j \alpha_{i,j} w_j$ for $i = 1, ..., n$ (with $\alpha_{i,j} \in K$). Whence $w_j r_i = \alpha_{i,j} w_j$. Moreover, the set

$$W = \{w_j \,|\, \alpha_{i,j} \neq 0 \text{ for some } i\}$$

is contained in I. Indeed, if $w_j \in W$, then there exists $r_i \in I$ such that $\alpha_{i,j} \neq 0$, hence $w_j = \alpha_{i,j}^{-1} w_j r_i \in I$. On the other hand, any generator of I is a K-linear combination of these elements and therefore the ideal generated by W coincides with I. Finally we show that the ideal generated by W is generated by the element $u = \sum_{w_j \in W} w_j \in I$. First observe that u is idempotent since $w_j w_i = 0$ if $i \neq j$ and

$$uu = \sum_{w_j, w_i \in W} w_j w_i = \sum_{w_j \in W} w_j w_j = \sum_{w_j \in W} w_j = u.$$

Moreover, $uw_j = w_j$ for each j. Hence, if $y \in I$ then $y = \sum_{w_j} b_j w_j$ with $b_j \in KS$, and therefore $yu = \sum_{w_j \in W} b_j w_j u = \sum_{w_j \in W} b_j w_j = y$, thus u acts as an identity for the elements of I. □

Remark 4.9. Recall that if A is a R-module then A is *flat* if for any finitely generated ideal J of R the map

$$J \otimes_R A \overset{j \otimes 1_A}{\to} R \otimes_R A$$

is injective where $j : J \to R$ is the inclusion map.

Lemma 4.10. *Any B-module X is flat.*

Proof. To use Remark 4.9 we have to show that for any finitely generated left ideal I of B, the morphism $I \otimes_B X \to B \otimes_B X \cong X$ is injective. By Lemma 4.8 and the fact that $B = KS$, where S is the commutative semigroup $S = \{e_{g_1} e_{g_2} ... e_{g_n} \,|\, g_i \in G,\, n \geq 1\}$, we have that I is generated by an idempotent u. Now assume that $\sum_i y_i \otimes x_i \in I \otimes_B X$ is such that $\sum_i y_i \otimes x_i = 0$ in $B \otimes_B X$, hence $\sum_i y_i \cdot x_i = 0$ in X. Since $y_i \in I$ for each i we have $y_i = uy_i$ and therefore

$$I \otimes_B X \ni \sum_i y_i \otimes_B x_i = \sum_i uy_i \otimes_B x_i = u \otimes_B \left(\sum_i y_i \cdot x_i \right) = 0,$$

which proves that $I \otimes_B X \to B \otimes_B X$ is injective. $\square$

Corollary 4.11. *The functor* $- \otimes_B (A \rtimes_\alpha G) : \operatorname{Rep} K_{par}(G) \to \operatorname{Rep}(A \rtimes_\alpha G)^e$ *is exact.*

Finally we can prove the next proposition which completes the proof of Theorem 4.4.

Proposition 4.12. *The functor* F_2 *is left exact and* $F_1(N)$ *is right* F_2*-acyclic for every injective object* N *in* $\operatorname{Rep}(A \rtimes_\alpha G)^e$.

Proof. We know that the Hom functor is left exact so that $F_2 = \operatorname{Hom}_{K_{par}(G)}(B, -)$ is left exact. On the other hand if N is an injective object in $\operatorname{Rep}(A \rtimes_\alpha G)^e$, then $\operatorname{Hom}_{(A \rtimes_\alpha G)^e}(-, N)$ is an exact functor and Corollary 4.11 says that $- \otimes_B (A \rtimes_\alpha G)$ is an exact functor, so the isomorphism of functors

$$\operatorname{Hom}_{K_{par}(G)}(-, \operatorname{Hom}_{A^e}(A, N)) \simeq \operatorname{Hom}_{(A \rtimes_\alpha G)^e}(- \otimes_B (A \rtimes_\alpha G), N)$$

implies that $\operatorname{Hom}_{K_{par}(G)}(-, \operatorname{Hom}_{A^e}(A, N))$ is exact. Hence $\operatorname{Ext}^n_{K_{par}(G)}(B, F_1(N)) = 0$ for any $n \geq 1$ and so $F_1(N)$ is F_2-acyclic. $\square$

CHAPTER 5

Globalization

Given a unital partial action α of G on a unital algebra $\mathcal{A}$, we work with an enveloping action $(\mathcal{B}, \beta)$ of $(\mathcal{A}, \alpha)$ and the multiplier algebra $\mathcal{M}(\mathcal{B})$ of $\mathcal{B}$. We study the globalization problem for the partial cohomology with values in $\mathcal{A}$. We reduce the globalization problem to an extendibility property of partial cocycles. Furthermore, we show that if $\mathcal{A}$ is a product of blocks then any cocycle from $Z^n_{par}(G, \mathcal{A})$ is globalizable and that globalizations of cohomologous cocycles are also cohomologous. Finally, under the above assumption of $\mathcal{A}$, we prove that $H^n_{par}(G, M)$ is isomorphic to the usual cohomology group $H^n(G, \mathcal{M}(\mathcal{B}))$. This chapter corresponds to the study of the final part of [5].

5.1 From globalization to an extendibility property

In this section α will be a unital partial action of a group G on an algebra $\mathcal{A}$. First we will recall some definitions and results extracted form [6] that will be necessary for the development of this work.

Definition 5.1. *An action β of a group G on an algebra $\mathcal{B}$ is said to be an* ***enveloping action*** *for the partial action α of G on an algebra $\mathcal{A}$ if there exists and algebra isomorphism φ of $\mathcal{A}$ onto an ideal of $\mathcal{B}$ such that for all $g \in G$ the following three properties are satisfied.*

(i) $\varphi(\mathcal{D}_g) = \varphi(\mathcal{A}) \cap \beta_g(\varphi(\mathcal{A}))$;

(ii) $\varphi \circ \alpha_g(x) = \beta_g \circ \varphi(x)$ *for all* x *in* $\mathcal{D}_{g^{-1}}$;

(iii) $\mathcal{B}$ *is generated by* $\bigcup_{g\in G} \beta_g(\varphi(\mathcal{A}))$.

From [6] we have the next theorem.

Theorem 5.2. *Let* $\mathcal{A}$ *be a unital algebra. Then a partial action* α *of a group* G *on* $\mathcal{A}$ *admits an enveloping action* β *if and only if each ideal* $\mathcal{D}_g$ $(g \in G)$ *is a unital algebra. Moreover,* β*, if it exists, is unique up to isomorphisms.*

Definition 5.3. *Let* L *and* R *be* K*-linear maps from* $\mathcal{A}$ *to itself. We will say that the pair* (L, R) *is a* ***multiplier*** *of* $\mathcal{A}$ *if, for every* a *and* b *in* $\mathcal{A}$*, one has that*

(i) $L(ab) = L(a)b$,

(ii) $R(ab) = aR(b)$,

(iii) $R(a)b = aL(b)$.

Remark 5.4. We will denote $L(a)$ by La and $R(a)$ by aR, thus Definition 5.3 says that the pair (L, R) is a multiplier of $\mathcal{A}$ if, for every a and b in $\mathcal{A}$, one has that

(i) $L(ab) = (La)b$,

(ii) $(ab)R = a(bR)$,

(iii) $(aR)b = a(Lb)$.

Moreover given another multiplier (L', R') of $\mathcal{A}$ we have that $LL'x = L \circ L'(x)$ and $xRR' = R' \circ R(x)$.

If $K = \mathcal{A}$, then every (L, R) in $End_{\mathcal{A}}(\mathcal{A}_{\mathcal{A}}) \times End_{\mathcal{A}}({}_{\mathcal{A}}\mathcal{A})$ satisfies conditions (i) and (ii) of Definition 5.3. Then (L, R) is a multiplier if, and only if, $(aR)b = a(Lb)$.

Definition 5.5. *The multiplier algebra of* $\mathcal{A}$ *is the set* $\mathcal{M}(\mathcal{A})$ *consisting of all multipliers* (L, R) *of* $\mathcal{A}$*. Given* (L, R) *and* (L', R') *in* $\mathcal{M}(\mathcal{A})$*, and* $\lambda \in K$ *we define*

$$\begin{aligned}
\lambda(L, R) &= (\lambda L, \lambda R),\\
(L, R) + (L', R') &= (L + L', R + R'),\\
(L, R)(L', R') &= (LL', RR').
\end{aligned}$$

Notice that by Remark 2.34 the algebra $\mathcal{A}$ has a natural structure of $K_{\mathrm{par}}(G)$-module. We also fix an enveloping action $(\mathcal{B}, \beta)$ of α with an injective morphism $\varphi : (\mathcal{A}, \alpha) \to (\mathcal{B}, \beta)$. Since the algebra $\mathcal{B}$ does not always have an identity element, and since the technique used in [5] needs to have a unital algebra we will work more generally with the multiplier algebra $\mathcal{M}(\mathcal{B})$ of $\mathcal{B}$. For a multiplier $\gamma = (L, R) \in \mathcal{M}(\mathcal{B})$ and $b \in \mathcal{B}$ we set $b\gamma = bR$ and $\gamma b = Lb$. Thus one always has $(a\gamma)b = a(\gamma b)$ for arbitrary $a, b \in \mathcal{B}$.

The action β induces an action β^* of G on $\mathcal{M}(\mathcal{B})$, where $\beta^*_g(u) = \beta_g u \beta_g^{-1}$, for $u \in \mathcal{M}(\mathcal{B})$ and $g \in G$, that is, if $u = (L, R)$ then

$$\beta^*_g(u) = (\beta_g \circ L \circ \beta_g^{-1}, \beta_g \circ R \circ \beta_g^{-1}) = (\beta_g L \beta_g^{-1}, \beta_g^{-1} R \beta_g).$$

Indeed, to prove that β^* is an action of G on $\mathcal{M}(\mathcal{B})$, observe that for $u = (L, R) \in \mathcal{M}(\mathcal{B})$, $g \in G$ and arbitrary $a, b \in \mathcal{B}$ we have

$$\begin{aligned} \beta^*(R)(ab) &= \beta_g \circ R \circ \beta_{g^{-1}}(ab) \\ &= \beta_g(R(\beta_{g^{-1}}(a)\beta_{g^{-1}}(b))) \\ &= \beta_g(\beta_{g^{-1}}(a)R(\beta_{g^{-1}}(b))) \\ &= a(\beta_g \circ R \circ \beta_{g^{-1}}(b)) \\ &= a\beta^*(R)(b), \end{aligned} \qquad \begin{aligned} \beta^*(L)(ab) &= \beta_g \circ L \circ \beta_{g^{-1}}(ab) \\ &= \beta_g(L(\beta_{g^{-1}}(a)\beta_{g^{-1}}(b))) \\ &= \beta_g(L(\beta_{g^{-1}}(a))\beta_{g^{-1}}(b)) \\ &= (\beta_g \circ L \circ \beta_{g^{-1}}(a))b \\ &= \beta^*(L)(a)b, \end{aligned}$$

and

$$\begin{aligned} \beta^*(R)(a)b &= \beta_g R \beta_{g^{-1}}(a)b \\ &= \beta_g(R(\beta_{g^{-1}}(a))\beta_{g^{-1}}(b)) \\ &= \beta_g(\beta_{g^{-1}}(a)L(\beta_{g^{-1}}(b))) \\ &= a(\beta_g L \beta_{g^{-1}}(b)) \\ &= a\beta^*(L)(b). \end{aligned}$$

Hence $\beta^*(u)$ is in $\mathcal{M}(\mathcal{A})$. Finally observe

$$\beta^*_h \beta^*_g(u) = \beta_h \beta_g u \beta_{g^{-1}} \beta_{h^{-1}} = \beta_{hg} u \beta_{(hg)^{-1}}$$

and

$$\beta^*_g(uv) = \beta_g uv \beta_{g^{-1}} = (\beta_g u \beta_{g^{-1}})(\beta_g v \beta_{g^{-1}}) = \beta^*_g(u)\beta^*_g(v).$$

Denote by $C^n(G, \mathcal{M}(\mathcal{B}))$, $Z^n(G, \mathcal{M}(\mathcal{B}))$, $B^n(G, \mathcal{M}(\mathcal{B}))$ and $H^n(G, \mathcal{M}(\mathcal{B}))$ the corresponding groups of n-cochains, n-cocycles, n-coboundaries and n-cohomologies of G with values in the additive group of $\mathcal{M}(\mathcal{B})$.

Definition 5.6. *Given $n \in \mathbb{N}$ and $u \in C^n(G, \mathcal{M}(\mathcal{B}))$, define the restriction of u to $\mathcal{A}$ to be the map $w : G^n \to \mathcal{A}$, such that*

$$\varphi\left(w\left(g_1, \ldots, g_n\right)\right) = \varphi\left(1_{(g_1, \ldots, g_n)}\right) u\left(g_1, \ldots, g_n\right)$$

where $g_1, \ldots, g_n \in G$. If $n = 0$ and $u \in C^0(G, \mathcal{M}(\mathcal{B})) = \mathcal{M}(\mathcal{B})$, then w is the element of $\mathcal{A}$, satisfying

$$\varphi(w) = \varphi(1_{\mathcal{A}})u.$$

Notice that we could replace

$$\varphi\left(w\left(g_1, \ldots, g_n\right)\right) = \varphi\left(1_{(g_1, \ldots, g_n)}\right) u\left(g_1, \ldots, g_n\right)$$

by

$$\varphi\left(w\left(g_1, \ldots, g_n\right)\right) = u\left(g_1, \ldots, g_n\right) \varphi\left(1_{(g_1, \ldots, g_n)}\right)$$

in Definition 5.6 because both options are equivalent. Indeed, since $\varphi(\mathcal{A})$ is an ideal of $\mathcal{B}$ and $\varphi(1_{(g_1,\ldots,g_n)})$ is a central idempotent of $\varphi(\mathcal{A})$

$$\begin{aligned}
\varphi(1_{(g_1,\ldots,g_n)})u(g_1, \ldots, g_n) &= \big(\varphi(1_{(g_1,\ldots,g_n)})\varphi(1_{(g_1,\ldots,g_n)})\big)u(g_1, \ldots, g_n) \\
&= \varphi(1_{(g_1,\ldots,g_n)}\big(\varphi(1_{(g_1,\ldots,g_n)})u(g_1, \ldots, g_n)\big) \in \varphi(\mathcal{A}) \\
&= \big(\varphi(1_{(g_1,\ldots,g_n)})u(g_1, \ldots, g_n)\big)\varphi(1_{(g_1,\ldots,g_n)}) \\
&= \varphi(1_{(g_1,\ldots,g_n)})\big(u(g_1, \ldots, g_n)\varphi(1_{(g_1,\ldots,g_n)})\big) \\
&= \big(u(g_1, \ldots, g_n)\varphi(1_{(g_1,\ldots,g_n)})\big)\varphi(1_{(g_1,\ldots,g_n)}) \\
&= u(g_1, \ldots, g_n)\varphi(1_{(g_1,\ldots,g_n)}).
\end{aligned}$$

We will write $\rho(u) = w$ when w is a restriction of u. Note that $\rho(u) \in C^n_{par}(G, \mathcal{A})$ since

$$\begin{aligned}
\rho(u)(g_1, \ldots, g_n) &= \varphi^{-1}\big(\varphi(1_{(g_1,\ldots,g_n)})(\varphi(1_{(g_1,\ldots,g_n)})u(g_1, \ldots, g_n))\big) \\
&= 1_{(g_1,\ldots,g_n)}\varphi^{-1}\big(\varphi(1_{(g_1,\ldots,g_n)})u(g_1, \ldots, g_n)\big) \in 1_{(g_1,\ldots,g_n)}\mathcal{A}
\end{aligned}$$

Proposition 5.7. *The restriction map $\rho : C^n(G, \mathcal{M}(\mathcal{B})) \to C^n_{par}(G, \mathcal{A})$ induces a homomorphism of the cohomology groups $H^n(G, \mathcal{M}(\mathcal{B})) \to H^n_{par}(G, \mathcal{A})$.*

Proof. First observe that ρ is an homomorphism since

$$\begin{aligned}
\varphi\left(\rho(u+v)\left(g_1, \ldots, g_n\right)\right) &= \varphi\left(1_{(g_1,\ldots,g_n)}\right)\left(u+v\right)\left(g_1, \ldots, g_n\right) \\
&= \varphi\left(1_{(g_1,\ldots,g_n)}\right) u\left(g_1, \ldots, g_n\right) + \varphi\left(1_{(g_1,\ldots,g_n)}\right) v\left(g_1, \ldots, g_n\right) \\
&= \varphi\left(\rho(u)\left(g_1, \ldots, g_n\right)\right) + \varphi\left(\rho(v)\left(g_1, \ldots, g_n\right)\right).
\end{aligned}$$

So it only remains to show that ρ commutes with the coboundary operators, that is, the next diagram commutes

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & C^{n+1}(G,\mathcal{M}(\mathcal{B})) & \xrightarrow{\delta^{n+1}} & C^{n}(G,\mathcal{M}(\mathcal{B})) & \xrightarrow{\delta^{n}} & C^{n-1}(G,\mathcal{M}(\mathcal{B})) & \xrightarrow{\delta^{n-1}} \cdots \\
& \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & \\
\cdots \longrightarrow & C^{n}_{par}(G,\mathcal{A}) & \xrightarrow{\delta^{n+1}} & C^{n}_{par}(G,\mathcal{A}) & \xrightarrow{\delta^{n}} & C^{n}_{par}(G,\mathcal{A}) & \xrightarrow{\delta^{n-1}} \cdots
\end{array}
$$

Let $n = 0$ and $u = (L, R) \in C^0(G, \mathcal{M}(\mathcal{B})) = \mathcal{M}(\mathcal{B})$. Then for all $g \in G$ by Definition 5.6 and the fact that φ is a morphism of partial actions we have

$$
\begin{aligned}
\varphi\left(\left(\delta^0\rho(u)\right)(g)\right) &= \varphi\left(\alpha_g\left(1_{g^{-1}}\rho(u)\right) - 1_g\rho(u)\right) \\
&= \beta_g\left(\varphi\left(1_{g^{-1}}\right)\varphi(\rho(u))\right) - \varphi\left(1_g\right)\varphi(\rho(u)) \\
&= \beta_g\left(\varphi\left(1_{g^{-1}}\right)\varphi\left(1_{\mathcal{A}}\right)u\right) - \varphi\left(1_g\right)\varphi\left(1_{\mathcal{A}}\right)u \\
&= \beta_g(\varphi(1_{g^{-1}})u) - \varphi(1_g)u \\
&= \beta_g(R(\varphi(1_{g^{-1}}))) - \varphi(1_g)u \\
&= \beta_g(R(\varphi(\alpha_{g^{-1}}(1_g)))) - \varphi(1_g)u \\
&= \beta_g(R(\beta_{g^{-1}}(\varphi(1_g)))) - \varphi(1_g)u \\
&= \varphi\left(1_g\right)\left(\beta^*_g(u) - u\right) \\
&= \varphi\left(1_g\right)\left(\delta^0 u\right)(g) \\
&= \varphi\left(\rho\left(\delta^0 u\right)(g)\right),
\end{aligned}
$$

whence $\delta^0\rho(u) = \rho(\delta^0 u)$.

