Variedades de Gelfand-Tsetlin

Germán Alonso Benitez Monsalve

TESE APRESENTADA
AO
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
DA
UNIVERSIDADE DE SÃO PAULO
PARA
OBTENÇÃO DO TÍTULO
DE
DOUTOR EM CIÊNCIAS

Programa: Doutorado em Matemática
Orientador: Prof. Dr. Vyacheslav Futorny

Durante o desenvolvimento deste trabalho o autor recebeu auxílio financeiro da CAPES

São Paulo, Novembro de 2016

Variedades de Gelfand-Tsetlin

Esta versão da tese contém as correções e alterações sugeridas pela Comissão Julgadora durante a defesa da versão original do trabalho, realizada em 21/11/2016. Uma cópia da versão original está disponível no Instituto de Matemática e Estatística da Universidade de São Paulo.

Comissão Julgadora:

- Prof. Dr. Vyacheslav Futorny (orientador) - IME-USP
- Prof. Dr. Cristian Ortiz Gonzalez- IME-USP
- Prof. Dr. Viktor Bekkert - UFMG
- Prof. Dr. Marcos Banevenuto Jardim - IMECC-UNICAMP
- Prof. Dr. Lucas Henrique Calixto - UFMG

A mi familia:
Papá, Mamá,
Suki, Henry, Fabian,
Thomas y Mathias.

Agradecimentos

Agradeço, primeiramente a Deus, por me permitir conquistar este sonho.

Muito obrigado a minha familia pelo carinho e apoio recebido durante todos esses anos, mesmo com todas minhas doideras, são eles meus motores e minha inspiração. Ao meu pai Osvaldo Benitez Tapias e a minha mãe Fabiola Monsalve Valencia, porque desde criança me mostraram o valor de estudar. Sou grato a Suki por ser minha parceira e fazer de minhas aventuras nossas aventuras. A meus irmãos Jhon Henry e Fabian pela confiança, a meu sobrinho Thomas por me ensinar a sorrir e brincar, mesmo nos momentos mais difíceis. Também, quero agradecer a Dom Jaime Palacio Escobar, muito querido e estimado pela familia Benitez Monsalve.

Gostaria agradecer a meu orientador Vyacheslav Futorny pela paciência, amizade e confiança depositada em mim durante este projeto de pesquisa.

À banca julgadora, composta pelos profesores Vyacheslav Futorny, Cristian Ortiz Gonzalez, Viktor Bekkert, Marcos Banevenuto Jardim e Lucas Henrique Calixto, agradeço pelas sugestões dadas visando melhorar meu trabalho de tese.

Sou grato ao grupo de trabalho Luis Enrique Ramirez, Wilson Mutis e Carlos Gomes por nossos seminários sobre a pesquisa, em que apresentei boa parte deste trabalho. Também ao pessoal dos seminários sobre álgebras de Lie, em especial a Cesar Rodriguez, Carlos Payares e Elkin Quintero. Sem esquecer meu agradecimento a Kostiantyn Iusenko pelos tantos outros seminários.

Agradeço ao Programa de Pós-Graduação do Instituto de Matemática e Estatística (IME) da Universidade de São Paulo (USP) pela qualidade acadêmica e à Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), pelo apoio financeiro.

Para terminar, gostaria de agradecer aos funcionários do Instituto de Matemática e Estatística da USP pela esforço, paciência e infraestrutura oferecidas.

Resumo

BENITEZ MONSALVE, G. A. **Variedades de Gelfand-Tsetlin**. 2016. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2016.

Serge Ovsienko provou que a variedade de Gelfand-Tsetlin para $\mathfrak{gl}_n$ é equidimensional (i.e., todas suas componentes irredutíveis têm a mesma dimensão) com dimensão $\frac{n(n-1)}{2}$. Este resultado é conhecido como *Teorema de Ovsienko* e tem importantes consequências na Teoria de Representacões de Álgebras. Neste trabalho, provamos uma versão fraca do Teorema de Ovsienko para $\mathfrak{gl}_n$ e estendemos tal versão fraca a uma estrutura que tem como caso particular $\mathfrak{gl}_3$, esse é o caso do grupo quântico Yangian $Y_p(\mathfrak{gl}_3)$ de nível p. Além disso, o Teorema de Ovsienko também tem consequências na Geometria Simplética, especificamente na equidimensionalidade das fibras em uma projeção da aplicação de Kostant-Wallach. Neste trabalho apresentamos a generalização deste resultado.

Palavras-chave: Variedades algébricas, dimensão, equidimensionalidade, Gelfand-Tsetlin, Kostant-Wallach, Yangians.

Abstract

BENITEZ MONSALVE, G. A. **Gelfand-Tsetlin varieties**. 2016. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2016.

Serge Ovsienko proved that the Gelfand-Tsetlin variety for $\mathfrak{gl}_n$ is equidimensional (i.e., all its irreducible components have the same dimension) with dimension $\frac{n(n-1)}{2}$. This result is known as *Ovsienko's Theorem* and it has important consequences in Representation Theory of Algebras. In this work, we prove a weak version of Ovsienko's Theorem for $\mathfrak{gl}_n$ and we extend that weak version to a structure which has as particular case $\mathfrak{gl}_3$, this case is the quantum group level p Yangian $Y_p(\mathfrak{gl}_3)$. Moreover, the theorem of Ovsienko also has consequences in Symplectic Geometry, more concretely in the equidimensionality of the fibers in a projection of the Kostant-Wallach map. In this work we will present the generalization of that result.

Keywords: Algebraic varieties, dimension, equidimensionality, Gelfand-Tsetlin, Kostant-Wallach, Yangians.

Sumário

Capítulo 1

Introdução

Consideremos a variedade de Gelfand-Tsetlin associada à subálgebra de Gelfand-Tsetlin Γ para a álgebra de Lie linear geral $\mathfrak{gl}_n$. Serge Ovsienko provou em [Ovsienko (2003)] que essa variedade é equidimensional (ou seja, todas suas componentes irredutíveis tem a mesma dimensão) com dimensão $\frac{n(n-1)}{2}$. Esse resultado é conhecido como *"Teorema de Ovsienko"* e, como consequência desse fato, a variedade de Gelfand-Tsetlin é uma interseção completa. De maneira independente, B. Kostant e N. Wallach em [Kostant e Wallach (2006a)] e [Kostant e Wallach (2006b)] provaram que todas as componentes regulares da variedade de Gelfand-Tsetlin para $\mathfrak{gl}_n$ têm dimensão $\frac{n(n-1)}{2}$.

Um famoso teorema de Kostant, afirma que a álgebra envolvente universal $U(\mathfrak{g})$ de uma álgebra de Lie semisimples e complexa $\mathfrak{g}$, é um módulo livre à esquerda (ou à direita) sobre seu centro (ver teorema 0.12 em [Kostant (1963)]). A razão por trás desse fato é que a variedade associada ao centro de $\mathfrak{g}$ é uma interseção completa. Por exemplo, a variedade associada ao centro de $\mathfrak{g} = \mathfrak{gl}_n$ coincide com o conjunto das matrizes nilpotentes, a qual é irredutível e possui dimensão $n^2 - n$.

A. Molev, M. Nazarov e G. Olshanskiĭ provaram em [Molev *et al.* (1996)] que o grupo quântico Yangian $Y(\mathfrak{gl}_n)$ é livre sobre seu centro. Também, S. Ovsienko provou em [Ovsienko (2002)] que $U(\mathfrak{gl}_n)$ é livre como módulo à esquerda (ou à direita) sobre sua subálgebra de Gelfand-Tsetlin.

Um resultado análogo ao teorema de Kostant em [Kostant (1963)] foi provado por V. Futorny e S. Ovsienko em [Futorny e Ovsienko (2005)] para a clase de álgebras filtradas especiais, onde eles estabelecem o seguinte resultado chave:

Teorema 1.1. *Seja U uma álgebra filtrada especial. Se $g_1, g_2, \dots, g_t \in U$ são elementos mutuamente comutativos cujas imagens graduadas formam uma interseção completa para a álgebra graduada associada de U, então U é livre como $k[g_1, g_2, \dots, g_t]$-módulo à esquerda (ou à direita).*

Como consequência desse teorema tem-se que $U(\mathfrak{g})$ e $Y(\mathfrak{gl}_n)$ são livres sobre seus centros e que $U(\mathfrak{gl}_n)$ é também livre como módulo sobre sua subálgebra de Gelfand-Tsetlin Γ. Ressaltando que esse último fato é importante pois Γ é uma subálgebra comutativa maximal de $U(\mathfrak{gl}_n)$ a qual contém o centro de $U(\mathfrak{gl}_n)$. Usando este mesmo raciocínio, V. Futorny e S. Ovsienko conseguiram provar em [Futorny e Ovsienko (2005)] que os Yangians restritos $Y_p(\mathfrak{gl}_n)$ e a envolvente universal $U(\mathfrak{g}_m)$ das álgebras current $\mathfrak{g}_m$ são também livres sobre seus centros.

Portanto, não é estranho conjecturar que os Yangians restritos $Y_p(\mathfrak{gl}_n)$ para qualquer nível p, é livre sobre sua subálgebra de Gelfand-Tsetlin. V. Futorny, A. Molev e S. Ovsienko em [Futorny *et al.* (2005)] provaram que esta conjectura é verdadeira para $n = 2$ e p arbitrário.

As subálgebras de Gelfand-Tsetlin foram consideradas por A. S. Miščenko e A. T. Fomenko em [Miščenko e Fomenko (1978)] em conexões com a solução da equação de Euler. Já È. B. Vinberg em [Vinberg (1990)] usaram as subálgebras de Gelfand-Tsetlin em trabalhos relacionados com subálgebras de dimensão de Gelfand-Kirillov maximais da álgebra envolvente universal de uma álgebra de Lie simples. B. Kostant e N. Wallach em [Kostant e Wallach (2006a)] e [Kostant e Wallach (2006b)] usaram as subálgebras em conexão com mecânica clássica e M. I. Graev em trabalhos relacionados com funções hipergeométricas gerais sobre o grupo de Lie GL(n, $\mathbb{C}$).

O objetivo deste trabalho é estudar a variedade associada à subálgebra de Gelfand-Tsetlin para $\mathfrak{gl}_n$ e $Y_p(\mathfrak{gl}_n)$ e sua equidimensionalidade. Nossa motivação vem da teoria de representações:

1. Sejam A uma álgebra e $B \subset A$ uma subálgebra. Em teoria de representações, frequentemente se estuda a relação entre as categorias de módulos A-mod e B-mod. A pergunta mais interessante neste sentido é: *Quando um B-módulo pode ser levantado num A-módulo?* pois claramente, cada A-módulo é um B-módulo. Uma condição suficiente para que isso ocorra é que A seja um B-módulo livre.

2. Sejam A uma álgebra associativa e $\Gamma \subset A$ uma subálgebra comutativa. Em teoria de representações frequentemente se estudam módulos de Harish-Chandra com respeito a Γ, i.e., A-módulos V que admitem uma decomposição como soma direta de Γ-módulos irredutíveis
$$V = \bigoplus_{\lambda \in \Omega} V_\lambda$$
em que Ω parametriza as classes de isomorfismos dos Γ-módulos irredutíveis. Um dos primeiros problemas no estudo dos módulos de Harish-Chandra é determinar quando um λ dado pode ser levantado a um módulo de Harish-Chandra irredutível V com $V_\lambda \neq 0$. O caso mais interessante é quando existe tal levantamento para qualquer λ. Isto é verdade quando A é um Γ-módulo livre.

Neste sentido, no segundo capítulo deste trabalho são apresentados os conceitos básicos necessários para o desenvolvimento da tese, tais como, equidimensionalidade de variedades, interseção completa, sequências regulares e a relação entre os três conceitos. Essa relação permite explorar várias técnicas que serão usadas para obter os principais resultados desta tese.

No terceiro capítulo, começamos definindo a variedade de Gelfand-Tsetlin para $\mathfrak{gl}_n$, para depois apresentar o Teorema de Ovsienko. Provaremos a seguinte versão fraca do Teorema de Ovsienko (teorema (3.18)) com técnicas diferentes das apresentadas por S. Ovsienko em [Ovsienko (2003)]

Versão fraca do Teorema de Ovsienko:
A variedade algébrica
$$V_n := V(\sigma_{n2}, \sigma_{n3}, \sigma_{n4}, \ldots, \sigma_{nn}) \subset \mathbb{A}_k^{\frac{(n+2)(n-1)}{2}}$$
é equidimensional com dimensão
$$\dim(V_n) = \frac{n(n-1)}{2}$$
onde
$$\sigma_{nj} = \sum_{n > t_1 > t_2 > \cdots > t_{j-1} \geq 1} X_{nt_1} X_{t_1 t_2} \cdots X_{t_{j-2} t_{j-1}} X_{t_{j-1} n}; \quad j = 2, 3, \ldots, n.$$

B. Kostant e N. Wallach em [Kostant e Wallach (2006a)] e [Kostant e Wallach (2006b)] afirmam que todas as componentes regulares da variedade de Gelfand-Tsetlin são equidimensionais com dimensão $\frac{n(n-1)}{2}$. No terceiro capítulo, também mostraremos que esse fato é consequência da versão fraca e será apresentado no corolário (3.23) da seguinte forma:

Corolário da versão fraca do Teorema de Ovsienko:
Todas as componentes regulares da variedade de Gelfand-Tsetlin são isomorfas. Em particular, são isomorfas à componente irredutível
$$V_{\leq} := V\left(\left\{X_{ij} : 1 \leq i \leq j \leq n\right\}\right).$$

Existe uma função conhecida como *"aplicação de Kostant-Wallach"* [ver Kostant e Wallach (2006a)]

$$\varphi : M_n(\mathbb{C}) \longrightarrow \mathbb{C}^{\frac{n(n+1)}{2}},$$

cuja fibra no ponto zero $\varphi^{-1}(0)$ coincide com a variedade de Gelfand-Tsetlin para $\mathfrak{gl}_n$. Recentemente, M. Colarusso e S. Evens provaram em [Colarusso e Evens (2015)] que a fibra em zero $\varphi_2^{-1}(0)$ da função, chamada de aplicação parcial de Kostant-Wallach,

$$\varphi_2 : M_n(\mathbb{C}) \longrightarrow \mathbb{C}^{n-1} \times \mathbb{C}^n$$

é equidimensional e nesse caso com dimensão $n^2 - 2n + 1$. Usando técnicas diferentes, no capítulo quatro, apresentamos uma generalização deste resultado. De forma precisa, definimos uma função que chamaremos de aplicação k-parcial de Kostant-Wallach

$$\varphi_k : M_n(\mathbb{C}) \longrightarrow \mathbb{C}^{n-k+1} \times \cdots \times \mathbb{C}^{n-1} \times \mathbb{C}^n,$$

unificando as definições de aplicação de Kostant-Wallach e de aplicação parcial de Kostant-Wallach e provamos que a fibra em zero $\varphi_k^{-1}(0)$ é equidimensional com dimensão $n^2-(k+1)n+\frac{k(k+1)}{2}$. Além disso, provamos que se fixarmos $\alpha \in \mathbb{C}^{n-k+1} \times \cdots \times \mathbb{C}^{n-1} \times \mathbb{C}^n$, então a fibra $\varphi_k^{-1}(\alpha)$ é também equidimensional com dimensão $n^2-(k+1)n+\frac{k(k+1)}{2}$. Tudo isso é consequência do seguinte teorema (teorema (4.15)), o qual é o resultado principal do capítulo quatro:

Teorema:
Para todo $k = 1, 2, \ldots, n$ e todo $\beta \in \mathbb{C}^{n-k+1} \times \cdots \times \mathbb{C}^{n-1} \times \mathbb{C}^n$ a variedade

$$\widetilde{V}_\beta^k = V\left(\left\{\overline{\gamma_{ij}} - \beta_{ij} : \ n-k+1 \leq i \leq n \ \text{ e } \ 1 \leq j \leq i\right\}\right) \subset \mathbb{C}^{n^2}$$

é equidimensional com dimensão

$$\dim\left(\widetilde{V}_\beta^k\right) = n^2 - nk + \frac{k(k-1)}{2},$$

onde os $\overline{\gamma_{ij}}$'s são os polinômios que determinam a variedade de Gelfand-Tsetlin para $\mathfrak{gl}_n$.

No quinto e último capítulo deste trabalho, apresentamos os Yangians restritos $Y_p(\mathfrak{gl}_n)$ de nível p para $\mathfrak{gl}_n$ e sua variedade de Gelfand-Tsetlin $\mathfrak{G}$. Provamos a equidimensionalidade da variedade de Gelfand-Tsetlin $\mathfrak{G}$ para $Y_2(\mathfrak{gl}_3)$ na proposição (5.12), apresentando sua decomposição em componentes irredutíveis. Também apresentamos na proposição (5.13), uma decomposição (não em componentes irredutíveis) da variedade de Gelfand-Tsetlin para $Y_p(\mathfrak{gl}_n)$ da forma

$$\mathfrak{G} = \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \cdots \cup \mathfrak{G}_{p+1}.$$

No corolário (5.19), mostramos uma decomposição da subvariedade $\mathfrak{G}_1$ de $\mathfrak{G}$ para $Y_p(\mathfrak{gl}_n)$ na forma

$$\mathfrak{G}_1 = \mathbb{W}_1 \cup \mathbb{W}_2, \ \text{ se } \ p = 1, 2$$

$$\mathfrak{G}_1 = \mathbb{W}^p \cup \mathbb{W}_1 \cup \mathbb{W}_2, \ \text{ se } \ p \geq 3.$$

Quando $n = 3$, a variedade $\mathfrak{G}_1$ será chamada de *versão fraca da variedade de Gelfand-Tsetlin* $\mathfrak{G}$ para $Y_p(\mathfrak{gl}_3)$. Encerramos o quinto capítulo provando a equidimensionalidade de $\mathbb{W}^p$, $\mathbb{W}_1$ e $\mathbb{W}_2$ e obtendo, portanto, o resultado principal (teorema (5.25)) desse capítulo

Teorema:
Para todo p, a variedade $\mathfrak{G}_1$ para $Y_p(\mathfrak{gl}_3)$ é equidimensional com dimensão

$$\dim \mathfrak{G}_1 = 3p.$$

Por fim, incluímos dois apêndices: O apêndice (A) contém alguns fatos que foram necessários para o capítulo 5, e no apêndice (B), estão todos os detalhes da decomposição em componentes irredutíveis da variedade de Gelfand-Tsetlin para $Y_2(\mathfrak{gl}_3)$.

Capítulo 2

Conceitos Preliminares

Neste capítulo introduziremos conceitos preliminares e algumas propiedades que usaremos ao longo deste trabalho. Começaremos com alguns conceitos para poder definir e dar exemplos de álgebras de Lie, seguindo com a definição de álgebra envolvente universal e o Teorema de Poincaré-Birkhoff-Witt. Seguidamente, apresentaremos algumas definições e fatos da Geometria Algébrica, tais como variedade algébrica, dimensão e equidimensionalidade. Continuaremos com conceitos de Álgebra Homológica, introduzindo o complexo de Koszul, sua homologia e o conceito de interseção completa. Finalmente, encerraremos este capítulo com definições e fatos da Álgebra Comutativa, tais como sequências regulares e a relação com equidimensionalidade e interseção completa.

Durante toda a tese, fixaremos k como um corpo algébricamente fechado de caracterítica zero. Alguns dos resultados mencionados neste trabalho valem em um contexto mais geral, mas por simplicidade vamos nos restringir a este caso.

Definição 2.1. *Sejam V e W dois k-espaços vetoriais.*

1. *Uma **aplicação n-linear** (ou **aplicação multilinear**) é uma aplicação*

$$B : \underbrace{V \times V \times \cdots \times V}_{n-vezes} \longrightarrow W,$$

a qual é k-linear em cada coordenada.

2. *Dizemos que a forma multilinear B é **anti-simétrica** se*

$$B(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = -B(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n).$$

3. *Uma **forma bilinear sobre** V é, simplesmente uma forma 2-linear $B : V \times V \longrightarrow k$.*

4. *Dizemos que uma forma bilinear B sobre V é **não-degenerada**, se para todo $u \in V$ com $u \neq 0$, existe $v \in V$ tal que $B(u, v) \neq 0$.*

Observação 2.2. *Se $\dim_k(V) = n$, podemos identificar V com k^n como k-espaço vetorial. Seja $A \in M_n(k)$ uma matriz $n \times n$. Claramente a aplicação*

$$\begin{array}{rccl} B: & V \times V & \longrightarrow & k \\ & (u, v) & \longmapsto & B(u, v) := u^T A v \end{array}$$

é uma forma bilinear. De fato, existe uma correspondência 1-1 entre as formas bilineares sobre V e as matrizes quadradas $n \times n$ (a menos de semelhança de matrizes), da seguinte forma:

Se B é uma forma bilinear sobre V e $\{e_1, e_2, \ldots, e_n\}$ é uma base de V, então a matriz

$$A = \begin{pmatrix} B(e_1,e_1) & B(e_1,e_2) & \cdots & B(e_1,e_n) \\ B(e_2,e_1) & B(e_2,e_2) & \cdots & B(e_2,e_n) \\ \vdots & \vdots & \ddots & \vdots \\ B(e_n,e_1) & B(e_n,e_2) & \cdots & B(e_n,e_n) \end{pmatrix},$$

*chamada de **matriz de B relativa à base** $\{e_1, e_2, \ldots, e_n\}$ satisfaz $B(u,v) = u^T A v$, $\forall u, v \in V$. E caso $\{f_1, f_2, \ldots, f_n\}$ seja outra base de V com matriz relativa C à forma bilinear B, então*

$$C = PAP^T,$$

em que P é a matriz mudança de base.

Definição 2.3. *Uma álgebra A é uma **álgebra graduada**, se existe uma coleção de subespaços vetoriais A_n (chamada de **graduação** de A), tais que $A_n \cdot A_m \subseteq A_{n+m}$ e*

$$A = \bigoplus_{n=0}^{\infty} A_n.$$

Definição 2.4. *Sejam A e B duas álgebras graduadas com graduações $\{A_n\}_{n=0}^{\infty}$ e $\{B_n\}_{n=0}^{\infty}$ respectivamente. Um **homomorfismo de álgebras graduadas** $f : A \longrightarrow B$ é um homomorfismo de álgebras tal que*

$$f(A_n) \subseteq B_n.$$

Exemplo 2.5.

A álgebra de polinômios em n variáveis $k[t_1, t_2, \ldots, t_n]$ é uma álgebra graduada, em que A_m são os polinômios de grau m.

Definição 2.6. *Uma álgebra A é uma **álgebra filtrada** se existe uma coleção de subespaços vetoriais $\{A_n\}_{n=0}^{\infty}$ (chamada de **filtração** de A), tais que*

a. $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$.

b. $A_n \cdot A_m \subseteq A_{n+m}$.

c. $A = \bigcup_{n=0}^{\infty} A_n$.

Observação 2.7. *Se A é uma álgebra associativa filtrada cuja filtração é $\{A_n\}_{n=0}^{\infty}$*

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots,$$

então:

1. *Observemos que esta filtração induz uma álgebra graduada, que denotaremos por* $\mathrm{gr}\,(A)$ *ou as vezes por simplicidade* $\overline{A}$

$$\mathrm{gr}\,(A) = \bigoplus_{n=0}^{\infty} \overline{A}_n,$$

onde

$$\overline{A}_n := A_n / A_{n-1}$$

e

$$A_{-1} = \{0\}.$$

*Esta graduação é chamada de **graduação associada à filtração de** A ou quando a filtração já está estabelecida é simplesmente chamada de **graduação associada de** A.*

2. *Se B é uma subálgebra de A, então B também é uma álgebra filtrada, cuja filtração induzida é $\{B_n = A_n \cap B\}_{n=0}^{\infty}$*
$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots,$$
portanto, do item anterior temos a álgebra graduada $\mathrm{gr}\,(B)$ *e a inclusão $i : B \hookrightarrow A$, também induz um morfismo de álgebras graduadas*
$$\begin{array}{rccc} \mathrm{gr}\,(i): & \mathrm{gr}\,(B) & \longrightarrow & \mathrm{gr}\,(A) \\ & v_n + B_{n-1} & \longmapsto & \mathrm{gr}\,(i)(v_n + B_{n-1}) := v_n + A_{n-1} \end{array},$$
o qual é injetor, pois se $v_n + B_{n-1}, w_n + B_{n-1} \in B_n/B_{n-1}$ são tais que
$$\mathrm{gr}\,(i)(v_n + B_{n-1}) = \mathrm{gr}\,(i)(w_n + B_{n-1}),$$
então $v_n + A_{n-1} = w_n + A_{n-1}$. Isto implica que $v_n - w_n \in A_{n-1}$, assim $v_n - w_n \in A_{n-1} \cap B$ e, como $A_{n-1} \cap B = B_{n-1}$, concluimos que
$$v_n + B_{n-1} = w_n + B_{n-1}.$$
Portanto, podemos identificar $\mathrm{gr}\,(B)$ *como a subálgebra de* $\mathrm{gr}\,(A)$ *cujos monômios v_n estão em B_n.*

2.1 Álgebras de Lie

Agora definiremos os conceitos de álgebras de Lie, subálgebras de Lie e daremos alguns exemplos que serão importantes ao longo do trabalho.

Definição 2.8. *Uma **álgebra de Lie** é um espaço vetorial $\mathfrak{g}$ munido de um produto*
$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g},$$
*chamado de **colchete de Lie** com as seguintes propriedades:*

a. *É bilinear.*

b. *É Antisimétrico (equivalentemente, $[x, x] = 0$ para todo $x \in \mathfrak{g}$).*

c. *Satisfaz a **Identidade de Jacobi**, i.e., para todo $x, y, z \in \mathfrak{g}$*
$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

Definição 2.9. *Uma álgebra de Lie $\mathfrak{g}$ é dita **abeliana** se*
$$[x, y] = 0,\ \forall x, y \in \mathfrak{g}.$$

Exemplo 2.10.

a. Para uma álgebra associativa A, denotemos por A^{-} a álgebra de Lie associada a A com colchete de Lie dado pelo comutador, i.e.,
$$[x, y] := xy - yx, \forall x, y \in A.$$

b. Sejam V um espaço vetorial e $End(V)$ o conjunto de todas as transformações lineares $V \longrightarrow V$ com estrutura de álgebra associativa. Denotemos a álgebra $End(V)$ vista como álgebra de Lie por $\mathfrak{gl}(V)$, i.e.,
$$\mathfrak{gl}(V) := End(V)^{-}$$
a qual é chamada de **Álgebra Linear Geral**.

Observemos que se $\dim_k(V) = n$, então podemos identificar $End(V)$ com o espaço vetorial $M_n(k)$ das matrices quadradas $n \times n$ e

$$\mathfrak{gl}_n := \mathfrak{gl}(n,k) := \mathfrak{gl}_n(k) := \mathfrak{gl}(V) = M_n(k)^-,$$

conhecida como **Álgebra de Lie linear**.

Definição 2.11. *Uma* ***subálgebra*** *de uma álgebra de Lie $\mathfrak{g}$ é um subespaço vetorial $\mathfrak{h}$ de $\mathfrak{g}$ tal que*

$$[x,y] \in \mathfrak{h}, \ \forall x,y \in \mathfrak{h}.$$

Exemplo 2.12.

a. **Álgebra Linear Especial**

$$\mathfrak{sl}(n,k) := \left\{X \in \mathfrak{gl}_n(k) \ : \ tr(X) = 0\right\},$$

onde $tr(X)$ denota o traço da matriz X. Também é denotada por $\mathfrak{sl}(n,k) := \mathfrak{sl}_n(k) := \mathfrak{sl}_n := \mathfrak{sl}(n)$.

b. **Álgebra Simplética**: Sejam V um espaço vetorial com $\dim_k(V) = 2n$ e $\{v_1, v_2, \ldots, v_{2n}\}$ uma base. Definamos como na observação (2.2) a forma bilinear B, anti-simétrica e não-degenerada sobre V dada pela matriz

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

onde I_n é a matriz identidade $n \times n$. Portanto a álgebra simplética é definida como

$$\mathfrak{sp}(V) := \{T \in End(V) \ : \ B(T(u),v) = -B(u,T(v))\}.$$

Em termos de matrizes temos que

$$\mathfrak{sp}(V) = \left\{X \in \mathfrak{gl}_{2n}(k) \ : \ XJ + JX^T = 0\right\}.$$

É comum ver na literátura as notações $\mathfrak{sp}(V) := \mathfrak{sp}(2n,k) := \mathfrak{sp}_{2n}(k) := \mathfrak{sp}_{2n} := \mathfrak{sp}(2n)$.

Definição 2.13. *Sejam $(\mathfrak{g}, [\ ,\ ]_1)$ e $(\mathfrak{h}, [\ ,\ ]_2)$ duas álgebras de Lie. Dizemos que uma transformação linear $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ é um* ***homomorfismo de álgebras de Lie*** *se satisfaz*

$$f([x,y]_1) = [f(x), f(y)]_2, \ \forall x,y \in \mathfrak{g}.$$

2.2 Álgebra Envolvente Universal

Nesta seção introduziremos a álgebra tensorial de um módulo, a qual permitirá definir as álgebras simétrica e exterior de um módulo e a álgebra envolvente universal de uma álgebra de Lie. Encerraremos esta seção com o necessário para enunciar o Teorema de Poincaré-Birkhoff-Witt. Para mais detalhes sobre a álgebra tensorial, simétrica e exterior ver [Bourbaki (1989)] nas páginas 484—522 e sobre a álgebra envolvente universal e o Teorema de Poincaré-Birkhoff-Witt ver [Humphreys (1978)] nas páginas 90 − 94.

2.2.1 Álgebras Tensorial, Simétrica e Exterior

Sejam R um anel comutativo, M um R-módulo e consideremos os seguintes R-módulos:

$$\begin{aligned} T^0M &= R, \\ T^1M &= M, \\ T^2M &= M \otimes_R M, \end{aligned}$$

$$\vdots$$
$$T^n M = \underbrace{M \otimes_R M \otimes_R \cdots \otimes_R M}_{n\ vezes},$$

e o R-módulo

$$T(M) = \bigoplus_{n=0}^{\infty} T^n M,$$

no qual podemos introduzir um produto associativo, definido sobre os geradores homogêneos de $T(M)$, da seguinte forma

$$\underbrace{(v_1 \otimes v_2 \otimes \cdots \otimes v_n)}_{\in T^n M}\underbrace{(w_1 \otimes w_2 \otimes \cdots \otimes w_m)}_{\in T^m M} = \underbrace{v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes w_1 \otimes w_2 \otimes \cdots \otimes w_m}_{\in T^{n+m} M},$$

pois todo elemento de $T(M)$ é uma soma finita $\sum z_n$, com $z_n \in T^n M$ e como z_n é uma combinação linear de monômios, podemos estender por R-bilinearidade o produto a todo o R-módulo $T(M)$. Desta forma, $T(M)$ é uma álgebra associativa graduada, pois

$$T^n M \cdot T^m M = T^{n+m} M, \ \forall n, m.$$

Esta álgebra é chamada de **álgebra tensorial de** M.

Proposição 2.14 (**Propriedade universal da álgebra tensorial**). *Sejam M um R-módulo e A uma R-álgebra associativa com unidade* 1. *Para qualquer homomorfismo de R-módulos $\phi : M \longrightarrow A$, existe um único homomorfismo de R-álgebras $\Phi : T(M) \longrightarrow A$ tal que, $\Phi(1) = 1$ e $\Phi \circ i = \phi$, onde $i : M \longrightarrow T(M)$ é a inclusão, i.e. o seguinte diagrama comuta*

$$\begin{tikzcd} M \arrow[rr, hook, "i"] \arrow[dr, "\phi"'] & & T(M) \arrow[dl, dashed, "\Phi"] \\ & A & \end{tikzcd}$$

Demonstração: Basta considerar

$$\Phi(v_1 \otimes v_2 \otimes \cdots \otimes v_n) := \phi(v_1)\phi(v_2)\ldots\phi(v_n).$$

□

Proposição 2.15. *Sejam M um R-módulo livre e $\{e_1, e_2, \ldots, e_n\}$ uma base de M, então:*

1. *Para qualquer $m \geq 1$, $T^m M$ é um R-módulo livre com posto n^m e os tensores da forma*

$$e_{i_1} \otimes e_{i_2} \otimes \cdots e_{i_m}$$

formam uma base para $T^m M$.

2. *A álgebra tensorial $T(M)$ é isomorfa à álgebra de polinômios $R\langle x_1, x_2, \ldots, x_n \rangle$ nas variáveis não comutativas $x_1, x_2, \ldots, x_n$ com coeficientes em R.*

Definição 2.16. *Sejam M um R-módulo, $T(M)$ sua álgebra tensorial e $C(M)$ o ideal bilateral de $T(M)$ gerado por todos os elementos da forma*

$$x \otimes y - y \otimes x, \ \forall x, y \in M.$$

A álgebra quociente

$$S(M) := T(M)/C(M)$$

*é dita **álgebra simétrica de** M. Denotemos por σ a projeção canônica de R-álgebras*

$$\begin{array}{rccc} \sigma: & T(M) & \longrightarrow & S(M) \\ & x & \longmapsto & \sigma(x) := \overline{x}. \end{array}$$

Observação 2.17.

1. *Para cada $n \geq 1$, seja $S^n M := \sigma(T^n M)$ a imagem de $T^n M$ em $S(M)$. Portanto, $S(M)$ é uma álgebra graduada com graduação*
$$S(M) = \bigoplus_{n=o}^{\infty} S^n M.$$

2. *$S(M)$ é uma R-álgebra comutativa.*

Proposição 2.18. *Sejam M um R-módulo livre e $\{e_1, e_2, \ldots, e_n\}$ uma base de M, então $S(M)$ é isomorfa a $R[x_1, x_2, \ldots, x_n]$ (álgebra de polinômios com coeficientes em R) como uma R-álgebra graduada.*

Definição 2.19. *Sejam M um R-módulo, $T(M)$ sua álgebra tensorial e $A(M)$ o ideal bilateral de $T(M)$ gerado por todos os elementos da forma*

$$x \otimes x, \ \forall x \in M.$$

A álgebra quociente
$$\bigwedge M := T(M)/A(M),$$
*é chamada de **álgebra exterior de** M.*

O produto em $\bigwedge M$ é denotado pelo símbolo $\wedge$, i.e., para $m_1, m_2, \ldots, m_n \in M$ denotamos por

$$m_1 \wedge m_2 \wedge \cdots \wedge m_n,$$

a classe de
$$m_1 \otimes m_2 \otimes \cdots \otimes m_n$$
em $\bigwedge(M)$.

Similar à álgebra simétrica, a álgebra exterior de M tem uma graduação natural

$$\bigwedge M = \bigoplus_{n=o}^{\infty} \bigwedge\nolimits^n M,$$

em que $\bigwedge^n M$ é o quociente de $T^n M$ em $\bigwedge M$, isto é,

$$\bigwedge\nolimits^n M = T^n M / A_n(M),$$

com
$$A_n(M) := \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \in T^n M \ : \ \text{existem } i \neq j \text{ tais que } v_i = v_j \right\rangle \subset A(M).$$

*$\bigwedge^n M$ é chamada de n**-ésima potência exterior** de M.*

Proposição 2.20 (Propriedade universal da n-ésima potência exterior de M).
Seja $n \in \mathbb{N}$. Se existir um R-módulo N e uma aplicação n-linear alternada $f : M^n \longrightarrow N$ (isto é, $f(x_1, x_2, \ldots, x, \ldots, x, \ldots, x_n) = 0$), então existe um único homomorfismo de R-módulos $g : \bigwedge^n M \longrightarrow N$, tal que $g \circ \wedge_n = f$, ou seja, o seguinte diagrama comuta

$$\begin{tikzcd} M^n \arrow[rr, "\wedge_n"] \arrow[dr, "f"'] & & \bigwedge^n M \arrow[dl, dashed, "\exists ! g"] \\ & N & \end{tikzcd}$$

em que, $\wedge_n$ *é a aplicação* n*-linear e anti-simétrica*

$$\begin{array}{rccl} \wedge_n : & M^n & \longrightarrow & \bigwedge^n M \\ & (m_1, m_2, \ldots, m_n) & \longmapsto & \wedge_n(m_1, m_2, \ldots, m_n) := m_1 \wedge m_2 \wedge \ldots \wedge m_n. \end{array}$$

Proposição 2.21. *Seja* M *um* R*-módulo livre, de posto* n *e com base* $\{e_1, e_2, \ldots, e_n\}$*, então:*

1. $\bigwedge M$ *é isomorfa a* $R\langle x_1, x_2, \ldots, x_n\rangle / I$ *como* R*-álgebra, em que* I *é o ideal bilateral gerado pelo conjunto*
$$\{x_i^2, x_i x_j + x_j x_i \ : \ i, j = 1, 2, \ldots, n\}.$$
Portanto,
$$\bigwedge M \cong \left\langle e_1, e_2, \ldots, e_n \ : \ e_i \wedge e_i = 0, \ e_i \wedge e_j = -e_j \wedge e_i \right\rangle.$$

2. *Para cada* $k \geq 1$, $\bigwedge^k M$ *é um* R*-módulo livre com base*
$$\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} \ : \ 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$
Em particular, seu posto é $\binom{n}{k}$.

3. $\bigwedge M$ *é um* R*-módulo livre finitamente gerado de posto* 2^n.

2.2.2 Álgebra Envolvente Universal

Nesta seção seguiremos [Humphreys (1978)], páginas 90−94.

Definição 2.22. *Seja* $\mathfrak{g}$ *uma álgebra de Lie sobre* k*. Uma* ***álgebra envolvente universal de*** $\mathfrak{g}$ *é um par* (U, i)*, em que* U *é uma* k*-álgebra associativa com* 1 *e* $i : \mathfrak{g} \longrightarrow U^-$ *um homomorfismo de álgebras de Lie, tais que* (U, i) *satisfaz a seguinte propriedade universal:*

Para qualquer k*-álgebra associativa* A *com* 1 *e qualquer homomorfismo de álgebras de Lie*
$$\phi : \mathfrak{g} \longrightarrow A^-$$
i.e., $\phi([x, y]) = \phi(x)\phi(y) - \phi(y)\phi(x)$ *para todo* $x, y \in \mathfrak{g}$*, existe um único* $\Psi : U \longrightarrow A$ *homomorfismo de álgebras associativas, tal que*
$$\Psi(1_U) = 1_A \ \text{ e } \ \Psi \circ i = \phi,$$
i.e. o seguinte diagrama comuta

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, hook, "i"] \arrow[dr, "\phi"'] & & U \arrow[dl, dashed, "\exists! \Psi"] \\ & A & \end{tikzcd}$$

Proposição 2.23 (Unicidade).
Sejam (U_1, i_1) *e* (U_2, i_2) *duas álgebras envolventes universais de uma álgebra de Lie* $\mathfrak{g}$*. Então, existe um único isomorfismo* $j : U_1 \longrightarrow U_2$ *de álgebras associativas tal que* $i_2 = j i_1$.

Demonstração: Segue-se da propriedade universal da álgebra envolvente universal.

□

Proposição 2.24 (Existência).
Seja $\mathfrak{g}$ *uma* k*-álgebra de Lie, então existe uma única (a menos de isomorfismo) álgebra envolvente universal de* $\mathfrak{g}$.

Demonstração: Consideremos a álgebra tensorial $T(\mathfrak{g})$ de $\mathfrak{g}$ e o ideal bilateral J de $T(\mathfrak{g})$ gerado por todos os elementos da forma
$$x \otimes y - y \otimes x - [x, y], \ \forall x, y \in \mathfrak{g}.$$

Agora, definamos a álgebra associativa $U(\mathfrak{g}) := T(\mathfrak{g})/J$ e a aplicação

$$\begin{array}{rccl} i: & \mathfrak{g} & \longrightarrow & U(\mathfrak{g})^- \\ & x & \longmapsto & i(x) := x + J, \end{array}$$

a qual claramente é k-linear, além disso é um homomorfismo de álgebras de Lie, pois

$$i([x,y]) = [x,y] + J = x \otimes y - y \otimes x + J, \ \forall x, y \in \mathfrak{g}.$$

Para provar que vale a propriedade universal, consideremos uma k-álgebra associativa A com 1 e um homorfismo de álgebras de Lie $\phi : \mathfrak{g} \longrightarrow A^-$. Pela propriedade universal da álgebra tensorial, existe um único homomorfismo de k-álgebras $\Phi : T(\mathfrak{g}) \longrightarrow A$, tal que $\Phi \circ j = \phi$, onde $j : \mathfrak{g} \longrightarrow T(\mathfrak{g})$ é a inclusão.

Observemos que $J \subset \ker \Phi$, pois como Φ é homomorfismo de álgebras

$$\begin{aligned} \Phi(x \otimes y - y \otimes x - [x,y]) &= \Phi(x)\Phi(y) - \Phi(y)\Phi(x) - \Phi([x,y]) \\ &= \phi(x)\phi(y) - \phi(y)\phi(x) - \phi([x,y]) \\ &= \phi(x \otimes y - y \otimes x - [x,y]) \end{aligned}$$

e como ϕ é um homomorfismo de álgebras de Lie temos

$$\Phi(x \otimes y - y \otimes x - [x,y]) = 0.$$

Agora, do fato que $J \subset \ker \Phi$, segue que existe um homomorfismo de k-álgebras $\Psi : U(\mathfrak{g}) \longrightarrow A$, tal que $\Psi(x + J) = \Phi(x), \forall x \in U(\mathfrak{g})$. Além disso, temos $\Psi(1) = 1$ e $\Psi i(x) = \phi(x), \forall x \in \mathfrak{g}$.

Para provar a unicidade (na propriedade universal da álgebra envolvente universal), consideremos outro homomorfismo de k-álgebras $\Psi' : U(\mathfrak{g}) \longrightarrow A$, tal que $\Psi'(1) = 1$ e $\Psi' i(x) = \phi(x), \forall x \in \mathfrak{g}$. Seja $w_1 \otimes w_2 \otimes \cdots \otimes w_n + J \in U(\mathfrak{g})$ arbitrário, logo

$$\begin{aligned} \Psi'(w_1 \otimes w_2 \otimes \cdots \otimes w_n + J) &= \Psi'((w_1 + J)(w_2 + J) \cdots (w_n + J)) \\ &= \Psi'(w_1 + J)\Psi'(w_2 + J) \cdots \Psi'(w_n + J) \\ &= \phi(w_1)\phi(w_2) \cdots \phi(w_n) \\ &= \Phi(w_1)\Phi(w_2) \cdots \Phi(w_n) \\ &= \Psi(w_1 \otimes w_2 \otimes \cdots \otimes w_n + J). \end{aligned}$$

□

Exemplo 2.25.

Seja $\mathfrak{g}$ uma álgebra de Lie abeliana, então J seria o ideal bilateral formado por todos os elementos da forma

$$x \otimes y - y \otimes x, \ \forall x, y \in \mathfrak{g}.$$

Portanto $U(\mathfrak{g}) \cong S(\mathfrak{g})$ como k-álgebras.

2.2.3 Teorema de Poincaré-Birkhoff-Witt

Em [Grivel (2004)] e [Shepler e Witherspoon (2014)], podem ser encontrados detalhes sobre a historia do Teorema de Poincaré-Birkhoff-Witt.

Notação 2.26. *Para facilitar a leitura, usaremos as seguintes notações:*

$$\begin{aligned} T &= T(\mathfrak{g}) \\ S &= S(\mathfrak{g}) \\ U &= U(\mathfrak{g}) \\ T^n &= T^n\mathfrak{g} \end{aligned}$$

$$S^n = S^n\mathfrak{g}$$

Consideremos uma filtração sobre T e sobre U dada por:

$$\begin{aligned} T_m :=& T^0 \oplus T^1 \oplus T^2 \oplus \cdots \oplus T^m \\ U_m :=& \pi(T_m) \\ U_{-1} :=& \{0\} \end{aligned}$$

onde $\pi : T(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ é a projeção canônica. Logo

$$\begin{aligned} U_m U_p \subset& U_{m+p} \\ U_m \subset& U_{m+1}. \end{aligned}$$

Também consideremos o k-espaço vetorial

$$G^m := U_m / U_{m-1}.$$

A multiplicação em U define uma aplicação k-bilinear

$$\begin{array}{ccc} G^m \times G^p & \longrightarrow & G^{m+p} \\ (\overline{u}, \overline{v}) & \longmapsto & \overline{uv}. \end{array}$$

Claramente, isso pode se estender para uma aplicação bilinear

$$\mathrm{gr}\,(U) \times \mathrm{gr}\,(U) \longrightarrow \mathrm{gr}\,(U),$$

em que

$$\mathrm{gr}\,(U) = \bigoplus_{n=0}^{\infty} G^m,$$

a qual é a álgebra graduada associativa com 1 associada à álgebra $U = U(\mathfrak{g})$.

Agora reparemos que a restrição de π a T^m é uma transformação k-linear

$$\pi|_{T^m} : T^m \longrightarrow \pi(T^m) \subset \pi(T_m) = U_m,$$

logo

$$\begin{array}{cccc} \phi_m : & T^m & \longrightarrow & G^m \\ & w & \longmapsto & \phi_m(w) := \pi(w) + U_{m-1} \end{array}$$

é uma transformação k-linear sobrejetora. Portanto, estendendo por k-linearidade, temos o homomorfismo de k-álgebras $\phi : T \longrightarrow \mathrm{gr}\,(U)$ sobrejetor, com $\phi(1) = 1$ e

$$\phi(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = \phi_m(v_1 \otimes v_2 \otimes \cdots \otimes v_m),$$

i.e., $\phi|_{T^m} = \phi_m$ e portanto ϕ é sobrejetora.

Lema 2.27. *$\phi : T \longrightarrow \mathrm{gr}\,(U)$ é um homomorfismo de álgebras. Além disso, $\phi(C(\mathfrak{g})) = 0$ e assim ϕ induz um homomorfismo de álgebras $\omega : S = T/C(\mathfrak{g}) \longrightarrow \mathrm{gr}\,(U)$.*

Demonstração: Basta provar que $\phi(x \otimes y - y \otimes x) = 0$, $\forall x, y \in \mathfrak{g}$. De fato, note que $x \otimes y - y \otimes x \in T^2 \subset T_2$, logo $\pi(x \otimes y - y \otimes x) \in U_2$, mas

$$\big(x \otimes y - y \otimes x\big) + J = [x, y] + J \quad \text{em } U,$$

logo em U

$$\pi\big(x \otimes y - y \otimes x\big) = \pi([x, y]) \in U_1,$$

assim
$$\phi\left(x\otimes y-y\otimes x\right)=\phi_2\left(x\otimes y-y\otimes x\right)=\pi\left(x\otimes y-y\otimes x\right)+U_1=\pi([x,y])+U_1=0.$$
Portanto, temos o homomorfismo de k-álgebras sobrejetor

$$\begin{array}{rccl}\omega: & S & \longrightarrow & \mathrm{gr}\,(U)\\ & v+C(\mathfrak{g}) & \longmapsto & \omega(v+C(\mathfrak{g})):=\phi(v).\end{array}$$

Além disso, notemos que

$$\begin{array}{rccl}\omega|_{S^m}: & S^m & \longrightarrow & G^m\\ & v+C(\mathfrak{g}) & \longmapsto & \omega(v+C(\mathfrak{g})):=\phi(v)=\phi_m(v)=\pi(v)+U_{m-1},\end{array}$$

também é sobrejetor e k-linear.

□

O seguinte teorema é o resultado principal desta seção sobre $U(\mathfrak{g})$, conhecido como ***Teorema de Poincaré-Birkhoff-Witt*** (ou ***Teorema PBW***).

Teorema 2.28 (**Teorema de Poincaré-Birkhoff-Witt (PBW)**).
O homomorfismo $\omega: S=T/C(\mathfrak{g})\longrightarrow \mathrm{gr}\,(U)$ é um isomorfismo de álgebras.

2.3 Geometria Algébrica

Como dito anteriormente, durante toda a tese fixaremos k como um corpo algébricamente fechado com característica zero, por exemplo, o corpo dos números complexos $k=\mathbb{C}$.

Nesta seção apresentaremos os conceitos e propriedades de variedades algébricas, ideal associado a uma variedade, componentes irredutíveis, dimensão de uma variedade e de um anel. Para esta seção, todos os anéis e as álgebras são comutativos.

Para começar fixemos o espaço onde queremos trabalhar.

Definição 2.29. *Definamos por $\mathbb{A}^n_k=k^n$ (ou simplesmente $\mathbb{A}^n$ ou k^n) o **espaço afim**. Aquí esqueceremos a estrutura de k^n como espaço vetorial.*

Definição 2.30. *Seja $S\subset k[x_1,x_2,\dots,x_n]$. Definamos o conjunto de zeros de S como*

$$V(S):=\left\{(a_1,a_2,\dots,a_n)\in\mathbb{A}^n\ :\ f(a_1,a_2,\dots,a_n)=0,\ \forall f\in S\right\}.$$

*Este conjunto é chamado de **variedade algébrica afim associada ao conjunto** S (ou simplesmente **variedade algébrica**).*

*Dizemos que $X\subset\mathbb{A}^n$, é um **conjunto algébrico** (ou **variedade algébrica**) se $X=V(S)$ para algum $S\subset k[x_1,x_2,\dots,x_n]$.*

Observação 2.31. *Sejam $S\subset k[x_1,x_2,\dots,x_n]$ e $I=\langle S\rangle$ o ideal de $k[x_1,x_2,\dots,x_n]$ gerado por S. Não é difícil ver que*
$$V(I)=V(S).$$

Teorema 2.32 (**Teorema da base de Hilbert**).
Sejam k um corpo e I um ideal de $k[x_1,x_2,\dots,x_n]$, então I é finitamente gerado, ou seja, existe $m\geq 0$ e $f_1,f_2,\dots,f_m\in I$ tais que
$$I=\left\langle f_1,f_2,\dots,f_m\right\rangle.$$

Demonstração: Ver [Atiyah e Macdonald (1969)] na página 81.

□

Observação 2.33. *Seja $S\subset k[x_1,x_2,\dots,x_n]$, da observação anterior e o Teorema da base de Hilbert tem-se*

$$V(S)=V(I)=V\left(\left\langle f_1,f_2,\dots,f_m\right\rangle\right)=V\left(f_1,f_2,\dots,f_m\right)$$

para alguns $f_1,f_2,\dots,f_m\in I=\langle S\rangle$.

Definição 2.34. *Seja I um ideal de um anel R. Definimos o **radical de** I como*

$$\sqrt{I} := \left\{x \in R \,:\, x^n \in I,\ \text{para algum } n \in \mathbb{N}\right\}.$$

*Dizemos, que I é um **ideal radical** se $I = \sqrt{I}$.*

Teorema 2.35 (Teorema dos zeros de Hilbert).
Sejam k um corpo algébricamente fechado e I um ideal de $k[x_1, x_2, \ldots, x_n]$, então

$$\sqrt{I} := \left\{f \in k[x_1, x_2, \ldots, x_n] \,:\, f(x) = 0,\ \forall x \in V(I)\right\}.$$

Demonstração: Ver o teorema 25 de [Matsumura (1970)] nas páginas $93-94$ ou o teorema 14 de [Zariski e Samuel (1975)] na página 164.

□

Definição 2.36. *Seja $X \subset \mathbb{A}^n_k$, definimos o **ideal associado** a X como*

$$I(X) := \left\{f \in k[x_1, x_2, \ldots, x_n] \,:\, f(x) = 0,\ \forall x \in X\right\}.$$

Observação 2.37.

1. *$I(X)$ é um ideal radical de $k[x_1, x_2, \ldots, x_n]$.*
2. *Temos as correspondências biunívocas*

$$\begin{array}{rccl} V: & \{\text{Ideais radicais de } k[x_1, x_2, \ldots, x_n]\} & \longrightarrow & \{\text{Variedades Algébricas em } \mathbb{A}^n\} \\ & I = \langle f_1, f_2, \ldots, f_n \rangle & \longmapsto & V(I), \end{array}$$

$$\begin{array}{rccl} I: & \{\text{Variedades Algébricas em } \mathbb{A}^n\} & \longrightarrow & \{\text{Ideais radicais de } k[x_1, x_2, \ldots, x_n]\} \\ & X & \longmapsto & I(X). \end{array}$$

Uma é a inversa da outra e tais correspondências invertem a inclusão.

Ver corolários 1 e 2 de [Zariski e Samuel (1975)] na página 167.

Definição 2.38. *Uma variedade algébrica X de $\mathbb{A}^n_k$ é dita **irredutível** se não existem X_1, X_2 variedades algébricas não vazias tais que*

$$X = X_1 \cup X_2.$$

Proposição 2.39. *Seja X uma variedade algébrica de $\mathbb{A}^n_k$, então*

$$X \text{ é irredutível} \Longleftrightarrow I(X) \text{ é um ideal primo de } k[x_1, x_2, \ldots, x_n].$$

Demonstração: Ver o teorema 12 de [Zariski e Samuel (1975)] na página 162.

□

Proposição 2.40.

$$\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{A}^n_k \Longrightarrow \boldsymbol{m_a} = I(\{a\}) \text{ é um ideal maximal de } k[x_1, x_2, \ldots, x_n].$$

$$V(\boldsymbol{m}) = \{\boldsymbol{a}\},\ \text{para algum } \boldsymbol{a} \in \mathbb{A}^n_k \Longleftarrow \boldsymbol{m} \text{ é um ideal maximal de } k[x_1, x_2, \ldots, x_n].$$

Demonstração: Ver corolário 2 de [Matsumura (1970)] na página 91.

□

Observação 2.41. *Temos as seguintes correspondências biunívocas induzidas por V e I:*

$$\{\textit{Conjuntos algébricos}\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{Ideais radicais de } k[x_1, x_2, \ldots, x_n]\}$$

$$\{\textit{Conjuntos algébricos irredutíveis}\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{Ideais primos de } k[x_1, x_2, \ldots, x_n]\}$$

$$\mathbb{A}^n_k = \{\textit{Pontos de } \mathbb{A}^n_k\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{Ideais maximais de } k[x_1, x_2, \ldots, x_n]\}$$

Definição 2.42. *Seja $X \subset \mathbb{A}^n_k$ um conjunto algébrico. Definimos o* ***anel de coordenadas de*** *X como*

$$A(X) := k[x_1, x_2, \ldots, x_n]/I(X).$$

Observação 2.43.

1. *Todo anel de coodenadas de $A(X)$ é uma k-álgebra finitamente gerada sem elementos nilpotentes ($a \in R$ é nilpotente em R se existe $n \in \mathbb{N}$ tal que $a^n = 0$).*

2. *Toda k-álgebra finitamente gerada sem elementos nilpotentes é o anel de coordenadas de algum conjunto algébrico.*

Definição 2.44. *Uma k-álgebra associativa, comutativa e finitamente gerada é chamada de k-**álgebra afim** . Uma k-álgebra afim é chamada de **reduzida** se não tem elementos nilpotentes.*

Observação 2.45. *Seja $X \subset \mathbb{A}^n_k$ um conjunto algébrico. A projeção natural*

$$\pi : k[x_1, x_2, \ldots, x_n] \longrightarrow A(X) = k[x_1, x_2, \ldots, x_n]/I(X),$$

induz a correspondência biunívoca

$$\{\textit{Ideais de } k[x_1, x_2, \ldots, x_n] \textit{ que contem } I(X)\} \xleftrightarrow{1-1} \{\textit{Ideais de } A(X)\},$$

esta correspondência preserva inclusões, ideais primos, ideais radicais, ideais maximais e portanto, temos as seguintes correspondências biunívocas:

$$\{\textit{Conjuntos algébricos contidos em } X\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{Ideais radicais de } A(X)\}$$

$$\{\textit{Conjuntos algébricos irredutíveis contidos em } X\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{Ideais primos de } A(X)\}$$

$$X = \{\textit{Pontos de } X\} \underset{V}{\overset{I}{\rightleftarrows}} \{\textit{Ideais maximais de } A(X)\}$$

Definição 2.46. *A **topologia de Zariski** é a topologia em $\mathbb{A}^n_k$ cujos fechados são os conjuntos algébricos afins.*

Proposição 2.47.

1. *$\mathbb{A}^n_k$ com a topologia de Zariski é um espaço topológico Noetheriano, ou seja, se $Y_1 \supset Y_2 \supset Y_3 \supset \cdots$ é uma cadeia de fechados, então existe n tal que $Y_i = Y_n$, $\forall i \geq n$.*

2. *Todo conjunto algébrico $X \subset \mathbb{A}^n_k$ se escreve de maneira única como*

$$X = X_1 \cup X_2 \cup \cdots \cup X_n,$$

onde cada X_i é uma variedade algébrica irredutível e $X_i \not\subset X_j$, $\forall i \neq j$.

*Cada X_i é chamada de **componente irredutível** de X.*

Demonstração: Ver o teorema 13 de [Zariski e Samuel (1975)] na página 162.

