Uncountable irredundant sets in nonseparable scattered C*-algebras

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Resumo

Hida, C. S. **Conjuntos irredundantes não enumeráveis em C*-álgebras dispersas não separáveis**. 2019. xii+ 61 f. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2019.

Dada uma C*-álgebra $\mathcal{A}$, um conjunto irredundante em $\mathcal{A}$ é um subconjunto $\mathcal{X}$ de $\mathcal{A}$ tal que nenhum elemento $a \in \mathcal{X}$ pertence à C*-subálgebra gerada por $\mathcal{X} \setminus \{a\}$. Toda C*-álgebra separável admite apenas conjuntos irredundantes enumeráveis e perguntamos se toda C*-álgebra não separável admite um conjunto irredundante não enumerável. Para o caso comutativo, se K é o espaço de Kunen então $C(K)$ é um exemplo consistente de uma C*-álgebra comutativa e não separável que não contém conjuntos irredundantes não enumeráveis. Por outro lado, um resultado devido a S. Todorcevic estabelece a consistência com ZFC de que toda C*-álgebra não separável e comutativa da forma $C(K)$, para K um espaço compacto 0-dimensional, possui um conjunto irredundante não enumerável.

Pelo método de forcing, construímos um exemplo de C*-álgebra dispersa não comutativa e não separável $\mathcal{A}$ que não possui conjuntos irredundantes não enumeráveis e não possui subálgebras abelianas não separáveis. Por outro lado, provamos a consistência do fato que toda C*-subálgebra de $\mathcal{B}(\ell_2)$ de densidade contínuo possui um conjunto irredundante de tamanho contínuo.

Palavras-chave: Conjuntos irredundantes, forcing, C*-álgebras dispersas.

Abstract

Hida, C. S. **Uncountable irredundant sets in nonseparable scattered C*-algebras**. 2019. xii+ 61 f. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2019.

Given a C*-algebra $\mathcal{A}$, an irredundant set in $\mathcal{A}$ is a subset $\mathcal{X}$ of $\mathcal{A}$ such that no $a \in \mathcal{X}$ belongs to the C*-subalgebra generated by $\mathcal{X} \setminus \{a\}$. Every separable C*-algebra has only countable irredundant sets and we ask if every nonseparable C*-algebra has an uncountable irredundant set. For commutative C*-algebras, if K is the Kunen line then $C(K)$ is a consistent example of a nonseparable commutative C*-algebra without uncountable irredundant sets. On the other hand, a result due to S. Todorcevic establishes that it is consistent with ZFC that every nonseparable C*-algebra of the form $C(K)$, for a compact 0-dimensional space K, has an uncountable irredundant set.

By the method of forcing, we construct a nonseparable and noncommutative scattered C*-algebra $\mathcal{A}$ without uncountable irredundant sets and with no nonseparable abelian subalgebras. On the other hand, we prove that it is consistent that every C*-subalgebra of $\mathcal{B}(\ell_2)$ of density continuum has an irredundant set of size continuum.

Keywords: Irredundant sets, forcing, scattered C*-algebras.

Contents

List of abbreviations

AF	approximately finite-dimensional
CH	Continuum Hypothesis
c.c.c	countable chain condition
MA	Martin's axiom
OCA	Open Coloring Axiom
PFA	Proper Forcing Axiom
SOT	strong operator topology
ZFC	Zermelo-Fraenkel set theory with the axiom of choice

List of Symbols

Set theory

$\mathbb{N}$	the set of natural numbers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of nonnegative real numbers
$\mathbb{C}$	the set of complex numbers
ω	the infinite countable cardinal
ω_1	the first uncountable cardinal
ω_2	the second uncountable cardinal
2^ω	the cardinality of the continuum
$\|X\|$	the cardinality of X
$f \restriction X$	the function f restricted to X
$\\|x\\|$	the norm of x
χ_X	the characteristic function of X
$w(x)$	the topological weight of X (see Definition 1.1)
$d(X)$	the topological density of X (see Definition 1.1)

C*-algebras

$irr(\mathcal{A})$	the irredundance of $\mathcal{A}$ (see Definition 2.1)
$M_n(\mathbb{C})$	the C*-algebra of all $n \times n$ matrices with complex entries
$C_0(X)$	the C*-algebra of all continuous function on X with values in $\mathbb{C}$ vanishing at infinity
$C(K)$	the C*-algebra of all continuous function on the compact space K with values in $\mathbb{C}$
$\mathcal{B}(H)$	the C*-algebra of all bounded linear operators on H
$\mathcal{K}(H)$	the C*-algebra of all compact operators on H
$C^*(X)$	the C*-subalgebra generated by a subset X (see Definition 1.22)
$C_1^*(X)$	the C*-subalgebra generated by $X \cup \{1\}$
$[A, B]$	the commutator of A and B
$\mathcal{A}_{sa}$	the set formed by all self-adjoint elements of $\mathcal{A}$
$\mathcal{A}_+$	the set formed by all positive elements of $\mathcal{A}$
$\widetilde{\mathcal{A}}$	the unitization of $\mathcal{A}$ (see Theorem 1.29)
$\mathcal{M}(\mathcal{A})$	the multiplier algebra of $\mathcal{A}$ (see Theorem 1.40)
$\mathbb{S}(\mathcal{A})$	the set of states of $\mathcal{A}$
$\mathbb{P}(\mathcal{A})$	the set of pure states of $\mathcal{A}$
$\hat{A}$	the spectrum of $\mathcal{A}$

Introduction

The purpose of this thesis is to apply combinatorial methods in the context of C*-algebras. More precisely, we apply extra set-theoretic axioms and the method of forcing to obtain examples of C*-algebras with some special properties. The interplay between C*-algebras and set theory is an active research field in mathematics. To mention a few examples, we have: 1 - Naimark's problem: The Naimark problem asks if it is true that every C*-algebra which has only one irreducible representation up to unitary equivalence is isomorphic to the C*-algebra of all compact operators on some Hilbert space. In 2004, C. Akemann and N. Weaver [3] constructed a consistent[1] counter-example assuming the Diamond Principle $\diamondsuit$. 2 - The independence of a problem about inner automorphisms[2] of the Calkin algebra: In 2006, N. C. Phillips and N. Weaver [33] proved under the Continuum Hypothesis (CH) that there are automorphisms of the Calkin algebra which are not inner and in 2011, I. Farah [14] proved the consistency of the fact that every automorphism of the Calkin algebra is inner. 4 - The existence of a nonseparable C*-algebra with no nonseparable abelian subalgebra: In 1978, C. Akemann and J.E. Doner [2] constructed a consistent example of a nonseparable C*-algebra with no nonseparable abelian subalgebra, assuming the Continuum Hypothesis (CH). In 1983, S. Popa [34] constructed the first such example in ZFC. Recently, in 2017, T. Bice e P. Koszmider [4] removed the assumption of the Continuum Hypothesis (CH) from the Akemann-Doner construction using a Luzin family.

In this thesis, we are mainly interested in the existence of irredundant sets:

Definition 1. *Let $\mathcal{A}$ be a C*-algebra. A subset $\mathcal{X} \subseteq \mathcal{A}$ is called irredundant if and only if for every $a \in \mathcal{X}$, the C*-subalgebra of $\mathcal{A}$ generated by $\mathcal{X} \setminus \{a\}$ does not contain a.*

The size of any irredundant set in a C*-algebra $\mathcal{A}$ is bounded by the density[3] (in the norm topology) of $\mathcal{A}$ (see Lemma 2.6). In particular, every separable C*-algebra admits only countable irredundant sets. One natural question is the following:

Question 2. *Is it true that every nonseparable C*-algebra admits an uncountable irredundant set?*

In the literature, Question 2 restricted to the class of commutative C*-algebras has been the subject of many papers, and the results depend on some extra set-theoretic axioms. Assuming CH, K. Kunen constructed a scattered[4] compact Hausdorff space K (known as the Kunen line) of weight ω_1 such that K^n is hereditarily separable for every $n \in \mathbb{N}$. In particular, the C*-algebra $C(K)$ is a nonseparable C*-algebra without uncountable irredundant sets (see Theorem 2.13).

[1]In this thesis, by "consistent" we mean "relatively consistent with ZFC", i.e., predicated upon the assumption that ZFC is consistent (see Chapter IV of [24]).

[2]An automorphism $\varphi : \mathcal{A} \to \mathcal{A}$ is inner if there is an unitary element $u \in \mathcal{A}$ such that $\varphi(a) = u^*au$.

[3]See Definition 1.1.

[4]Given a compact Hausdorff space K, we say that K is scattered if every nonempty subspace of K has a relative isolated point.

One motivation of this thesis is to investigate if the above phenomenon also happens in the noncommutative situation. In this direction we obtain the following result:

Theorem 3. *It is consistent that there is a scattered fully noncommutative C*-algebra $\mathcal{A}$ with the following properties:*

1. *There is a directed family of finite-dimensional algebras whose union $\mathcal{B}$ is dense in $\mathcal{A}$ such that whenever $(P_\xi : \xi < \omega_1) \subseteq \mathcal{B}$ is a family of projections which generate a nonseparable subalgebra of $\mathcal{A}$, then for every $\varepsilon > 0$*

   (a) there are $\xi_1 < \xi_2 < \xi_3 < \omega_1$ such that $\|P_{\xi_1} - P_{\xi_2}P_{\xi_3}\| < \varepsilon$,

   (b) there are[5] $\xi_1 < \xi_2 < \omega_1$ such that $\|[P_{\xi_1}, P_{\xi_2}]\| < \varepsilon$,

   (c) there are $\xi_1 < \xi_2 < \omega_1$ such that $\|[P_{\xi_1}, P_{\xi_2}]\| > 1/2 - \varepsilon$.

2. *$\mathcal{A}$ has no uncountable irredundant sets.*

3. *$\mathcal{A}$ has no nonseparable abelian subalgebra.*

Proof. See Theorem 4.1 □

The class of scattered C*-algebras is the noncommutative version of the class of scattered compact Hausdorff spaces (see Section 1.2.4). The C*-algebra consistently constructed in Theorem 3 can be considered as being the noncommutative version of the Kunen line.

At the moment, there is no absolute example of a nonseparable commutative C*-algebra without uncountable irredundant sets nor is there a consistent proof that every nonseparable commutative C*-algebra admits an uncountable irredundant set.