Consider now $n \in \mathbb{N}$ and $u \in C^n(G, \mathcal{M}(\mathcal{B}))$. For arbitrary $g_1, \ldots, g_{n+1} \in G$, first notice that

$$
\begin{aligned}
&\beta_{g_1}\left(\varphi(1_{g_1^{-1}})\varphi\left(1_{(g_2,\ldots,g_{n+1})}\right)u\left(g_2,\ldots,g_{n+1}\right)\right) \\
&\quad= \beta_{g_1}\left(R_{(g_2,\ldots,g_{n+1})}\left(\varphi(1_{g_1^{-1}})\varphi\left(1_{(g_2,\ldots,g_{n+1})}\right)\right)\right) \\
&\quad= \beta_{g_1}\left(R_{(g_2,\ldots,g_{n+1})}\left(\beta_{g_1^{-1}}\varphi\left(1_{(g_1,g_2,\ldots,g_{n+1})}\right)\right)\right) \\
&\quad= \varphi\left(1_{(g_1,g_2,\ldots,g_{n+1})}\right)\beta^*_{g_1}(u(g_2,\ldots,g_{n+1})),
\end{aligned}
$$

where $u_{(g_2,\ldots,g_{n+1})} = (L_{(g_2,\ldots,g_{n+1})}, R_{(g_2,\ldots,g_{n+1})})$. Thus

$$\begin{aligned}
\varphi\left(\left(\delta^n\rho(u)\right)(g_1,\ldots,g_{n+1})\right) =& \varphi\Big(\alpha_{g_1}\left(1_{g_1^{-1}}\rho(u)\left(g_2,\ldots,g_{n+1}\right)\right) \\
&+\sum_{i=1}^{n}(-1)^i 1_{g_1\ldots g_i}\rho(u)\left(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1}\right) \\
&+(-1)^{n+1}1_{g_1\ldots g_{n+1}}\rho(u)\left(g_1,\ldots,g_n\right)\Big) \\
=& \beta_{g_1}\left(\varphi\left(1_{g_1-1}\right)\varphi\left(\rho(u)\left(g_2,\ldots,g_{n+1}\right)\right)\right) \\
&+\sum_{i=1}^{n}(-1)^i\varphi\left(1_{g_1\ldots g_i}\right)\varphi\left(\rho(u)\left(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1}\right)\right) \\
&+(-1)^{n+1}\varphi\left(1_{g_1\ldots g_{n+1}}\right)\varphi\left(\rho(u)\left(g_1,\ldots,g_n\right)\right) \\
=& \beta_{g_1}\left(\varphi(1_{g_1^{-1}})\varphi\left(1_{(g_2,\ldots,g_{n+1})}\right)u\left(g_2,\ldots,g_{n+1}\right)\right) \\
&+\sum_{i=1}^{n}(-1)^i\varphi\left(1_{g_1\ldots g_i}\right)\varphi\left(1_{(g_1,\ldots,g_1g_1+1,\ldots,g_{n+1})}\right) \\
&\times u\left(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1}\right) \\
&+(-1)^{n+1}\varphi\left(1_{g_1\ldots g_{n+1}}\right)\varphi\left(1_{(g_1,\ldots,g_n)}\right)u\left(g_1,\ldots,g_n\right) \\
=& \varphi(1_{(g_1,\ldots,g_{n+1})})\left(\beta^*_{g_1}\left(u\left(g_2,\ldots,g_{n+1}\right)\right)\right. \\
&+\sum_{i=1}^{n}(-1)^i u\left(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1}\right) \\
&\left.+(-1)^{n+1}u\left(g_1,\ldots,g_n\right)\right) \\
=& \varphi(1_{(g_1,\ldots,g_{n+1})})\left(\delta^n u\right)\left(g_1,\ldots,g_{n+1}\right) \\
=& \varphi\left(\rho\left(\delta^n u\right)\left(g_1,\ldots,g_{n+1}\right)\right).
\end{aligned}$$

so that $\delta^n\rho(u) = \rho(\delta^n u)$.

□

Definition 5.8. *Given $w \in Z^n_{par}(G,\mathcal{A})$, by a* ***globalization*** *of w we mean $u \in Z^n(G,\mathcal{M}(\mathcal{B}))$ satisfying*

$$\varphi\left(w\left(g_1,\ldots,g_n\right)\right) = \varphi\left(1_{(g_1,\ldots,g_n)}\right)u\left(g_1,\ldots,g_n\right).$$

If w admits a globalization, then we say that w is ***globalizable****.*

In the proof of Theorem 5.2 given in [6] the enveloping action $(\mathcal{B},\beta)$ for $(\mathcal{A},\alpha)$ was constructed as the restriction of the global action $(\mathcal{F},\beta)$ to the subalgebra

$$\mathcal{B} = \sum_{g\in G}\beta_g(\varphi(\mathcal{A})).$$

Here $\mathcal{F}$ is the ring of functions $G \to \mathcal{A}$ and

$$\beta_g(f)|_t = f(g^{-1}t)$$

for all $x, t \in G$, where the notation $f|_t$ is used for the value $f(t)$ if $f \in \mathcal{F}$. The injective morphism $\varphi : \mathcal{A} \to \mathcal{B}$ is then defined by the formula

$$\varphi(a)|_t = \alpha_{t^{-1}}(1_t a).$$

Notice that

$$\begin{aligned}
\varphi(\alpha_g(1_{g^{-1}}a))|_t &= \alpha_{t^{-1}}(1_t\alpha_g(1_{g^{-1}}a)) \\
&= \alpha_{t^{-1}}(\alpha_g(1_{g^{-1}}1_{g^{-1}t}a)) \\
&= \alpha_{t^{-1}g}(1_{g^{-1}}1_{g^{-1}t}a) \\
&= \varphi(1_{g^{-1}}a)|_{g^{-1}t} \\
&= \beta_g(\varphi(1_{g^{-1}}a))|_t,
\end{aligned}$$

then $\varphi(\alpha_g(1_{g^{-1}}a)) = \beta_g(\varphi(1_{g^{-1}}a))$. Clearly, $\varphi(\mathcal{A}) \subseteq \mathcal{B}$, so φ is a morphism $(\mathcal{A}, \alpha) \to (\mathcal{B}, \beta)$ too. Since all enveloping actions of $(\mathcal{A}, \alpha)$ are isomorphic to each other by Theorem 5.2, we may assume that $(\mathcal{B}, \beta)$ and φ are of this form.

Remark 5.9. Notice that if $u \in C^n(G, \mathcal{F})$ is such that $u(g_1, \dots, g_n)\mathcal{B}, \mathcal{B}u(g_1, \dots, g_n) \subseteq \mathcal{B}$ then $u(g_1, \dots, g_n) \in \mathcal{M}(\mathcal{B})$ in the next sense, define $R : b \in \mathcal{B} \mapsto bu(g_1, \dots, g_n) \in \mathcal{B}$ and $L : b \in \mathcal{B} \mapsto u(g_1, \dots, g_n)b \in \mathcal{B}$, thus the pair (L, R) is in $\mathrm{End}(\mathcal{B}_{\mathcal{B}}) \times \mathrm{End}({}_{\mathcal{B}}\mathcal{B})$. So identify $u(g_1, \dots, g_n)$ with (L, R), finally notice that for arbitrary $a, b \in \mathcal{B}$ we have

$$(aR)b = (au(g_1, \dots, g_n))b = a(u(g_1, \dots, g_n)b) = a(Lb).$$

Hence $u(g_1, \dots, g_n) \in \mathcal{M}(\mathcal{B})$.

Lemma 5.10. *Any $w \in Z^0_{par}(G, \mathcal{A})$ is uniquely globalizable.*

Proof. Define $u \in C^0(G, \mathcal{F}) = \mathcal{F}$ to be the constant function taking the value $w \in \mathcal{A}$ at any $t \in G$. Using Remark 3.40 we have $1_{t^{-1}}w = \alpha_{t^{-1}}(1_t w)$ since $w \in Z^0_{par}(G, \mathcal{A})$, then using that $\varphi(a)|_t = \alpha_{t^{-1}}(1_t a)$, we obtain

$$\varphi(1_{\mathcal{A}})|_t \, u|_t = 1_{t^{-1}}w = \alpha_{t^{-1}}(1_t w) = \varphi(w)|_t,$$

yielding Definition 5.8.

$$\begin{aligned}
\beta_g(\varphi(a))|_t u|_t &= \varphi(a)|_{g^{-1}t}w = \alpha_{t^{-1}g}(1_{g^{-1}t}a)w = \alpha_{t^{-1}g}(1_{g^{-1}t}a) \cdot 1_{t^{-1}g}w \\
&= \alpha_{t^{-1}g}(1_{g^{-1}t}aw) = \beta_g(\varphi(aw))|_t.
\end{aligned}$$

Then $\beta_g(\varphi(a))|_t u|_t = \beta_g(\varphi(aw))|_t$, so $\beta_g(\varphi(\mathcal{A}))u \subseteq \beta_g(\varphi(\mathcal{A}))$. Hence $\mathcal{B}u \subseteq \mathcal{B}$ since $\mathcal{B} = \sum_{g\in G}\beta_g(\varphi(\mathcal{A}))$. In a similar way $u|_t\,\beta_g(\varphi(a))|_t = \beta_g(\varphi(wa))|_t$, which implies $u\mathcal{B} \subseteq \mathcal{B}$, and thus $u \in C^0(G, \mathcal{M}(\mathcal{B}))$.

To prove the 0-cocycle identity $\beta_g^*(u) = u$ for u, it suffices to show that $\beta_g(uf) = u\beta_g(f)$ for any $f \in \mathcal{F}$. We have that

$$\beta_g(uf)|_t = (uf)|_{g^{-1}t} = u|_{g^{-1}t} f|_{g^{-1}t} = u|_t \beta_g(f)|_t,$$

whence $u \in Z^0(G, \mathcal{M}(\mathcal{B}))$. Now if u_1 and u_2 in $Z^0(G, \mathcal{M}(\mathcal{B}))$ are globalizations of w, then $\varphi(1_\mathcal{A})u_1 = \varphi(1_\mathcal{A})u_2$, using the 0-cocycle identity we have

$$\begin{aligned}\beta_g(\varphi(1_\mathcal{A})u_i) &= \beta_g(\varphi(1_\mathcal{A})\beta_g u_i \beta_{g^{-1}})\\ &= \varphi(1_\mathcal{A})\beta_g u_i\\ &= \beta_g(\varphi(1_\mathcal{A}))u_i,\end{aligned}$$

for $i = 1, 2$. Then $\beta_g(\varphi(a))u_1 = \beta_g(\varphi(a))u_2$ for all $g \in G$ and all $a \in \mathcal{A}$, hence $u_1 = u_2$ since $\mathcal{B} = \sum_{g\in G}\beta_g(\varphi(\mathcal{A}))$. □

Remark 5.11. Recall that $\mathcal{A}$ has a trivial G-module structure given by the trivial action $g \cdot x = x$, for all $g \in G$ and $x \in \mathcal{A}$.

For the case $w \in Z^n_{par}(G, \mathcal{A})$, $n \in N$, we will need the next lemma.

Lemma 5.12. *Let $\widetilde{w} \in C^n(G, \mathcal{A})$. Then $u \in C^n(G, \mathcal{F})$, defined by*

$$\begin{aligned}u\left(g_1, \dots, g_n\right)|_t =&(-1)^n \widetilde{w}\left(t^{-1}, g_1, \dots, g_{n-1}\right) + \widetilde{w}\left(t^{-1}g_1, g_2, \dots, g_n\right)\\ &+\sum_{i=1}^{n-1}(-1)^i \widetilde{w}\left(t^{-1}, g_1, \dots, g_i g_{i+1}, \dots, g_n\right)\end{aligned}$$

is an n-cocycle with respect to the action β of G on $\mathcal{F}$.

Proof. Observe that

$$u\left(g_1, \dots, g_n\right)|_t = \widetilde{w}\left(g_1, \dots, g_n\right) - \left(\tilde{\delta}^n \widetilde{w}\right)\left(t^{-1}, g_1, \dots, g_n\right) \quad (1)$$

where $\tilde{\delta}^n : C^n(G, \mathcal{A}) \to C^{n+1}(G, \mathcal{A})$ is the coboundary operator which corresponds to the trivial G-module, i.e.

$$\begin{aligned}\left(\tilde{\delta}^n \widetilde{w}\right)\left(g_1, \dots, g_{n+1}\right) =& \widetilde{w}\left(g_2, \dots, g_{n+1}\right)\\ &+\sum_{i=1}^{n}(-1)^i \widetilde{w}\left(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}\right)\\ &+(-1)^{n+1}\widetilde{w}\left(g_1, \dots, g_n\right)\end{aligned}$$

Calculating the value of $(\delta^n u)(g_1,\dots,g_{n+1})$ at $t\in G$, we obtain using that $\beta_g(f)|_t = f(g^{-1}t)$

$$u(g_2,\dots,g_{n+1})|_{g_1^{-1}t} + \sum_{i=1}^{n}(-1)^i u(g_1,\dots,g_ig_{i+1},\dots,g_{n+1})|_t \\ + (-1)^{n+1}u(g_1,\dots,g_n)\big|_t ,$$

which in view of (1) equals

$$\begin{aligned}&\tilde{w}(g_2,\dots,g_{n+1}) - (\tilde{\delta}^n\tilde{w})\left(t^{-1}g_1,g_2,\dots,g_{n+1}\right)\\ &+\sum_{i=1}^{n}(-1)^i\tilde{w}(g_1,\dots,g_ig_{i+1},\dots,g_{n+1})\\ &+\sum_{i=1}^{n}(-1)^{i+1}(\tilde{\delta}^n\tilde{w})\left(t^{-1},g_1,\dots,g_ig_{i+1},\dots,g_{n+1}\right)\\ &+(-1)^{n+1}\tilde{w}(g_1,\dots,g_n)+(-1)^n(\tilde{\delta}^n\tilde{w})\left(t^{-1},g_1,\dots,g_n\right).\end{aligned}$$

The latter is readily seen to be $\left(\tilde{\delta}^{n+1}\tilde{\delta}^n\tilde{w}\right)(t^{-1},g_1,\dots,g_{n+1}) = 0_{\mathcal{A}}$. □

The existence of a globalization of a globalization of $w\in Z^n_{par}(G,\mathcal{A})$ is equivalent to certain extendibility property.

Definition 5.13. *For any $f\in C^n(G,\mathcal{A})$ define $\tilde{\delta}^n : C^n(G,\mathcal{A})\to C^{n+1}(G,\mathcal{A})$ by*

$$\begin{aligned}(\tilde{\delta}^n f)(g_1,\dots,g_{n+1}) =&\alpha_{g_1}\left(1_{g_1^{-1}}f(g_2,\dots,g_{n+1})\right)\\ &+\sum_{i=1}^{n}(-1)^i 1_{g_1}f(g_1,\dots,g_ig_{i+1},\dots,g_{n+1})\\ &+(-1)^{n+1}1_{g_1}f(g_1,\dots,g_n).\end{aligned}$$

Theorem 5.14. *A cocycle $w\in Z^n_{par}(G,\mathcal{A})$, $n\in\mathbb{N}$, is globalizable if, and only if, there exists $\tilde{w}\in C^n(G,\mathcal{A})$ such that*

$$\tilde{\delta}^n\tilde{w} = 0 \tag{5.1}$$

and

$$w(g_1,\dots,g_n) = 1_{(g_1,\dots,g_n)}\tilde{w}(g_1,\dots,g_n), \tag{5.2}$$

for all $g_1,\dots,g_n\in G$.

Proof. If w is globalizable and $u \in Z^n(G, \mathcal{M}(\mathcal{B}))$ is its globalization, then we define $\tilde{w} \in C^n(G, \mathcal{A})$ such that

$$\varphi\left(\tilde{w}\left(g_1, \ldots, g_n\right)\right) = \varphi\left(1_{\mathcal{A}}\right) u\left(g_1, \ldots, g_n\right) = u\left(g_1, \ldots, g_n\right) \varphi\left(1_{\mathcal{A}}\right).$$

Clearly, $\tilde{w}\left(g_1, \ldots, g_n\right) \in \mathcal{A}$, since $\varphi(\mathcal{A})$ is an ideal in $\mathcal{B}$, and moreover (5.2) is satisfied. Note that

$$\begin{aligned}
\big(\beta_g(\varphi(1_{\mathcal{A}}))\varphi(1_{\mathcal{A}})\big)|_t &= \beta_g(\varphi(1_{\mathcal{A}}))|_t \varphi(1_{\mathcal{A}})|_t \\
&= \varphi(1_{\mathcal{A}})|_{g^{-1}t}\, \alpha_{t^{-1}}(1_t 1_{\mathcal{A}}) \\
&= \alpha_{t^{-1}g}(1_{g^{-1}t})\, \alpha_{t^{-1}}(1_t) \\
&= 1_{t^{-1}g} 1_{t^{-1}} \\
&= \alpha_{t^{-1}}(1_g 1_t) \\
&= \varphi(1_g)|_t,
\end{aligned}$$

thus $\beta_g(\varphi(1_{\mathcal{A}}))\varphi(1_{\mathcal{A}}) = \varphi(1_g)$. Hence

$$\begin{aligned}
\varphi(\alpha_g(1_{g^{-1}}a)) &= \varphi(\alpha_g(1_{g^{-1}}a))\varphi(1_g) \\
&= \beta_g(\varphi(1_{g^{-1}}a))\beta_g(\varphi(1_{\mathcal{A}}))\varphi(1_{\mathcal{A}}) \\
&= \beta_g(\varphi(1_{g^{-1}}a))\varphi(1_{\mathcal{A}}),
\end{aligned}$$

then $\varphi(\alpha_g(1_{g^{-1}}a)) = \beta_g(\varphi(1_{g^{-1}}a))\varphi(1_{\mathcal{A}})$. Therefore

$$\begin{aligned}
\beta^*_{g_1}\left(u\left(g_2, \ldots, g_{n+1}\right)\right)\varphi\left(1_{g_1}\right) &= \left(\beta_{g_1} u\left(g_2, \ldots, g_{n+1}\right)\beta_{g_1}^{-1}\right)\left(\beta_{g_1}\left(\varphi\left(1_{\mathcal{A}}\right)\right)\varphi\left(1_{\mathcal{A}}\right)\right) \\
&= \left(\beta_{g_1}\left(u\left(g_2, \ldots, g_{n+1}\right)\varphi\left(1_{\mathcal{A}}\right)\right)\right)\varphi\left(1_{\mathcal{A}}\right) \\
&= \left(\beta_{g_1}\left[\varphi\left(\widetilde{w}\left(g_2, \ldots, g_{n+1}\right)\right)\right]\right)\varphi\left(1_{\mathcal{A}}\right) \\
&= \varphi(\alpha_{g_1}(1_{g_1^{-1}}\tilde{w}(g_2, \ldots, g_{n+1}))).
\end{aligned}$$

Thus using the formula

$$\beta^*_{g_1}\left(u\left(g_2, \ldots, g_{n+1}\right)\right)\varphi\left(1_{g_1}\right) = \varphi(\alpha_{g_1}(1_{g_1^{-1}}\tilde{w}(g_2, \ldots, g_{n+1})))$$

we obtain (5.1) by applying both sides of the cocycle identity

$$\begin{aligned}
\beta^*_{g_1}\left(u\left(g_2, \ldots, g_{n+1}\right)\right) + \sum_{i=1}^{n}(-1)^i u\left(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}\right)& \\
+ (-1)^{n+1} u\left(g_1, \ldots, g_n\right) = 0&
\end{aligned}$$

to $\varphi(1_{g_1})$.