□

Observação 2.48. *Uma componente irredutível é uma subvariedade irredutível e maximal.*

Definição 2.49. *Seja X um conjunto algébrico. Para cada cadeia estritamente crescente*

$$X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_d$$

*de subconjuntos fechados e irredutíveis de X, o número d é dito **comprimento da cadeia** e o supremo dos comprimentos dessas cadeias estritamente crescentes de X é chamada de **dimensão de** X e será denotado por* $\dim X$.

Similarmente definiremos a dimensão de um anel.

Definição 2.50. *Seja R um anel. Consideremos as cadeias estritamente crescentes de ideais primos de R*

$$\boldsymbol{p}_0 \subsetneq \boldsymbol{p}_1 \subsetneq \boldsymbol{p}_2 \subsetneq \cdots \subsetneq \boldsymbol{p}_r.$$

*A **dimensão de Krull de** R é o supremo dos comprimentos dessas cadeias e será denotado por* $\dim(R)$.

Exemplo 2.51. $\dim(k[x_1, x_2, \ldots, x_n]) = n$.

Teorema 2.52. *A dimensão de um conjunto algébrico X é igual à dimensão de Krull do anel de coordenadas de X, i. e.,*

$$\dim X = \dim(A(X)).$$

Definição 2.53. *Uma variedade é dita **equidimensional** se todas suas componentes irredutíveis tem a mesma dimensão.*

Exemplo 2.54. *Consideremos a variedade algébrica $V(xy, xz) \subset \mathbb{A}_k^3$. Claramente,*

$$V(xy, xz) = V((x)(y, z)) = V(x) \cup V(y, z).$$

Agora, observemos que como os ideais (x) e (y, z) são primos, então $V(x)$ e $V(y, z)$ são irredutíveis. Além disso, $\dim(A(V(x))) = \dim(k[y, z]) = 2$ *e* $\dim(A(V(y, z))) = \dim(k[x]) = 1$.

Geométricamente, notemos que $V(x)$ é o plano yz enquanto $V(y, z)$ é o eixo x.

2.4 Álgebra Homológica

Nesta seção apresentaremos os conceitos de complexos de módulos, complexos de Koszul e suas homologias, para assim definir interseção completa. Para esta seção fixaremos R um anel associativo.

Definição 2.55. *Um **complexo** $X_\bullet$ está dado por sequências $(X_n)_{n\in\mathbb{Z}}$ e $\left(d_n^X\right)_{n\in\mathbb{Z}}$, em que cada X_n é um R-módulo e cada $d_n^X : X_n \longrightarrow X_{n-1}$ (**diferencial**) é um homomorfismo de R-módulos, com $d_n^X d_{n+1}^X = 0$ para todo $n \in \mathbb{Z}$. Um complexo será representado pela seguinte sequência*

$$X_\bullet := \cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}^X} X_n \xrightarrow{d_n^X} X_{n-1} \longrightarrow \cdots$$

Definição 2.56. *Seja $X_\bullet$ um complexo. Para cada $n \in \mathbb{Z}$ definimos sua n-**ésima homologia** como*

$$H_n(X^\bullet) = \ker(d_n^X)/\mathrm{Im}\,(d_{n+1}^X).$$

2.4.1 Complexos de Koszul

Comecemos por dar uma ideia básica de complexo de Koszul. Sejam R um anel, $x \in R$ não nulo, M um R-módulo e consideremos o homomorfismo de R-módulos

$$\begin{array}{rccc} x\cdot: & M & \longrightarrow & M \\ & m & \longmapsto & xm. \end{array}$$

Por definição

$$\ker(x\cdot) = \{m \in M \ : \ xm = 0\} =: Ann_M(x).$$

Além disso,

$$\text{Im}\,(x\cdot) = xM \ \ e \ \ \text{Coker}\,(x\cdot) = M/xM.$$

E se considerarmos o complexo

$$K_\bullet(x,M) := \cdots \longrightarrow 0 \longrightarrow \underbrace{M}_{1} \xrightarrow{x\cdot} \underbrace{M}_{0} \longrightarrow 0 \longrightarrow \cdots$$

temos que

$$H_1(K_\bullet(x,M)) = \ker(x\cdot) = Ann_M(x),$$
$$H_0(K_\bullet(x,M)) = \text{Coker}\,(x\cdot) = M/xM.$$

Agora introduziremos o conceito de complexo de Koszul, para maiores detalhes ver [Bourbaki (1980)] pag. 147, [Bruns e Herzog (1993)] pag. 39 − 46 e [Matsumura (1970)] pag. 132 − 135.

Sejam R um anel comutativo, M um R-módulo e $f : M \longrightarrow R$ uma aplicação R-linear. A aplicação

$$\begin{array}{ccc} M^n & \longrightarrow & \bigwedge^{n-1} M \\ (x_1, x_2, \ldots, x_n) & \longmapsto & \sum_{i=1}^{n} (-1)^{i+1} f(x_i) x_1 \wedge x_2 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n \end{array}$$

é n-linear e alternada, onde $\widehat{x_i}$ indica que estamos removendo x_i do produto exterior. Pela propriedade universal da n-ésima potência exterior (proposição (2.20)) existe um único $d_n^f : \bigwedge^n M \longrightarrow \bigwedge^{n-1} M$ homomorfismo de R-módulos, tal que

$$\begin{array}{rccc} d_n^f : & \bigwedge^n M & \longrightarrow & \bigwedge^{n-1} M \\ & x_1 \wedge x_2 \wedge \ldots \wedge x_n & \longmapsto & \sum_{i=1}^{n} (-1)^{i+1} f(x_i) x_1 \wedge x_2 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n. \end{array}$$

Proposição 2.57. *Sejam R um anel comutativo, M um R-módulo e $f : M \longrightarrow R$ uma aplicação R-linear. A cadeia*

$$K_\bullet := \cdots \xrightarrow{d_{n+2}^f} \bigwedge^{n+1} M \xrightarrow{d_{n+1}^f} \bigwedge^n M \xrightarrow{d_n^f} \bigwedge^{n-1} M \xrightarrow{d_{n-1}^f} \cdots \xrightarrow{d_3^f} \bigwedge^2 M \xrightarrow{d_2^f} M \xrightarrow{f} R \longrightarrow 0$$

é um complexo de R-módulos.

Demonstração:

$$f d_2^f(x_1 \wedge x_2) = f(f(x_1)x_2 - f(x_2)x_1) = f(x_1)f(x_2) - f(x_2)f(x_1) = 0$$

$$\begin{aligned} d_n^f d_{n+1}^f(x_1 \wedge x_2 \wedge \cdots \wedge x_{n+1}) &= d_n^f\left(\sum_{i=1}^{n+1} (-1)^{i+1} f(x_i) x_1 \wedge x_2 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{n+1}\right) \\ &= \sum_{i=1}^{n+1} (-1)^{i+1} f(x_i) d_n^f(x_1 \wedge x_2 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{n+1}) \\ &= 0 \end{aligned}$$

□

Definição 2.58. *O complexo acima é chamado de* ***complexo de Koszul de*** f *e denotado por* $K_\bullet(f)$ *(ou simplesmente* $K_\bullet$*) e sua homologia é chamada de* ***homologia de Koszul de*** f *e denotada por* $H_n(f)$.

Agora se N *é um* R*-módulo, então* $K_\bullet(f,N)$ *é o complexo* $K_\bullet(f)\otimes_R N$ *chamado de* ***complexo de Koszul de*** f ***com coeficientes em*** N. *Seus diferenciais são denotados por* $d_n^{f,N}$, *sua homologia por* $H_n(f,N)$ *e será chamada de* ***homologia de Koszul de*** f ***com coeficientes em*** N.

Observação 2.59. *Seja* M *um* R*-módulo livre de posto finito e com base* $e_1, e_2, \ldots, e_n$. *Então, a aplicação* R*-linear* f *está únicamente determinada pelos valores* $x_i = f(e_i) \in R$ *com* $i = 1,2,\ldots,n$. *Reciprocamente, dada uma sequência* $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ *em* R, *existe uma aplicação* R*-linear* f *sobre* M *com* $f(e_i) = x_i$, *portanto denotemos*

$$K_\bullet(\mathbf{x}) := K_\bullet(f).$$

Definição 2.60. *Para cada sequência* $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ *num anel comutativo* R, *o complexo* $K_\bullet(\mathbf{x})$ *é chamado de* ***complexo de Koszul de*** $\mathbf{x}$, *sua a homologia de* ***homologia de Koszul de*** $\mathbf{x}$ *e denotada por* $H_n(\mathbf{x})$.

Agora, se N *é um* R*-módulo, então* $K_\bullet(\mathbf{x},N)$ *é o complexo* $K_\bullet(\mathbf{x})\otimes_R N$ *chamado de* ***complexo de Koszul de*** $\mathbf{x}$ ***com coeficientes em*** N. *Seus diferenciais são denotados por* $d_n^{\mathbf{x},N}$, *sua homologia por* $H_n(\mathbf{x},N)$ *e será chamada de* ***homologia de Koszul de*** $\mathbf{x}$ ***com coeficientes em*** N.

Exemplo 2.61. *Se considerarmos* $x \neq 0$ *um elemento num anel* R *e a aplicação*

$$\begin{array}{rccc} f: & R & \longrightarrow & R \\ & r & \longmapsto & xr \end{array}$$

então, para cada R*-módulo* M *temos o complexo inicial* $K_\bullet(x,M) := K_\bullet(f,M)$.

Definição 2.62. *Sejam* $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ *uma sequência de elementos num anel* R *e* M *um* R*-módulo. Diremos que a sequência é* ***interseção completa para*** M *se*

$$H_n(\mathbf{x}, M) = 0, \ \ \forall n > 0.$$

Exemplo 2.63. *Voltando ao primeiro complexo, temos que a sequência* $\mathbf{x} = \{x\}$ *é interseção completa sobre o anel* R *se, e somente se,* x *não é um divisor de zero sobre* R.

Com alguns dos resultados da próxima seção, vamos ter uma ideia muito mais clara sobre o que significa que uma sequência seja interseção completa. Para maiores detalhes sobre interseção completa, ver [Bourbaki (1980)] pag. $157-161$.

2.5 Álgebra Comutativa

Nesta última seção apresentaremos o conceito de sequência regular, algumas propriedades úteis e a relação com equidimensionalidade e interseção completa.

Definição 2.64. *Uma sequência* $g_1, g_2, \ldots, g_t$ *em um anel* R *é dita* ***regular*** *se a imagem da classe de* g_i *não é divisor de zero e nem invertível em* $R/(g_1, g_2, \ldots, g_{i-1})$ *para qualquer* $i = 1,2,\ldots,t$.

Exemplo 2.65. *A sequência* $x, y(x-1), z(x-1)$ *em* $k[x,y,z]$ *é uma sequência regular, mas a sequência* $y(x-1), z(x-1), x$ *não é regular em* $k[x,y,z]$, *pois* $\overline{z(x-1)}$ *é um divisor de zero em* $k[x,y,z]/\langle y(x-1)\rangle$. *Para ver isto, basta multiplicar por* $\overline{y} \neq 0$ *em* $k[x,y,z]/\langle y(x-1)\rangle$.

Deste exemplo, podemos concluir que permutações e subsequências de sequências regulares, não necessariamente são regulares.

Proposição 2.66. *Sejam* R *um anel Noetheriano e* $g_1, g_2, \ldots, g_t$ *uma sequência regular em* R. *Se* R *é um anel graduado e cada* g_i *é homogêneo de grau positivo, então qualquer permutação de* $g_1, g_2, \ldots, g_t$ *é também regular em* R.

Demonstração: Ver teorema 28 em [Matsumura (1970)] página 102 ou [Matsumura (1989)] página 127.
□

Corolário 2.67. *Sejam R um anel Noetheriano e $g_1, g_2, \dots, g_t$ uma sequência regular em R. Se R é um anel graduado e cada g_i é homogêneo de grau positivo, então qualquer subsequência de $g_1, g_2, \dots, g_t$ é também regular em R.*

Demonstração: Segue da proposição (2.66) e da definição de sequência regular.
□

As seguintes proposições relacionam os conceitos de equidimensionalidade, interseąo completa e sequências regulares.

Proposição 2.68. *Seja $g_1, g_2, \dots, g_t$ uma sequência regular de um anel R e seja $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_t$ a base canônica do módulo livre R^t, então a sequência $g_1, g_2, \dots, g_t$ é interseção completa para R, obtendo assim, a sequência exata*

$$\cdots \longrightarrow \bigwedge^2(R^t) \xrightarrow{\partial} R^t \xrightarrow{\eta} R \xrightarrow{\pi} R/(g_1, g_2, \dots, g_t) \longrightarrow 0$$

chamada de ***resolução de Koszul****, em que π é a projeção, $\bigwedge^i$ é a i-ésima potência exterior,*

$$\partial(\boldsymbol{e}_i \wedge \boldsymbol{e}_j) = g_i \boldsymbol{e}_j - g_j \boldsymbol{e}_i,$$

$$\eta(f_1, f_2, \dots, f_t) = \sum_{i=1}^{t} f_i g_i.$$

Demonstração: Ver proposição 5 de [Bourbaki (1980)] página 157.
□

Proposição 2.69. *Sejam R uma álgebra afim com dimensão de Krull n e $g_1, g_2, \dots, g_t$ uma sequência em R com $0 \le t \le n$.*

1. *Se R é graduada e $g_1, g_2, \dots, g_t$ homogêneos, então $g_1, g_2, \dots, g_t$ é uma sequência regular em R se, e somente se, a sequência $g_1, g_2, \dots, g_t$ é uma interseção completa em R.*

2. *Se R é Cohen-Macaulay[1], então a sequência $g_1, g_2, \dots, g_t$ é interseção completa para R se, e somente se, a variedade $V(g_1, g_2, \dots, g_t)$ é equidimensional de dimensão $n - t$.*

Demonstração: Proposition 2.1 em [Futorny e Ovsienko (2005)].
□

Proposição 2.70. *Sejam $R = k[X_1, X_2, \dots, X_n]$ a álgebra de polinômios e $G_1, G_2, \dots, G_t \in R$. A sequência*

$$X_1, X_2, \dots, X_r, G_1, G_2, \dots, G_t,$$

é interseção completa para R se, e somente se, a sequência $g_1, g_2, \dots, g_t$ é também interseção completa para $k[X_{r+1}, X_{r+2}, \dots, X_n]$, em que

$$g_i(X_{r+1}, X_{r+2}, \dots, X_n) = G_i(0, 0, \dots, 0, X_{r+1}, X_{r+2}, \dots, X_n), \quad \forall i = 1, 2, \dots, t.$$

Demonstração: Lema 2.2 de [Futorny e Ovsienko (2005)].
□

[1] Para detalhes sobre anéis Cohen-Macaulay ver [Bruns e Herzog (1993)] página 57 ou em [Matsumura (1989)] página 134. Neste trabalho, só vamos usar o fato que o anel de polinômios é Cohen-Macaulay (teorema 17.7 em [Matsumura (1989)]).

Capítulo 3

Teorema de Ovsienko e sua versão fraca

Neste capítulo apresentaremos a subálgebra de Gelfand-Tsetlin Γ para $\mathfrak{gl}_n$, algumas famílias de geradores de Γ e a variedade de Gelfand-Tsetlin como uma variedade algébrica associada a Γ. Com isto podemos enunciar o Teorema de Ovsienko, o qual fala sobre a equidimensionalidade da variedade de Gelfand-Tsetlin e sua dimensão. Os resultados principais neste capítulo são:

Versão fraca do Teorema de Ovsienko:

A variedade

$$V_n := V(\sigma_{n2}, \sigma_{n3}, \sigma_{n4}, \ldots, \sigma_{nn}) \subset \mathbb{A}_k^{\frac{(n+2)(n-1)}{2}},$$

é equidimensional de dimensão

$$\dim(V_n) = \frac{(n+2)(n-1)}{2} - (n-1) = \frac{n(n-1)}{2},$$

em que

$$\sigma_{nj} = \sum_{n>t_1>t_2>\cdots>t_{j-1}\geq 1} X_{nt_1} X_{t_1t_2} \cdots X_{t_{j-2}t_{j-1}} X_{t_{j-1}n}; \quad j = 2, 3, \ldots, n.$$

Corolário da versão fraca do Teorema de Ovsienko:

Todas as componentes regulares da variedade de Gelfand-Tsetlin são isomorfas. Em particular, são isomorfas à componente irredutível

$$V_{\leq} := V\left(\left\{X_{ij} : 1 \leq i \leq j \leq n\right\}\right) \subset \mathbb{A}_k^{\frac{(n+2)(n-1)}{2}}.$$

Antes de demonstrar a versão fraca do Teorema de Ovsienko, veremos a prova para os casos particulares $n = 2, 3, 4, 5$, com a finalidade de fixar a ideia da demonstração para o caso em que n é arbitrário.

3.1 Teorema de Ovsienko

3.1.1 Subálgebra de Gelfand-Tsetlin para $\mathfrak{gl}_n$

Fixemos $n \in \mathbb{N}$, para cada $i \in \{1, 2, \ldots, n\}$ denotemos por $Z_i := Z(U(\mathfrak{gl}_i))$ o centro de $U(\mathfrak{gl}_i)$.

Consideremos a álgebra $U(\mathfrak{gl}_n)$ e a base canônica $\{E_{ij} \mid 1 \leq i, j \leq n\}$ de $\mathfrak{gl}_n$. Claramente, para cada $i \in \{1, 2, \ldots, n-1\}$ tem-se a identificação de $\mathfrak{gl}_i$ como uma subálgebra de $\mathfrak{gl}_{i+1}$ e $\mathfrak{gl}_n$, gerada pelo conjunto $\{E_{jk} \mid 1 \leq j, k \leq i\}$ e portanto, temos a cadeia de subálgebras de Lie de $\mathfrak{gl}_n$

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n,$$

a qual induz uma cadeia de subálgebras associativas de $U(\mathfrak{gl}_n)$

$$U(\mathfrak{gl}_1) \subset U(\mathfrak{gl}_2) \subset \cdots \subset U(\mathfrak{gl}_{n-1}) \subset U(\mathfrak{gl}_n).$$

Definição 3.1. *A subálgebra de $U(\mathfrak{gl}_n)$ gerada por $\{Z_1, Z_2, \ldots, Z_n\}$ é chamada de* ***subálgebra de Gelfand-Tsetlin de*** *$U(\mathfrak{gl}_n)$ e será denotada por Γ.*

Proposição 3.2. *Para qualquer $i \in \{1,2,\ldots,n\}$, o centro Z_i é uma álgebra de polinômios nas i variáveis $\{\gamma_{ij} \,:\, j = 1,2,\ldots,i\}$, com*

$$\gamma_{ij} = \sum_{t_1,t_2,\ldots,t_j \in \{1,2,\ldots,i\}} E_{t_1 t_2} E_{t_2 t_3} \cdots E_{t_{j-1} t_j} E_{t_j t_1}.$$

Além disso, a subálgebra Γ é uma álgebra de polinômios nas $\dfrac{n(n+1)}{2}$ variáveis $\{\gamma_{ij} \,:\, 1 \le j \le i \le n\}$.

Demonstração: Ver [Želobenko (1973)] página 169.

□

Observação 3.3. *Consideremos a matriz*

$$E := \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n} \\ E_{21} & E_{22} & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn} \end{pmatrix} \in M_n\big(U(\mathfrak{gl}_n)\big)$$

e para cada $i \in \{1,2,\ldots,n\}$ a i-ésima submatriz principal

$$E_i := \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1i} \\ E_{21} & E_{22} & \cdots & E_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ E_{i1} & E_{i2} & \cdots & E_{ii} \end{pmatrix} \in M_i\big(U(\mathfrak{gl}_i)\big).$$

Observemos que para $1 \le j \le i \le n$

$$\gamma_{ij} = tr(E_i^j) = \sum_{t_1=1}^{i} \sum_{t_2=1}^{i} \cdots \sum_{t_j=1}^{i} E_{t_1 t_2} E_{t_2 t_3} \cdots E_{t_{j-1} t_j} E_{t_j t_1}.$$

Para visualizar um pouco a dificuldade da combinatória dos geradores, vejamos que os primeiros 9 geradores de Γ são:

$$\begin{aligned}
\gamma_{11} =& E_{11}, \\
\gamma_{21} =& E_{11} + E_{22}, \\
\gamma_{22} =& E_{11}^2 + E_{12}E_{21} + E_{21}E_{12} + E_{22}^2, \\
\gamma_{31} =& E_{11} + E_{22} + E_{33}, \\
\gamma_{32} =& E_{11}^2 + E_{12}E_{21} + E_{13}E_{31} + E_{21}E_{12} + E_{22}^2 + E_{23}E_{32} + E_{31}E_{13} + E_{32}E_{23} + E_{33}^2, \\
\gamma_{33} =& E_{11}^3 + E_{11}E_{12}E_{21} + E_{11}E_{13}E_{31} + E_{12}E_{21}E_{11} + E_{12}E_{22}E_{21} + E_{12}E_{23}E_{31} + E_{13}E_{31}E_{11} + \\
& + E_{13}E_{32}E_{21} + E_{13}E_{33}E_{31} + E_{21}E_{11}E_{12} + E_{21}E_{12}E_{22} + E_{21}E_{13}E_{32} + E_{22}E_{21}E_{12} + E_{22}^3 + \\
& + E_{22}E_{23}E_{32} + E_{23}E_{31}E_{12} + E_{23}E_{32}E_{22} + E_{23}E_{33}E_{32} + E_{31}E_{11}E_{13} + E_{31}E_{12}E_{23} + E_{31}E_{13}E_{33} + \\
& + E_{32}E_{21}E_{13} + E_{32}E_{22}E_{23} + E_{32}E_{23}E_{33} + E_{33}E_{31}E_{13} + E_{33}E_{32}E_{23} + E_{33}^3, \\
\gamma_{41} =& E_{11} + E_{22} + E_{33} + E_{44}, \\
\gamma_{42} =& E_{11}^2 + E_{12}E_{21} + E_{13}E_{31} + E_{14}E_{41} + E_{21}E_{12} + E_{22}^2 + E_{23}E_{32} + E_{24}E_{42} + E_{31}E_{13} + E_{32}E_{23} + \\
& + E_{33}^2 + E_{34}E_{43} + E_{41}E_{14} + E_{42}E_{24} + E_{43}E_{34} + E_{44}^2,
\end{aligned}$$

$$\begin{aligned}\gamma_{43} = & E_{11}^3 + E_{11}E_{12}E_{21} + E_{11}E_{13}E_{31} + E_{11}E_{14}E_{41} + E_{12}E_{21}E_{11} + E_{12}E_{22}E_{21} + E_{12}E_{23}E_{31} + E_{12}E_{24}E_{41} +\\
& + E_{13}E_{31}E_{11} + E_{13}E_{32}E_{21} + E_{13}E_{33}E_{31} + E_{13}E_{34}E_{41} + E_{14}E_{41}E_{11} + E_{14}E_{42}E_{21} + E_{14}E_{43}E_{31} +\\
& + E_{14}E_{44}E_{41} + E_{21}E_{11}E_{12} + E_{21}E_{12}E_{22} + E_{21}E_{13}E_{32} + E_{21}E_{14}E_{42} + E_{22}E_{21}E_{12} + E_{22}^3 + E_{22}E_{23}E_{32} +\\
& + E_{22}E_{24}E_{42} + E_{23}E_{31}E_{12} + E_{23}E_{32}E_{22} + E_{23}E_{33}E_{32} + E_{23}E_{34}E_{42} + E_{24}E_{41}E_{12} + E_{24}E_{42}E_{22} +\\
& + E_{24}E_{43}E_{32} + E_{24}E_{44}E_{42} + E_{31}E_{11}E_{13} + E_{31}E_{12}E_{23} + E_{31}E_{13}E_{33} + E_{31}E_{14}E_{43} + E_{32}E_{21}E_{13} +\\
& + E_{32}E_{22}E_{23} + E_{32}E_{23}E_{33} + E_{32}E_{24}E_{43} + E_{33}E_{31}E_{13} + E_{33}E_{32}E_{23} + E_{33}^3 + E_{33}E_{34}E_{43} + E_{34}E_{41}E_{13} +\\
& + E_{34}E_{42}E_{23} + E_{34}E_{43}E_{33} + E_{34}E_{44}E_{43} + E_{41}E_{11}E_{14} + E_{41}E_{12}E_{24} + E_{41}E_{13}E_{34} + E_{41}E_{14}E_{44} +\\
& + E_{42}E_{21}E_{14} + E_{42}E_{22}E_{24} + E_{42}E_{23}E_{34} + E_{42}E_{24}E_{44} + E_{43}E_{31}E_{14} + E_{43}E_{32}E_{24} + E_{43}E_{33}E_{34} +\\
& + E_{43}E_{34}E_{44} + E_{44}E_{41}E_{14} + E_{44}E_{42}E_{24} + E_{44}E_{43}E_{34} + E_{44}^3.\end{aligned}$$

3.1.2 Outros geradores da subálgebra de Gelfand-Tsetlin Γ

Como vimos, a combinatória dos geradores é bastante complicada para n suficientemente grande, vejamos outros geradores que vão ser muito úteis. Para mais detalhes ver [Gelfand *et al.* (1995)] ou [Molev (2007)] páginas $246-250$.

Sejam R um anel arbitrário com $1 \in R$ e $A = \left(a_{ij}\right)_{i,j=1}^n \in M_n(R)$. Denote por A^{ij} a matriz obtida de A tirando a i-ésima linha e a j-ésima coluna, por r_i^j a matriz linha $1 \times n-1$ obtida da i-ésima linha de A tirando o elemento a_{ij} e por c_j^i a matriz coluna $n-1 \times 1$ obtida da j-ésima coluna de A tirando o elemento a_{ij}.

Definição 3.4. *Sejam $A = \left(a_{ij}\right)_{i,j=1}^n \in M_n(R)$ uma matriz $n \times n$ sobre um anel arbitrário R com $1 \in R$ e $i, j \in \{1, 2, \ldots, n\}$ tais que, a matriz A^{ij} é invertível. O ij-ésimo* ***quasideterminante*** *de A é definido pela fórmula*

$$|A|_{ij} := a_{ij} - r_i^j \left(A^{ij}\right)^{-1} c_j^i.$$

Exemplo 3.5. *Considere a matriz*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Então, os 4 quasideterminantes de A são:

$$\begin{aligned}|A|_{11} &= a_{11} - r_1^1 \left(A^{11}\right)^{-1} c_1^1 = a_{11} - a_{12}a_{22}^{-1}a_{21},\\
|A|_{12} &= a_{12} - r_1^2 \left(A^{12}\right)^{-1} c_2^1 = a_{12} - a_{11}a_{21}^{-1}a_{22},\\
|A|_{21} &= a_{21} - r_2^1 \left(A^{21}\right)^{-1} c_1^2 = a_{21} - a_{22}a_{12}^{-1}a_{11},\\
|A|_{22} &= a_{22} - r_2^2 \left(A^{22}\right)^{-1} c_2^2 = a_{22} - a_{21}a_{11}^{-1}a_{12}.\end{aligned}$$

Para mais detalhes sobre quasideterminante ver [Molev (2007)] páginas $23-27$.

Agora, sejam q uma variável formal e I a matriz identidade $n \times n$.

Definição 3.6. *Sejam $A = \left(a_{ij}\right)_{i,j=1}^n \in M_n(R)$ uma matriz $n \times n$ sobre um anel arbitrário R com $1 \in R$ e $i \in \{1, 2, \ldots, n\}$ fixo. As* ***funções simétricas elementares não comutativas*** *$\Lambda_k^{(i)}$* ***associadas à matriz*** *A são definidas como os coeficientes da expansão do seguinte quasideterminante*

$$1 + \sum_{k \geq 1} \Lambda_k^{(i)} q^k = \left|I + qA\right|_{ii}.$$

Outras famílias de funções simétricas não comutativas de A, chamadas respectivamente ***completas*** *$S_k^{(i)}$,* ***séries de potências de primeiro tipo*** *$\Psi_k^{(i)}$ e* ***séries de potências de segundo tipo*** *$\Phi_k^{(i)}$ são definidas respectivamente como:*

$$1+\sum_{k\geq 1} S_k^{(i)} q^k = \left|I-qA\right|_{ii}^{-1},$$

$$\sum_{k\geq 1} \Psi_k^{(i)} q^{k-1} = \left|I-qA\right|_{ii} \frac{d}{dq} \left|I-qA\right|_{ii}^{-1},$$

$$\sum_{k\geq 1} \Phi_k^{(i)} q^{k-1} = -\frac{d}{dq} \log \left|I-qA\right|_{ii},$$

em que, a derivada sobre q e o logaritmo são consideradas como operações formais sobre as séries de potências em q com coeficientes no anel R. Em particular, o logaritmo está bem definido sobre as séries quando começam com 1.

Continuando com a matriz A, considere o grafo orientado completo $\mathscr{A}$ com n vértices $\{1,2,\ldots,n\}$, e a flecha de i para j será indexada por a_{ij}. Cada caminho dirigido no grafo, do vértice i para o vértice j define um monômio da forma

$$a_{ir_1} a_{r_1 r_2} a_{r_2 r_3} \cdots a_{r_{k-2} r_{k-1}} a_{r_{k-1} j},$$

o qual é obtido considerando o produto de indexações das flechas consecutivas do caminho. O inteiro positivo k é o **comprimento do caminho** . Um **caminho simples** , é um caminho tal que $r_s \neq i, j$ para todo $s \in \{1,2,\ldots,k-1\}$.

Existe a seguinte descrição das funções simétricas não comutativas da definição (3.6), em termos do grafo $\mathscr{A}$.

Proposição 3.7.

1. $(-1)^{k-1}\Lambda_k^{(i)}$ *é a soma de todos os monômios indexando caminhos simples de i para i em $\mathscr{A}$ de comprimento k.*

2. $S_k^{(i)}$ *é a soma de todos os monômios indexando caminhos de i para i em $\mathscr{A}$ de comprimento k.*

3. $\Psi_k^{(i)}$ *é a soma de todos os monômios indexando caminhos de i para i em $\mathscr{A}$ de comprimento k, em que o coeficiente de cada monômio é o comprimento do primeiro retorno ao vértice i.*

4. $\Phi_k^{(i)}$ *é a soma de todos os monômios indexando caminhos de i para i em $\mathscr{A}$ de comprimento k, em que o coeficiente de cada monômio é o quociente de k pelo número de retornos ao vértice i.*

Demonstração: Ver [Gelfand *et al.* (1995)] ou [Molev (2007)] proposição 7.3.1 (página 247).

□

Agora considere em $\mathfrak{gl}_n$, as matrizes $E_m \in M_m\left(U(\mathfrak{gl}_m)\right)$ para cada $1 \leq m \leq n$ como na observação (3.3) e as funções simétricas não comutativas $\Lambda_k^{(m)}, S_k^{(m)}, \Psi_k^{(m)}$ e $\Phi_k^{(m)}$ associadas à matriz $E_m+(-m+1)I_m$. Pela proposição (3.7), temos expressões explícitas para essas funções. Considere para cada $k \geq 1$

$$\Lambda_k := \sum_{i_1+i_2+\cdots+i_n=k} \Lambda_{i_1}^{(1)}\Lambda_{i_2}^{(2)}\cdots\Lambda_{i_n}^{(n)},$$

$$S_k := \sum_{i_1+i_2+\cdots+i_n=k} S_{i_1}^{(1)}S_{i_2}^{(2)}\cdots S_{i_n}^{(n)},$$

$$\Psi_{nk} := \sum_{i=1}^{n} \Psi_k^{(i)},$$

$$\Phi_k := \sum_{i=1}^{n} \Phi_k^{(i)},$$

em que os índices i_j são inteiros não negativos e $\Lambda_0^{(j)} = S_0^{(j)} = 1$.

Teorema 3.8. *Todos os elementos Λ_k, S_k, Ψ_k e Φ_k com $k \geq 1$ pertencem ao centro $Z(U(\mathfrak{gl}_n))$ da álgebra envolvente universal $U(\mathfrak{gl}_n)$. Além disso, $\Psi_{nk} = \Phi_k$ para todo k.*

Demonstração: Ver [Gelfand *et al.* (1995)] ou [Molev (2007)] teorema 7.3.2 (página 248) .

□

Corolário 3.9. *Cada família $\{\Lambda_k\}_{k=1}^n$, $\{S_k\}_{k=1}^n$ e $\{\Psi_{nk}\}_{k=1}^n$ gera $Z(U(\mathfrak{gl}_n))$.*

Demonstração: Ver [Gelfand *et al.* (1995)] ou [Molev (2007)] corolário 7.3.3 (página 250).

□

Exemplo 3.10.

$$\Psi_{n1} = \sum_{i=1}^{n} (E_{ii} - i + 1)$$

$$\Psi_{n2} = \sum_{i=1}^{n} (E_{ii} - i + 1)^2 + 2 \sum_{1 \leq t < i \leq n} E_{it} E_{ti}$$

Corolário 3.11. *A subálgebra de Gelfand-Tsetlin Γ é gerada por cada família de elementos $\{\Lambda_k^{(i)}\}$, $\{S_k^{(i)}\}$ e $\{\Psi_k^{(i)}\}$ com $1 \leq k \leq i \leq n$.*

Demonstração: Ver [Gelfand *et al.* (1995)] ou [Molev (2007)] corolário 7.3.5 (página 250).

□

3.1.3 Variedade de Gelfand-Tsetlin para $\mathfrak{gl}_n$

Considere a filtração em $U(\mathfrak{gl}_n)$ vista em (2.2.3). Pelo Teorema de Poincaré-Birkhoff-Witt (2.28)

$$\operatorname{gr}(U(\mathfrak{gl}_n)) \cong S(\mathfrak{gl}_n).$$

Agora considere $\operatorname{gr}(\Gamma)$, a qual pela observação (2.7) é uma subálgebra de $\operatorname{gr}(U(\mathfrak{gl}_n))$ e como consequência da proposição (2.18), os geradores $\overline{\gamma}_{ij}$ são polinômios nas variáveis comutativas $\overline{E}_{ij}$

$$\overline{\gamma}_{ij} = \sum_{t_1, t_2, \ldots, t_j \in \{1,2,\ldots,i\}} \overline{E}_{t_1 t_2} \overline{E}_{t_2 t_3} \cdots \overline{E}_{t_{j-1} t_j} \overline{E}_{t_j t_1}.$$

Similarmente com os geradores $\overline{\Psi}_{ij}$ do corolário (3.9).

Exemplo 3.12.

$$\overline{\Psi}_{n1} = \sum_{i=1}^{n} \overline{E}_{ii}$$

$$\overline{\Psi}_{n2} = \sum_{i=1}^{n} \overline{E}_{ii}^2 + 2 \sum_{1 \leq t < i \leq n} \overline{E}_{it} \overline{E}_{ti}$$

Para facilitar a notação, da proposição (2.18) podemos visualizar $\overline{E}_{ij}$ como variáveis X_{ij}. Logo

$$\overline{\Psi}_{ij}, \overline{\gamma}_{ij} \in k[X_{ij}]_{i,j=1,2,\ldots,n}$$

são polinômios nas variáveis comutativas X_{ij}.

Definição 3.13. *Chamaremos de **variedade de Gelfand-Tsetlin para** $\mathfrak{gl}_n$ a variedade algébrica*

$$V\left(\{\overline{\gamma}_{ij} \,:\, i = 1,2,\ldots,n;\; j = 1,2,\ldots,i\}\right) \subset \mathbb{A}_k^{n^2}.$$

Da graduação de $U(\mathfrak{gl}_n)$ e corolário (3.9), temos que a variedade de Gelfand-Tsetlin é igual a

$$V\left(\left\{\overline{\Psi}_{ij}\ :\ i=1,2,\ldots,n;\ j=1,2,\ldots,i\right\}\right)\subset\mathbb{A}_k^{n^2},$$

em que,

$$\begin{aligned}\overline{\Psi}_{i1} =& X_{ii},\ \forall i=1,2,\ldots,n,\\ \overline{\Psi}_{ij} =& \sum_{t_1,t_2,\ldots,t_{j-1}\in\{1,2,\ldots,i-1\}} X_{it_1}X_{t_1t_2}\cdots X_{t_{j-2}t_{j-1}}X_{t_{j-1}i},\ \forall i=1,2,\ldots,n;\ j=1,2,\ldots,i\end{aligned}$$

com $t_r\neq t_s$ para todo $r\neq s$.

Exemplo 3.14. *Alguns desses polinômios $\overline{\Psi}_{ij}$ são:*

Ciclos de comprimento 1:

$$\overline{\Psi}_{i1}=X_{ii},\ \forall i=1,2,\ldots,n.$$

Ciclos de comprimento 2:

$$\begin{aligned}\overline{\Psi}_{22}&=X_{21}X_{12},\\ \overline{\Psi}_{32}&=X_{31}X_{13}+X_{32}X_{23},\\ \overline{\Psi}_{42}&=X_{41}X_{14}+X_{42}X_{24}+X_{43}X_{34},\\ &\ \vdots\\ \overline{\Psi}_{n2}&=X_{n1}X_{1n}+X_{n2}X_{2n}+X_{n3}X_{3n}+\cdots+X_{n\,n-1}X_{n-1\,n}.\end{aligned}$$

Ciclos de comprimento 3:

$$\begin{aligned}\overline{\Psi}_{33}&=X_{31}X_{12}X_{23}+X_{32}X_{21}X_{13},\\ \overline{\Psi}_{43}&=X_{41}X_{12}X_{24}+X_{41}X_{13}X_{34}+X_{42}X_{21}X_{14}+X_{42}X_{23}X_{34}+X_{43}X_{31}X_{14}+X_{43}X_{32}X_{24},\\ \overline{\Psi}_{53}&=X_{51}X_{12}X_{25}+X_{51}X_{13}X_{35}+X_{51}X_{14}X_{45}+X_{52}X_{21}X_{15}+X_{52}X_{23}X_{35}+X_{52}X_{24}X_{45}+\\ &\quad+X_{53}X_{31}X_{15}+X_{53}X_{32}X_{25}+X_{53}X_{34}X_{45}+X_{54}X_{41}X_{15}+X_{54}X_{42}X_{25}+X_{54}X_{43}X_{35},\\ &\ \vdots\end{aligned}$$

Teorema 3.15 (Teorema de Ovsienko).
A variedade de Gelfand-Tsetlin é equidimensional com dimensão

$$n^2-\frac{n(n+1)}{2}=\frac{n(n-1)}{2}.$$

Demonstração: Ver [Ovsienko (2003)]. □

3.2 Versão fraca do Teorema de Ovsienko

Antes de enunciar e provar a versão fraca do Teorema de Ovsienko vamos precisar introduzir algumas notações e do seguinte lema:

Lema 3.16. *A variedade de Gelfand-Tsetlin é equidimensional de dimensão*

$$n^2-\frac{n(n+1)}{2}=\frac{n(n-1)}{2}$$

se, e somente se, a variedade

$$V_*:=V\left(\left\{\overline{\Psi}_{ij}\ :\ i=2,3,\ldots,n;\ j=2,3,\ldots,i\right\}\right)\subset\mathbb{A}_k^{n^2-n}$$

é equidimensional de dimensão

$$\dim(V_*) = n^2 - n - \left(\frac{n(n+1)}{2} - n\right) = \frac{n(n-1)}{2}.$$

Demonstração: Isto segue do fato que $\overline{\Psi}_{i1} = X_{ii} = 0\ ,\ \ \forall i = 1,2,\ldots,n.$

□

Notação 3.17. *Para facilitar a escrita deste capítulo, vamos considerar a seguinte notação:*

1. *Para um polinômio $P \in k[x_1, x_2, \ldots, x_n]$ e um conjunto de variáveis $\mathbf{X} = \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ denotemos*

$$P^{\mathbf{X}} := P\Big|_{x_{i_1}=x_{i_2}=\cdots=x_{i_r}=0}.$$

   Ou seja, $P^{\mathbf{X}}$ é o polinômio obtido de P fazendo a substituição $x_{i_1} = x_{i_2} = \cdots = x_{i_r} = 0$.

2. *Para $n \in \mathbb{N}$, definimos o conjunto de variáveis*

$$I_n := \left\{X_{ij} : n \geq i > j \geq 1\right\} \cup \{X_{in} : i = 1,2,\ldots,n-1\}.$$

3. *Para $t \in \mathbb{N}$ com $t \leq n$, denotemos*

$$I_n^{(t)} := \left\{X_{ij} \in I_n : i, j \neq t\right\}.$$

Agora, podemos apresentar o resultado principal deste capítulo.

Teorema 3.18 (Versão fraca do Teorema de Ovsienko).
A variedade

$$V_n := V(\sigma_{n2}, \sigma_{n3}, \sigma_{n4}, \ldots, \sigma_{nn}) \subset \mathbb{A}_k^{n^2-n-\frac{(n-2)(n-1)}{2}} = \mathbb{A}_k^{\frac{(n+2)(n-1)}{2}}$$

é equidimensional com dimensão

$$\dim(V_n) = \frac{(n+2)(n-1)}{2} - (n-1) = \frac{n(n-1)}{2},$$

com

$$\sigma_{nj} = \sum_{n>t_1>t_2>\cdots>t_{j-1}\geq 1} X_{nt_1} X_{t_1t_2} \cdots X_{t_{j-2}t_{j-1}} X_{t_{j-1}n};\quad j = 2,3,\ldots,n.$$

Observação 3.19. *Vale a pena notar que:*

1. *Se considerarmos a matriz*

$$X = \left(X_{ij}\right)_{i,j=1}^{n},$$

   o conjunto I_n está formado pelas entradas da parte triangular estritamente inferior, junto com a última coluna da matriz X e $I_n^{(t)}$ está formado pelos elementos de I_n tirando a t-ésima linha e a t-ésima coluna. Intuitivamente, podem ser visualizados como matriz, por exemplo

$$I_4 = \begin{pmatrix} & & & X_{14} \\ X_{21} & & & X_{24} \\ X_{31} & X_{32} & & X_{34} \\ X_{41} & X_{42} & X_{43} & \end{pmatrix} \quad e \quad I_5^{(1)} = \begin{pmatrix} & & & X_{25} \\ X_{32} & & & X_{35} \\ X_{42} & X_{43} & & X_{45} \\ X_{52} & X_{53} & X_{54} & \end{pmatrix}.$$

2. $\sigma_{nj} \in k[I_n]\ ,\ \ \forall j = 2,3,\ldots,n.$

A motivação da versão fraca é que acreditamos que ela implica o Teorema de Ovsienko, por exemplo, V_2 é exatamente a variedade de Gelfand-Tsetlin para $\mathfrak{gl}_2$ e a variedade de Gelfand-Tsetlin para $\mathfrak{gl}_3$ é igual à união de V_3 e mais uma variedade isomorfa a V_3. Mas, o nome de *Versão fraca* se deve ao fato que V_n é uma subvariedade da variedade de Gelfand-Tsetlin para $\mathfrak{gl}_n$.

3.2.1 Casos particulares da Versão fraca do Teorema de Ovsienko

A prova da versão fraca será por indução sobre n. Começaremos provando, os casos particulares $n = 2,3,4,5$. O objetivo disso é que $n = 2,3$ são casos muitos simples, mas $n = 4,5$ são mais interessantes, pois nesses casos se apresenta a técnica que usaremos na demonstração da versão fraca do Teorema de Ovsienko para n arbitrário.

n=2: $V_2 = V(\sigma_{22}) = V(X_{21}X_{12}) = V(X_{21}) \cup V(X_{12}) \subset \mathbb{A}_k^2$ é equidimensional com

$$\dim(V(X_{21})) = 1 = \dim(V(X_{12})).$$

n=3: $V_3 := V(\sigma_{32}, \sigma_{33}) \subset \mathbb{A}_k^{3^2-3-1} = \mathbb{A}_k^5$

$$\sigma_{32} = X_{31}X_{13} + X_{32}X_{23},$$
$$\sigma_{33} = X_{32}X_{21}X_{13}.$$

Vejamos que esta variedade é equidimensional com $\dim(V_3) = 5-2 = 3$. Como

$$\sigma_{33} = X_{32}X_{21}X_{13},$$

então

$$V_3 = V(\sigma_{32}, X_{32}) \cup V(\sigma_{32}, X_{21}) \cup V(\sigma_{32}, X_{13}).$$

Para $V(\sigma_{32}, X_{13}) \subset \mathbb{A}_k^5$, pelas proposições (2.69) e (2.70) basta provar que a variedade

$$V_3^{(1)} := V\left(\sigma_{32}^{\mathbf{X}}\right) \subset \mathbb{A}_k^4\,; \quad \text{onde} \quad \mathbf{X} = \{X_{13}\}$$

é equidimensional com $\dim\left(V_3^{(1)}\right) = 4-1 = 3$. O qual é claro, pois

$$V_3^{(1)} = V\left(\sigma_{32}^{\mathbf{X}}\right) = V(X_{32}X_{23}) = V(X_{32}) \cup V(X_{23}) \subset \mathbb{A}_k^4$$

é equidimensional com $\dim(V(X_{32})) = 3 = \dim(V(X_{23}))$.

Para $V(\sigma_{32}, X_{21}) \subset \mathbb{A}_k^5$, segue da proposição (2.69) e do corolário (2.67) que basta provar que

$$V_3^{(1)} := V(\sigma_{32}, X_{21}, X_{23}, X_{32}) \subset \mathbb{A}_k^5$$

é equidimensional com $\dim\left(V_3^{(1)}\right) = 5-4 = 1$. Para isto, pelas proposições (2.69) e (2.70) é equivalente provar que a variedade

$$V_3^{(2)} := V\left(\sigma_{32}^{\mathbf{X}}\right) \subset \mathbb{A}_k^2\,, \quad \text{com} \ \mathbf{X} = \{X_{21}, X_{23}, X_{32}\}$$

é equidimensional com $\dim\left(V_3^{(2)}\right) = 2-1 = 1$. O qual é claro, pois

$$V_3^{(2)} = V\left(\sigma_{32}^{\mathbf{X}}\right) = V(X_{31}X_{13}) = V(X_{31}) \cup V(X_{13}) \subset \mathbb{A}_k^2$$

é equidimensional com $\dim(V(X_{31})) = 1 = \dim(V(X_{13}))$.

Para $V(\sigma_{32}, X_{32}) \subset \mathbb{A}_k^5$, pelas proposições (2.69) e (2.70) basta provar que a variedade

$$V_3^{(1)} := V\left(\sigma_{32}^{\mathbf{X}}\right) \subset \mathbb{A}_k^4\,; \quad \text{onde} \quad \mathbf{X} = \{X_{32}\}$$

é equidimensional com $\dim\left(V_3^{(1)}\right)=4-1=3$. O qual segue do fato que

$$V_3^{(1)} = V\left(\sigma_{32}^{\mathbf{X}}\right) = V\left(X_{31}X_{13}\right) = V(X_{31}) \cup V(X_{13}) \subset \mathbb{A}_k^4$$

é equidimensional com $\dim(V(X_{31})) = 3 = \dim(V(X_{13}))$.
Finalmente concluimos que a variedade V_3 é equidimensional com

$$\dim(V_3) = 3.$$

n=4: $V_4 := V\left(\sigma_{42}, \sigma_{43}, \sigma_{44}\right) \subset \mathbb{A}_k^{4^2-4-3} = \mathbb{A}_k^9$

$$\begin{aligned}
\sigma_{42} &= X_{41}X_{14} + X_{42}X_{24} + X_{43}X_{34},\\
\sigma_{43} &= X_{42}X_{21}X_{14} + X_{43}X_{31}X_{14} + X_{43}X_{32}X_{24},\\
\sigma_{44} &= X_{43}X_{32}X_{21}X_{14}.
\end{aligned}$$

Vejamos que esta variedade é equidimensional de $\dim(V_4) = 9-3 = 6$. Como

$$\sigma_{44} = X_{43}X_{32}X_{21}X_{14},$$

então

$$V_4 = V\left(\sigma_{42}, \sigma_{43}, X_{43}\right) \cup \bigcup_{t=2}^{3} V\left(\sigma_{42}, \sigma_{43}, X_{t\,t-1}\right) \cup V\left(\sigma_{42}, \sigma_{43}, X_{14}\right).$$

Para $V\left(\sigma_{42}, \sigma_{43}, X_{14}\right) \subset \mathbb{A}_k^9$, segue da proposição (2.69) e do corolário (2.67) que basta provar que

$$V_4^{(1)} := V\left(\sigma_{42}, \sigma_{43}, X_{14}, X_{21}, X_{31}, X_{41}\right) \subset \mathbb{A}_k^9$$

é equidimensional com $\dim\left(V_4^{(1)}\right) = 9-6 = 3$. Para isso, pelas proposições (2.69) e (2.70) é equivalente provar que[1]

$$V_4^{(2)} := V\left(\sigma_{42}^{\mathbf{X}}, \sigma_{43}^{\mathbf{X}}\right) \subset \mathbb{A}_k^5\,, \quad \mathbf{X} = \{X_{14}, X_{21}, X_{31}, X_{41}\}$$

é equidimensional com $\dim\left(V_4^{(2)}\right) = 3$. Mas,

$$\begin{aligned}
\sigma_{42}^{\mathbf{X}} &= X_{42}X_{24} + X_{43}X_{34},\\
\sigma_{43}^{\mathbf{X}} &= X_{43}X_{32}X_{24}.
\end{aligned}$$

Agora, tomemos o isomorfismo de k-álgebras

$$\begin{array}{rccl}
\varphi: & k[I_3] & \longrightarrow & k[\,I_4^{(1)}\,]\\
& X_{ij} & \longmapsto & \varphi(X_{ij}) := X_{i+1,j+1},
\end{array}$$

isto é,

$$\begin{pmatrix} & & X_{13}\\ X_{21} & & X_{23}\\ X_{31} & X_{32} & \end{pmatrix} \longmapsto \begin{pmatrix} & & X_{24}\\ X_{32} & & X_{34}\\ X_{42} & X_{43} & \end{pmatrix},$$

temos que

$$\begin{aligned}
\varphi\left(\sigma_{32}\right) &= \varphi\left(X_{31}X_{13} + X_{32}X_{23}\right) = X_{42}X_{24} + X_{43}X_{34},\\
\varphi\left(\sigma_{33}\right) &= \varphi\left(X_{32}X_{21}X_{13}\right) = X_{43}X_{32}X_{24}.
\end{aligned}$$

[1] Observemos que se visualizarmos I_4 como matriz, então $\mathbf{X}$ está formado pela primeira linha e a primeira coluna de I_4.

Portanto $V_4^{(2)} \cong V_3$, que é equidimensional com dimensão

$$\dim\left(V_4^{(2)}\right) = \dim(V_3) = 3.$$

Para $V(\sigma_{42}, \sigma_{43}, X_{t\,t-1}) \subset \mathbb{A}_k^9$ com $t \in \{2,3\}$, segue da proposição (2.69) e do corolário (2.67) que é suficiente provar que

$$V_4^{(1)} := V(\sigma_{42}, \sigma_{43}, X_{21}, X_{24}, X_{32}, X_{42}) \subset \mathbb{A}_k^9 \quad (\text{se} \;\; t = 2),$$

$$V_4^{(1)} := V(\sigma_{42}, \sigma_{43}, X_{32}, X_{31}, X_{34}, X_{43}) \subset \mathbb{A}_k^9 \quad (\text{se} \;\; t = 3)$$

é equidimensional com $\dim\left(V_4^{(1)}\right) = 9 - 6 = 3$. Para isso, pelas proposições (2.69) e (2.70) é equivalente provar que

$$V_4^{(2)} := V\left(\sigma_{42}^{\mathbf{X}}, \sigma_{43}^{\mathbf{X}}\right) \subset \mathbb{A}_k^5$$

é equidimensional com $\dim\left(V_4^{(2)}\right) = 5 - 2 = 3$, em que[2]

$$\mathbf{X} = \{X_{21}, X_{24}, X_{32}, X_{42}\} \quad (\text{se} \;\; t = 2),$$

$$\mathbf{X} = \{X_{32}, X_{31}, X_{34}, X_{43}\} \quad (\text{se} \;\; t = 3).$$

Observemos que

	t=2	t=3
$\sigma_{42}^{\mathbf{X}}$	$X_{41}X_{14} + X_{43}X_{34}$	$X_{41}X_{14} + X_{42}X_{24}$
$\sigma_{43}^{\mathbf{X}}$	$X_{43}X_{31}X_{14}$	$X_{42}X_{21}X_{14}$

e se considerarmos o isomorfismo de k-álgebras

$$\begin{array}{rccl} \varphi: & k[I_3] & \longrightarrow & k[\,I_4^{(t)}\,] \\ & X_{ij} & \longmapsto & \varphi(X_{ij}) := \begin{cases} X_{ij}; & \text{se } i, j < t \\ X_{i+1,j}; & \text{se } j < t \leq i \\ X_{i,j+1}; & \text{se } i < t \leq j \\ X_{i+1,j+1}; & \text{se } t \leq i, j \end{cases}, \end{array}$$

isto é,

$$\begin{pmatrix} & & X_{13} \\ X_{21} & & X_{23} \\ X_{31} & X_{32} & \end{pmatrix} \longmapsto \begin{pmatrix} & & X_{14} \\ X_{31} & & X_{34} \\ X_{41} & X_{43} & \end{pmatrix} \quad (\text{se} \;\; t = 2),$$

$$\begin{pmatrix} & & X_{13} \\ X_{21} & & X_{23} \\ X_{31} & X_{32} & \end{pmatrix} \longmapsto \begin{pmatrix} & & X_{14} \\ X_{21} & & X_{24} \\ X_{41} & X_{42} & \end{pmatrix} \quad (\text{se} \;\; t = 3),$$

temos que

	t=2	t=3
$\varphi(\sigma_{32})$	$\varphi(X_{31}X_{13} + X_{32}X_{23}) = X_{41}X_{14} + X_{43}X_{34}$	$\varphi(X_{31}X_{13} + X_{32}X_{23}) = X_{41}X_{14} + X_{42}X_{24}$
$\varphi(\sigma_{33})$	$\varphi(X_{32}X_{21}X_{13}) = X_{43}X_{31}X_{14}$	$\varphi(X_{32}X_{21}X_{13}) = X_{42}X_{21}X_{14}$

Assim,

$$\varphi(\sigma_{32}) = \sigma_{42}^{\mathbf{X}},$$

$$\varphi(\sigma_{33}) = \sigma_{43}^{\mathbf{X}}.$$

[2] Observemos que se visualizarmos I_4 como matriz, então $\mathbf{X}$ está formado pela t-ésima linha e a t-ésima coluna de I_4.

Portanto $V_4^{(2)} \cong V_3$, a qual é equidimensional com dimensão $\dim\left(V_4^{(2)}\right) = \dim(V_3) = 3$.

Para $V(\sigma_{42}, \sigma_{43}, X_{43}) \subset \mathbb{A}_k^9$, segue da proposição (2.69) e do corolário (2.67) que é suficiente provar que[3]

$$V_4^{(1)} := V(\sigma_{42}, \sigma_{43}, X_{43}, X_{31}, X_{32}, X_{34}) \subset \mathbb{A}_k^9$$

é equidimensional com $\dim\left(V_4^{(1)}\right) = 9 - 6 = 3$. Para isso, pelas proposições (2.69) e (2.70) é equivalente provar que[4]

$$V_4^{(2)} := V\left(\sigma_{42}^{\mathbf{X}}, \sigma_{43}^{\mathbf{X}}\right) \subset \mathbb{A}_k^{9-4} = \mathbb{A}_k^5 \quad \text{com} \quad \mathbf{X} = \{X_{43}, X_{31}, X_{32}, X_{34}\}$$

é equidimensional com $\dim\left(V_4^{(2)}\right) = 5 - 2 = 3$. Observemos que

$$\begin{aligned}
\sigma_{42}^{\mathbf{X}} &= X_{41}X_{14} + X_{42}X_{24}, \\
\sigma_{43}^{\mathbf{X}} &= X_{42}X_{21}X_{14}.
\end{aligned}$$

Agora, consideremos o isomomorfismo de k-álgebras

$$\begin{array}{rccl}
\varphi: & k[I_3] & \longrightarrow & k\left[I_4^{(3)}\right] \\
& X_{ij} & \longmapsto & \varphi(X_{ij}) := \begin{cases} X_{ij}; & \text{se } i,j \neq 3 \\ X_{4j}; & \text{se } i = 3 \\ X_{i4}; & \text{se } j = 3 \end{cases},
\end{array}$$

isto é,

$$\begin{pmatrix} & & X_{13} \\ X_{21} & & X_{23} \\ X_{31} & X_{32} & \end{pmatrix} \longmapsto \begin{pmatrix} & & X_{14} \\ X_{21} & & X_{24} \\ X_{41} & X_{42} & \end{pmatrix}.$$

Logo

$$\begin{aligned}
\varphi(\sigma_{32}) &= \varphi(X_{31}X_{13} + X_{32}X_{23}) = X_{41}X_{14} + X_{42}X_{24} = \sigma_{42}^{\mathbf{X}}, \\
\varphi(\sigma_{33}) &= \varphi(X_{32}X_{21}X_{13}) = X_{42}X_{21}X_{14} = \sigma_{42}^{\mathbf{X}}.
\end{aligned}$$

Portanto $V_4^{(2)} \cong V_3$, a qual é equidimensional com $\dim\left(V_4^{(2)}\right) = \dim(V_3) = 3$.