Another motivation of this thesis is a result due to S. Todorcevic (see [40]) which states that under MA+¬ CH, every nonseparable commutative C*-algebra of the form $C(K)$, for a compact 0-dimensional space K (in particular, for any compact scattered space), admits an uncountable irredundant set. Here, the related question for noncommutative scattered C*-algebra remains open:

Question 4. *Is it consistent that every nonseparable scattered C*-algebra admits an uncountable irredundant set?*

On the other hand, for C*-algebras of density 2^ω in $\mathcal{B}(\ell_2)$, we obtain the following:

Theorem 5. *It is relatively consistent that whenever $(T_\alpha : \alpha < 2^\omega)$ is a collection of operators in $\mathcal{B}(\ell_2)$ which generates a C*-algebra of density continuum, then there is a set $I \subsetneq 2^\omega$ of cardinality continuum such that $(T_\alpha : \alpha \in I)$ is irredundant.*

Proof. See Theorem 2.32. □

From the above result together with the fact that our algebra from Theorem 3 is a consistent example of a C*-algebra in $\mathcal{B}(\ell_2)$ of density $\omega_1 = 2^\omega$ without uncountable (of size 2^ω) irredundant set, we conclude that:

Theorem 6. *It is independent from ZFC whether there is C*-subalgebra of $\mathcal{B}(\ell_2)$ of density 2^ω with no uncountable (of size 2^ω) irredundant set.*

Proof. It follows from Theorem 2.32 and Theorem 4.1. □

[5] $[A, B]$ is the commutator of A and B defined as $[A, B] = AB - BA$.

In the pursuit of an answer to Question 4 we consider a weaker version of the notion of irredundant sets. In this direction, we consider almost irredundant sets (see Definition 2.33) and we obtain the following result:

Theorem 7. *Assume PFA. Then every nonseparable scattered C*-algebra has an uncountable almost irredundant set.*

Proof. See Theorem 2.41. □

All the results in this thesis (except the results from Section 2.6) were published in the article *Large irredundant sets in operator algebras* [18] with the co-advisor Piotr Koszmider.

This thesis is divided in four chapters and is organized as follows:

In Chapter 1 - **Preliminaries**, we review some definitions and results from set theory and C*-algebras that will be used in this thesis. We also introduce notation, in particular in Section 1.2.4, where we consider the class of scattered C*-algebras. Most of the consistent results obtained in this thesis use the method of forcing, which is not discussed in Chapter 1. See *Set theory: An introduction to independence proofs* [24], for details about forcing.

Chapter 2 - **Irredundant sets in C*-algebras**, is devoted to the main notion of this thesis, that of an irredundant set. We explore some examples and some properties of irredundant sets and look at the commutative case to give us some motivation for the noncommutative one. Then in Section 2.5 we deal with the problem of extracting irredundant sets from a given collection of operators. We prove Theorem 2.32, which guarantees that it is consistent with ZFC that every subalgebra of $\mathcal{B}(\ell_2)$ of density continuum has an irredundant set of size continuum. We conclude this chapter with the study of almost irredundant sets, which is a weak version of the notion of irredundance. We prove Theorem 2.41, which tells us that under PFA, every nonseparable scattered C*-algebra has an uncountable almost irredundant set.

In Chapter 3 - **The forcing notion**, we define the forcing notion which will be used in Chapter 4 to construct a nonseparable scattered C*-algebra without uncountable irredundant set. The forcing notion will be formed by finite approximations of a set of generators for our algebra. We define some dense sets and we conclude this chapter proving that our forcing notion is c.c.c.

Finally, in Chapter 4 - **Scattered C*-algebra without uncountable irredundant sets**, we prove the main result of this thesis, Theorem 4.1, which gives a consistent example of a nonseparable scattered C*-algebra $\mathcal{A}$ without uncountable irredundant sets. Our algebra $\mathcal{A}$ has some other interesting properties. For example, it is a noncommutative nonseparable C*-algebra such that every commutative C*-subalgebra is separable. In the last section of this chapter we combine some of the consistent results obtained in this thesis in the form of independence results and we prove a new consistent result about commutators in $\mathcal{B}(\ell_2)$ under OCA.

Chapter 1

Preliminaries

In this thesis we apply set-theoretic methods to obtain some new results in the context of C*-algebras. With this purpose in mind, in this chapter we review some concepts and results from set theory and from C*-algebras.

1.1 Set theory and Topology

This section is devoted to the review of some definitions and results from set theory and topology.

Definition 1.1. *Let X be a topological space.*

1. *The weight of X is defined as*

$$w(X) = \inf\{|B| : B \text{ is a base of open sets for } X\}.$$

2. *The density of X is defined as*

$$d(X) = \inf\{|D| : D \text{ is a dense set in } X\}.$$

If $d(X) \leq \omega$, we say that X is separable. Otherwise, we say that X is nonseparable.

Definition 1.2. *Let X be a topological space.*

- *X is hereditarily separable if and only if every subspace of X is separable.*
- *X is Lindelöf if and only if every open cover of X admits a countable subcover.*
- *X is hereditarily Lindelöf if and only if every subspace of X is Lindelöf.*

1.1.1 Proper Forcing Axiom and S-spaces

The Proper Forcing Axiom (PFA) is a strengthening of Martin's Axiom (MA). In this thesis we only need one consequence of PFA concerning S-spaces. For the statement and other consequences of PFA, see [39].

Definition 1.3. *Let X be a topological space. We say that X is an S-space if and only if X is a regular Hausdorff space which is hereditarily separable and not hereditarily Lindelöf.*

Definition 1.4. *Let X be a topological space and $(x_\alpha)_{\alpha<\kappa}$ a sequence of elements of X. Then $(x_\alpha)_{\alpha<\kappa}$ is*

1. *Right-separated if there is a sequence of open sets $(U_\alpha)_{\alpha<\kappa}$ such that $x_\alpha \in U_\alpha$ and $x_\beta \notin U_\alpha$ for $\alpha < \beta < \kappa$.*

2. *Left-separated if there is a sequence of open sets $(U_\alpha)_{\alpha<\kappa}$ such that $x_\alpha \in U_\alpha$ and $x_\beta \notin U_\alpha$ for $\beta < \alpha < \kappa$.*

Theorem 1.5. *Let X be a regular Hausdorff space. Then X is hereditarily separable if and only if it has no uncountable left-separated sequence and X is hereditarily Lindelöf if and only if it has no uncountable right-separated sequence.*

Proof. See Theorem 3.1 of [35]. □

Theorem 1.6. *Assume PFA. There are no S-spaces, i.e., every regular topological space which has an uncountable right-separated sequence has an uncountable left-separated sequence.*

Proof. See Theorem 8.9 of [39]. □

In this thesis, we use the following consequence of Theorem 1.6:

Corollary 1.7. *Assume PFA. Suppose X is a regular Hausdorff space such that X has an uncountable right-separated sequence. Then X has an uncountable discrete set.*

Proof. This follows from the fact that a right-separated sequence which is also a left-separated sequence is a discrete set. □

Remark 1.8. *Under the Continuum Hypothesis (CH), there is an S-space (see Theorem 7.1.10 of [35]). In particular, this result together with Theorem 1.6 imply that the existence of an S-space is independent from ZFC.*

1.1.2 Open Coloring Axiom

The Open Coloring Axiom (OCA) was introduced by S. Todorcevic in [39]. The OCA states that for every subspace X of the real line and for every partition of[1] $[X]^2$ of the form $[X]^2 = K_0 \cup K_1$ satisfying some properties, either there is an uncountable subset Y of X such that $[Y]^2 \subseteq K_0$, or else there is a sequence $(Z_n)_n$ of subspaces of X such that $X = \bigcup_n Z_n$ and $[Z_n]^2 \subseteq K_1$ for every $n \in \mathbb{N}$.

Definition 1.9. *Let X be a topological space.*

1. *We say that $Y \subseteq [X]^2$ is open if for every $\{x, y\} \in Y$ there are disjoint open sets U, V in X such that $x \in U$, $y \in V$ and $\{\{z, w\} : z \in U$ and $w \in V\} \subseteq Y$.*

2. *Given a partition $[X]^2 = K_0 \cup K_1$, we say that a subset $Y \subseteq X$ is 0-homogeneous (1-homogeneous) if $[Y]^2 \subseteq K_0$ ($[Y]^2 \subseteq K_1$, respectively).*

The original form of OCA from [39] was stated for subsets of the real line. In this thesis, we use the following equivalent version as in [13]:

Definition 1.10. *OCA denotes the following statement: If X is a separable metric space and $[X]^2 = K_0 \cup K_1$ is a given partition with K_0 open, then either there is an uncountable 0-homogeneous set, or else X is the union of countably many 1-homogeneous sets.*

Theorem 1.11 (S. Todorcevic [39])**.** *The Open Coloring Axiom is relatively consistent with ZFC.*

[1] $[X]^2$ is the set of all subsets of X that have exactly two elements.

1.1.3 Cohen forcing

In this section we review some definitions and results about Cohen models that will be used in Section 2.5.

Definition 1.12. *Let A be a set and define $Fn(A)$ to be the set of all finite functions $p : a \to 2$, where $a \subseteq A$. The order in $Fn(A)$ is defined to be the extension of functions, i.e., for $p, q \in Fn(A)$ we define $p \leq q$ if and only if $q \subseteq p$.*

Denote by $V^{Fn(A)}$ the class of all $Fn(A)$-names.

One important feature of Cohen models is the fact that every bijection $f : A \to B$ between two sets A and B induces an order isomorphism between $Fn(A)$ and $Fn(B)$.