Conversely, given $\tilde{w} \in C^n(G, \mathcal{A})$ satisfying (5.1) and (5.2), define $u \in C^n(G, \mathcal{F})$ using Lemma 5.12 by

$$\begin{aligned} u(g_1, \ldots, g_n)\big|_t = &(-1)^n \widetilde{w}\left(t^{-1}, g_1, \ldots, g_{n-1}\right) + \widetilde{w}\left(t^{-1}g_1, g_2, \ldots, g_n\right) \\ &+ \sum_{i=1}^{n-1} (-1)^i \widetilde{w}\left(t^{-1}, g_1, \ldots, g_i g_{i+1}, \ldots, g_n\right). \end{aligned}$$

Then $u \in Z^n(G, \mathcal{F})$. Now using that $\varphi(a)|_t = \alpha_{t^{-1}}(1_t a)$, (5.2) and the cocycle identity for w, we obtain

$$\begin{aligned} \varphi\left(w(g_1, \ldots, g_n)\right)\big|_t &= \alpha_{t^{-1}}\left(1_t w(g_1, \ldots, g_n)\right) \\ &= 1_{t^{-1}} w\left(t^{-1}g_1, g_2, \ldots, g_n\right) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i 1_{t^{-1}g_1 \ldots g_i} w\left(t^{-1}, g_1, \ldots, g_i g_{i+1}, \ldots, g_n\right) \\ &\quad + (-1)^n 1_{t^{-1}g_1 \ldots g_n} w\left(t^{-1}, g_1, \ldots, g_{n-1}\right) \\ &\quad = 1_{t^{-1}} 1_{(t^{-1}g_1, g_2, \ldots, g_n)} \tilde{w}\left(t^{-1}g_1, g_2, \ldots, g_n\right) \\ &\qquad + \sum_{i=1}^{n-1} (-1)^i 1_{t^{-1}} g_1 \ldots g_i 1_{(t^{-1}, g_1, \ldots, g_i g_{i+1}, \ldots, g_n)} \\ &\qquad \times \tilde{w}\left(t^{-1}, g_1, \ldots, g_i g_{i+1}, \ldots, g_n\right) \\ &\qquad + (-1)^n 1_{t^{-1}g_1 \ldots g_n} 1_{(t^{-1}, g_1, \ldots, g_{n-1})} \tilde{w}\left(t^{-1}, g_1, \ldots, g_{n-1}\right) \\ &= 1_{(t^{-1}, g_1, \ldots, g_n)} u(g_1, \ldots, g_n)\big|_t \\ &= \varphi\left(1_{(g_1, \ldots, g_n)}\right)\big|_t u(g_1, \ldots, g_n)\big|_t, \end{aligned}$$

whence

$$\varphi(w(g_1, \ldots, g_n))|_t = \varphi(1_{(g_1, \ldots, g_n)} t)|_t u(g_1, \ldots, g_n).$$

We have yet to prove that $u(g_1, \ldots, g_n) \in \mathcal{M}(\mathcal{B})$, i.e.

$$u(g_1, \ldots, g_n)\mathcal{B}, \mathcal{B}u(g_1, \ldots, g_n) \subseteq \mathcal{B}$$

for all $g_1, \ldots, g_n$. Since (5.1)

$$\begin{aligned} 1_{t-1} u(g_1, \ldots, g_n)\big|_t &= 1_{t^{-1}} \widetilde{w}\left(t^{-1}g_1, g_2, \ldots, g_n\right) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i 1_{t-1} \widetilde{w}\left(t^{-1}, g_1, \ldots, g_i g_{i+1}, \ldots, g_n\right) \\ &\quad + (-1)^n 1_{t^{-1}} \widetilde{w}\left(t^{-1}, g_1, \ldots, g_{n-1}\right) \\ &= \alpha_{t^{-1}}\left(1_t \tilde{w}(g_1, \ldots, g_n)\right), \end{aligned}$$

it follows that

$$u(g_1,\dots,g_n)|_t\,\varphi(a)|_t = \alpha_{t^{-1}}\left(1_t\widetilde{w}(g_1,\dots,g_n)\right)\alpha_{t^{-1}}(1_t a) = \varphi\left(\widetilde{w}(g_1,\dots,g_n)\,a\right)|_t\,,$$

whence

$$u(g_1,\dots,g_n)\,\varphi(\mathcal{A}) \subseteq \varphi(\mathcal{A}).$$

Now u being an n-cocycle with values in $(\mathcal{F},\beta)$ satisfies

$$\begin{aligned}\beta_{t^{-1}}\left(u(g_1,\dots,g_n)\right)\varphi(a) = {} & u\left(t^{-1}g_1,g_2,\dots,g_n\right)\varphi(a)\\ & + \sum_{i=1}^{n-1}(-1)^i u\left(t^{-1},g_1,\dots,g_ig_{i+1},\dots,g_{n+1}\right)\varphi(a)\\ & + (-1)^n u\left(t^{-1},g_1,\dots,g_{n-1}\right)\varphi(a),\end{aligned}$$

where the right-hand side is an element of $\varphi(\mathcal{A})$ thanks to the previous statement. Therefore, $\beta_{t^{-1}}(u(g_1,\dots,g_n))\varphi(\mathcal{A}) \subseteq \varphi(\mathcal{A})$, so, applying β_t, we obtain

$$u(g_1,\dots,g_n)\,\beta_t(\varphi(\mathcal{A})) \subseteq \beta_t(\varphi(\mathcal{A})).$$

Similarly, $\beta_t(\varphi(\mathcal{A}))u(g_1,\dots,g_n) \subseteq \beta_t(\varphi(\mathcal{A}))$, proving $u(g_1,\dots,g_n)\mathcal{B}, \mathcal{B}u(g_1,\dots,g_n) \subseteq \mathcal{B}$ since $\mathcal{B} = \sum_{g\in G}\beta_g(\varphi(\mathcal{A}))$.

□

5.2 The construction of w'

From now on we assume that $\mathcal{A} = \prod_{\lambda\in\Lambda}\mathcal{A}_\lambda$, where each $\mathcal{A}_\lambda$ is an indecomposable unital ring, i.e. $\mathcal{A}_\lambda$ cannot be written as $\mathcal{A}_\lambda \cong \mathcal{A}_\lambda^1 \times \mathcal{A}_\lambda^2$ with non-zero $\mathcal{A}_\lambda^1$ or $\mathcal{A}_\lambda^2$ ideals of $\mathcal{A}_\lambda$. Each $\mathcal{A}_\lambda$ is called a *block* of $\mathcal{A}$. The main objective is to show that every $w \in Z^n_{par}(G,\mathcal{A})$ can be replaced by a more manageable $w' \in Z^n_{par}(G,\mathcal{A})$ which will be used in the construction of $\tilde{w}$ satisfying the conditions of Theorem 5.14. Let us identify the identity of $\mathcal{A}_\mu$, $\mu\in\Lambda$, with the primitive idempotent $1_{\mathcal{A}_\mu}$ of $\mathcal{A}$ which is the function $\Lambda \to \bigcup_{\lambda\in\Lambda}\mathcal{A}_\lambda$ whose value at μ is the identity of $\mathcal{A}_\mu$ and the value in $\lambda\neq\mu$ is the zero of $\mathcal{A}_\lambda$, then the block $\mathcal{A}_\mu$ is identified with the ideal generated by $1_{\mathcal{A}_\mu}$, and the canonical projection $\mathrm{pr}_\lambda : \mathcal{A}\to\mathcal{A}_\lambda$ with the multiplication by $1_{\mathcal{A}_\lambda}$ in $\mathcal{A}$. We write $a = \prod_{\lambda\in\Lambda_1} a_\lambda$, where $\Lambda_1\subseteq\Lambda$ and $a_\lambda\in\mathcal{A}_\lambda$ for all $\lambda\in\Lambda_1$, if

$$\mathrm{pr}_\lambda(a) = \begin{cases} a_\lambda & \lambda\in\Lambda_1,\\ 0_{\mathcal{A}} & \text{otherwise.}\end{cases}$$

Thus, each idempotent e of $\mathcal{A}$ is central and is of the form $\prod_{\lambda\in\Lambda_1}1_{\mathcal{A}_\lambda}$, so that $e\mathcal{A} = \prod_{\lambda\in\Lambda_1}\mathcal{A}_\lambda$, therefore each $\mathcal{D}_g$ is of the form $\prod_{\lambda\in\Lambda_g}\mathcal{A}_\lambda$, for some $\Lambda_g\subseteq\Lambda$. Moreover we have the next lemma

Lemma 5.15. *Let $I = \prod_{\lambda \in \Lambda_1} \mathcal{A}_\lambda$ and $J = \prod_{\lambda \in \Lambda_2} \mathcal{A}_\lambda$ be unital ideals of $\mathcal{A}$ and $\varphi : I \to J$ an isomorphism. Then there exists a bijection $\sigma : \Lambda_1 \to \Lambda_2$, such that $\varphi(\mathrm{pr}_\lambda(a)) = \mathrm{pr}_{\sigma(\lambda)}(\varphi(a))$ for all $a \in I$ and $\lambda \in \Lambda_1$.*

Proof. Note that $\{1_{\mathcal{A}_\lambda}\}_{\lambda \in \Lambda_1}$ and $\{1_{\mathcal{A}_\lambda}\}_{\lambda \in \Lambda_2}$ are the sets of centrally primitive idempotents of I and J, respectively. Since φ is an isomorphism,

$$\varphi(1_{\mathcal{A}_\lambda}) = 1_{\mathcal{A}_{\sigma(\lambda)}},$$

for some bijection $\sigma : \Lambda_1 \to \Lambda_2$. Then

$$\varphi(\mathrm{pr}_\lambda(a)) = \varphi(1_{\mathcal{A}_\lambda} a) = 1_{\mathcal{A}_{\sigma(\lambda)}} \varphi(a) = \mathrm{pr}_{\sigma(\lambda)}(\varphi(a)).$$

$\square$

Definition 5.16. *A unital partial action α of a group G on $\mathcal{A}$ is called **transitive**, if for all $\lambda', \lambda'' \in \Lambda$ there exists $x \in G$, such that $\mathcal{A}_{\lambda'} \subseteq \mathcal{D}_{x^{-1}}$ and $\alpha_x(\mathcal{A}_{\lambda'}) = \mathcal{A}_{\lambda''}$.*

Assume that α is a transitive partial action. We fix $\lambda_0 \in \Lambda$, note that for all $\lambda \in \Lambda$ there exists $x \in G$ such that $\alpha_x(\mathcal{A}_{\lambda_0}) = \mathcal{A}_\lambda$. Denote by H the **stabilizer** of the block $\mathcal{A}_{\lambda_0}$, i.e. the subgroup

$$H = \{x \in G \mid \mathcal{A}_{\lambda_0} \subseteq \mathcal{D}_{x^{-1}} \text{ and } \alpha_x(\mathcal{A}_{\lambda_0}) = \mathcal{A}_{\lambda_0}\}$$

of G. Let Λ' be a left transversal of H in G containing the identity element 1 of G, i.e. $G = \bigcup_{g \in \Lambda'} gH$, a disjoint union. Then Λ can be identified with a subset of Λ', namely, λ_0 is identified with 1, and

$$\mathcal{A}_g = \alpha_g(\mathcal{A}_1) \text{ for } g \in \Lambda \subseteq \Lambda'.$$

Indeed, note that for all $\lambda \in \Lambda$ there exists $g \in G$ such that $\mathcal{A}_1 \subseteq \mathcal{D}_{g^{-1}}$ and $\mathcal{A}_\lambda = \alpha_g(\mathcal{A}_1)$, then for any $x \in H$ since $\alpha_x(\mathcal{A}_1) = \mathcal{A}_1 = \alpha_{x^{-1}}(\mathcal{A}_1)$, $\mathcal{A}_1 \subseteq \mathcal{D}_x \cap \mathcal{D}_{x^{-1}}$ and $\mathcal{A}_1 \subseteq \mathcal{D}_{g^{-1}}$, thus $\mathcal{A}_1 \subseteq \alpha_{x^{-1}}(\mathcal{D}_x \cap \mathcal{D}_{g^{-1}})$. Hence

$$\mathcal{A}_\lambda = \alpha_g(\mathcal{A}_1) = \alpha_g(\alpha_x(\mathcal{A}_1)) = \alpha_{gx}(\mathcal{A}_1).$$

Therefore $\alpha_w(\mathcal{A}_1) = \mathcal{A}_\lambda$ for all w in gH. If $g, g' \in G$ are such that $\alpha_g(\mathcal{A}_1) = \mathcal{A}_\lambda = \alpha_{g'}(\mathcal{A}_1)$, then $g^{-1}g' \in H$ and $gH = g'H$.

Given $x \in G$, denote by $\bar{x}$ the (unique) element of Λ', such that $x \in \bar{x}H$. Observe that $\overline{x\bar{y}} = \overline{xy}$ for all $x, y \in G$. Indeed, note that $xyH = x\bar{y}H$, thus $\overline{x\bar{y}}, \overline{xy} \in \Lambda'$ are such that $\overline{x\bar{y}}H = xyH = \overline{xy}H$. Hence $\overline{x\bar{y}} = \overline{xy}$.

We will use the following easy fact throughout the text.

Lemma 5.17. *Given $x \in G$ and $g \in \Lambda'$, one has*

(i) $g \in \Lambda \Leftrightarrow \mathcal{A}_1 \subseteq \mathcal{D}_{g^{-1}}$;

(ii) *if $g \in \Lambda$, then $\overline{xg} \in \Lambda \Leftrightarrow \mathcal{A}_g \subseteq \mathcal{D}_{x^{-1}}$, and in this situation $\alpha_x(\mathcal{A}_g) = \mathcal{A}_{\overline{xg}}$.*

Proof.

(i) We only need to see the "$\Longleftarrow$" part, so let $\mathcal{A}_1 \subseteq \mathcal{D}_{g^{-1}}$ for some $g \in \Lambda'$. Then $\alpha_g(\mathcal{A}_1)$ must be a block of $\mathcal{A}$, so it equals $\mathcal{A}_t$ for some $t \in \Lambda$. Hence $\alpha_t(\mathcal{A}_1) \subseteq \mathcal{D}_g \cap \mathcal{D}_t$, thus $\mathcal{A}_1 \subseteq \alpha_{t^{-1}}(\mathcal{D}_g \cap \mathcal{D}_t) = \mathrm{dom}(\alpha_{g^{-1}} \circ \alpha_t)$. Then

$$\mathcal{A}_1 = \alpha_{g^{-1}} \circ \alpha_t(\mathcal{A}_1) = \alpha_{g^{-1}t}(\mathcal{A}_1).$$

Consequently, $g^{-1}t \in H$, and $g = t \in \Lambda$.

(ii) Let $g, \overline{xg} \in \Lambda$. Then $\mathcal{A}_1 \subseteq \mathcal{D}_{(\overline{xg})^{-1}} \cap \mathcal{D}_{g^{-1}}$, and since $(\overline{xg})^{-1}xg \in H$, one has that $\alpha_{(\overline{xg})^{-1}xg}(\mathcal{A}_1) = \mathcal{A}_1$, then $\alpha_{(\overline{xg})^{-1}xg}(\mathcal{A}_1) \subseteq \mathcal{D}_{\overline{xg}(xg)^{-1}} \cap \mathcal{D}_{(\overline{xg})^{-1}}$. Hence $\mathcal{A}_1 \subseteq \alpha_{\overline{xg}(xg)^{-1}}(\mathcal{D}_{\overline{xg}(xg)^{-1}} \cap \mathcal{D}_{(\overline{xg})^{-1}})$. Therefore $\alpha_{\overline{xg}} \circ \alpha_{(\overline{xg})^{-1}xg}$ is applicable to $\mathcal{A}_1$ and as α is a partial action so too is $\alpha_{xg} = \alpha_{\overline{xg}\cdot(\overline{xg})^{-1}xg}$. Thus $\mathcal{A}_1 \subseteq \mathcal{D}_{(xg)^{-1}}$ and using again that α is a partial action we see that $\mathcal{A}_g = \alpha_g(\mathcal{A}_1) \subseteq \alpha_g(\mathcal{D}_{(xg)^{-1}} \cap \mathcal{D}_{g^{-1}}) = \mathcal{D}_{x^{-1}} \cap \mathcal{D}_g$, so that α_x is applicable to $\mathcal{A}_g$. Moreover, since $(\overline{xg})^{-1}xg \in H$ we have

$$\alpha_x(\mathcal{A}_g) = \alpha_{xg}(\mathcal{A}_1) = \alpha_{\overline{xg}} \circ \alpha_{(\overline{xg})^{-1}xg}(\mathcal{A}_1) = \alpha_{\overline{xg}}(\mathcal{A}_1),$$

and consequently $\alpha_x(\mathcal{A}_g) = \mathcal{A}_{\overline{xg}}$, as $\alpha_x(\mathcal{A}_g)$ must be a block.

Conversely,

$$\begin{aligned}
&\mathcal{A}_g \subseteq \mathcal{D}_{x^{-1}} \Longrightarrow g \in \Lambda, \mathcal{A}_g \subseteq \mathcal{D}_{x^{-1}} \cap \mathcal{D}_g = \mathcal{D}_{x^{-1}}\mathcal{D}_g \Longrightarrow \\
&\mathcal{A}_1 = \alpha_g^{-1}(\mathcal{A}_g) \subseteq \alpha_g^{-1}(\mathcal{D}_{x^{-1}}\mathcal{D}_g) \subseteq \mathcal{D}_{g^{-1}x^{-1}} \Longrightarrow \mathcal{A}_1 \subseteq \mathcal{D}_{g^{-1}x^{-1}} \cap \mathcal{D}_{g^{-1}x^{-1}\overline{xg}} \Longrightarrow \\
&\mathcal{A}_1 = \alpha_{(\overline{xg})^{-1}xg}(\mathcal{A}_1) \subseteq \alpha_{(\overline{xg})^{-1}xg}(\mathcal{D}_{g^{-1}x^{-1}} \cap \mathcal{D}_{g^{-1}x^{-1}\overline{xg}}) \subseteq \mathcal{D}_{(\overline{xg})^{-1}}
\end{aligned}$$

which gives $\overline{xg} \in \Lambda$.

□

If follows that

$$\mathcal{A}_g \subseteq \mathcal{D}_{x^{-1}} \Leftrightarrow \overline{x^{-1}g} \in \Lambda \Leftrightarrow \mathcal{A}_1 \subseteq \mathcal{D}_{g^{-1}x}.$$

In particular,

$$\mathcal{A}_{\overline{x}} \subseteq \mathcal{D}_x,$$

for all $x \in G$, such that $\overline{x} \in \Lambda$.

Remark 5.18. Let $(\mathcal{B}, \beta)$ be a globalization of $(\mathcal{A}, \alpha)$, assume that $\mathcal{A} \subseteq \mathcal{B}$, and set $\mathcal{A}_g = \beta_g(\mathcal{A}_1)$. Then $\beta_x(\mathcal{A}_g) = \mathcal{A}_{\overline{xg}}$, for $g \in \Lambda'$ and $x \in G$. Indeed,

$$\mathcal{A}_{\overline{xg}} = \beta_{xg}(\mathcal{A}_1) = \beta_x(\mathcal{A}_g).$$

Definition 5.19. *For $g \in \Lambda$ and $a \in \mathcal{A}$, define the homomorphism $\theta_g : \mathcal{A} \to \mathcal{A}_g$ given by*

$$\theta_g(a) = \alpha_g(\mathrm{pr}_1(a)) = \mathrm{pr}_g(\alpha_g(1_{g^{-1}}a)).$$

Remark 5.20. Notice that $1_g = \prod_{h\in\Lambda_g} 1_{\mathcal{A}_h}$, for some $\Lambda_g \subseteq \Lambda$, then if $a = \prod_{s\in\Lambda} a_s$, where $a_s \in \mathcal{A}_s$. Then $1_g a = \prod_{h\in\Lambda_g} a_h$. Then by Lemma 5.17 we have $\alpha_g(1_{g^{-1}}a_h) \in \mathcal{D}_{\overline{gh}}$. Thus

$$\mathrm{pr}_g(\alpha_g(1_{g^{-1}}a)) = \alpha_g(a_1) = \alpha_g(\mathrm{pr}_1(a)).$$

It follows that, since $\theta_g(\alpha_g(1_{g^{-1}}a)) = \mathrm{pr}_g(a)$ then

$$a = \prod_{g\in\Lambda} \theta_g(\alpha_g(1_{g^{-1}}a)).$$

Moreover, if $x \in G$ is such that $\mathcal{A}_1 \subseteq \mathcal{D}_x$, then $\mathrm{pr}_1(1_x a) = \mathrm{pr}_1(a) = a_1$, whence $\theta_g(a) = \theta_g(1_x a)$. In particular, this holds if $x \in H$ and $x = g^{-1}$.