Finalmente concluimos que a variedade V_4 é equidimensional com

$$\dim(V_4) = 6.$$

n=5: $V_5 := V(\sigma_{52}, \sigma_{53}, \sigma_{54}, \sigma_{55}) \subset \mathbb{A}_k^{5^2-5-6} = \mathbb{A}_k^{14}$

$$\begin{aligned}
\sigma_{52} &= X_{51}X_{15} + X_{52}X_{25} + X_{53}X_{35} + X_{54}X_{45}, \\
\sigma_{53} &= X_{52}X_{21}X_{15} + X_{53}X_{31}X_{15} + X_{53}X_{32}X_{25} + X_{54}X_{41}X_{15} + X_{54}X_{42}X_{25} + X_{54}X_{43}X_{35}, \\
\sigma_{54} &= X_{53}X_{32}X_{21}X_{15} + X_{54}X_{42}X_{21}X_{15} + X_{54}X_{43}X_{31}X_{15} + X_{54}X_{43}X_{32}X_{25}, \\
\sigma_{55} &= X_{54}X_{43}X_{32}X_{21}X_{15}.
\end{aligned}$$

Provemos que esta variedade é equidimensional de $\dim(V_5) = 14 - 4 = 10$. Como

$$\sigma_{55} = X_{54}X_{43}X_{32}X_{21}X_{15},$$

[3]Observemos que aqui não podemos usar a mesma técnica dos dois casos anteriores, porque se acrescentarmos a quarta coluna ou a quarta linha estaríamos eliminando todas as equações, i.e., $\sigma_{42}^{\mathbf{X}} = \sigma_{43}^{\mathbf{X}} = 0$ e portanto a sequência $\sigma_{42}^{\mathbf{X}}, \sigma_{43}^{\mathbf{X}}$ não seria uma sequência regular. Neste caso, só acrescentaremos a terceira linha de I_4.

[4]Observemos que se visualizarmos I_4 como matriz, então $\mathbf{X}$ está formado pela terceira linha e a terceira coluna de I_4.

então,

$$V_5 = V(\sigma_{52},\sigma_{53},\sigma_{54},X_{54}) \cup \bigcup_{t=2}^{4} V(\sigma_{52},\sigma_{53},\sigma_{54},X_{t\,t-1}) \cup V(\sigma_{52},\sigma_{53},\sigma_{54},X_{15}).$$

Para $V(\sigma_{52},\sigma_{53},\sigma_{54},X_{15}) \subset \mathbb{A}_k^{14}$, segue da proposição (2.69) e do corolário (2.67) que é suficiente provar que a variedade

$$V_5^{(1)} := V(\sigma_{52},\sigma_{53},\sigma_{54},X_{15},X_{21},X_{31},X_{41},X_{51}) \subset \mathbb{A}_k^{14}$$

é equidimensional com $\dim\left(V_5^{(1)}\right) = 14-8 = 6$. Para isso, pelas proposições (2.69) e (2.70) é equivalente provar que

$$V_5^{(2)} := V\left(\sigma_{52}^{\mathbf{X}},\sigma_{53}^{\mathbf{X}},\sigma_{54}^{\mathbf{X}}\right) \subset \mathbb{A}_k^{9}\,, \quad \mathbf{X} = \{X_{15},X_{21},X_{31},X_{41},X_{51}\}$$

é equidimensional com $\dim\left(V_5^{(2)}\right) = 9-3 = 6$. Observemos que

$$\begin{aligned}
\sigma_{52}^{\mathbf{X}} &= X_{52}X_{25} + X_{53}X_{35} + X_{54}X_{45},\\
\sigma_{53}^{\mathbf{X}} &= X_{53}X_{32}X_{25} + X_{54}X_{42}X_{25} + X_{54}X_{43}X_{35},\\
\sigma_{54}^{\mathbf{X}} &= X_{54}X_{43}X_{32}X_{25}.
\end{aligned}$$

Agora, considerando o isomorfismo de k-álgebras

$$\begin{array}{rccc}
\varphi: & k[I_4] & \longrightarrow & k[\,I_5^{(1)}\,] \\
 & X_{ij} & \longmapsto & \varphi(X_{ij}) := X_{i+1,j+1}
\end{array},$$

isto é,

$$\begin{pmatrix} & & & X_{14}\\ X_{21} & & & X_{24}\\ X_{31} & X_{32} & & X_{34}\\ X_{41} & X_{42} & X_{43} & \end{pmatrix} \longmapsto \begin{pmatrix} & & & X_{25}\\ X_{32} & & & X_{35}\\ X_{42} & X_{43} & & X_{45}\\ X_{52} & X_{53} & X_{54} & \end{pmatrix},$$

temos que

$$\begin{aligned}
\varphi(\sigma_{42}) &= \varphi(X_{41}X_{14} + X_{42}X_{24} + X_{43}X_{34}) = X_{52}X_{25} + X_{53}X_{35} + X_{54}X_{45} = \sigma_{52}^{\mathbf{X}},\\
\varphi(\sigma_{43}) &= \varphi(X_{42}X_{21}X_{14} + X_{43}X_{31}X_{14} + X_{43}X_{32}X_{24}) = X_{53}X_{32}X_{25} + X_{54}X_{42}X_{25} + X_{54}X_{43}X_{35} = \sigma_{53}^{\mathbf{X}},\\
\varphi(\sigma_{44}) &= \varphi(X_{43}X_{32}X_{21}X_{14}) = X_{54}X_{43}X_{32}X_{25} = \sigma_{54}^{\mathbf{X}}.
\end{aligned}$$

Portanto $V_5^{(2)} \cong V_4$, a qual é equidimensional com dimensão $\dim\left(V_5^{(2)}\right) = \dim(V_4) = 6$.

Para $V(\sigma_{52},\sigma_{53},\sigma_{54},X_{t\,t-1}) \subset \mathbb{A}_k^{14}$ com $t \in \{2,3,4\}$, segue da proposição (2.69) e do corolário (2.67) que é suficiente provar que[5]

$$\begin{aligned}
V_5^{(1)} &:= V(\sigma_{52},\sigma_{53},\sigma_{54},X_{21},X_{25},X_{32},X_{42},X_{52}) \subset \mathbb{A}_k^{14}\,, \quad \text{se } t=2\\
V_5^{(1)} &:= V(\sigma_{52},\sigma_{53},\sigma_{54},X_{32},X_{31},X_{35},X_{43},X_{53}) \subset \mathbb{A}_k^{14}\,, \quad \text{se } t=3\\
V_5^{(1)} &:= V(\sigma_{52},\sigma_{53},\sigma_{54},X_{43},X_{41},X_{42},X_{45},X_{54}) \subset \mathbb{A}_k^{14}\,, \quad \text{se } t=4
\end{aligned}$$

é equidimensional com $\dim\left(V_5^{(1)}\right) = 14-8 = 6$. Para isso, pelas proposições (2.69) e (2.70) é equivalente mostrar que

$$V_5^{(2)} := V\left(\sigma_{52}^{\mathbf{X}},\sigma_{53}^{\mathbf{X}},\sigma_{54}^{\mathbf{X}}\right) \subset \mathbb{A}_k^{9}$$

[5] Aqui estamos acrescentando a t-ésima linha e a t-ésima coluna da matriz I_5.

é equidimensional com $\dim\left(V_5^{(2)}\right) = 9-3 = 6$, em que

$$\mathbf{X} = (X_{21}, X_{25}, X_{32}, X_{42}, X_{52}) \ , \quad \text{se} \ \ t = 2$$
$$\mathbf{X} = (X_{32}, X_{31}, X_{35}, X_{43}, X_{53}) \ , \quad \text{se} \ \ t = 3$$
$$\mathbf{X} = (X_{43}, X_{41}, X_{42}, X_{45}, X_{54}) \ , \quad \text{se} \ \ t = 4.$$

Agora, observemos que

	$t=2$	$t=3$
$\sigma_{52}^{\mathbf{X}}$	$X_{51}X_{15}+X_{53}X_{35}+X_{54}X_{45}$	$X_{51}X_{15}+X_{52}X_{25}+X_{54}X_{45}$
$\sigma_{53}^{\mathbf{X}}$	$X_{53}X_{31}X_{15}+X_{54}X_{41}X_{15}+X_{54}X_{43}X_{35}$	$X_{52}X_{21}X_{15}+X_{54}X_{41}X_{15}+X_{54}X_{42}X_{25}$
$\sigma_{54}^{\mathbf{X}}$	$X_{54}X_{43}X_{31}X_{15}$	$X_{54}X_{42}X_{21}X_{15}$

	$t=4$
$\sigma_{52}^{\mathbf{X}}$	$X_{51}X_{15}+X_{52}X_{25}+X_{53}X_{35}$
$\sigma_{53}^{\mathbf{X}}$	$X_{52}X_{21}X_{15}+X_{53}X_{31}X_{15}+X_{53}X_{32}X_{25}$
$\sigma_{54}^{\mathbf{X}}$	$X_{53}X_{32}X_{21}X_{15}$

Agora, consideremos o isomorfismo de k-álgebras [6]

$$\begin{array}{cccc} \varphi: & k[I_4] & \longrightarrow & k[\,I_5^{(t)}\,] \\ & X_{ij} & \longmapsto & \varphi(X_{ij}) := \begin{cases} X_{ij}; & \text{se } i,j<t \\ X_{i+1,j}; & \text{se } j<t\le i \\ X_{i,j+1}; & \text{se } i<t\le j \\ X_{i+1,j+1}; & \text{se } t\le i,j \end{cases} , \end{array}$$

isto é,

$$\begin{pmatrix} & & & X_{14} \\ X_{21} & & & X_{24} \\ X_{31} & X_{32} & & X_{34} \\ X_{41} & X_{42} & X_{43} & \end{pmatrix} \longmapsto \begin{pmatrix} & & & X_{15} \\ X_{31} & & & X_{35} \\ X_{41} & X_{43} & & X_{45} \\ X_{51} & X_{53} & X_{54} & \end{pmatrix} , \quad \text{se} \ \ t=2$$

$$\begin{pmatrix} & & & X_{14} \\ X_{21} & & & X_{24} \\ X_{31} & X_{32} & & X_{34} \\ X_{41} & X_{42} & X_{43} & \end{pmatrix} \longmapsto \begin{pmatrix} & & & X_{15} \\ X_{21} & & & X_{25} \\ X_{41} & X_{42} & & X_{45} \\ X_{51} & X_{52} & X_{54} & \end{pmatrix} , \quad \text{se} \ \ t=3$$

$$\begin{pmatrix} & & & X_{14} \\ X_{21} & & & X_{24} \\ X_{31} & X_{32} & & X_{34} \\ X_{41} & X_{42} & X_{43} & \end{pmatrix} \longmapsto \begin{pmatrix} & & & X_{15} \\ X_{21} & & & X_{25} \\ X_{31} & X_{32} & & X_{35} \\ X_{51} & X_{52} & X_{53} & \end{pmatrix} , \quad \text{se} \ \ t=4$$

segue que

	$t=2$	$t=3$
$\varphi(\sigma_{42})$	$X_{51}X_{15}+X_{53}X_{35}+X_{54}X_{45}$	$X_{51}X_{15}+X_{52}X_{25}+X_{54}X_{45}$
$\varphi(\sigma_{43})$	$X_{53}X_{31}X_{15}+X_{54}X_{41}X_{15}+X_{54}X_{43}X_{35}$	$X_{52}X_{21}X_{15}+X_{54}X_{41}X_{15}+X_{54}X_{42}X_{25}$
$\varphi(\sigma_{44})$	$X_{54}X_{43}X_{31}X_{15}$	$X_{54}X_{42}X_{21}X_{15}$

[6] Aqui estamos associando a matriz I_4 com a matriz $I_5^{(t)}$, i.e., com a matriz I_5 tirando a t-ésima linha e a t-ésima coluna.

	$t=4$
$\varphi(\sigma_{42})$	$X_{51}X_{15}+X_{52}X_{25}+X_{53}X_{35}$
$\varphi(\sigma_{43})$	$X_{52}X_{21}X_{15}+X_{53}X_{31}X_{15}+X_{53}X_{32}X_{25}$
$\varphi(\sigma_{44})$	$X_{53}X_{32}X_{21}X_{15}$

Assim

$$\varphi(\sigma_{42})=\sigma^{\mathbf{X}}_{52},$$
$$\varphi(\sigma_{43})=\sigma^{\mathbf{X}}_{53},$$
$$\varphi(\sigma_{44})=\sigma^{\mathbf{X}}_{54}.$$

Portanto $V_5^{(2)}\cong V_4$, a qual é equidimensional com dimensão $\dim\left(V_5^{(2)}\right)=\dim(V_4)=6$.

Para $V(\sigma_{52},\sigma_{53},\sigma_{54},X_{54})\subset\mathbb{A}^{14}_k$, segue da proposição (2.69) e do corolário (2.67) que basta provar que[7]

$$V_5^{(1)}:=V(\sigma_{52},\sigma_{53},\sigma_{54},X_{54},X_{41},X_{42},X_{43},X_{45})\subset\mathbb{A}^{14}_k$$

é equidimensional com $\dim\left(V_5^{(1)}\right)=14-8=6$. Para isso, pelas proposições (2.69) e (2.70) é equivalente mostrar que

$$V_5^{(2)}:=V\left(\sigma^{\mathbf{X}}_{52},\sigma^{\mathbf{X}}_{53},\sigma^{\mathbf{X}}_{54}\right)\subset\mathbb{A}^{14-5}_k=\mathbb{A}^9_k \quad \text{com} \quad \mathbf{X}=\{X_{54},X_{41},X_{42},X_{43},X_{45}\}$$

é equidimensional com $\dim\left(V_5^{(2)}\right)=9-3=6$. Observemos que

$$\begin{aligned}
\sigma^{\mathbf{X}}_{52}&=X_{51}X_{15}+X_{52}X_{25}+X_{53}X_{35},\\
\sigma^{\mathbf{X}}_{53}&=X_{52}X_{21}X_{15}+X_{53}X_{31}X_{15}+X_{53}X_{32}X_{25},\\
\sigma^{\mathbf{X}}_{54}&=X_{53}X_{32}X_{21}X_{15}.
\end{aligned}$$

Agora, consideremos o isomomorfismo de k-álgebras [8]

$$\begin{array}{rccl}
\varphi: & k[I_4] & \longrightarrow & k\left[I_5^{(4)}\right]\\
& X_{ij} & \longmapsto & \varphi(X_{ij}):=\begin{cases} X_{ij}; & \text{se } i,j\neq 4\\ X_{5j}; & \text{se } i=4\\ X_{i5}; & \text{se } j=4\end{cases},
\end{array}$$

isto é,

$$\begin{pmatrix} & & & X_{14}\\ X_{21} & & & X_{24}\\ X_{31} & X_{32} & & X_{34}\\ X_{41} & X_{42} & X_{43} & \end{pmatrix}\longmapsto\begin{pmatrix} & & & X_{15}\\ X_{21} & & & X_{25}\\ X_{31} & X_{32} & & X_{35}\\ X_{51} & X_{52} & X_{53} & \end{pmatrix}.$$

Assim

$$\begin{aligned}
&\varphi(\sigma_{42})=\varphi(X_{41}X_{14}+X_{42}X_{24}+X_{43}X_{34})=X_{51}X_{15}+X_{52}X_{25}+X_{53}X_{35}=\sigma^{\mathbf{X}}_{52},\\
&\varphi(\sigma_{43})=\varphi(X_{42}X_{21}X_{14}+X_{43}X_{31}X_{14}+X_{43}X_{32}X_{24})=X_{52}X_{21}X_{15}+X_{53}X_{31}X_{15}+X_{53}X_{32}X_{25}=\sigma^{\mathbf{X}}_{53},\\
&\varphi(\sigma_{44})=\varphi(X_{43}X_{32}X_{21}X_{14})=X_{53}X_{32}X_{21}X_{15}=\sigma^{\mathbf{X}}_{54}.
\end{aligned}$$

Portanto $V_5^{(2)}\cong V_4$, a qual é equidimensional com $\dim\left(V_5^{(2)}\right)=\dim(V_4)=6$.

[7] Observemos que aqui não podemos usar a mesma técnica dos dois casos anteriores, porque se acrescentarmos a quinta coluna ou a quinta linha estaríamos eliminando todas as equações, i.e., $\sigma^{\mathbf{X}}_{52}=\sigma^{\mathbf{X}}_{53}=\sigma^{\mathbf{X}}_{54}=0$ e portanto a sequência $\sigma^{\mathbf{X}}_{52},\sigma^{\mathbf{X}}_{53},\sigma^{\mathbf{X}}_{54}$ não seria uma sequência regular. Neste caso, só acrescentamos a quarta linha de I_5.

[8] Aqui estamos só associando I_4 com $I_5^{(4)}$, i.e., tirando a quarta coluna e a quarta linha da matriz I_5.

Finalmente, dos casos anteriores concluimos que a variedade V_5 é equidimensional com

$$\dim(V_5) = 10.$$

3.2.2 Demonstração da versão fraca do Teorema de Ovsienko

Demonstração: Provemos por indução sobre n. Claramente, como visto anteriormente

$$V_2 = V(\sigma_{22}) = V(X_{21}X_{12}) = V(X_{21}) \cup V(X_{12}) \subset \mathbb{A}_k^2$$

é equidimensional com

$$\dim(V(X_{21})) = 1 = \dim(V(X_{12})).$$

Suponhamos que

$$V_{n-1} = V\left(\sigma_{n-1,2}, \sigma_{n-1,3}, \ldots, \sigma_{n-1,n-1}\right) \subset \mathbb{A}_k^{\frac{(n+1)(n-2)}{2}}$$

é equidimensional com

$$\dim(V_{n-1}) = \frac{(n-1)(n-2)}{2},$$

onde

$$\sigma_{n-1,j} = \sum_{n-1>t_1>t_2>\cdots>t_{j-1}\geq 1} X_{n-1,t_1} X_{t_1 t_2} \cdots X_{t_{j-2} t_{j-1}} X_{t_{j-1},n-1}; \quad j = 2, 3, \ldots, n-1.$$

Mostremos que

$$V_n = V(\sigma_{n2}, \sigma_{n3}, \sigma_{n4}, \ldots, \sigma_{nn}) \subset \mathbb{A}_k^{\frac{(n+2)(n-1)}{2}}$$

é equidimensional com

$$\dim(V_n) = \frac{(n+2)(n-1)}{2} - (n-1) = \frac{n(n-1)}{2},$$

onde

$$\sigma_{nj} = \sum_{n>t_1>t_2>\cdots>t_{j-1}\geq 1} X_{nt_1} X_{t_1 t_2} \cdots X_{t_{j-2} t_{j-1}} X_{t_{j-1} n}; \quad j = 2, 3, \ldots, n.$$

De fato, como

$$\sigma_{nn} = X_{n,n-1} X_{n-1,n-2} \cdots X_{32} X_{21} X_{1n},$$

então

$$V_n = V(\sigma_{n2}, \sigma_{n3}, \ldots, \sigma_{nn-1}, X_{nn-1}) \cup \bigcup_{t=2}^{n-1} V(\sigma_{n2}, \sigma_{n3}, \ldots, \sigma_{nn-1}, X_{tt-1}) \cup V(\sigma_{n2}, \sigma_{n3}, \ldots, \sigma_{nn-1}, X_{1n}).$$

Para $V(\sigma_{n2}, \sigma_{n3}, \ldots, \sigma_{nn-1}, X_{1n}) \subset \mathbb{A}_k^{\frac{(n+2)(n-1)}{2}}$, como consequência da proposição (2.69) e do corolário (2.67), é suficiente provar que[9]

$$V_n^{(1)} := V(\sigma_{n2}, \sigma_{n3}, \ldots, \sigma_{nn-1}, X_{1n}, X_{21}, X_{31}, X_{41}, \ldots, X_{n1}) \subset \mathbb{A}_k^{\frac{(n+2)(n-1)}{2}}$$

é equidimensional com

$$\dim\left(V_n^{(1)}\right) = \frac{(n+2)(n-1)}{2} - (2n-2) = \frac{(n-2)(n-1)}{2}.$$

Para isso, pelas proposições (2.69) e (2.70), é equivalente provar que a variedade

$$V_n^{(2)} := V\left(\sigma_{n2}^{\mathbf{X}}, \sigma_{n3}^{\mathbf{X}}, \ldots, \sigma_{nn-1}^{\mathbf{X}}\right) \subset \mathbb{A}_k^{\frac{(n+2)(n-1)}{2} - n} = \mathbb{A}_k^{\frac{(n-2)(n+1)}{2}}$$

[9]Observemos que as variáveis X_{i1} com $i = 2, 3, \ldots, n$ não aparecem nos polinômios $\sigma_{nj}^{\mathbf{X}}$ para qualquer $j = 2, 3, \ldots, n$.

é equidimensional com

$$\dim\left(V_n^{(2)}\right) = \frac{(n-2)(n+1)}{2} - (n-2) = \frac{(n-2)(n-1)}{2},$$

onde

$$\mathbf{X} = \{X_{1n}, X_{21}, X_{31}, X_{41}, \ldots, X_{n1}\}.$$

Observemos que [10]

$$\sigma_{nj}^{\mathbf{X}} = \sum_{n>t_1>t_2>\cdots>t_{j-1}>1} X_{nt_1}X_{t_1t_2}\cdots X_{t_{j-2}t_{j-1}}X_{t_{j-1}n} \in k[\,I_n^{(1)}\,]; \quad j=2,3,\ldots,n-1$$

pois $t_s \neq 1,\ \forall s = 1,2,\ldots,j-1$ e se considerarmos o isomorfismo de k-álgebras

$$\begin{array}{rccc} \varphi: & k[I_{n-1}] & \longrightarrow & k[\,I_n^{(1)}\,] \\ & X_{ij} & \longmapsto & \varphi(X_{ij}) := X_{i+1,j+1} \end{array},$$

temos que para qualquer $j = 2,3,\ldots,n-1$

$$\begin{aligned}
\varphi(\sigma_{n-1,j}) &= \varphi\left(\sum_{n-1>t_1>t_2>\cdots>t_{j-1}\geq 1} X_{n-1,t_1}X_{t_1t_2}\cdots X_{t_{j-2}t_{j-1}}X_{t_{j-1},n-1}\right) \\
&= \sum_{n-1>t_1>t_2>\cdots>t_{j-1}\geq 1} X_{n,t_1+1}X_{t_1+1,t_2+1}\cdots X_{t_{j-2}+1,t_{j-1}+1}X_{t_{j-1}+1,n} \\
&= \sum_{n>t_1>t_2>\cdots>t_{j-1}>1} X_{nt_1}X_{t_1,t_2}\cdots X_{t_{j-2},t_{j-1}}X_{t_{j-1},n} \\
&= \sigma_{nj}^{\mathbf{X}}.
\end{aligned}$$

Portanto $V_n^{(2)} \cong V_{n-1}$, que por hipótese indutiva é equidimensional com dimensão

$$\dim\left(V_n^{(2)}\right) = \dim(V_{n-1}) = \frac{(n-1)(n-2)}{2}.$$

Para $V(\sigma_{n2}, \sigma_{n3}, \ldots, \sigma_{nn-1}, X_{tt-1}) \subset \mathbb{A}_k^{\frac{(n+2)(n-1)}{2}}$ com $t \in \{2,3,\cdots,n-1\}$, segue da proposição (2.69) e do corolário (2.67), que é suficiente provar que a variedade[11]

$$V_n^{(1)} := V\left(\sigma_{n2}, \sigma_{n3}, \ldots, \sigma_{nn-1}, X_{tt-1}, X_{t1}, X_{t2}, \ldots,, X_{t,t-2}, X_{tn}, X_{t+1,t}, X_{t+2,t}, \ldots,, X_{nt}\right) \subset \mathbb{A}_k^{\frac{(n+2)(n-1)}{2}}$$

é equidimensional com

$$\dim\left(V_n^{(1)}\right) = \frac{(n+2)(n-1)}{2} - (2n-2) = \frac{(n-2)(n-1)}{2}.$$

Para isso, pelas proposições (2.69) e (2.70), é equivalente mostrar que

$$V_n^{(2)} := V\left(\sigma_{n2}^{\mathbf{X}}, \sigma_{n3}^{\mathbf{X}}, \ldots, \sigma_{nn-1}^{\mathbf{X}}\right) \subset \mathbb{A}_k^{\frac{(n+2)(n-1)}{2}-n} = \mathbb{A}_k^{\frac{(n-2)(n+1)}{2}}$$

é equidimensional com

$$\dim\left(V_n^{(2)}\right) = \frac{(n-2)(n+1)}{2} - (n-2) = \frac{(n-2)(n-1)}{2},$$

[10] $\sigma_{nj}^{\mathbf{X}} \neq 0,\ \forall j = 2,3,\ldots,n-1.$

[11] Note que, estamos acrescentando a t-ésima linha e a t-ésima coluna da matriz I_n.

onde
$$\mathbf{X}=\{X_{t\,t-1},X_{t1},X_{t2},\ldots,,X_{t,t-2},X_{tn},X_{t+1,t},X_{t+2,t},\ldots,,X_{nt}\}.$$
Observemos que [12]
$$\sigma^{\mathbf{X}}_{nj}=\sum_{\substack{n>t_1>t_2>\cdots>t_{j-1}\geq 1\\ t_s\neq t}} X_{nt_1}X_{t_1t_2}\cdots X_{t_{j-2}t_{j-1}}X_{t_{j-1}n};\quad j=2,3,\ldots,n-1$$

pois $t_s\neq t,\ \forall s=1,2,\ldots,j-1$ e se considerarmos o isomorfismo de k-álgebras [13]
$$\begin{array}{rccl}\varphi: & k[I_{n-1}] & \longrightarrow & k[\,I_n^{(t)}\,]\\ & X_{ij} & \longmapsto & \varphi(X_{ij}):=\begin{cases} X_{ij}; & \text{se } i,j<t\\ X_{i+1,j}; & \text{se } j<t\leq i\\ X_{i,j+1}; & \text{se } i<t\leq j\\ X_{i+1,j+1}; & \text{se } t\leq i,j\end{cases},\end{array}$$
temos que para qualquer $j=2,3,\ldots,n-1$
$$\begin{aligned}\varphi(\sigma_{n-1,j})&=\varphi\left(\sum_{n-1>t_1>t_2>\cdots>t_{j-1}\geq 1} X_{n-1,t_1}X_{t_1t_2}\cdots X_{t_{j-2}t_{j-1}}X_{t_{j-1},n-1}\right)\\ &=\sum_{\substack{n>t_1>t_2>\cdots>t_{j-1}\geq 1\\ t_s\neq t}} X_{nt_1}X_{t_1,t_2}\cdots X_{t_{j-2},t_{j-1}}X_{t_{j-1},n}\\ &=\sigma^{\mathbf{X}}_{nj}.\end{aligned}$$
Portanto $V_n^{(2)}\cong V_{n-1}$, que por hipótese indutiva é equidimensional com dimensão
$$\dim\left(V_n^{(2)}\right)=\dim(V_{n-1})=\frac{(n-1)(n-2)}{2}.$$

Para $V(\sigma_{n2},\sigma_{n3},\ldots,\sigma_{nn-1},X_{nn-1})\subset\mathbb{A}_k^{\frac{(n+2)(n-1)}{2}}$, como consequência da proposição (2.69) e do corolário (2.67), é suficiente provar que[14]
$$V_n^{(1)}:=V\left(\sigma_{n2},\sigma_{n3},\ldots,\sigma_{nn-1},X_{nn-1},X_{n-1,1},X_{n-1,2},\ldots,X_{n-1,n-2},X_{n-1,n}\right)\subset\mathbb{A}_k^{\frac{(n+2)(n-1)}{2}}$$
é equidimensional com
$$\dim\left(V_n^{(1)}\right)=\frac{(n+2)(n-1)}{2}-(2n-2)=\frac{(n-1)(n-2)}{2}.$$
Pelas proposições (2.69) e (2.70), é equivalente a mostrar que
$$V_n^{(2)}:=V\left(\sigma^{\mathbf{X}}_{n2},\sigma^{\mathbf{X}}_{n3},\ldots,\sigma^{\mathbf{X}}_{nn-1}\right)\subset\mathbb{A}_k^{\frac{(n+2)(n-1)}{2}-n}=\mathbb{A}_k^{\frac{(n-2)(n+1)}{2}},$$

[12] $\sigma^{\mathbf{X}}_{nj}\neq 0$, pois se $j\leq t$, então $\sigma^{\mathbf{X}}_{nj}$ tem o monômio
$$X_{n,j-1}X_{j-1,j-2}X_{j-2,j-3}\cdots X_{32}X_{21}X_{1n}$$
e se $j>t$, então $\sigma^{\mathbf{X}}_{nj}$ tem o monômio
$$X_{n,n-1}X_{n-1,n-2}\cdots X_{n-(j-t+1),n-(j-t)}X_{n-(j-t),t-1}X_{t-1,t-2}X_{t-2,t-3}\cdots X_{32}X_{21}X_{1n}.$$

[13] Aqui estamos usando a correspondência natural de tirar da matriz I_n a t-ésima linha e a t-ésima coluna.

[14] Aqui não podemos usar a mesma técnica dos dois casos anteriores, porque se acrescentarmos a n-ésima coluna ou a n-ésima linha estaríamos eliminando todas as equações, i.e., $\sigma^{\mathbf{X}}_{nj}=0$ para qualquer $j=2,3,\ldots,n-1$ e portanto a sequência $\sigma^{\mathbf{X}}_{n2},\sigma^{\mathbf{X}}_{n3},\sigma^{\mathbf{X}}_{n4},\ldots,\sigma^{\mathbf{X}}_{n,n-1}$ não seria uma sequência regular. Neste caso, só acrescentamos a linha $n-1$ de I_n.

onde $\mathbf{X} = \{X_{n,n-1}, X_{n-1,1}, X_{n-1,2}, \dots, X_{n-1,n-2}, X_{n-1,n}\}$ é equidimensional com

$$\dim\left(V_n^{(2)}\right) = \frac{(n-2)(n+1)}{2} - (n-2) = \frac{(n-2)(n-1)}{2}.$$

Não é difícil notar que para $j = 2, 3, \dots, n-1$

$$\sigma_{nj}^{\mathbf{X}} = \sum_{n-1>t_1>t_2>\cdots>t_{j-1}\geq 1} X_{nt_1} X_{t_1t_2} \cdots X_{t_{j-2}t_{j-1}} X_{t_{j-1}n}.$$

Agora, consideremos o isomomorfismo de k-álgebras [15]

$$\begin{array}{rccl} \varphi: & k[I_{n-1}] & \longrightarrow & k\left[I_n^{(n-1)}\right] \\ & X_{ij} & \longmapsto & \varphi(X_{ij}) := \begin{cases} X_{ij}; & \text{se } i, j \neq n-1 \\ X_{nj}; & \text{se } i = n-1 \\ X_{in}; & \text{se } j = n-1 \end{cases} \end{array}.$$

Assim, para qualquer $j = 2, 3, \dots, n-1$

$$\begin{aligned} \varphi(\sigma_{n-1,j}) &= \varphi\left(\sum_{n-1>t_1>t_2>\cdots>t_{j-1}\geq 1} X_{n-1,t_1} X_{t_1t_2} \cdots X_{t_{j-2}t_{j-1}} X_{t_{j-1},n-1} \right) \\ &= \sum_{n-1>t_1>t_2>\cdots>t_{j-1}\geq 1} X_{nt_1} X_{t_1,t_2} \cdots X_{t_{j-2},t_{j-1}} X_{t_{j-1},n} \\ &= \sigma_{nj}^{\mathbf{X}}. \end{aligned}$$

Portanto $V_n^{(2)} \cong V_{n-1}$. Assim, da hipótese indutiva temos que a variedade $V_n^{(2)}$ é equidimensional com

$$\dim\left(V_n^{(2)}\right) = \dim(V_{n-1}) = \frac{(n-1)(n-2)}{2}.$$

Finalmente, dos três casos concluimos que a variedade V_n é equidimensional com

$$\dim(V_n) = \frac{n(n-1)}{2}.$$

□

3.2.3 Teorema de Kostant-Wallach

Definição 3.20. *Diremos que a variedade algébrica $V(f_1, f_2, \dots, f_t) \subset \mathbb{A}_k^n$ é uma **variedade regular**, se os polinômios $f_1, f_2, \dots, f_t \in k[X_1, X_2, \dots, X_n]$ são todos variáveis, ou seja, para qualquer i*

$$f_i(X_1, X_2, \dots, X_n) = X_j \ , \ \text{para algum } j.$$

Sem perda de generalidade assumiremos que

$$f_i \neq f_j \ , \ \forall i \neq j.$$

Exemplo 3.21. *Claramente as seguintes variedades são regulares:*

$$V(X) \subset \mathbb{A}_k^2,$$

$$V(X_{s_1}, X_{s_2}, \dots, X_{s_t}) \subset \mathbb{A}_k^n \ , \ \text{onde } n \geq t.$$

[15] Aqui estamos só transladando a $(n-1)$-ésima coluna e a $(n-1)$-ésima linha da matriz I_{n-1} para a n-ésima coluna e a n-ésima linha da matriz $I_n^{(n-1)}$, já que em $\sigma_{nj}^{\mathbf{X}}$ não aparece nenhuma variável do tipo $X_{n-1,i}$ ou $X_{i,n-1}$.

Observação 3.22. *Sejam $V(X_{i_1}, X_{i_2}, \dots, X_{i_r})$ e $V(X_{j_1}, X_{j_2}, \dots, X_{j_r})$ duas subvariedades regulares de uma variedade $V \in \mathbb{A}^n$, então existe uma permutação $\sigma \in S_n$ tal que*

$$X_{\sigma(i_t)} = X_t, \ \forall t = 1, 2, \dots, r.$$

Portanto, temos o isomorfismo de k-álgebras

$$\begin{array}{rccc} \varphi_\sigma : & k[X_1, X_2, \dots, X_n] & \longrightarrow & k[X_1, X_2, \dots, X_n] \\ & X_i & \longmapsto & \varphi_\sigma(X_i) := X_{\sigma(i)} \end{array},$$

o qual induz o isomorfismo de variedades

$$\begin{array}{rccc} f : & V(X_{i_1}, X_{i_2}, \dots, X_{i_r}) & \longrightarrow & V(X_{j_1}, X_{j_2}, \dots, X_{j_r}) \\ & (a_1, a_2, \dots, a_n) & \longmapsto & f(a_1, a_2, \dots, a_n) := (a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}) \end{array}.$$

Observemos que $\varphi_\sigma\big(I\big(V(X_{j_1}, X_{j_2}, \dots, X_{j_r})\big)\big) = I\big(V(X_{i_1}, X_{i_2}, \dots, X_{i_r})\big)$.

"$\subseteq$" *Sejam $p \in \varphi_\sigma\big(I\big(V(X_{j_1}, X_{j_2}, \dots, X_{j_r})\big)\big)$ e $a = (a_1, a_2, \dots, a_n) \in V(X_{i_1}, X_{i_2}, \dots, X_{i_r})$, segue-se que existe $q \in I\big(V(X_{j_1}, X_{j_2}, \dots, X_{j_r})\big)$ tal que $\varphi_\sigma(q) = p$, e como $f(a) \in V(X_{j_1}, X_{j_2}, \dots, X_{j_r})$, então*

$$p(a) = \varphi_\sigma(q)(a) = q(X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(n)})(a) = q(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}) = q(f(a)) = 0.$$

"$\supseteq$" *Seja $p \in I\big(V(X_{i_1}, X_{i_2}, \dots, X_{i_r})\big)$, vejamos que $q = \varphi_\sigma^{-1}(p) \in I\big(V(X_{j_1}, X_{j_2}, \dots, X_{j_r})\big)$, para isto consideremos $a \in V(X_{j_1}, X_{j_2}, \dots, X_{j_r})$, logo $f^{-1}(a) \in V(X_{i_1}, X_{i_2}, \dots, X_{i_r})$ e portanto*

$$q(a) = \varphi^{-1}(p)(a) = p(X_{\sigma^{-1}(1)}, X_{\sigma^{-1}(2)}, \dots, X_{\sigma^{-1}(n)})(a) = p(a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \dots, a_{\sigma^{-1}(n)})$$

$$q(a) = p(f^{-1}(a)) = 0.$$

Agora, como $\varphi_\sigma\big(I\big(V(X_{j_1}, X_{j_2}, \dots, X_{j_r})\big)\big) = I\big(V(X_{i_1}, X_{i_2}, \dots, X_{i_r})\big)$, então $I\big(V(X_{j_1}, X_{j_2}, \dots, X_{j_r})\big)$ é um ideal primo minimal se, e somente se, $I\big(V(X_{i_1}, X_{i_2}, \dots, X_{i_r})\big)$ é um ideal primo minimal. Equivalentemente, $V(X_{j_1}, X_{j_2}, \dots, X_{j_r})$ é uma componente irredutível se, e somente se, $V(X_{i_1}, X_{i_2}, \dots, X_{i_r})$ é uma componente irredutível.

O Teorema de Kostant-Wallach em [Kostant e Wallach (2006a)] e [Kostant e Wallach (2006b)] afirma que todas as componentes regulares são equidimensionais com dimensão $\frac{n(n-1)}{2}$. Mas, como consequência da versão fraca temos o seguinte corolário:

Corolário 3.23. *Todas as componentes regulares da variedade de Gelfand-Tsetlin são isomorfas. Em particular, são isomorfas à componente irredutível*

$$V_{\leq} := V\big(\{X_{ij} : 1 \leq i \leq j \leq n\}\big).$$

Demonstração:
Consideremos uma componente regular da variedade de Gelfand-Tsetlin da forma

$$V\big(f_1, f_2, \dots, f_{d(n)}\big) \subset \mathbb{A}_k^{n^2},$$

vejamos que

$$d(n) = \frac{n(n+1)}{2}.$$

Claramente para qualquer $i \in \{1, 2, \dots, n\}$, existe $t \in \{1, 2, \dots, d(n)\}$ tal que

$$f_t = X_{ii},$$

caso contrário existiria $A \in V\left(f_1, f_2, \ldots, f_{d(n)}\right) \subset \mathbb{A}_k^{n^2}$ tal que

$$\overline{\gamma}_{i1}(A) = A_{ii} \neq 0,$$

por exemplo a matriz elementar [16] E_{ii}.
Também temos que, para quaisquer $i, j \in \{1, 2, \ldots, n\}$ com $i \neq j$, existe $t \in \{1, 2, \ldots, d(n)\}$ tal que

$$f_t = X_{ij} \ \text{ ou } \ f_t = X_{ji},$$

caso contrário, se $i > j$ existiria $A \in V\left(f_1, f_2, \ldots, f_{d(n)}\right) \subset \mathbb{A}_k^{n^2}$ tal que

$$\overline{\gamma}_{i2}(A) = \sum_{t=1}^{i-1} A_{it}A_{ti} = \sum_{t=1}^{j-1} A_{it}A_{ti} + \underbrace{A_{ij}A_{ji}}_{\neq 0} + \sum_{t=j+1}^{i-1} A_{it}A_{ti} \neq 0,$$

por exemplo $A = E_{ij} + E_{ji}$. De maneira similar, se $i < j$ então

$$d(n) \geq \frac{n(n+1)}{2}.$$

Agora reparemos que se $d(n) > \dfrac{n(n+1)}{2}$, então existem $i, j \in \{1, 2, \ldots, N\}$ com $i > j$ tais que

$$X_{ij}, X_{ji} \in \left\{f_1, f_2, \ldots, f_{d(N)}\right\}.$$

Consideremos um conjunto $S \subset \left\{X_{ij} : i, j = 1, 2, \ldots, N\right\}$ com as seguintes condições:

1. S tem $d(n)$ elementos.
2. $\left\{X_{ij} : 1 \leq i \leq j \leq n\right\} \subset S$.

Claramente $V(S)$ é uma subvariedade da variedade de Gelfand-Tsetlin e da observação anterior (3.22)

$$V\left(f_1, f_2, \ldots, f_{d(n)}\right) \cong V(S).$$

Notemos que $V(S)$ é irredutível, mas, não é uma componente, já que não é uma subvariedade maximal

$$V(S) \subsetneq V_{\leq},$$

o qual contradiz a hipótese de que $V\left(f_1, f_2, \ldots, f_{d(n)}\right)$ seja uma componente regular.
Assim temos que se $V\left(f_1, f_2, \ldots, f_{d(n)}\right)$ é uma componente regular, então

$$d(n) = \frac{n(n+1)}{2}$$

e portanto,

$$\dim\left(V\left(f_1, f_2, \ldots, f_{d(n)}\right)\right) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Além disso, da observação (3.22)

$$V\left(f_1, f_2, \ldots, f_{d(n)}\right) \cong V_{\leq}.$$

□

[16] Aqui estamos visualizando os elementos no espaço afim $\mathbb{A}_k^{n^2}$ como matrizes quadradas $n \times n$ e, por matriz elementar E_{ij} entenderemos a matriz com 1 na entrada (i, j) e zero no resto.

Capítulo 4

Equidimensionalidade das fibras da aplicação de Kostant-Wallach e da aplicação parcial de Kostant-Wallach

Começaremos este capítulo definindo PBW álgebras e álgebras filtradas especiais, além de mostrar alguns resultados sobre estas álgebras. Seguiremos com a aplicação de Kostant-Wallach

$$\varphi : M_n(\mathbb{C}) \longrightarrow \mathbb{C}^{\frac{n(n+1)}{2}},$$

a aplicação de k-parcial de Kostant-Wallach

$$\varphi_k : M_n(\mathbb{C}) \longrightarrow \mathbb{C}^{n-k+1} \times \cdots \times \mathbb{C}^{n-1} \times \mathbb{C}^n$$

e a relação das fibras em zero $\varphi^{-1}(0)$ e $\varphi_k^{-1}(0)$ com a variedade de Gelfand-Tsetlin. Encerraremos este capítulo provando o seguinte teorema:

Teorema:

Para todo $k = 1, 2, \ldots, n$ e todo $\beta \in \mathbb{C}^{n-k+1} \times \cdots \times \mathbb{C}^{n-1} \times \mathbb{C}^n$ a variedade

$$\widetilde{V}_\beta^k = V\left(\left\{\overline{\gamma_{ij}} - \beta_{ij} : \ n-k+1 \leq i \leq n \ \text{ e } \ 1 \leq j \leq i\right\}\right) \subset \mathbb{C}^{n^2}$$

é equidimensional com dimensão

$$\dim\left(\widetilde{V}_\beta^k\right) = n^2 - nk + \frac{k(k-1)}{2}.$$

O qual, claramente é uma extensão do Teorema de Ovsienko. Com este teorema obtemos como corolários a equidimensionalidade das fibras $\varphi^{-1}(\alpha)$ e $\varphi_k^{-1}(\alpha)$.

Este teorema e seus corolários, são os resultados principais deste capítulo.

4.1 PBW álgebras e álgebras filtradas especiais

Definição 4.1. *Seja U uma álgebra associativa filtrada sobre k, isto é, possui uma filtração crescente $\{U_i\}_{i\geq 0}$, em que*

$$U_0 = k,$$

$$U_i U_j \subseteq U_{i+j}$$

e

$$U = \bigcup_{i=0}^{\infty} U_i.$$

Para $u \in U_i/U_{i-1}$ *denotemos o grau de* u *por* $\deg(u) = i$ *e por convenção consideraremos* $U_{-1} = \{0\}$.

Da observação (2.7)*, a álgebra graduada associada a* U *é*

$$\overline{U} = \mathrm{gr}(U) = \bigoplus_{i=0}^{\infty} U_i/U_{i-1}.$$

Para $u \in U$ *denote por* $\overline{u}$ *sua imagem em* $\overline{U}$.

Uma álgebra U *é chamada de* ***PBW álgebra****, se qualquer elemento de* U *pode ser escrito de forma única como uma combinação linear de monômios ordenados em alguns geradores fixados de* U*. Se* U *é uma PBW álgebra tal que* $\mathrm{gr}(U)$ *é uma álgebra de polinômios, então* U *será chamada de* ***filtrada especial*** *ou* ***PBW álgebra especial****.*

Observação 4.2. *Note que pelo Teorema de Poincaré-Birkhoff-Witt* (2.28)*, a álgebra envolvente universal de qualquer álgebra de Lie de dimensão finita é uma álgebra filtrada especial.*

Consideremos a seguinte aplicação

$$(\ ,\ )_U : U^t \times U^t \longrightarrow U,$$

tal que, para $\mathbf{u} = (u_1, u_2, \ldots, u_t)$ e $\mathbf{v} = (v_1, v_2, \ldots, v_t)$ em U^t está dada por

$$(\mathbf{u}, \mathbf{v})_U = \sum_{i=1}^{t} u_i v_i.$$

Lema 4.3. *Sejam* U *uma álgebra filtrada especial,* $g_1, g_2, \ldots, g_t \in U$ *elementos mutuamente comutativos tais que,* $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_t$ *é uma sequência regular em* $\mathrm{gr}(U)$ *e* $I \subseteq U$ *o ideal à esquerda de* U *gerado por* $g_1, g_2, \ldots, g_t$*. Então, qualquer* $f \in I$ *pode ser escrito na forma*

$$f = \sum_{j=1}^{t} f_j g_j,$$

para alguns $f_j \in U$, $j = 1, 2, \ldots, t$*, tais que*

$$\deg(f) = \max_{1 \le j \le t} \deg(f_j g_j).$$

Demonstração: Seja $f \in I$ e consideremos $f = f_1 g_1 + f_2 g_2 + \cdots + f_t g_t$ com o menor possível

$$d = \max_{1 \le i \le t} \deg(f_i g_i).$$

Podemos assumir que $d = \deg(f_i g_i)$ se, e somente se, $i = 1, 2, \ldots, r$ para algum $r \le t$. Consideremos

$$\mathbf{f} = (f_1, f_2, \ldots, f_t),$$

$$\mathbf{g} = (g_1, g_2, \ldots, g_t)$$

e para um vetor $\mathbf{s} = (s_1, s_2, \ldots, s_t) \in U^t$ denotemos $\overline{\mathbf{s}} = (\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_r, 0, 0, \ldots, 0) \in \overline{U}^t$.

Se $d > \deg(f)$, do fato que $(\mathbf{f}, \mathbf{g})_U = f$ segue que $(\overline{\mathbf{f}}, \overline{\mathbf{g}})_{\overline{U}} = 0$ e da resolução de Koszul (prop. (2.68)) temos $\eta(\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_r) = 0$, isto implica que

$$(\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_r) \in \ker(\eta).$$

Dado que U é uma álgebra filtrada especial, então pelo corolário (2.67) as sequências $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_r$ são também regulares em $\overline{U} = \mathrm{gr}\,(U)$. Novamente, da resolução de Koszul (prop. (2.68)) temos

$$(\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_r) \in \partial\left(\bigwedge^2 \overline{U}^r\right),$$

portanto, existem $g_{ij} \in U$ com $1 \le i < j \le r$ tais que

$$\bar{\mathbf{f}} = \sum_{1 \le i < j \le r} \overline{g}_{ij}(\overline{g}_i \mathbf{e}_j - \overline{g}_j \mathbf{e}_i),$$

em que $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r$ é a base canônica de $\overline{U}^r$.

Agora, consideremos

$$\mathbf{k} = \sum_{1 \le i < j \le r} g_{ij}(g_i \mathbf{e}_j - g_j \mathbf{e}_i) \in U^t,$$

logo $\bar{\mathbf{f}} = \overline{\mathbf{k}}$, assim para $\mathbf{h} = (h_1, h_2, \ldots, h_t) = \mathbf{f} - \mathbf{k}$ tem-se que

$$\deg(h_i) \le \deg(f_i) \text{ para cada } i = 1, 2, \ldots, t$$

e

$$\deg(h_i) < \deg(f_i) \text{ para cada } i = 1, 2, \ldots, r.$$

Note que $(g_i \mathbf{e}_j - g_j \mathbf{e}_i, \mathbf{g})_U = g_i g_j - g_j g_i = 0$, logo

$$(\mathbf{k}, \mathbf{g})_U = 0.$$

Então

$$\sum_{i=1}^{t} h_i g_i = (\mathbf{h}, \mathbf{g})_U = (\mathbf{f} - \mathbf{k}, \mathbf{g})_U = (\mathbf{f}, \mathbf{g})_U = f.$$

Dado que

$$\max_{1 \le i \le t} \deg\,(h_i g_i) < \max_{1 \le i \le t} \deg\,(f_i g_i),$$

temos que isso contradiz a minimalidade de d, e portanto obtemos que

$$d = \deg\,(f).$$

□

Observação 4.4. *Antes da seguinte proposição, note-se que se U é uma álgebra filtrada especial, a graduação sobre $\overline{U} = \mathrm{gr}\,(U)$ em geral não coincide com a graduação standard da álgebra de polinômios. Em particular, se $d_1, d_2, \ldots, d_n$ são inteiros positivos então $\Lambda = \overline{\Lambda} = \mathbb{K}[X_1, X_2, \ldots, X_n]$, munida com uma graduação dada por $\deg\,(X_i) = d_i$, é uma álgebra filtrada especial com sua correspondente filtração. Quando todos os $d_i = 1$, claramente obtemos a graduação standard sobre Λ.*

Proposição 4.5. *Sejam $\Lambda = k[X_1, X_2, \ldots, X_n]$ e $g_1, g_2, \ldots, g_t \in \Lambda$ tais que a sequência $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_t$ é regular em $\Lambda = \mathrm{gr}\,(\Lambda)$. Então, a sequência $g_1, g_2, \ldots, g_t$ é também regular em Λ.*

Demonstração: Suponha que a sequência $g_1, g_2, \ldots, g_t$ não é regular em Λ, da definição de sequência regular (2.64), existe $i \in \{1, 2, \ldots, t\}$ tal que g_i é um divisor de zero ou um elemento inversível em $\Lambda/(g_1, g_2, \ldots, g_{i-1})$.

Se g_i é um divisor de zero em $\Lambda/(g_1, g_2, \ldots, g_{i-1})$, então existe $f \in \Lambda$ tal que $f \notin (g_1, g_2, \ldots, g_{i-1})$ e $f g_i \in (g_1, g_2, \ldots, g_{i-1})$, isto implica que

$$f g_i = \sum_{j=1}^{i-1} f_j g_j, \text{ para alguns } f_j \in \Lambda.$$

Consideremos f com grau o menor possível, pelo lema (4.3)

$$\deg(fg_i) = \max_{1\leq j\leq i-1} \deg(f_j g_j),$$

assim $\overline{f}\,\overline{g}_i$ é zero em $\Lambda/(\overline{g}_1, \overline{g}_2, \dots, \overline{g}_{i-1})$. Como a sequência $\overline{g}_1, \overline{g}_2, \dots, \overline{g}_t$ é regular, então da definição de sequência regular (2.64), tem-se que $\overline{g}_1, \overline{g}_2, \dots, \overline{g}_{i-1}$ é também uma sequência regular. Portanto,

$$\overline{f} = 0 \quad \text{em} \quad \Lambda/(\overline{g}_1, \overline{g}_2, \dots, \overline{g}_{i-1}),$$

disso

$$\overline{f} = \sum_{j=1}^{i-1} \overline{h}_j \overline{g}_j, \quad \text{para alguns} \quad h_j \in \Lambda.$$

Como consequência

$$f' = f - \sum_{j=1}^{i-1} h_j g_j$$

tem um grau menor que f. Por outro lado, como $f \notin (g_1, g_2, \dots, g_{i-1})$ então $f' \notin (g_1, g_2, \dots, g_{i-1})$ e além disso $f'g_i \in (g_1, g_2, \dots, g_{i-1})$. Isso contradiz a minimalidade de f e portanto g_i não é um divisor de zero em $\Lambda/(g_1, g_2, \dots, g_{i-1})$. O caso quando a imagem de g_i em $\Lambda/(g_1, g_2, \dots, g_{i-1})$ seja inversível é análogo.

□

4.2 Aplicação de Kostant-Wallach e aplicação parcial de Kostant-Wallach

Primeiro fixaremos a notação. Seja $n \in \mathbb{Z}_{>0}$. Para cada matriz

$$X = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{pmatrix} \in M_n(\mathbb{C}).$$

Denotemos:

1. A i-ésima submatriz principal de X com $i = 1, 2, \dots, n$ por

$$X_i := \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1i} \\ X_{21} & X_{22} & \cdots & X_{2i} \\ \vdots & \vdots & \ddots & \vdots \\ X_{i1} & X_{i2} & \cdots & X_{ii} \end{pmatrix} \in M_i(\mathbb{C}).$$

2. O coeficiente de t^{i-j} do polinômio carácteristico de X_i com $i = 1, 2, \dots, n$ e $j = 1, 2, \dots, i$ por

$$\chi_{ij}(X),$$

i.e., o polinômio característico da i-ésima submatriz principal X_i é

$$t^i + \chi_{i1}(X)t^{i-1} + \chi_{i2}(X)t^{i-2} + \cdots + \chi_{i\,i-2}(X)t^2 + \chi_{i\,i-1}(X)t^1 + \chi_{ii}(X)t^0.$$

3. E por $\chi_i(X)$ o vetor

$$\chi_i(X) := \big(\chi_{i1}(X), \chi_{i2}(X), \dots, \chi_{ii}(X)\big) \in \mathbb{C}^i.$$

Definição 4.6.

1. *A aplicação definida como*

$$\begin{array}{rccl} \varphi: & M_n(\mathbb{C}) & \longrightarrow & \mathbb{C}^{\frac{n(n+1)}{2}} \\ & X & \longmapsto & \varphi(X) := \big(\chi_1(X), \chi_2(X), \dots, \chi_n(X)\big) \end{array}$$

   é conhecida como ***aplicação de Kostant-Wallach****.*

2. *Para cada $k = 1, 2, \dots, n$ definiremos a* ***aplicação** k**-parcial de Kostant-Wallach*** *pelo morfismo*

$$\varphi_k := \pi_k \circ \varphi,$$

   em que $\pi_k : \mathbb{C} \times \mathbb{C}^2 \times \cdots \mathbb{C}^n \longrightarrow \mathbb{C}^{n-k+1} \times \cdots \times \mathbb{C}^{n-1} \times \mathbb{C}^n$ é a projeção canônica sobre os últimos k vetores. Ou seja,

$$\begin{array}{rccl} \varphi_k: & M_n(\mathbb{C}) & \longrightarrow & \mathbb{C}^{n-k+1} \times \cdots \times \mathbb{C}^{n-1} \times \mathbb{C}^n \\ & X & \longmapsto & \varphi_k(X) := \big(\chi_{n-k+1}(X), \dots, \chi_{n-1}(X), \chi_n(X)\big). \end{array}$$

Observação 4.7. *Claramente $\varphi_n = \varphi$.*

Definição 4.8.

1. *Para cada $\alpha \in \mathbb{C}^{\frac{n(n+1)}{2}}$, a* ***fibra em** α **de*** *φ é dada por*

$$\varphi^{-1}(\alpha) := \big\{X \in M_n(\mathbb{C}) \ / \ \varphi(X) = \alpha\big\} \subset \mathbb{C}^{n^2},$$

   a qual claramente é uma variedade algébrica em $\mathbb{C}^{n^2}$.

2. *Similarmente, para $\alpha \in \mathbb{C}^{n-k+1} \times \cdots \times \mathbb{C}^{n-1} \times \mathbb{C}^n$ a* ***fibra em** α **de*** *φ_k é dada por*

$$\varphi_k^{-1}(\alpha) := \big\{X \in M_n(\mathbb{C}) \ / \ \varphi_k(X) = \alpha\big\} \subset \mathbb{C}^{n^2},$$

   a qual claramente também é uma variedade algébrica em $\mathbb{C}^{n^2}$.

Proposição 4.9 (Teorema de Colarusso-Evens, 2015).
A fibra em $\alpha \in \mathbb{C}^{n-1} \times \mathbb{C}^n$ da aplicação 2-parcial de Kostant-Wallach φ_2 é equidimensional com dimensão

$$\dim(\varphi_2^{-1}(\alpha)) = n^2 - 2n + 1.$$

Demonstração: Ver [Colarusso e Evens (2015)].