Lemma 1.13. *Consider a bijection $f : A \to B$. Then f induces an order isomorphism $f^c : Fn(B) \to Fn(A)$.*

Proof. For each element $p : b \to 2$ in $Fn(B)$, define $f^c(p) : f^{-1}[b] \to 2$ in $Fn(A)$ by

$$f^c(p)(x) = p \circ f(x)$$

for every $x \in f^{-1}[b]$. Then the map $p \to f^c(p)$ is an order isomorphism. □

Consider now an order isomorphism $i : Fn(B) \to Fn(A)$. Then we can define by recursion a map i_* from the class of all $Fn(B)$-names into the class of all $Fn(A)$-names in the following way:

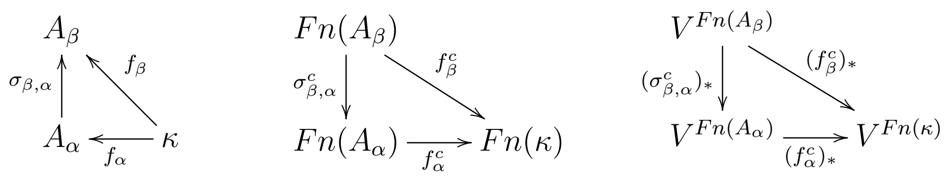
$$i_*(\tau) = \{\langle i_*(\sigma), i(p)\rangle : \langle \sigma, p\rangle \in \tau\}.$$

Theorem 1.14. *Let $\varphi(x_1, \cdots, x_n)$ be a formula, p an element of $Fn(B)$, $i : Fn(B) \to Fn(A)$ an order isomorphism and $\tau_1, \cdots, \tau_n$ a sequence of $Fn(B)$-names. Then*

$$p \Vdash \varphi(\tau_1, \cdots, \tau_n) \Leftrightarrow i(p) \Vdash \varphi(i_*(\tau_1), \cdots, i_*(\tau_n)).$$

Proof. See Lemma 7.13 of [24]. □

Theorem 1.15. *Consider $(A_\alpha)_{\alpha<\theta}$ a family of subsets of some ordinal. Suppose that for every $\alpha < \theta$, the order type of A_α is κ (witnessed by $f_\alpha : \kappa \to A_\alpha$) and for every $\alpha < \beta < \theta$, there is an order preserving bijection $\sigma_{\beta,\alpha} : A_\alpha \to A_\beta$. Then we have the following commutative diagrams:*



where in the second diagram we have the induced automorphism of Cohen forcings, and in the third diagram we have maps between classes of names induced by the automorphisms in the second diagram[2]*.*

Definition 1.16. *A family $\mathcal{A}$ of sets is called a Δ-system if and only if there is a fixed set Δ, called the root of the Δ-system, such that $a \cap b = \Delta$ whenever a and b are distinct members of $\mathcal{A}$.*

[2]For details, see Definition 7.12 of [24].

Theorem 1.17. *Assume CH. Suppose $\mathcal{A}$ is a family of sets such that $|\mathcal{A}| = \omega_2$ and for every $x \in \mathcal{A}$ we have that $|x| \leq \omega$. Then there is $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| = \omega_2$ and $\mathcal{B}$ forms a Δ-system.*

Proof. Assume CH. Consider $\alpha < \omega_2$, then

$$|\alpha^{<\omega_1}| \leq \omega_1^{<\omega_1} = \omega_1^{\omega} = (2^{\omega})^{\omega} = 2^{\omega} = \omega_1 < \omega_2.$$

Now, the result follows from Theorem 1.6 of [24]. □

1.2 C*-algebras

In this section we review some definitions and results in the theory of C*-algebras. See [9], [10], [26], [28] and [32] for other definitions and results from C*-algebras.

1.2.1 Definitions and results

Definition 1.18. *Let $\mathcal{A}$ be a Banach algebra. We say that $\mathcal{A}$ is a C*-algebra if there is a conjugate-linear map $* : \mathcal{A} \to \mathcal{A}$ such that, for every $a, b \in \mathcal{A}$,*

- $a^{**} = a$,
- $(ab)^* = b^* a^*$,
- $\|a^*\| = \|a\|$,
- $\|a^* a\| = \|a\|^2$.

The map $*$ will be called the involution of the algebra $\mathcal{A}$.

As classical examples of C*-algebras, we have the following:

Example 1.19. *Given $n \in \mathbb{N}$, we denote by $M_n(\mathbb{C})$ the C*-algebra of all $n \times n$ matrices over $\mathbb{C}$. The sum and the product are the usual sum and product of matrices and given $a \in M_n(\mathbb{C})$, a^* is the transpose conjugate of a.*

Example 1.20. *Let X be a locally compact Hausdorff space. We denote by $C_0(X)$ the C*-algebra of all continuous functions on X with values in $\mathbb{C}$ vanishing at infinity. The norm is the supremum norm and the sum and product are defined pointwise. Given $f \in C_0(X)$, f^* is the conjugate of f, i.e., the function on X defined by $f^*(x) = \overline{f(x)}$, where $\overline{\lambda}$ denotes de complex conjugate of λ. In the case where K is a compact Hausdorff space, we denote the C*-algebra $C_0(K)$ by $C(K)$.*

Example 1.21. *Let H be a Hilbert space. The C*-algebra $\mathcal{B}(H)$ is formed by all bounded linear operators on H. The sum is defined as the usual sum of operators and the product of $T, S \in \mathcal{B}(H)$ is defined as the composition $T \circ S$. Given $T \in \mathcal{B}(H)$, we define T^* to be the adjoint of T, i.e., the unique operator T^* in $\mathcal{B}(H)$ such that*

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

for every $x, y \in H$.

Definition 1.22. *Let $\mathcal{A}$ be a C*-algebra.*

1. *We say that $\mathcal{B} \subseteq \mathcal{A}$ is a C*-subalgebra of $\mathcal{A}$ if $\mathcal{B}$ is a Banach subalgebra of $\mathcal{A}$ and $b^* \in \mathcal{B}$ for every $b \in \mathcal{B}$.*

2. *Given $X \subseteq \mathcal{A}$, the C*-subalgebra of $\mathcal{A}$ generated by X is the smallest C*-subalgebra of $\mathcal{A}$ which contains X. We use the notation $C^*(X)$ for the C*-subalgebra generated by X. In the special case where F is finite, $F = \{a_1, \cdots, a_n\}$, we use the notation $C^*(a_1, \cdots, a_n)$ for $C^*(F)$.*

Example 1.23. *Let H be a Hilbert space. An operator $T \in \mathcal{B}(H)$ is compact if $T(B_H)$ is compact in H, where B_H denotes de closed unit ball of H. We denote by $\mathcal{K}(H)$ the C*-subalgebra of $\mathcal{B}(H)$ formed by all compact operators on H.*

Definition 1.24. *A map $\varphi : \mathcal{A} \to \mathcal{B}$ between two C*-algebras is a *-homomorphism if it is linear and preserves the product and the involution. If φ is a bijection, we say that φ is a *-isomorphism and that $\mathcal{A}$ and $\mathcal{B}$ are *-isomorphic.*

If $\mathcal{A}$ is a C*-algebra such that $ab = ba$ for every $a, b \in \mathcal{A}$, we say that $\mathcal{A}$ is a commutative (or abelian) C*-algebra. For instance, for every locally compact Hausdorff space X, the C*-algebra $C_0(X)$ is a commutative C*-algebra. The following theorem due to Gelfand establishes that every commutative C*-algebra is of this form:

Theorem 1.25 (I. Gelfand)**.** *For every commutative C*-algebra $\mathcal{A}$, there is a locally compact Hausdorff space X such that $\mathcal{A}$ is *-isomorphic to $C_0(X)$.*

Proof. See Theorem 2.1.10 of [28]. □

In every C*-algebra, there are elements which have special properties with respect to the involution. In this thesis, the following ones will be useful

Definition 1.26. *Let $\mathcal{A}$ be a C*-algebra and $a \in \mathcal{A}$. We say that a is:*

1. *normal, if $a^*a = aa^*$,*

2. *self-adjoint (or hermitian), if $a^* = a$,*

3. *a projection if $a^* = a = a^2$,*

4. *positive if a is self-adjoint and the spectrum of a, defined as $spec(a) := \{\lambda \in \mathbb{C} : a - \lambda 1_{\widetilde{\mathcal{A}}} \text{ is not invertible}\}$, is contained in $\mathbb{R}_+$, where $1_{\widetilde{\mathcal{A}}}$ is the unit of the unitization $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ (see Theorem 1.29 below).*

We denote by $\mathcal{A}_{sa}$ the set formed by all self-adjoint elements and $\mathcal{A}_+$ the set formed by all positive elements of $\mathcal{A}$.

As a consequence of Theorem 1.25 we have the following:

Corollary 1.27. *Let $\mathcal{A}$ be a C*-algebra and $a \in \mathcal{A}$ be a normal element. Then $C^*(a)$ is *-isomorphic to $C_0(spec(a) \setminus \{0\})$, where the *-isomorphism sends a to the identity function on $spec(a) \setminus \{0\}$.*

Proof. See Theorem 2.1.13 of [28]. □

Definition 1.28. *Let $\mathcal{A}$ be a C*-algebra. A C*-subalgebra $\mathcal{I} \subseteq \mathcal{A}$ is a left ideal (resp. right ideal) if $ab \in \mathcal{I}$ (resp. $ba \in \mathcal{I}$) whenever $a \in \mathcal{A}$ and $b \in \mathcal{I}$. In case $\mathcal{I}$ is a left and right ideal, we simply say that $\mathcal{I}$ is an ideal.*

For instance, the kernel $Ker(\varphi) := \{a : \varphi(a) = 0\}$ is an ideal in $\mathcal{A}$ for every *-homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$.

If K is a compact Hausdorff space, then $\mathcal{I} \subseteq C(K)$ is an ideal of $C(K)$ if and only if there is a closed set $F \subseteq K$ such that

$$\mathcal{I} = \{f \in C(K) : f \restriction F \equiv 0\}.$$

We say that a C*-algebra $\mathcal{A}$ is unital if it has a unit (denoted by $1_{\mathcal{A}}$ or just 1 when no confusion may arise). In this case, we have that $1_{\mathcal{A}}^* = 1_{\mathcal{A}}$ and $\|1_{\mathcal{A}}\| = 1$.

If X is a locally compact noncompact Hausdorff space, then $C_0(X)$ is a non unital C*-algebra. Observe that if Y is the Alexandroff compactification of X, then $C(Y)$ is a unital C*-algebra such that $C_0(X)$ is an ideal of $C(Y)$.

The next result tells us that a minimal unitization like in the previous remark also exists for general C*-algebras without unit:

Theorem 1.29. *Let $\mathcal{A}$ be a C*-algebra without unit. Then there is a unital C*-algebra $\widetilde{\mathcal{A}}$ which contains $\mathcal{A}$ as an ideal and such that $\widetilde{\mathcal{A}}/\mathcal{A} \equiv \mathbb{C}$.*

Proof. Define $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ as a vector space. Define the product of $(a, \lambda), (b, \mu) \in \widetilde{\mathcal{A}}$ by

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu).$$

Finally, define $(a, \lambda)^* = (a^*, \overline{\lambda})$ and

$$\|(a, \lambda)\| := \sup\{\|ab + \lambda b\| : \|b\| = 1\}.$$

It is straightforward to check that $\widetilde{A}$ is as required. □

If $\mathcal{A}$ is a unital C*-algebra and $X \subseteq \mathcal{A}$, then we denote by $C_1^*(X)$ the C*-subalgebra of $\mathcal{A}$ generated by $X \cup \{1\}$.