Lemma 5.21. *Let $n > 0$ and $w \in Z^n_{par}(G, \mathcal{A})$. Then*

$$\begin{aligned} w(x_1, \ldots, x_n) = 1_{(x_1,\ldots,x_n)} \prod_{g\in\Lambda} \theta_g \big[& w\left(g^{-1}x_1, x_2, \ldots, x_n\right) \\ & + \sum_{k=1}^{n-1} (-1)^k w\left(g^{-1}, x_1, \ldots, x_k x_{k+1}, \ldots, x_n\right) \\ & + (-1)^n w\left(g^{-1}, x_1, \ldots, x_{n-1}\right) \big]. \end{aligned} \tag{5.3}$$

Proof. Using the partial n-cocycle identity we have that

$$\begin{aligned} \alpha_{g^{-1}}(1_g w(x_1, \ldots, x_n)) = & 1_{g^{-1}} w(g^{-1}x_1, x_2, \ldots, x_n) \\ & + \sum_{k=1}^{n-1} (-1)^k 1_{g^{-1}x_1\ldots x_k} w(g^{-1}, x_1, \ldots, x_k x_{k+1}, \ldots, x_n) \\ & + (-1)^n 1_{g^{-1}x_1\ldots x_n} w(g^{-1}, x_1, \ldots, x_{n-1}), \end{aligned}$$

and by Remark 5.20

$$w(x_1, \ldots, x_n) = \prod_{g\in\Lambda} \theta_g(\alpha_{g^{-1}}(1_g w(x_1, \ldots, x_n))).$$

Thus

$$
\begin{aligned}
w(x_1,\dots,x_n) &= \prod_{g\in\Lambda}\theta_g(\alpha_{g^{-1}}(1_g w(x_1,\dots,x_n)))\\
&= \prod_{g\in\Lambda}\theta_g[1_{g^{-1}}w(g^{-1}x_1,x_2,\dots,x_n)\\
&\quad+\sum_{k=1}^{n-1}(-1)^k 1_{g^{-1}x_1\dots x_k}w(g^{-1},x_1,\dots,x_kx_{k+1},\dots,x_n)\\
&\quad+(-1)^n 1_{g^{-1}x_1\dots x_n}w(g^{-1},x_1,\dots,x_{n-1})]\\
&= \prod_{g\in\Lambda}\theta_g[1_{(g^{-1},x_1,\dots,x_n)}(w(g^{-1}x_1,x_2,\dots,x_n)\\
&\quad+\sum_{k=1}^{n-1}(-1)^k w\left(g^{-1},x_1,\dots,x_kx_{k+1},\dots,x_n\right)\\
&\quad+(-1)^n w(g^{-1},x_1,\dots,x_{n-1}))].
\end{aligned}
$$

It remains to observe

$$
\theta_g\left(1_{(g^{-1},x_1,\dots,x_n)}\right)=\mathrm{pr}_g\left(\alpha_g\left(1_{g^{-1}}1_{(g^{-1},x_1,\dots,x_n)}\right)\right)=\mathrm{pr}_g\left(1_g 1_{(x_1,\dots,x_n)}\right)=\mathrm{pr}_g\left(1_{(x_1,\dots,x_n)}\right)
$$

so that

$$
\prod_{g\in\Lambda}\theta_g\left(1_{(g^{-1},x_1,\dots,x_n)}\right)=\prod_{g\in\Lambda}\mathrm{pr}_g\left(1_{(x_1,\dots,x_n)}\right)=1_{(x_1,\dots,x_n)}.
$$

□

Define the map $\eta: G\to H$ such that for $x\in G$, we have $\eta(x)=x^{-1}\bar{x}\in H$. Let $n>0$ and $g\in\Lambda'$. Define $\eta_n^g: G^n\to H$ by

$$
\eta_n^g(x_1,\dots,x_n)=\eta\left(x_n^{-1}\overline{x_{n-1}^{-1}\dots x_1^{-1}g}\right)
$$

and $\tau_n^g: G^n\to H^n$ by

$$
\tau_n^g(x_1,\dots,x_n)=\left(\eta_1^g(x_1),\eta_2^g(x_1,x_2),\dots,\eta_n^g(x_1,\dots,x_n)\right).
$$

Note that

$$
\eta_1^g(x_1)\,\eta_2^g(x_1,x_2)\dots\eta_n^g(x_1,\dots,x_n)=\eta\left(x_n^{-1}\dots x_1^{-1}g\right)=\eta_1^g(x_1\dots x_n).
$$

Indeed,

$$
\eta_1^g(x_1)=g^{-1}x_1\overline{x_1^{-1}g}
$$

and
$$\eta_2^g(x_1,x_2) = (\overline{x_1^{-1}g})^{-1}x_2\overline{x_2^{-1}x_1^{-1}g}.$$
Thus
$$\eta_1^g(x_1)\eta_2^g(x_1,x_2) = g^{-1}x_1x_2\overline{x_2^{-1}x_1^{-1}g} = \eta(x_2^{-1}x_1^{-1}g).$$
Now using an inductive argument over n we have that

$$\begin{aligned}
&\eta_1^g(x_1)\,\eta_2^g(x_1,x_2)\dots\eta_{n-1}^g(x_1,\dots,x_{n-1})\,\eta_n^g(x_1,\dots,x_n)\\
&\quad= \eta(x_{n-1}^{-1}x_{n-2}^{-1}\dots x_1^{-1}g)\eta_n^g(x_1,\dots,x_n)\\
&\quad= (g^{-1}x_1x_2\dots x_{n-1})(\overline{x_{n-1}^{-1}x_{n-2}^{-1}\dots x_1^{-1}g})(\overline{x_{n-1}^{-1}\dots x_1^{-1}g})^{-1}x_n(\overline{x_n^{-1}x_{n-1}^{-1}\dots x_1^{-1}g})\\
&\quad= g^{-1}x_1x_2\dots x_{n-1}x_n(\overline{x_n^{-1}x_{n-1}^{-1}\dots x_1^{-1}g})\\
&\quad= \eta\left(x_n^{-1}\dots x_1^{-1}g\right).
\end{aligned}$$

Moreover, we will define the maps $\sigma_{n,i}^g : G^n \to G^{n+1}, n>0, 0\le i\le n$, by

$$\begin{aligned}
\sigma_{n,0}^g(x_1,\dots,x_n) &= \left(g^{-1},x_1,\dots,x_n\right),\\
\sigma_{n,i}^g(x_1,\dots,x_n) &= \left(\tau_i^g(x_1,\dots,x_i),(\overline{x_i^{-1}\dots x_1^{-1}g})^{-1},x_{i+1},\dots,x_n\right), \quad 0<i<n,\\
\sigma_{n,n}^g(x_1,\dots,x_n) &= \left(\tau_n^g(x_1,\dots,x_n),(\overline{x_n^{-1}\dots x_1^{-1}g})^{-1}\right).
\end{aligned}$$

If $n=0$, then we set
$$\sigma_{0,0}^g = g^{-1}\in G.$$

Definition 5.22. *Given $n>0$ and $w\in C^n_{par}(G,\mathcal{A})$, define $w'\in C^n_{par}(G,\mathcal{A})$ and $\varepsilon\in C^{n-1}_{par}(G,\mathcal{A})$ by*

$$w'(x_1,\dots,x_n) = 1_{(x_1,\dots,x_n)}\prod_{g\in\Lambda}\theta_g\circ w\circ\tau_n^g(x_1,\dots,x_n),$$

$$\varepsilon(x_1,\dots,x_{n-1}) = 1_{(x_1,\dots,x_{n-1})}\prod_{g\in\Lambda}\theta_g\left(\sum_{i=0}^{n-1}(-1)^i w\circ\sigma_{n-1,i}^g(x_1,\dots,x_{n-1})\right).$$

When $n=1$, Definition 5.22 for ε should be understood as

$$\varepsilon = \prod_{g\in\Lambda}\theta_g\left(w\left(g^{-1}\right)\right)\in\mathcal{A}.$$

Let us introduce the following notation that will be used in the results below.

$$\Sigma(l,m)=\sum_{k=l,i=m}^{n-1}(-1)^{k+i}w\circ\sigma^{g}_{n-1,i}\left(x_1,\ldots,x_kx_{k+1},\ldots,x_n\right)$$
$$+\sum_{i=m}^{n-1}(-1)^{n+i}w\circ\sigma^{g}_{n-1,i}\left(x_1,\ldots,x_{n-1}\right),$$

where $1\leq l\leq n-1$ and $0\leq m\leq n-1$(n is assumed to be fixed).

Lemma 5.23. *For all $w\in Z^1_{par}(G,\mathcal{A})$ and $x\in G$ we have:*

$$\left(\delta^0\varepsilon\right)(x)-\alpha_x\left(1_{x^{-1}}\varepsilon\right)-w(x)=1_x\prod_{g\in\Lambda}\theta_g\left(-w\left(g^{-1}x\right)\right).$$

Moreover, for $n>1, w\in Z^n(G,\mathcal{A})$ and $x_1,\ldots,x_n\in G$:

$$\left(\delta^{n-1}\varepsilon\right)(x_1,\ldots,x_n)-\alpha_{x_1}\Big(1_{x_1^{-1}}\varepsilon\left(x_2,\ldots,x_n\right)\Big)-w\left(x_1,\ldots,x_n\right)$$
$$=1_{(x_1,\ldots,x_n)}\prod_{g\in\Lambda}\theta_g\left(-w\left(g^{-1}x_1,x_2,\ldots,x_n\right)+\Sigma(1,1)\right).$$

Proof. Recall that by Remark 3.40 we have that

$$\left(\delta^0a\right)(g)=\alpha_g\left(1_{g^{-1}}a\right)-1_ga,$$

thus

$$\left(\delta^0\varepsilon\right)(x)-\alpha_x\left(1_{x^{-1}}\varepsilon\right)-w(x)=-1_x\varepsilon-w(x).$$

By Lemma 5.21

$$w(x)=1_x\prod_{g\in\Lambda}\theta_g(w(g^{-1}x)-w(g^{-1})).$$

Whence, and using Definition of ε we have

$$\left(\delta^0\varepsilon\right)(x)-\alpha_x\left(1_{x^{-1}}\varepsilon\right)-w(x)=-1_x\prod_{g\in\Lambda}\theta_g\left(w\left(g^{-1}\right)\right)-1_x\prod_{g\in\Lambda}\theta_g(w(g^{-1}x)-w(g^{-1}))$$
$$=1_x\prod_{g\in\Lambda}\theta_g\left(-w\left(g^{-1}x\right)\right).$$

Now for $n>0$, using Remark 3.40 first, Definition 5.22, and finally definition of $\Sigma(1,0)$ we have

$$\begin{aligned}
&\left(\delta^{n-1}\varepsilon\right)(x_1,\ldots,x_n)-\alpha_{x_1}\left(1_{x_1^{-1}}\varepsilon(x_2,\ldots,x_n)\right)\\
&\quad=\sum_{k=1}^{n-1}(-1)^k 1_{x_1\ldots x_k}\varepsilon(x_1,\ldots,x_kx_{k+1},\ldots,x_n)+(-1)^n 1_{x_1\ldots x_n}\varepsilon(x_1,\ldots,x_{n-1})\\
&\quad=1_{(x_1,\ldots,x_n)}\prod_{g\in\Lambda}\theta_g\left(\sum_{k=1,i=0}^{n-1}(-1)^{k+i}w\circ\sigma^g_{n-1,i}(x_1,\ldots,x_kx_{k+1},\ldots,x_n)\right)\\
&\qquad+1_{(x_1,\ldots,x_n)}\prod_{g\in\Lambda}\theta_g\left(\sum_{i=0}^{n-1}(-1)^{n+i}w\circ\sigma^g_{n-1,i}(x_1,\ldots,x_{n-1})\right)\\
&\quad=1_{(x_1,\ldots,x_n)}\prod_{g\in\Lambda}\theta_g(\Sigma(1,0)).
\end{aligned}$$

Now notice that

$$\begin{aligned}
(-1)^k w\left(g^{-1},x_1,\ldots,x_kx_{k+1},\ldots,x_n\right)&=(-1)^{k+0}w\circ\sigma^g_{n-1,0}(x_1,\ldots,x_kx_{k+1},\ldots,x_n)\\
(-1)^n w\left(g^{-1},x_1,\ldots,x_{n-1}\right)&=(-1)^{n+0}w\circ\sigma^g_{n-1,0}(x_1,\ldots,x_{n-1}).
\end{aligned}$$

Then the formula (5.3) in Lemma 5.21 becomes

$$\begin{aligned}
w(x_1,\ldots,x_n)=1_{(x_1,\ldots,x_n)}\prod_{g\in\Lambda}\theta_g\Big[&w\left(g^{-1}x_1,x_2,\ldots,x_n\right)\\
&+\sum_{k=1}^{n-1}(-1)^{k+0}w\circ\sigma^g_{n-1,0}(x_1,\ldots,x_kx_{k+1},\ldots,x_n)\\
&+(-1)^{n+0}w\circ\sigma^g_{n-1,0}(x_1,\ldots,x_{n-1})\Big].
\end{aligned}$$

Observe that

$$\begin{aligned}
\Sigma(1,0)-\Sigma(1,1)=&\sum_{k=1}^{n-1}(-1)^{k+0}w\circ\sigma^g_{n-1,0}(x_1,\ldots,x_kx_{k+1},\ldots,x_n)\\
&+(-1)^{n+0}w\circ\sigma^g_{n-1,0}(x_1,\ldots,x_{n-1}).
\end{aligned}$$

Hence,

$$w(x_1,\ldots,x_n)=1_{(x_1,\ldots,x_n)}\prod_{g\in\Lambda}\theta_g[w(g^{-1}x_1,x_2,\ldots,x_n)+\Sigma(1,0)-\Sigma(1,1)],$$

whence

$$\begin{aligned} &1_{(x_1,\dots,x_n)} \prod_{g\in\Lambda} \theta_g(\Sigma(1,0)) \\ &\qquad = w\left(x_1,\dots,x_n\right) + 1_{(x_1,\dots,x_n)} \prod_{g\in\Lambda} \theta_g(-w(g^{-1}x_1,x_2,\dots,x_n)+\Sigma(1,1)). \end{aligned}$$

□

Lemma 5.24. *For all $n>1, w\in Z^n_{par}(G,\mathcal{A}), g\in\Lambda$ and $x_1,\dots,x_n\in G$:*

$$\begin{aligned} &1_{\sigma^g_{n,1}(x_1,\dots,x_n)}(-w(g^{-1}x_1,x_2,\dots,x_n)+\Sigma(1,1)) \\ &\qquad = -\,\alpha_{\eta^g_1(x_1)}(1_{\eta^g_1(x_1)^{-1}}w\circ\sigma^{\overline{x_1^{-1}g}}_{n-1,0}(x_2,\dots,x_n)) \\ &\qquad\quad + 1_{\sigma^g_{n,1}(x_1,\dots,x_n)}(-w(\tau^g_1(x_1),(\overline{x_1^{-1}g})^{-1}x_2,x_3,\dots,x_n)+\Sigma(2,2) \\ &\qquad\quad + \sum_{i=1}^{n-1}(-1)^{i+1}w\circ\sigma^g_{n-1,i}(x_1x_2,x_3,\dots,x_n)). \end{aligned}$$

Proof. Note that since $w\in Z^n_{par}(G,\mathcal{A})$ we have that

$$\begin{aligned} 0 &= (\delta^n w)\circ\sigma^g_{n,1}\left(x_1,\dots,x_n\right) \\ &= (\delta^n w)\left(g^{-1}x_1\cdot\overline{x_1^{-1}g},(\overline{x_1^{-1}g})^{-1},x_2,\dots,x_n\right) \\ &= \alpha_{g^{-1}x_1\cdot\overline{x_1^{-1}g}}(1_{(\overline{x_1^{-1}g})^{-1}x_1^{-1}g}w((\overline{x_1^{-1}g})^{-1},x_2,\dots,x_n)) \\ &\quad - 1_{g^{-1}x_1\cdot\overline{x_1^{-1}g}}w\left(g^{-1}x_1,x_2,\dots,x_n\right) \\ &\quad + 1_{g^{-1}x_1}w\left(g^{-1}x_1\cdot\overline{x_1^{-1}g},(\overline{x_1^{-1}g})^{-1}x_2,x_3,\dots,x_n\right) \\ &\quad + \sum_{k=2}^{n-1}(-1)^{k+1}1_{g^{-1}x_1\dots x_k}w\left(g^{-1}x_1\cdot\overline{x_1^{-1}g},(\overline{x_1^{-1}g})^{-1},x_2,\dots,x_kx_{k+1},\dots x_n\right) \\ &\quad + (-1)^{n+1}1_{g^{-1}x_1\dots x_n}w(g^{-1}x_1\cdot\overline{x_1^{-1}g},(\overline{x_1^{-1}g})^{-1},x_2,\dots,x_{n-1}). \end{aligned}$$

Observe that we can rewrite some factors of the latter equality.

$$\begin{aligned} \eta^g_1(x_1) &= g^{-1}x_1\cdot\overline{x_1^{-1}g}, \\ \sigma^{\overline{x_1^{-1}g}}_{n-1,0}(x_2,\dots,x_n) &= ((\overline{x_1^{-1}g})^{-1},x_2,\dots,x_n), \\ \tau^g_1(x_1) &= g^{-1}x_1\cdot\overline{x_1^{-1}g}, \\ \sigma^g_{n-1,1}\left(x_1,\dots,x_kx_{k+1},\dots x_n\right) &= \left(g^{-1}x_1\cdot\overline{x_1^{-1}g},(\overline{x_1^{-1}g})^{-1},x_2,\dots,x_kx_{k+1},\dots x_n\right) \\ \sigma^g_{n-1,1}\left(x_1,\dots,x_{n-1}\right) &= (g^{-1}x_1\cdot\overline{x_1^{-1}g},(\overline{x_1^{-1}g})^{-1},x_2,\dots,x_{n-1}). \end{aligned}$$

Therefore,

$$
\begin{aligned}
-1_{\eta_1^g(x_1)} w(g^{-1}x_1, x_2, \dots, x_n) = &- \alpha_{\eta_1^g(x_1)}(1_{\eta_1^g(x_1)^{-1}} w \circ \sigma_{n-1,0}^{\overline{x_1^{-1}g}}(x_2, \dots, x_n)) \\
&- 1_{g^{-1}x_1} w(\tau_1^g(x_1), (\overline{x_1^{-1}g})^{-1}x_2, x_3, \dots, x_n) \\
&+ \sum_{k=2}^{n-1} (-1)^k 1_{g^{-1}x_1\dots x_k} w \circ \sigma_{n-1,1}^g (x_1, \dots, x_k x_{k+1}, \dots x_n) \\
&+ (-1)^n 1_{g^{-1}x_1\dots x_n} w \circ \sigma_{n-1,1}^g (x_1, \dots, x_{n-1}) . \qquad (1)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\Sigma(1,1) - \Sigma(2,2) = &\sum_{k=2}^{n-1} (-1)^{k+1} w \circ \sigma_{n-1,1}^g(x_1, \dots, x_k x_{k+1}, \dots, x_n) \\
&+ \sum_{i=1}^{n-1} (-1)^{1+i} w \circ \sigma_{n-1,i}^g(x_1 x_2, \dots, x_n) \\
&+ (-1)^{n+1} w \circ \sigma_{n-1,1}^g(x_1, \dots, x_{n-1}).
\end{aligned}
$$

Whence,

$$
\begin{aligned}
\Sigma(2,2) + &\sum_{i=1}^{n-1} (-1)^{1+i} w \circ \sigma_{n-1,i}^g(x_1 x_2, \dots, x_n) \\
&= \Sigma(1,1) + \sum_{k=2}^{n-1} (-1)^k w \circ \sigma_{n-1,1}^g(x_1, \dots, x_k x_{k+1}, \dots, x_n) \\
&\quad + (-1)^n w \circ \sigma_{n-1,1}^g(x_1, \dots, x_{n-1}).
\end{aligned}
$$

Thus adding $\Sigma(1,1)$ then multiplying both side of equality (1) by $1_{\sigma_{n+1}^g(x_1,\dots,x_n)} = 1_{\eta_1^g(x_1)} 1_{(g^{-1}x_1, x_2, \dots, x_n)}$, we get

$$
\begin{aligned}
1_{\sigma_{n,1}^g(x_1,\dots,x_n)}&(-w(g^{-1}x_1, x_2, \dots, x_n) + \Sigma(1,1)) \\
&= - \alpha_{\eta_1^g(x_1)}(1_{\eta_1^g(x_1)^{-1}} w \circ \sigma_{n-1,0}^{\overline{x_1^{-1}g}}(x_2, \dots, x_n)) \\
&\quad + 1_{\sigma_{n,1}^g(x_1,\dots,x_n)}(-w(\tau_1^g(x_1), (\overline{x_1^{-1}g})^{-1}x_2, x_3, \dots, x_n) + \Sigma(2,2) \\
&\quad + \sum_{i=1}^{n-1} (-1)^{i+1} w \circ \sigma_{n-1,i}^g(x_1 x_2, x_3, \dots, x_n)).
\end{aligned}
$$

$\square$

Lemma 5.25. *For all $1 < j < n, w \in Z^n_{par}(G,\mathcal{A}), g \in \Lambda$ and $x_1,\dots,x_n \in G$:*

$$\begin{aligned}
1_{\sigma^g_{n,j}(x_1,\dots,x_n)}&(-w(\tau^g_{j-1}(x_1,\dots,x_{j-1}),(\overline{x^{-1}_{j-1}\dots x^{-1}_1 g})^{-1}x_j,x_{j+1},\dots,x_n)+\Sigma(j,j))\\
=&(-1)^j\alpha_{\eta^g_1(x_1)}(1_{\eta^g_1(x_1)^{-1}}w\circ\sigma^{-1_g}_{n-1,j-1}(x_2,\dots,x_n))\\
&+1_{\sigma^g_{n,j}(x_1,\dots,x_n)}(-w(\tau^g_j(x_1,\dots,x_j),(\overline{x^{-1}_j\dots x^{-1}_1 g})^{-1}x_{j+1},x_{j+2},\dots,x_n)\\
&+\Sigma(j+1,j+1)\\
&+\sum_{i=j}^{n-1}(-1)^{i+j}w\circ\sigma^g_{n-1,i}(x_1,\dots,x_jx_{j+1},\dots,x_n)\\
&+\sum_{s=1}^{j-1}(-1)^{s+j}w\circ\sigma^g_{n-1,j-1}(x_1,\dots,x_sx_{s+1},\dots,x_n))
\end{aligned}$$

(here by $\Sigma(n,n)$ we mean $0_{\mathcal{A}}$).