□

4.3 Aplicação de Kostant-Wallach vs Variedades de Gelfand-Tseltin

Podemos visualizar as fibras em zero da aplicação de Kostant-Wallach e das aplicações parciais de Kostant-Wallach como variedades de Gelfand-Tsetlin da seguinte forma:

Definição 4.10. *Para $k = 1, 2, \dots, n$ e $\beta \in \mathbb{C}^{n-k+1} \times \cdots \times \mathbb{C}^{n-1} \times \mathbb{C}^n$, definiremos a* ***variedade** k**-parcial de Gelfand-Tsetlin em*** *β como a variedade algébrica*

$$\widetilde{V}_\beta^k = V\big(\big\{\overline{\gamma}_{ij} - \beta_{ij} : \ n-k+1 \le i \le n \ \ e \ \ 1 \le j \le i\big\}\big) \subset \mathbb{C}^{n^2}.$$

Observação 4.11. *Claramente*

$$\widetilde{V}_0^n = V\big(\big\{\overline{\gamma}_{ij} : \ 1 \le j \le i \le n\big\}\big)$$

é a variedade de Gelfand-Tsetlin para $\mathfrak{gl}_n$.

Proposição 4.12. *Para cada* $i = 1, 2, \dots, n$

$$V\left(\chi_{i1}, \chi_{i2}, \dots, \chi_{ii}\right) = V\left(\overline{\gamma}_{i1}, \overline{\gamma}_{i2}, \dots, \overline{\gamma}_{ii}\right).$$

Demonstração: Ver [Želobenko (1973)] página 170 ou [Ovsienko (2003)].

□

A relação entre a aplicação de Kostant-Wallach e a variedade de Gelfand-Tseltin está determinada pelo seguinte corolário.

Corolário 4.13. *Para todo* $k = 1, 2, \dots, n$*, a variedade* k*-parcial de Gelfand-Tsetlin em zero coincide com a fibra em zero da aplicação* k*-parcial de Kostant-Wallach, i.e.,*

$$\widetilde{V}_0^k = \varphi_k^{-1}(0).$$

Demonstração: Dado que

$$\varphi_k^{-1}(0) = \left\{X \in M_n(\mathbb{C}) \ / \ \varphi_k(X) = 0\right\}$$

e

$$\begin{array}{rccl} \varphi_k: & M_n(\mathbb{C}) & \longrightarrow & \mathbb{C}^{n-k+1} \times \cdots \times \mathbb{C}^{n-1} \times \mathbb{C}^n \\ & X & \longmapsto & \varphi(X) := \left(\chi_{n-k+1}(X), \dots, \chi_{n-1}(X), \chi_n(X)\right), \end{array}$$

então

$$\begin{aligned} \varphi_k^{-1}(0) &= V\left(\left\{\chi_{ij} : \ n-k+1 \le i \le n \ \text{ e } \ 1 \le j \le i\right\}\right) \\ &= \bigcap_{i=n-k+1}^{n} V\left(\chi_{i1}, \chi_{i2}, \dots, \chi_{ii}\right) \\ &= \bigcap_{i=n-k+1}^{n} V\left(\overline{\gamma}_{i1}, \overline{\gamma}_{i2}, \dots, \overline{\gamma}_{ii}\right) \\ &= V\left(\left\{\overline{\gamma_{ij}} : \ n-k+1 \le i \le n \ \text{ e } \ 1 \le j \le i\right\}\right) \\ &= \widetilde{V}_0^k. \end{aligned}$$

□

Observação 4.14. *Segue do corolário anterior, que a variedade de Gelfand-Tsetlin coincide com a fibra em zero da aplicação de Kostant-Wallach* $\varphi^{-1}(0)$*. Como consequência desse fato, a variedade de Gelfand-Tsetlin para* $\mathfrak{gl}_n(\mathbb{C})$ *é exatamente o conjunto das matrizes quadradas* $n \times n$ *fortemente nilpotentes (em que, uma matriz* $X \in M_n(\mathbb{C})$ *é dita* ***fortemente nilpotente****, se todas suas* i*-ésimas submatrizes principais* X_i *são nilpotentes).*

4.4 Uma generalização do Teorema de Colarusso-Evens

Nesta seção vamos provar uma generalização do Teorema de Colarusso-Evens.

Teorema 4.15. *Para todo* $k = 1, 2, \dots, n$ *e todo* $\beta \in \mathbb{C}^{n-k+1} \times \cdots \times \mathbb{C}^{n-1} \times \mathbb{C}^n$*, a variedade* k*-parcial de Gelfand-Tsetlin* $\widetilde{V}_\beta^k$ *é equidimensional com dimensão*

$$\dim\left(\widetilde{V}_\beta^k\right) = n^2 - nk + \frac{k(k-1)}{2}.$$

Em particular, para todo $\alpha \in \mathbb{C}^{\frac{n(n+1)}{2}}$ *a variedade* n*-parcial de Gelfand-Tsetlin*

$$\widetilde{V}_\alpha^n = \left\{x \in \mathbb{C}^{n^2} : \overline{\gamma}_{ij}(x) = \alpha_{ij}, \ \ \forall i = 1, 2, \dots, n; \ \forall j = 1, 2, \dots, i\right\}$$

é equidimensional com dimensão

$$\dim\left(\widetilde{V}_\alpha^n\right) = \frac{n(n-1)}{2}.$$

Demonstração: Consideremos a sequência em $\mathrm{gr}\,(U(\mathfrak{gl}_n(\mathbb{C}))$

$$\left\{\sigma_{ij} := \overline{\gamma_{ij}} - \beta_{ij} \,:\, 1 \le j \le i,\ n-k+1 \le i \le n\right\}.$$

Notemos que para $1 \le j \le i$ e $n-k+1 \le i \le n$ tem-se

$$\overline{\sigma_{ij}} = \overline{\overline{\gamma_{ij}} - \beta_{ij}} = \overline{\overline{\gamma_{ij}}} = \overline{\gamma_{ij}}$$

em $\mathrm{gr}\,(U(\mathfrak{gl}_n(\mathbb{C})) = \mathrm{gr}\,(\mathrm{gr}\,(U(\mathfrak{gl}_n(\mathbb{C})))$. Assim, a sequência

$$\left\{\overline{\sigma_{ij}} = \overline{\gamma_{ij}} \,:\, 1 \le j \le i,\ n-k+1 \le i \le n\right\}$$

é uma subsequência de

$$\left\{\overline{\gamma_{ij}} \,:\, 1 \le j \le i \le n\right\},$$

a qual, pelo Teorema de Ovsienko (3.15) e pela proposição (2.69), é regular em $\mathrm{gr}\,(U(\mathfrak{gl}_n(\mathbb{C})))$. Logo, do corolário (2.67), tem-se que

$$\left\{\overline{\sigma_{ij}} = \overline{\gamma_{ij}} \,:\, 1 \le j \le i,\ n-k+1 \le i \le n\right\}$$

é uma sequência regular em $\mathrm{gr}\,(U(\mathfrak{gl}_n(\mathbb{C})) = \mathrm{gr}\,(\mathrm{gr}\,(U(\mathfrak{gl}_n(\mathbb{C})))$.

Como consequência da proposição (4.5), temos que

$$\left\{\sigma_{ij} := \overline{\gamma_{ij}} - \beta_{ij} \,:\, 1 \le j \le i,\ n-k+1 \le i \le n\right\}$$

é uma sequência regular em $\mathrm{gr}\,(U(\mathfrak{gl}_n(\mathbb{C})))$. Portanto, das proposições (2.68) e (2.69),

$$\widetilde{V}^k_\beta = V\left(\left\{\overline{\gamma_{ij}} - \beta_{ij} :\ 1 \le j \le i,\ n-k+1 \le i \le n\right\}\right) \subset \mathbb{C}^{n^2}$$

é uma variedade equidimensional com dimensão

$$\dim\left(\widetilde{V}^k_\beta\right) = n^2 - n - (n-1) - (n-2) - \cdots - (n-k+1) = n^2 - nk + \frac{k(k-1)}{2}.$$

□

Agora podemos concluir a equidimensionalidade das fibras da aplicação de Kostant-Wallach e de suas aplicações parciais.

Corolário 4.16. *Para cada $k = 1, 2, \ldots, n$ e cada $\beta \in \mathbb{C}^{n-k+1} \times \cdots \times \mathbb{C}^{n-1} \times \mathbb{C}^n$, a fibra $\varphi_k^{-1}(\beta)$ da aplicação k-parcial de Kostant-Wallach φ_k é equidimensional com dimensão*

$$\dim(\varphi_k^{-1}(\beta)) = n^2 - nk + \frac{k(k-1)}{2}.$$

Demonstração: Note-se que os polinômios χ_{ij} são todos homogêneos em $\Lambda = \mathbb{C}[X_{ij} : 1 \le i, j \le n]$ e portanto podemos fazer o mesmo raciocinio feito na prova do teorema anterior (4.15).

Consideremos a sequência

$$\left\{\chi_{ij} - \beta_{ij} : n-k+1 \le i \le n \ \text{ e } \ 1 \le j \le i\right\}$$

e a graduação $\overline{\Lambda} = \Lambda$ de Λ dada pelo grau do polinômio (i.e., $\deg(X_i^t) = t$).

Para $n-k+1 \le i \le n$ e $1 \le j \le i$ tem-se

$$\overline{\chi_{ij} - \beta_{ij}} = \overline{\chi_{ij}} = \chi_{ij}.$$

Segue do corolário (4.13), que

$$\widetilde{V}^k_0 = \varphi_k^{-1}(0) = V\left(\left\{\chi_{ij} : n-k+1 \le i \le n \ \text{ e } \ 1 \le j \le i\right\}\right),$$

a qual pelo teorema (4.15), é equidimensional com dimensão

$$n^2 - nk + \frac{k(k-1)}{2}.$$

Logo da proposição (2.69) a sequência

$$\left\{\overline{\chi_{ij} - \beta_{ij}} = \chi_{ij} : n-k+1 \leq i \leq n \text{ e } 1 \leq j \leq i\right\}$$

é regular em $\overline{\Lambda}$ e da proposição (4.5) podemos concluir que a sequência

$$\left\{\chi_{ij} - \beta_{ij} : n-k+1 \leq i \leq n \text{ e } 1 \leq j \leq i\right\}$$

é regular em Λ. Finalmente das proposições (2.68) e (2.69), a variedade

$$\varphi_k^{-1}(\beta) = V\left(\left\{\chi_{ij} - \beta_{ij} : n-k+1 \leq i \leq n \text{ e } 1 \leq j \leq i\right\}\right)$$

é equidimensional com dimensão

$$\dim(\varphi_k^{-1}(\beta)) = n^2 - nk + \frac{k(k-1)}{2}.$$

□

Corolário 4.17. *Para cada $\alpha \in \mathbb{C}^{\frac{n(n+1)}{2}}$, a fibra $\varphi^{-1}(\alpha)$ da aplicação de Kostant-Wallach φ é equidimensional com dimensão*

$$\dim(\varphi^{-1}(\alpha)) = \frac{n(n-1)}{2}.$$

Demonstração: Segue do fato que $\varphi_n = \varphi$.

□

Capítulo 5

Versão fraca $\mathfrak{G}_1$ para Yangians $Y_p(\mathfrak{gl}_3)$

Começaremos este capítulo com uma seção onde definiremos Yangians restritos $Y_p(\mathfrak{gl}_n)$ de nível p para $\mathfrak{gl}_n$, sua subálgebra de Gelfand-Tsetlin Γ e sua variedade de Gelfand-Tsetlin $\mathfrak{G}$. Dada a dificuldade na combinatória dos polinômios que determinam $\mathfrak{G}$, na segunda seção apresentamos uns polinômios um pouco mais acessíveis, os quais também determinam a variedade de Gelfand-Tsetlin $\mathfrak{G}$ para $Y_p(\mathfrak{gl}_3)$ e aproveitamos tais polinômios para apresentar a decomposição em componentes irredutíves de $\mathfrak{G}$ para Yangians $Y_1(\mathfrak{gl}_3)$ e $Y_2(\mathfrak{gl}_3)$. Esta decomposição permitirá concluir nesses casos a equidimensionalidade de $\mathfrak{G}$. Na terceira seção, apresentamos uma decomposição de $\mathfrak{G}$ para $Y_p(\mathfrak{gl}_n)$ da forma

$$\mathfrak{G} = \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \cdots \cup \mathfrak{G}_{p+1}.$$

Essa decomposição não é em componentes irredutíveis, pois na mesma seção mostraremos uma decomposição da subvariedade $\mathfrak{G}_1$ de $\mathfrak{G}$ para $Y_p(\mathfrak{gl}_n)$ na forma

$$\mathfrak{G}_1 = \mathbb{W}_1 \cup \mathbb{W}_2, \text{ se } p = 1,2$$

$$\mathfrak{G}_1 = \mathbb{W}^p \cup \mathbb{W}_1 \cup \mathbb{W}_2, \text{ se } p \geq 3.$$

Quando $n = 3$, a variedade $\mathfrak{G}_1$ será chamada de *versão fraca da variedade de Gelfand-Tsetlin* $\mathfrak{G}$ para $Y_p(\mathfrak{gl}_3)$. Encerramos este capítulo provando a equidimensionalidade de $\mathbb{W}^p$, $\mathbb{W}_1$ e $\mathbb{W}_2$ e obtendo portanto o resultado principal deste capítulo.

Teorema: Para todo p, a variedade $\mathfrak{G}_1$ para $Y_p(\mathfrak{gl}_3)$ é equidimensional com dimensão

$$\dim \mathfrak{G}_1 = 3p.$$

5.1 Yangians restritos $Y_p(\mathfrak{gl}_n)$ de nível p para $\mathfrak{gl}_n$

Agora vamos introduzir o objeto que queremos estudar, já que tudo o apresentado nos capítulos 3 e 4 foi encontrado, na procura de técnicas e estratégias para nosso objetivo principal, que é Yangians.

Sejam p e n inteiros positivos. O **Yangian**[1] $Y_p(\mathfrak{gl}_n)$ **de nível** p para a álgebra de Lie $\mathfrak{gl}_n$, pode ser definido como a álgebra associativa com geradores $t_{ij}^{(1)}, t_{ij}^{(2)}, \ldots, t_{ij}^{(p)}$, $i, j = 1, 2, \ldots, n$ e com as relações

$$[T_{ij}(u), T_{kl}(v)] = \frac{1}{u-v}\big(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)\big),$$

onde u, v são variáveis formais e

$$T_{ij}(u) = \delta_{ij} u^p + \sum_{k=1}^{p} t_{ij}^{(k)} u^{p-k} \in Y_p(\mathfrak{gl}_n)[u].$$

[1] Escrevemos *"O Yangian"*, porque $Y_p(\mathfrak{gl}_n)$ mais do que uma álgebra é um grupo quântico.

Essas relações são equivalentes à equação

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min\{r,s\}} \left(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)} \right),$$

onde $t_{ij}^{(0)} = \delta_{ij}$ e $t_{ij}^{(r)} = 0$ para $r \geq p+1$.

Para mais detalhes sobre Yangians ver [Cherednik (1989)] e [Drinfel′d (1985)].

Observação 5.1.

1. *O Yangian $Y_1(\mathfrak{gl}_n)$ de nível 1 coincide com a álgebra envolvente universal $U(\mathfrak{gl}_n)$.*
2. *Além disso,*
$$\deg\left(t_{ij}^{(k)}\right) = k, \quad para \ \ i,j = 1,2,\dots,n \ \ e \ \ k = 1,2,\dots,p$$
define uma filtração natural sobre $Y_p(\mathfrak{gl}_n)$. Denotaremos a correspondente álgebra graduada por $\overline{Y}_p(\mathfrak{gl}_n)$ ou por $\mathrm{gr}\left(Y_p(\mathfrak{gl}_n)\right)$.

A seguinte proposição é a versão análoga do Teorema de Poincaré-Birkhoff-Witt para $Y_p(\mathfrak{gl}_n)$, o qual será muito importante para nossa abordagem, pois ele nos permite usar a linguagem e resultados de álgebra comutativa da mesma forma que foi usada no capítulo 3, na demonstração da versão fraca do Teorema de Ovsienko (3.18).

Proposição 5.2 (**PBW para Yangians**)**.**
A álgebra associativa graduada $\overline{Y}_p(\mathfrak{gl}_n) = \mathrm{gr}\left(Y_p(\mathfrak{gl}_n)\right)$ é uma álgebra de polinômios nas variáveis $\overline{t}_{ij}^{(k)}$, com $i,j = 1,2,\dots,n$ e $k = 1,2,\dots,p$.

Demonstração: Ver [Cherednik (1989)] ou teorema 2.1 em [Molev (1997)].

□

Considere a matriz $T(u) = \left(T_{ij}(u)\right)_{i,j=1}^{n}$ e o elemento em $Y_p(\mathfrak{gl}_n)[u]$, chamado de **determinante quântico** para $Y_p(\mathfrak{gl}_n)$, definido por

$$\mathrm{qdet}\ T(u) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) T_{1\sigma(1)}(u) T_{2\sigma(2)}(u-1) \cdots T_{n\sigma(n)}(u-n+1).$$

Proposição 5.3 (Cherednik (1989))**.** *Sejam p e n inteiros positivos. O centro $Z(Y_p(\mathfrak{gl}_n))$ de $Y_p(\mathfrak{gl}_n)$ é gerado pelos coeficientes d_{ni}, das potências u^{np-i} do determinante quântico* $\mathrm{qdet}\ T(u)$*, o qual é um polinômio mônico em u e de grau np, i.e.,*

$$Z(Y_p(\mathfrak{gl}_n)) = \langle d_{n1}, d_{n2}, \dots, d_{n\,np} \rangle,$$

em que

$$\mathrm{qdet}\ T(u) = u^{np} + d_{n1} u^{np-1} + d_{n2} u^{np-2} + \cdots + d_{n\,np-1} u + d_{n\,np}, \quad d_{ni} \in Y_p(\mathfrak{gl}_n).$$

Demonstração: Ver teorema (1) em [Cherednik (1989)] ou Corolário (4.1) em [Molev (1997)].

□

Agora, considere a cadeia de subálgebras

$$Y_p(\mathfrak{gl}_1) \subset Y_p(\mathfrak{gl}_2) \subset \cdots \subset Y_p(\mathfrak{gl}_{n-1}) \subset Y_p(\mathfrak{gl}_n).$$

A **subálgebra de Gelfand-Tsetlin** Γ para Yangians $Y_p(\mathfrak{gl}_n)$ é a subálgebra de $Y_p(\mathfrak{gl}_n)$ gerada pelos centros $Z(Y_p(\mathfrak{gl}_i))$ de cada subálgebra $Y_p(\mathfrak{gl}_i)$ $(i = 1,2,\dots,n)$. Pela proposição (5.3), os coeficientes do determinante quântico

$$\{d_{ij}\}_{\substack{i=1,2,\dots,n \\ j=1,2,\dots,ip}}$$

geram a subálgebra Γ.

Observação 5.4. *Para*

$$F = \sum_i f_i u^i \in Y_p(\mathfrak{gl}_n)[u],$$

denote

$$\overline{F} := \sum_i \overline{f_i} u^i \in \overline{Y}_p(\mathfrak{gl}_n)[u].$$

Também, denote

$$X_{ij}^{(k)} := \overline{t}_{ij}^{(k)},$$
$$X_{ij}(u) := \overline{T}_{ij}(u),$$

e

$$X(u) := \left(X_{ij}(u)\right)_{i,j=1}^{n}.$$

Portanto, como

$$\overline{T_{ij}(u-\alpha)} = X_{ij}(u), \quad \forall \alpha \in k,$$

podemos facilmente ver que

$$\operatorname{gr} \operatorname{qdet} T(u) = \det X(u).$$

Pelo PBW para Yangians (prop. (5.2)), cada $\overline{d}_{ij}$ é um polinômio nas variáveis comutativas $\overline{t}_{ij}^{(k)}$ com $i, j = 1, 2, \ldots, n$ e $k = 1, 2, \ldots, p$.

Definição 5.5. *A **variedade de Gelfand-Tsetlin** para $Y_p(\mathfrak{gl}_n)$ é a variedade algébrica*

$$\mathfrak{G} := V\left(\{\overline{d_{ij}}\}_{\substack{i=1,2,\ldots,n \\ j=1,2,\ldots,ip}}\right) \subset k^{n^2 p}.$$

Observação 5.6. *Segue do corolário* (4.13)*, que a variedade de Gelfand-Tsetlin $\mathfrak{G}$ para $Y_1(\mathfrak{gl}_n)$ coincide com a variedade de Gelfand-Tsetlin para $\mathfrak{gl}_n$.*

5.1.1 Alguns polinômios da variedade de Gelfand-Tsetlin

Dado que

$$\operatorname{gr} \operatorname{qdet} T(u) = \det X(u) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) X_{1\sigma(1)}(u) X_{2\sigma(2)}(u) \cdots X_{n\sigma(n)}(u)$$
$$\det X(u) = u^{np} + \overline{d}_{n1} u^{np-1} + \overline{d}_{n2} u^{np-2} + \cdots + \overline{d}_{n\,np-1} u + \overline{d}_{n\,np}, \quad \overline{d}_{ni} \in \overline{Y}_p(\mathfrak{gl}_n)[u]$$

e da observação (5.4)

$$X_{ij}(u) = \delta_{ij} u^p + \sum_{k=1}^{p} X_{ij}^{(k)} u^{p-k} \in \overline{Y}_p(\mathfrak{gl}_n)[u],$$

com $X_{ij}^{(0)} = \delta_{ij}$ e $X_{ij}^{(r)} = 0$ para $r \geq p+1$, então

$$\overline{d}_{ni} = \sum_{\sigma \in S_n} \sum_{t_1+t_2+\cdots+t_n=i} \operatorname{sgn}(\sigma) X_{1\sigma(1)}^{(t_1)} X_{2\sigma(2)}^{(t_2)} \cdots X_{n\sigma(n)}^{(t_n)}.$$

Para dar alguns exemplos:

- O primeiro polinômio é:

$$\begin{aligned}\overline{d}_{n1} &= \sum_{\sigma \in S_n} \sum_{t_1+t_2+\cdots+t_n=1} \operatorname{sgn}(\sigma) X_{1\sigma(1)}^{(t_1)} X_{2\sigma(2)}^{(t_2)} \cdots X_{n\sigma(n)}^{(t_n)} \\ &= X_{11}^{(1)} + X_{22}^{(1)} + \cdots + X_{nn}^{(1)}.\end{aligned}$$

- O segundo polinômio é:

$$\begin{aligned}\overline{d}_{n2} &= \sum_{\sigma\in S_n}\sum_{t_1+t_2+\cdots+t_n=2}\operatorname{sgn}(\sigma)X^{(t_1)}_{1\sigma(1)}X^{(t_2)}_{2\sigma(2)}\cdots X^{(t_n)}_{n\sigma(n)}\\ &= \sum_{i=1}^{n}X^{(2)}_{ii}+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}X^{(1)}_{ii}X^{(1)}_{jj}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}X^{(1)}_{ij}X^{(1)}_{ji}\\ &= \sum_{i=1}^{n}X^{(2)}_{ii}+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(X^{(1)}_{ii}X^{(1)}_{jj}-X^{(1)}_{ij}X^{(1)}_{ji}\right).\end{aligned}$$

- O terceiro polinômio é:

$$\begin{aligned}\overline{d}_{n3} &= \sum_{\sigma\in S_n}\sum_{t_1+t_2+\cdots+t_n=3}\operatorname{sgn}(\sigma)X^{(t_1)}_{1\sigma(1)}X^{(t_2)}_{2\sigma(2)}\cdots X^{(t_n)}_{n\sigma(n)}\\ &= \sum_{i=1}^{n}X^{(3)}_{ii}+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(X^{(2)}_{ii}X^{(1)}_{jj}+X^{(1)}_{ii}X^{(2)}_{jj}\right)-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(X^{(2)}_{ij}X^{(1)}_{ji}+X^{(1)}_{ij}X^{(2)}_{ji}\right)+\\ &\quad+\sum_{\substack{A\subset\{1,2,\ldots,n\}\\ |A|=3\\ \sigma\in S_3}}\operatorname{sgn}(\sigma)X^{(1)}_{a_1a_{\sigma(1)}}X^{(1)}_{a_2a_{\sigma(2)}}X^{(1)}_{a_3a_{\sigma(3)}}\\ &= \sum_{i=1}^{n}X^{(3)}_{ii}+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(X^{(2)}_{ii}X^{(1)}_{jj}+X^{(1)}_{ii}X^{(2)}_{jj}-X^{(2)}_{ij}X^{(1)}_{ji}-X^{(1)}_{ij}X^{(2)}_{ji}\right)+\\ &\quad+\sum_{\substack{A\subset\{1,2,\ldots,n\}\\ |A|=3\\ \sigma\in S_3}}\operatorname{sgn}(\sigma)X^{(1)}_{a_1a_{\sigma(1)}}X^{(1)}_{a_2a_{\sigma(2)}}X^{(1)}_{a_3a_{\sigma(3)}},\end{aligned}$$

  onde $A=\{a_1,a_2,a_3\}$ com $a_1<a_2<a_3$.

Observação 5.7. *Vale a pena ressaltar, que temos a seguinte expressão*[2] *para os polinômios* $\overline{d}_{ni}$*. Para qualquer* $i\in\{1,2,\ldots,np\}$ *consideremos os diagramas de Young para* i

$$\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_r)$$

com $r\le n$ *e* $\lambda_1\le p$*, ou seja,* $\lambda_1,\lambda_2,\ldots,\lambda_r\in\mathbb{Z}_{>0}$ *tais que*

$$p\ge\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_r>0,$$

$$\lambda_1+\lambda_2+\ldots+\lambda_r=i,$$

$$1\le r\le n.$$

Denotemos o conjunto de todos os diagramas de Young $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_r)$ *de* i *com comprimento* r *e* $\lambda_1\le p$ *por*

$$\Omega_r(i).$$

Assim podemos ver que

$$\overline{d}_{ni}=\sum_{j=1}^{n}\sum_{\substack{A\subset\{1,2,\ldots,n\}\\ |A|=j\\ \sigma\in S_j}}\sum_{\substack{\lambda\in\Omega_j(i)\\ \sigma'\in S_j}}\operatorname{sgn}(\sigma)X^{(\lambda_{\sigma'(1)})}_{a_1a_{\sigma(1)}}X^{(\lambda_{\sigma'(2)})}_{a_2a_{\sigma(2)}}\cdots X^{(\lambda_{\sigma'(j)})}_{a_ja_{\sigma(j)}},$$

onde $A=\{a_1,a_2,\ldots,a_j\}$ *com* $a_1<a_2<\cdots<a_j$*,* $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_r)$ *e* S_j *as permutações de* $\{1,2,\ldots,j\}$.

[2] Neste trabalho não iremos usar esta expressão dos polinômios $\overline{d}_{ni}$.

5.2 Variedade de Gelfand-Tsetlin para Yangians restritos $Y_p(\mathfrak{gl}_3)$

Notemos que a variedade de Gelfand-Tsetlin $\mathfrak{G}$ para $Y_p(\mathfrak{gl}_n)$ tem polinômios da forma

$$\begin{aligned}
\overline{d}_{11} &= X_{11}^{(1)},\\
\overline{d}_{21} &= X_{11}^{(1)} + X_{22}^{(1)},\\
\overline{d}_{31} &= X_{11}^{(1)} + X_{22}^{(1)} + X_{33}^{(1)},\\
&\vdots\\
\overline{d}_{n1} &= X_{11}^{(1)} + X_{22}^{(1)} + \cdots + X_{nn}^{(1)},
\end{aligned}$$

mas, claramente

$$V\left(\overline{d}_{11}, \overline{d}_{21}, \ldots, \overline{d}_{n1}\right) = V\left(X_{11}^{(1)}, X_{22}^{(1)}, \ldots, X_{nn}^{(1)}\right).$$

Portanto, gostaríamos de melhorar os polinômios que determinam a variedade de Gelfand-Tsetlin. Com este objetivo, vamos entender como é a expressão dos polinômios $\overline{d}_{ij}$ e para isso estudaremos os polinômios até a subálgebra $Y_p(\mathfrak{gl}_3)$, pois como vamos ver, a combinatória apartir deste caso não é nada simples.

Lema 5.8. *Os polinômios que determinam a variedade de Gelfand-Tsetlin $\mathfrak{G}$ para $Y_p(\mathfrak{gl}_3)$ são:*

$$\begin{aligned}
\overline{d}_{1i} &= X_{11}^{(i)}, \quad i = 1, 2, \ldots, p\\
\overline{d}_{21} &= X_{11}^{(1)} + X_{22}^{(1)}\\
\overline{d}_{2i} &= X_{11}^{(i)} + X_{22}^{(i)} + \sum_{t=1}^{i-1} X_{11}^{(t)} X_{22}^{(i-t)} - \sum_{t=1}^{i-1} X_{12}^{(t)} X_{21}^{(i-t)}, \quad i = 2, 3, \ldots, p\\
\overline{d}_{2p+i} &= \sum_{t=i}^{p} X_{11}^{(t)} X_{22}^{(p+i-t)} - \sum_{t=i}^{p} X_{12}^{(t)} X_{21}^{(p+i-t)}, \quad i = 1, 2, 3, \ldots, p\\
\overline{d}_{31} &= X_{11}^{(1)} + X_{22}^{(1)} + X_{33}^{(1)}\\
\overline{d}_{32} &= X_{11}^{(2)} + X_{22}^{(2)} + X_{33}^{(2)} + X_{11}^{(1)} X_{22}^{(1)} + X_{11}^{(1)} X_{33}^{(1)} + X_{22}^{(1)} X_{33}^{(1)} - X_{23}^{(1)} X_{32}^{(1)} - X_{12}^{(1)} X_{21}^{(1)} - X_{13}^{(1)} X_{31}^{(1)}\\
\overline{d}_{3i} &= X_{33}^{(i)} + X_{22}^{(i)} + X_{11}^{(i)} + \sum_{s=1}^{i-1} \left(X_{22}^{(s)} X_{33}^{(i-s)} - X_{23}^{(s)} X_{32}^{(i-s)} \right) +\\
&\quad + \sum_{s=1}^{i-1} \left(X_{11}^{(s)} X_{33}^{(i-s)} - X_{13}^{(s)} X_{31}^{(i-s)} \right) + \sum_{s=1}^{i-1} \left(X_{11}^{(s)} X_{22}^{(i-s)} - X_{12}^{(s)} X_{21}^{(i-s)} \right) +\\
&\quad + \sum_{r=1}^{i-2} \sum_{s=1}^{i-r-1} \left\{ X_{11}^{(r)} X_{22}^{(s)} X_{33}^{(i-r-s)} - X_{11}^{(r)} X_{23}^{(s)} X_{32}^{(i-r-s)} - X_{12}^{(r)} X_{21}^{(s)} X_{33}^{(i-r-s)} - \right.\\
&\quad \left. - X_{13}^{(r)} X_{22}^{(s)} X_{31}^{(i-r-s)} + X_{12}^{(r)} X_{23}^{(s)} X_{31}^{(i-r-s)} + X_{13}^{(r)} X_{21}^{(s)} X_{32}^{(i-r-s)} \right\}, \quad i = 3, 4, \ldots, p\\
\overline{d}_{3p+1} &= \sum_{s=1}^{p} \left(X_{22}^{(s)} X_{33}^{(p+1-s)} - X_{23}^{(s)} X_{32}^{(p+1-s)} \right) + \sum_{s=1}^{p} \left(X_{11}^{(s)} X_{33}^{(p+1-s)} - X_{13}^{(s)} X_{31}^{(p+1-s)} \right) +\\
&\quad + \sum_{s=1}^{p} \left(X_{11}^{(s)} X_{22}^{(p+1-s)} - X_{12}^{(s)} X_{21}^{(p+1-s)} \right) +\\
&\quad + \sum_{r=1}^{p-1} \sum_{s=1}^{p-r} \left\{ X_{11}^{(r)} X_{22}^{(s)} X_{33}^{(p+1-r-s)} - X_{11}^{(r)} X_{23}^{(s)} X_{32}^{(p+1-r-s)} - X_{12}^{(r)} X_{21}^{(s)} X_{33}^{(p+1-r-s)} - \right.\\
&\quad \left. - X_{13}^{(r)} X_{22}^{(s)} X_{31}^{(p+1-r-s)} + X_{12}^{(r)} X_{23}^{(s)} X_{31}^{(p+1-r-s)} + X_{13}^{(r)} X_{21}^{(s)} X_{32}^{(p+1-r-s)} \right\}
\end{aligned}$$

$$\begin{aligned}
\overline{d}_{3\,p+i} =& \sum_{s=i}^{p} X_{22}^{(s)} X_{33}^{(p+i-s)} - \sum_{s=i}^{p} X_{23}^{(s)} X_{32}^{(p+i-s)} + \sum_{s=i}^{p} X_{11}^{(s)} X_{33}^{(p+i-s)} - \\
&- \sum_{s=i}^{p} X_{13}^{(s)} X_{31}^{(p+i-s)} + \sum_{s=i}^{p} X_{11}^{(s)} X_{22}^{(p+i-s)} - \sum_{s=i}^{p} X_{12}^{(s)} X_{21}^{(p+i-s)} + \\
&+ \sum_{r=1}^{i-1} \sum_{s=i-r}^{p} \Big\{ X_{11}^{(r)} X_{22}^{(s)} X_{33}^{(p+i-r-s)} - X_{11}^{(r)} X_{23}^{(s)} X_{32}^{(p+i-r-s)} - X_{12}^{(r)} X_{21}^{(s)} X_{33}^{(p+i-r-s)} - \\
&- X_{13}^{(r)} X_{22}^{(s)} X_{31}^{(p+i-r-s)} + X_{12}^{(r)} X_{23}^{(s)} X_{31}^{(p+i-r-s)} + X_{13}^{(r)} X_{21}^{(s)} X_{32}^{(p+i-r-s)} \Big\} + \\
&+ \sum_{r=i}^{p} \sum_{s=1}^{p+i-r-1} \Big\{ X_{11}^{(r)} X_{22}^{(s)} X_{33}^{(p+i-r-s)} - X_{11}^{(r)} X_{23}^{(s)} X_{32}^{(p+i-r-s)} - X_{12}^{(r)} X_{21}^{(s)} X_{33}^{(p+i-r-s)} - \\
&- X_{13}^{(r)} X_{22}^{(s)} X_{31}^{(p+i-r-s)} + X_{12}^{(r)} X_{23}^{(s)} X_{31}^{(p+i-r-s)} + X_{13}^{(r)} X_{21}^{(s)} X_{32}^{(p+i-r-s)} \Big\}, \quad i = 2, 3, \ldots, p \\
\overline{d}_{3\,2p+i} =& \sum_{r=i}^{p} \sum_{s=p+i-r}^{p} \Big\{ X_{11}^{(r)} X_{22}^{(s)} X_{33}^{(2p+i-r-s)} - X_{11}^{(r)} X_{23}^{(s)} X_{32}^{(2p+i-r-s)} - X_{12}^{(r)} X_{21}^{(s)} X_{33}^{(2p+i-r-s)} - \\
&- X_{13}^{(r)} X_{22}^{(s)} X_{31}^{(2p+i-r-s)} + X_{12}^{(r)} X_{23}^{(s)} X_{31}^{(2p+i-r-s)} + X_{13}^{(r)} X_{21}^{(s)} X_{32}^{(2p+i-r-s)} \Big\}, \quad i = 1, 2, \ldots, p
\end{aligned}$$

Demonstração: Claramente

$$\overline{d}_{1i} = X_{11}^{(i)}, \quad i = 1, 2, \ldots, p$$

e

$$\begin{aligned}
\overline{d}_{21} &= X_{11}^{(1)} + X_{22}^{(1)} \\
\overline{d}_{2i} &= X_{11}^{(i)} + X_{22}^{(i)} + \sum_{\substack{t_1+t_2=i \\ t_1,t_2 \neq 0}} X_{11}^{(t_1)} X_{22}^{(t_2)} - \sum_{\substack{t_1+t_2=i \\ t_1,t_2 \neq 0}} X_{12}^{(t_1)} X_{21}^{(t_2)}, \quad i = 2, 3, \ldots, p \\
&= X_{11}^{(i)} + X_{22}^{(i)} + \sum_{t=1}^{i-1} X_{11}^{(t)} X_{22}^{(i-t)} - \sum_{t=1}^{i-1} X_{12}^{(t)} X_{21}^{(i-t)}, \quad i = 2, 3, \ldots, p.
\end{aligned}$$

Para $i = 1, 2, 3, \ldots, p$

$$\begin{aligned}
\overline{d}_{2\,p+i} &= \sum_{\substack{t_1+t_2=p+i \\ i \leq t_1, t_2 \leq p}} X_{11}^{(t_1)} X_{22}^{(t_2)} - \sum_{\substack{t_1+t_2=p+i \\ i \leq t_1, t_2 \leq p}} X_{12}^{(t_1)} X_{21}^{(t_2)} \quad (\text{pois } \ X_{ij}^{(r)} = 0, \ \forall r > p) \\
&= \sum_{t=i}^{p} X_{11}^{(t)} X_{22}^{(p+i-t)} - \sum_{t=i}^{p} X_{12}^{(t)} X_{21}^{(p+i-t)}.
\end{aligned}$$

Dado que, para $i = 1, 2, \ldots, p$

$$\begin{aligned}
\overline{d}_{3i} &= \sum_{\sigma \in S_3} \sum_{t_1+t_2+t_3=i} \operatorname{sgn}(\sigma) X_{1\sigma(1)}^{(t_1)} X_{2\sigma(2)}^{(t_2)} X_{3\sigma(3)}^{(t_3)} \\
&= \sum_{t_1+t_2+t_3=i} \Big\{ X_{11}^{(t_1)} X_{22}^{(t_2)} X_{33}^{(t_3)} - X_{11}^{(t_1)} X_{23}^{(t_2)} X_{32}^{(t_3)} - X_{12}^{(t_1)} X_{21}^{(t_2)} X_{33}^{(t_3)} - X_{13}^{(t_1)} X_{22}^{(t_2)} X_{31}^{(t_3)} + \\
&\quad + X_{12}^{(t_1)} X_{23}^{(t_2)} X_{31}^{(t_3)} + X_{13}^{(t_1)} X_{21}^{(t_2)} X_{32}^{(t_3)} \Big\},
\end{aligned}$$

então, pela proposição[3] (A.2),

$$\begin{aligned}
\overline{d}_{31} &= X_{11}^{(1)} + X_{22}^{(1)} + X_{33}^{(1)}\\
\overline{d}_{32} &= X_{11}^{(2)} + X_{22}^{(2)} + X_{33}^{(2)} + X_{11}^{(1)}X_{22}^{(1)} + X_{11}^{(1)}X_{33}^{(1)} + X_{22}^{(1)}X_{33}^{(1)} - X_{23}^{(1)}X_{32}^{(1)} - X_{12}^{(1)}X_{21}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}\\
\overline{d}_{3i} &= X_{33}^{(i)} + X_{22}^{(i)} + X_{11}^{(i)} + \sum_{s=1}^{i-1}\Big(X_{22}^{(s)}X_{33}^{(i-s)} - X_{23}^{(s)}X_{32}^{(i-s)}\Big) +\\
&\quad + \sum_{s=1}^{i-1}\Big(X_{11}^{(s)}X_{33}^{(i-s)} - X_{13}^{(s)}X_{31}^{(i-s)}\Big) + \sum_{s=1}^{i-1}\Big(X_{11}^{(s)}X_{22}^{(i-s)} - X_{12}^{(s)}X_{21}^{(i-s)}\Big) +\\
&\quad + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}\Big\{X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(i-r-s)} - X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(i-r-s)} - X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(i-r-s)} -\\
&\quad - X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(i-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(i-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(i-r-s)}\Big\}, \quad i = 3, 4, \ldots, p.
\end{aligned}$$

Similarmente, dado que, para $i = 1, 2, \ldots, p$

$$\begin{aligned}
\overline{d}_{3\,p+i} &= \sum_{\sigma\in S_3}\sum_{\substack{t_1+t_2+t_3=p+i\\ 0\le t_1,t_2,t_3\le p}} \operatorname{sgn}(\sigma)X_{1\sigma(1)}^{(t_1)}X_{2\sigma(2)}^{(t_2)}X_{3\sigma(3)}^{(t_3)} \quad (\text{pois} \quad X_{ij}^{(r)} = 0, \ \forall r > p)\\
&= \sum_{\substack{t_1+t_2+t_3=p+i\\ 0\le t_1,t_2,t_3\le p}} \Big\{X_{11}^{(t_1)}X_{22}^{(t_2)}X_{33}^{(t_3)} - X_{11}^{(t_1)}X_{23}^{(t_2)}X_{32}^{(t_3)} - X_{12}^{(t_1)}X_{21}^{(t_2)}X_{33}^{(t_3)} -\\
&\quad - X_{13}^{(t_1)}X_{22}^{(t_2)}X_{31}^{(t_3)} + X_{12}^{(t_1)}X_{23}^{(t_2)}X_{31}^{(t_3)} + X_{13}^{(t_1)}X_{21}^{(t_2)}X_{32}^{(t_3)}\Big\},
\end{aligned}$$

então, da proposição (A.3), segue que

$$\begin{aligned}
\overline{d}_{3\,p+1} &= \sum_{\substack{t_1+t_2+t_3=p+1\\ 0\le t_1,t_2,t_3\le p}} \Big\{X_{11}^{(t_1)}X_{22}^{(t_2)}X_{33}^{(t_3)} - X_{11}^{(t_1)}X_{23}^{(t_2)}X_{32}^{(t_3)} - X_{12}^{(t_1)}X_{21}^{(t_2)}X_{33}^{(t_3)} -\\
&\quad - X_{13}^{(t_1)}X_{22}^{(t_2)}X_{31}^{(t_3)} + X_{12}^{(t_1)}X_{23}^{(t_2)}X_{31}^{(t_3)} + X_{13}^{(t_1)}X_{21}^{(t_2)}X_{32}^{(t_3)}\Big\}\\
&= X_{11}^{(0)}\sum_{s=1}^{p}\Big(X_{22}^{(s)}X_{33}^{(p+1-s)} - X_{23}^{(s)}X_{32}^{(p+1-s)}\Big) + X_{22}^{(0)}\sum_{s=1}^{p}\Big(X_{11}^{(s)}X_{33}^{(p+1-s)} - X_{13}^{(s)}X_{31}^{(p+1-s)}\Big) +\\
&\quad + X_{33}^{(0)}\sum_{s=1}^{p}\Big(X_{11}^{(s)}X_{22}^{(p+1-s)} - X_{12}^{(s)}X_{21}^{(p+1-s)}\Big) +\\
&\quad + \sum_{r=1}^{p-1}\sum_{s=1}^{p-r}\Big\{X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(p+1-r-s)} - X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(p+1-r-s)} - X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+1-r-s)} -\\
&\quad - X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+1-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+1-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+1-r-s)}\Big\}\\
&= \sum_{s=1}^{p}\Big(X_{22}^{(s)}X_{33}^{(p+1-s)} - X_{23}^{(s)}X_{32}^{(p+1-s)}\Big) + \sum_{s=1}^{p}\Big(X_{11}^{(s)}X_{33}^{(p+1-s)} - X_{13}^{(s)}X_{31}^{(p+1-s)}\Big) +\\
&\quad + \sum_{s=1}^{p}\Big(X_{11}^{(s)}X_{22}^{(p+1-s)} - X_{12}^{(s)}X_{21}^{(p+1-s)}\Big) +\\
&\quad + \sum_{r=1}^{p-1}\sum_{s=1}^{p-r}\Big\{X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(p+1-r-s)} - X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(p+1-r-s)} - X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+1-r-s)} -
\end{aligned}$$

[3]Quando aparecer uma referência na forma (A.–), significa que esta referência está no apêndice (A).

$$
\begin{aligned}
&-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+1-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+1-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+1-r-s)}\Big\}\\
\overline{d}_{3p+i}=&\sum_{s=i}^{p}X_{22}^{(s)}X_{33}^{(p+i-s)}-\sum_{s=i}^{p}X_{23}^{(s)}X_{32}^{(p+i-s)}+\sum_{s=i}^{p}X_{11}^{(s)}X_{33}^{(p+i-s)}-\\
&-\sum_{s=i}^{p}X_{13}^{(s)}X_{31}^{(p+i-s)}+\sum_{s=i}^{p}X_{11}^{(s)}X_{22}^{(p+i-s)}-\sum_{s=i}^{p}X_{12}^{(s)}X_{21}^{(p+i-s)}+\\
&+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p}\Big\{X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(p+i-r-s)}-X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+i-r-s)}-\\
&-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\Big\}+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1}\Big\{X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(p+i-r-s)}-X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+i-r-s)}-\\
&-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\Big\},\quad i=2,3,\dots,p.
\end{aligned}
$$

Finalmente, pela proposição (A.4), para cada $i=1,2,\dots,p$

$$
\begin{aligned}
\overline{d}_{32p+i}&=\sum_{\sigma\in S_3}\sum_{\substack{t_1+t_2+t_3=2p+i\\0\le t_1,t_2,t_3\le p}}\operatorname{sgn}(\sigma)X_{1\sigma(1)}^{(t_1)}X_{2\sigma(2)}^{(t_2)}X_{3\sigma(3)}^{(t_3)}\quad(\text{pois}\quad X_{ij}^{(r)}=0,\ \forall r>p)\\
&=\sum_{\substack{t_1+t_2+t_3=2p+i\\0\le t_1,t_2,t_3\le p}}\Big\{X_{11}^{(t_1)}X_{22}^{(t_2)}X_{33}^{(t_3)}-X_{11}^{(t_1)}X_{23}^{(t_2)}X_{32}^{(t_3)}-X_{12}^{(t_1)}X_{21}^{(t_2)}X_{33}^{(t_3)}-\\
&\quad-X_{13}^{(t_1)}X_{22}^{(t_2)}X_{31}^{(t_3)}+X_{12}^{(t_1)}X_{23}^{(t_2)}X_{31}^{(t_3)}+X_{13}^{(t_1)}X_{21}^{(t_2)}X_{32}^{(t_3)}\Big\}\\
&=\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}\Big\{X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(2p+i-r-s)}-X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(2p+i-r-s)}-X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(2p+i-r-s)}-\\
&\quad-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(2p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(2p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p+i-r-s)}\Big\}.
\end{aligned}
$$

□

Como consequência deste lema, vamos poder apresentar a variedade de Gelfand-Tsetlin até a subálgebra $Y_p(\mathfrak{gl}_3)$ de uma forma um pouco mais acessível.

Proposição 5.9.

$$V\left(\{\overline{d}_{ij}\}_{\substack{i=1,2,3\\j=1,2,\dots,ip}}\right)=V\left(\{p_{ij}\}_{\substack{i=1,2,3\\j=1,2,\dots,ip}}\right)\subset k^{9p},$$

em que

$$
\begin{aligned}
p_{1i}&=X_{11}^{(i)},\quad i=1,2,\dots,p\\
p_{21}&=X_{22}^{(1)}\\
p_{2i}&=X_{22}^{(i)}-\sum_{t=1}^{i-1}X_{12}^{(t)}X_{21}^{(i-t)},\quad i=2,3,\dots,p\\
p_{2p+i}&=\sum_{t=i}^{p}X_{12}^{(t)}X_{21}^{(p+i-t)},\quad i=1,2,\dots,p\\
p_{22p}&=X_{12}^{(p)}X_{21}^{(p)}\\
p_{31}&=X_{33}^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
p_{32} &= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}\\
p_{33} &= X_{33}^{(3)} - X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(1)} + X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)}\\
p_{3i} &= -\sum_{r=1}^{i-2}\sum_{s=2}^{i-r-1} X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(i-r-s)} + X_{33}^{(i)} - \sum_{s=1}^{i-1}\Big(X_{13}^{(s)}X_{31}^{(i-s)} + X_{23}^{(s)}X_{32}^{(i-s)}\Big)+\\
&\quad+\sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}\Big\{X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(i-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(i-r-s)}\Big\},\quad i=4,5,\dots,p\\
p_{3\,p+1} &= \sum_{r=1}^{p-1}\sum_{s=1}^{p-r}\Big\{X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+1-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+1-r-s)}\Big\}-\\
&\quad-\sum_{s=1}^{p}\Big(X_{13}^{(s)}X_{31}^{(p+1-s)} + X_{23}^{(s)}X_{32}^{(p+1-s)}\Big) - \sum_{r=1}^{p-1}\sum_{s=2}^{p-r} X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+1-r-s)}\\
p_{3\,p+i} &= -\sum_{s=i}^{p} X_{23}^{(s)}X_{32}^{(p+i-s)} - \sum_{s=i}^{p} X_{13}^{(s)}X_{31}^{(p+i-s)}+\\
&\quad+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p}\Big\{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\Big\}+\\
&\quad+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1}\Big\{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\Big\},\\
&\quad i=2,3,\dots,p\\
p_{3\,2p+i} &= \sum_{r=i}^{p}\sum_{s=p+i-r}^{p}\Big\{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(2p+i-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(2p+i-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p+i-r-s)}\Big\},\quad i=1,2,\dots,p.
\end{aligned}
$$

Demonstração: Claramente

$$
\begin{aligned}
\overline{d}_{1i} &= X_{11}^{(i)} = p_{1i},\quad i=1,2,\dots,p.\\
\overline{d}_{21} &= X_{11}^{(1)} + X_{22}^{(1)} = p_{11} + p_{21}.
\end{aligned}
$$

Para $i=2,3,\dots,p$

$$
\begin{aligned}
\overline{d}_{2i} &= X_{11}^{(i)} + X_{22}^{(i)} + \sum_{t=1}^{i-1} X_{11}^{(t)}X_{22}^{(i-t)} - \sum_{t=1}^{i-1} X_{12}^{(t)}X_{21}^{(i-t)}\\
&= X_{11}^{(i)} + \sum_{t=1}^{i-1} X_{11}^{(t)}X_{22}^{(i-t)} + X_{22}^{(i)} - \sum_{t=1}^{i-1} X_{12}^{(t)}X_{21}^{(i-t)}\\
&= p_{1i} + \sum_{t=1}^{i-1} p_{1t}X_{22}^{(i-t)} + p_{2i}.
\end{aligned}
$$

Similarmente

$$
\begin{aligned}
\overline{d}_{2\,p+i} &= \sum_{t=i}^{p} X_{11}^{(t)}X_{22}^{(p+i-t)} - \sum_{t=i}^{p} X_{12}^{(t)}X_{21}^{(p+i-t)} = \sum_{t=i}^{p} p_{1t}X_{22}^{(p+i-t)} - p_{2\,p+i},\quad i=1,2,\dots,p\\
\overline{d}_{22p} &= X_{11}^{(p)}X_{22}^{(p)} - X_{12}^{(p)}X_{21}^{(p)} = p_{1p}X_{22}^{(p)} - p_{22p}.
\end{aligned}
$$

Continuando com o mesmo raciocinio

$$
\begin{aligned}
\overline{d}_{31} &= X_{11}^{(1)} + X_{22}^{(1)} + X_{33}^{(1)} = p_{11} + p_{21} + p_{31}\\
\overline{d}_{32} &= X_{11}^{(2)} + X_{22}^{(2)} + X_{33}^{(2)} + X_{11}^{(1)}X_{22}^{(1)} + X_{11}^{(1)}X_{33}^{(1)} + X_{22}^{(1)}X_{33}^{(1)} - X_{23}^{(1)}X_{32}^{(1)} - X_{12}^{(1)}X_{21}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
&= X_{11}^{(2)} + X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)} + X_{11}^{(1)}X_{22}^{(1)} + X_{11}^{(1)}X_{33}^{(1)} + X_{22}^{(1)}X_{33}^{(1)} + X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)} \\
&= p_{12} + p_{22} + p_{11}X_{22}^{(1)} + p_{11}X_{33}^{(1)} + p_{21}X_{33}^{(1)} + p_{32}.
\end{aligned}
$$

Para cada $i = 3, 4, \dots, p$

$$
\begin{aligned}
\overline{d}_{3i} &= X_{33}^{(i)} + X_{22}^{(i)} + X_{11}^{(i)} + \sum_{s=1}^{i-1}\Big(X_{22}^{(s)}X_{33}^{(i-s)} - X_{23}^{(s)}X_{32}^{(i-s)}\Big) + \\
&\quad + \sum_{s=1}^{i-1}\Big(X_{11}^{(s)}X_{33}^{(i-s)} - X_{13}^{(s)}X_{31}^{(i-s)}\Big) + \sum_{s=1}^{i-1}\Big(X_{11}^{(s)}X_{22}^{(i-s)} - X_{12}^{(s)}X_{21}^{(i-s)}\Big) + \\
&\quad + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}\Big\{X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(i-r-s)} - X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(i-r-s)} - X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(i-r-s)} - \\
&\quad - X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(i-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(i-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(i-r-s)}\Big\} \\
&= X_{33}^{(i)} + X_{22}^{(i)} + X_{11}^{(i)} + \sum_{s=1}^{i-1}X_{22}^{(s)}X_{33}^{(i-s)} - \sum_{s=1}^{i-1}X_{23}^{(s)}X_{32}^{(i-s)} + \\
&\quad + \sum_{s=1}^{i-1}\Big(X_{11}^{(s)}X_{33}^{(i-s)} + X_{11}^{(s)}X_{22}^{(i-s)}\Big) - \sum_{s=1}^{i-1}X_{13}^{(s)}X_{31}^{(i-s)} - \sum_{s=1}^{i-1}X_{12}^{(s)}X_{21}^{(i-s)} + \\
&\quad + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}\Big\{X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(i-r-s)} - X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(i-r-s)}\Big\} - \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(i-r-s)} + \\
&\quad + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}\Big\{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(i-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(i-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(i-r-s)}\Big\} \\
&= X_{11}^{(i)} + \sum_{s=1}^{i-1}\Big(X_{11}^{(s)}X_{33}^{(i-s)} + X_{11}^{(s)}X_{22}^{(i-s)}\Big) + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}\Big\{X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(i-r-s)} - X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(i-r-s)}\Big\} + \\
&\quad + X_{22}^{(i)} - \sum_{s=1}^{i-1}X_{12}^{(s)}X_{21}^{(i-s)} + \sum_{s=1}^{i-1}X_{22}^{(s)}X_{33}^{(i-s)} - \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(i-r-s)} + X_{33}^{(i)} - \sum_{s=1}^{i-1}X_{13}^{(s)}X_{31}^{(i-s)} - \\
&\quad - \sum_{s=1}^{i-1}X_{23}^{(s)}X_{32}^{(i-s)} + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}\Big\{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(i-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(i-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(i-r-s)}\Big\} \\
&= p_{1i} + \sum_{s=1}^{i-1}p_{1s}\Big(X_{33}^{(i-s)} + X_{22}^{(i-s)}\Big) + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}p_{1r}\Big(X_{22}^{(s)}X_{33}^{(i-r-s)} - X_{23}^{(s)}X_{32}^{(i-r-s)}\Big) + \\
&\quad + p_{2i} + \sum_{s=1}^{i-1}X_{22}^{(s)}X_{33}^{(i-s)} - \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(i-r-s)} + p_{3i} \\
&= p_{1i} + \sum_{s=1}^{i-1}p_{1s}\Big(X_{33}^{(i-s)} + X_{22}^{(i-s)}\Big) + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}p_{1r}\Big(X_{22}^{(s)}X_{33}^{(i-r-s)} - X_{23}^{(s)}X_{32}^{(i-r-s)}\Big) + \\
&\quad + p_{2i} + \sum_{s=1}^{i-1}X_{22}^{(s)}X_{33}^{(i-s)} - \sum_{\substack{r+s+t=i\\ r,s,t\neq 0}}X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(t)} + p_{3i} \quad \text{(proposição (A.1))} \\
&= p_{1i} + \sum_{s=1}^{i-1}p_{1s}\Big(X_{33}^{(i-s)} + X_{22}^{(i-s)}\Big) + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}p_{1r}\Big(X_{22}^{(s)}X_{33}^{(i-r-s)} - X_{23}^{(s)}X_{32}^{(i-r-s)}\Big) + \\
&\quad + p_{2i} + \sum_{r=1}^{i-1}X_{22}^{(i-r)}X_{33}^{(r)} - \sum_{\substack{r+s+t=i\\ r,s,t\neq 0}}X_{33}^{(r)}X_{12}^{(s)}X_{21}^{(t)} + p_{3i} \quad \text{(por comutatividade)}
\end{aligned}
$$

$$
\begin{aligned}
&= p_{1i} + \sum_{s=1}^{i-1} p_{1s}\left(X_{33}^{(i-s)} + X_{22}^{(i-s)}\right) + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1} p_{1r}\left(X_{22}^{(s)}X_{33}^{(i-r-s)} - X_{23}^{(s)}X_{32}^{(i-r-s)}\right) + \\
&\quad + p_{2i} + X_{22}^{(1)}X_{33}^{(i-1)} + \sum_{r=1}^{i-2} X_{22}^{(i-r)}X_{33}^{(r)} - \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1} X_{33}^{(r)}X_{12}^{(s)}X_{21}^{(i-r-s)} + p_{3i} \quad \text{(proposição (A.1))} \\
&= p_{1i} + \sum_{s=1}^{i-1} p_{1s}\left(X_{33}^{(i-s)} + X_{22}^{(i-s)}\right) + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1} p_{1r}\left(X_{22}^{(s)}X_{33}^{(i-r-s)} - X_{23}^{(s)}X_{32}^{(i-r-s)}\right) + \\
&\quad + p_{2i} + p_{21}X_{33}^{(i-1)} + \sum_{r=1}^{i-2}\left(X_{22}^{(i-r)} - \sum_{s=1}^{i-r-1} X_{12}^{(s)}X_{21}^{(i-r-s)}\right)X_{33}^{(r)} + p_{3i} \\
&= p_{1i} + \sum_{s=1}^{i-1} p_{1s}\left(X_{33}^{(i-s)} + X_{22}^{(i-s)}\right) + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1} p_{1r}\left(X_{22}^{(s)}X_{33}^{(i-r-s)} - X_{23}^{(s)}X_{32}^{(i-r-s)}\right) + \\
&\quad + p_{2i} + p_{21}X_{33}^{(i-1)} + \sum_{r=1}^{i-2} p_{2\,i-r}X_{33}^{(r)} + p_{3i}.
\end{aligned}
$$