Lemma 1.30. *Let $\mathcal{A}$ be a unital C*-algebra and $X \subseteq \mathcal{A}$ be such that $1 \notin C^*(X)$. Then $C_1^*(X)$ is *-isomorphic to $1 \oplus C^*(X) = \{\lambda + a : \lambda \in \mathbb{C}, a \in C^*(X)\}$.*

Proof. Since $1 \oplus C^*(X) \subseteq C_1^*(X)$ and $X \cup \{1\} \subseteq 1 \oplus C^*(X)$, it is enough to show that $1 \oplus C^*(X)$ is closed. Consider a sequence $(\lambda_n + a_n)_n$ in $1 \oplus C^*(X)$ converging to an element $a \in \mathcal{A}$. Then

$$\|\lambda_n + a_n - \lambda_m - a_m\| = \|(\lambda_n - \lambda_m) + (a_n - a_m)\| =$$
$$= |\lambda_n - \lambda_m|\|1 + (a_n - a_m)/(\lambda_n - \lambda_m)\| \geq |\lambda_n - \lambda_m| d(1, C^*(X)).$$

This shows that $(\lambda_n)_n$ is a Cauchy sequence. In particular it converges to some $\lambda \in \mathbb{C}$. Then $(a_n)_n$ also converges and $a = \lambda + b$, where $b = \lim a_n \in C^*(X)$. This shows that $a = \lim_n(\lambda_n + a_n) \in 1 \oplus C^*(X)$. □

The following lemma characterizes the projections of $\widetilde{\mathcal{A}}$:

Lemma 1.31. *Let $\mathcal{A}$ be a C*-algebra and $\widetilde{\mathcal{A}}$ its unitization. If $(a, \lambda) \in \widetilde{\mathcal{A}}$ is a projection, then $(a, \lambda) = (0, 1)$ or $(a, \lambda) = (a, 0)$ with $a \in \mathcal{A}$ a projection.*

Proof. Suppose $(a, \lambda) \in \widetilde{\mathcal{A}}$ is a projection. Then $(a, \lambda)(a, \lambda) = (a, \lambda)$ and $(a, \lambda)^* = (a^*, \overline{\lambda}) = (a, \lambda)$. It follows that $\lambda \in \mathbb{R}$, $\lambda^2 = \lambda$ and $a^* = a$, $a^2 + 2\lambda a = a$. If $\lambda = 0$, then $a^2 = a$ and therefore, $(a, \lambda) = (a, 0)$ where $a \in \mathcal{A}$ is a projection. If $\lambda = 1$, then $a^2 + a = 0$ and therefore $a = 0$ (consequence of Lemma 1.27). In particular, $(a, \lambda) = (0, 1)$. This concludes the proof. □

Definition 1.32. *Let $\mathcal{A}$ be a C*-algebra and $\mathcal{I}$ an ideal of $\mathcal{A}$. We say that $\mathcal{I}$ is an essential ideal of $\mathcal{A}$ if $\mathcal{I} \cap \mathcal{J} \neq \{0\}$ for every nonzero ideal $\mathcal{J}$ of $\mathcal{A}$. Equivalently, $\mathcal{I}$ is an essential ideal of $\mathcal{A}$ if for every nonzero element $a \in \mathcal{A}$, there is an element $b \in \mathcal{I}$ such that $ab \neq 0$.*

For C*-algebras of the form $C(K)$, an ideal $\mathcal{I} = \{f \in C(K) : f \upharpoonright F \equiv 0\}$ is essential if and only if F is nowhere dense.

Definition 1.33. *Let $\mathcal{A}$ be a C*-algebra.*

1. *A positive linear functional is a linear functional $\tau \in \mathcal{A}^*$ such that $\tau[\mathcal{A}_+] \subseteq \mathbb{R}_+$.*
2. *If τ is a positive linear functional such that $\|\tau\| = 1$, we say that τ is a state on $\mathcal{A}$. We denote by $\mathbb{S}(\mathcal{A})$ the set formed by all states on $\mathcal{A}$.*

Proposition 1.34. *Let $\mathcal{A}$ be a C*-algebra and $\tau \in \mathcal{A}^*$ a positive linear functional. Then*

1. *$|\tau(a)|^2 \leq \|\tau\|\tau(a^*a)$ for every $a \in \mathcal{A}$.*
2. *$|\tau(b^*a)|^2 \leq \tau(a^*a)\tau(b^*b)$ for every $a, b \in \mathcal{A}$.*
3. *$\tau(b^*a^*ab) \leq \|a^*a\|\tau(b^*b)$ for every $a, b \in \mathcal{A}$.*
4. *$N_\tau := \{a : \tau(a^*a) = 0\}$ is a left ideal of $\mathcal{A}$.*

Proof. See [28], Theorem 3.3.2 and Theorem 3.3.7. □

Recall that given a convex set C in a Banach space, we say that $x_0 \in C$ is an extreme point of C if $x_1 = x_2 = x_0$ whenever $x_1, x_2 \in C$ and $x_0 = \frac{x_1 + x_2}{2}$.

Definition 1.35. *Let $\mathcal{A}$ be a C*-algebra and denote by $Q \subseteq \mathcal{A}^*$ the set formed by all positive linear functionals with norm not greater than 1. Then Q is a convex set and weakly* compact. By the Krein-Milman theorem (Theorem 3.65 of [12]), Q has extreme points. Denote by $\mathbb{P}(\mathcal{A})$ the set formed by all extreme points of Q different from zero. The elements of $\mathbb{P}(\mathcal{A})$ are called pure states of $\mathcal{A}$.*

Example 1.36.

1. *A state τ on $C(K)$ is a pure state if and only if there is $x \in K$ such that $\tau(f) = f(x)$ for every $f \in C(K)$.*
2. *A state τ on $\mathcal{K}(H)$ is a pure state if and only if there is a norm-one vector $x \in H$ such that*
$$\tau(T) = \langle Tx, x \rangle$$
for every $T \in \mathcal{K}(H)$.

Definition 1.37. *Let $\mathcal{A}$ be a C*-algebra. A representation of $\mathcal{A}$ is a pair (π, H) where H is a Hilbert space and $\pi : \mathcal{A} \to \mathcal{B}(H)$ is a *-homomorphism.*

Theorem 1.38 (Hoffman - Neeb). *Let $\mathcal{B}$ be a C*-subalgebra of the C*-algebra $\mathcal{A}$. Then for each $a \in \mathcal{A} \backslash \mathcal{B}$, there are representations (π_1, H) and (π_2, H) of $\mathcal{A}$ such that $\pi_1 \restriction \mathcal{B} \equiv \pi_2 \restriction \mathcal{B}$ but $\pi_1(a) \neq \pi_2(a)$.*

Proof. See Theorem 3 of [19]. □

Theorem 1.39 (Gelfand-Naimark). *Let $\mathcal{A}$ be a C*-algebra. Then $\mathcal{A}$ has a representation (π, H) such that π is injective. In particular, every C*-algebra is *-isomorphic to a C*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.*

Proof. See Theorem 3.4.1 of [28]. □

Now we introduce the notion of the multiplier algebra of a C*-algebra. We start looking at the commutative case. Consider X a locally compact Haudorff space. If βX is the Čech-Stone compactification of X, then X is an open dense set in βX and therefore, $C_0(X)$ is an essential ideal of $C(\beta X)$. Moreover, $C(\beta X)$ is a unital C*-algebra and for every C*-algebra $\mathcal{B}$ which contains $C_0(X)$ as an essential ideal, there is a unique *-homomorphism $\varphi : \mathcal{B} \to C(\beta X)$ which is the identity on $C_0(X)$.

In the general case, given a C*-algebra $\mathcal{A}$, there is a C*-algebra which plays the role of $C(\beta X)$ in the commutative case. More precisely:

Theorem 1.40. *Let $\mathcal{A}$ be a C*-algebra. Then there is a unital C*-algebra $\mathcal{M}(\mathcal{A})$ containing $\mathcal{A}$ as an essential ideal, which is universal in the sense that whenever $\mathcal{A}$ sits as an ideal in a C*-algebra $\mathcal{B}$, the identity map on $\mathcal{A}$ extends uniquely to a *-homomorphism $\varphi : \mathcal{B} \to \mathcal{M}(\mathcal{A})$ with kernel $\{b \in \mathcal{B} : \mathcal{A}b = \{0\}\}$.*

Proof. See Theorem II.7.3.1 of [5]. □

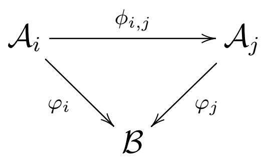
The C*-algebra $\mathcal{M}(\mathcal{A})$ is called the multiplier algebra of $\mathcal{A}$. For instance, if $\mathcal{A} = \mathcal{K}(H)$, then $\mathcal{M}(\mathcal{A}) = \mathcal{B}(H)$.

While the unitization $\widetilde{\mathcal{A}}$ of Theorem 1.29 can be thought of as a minimal unitization, the multiplier algebra $\mathcal{M}(\mathcal{A})$ can be thought of as a maximal unitization.

1.2.2 Inductive limits

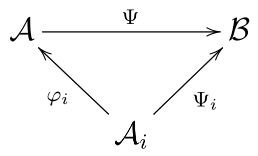
Definition 1.41. *An inductive system of C*-algebras is a collection $\{(\mathcal{A}_i, \phi_{i,j}) : i, j \in I, i \leq j\}$, where I is a directed set, $\mathcal{A}_i$ is a C*-algebra and $\phi_{i,j} : \mathcal{A}_i \to \mathcal{A}_j$ is a *-monomorphism satisfying $\phi_{i,k} = \phi_{j,k}\phi_{i,j}$ for every $i \leq j \leq k$.*

Definition 1.42. *A family of *-homomorphisms $\{\varphi_i : A_i \to \mathcal{B}\}_{i \in I}$ is compatible with the inductive system of C*-algebras $\{(\mathcal{A}_i, \phi_{i,j}) : i, j \in I, i \leq j\}$ if the diagram*



commutes for every $i, j \in I$.