Proof. This proof is analogous to that of Lemma 5.24

$$\begin{aligned}
0=&(\delta^n w)\circ\sigma^g_{n,j}(x_1,\dots,x_n)\\
=&\alpha_{\eta^g_1(x_1)}\left(1_{\eta^g_1(x_1)^{-1}}w\circ\sigma^{\overline{x_1^{-1}g}}_{n-1,j-1}(x_2,\dots,x_n)\right)\\
&+\sum_{s=1}^{j-1}(-1)^s 1_{\eta^g_1(x_1\dots x_s)}w\circ\sigma^g_{n-1,j-1}(x_1,\dots,x_sx_{s+1},\dots,x_n)\\
&+(-1)^j 1_{\eta^g_1(x_1\dots x_j)}w\left(\tau^g_{j-1}(x_1,\dots,x_{j-1}),(\overline{x^{-1}_{j-1}\dots x^{-1}_1 g})^{-1}x_j,x_{j+1},\dots,x_n\right)\\
&+(-1)^{j+1}1_{g^{-1}x_1\dots x_j}w\left(\tau^g_j(x_1,\dots,x_j),(\overline{x^{-1}_j\dots x^{-1}_1 g})^{-1}x_{j+1},x_{j+2},\dots,x_n\right)\\
&+\sum_{t=j+1}^{n-1}(-1)^{t+1}1_{g^{-1}x_1\dots x_t}w\circ\sigma^g_{n-1,j}(x_1,\dots,x_tx_{t+1},\dots,x_n)\\
&+(-1)^{n+1}1_{g^{-1}x_1\dots x_n}w\circ\sigma^g_{n-1,j}(x_1,\dots,x_{n-1})
\end{aligned}$$

Therefore,

$$\begin{aligned}
&-1_{\eta_1^g(x_1\ldots x_j)}w\left(\tau_{j-1}^g\left(x_1,\ldots,x_{j-1}\right),(\overline{x_{j-1}^{-1}\ldots x_1^{-1}g})^{-1}x_j,x_{j+1},\ldots,x_n\right)\\
&=(-1)^j\alpha_{\eta_1^g(x_1)}(1_{\eta_1^g(x_1)^{-1}}w\circ\sigma_{n-1,j-1}^{\overline{x_1^{-1}g}}(x_2,\ldots,x_n))\\
&\quad+\sum_{s=1}^{j-1}(-1)^{s+j}1_{\eta_1^g(x_1\ldots x_s)}w\circ\sigma_{n-1,j-1}^g(x_2,\ldots,x_sx_{s+1},\ldots,x_n))\\
&\quad-1_{g^{-1}x_1\ldots x_j}w\left(\tau_j^g\left(x_1,\ldots,x_j\right),(\overline{x_j^{-1}\ldots x_1^{-1}g})^{-1}x_{j+1},x_{j+2},\ldots,x_n\right)\\
&\quad+\sum_{t=j+1}^{n-1}(-1)^{t+j+1}1_{g^{-1}x_1\ldots x_t}w\circ\sigma_{n-1,j}^g\left(x_1,\ldots,x_tx_{t+1},\ldots,x_n\right)\\
&\quad+(-1)^{n+j+1}1_{g^{-1}x_1\ldots x_n}w\circ\sigma_{n-1,j}^g\left(x_1,\ldots,x_{n-1}\right).
\end{aligned}\tag{1}$$

Notice that

$$\begin{aligned}
\Sigma(j,j)-\Sigma(j+1,j+1)=&\sum_{t=j+1}^{n-1}(-1)^{t+j}w\circ\sigma_{n-1,j}^g(x_1,\ldots,x_tx_{t+1},\ldots,x_n)\\
&+\sum_{i=j}^{n-1}(-1)^{j+i}w\circ\sigma_{n-1,i}^g(x_1,x_2,\ldots,x_jx_{j+1},\ldots,x_n)\\
&+(-1)^{n+j}w\circ\sigma_{n-1,j}^g(x_1,\ldots,x_{n-1}).
\end{aligned}$$

Thus adding $\Sigma(j,j)$ to equality (1) and then multiplying both sides by the idempotent element

$$1_{\sigma_{n,j}^g}(x_1,\ldots,x_n)=1_{\eta_1^g(x_1)}1_{\eta_1^g(x_1x_2)}\ldots1_{\eta_1^g(x_1x_2\ldots x_j)}1_{\eta_1^g(g^{-1}x_1\ldots x_j,x_{j+1},\ldots x_n)}$$

we get

$$\begin{aligned}
1_{\sigma_{n,j}^g(x_1,\ldots,x_n)}&(-w(\tau_{j-1}^g(x_1,\ldots,x_{j-1}),(\overline{x_{j-1}^{-1}\ldots x_1^{-1}g})^{-1}x_j,x_{j+1},\ldots,x_n)+\Sigma(j,j))\\
=&(-1)^j\alpha_{\eta_1^g(x_1)}(1_{\eta_1^g(x_1)^{-1}}w\circ\sigma_{n-1,j-1}^{-1_g}(x_2,\ldots,x_n))\\
&+1_{\sigma_{n,j}^g(x_1,\ldots,x_n)}(-w(\tau_j^g(x_1,\ldots,x_j),(\overline{x_j^{-1}\ldots x_1^{-1}g})^{-1}x_{j+1},x_{j+2},\ldots,x_n)\\
&+\Sigma(j+1,j+1)\\
&+\sum_{i=j}^{n-1}(-1)^{i+j}w\circ\sigma_{n-1,i}^g(x_1,\ldots,x_jx_{j+1},\ldots,x_n)\\
&+\sum_{s=1}^{j-1}(-1)^{s+j}w\circ\sigma_{n-1,j-1}^g(x_1,\ldots,x_sx_{s+1},\ldots,x_n)).
\end{aligned}$$

$\square$

Lemma 5.26. *For all $w \in Z^1_{par}(G,\mathcal{A}), g \in \Lambda$ and $x \in G$:*

$$-1_{\eta^g_1(x)} w\left(g^{-1}x\right) = -\alpha_{\eta^g_1(x)}\left(1_{\eta^g_1(x)^{-1}} w\left((\overline{x^{-1}g})^{-1}\right)\right) - 1_{g^{-1}x} w \circ \tau^g_1(x)$$

Moreover, for all $n > 1, w \in Z^n_{par}(G,\mathcal{A}), g \in \Lambda$ and $x_1,\ldots,x_n \in G$

$$\begin{aligned}
-1_{\eta^g_1(x_1\ldots x_n)} w & \left(\tau^g_{n-1}(x_1,\ldots,x_{n-1}), (\overline{x^{-1}_{n-1}\ldots x^{-1}_1 g})^{-1}x_n\right) \\
&= (-1)^n \alpha_{\eta^g_1(x_1)}\left(1_{\eta^g_1(x_1)^{-1}} w \circ \sigma^{\overline{x^{-1}_1 g}}_{n-1,n-1}(x_2,\ldots,x_n)\right) \\
&\quad + 1_{\sigma^g_{n,n}(x_1,\ldots,x_n)}\Bigg(\sum_{s=1}^{n-1}(-1)^{s+n} w \circ \sigma^g_{n-1,n-1}(x_1,\ldots,x_s x_{s+1},\ldots,x_n) \\
&\quad - w \circ \tau^g_n(x_1,\ldots,x_n)\Bigg)
\end{aligned}$$

Proof. We first observe that for $n = 1$, since $w \in Z^1_{par}(G,\mathcal{A})$ we have

$$\begin{aligned}
0 &= (\delta^1 w)(\eta^g_1(x), (\overline{x^{-1}g})^{-1}) \\
&= \alpha_{\eta^g_1(x)}(1_{\eta^g_1(x)^{-1}} w((\overline{x^{-1}g})^{-1})) - 1_{\eta^g_1(x)} w\left(g^{-1}x\right) + 1_{g^{-1}x} w \circ \tau^g_1(x).
\end{aligned}$$

Next for $n > 1$, we have that

$$\begin{aligned}
0 &= (\delta^n w) \circ \sigma^g_{n,n}(x_1,\ldots,x_n) \\
&= \alpha_{\eta^g_1(x_1)}\left(1_{\eta^g_1(x_1)^{-1}} w \circ \sigma^{\overline{x^{-1}_1 g}}_{n-1,n-1}(x_2,\ldots,x_n)\right) \\
&\quad + \sum_{s=1}^{n-1}(-1)^s 1_{\eta^g_1(x_1\ldots x_s)} w \circ \sigma^g_{n-1,n-1}(x_1,\ldots,x_s x_{s+1},\ldots,x_n) \\
&\quad + (-1)^n 1_{\eta^g_1(x_1\ldots x_n)} w\left(\tau^g_{n-1}(x_1,\ldots,x_{n-1}), (\overline{x^{-1}_{n-1}\ldots x^{-1}_1 g})^{-1}x_n\right) \\
&\quad + (-1)^{n+1} 1_{g^{-1}x_1\ldots x_n} w \circ \tau^g_n(x_1,\ldots,x_n).
\end{aligned}$$

Thus to obtain the desire equality we have to multiply both sides by $1_{\sigma^g_{n,n}(x_1,\ldots,x_n)}$. $\square$

Lemma 5.27. *For all $n > 0, w \in Z^n_{par}(G,\mathcal{A})$ and $x_1,\ldots,x_n \in G$, define the idempotent element*

$$e = \prod_{i=1}^n 1_{\sigma^g_{n,i}(x_1,\ldots,x_n)} = 1_{(g^{-1},x_1,\ldots,x_n)} \prod_{i=1}^n 1_{\eta^g_i(x_1,\ldots,x_i)}.$$

Then

$$\begin{aligned}
&e(-w(g^{-1}x_1, x_2, \dots, x_n) + \Sigma(1,1)) \\
&\qquad = e\alpha_{\eta_1^g(x_1)}\Big(1_{\eta_1^g(x_1)^{-1}} \sum_{j=0}^{n-1} (-1)^{j+1} w \circ \sigma_{n-1,j}^{\overline{x_1^{-1}g}}(x_2, \dots, x_n)\Big) \\
&\qquad\quad - e(w \circ \tau_n^g)\,(x_1, \dots, x_n) \\
&\qquad\quad + e\sum_{j=1}^{n-1}\sum_{i=j}^{n-1} (-1)^{i+j} w \circ \sigma_{n-1,i}^g(x_1, \dots, x_j x_{j+1}, \dots, x_n) \\
&\qquad\quad + e\sum_{j=2}^{n}\sum_{s=1}^{j-1} (-1)^{s+j} w \circ \sigma_{n-1,j-1}^g(x_1, \dots, x_s x_{s+1}, \dots, x_n).
\end{aligned}$$

Proof. By Lemma 5.24

$$\begin{aligned}
&1_{\sigma_{n,1}^g(x_1,\dots,x_n)}(-w(g^{-1}x_1, x_2, \dots, x_n) + \Sigma(1,1)) \\
&\qquad = -\,\alpha_{\eta_1^g(x_1)}(1_{\eta_1^g(x_1)^{-1}} w \circ \sigma_{n-1,0}^{\overline{x_1^{-1}g}}(x_2, \dots, x_n)) \\
&\qquad\quad + 1_{\sigma_{n,1}^g(x_1,\dots,x_n)}(-w(\tau_1^g(x_1), (\overline{x_1^{-1}g})^{-1}x_2, x_3, \dots, x_n) + \Sigma(2,2) \\
&\qquad\quad + \sum_{i=1}^{n-1} (-1)^{i+1} w \circ \sigma_{n-1,i}^g(x_1x_2, x_3, \dots, x_n)).
\end{aligned}$$

Observe that

$$\begin{aligned}
&-\alpha_{\eta_1^g(x_1)}(1_{\eta_1^g(x_1)^{-1}} w \circ \sigma_{n-1,0}^{\overline{x_1^{-1}g}}(x_2, \dots, x_n)) \\
&\qquad = \alpha_{\eta_1^g(x_1)} 1_{\eta_1^g(x_1)^{-1}} \Big[\sum_{j=0}^{n-1} (-1)^{j+1} w \circ \sigma_{n-1,j}^{\overline{x_1^{-1}g}}(x_2, \dots, x_n) \\
&\qquad\quad + \sum_{j=2}^{n} (-1)^{j+1} w \circ \sigma_{n-1,j-1}^{\overline{x_1^{-1}g}}(x_2, \dots, x_n)\Big].
\end{aligned}$$

By Lemma 5.25

$$
\begin{aligned}
&\sum_{j=2}^{n}(-1)^{j+1}\alpha_{\eta_1^g(x_1)}(1_{\eta_1^g(x_1)^{-1}}w\circ\sigma^{\overline{x_1^{-1}g}}_{n-1,j-1}(x_2,\dots,x_n))\\
&\quad=\sum_{j=2}^{n-1}1_{\sigma^g_{n,j}(x_1,\dots,x_n)}\Big[w(\tau^g_{j-1}(x_1,\dots,x_{j-1}),(\overline{x_{j-1}^{-1}\dots x_1^{-1}g})^{-1}x_j,x_{j+1},\dots,x_n)-\Sigma(j,j)\\
&\qquad-w(\tau^g_j(x_1,\dots,x_j),(\overline{x_j^{-1}\dots x_1^{-1}g})^{-1}x_{j+1},x_{j+2},\dots,x_n)+\Sigma(j+1,j+1)\\
&\qquad+\sum_{i=j}^{n-1}(-1)^{i+j}w\circ\sigma^g_{n-1,i}(x_1,\dots,x_jx_{j+1},\dots,x_n)\\
&\qquad+\sum_{s=1}^{j-1}(-1)^{s+j}w\circ\sigma^g_{n-1,j-1}(x_1,\dots,x_sx_{s+1},\dots,x_n))\Big]\\
&\qquad+(-1)^{n+1}\alpha_{\eta_1^g(x_1)}(1_{\eta_1^g(x_1)^{-1}}w\circ\sigma^{\overline{x_1^{-1}g}}_{n-1,n-1}(x_2,\dots,x_n))
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\sum_{j=2}^{n-1}\Big[w(\tau^g_{j-1}(x_1,\dots,x_{j-1}),(\overline{x_{j-1}^{-1}\dots x_1^{-1}g})^{-1}x_j,x_{j+1},\dots,x_n)-\Sigma(j,j)\\
&\qquad-w(\tau^g_j(x_1,\dots,x_j),(\overline{x_j^{-1}\dots x_1^{-1}g})^{-1}x_{j+1},x_{j+2},\dots,x_n)+\Sigma(j+1,j+1)\Big]\\
&\quad=w(\tau^g_1(x_1),(\overline{x_1^{-1}g})^{-1}x_2,x_3,\dots,x_n)-\Sigma(2,2)\\
&\qquad-w(\tau^g_{n-1}(x_1,\dots,x_n),(\overline{x_{n-1}^{-1}\dots x_1^{-1}g})^{-1}x_n).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&e(-w(g^{-1}x_1, x_2, \ldots, x_n) + \Sigma(1,1))\\
&\quad= e\alpha_{\eta_1^g(x_1)}\Big(1_{\eta_1^g(x_1)^{-1}}\sum_{j=0}^{n-1}(-1)^{j+1} w\circ\sigma_{n-1,j}^{\overline{x_1^{-1}g}}(x_2,\ldots,x_n)\Big)\\
&\qquad+ e\sum_{j=2}^{n-1}\sum_{i=j}^{n-1}(-1)^{i+j} w\circ\sigma_{n-1,i}^{g}(x_1,\ldots,x_jx_{j+1},\ldots,x_n)\\
&\qquad+ e\sum_{j=2}^{n-1}\sum_{s=1}^{j-1}(-1)^{s+j} w\circ\sigma_{n-1,j-1}^{g}(x_1,\ldots,x_sx_{s+1},\ldots,x_n))\Big]\\
&\qquad+ e(-1)^{n+1}\alpha_{\eta_1^g(x_1)}(1_{\eta_1^g(x_1)^{-1}} w\circ\sigma_{n-1,n-1}^{\overline{x_1^{-1}g}}(x_2,\ldots,x_n))\\
&\qquad- ew(\tau_{n-1}^g(x_1,\ldots,x_n), (\overline{x_{n-1}^{-1}\ldots x_1^{-1}g})^{-1}x_n)\\
&\qquad+ e\sum_{i=1}^{n-1}(-1)^{i+1} w\circ\sigma_{n-1,i}^{g}(x_1x_2, x_3,\ldots,x_n)).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&e(-w(g^{-1}x_1, x_2, \ldots, x_n) + \Sigma(1,1))\\
&\quad= e\alpha_{\eta_1^g(x_1)}\Big(1_{\eta_1^g(x_1)^{-1}}\sum_{j=0}^{n-1}(-1)^{j+1} w\circ\sigma_{n-1,j}^{\overline{x_1^{-1}g}}(x_2,\ldots,x_n)\Big)\\
&\qquad+ e\sum_{j=1}^{n-1}\sum_{i=j}^{n-1}(-1)^{i+j} w\circ\sigma_{n-1,i}^{g}(x_1,\ldots,x_jx_{j+1},\ldots,x_n)\\
&\qquad+ e\sum_{j=2}^{n-1}\sum_{s=1}^{j-1}(-1)^{s+j} w\circ\sigma_{n-1,j-1}^{g}(x_1,\ldots,x_sx_{s+1},\ldots,x_n))\Big]\\
&\qquad+ e(-1)^{n+1}\alpha_{\eta_1^g(x_1)}(1_{\eta_1^g(x_1)^{-1}} w\circ\sigma_{n-1,n-1}^{\overline{x_1^{-1}g}}(x_2,\ldots,x_n))\\
&\qquad- ew(\tau_{n-1}^g(x_1,\ldots,x_n), (\overline{x_{n-1}^{-1}\ldots x_1^{-1}g})^{-1}x_n).
\end{aligned}
$$

Now by Lemma 5.26 we have that

$$\begin{aligned}
& ew\left(\tau^g_{n-1}(x_1,\dots,x_{n-1}), (\overline{x^{-1}_{n-1}\dots x^{-1}_1 g})^{-1}x_n\right) \\
& + (-1)^{n+1} e\alpha_{\eta^g_1(x_1)}\left(1_{\eta^g_1(x_1)^{-1}} w\circ\sigma^{\overline{x_1^{-1}g}}_{n-1,n-1}(x_2,\dots,x_n)\right) \\
= {} & e\sum_{s=1}^{n-1}(-1)^{s+n} w\circ\sigma^g_{n-1,n-1}(x_1,\dots,x_s x_{s+1},\dots,x_n) \\
& - e(w\circ\tau^g_n)(x_1,\dots,x_n)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& e(-w(g^{-1}x_1,x_2,\dots,x_n)+\Sigma(1,1)) \\
& \quad = e\alpha_{\eta^g_1(x_1)}\left(1_{\eta^g_1(x_1)^{-1}}\sum_{j=0}^{n-1}(-1)^{j+1} w\circ\sigma^{\overline{x_1^{-1}g}}_{n-1,j}(x_2,\dots,x_n)\right) \\
& \quad - e(w\circ\tau^g_n)(x_1,\dots,x_n) \\
& \quad + e\sum_{j=1}^{n-1}\sum_{i=j}^{n-1}(-1)^{i+j} w\circ\sigma^g_{n-1,i}(x_1,\dots,x_j x_{j+1},\dots,x_n) \\
& \quad + e\sum_{j=2}^{n}\sum_{s=1}^{j-1}(-1)^{s+j} w\circ\sigma^g_{n-1,j-1}(x_1,\dots,x_s x_{s+1},\dots,x_n).
\end{aligned}$$