Análogamente

$$
\begin{aligned}
\overline{d}_{3\,p+1} &= \sum_{s=1}^{p}\left(X_{22}^{(s)}X_{33}^{(p+1-s)} - X_{23}^{(s)}X_{32}^{(p+1-s)}\right) + \sum_{s=1}^{p}\left(X_{11}^{(s)}X_{33}^{(p+1-s)} - X_{13}^{(s)}X_{31}^{(p+1-s)}\right) + \\
&\quad + \sum_{s=1}^{p}\left(X_{11}^{(s)}X_{22}^{(p+1-s)} - X_{12}^{(s)}X_{21}^{(p+1-s)}\right) + \\
&\quad + \sum_{r=1}^{p-1}\sum_{s=1}^{p-r}\Big\{X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(p+1-r-s)} - X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(p+1-r-s)} - X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+1-r-s)} - \\
&\quad - X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+1-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+1-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+1-r-s)}\Big\} \\
&= \sum_{s=1}^{p} X_{22}^{(s)}X_{33}^{(p+1-s)} + \sum_{s=1}^{p} X_{11}^{(s)}X_{33}^{(p+1-s)} - \sum_{s=1}^{p}\left(X_{23}^{(s)}X_{32}^{(p+1-s)} + X_{13}^{(s)}X_{31}^{(p+1-s)}\right) + \\
&\quad + \sum_{s=1}^{p} X_{11}^{(s)}X_{22}^{(p+1-s)} - \sum_{s=1}^{p} X_{12}^{(s)}X_{21}^{(p+1-s)} + \\
&\quad + \sum_{r=1}^{p-1}\sum_{s=1}^{p-r}\left(X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(p+1-r-s)} - X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(p+1-r-s)}\right) - \sum_{r=1}^{p-1}\sum_{s=1}^{p-r} X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+1-r-s)} + \\
&\quad + \sum_{r=1}^{p-1}\sum_{s=1}^{p-r}\left(-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+1-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+1-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+1-r-s)}\right) \\
&= \sum_{s=1}^{p} X_{11}^{(s)}X_{33}^{(p+1-s)} + \sum_{s=1}^{p} X_{11}^{(s)}X_{22}^{(p+1-s)} + \sum_{r=1}^{p-1}\sum_{s=1}^{p-r}\left(X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(p+1-r-s)} - X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(p+1-r-s)}\right) + \\
&\quad + \sum_{s=1}^{p} X_{22}^{(s)}X_{33}^{(p+1-s)} - \sum_{r=1}^{p-1}\sum_{s=1}^{p-r} X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+1-r-s)} - \sum_{s=1}^{p} X_{12}^{(s)}X_{21}^{(p+1-s)} + \\
&\quad - \sum_{s=1}^{p}\left(X_{23}^{(s)}X_{32}^{(p+1-s)} + X_{13}^{(s)}X_{31}^{(p+1-s)}\right) + \\
&\quad + \sum_{r=1}^{p-1}\sum_{s=1}^{p-r}\left(-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+1-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+1-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+1-r-s)}\right) \\
&= \sum_{s=1}^{p} p_{1s}\left(X_{33}^{(p+1-s)} + X_{22}^{(p+1-s)}\right) + \sum_{r=1}^{p-1}\sum_{s=1}^{p-r} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+1-r-s)} - X_{23}^{(s)}X_{32}^{(p+1-r-s)}\right) +
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{s=1}^{p} X_{22}^{(s)}X_{33}^{(p+1-s)}-\sum_{r=1}^{p-1}\sum_{s=1}^{p-r} X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+1-r-s)}-p_{2\,p+1}+p_{3\,p+1}\\
=&\sum_{s=1}^{p} p_{1s}\left(X_{33}^{(p+1-s)}+X_{22}^{(p+1-s)}\right)+\sum_{r=1}^{p-1}\sum_{s=1}^{p-r} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+1-r-s)}-X_{23}^{(s)}X_{32}^{(p+1-r-s)}\right)+\\
&+\sum_{s=1}^{p} X_{22}^{(s)}X_{33}^{(p+1-s)}-\sum_{\substack{r+s+t=p+1\\1\le r,s,t\le p}} X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(t)}-p_{2\,p+1}+p_{3\,p+1}\quad\text{(proposição (A.1))}\\
=&\sum_{s=1}^{p} p_{1s}\left(X_{33}^{(p+1-s)}+X_{22}^{(p+1-s)}\right)+\sum_{r=1}^{p-1}\sum_{s=1}^{p-r} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+1-r-s)}-X_{23}^{(s)}X_{32}^{(p+1-r-s)}\right)+\\
&+\sum_{r=1}^{p} X_{22}^{(p+1-r)}X_{33}^{(r)}-\sum_{\substack{r+s+t=p+1\\1\le r,s,t\le p}} X_{33}^{(r)}X_{12}^{(s)}X_{21}^{(t)}-p_{2\,p+1}+p_{3\,p+1}\quad\text{(por comutatividade)}\\
=&\sum_{s=1}^{p} p_{1s}\left(X_{33}^{(p+1-s)}+X_{22}^{(p+1-s)}\right)+\sum_{r=1}^{p-1}\sum_{s=1}^{p-r} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+1-r-s)}-X_{23}^{(s)}X_{32}^{(p+1-r-s)}\right)+\\
&+X_{22}^{(1)}X_{33}^{(p)}+\sum_{r=1}^{p-1} X_{22}^{(p+1-r)}X_{33}^{(r)}-\sum_{r=1}^{p-1}\sum_{s=1}^{p-r} X_{33}^{(r)}X_{12}^{(s)}X_{21}^{(p+1-r-s)}-p_{2\,p+1}+p_{3\,p+1}\quad\text{(proposição (A.1))}\\
=&\sum_{s=1}^{p} p_{1s}\left(X_{33}^{(p+1-s)}+X_{22}^{(p+1-s)}\right)+\sum_{r=1}^{p-1}\sum_{s=1}^{p-r} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+1-r-s)}-X_{23}^{(s)}X_{32}^{(p+1-r-s)}\right)+\\
&+p_{21}X_{33}^{(p)}+\sum_{r=1}^{p-1} X_{33}^{(r)}\left(X_{22}^{(p+1-r)}-\sum_{s=1}^{p-r} X_{12}^{(s)}X_{21}^{(p+1-r-s)}\right)-p_{2\,p+1}+p_{3\,p+1}\\
=&\sum_{s=1}^{p} p_{1s}\left(X_{33}^{(p+1-s)}+X_{22}^{(p+1-s)}\right)+\sum_{r=1}^{p-1}\sum_{s=1}^{p-r} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+1-r-s)}-X_{23}^{(s)}X_{32}^{(p+1-r-s)}\right)+\\
&+p_{21}X_{33}^{(p)}+\sum_{r=1}^{p-1} X_{33}^{(r)}p_{2\,p+1-r}-p_{2\,p+1}+p_{3\,p+1}.
\end{aligned}
$$

Para $i=2,3,\ldots,p$

$$
\begin{aligned}
\overline{d}_{3\,p+i}=&\sum_{s=i}^{p} X_{22}^{(s)}X_{33}^{(p+i-s)}-\sum_{s=i}^{p} X_{23}^{(s)}X_{32}^{(p+i-s)}+\sum_{s=i}^{p} X_{11}^{(s)}X_{33}^{(p+i-s)}-\\
&-\sum_{s=i}^{p} X_{13}^{(s)}X_{31}^{(p+i-s)}+\sum_{s=i}^{p} X_{11}^{(s)}X_{22}^{(p+i-s)}-\sum_{s=i}^{p} X_{12}^{(s)}X_{21}^{(p+i-s)}+\\
&+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p}\Big\{X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(p+i-r-s)}-X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+i-r-s)}-\\
&-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\Big\}+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1}\Big\{X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(p+i-r-s)}-X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+i-r-s)}-\\
&-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\Big\}\\
=&\sum_{s=i}^{p} X_{22}^{(s)}X_{33}^{(p+i-s)}-\sum_{s=i}^{p} X_{23}^{(s)}X_{32}^{(p+i-s)}+\sum_{s=i}^{p} X_{11}^{(s)}X_{33}^{(p+i-s)}-
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{s=i}^{p} X_{13}^{(s)}X_{31}^{(p+i-s)}+\sum_{s=i}^{p} X_{11}^{(s)}X_{22}^{(p+i-s)}-\sum_{s=i}^{p} X_{12}^{(s)}X_{21}^{(p+i-s)}+\\
&+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p}\left(X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)-\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+i-r-s)}+\\
&\sum_{r=1}^{i-1}\sum_{s=i-r}^{p}\left(-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\right)+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1}\left(X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)-\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+i-r-s)}+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1}\left(-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\right)\\
=&\sum_{s=i}^{p} X_{11}^{(s)}X_{22}^{(p+i-s)}+\sum_{s=i}^{p} X_{11}^{(s)}X_{33}^{(p+i-s)}+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p}\left(X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1}\left(X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)-\sum_{s=i}^{p} X_{12}^{(s)}X_{21}^{(p+i-s)}+\\
&+\sum_{s=i}^{p} X_{22}^{(s)}X_{33}^{(p+i-s)}-\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+i-r-s)}-\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+i-r-s)}+\\
&-\sum_{s=i}^{p} X_{13}^{(s)}X_{31}^{(p+i-s)}-\sum_{s=i}^{p} X_{23}^{(s)}X_{32}^{(p+i-s)}+\\
&+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p}\left(-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\right)+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1}\left(-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\right)\\
=&\sum_{s=i}^{p} p_{1s}X_{22}^{(p+i-s)}+\sum_{s=i}^{p} p_{1s}X_{33}^{(p+i-s)}+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p}\left(p_{1r}X_{22}^{(s)}X_{33}^{(p+i-r-s)}-p_{1r}X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1}\left(p_{1r}X_{22}^{(s)}X_{33}^{(p+i-r-s)}-p_{1r}X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)-p_{2i}+\\
&+\sum_{s=i}^{p} X_{22}^{(s)}X_{33}^{(p+i-s)}-\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+i-r-s)}-\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+i-r-s)}+p_{3\,p+i}\\
=&\sum_{s=i}^{p} p_{1s}\left(X_{22}^{(p+i-s)}+X_{33}^{(p+i-s)}\right)+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)-p_{2i}+\\
&+\sum_{s=i}^{p} X_{22}^{(s)}X_{33}^{(p+i-s)}-\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+i-r-s)}-\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(p+i-r-s)}+p_{3\,p+i}\\
=&\sum_{s=i}^{p} p_{1s}\left(X_{22}^{(p+i-s)}+X_{33}^{(p+i-s)}\right)+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)-p_{2i}+
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{s=i}^{p} X_{22}^{(s)}X_{33}^{(p+i-s)} - \sum_{\substack{r+s+t=p+i\\ 1\le r,s,t\le p}} X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(t)} + p_{3\,p+i} \quad \text{(proposição (A.1))}\\
&=\sum_{s=i}^{p} p_{1s}\left(X_{22}^{(p+i-s)}+X_{33}^{(p+i-s)}\right)+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)-p_{2i}+\\
&+\sum_{s=i}^{p} X_{22}^{(s)}X_{33}^{(p+i-s)} - \sum_{\substack{r+s+t=p+i\\ 1\le r,s,t\le p}} X_{33}^{(r)}X_{12}^{(s)}X_{21}^{(t)} + p_{3\,p+i} \quad \text{(por comutatividade)}\\
&=\sum_{s=i}^{p} p_{1s}\left(X_{22}^{(p+i-s)}+X_{33}^{(p+i-s)}\right)+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)-p_{2i}+\sum_{r=i}^{p} X_{22}^{(p+i-r)}X_{33}^{(r)}-\\
&-\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} X_{33}^{(r)}X_{12}^{(s)}X_{21}^{(p+i-r-s)}-\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} X_{33}^{(r)}X_{12}^{(s)}X_{21}^{(p+i-r-s)}+p_{3\,p+i} \quad \text{(proposição (A.1))}\\
&=\sum_{s=i}^{p} p_{1s}\left(X_{22}^{(p+i-s)}+X_{33}^{(p+i-s)}\right)+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)-p_{2i}+\sum_{r=i}^{p} X_{22}^{(p+i-r)}X_{33}^{(r)}-\\
&-\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} X_{33}^{(r)}X_{12}^{(s)}X_{21}^{(p+i-r-s)}-\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} X_{33}^{(r)}X_{12}^{(s)}X_{21}^{(p+i-r-s)}+p_{3\,p+i}\\
&=\sum_{s=i}^{p} p_{1s}\left(X_{22}^{(p+i-s)}+X_{33}^{(p+i-s)}\right)+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)-p_{2i}+\\
&+\sum_{r=i}^{p} X_{33}^{(r)}\left(X_{22}^{(p+i-r)}-\sum_{s=1}^{p+i-r-1} X_{12}^{(s)}X_{21}^{(p+i-r-s)}\right)-\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} X_{33}^{(r)}X_{12}^{(s)}X_{21}^{(p+i-r-s)}+p_{3\,p+i}\\
&=\sum_{s=i}^{p} p_{1s}\left(X_{22}^{(p+i-s)}+X_{33}^{(p+i-s)}\right)+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} p_{1r}\left(X_{22}^{(s)}X_{33}^{(p+i-r-s)}-X_{23}^{(s)}X_{32}^{(p+i-r-s)}\right)-p_{2i}+\\
&+\sum_{r=i}^{p} X_{33}^{(r)}p_{2\,p+i-r}-\sum_{r=1}^{i-1} X_{33}^{(r)}p_{2\,p+i-r}+p_{3\,p+i}.
\end{aligned}
$$

Finalmente, para cada $i=1,2,\ldots,p$

$$
\begin{aligned}
\overline{d}_{3\,2p+i}=\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}&\Big\{X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(2p+i-r-s)}-X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(2p+i-r-s)}-X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(2p+i-r-s)}-\\
&-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(2p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(2p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p+i-r-s)}\Big\}
\end{aligned}
$$

$$=\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}\left(X_{11}^{(r)}X_{22}^{(s)}X_{33}^{(2p+i-r-s)}-X_{11}^{(r)}X_{23}^{(s)}X_{32}^{(2p+i-r-s)}\right)-\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(2p+i-r-s)}+$$

$$+\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}\left(-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(2p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(2p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p+i-r-s)}\right)$$

$$=\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}\left(p_{1r}X_{22}^{(s)}X_{33}^{(2p+i-r-s)}-p_{1r}X_{23}^{(s)}X_{32}^{(2p+i-r-s)}\right)-\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(2p+i-r-s)}+p_{3\,2p+i}$$

$$=\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}p_{1r}\left(X_{22}^{(s)}X_{33}^{(2p+i-r-s)}-X_{23}^{(s)}X_{32}^{(2p+i-r-s)}\right)-\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(2p+i-r-s)}+p_{3\,2p+i}$$

$$=\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}p_{1r}\left(X_{22}^{(s)}X_{33}^{(2p+i-r-s)}-X_{23}^{(s)}X_{32}^{(2p+i-r-s)}\right)-\sum_{\substack{r+s+t=2p+i\\1\le r,s,t\le p}}X_{12}^{(r)}X_{21}^{(s)}X_{33}^{(t)}+p_{3\,2p+i}$$

(proposição (A.1))

$$=\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}p_{1r}\left(X_{22}^{(s)}X_{33}^{(2p+i-r-s)}-X_{23}^{(s)}X_{32}^{(2p+i-r-s)}\right)-\sum_{\substack{r+s+t=2p+i\\1\le r,s,t\le p}}X_{33}^{(r)}X_{12}^{(s)}X_{21}^{(t)}+p_{3\,2p+i}$$

(por comutatividade)

$$=\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}p_{1r}\left(X_{22}^{(s)}X_{33}^{(2p+i-r-s)}-X_{23}^{(s)}X_{32}^{(2p+i-r-s)}\right)-\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}X_{33}^{(r)}X_{12}^{(s)}X_{21}^{(2p+i-r-s)}+p_{3\,2p+i}$$

(proposição (A.1))

$$=\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}p_{1r}\left(X_{22}^{(s)}X_{33}^{(2p+i-r-s)}-X_{23}^{(s)}X_{32}^{(2p+i-r-s)}\right)-\sum_{r=i}^{p}X_{33}^{(r)}p_{2\,p+i-r}+p_{3\,2p+i}.$$

□

Corolário 5.10. *A variedade de Gelfand-Tsetlin $\mathfrak{G}$ para $Y_p(\mathfrak{gl}_n)$ pode ser descrita na forma:*

$$\mathfrak{G}=V\left(\{p_{ij}\}_{\substack{i=1,2,\dots,n\\ j=1,2,\dots,ip}}\right)\subset k^{9p},$$

em que

$$p_{ij}=\overline{d}_{ij},\quad para\ \ 4\le i\le n\ \ e\ \ 1\le j\le ip.$$

Demonstração: Segue da definição (5.5) e da proposição (5.9).

□

5.2.1 Equidimensionalidade para $Y_1(\mathfrak{gl}_3)$ e $Y_2(\mathfrak{gl}_3)$

Como já dito, do corolário (4.13) segue que a variedade de Gelfand-Tsetlin $\mathfrak{G}$ para $Y_1(\mathfrak{gl}_n)$ coincide com a variedade de Gelfand-Tsetlin para $\mathfrak{gl}_n$. Assim, pelo Teorema de Ovsienko (3.15) temos que esta variedade é equidimensional de dimensão

$$\dim\mathfrak{G}=\frac{n(n-1)}{2}.$$

O Teorema de Ovsienko (3.15) afirma que todas as componentes irredutíveis têm a mesma dimensão, mas não diz nada sobre como é sua decomposição em componentes irredutíveis. Com o intuito de apresentar a dificuldade desse problema, temos o seguinte resultado:

Proposição 5.11. *A decomposição em componentes irredutíveis da variedade de Gelfand-Tsetlin $\mathfrak{G}$ para $Y_1(\mathfrak{gl}_3)$ é*

$$\begin{aligned}\mathfrak{G} = V&\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{23}^{(1)},X_{13}^{(1)}\right)\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{32}^{(1)},X_{13}^{(1)}\right)\cup\\ &\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{13}^{(1)}X_{31}^{(1)}+X_{23}^{(1)}X_{32}^{(1)},X_{21}^{(1)}\right)\cup\\ &\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{31}^{(1)},X_{32}^{(1)}\right)\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{21}^{(1)},X_{33}^{(1)},X_{13}^{(1)},X_{23}^{(1)}\right)\cup\\ &\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{21}^{(1)},X_{33}^{(1)},X_{31}^{(1)},X_{23}^{(1)}\right)\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{21}^{(1)},X_{33}^{(1)},X_{32}^{(1)},X_{31}^{(1)}\right).\end{aligned}$$

Portanto, $\mathfrak{G}$ para $Y_1(\mathfrak{gl}_3)$ é equidimensional com

$$\dim\mathfrak{G}=3.$$

Demonstração: Segue da proposição (5.9) ou do corolário (5.10) que

$$\mathfrak{G}=V\left(p_{11},p_{21},p_{22},p_{31},p_{32},p_{33}\right)\subset k^9,$$

em que

$$\begin{aligned}p_{11}&=X_{11}^{(1)},\\ p_{21}&=X_{22}^{(1)},\\ p_{22}&=X_{12}^{(1)}X_{21}^{(1)},\\ p_{31}&=X_{33}^{(1)},\\ p_{32}&=X_{13}^{(1)}X_{31}^{(1)}+X_{23}^{(1)}X_{32}^{(1)},\\ p_{33}&=-X_{13}^{(1)}\underbrace{X_{22}^{(1)}}_{=p_{21}}X_{31}^{(1)}+X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)}+X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)},\end{aligned}$$

logo

$$\begin{aligned}\mathfrak{G}&=V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)}X_{21}^{(1)},X_{33}^{(1)},X_{13}^{(1)}X_{31}^{(1)}+X_{23}^{(1)}X_{32}^{(1)},X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)}+X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)}\right)\\ &=V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{13}^{(1)}X_{31}^{(1)}+X_{23}^{(1)}X_{32}^{(1)},X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)}\right)\cup\\ &\quad\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{21}^{(1)},X_{33}^{(1)},X_{13}^{(1)}X_{31}^{(1)}+X_{23}^{(1)}X_{32}^{(1)},X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)}\right)\\ &=V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{23}^{(1)}X_{32}^{(1)},X_{13}^{(1)}\right)\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{13}^{(1)}X_{31}^{(1)}+X_{23}^{(1)}X_{32}^{(1)},X_{21}^{(1)}\right)\cup\\ &\quad\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{13}^{(1)}X_{31}^{(1)},X_{32}^{(1)}\right)\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{21}^{(1)},X_{33}^{(1)},X_{13}^{(1)}X_{31}^{(1)}+X_{23}^{(1)}X_{32}^{(1)},X_{12}^{(1)}\right)\cup\\ &\quad\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{21}^{(1)},X_{33}^{(1)},X_{13}^{(1)}X_{31}^{(1)},X_{23}^{(1)}\right)\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{21}^{(1)},X_{33}^{(1)},X_{23}^{(1)}X_{32}^{(1)},X_{31}^{(1)}\right)\\ &=V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{23}^{(1)}X_{32}^{(1)},X_{13}^{(1)}\right)\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{13}^{(1)}X_{31}^{(1)}+X_{23}^{(1)}X_{32}^{(1)},X_{21}^{(1)}\right)\cup\\ &\quad\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{13}^{(1)}X_{31}^{(1)},X_{32}^{(1)}\right)\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{21}^{(1)},X_{33}^{(1)},X_{13}^{(1)}X_{31}^{(1)},X_{23}^{(1)}\right)\cup\\ &\quad\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{21}^{(1)},X_{33}^{(1)},X_{23}^{(1)}X_{32}^{(1)},X_{31}^{(1)}\right)\\ &=V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{23}^{(1)},X_{13}^{(1)}\right)\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{32}^{(1)},X_{13}^{(1)}\right)\cup\\ &\quad\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{13}^{(1)}X_{31}^{(1)}+X_{23}^{(1)}X_{32}^{(1)},X_{21}^{(1)}\right)\cup\\ &\quad\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{13}^{(1)},X_{32}^{(1)}\right)\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{12}^{(1)},X_{33}^{(1)},X_{31}^{(1)},X_{32}^{(1)}\right)\cup\\ &\quad\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{21}^{(1)},X_{33}^{(1)},X_{13}^{(1)},X_{23}^{(1)}\right)\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{21}^{(1)},X_{33}^{(1)},X_{31}^{(1)},X_{23}^{(1)}\right)\cup\\ &\quad\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{21}^{(1)},X_{33}^{(1)},X_{23}^{(1)},X_{31}^{(1)}\right)\cup V\left(X_{11}^{(1)},X_{22}^{(1)},X_{21}^{(1)},X_{33}^{(1)},X_{32}^{(1)},X_{31}^{(1)}\right)\end{aligned}$$

$$
\begin{aligned}
&= V\left(X_{11}^{(1)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{33}^{(1)}, X_{23}^{(1)}, X_{13}^{(1)}\right) \cup V\left(X_{11}^{(1)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{33}^{(1)}, X_{32}^{(1)}, X_{13}^{(1)}\right) \cup \\
&\cup V\left(X_{11}^{(1)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{33}^{(1)}, X_{13}^{(1)}X_{31}^{(1)} + X_{23}^{(1)}X_{32}^{(1)}, X_{21}^{(1)}\right) \cup \\
&\cup V\left(X_{11}^{(1)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{33}^{(1)}, X_{31}^{(1)}, X_{32}^{(1)}\right) \cup V\left(X_{11}^{(1)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{33}^{(1)}, X_{13}^{(1)}, X_{23}^{(1)}\right) \cup \\
&\cup V\left(X_{11}^{(1)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{33}^{(1)}, X_{31}^{(1)}, X_{23}^{(1)}\right) \cup V\left(X_{11}^{(1)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{33}^{(1)}, X_{32}^{(1)}, X_{31}^{(1)}\right).
\end{aligned}
$$

Portanto, $\mathfrak{G}$ para $Y_1(\mathfrak{gl}_3)$ tem 7 componentes irredutíveis e cada componente tem dimensão 3.

□

Com o mesmo intuito, de apresentar a dificuldade da equidimensionalidade de $\mathfrak{G}$, a seguinte proposição garante a equidimensionalidade da variedade de Gelfand-Tsetlin $\mathfrak{G}$ para $Y_2(\mathfrak{gl}_3)$, apresentando sua decomposição em componentes irredutíveis. Todos os detalhes da decomposição estão no apêndice (B).

Proposição 5.12. *A variedade de Gelfand-Tsetlin $\mathfrak{G}$ para $Y_2(\mathfrak{gl}_3)$ é equidimensional com*

$$\dim \mathfrak{G} = 6.$$

Demonstração. A proposição (5.9) ou corolário (5.10), garantem que

$$\mathfrak{G} = V\left(\left\{p_{ij}\right\}_{\substack{i=1,2,3 \\ j=1,2,\ldots,2i}}\right) \subset k^{18},$$

em que

$$
\begin{aligned}
p_{11} &= X_{11}^{(1)}, \\
p_{12} &= X_{11}^{(2)}, \\
p_{21} &= X_{22}^{(1)}, \\
p_{22} &= X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, \\
p_{23} &= X_{12}^{(1)}X_{21}^{(2)} + X_{12}^{(2)}X_{21}^{(1)}, \\
p_{24} &= X_{12}^{(2)}X_{21}^{(2)}, \\
p_{31} &= X_{33}^{(1)}, \\
p_{32} &= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
p_{33} &= X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(1)} - X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(1)}\underbrace{X_{22}^{(1)}}_{=p_{21}}X_{31}^{(1)}, \\
p_{34} &= -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)} - X_{13}^{(1)}\underbrace{X_{22}^{(1)}}_{=p_{21}}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(1)} + \\
&\quad + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)} - X_{13}^{(2)}\underbrace{X_{22}^{(1)}}_{=p_{21}}X_{31}^{(1)} + X_{12}^{(2)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}, \\
p_{35} &= -X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}\underbrace{X_{22}^{(1)}}_{=p_{21}}X_{31}^{(2)} + X_{12}^{(2)}X_{23}^{(1)}X_{31}^{(2)} + \\
&\quad + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(1)} + X_{12}^{(2)}X_{23}^{(2)}X_{31}^{(1)} + X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(1)}, \\
p_{36} &= -X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(2)} + X_{12}^{(2)}X_{23}^{(2)}X_{31}^{(2)} + X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(2)}.
\end{aligned}
$$

Pelo apêndice (B) temos que a decomposição em componentes irredutíveis é

$$\mathfrak{G} = \bigcup_{i=1}^{22} C_i,$$

em que

$$
\begin{aligned}
C_1 &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{23}^{(1)}, X_{23}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}\right),\\
C_2 &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)}, X_{32}^{(2)}, X_{23}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}\right),\\
C_3 &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}\right),\\
C_4 &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{32}^{(2)}, X_{21}^{(2)}, X_{13}^{(2)}\Big),\\
C_5 &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad -X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{31}^{(2)}, -X_{23}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{13}^{(2)}\Big),\\
C_6 &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big),\\
C_7 &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{23}^{(1)}X_{32}^{(2)} + X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{31}^{(2)} + X_{13}^{(2)}X_{31}^{(1)}, X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(2)}X_{31}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}\Big),\\
C_8 &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{21}^{(1)}X_{32}^{(1)} - X_{31}^{(2)}, X_{32}^{(2)}, X_{21}^{(2)}\Big),\\
C_9 &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{31}^{(1)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}\Big),\\
C_{10} &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{23}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big),\\
C_{11} &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big),\\
C_{12} &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{32}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big),\\
C_{13} &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{13}^{(1)}, X_{23}^{(1)}, X_{23}^{(2)}, X_{13}^{(2)}\Big),\\
C_{14} &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{23}^{(1)}X_{32}^{(2)} - X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{31}^{(2)}, X_{23}^{(2)}, X_{22}^{(2)}\Big),\\
C_{15} &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{23}^{(1)}X_{32}^{(2)} + X_{13}^{(1)}X_{31}^{(2)}, X_{12}^{(1)}X_{23}^{(1)} - X_{13}^{(2)}, X_{23}^{(2)}, X_{22}^{(2)}\Big),\\
C_{16} &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{22}^{(2)}\Big),
\end{aligned}
$$

$$C_{17}=V\left(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)}-X_{12}^{(1)}X_{21}^{(1)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)},X_{31}^{(1)},X_{32}^{(1)},X_{32}^{(2)},X_{31}^{(2)}\right),$$
$$C_{18}=V\left(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)},X_{21}^{(1)},X_{21}^{(2)},X_{33}^{(1)},X_{33}^{(2)},X_{32}^{(1)},X_{32}^{(2)},X_{31}^{(1)},X_{31}^{(2)}\right),$$
$$C_{19}=V\left(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)},X_{21}^{(1)},X_{21}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{32}^{(1)}X_{23}^{(1)},X_{23}^{(2)},X_{32}^{(2)},X_{31}^{(1)},X_{31}^{(2)}\right),$$
$$C_{20}=V\left(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)},X_{21}^{(1)},X_{21}^{(2)},X_{33}^{(1)},X_{33}^{(2)},X_{23}^{(1)},X_{23}^{(2)},X_{31}^{(1)},X_{31}^{(2)}\right),$$
$$C_{21}=V\left(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)},X_{21}^{(1)},X_{21}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{31}^{(1)}X_{13}^{(1)},X_{13}^{(2)},X_{23}^{(1)},X_{23}^{(2)},X_{31}^{(2)}\right),$$
$$C_{22}=V\left(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)},X_{21}^{(1)},X_{21}^{(2)},X_{33}^{(1)},X_{33}^{(2)},X_{13}^{(1)},X_{13}^{(2)},X_{23}^{(1)},X_{23}^{(2)}\right).$$

Cada componente irredutível de $\mathfrak{G}$ tem dimensão 6.

□

5.3 Uma primeira decomposição de $\mathfrak{G}$ para Yangians restritos $Y_p(\mathfrak{gl}_n)$

Nesta seção apresentaremos uma decomposição da variedade de Gelfand-Tsetlin $\mathfrak{G}$ para Yangians $Y_p(\mathfrak{gl}_n)$. Mas, vale a pena ressaltar que como veremos nesta seção, tal decomposição não é em componentes irredutíveis.

Proposição 5.13. *Para $Y_p(\mathfrak{gl}_n)$*

$$\mathfrak{G}=\bigcup_{s=1}^{p+1}\mathfrak{G}_s,$$

em que

$$\mathfrak{G}_1:=V\left(\{p_{ij}\}_{\substack{i=1,2\\ j=1,2,\dots,p}}\cup\left\{X_{12}^{(i)}\right\}_{i=1}^{p}\cup\{p_{ij}\}_{\substack{i=3,4,\dots,n\\ j=1,2,\dots,ip}}\right)\subset k^{n^2p},$$

$$\mathfrak{G}_s:=V\left(\{p_{ij}\}_{\substack{i=1,2\\ j=1,2,\dots,p}}\cup\left\{X_{12}^{(i)}\right\}_{i=s}^{p}\cup\left\{X_{21}^{(i)}\right\}_{i=p-(s-2)}^{p}\cup\{p_{ij}\}_{\substack{i=3,4,\dots,n\\ j=1,2,\dots,ip}}\right)\subset k^{n^2p},\ \textit{para}\ 2\le s\le p,$$

$$\mathfrak{G}_{p+1}:=V\left(\{p_{ij}\}_{\substack{i=1,2\\ j=1,2,\dots,p}}\cup\left\{X_{21}^{(i)}\right\}_{i=1}^{p}\cup\{p_{ij}\}_{\substack{i=3,4,\dots,n\\ j=1,2,\dots,ip}}\right)\subset k^{n^2p}.$$

Demonstração. Claramente $\mathfrak{G}_s\subseteq\mathfrak{G}$ para cada $s=1,2,\dots,p+1$, portanto

$$\mathfrak{G}\supseteq\bigcup_{s=1}^{p+1}\mathfrak{G}_s.$$

Agora, para

$$\mathfrak{G}\subseteq\bigcup_{s=1}^{p+1}\mathfrak{G}_s,$$

consideremos $A=\left(a_{ij}^{(t)}\right)_{\substack{i,j=1,2,3\\ t=1,2,\dots,p}}\in\mathfrak{G}$ e o polinômio

$$p_{22p}=X_{12}^{(p)}X_{21}^{(p)},$$

pela proposição (5.9) ou corolário (5.10),

$$p_{22p}(A)=a_{12}^{(p)}a_{21}^{(p)}=0,$$

logo

$$a_{12}^{(p)}=0\ \text{ ou }\ a_{21}^{(p)}=0.$$

Portanto, se $p=1$ então

$$\mathfrak{G}=\mathfrak{G}_1\cup\mathfrak{G}_2.$$

Novamente, se $p>1$ consideremos também o polinômio

$$p_{2\,2p-1}=\sum_{t=p-1}^{p}X_{12}^{(t)}X_{21}^{(2p-1-t)}=X_{12}^{(p-1)}X_{21}^{(p)}+X_{12}^{(p)}X_{21}^{(p-1)},$$

como consequência da proposição (5.9) ou do corolário (5.10), temos

$$p_{2\,2p-1}(A)=a_{12}^{(p-1)}a_{21}^{(p)}+a_{12}^{(p)}a_{21}^{(p-1)}=0,$$

assim

$$a_{12}^{(p)}=a_{12}^{(p-1)}=0 \ \text{ ou } \ a_{12}^{(p)}=a_{21}^{(p)}=0 \ \text{ ou } \ a_{21}^{(p)}=a_{21}^{(p-1)}=0.$$

Portanto, se $p=2$ então

$$\mathfrak{G}=\mathfrak{G}_1\cup\mathfrak{G}_2\cup\mathfrak{G}_3.$$

Agora, para o caso que $p>2$. Dado que $A\in\mathfrak{G}_1$ quando $a_{12}^{(t)}=0$ para todo $t=1,2,\dots,p$ e $A\in\mathfrak{G}_{p+1}$ quando $a_{21}^{(t)}=0$ para todo $t=1,2,\dots,p$, então podemos assumir que, existe $s\in\{2,3,\dots,p\}$ tal que

$$a_{12}^{(t)}=0,\ \ \forall t\geq s \quad \text{e} \quad a_{12}^{(s-1)}\neq 0.$$

Como consequência do polinômio

$$p_{p\,p+s-1}=\sum_{t=s-1}^{p}X_{12}^{(t)}X_{21}^{(p+s-1-t)}=X_{12}^{(s-1)}X_{21}^{(p)}+\sum_{t=s}^{p}X_{12}^{(t)}X_{21}^{(p+s-1-t)}$$

e da proposição (5.9) ou corolário (5.10), tem-se

$$p_{p\,p+s-1}(A)=a_{12}^{(s-1)}a_{21}^{(p)}+\sum_{t=s}^{p}a_{12}^{(t)}a_{21}^{(p+s-1-t)}=0$$

e assim

$$a_{21}^{(p)}=0$$

implica que, se $s=2$ então $A\in\mathfrak{G}_2$.

Continuando com o mesmo raciocinio para $3\leq s\leq p$, se

$$a_{21}^{(t)}=0,\ \ \forall t\geq j \quad \text{para algum} \quad j\in\{p-s+3,p-s+4,\dots,p\}.$$

Dado que[4] $1\leq s-p+j-2\leq p$, então temos o polinômio

$$\begin{aligned}p_{p\,s+j-2}=p_{p\,p+s-p+j-2}&=\sum_{t=s-p+j-2}^{p}X_{12}^{(t)}X_{21}^{(p+s-p+j-2-t)}=\sum_{t=s-p+j-2}^{p}X_{12}^{(t)}X_{21}^{(s+j-2-t)}\\&=\sum_{t=s-p+j-2}^{s-2}X_{12}^{(t)}X_{21}^{(s+j-2-t)}+X_{12}^{(s-1)}X_{21}^{(j-1)}+\sum_{t=s}^{p}X_{12}^{(t)}X_{21}^{(s+j-2-t)}\\&=\sum_{m=j}^{p}X_{12}^{(s+j-2-m)}X_{21}^{(m)}+X_{12}^{(s-1)}X_{21}^{(j-1)}+\sum_{t=s}^{p}X_{12}^{(t)}X_{21}^{(s+j-2-t)}\ \text{(fazendo } m=s+j-2-t\text{)},\end{aligned}$$

[4]Como $p-s+3\leq j\leq p$ e $3\leq s\leq p$, segue-se $p+1\leq s+j-2\leq p+s-2\leq 2p$ e portanto, $1\leq s-p+j-2\leq p$.

logo da proposição (5.9) ou corolário (5.10),

$$p_{p\,s+j-2}(A) = \underbrace{\sum_{m=j}^{p} a_{12}^{(s+j-2-m)} a_{21}^{(m)}}_{=0} + a_{12}^{(s-1)} a_{21}^{(j-1)} + \underbrace{\sum_{t=s}^{p} a_{12}^{(t)} a_{21}^{(s+j-2-t)}}_{=0} = 0,$$

disso $a_{21}^{(j-1)} = 0$ e portanto

$$A \in \mathfrak{G}_s, \;\; 3 \leq s \leq p.$$

□

5.3.1 Uma decomposição para $\mathfrak{G}_1$

Note que, como

$$\mathfrak{G}_1 = V\left(\left\{p_{ij}\right\}_{\substack{i=1,2\\ j=1,2,\dots,p}} \cup \left\{X_{12}^{(i)}\right\}_{i=1}^{p} \cup \left\{p_{ij}\right\}_{\substack{i=3,4,\dots,n\\ j=1,2,\dots,ip}}\right) \subset k^{n^2p},$$

então da proposição (5.9) ou corolário (5.10), pode concluirse que

$$\mathfrak{G}_1 = V\left(\mathbf{W} \cup \left\{p_{ij}\right\}_{\substack{i=3,4,\dots,n\\ j=1,2,\dots,ip}}\right) \subset k^{n^2p},$$

com

$$\mathbf{W} := \bigcup_{i=1}^{p} \left\{X_{11}^{(i)}, X_{12}^{(i)}, X_{22}^{(i)}\right\}.$$

Quando $n = 3$, chamaremos $\mathfrak{G}_1$ de **versão fraca da variedade de Gelfand-Tsetlin** $\mathfrak{G}$ para $Y_p(\mathfrak{gl}_3)$.

No que segue, iremos procurar uma decomposição (não necessariamente em componentes irredutíveis) para $\mathfrak{G}_1$. Para isso precisamos do seguinte lema:

Lema 5.14. *Suponha que $p > 2$ e $A = \left(a_{ij}^{(t)}\right)_{\substack{i,j=1,2,\dots,n\\ t=1,2,\dots,p}} \in \mathfrak{G}_1$*

1.

(a) Se $1 \leq i \leq p-2$ e

$$a_{13}^{(p)} = a_{13}^{(p-1)} = a_{13}^{(p-2)} = \cdots = a_{13}^{(p-i)} = 0,$$

então

$$a_{13}^{(p-i-1)} = 0 \;\; ou \;\; a_{21}^{(p)} = 0 \;\; ou \;\; a_{32}^{(p)} = 0.$$

(b) Se $1 \leq j \leq i \leq p-2$ e

$$a_{13}^{(p)} = a_{13}^{(p-1)} = a_{13}^{(p-2)} = \cdots = a_{13}^{(p-i+j)} = a_{21}^{(p)} = a_{21}^{(p-1)} = a_{21}^{(p-2)} = \cdots = a_{21}^{(p-j+1)} = 0,$$

então

$$a_{13}^{(p-i+j-1)} = 0 \;\; ou \;\; a_{21}^{(p-j)} = 0 \;\; ou \;\; a_{32}^{(p)} = 0.$$

(c) Se $1 \leq i \leq p-2$ e

$$a_{21}^{(p)} = a_{21}^{(p-1)} = a_{21}^{(p-2)} = \cdots = a_{21}^{(p-i)} = 0,$$

então

$$a_{13}^{(p)} = 0 \;\; ou \;\; a_{21}^{(p-i-1)} = 0 \;\; ou \;\; a_{32}^{(p)} = 0.$$

2.

(a) *Se* $1 \leq j \leq i \leq p-2$ *e*

$$a_{21}^{(p)} = a_{21}^{(p-1)} = a_{21}^{(p-2)} = \cdots = a_{21}^{(p-i+j)} = a_{32}^{(p)} = a_{32}^{(p-1)} = a_{32}^{(p-2)} = \cdots = a_{32}^{(p-j+1)} = 0,$$

então

$$a_{13}^{(p)} = 0 \;\; ou \;\; a_{21}^{(p-i+j-1)} = 0 \;\; ou \;\; a_{32}^{(p-j)} = 0.$$

(b) *Se* $1 \leq i \leq p-2$ *e*

$$a_{32}^{(p)} = a_{32}^{(p-1)} = a_{32}^{(p-2)} = \cdots = a_{32}^{(p-i)} = 0,$$

então

$$a_{13}^{(p)} = 0 \;\; ou \;\; a_{21}^{(p)} = 0 \;\; ou \;\; a_{32}^{(p-i-1)} = 0.$$

3. *Se* $1 \leq j \leq i \leq p-2$ *e*

$$a_{13}^{(p)} = a_{13}^{(p-1)} = a_{13}^{(p-2)} = \cdots = a_{13}^{(p-i+j)} = a_{32}^{(p)} = a_{32}^{(p-1)} = a_{32}^{(p-2)} = \cdots = a_{32}^{(p-j+1)} = 0,$$

então

$$a_{13}^{(p-i+j-1)} = 0 \;\; ou \;\; a_{21}^{(p)} = 0 \;\; ou \;\; a_{32}^{(p-j)} = 0.$$

Demonstração: Como $1 \leq i \leq p-2$, então $1 \leq p-i-1 \leq p-2$. Podemos considerar o polinômio

$$p_{33p-i-1} = \sum_{r=p-i-1}^{p} \sum_{s=2p-i-1-r}^{p} \left\{ -X_{13}^{(r)} X_{22}^{(s)} X_{31}^{(3p-i-1-r-s)} + X_{12}^{(r)} X_{23}^{(s)} X_{31}^{(3p-i-1-r-s)} + X_{13}^{(r)} X_{21}^{(s)} X_{32}^{(3p-i-1-r-s)} \right\},$$

do fato que $A \in \mathfrak{G}_1$ e da proposição (5.9) ou corolário (5.10), temos a equação

$$\begin{aligned} p_{33p-i-1}(A) &= \sum_{r=p-i-1}^{p} \sum_{s=2p-i-1-r}^{p} \left\{ \underbrace{-a_{13}^{(r)} a_{22}^{(s)} a_{31}^{(3p-i-1-r-s)} + a_{12}^{(r)} a_{23}^{(s)} a_{31}^{(3p-i-1-r-s)}}_{=0} + a_{13}^{(r)} a_{21}^{(s)} a_{32}^{(3p-i-1-r-s)} \right\} \\ 0 &= \sum_{r=p-i-1}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(s)} a_{32}^{(3p-i-1-r-s)} \end{aligned}$$

1.

(a) Se $1 \leq i \leq p-2$ e

$$a_{13}^{(p)} = a_{13}^{(p-1)} = a_{13}^{(p-2)} = \cdots = a_{13}^{(p-i)} = 0,$$

então

$$\begin{aligned} p_{33p-i-1}(A) &= \sum_{r=p-i-1}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(s)} a_{32}^{(3p-i-1-r-s)} = 0 \\ 0 &= a_{13}^{(p-i-1)} a_{21}^{(p)} a_{32}^{(p)} + \underbrace{\sum_{r=p-i}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(s)} a_{32}^{(3p-i-1-r-s)}}_{=0} \\ 0 &= a_{13}^{(p-i-1)} a_{21}^{(p)} a_{32}^{(p)}, \end{aligned}$$

assim

$$a_{13}^{(p-i-1)} = 0 \quad \text{ou} \quad a_{21}^{(p)} = 0 \quad \text{ou} \quad a_{32}^{(p)} = 0.$$

(b) Se $1 \le j \le i \le p-2$ e

$$a_{13}^{(p)} = a_{13}^{(p-1)} = a_{13}^{(p-2)} = \cdots = a_{13}^{(p-i+j)} = a_{21}^{(p)} = a_{21}^{(p-1)} = a_{21}^{(p-2)} = \cdots = a_{21}^{(p-j+1)} = 0,$$

então

$$\begin{aligned}
p_{33p-i-1}(A) &= \sum_{r=p-i-1}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(s)} a_{32}^{(3p-i-1-r-s)} = 0 \\
0 &= \underbrace{\sum_{r=p-i-1}^{p-i+j-2} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(s)} a_{32}^{(3p-i-1-r-s)}}_{=0} + \sum_{s=p-j}^{p} a_{13}^{(p-i+j-1)} a_{21}^{(s)} a_{32}^{(2p-j-s)} + \\
&\quad + \underbrace{\sum_{r=p-i+j}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(s)} a_{32}^{(3p-i-1-r-s)}}_{=0} \\
0 &= \sum_{s=p-j}^{p} a_{13}^{(p-i+j-1)} a_{21}^{(s)} a_{32}^{(2p-j-s)} \\
0 &= a_{13}^{(p-i+j-1)} a_{21}^{(p-j)} a_{32}^{(p)} + \underbrace{\sum_{s=p-j+1}^{p} a_{13}^{(p-i+j-1)} a_{21}^{(s)} a_{32}^{(2p-j-s)}}_{=0} \\
0 &= a_{13}^{(p-i+j-1)} a_{21}^{(p-j)} a_{32}^{(p)},
\end{aligned}$$

logo

$$a_{13}^{(p-i+j-1)} = 0 \ \text{ ou } \ a_{21}^{(p-j)} = 0 \ \text{ ou } \ a_{32}^{(p)} = 0.$$

(c) Se $1 \le i \le p-2$ e

$$a_{21}^{(p)} = a_{21}^{(p-1)} = a_{21}^{(p-2)} = \cdots = a_{21}^{(p-i)} = 0,$$

então

$$\begin{aligned}
p_{33p-i-1}(A) &= \sum_{r=p-i-1}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(s)} a_{32}^{(3p-i-1-r-s)} = 0 \\
0 &= \underbrace{\sum_{r=p-i-1}^{p-1} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(s)} a_{32}^{(3p-i-1-r-s)}}_{=0} + \sum_{s=p-i-1}^{p} a_{13}^{(p)} a_{21}^{(s)} a_{32}^{(2p-i-1-s)} \\
0 &= \sum_{s=p-i-1}^{p} a_{13}^{(p)} a_{21}^{(s)} a_{32}^{(2p-i-1-s)} \\
0 &= a_{13}^{(p)} a_{21}^{(p-i-1)} a_{32}^{(p)} + \underbrace{\sum_{s=p-i}^{p} a_{13}^{(p)} a_{21}^{(s)} a_{32}^{(2p-i-1-s)}}_{=0} \\
0 &= a_{13}^{(p)} a_{21}^{(p-i-1)} a_{32}^{(p)},
\end{aligned}$$

logo

$$a_{13}^{(p)} = 0 \ \text{ ou } \ a_{21}^{(p-i-1)} = 0 \ \text{ ou } \ a_{32}^{(p)} = 0.$$

2.

(a) Se $1 \leq j \leq i \leq p-2$ e

$$a_{21}^{(p)} = a_{21}^{(p-1)} = a_{21}^{(p-2)} = \cdots = a_{21}^{(p-i+j)} = a_{32}^{(p)} = a_{32}^{(p-1)} = a_{32}^{(p-2)} = \cdots = a_{32}^{(p-j+1)} = 0,$$

então

$$\begin{aligned}
p_{33p-i-1}(A) = \sum_{r=p-i-1}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(s)} a_{32}^{(3p-i-1-r-s)} &= 0 \\
0 &= \sum_{r=p-i-1}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(3p-i-1-r-s)} a_{21}^{(r)} a_{32}^{(s)} \\
0 &= \underbrace{\sum_{r=p-i-1}^{p-i+j-2} \sum_{s=2p-i-1-r}^{p} a_{13}^{(3p-i-1-r-s)} a_{21}^{(r)} a_{32}^{(s)}}_{=0} + \sum_{s=p-j}^{p} a_{13}^{(2p-j-s)} a_{21}^{(p-i+j-1)} a_{32}^{(s)} \\
&\quad + \underbrace{\sum_{r=p-i+j}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(3p-i-1-r-s)} a_{21}^{(r)} a_{32}^{(s)}}_{=0} \\
0 &= \sum_{s=p-j}^{p} a_{13}^{(2p-j-s)} a_{21}^{(p-i+j-1)} a_{32}^{(s)} \\
0 &= a_{13}^{(p)} a_{21}^{(p-i+j-1)} a_{32}^{(p-j)} + \underbrace{\sum_{s=p-j+1}^{p} a_{13}^{(2p-j-s)} a_{21}^{(p-i+j-1)} a_{32}^{(s)}}_{=0} \\
0 &= a_{13}^{(p)} a_{21}^{(p-i+j-1)} a_{32}^{(p-j)},
\end{aligned}$$

logo

$$a_{13}^{(p)} = 0 \quad \text{ou} \quad a_{21}^{(p-i+j-1)} = 0 \quad \text{ou} \quad a_{32}^{(p-j)} = 0.$$

(b) Se $1 \leq i \leq p-2$ e

$$a_{32}^{(p)} = a_{32}^{(p-1)} = a_{32}^{(p-2)} = \cdots = a_{32}^{(p-i)} = 0,$$

então

$$\begin{aligned}
p_{33p-i-1}(A) = \sum_{r=p-i-1}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(s)} a_{32}^{(3p-i-1-r-s)} &= 0 \\
0 &= \sum_{r=p-i-1}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(3p-i-1-r-s)} a_{21}^{(r)} a_{32}^{(s)} \\
0 &= \underbrace{\sum_{r=p-i-1}^{p-1} \sum_{s=2p-i-1-r}^{p} a_{13}^{(3p-i-1-r-s)} a_{21}^{(r)} a_{32}^{(s)}}_{=0} + \sum_{s=p-i-1}^{p} a_{13}^{(2p-i-1-s)} a_{21}^{(p)} a_{32}^{(s)} \\
0 &= \sum_{s=p-i-1}^{p} a_{13}^{(2p-i-1-s)} a_{21}^{(p)} a_{32}^{(s)} \\
0 &= a_{13}^{(p)} a_{21}^{(p)} a_{32}^{(p-i-1)} + \underbrace{\sum_{s=p-i}^{p} a_{13}^{(2p-i-1-s)} a_{21}^{(p)} a_{32}^{(s)}}_{=0}
\end{aligned}$$

$$0 = a_{13}^{(p)} a_{21}^{(p)} a_{32}^{(p-i-1)},$$

assim

$$a_{13}^{(p)} = 0 \quad \text{ou} \quad a_{21}^{(p)} = 0 \quad \text{ou} \quad a_{32}^{(p-i-1)} = 0.$$

3. Se $1 \leq j \leq i \leq p-2$ e

$$a_{13}^{(p)} = a_{13}^{(p-1)} = a_{13}^{(p-2)} = \cdots = a_{13}^{(p-i+j)} = a_{32}^{(p)} = a_{32}^{(p-1)} = a_{32}^{(p-2)} = \cdots = a_{32}^{(p-j+1)} = 0,$$

então

$$\begin{aligned}
p_{3\,3p-i-1}(A) &= \sum_{r=p-i-1}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(s)} a_{32}^{(3p-i-1-r-s)} = 0 \\
0 &= \sum_{r=p-i-1}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(3p-i-1-r-s)} a_{32}^{(s)} \\
0 &= \underbrace{\sum_{r=p-i-1}^{p-i+j-2} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(3p-i-1-r-s)} a_{32}^{(s)}}_{=0} + \sum_{s=p-j}^{p} a_{13}^{(p-i+j-1)} a_{21}^{(2p-j-s)} a_{32}^{(s)} + \\
&\quad + \underbrace{\sum_{r=p-i+j}^{p} \sum_{s=2p-i-1-r}^{p} a_{13}^{(r)} a_{21}^{(3p-i-1-r-s)} a_{32}^{(s)}}_{=0} \\
0 &= \sum_{s=p-j}^{p} a_{13}^{(p-i+j-1)} a_{21}^{(2p-j-s)} a_{32}^{(s)} \\
0 &= a_{13}^{(p-i+j-1)} a_{21}^{(p)} a_{32}^{(p-j)} + \underbrace{\sum_{s=p-j+1}^{p} a_{13}^{(p-i+j-1)} a_{21}^{(2p-j-s)} a_{32}^{(s)}}_{=0} \\
0 &= a_{13}^{(p-i+j-1)} a_{21}^{(p)} a_{32}^{(p-j)},
\end{aligned}$$

logo

$$a_{13}^{(p-i+j-1)} = 0 \quad \text{ou} \quad a_{21}^{(p)} = 0 \quad \text{ou} \quad a_{32}^{(p-j)} = 0.$$

□

Notação 5.15. *Para $Y_p(\mathfrak{gl}_n)$ usaremos as seguintes subvariedades de $\mathfrak{G}_1$:*

1.

$$\mathbb{W}^p := V\left(\boldsymbol{W} \cup \{p_{3j}\}_{j=1}^{3p-3} \cup \left\{X_{32}^{(p)}, X_{21}^{(p)}, X_{13}^{(p)}\right\} \cup \{p_{ij}\}_{\substack{i=4,5,\ldots,n \\ j=1,2,\ldots,ip}}\right).$$

2.

$$\mathbb{W}_{11} := V\left(\boldsymbol{W} \cup \{p_{3j}\}_{j=1}^{2p} \cup \left\{X_{13}^{(t)}\right\}_{t=1}^{p} \cup \{p_{ij}\}_{\substack{i=4,5,\ldots,n \\ j=1,2,\ldots,ip}}\right).$$

3. Para cada $s = 2,3,\ldots,p$,

$$\mathbb{W}_{1s} := V\left(\boldsymbol{W} \cup \{p_{3j}\}_{j=1}^{2p} \cup \left\{X_{13}^{(t)}\right\}_{t=s}^{p} \cup \left\{X_{21}^{(t)}\right\}_{t=p-(s-2)}^{p} \cup \{p_{ij}\}_{\substack{i=4,5,\ldots,n \\ j=1,2,\ldots,ip}}\right).$$

4.

$$\mathbb{W}_{1\,p+1} := V\left(\boldsymbol{W} \cup \{p_{3j}\}_{j=1}^{2p} \cup \left\{X_{21}^{(t)}\right\}_{t=1}^{p} \cup \{p_{ij}\}_{\substack{i=4,5,\ldots,n \\ j=1,2,\ldots,ip}}\right).$$

5.
$$\mathbb{W}_1 := \bigcup_{s=1}^{p+1} \mathbb{W}_{1s}.$$

6. *Para cada $s = 2, 3, \ldots, p$,*
$$\mathbb{W}_{2s} := V\left(\boldsymbol{W} \cup \left\{p_{3j}\right\}_{j=1}^{2p} \cup \left\{X_{21}^{(t)}\right\}_{t=s}^{p} \cup \left\{X_{32}^{(t)}\right\}_{t=p-(s-2)}^{p} \cup \left\{p_{ij}\right\}_{\substack{i=4,5,\ldots,n \\ j=1,2,\ldots,ip}}\right).$$

7.
$$\mathbb{W}_{2p+1} := V\left(\boldsymbol{W} \cup \left\{p_{3j}\right\}_{j=1}^{2p} \cup \left\{X_{32}^{(t)}\right\}_{t=1}^{p} \cup \left\{p_{ij}\right\}_{\substack{i=4,5,\ldots,n \\ j=1,2,\ldots,ip}}\right).$$

8.
$$\mathbb{W}_2 := \bigcup_{s=2}^{p+1} \mathbb{W}_{2s}.$$

9. *Para cada $s = 2, 3, \ldots, p$,*
$$\mathbb{W}_{3s} := V\left(\boldsymbol{W} \cup \left\{p_{3j}\right\}_{j=1}^{2p} \cup \left\{X_{13}^{(t)}\right\}_{t=s}^{p} \cup \left\{X_{32}^{(t)}\right\}_{t=p-(s-2)}^{p} \cup \left\{p_{ij}\right\}_{\substack{i=4,5,\ldots,n \\ j=1,2,\ldots,ip}}\right).$$

10.
$$\mathbb{W}_3 := \bigcup_{s=2}^{p} \mathbb{W}_{3s}.$$

Proposição 5.16. *Para $Y_p(\mathfrak{gl}_n)$*
$$\mathfrak{G}_1 = \mathbb{W}_1 \cup \mathbb{W}_2, \ \ se \ \ p = 1;$$
$$\mathfrak{G}_1 = \mathbb{W}_1 \cup \mathbb{W}_2 \cup \mathbb{W}_3, \ \ se \ \ p = 2;$$
$$\mathfrak{G}_1 = \mathbb{W}^p \cup \mathbb{W}_1 \cup \mathbb{W}_2 \cup \mathbb{W}_3, \ \ se \ \ p \geq 3.$$

Demonstração: Claramente, cada $\mathbb{W}$'s está contido em $\mathfrak{G}_1$, então só temos que provar que
$$\mathfrak{G}_1 \subseteq \mathbb{W}_1 \cup \mathbb{W}_2, \ \text{ para } \ p = 1,$$
$$\mathfrak{G}_1 \subseteq \mathbb{W}_1 \cup \mathbb{W}_2 \cup \mathbb{W}_3, \ \text{ para } \ p = 2,$$
$$\mathfrak{G}_1 \subseteq \mathbb{W}^p \cup \mathbb{W}_1 \cup \mathbb{W}_2 \cup \mathbb{W}_3, \ \text{ para } \ p \geq 3.$$
De fato, consideremos $A = \left(a_{ij}^{(t)}\right)_{\substack{i,j=1,2,\ldots,n \\ t=1,2,\ldots,p}} \in \mathfrak{G}_1$. Para esta prova usaremos os polinômios
$$p_{32p+i} = \sum_{r=i}^{p} \sum_{s=p+i-r}^{p} \left\{-X_{13}^{(r)} X_{22}^{(s)} X_{31}^{(2p+i-r-s)} + X_{12}^{(r)} X_{23}^{(s)} X_{31}^{(2p+i-r-s)} + X_{13}^{(r)} X_{21}^{(s)} X_{32}^{(2p+i-r-s)}\right\}, \quad i = p-1, p.$$

Comecemos com o polinômio
$$p_{33p} = -X_{13}^{(p)} X_{22}^{(p)} X_{31}^{(p)} + X_{12}^{(p)} X_{23}^{(p)} X_{31}^{(p)} + X_{13}^{(p)} X_{21}^{(p)} X_{32}^{(p)},$$
com isso e a proposição (5.9) ou corolário (5.10),
$$p_{33p}(A) = \underbrace{-a_{13}^{(p)} a_{22}^{(p)} a_{31}^{(p)} + a_{12}^{(p)} a_{23}^{(p)} a_{31}^{(p)}}_{=0} + a_{13}^{(p)} a_{21}^{(p)} a_{32}^{(p)} = 0,$$

logo

$$a_{13}^{(p)}=0 \quad \text{ou} \quad a_{21}^{(p)}=0 \quad \text{ou} \quad a_{32}^{(p)}=0,$$

como consequência disso, se $p=1$ então $A\in\mathbb{W}_1\cup\mathbb{W}_2$. Para $p>1$, consideremos o polinômio

$$p_{33p-1}=\sum_{r=p-1}^{p}\sum_{s=2p-1-r}^{p}\left\{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(3p-1-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(3p-1-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(3p-1-r-s)}\right\},$$

segue da proposição (5.9) ou corolário (5.10) que

$$\begin{aligned}
p_{33p-1}(A)&=\sum_{r=p-1}^{p}\sum_{s=2p-1-r}^{p}\left\{\underbrace{-a_{13}^{(r)}a_{22}^{(s)}a_{31}^{(3p-1-r-s)}+a_{12}^{(r)}a_{23}^{(s)}a_{31}^{(3p-1-r-s)}}_{=0}+a_{13}^{(r)}a_{21}^{(s)}a_{32}^{(3p-1-r-s)}\right\}=0\\
&=\sum_{r=p-1}^{p}\sum_{s=2p-1-r}^{p}a_{13}^{(r)}a_{21}^{(s)}a_{32}^{(3p-1-r-s)}=0\\
&=\sum_{s=p}^{p}a_{13}^{(p-1)}a_{21}^{(s)}a_{32}^{(2p-s)}+\sum_{s=p-1}^{p}a_{13}^{(p)}a_{21}^{(s)}a_{32}^{(2p-1-s)}=0\\
&=a_{13}^{(p-1)}a_{21}^{(p)}a_{32}^{(p)}+a_{13}^{(p)}a_{21}^{(p-1)}a_{32}^{(p)}+a_{13}^{(p)}a_{21}^{(p)}a_{32}^{(p-1)}=0,
\end{aligned}$$

assim

$$a_{13}^{(p)}=a_{13}^{(p-1)}=0 \ \text{ ou } \ a_{13}^{(p)}=a_{21}^{(p)}=0 \ \text{ ou } \ a_{13}^{(p)}=a_{32}^{(p)}=0$$

ou

$$a_{21}^{(p)}=a_{21}^{(p-1)}=0 \ \text{ ou } \ a_{21}^{(p)}=a_{32}^{(p)}=0$$

ou

$$a_{32}^{(p)}=a_{32}^{(p-1)}=0,$$

portanto, no caso que $p=2$ tem-se $A\in\mathbb{W}_1\cup\mathbb{W}_2\cup\mathbb{W}_3$.