Theorem 1.43. *Given an inductive system of C*-algebras* $\{(\mathcal{A}_i, \phi_{i,j}) : i, j \in I, i \leq j\}$*, there is a C*-algebra* $\mathcal{A} = \varinjlim(\mathcal{A}_i, \phi_{i,j})$ *together with a family of compatible *-homomorphisms* $\{\varphi_i : \mathcal{A}_i \to \mathcal{A}\}_{i \in I}$ *with the following universal property: For every other C*-algebra* $\mathcal{B}$ *with a family of compatible *-homomorphisms* $\{\Psi_i : A_i \to \mathcal{B}\}_{i \in I}$*, there is a unique *-homomorphism* $\Psi : \mathcal{A} \to \mathcal{B}$ *such that the diagram*



commutes for every $i \in I$*. The C*-algebra* $\varinjlim(\mathcal{A}_i, \phi_{i,j})$ *is called the inductive limit of the system* $\{(\mathcal{A}_i, \phi_{i,j}) : i, j \in I, i \leq j\}$*.*

Proof. See Section 3.7 of [26]. □

Theorem 1.44. *Let* $\mathcal{A} = \varinjlim(\mathcal{A}_i, \phi_{i,j})$ *be the inductive limit of an inductive system of C*-algebras* $\{(\mathcal{A}_i, \phi_{i,j}) : i, j \in I, i \leq j\}$*. Then there is a family* $\{\mathcal{A}'_i : i \in I\}$ *of C*-subalgebras of* $\mathcal{A}$*, and a *-isomorphism* ϕ_i *from* $\mathcal{A}_i$ *onto* $\mathcal{A}'_i$ *for each* $i \in I$ *such that:*

1. $\mathcal{A}'_i \subseteq \mathcal{A}'_j$ *for all* $i \leq j$*,*
2. $\phi_i = \phi_j \phi_{i,j}$ *for all* $i \leq j$*,*
3. $\bigcup_{i \in I} \mathcal{A}'_i$ *is dense in* $\mathcal{A}$*.*

Proof. See Theorem 3.7.2 of [26]. □

Theorem 1.45. *Let* $\mathcal{A} = \varinjlim(\mathcal{A}_i, \phi_{i,j})$ *be the inductive limit of an inductive system of C*-algebras* $\{(\mathcal{A}_i, \phi_{i,j}) : i, j \in I, i \leq j\}$*. Suppose* $\mathcal{B}$ *is a C*-algebra,* $\{\mathcal{B}_i : i \in I\}$ *is a family of C*-subalgebras of* $\mathcal{B}$*, and* Ψ_i *is a *-isomorphism from* $\mathcal{A}_i$ *onto* $\mathcal{B}_i$*, for every* $i \in I$*. Assume that*

1. $B_i \subseteq B_j$ *for all* $i \leq j$*,*
2. $\Psi_i = \Psi_j \phi_{i,j}$ *for all* $i \leq j$*,*
3. $\bigcup_{i \in I} \mathcal{B}_i$ *is dense in* $\mathcal{B}$*.*

*Then there is a *-isomorphism* Ψ *from* $\mathcal{A}$ *onto* $\mathcal{B}$ *such that* $\Psi \phi_i = \Psi_i$ *for all* $i \in I$*, where* $(\phi_i : i \in I)$ *is defined by Theorem 1.44.*

Proof. See Theorem 3.7.3 of [26]. □

1.2.3 AF and LF- algebras

In this section we introduce the notion of AF and LF algebras. In [9], a C*-algebra $\mathcal{A}$ is AF if it is the closure of an increasing union of finite dimensional subalgebras $(\mathcal{A}_n)_{n \in \mathbb{N}}$. In particular, if $\mathcal{A}$ is AF, then $\mathcal{A}$ is separable. Since we want to work with nonseparable C*-algebras, we will consider the following version of the definition of an AF algebra:

Definition 1.46 ([15])**.** *A C*-algebra* $\mathcal{A}$ *is said to be*

- *approximately finite-dimensional (AF) if it has a directed family of finite dimensional subalgebras with dense union.*

- *locally finite-dimensional (LF) if for any finite set $\mathcal{F}$ of $\mathcal{A}$ and any $\varepsilon > 0$, there exists a finite-dimensional subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that for every element $a \in \mathcal{F}$, there is an element $b \in \mathcal{B}$ so that $\|a - b\| \leq \varepsilon$.*

Observe that if $\mathcal{A}$ is an AF C*-algebra, then $\mathcal{A}$ is LF. For a separable C*-algebra $\mathcal{A}$, we have that $\mathcal{A}$ is AF if and only if $\mathcal{A}$ is LF (see Theorem 2.2 of [6]).

For nonseparable C*-algebras, we have the following result due to I. Farah and T. Katsura:

Theorem 1.47 (I. Farah and T. Katsura)**.**

1. *For a C*-algebra of density at most ω_1, AF and LF are equivalent.*

2. *For every cardinal $\kappa > \omega_1$, there exists an LF algebra with density κ which is not AF.*

Proof. See Theorem 1.5 of [15]. □

1.2.4 Scattered C*-algebras

In this section we review the definition and some properties of scattered C*-algebras. The notion of a scattered C*-algebra is the noncommutative version of that of a locally compact scattered space. More precisely, a commutative C*-algebra $C_0(X)$ is scattered if and only if X is a locally compact scattered space. For this reason, before going to the formal definition of a scattered C*-algebra, we start by analysing the commutative case $C(K)$.

By definition, a topological space X is scattered if every nonempty subspace of X has a relative isolated point. Consider K a scattered compact Hausdorff space. Then a point $x \in K$ is isolated if and only if $\{x\}$ is a clopen set. In particular, $\chi_{\{x\}} \in C(K)$, where $\chi_{\{x\}}$ is the characteristic function on $\{x\}$. Moreover $p := \chi_{\{x\}}$ is a nonzero projection in $C(K)$ such that the C*-subalgebra $pC(K)p := \{pfp : f \in C(K)\}$ is equal to the C*-subalgebra $\mathbb{C}p := \{\lambda p : \lambda \in \mathbb{C}\}$.

Consider now $\mathcal{A}$ a nonzero quotient of $C(K)$. Then $\mathcal{A}$ is isomorphic to $C(L)$, where L is a nonempty closed subspace[3] of K. Since K is scattered, L has a relative isolated point and therefore there is a nonzero projection $p \in C(L)$ such that $pC(Y)p = \mathbb{C}p$. Since $\mathcal{A}$ is isomorphic to $C(Y)$, we conclude that there is a nonzero projection $q \in \mathcal{A}$ such that $q\mathcal{A}q = \mathbb{C}q$.

In summary, if K is a scattered compact Hausdorff space, then $\mathcal{A} = C(K)$ is a C*-algebra with the following property:

(*) For every nonzero quotient $\mathcal{B}$ of $\mathcal{A}$, there is a nonzero projection $p \in \mathcal{B}$ such that $p\mathcal{B}p = \mathbb{C}p$.

The class of scattered C*-algebras will be exactly the class of C*-algebras satisfying property (*).

Let us go now to the formal definition:

Definition 1.48. *Let $\mathcal{A}$ be a C*-algebra. A projection $p \in \mathcal{A}$ is a minimal projection if p is nonzero and*

$$p\mathcal{A}p = \mathbb{C}p.$$

Example 1.49. *Let X be a locally compact Hausdorff space and $x \in X$ an isolated point. We have seen that $p := \chi_{\{x\}}$ is a minimal projection of $C_0(X)$. Moreover, every minimal*

[3]This follows from the fact that every ideal of $C(K)$ is of the form $I_L := \{f : f \upharpoonright L \equiv 0\}$ and from the fact that $C(K)/I_L \equiv C(L)$.

projection $p \in C_0(X)$ is of the form $\chi_{\{x\}}$ for some $x \in X$ an isolated point. In fact, let $p \in C_0(X)$ be a projection. Since p is a projection, there is a clopen set $B \subseteq K$ such that $p = \chi_B$. Suppose that B has at least two elements. Consider $x, y \in B$ two distinct elements and $f \in C_0(X)$ such that $f(x) = 1$ and $f(y) = 0$. Then there is no $\lambda \in \mathbb{C}$ such that $pfp = \lambda p$. In particular, $pC_0(X)p \neq \mathbb{C}p$ and therefore, p is not a minimal projection.

Example 1.50. *Let H be a Hilbert space. Then p is a minimal projection in $\mathcal{B}(H)$ if and only if p is an orthogonal projection onto a one-dimensional subspace of H (see Lemma 3.1 of [16]).*

Notation 1.51. *Let $\mathcal{A}$ be a C*-algebra. Denote by $At(\mathcal{A})$ the set formed by all minimal projections of $\mathcal{A}$ and denote by $\mathcal{I}^{At}(\mathcal{A})$ the C*-subalgebra generated by $At(\mathcal{A})$.*

Example 1.52. *Let H be a Hilbert space. Then $At(\mathcal{B}(H))$ is formed by all orthogonal projections onto one-dimensional subspaces. In particular, $\mathcal{I}^{At}(\mathcal{B}(H)) = \mathcal{K}(H)$.*

Now, we can formally define the class of scattered C*-algebras:

Definition 1.53. *A C*-algebra is scattered if and only if for every nonzero quotient $\mathcal{B}$ of $\mathcal{A}$, the C*-subalgebra $\mathcal{I}^{At}(\mathcal{B})$ is nonzero. Equivalently, $\mathcal{A}$ is scattered if and only if every nonzero quotient of $\mathcal{A}$ has a minimal projection.*

By definition, $\mathcal{I}^{At}(\mathcal{A})$ is a C*-subalgebra of $\mathcal{A}$. In the example where $\mathcal{A} = \mathcal{B}(H)$, we have seen that $\mathcal{I}^{At}(\mathcal{A}) = \mathcal{K}(H)$ is an essential ideal of $\mathcal{B}(H)$. The next result tells us that this is always the case:

Theorem 1.54. *Let $\mathcal{A}$ be a scattered C*-algebra. Then*

1. *$\mathcal{I}^{At}(\mathcal{A})$ is an essential ideal of $\mathcal{A}$.*

2. *$\mathcal{I}^{At}(\mathcal{A})$ is isomorphic to a C*-subalgebra of $\mathcal{K}(H)$ for some Hilbert space H.*

Proof. See Proposition 3.15 and Proposition 3.16 of [16]. □

The next theorem summarizes some properties of being scattered:

Theorem 1.55 ([20, 21, 42, 27, 25, 41, 16])**.** *Suppose that $\mathcal{A}$ is a C*-algebra. The following conditions are equivalent:*

1. *Every nonzero quotient of $\mathcal{A}$ has a minimal projection.*

2. *There is an ordinal $ht(\mathcal{A})$ and a continuous increasing sequence of ideals $(\mathcal{I}^{At}_\alpha(\mathcal{A}))_{\alpha \leq ht(\mathcal{A})}$ called the Cantor-Bendixson composition series for $\mathcal{A}$ such that $\mathcal{I}_0 = \{0\}$, $\mathcal{I}_{ht(\mathcal{A})} = \mathcal{A}$ and*
$$\mathcal{I}^{At}(\mathcal{A}/\mathcal{I}^{At}_\alpha(\mathcal{A})) = \{[a]_{\mathcal{I}^{At}_\alpha(\mathcal{A})} : a \in \mathcal{I}^{At}_{\alpha+1}(\mathcal{A})\},$$
for every $\alpha < ht(\mathcal{A})$.