□

Lemma 5.28. *For all $n>0, w\in Z^n_{par}(G,\mathcal{A})$ and $x_1,\dots,x_n\in G$*

$$\begin{aligned}
& \left(\delta^{n-1}\varepsilon\right)(x_1,\dots,x_n) - \alpha_{x_1}\left(1_{x_1^{-1}}\varepsilon(x_2,\dots,x_n)\right) - w(x_1,\dots,x_n) \\
& = 1_{(x_1,\dots,x_n)}\prod_{g\in\Lambda}\theta_g\circ\alpha_{\eta^g_1(x_1)}\left(1_{\eta^g_1(x_1)^{-1}}\sum_{j=0}^{n-1}(-1)^{j+1} w\circ\sigma^{\overline{x_1^{-1}g}}_{n-1,j}(x_2,\dots,x_n)\right) \\
& \quad - w'(x_1,\dots,x_n)
\end{aligned}$$

Proof. Let $n=1$. By Remark 5.20 $1_{\eta^g_1(x)}\in H$, then

$$\theta_g(-w(g^{-1}x)) = \theta_g(-1_{\eta^g_1(x)} w(g^{-1}x)).$$

Thus using that fact and Lemma 5.23 we have

$$\begin{aligned}(\delta^0\varepsilon)(x) - \alpha_x(1_{x^{-1}\varepsilon}) - w(x) &= 1_x \prod_{g\in\Lambda} \theta_g(-w(g^{-1}x)) \\ &= 1_x \prod_{g\in\Lambda} \theta_g(-1_{\eta_1^g(x)} w(g^{-1}x)),\end{aligned}$$

by Lemma 5.26

$$\begin{aligned}1_x \prod_{g\in\Lambda} \theta_g(-1_{\eta_1^g(x)} w(g^{-1}x)) =& -1_x \prod_{g\in\Lambda} \theta_g(\alpha_{\eta_1^g(x)}(1_{\eta_1^g(x)^{-1}} w(\overline{x^{-1}g})^{-1}))) \\ & - 1_x \prod_{g\in\Lambda} \theta_g(1_{g^{-1}x} w \circ \tau_1^g(x)).\end{aligned}$$

By Remark 5.20 we obtain

$$\theta_g(1_{g^{-1}x} w \circ \tau_1^g(x)) = \theta_g(1_{g^{-1}} 1_{g^{-1}x} w \circ \tau_1^g(x)) = \theta_g(1_{(g^{-1},x)} w \circ \tau_1^g(x)).$$

In that proof of Lemma 5.21 we show that

$$\prod_{g\in\Lambda} \theta_g(1_{(g^{-1},x_1,\ldots,x_n)}) = 1_{(x_1,\ldots,x_n)}.$$

Hence,

$$1_x \prod_{g\in\Lambda} \theta_g(1_{g^{-1}x} w \circ \tau_1^g(x)) = 1_x \prod_{g\in\Lambda} \theta_g(w \circ \tau_1^g(x)).$$

Therefore, by Definition 5.22 and the fact that $(\overline{x^{-1}g})^{-1} = w(\sigma_{0,0}^{\overline{x^{-1}g}})$ we have that

$$\begin{aligned}(\delta^0\varepsilon)(x) - \alpha_x(1_{x^{-1}\varepsilon}) - w(x) =& -1_x \prod_{g\in\Lambda} \theta_g \circ \alpha_{\eta_1^g(x)} \left(1_{\eta_1^g(x)^{-1}} w\left(\sigma_{0,0}^{\overline{x^{-1}g}}\right)\right) \\ & - w'(x).\end{aligned}$$

For $n > 1$. By Lemma 5.27 we get

$$\begin{aligned} & e(-w(g^{-1}x_1, x_2, \ldots, x_n) + \Sigma(1,1)) \\ &= e\alpha_{\eta_1^g(x_1)}\Big(1_{\eta_1^g(x_1)^{-1}} \sum_{j=0}^{n-1} (-1)^{j+1} w \circ \sigma_{n-1,j}^{\overline{x_1^{-1}g}}(x_2,\ldots,x_n)\Big) \\ &\quad - e(w\circ\tau_n^g)(x_1,\ldots,x_n) \\ &\quad + e\sum_{j=1}^{n-1}\sum_{i=j}^{n-1} (-1)^{i+j} w\circ\sigma_{n-1,i}^g(x_1,\ldots,x_jx_{j+1},\ldots,x_n) \qquad (1) \\ &\quad + e\sum_{j=2}^{n}\sum_{s=1}^{j-1} (-1)^{s+j} w\circ\sigma_{n-1,j-1}^g(x_1,\ldots,x_sx_{s+1},\ldots,x_n), \qquad (2)\end{aligned}$$

where

$$e=\prod_{i=1}^{n}1_{\sigma^g_{n,i}(x_1,\ldots,x_n)}=1_{(g^{-1},x_1,\ldots,x_n)}\prod_{i=1}^{n}1_{\eta^g_i(x_1,\ldots,x_i)}.$$

Taking $j'=j-1$ in the sum (2), we rewrite it as

$$\sum_{j'=1}^{n-1}\sum_{s=1}^{j'}(-1)^{s+j'+1}w\circ\sigma^g_{n-1,j'}\left(x_1,\ldots,x_sx_{s+1},\ldots,x_n\right),$$

switching the order of summation we get

$$\sum_{s=1}^{n-1}\sum_{j'=s}^{n-1}(-1)^{s+j'+1}w\circ\sigma^g_{n-1,j'}\left(x_1,\ldots,x_sx_{s+1},\ldots,x_n\right),$$

and that is the opposite of the formula (1). Then

$$\begin{aligned}
&e(-w(g^{-1}x_1,x_2,\ldots,x_n)+\Sigma(1,1))\\
&\quad=e\alpha_{\eta^g_1(x_1)}\Big(1_{\eta^g_1(x_1)^{-1}}\sum_{j=0}^{n-1}(-1)^{j+1}w\circ\sigma^{\overline{x_1^{-1}g}}_{n-1,j}(x_2,\ldots,x_n)\Big)\\
&\qquad-e(w\circ\tau^g_n)\left(x_1,\ldots,x_n\right).
\end{aligned}$$

Applying θ_g to the both sides of previous formula, and since $\eta^g_i(x_1,\ldots,x_i)\in H$ for all i, we may remove $\prod_{i=1}^{n}1_{\eta^g_i(x_1,\ldots,x_i)}$ from e. Moreover using the fact that $\prod_{g\in\Lambda}\theta_g(1_{(g^{-1},x_1,\ldots,x_n)})=1_{(x_1,\ldots,x_n)}$, we obtain

$$\begin{aligned}
&1_{(x_1,\ldots,x_n)}\prod_{g\in\Lambda}\theta_g(-w(g^{-1}x_1,x_2,\ldots,x_n)+\Sigma(1,1))\\
&=1_{t(x_1,\ldots,x_n)}\prod_{g\in\Lambda}\theta_g\circ\alpha_{\eta^g_1(x_1)}(1_{\eta^g_1(x_1)^{-1}}\\
&\quad\sum_{j=0}^{n-1}(-1)^{j+1}w\circ\sigma^{\overline{x_1^{-1}g}}_{n-1,j}(x_2,\ldots,x_n))\\
&\quad-1_{(x_1,\ldots,x_n)}\prod_{g\in\Lambda}\theta_g\circ w\circ\tau^g_n(x_1,\ldots,x_n).
\end{aligned}$$

Finally, using Definition 5.22 and Lemma 5.23 we obtain the desire result. □

Lemma 5.29. *For all $x\in G$ and $a:\Lambda'\to\mathcal{A}$ one has*

$$\alpha_x\left(1_{x^{-1}}\prod_{g\in\Lambda}\theta_g(a(g))\right)=1_x\prod_{g\in\Lambda}\theta_g\circ\alpha_{\eta^g_1(x)}\left(1_{\eta^g_1(x)^{-1}}a(\overline{x^{-1}g})\right).$$

Proof. First observe using (ii) of Lemma 5.17 that

$$1_x \prod_{g\in\Lambda} a_g = \prod_{g\in\Lambda, \mathcal{A}_g\subseteq\mathcal{D}_x} a_g = \prod_{g,\overline{x^{-1}g}\in\Lambda} a_g,$$

where a_g is an arbitrary element of $\mathcal{A}_g$. We may replace the condition $g\in\Lambda$ by a stronger one $g, \overline{x^{-1}g}\in\Lambda$, and since we may put $1_{g^{-1}x}$ inside of θ_g. Thus,

$$1_x \prod_{g\in\Lambda} \theta_g \circ \alpha_{\eta_1^g(x)}\left(1_{\eta_1^g(x)^{-1}} a(\overline{x^{-1}g})\right) = 1_x \prod_{g,\overline{x^{-1}g}\in\Lambda} \theta_g\left(1_{g^{-1}x}\alpha_{\eta_1^g(x)}\left(1_{\eta_1^g(x)^{-1}} a(\overline{x^{-1}g})\right)\right).$$

Observe that

$$1_{g^{-1}x}\alpha_{\eta_1^g(x)}\left(1_{\eta_1^g(x)^{-1}} a(\overline{x^{-1}g})\right) = \alpha_{g^{-1}x_1}\circ\alpha_{\overline{x_1^{-1}g}}\left(1_{(\overline{x^{-1}g})^{-1}}1_{\eta_1^g(x)^{-1}}, a(\overline{x^{-1}g})\right)$$

denote the argument of $\alpha_{x_1^{-1}g}$ in the previous equality by $b=b(g,x)$, then by Remark 5.20 we get

$$\begin{aligned}\theta_g\circ\alpha_{g^{-1}x}\circ\alpha_{\overline{x^{-1}g}}(b) &= \operatorname{pr}_g\circ\alpha_g\left(1_{g^{-1}}\alpha_{g^{-1}x}\circ\alpha_{\overline{x^{-1}g}}(b)\right)\\ &= \operatorname{pr}_g\circ\alpha_g\circ\alpha_{g^{-1}}\circ\alpha_x\left(1_{x^{-1}}1_{x^{-1}g}\alpha_{\overline{x^{-1}g}}(b)\right)\\ &= \operatorname{pr}_g\circ\alpha_x\left(1_{x^{-1}}1_{x^{-1}g}\alpha_{x^{-1}g}(b)\right).\end{aligned}$$

As $\overline{x\overline{x^{-1}g}} = g\in\Lambda$, by (ii) of Lemma 5.17 we have $\mathcal{A}_{\overline{x^{-1}g}}\subseteq\mathcal{D}_{x^{-1}}$ and $\alpha_x\left(\mathcal{A}_{\overline{x^{-1}g}}\right) = \mathcal{A}_g$. Moreover, $\mathcal{A}_{\overline{x^{-1}g}}\subseteq\mathcal{D}_{x^{-1}g}$. Hence, in view of Lemma 5.15

$$\begin{aligned}\operatorname{pr}_g\circ\alpha_x\left(1_{x^{-1}}1_{x^{-1}g}\alpha_{\overline{x^{-1}g}}(b)\right) &= \alpha_x\circ\operatorname{pr}_{\overline{x^{-1}g}}\left(1_{x^{-1}}1_{x^{-1}g}\alpha_{\overline{x^{-1}g}}(b)\right)\\ &= \alpha_x\circ\operatorname{pr}_{\overline{x^{-1}g}}\circ\alpha_{\overline{x^{-1}g}}(b)\\ &= \alpha_x\circ\theta_{\overline{x^{-1}g}}(b),\end{aligned}$$

and consequently

$$\theta_g\circ\alpha_{\eta_1^g(x)}\left(1_{\eta_1^g(x)^{-1}} a\left(\overline{x^{-1}g}\right)\right) = \alpha_x\circ\theta_{\overline{x^{-1}g}}(b) = \alpha_x\circ\theta_{\overline{x^{-1}g}}\left(a\left(\overline{x^{-1}g}\right)\right).$$

Here we used Remark 5.20 to remove $1_{\eta_1^g(x)-1}$ and $1_{(\overline{x^{-1}g})^{-1}}$ from b. It follows that

$$\begin{aligned}&1_x\prod_{g,\overline{x^{-1}g}\in\Lambda}\theta_g\left(1_{g^{-1}x}\alpha_{\eta_1^g(x)}\left(1_{\eta_1^g(x)^{-1}}a(\overline{x^{-1}g})\right)\right)\\ &= \prod_{g,\overline{x^{-1}g}\in\Lambda}\alpha_x\circ\theta_{\overline{x^{-1}g}}(a(\overline{x^{-1}g})) = \alpha_x\left(\prod_{g,\overline{x^{-1}g}\in\Lambda}\theta_{\overline{x^{-1}g}}(a(\overline{x^{-1}g}))\right).\end{aligned}$$

To check the latter equality we have to check it in every block $\mathcal{A}_t$ of $\mathcal{A}$, where $t \in \Lambda$, it is easy to check since $\alpha_x\left(\mathcal{A}_{\overline{x^{-1}g}}\right) = \mathcal{A}_g$ and $\theta_{\overline{x^{-1}g}} : \mathcal{A} \to \mathcal{A}_{\overline{x^{-1}g}}$. Finally, let $g' = \overline{x^{-1}g} \in \Lambda$. Then $g = \overline{x\overline{x^{-1}g}} = \overline{xg'} \in \Lambda$, then

$$\alpha_x\left(\prod_{g',\overline{xg'}\in\Lambda} \theta_{g'}\left(a\left(g'\right)\right)\right) = \alpha_x\left(1_{x^{-1}}\prod_{g'\in\Lambda} \theta_{g'}\left(a\left(g'\right)\right)\right),$$

what give us the desire result. □

Lemma 5.30. *For all $n > 0, w \in Z^n_{par}(G,\mathcal{A})$ and $x_1,\ldots,x_n \in G$*

$$\begin{aligned}&1_{(x_1\ldots,x_n)}\prod_{g\in\Lambda}\theta_g\circ\alpha_{\eta_1^g(x_1)}\left(1_{\eta_1^g(x_1)-1}\sum_{j=0}^{n-1}(-1)^j w\circ\sigma_{n-1,j}^{\overline{x_1^{-1}g}}\left(x_2,\ldots,x_n\right)\right)\\&=\alpha_{x_1}(1_{x_1^{-1}}\varepsilon(x_2,\ldots,x_n))\end{aligned}$$

Proof. Using Lemma 5.29 with

$$a(g) = \sum_{j=0}^{n-1}(-1)^j w\circ\sigma_{n-1,j}^{g}\left(x_2,\ldots,x_n\right),$$

where n, w and $x_2,\ldots,x_n$ are fixed and $g \in \Lambda'$, we see that

$$\begin{aligned}&1_{(x_1\ldots,x_n)}\prod_{g\in\Lambda}\theta_g\circ\alpha_{\eta_1^g(x_1)}\left(1_{\eta_1^g(x_1)-1}\sum_{j=0}^{n-1}(-1)^j w\circ\sigma_{n-1,j}^{\overline{x_1^{-1}g}}\left(x_2,\ldots,x_n\right)\right)\\&=1_{(x_1\ldots,x_n)}\alpha_{x_1}\left(1_{x_1^{-1}}\prod_{g\in\Lambda}\theta_g\left(\sum_{j=0}^{n-1}(-1)^j w\circ\sigma_{n-1,j}^{g}\left(x_2,\ldots,x_n\right)\right)\right),\end{aligned}$$

since $1_{(x_1,\ldots,x_n)} = \alpha_{x_1}\left(1_{x_1^{-1}}1_{(x_2,\ldots,x_n)}\right)$, by Definition 5.22 we obtain

$$\begin{aligned}&1_{(x_1\ldots,x_n)}\prod_{g\in\Lambda}\theta_g\circ\alpha_{\eta_1^g(x_1)}\left(1_{\eta_1^g(x_1)-1}\sum_{j=0}^{n-1}(-1)^j w\circ\sigma_{n-1,j}^{\overline{x_1^{-1}g}}\left(x_2,\ldots,x_n\right)\right)\\&=\alpha_{x_1}(1_{x_1^{-1}}\varepsilon(x_2,\ldots,x_n)).\end{aligned}$$

□

Theorem 5.31. *Let $n > 0$ and $w \in Z^n_{par}(G,\mathcal{A})$. Then $w = \delta^{n-1}\varepsilon + w'$. In particular $w' \in Z^n_{par}(G,\mathcal{A})$.*

Proof. By Lemma 5.28 we have that

$$\begin{aligned}
&w'\left(x_1,\dots,x_n\right)+\left(\delta^{n-1}\varepsilon\right)\left(x_1,\dots,x_n\right)\\
&=w\left(x_1,\dots,x_n\right)\\
&\quad+1_{(x_1,\dots,x_n)}\prod_{g\in\Lambda}\theta_g\circ\alpha_{\eta_1^g(x_1)}\left(1_{\eta_1^g(x_1)^{-1}}\sum_{j=0}^{n-1}(-1)^{j+1}w\circ\sigma_{n-1,j}^{\overline{x_1^{-1}g}}\left(x_2,\dots,x_n\right)\right)\\
&\quad+\alpha_{x_1}\left(1_{x_1^{-1}}\varepsilon\left(x_2,\dots,x_n\right)\right),
\end{aligned}$$

and by Lemma 5.30 we get

$$w'\left(x_1,\dots,x_n\right)+\left(\delta^{n-1}\varepsilon\right)\left(x_1,\dots,x_n\right)=w\left(x_1,\dots,x_n\right).$$

□

5.3 Existence and uniqueness of a globalization

The aim in this section is to complete the construction of $\tilde{w}$ satisfying the conditions of Theorem 5.14. We will introduce some formulas which will be used here as well.