Agora, se $p>2$ pelo lema (5.14) temos os casos:

$$a_{13}^{(p)}=a_{13}^{(p-1)}=a_{13}^{(p-2)}=0 \ \text{ ou } \ a_{13}^{(p)}=a_{13}^{(p-1)}=a_{21}^{(p)}=0 \ \text{ ou } \ a_{13}^{(p)}=a_{13}^{(p-1)}=a_{32}^{(p)}=0$$

ou

$$a_{13}^{(p)}=a_{21}^{(p)}=a_{21}^{(p-1)}=0 \ \text{ ou } \ a_{13}^{(p)}=a_{21}^{(p)}=a_{32}^{(p)}=0$$

ou

$$a_{13}^{(p)}=a_{32}^{(p)}=a_{32}^{(p-1)}=0$$

ou

$$a_{21}^{(p)}=a_{21}^{(p-1)}=a_{21}^{(p-2)}=0 \ \text{ ou } \ a_{21}^{(p)}=a_{21}^{(p-1)}=a_{32}^{(p)}=0$$

ou

$$a_{21}^{(p)}=a_{32}^{(p)}=a_{32}^{(p-1)}=0$$

ou

$$a_{32}^{(p)}=a_{32}^{(p-1)}=a_{32}^{(p-2)}=0.$$

Quando

$$a_{13}^{(p)}=a_{21}^{(p)}=a_{32}^{(p)}=0,$$

temos que $A\in\mathbb{W}^p$ e portanto podemos assumir que

$$A\notin\mathbb{W}^p.$$

Similarmente, quando

$$a_{13}^{(t)} = 0, \text{ para todo } t = 1, 2, \ldots, p$$

ou

$$a_{21}^{(t)} = 0, \text{ para todo } t = 1, 2, \ldots, p$$

ou

$$a_{32}^{(t)} = 0, \text{ para todo } t = 1, 2, \ldots, p$$

temos que $A \in \mathbb{W}_{11} \cup \mathbb{W}_{1\,p+1} \cup \mathbb{W}_{2\,p+1}$ e portanto podemos assumir que

$$A \notin \mathbb{W}_{11} \cup \mathbb{W}_{1\,p+1} \cup \mathbb{W}_{2\,p+1}.$$

Continuando com este raciocinio, suponha que

$$A \notin \mathbb{W}^p \cup \mathbb{W}_1 \cup \mathbb{W}_2$$

e provemos que

$$A \in \mathbb{W}_3.$$

Claramente, existe $s \in \{2, 3, \ldots, p\}$ tal que

$$a_{13}^{(t)} = 0, \text{ para cada } t = s, s+1, \ldots, p \text{ e } a_{13}^{(s-1)} \neq 0$$

pois $A \notin \mathbb{W}_{11}$, segue do lema (5.14) item $(1.a)$ que

$$a_{21}^{(p)} = 0 \text{ ou } a_{32}^{(p)} = 0.$$

Notemos que, se $a_{21}^{(p)} = 0$ existe $j \in \{p-(s-3), p-(s-4), \ldots, p\}$ tal que

$$a_{21}^{(t)} = 0, \text{ para cada } t = j, j+1, \ldots, p \text{ e } a_{21}^{(j-1)} \neq 0$$

pois $A \notin \mathbb{W}_{1s}$ e como consequência do lema (5.14) item $(1.b)$, temos

$$a_{32}^{(p)} = 0$$

o qual contradiz o fato que $A \notin \mathbb{W}^p$. Portanto

$$a_{21}^{(p)} \neq 0 \text{ e } a_{32}^{(p)} = 0.$$

Novamente, pelo lema (5.14) item (3)

$$a_{32}^{(p-1)} = 0$$

e continuando desta forma temos que

$$A \in \mathbb{W}_{3s} \subseteq \mathbb{W}_3.$$

□

Agora o que vamos fazer, é melhorar a decomposição da proposição (5.16). Com este objetivo, iremos precisar do seguinte lema:

Lema 5.17. *Suponha que $p > 2$ e $A = \left(a_{ij}^{(t)}\right)_{\substack{i,j=1,2,\ldots,n \\ t=1,2,\ldots,p}} \in \mathfrak{G}_1$.*

Se $2 \leq s \leq p-1$, $0 \leq j \leq p-s$ e

$$a_{13}^{(p)} = a_{13}^{(p-1)} = a_{13}^{(p-2)} = \cdots = a_{13}^{(s+1)} = a_{13}^{(s)} = 0,$$

$$a_{32}^{(p)} = a_{32}^{(p-1)} = a_{32}^{(p-2)} = \cdots = a_{32}^{(p-(s-2))} = a_{32}^{(p-(s-2)-1)} = a_{32}^{(p-(s-2)-2)} = \cdots = a_{32}^{(p-(s-2)-j)} = 0,$$

então

$$a_{13}^{(s-1)}=0 \ \ ou \ \ a_{21}^{(p)}=0 \ \ ou \ \ a_{32}^{(p-(s-2)-j-1)}=0.$$

Demonstração: Consideremos os polinômios[5]

$$\begin{aligned}
p_{3\,2p-j} = &-\sum_{t=p-j}^{p} X_{23}^{(t)}X_{32}^{(2p-j-t)}-\sum_{t=p-j}^{p} X_{13}^{(t)}X_{31}^{(2p-j-t)}+\\
&+\sum_{r=1}^{p-j-1}\sum_{t=p-j-r}^{p}\left\{-X_{13}^{(r)}X_{22}^{(t)}X_{31}^{(2p-j-r-t)}+X_{12}^{(r)}X_{23}^{(t)}X_{31}^{(2p-j-r-t)}+X_{13}^{(r)}X_{21}^{(t)}X_{32}^{(2p-j-r-t)}\right\}+\\
&+\sum_{r=p-j}^{p}\sum_{t=1}^{2p-j-r-1}\left\{-X_{13}^{(r)}X_{22}^{(t)}X_{31}^{(2p-j-r-t)}+X_{12}^{(r)}X_{23}^{(t)}X_{31}^{(2p-j-r-t)}+X_{13}^{(r)}X_{21}^{(t)}X_{32}^{(2p-j-r-t)}\right\},
\end{aligned}$$

do fato que $A\in\mathfrak{G}_1$ temos a equação[6]

$$\begin{aligned}
p_{3\,2p-j}(A) = &\underbrace{-\sum_{t=p-j}^{p} a_{23}^{(t)}a_{32}^{(2p-j-t)}}_{=0}\underbrace{-\sum_{t=p-j}^{p} a_{13}^{(t)}a_{31}^{(2p-j-t)}}_{=0}+\\
&+\sum_{r=1}^{p-j-1}\sum_{t=p-j-r}^{p}\left\{\underbrace{-a_{13}^{(r)}a_{22}^{(t)}a_{31}^{(2p-j-r-t)}+a_{12}^{(r)}a_{23}^{(t)}a_{31}^{(2p-j-r-t)}}_{=0}+a_{13}^{(r)}a_{21}^{(t)}a_{32}^{(2p-j-r-t)}\right\}+\\
&+\sum_{r=p-j}^{p}\sum_{t=1}^{2p-j-r-1}\left\{\underbrace{-a_{13}^{(r)}a_{22}^{(t)}a_{31}^{(2p-j-r-t)}+a_{12}^{(r)}a_{23}^{(t)}a_{31}^{(2p-j-r-t)}}_{=0}+\underbrace{a_{13}^{(r)}a_{21}^{(t)}a_{32}^{(2p-j-r-t)}}_{=0}\right\}=0\\
=&\sum_{r=1}^{p-j-1}\sum_{t=p-j-r}^{p} a_{13}^{(r)}a_{21}^{(t)}a_{32}^{(2p-j-r-t)}=0.
\end{aligned}$$

Se $s=2$

$$\begin{aligned}
p_{3\,2p-j}(A) &= \sum_{r=1}^{p-j-1}\sum_{t=p-j-r}^{p} a_{13}^{(r)}a_{21}^{(t)}a_{32}^{(2p-j-r-t)}=0\\
&=\sum_{t=p-j-1}^{p} a_{13}^{(1)}a_{21}^{(t)}a_{32}^{(2p-j-1-t)}+\underbrace{\sum_{r=2}^{p-j-1}\sum_{t=p-j-r}^{p} a_{13}^{(r)}a_{21}^{(t)}a_{32}^{(2p-j-r-t)}}_{=0}=0\\
&=\sum_{t=p-j-1}^{p} a_{13}^{(1)}a_{21}^{(t)}a_{32}^{(2p-j-1-t)}=0.
\end{aligned}$$

Se $2<s<p-j$

$$\begin{aligned}
p_{3\,2p-j}(A)=&\sum_{r=1}^{s-2}\sum_{t=p-j-r}^{p} a_{13}^{(r)}a_{21}^{(t)}a_{32}^{(2p-j-r-t)}+\sum_{t=p-j-s+1}^{p} a_{13}^{(s-1)}a_{21}^{(t)}a_{32}^{(2p-j-s+1-t)}+\\
&+\underbrace{\sum_{r=s}^{p-j-1}\sum_{t=p-j-r}^{p} a_{13}^{(r)}a_{21}^{(t)}a_{32}^{(2p-j-r-t)}}_{=0}=0
\end{aligned}$$

[5] Note que $s\le p-j$.

[6] Aqui usamos o fato que $s\le p-j$ e $p-(s-2)-j\le p-j$.

$$
\begin{aligned}
&= \sum_{r=1}^{s-2} \sum_{t=p-j-r}^{p} a_{13}^{(r)} a_{21}^{(t)} a_{32}^{(2p-j-r-t)} + \sum_{t=p-j-s+1}^{p} a_{13}^{(s-1)} a_{21}^{(t)} a_{32}^{(2p-j-s+1-t)} = 0 \\
&= \underbrace{\sum_{r=1}^{s-2} \sum_{m=p-j-r}^{p} a_{13}^{(r)} a_{21}^{(2p-j-r-m)} a_{32}^{(m)}}_{=0} + \sum_{t=p-j-s+1}^{p} a_{13}^{(s-1)} a_{21}^{(t)} a_{32}^{(2p-j-s+1-t)} = 0, \\
&= \sum_{t=p-j-s+1}^{p} a_{13}^{(s-1)} a_{21}^{(t)} a_{32}^{(2p-j-s+1-t)} = 0,
\end{aligned}
$$

onde na passagem da linha três para a linha quatro fizemos $m = 2p - j - r - t$.

Se $s = p - j$

$$
\begin{aligned}
p_{32p-j}(A) &= \sum_{r=1}^{p-j-1} \sum_{t=p-j-r}^{p} a_{13}^{(r)} a_{21}^{(t)} a_{32}^{(2p-j-r-t)} = 0 \\
&= \sum_{r=1}^{p-j-2} \sum_{t=p-j-r}^{p} a_{13}^{(r)} a_{21}^{(t)} a_{32}^{(2p-j-r-t)} + \sum_{t=1}^{p} a_{13}^{(p-j-1)} a_{21}^{(t)} a_{32}^{(p+1-t)} = 0 \\
&= \underbrace{\sum_{r=1}^{p-j-2} \sum_{m=p-j-r}^{p} a_{13}^{(r)} a_{21}^{(2p-j-r-m)} a_{32}^{(m)}}_{=0} + \sum_{t=1}^{p} a_{13}^{(p-j-1)} a_{21}^{(t)} a_{32}^{(p+1-t)} = 0 \\
&= \sum_{t=1}^{p} a_{13}^{(p-j-1)} a_{21}^{(t)} a_{32}^{(p+1-t)} = 0,
\end{aligned}
$$

onde na passagem da linha dois para a linha três fizemos $m = 2p - j - r - t$.

Agora, por todos os casos anteriores temos

$$
\begin{aligned}
p_{32p-j}(A) &= \sum_{t=p-j-s+1}^{p} a_{13}^{(s-1)} a_{21}^{(t)} a_{32}^{(2p-j-s+1-t)} = 0 \\
&= \sum_{t=p-j-s+1}^{p-1} a_{13}^{(s-1)} a_{21}^{(t)} a_{32}^{(2p-j-s+1-t)} + a_{13}^{(s-1)} a_{21}^{(p)} a_{32}^{(p-j-s+1)} = 0 \\
&= \underbrace{\sum_{m=p-j-s+2}^{p} a_{13}^{(s-1)} a_{21}^{(2p-j-s+1-m)} a_{32}^{(m)}}_{=0} + a_{13}^{(s-1)} a_{21}^{(p)} a_{32}^{(p-j-s+1)} = 0 \\
&= a_{13}^{(s-1)} a_{21}^{(p)} a_{32}^{(p-j-s+1)} = 0.
\end{aligned}
$$

onde na passagem da linha dois para a linha três fizemos $m = 2p - j - s + 1 - t$.

□

Proposição 5.18. *Para $Y_p(\mathfrak{gl}_n)$ com $p > 1$*

$$\mathbb{W}_3 \subseteq \mathbb{W}^p \cup \mathbb{W}_1 \cup \mathbb{W}_2.$$

Demonstração: Suponha $s \in \{2, 3, \dots, p\}$ e $A = \left(a_{ij}^{(t)}\right)_{\substack{i,j=1,2,\dots,n \\ t=1,2,\dots,p}} \in \mathbb{W}_{3s}$, isto implica que

$$a_{13}^{(p)} = a_{13}^{(p-1)} = \cdots = a_{13}^{(s)} = 0,$$

$$a_{32}^{(p)} = a_{32}^{(p-1)} = \cdots = a_{32}^{(p-(s-2))} = 0.$$

Pelo lema (5.17)

$$a_{13}^{(s-1)} = 0 \text{ ou } a_{21}^{(p)} = 0 \text{ ou } a_{32}^{(p-(s-2)-1)} = 0.$$

Se $a_{21}^{(p)} = 0$, então $A \in \mathbb{W}^p$.
Se $a_{13}^{(t)} = 0 \ \forall t = 1, 2, \ldots, p$, então $A \in \mathbb{W}_{11} \subseteq \mathbb{W}_1$.
Se existe $t \in \{2, 3, \ldots, s\}$ tal que

$$a_{13}^{(j)} = 0 \ \forall j = t, t+1, \ldots, p \text{ e } a_{13}^{(t-1)} \neq 0,$$

então do lema (5.17)

$$A \in \mathbb{W}_{2p+1} \subseteq \mathbb{W}_2.$$

□

Corolário 5.19. *Para $Y_p(\mathfrak{gl}_n)$*

$$\mathfrak{G}_1 = \mathbb{W}_1 \cup \mathbb{W}_2, \ \textit{se} \ p = 1, 2;$$

$$\mathfrak{G}_1 = \mathbb{W}^p \cup \mathbb{W}_1 \cup \mathbb{W}_2, \ \textit{se} \ p \geq 3.$$

Demonstração: Segue-se das proposições (5.16) e (5.18).

□

5.4 Equidimensionalidade da versão fraca $\mathfrak{G}_1$ para $Y_p(\mathfrak{gl}_3)$

Nesta seção demonstraremos que a versão fraca $\mathfrak{G}_1$ de $\mathfrak{G}$ para $Y_p(\mathfrak{gl}_3)$ é equidimensional com

$$\dim \mathfrak{G}_1 = 3p,$$

provando a equidimensionalidade de cada subvariedade $\mathbb{W}$'s da decomposição do corolário (5.19). Con este objetivo o seguinte teorema de Futorny, Molev e Ovsienko será muito útil.

Teorema 5.20. *A variedade de Gelfand-Tsetlin $\mathfrak{G}$ para $Y_p(\mathfrak{gl}_2)$ é equidimensional com dimensão p.*

Demonstração: Ver [Futorny *et al.* (2005)].

□

Proposição 5.21. *Para todo $s = 1, 2, \ldots, p, p+1$ com $p > 2$, a subvariedade $\mathbb{W}_{1s}$ da versão fraca $\mathfrak{G}_1$ da variedade de Gelfand-Tsetlin para $Y_p(\mathfrak{gl}_3)$ é equidimensional com dimensão $3p$.*

Demonstração: Usando a notação (5.15) as subvariedades $\mathbb{W}_{1s}$'s são:

$$\mathbb{W}_{1s} = V\left(\mathbf{X}_s \cup \{p_{3j}\}_{j=1}^{2p}\right) \subset k^{9p}, \quad s = 1, 2, \ldots, p, p+1,$$

em que

$$\begin{aligned}
\mathbf{X}_1 &= \left\{X_{11}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{12}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{22}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{13}^{(i)}\right\}_{i=1}^{p} \\
\mathbf{X}_s &= \left\{X_{11}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{12}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{22}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{13}^{(i)}\right\}_{i=s}^{p} \cup \left\{X_{21}^{(i)}\right\}_{i=p-(s-2)}^{p}, \quad s = 2, 3, \ldots, p \\
\mathbf{X}_{p+1} &= \left\{X_{11}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{12}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{22}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{21}^{(i)}\right\}_{i=1}^{p}.
\end{aligned}$$

Provaremos que cada $\mathbb{W}_{1s}$ é equidimensional com

$$\dim(\mathbb{W}_{1s}) = 9p - (\underbrace{4p}_{=|\mathbf{X}_s|} + 2p) = 3p.$$

Para isso, pelo corolário (2.67) e a proposição (2.69), é suficiente provar que

$$V\left(\mathbf{X}_s \cup \{p_{3j}\}_{j=1}^{2p} \cup \mathbf{Y}_s\right) \subset k^{9p}$$

com

$$\begin{aligned}
\mathbf{Y}_1 &= \left\{X_{21}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{31}^{(i)}\right\}_{i=1}^{p}\\
\mathbf{Y}_s &= \left\{X_{13}^{(i)}\right\}_{i=1}^{s-1} \cup \left\{X_{21}^{(i)}\right\}_{i=1}^{p-(s-2)-1} \cup \left\{X_{31}^{(i)}\right\}_{i=1}^{p}, \quad s = 2,3,\dots,p\\
\mathbf{Y}_{p+1} &= \left\{X_{13}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{31}^{(i)}\right\}_{i=1}^{p}
\end{aligned}$$

é equidimensional com dimensão $9p - (\underbrace{4p}_{=|\mathbf{X}_s|} + 2p + \underbrace{2p}_{=|\mathbf{Y}_s|}) = p$. Para cada $s = 1,2,\dots,p,p+1$

$$V\left(\mathbf{X}_s \cup \{p_{3j}\}_{j=1,}^{2p} \cup \mathbf{Y}_s\right) = V\left(\mathbf{Z} \cup \left\{p_{3j}^{\mathbf{Z}}\right\}_{j=1}^{2p}\right) \subset k^{9p},$$

em que, pela proposição (5.9) e a notação (3.17)

$$\begin{aligned}
\mathbf{Z} = \mathbf{X}_s \cup \mathbf{Y}_s &= \left\{X_{11}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{12}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{22}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{13}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{21}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{31}^{(i)}\right\}_{i=1}^{p},\\
p_{31}^{\mathbf{Z}} &= X_{33}^{(1)},\\
p_{32}^{\mathbf{Z}} &= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - \underbrace{X_{13}^{(1)}X_{31}^{(1)}}_{=0}\\
&= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)},\\
p_{33}^{\mathbf{Z}} &= X_{33}^{(3)} - X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} \underbrace{-X_{13}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(1)} + X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)}}_{=0}\\
&= X_{33}^{(3)} - X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)},\\
p_{3i}^{\mathbf{Z}} &= -\sum_{r=1}^{i-2}\sum_{s=2}^{i-r-1} \underbrace{X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(i-r-s)}}_{=0} + X_{33}^{(i)} - \sum_{s=1}^{i-1}\left(\underbrace{X_{13}^{(s)}X_{31}^{(i-s)}}_{=0} + X_{23}^{(s)}X_{32}^{(i-s)}\right) +\\
&\quad + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}\left\{\underbrace{X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(i-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(i-r-s)}}_{=0}\right\}\\
&= X_{33}^{(i)} - \sum_{s=1}^{i-1} X_{23}^{(s)}X_{32}^{(i-s)}, \quad i = 4,5,\dots,p,\\
p_{3\,p+1}^{\mathbf{Z}} &= \sum_{r=1}^{p-1}\sum_{s=1}^{p-r}\left\{\underbrace{X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+1-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+1-r-s)}}_{=0}\right\} -\\
&\quad - \sum_{s=1}^{p}\left(\underbrace{X_{13}^{(s)}X_{31}^{(p+1-s)}}_{=0} + X_{23}^{(s)}X_{32}^{(p+1-s)}\right) - \sum_{r=1}^{p-1}\sum_{s=2}^{p-r}\underbrace{X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+1-r-s)}}_{=0}\\
&= -\sum_{s=1}^{p} X_{23}^{(s)}X_{32}^{(p+1-s)},\\
p_{3\,p+i}^{\mathbf{Z}} &= -\sum_{s=i}^{p} X_{23}^{(s)}X_{32}^{(p+i-s)} - \sum_{s=i}^{p}\underbrace{X_{13}^{(s)}X_{31}^{(p+i-s)}}_{=0} +\\
&\quad + \sum_{r=1}^{i-1}\sum_{s=i-r}^{p}\left\{\underbrace{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}}_{=0}\right\} +
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1}\left\{\underbrace{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}}_{=0}\right\}\\
&=-\sum_{s=i}^{p}X_{23}^{(s)}X_{32}^{(p+i-s)},\qquad i=2,3,\dots,p.
\end{aligned}
$$

Agora, projetando esta variedade sobre as variáveis

$$X_{i1}^{(t)},\ \ i=1,2,3;\ \ t=1,2,\dots,p$$

e

$$X_{1j}^{(t)},\ \ j=1,2,3;\ \ t=1,2,\dots,p$$

temos, pelas proposições (2.69) e (2.70), que para mostrar a equidimensionalidade de

$$V\left(\mathbf{Z}\cup\left\{p_{3j}^{\mathbf{Z}}\right\}_{j=1}^{2p}\right)\subset k^{9p}$$

com dimensão p é equivalente provar que

$$\widetilde{\mathbb{W}}_1 := V\left(\mathbf{A}\cup\left\{q_{3j}\right\}_{j=1}^{2p}\right)=V\left(\mathbf{A}\cup\left\{p_{3j}^{\mathbf{Z}}\right\}_{j=1}^{2p}\right)\subset k^{9p-5p}=k^{4p}$$

é equidimensional com dimensão

$$\dim\widetilde{\mathbb{W}}_1=4p-(p+2p)=p,$$

em que

$$
\begin{aligned}
\mathbf{A} &:= \left\{X_{22}^{(i)}\right\}_{i=1}^{p},\\
q_{31} &:= p_{31}^{\mathbf{Z}} = X_{33}^{(1)},\\
q_{32} &:= p_{32}^{\mathbf{Z}} = X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)},\\
q_{33} &:= p_{33}^{\mathbf{Z}} = X_{33}^{(3)}-X_{23}^{(1)}X_{32}^{(2)}-X_{23}^{(2)}X_{32}^{(1)},\\
q_{3i} &:= p_{3i}^{\mathbf{Z}} = X_{33}^{(i)}-\sum_{s=1}^{i-1}X_{23}^{(s)}X_{32}^{(i-s)},\quad i=4,5,\dots,p-2,\\
q_{3\,p-1} &:= p_{3\,p-1}^{\mathbf{Z}} = X_{33}^{(p-1)}-\sum_{s=1}^{p-2}X_{23}^{(s)}X_{32}^{(p-1-s)},\\
q_{3p} &:= p_{3p}^{\mathbf{Z}} = X_{33}^{(p)}-\sum_{s=1}^{p-1}X_{23}^{(s)}X_{32}^{(p-s)},\\
q_{3p+1} &:= -p_{3p+1}^{\mathbf{Z}} = \sum_{s=1}^{p}X_{23}^{(s)}X_{32}^{(p+1-s)},\\
q_{3p+i} &:= -p_{3p+i}^{\mathbf{Z}} = \sum_{s=i}^{p}X_{23}^{(s)}X_{32}^{(p+i-s)},\quad i=2,3,\dots,p-3,\\
q_{3\,2p-2} &:= -p_{3\,2p-2}^{\mathbf{Z}} = \sum_{s=p-2}^{p}X_{23}^{(s)}X_{32}^{(2p-2-s)},\\
q_{3\,2p-1} &:= -p_{3\,2p-1}^{\mathbf{Z}} = X_{23}^{(p-1)}X_{32}^{(p)}+X_{23}^{(p)}X_{32}^{(p-1)},\\
q_{3\,2p} &:= -p_{3\,2p}^{\mathbf{Z}} = X_{23}^{(p)}X_{32}^{(p)}
\end{aligned}
$$

e usando a mudança variável φ

$$\begin{aligned}
X_{22}^{(t)} &\longmapsto X_{11}^{(t)}, \quad t=1,2,\dots,p\\
X_{23}^{(t)} &\longmapsto X_{12}^{(t)}, \quad t=1,2,\dots,p\\
X_{32}^{(t)} &\longmapsto X_{21}^{(t)}, \quad t=1,2,\dots,p\\
X_{33}^{(t)} &\longmapsto X_{22}^{(t)}, \quad t=1,2,\dots,p
\end{aligned}$$

temos $\widetilde{\mathbb{W}}_1 \cong \varphi\left(\widetilde{\mathbb{W}}_1\right)$ e

$$\varphi\left(\widetilde{\mathbb{W}}_1\right)=\varphi\left(V\left(\mathbf{A}\cup\{q_{3j}\}_{j=1}^{2p}\right)\right)=V\left(\varphi(\mathbf{A})\cup\left\{\varphi\left(q_{3j}\right)\right\}_{j=1}^{2p}\right)\subset k^{4p},$$

com

$$\begin{aligned}
\varphi(\mathbf{A}) &= \left\{\varphi\left(X_{22}^{(i)}\right)\right\}_{i=1}^{p}=\left\{X_{11}^{(i)}\right\}_{i=1}^{p}=\{p_{1i}\}_{i=1}^{p},\\
\varphi\left(q_{31}\right) &= \varphi\left(X_{33}^{(1)}\right)=X_{22}^{(1)}=p_{21},\\
\varphi\left(q_{32}\right) &= \varphi\left(X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}\right)=X_{22}^{(2)}-X_{12}^{(1)}X_{21}^{(1)}=p_{22},\\
\varphi\left(q_{33}\right) &= \varphi\left(X_{33}^{(3)}-X_{23}^{(1)}X_{32}^{(2)}-X_{23}^{(2)}X_{32}^{(1)}\right)=X_{22}^{(3)}-X_{12}^{(1)}X_{21}^{(2)}-X_{12}^{(2)}X_{21}^{(1)}=p_{23},\\
\varphi\left(q_{3i}\right) &= \varphi\left(X_{33}^{(i)}-\sum_{s=1}^{i-1}X_{23}^{(s)}X_{32}^{(i-s)}\right)=X_{22}^{(i)}-\sum_{s=1}^{i-1}X_{12}^{(s)}X_{21}^{(i-s)}=p_{2i}, \quad i=4,5,\dots,p-2,\\
\varphi\left(q_{3\,p-1}\right) &= \varphi\left(X_{33}^{(p-1)}-\sum_{s=1}^{p-2}X_{23}^{(s)}X_{32}^{(p-1-s)}\right)=X_{22}^{(p-1)}-\sum_{s=1}^{p-2}X_{12}^{(s)}X_{21}^{(p-1-s)}=p_{2\,p-1},\\
\varphi\left(q_{3p}\right) &= \varphi\left(X_{33}^{(p)}-\sum_{s=1}^{p-1}X_{23}^{(s)}X_{32}^{(p-s)}\right)=X_{22}^{(p)}-\sum_{s=1}^{p-1}X_{12}^{(s)}X_{21}^{(p-s)}=p_{2p},\\
\varphi\left(q_{3\,p+1}\right) &= \varphi\left(\sum_{s=1}^{p}X_{23}^{(s)}X_{32}^{(p+1-s)}\right)=\sum_{s=1}^{p}X_{12}^{(s)}X_{21}^{(p+1-s)}=p_{2\,p+1},\\
\varphi\left(q_{3\,p+i}\right) &= \varphi\left(\sum_{s=i}^{p}X_{23}^{(s)}X_{32}^{(p+i-s)}\right)=\sum_{s=i}^{p}X_{12}^{(s)}X_{21}^{(p+i-s)}=p_{2\,p+i}, \quad i=2,3,\dots,p-3,\\
\varphi\left(q_{3\,2p-2}\right) &= \varphi\left(\sum_{s=p-2}^{p}X_{23}^{(s)}X_{32}^{(2p-2-s)}\right)=\sum_{s=p-2}^{p}X_{12}^{(s)}X_{21}^{(2p-2-s)}=p_{2\,2p-2},\\
\varphi\left(q_{3\,2p-1}\right) &= \varphi\left(X_{23}^{(p-1)}X_{32}^{(p)}+X_{23}^{(p)}X_{32}^{(p-1)}\right)=X_{12}^{(p-1)}X_{21}^{(p)}+X_{12}^{(p)}X_{21}^{(p-1)}=p_{2\,2p-1},\\
\varphi\left(q_{3\,2p}\right) &= \varphi\left(X_{23}^{(p)}X_{32}^{(p)}\right)=X_{12}^{(p)}X_{21}^{(p)}=p_{2\,2p}.
\end{aligned}$$

Portanto, $\widetilde{\mathbb{W}}_1 \cong \varphi\left(\widetilde{\mathbb{W}}_1\right)$ é a variedade de Gelfand-Tsetlin $\mathfrak{G}$ para Yangian $Y_p(\mathfrak{gl}_2)$, a qual pelo teorema (5.20) é equidimensional com dimensão

$$\dim\varphi\left(\widetilde{\mathbb{W}}_1\right)=4p-3p=p.$$

□

Corolário 5.22. *Para $p>2$, a subvariedade $\mathbb{W}_1$ da versão fraca $\mathfrak{G}_1$ da variedade de Gelfand-Tsetlin para $Y_p(\mathfrak{gl}_3)$ é equidimensional com dimensão $3p$.*

Demonstração: Pela proposição (5.21) e a notação (5.15)

$$\mathbb{W}_1 = \bigcup_{s=1}^{p+1} \mathbb{W}_{1s}.$$

□

Proposição 5.23. *Para todo $s = 2,3,\dots,p,p+1$ e $p > 2$, a subvariedade $\mathbb{W}_{2s}$ da versão fraca da variedade de Gelfand-Tsetlin para $Y_p(\mathfrak{gl}_3)$ é equidimensional com dimensão $3p$.*

Demonstração: Usando a notação (5.15) as subvariedades $\mathbb{W}_{2s}$'s são:

$$\mathbb{W}_{2s} = V\left(\mathbf{X}_s \cup \left\{p_{3j}^{\mathbf{X}_s}\right\}_{j=1}^{2p}\right) \subset k^{9p}, \quad s = 2,3,\dots,p,p+1,$$

em que

$$\begin{aligned}
\mathbf{X}_s &= \left\{X_{11}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{12}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{22}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{21}^{(i)}\right\}_{i=s}^{p} \cup \left\{X_{32}^{(i)}\right\}_{i=p-(s-2)}^{p}, \quad s = 2,3,\dots,p\\
\mathbf{X}_{p+1} &= \left\{X_{11}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{12}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{22}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{32}^{(i)}\right\}_{i=1}^{p}.
\end{aligned}$$

Mostraremos que $\mathbb{W}_{2s}$ é equidimensional com

$$\dim(\mathbb{W}_{2s}) = 9p - (\underbrace{4p}_{=|\mathbf{X}_s|} + 2p) = 3p.$$

Para isso, pelo corolário (2.67) e a proposição (2.69) é suficiente provar que

$$V\left(\mathbf{X}_s \cup \left\{p_{3j}^{\mathbf{X}_s}\right\}_{j=1,2,\dots,2p} \cup \mathbf{Y}_s\right) \subset k^{9p},$$

com

$$\begin{aligned}
\mathbf{Y}_s &= \left\{X_{21}^{(i)}\right\}_{i=1}^{s-1} \cup \left\{X_{32}^{(i)}\right\}_{i=1}^{p-(s-2)-1} \cup \left\{X_{23}^{(i)}\right\}_{i=1}^{p}, \quad s = 2,3,\dots,p\\
\mathbf{Y}_{p+1} &= \left\{X_{21}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{23}^{(i)}\right\}_{i=1}^{p}
\end{aligned}$$

é equidimensional com dimensão $9p - (\underbrace{4p}_{=|\mathbf{X}_s|} + 2p + \underbrace{2p}_{=|\mathbf{Y}_s|}) = p$. Para cada $s = 2,3,\dots,p,p+1$

$$V\left(\mathbf{X}_s \cup \left\{p_{3j}^{\mathbf{X}_s}\right\}_{j=1}^{2p} \cup \mathbf{Y}_s\right) = V\left(\mathbf{Z} \cup \left\{p_{3j}^{\mathbf{Z}}\right\}_{j=1}^{2p}\right) \subset k^{9p},$$

em que, pela proposição (5.9) e a notação (3.17)

$$\begin{aligned}
\mathbf{Z} &:= \mathbf{X}_s \cup \mathbf{Y}_s = \left\{X_{11}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{12}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{22}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{21}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{23}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{32}^{(i)}\right\}_{i=1}^{p},\\
p_{31}^{\mathbf{Z}} &= X_{33}^{(1)},\\
p_{32}^{\mathbf{Z}} &= X_{33}^{(2)} - \underbrace{X_{23}^{(1)}X_{32}^{(1)}}_{=0} - X_{13}^{(1)}X_{31}^{(1)}\\
&= X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)},\\
p_{33}^{\mathbf{Z}} &= X_{33}^{(3)} - \underbrace{X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)}}_{=0} - \sum_{s=1}^{2} X_{13}^{(s)}X_{31}^{(3-s)} + \underbrace{X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)}}_{=0}\\
&= X_{33}^{(3)} - \sum_{s=1}^{2} X_{13}^{(s)}X_{31}^{(3-s)},\\
p_{3i}^{\mathbf{Z}} &= -\sum_{r=1}^{i-2}\sum_{s=2}^{i-r-1} \underbrace{X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(i-r-s)}}_{=0} + X_{33}^{(i)} - \sum_{s=1}^{i-1}\left(X_{13}^{(s)}X_{31}^{(i-s)} + \underbrace{X_{23}^{(s)}X_{32}^{(i-s)}}_{=0}\right) +
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}\left\{\underbrace{X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(i-r-s)}}_{=0}\right\}\\
&=X_{33}^{(i)}-\sum_{s=1}^{i-1}X_{13}^{(s)}X_{31}^{(i-s)},\quad i=4,5,\ldots,p,\\
p_{3\,p+1}^{\mathbf{Z}}&=\sum_{r=1}^{p-1}\sum_{s=1}^{p-r}\left\{\underbrace{X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+1-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+1-r-s)}}_{=0}\right\}-\\
&-\sum_{s=1}^{p}\left(X_{13}^{(s)}X_{31}^{(p+1-s)}+\underbrace{X_{23}^{(s)}X_{32}^{(p+1-s)}}_{=0}\right)-\sum_{r=1}^{p-1}\sum_{s=2}^{p-r}\underbrace{X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+1-r-s)}}_{=0}\\
&=-\sum_{s=1}^{p}X_{13}^{(s)}X_{31}^{(p+1-s)},\\
p_{3\,p+i}^{\mathbf{Z}}&=-\sum_{s=i}^{p}\underbrace{X_{23}^{(s)}X_{32}^{(p+i-s)}}_{=0}-\sum_{s=i}^{p}X_{13}^{(s)}X_{31}^{(p+i-s)}+\\
&+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p}\left\{\underbrace{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}}_{=0}\right\}+\\
&+\sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1}\left\{\underbrace{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}}_{=0}\right\},\\
&=-\sum_{s=i}^{p}X_{13}^{(s)}X_{31}^{(p+i-s)},\quad i=2,3,\ldots,p.
\end{aligned}
$$

Projetando esta variedade sobre as variáveis

$$X_{i2}^{(t)},\ \ i=1,2,3;\ \ t=1,2,\ldots,p$$

e

$$X_{2j}^{(t)},\ \ j=1,2,3;\ \ t=1,2,\ldots,p$$

temos, pelas proposições (2.69) e (2.70), que para mostrar a equidimensionalidade de

$$V\left(\mathbf{Z}\cup\left\{p_{3j}^{\mathbf{Z}}\right\}_{j=1}^{2p}\right)\subset k^{9p}$$

com dimensão p é equivalente provar que

$$\widetilde{\mathbb{W}}_2:=V\left(\mathbf{A}\cup\left\{q_{3j}\right\}_{j=1}^{2p}\right)=V\left(\mathbf{A}\cup\left\{p_{3j}^{\mathbf{Z}}\right\}_{j=1}^{2p}\right)\subset k^{9p-5p}=k^{4p}$$

é equidimensional com dimensão

$$\dim\widetilde{\mathbb{W}}_2=4p-(p+2p)=p,$$

em que

$$
\begin{aligned}
\mathbf{A}&:=\left\{X_{11}^{(i)}\right\}_{i=1}^{p},\\
q_{31}&:=p_{31}^{\mathbf{Z}}=X_{33}^{(1)},\\
q_{32}&:=p_{32}^{\mathbf{Z}}=X_{33}^{(2)}-X_{13}^{(1)}X_{31}^{(1)},
\end{aligned}
$$

$$q_{33} := p_{33}^{\mathbf{Z}} = X_{33}^{(3)} - \sum_{s=1}^{2} X_{13}^{(s)} X_{31}^{(3-s)},$$
$$q_{3i} := p_{3i}^{\mathbf{Z}} = X_{33}^{(i)} - \sum_{s=1}^{i-1} X_{13}^{(s)} X_{31}^{(i-s)}, \quad i = 4,5,\ldots,p,$$
$$q_{3p+1} := -p_{3p+1}^{\mathbf{Z}} = \sum_{s=1}^{p} X_{13}^{(s)} X_{31}^{(p+1-s)},$$
$$q_{3p+i} := -p_{3p+i}^{\mathbf{Z}} = \sum_{s=i}^{p} X_{13}^{(s)} X_{31}^{(p+i-s)}, \quad i = 2,3,\ldots,p$$

e usando a mudança de variável φ

$$X_{11}^{(t)} \longmapsto X_{11}^{(t)}, \quad t = 1,2,\ldots,p$$
$$X_{13}^{(t)} \longmapsto X_{12}^{(t)}, \quad t = 1,2,\ldots,p$$
$$X_{31}^{(t)} \longmapsto X_{21}^{(t)}, \quad t = 1,2,\ldots,p$$
$$X_{33}^{(t)} \longmapsto X_{22}^{(t)}, \quad t = 1,2,\ldots,p,$$

temos $\widetilde{\mathbb{W}}_2 \cong \varphi\left(\widetilde{\mathbb{W}}_2\right)$ e

$$\varphi\left(\widetilde{\mathbb{W}}_2\right) = \varphi\left(V\left(\mathbf{A} \cup \{q_{3j}\}_{j=1}^{2p}\right)\right) = V\left(\varphi(\mathbf{A}) \cup \{\varphi(q_{3j})\}_{j=1}^{2p}\right) \subset k^{4p},$$

com

$$\varphi(\mathbf{A}) = \left\{X_{11}^{(i)}\right\}_{i=1}^{p} = \{p_{1i}\}_{i=1}^{p},$$
$$\varphi(q_{31}) = \varphi\left(X_{33}^{(1)}\right) = X_{22}^{(1)} = p_{21},$$
$$\varphi(q_{32}) = \varphi\left(X_{33}^{(2)} - X_{13}^{(1)} X_{31}^{(1)}\right) = X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)} = p_{22},$$
$$\varphi(q_{33}) = \varphi\left(X_{33}^{(3)} - \sum_{s=1}^{2} X_{13}^{(s)} X_{31}^{(3-s)}\right) = X_{22}^{(3)} - \sum_{s=1}^{2} X_{12}^{(s)} X_{21}^{(3-s)} = p_{23},$$
$$\varphi(q_{3i}) = \varphi\left(X_{33}^{(i)} - \sum_{s=1}^{i-1} X_{13}^{(s)} X_{31}^{(i-s)}\right) = X_{22}^{(i)} - \sum_{s=1}^{i-1} X_{12}^{(s)} X_{21}^{(i-s)} = p_{2i}, \quad i = 4,5,\ldots,p,$$
$$\varphi(q_{3p+1}) = \varphi\left(\sum_{s=1}^{p} X_{13}^{(s)} X_{31}^{(p+1-s)}\right) = \sum_{s=1}^{p} X_{12}^{(s)} X_{21}^{(p+1-s)} = p_{2p+1},$$
$$\varphi(q_{3p+i}) = \varphi\left(\sum_{s=i}^{p} X_{13}^{(s)} X_{31}^{(p+i-s)}\right) = \sum_{s=i}^{p} X_{12}^{(s)} X_{21}^{(p+i-s)} = p_{2p+i}, \quad i = 2,3,\ldots,p.$$

Portanto, $\widetilde{\mathbb{W}}_2 \cong \varphi\left(\widetilde{\mathbb{W}}_2\right)$ é a variedade de Gelfand-Tsetlin $\mathfrak{G}$ para Yangian $Y_p(\mathfrak{gl}_2)$, a qual pelo teorema (5.20) é equidimensional com dimensão

$$\dim \varphi\left(\widetilde{\mathbb{W}}_2\right) = 4p - 3p = p.$$

□

Corolário 5.24. *Para $p > 2$, a subvariedade $\mathbb{W}_2$ da versão fraca $\mathfrak{G}_1$ da variedade de Gelfand-Tsetlin para $Y_p(\mathfrak{gl}_3)$ é equidimensional com dimensão $3p$.*

Demonstração: Pela proposição (5.23) e a notação (5.15)

$$\mathbb{W}_2 = \bigcup_{s=2}^{p+1} \mathbb{W}_{2s}.$$

□

Agora estamos em condições de provar nosso resultado principal.

Teorema 5.25. *A variedade $\mathfrak{G}_1$ para $Y_p(\mathfrak{gl}_3)$ é equidimensional com dimensão*

$$\dim \mathfrak{G}_1 = 3p.$$

Demonstração: Provemos por indução sobre p.

Pela proposição (5.11), temos que $\mathfrak{G}_1$ para $Y_1(\mathfrak{gl}_3)$ é equidimensional com dimensão

$$\dim \mathfrak{G}_1 = 3.$$

Similarmente, da proposição (5.12), temos que $\mathfrak{G}_1$ para $Y_2(\mathfrak{gl}_3)$ é equidimensional com dimensão

$$\dim \mathfrak{G}_1 = 6.$$

Agora suponha $p > 2$ e que a variedade $\mathfrak{G}_1$ para $Y_{p-1}(\mathfrak{gl}_3)$ é equidimensional com dimensão $3(p-1)$.

Como consequência do corolário (5.19), temos a decomposição

$$\mathfrak{G}_1 = \mathbb{W}^p \cup \mathbb{W}_1 \cup \mathbb{W}_2, \ \text{ para } \ p \geq 3.$$

Pelos corolários (5.22) e (5.24), as componentes $\mathbb{W}_1$ e $\mathbb{W}_2$ são equidimensionais com dimensão $3p$.