3. *Every nonzero C*-subalgebra of $\mathcal{A}$ has a minimal projection.*

4. *Every nonzero C*-subalgebra of $\mathcal{A}$ has a projection.*

5. *Every C*-subalgebra of $\mathcal{A}$ has real rank zero*[4]*.*

[4]We say that a C*-algebra $\mathcal{A}$ has real rank zero if the set of all self-adjoint elements with finite spectrum is dense in $\mathcal{A}_{sa}$.

6. $\mathcal{A}$ does not contain a copy of the C^*-algebra $C_0((0,1])$.

7. The spectrum of every self-adjoint element is countable.

Definition 1.56. *The ordinal $ht(\mathcal{A})$ is called the height of $\mathcal{A}$. When the context is clear, we use the notation $(\mathcal{I}_\alpha)_{\alpha\leq ht(\mathcal{A})}$ for the Cantor-Bendixson composition series for $\mathcal{A}$.*

Let $\mathcal{A}$ be a scattered C*-algebra and consider its Cantor-Bendixson composition series $(\mathcal{I}^{At}_\alpha(\mathcal{A}))_{\alpha\leq ht(\mathcal{A})}$. Then for every $\alpha < ht(\mathcal{A})$, the ideal $\mathcal{I}^{At}_{\alpha+1}(\mathcal{A})/\mathcal{I}^{At}_\alpha(\mathcal{A})$ is the ideal in $\mathcal{A}/\mathcal{I}^{At}_\alpha(\mathcal{A})$ generated by the minimal projections of $\mathcal{A}/\mathcal{I}^{At}_\alpha(\mathcal{A})$. In particular, there is a Hilbert space H_α such that $\mathcal{I}^{At}_{\alpha+1}(\mathcal{A})/\mathcal{I}^{At}_\alpha(\mathcal{A})$ is isomorphic to a nondegenerated[5] *-subalgebra of $\mathcal{K}(H_\alpha)$. We can assume that $H_\alpha = \ell_2(\kappa_\alpha)$ for some cardinal κ_α.

Definition 1.57. *We define the width (denoted by $wd(\mathcal{A})$) of $\mathcal{A}$ by*

$$wd(\mathcal{A}) = \sup_\alpha \kappa_\alpha$$

Definition 1.58 (Definition 1.5 of [16])**.** *Given a cardinal κ, a scattered C*-algebra $\mathcal{A}$ is called κ-thin-tall if and only if $ht(\mathcal{A}) = \kappa^+$ and $wd(\mathcal{A}) = \kappa$. An ω-thin-tall C*-algebra is called thin-tall.*

Example 1.59. *We have seen that every C*-algebra of the form $C_0(X)$ for X a scattered locally compact Hausdorff space is a scattered C*-algebra. Moreover, every commutative scattered C*-algebra is of the form $C_0(X)$ for some scattered locally compact Hausdorff space X.*

Example 1.60. *Let H be a Hilbert space. Then $\mathcal{K}(H)$ is a scattered C*-algebra.*

Lemma 1.61. *Let $\mathcal{A}$ be a scattered C*-algebra such that $wd(\mathcal{A}) = \kappa$. Then $\mathcal{A}$ embeds into $\mathcal{B}(\ell_2(\kappa))$.*

Proof. See Proposition 3.19 of [16]. □

Observe that, by Lemma 1.61, every thin-tall C*-algebra can be embedded in $\mathcal{B}(\ell_2)$.

Lemma 1.62. *Let $\mathcal{A}$ be a scattered C*-algebra and $\mathcal{J}$ an ideal in $\mathcal{A}$. Then for every projection $[a]_\mathcal{J}$ in $\mathcal{A}/\mathcal{J}$, there is a projection $p \in \mathcal{A}$ such that*

$$[p]_\mathcal{J} = [a]_\mathcal{J}.$$

In other words, every projection in $\mathcal{A}/\mathcal{J}$ lifts to a projection in $\mathcal{A}$.

Proof. The result is true for every C*-algebra of real rank zero (See Theorem 3.14 of [8]). Since every scattered C*-algebra is of real rank zero (see Theorem 2.3 of [25] or (5) of Theorem 1.55), we conclude the proof. □

With respect to the unitization, we have the following:

Proposition 1.63. *Let $\mathcal{A}$ be a scattered C*-algebra and $\widetilde{\mathcal{A}}$ its unitization. Then $\widetilde{\mathcal{A}}$ is scattered if $\mathcal{A}$ is scattered.*

Proof. See Proposition 2.4 of [20]. □

[5]A C*-algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ is nondegenerated if $\{Tx : T \in \mathcal{A}, x \in H\}$ is dense in H.

The C*-algebra constructed in Chapter 4 is a thin-tall C*-algebra which is also fully noncommutative:

Definition 1.64. *A scattered C*-algebra is called fully noncommutative if and only if for all* $\alpha < ht(\mathcal{A})$ *the algebra* $\mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha)$ *is *-isomorphic to the algebra of all compact operators on a Hilbert space.*

The following result explains the terminology "fully noncommutative" :

Proposition 1.65. *Suppose that* $\mathcal{A}$ *is a scattered* C^**-algebra. The following are equivalent:*

1. $\mathcal{A}$ *is fully noncommutative.*
2. *The ideals of* $\mathcal{A}$ *form a chain.*
3. *The centers*[6] *of the multiplier algebra of any quotient of* $\mathcal{A}$ *are all trivial.*

Proof. See [16], Proposition 6.3. □

[6]The center $Z(\mathcal{A})$ of a C*-algebra is the set $\{a \in \mathcal{A} : \forall b \in \mathcal{A}(ab = ba)\}$.

Chapter 2

Irredundant sets in C*-algebras

In this chapter we introduce the notion of an irredundant set and develop some of its properties.

Given any structure S with a fixed class of substructures we can say that a set $X \subseteq S$ is irredundant if and only if no $x \in X$ belongs to the substructure generated by $X \setminus \{x\}$. For example, in the class of vector spaces this notion of irredundance corresponds to the notion of linearly independent sets. When applied to the class of Banach spaces, the notion of irredundance corresponds to the notion of biorthogonal systems [1], and it is known that the existence of uncountable biorthogonal systems in every nonseparable Banach space is independent from ZFC. In this thesis we focus on irredundant sets in the context of C*-algebras. Let us start with the formal definition and some examples:

2.1 Definitions and examples

Definition 2.1. *Let $\mathcal{A}$ be a C*-algebra. A subset $\mathcal{X} \subseteq \mathcal{A}$ is called irredundant if and only if for every $a \in \mathcal{X}$, the C*-subalgebra of $\mathcal{A}$ generated by $\mathcal{X} \setminus \{a\}$ does not contain a. We define*

$$irr(\mathcal{A}) := \sup\{|\mathcal{X}| : \mathcal{X} \text{ is an irredundant set in } \mathcal{A}\}.$$

In this thesis we are interested in the behaviour of the cardinal characteristic $irr(*)$. Let us look at some examples.

Example 2.2. *Let X be an infinite locally compact Hausdorff space. Then the C*-algebra $C_0(X)$ contains an infinite irredundant set. Indeed, let $(O_n)_{n<\omega}$ be a pairwise disjoint family of open sets in X. For each $n < \omega$, let $f_n : X \to [0, 1]$ be a continuous function such that f_n vanishes outside O_n and takes the value 1 in some point of O_n. Then $\mathcal{X} := \{f_n : n < \omega\}$ is an irredundant set in $C_0(X)$.*

Example 2.3. *Let $\mathcal{K}(\ell_2)$ be the C*-algebra of all compact operators on ℓ_2. Consider $(e_n)_n$ the canonical basis of ℓ_2. For each $n < \omega$, let P_n be the projection onto the space generated by e_n. Then $\mathcal{X} := \{P_n : n < \omega\}$ is an irredundant set in $\mathcal{K}(\ell_2)$.*

Examples 2.2 and 2.3 are consequences of the following lemma:

Lemma 2.4. *Let $\mathcal{A}$ be a C*-algebra. Suppose $(a_\alpha)_{\alpha<\kappa}$ is a family of self-adjoint elements in $\mathcal{A}$ such that $a_\alpha a_\beta = 0$ for every $\alpha \neq \beta$. Then $(a_\alpha)_{\alpha<\kappa}$ is an irredundant set.*

[1] Given a Banach space X, a biorthogonal system for X is a sequence of the form $(x_\alpha, x^*_\alpha)_{\alpha<\kappa}$ in $X \times X^*$ such that $x^*_\alpha(x_\beta) = 1$ if $\alpha = \beta$ and 0 otherwise. See [23] for a discussion about biorthogonal systems and irredundant sets in Banach spaces.