Lemma 5.32. *Let $g\in\Lambda'$. Then*

$$\begin{gathered}
\eta_n^g\left(x_1,\dots,x_n\right)=\eta_{n-1}^{\overline{x_1^{-1}g}}\left(x_2,\dots,x_n\right),n\geq 2,\\
\eta_n^g\left(x_1,\dots,x_i,x_{i+1},\dots,x_n\right)=\eta_{n-1}^g\left(x_1,\dots,x_ix_{i+1},\dots,x_n\right),1\leq i\leq n-2,\\
\eta_n^g\left(x_1,\dots,x_{n-1},x_nx_{n+1}\right)=\eta_n^g\left(x_1,\dots,x_n\right)\eta_{n+1}^g\left(x_1,\dots,x_{n+1}\right),n\geq 1.
\end{gathered}$$

Proof. For the very first equality

$$\begin{aligned}
\eta_{n-1}^{\overline{x_1^{-1}g}}\left(x_2,\dots,x_n\right)&=\eta(x_n^{-1}\overline{x_{n-1}^{-1}\dots\overline{x_1^{-1}g}})\\
&=\eta(x_n^{-1}\overline{x_{n-1}^{-1}\dots x_1^{-1}g})\\
&=\eta_n^g\left(x_1,\dots,x_n\right).
\end{aligned}$$

The second equality

$$\begin{aligned}
\eta_{n-1}^g\left(x_1,\dots,x_ix_{i+1},\dots,x_n\right)&=\eta(x_n^{-1}\overline{x_{n-1}^{-1}\dots(x_ix_{i+1})^{-1}\dots x_1^{-1}g})\\
&=\eta(x_n^{-1}\overline{x_{n-1}^{-1}\dots x_{i+1}^{-1}x_i^{-1}\dots x_1^{-1}g})\\
&=\eta_n^g\left(x_1,\dots,x_i,x_{i+1},\dots,x_n\right).
\end{aligned}$$

Finally, for the last one equality we have

$$\begin{aligned}\eta_n^g(x_1,\ldots,x_n)\,\eta_{n+1}^g(x_1,\ldots,x_{n+1}) &= (\overline{x_{n-1}^{-1}\ldots x_1^{-1}g})^{-1}x_n(\overline{x_n^{-1}\ldots x_1^{-1}g})\\ &\quad\cdot(\overline{x_n^{-1}\ldots x_1^{-1}g})^{-1}x_{n+1}(\overline{x_{n+1}^{-1}\ldots x_1^{-1}g})\\ &= (\overline{x_{n-1}^{-1}\ldots x_1^{-1}g})^{-1}x_nx_{n+1}(\overline{x_{n+1}^{-1}\ldots x_1^{-1}g})\\ &= \eta_n^g(x_1,\ldots,x_{n-1},x_nx_{n+1}).\end{aligned}$$

□

Definition 5.33. *Define* $\widetilde{w'}: G^n \to \mathcal{A}$ *by removing* $1_{(x_1,\ldots,x_n)}$ *from the definition of* w' *in Definition 5.22, i.e.*

$$\widetilde{w'}(x_1,\ldots,x_n) = \prod_{g\in\Lambda}\theta_g\circ w\circ\tau_n^g(x_1,\ldots,x_n).$$

Lemma 5.34. *Let* $n>0, w\in Z^n_{par}(G,A)$ *and* $x_1,\ldots,x_n\in G$. *Then*

$$\tilde{\delta}^n\widetilde{w'} = 0.$$

Proof. By Definition 5.13 we have

$$\begin{aligned}(\tilde{\delta}^n\widetilde{w'})(x_1,\ldots,x_{n+1}) = &\,\alpha_{x_1}\left(1_{x_1^{-1}}\widetilde{w'}(x_2,\ldots,x_{n+1})\right)\\ &+\sum_{i=1}^n(-1)^i1_{x_1}\widetilde{w'}(x_1,\ldots,x_ix_{i+1},\ldots,x_{n+1})\\ &+(-1)^{n+1}1_{x_1}\widetilde{w'}(x_1,\ldots,x_n).\end{aligned}$$

Now switching $\widetilde{w'}$ by its definition we get

$$\begin{aligned}(\tilde{\delta}^n\widetilde{w'})(x_1,\ldots,x_{n+1}) = &\,\alpha_{x_1}\left(1_{x_1^{-1}}\prod_{g\in\Lambda}\theta_g\left(w\circ\tau_n^g(x_2,\ldots,x_{n+1})\right)\right)\\ &+1_{x_1}\prod_{g\in\Lambda}\theta_g\left(\sum_{i=1}^n(-1)^iw\circ\tau_n^g(x_1,\ldots,x_ix_{i+1},\ldots,x_{n+1})\right)\\ &+1_{x_1}\prod_{g\in\Lambda}\theta_g\left((-1)^{n+1}w\circ\tau_n^g(x_1,\ldots,x_n)\right).\end{aligned}$$

Thus using Lemma 5.29 we have

$$\begin{aligned}
&\alpha_{x_1}\left(1_{x_1^{-1}}\prod_{g\in\Lambda}\theta_g(w\circ\tau_n^g(x_2,\dots,x_{n+1}))\right)\\
&\quad=1_{x_1}\prod_{g\in\Lambda}\theta_g\circ\alpha_{\eta_1^g(x_1)}\left(1_{\eta_1^g(x_1)^{-1}}(w\circ\tau_n^{\overline{x_1^{-1}g}}(x_2,\dots,x_{n+1}))\right).
\end{aligned}$$

Then

$$\begin{aligned}
(\tilde{\delta}^n\widetilde{w'})(x_1,\dots,x_{n+1})=&1_{x_1}\prod_{g\in\Lambda}\theta_g\circ\alpha_{\eta_1^g(x_1)}\left(1_{\eta_1^g(x_1)^{-1}}(w\circ\tau_n^{\overline{x_1^{-1}g}}(x_2,\dots,x_{n+1}))\right)\\
&+1_{x_1}\prod_{g\in\Lambda}\theta_g\left(\sum_{i=1}^{n}(-1)^i w\circ\tau_n^g(x_1,\dots,x_ix_{i+1},\dots,x_{n+1})\right)\\
&+1_{x_1}\prod_{g\in\Lambda}\theta_g\left((-1)^{n+1}w\circ\tau_n^g(x_1,\dots,x_n)\right).
\end{aligned}$$

By Remark 5.20 we have

$$\begin{aligned}
(\tilde{\delta}^n\widetilde{w'})(x_1,\dots,x_{n+1})=&1_{x_1}\prod_{g\in\Lambda}\theta_g\circ\alpha_{\eta_1^g(x_1)}\left(1_{\eta_1^g(x_1)^{-1}}(w\circ\tau_n^{\overline{x_1^{-1}g}}(x_2,\dots,x_{n+1}))\right)\\
&+1_{x_1}\prod_{g\in\Lambda}\theta_g\left(\sum_{i=1}^{n}(-1)^i 1_{\eta_1^g(x_1\dots x_i)}w\circ\tau_n^g(x_1,\dots,x_ix_{i+1},\dots,x_{n+1})\right)\\
&+1_{x_1}\prod_{g\in\Lambda}\theta_g\left((-1)^{n+1}1_{\eta_1^g(x_1\dots x_n)}w\circ\tau_n^g(x_1,\dots,x_n)\right).
\end{aligned}$$

Therefore $\tilde{\delta}^n\widetilde{w'}=0$ if

$$\begin{aligned}
0=&\alpha_{\eta_1^g(x_1)}\left(1_{\eta_1^g(x_1)^{-1}}(w\circ\tau_n^{\overline{x_1^{-1}g}}(x_2,\dots,x_{n+1}))\right)\\
&+\sum_{i=1}^{n}(-1)^i 1_{\eta_1^g(x_1\dots x_i)}w\circ\tau_n^g(x_1,\dots,x_ix_{i+1},\dots,x_{n+1})\\
&+(-1)^{n+1}1_{\eta_1^g(x_1\dots x_n)}w\circ\tau_n^g(x_1,\dots,x_n).
\end{aligned}$$

The right part of the previous formula is an expansion of cocycle identity

$$(\delta^n w)\circ\tau_{n+1}^g(x_1,\dots,x_{n+1})=0.$$

Indeed,

$$\begin{aligned}
&(\delta^n w)\circ \tau^g_{n+1}(x_1,\dots,x_{n+1})\\
&\quad =\alpha_{\eta^g_1(x_1)}(1_{\eta^g_1(x_1)^{-1}}w(\eta^g_2(x_1,x_2),\dots,\eta^g_{n+1}(x_1,\dots,x_{n+1})))\\
&\qquad +\sum_{i=1}^{n}(-1)^i 1_{\eta^g_1(x_1)\dots\eta^g_i(\cdots)}w\left(\eta^g_1(x_1),\dots,\eta^g_i(\cdots)\eta^g_{i+1}(\cdots),\dots,\eta^g_{n+1}(\cdots)\right)\\
&\qquad +(-1)^{n+1}1_{\eta^g_1(x_1)\dots\eta^g_{n+1}(\cdots)}w\left(\eta^g_1(x_1),\dots,\eta^g_n(\cdots)\right),
\end{aligned}$$

where $\eta^g_i(\cdots)$ means $\eta^g_i(x_1,\dots,x_i)$. By Lemma 5.32 we have

1. $(\eta^g_2(x_1,x_2),\dots,\eta^g_{n+1}(x_1,\dots,x^g_{n+1}))=\tau^{\overline{x_1^{-1}g}}_n(x_2,\dots,x_{n+1})$;
2. $(\eta^g_1(x_1),\dots,\eta^g_i(\cdots)\eta^g_{i+1}(\cdots),\dots,\eta^g_{n+1}(\cdots))=\tau^g_n\left(x_1,\dots,x_ix_{i+1},\dots,x_{n+1}\right)$;
3. $1_{\eta^g_1(x_1)\dots\eta^g_i(x_1,\dots,x_i)}=1_{\eta^g_1(x_1\dots x_i)}$.

Hence,

$$\begin{aligned}
0&=(\delta^n w)\circ \tau^g_{n+1}(x_1,\dots,x_{n+1})\\
&=\alpha_{\eta^g_1(x_1)}\left(1_{\eta^g_1(x_1)^{-1}}(w\circ\tau^{\overline{x_1^{-1}g}}_n(x_2,\dots,x_{n+1}))\right)\\
&\quad+\sum_{i=1}^{n}(-1)^i 1_{\eta^g_1(x_1\dots x_i)}w\circ\tau^g_n\left(x_1,\dots,x_ix_{i+1},\dots,x_{n+1}\right)\\
&\quad+(-1)^{n+1}1_{\eta^g_1(x_1\dots x_n)}w\circ\tau^g_n\left(x_1,\dots,x_n\right).
\end{aligned}$$

□

Definition 5.35. *For arbitrary $n\in\mathbb{Z}^+$ and $x_1,\dots,x_n\in G$, define $\widetilde{w}:G^n\to\mathcal{A}$*

$$\widetilde{w}=\widetilde{w'}+\hat{\delta}^{n-1}\varepsilon,$$

where

$$\begin{aligned}
(\hat{\delta}^{n-1}\varepsilon)(x_1,\dots,x_n)&=\alpha_{x_1}(1_{x^{-1}}\varepsilon(x_2,\dots,x_n))\\
&\quad+\sum_{i=1}^{n-1}(-1)^i\varepsilon(x_1,\dots,x_ix_{i+1},\dots,x_n)\\
&\quad+(-1)^n\varepsilon(x_1,\dots,x_{n-1}).
\end{aligned}$$

Now we can proof the next theorem which establish the existence of a globalization.

Theorem 5.36. *Let $\mathcal{A}$ be a direct product of indecomposable unital rings and $\alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g \mid g \in G\}$ a (non-necessarily transitive) unital partial action of G on $\mathcal{A}$. Then for any $n \geq 0$ each cocycle $w \in Z^n_{par}(G, \mathcal{A})$ with values in the induced $K_{par}(G)$-module is globalizable.*

Proof. By Lemma 5.10 we have the case $n = 0$. Now take $n > 0$. First consider the transitive case, the map $\widetilde{w} : G^n \to \mathcal{A}$ satisfies $w(g_1, \dots, g_n) = 1_{(g_1,\dots,g_n)}\tilde{w}(g_1, \dots, g_n)$ which is a condition of Theorem 5.14. Indeed, recall that $w' = 1_{(x_1,\dots,x_n)}\widetilde{w'}$ by Definition 5.33 and $1_{(x_1,\dots,x_n)}(\hat{\delta}^{n-1}\varepsilon) = \delta^{n-1}\varepsilon$ by Remark 3.40 and Definition 5.13, thus using Theorem 5.31 we have

$$\begin{aligned}1_{(x_1,\dots,x_n)}\widetilde{w}\,(x_1,\dots,x_n) &= 1_{(x_1,\dots,x_n)}\widetilde{w'}\,(x_1,\dots,x_n) + 1_{(x_1,\dots,x_n)}(\hat{\delta}^{n-1}\varepsilon)\,(x_1,\dots,x_n)\\ &= w'\,(x_1,\dots,x_n) + (\delta^{n-1}\varepsilon)\,(x_1,\dots,x_n)\\ &= w\,(x_1,\dots,x_n).\end{aligned}$$

Then to use Theorem 5.14 we only have to show that $\widehat{\delta}^n\widetilde{w} = 0$. Notice that by Definition 5.35

$$(\tilde{\delta}^n\widetilde{w})(x_1,\dots,x_{n+1}) = (\tilde{\delta}^n\widetilde{w'})(x_1,\dots,x_{n+1}) + (\tilde{\delta}^n\hat{\delta}^{n-1}\varepsilon)(x_1,\dots,x_{n+1})$$

By Lemma 5.34 $\tilde{\delta}^n\widetilde{w'} = 0$. Hence,

$$(\tilde{\delta}^n\widetilde{w})(x_1,\dots,x_{n+1}) = (\tilde{\delta}^n\hat{\delta}^{n-1}\varepsilon)(x_1,\dots,x_{n+1}).$$

Then observe that

$$(\tilde{\delta}^n\hat{\delta}^{n-1}\varepsilon)(x_1,\dots,x_{n+1}) = \alpha_{x_1}\left(1_{x_1^{-1}}(\hat{\delta}^{n-1}\varepsilon)\,(x_2,\dots,x_{n+1})\right) \tag{5.4}$$

$$+\sum_{i=1}^{n}(-1)^i 1_{x_1}(\hat{\delta}^{n-1}\varepsilon)\,(x_1,\dots,x_ix_{i+1},\dots,x_{n+1}) \tag{5.5}$$

$$+(-1)^{n+1}1_{x_1}(\hat{\delta}^{n-1}\varepsilon)\,(x_1,\dots,x_n). \tag{5.6}$$

Now observe that (5.5) is equal to

$$\begin{aligned}\sum_{i=1}^{n}(-1)^i 1_{x_1}\Big[\,\alpha_{x_1}\left(1_{x_1^{-1}}\varepsilon\left(w_{(i,2)},\dots,w_{(i,i)}\dots,w_{(i,n)}\right)\right)&\\ +\sum_{j=1}^{n-1}(-1)^j\varepsilon\left(w_{(i,1)},\dots,w_{(i,j)}w_{(i,j+1)},\dots,w_{(i,n)}\right)&\\ +(-1)^n\varepsilon\left(w_{(i,1)},\dots,w_{(i,i)}\dots,w_{(i,n-1)}\right)\Big],&\end{aligned}$$

where

$$w_{(i,j)} = \begin{cases} x_j & \text{if } j < i \\ x_i x_{i+1} & \text{if } j = i \\ x_{j+1} & \text{if } j > i \end{cases}$$

for $1 \le i \le n$ and $1 \le j \le n-1$. Define $\gamma(i,j) = \left(w_{(i,1)}, \ldots, w_{(i,j)}w_{(i,j+1)}, \ldots, w_{(i,n)}\right)$, note that if $j > i$ then

$$\begin{aligned}\gamma(i,j) &= \left(w_{(i,1)}, \ldots, w_{(i,i)}, \ldots, w_{(i,j)}w_{(i,j+1)}, \ldots, w_{(i,n)}\right)\\ &= \left(x_1, \ldots, x_i x_{i+1}, \ldots, x_j x_{j+1}, \ldots, x_{n+1}\right)\\ &= \left(w_{(j+1,1)}, \ldots, w_{(j+1,i)}w_{(j+1,i+1)}, \ldots, w_{(j+1,j+1)}, \ldots, w_{(i,n)}\right)\\ &= \gamma(j+1,i),\end{aligned}$$

and if $j = i$

$$\gamma(i,i) = (x_1, \ldots, x_i x_{i+1} x_{i+2}, \ldots, x_{n+1}) = \gamma(i+1,i).$$

Thus if $j \ge i$ we have $\gamma(i,j) = \gamma(j+1,i)$ and if $j < i$ $\gamma(i,j) = \gamma(j,i-1)$. Hence if $j \ge i$

$$(-1)^{i+j} 1_{x_1} \varepsilon\left(\gamma(i,j)\right) + (-1)^{i+j+1} 1_{x_1} \varepsilon\left(\gamma(j+1,i)\right) = 0.$$

Therefore,

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n-1}(-1)^{i+j} 1_{x_1} \varepsilon\left(w_{(i,1)}, \ldots, w_{(i,j)}w_{(i,j+1)}, \ldots, w_{(i,n)}\right)\\ &\qquad = 1_{x_1}\sum_{i=1}^{n}\sum_{j=1}^{n-1}(-1)^{i+j}\varepsilon(\gamma(i,j))\\ &\qquad = 1_{x_1}\sum_{i,j=1,\, j\ge i}^{n-1}(-1)^{i+j}\varepsilon(\gamma(i,j)) + (-1)^{i+j+1}\varepsilon(\gamma(j+1,i))\\ &\qquad = 0.\end{aligned}$$

Hence we have that (5.5) is equal to

$$\begin{aligned}\sum_{i=1}^{n}(-1)^i 1_{x_1}\Big[\, &\alpha_{x_1}\left(1_{x_1^{-1}}\varepsilon\left(w_{(i,2)}, \ldots, w_{(i,i)} \ldots, w_{(i,n)}\right)\right)\\ &+ (-1)^n \varepsilon\left(w_{(i,1)}, \ldots, w_{(i,i)} \ldots, w_{(i,n-1)}\right)\Big].\end{aligned}$$

On the other hand (5.6) is equal to

$$
\begin{aligned}
(-1)^{n+1}1_{x_1}(\hat{\delta}^{n-1}\varepsilon)(x_1,\dots,x_n) = {} & (-1)^{n+1}\alpha_{x_1}(1_{x^{-1}}\varepsilon(x_2,\dots,x_n)) \\
& + \sum_{i=1}^{n-1}(-1)^{i+n+1}1_{x_1}\varepsilon(x_1,\dots,x_ix_{i+1},\dots,x_n) \\
& - 1_{x_1}\varepsilon(x_1,\dots,x_{n-1}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(\tilde{\delta}^n\hat{\delta}^{n-1}\varepsilon)(x_1,\dots,x_{n+1}) = {} & \alpha_{x_1}\left(1_{x_1^{-1}}(\hat{\delta}^{n-1}\varepsilon)\,(x_2,\dots,x_{n+1})\right) \\
& + \sum_{i=1}^{n}(-1)^i 1_{x_1}\Big[\alpha_{x_1}\left(1_{x_1^{-1}}\varepsilon\,(x_2,\dots,x_ix_{i+1},\dots,x_{n+1})\right) \\
& + (-1)^n\varepsilon\left(w_{(i,1)},\dots,w_{(i,i)}\dots,w_{(i,n-1)}\right)\Big] \\
& + (-1)^{n+1}\alpha_{x_1}(1_{x^{-1}}\varepsilon(x_2,\dots,x_n)) \\
& + \sum_{i=1}^{n-1}(-1)^{i+n+1}1_{x_1}\varepsilon(w_{(i,1)},\dots,w_{(i,i)}\dots,w_{(i,n-1)}) \\
& - 1_{x_1}\varepsilon(x_1,\dots,x_{n-1}) \\
= {} & \alpha_{x_1}\left(1_{x_1^{-1}}(\hat{\delta}^{n-1}\varepsilon)\,(x_2,\dots,x_{n+1})\right) \\
& + \sum_{i=1}^{n}(-1)^i\alpha_{x_1}\left(1_{x_1^{-1}}\varepsilon\,(x_2,\dots,x_ix_{i+1},\dots,x_{n+1})\right) \\
& + 1_{x_1}\varepsilon(x_1,\dots,x_nx_{n+1}) \\
& + (-1)^{n+1}\alpha_{x_1}(1_{x^{-1}}\varepsilon(x_2,\dots,x_n)) \\
& - 1_{x_1}\varepsilon(x_1,\dots,x_{n-1}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(\tilde{\delta}^n\hat{\delta}^{n-1}\varepsilon)(x_1,\dots,x_{n+1}) = {} & \alpha_{x_1}\left(1_{x_1^{-1}}(\hat{\delta}^{n-1}\varepsilon)\,(x_2,\dots,x_{n+1})\right) && (5.7)\\
& + \sum_{i=1}^{n}(-1)^i\alpha_{x_1}\left(1_{x_1^{-1}}\varepsilon\,(x_2,\dots,x_ix_{i+1},\dots,x_{n+1})\right) && (5.8)\\
& + (-1)^{n+1}\alpha_{x_1}(1_{x^{-1}}\varepsilon(x_2,\dots,x_n)). && (5.9)
\end{aligned}
$$

The terms of the expansion of (5.7) are

$$\begin{aligned}
&\alpha_{x_1}\left(1_{x_1^{-1}}\alpha_{x_2}\left(1_{x_2^{-1}}\varepsilon(x_3,\dots,x_{n+1})\right)\right),\\
&\alpha_{x_1}\left((-1)^{i-1}1_{x_1^{-1}}\varepsilon(x_2,\dots,x_ix_{i+1},\dots,x_{n+1})\right), 2\le i\le n,\\
&(-1)^n\alpha_{x_1}\left(1_{x_1^{-1}}\varepsilon(x_2,\dots,x_n)\right),
\end{aligned}$$

while the summands in (5.8) and (5.9) are

$$\begin{aligned}
&-1_{x_1}\alpha_{x_1x_2}\left(1_{x_2^{-1}x_1^{-1}\varepsilon(x_3,\dots,x_{n+1})}\right),\\
&(-1)^i\alpha_{x_1}\left(1_{x_1^{-1}}\varepsilon\left(x_2,\dots,x_ix_{i+1},\dots,x_{n+1}\right)\right), 2\le i\le n,\\
&(-1)^{n+1}\alpha_{x_1}\left(1_{x_1^{-1}\varepsilon(x_2,\dots,x_n)}\right)
\end{aligned}$$