Mostremos que $\mathbb{W}^p$ é equidimensional com dimensão $3p$. Usando a notação (5.15), temos

$$\mathbb{W}^p = V\left(\mathbf{X} \cup \left\{p_{3j}\right\}_{j=1}^{3p-3}\right) \subset k^{9p},$$

com

$$\mathbf{X} = \left\{X_{11}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{12}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{22}^{(i)}\right\}_{i=1}^{p} \cup \left\{X_{32}^{(p)}, X_{21}^{(p)}, X_{13}^{(p)}\right\}.$$

Para provar que $\mathbb{W}^p$ é equidimensional com $\dim(\mathbb{W}^p) = 3p$, pelo corolário (2.67) e a proposição (2.69), é suficiente mostrar que

$$V\left(\mathbf{X} \cup \left\{p_{3j}\right\}_{j=1,2}^{3p-3} \cup \left\{X_{23}^{(p)}, X_{31}^{(p)}, X_{33}^{(p)}\right\}\right) \subset k^{9p}$$

é equidimensional com dimensão $9p - (6p+3) = 3p - 3 = 3(p-1)$. Claramente

$$V\left(\mathbf{Y} \cup \left\{p_{3j}^{\mathbf{Y}}\right\}_{j=1}^{3p-3}\right) = V\left(\mathbf{X} \cup \left\{p_{3j}\right\}_{j=1}^{3p-3} \cup \left\{X_{23}^{(p)}, X_{31}^{(p)}, X_{33}^{(p)}\right\}\right) \subset k^{9p},$$

em que, como consequência da proposição (5.9) e a notação (3.17)

$$\begin{aligned}
\mathbf{Y} &= \left\{X_{11}^{(i)}\right\}_{i=1}^{p-1} \cup \left\{X_{12}^{(i)}\right\}_{i=1}^{p-1} \cup \left\{X_{22}^{(i)}\right\}_{i=1}^{p-1} \cup \left\{X_{ij}^{(p)}\right\}_{i,j=1}^{3}, \\
p_{31}^{\mathbf{Y}} &= X_{33}^{(1)}, \\
p_{32}^{\mathbf{Y}} &= X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)}, \\
p_{33}^{\mathbf{Y}} &= X_{33}^{(3)} - X_{23}^{(1)} X_{32}^{(2)} - X_{23}^{(2)} X_{32}^{(1)} - \sum_{s=1}^{2} X_{13}^{(s)} X_{31}^{(3-s)} + \underbrace{X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)}}_{=0} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(1)} \\
&= X_{33}^{(3)} - X_{23}^{(1)} X_{32}^{(2)} - X_{23}^{(2)} X_{32}^{(1)} - \sum_{s=1}^{2} X_{13}^{(s)} X_{31}^{(3-s)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(1)},
\end{aligned}$$

para cada $i = 4, 5, \ldots, p-1$ temos

$$\begin{aligned}
p^{\mathbf{Y}}_{3i} &= -\sum_{r=1}^{i-2}\sum_{s=2}^{i-r-1} \underbrace{X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(i-r-s)}}_{=0} + X_{33}^{(i)} - \sum_{s=1}^{i-1}\Big(X_{13}^{(s)}X_{31}^{(i-s)} + X_{23}^{(s)}X_{32}^{(i-s)}\Big) + \\
&\quad + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1}\Big\{\underbrace{X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(i-r-s)}}_{=0} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(i-r-s)}\Big\} \\
&= X_{33}^{(i)} - \sum_{s=1}^{i-1}\Big(X_{13}^{(s)}X_{31}^{(i-s)} + X_{23}^{(s)}X_{32}^{(i-s)}\Big) + \sum_{r=1}^{i-2}\sum_{s=1}^{i-r-1} X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(i-r-s)},
\end{aligned}$$

análogamente, temos

$$\begin{aligned}
p^{\mathbf{Y}}_{3p} &= -\sum_{r=1}^{p-2}\sum_{s=2}^{p-r-1} \underbrace{X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p-r-s)}}_{=0} + \underbrace{X_{33}^{(p)}}_{=0} - \sum_{s=1}^{p-1}\Big(X_{13}^{(s)}X_{31}^{(p-s)} + X_{23}^{(s)}X_{32}^{(p-s)}\Big) + \\
&\quad + \sum_{r=1}^{p-2}\sum_{s=1}^{p-r-1}\Big\{\underbrace{X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p-r-s)}}_{=0} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p-r-s)}\Big\} \\
&= -\sum_{s=1}^{p-1}\Big(X_{13}^{(s)}X_{31}^{(p-s)} + X_{23}^{(s)}X_{32}^{(p-s)}\Big) + \sum_{r=1}^{p-2}\sum_{s=1}^{p-r-1} X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p-r-s)}, \\
p^{\mathbf{Y}}_{3\,p+1} &= \sum_{r=1}^{p-1}\sum_{s=1}^{p-r}\Big\{\underbrace{X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+1-r-s)}}_{=0} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+1-r-s)}\Big\} - \\
&\quad - \sum_{s=1}^{p}\Big(X_{13}^{(s)}X_{31}^{(p+1-s)} + X_{23}^{(s)}X_{32}^{(p+1-s)}\Big) - \sum_{r=1}^{p-1}\sum_{s=2}^{p-r}\underbrace{X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+1-r-s)}}_{=0} \\
&= \sum_{r=1}^{p-1}\sum_{s=1}^{p-r} X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+1-r-s)} - \sum_{s=1}^{p}\Big(X_{13}^{(s)}X_{31}^{(p+1-s)} + X_{23}^{(s)}X_{32}^{(p+1-s)}\Big) \\
&= \sum_{r=1}^{p-1}\sum_{s=1}^{p-r} X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+1-r-s)} - \sum_{s=2}^{p-1}\Big(X_{13}^{(s)}X_{31}^{(p+1-s)} + X_{23}^{(s)}X_{32}^{(p+1-s)}\Big) - \\
&\quad - \left(\underbrace{X_{13}^{(1)}X_{31}^{(p)} + X_{23}^{(1)}X_{32}^{(p)} + X_{13}^{(p)}X_{31}^{(1)} + X_{23}^{(p)}X_{32}^{(1)}}_{=0}\right) \\
&= \sum_{r=1}^{p-1}\sum_{s=1}^{p-r} X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+1-r-s)} - \sum_{s=2}^{p-1}\Big(X_{13}^{(s)}X_{31}^{(p+1-s)} + X_{23}^{(s)}X_{32}^{(p+1-s)}\Big),
\end{aligned}$$

para cada $i = 2, 3, \ldots, p-2$ temos

$$\begin{aligned}
p^{\mathbf{Y}}_{3\,p+i} &= -\sum_{s=i}^{p} X_{23}^{(s)}X_{32}^{(p+i-s)} - \sum_{s=i}^{p} X_{13}^{(s)}X_{31}^{(p+i-s)} + \\
&\quad + \sum_{r=1}^{i-1}\sum_{s=i-r}^{p}\Big\{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\Big\} + \\
&\quad + \sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1}\Big\{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)} + X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)} + X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\Big\} \\
&= -\sum_{s=i+1}^{p-1} X_{23}^{(s)}X_{32}^{(p+i-s)} \underbrace{-X_{23}^{(i)}X_{32}^{(p)} - X_{23}^{(p)}X_{32}^{(i)}}_{=0} - \sum_{s=i+1}^{p-1} X_{13}^{(s)}X_{31}^{(p+i-s)} \underbrace{-X_{13}^{(i)}X_{31}^{(p)} - X_{13}^{(p)}X_{31}^{(i)}}_{=0} +
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=1}^{i-1}\sum_{s=i-r+1}^{p-1}\Big\{\underbrace{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)}}_{=0}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\Big\}+\\
&+\sum_{r=1}^{i-1}\Big\{\underbrace{-X_{13}^{(r)}X_{22}^{(i-r)}X_{31}^{(p)}+X_{12}^{(r)}X_{23}^{(i-r)}X_{31}^{(p)}+X_{13}^{(r)}X_{21}^{(i-r)}X_{32}^{(p)}}_{=0}\Big\}+\\
&+\sum_{r=1}^{i-1}\Big\{\underbrace{-X_{13}^{(r)}X_{22}^{(p)}X_{31}^{(i-r)}+X_{12}^{(r)}X_{23}^{(p)}X_{31}^{(i-r)}+X_{13}^{(r)}X_{21}^{(p)}X_{32}^{(i-r)}}_{=0}\Big\}+\\
&+\sum_{r=i}^{p-1}\sum_{s=1}^{p+i-r-1}\Big\{\underbrace{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(p+i-r-s)}}_{=0}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}\Big\}+\\
&+\sum_{s=1}^{i-1}\Big\{\underbrace{-X_{13}^{(p)}X_{22}^{(s)}X_{31}^{(i-s)}+X_{12}^{(p)}X_{23}^{(s)}X_{31}^{(i-s)}+X_{13}^{(p)}X_{21}^{(s)}X_{32}^{(i-s)}}_{=0}\Big\},\\
=&-\sum_{s=i+1}^{p-1}X_{23}^{(s)}X_{32}^{(p+i-s)}-\sum_{s=i+1}^{p-1}X_{13}^{(s)}X_{31}^{(p+i-s)}+\\
&+\sum_{r=1}^{i-1}\sum_{s=i-r+1}^{p-1}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)}+\sum_{r=i}^{p-1}\sum_{s=1}^{p+i-r-1}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(p+i-r-s)},
\end{aligned}
$$

continuando com o mesmo raciocinio

$$
\begin{aligned}
p_{3\,2p-1}^{\mathbf{Y}}=&\underbrace{-\sum_{s=p-1}^{p}X_{23}^{(s)}X_{32}^{(2p-1-s)}-\sum_{s=p-1}^{p}X_{13}^{(s)}X_{31}^{(2p-1-s)}+}_{=0}\\
&+\sum_{r=1}^{p-2}\sum_{s=p-1-r}^{p}\Big\{\underbrace{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(2p-1-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(2p-1-r-s)}}_{=0}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p-1-r-s)}\Big\}+\\
&+\sum_{r=p-1}^{p}\sum_{s=1}^{2p-r-2}\Big\{\underbrace{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(2p-1-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(2p-1-r-s)}}_{=0}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p-1-r-s)}\Big\}\\
=&\sum_{r=1}^{p-2}\sum_{s=p-1-r}^{p}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p-1-r-s)}+\sum_{r=p-1}^{p}\sum_{s=1}^{2p-r-2}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p-1-r-s)}\\
=&\sum_{r=1}^{p-2}\sum_{s=p-r}^{p-1}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p-1-r-s)}+\underbrace{\sum_{r=1}^{p-2}X_{13}^{(r)}X_{21}^{(p-1-r)}X_{32}^{(p)}+\sum_{r=1}^{p-2}X_{13}^{(r)}X_{21}^{(p)}X_{32}^{(p-1-r)}+}_{=0}\\
&+\sum_{s=1}^{p-1}X_{13}^{(p-1)}X_{21}^{(s)}X_{32}^{(p-s)}+\sum_{s=1}^{p-2}\underbrace{X_{13}^{(p)}X_{21}^{(s)}X_{32}^{(p-1-s)}}_{=0}\\
=&\sum_{r=1}^{p-2}\sum_{s=p-r}^{p-1}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p-1-r-s)}+\sum_{s=1}^{p-1}X_{13}^{(p-1)}X_{21}^{(s)}X_{32}^{(p-s)},\\
p_{3\,2p}^{\mathbf{Y}}=&\underbrace{-X_{23}^{(p)}X_{32}^{(p)}-X_{13}^{(p)}X_{31}^{(p)}}_{=0}+\\
&+\sum_{r=1}^{p-1}\sum_{s=p-r}^{p}\Big\{\underbrace{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(2p-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(2p-r-s)}}_{=0}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p-r-s)}\Big\}+
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{s=1}^{p-1}\Big\{\underbrace{-X_{13}^{(p)}X_{22}^{(s)}X_{31}^{(p-s)}+X_{12}^{(p)}X_{23}^{(s)}X_{31}^{(p-s)}+X_{13}^{(p)}X_{21}^{(s)}X_{32}^{(p-s)}}_{=0}\Big\}\\
&=\sum_{r=1}^{p-1}\sum_{s=p-r}^{p}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p-r-s)}\\
&=\sum_{r=1}^{p-1}\sum_{s=p-r+1}^{p-1}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p-r-s)}+\underbrace{\sum_{r=1}^{p-1}X_{13}^{(r)}X_{21}^{(p-r)}X_{32}^{(p)}+\sum_{r=1}^{p-1}X_{13}^{(r)}X_{21}^{(p)}X_{32}^{(p-r)}}_{=0}\\
&=\sum_{r=1}^{p-1}\sum_{s=p-r+1}^{p-1}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p-r-s)},
\end{aligned}
$$

para cada $i=1,2,\dots,p-4$ temos

$$
\begin{aligned}
p_{32p+i}^{\mathbf{Y}}&=\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}\Big\{\underbrace{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(2p+i-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(2p+i-r-s)}}_{=0}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p+i-r-s)}\Big\}\\
&=\sum_{r=i}^{p}\sum_{s=p+i-r}^{p}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p+i-r-s)}\\
&=\sum_{r=i+1}^{p-1}\sum_{s=p+i-r}^{p}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p+i-r-s)}+\underbrace{X_{13}^{(i)}X_{21}^{(p)}X_{32}^{(p)}+\sum_{s=i}^{p}X_{13}^{(p)}X_{21}^{(s)}X_{32}^{(p+i-s)}}_{=0}\\
&=\sum_{r=i+1}^{p-1}\sum_{s=p+i-r}^{p}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p+i-r-s)}\\
&=\sum_{r=i+1}^{p-1}\sum_{s=p+i-r+1}^{p-1}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p+i-r-s)}+\underbrace{\sum_{r=i+1}^{p-1}X_{13}^{(r)}X_{21}^{(p+i-r)}X_{32}^{(p)}+\sum_{r=i+1}^{p-1}X_{13}^{(r)}X_{21}^{(p)}X_{32}^{(p+i-r)}}_{=0}\\
&=\sum_{r=i+1}^{p-1}\sum_{s=p+i-r+1}^{p-1}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(2p+i-r-s)},
\end{aligned}
$$

análogamente tem-se

$$
\begin{aligned}
p_{33p-3}^{\mathbf{Y}}&=\sum_{r=p-3}^{p}\sum_{s=2p-3-r}^{p}\Big\{\underbrace{-X_{13}^{(r)}X_{22}^{(s)}X_{31}^{(3p-3-r-s)}+X_{12}^{(r)}X_{23}^{(s)}X_{31}^{(3p-3-r-s)}}_{=0}+X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(3p-3-r-s)}\Big\}\\
&=\sum_{r=p-3}^{p}\sum_{s=2p-3-r}^{p}X_{13}^{(r)}X_{21}^{(s)}X_{32}^{(3p-3-r-s)}\\
&=\underbrace{X_{13}^{(p-3)}X_{21}^{(p)}X_{32}^{(p)}+\sum_{s=p-1}^{p}X_{13}^{(p-2)}X_{21}^{(s)}X_{32}^{(2p-1-s)}}_{=0}+\sum_{s=p-2}^{p}X_{13}^{(p-1)}X_{21}^{(s)}X_{32}^{(2p-2-s)}\\
&\quad+\underbrace{\sum_{s=p-3}^{p}X_{13}^{(p)}X_{21}^{(s)}X_{32}^{(2p-3-s)}}_{=0}\\
&=\sum_{s=p-2}^{p}X_{13}^{(p-1)}X_{21}^{(s)}X_{32}^{(2p-2-s)}
\end{aligned}
$$

$$
\begin{aligned}
&= \underbrace{X_{13}^{(p-1)} X_{21}^{(p-2)} X_{32}^{(p)}}_{=0} + X_{13}^{(p-1)} X_{21}^{(p-1)} X_{32}^{(p-1)} + \underbrace{X_{13}^{(p-1)} X_{21}^{(p)} X_{32}^{(p-2)}}_{=0} \\
&= X_{13}^{(p-1)} X_{21}^{(p-1)} X_{32}^{(p-1)}.
\end{aligned}
$$

Agora, projetando a variedade sobre as variáveis

$$X_{ij}^{(p)}, \ \ i = 1, 2, 3,$$

temos, pela proposição (2.70), que para mostrar a equidimensionalidade de

$$V\left(\mathbf{Y} \cup \left\{ p_{3j}^{\mathbf{Y}} \right\}_{j=1}^{3p-3}\right) \subset k^{9p}$$

com dimensão $3(p-1)$, é equivalente provar que

$$\widetilde{\mathbb{W}}^p = V\left(\mathbf{Z} \cup \left\{ q_{3j}^{\mathbf{Z}} \right\}_{j=1}^{3p-3}\right) \subset k^{9p-9},$$

com

$$\mathbf{Z} = \left\{ X_{11}^{(i)} \right\}_{i=1}^{p-1} \cup \left\{ X_{12}^{(i)} \right\}_{i=1}^{p-1} \cup \left\{ X_{22}^{(i)} \right\}_{i=1}^{p-1}$$

e

$$q_{3j}^{\mathbf{Z}} = p_{3j}^{\mathbf{Y}}$$

é equidimensional com $\dim\left(\widetilde{\mathbb{W}}^p\right) = 9p - 9 - 6(p-1) = 3(p-1)$. Mas, notemos que $\widetilde{\mathbb{W}}^p$ é a versão fraca $\mathfrak{G}_1$ para $Y_{p-1}(\mathfrak{gl}_3)$.

□

Apêndice A

Contas

Neste apêndice apresentaremos alguns resultados necessários para o capítulo 5.
Todas as operações do tipo soma e produto neste apêndice são sobre um anel comutativo.

Proposição A.1. *Seja $p \in \mathbb{Z}_{>0}$:*

1. *Para todo $3 \leq i \leq p$*

$$\sum_{\substack{r+s+t=i \\ r,s,t\neq 0}} x_s y_s z_t = \sum_{r=1}^{i-2} \sum_{s=1}^{i-r-1} x_r y_s z_{i-r-s}.$$

2.

$$\sum_{\substack{r+s+t=p+1 \\ 1\leq r,s,t\leq p}} x_s y_s z_t = \sum_{r=1}^{p-1} \sum_{s=1}^{p-r} x_r y_s z_{p+1-r-s}.$$

3. *Para todo $2 \leq i \leq p$*

$$\sum_{\substack{r+s+t=p+i \\ 1\leq r,s,t\leq p}} x_s y_s z_t = \sum_{r=1}^{i-1} \sum_{s=i-r}^{p} x_r y_s z_{p+i-r-s} + \sum_{r=i}^{p} \sum_{s=1}^{p+i-r-1} x_r y_s z_{p+i-r-s}.$$

4. *Para todo $1 \leq i \leq p$*

$$\sum_{\substack{r+s+t=2p+i \\ 1\leq r,s,t\leq p}} x_s y_s z_t = \sum_{r=i}^{p} \sum_{s=p+i-r}^{p} x_r y_s z_{2p+i-r-s}.$$

Demonstração:

1.

$$\begin{aligned}
\sum_{\substack{r+s+t=i \\ r,s,t\neq 0}} x_s y_s z_t &= x_1 \sum_{\substack{s+t=i-1 \\ s,t\neq 0}} y_s z_t + x_2 \sum_{\substack{s+t=i-2 \\ s,t\neq 0}} y_s z_t + x_3 \sum_{\substack{s+t=i-3 \\ s,t\neq 0}} y_s z_t + \cdots + x_{i-2} \sum_{\substack{s+t=2 \\ s,t\neq 0}} y_s z_t \\
&= x_1 \sum_{s=1}^{i-2} y_s z_{i-1-s} + x_2 \sum_{s=1}^{i-3} y_s z_{i-2-s} + x_3 \sum_{s=1}^{i-4} y_s z_{i-3-s} + \cdots + x_{i-2} \sum_{s=1}^{1} y_s z_{2-s} \\
&= \sum_{r=1}^{i-2} x_r \sum_{s=1}^{i-r-1} y_s z_{i-r-s}.
\end{aligned}$$

2.

$$\sum_{\substack{r+s+t=p+1\\1\le r,s,t\le p}} x_s y_s z_t = x_1 \sum_{\substack{s+t=p\\1\le s,t\le p-1}} y_s z_t + x_2 \sum_{\substack{s+t=p-1\\1\le s,t\le p-2}} y_s z_t + x_3 \sum_{\substack{s+t=p-2\\1\le s,t\le p-3}} y_s z_t + \cdots + x_{p-1} \sum_{\substack{s+t=2\\1\le s,t\le 1}} y_s z_t$$
$$= x_1 \sum_{s=1}^{p-1} y_s z_{p-s} + x_2 \sum_{s=1}^{p-2} y_s z_{p-1-s} + x_3 \sum_{s=1}^{p-3} y_s z_{p-2-s} + \cdots + x_{p-1} \sum_{s=1}^{1} y_s z_{2-s}$$
$$= \sum_{r=1}^{p-1} x_r \sum_{s=1}^{p-r} y_s z_{p+1-r-s}.$$

3. Para $2 \le i \le p$

$$\sum_{\substack{r+s+t=p+i\\1\le r,s,t\le p}} x_s y_s z_t = x_1 \sum_{\substack{s+t=p+i-1\\i-1\le s,t\le p}} y_s z_t + x_2 \sum_{\substack{s+t=p+i-2\\i-2\le s,t\le p}} y_s z_t + \cdots + x_{i-1} \sum_{\substack{s+t=p+1\\1\le s,t\le p}} y_s z_t +$$
$$+ x_i \sum_{\substack{s+t=p\\1\le s,t\le p-1}} y_s z_t + x_{i+1} \sum_{\substack{s+t=p-1\\1\le s,t\le p-2}} y_s z_t + \cdots + x_p \sum_{\substack{s+t=i\\1\le s,t\le i-1}} y_s z_t$$
$$= x_1 \sum_{s=i-1}^{p} y_s z_{p+i-1-s} + x_2 \sum_{s=i-2}^{p} y_s z_{p+i-2-s} + \cdots + x_{i-1} \sum_{s=1}^{p} y_s z_{p+1-s} +$$
$$+ x_i \sum_{s=1}^{p-1} y_s z_{p-s} + x_{i+1} \sum_{s=1}^{p-2} y_s z_{p-1-s} + \cdots + x_p \sum_{s=1}^{i-1} y_s z_{i-s}$$
$$= \sum_{r=1}^{i-1} x_r \sum_{s=i-r}^{p} y_s z_{p+i-r-s} + \sum_{r=i}^{p} x_r \sum_{s=1}^{p+i-r-1} y_s z_{p+i-r-s}.$$

4. Para $1 \le i \le p$

$$\sum_{\substack{r+s+t=2p+i\\1\le r,s,t\le p}} x_s y_s z_t = x_i y_p z_p + x_{i+1} \sum_{\substack{s+t=2p-1\\p-1\le s,t\le p}} y_s z_t + x_{i+2} \sum_{\substack{s+t=2p-2\\p-2\le s,t\le p}} y_s z_t + \cdots +$$
$$+ x_{p-2} \sum_{\substack{s+t=p+i+2\\i+2\le s,t\le p}} y_s z_t + x_{p-1} \sum_{\substack{s+t=p+i+1\\i+1\le s,t\le p}} y_s z_t + x_p \sum_{\substack{s+t=p+i\\i\le s,t\le p}} y_s z_t$$
$$= x_i y_p z_p + x_{i+1} \sum_{s=p-1}^{p} y_s z_{2p-1-s} + x_{i+2} \sum_{s=p-2}^{p} y_s z_{2p-2-s} + \cdots +$$
$$+ x_{p-2} \sum_{s=i+2}^{p} y_s z_{p+i+2-s} + x_{p-1} \sum_{s=i+1}^{p} y_s z_{p+i+1-s} + x_p \sum_{s=i}^{p} y_s z_{p+i-s}$$
$$= \sum_{r=i}^{p} x_r \sum_{s=p+i-r}^{p} y_s z_{2p+i-r-s}.$$

$\square$

Proposição A.2. *Para quaisquer $i, p \in \mathbb{Z}_{>0}$, $1 \leq i \leq p$*

$$\sum_{r+s+t=1} x_s y_s z_t = x_0 y_0 z_1 + x_0 y_1 z_0 + x_1 y_0 z_0,$$

$$\sum_{r+s+t=2} x_s y_s z_t = x_0 y_0 z_2 + x_0 y_2 z_0 + x_2 y_0 z_0 + x_0 y_1 z_1 + y_0 x_1 z_1 + z_0 x_1 y_1,$$

$$\begin{aligned}\sum_{r+s+t=i} x_s y_s z_t = x_0 y_0 z_i + x_0 y_i z_0 + x_i y_0 z_0 + x_0 \sum_{s=1}^{i-1} y_s z_{i-s} + y_0 \sum_{s=1}^{i-1} x_s z_{i-s} + \\ + z_0 \sum_{s=1}^{i-1} x_s y_{i-s} + \sum_{r=1}^{i-2} \sum_{s=1}^{i-r-1} x_r y_s z_{i-r-s}, \quad i = 3, 4, 5, \ldots, p.\end{aligned}$$

Demonstração: Para $i \geq 3$

$$\begin{aligned}
\sum_{r+s+t=i} x_s y_s z_t &= x_0 \sum_{s+t=i} y_s z_t + x_1 \sum_{s+t=i-1} y_s z_t + x_2 \sum_{s+t=i-2} y_s z_t + \cdots + x_{i-2} \sum_{s+t=2} y_s z_t + \\
&\quad + x_{i-1} \sum_{s+t=1} y_s z_t + x_i \sum_{s+t=0} y_s z_t \\
&= x_0 \sum_{s=0}^{i} y_s z_{i-s} + x_1 \sum_{s=0}^{i-1} y_s z_{i-1-s} + x_2 \sum_{s=0}^{i-2} y_s z_{i-2-s} + \cdots + \\
&\quad + x_{i-2} \sum_{s=0}^{2} y_s z_{2-s} + x_{i-1} y_0 z_1 + x_{i-1} y_1 z_0 + x_i y_0 z_0 \\
&= x_0 \sum_{s=0}^{i} y_s z_{i-s} + \sum_{r=1}^{i-2} x_r \sum_{s=0}^{i-r} y_s z_{i-r-s} + x_{i-1} y_0 z_1 + x_{i-1} y_1 z_0 + x_i y_0 z_0 \\
&= x_0 y_0 z_i + x_0 y_i z_0 + x_0 \sum_{s=1}^{i-1} y_s z_{i-s} + x_{i-1} y_0 z_1 + x_{i-1} y_1 z_0 + x_i y_0 z_0 + \\
&\quad + \sum_{r=1}^{i-2} \left(x_r y_0 z_{i-r} + x_r y_{i-r} z_0 + x_r \sum_{s=1}^{i-r-1} y_s z_{i-r-s} \right) \\
&= x_0 y_0 z_i + x_0 y_i z_0 + x_0 \sum_{s=1}^{i-1} y_s z_{i-s} + x_{i-1} y_0 z_1 + x_{i-1} y_1 z_0 + x_i y_0 z_0 + \\
&\quad + \sum_{r=1}^{i-2} x_r y_0 z_{i-r} + \sum_{r=1}^{i-2} x_r y_{i-r} z_0 + \sum_{r=1}^{i-2} x_r \sum_{s=1}^{i-r-1} y_s z_{i-r-s} \\
&= x_0 y_0 z_i + x_0 y_i z_0 + x_i y_0 z_0 + x_0 \sum_{s=1}^{i-1} y_s z_{i-s} + x_{i-1} y_0 z_1 + \sum_{r=1}^{i-2} x_r y_0 z_{i-r} + \\
&\quad + x_{i-1} y_1 z_0 + \sum_{r=1}^{i-2} x_r y_{i-r} z_0 + \sum_{r=1}^{i-2} x_r \sum_{s=1}^{i-r-1} y_s z_{i-r-s} \\
&= x_0 y_0 z_i + x_0 y_i z_0 + x_i y_0 z_0 + x_0 \sum_{s=1}^{i-1} y_s z_{i-s} + \sum_{r=1}^{i-1} x_r y_0 z_{i-r} + \\
&\quad + \sum_{r=1}^{i-1} x_r y_{i-r} z_0 + \sum_{r=1}^{i-2} x_r \sum_{s=1}^{i-r-1} y_s z_{i-r-s}.
\end{aligned}$$

□

Proposição A.3. *Para quaisquer $i, p \in \mathbb{Z}_{>0}$, $1 \le i \le p$*

$$\sum_{\substack{r+s+t=p+1\\0\le r,s,t\le p}} x_s y_s z_t = x_0\sum_{s=1}^{p} y_s z_{p+1-s} + y_0\sum_{s=1}^{p} x_s z_{p+1-s} + z_0\sum_{s=1}^{p} x_s y_{p+1-s} + \sum_{r=1}^{p-1}\sum_{s=1}^{p-r} x_r y_s z_{p+1-r-s},$$

$$\sum_{\substack{r+s+t=p+i\\0\le r,s,t\le p}} x_s y_s z_t = x_0\sum_{s=i}^{p} y_s z_{p+i-s} + y_0\sum_{s=i}^{p} x_s z_{p+i-s} + z_0\sum_{s=i}^{p} x_s y_{p+i-s} +$$

$$+\sum_{r=1}^{i-1}\sum_{s=i-r}^{p} x_r y_s z_{p+i-r-s} + \sum_{r=i}^{p}\sum_{s=1}^{p+i-r-1} x_r y_s z_{p+i-r-s}, \quad i \ge 2.$$

Demonstração:

$$\sum_{\substack{r+s+t=p+i\\0\le r,s,t\le p}} x_s y_s z_t = x_0 \sum_{\substack{s+t=p+i\\0\le s,t\le p}} y_s z_t + x_1 \sum_{\substack{s+t=p+i-1\\0\le s,t\le p}} y_s z_t + \cdots + x_{i-1} \sum_{\substack{s+t=p+1\\0\le s,t\le p}} y_s z_t +$$

$$+ x_i \sum_{s+t=p} y_s z_t + x_{i+1} \sum_{s+t=p-1} y_s z_t + x_{i+2} \sum_{s+t=p-2} y_s z_t +$$

$$+ \cdots + x_{p-2} \sum_{s+t=i+2} y_s z_t + x_{p-1} \sum_{s+t=i+1} y_s z_t + x_p \sum_{s+t=i} y_s z_t$$

$$= x_0\sum_{s=i}^{p} y_s z_{p+i-s} + x_1 \sum_{s=i-1}^{p} y_s z_{p+i-1-s} + \cdots + x_{i-1}\sum_{s=1}^{p} y_s z_{p+1-s} +$$

$$+ x_i\sum_{s=0}^{p} y_s z_{p-s} + x_{i+1}\sum_{s=0}^{p-1} y_s z_{p-1-s} + x_{i+2}\sum_{s=0}^{p-2} y_s z_{p-2-s} +$$

$$+ \cdots + x_{p-2}\sum_{s=0}^{i+2} y_s z_{i+2-s} + x_{p-1}\sum_{s=0}^{i+1} y_s z_{i+1-s} + x_p \sum_{s=0}^{i} y_s z_{i-s}.$$

Agora, se $i = 1$,

$$\sum_{\substack{r+s+t=p+1\\0\le r,s,t\le p}} x_s y_s z_t = x_0\sum_{s=1}^{p} y_s z_{p+1-s} + x_1\sum_{s=0}^{p} y_s z_{p-s} + x_2\sum_{s=0}^{p-1} y_s z_{p-1-s} + \cdots +$$

$$+ x_{p-2}\sum_{s=0}^{3} y_s z_{3-s} + x_{p-1}\sum_{s=0}^{2} y_s z_{2-s} + x_p\sum_{s=0}^{1} y_s z_{1-s}$$

$$= x_0\sum_{s=1}^{p} y_s z_{p+1-s} + \sum_{r=1}^{p-1} x_r \sum_{s=0}^{p+1-r} y_s z_{p+1-r-s} + x_p y_0 z_1 + x_p y_1 z_0$$

$$= x_p y_0 z_1 + x_p y_1 z_0 + x_0\sum_{s=1}^{p} y_s z_{p+1-s} + \sum_{r=1}^{p-1} x_r \left(y_0 z_{p+1-r} + \sum_{s=1}^{p-r} y_s z_{p+1-r-s} + y_{p+1-r} z_0 \right)$$

$$= x_p y_0 z_1 + x_p y_1 z_0 + x_0\sum_{s=1}^{p} y_s z_{p+1-s} + \sum_{r=1}^{p-1} x_r y_0 z_{p+1-r} +$$

$$+ \sum_{r=1}^{p-1} x_r y_{p+1-r} z_0 + \sum_{r=1}^{p-1} x_r \sum_{s=1}^{p-r} y_s z_{p+1-r-s}$$

$$= x_0\sum_{s=1}^{p} y_s z_{p+1-s} + \sum_{r=1}^{p} x_r y_0 z_{p+1-r} + \sum_{r=1}^{p} x_r y_{p+1-r} z_0 + \sum_{r=1}^{p-1} x_r \sum_{s=1}^{p-r} y_s z_{p+1-r-s}$$

e se $2 \le i \le p$

$$
\begin{aligned}
\sum_{\substack{r+s+t=p+i\\0\le r,s,t\le p}} x_s y_s z_t &= x_0 \sum_{s=i}^{p} y_s z_{p+i-s} + \sum_{r=1}^{i-1} x_r \sum_{s=i-r}^{p} y_s z_{p+i-r-s} + x_i \sum_{s=0}^{p} y_s z_{p-s} + \sum_{r=i+1}^{p} x_r \sum_{s=0}^{p+i-r} y_s z_{p+i-r-s}\\
&= x_0 \sum_{s=i}^{p} y_s z_{p+i-s} + x_i \sum_{s=0}^{p} y_s z_{p-s} + \sum_{r=1}^{i-1} x_r \sum_{s=i-r}^{p} y_s z_{p+i-r-s} +\\
&\quad + \sum_{r=i+1}^{p} x_r \left(y_0 z_{p+i-r} + \sum_{s=1}^{p+i-r-1} y_s z_{p+i-r-s} + y_{p+i-r} z_0 \right)\\
&= x_0 \sum_{s=i}^{p} y_s z_{p+i-s} + x_i y_0 z_p + x_i \sum_{s=1}^{p-1} y_s z_{p-s} + x_i y_p z_0 +\\
&\quad + \sum_{r=1}^{i-1} x_r \sum_{s=i-r}^{p} y_s z_{p+i-r-s} + \sum_{r=i+1}^{p} x_r y_0 z_{p+i-r} +\\
&\quad + \sum_{r=i+1}^{p} x_r \sum_{s=1}^{p+i-r-1} y_s z_{p+i-r-s} + \sum_{r=i+1}^{p} x_r y_{p+i-r} z_0\\
&= x_0 \sum_{s=i}^{p} y_s z_{p+i-s} + x_i y_0 z_p + \sum_{r=i+1}^{p} x_r y_0 z_{p+i-r} + x_i y_p z_0 +\\
&\quad + \sum_{r=i+1}^{p} x_r y_{p+i-r} z_0 + \sum_{r=1}^{i-1} x_r \sum_{s=i-r}^{p} y_s z_{p+i-r-s} + x_i \sum_{s=1}^{p-1} y_s z_{p-s} +\\
&\quad + \sum_{r=i+1}^{p} x_r \sum_{s=1}^{p+i-r-1} y_s z_{p+i-r-s}\\
&= x_0 \sum_{s=i}^{p} y_s z_{p+i-s} + y_0 \sum_{r=i}^{p} x_r z_{p+i-r} + z_0 \sum_{r=i}^{p} x_r y_{p+i-r} +\\
&\quad + \sum_{r=1}^{i-1} x_r \sum_{s=i-r}^{p} y_s z_{p+i-r-s} + \sum_{r=i}^{p} x_r \sum_{s=1}^{p+i-r-1} y_s z_{p+i-r-s}.
\end{aligned}
$$

□

Proposição A.4. *Para quaisquer $i, p \in \mathbb{Z}_{>0}$, $1 \le i \le p$*

$$\sum_{\substack{r+s+t=2p+i\\0\le r,s,t\le p}} x_s y_s z_t = \sum_{r=i}^{p} \sum_{s=p+i-r}^{p} x_r y_s z_{2p+i-r-s}.$$

Demonstração:

$$
\begin{aligned}
\sum_{\substack{r+s+t=2p+i\\0\le r,s,t\le p}} x_s y_s z_t &= x_i y_p z_p + x_{i+1} \sum_{\substack{s+t=2p-1\\0\le s,t\le p}} y_s z_t + x_{i+2} \sum_{\substack{s+t=2p-2\\0\le s,t\le p}} y_s z_t + \cdots +\\
&\quad + x_{p-2} \sum_{\substack{s+t=p+i+2\\0\le s,t\le p}} y_s z_t + x_{p-1} \sum_{\substack{s+t=p+i+1\\0\le s,t\le p}} y_s z_t + x_p \sum_{\substack{s+t=p+i\\0\le s,t\le p}} y_s z_t
\end{aligned}
$$

$$
\begin{aligned}
&= x_i y_p z_p + x_{i+1} \sum_{s=p-1}^{p} y_s z_{2p-1-s} + x_{i+2} \sum_{s=p-2}^{p} y_s z_{2p-2-s} + \cdots + \\
&\quad + x_{p-2} \sum_{s=i+2}^{p} y_s z_{p+i+2-s} + x_{p-1} \sum_{s=i+1}^{p} y_s z_{p+i+1-s} + x_p \sum_{s=i}^{p} y_s z_{p+i-s} \\
&= \sum_{r=i}^{p} x_r \sum_{s=p+i-r}^{p} y_s z_{2p+i-r-s}.
\end{aligned}
$$

□

Apêndice B

Decomposição da variedade de Gelfand-Tsetlin $\mathfrak{G}$ para $Y_2(\mathfrak{gl}_3)$

Neste apêndice apresentaremos todos os detalhes da decomposição em componentes irredutíveis, da variedade de Gelfand-Tsetlin $\mathfrak{G}$ para Yangian $Y_2(\mathfrak{gl}_3)$, tal decomposição foi usada na prova da proposição (5.12).

B.1 $\mathfrak{G} = W_1 \cup W_2$

Como consequência da proposição (5.9), pode concluirse que a variedade de Gelfand-Tsetlin $\mathfrak{G}$ para $Y_2(\mathfrak{gl}_3)$ é

$$\mathfrak{G} = V\left(\{p_{ij}\}_{\substack{i=1,2,3\\ j=1,2,\dots,2i}}\right) \subset k^{18}$$

com

$$\begin{aligned}
p_{11} &= X_{11}^{(1)},\\
p_{12} &= X_{11}^{(2)},\\
p_{21} &= X_{22}^{(1)},\\
p_{22} &= X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)},\\
p_{23} &= X_{12}^{(1)}X_{21}^{(2)} + X_{12}^{(2)}X_{21}^{(1)},\\
p_{24} &= X_{12}^{(2)}X_{21}^{(2)},\\
p_{31} &= X_{33}^{(1)},\\
p_{32} &= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
p_{33} &= X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(1)} - X_{23}^{(2)}X_{32}^{(1)},\\
p_{34} &= -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)} - X_{13}^{(1)}\underbrace{X_{22}^{(1)}}_{=0}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(1)} +\\
&\quad + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)} - X_{13}^{(2)}\underbrace{X_{22}^{(1)}}_{=0}X_{31}^{(1)} + X_{12}^{(2)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}\\
&= -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(1)} +\\
&\quad + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)} + X_{12}^{(2)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)},\\
p_{35} &= -X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}\underbrace{X_{22}^{(1)}}_{=0}X_{31}^{(2)} + X_{12}^{(2)}X_{23}^{(1)}X_{31}^{(2)} +\\
&\quad + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(1)} + X_{12}^{(2)}X_{23}^{(2)}X_{31}^{(1)} + X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(1)}
\end{aligned}$$

$$
\begin{aligned}
&= -X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(2)} + X_{12}^{(2)}X_{23}^{(1)}X_{31}^{(2)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)} - \\
&\quad - X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(1)} + X_{12}^{(2)}X_{23}^{(2)}X_{31}^{(1)} + X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(1)}, \\
p_{36} &= -X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(2)} + X_{12}^{(2)}X_{23}^{(2)}X_{31}^{(2)} + X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(2)}.
\end{aligned}
$$

Usando a notação (3.17), tem-se $V = W_1 \cup W_2$ com

$$
\begin{aligned}
W_1 &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, p_{23}^{\mathbf{X}}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, p_{36}^{\mathbf{X}}\right) \subset k^{18} \\
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}\right\}
\end{aligned}
$$

e

$$
\begin{aligned}
W_2 &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, p_{23}^{\mathbf{X}}, X_{21}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, p_{36}^{\mathbf{X}}\right) \subset k^{18} \\
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(2)}, X_{33}^{(1)}\right\}.
\end{aligned}
$$

B.2 $W_1 = W_{11} \cup W_{12} \cup W_{13} \cup W_{14}$

Para

$$
W_1 = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, p_{23}^{\mathbf{X}}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, p_{36}^{\mathbf{X}}\right) \subset k^{18},
$$

temos

$$
\begin{aligned}
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}\right\}, \\
p_{22}^{\mathbf{X}} &= X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, \\
p_{23}^{\mathbf{X}} &= X_{12}^{(1)}X_{21}^{(2)} + \underbrace{X_{12}^{(2)}X_{21}^{(1)}}_{=0} \\
&= X_{12}^{(1)}X_{21}^{(2)}, \\
p_{32}^{\mathbf{X}} &= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
p_{33}^{\mathbf{X}} &= X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(1)} - X_{23}^{(2)}X_{32}^{(1)}, \\
p_{34}^{\mathbf{X}} &= -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(1)} + \\
&\quad + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)} + \underbrace{X_{12}^{(2)}X_{23}^{(1)}X_{31}^{(1)}}_{=0} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)} \\
&= -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(1)} + \\
&\quad + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}, \\
p_{35}^{\mathbf{X}} &= -X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(2)} + \underbrace{X_{12}^{(2)}X_{23}^{(1)}X_{31}^{(2)}}_{=0} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)} - \\
&\quad - X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(1)} + \underbrace{X_{12}^{(2)}X_{23}^{(2)}X_{31}^{(1)}}_{=0} + X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(1)} \\
&= -X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(2)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(1)} + X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(1)}, \\
p_{36}^{\mathbf{X}} &= -X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(2)} + \underbrace{X_{12}^{(2)}X_{23}^{(2)}X_{31}^{(2)}}_{=0} + X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(2)} \\
&= -X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(2)} + X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(2)} \\
&= X_{13}^{(2)}\left(X_{21}^{(2)}X_{32}^{(2)} - X_{22}^{(2)}X_{31}^{(2)}\right).
\end{aligned}
$$

Logo $W_1 = W_{11} \cup W_{12} \cup W_{13} \cup W_{14}$, com

$$\begin{aligned}
W_{11} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, X_{13}^{(2)}\right) \subset k^{18} \\
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{33}^{(1)}, X_{12}^{(2)}, X_{12}^{(1)}, X_{13}^{(2)}\right\},
\end{aligned}$$

$$\begin{aligned}
W_{12} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, \frac{p_{36}^{\mathbf{X}}}{X_{13}^{(2)}}\right) \subset k^{18} \\
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{33}^{(1)}, X_{12}^{(2)}, X_{12}^{(1)}\right\},
\end{aligned}$$

$$\begin{aligned}
W_{13} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, X_{13}^{(2)}\right) \subset k^{18} \\
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{33}^{(1)}, X_{12}^{(2)}, X_{21}^{(2)}, X_{13}^{(2)}\right\},
\end{aligned}$$

e

$$\begin{aligned}
W_{14} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, \frac{p_{36}^{\mathbf{X}}}{X_{13}^{(2)}}\right) \subset k^{18} \\
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{33}^{(1)}, X_{12}^{(2)}, X_{21}^{(2)}\right\}.
\end{aligned}$$

Vale a pena ressaltar que pelo quociente $\frac{p_{36}^{\mathbf{X}}}{X_{13}^{(2)}}$, queremos dizer

$$\frac{p_{36}^{\mathbf{X}}}{X_{13}^{(2)}} = X_{21}^{(2)} X_{32}^{(2)} - X_{22}^{(2)} X_{31}^{(2)}.$$

B.2.1 Decomposição de W_{11}

Fazendo o mesmo raciocinio das variedades anteriores deste apêndice, temos que para

$$W_{11} = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, X_{13}^{(2)}\right) \subset k^{18},$$

com

$$\begin{aligned}
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{13}^{(2)}\right\}, \\
p_{22}^{\mathbf{X}} &= X_{22}^{(2)} - \underbrace{X_{12}^{(1)} X_{21}^{(1)}}_{=0} \\
&= X_{22}^{(2)}, \\
p_{32}^{\mathbf{X}} &= X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)}, \\
p_{33}^{\mathbf{X}} &= \underbrace{X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)}}_{=0} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(2)} - X_{23}^{(1)} X_{32}^{(2)} - \underbrace{X_{13}^{(2)} X_{31}^{(1)}}_{=0} - X_{23}^{(2)} X_{32}^{(1)} \\
&= X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(2)} - X_{23}^{(1)} X_{32}^{(2)} - X_{23}^{(2)} X_{32}^{(1)}, \\
p_{34}^{\mathbf{X}} &= -X_{23}^{(2)} X_{32}^{(2)} - \underbrace{X_{13}^{(2)} X_{31}^{(2)} + X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(2)}}_{=0} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(2)} - \underbrace{X_{13}^{(1)} X_{22}^{(2)} X_{31}^{(1)}}_{=0} + \\
&\quad + \underbrace{X_{12}^{(1)} X_{23}^{(2)} X_{31}^{(1)}}_{=0} + X_{13}^{(1)} X_{21}^{(2)} X_{32}^{(1)} + \underbrace{X_{13}^{(2)} X_{21}^{(1)} X_{32}^{(1)}}_{=0} \\
&= -X_{23}^{(2)} X_{32}^{(2)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(2)} + X_{13}^{(1)} X_{21}^{(2)} X_{32}^{(1)},
\end{aligned}$$

$$
\begin{aligned}
p_{35}^{\mathbf{X}} &= \underbrace{-X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)}}_{=0} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(2)} + \underbrace{X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(1)} + X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(1)}}_{=0} \\
&= X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(2)}.
\end{aligned}
$$

Novamente, $W_{11} = W_{111} \cup W_{112} \cup W_{113}$, onde

$$
\begin{aligned}
W_{111} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, X_{13}^{(1)}, X_{13}^{(2)}\right) \subset k^{18} \\
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{13}^{(1)}, X_{13}^{(2)}\right\},
\end{aligned}
$$

$$
\begin{aligned}
W_{112} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, X_{21}^{(2)}, X_{13}^{(2)}\right) \subset k^{18} \\
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{21}^{(2)}, X_{13}^{(2)}\right\},
\end{aligned}
$$

e

$$
\begin{aligned}
W_{113} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, X_{32}^{(2)}, X_{13}^{(2)}\right) \subset k^{18} \\
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\right\}.
\end{aligned}
$$

Notemos que para

$$
W_{111} = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, X_{13}^{(1)}, X_{13}^{(2)}\right) \subset k^{18},
$$

as equações são

$$
\begin{aligned}
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{13}^{(1)}, X_{13}^{(2)}\right\}, \\
p_{32}^{\mathbf{X}} &= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - \underbrace{X_{13}^{(1)}X_{31}^{(1)}}_{=0} \\
&= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)}, \\
p_{33}^{\mathbf{X}} &= \underbrace{X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}}_{=0} - X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} \\
&= -X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)}, \\
p_{34}^{\mathbf{X}} &= -X_{23}^{(2)}X_{32}^{(2)} + \underbrace{X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)}}_{=0} \\
&= -X_{23}^{(2)}X_{32}^{(2)}
\end{aligned}
$$

e portanto podemos obter a decomposição para W_{111} da seguinte forma

$$
\begin{aligned}
W_{111} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, X_{13}^{(1)}, X_{13}^{(2)}\right) \\
&= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)}, X_{23}^{(1)}X_{32}^{(2)}, X_{23}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}\right) \cup \\
&\quad \cup V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)}, X_{23}^{(2)}X_{32}^{(1)}, X_{32}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}\right) \\
&= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{23}^{(1)}, X_{23}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}\right) \cup \\
&\quad \cup V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)}, X_{32}^{(2)}, X_{23}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}\right) \cup \\
&\quad \cup V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}\right).
\end{aligned}
$$

Denotando por

$$\begin{aligned}
C_1 &:= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{23}^{(1)}, X_{23}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}\right),\\
C_2 &:= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)}, X_{32}^{(2)}, X_{23}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}\right),\\
C_3 &:= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}\right),
\end{aligned}$$

tem-se

$$W_{111} = C_1 \cup C_2 \cup C_3. \tag{B.1}$$

Agora, notemos que para

$$W_{112} = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, X_{21}^{(2)}, X_{13}^{(2)}\right) \subset k^{18},$$

as equações são

$$\begin{aligned}
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{21}^{(2)}, X_{13}^{(2)}\right\},\\
p_{32}^{\mathbf{X}} &= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
p_{33}^{\mathbf{X}} &= X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)},\\
p_{34}^{\mathbf{X}} &= -X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} + \underbrace{X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)}}_{=0}\\
&= -X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)}\\
&= \left(X_{13}^{(1)}X_{21}^{(1)} - X_{23}^{(2)}\right)X_{32}^{(2)}
\end{aligned}$$

e portanto obtemos a seguinte decomposição para W_{112}

$$\begin{aligned}
W_{112} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, X_{21}^{(2)}, X_{13}^{(2)}\right)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, \left(-X_{23}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}\right)X_{32}^{(2)}, X_{21}^{(2)}, X_{13}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{32}^{(2)}, X_{21}^{(2)}, X_{13}^{(2)}\Big) \cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{31}^{(2)}, -X_{23}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{13}^{(2)}\Big).
\end{aligned}$$

Denotemos por

$$\begin{aligned}
C_4 &:= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{32}^{(2)}, X_{21}^{(2)}, X_{13}^{(2)}\Big),\\
C_5 &:= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{31}^{(2)}, -X_{23}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{13}^{(2)}\Big).
\end{aligned}$$

Portanto

$$W_{112} = C_4 \cup C_5. \tag{B.2}$$

Novamente, notemos que para

$$W_{113} = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, X_{32}^{(2)}, X_{13}^{(2)}\right) \subset k^{18}$$

as equações são:

$$\begin{aligned}
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\right\},\\
p_{32}^{\mathbf{X}} &= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
p_{33}^{\mathbf{X}} &= X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - \underbrace{X_{23}^{(1)}X_{32}^{(2)}}_{=0} - X_{23}^{(2)}X_{32}^{(1)}\\
&= X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{23}^{(2)}X_{32}^{(1)},\\
p_{34}^{\mathbf{X}} &= \underbrace{-X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)}}_{=0} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)}\\
&= X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)},
\end{aligned}$$

obtendo a seguinte decomposição para W_{113}

$$\begin{aligned}
W_{113} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, X_{32}^{(2)}, X_{13}^{(2)}\right)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad - X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)}, X_{23}^{(2)}X_{32}^{(1)}, X_{13}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big) \cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad - X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{21}^{(2)}, X_{32}^{(2)}, X_{13}^{(2)}\Big) \cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{13}^{(1)}X_{31}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)}, X_{23}^{(2)}, X_{13}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big) \cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{32}^{(1)}, X_{13}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big) \cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad - X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{21}^{(2)}, X_{32}^{(2)}, X_{13}^{(2)}\Big) \cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{13}^{(1)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big) \cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)}, X_{23}^{(2)}, X_{13}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big) \cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{32}^{(1)}, X_{13}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big) \cup
\end{aligned}$$

$$\begin{aligned}
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{21}^{(2)}, X_{32}^{(2)}, X_{13}^{(2)}\Big) \cup\\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big).
\end{aligned}$$

Como

$$\begin{aligned}
C_2 &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)}, X_{23}^{(2)}, X_{13}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big),\\
C_3 &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{32}^{(1)}, X_{13}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big),\\
C_4 &= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{32}^{(2)}, X_{21}^{(2)}, X_{13}^{(2)}\Big),
\end{aligned}$$

e, denotando por

$$C_6 := V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{13}^{(2)}\Big),$$

tem-se

$$W_{113} = C_2 \cup C_3 \cup C_4 \cup C_6. \tag{B.3}$$

Agora, das equações (B.1), (B.2) e (B.3), temos

$$W_{11} = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6. \tag{B.4}$$

B.2.2 Decomposição de W_{12}

Continuaremos com o mesmo raciocinio feito para determinar a decomposição em componentes irredutíveis de W_{11}. Lembremos que

$$W_{12} = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, \frac{p_{36}^{\mathbf{X}}}{X_{13}^{(2)}}\right) \subset k^{18},$$

com

$$\begin{aligned}
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}\right\},\\
p_{22}^{\mathbf{X}} &= X_{22}^{(2)} - \underbrace{X_{12}^{(1)}X_{21}^{(1)}}_{=0}\\
&= X_{22}^{(2)},\\
p_{32}^{\mathbf{X}} &= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
p_{33}^{\mathbf{X}} &= \underbrace{X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)}}_{=0} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(1)} - X_{23}^{(2)}X_{32}^{(1)}\\
&= X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(1)} - X_{23}^{(2)}X_{32}^{(1)},\\
p_{34}^{\mathbf{X}} &= -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)} + \underbrace{X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)}}_{=0} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - \underbrace{X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(1)}}_{=0} +\\
&\quad + \underbrace{X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)}}_{=0} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}\\
&= -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)},
\end{aligned}$$

$$\begin{aligned}
p_{35}^{\mathbf{X}} &= \underbrace{-X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)}}_{=0} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(2)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)} - \underbrace{X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(1)}}_{=0} + X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(1)} \\
&= X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(2)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)} + X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(1)}, \\
\frac{p_{36}^{\mathbf{X}}}{X_{13}^{(2)}} &= X_{21}^{(2)}X_{32}^{(2)} - \underbrace{X_{22}^{(2)}X_{31}^{(2)}}_{=0} \\
&= X_{21}^{(2)}X_{32}^{(2)},
\end{aligned}$$

segue da última equação que $W_{12} = W_{121} \cup W_{122}$, em que

$$\begin{aligned}
W_{121} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, X_{21}^{(2)}\right) \subset k^{18} \\
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{21}^{(2)}\right\} \\
&\text{e} \\
W_{122} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, X_{32}^{(2)}\right) \subset k^{18} \\
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{32}^{(2)}\right\}.
\end{aligned}$$

Notemos que as equações para

$$W_{121} = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, X_{21}^{(2)}\right) \subset k^{18}$$

são:

$$\begin{aligned}
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{21}^{(2)}\right\}, \\
p_{32}^{\mathbf{X}} &= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
p_{33}^{\mathbf{X}} &= X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(1)} - X_{23}^{(2)}X_{32}^{(1)}, \\
p_{34}^{\mathbf{X}} &= -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} + \underbrace{X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)}}_{=0} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}, \\
&= -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}, \\
p_{35}^{\mathbf{X}} &= \underbrace{X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(2)}}_{=0} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)} + \underbrace{X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(1)}}_{=0} \\
&= X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)}
\end{aligned}$$

e portanto a decomposição para W_{121} é

$$\begin{aligned}
W_{121} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, X_{21}^{(2)}\right) \\
&= V\Bigg(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
&\quad -X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(1)}, \\
&\quad -X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(2)}X_{31}^{(2)}, X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)}, X_{21}^{(2)}\Bigg) \\
&= V\Bigg(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
&\quad -X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, -X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)}, X_{13}^{(2)}, X_{21}^{(2)}\Bigg) \cup \\
&\quad \cup V\Bigg(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
&\quad -X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(1)}, -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}\Bigg) \cup
\end{aligned}$$

$$
\begin{aligned}
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(1)}, X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(2)}X_{31}^{(2)}, X_{32}^{(2)}, X_{21}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{32}^{(2)}, X_{13}^{(2)}, X_{21}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{31}^{(2)}, -X_{23}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}, X_{13}^{(2)}, X_{21}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(1)}, -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{13}^{(2)}, X_{32}^{(2)}, X_{21}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(2)}X_{31}^{(1)}, X_{21}^{(1)}X_{32}^{(1)} - X_{31}^{(2)}, X_{32}^{(2)}, X_{21}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{32}^{(2)}, X_{13}^{(2)}, X_{21}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{31}^{(2)}, -X_{23}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}, X_{13}^{(2)}, X_{21}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(1)}, -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(2)}X_{31}^{(1)}, X_{21}^{(1)}X_{32}^{(1)} - X_{31}^{(2)}, X_{32}^{(2)}, X_{21}^{(2)}\Big).
\end{aligned}
$$

Como

$$
\begin{aligned}
C_4 = V\Big(&X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&-X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{32}^{(2)}, X_{13}^{(2)}, X_{21}^{(2)}\Big),\\
C_5 = V\Big(&X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&-X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{31}^{(2)}, -X_{23}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}, X_{13}^{(2)}, X_{21}^{(2)}\Big)
\end{aligned}
$$

e denotando

$$\begin{aligned}
C_7 = V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
& -X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(1)}, -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}\Big),\\
C_8 = V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
& -X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(2)}X_{31}^{(1)}, X_{21}^{(1)}X_{32}^{(1)} - X_{31}^{(2)}, X_{32}^{(2)}, X_{21}^{(2)}\Big),
\end{aligned}$$

segue que

$$W_{121} = C_4 \cup C_5 \cup C_7 \cup C_8. \tag{B.5}$$

Por outro lado, as equações de

$$W_{122} = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, X_{32}^{(2)}\right) \subset k^{18}$$

são:

$$\begin{aligned}
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{32}^{(2)}\right\},\\
p_{32}^{\mathbf{X}} &= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
p_{33}^{\mathbf{X}} &= X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - \underbrace{X_{23}^{(1)}X_{32}^{(2)}}_{=0} - X_{13}^{(2)}X_{31}^{(1)} - X_{23}^{(2)}X_{32}^{(1)}\\
&= X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(1)} - X_{23}^{(2)}X_{32}^{(1)},\\
p_{34}^{\mathbf{X}} &= -\underbrace{X_{23}^{(2)}X_{32}^{(2)}}_{=0} - X_{13}^{(2)}X_{31}^{(2)} + \underbrace{X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)}}_{=0} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}\\
&= -X_{13}^{(2)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)},\\
p_{35}^{\mathbf{X}} &= \underbrace{X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(2)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)}}_{=0} + X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(1)}\\
&= X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(1)},
\end{aligned}$$

obtendo a seguinte decomposição

$$\begin{aligned}
W_{122} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, X_{32}^{(2)}\right)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(1)},\\
&\qquad X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(2)}X_{31}^{(2)}, X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(1)}, X_{32}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\Big) \cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(1)}, X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(2)}X_{31}^{(2)}, X_{21}^{(2)}, X_{32}^{(2)}\Big) \cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{13}^{(1)}X_{31}^{(2)} + X_{13}^{(2)}X_{31}^{(1)}, X_{13}^{(2)}X_{31}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)}, X_{23}^{(2)}X_{32}^{(1)}, X_{13}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{21}^{(2)}, X_{13}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{13}^{(1)}X_{31}^{(2)}, X_{32}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{13}^{(2)}, X_{21}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{21}^{(1)}X_{32}^{(1)} - X_{31}^{(2)}, X_{21}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{13}^{(1)}X_{31}^{(2)}, X_{13}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{13}^{(2)}X_{31}^{(1)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)}, X_{23}^{(2)}X_{32}^{(1)}, X_{13}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{21}^{(2)}, X_{13}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{13}^{(1)}X_{31}^{(2)}, X_{32}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{21}^{(1)}X_{32}^{(1)} - X_{31}^{(2)}, X_{21}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{13}^{(2)}X_{31}^{(1)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)}, X_{23}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{32}^{(1)}, X_{13}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad -X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{21}^{(2)}, X_{13}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{13}^{(1)}, X_{32}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{21}^{(1)}X_{32}^{(1)} - X_{31}^{(2)}, X_{21}^{(2)}, X_{32}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}\Big)\cup
\end{aligned}
$$

$$
\begin{aligned}
&\cup V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{31}^{(1)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}\right)\\
=\,& V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}, X_{23}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\right)\cup\\
&\cup V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{32}^{(1)}, X_{13}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\right)\cup\\
&\cup V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\right.\\
&\quad\left.-X_{23}^{(2)}X_{32}^{(1)}+X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(2)}, X_{21}^{(2)}, X_{13}^{(2)}, X_{32}^{(2)}\right)\cup\\
&\cup V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}-X_{13}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\right)\cup\\
&\cup V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\right.\\
&\quad\left.X_{23}^{(2)}X_{32}^{(1)}+X_{13}^{(2)}X_{31}^{(1)}, X_{21}^{(1)}X_{32}^{(1)}-X_{31}^{(2)}, X_{21}^{(2)}, X_{32}^{(2)}\right)\cup\\
&\cup V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{31}^{(1)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}\right).
\end{aligned}
$$

Como

$$
\begin{aligned}
C_2 &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}, X_{23}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\right),\\
C_3 &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{32}^{(1)}, X_{13}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\right),\\
C_4 &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\right.\\
&\qquad\left.-X_{23}^{(2)}X_{32}^{(1)}+X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(2)}, X_{21}^{(2)}, X_{13}^{(2)}, X_{32}^{(2)}\right),\\
C_6 &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}-X_{13}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}\right),\\
C_8 &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\right.\\
&\qquad\left.X_{23}^{(2)}X_{32}^{(1)}+X_{13}^{(2)}X_{31}^{(1)}, X_{21}^{(1)}X_{32}^{(1)}-X_{31}^{(2)}, X_{21}^{(2)}, X_{32}^{(2)}\right)
\end{aligned}
$$

e denotando

$$C_9 := V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{31}^{(1)}, X_{31}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}\right),$$

tem-se

$$W_{122} = C_2 \cup C_3 \cup C_4 \cup C_6 \cup C_8 \cup C_9. \tag{B.6}$$

Portanto, pelas equações (B.5) e (B.6)

$$W_{12} = C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \cup C_8 \cup C_9. \tag{B.7}$$

B.2.3 Decomposição de W_{13}

Continuando da mesma forma, para as variedades W_{11} e W_{12}, as equações de

$$W_{13} = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, X_{13}^{(2)}\right) \subset k^{18}$$

são:

$$\mathbf{X} = \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{13}^{(2)}\right\},$$

$$
\begin{aligned}
p_{22}^{\mathbf{X}} &= X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)},\\
p_{32}^{\mathbf{X}} &= X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},\\
p_{33}^{\mathbf{X}} &= X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(2)} - X_{23}^{(1)} X_{32}^{(2)} - \underbrace{X_{13}^{(2)} X_{31}^{(1)}}_{=0} - X_{23}^{(2)} X_{32}^{(1)}\\
&= X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(2)} - X_{23}^{(1)} X_{32}^{(2)} - X_{23}^{(2)} X_{32}^{(1)},\\
p_{34}^{\mathbf{X}} &= -X_{23}^{(2)} X_{32}^{(2)} - \underbrace{X_{13}^{(2)} X_{31}^{(2)}}_{=0} + X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(2)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(2)} - X_{13}^{(1)} X_{22}^{(2)} X_{31}^{(1)} +\\
&\quad + X_{12}^{(1)} X_{23}^{(2)} X_{31}^{(1)} + \underbrace{X_{13}^{(1)} X_{21}^{(2)} X_{32}^{(1)} + X_{13}^{(2)} X_{21}^{(1)} X_{32}^{(1)}}_{=0}\\
&= -X_{23}^{(2)} X_{32}^{(2)} + X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(2)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(2)} - X_{13}^{(1)} X_{22}^{(2)} X_{31}^{(1)} + X_{12}^{(1)} X_{23}^{(2)} X_{31}^{(1)},\\
p_{35}^{\mathbf{X}} &= -X_{13}^{(1)} X_{22}^{(2)} X_{31}^{(2)} + X_{12}^{(1)} X_{23}^{(2)} X_{31}^{(2)} + \underbrace{X_{13}^{(1)} X_{21}^{(2)} X_{32}^{(2)} + X_{13}^{(2)} X_{21}^{(1)} X_{32}^{(2)} - X_{13}^{(2)} X_{22}^{(2)} X_{31}^{(1)} + X_{13}^{(2)} X_{21}^{(2)} X_{32}^{(1)}}_{=0}\\
&= -X_{13}^{(1)} X_{22}^{(2)} X_{31}^{(2)} + X_{12}^{(1)} X_{23}^{(2)} X_{31}^{(2)}\\
&= \left(X_{12}^{(1)} X_{23}^{(2)} - X_{13}^{(1)} X_{22}^{(2)} \right) X_{31}^{(2)},
\end{aligned}
$$

segue da última equação que $W_{13} = W_{131} \cup W_{132}$, com

$$
\begin{aligned}
W_{131} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, X_{31}^{(2)}, X_{13}^{(2)} \right) \subset k^{18}\\
\mathbf{X} &= \left\{ X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)} \right\}\\
&\text{e}\\
W_{132} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, \frac{p_{35}^{\mathbf{X}}}{X_{31}^{(2)}}, X_{13}^{(2)} \right) \subset k^{18}\\
\mathbf{X} &= \left\{ X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{13}^{(2)} \right\},
\end{aligned}
$$

reparando que com $\frac{p_{35}^{\mathbf{X}}}{X_{31}^{(2)}}$, queremos dizer

$$
\frac{p_{35}^{\mathbf{X}}}{X_{31}^{(2)}} = X_{12}^{(1)} X_{23}^{(2)} - X_{13}^{(1)} X_{22}^{(2)}.
$$