Proof. The proof follows from the fact that for every self-adjoint element $a \in \mathcal{A}$, the set $\{b \in \mathcal{A} : ab = ba = 0\}$ is a C*-subalgebra of $\mathcal{A}$. □

Every irredundant set is linearly independent. In particular, if a C*-algebra is finite-dimensional, then it admits only finite irredundant sets. For infinite-dimensional C*-algebras we have the following:

Theorem 2.5. *Every infinite-dimensional C*-algebra admits an infinite irredundant set.*

Proof. Let $\mathcal{A}$ be an infinite-dimensional C*-algebra. By a result of T. Ogasawara ([30]), there is an infinite-dimensional commutative C*-subalgebra $\mathcal{B}$ of $\mathcal{A}$. Since $\mathcal{B}$ is commutative, by Theorem 1.25 we can assume that $\mathcal{B}$ is of the form $C_0(X)$ for some infinite locally compact Hausdorff space X. As we have seen in Example 2.2, $\mathcal{B}$ has an infinite irredundant set and therefore, $\mathcal{A}$ has an infinite irredundant set. □

The above theorem gives us a lower bound for $irr(\mathcal{A})$ in the case where $\mathcal{A}$ is an infinite-dimensional C*-algebra. The following lemma will give us an upper bound:

Lemma 2.6. *Let $\mathcal{A}$ be a C*-algebra and $\mathcal{X}$ an irredundant set in $\mathcal{A}$. Then $\mathcal{X}$ is a norm-discrete set in $\mathcal{A}$. In particular, $irr(\mathcal{A}) \leq d(\mathcal{A})$.*

Proof. The proof follows from the fact that a C*-subalgebra is in particular norm-closed. □

In this thesis, we are interested in the possible gap between $irr(\mathcal{A})$ and $d(\mathcal{A})$. We will see that the existence of a gap between these two cardinal characteristics is sensitive to set-theoretic axioms.

2.2 Irredundant sets of special elements

This section deals with the problem of extracting a special type of irredundant set from a given one.

Lemma 2.7. *Let $\mathcal{A}$ be a C*-algebra and $\mathcal{X}$ an irredundant set in $\mathcal{A}$. Let $a \in \mathcal{X}$ be such that a is a linear combination of elements $a_1, \dots, a_n \in C^*(a)$. Then there is $i \in \{1, \dots, n\}$ such that $\mathcal{X}_i' = (\mathcal{X} \cup \{a_i\}) \setminus \{a\}$ is an irredundant set.*

Proof. Suppose that $\mathcal{X}_i'$ is not an irredundant set for every $i \in \{1, \dots, n\}$ and let us get a contradiction. From the fact that $\mathcal{X}_i'$ is not irredundant but $\mathcal{X}$ is irredundant, it follows that we must have $a_i \in C^*(\mathcal{X} \setminus \{a\})$. As a is a linear combination of $a_1, \dots, a_n$, we conclude that $a \in C^*(\mathcal{X} \setminus \{a\})$ which is a contradiction with the fact that $\mathcal{X}$ is an irredundant set. □

Theorem 2.8. *Let $\mathcal{A}$ be a C*-algebra and $\mathcal{X}$ an infinite irredundant set in $\mathcal{A}$. Then*

1. *There is an irredundant set $\mathcal{X}'$ formed by self-adjoint elements and such that $|\mathcal{X}'| = |\mathcal{X}|$.*
2. *There is an irredundant set $\mathcal{X}'$ formed by positive elements and such that $|\mathcal{X}'| = |\mathcal{X}|$.*

Proof. Fix an enumeration $\mathcal{X} = \{a_\alpha : \alpha < \kappa\}$.

1. For every $\alpha < \kappa$, decompose a_α as $a_\alpha = a_{\alpha,1} + i a_{\alpha,2}$, where $a_{\alpha,1} = \frac{ia_\alpha^* - ia_\alpha}{2}$ and $a_{\alpha,2} = \frac{a_\alpha + a_\alpha^*}{2}$ are self-adjoint elements such that $a_{\alpha,1}, a_{\alpha,2} \in C^*(a_\alpha)$. By Lemma 2.7, for every $\alpha < \kappa$, there is $j \in \{1, 2\}$ such that $(\mathcal{X} \cup \{a_{\alpha,j}\}) \setminus \{a_\alpha\}$ is irredundant. An inductive argument gives us the desired irredundant set $\mathcal{X}'$ formed by self-adjoint elements.

2. The proof of the second statement follows the same argument as above and using the fact that every self-adjoint element a can be written as a linear combination of two positive elements in $C^*(a)$.

□

Theorem 2.8 allows us to restrict our attention to those irredundant sets formed by positive elements. For instance, to prove that a C*-algebra does not have any uncountable irredundant set, it is enough to prove that it does not contain any uncountable irredundant set formed by positive elements. Since positive elements have special properties, dealing with positive elements can make this task less complicated.

In the case of scattered C*-algebras, we will prove that we can go a little bit further and restrict irredundant sets to irredundant sets formed by projections.

Theorem 2.9. *Let $\mathcal{A}$ be a scattered C*-algebra and let $\mathcal{X}$ be an infinite irredundant set in $\mathcal{A}$. Then there is an irredundant set $\mathcal{X}'$ formed by projections and such that $|\mathcal{X}'| = |\mathcal{X}|$.*

Proof. By Theorem 2.8 we can assume that $\mathcal{X}$ is an irredundant set of self-adjoint elements. Since every C*-subalgebra of a scattered C*-algebra is also scattered (consequence of Theorem 1.55), it follows that for every element $a \in \mathcal{X}$, the C*-subalgebra $C^*(a)$ is of the form $C_0(X_a)$ for some scattered locally compact space X_a. Hence, the linear combinations of projections are norm dense in $C^*(a)$. In particular, there is a projection $p_a \in C^*(a)$ such that $p_a \notin C^*(\mathcal{X} \setminus \{a\})$. Define $\mathcal{X}' := \{p_a : a \in \mathcal{X}\}$. Then $\mathcal{X}'$ is an irredundant set. Indeed, suppose there is $a \in \mathcal{X}$ such that $p_a \in C^*(\{p_b : b \in \mathcal{X} \setminus \{a\}\})$. Since $p_b \in C^*(b)$ for every $b \in \mathcal{X}$ we conclude that $p_a \in C^*(\mathcal{X} \setminus \{a\})$, which is a contradiction with the choice of p_a. □

2.3 Commutative C*-algebras

In this section we review some classical results about irredundant sets in commutative C*-algebras. This review will serve as a motivation for the study of irredundant sets in the noncommutative context.

In the context of commutative C*-algebras, the Stone-Weierstrass theorem holds and in many of its applications we need to consider C*-subalgebras with unit. For this, the following lemma will be useful:

Lemma 2.10. *Let $\mathcal{A}$ be a unital C*-algebra. If $\mathcal{X} \subseteq \mathcal{A}$ is an irredundant set, then there is $a \in \mathcal{X}$ such that for every element $b \in \mathcal{X} \setminus \{a\}$, b does not belong to the C*-subalgebra $C_1^*(\mathcal{X} \setminus \{a, b\})$.*

Proof. If there is no $a \in \mathcal{X}$ such that $a \in C_1^*(\mathcal{X} \setminus \{a\})$, then consider a to be any element in $\mathcal{X}$.

Suppose now that there is $a \in \mathcal{X}$ such that $a \in C_1^*(\mathcal{X} \setminus \{a\})$. Then by Lemma 1.30, $a = \lambda \mathbf{1} + a'$ where $\lambda \neq 0$ and $a' \in C^*(\mathcal{X} \setminus \{a\})$.

Suppose that there is $b \in \mathcal{X} \setminus \{a\}$ such that $b \in C_1^*(\mathcal{X} \setminus \{a, b\})$ and let us get a contradiction. Since $b \in C_1^*(\mathcal{X} \setminus \{a, b\})$, there are $\lambda' \neq 0$ and $b' \in C^*(\mathcal{X} \setminus \{a, b\})$ such that $b = \lambda' \mathbf{1} + b'$. In particular we can write $\mathbf{1} = \frac{1}{\lambda'}(b - b')$. This shows that

$$a = \lambda \mathbf{1} + a' = \frac{\lambda}{\lambda'}(b - b') + a' \in C^*(\mathcal{X} \setminus \{a\})$$

which is a contradiction with the fact that $\mathcal{X}$ is an irredundant set. □

The following lemma characterizes C*-subalgebras of $C(K)$:

Proposition 2.11. *Suppose that $\mathcal{A} \subseteq C(K)$ is a unital subalgebra of $C(K)$. Then there is a compact Hausdorff space Y and a continuous surjective map $\varphi : K \to Y$ such that*

$$\mathcal{A} = \{f \circ \varphi : f \in C(L)\}.$$

Proof. Define in K the relation $x \sim y$ if and only if $f(x) = f(y)$ for every $f \in \mathcal{A}$. Consider $Y = K/\sim$ and let $\varphi : K \to Y$ be the quotient map. Observe that the relation $\sim$ is closed in $K \times K$, and therefore, Y is a compact Hausdorff space.

For every $g \in \mathcal{A}$, there is a well defined $[g] : Y \to \mathbb{C}$ such that $[g] \circ \varphi = g$. As g is continuous it follows that $[g] : Y \to \mathbb{C}$ is continuous, i.e., $[g] \in C(Y)$ (see Proposition 2.4.2 of [11]). Observe that $\{[g] : g \in \mathcal{A}\}$ is a C*-subalgebra of $C(Y)$ which contains the constant functions and separates the points of Y. By the Stone-Weierstrass theorem, we conclude that $C(Y) = \{[g] : g \in \mathcal{A}\}$. Moreover, we have that $\{f \circ \varphi : f \in C(L)\} = \{[g] \circ \varphi : g \in \mathcal{A}\} = \mathcal{A}$. □

For a commutative C*-algebra of the form $C(K)$, we have the following characterization of irredundant sets:

Theorem 2.12. *Let K be a compact Hausdorff space.*

1. *Suppose $\mathcal{X} \subseteq C(K)$ is such that for each $f \in \mathcal{X}$, there are two points $x_f, y_f \in K$ satisfying*
   - $f(x_f) \neq f(y_f)$;
   - $g(x_f) = g(y_f)$ *for every* $g \neq f$ *in* $\mathcal{X}$.

   Then $\mathcal{X}$ is an irredundant set.

2. *Suppose $\mathcal{X} \subseteq C(K)$ is such that no $f \in \mathcal{X}$ belongs to $C_1^*(\mathcal{X} \setminus \{f\})$. Then for each $f \in \mathcal{X}$, there are two points $x_f, y_f \in K$ such that*
   - $f(x_f) \neq f(y_f)$;
   - $g(x_f) = g(y_f)$ *for every* $g \neq f$ *in* $\mathcal{X}$.

   Consequently, if $\mathcal{X}$ is an irredundant set in $C(K)$, then there is $h \in \mathcal{X}$ such that $\mathcal{X} \setminus \{h\}$ has the above property.

Proof.

1. The proof follows from the fact that for every $x, y \in K$, the set $\{f \in C(K) : f(x) = f(y)\}$ is actually a C*-subalgebra of $C(K)$.