Thus the terms of the expansion of $(\tilde{\delta}^n\hat{\delta}^{n-1}\varepsilon)(x_1,\dots,x_{n+1})$ cancels and $\tilde{\delta}^n\hat{\delta}^{n-1}\varepsilon=0$. Finally, for the non transitive case suppose that $\mathcal{A}$ is a product of blocks:

$$\mathcal{A}=\prod_{\lambda\in\Lambda}\mathcal{A}_\lambda$$

i.e. each $\mathcal{A}_\lambda$ is an indecomposable unital ring, and let α be a unital partial action of G on $\mathcal{A}$. If α is not necessarily transitive, then for a given block $\mathcal{A}_\lambda$ define its orbit by

$$o_\lambda=\left\{\mathcal{A}_{\lambda'}:\exists g\in G,\mathcal{A}_\lambda\subseteq\mathcal{D}_{g^{-1}},\alpha_g\left(\mathcal{A}_\lambda\right)=\mathcal{A}_{\lambda'}\right\}.$$

These are the *block-orbits* of $\mathcal{A}$ with respect to α. Note that for any pair $\lambda,\lambda'\in\Lambda$ such that $o_\lambda\cap o_{\lambda'}\neq\varnothing$ there exist $\lambda''\in\Lambda$ and $g,g'\in G$ such that $\mathcal{A}_\lambda\subseteq\mathcal{D}_{g^{-1}}$, $\mathcal{A}_{\lambda'}\subseteq\mathcal{D}_{g'^{-1}}$ and $\alpha_g(\mathcal{A}_\lambda)=\mathcal{A}_{\lambda''}=\alpha_{g'}(\mathcal{A}_{\lambda'})$, then since $\mathcal{A}_{\lambda''}\subseteq\mathcal{D}_g\cap\mathcal{D}_{g'}$ we have $\mathcal{A}_{\lambda'}\subseteq\mathcal{D}_{(g^{-1}g')^{-1}}$ and $\mathcal{A}_\lambda=\alpha_{g^{-1}g'}(\mathcal{A}_{\lambda'})$, therefore $\mathcal{A}_\lambda\in o_{\lambda'}$, whence $o_\lambda=o_{\lambda'}$. So we can take a partition of $\{\mathcal{A}_\lambda\mid\lambda\in\Lambda\}$ of block-orbits. Let $\Upsilon\subseteq\Lambda$ be such that $\{o_\mu:\mu\in\Upsilon\}$ is a partition of $\{\mathcal{A}_\lambda\mid\lambda\in\Lambda\}$. For any $\mu\in\Upsilon$, put $\mathcal{O}_\mu=\prod_{\mathcal{A}_\lambda\in o_\mu}\mathcal{A}_\lambda$. Therefore

$$\mathcal{A}=\prod_{\mu\in\Upsilon}\mathcal{O}_\mu.$$

The ring $\mathcal{O}_\mu$ will be called the orbit ideal corresponding to μ. Due to the way we construct each orbit ideal $\mathcal{O}_\mu$ we have that α restricted to $\mathcal{O}_\mu$ is a transitive unital partial action of G on $\mathcal{O}_\mu$. So the construction of $\widetilde{w}$ reduces to the transitive case over each $\mathcal{O}_\mu$.

□

For the following theorem we will use the next results from [7].

Remark 5.37. Let $\mathcal{R}$ be a ring and $\{\mathcal{R}_\mu\}_{\mu\in M}$ a family of unital ideals of $\mathcal{R}$. Now define the homomorphism $\phi : \mathcal{R} \to \prod_{\mu\in M} \mathcal{R}_\mu$ given by $r \mapsto (1_\mu \cdot r)_{\mu\in M}$, where 1_μ is the unity of $\mathcal{R}_\mu$. Then ϕ satisfies $\pi_{\mu'} \circ \phi(r) = 1_{\mu'} \cdot r$, where $\pi_{\mu'} : \prod_{\mu\in M} \mathcal{R}_\mu \to \mathcal{R}_{\mu'}$ is the natural projection. When a homomorphism satisfies the previous condition we say that it **respects projections**, moreover ϕ is the unique homomorphism $\mathcal{R} \to \prod_{\mu\in M} \mathcal{R}_\mu$ whose respects projections.

Lemma 5.38. *Let $\mathcal{C}$ be a not necessarily unital ring and $\{\mathcal{C}_\mu \mid \mu \in M\}$ a family of pairwise distinct unital ideals in $\mathcal{C}$. Suppose that I and J are unital ideals in $\mathcal{C}$ such that*

$$I \cong \prod_{\mu\in M_1} \mathcal{C}_\mu \text{ and } J \cong \prod_{\mu\in M_2} \mathcal{C}_\mu,$$

where $M_1, M_2 \subseteq M, \mathcal{C}_\mu \subseteq I$ for all $\mu \in M_1$ and $\mathcal{C}_{\mu'} \subseteq J$ for all $\mu' \in M_2$. If the above isomorphisms respect projections, then there is an isomorphism

$$I + J \cong \prod_{\mu\in M_1\cup M_2} \mathcal{C}_\mu,$$

which also respects projections.

Proof. First observe that $I + J$ is a unital ring with unity element $1_I + 1_J - 1_I1_J$. Indeed, for any $v \in I$ and $w \in J$ we have that

$$\begin{aligned}(v + w)(1_I + 1_J - 1_I1_J) &= v1_I + v1_J - v1_I1_J + w1_I + w1_J - w1_I1_J\\ &= v + v1_J - v1_J + w1_I + w - w1_I\\ &= v + w\end{aligned}$$

and

$$\begin{aligned}(1_I + 1_J - 1_I1_J)(v + w) &= 1_Iv + 1_Jv - 1_I1_Jv + 1_Iw + 1_Jw - 1_I1_Jw\\ &= v + 1_Jv - 1_Jv + 1_Iw + w - 1_Iw\\ &= v + w.\end{aligned}$$

Now define $J' = J\,(1_J - 1_I1_J)$. Observe that for any $u \in I \cap J'$ we have that $u = 0$. Indeed, for $u \in J'$ there exist $j \in J$ such that $u = j\,(1_J - 1_I1_J)$, so if $u \in I$ then $j - j1_I \in I$, whence $j \in I$ and $u = 0$. Hence, $I + J = I \oplus J'$ and $J = (I \cap J) \oplus J'$. Notice that for any $\mu \in M_1$ and $j \in J$, we have that $1_\mu j(1_I - 1_I1_J) = 1_\mu j - 1_\mu j = 0$. Therefore, since the isomorphism $J \cong \prod_{\mu\in M_2} \mathcal{C}_\mu$ respects projections it can be restricted to the isomorphism

$$J' \cong \prod_{\mu\in M_2\setminus M_1} \mathcal{C}_\mu \subseteq \prod_{\mu\in M_2} \mathcal{C}_\mu,$$

such that respects projections. Then

$$I + J = I \oplus J' \cong \left(\prod_{\mu \in M_1} \mathcal{C}_\mu\right) \oplus \left(\prod_{\mu \in M_2 \setminus M_1} \mathcal{C}_\mu\right) \cong \left(\prod_{\mu \in M_1} \mathcal{C}_\mu\right) \times \left(\prod_{\mu \in M_2 \setminus M_1} \mathcal{C}_\mu\right),$$

the latter being isomorphic to $\prod_{\mu \in M_1 \sqcup (M_2 \setminus M_1)} \mathcal{C}_\mu$, which proves

$$I + J \cong \prod_{\mu \in M_1 \cup M_2} \mathcal{C}_\mu.$$

Furthermore, since all the isomorphisms used respect projections, the latter one too. □

Proposition 5.39. *Let $\mathcal{A}$ be a direct product $\prod_{g \in \Lambda} \mathcal{A}_g$ of indecomposable unital rings, α a transitive unital partial action of G on $\mathcal{A}$ and $(\beta, \mathcal{B})$ an enveloping action of $(\alpha, \mathcal{A})$ with $\mathcal{A} \subseteq \mathcal{B}$. Then $\mathcal{B}$ embeds as an ideal into $\prod_{g \in \Lambda'} \mathcal{A}_g$, where $\mathcal{A}_g$ denotes the ideal $\beta_g(\mathcal{A}_1)$ in $\mathcal{B}$. Moreover, $\mathcal{M}(\mathcal{B}) \cong \prod_{g \in \Lambda'} \mathcal{A}_g$, and β^* is transitive, when seen as a partial action of G on $\prod_{g \in \Lambda'} \mathcal{A}_g$.*

Proof. By Remark 5.37 there is a unique homomorphism $\phi : \mathcal{B} \to \prod_{g \in \Lambda'} \mathcal{A}_g$, which respects projections. We will prove that ϕ is injective. Since $\mathcal{B} = \sum_{g \in G} \beta_g(\mathcal{A})$, each element of $\mathcal{B}$ belongs to an ideal I of $\mathcal{B}$ of the form $\sum_{i=1}^{k} \beta_{x_i}(\mathcal{A}), x_1, \dots, x_k \in G$. Therefore, it suffices to show that the restriction of ϕ to any such I is injective. Using Remark 5.18, we may construct for any $i = 1, \dots, k$ an isomorphism

$$\beta_{x_i}(\mathcal{A}) = \beta_{x_i}\left(\prod_{g \in \Lambda} \mathcal{A}_g\right) \cong \prod_{g \in \Lambda} \beta_{x_i}(\mathcal{A}_g) = \prod_{g \in \Lambda} \mathcal{A}_{\overline{x_i g}}$$

which respects projections. Notice that it follows from the definition of Λ' that the ideals $\mathcal{A}_g, g \in \Lambda'$, are pairwise distinct. Hence by Lemma 5.38 there is an isomorphism

$$\psi : I \to \prod_{g \in \Lambda''} \mathcal{A}_g,$$

where $\Lambda'' = \{\overline{x_i g} \mid g \in \Lambda, i = 1, \dots, k\} \subseteq \Lambda'$, and it also respects projections. We claim that the restriction of ϕ to I coincides with ψ if one understands the product in the right-hand side of

$$\psi : I \to \prod_{g \in \Lambda''} \mathcal{A}_g$$

as an ideal in $\prod_{g\in\Lambda'}\mathcal{A}_g$. Indeed, for all $g\in\Lambda''$ and $b\in I$ one has

$$\mathrm{pr}_g\circ\psi(b)=1_{\mathcal{A}_g}b=\mathrm{pr}_g\circ\phi(b),$$

because ϕ and ψ respect projections. Now if $g\in\Lambda'\backslash\Lambda''$, then $\overline{x_i^{-1}g}\notin\Lambda$ for all $i=1,\dots,k$, since otherwise $g=\overline{x_i\overline{x_i^{-1}g}}\in\Lambda''$. Hence, for all $b=\sum_{i=1}^k\beta_{x_i}(a_i)\in I$ $(a_i\in\mathcal{A})$ in view of Remark 5.18

$$\mathrm{pr}_g\circ\phi(b)=1_{\mathcal{A}_g}b=\sum_{i=1}^k\beta_{x_i}\left(1_{\mathcal{A}_{\overline{x_i^{-1}g}}}a_i\right)=\sum_{i=1}^k\beta_{x_i}(0)=0.$$

This proves the claim, and thus injectivity of ϕ. Moreover, since $\phi(I)=\prod_{g\in\Lambda''}\mathcal{A}_g$ is an ideal in $\prod_{g\in\Lambda'}\mathcal{A}_g$, it follows that $\phi(\mathcal{B})$ is also an ideal in $\prod_{g\in\Lambda'}\mathcal{A}_g$.

Regarding the second statement of the proposition, notice that each element of $\prod_{g\in\Lambda'}\mathcal{A}_g$ acts as a multiplier of $\mathcal{B}$, as $\phi(\mathcal{B})$ is an ideal in $\prod_{g\in\Lambda'}\mathcal{A}_g$. Conversely, let $w\in\mathcal{M}(\mathcal{B})$. Then $w1_{\mathcal{A}_g}=w1_{\mathcal{A}_g}\cdot1_{\mathcal{A}_g}\in\mathcal{A}_g$ for all $g\in\Lambda'$. Define $a\in\prod_{g\in\Lambda'}\mathcal{A}_g$ by $\mathrm{pr}_g(a)=w1_{\mathcal{A}_g}$. We need to show that $\phi(wb)=a\phi(b)$ and $\phi(bw)=\phi(b)a$. Indeed using the fact that ϕ respects projections, we get

$$\mathrm{pr}_g(\phi(wb))=1_{\mathcal{A}_g}\cdot wb=w1_{\mathcal{A}_g}\cdot1_{\mathcal{A}_g}b=w1_{\mathcal{A}_g}\cdot\mathrm{pr}_g(\phi(b))=\mathrm{pr}_g(a\phi(b))$$

for all $g\in\Lambda'$. Similarly $\mathrm{pr}_g(\phi(bw))=\mathrm{pr}_g(a\phi(b))$ for arbitrary $g\in\Lambda'$. The transitivity of β^* easily follows from the definition of $\mathcal{A}_g$ for $g\in\Lambda'$. $\square$

Finally the next theorem allow us to obtain the uniqueness of a globalization.

Theorem 5.40. *Let $\mathcal{A}$ be a direct product $\prod_{g\in\Lambda}\mathcal{A}_g$ of indecomposable unital rings, α a unital partial action of G on $\mathcal{A}$ and $w_i\in Z^n_{par}(G,\mathcal{A}), i=1,2\,(n>0)$. Suppose that $(\beta,\mathcal{B})$ is an enveloping action of $(\alpha,\mathcal{A})$ and $u_i\in Z^n(G,\mathcal{M}(\mathcal{B}))$ is a globalization of w_i $i=1,2$. If w_1 is cohomologous to w_2, then u_1 is cohomologous to u_2. In particular, any two globalizations of the same partial n-cocycle are cohomologous.*

Proof. Using the same argument used in the proof of Theorem 5.36 we can consider only the transitive case and by Proposition 5.39 we can assume, without loss of generality, that $\mathcal{M}(\mathcal{B})=\prod_{g\in\Lambda'}\mathcal{A}_g\supseteq\mathcal{A}$. Define homomorphism $\vartheta_g:\mathcal{M}(\mathcal{B})\to\mathcal{M}(\mathcal{B})$ by

$$\vartheta_g=\beta_g\circ\mathrm{pr}_1$$

and $u_i'\in C^n(G,\mathcal{M}(\mathcal{B}))$ by

$$u_i'(x_1,\dots,x_n)=\prod_{g\in\Lambda'}\vartheta_g\circ u_i\circ\tau_n^g(x_1,\dots,x_n),\ i=1,2.$$

Note that the definition of u'_i is analogous to that of w' in Definition 5.22, and the definition of ϑ_g is analogous to that of 5.19 then using Theorem 5.31 we have that $u'_i \in Z^n(G, \mathcal{M}(\mathcal{B}))$ and u_i is cohomologous to u'_i, $i = 1, 2$. Suppose that w_1 is cohomologous to w_2. Then if we prove that u'_1 is cohomologous to u'_2 then u'_1 and u'_2 are cohomologous. First observe that since u_i is a globalization of w_i we have $1_{(x_1,\dots,x_n)} u_i(x_1, \dots, x_n) = w_i(x_1, \dots, x_n)$, then for arbitrary $h_1, \dots, h_n \in H$

$$\mathrm{pr}_1 \circ u_i(h_1, \dots, h_n) = \mathrm{pr}_1(1_{(h_1,\dots,h_n)} u_i(h_1, \dots, h_n)) = \mathrm{pr}_1 \circ w_i(h_1, \dots, h_n)$$

whence

$$u'_i(x_1, \dots, x_n) = \prod_{g \in \Lambda'} \vartheta_g \circ w_i \circ \tau_n^g(x_1, \dots, x_n),\ i = 1, 2.$$

If $w_2 = w_1 + \delta^{n-1}\xi$ for some $\xi \in C^{n-1}_{par}(G, \mathcal{A})$, then we get $u'_2 = u'_1 + (\delta^{n-1}\xi)'$, where

$$\left(\delta^{n-1}\xi\right)'(x_1, \dots, x_n) = \prod_{g \in \Lambda'} \vartheta_g \circ \left(\delta^{n-1}\xi\right) \circ \tau_n^g (x_1, \dots, x_n).$$

Then is enough to prove that

$$(\delta^{n-1}\xi)' = \delta^{n-1}\xi',$$

where

$$\xi'(x_1, \dots, x_{n-1}) = \prod_{g \in \Lambda'} \vartheta_g \circ \xi \circ \tau_{n-1}^g (x_1, \dots, x_{n-1}).$$

Observe that since we can omit the idempotents $1_{\eta_1^g(x_1)\dots\eta_i^g(x_1,\dots,x_i)}$ in ϑ_g, then $(\delta^{n-1}\xi)'(x_1, \dots, x_n)$ is equal to

$$\begin{aligned}\prod_{g \in \Lambda'} \vartheta_g \Bigg[& \beta_{\eta_i^g(x_1)}(\xi \circ \tau_{n-1}^{\overline{x^{-1}g}}(x_2, \dots, x_n)) \\ & + \sum_{i=1}^{n-1} (-1)^i \xi(\eta_1^g(x_1), \dots, \eta_i^g(\cdots)\eta_{i+1}^g(\cdots), \dots, \eta_n^g(x_1, \dots, x_n)) \\ & + (-1)^i \xi(\eta_1^g(x_1), \dots, \eta_{n-1}^g(x_1, \dots, x_{n-1})) \Bigg].\end{aligned}$$

On the other hand $(\delta^{n-1}\xi')(x_1, \ldots, x_n)$ is equal to

$$\beta_{x_1}\left(\prod_{g\in\Lambda'} \vartheta_g \circ \xi \circ \tau^g_{n-1}(x_2, \ldots, x_n)\right)$$
$$+ \sum_{i=1}^{n-1}(-1)^i \prod_{g\in\Lambda'} \vartheta_g \circ \xi(\eta^g_1(x_1), \ldots, \eta^g_i(\cdots)\eta^g_{i+1}(\cdots), \ldots, \eta^g_n(x_1, \ldots, x_n))$$
$$+ (-1)^i \prod_{g\in\Lambda'} \vartheta_g \circ \xi(\eta^g_1(x_1), \ldots, \eta^g_{n-1}(x_1, \ldots, x_{n-1})).$$

Hence to prove that $(\delta^{n-1}\xi')(x_1, \ldots, x_n) = (\delta^{n-1}\xi)'(x_1, \ldots, x_n)$, since ϑ_g is an homomorphism, we only have to show that

$$\beta_{x_1}\left(\prod_{g\in\Lambda'} \vartheta_g \circ \xi \circ \tau^g_{n-1}(x_2, \ldots, x_n)\right) = \prod_{g\in\Lambda'} \vartheta_g \circ \beta_{\eta^g_i(x_1)} \circ \xi \circ \tau^{\overline{x^{-1}g}}_{n-1}(x_2, \ldots, x_n)$$

which is consequence of the global version of Lemma 5.29. □

Corollary 5.41. *Let $\mathcal{A}$ be a direct product $\prod_{g\in\Lambda} \mathcal{A}_g$ of indecomposable unital rings, α a partial action of G on $\mathcal{A}$ and $(\beta, \mathcal{B})$ an enveloping action of $(\alpha, \mathcal{A})$. Then $H^n_{par}(G, \mathcal{A})$ is isomorphic to the classical cohomology group $H^n(G, \mathcal{M}(\mathcal{B}))$.*

Proof. By Proposition 5.7 the map $\rho : H^n_{par}(G, \mathcal{M}(\mathcal{B})) \to H^n_{par}(G, \mathcal{A})$ is an homomorphism, thus the case $n = 0$ is Lemma 5.10, finally for $n > 0$ we have that ρ is invertible by Theorems 5.36 and 5.40. □

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Index