Agora, as equações que determinam

$$
W_{131} = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, X_{31}^{(2)}, X_{13}^{(2)} \right) \subset k^{18}
$$

são:

$$
\begin{aligned}
\mathbf{X} &= \left\{ X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)} \right\},\\
p_{22}^{\mathbf{X}} &= X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)},\\
p_{32}^{\mathbf{X}} &= X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},\\
p_{33}^{\mathbf{X}} &= X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(1)} - \underbrace{X_{13}^{(1)} X_{31}^{(2)}}_{=0} - X_{23}^{(1)} X_{32}^{(2)} - X_{23}^{(2)} X_{32}^{(1)}\\
&= X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(1)} - X_{23}^{(1)} X_{32}^{(2)} - X_{23}^{(2)} X_{32}^{(1)}\\
&= X_{23}^{(1)} \left(X_{12}^{(1)} X_{31}^{(1)} - X_{32}^{(2)} \right) - \left(X_{23}^{(2)} - X_{13}^{(1)} X_{21}^{(1)} \right) X_{32}^{(1)},\\
p_{34}^{\mathbf{X}} &= -X_{23}^{(2)} X_{32}^{(2)} + \underbrace{X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(2)}}_{=0} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(2)} - X_{13}^{(1)} X_{22}^{(2)} X_{31}^{(1)} + X_{12}^{(1)} X_{23}^{(2)} X_{31}^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
&= -X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(1)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)}\\
&= -X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{12}^{(1)}X_{21}^{(1)}X_{31}^{(1)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)}\\
&= -\left(X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}\right)X_{32}^{(2)} + \left(-X_{13}^{(1)}X_{21}^{(1)} + X_{23}^{(2)}\right)X_{12}^{(1)}X_{31}^{(1)}\\
&= \left(X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}\right)\left(-X_{32}^{(2)} + X_{12}^{(1)}X_{31}^{(1)}\right).
\end{aligned}
$$

Portanto, tem-se a decomposição

$$
\begin{aligned}
W_{131} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, X_{31}^{(2)}, X_{13}^{(2)}\right)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{23}^{(1)}\left(X_{12}^{(1)}X_{31}^{(1)} - X_{32}^{(2)}\right) - \left(X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}\right)X_{32}^{(1)}, \left(X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}\right)\left(-X_{32}^{(2)} + X_{12}^{(1)}X_{31}^{(1)}\right), X_{31}^{(2)}, X_{13}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{23}^{(1)}\left(X_{12}^{(1)}X_{31}^{(1)} - X_{32}^{(2)}\right), X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad \left(X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}\right)X_{32}^{(1)}, -X_{32}^{(2)} + X_{12}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{23}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{32}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{23}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big)\cup\\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{32}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big).
\end{aligned}
$$

Denotando

$$
\begin{aligned}
C_{10} &:= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{23}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big),\\
C_{11} &:= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big),\\
C_{12} &:= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{32}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{13}^{(2)}\Big),
\end{aligned}
$$

tem-se

$$W_{131} = C_{10} \cup C_{11} \cup C_{12}. \tag{B.8}$$

Por outro lado, os polinômios que determinam

$$W_{132} = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, \frac{p_{35}^{\mathbf{X}}}{X_{31}^{(2)}}, X_{13}^{(2)}\right) \subset k^{18}$$

são

$$\begin{aligned}
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{13}^{(2)}\right\},\\
p_{22}^{\mathbf{X}} &= X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)},\\
p_{32}^{\mathbf{X}} &= X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},\\
p_{33}^{\mathbf{X}} &= X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(2)} - X_{23}^{(1)} X_{32}^{(2)} - X_{23}^{(2)} X_{32}^{(1)},\\
p_{34}^{\mathbf{X}} &= -X_{23}^{(2)} X_{32}^{(2)} + X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(2)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(2)} - X_{13}^{(1)} X_{22}^{(2)} X_{31}^{(1)} + X_{12}^{(1)} X_{23}^{(2)} X_{31}^{(1)}\\
&= -X_{23}^{(2)} X_{32}^{(2)} + X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(2)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(2)} + \underbrace{\left(-X_{13}^{(1)} X_{22}^{(2)} + X_{12}^{(1)} X_{23}^{(2)}\right)}_{=0} X_{31}^{(1)}\\
&= -X_{23}^{(2)} X_{32}^{(2)} + X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(2)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(2)}\\
&= -\left(X_{23}^{(2)} - X_{13}^{(1)} X_{21}^{(1)}\right) X_{32}^{(2)} + X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(2)},\\
\frac{p_{35}^{\mathbf{X}}}{X_{31}^{(2)}} &= X_{12}^{(1)} X_{23}^{(2)} - X_{13}^{(1)} X_{22}^{(2)}\\
&= X_{12}^{(1)} X_{23}^{(2)} - X_{13}^{(1)} X_{12}^{(1)} X_{21}^{(1)}\\
&= X_{12}^{(1)}\left(X_{23}^{(2)} - X_{13}^{(1)} X_{21}^{(1)}\right),
\end{aligned}$$

obtendo a decomposição

$$\begin{aligned}
W_{132} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, \frac{p_{35}^{\mathbf{X}}}{X_{31}^{(2)}}, X_{13}^{(2)}\right)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},\\
&\quad X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(2)} - X_{23}^{(1)} X_{32}^{(2)} - X_{23}^{(2)} X_{32}^{(1)},\\
&\quad -\left(X_{23}^{(2)} - X_{13}^{(1)} X_{21}^{(1)}\right) X_{32}^{(2)} + X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(2)}, X_{12}^{(1)}\left(X_{23}^{(2)} - X_{13}^{(1)} X_{21}^{(1)}\right), X_{13}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},\\
&\quad -X_{23}^{(1)} X_{32}^{(2)} - X_{23}^{(2)} X_{32}^{(1)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(2)}, \left(X_{23}^{(2)} - X_{13}^{(1)} X_{21}^{(1)}\right) X_{32}^{(2)}, X_{12}^{(1)}, X_{13}^{(2)}\Big) \cup\\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},\\
&\quad -X_{23}^{(1)} X_{32}^{(2)} + X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)} - X_{13}^{(1)} X_{31}^{(2)}, X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(2)}, X_{23}^{(2)} - X_{13}^{(1)} X_{21}^{(1)}, X_{13}^{(2)}\Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},\\
&\quad -X_{23}^{(2)} X_{32}^{(1)} + X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(2)}, X_{32}^{(2)}, X_{12}^{(1)}, X_{13}^{(2)}\Big) \cup
\end{aligned}$$

$$
\begin{aligned}
&\cup V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(1)}X_{32}^{(2)}+X_{13}^{(1)}X_{31}^{(2)},X_{23}^{(2)}-X_{13}^{(1)}X_{21}^{(1)},X_{12}^{(1)},X_{13}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(1)}X_{32}^{(2)}+X_{13}^{(1)}X_{31}^{(2)},X_{12}^{(1)},X_{23}^{(2)}-X_{13}^{(1)}X_{21}^{(1)},X_{13}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)}-X_{12}^{(1)}X_{21}^{(1)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{13}^{(1)}X_{31}^{(1)},X_{13}^{(1)}X_{31}^{(2)},\\
&\quad X_{23}^{(1)},X_{23}^{(2)}-X_{13}^{(1)}X_{21}^{(1)},X_{13}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)}-X_{12}^{(1)}X_{21}^{(1)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(1)}X_{32}^{(2)}+X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)},X_{31}^{(2)},X_{23}^{(2)}-X_{13}^{(1)}X_{21}^{(1)},X_{13}^{(2)}\Big)\\
=\,& V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(2)}X_{32}^{(1)}+X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(2)},X_{32}^{(2)},X_{12}^{(1)},X_{13}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(1)}X_{32}^{(2)}+X_{13}^{(1)}X_{31}^{(2)},X_{23}^{(2)}-X_{13}^{(1)}X_{21}^{(1)},X_{12}^{(1)},X_{13}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(1)}X_{32}^{(2)}+X_{13}^{(1)}X_{31}^{(2)},X_{12}^{(1)},X_{23}^{(2)}-X_{13}^{(1)}X_{21}^{(1)},X_{13}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)}-X_{12}^{(1)}X_{21}^{(1)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)},X_{13}^{(1)},X_{23}^{(1)},X_{23}^{(2)},X_{13}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)}-X_{12}^{(1)}X_{21}^{(1)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{13}^{(1)}X_{31}^{(1)},X_{31}^{(2)},\\
&\quad X_{23}^{(1)},X_{23}^{(2)}-X_{13}^{(1)}X_{21}^{(1)},X_{13}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)}-X_{12}^{(1)}X_{21}^{(1)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{13}^{(1)}X_{31}^{(1)},X_{23}^{(1)},X_{31}^{(2)},X_{23}^{(2)}-X_{13}^{(1)}X_{21}^{(1)},X_{13}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)}-X_{12}^{(1)}X_{21}^{(1)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{32}^{(2)}-X_{12}^{(1)}X_{31}^{(1)},X_{31}^{(2)},X_{23}^{(2)}-X_{13}^{(1)}X_{21}^{(1)},X_{13}^{(2)}\Big)\\
=\,& V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\\
&\quad -X_{23}^{(2)}X_{32}^{(1)}+X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(2)},X_{32}^{(2)},X_{12}^{(1)},X_{13}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(1)}X_{32}^{(2)}+X_{13}^{(1)}X_{31}^{(2)},X_{23}^{(2)}-X_{13}^{(1)}X_{21}^{(1)},X_{12}^{(1)},X_{13}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)}-X_{12}^{(1)}X_{21}^{(1)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)},X_{13}^{(1)},X_{23}^{(1)},X_{23}^{(2)},X_{13}^{(2)}\Big)\cup\\
&\cup V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{22}^{(2)}-X_{12}^{(1)}X_{21}^{(1)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{13}^{(1)}X_{31}^{(1)},X_{31}^{(2)},X_{23}^{(1)},X_{23}^{(2)}-X_{13}^{(1)}X_{21}^{(1)},X_{13}^{(2)}\Big)\cup
\end{aligned}
$$

$$\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},$$
$$X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{13}^{(2)}\Big).$$

Como

$$C_4 = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},$$
$$-X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{32}^{(2)}, X_{12}^{(1)}, X_{13}^{(2)}\Big),$$
$$C_5 = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},$$
$$X_{23}^{(1)}X_{32}^{(2)} + X_{13}^{(1)}X_{31}^{(2)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{12}^{(1)}, X_{13}^{(2)}\Big),$$
$$C_{10} = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{23}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{13}^{(2)}\Big),$$
$$C_{11} = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},$$
$$X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{13}^{(2)}\Big)$$

e denotando

$$C_{13} := V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{13}^{(1)}, X_{23}^{(1)}, X_{23}^{(2)}, X_{13}^{(2)}\Big),$$

temos

$$W_{132} = C_4 \cup C_5 \cup C_{10} \cup C_{11} \cup C_{13} \tag{B.9}$$

e das equaçõess (B.8) e (B.9)

$$W_{13} = C_4 \cup C_5 \cup C_{10} \cup C_{11} \cup C_{12} \cup C_{13}. \tag{B.10}$$

B.2.4 Decomposição de W_{14}

Similarmente aos casos anteriores, temos que

$$W_{14} = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, \frac{p_{36}^{\mathbf{X}}}{X_{13}^{(2)}}\right) \subset k^{18},$$

com

$$\mathbf{X} = \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}\right\},$$
$$p_{22}^{\mathbf{X}} = X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)},$$
$$p_{32}^{\mathbf{X}} = X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},$$
$$p_{33}^{\mathbf{X}} = X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(1)} - X_{23}^{(2)}X_{32}^{(1)},$$
$$p_{34}^{\mathbf{X}} = -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(1)} +$$
$$+ X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)} + \underbrace{X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(1)}}_{=0} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}$$
$$= -X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(1)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)},$$

$$
\begin{aligned}
p_{35}^{\mathbf{X}} &= -X_{13}^{(1)}\underbrace{X_{22}^{(2)}X_{31}^{(2)}}_{=0}+X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)}+\underbrace{X_{13}^{(1)}X_{21}^{(2)}X_{32}^{(2)}}_{=0}+X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)}-X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(1)}+\underbrace{X_{13}^{(2)}X_{21}^{(2)}X_{32}^{(1)}}_{=0}\\
&= X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)}+X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)}-X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(1)},\\
\frac{p_{36}^{\mathbf{X}}}{X_{13}^{(2)}} &= \underbrace{X_{21}^{(2)}X_{32}^{(2)}}_{=0}-X_{22}^{(2)}X_{31}^{(2)}\\
&= -X_{22}^{(2)}X_{31}^{(2)},
\end{aligned}
$$

segue da última equação que $W_{14}=W_{141}\cup W_{142}$, com

$$
\begin{aligned}
&W_{141}=V\left(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},p_{22}^{\mathbf{X}},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},p_{32}^{\mathbf{X}},p_{33}^{\mathbf{X}},p_{34}^{\mathbf{X}},p_{35}^{\mathbf{X}},X_{22}^{(2)}\right)\subset k^{18}\\
&\quad \mathbf{X}=\left\{X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{22}^{(2)}\right\}\\
&\quad \text{e}\\
&W_{142}=V\left(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},p_{22}^{\mathbf{X}},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},p_{32}^{\mathbf{X}},p_{33}^{\mathbf{X}},p_{34}^{\mathbf{X}},p_{35}^{\mathbf{X}},X_{31}^{(2)}\right)\subset k^{18}\\
&\quad \mathbf{X}=\left\{X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{31}^{(2)}\right\},
\end{aligned}
$$

Os polinômios que determinam

$$
W_{141}=V\left(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},p_{22}^{\mathbf{X}},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},p_{32}^{\mathbf{X}},p_{33}^{\mathbf{X}},p_{34}^{\mathbf{X}},p_{35}^{\mathbf{X}},X_{22}^{(2)}\right)\subset k^{18}
$$

são

$$
\begin{aligned}
\mathbf{X} &= \left\{X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{22}^{(2)}\right\},\\
p_{22}^{\mathbf{X}} &= \underbrace{X_{22}^{(2)}}_{=0}-X_{12}^{(1)}X_{21}^{(1)}\\
&= -X_{12}^{(1)}X_{21}^{(1)},\\
p_{32}^{\mathbf{X}} &= X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\\
p_{33}^{\mathbf{X}} &= X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)}+X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(2)}-X_{23}^{(1)}X_{32}^{(2)}-X_{13}^{(2)}X_{31}^{(1)}-X_{23}^{(2)}X_{32}^{(1)},\\
p_{34}^{\mathbf{X}} &= -X_{23}^{(2)}X_{32}^{(2)}-X_{13}^{(2)}X_{31}^{(2)}+X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)}+X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)}-\underbrace{X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(1)}}_{=0}+X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)}+X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}\\
&= -X_{23}^{(2)}X_{32}^{(2)}-X_{13}^{(2)}X_{31}^{(2)}+X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)}+X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)}+X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)}+X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)},\\
p_{35}^{\mathbf{X}} &= X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)}+X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)}-\underbrace{X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(1)}}_{=0}\\
&= X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)}+X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)}
\end{aligned}
$$

e portanto podemos obter a decomposição para W_{141} da seguinte forma

$$
\begin{aligned}
W_{141} &= V\left(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},p_{22}^{\mathbf{X}},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},p_{32}^{\mathbf{X}},p_{33}^{\mathbf{X}},p_{34}^{\mathbf{X}},p_{35}^{\mathbf{X}},X_{22}^{(2)}\right)\\
&= V\Big(X_{11}^{(1)},X_{11}^{(2)},X_{22}^{(1)},X_{12}^{(1)}X_{21}^{(1)},X_{21}^{(2)},X_{12}^{(2)},X_{33}^{(1)},X_{33}^{(2)}-X_{23}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(1)},\\
&\qquad -X_{23}^{(1)}X_{32}^{(2)}-X_{23}^{(2)}X_{32}^{(1)}+X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)}+X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)}-X_{13}^{(1)}X_{31}^{(2)}-X_{13}^{(2)}X_{31}^{(1)},\\
&\qquad -X_{23}^{(2)}X_{32}^{(2)}+X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)}+X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)}+X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)}+X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}-X_{13}^{(2)}X_{31}^{(2)},\\
&\qquad X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)}+X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)},X_{22}^{(2)}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad - X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(1)},\\
&\quad - X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(2)}X_{31}^{(2)}, X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)}, X_{22}^{(2)} \Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad - X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} + X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} - X_{13}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(1)},\\
&\quad - X_{23}^{(2)}X_{32}^{(2)} + X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)} - X_{13}^{(2)}X_{31}^{(2)}, X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)}, X_{22}^{(2)} \Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad - X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(2)}, X_{23}^{(2)}X_{32}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)}, X_{13}^{(2)}, X_{22}^{(2)} \Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(1)}X_{32}^{(2)} + X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{31}^{(2)} + X_{13}^{(2)}X_{31}^{(1)}, X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(2)}X_{31}^{(2)}, X_{21}^{(1)}, X_{22}^{(2)} \Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} + X_{13}^{(1)}X_{31}^{(2)} + X_{13}^{(2)}X_{31}^{(1)}, X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(2)}X_{31}^{(2)}, X_{32}^{(2)}, X_{22}^{(2)} \Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(1)}X_{32}^{(2)} + X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{31}^{(2)} + X_{13}^{(2)}X_{31}^{(1)}, X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(2)}X_{31}^{(2)}, X_{12}^{(1)}, X_{22}^{(2)} \Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(1)}X_{32}^{(2)} - X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(1)}X_{31}^{(2)} + X_{13}^{(2)}X_{31}^{(1)}, X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)} - X_{13}^{(2)}X_{31}^{(2)}, X_{23}^{(2)}, X_{22}^{(2)} \Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(1)}X_{32}^{(2)} + X_{23}^{(2)}X_{32}^{(1)} - X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{23}^{(2)}X_{32}^{(2)} - X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)}, X_{31}^{(2)}, X_{22}^{(2)} \Big)\\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} + X_{13}^{(1)}X_{31}^{(2)}, X_{32}^{(2)}, X_{13}^{(2)}, X_{22}^{(2)} \Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(1)}X_{32}^{(2)} + X_{13}^{(1)}X_{31}^{(2)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{13}^{(2)}, X_{22}^{(2)} \Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(1)}X_{32}^{(2)} + X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{31}^{(2)} + X_{13}^{(2)}X_{31}^{(1)}, X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(2)}X_{31}^{(2)}, X_{21}^{(1)}, X_{22}^{(2)} \Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
&\quad X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} + X_{13}^{(1)}X_{31}^{(2)}, X_{13}^{(2)}, X_{32}^{(2)}, X_{22}^{(2)} \Big) \cup
\end{aligned}
$$

$$\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},$$

$$X_{23}^{(2)} X_{32}^{(1)} + X_{13}^{(2)} X_{31}^{(1)}, X_{21}^{(1)} X_{32}^{(1)} - X_{31}^{(2)}, X_{32}^{(2)}, X_{22}^{(2)}\Big) \cup$$

$$\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},$$

$$X_{23}^{(1)} X_{32}^{(2)} + X_{23}^{(2)} X_{32}^{(1)} + X_{13}^{(1)} X_{31}^{(2)} + X_{13}^{(2)} X_{31}^{(1)}, X_{23}^{(2)} X_{32}^{(2)} + X_{13}^{(2)} X_{31}^{(2)}, X_{12}^{(1)}, X_{22}^{(2)}\Big) \cup$$

$$\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},$$

$$X_{23}^{(1)} X_{32}^{(2)} - X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)} + X_{13}^{(2)} X_{31}^{(1)}, X_{31}^{(2)}, X_{23}^{(2)}, X_{22}^{(2)}\Big) \cup$$

$$\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},$$

$$X_{23}^{(1)} X_{32}^{(2)} + X_{13}^{(1)} X_{31}^{(2)}, X_{12}^{(1)} X_{23}^{(1)} - X_{13}^{(2)}, X_{23}^{(2)}, X_{22}^{(2)}\Big) \cup$$

$$\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},$$

$$X_{23}^{(1)} X_{32}^{(2)} - X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)} + X_{13}^{(2)} X_{31}^{(1)}, X_{23}^{(2)}, X_{31}^{(2)}, X_{22}^{(2)}\Big) \cup$$

$$\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},$$

$$X_{23}^{(2)} X_{32}^{(1)} + X_{13}^{(2)} X_{31}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)} X_{31}^{(1)}, X_{31}^{(2)}, X_{22}^{(2)}\Big)$$

$$= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},$$

$$X_{23}^{(2)} X_{32}^{(1)} - X_{13}^{(1)} X_{21}^{(1)} X_{32}^{(1)} + X_{13}^{(1)} X_{31}^{(2)}, X_{32}^{(2)}, X_{13}^{(2)}, X_{22}^{(2)}\Big) \cup$$

$$\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},$$

$$X_{23}^{(1)} X_{32}^{(2)} + X_{13}^{(1)} X_{31}^{(2)}, X_{23}^{(2)} - X_{13}^{(1)} X_{21}^{(1)}, X_{13}^{(2)}, X_{22}^{(2)}\Big) \cup$$

$$\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},$$

$$X_{23}^{(1)} X_{32}^{(2)} + X_{23}^{(2)} X_{32}^{(1)} + X_{13}^{(1)} X_{31}^{(2)} + X_{13}^{(2)} X_{31}^{(1)}, X_{23}^{(2)} X_{32}^{(2)} + X_{13}^{(2)} X_{31}^{(2)}, X_{21}^{(1)}, X_{22}^{(2)}\Big) \cup$$

$$\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},$$

$$X_{23}^{(2)} X_{32}^{(1)} + X_{13}^{(2)} X_{31}^{(1)}, X_{21}^{(1)} X_{32}^{(1)} - X_{31}^{(2)}, X_{32}^{(2)}, X_{22}^{(2)}\Big) \cup$$

$$\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},$$

$$X_{23}^{(1)} X_{32}^{(2)} - X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)} + X_{13}^{(2)} X_{31}^{(1)}, X_{31}^{(2)}, X_{23}^{(2)}, X_{22}^{(2)}\Big) \cup$$

$$\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},$$

$$X_{23}^{(1)} X_{32}^{(2)} + X_{13}^{(1)} X_{31}^{(2)}, X_{12}^{(1)} X_{23}^{(1)} - X_{13}^{(2)}, X_{23}^{(2)}, X_{22}^{(2)}\Big) \cup$$

$$\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)},$$

$$X_{23}^{(2)} X_{32}^{(1)} + X_{13}^{(2)} X_{31}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)} X_{31}^{(1)}, X_{31}^{(2)}, X_{22}^{(2)}\Big).$$

Como

$$\begin{aligned}
C_4 = V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
& X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} + X_{13}^{(1)}X_{31}^{(2)}, X_{32}^{(2)}, X_{13}^{(2)}, X_{22}^{(2)}\Big),\\
C_5 = V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
& X_{23}^{(1)}X_{32}^{(2)} + X_{13}^{(1)}X_{31}^{(2)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{13}^{(2)}, X_{22}^{(2)}\Big),\\
C_7 = V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
& X_{23}^{(1)}X_{32}^{(2)} + X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(1)}X_{31}^{(2)} + X_{13}^{(2)}X_{31}^{(1)}, X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(2)}X_{31}^{(2)}, X_{21}^{(1)}, X_{22}^{(2)}\Big),\\
C_8 = V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
& X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{21}^{(1)}X_{32}^{(1)} - X_{31}^{(2)}, X_{32}^{(2)}, X_{22}^{(2)}\Big)
\end{aligned}$$

e denotando

$$\begin{aligned}
C_{14} \coloneqq V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
& X_{23}^{(1)}X_{32}^{(2)} - X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{31}^{(2)}, X_{23}^{(2)}, X_{22}^{(2)}\Big),\\
C_{15} \coloneqq V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
& X_{23}^{(1)}X_{32}^{(2)} + X_{13}^{(1)}X_{31}^{(2)}, X_{12}^{(1)}X_{23}^{(1)} - X_{13}^{(2)}, X_{23}^{(2)}, X_{22}^{(2)}\Big),\\
C_{16} \coloneqq V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
& X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}, X_{22}^{(2)}\Big),
\end{aligned}$$

temos

$$W_{141} = C_4 \cup C_5 \cup C_7 \cup C_8 \cup C_{14} \cup C_{15} \cup C_{16}. \tag{B.11}$$

Por outro lado, as equações que determinam

$$W_{142} = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, X_{31}^{(2)}\right) \subset k^{18}$$

são:

$$\begin{aligned}
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{31}^{(2)}\right\},\\
p_{22}^{\mathbf{X}} &= X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)},\\
p_{32}^{\mathbf{X}} &= X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
p_{33}^{\mathbf{X}} &= X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - \underbrace{X_{13}^{(1)}X_{31}^{(2)}}_{=0} - X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(1)} - X_{23}^{(2)}X_{32}^{(1)}\\
&= X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{23}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{31}^{(1)} - X_{23}^{(2)}X_{32}^{(1)},\\
p_{34}^{\mathbf{X}} &= -X_{23}^{(2)}X_{32}^{(2)} \underbrace{- X_{13}^{(2)}X_{31}^{(2)} + X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(2)}}_{=0} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(1)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}\\
&= -X_{23}^{(2)}X_{32}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(1)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}
\end{aligned}$$

$$
\begin{aligned}
&= -X_{23}^{(2)}X_{32}^{(2)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{22}^{(2)}X_{31}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)} \\
&= -X_{23}^{(2)}X_{32}^{(2)} + X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(1)}X_{12}^{(1)}X_{21}^{(1)}X_{31}^{(1)} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)} \\
&= -X_{23}^{(2)}\left(X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}\right) + X_{13}^{(1)}X_{21}^{(1)}\left(X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}\right) + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)} \\
&= \left(-X_{23}^{(2)} + X_{13}^{(1)}X_{21}^{(1)}\right)\left(X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}\right) + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)} \\
&= -\left(X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}\right)\left(X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}\right) + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}, \\
p_{35}^{\mathbf{X}} &= \underbrace{X_{12}^{(1)}X_{23}^{(2)}X_{31}^{(2)}}_{=0} + X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(1)} \\
&= X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{22}^{(2)}X_{31}^{(1)} \\
&= X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(2)} - X_{13}^{(2)}X_{12}^{(1)}X_{21}^{(1)}X_{31}^{(1)} \\
&= X_{13}^{(2)}X_{21}^{(1)}\left(X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}\right).
\end{aligned}
$$

Portanto a decomposição pode ser obtida da seguinte forma

$$
\begin{aligned}
W_{142} &= V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, X_{31}^{(2)}\right) \\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
&\qquad -X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)} + X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{13}^{(2)}X_{31}^{(1)}, \\
&\qquad \left(X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}\right)\left(X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}\right) - X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}, X_{13}^{(2)}X_{21}^{(1)}\left(X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}\right), X_{31}^{(2)}\Big) \\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
&\qquad X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} - X_{23}^{(1)}X_{32}^{(2)} - X_{23}^{(2)}X_{32}^{(1)}, \left(X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}\right)\left(X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}\right), X_{13}^{(2)}, X_{31}^{(2)}\Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
&\qquad X_{23}^{(1)}X_{32}^{(2)} + X_{23}^{(2)}X_{32}^{(1)} - X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{23}^{(2)}\left(X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}\right), X_{21}^{(1)}, X_{31}^{(2)}\Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
&\qquad X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{13}^{(2)}X_{21}^{(1)}X_{32}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{31}^{(2)}\Big) \\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
&\qquad X_{23}^{(1)}X_{32}^{(2)} - X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
&\qquad X_{23}^{(2)}X_{32}^{(1)} - X_{13}^{(1)}X_{21}^{(1)}X_{32}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
&\qquad X_{23}^{(1)}X_{32}^{(2)} - X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{23}^{(2)}, X_{21}^{(1)}, X_{31}^{(2)}\Big) \cup \\
&\quad \cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)}, \\
&\qquad X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{21}^{(1)}, X_{31}^{(2)}\Big) \cup
\end{aligned}
$$

$$
\begin{aligned}
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)} X_{31}^{(1)}, X_{13}^{(2)} X_{31}^{(1)}, \\
&\quad X_{32}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)} X_{31}^{(1)}, X_{31}^{(2)}\Big) \\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)} X_{31}^{(1)}, X_{23}^{(1)}, \\
&\quad X_{23}^{(2)} - X_{13}^{(1)} X_{21}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big) \cup \\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)}, \\
&\quad X_{32}^{(2)} - X_{12}^{(1)} X_{31}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)} X_{21}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big) \cup \\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)} X_{31}^{(1)}, X_{32}^{(1)}, \\
&\quad X_{32}^{(2)} - X_{12}^{(1)} X_{31}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big) \cup \\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)}, \\
&\quad X_{23}^{(2)} - X_{13}^{(1)} X_{21}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)} X_{31}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big) \cup \\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)}, \\
&\quad X_{23}^{(1)} X_{32}^{(2)} - X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)} + X_{13}^{(2)} X_{31}^{(1)}, X_{23}^{(2)}, X_{21}^{(1)}, X_{31}^{(2)}\Big) \cup \\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)}, \\
&\quad X_{23}^{(2)} X_{32}^{(1)} + X_{13}^{(2)} X_{31}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)} X_{31}^{(1)}, X_{21}^{(1)}, X_{31}^{(2)}\Big) \cup \\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)} X_{31}^{(1)}, X_{13}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)} X_{31}^{(1)}, X_{31}^{(2)}\Big) \cup \\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{31}^{(1)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{31}^{(2)}\Big) \\
&= V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)} X_{31}^{(1)}, X_{23}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)} X_{21}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big) \cup \\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)}, \\
&\quad X_{32}^{(2)} - X_{12}^{(1)} X_{31}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)} X_{21}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big) \cup \\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)} X_{31}^{(1)}, X_{32}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)} X_{31}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big) \cup \\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)}, \\
&\quad X_{23}^{(1)} X_{32}^{(2)} - X_{12}^{(1)} X_{23}^{(1)} X_{31}^{(1)} + X_{13}^{(2)} X_{31}^{(1)}, X_{23}^{(2)}, X_{21}^{(1)}, X_{31}^{(2)}\Big) \cup \\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)} X_{32}^{(1)} - X_{13}^{(1)} X_{31}^{(1)}, \\
&\quad X_{23}^{(2)} X_{32}^{(1)} + X_{13}^{(2)} X_{31}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)} X_{31}^{(1)}, X_{21}^{(1)}, X_{31}^{(2)}\Big) \cup \\
&\cup V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)} X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{31}^{(1)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{31}^{(2)}\Big).
\end{aligned}
$$

Note que

$$\begin{aligned}
C_{10} = V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{23}^{(1)},\\
& X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big),\\
C_{11} = V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
& X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{23}^{(2)} - X_{13}^{(1)}X_{21}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big),\\
C_{12} = V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{13}^{(1)}X_{31}^{(1)}, X_{32}^{(1)},\\
& X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big),\\
C_{14} = V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
& X_{23}^{(1)}X_{32}^{(2)} - X_{12}^{(1)}X_{23}^{(1)}X_{31}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{23}^{(2)}, X_{21}^{(1)}, X_{31}^{(2)}\Big),\\
C_{16} = V\Big(& X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{23}^{(1)}X_{32}^{(1)} - X_{13}^{(1)}X_{31}^{(1)},\\
& X_{23}^{(2)}X_{32}^{(1)} + X_{13}^{(2)}X_{31}^{(1)}, X_{32}^{(2)} - X_{12}^{(1)}X_{31}^{(1)}, X_{21}^{(1)}, X_{31}^{(2)}\Big)
\end{aligned}$$

e denotando

$$C_{17} := V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)}, X_{21}^{(2)}, X_{12}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{31}^{(1)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{31}^{(2)}\Big),$$

tem-se

$$W_{142} = C_{10} \cup C_{11} \cup C_{12} \cup C_{14} \cup C_{16} \cup C_{17}. \tag{B.12}$$

Pelas equações (B.11) e (B.12), segue que

$$W_{14} = C_4 \cup C_5 \cup C_7 \cup C_8 \cup C_{10} \cup C_{11} \cup C_{12} \cup C_{14} \cup C_{15} \cup C_{16} \cup C_{17}. \tag{B.13}$$

Finalmente, das equações (B.4), (B.7), (B.10) e (B.13) temos que,

$$W_1 = \bigcup_{i=1}^{17} C_i. \tag{B.14}$$

B.3 Decomposição de W_2

Para determinar a decomposição de W_2 podemos fazer o mesmo raciocinio feito com a variedade W_1, mas, neste caso consideraremos o automorfismo τ definido por

$$\tau(X_{ij}^{p}) = X_{ji}^{p}, \quad i,j = 1,2,3 \quad p = 1,2.$$

Claramente $W_2 = \tau(W_1)$, pois

$$W_1 = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, p_{22}^{\mathbf{X}}, p_{23}^{\mathbf{X}}, X_{12}^{(2)}, X_{33}^{(1)}, p_{32}^{\mathbf{X}}, p_{33}^{\mathbf{X}}, p_{34}^{\mathbf{X}}, p_{35}^{\mathbf{X}}, p_{36}^{\mathbf{X}}\right) \subset k^{18},$$

com

$$\begin{aligned}
\mathbf{X} &= \left\{X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(2)}, X_{33}^{(1)}\right\},\\
p_{22}^{\mathbf{X}} &= X_{22}^{(2)} - X_{12}^{(1)}X_{21}^{(1)},
\end{aligned}$$

$$
\begin{aligned}
p^{\mathbf{X}}_{23} &= X^{(1)}_{12}X^{(2)}_{21},\\
p^{\mathbf{X}}_{32} &= X^{(2)}_{33} - X^{(1)}_{23}X^{(1)}_{32} - X^{(1)}_{13}X^{(1)}_{31},\\
p^{\mathbf{X}}_{33} &= X^{(1)}_{12}X^{(1)}_{23}X^{(1)}_{31} + X^{(1)}_{13}X^{(1)}_{21}X^{(1)}_{32} - X^{(1)}_{13}X^{(2)}_{31} - X^{(1)}_{23}X^{(2)}_{32} - X^{(2)}_{13}X^{(1)}_{31} - X^{(2)}_{23}X^{(1)}_{32},\\
p^{\mathbf{X}}_{34} &= -X^{(2)}_{23}X^{(2)}_{32} - X^{(2)}_{13}X^{(2)}_{31} + X^{(1)}_{12}X^{(1)}_{23}X^{(2)}_{31} + X^{(1)}_{13}X^{(1)}_{21}X^{(2)}_{32} - X^{(1)}_{13}X^{(2)}_{22}X^{(1)}_{31} +\\
&\quad + X^{(1)}_{12}X^{(2)}_{23}X^{(1)}_{31} + X^{(1)}_{13}X^{(2)}_{21}X^{(1)}_{32} + X^{(2)}_{13}X^{(1)}_{21}X^{(1)}_{32},\\
p^{\mathbf{X}}_{35} &= -X^{(1)}_{13}X^{(2)}_{22}X^{(2)}_{31} + X^{(1)}_{12}X^{(2)}_{23}X^{(2)}_{31} + X^{(1)}_{13}X^{(2)}_{21}X^{(2)}_{32} + X^{(2)}_{13}X^{(1)}_{21}X^{(2)}_{32} - X^{(2)}_{13}X^{(2)}_{22}X^{(1)}_{31} + X^{(2)}_{13}X^{(2)}_{21}X^{(1)}_{32},\\
p^{\mathbf{X}}_{36} &= X^{(2)}_{13}\left(X^{(2)}_{21}X^{(2)}_{32} - X^{(2)}_{22}X^{(2)}_{31}\right)
\end{aligned}
$$

e

$$W_2 = V\left(X^{(1)}_{11}, X^{(2)}_{11}, X^{(1)}_{22}, p^{\mathbf{X}}_{22}, p^{\mathbf{X}}_{23}, X^{(2)}_{21}, X^{(1)}_{33}, p^{\mathbf{X}}_{32}, p^{\mathbf{X}}_{33}, p^{\mathbf{X}}_{34}, p^{\mathbf{X}}_{35}, p^{\mathbf{X}}_{36}\right) \subset k^{18},$$

com

$$
\begin{aligned}
\mathbf{X} &= \left\{X^{(1)}_{11}, X^{(2)}_{11}, X^{(1)}_{22}, X^{(2)}_{21}, X^{(1)}_{33}\right\},\\
p^{\mathbf{X}}_{22} &= X^{(2)}_{22} - X^{(1)}_{12}X^{(1)}_{21},\\
p^{\mathbf{X}}_{23} &= \underbrace{X^{(1)}_{12}X^{(2)}_{21}}_{=0} + X^{(2)}_{12}X^{(1)}_{21}\\
&= X^{(2)}_{12}X^{(1)}_{21},\\
p^{\mathbf{X}}_{32} &= X^{(2)}_{33} - X^{(1)}_{23}X^{(1)}_{32} - X^{(1)}_{13}X^{(1)}_{31},\\
p^{\mathbf{X}}_{33} &= X^{(1)}_{12}X^{(1)}_{23}X^{(1)}_{31} + X^{(1)}_{13}X^{(1)}_{21}X^{(1)}_{32} - X^{(1)}_{13}X^{(2)}_{31} - X^{(1)}_{23}X^{(2)}_{32} - X^{(2)}_{13}X^{(1)}_{31} - X^{(2)}_{23}X^{(1)}_{32},\\
p^{\mathbf{X}}_{34} &= -X^{(2)}_{23}X^{(2)}_{32} - X^{(2)}_{13}X^{(2)}_{31} + X^{(1)}_{12}X^{(1)}_{23}X^{(2)}_{31} + X^{(1)}_{13}X^{(1)}_{21}X^{(2)}_{32} - X^{(1)}_{13}X^{(2)}_{22}X^{(1)}_{31} +\\
&\quad + X^{(1)}_{12}X^{(2)}_{23}X^{(1)}_{31} + \underbrace{X^{(1)}_{13}X^{(2)}_{21}X^{(1)}_{32}}_{=0} + X^{(2)}_{12}X^{(1)}_{23}X^{(1)}_{31} + X^{(2)}_{13}X^{(1)}_{21}X^{(1)}_{32}\\
&= -X^{(2)}_{23}X^{(2)}_{32} - X^{(2)}_{13}X^{(2)}_{31} + X^{(1)}_{12}X^{(1)}_{23}X^{(2)}_{31} + X^{(1)}_{13}X^{(1)}_{21}X^{(2)}_{32} - X^{(1)}_{13}X^{(2)}_{22}X^{(1)}_{31} +\\
&\quad + X^{(1)}_{12}X^{(2)}_{23}X^{(1)}_{31} + X^{(2)}_{12}X^{(1)}_{23}X^{(1)}_{31} + X^{(2)}_{13}X^{(1)}_{21}X^{(1)}_{32},\\
p^{\mathbf{X}}_{35} &= -X^{(1)}_{13}X^{(2)}_{22}X^{(2)}_{31} + X^{(1)}_{12}X^{(2)}_{23}X^{(2)}_{31} + \underbrace{X^{(1)}_{13}X^{(2)}_{21}X^{(2)}_{32}}_{=0} + X^{(2)}_{12}X^{(1)}_{23}X^{(2)}_{31} + X^{(2)}_{13}X^{(1)}_{21}X^{(2)}_{32} -\\
&\quad - X^{(2)}_{13}X^{(2)}_{22}X^{(1)}_{31} + X^{(2)}_{12}X^{(2)}_{23}X^{(1)}_{31} + \underbrace{X^{(2)}_{13}X^{(2)}_{21}X^{(1)}_{32}}_{=0}\\
&= -X^{(1)}_{13}X^{(2)}_{22}X^{(2)}_{31} + X^{(1)}_{12}X^{(2)}_{23}X^{(2)}_{31} + X^{(2)}_{12}X^{(1)}_{23}X^{(2)}_{31} + X^{(2)}_{13}X^{(1)}_{21}X^{(2)}_{32} - X^{(2)}_{13}X^{(2)}_{22}X^{(1)}_{31} + X^{(2)}_{12}X^{(2)}_{23}X^{(1)}_{31},\\
p^{\mathbf{X}}_{36} &= -X^{(2)}_{13}X^{(2)}_{22}X^{(2)}_{31} + X^{(2)}_{12}X^{(2)}_{23}X^{(2)}_{31} + \underbrace{X^{(2)}_{13}X^{(2)}_{21}X^{(2)}_{32}}_{=0}\\
&= -X^{(2)}_{13}X^{(2)}_{22}X^{(2)}_{31} + X^{(2)}_{12}X^{(2)}_{23}X^{(2)}_{31}\\
&= \left(X^{(2)}_{12}X^{(2)}_{23} - X^{(2)}_{13}X^{(2)}_{22}\right)X^{(2)}_{31}.
\end{aligned}
$$

Portanto

$$W_2 = \bigcup_{i=1}^{17} \tau(C_i)$$

e

$$
\begin{aligned}
\tau(C_1) &= V\left(X^{(1)}_{11}, X^{(2)}_{11}, X^{(1)}_{22}, X^{(2)}_{22}, X^{(1)}_{21}, X^{(2)}_{21}, X^{(1)}_{33}, X^{(2)}_{33}, X^{(1)}_{32}, X^{(2)}_{32}, X^{(1)}_{31}, X^{(2)}_{31}\right),\\
\tau(C_2) &= V\left(X^{(1)}_{11}, X^{(2)}_{11}, X^{(1)}_{22}, X^{(2)}_{22}, X^{(1)}_{21}, X^{(2)}_{21}, X^{(1)}_{33}, X^{(2)}_{33} - X^{(1)}_{32}X^{(1)}_{23}, X^{(2)}_{23}, X^{(2)}_{32}, X^{(1)}_{31}, X^{(2)}_{31}\right),\\
\tau(C_3) &= V\left(X^{(1)}_{11}, X^{(2)}_{11}, X^{(1)}_{22}, X^{(2)}_{22}, X^{(1)}_{21}, X^{(2)}_{21}, X^{(1)}_{33}, X^{(2)}_{33}, X^{(1)}_{23}, X^{(2)}_{23}, X^{(1)}_{31}, X^{(2)}_{31}\right),
\end{aligned}
$$

$$\tau(C_4) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{32}^{(1)}X_{23}^{(1)} - X_{31}^{(1)}X_{13}^{(1)},$$
$$-X_{32}^{(2)}X_{23}^{(1)} + X_{31}^{(1)}X_{12}^{(1)}X_{23}^{(1)} - X_{31}^{(1)}X_{13}^{(2)}, X_{23}^{(2)}, X_{12}^{(2)}, X_{31}^{(2)}\Big),$$
$$\tau(C_5) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{32}^{(1)}X_{23}^{(1)} - X_{31}^{(1)}X_{13}^{(1)},$$
$$-X_{32}^{(1)}X_{23}^{(2)} - X_{31}^{(1)}X_{13}^{(2)}, -X_{32}^{(2)} + X_{31}^{(1)}X_{12}^{(1)}, X_{12}^{(2)}, X_{31}^{(2)}\Big),$$
$$\tau(C_6) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{31}^{(1)}X_{13}^{(1)}, X_{13}^{(2)}, X_{23}^{(1)}, X_{23}^{(2)}, X_{31}^{(2)}\Big),$$
$$\tau(C_7) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{32}^{(1)}X_{23}^{(1)} - X_{31}^{(1)}X_{13}^{(1)},$$
$$X_{32}^{(1)}X_{23}^{(2)} + X_{32}^{(2)}X_{23}^{(1)} + X_{31}^{(1)}X_{13}^{(2)} + X_{31}^{(2)}X_{13}^{(1)}, X_{32}^{(2)}X_{23}^{(2)} + X_{31}^{(2)}X_{13}^{(2)}, X_{12}^{(1)}, X_{12}^{(2)}\Big),$$
$$\tau(C_8) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{32}^{(1)}X_{23}^{(1)} - X_{31}^{(1)}X_{13}^{(1)},$$
$$X_{32}^{(2)}X_{23}^{(1)} + X_{31}^{(2)}X_{13}^{(1)}, X_{12}^{(1)}X_{23}^{(1)} - X_{13}^{(2)}, X_{23}^{(2)}, X_{12}^{(2)}\Big),$$
$$\tau(C_9) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}, X_{23}^{(1)}, X_{23}^{(2)}\Big),$$
$$\tau(C_{10}) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{21}^{(1)}X_{12}^{(1)}, X_{12}^{(2)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{31}^{(1)}X_{13}^{(1)}, X_{32}^{(1)}, X_{32}^{(2)} - X_{31}^{(1)}X_{12}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big),$$
$$\tau(C_{11}) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{21}^{(1)}X_{12}^{(1)}, X_{12}^{(2)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{32}^{(1)}X_{23}^{(1)} - X_{31}^{(1)}X_{13}^{(1)},$$
$$X_{23}^{(2)} - X_{21}^{(1)}X_{13}^{(1)}, X_{32}^{(2)} - X_{31}^{(1)}X_{12}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big),$$
$$\tau(C_{12}) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{21}^{(1)}X_{12}^{(1)}, X_{12}^{(2)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{31}^{(1)}X_{13}^{(1)}, X_{23}^{(1)}, X_{23}^{(2)} - X_{21}^{(1)}X_{13}^{(1)}, X_{13}^{(2)}, X_{31}^{(2)}\Big),$$
$$\tau(C_{13}) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{21}^{(1)}X_{12}^{(1)}, X_{12}^{(2)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{31}^{(1)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{31}^{(2)}\Big),$$
$$\tau(C_{14}) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{32}^{(1)}X_{23}^{(1)} - X_{31}^{(1)}X_{13}^{(1)},$$
$$X_{32}^{(1)}X_{23}^{(2)} - X_{21}^{(1)}X_{32}^{(1)}X_{13}^{(1)} + X_{31}^{(2)}X_{13}^{(1)}, X_{13}^{(2)}, X_{32}^{(2)}, X_{22}^{(2)}\Big),$$
$$\tau(C_{15}) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{32}^{(1)}X_{23}^{(1)} - X_{31}^{(1)}X_{13}^{(1)},$$
$$X_{32}^{(1)}X_{23}^{(2)} + X_{31}^{(1)}X_{13}^{(2)}, X_{21}^{(1)}X_{32}^{(1)} - X_{31}^{(2)}, X_{32}^{(2)}, X_{22}^{(2)}\Big),$$
$$\tau(C_{16}) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{12}^{(1)}, X_{12}^{(2)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{32}^{(1)}X_{23}^{(1)} - X_{31}^{(1)}X_{13}^{(1)},$$
$$X_{32}^{(2)}X_{23}^{(1)} + X_{31}^{(2)}X_{13}^{(1)}, X_{23}^{(2)} - X_{21}^{(1)}X_{13}^{(1)}, X_{13}^{(2)}, X_{22}^{(2)}\Big),$$
$$\tau(C_{17}) = V\Big(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)} - X_{21}^{(1)}X_{12}^{(1)}, X_{12}^{(2)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{13}^{(1)}, X_{23}^{(1)}, X_{23}^{(2)}, X_{13}^{(2)}\Big).$$

Note que

$$\tau(C_4) = C_{14},$$
$$\tau(C_5) = C_{16},$$
$$\tau(C_7) = C_7,$$
$$\tau(C_8) = C_{15},$$

$$\begin{aligned}
\tau(C_{10}) &= C_{12},\\
\tau(C_{11}) &= C_{11},\\
\tau(C_{12}) &= C_{10},\\
\tau(C_{13}) &= C_{17},\\
\tau(C_{14}) &= C_{4},\\
\tau(C_{15}) &= C_{8},\\
\tau(C_{16}) &= C_{5},\\
\tau(C_{17}) &= C_{13}.
\end{aligned}$$

Finalizamos denotando

$$\begin{aligned}
C_{18} &:= \tau(C_1) = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{32}^{(1)}, X_{32}^{(2)}, X_{31}^{(1)}, X_{31}^{(2)}\right),\\
C_{19} &:= \tau(C_2) = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{32}^{(1)} X_{23}^{(1)}, X_{23}^{(2)}, X_{32}^{(2)}, X_{31}^{(1)}, X_{31}^{(2)}\right),\\
C_{20} &:= \tau(C_3) = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{23}^{(1)}, X_{23}^{(2)}, X_{31}^{(1)}, X_{31}^{(2)}\right),\\
C_{21} &:= \tau(C_6) = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)} - X_{31}^{(1)} X_{13}^{(1)}, X_{13}^{(2)}, X_{23}^{(1)}, X_{23}^{(2)}, X_{31}^{(2)}\right),\\
C_{22} &:= \tau(C_9) = V\left(X_{11}^{(1)}, X_{11}^{(2)}, X_{22}^{(1)}, X_{22}^{(2)}, X_{21}^{(1)}, X_{21}^{(2)}, X_{33}^{(1)}, X_{33}^{(2)}, X_{13}^{(1)}, X_{13}^{(2)}, X_{23}^{(1)}, X_{23}^{(2)}\right).
\end{aligned}$$

Referências Bibliográficas

Atiyah e Macdonald (1969) M. F. Atiyah e I. G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. Citado na pág. 14

Bourbaki (1980) Nicolas Bourbaki. *Éléments de mathématique.* Masson, Paris. ISBN 2-225-65516-2. Algèbre. Chapitre 10. Algèbre homologique. [Algebra. Chapter 10. Homological algebra]. Citado na pág. 18, 19, 20

Bourbaki (1989) Nicolas Bourbaki. *Algebra. I. Chapters 1–3.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin. ISBN 3-540-19373-1. Translated from the French, Reprint of the 1974 edition. Citado na pág. 8

Bruns e Herzog (1993) Winfried Bruns e Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge. ISBN 0-521-41068-1. Citado na pág. 18, 20

Cherednik (1989) Ivan Cherednik. Quantum groups as hidden symmetries of classic representation theory. Em *Differential geometric methods in theoretical physics (Chester, 1988)*, páginas 47–54. World Sci. Publ., Teaneck, NJ. Citado na pág. 50

Colarusso e Evens (2015) Mark Colarusso e Sam Evens. Eigenvalue coincidences and K-orbits, I. *J. Algebra*, 422:611–632. ISSN 0021-8693. doi: 10.1016/j.jalgebra.2014.08.051. URL http://dx.doi.org/10.1016/j.jalgebra.2014.08.051. Citado na pág. 3, 45

Drinfel′d (1985) V. G. Drinfel′d. Hopf algebras and the quantum Yang-Baxter equation. *Dokl. Akad. Nauk SSSR*, 283(5):1060–1064. ISSN 0002-3264. Citado na pág. 50

Futorny e Ovsienko (2005) Vyacheslav Futorny e Serge Ovsienko. Kostant's theorem for special filtered algebras. *Bull. London Math. Soc.*, 37(2):187–199. ISSN 0024-6093. doi: 10.1112/S0024609304003844. URL http://dx.doi.org/10.1112/S0024609304003844. Citado na pág. 1, 20

Futorny *et al.* (2005) Vyacheslav Futorny, Alexander Molev e Serge Ovsienko. Harish-Chandra modules for Yangians. *Represent. Theory*, 9:426–454. ISSN 1088-4165. doi: 10.1090/S1088-4165-05-00195-0. URL http://dx.doi.org/10.1090/S1088-4165-05-00195-0. Citado na pág. 1, 79

Gelfand *et al.* (1995) Israel M. Gelfand, Daniel Krob, Alain Lascoux, Bernard Leclerc, Vladimir S. Retakh e Jean-Yves Thibon. Noncommutative symmetric functions. *Adv. Math.*, 112(2):218–348. ISSN 0001-8708. doi: 10.1006/aima.1995.1032. URL http://dx.doi.org/10.1006/aima.1995.1032. Citado na pág. 23, 24, 25

Grivel (2004) Pierre-Paul Grivel. Une histoire du théorème de Poincare-Birkhoff-Witt. *Expositiones Mathematicae*, 22(2):145 – 184. ISSN 0723-0869. doi: http://dx.doi.org/10.1016/S0723-0869(04)80010-0. URL http://www.sciencedirect.com/science/article/pii/S0723086904800100. Citado na pág. 12

Humphreys (1978) James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin. ISBN 0-387-90053-5. Second printing, revised. Citado na pág. 8, 11

Kostant (1963) Bertram Kostant. Lie group representations on polynomial rings. *Bull. Amer. Math. Soc.*, 69:518–526. ISSN 0002-9904. Citado na pág. 1

Kostant e Wallach (2006a) Bertram Kostant e Nolan Wallach. Gelfand-Zeitlin theory from the perspective of classical mechanics. I. Em *Studies in Lie theory*, volume 243 of *Progr. Math.*, páginas 319–364. Birkhäuser Boston, Boston, MA. doi: 10.1007/0-8176-4478-4_12. URL http://dx.doi.org/10.1007/0-8176-4478-4_12. Citado na pág. 1, 2, 3, 39

Kostant e Wallach (2006b) Bertram Kostant e Nolan Wallach. Gelfand-Zeitlin theory from the perspective of classical mechanics. II. Em *The unity of mathematics*, volume 244 of *Progr. Math.*, páginas 387–420. Birkhäuser Boston, Boston, MA. doi: 10.1007/0-8176-4467-9_10. URL http://dx.doi.org/10.1007/0-8176-4467-9_10. Citado na pág. 1, 2, 39

Matsumura (1970) Hideyuki Matsumura. *Commutative algebra*. W. A. Benjamin, Inc., New York. Citado na pág. 15, 18, 20

Matsumura (1989) Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second ed. ISBN 0-521-36764-6. Translated from the Japanese by M. Reid. Citado na pág. 20

Miščenko e Fomenko (1978) A. S. Miščenko e A. T. Fomenko. Euler equation on finite-dimensional Lie groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 42(2):396–415, 471. ISSN 0373-2436. Citado na pág. 1

Molev *et al.* (1996) A. Molev, M. Nazarov e G. Olʹshanskiĭ. Yangians and classical Lie algebras. *Uspekhi Mat. Nauk*, 51(2(308)):27–104. ISSN 0042-1316. doi: 10.1070/RM1996v051n02ABEH002772. URL http://dx.doi.org/10.1070/RM1996v051n02ABEH002772. Citado na pág. 1

Molev (1997) Alexander Molev. Casimir elements for certain polynomial current lie algerbas. Em *Group 21, Physical Applications and Mathematical Aspects of Geometry, Groups, and Algebras, Vol. 1, (H.-D. Doebner, W. Scherer, P. Nattermann, Eds)*, páginas 172–176. World Scientific, Singapore. URL http://www.maths.usyd.edu.au/u/alexm/publications/current.pdf. Citado na pág. 50

Molev (2007) Alexander Molev. *Yangians and classical Lie algebras*, volume 143 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI. ISBN 978-0-8218-4374-1. doi: 10.1090/surv/143. URL http://dx.doi.org/10.1090/surv/143. Citado na pág. 23, 24, 25

Ovsienko (2002) Serge Ovsienko. Finiteness statements for Gelfand-Zetlin modules. Em *Third International Algebraic Conference in the Ukraine (Ukrainian)*, páginas 323–338. Natsīonal. Akad. Nauk Ukraïni, Īnst. Mat., Kiev. Citado na pág. 1

Ovsienko (2003) Serge Ovsienko. Strongly nilpotent matrices and Gelfand-Zetlin modules. *Linear Algebra Appl.*, 365:349–367. ISSN 0024-3795. doi: 10.1016/S0024-3795(02)00675-4. URL http://dx.doi.org/10.1016/S0024-3795(02)00675-4. Special issue on linear algebra methods in representation theory. Citado na pág. 1, 2, 26, 46

Shepler e Witherspoon (2014) A. V. Shepler e S. Witherspoon. Poincare-Birkhoff-Witt Theorems. *ArXiv e-prints*. Citado na pág. 12

Vinberg (1990) È. B. Vinberg. Some commutative subalgebras of a universal enveloping algebra. *Izv. Akad. Nauk SSSR Ser. Mat.*, 54(1):3–25, 221. ISSN 0373-2436. Citado na pág. 1

Zariski e Samuel (1975) Oscar Zariski e Pierre Samuel. *Commutative algebra. Vol. II*. Springer-Verlag, New York-Heidelberg. Reprint of the 1960 edition, Graduate Texts in Mathematics, Vol. 29. Citado na pág. 15, 17

Želobenko (1973) D. P. Želobenko. *Compact Lie groups and their representations*. American Mathematical Society, Providence, R.I. Translated from the Russian by Israel Program for Scientific Translations, Translations of Mathematical Monographs, Vol. 40. Citado na pág. 22, 46

Índice Remissivo