2. Consider $\mathcal{B}$ as the C*-subalgebra of $C(K)$ generated by $\mathcal{X} \cup \{1\}$. By Lemma 2.11 there is a compact Hausdorff space L and a continuous surjection map $\varphi : K \to L$ such that $\mathcal{B}$ is *-isomorphic to $\{f \circ \varphi : f \in C(L)\}$. In particular, $\mathcal{X}' := \{f_L \in C(L) : f_L \circ \varphi \in \mathcal{X}\}$ is an irredundant set which generates $C(L)$. By the Stone-Weierstrass theorem, for every $f_L \in \mathcal{X}'$ there are $x_{f_L}, y_{f_L} \in L$ such that $g_L(x_{f_L}) = g_L(y_{f_L})$ for every $g_L \in \mathcal{X}' \setminus \{f_L\}$ and $f_L(x_{f_L}) \neq f_L(y_{f_L})$. To conclude the proof of the first part, consider for each $f_L \in C(L)$ two points $x_f, y_f \in K$ such that $\varphi(x_f) = x_{f_L}$ and $\varphi(y_f) = y_{f_L}$.

   For the second part, use Lemma 2.10.

□

The next two theorems tell that the gap between $d(C(K)) = w(K)$ and $irr(C(K))$ is sensitive to extra set-theoretic axioms:

Theorem 2.13. *Assume CH. Then there is a nonseparable commutative C*-algebra $\mathcal{A}$ such that $irr(\mathcal{A})$ is countable.*

Proof. Consider K as the Kunen line [29]. Then K is a scattered compact Hausdorff space of weight ω_1 such that K^n is hereditarily separable for every $n \in \mathbb{N}$. If $\mathcal{X}$ is an uncountable irredundant set in $C(K)$, then as a consequence of Lemma 2.12 we would have an uncountable discrete set in K^2, which is a contradiction with the fact that K^2 is hereditarily separable. From this, we conclude that $irr(C(K))$ is countable. □

The question if every nonseparable commutative C*-algebra $C(K)$ admits an uncountable irredundant set is still open. But in the case where K is 0-dimensional, we have the following:

Theorem 2.14 (S. Todorcevic [38] and [40])**.** *Assume MA+¬CH. Then every nonseparable commutative C*-algebra of form $C(K)$ for K a 0-dimensional compact Hausdorff space admits an uncountable irredundant set.*

In the literature, there has been a special interest for those compact Hausdorff spaces $K_{\mathbb{A}}$ which are the Stone space of some Boolean algebra $\mathbb{A}$. This interest relies on the fact that in this situation, whenever $\mathbb{A}$ has an irredundant set[2] there is an irredundant set in $C(K_{\mathbb{A}})$ of the same size. This relation is explained in the next lemma:

Lemma 2.15. *Let $\mathbb{A}$ be a Boolean algebra and $K_{\mathbb{A}}$ its Stone space. If we define $irr_{ba}(\mathbb{A}) := \sup\{|\mathcal{X}| : \mathcal{X} \subseteq \mathbb{A}$ is an irredundant set$\}$, then*

$$irr_{ba}(\mathbb{A}) \leq irr(C(K_{\mathbb{A}})).$$

Proof. For every element $a \in \mathbb{A}$ define the clopen set $[a] := \{x \in K : a \in x\}$. Now, given an irredundant set $(a_\alpha)_{\alpha<\kappa}$ in $\mathbb{A}$ the set

$$\mathcal{X} := \{\chi_{[a_\alpha]} : \alpha < \kappa\},$$

is an irredundant set in $C(K_{\mathbb{A}})$, where $\chi_{[a_\alpha]} : K \to \{0,1\}$ is the characteristic function of $[a_\alpha]$. □

A question motivated by Lemma 2.15 is whether the equality $irr_{ba}(\mathbb{A}) = irr(C(K_{\mathbb{A}}))$ holds for every Boolean algebra $\mathbb{A}$. This question was raised in [23], Question 3.10(3). The next proposition answers positively this question in the case of superatomic Boolean algebras:

Proposition 2.16. *Let $\mathcal{A}$ be a superatomic Boolean algebra. Then $irr_{ba}(\mathbb{A}) = irr(C(K_{\mathbb{A}}))$*

Proof. By the Stone duality, $\mathbb{A}$ is superatomic if and only if $K_{\mathbb{A}}$ is scattered. In particular, if $\mathbb{A}$ is a superatomic Boolean algebra, then $C(K_{\mathbb{A}})$ is a scattered C*-algebra. Consider $\mathcal{X}$ an irredundant set in $C(K_{\mathbb{A}})$ of size κ. By Proposition 2.9, we can assume that $\mathcal{X}$ is a set of projections $\mathcal{X} := \{p_\alpha : \alpha < \kappa\}$. Since $p \in C(K_{\mathbb{A}})$ is a projection if and only if $p = \chi_B$ for some clopen B, consider for every $\alpha < \kappa$ a clopen $B_\alpha \subseteq K_{\mathbb{A}}$ such that $p_\alpha = \chi_{B_\alpha}$. Then $\{B_\alpha : \alpha < \kappa\}$ is an irredundant set in $Clop(K_{\mathbb{A}})$. Since $Clop(K_{\mathbb{A}})$ is *-isomorphic to $\mathbb{A}$ we conclude that $\mathbb{A}$ has an irredundant set of size κ. □

[2]An irredundant set in a Boolean algebra is defined in the same spirit of Definition 2.1.

2.4 Conditions to ensure irredundance

In this section we present some sufficient conditions to guarantee that a given set is irredundant. More precisely, we are interested in results like Theorem 2.5, which tells that every orthogonal family of self-adjoint elements is an irredundant set.

For a commutative C*-algebra of the form $C(K)$, we have seen in Theorem 2.12 that irredundant sets in $C(K)$ are related to families of pairs of points in K. In the general noncommutative situation, there are some topological spaces which correspond to the non-commutative version of K. Namely, for a C*-algebra $\mathcal{A}$ we have the space of pure states $\mathbb{P}(\mathcal{A})$ with the weak* topology and the spectrum[3] $\hat{\mathcal{A}}$ of $\mathcal{A}$ with the Jacobson topology.

In terms of C*-algebras, Theorem 2.12 states that a set $\mathcal{X} \subseteq C(K)$ is irredundant if and only if for every $f \in \mathcal{X}$, there are two pure states $\tau_f, \sigma_f \in \mathbb{P}(C(K))$ such that $\tau_f(f) \neq \sigma_f(f)$ but $\tau_f(g) = \sigma_f(g)$ for every $g \in \mathcal{X} \setminus \{f\}$. This follows from the fact that τ is a pure state in $C(K)$ if and only if there is $x \in K$ such that τ is the Dirac measure concentrated on x (see Example 1.36).

Part of the proof of Theorem 2.12 follows from the fact that for two pure states $\tau, \sigma \in \mathbb{P}(C(K))$ we have that $\{f : \tau(f) = \sigma(f)\}$ is a C*-subalgebra of $C(K)$. This is not the case for any given C*-algebra:

Remark 2.17. *Let $\mathcal{A}$ be a C*-algebra and τ, σ two pure states. It is not true in general that $B := \{a : \tau(a) = \sigma(a)\}$ is a C*- subalgebra.*

In fact, consider $\mathcal{A} = \mathcal{K}(\ell_2)$ the C-algebra of compact operators on ℓ_2. Consider $(e_n)_n$ an orthonormal basis for ℓ_2. Define $\tau, \sigma \in \mathbb{P}(\mathcal{A})$ as*

$$\tau(T) = \langle Te_1, e_1\rangle \text{ and } \sigma(T) = \langle Te_2, e_2\rangle$$

and $B := \{T : \tau(T) = \sigma(T)\}$.

Define operators $T, S \in \mathcal{K}(\ell_2)$ such that $T(e_1) = e_2, T(e_n) = 0$ for $n \neq 1$ and $S(e_2) = e_1, S(e_n) = 0$ for $n \neq 2$. Then $\tau(S) = \sigma(S) = 0$ and $\tau(T) = \sigma(T) = 0$. In particular $T, S \in B$ but $TS \notin B$ because $\tau(TS) = 0 \neq 1 = \sigma(TS)$.

For general C*-algebras, we have the following:

Lemma 2.18. *Suppose there exist a family $(a_\alpha)_{\alpha<\kappa}$ in $\mathcal{A}_+$ and a family $(\tau_\alpha)_{\alpha<\kappa}$ of positive linear functionals such that for every $\alpha \neq \beta < \kappa$:*

- $\tau_\alpha(a_\alpha) > 0$,
- $\tau_\alpha(a_\beta) = 0$.

Then $(a_\alpha)_{\alpha<\kappa}$ is irredundant in $\mathcal{A}$.

Proof. For every $\alpha < \kappa$, there is $c_\alpha \in \mathcal{A}$ such that[4] $a_\alpha = c_\alpha^* c_\alpha$, since $a_\alpha \in \mathcal{A}_+$. By the hypothesis, for every $\beta \neq \alpha$, $c_\beta \in N_{\tau_\alpha} := \{a : \tau_\alpha(a^*a) = 0\}$, because $\tau_\alpha(c_\beta^* c_\beta) = \tau_\alpha(a_\beta) = 0$.

By (4) of Proposition 1.34, N_{τ_α} is a left ideal. In particular, $a_\beta = c_\beta^* c_\beta \in N_{\tau_\alpha}$. Then for every $\alpha \neq \beta < \kappa$ we have that $a_\beta \in N_{\tau_\alpha}$. For $\alpha < \kappa$, $a_\alpha \notin N_{\tau_\alpha}$ because $0 \neq |\tau_\alpha(a_\alpha)|^2 \leq \|\tau_\alpha\|\tau_\alpha(a_\alpha^* a_\alpha)$ (by (1) of Proposition 1.34). This finishes the proof. □

Remark 2.19. *Let K be a compact Hausdorff space. Then for every discrete set in K, there is an irredundant set in $C(K)$ of the same size. In fact, suppose $\{x_\alpha : \alpha < \kappa\}$ is a discrete set*

[3]See Chapter 5 of [28] for the definition of the spectrum.
[4]See Theorem 2.2.5 of [28].

witnessed by open sets $(O_\alpha)_{\alpha<\kappa}$. For each $\alpha < \kappa$, consider $f_\alpha \in C(K)$ such that $f_\alpha(x_\alpha) = 1$ and f_α vanishes outside O_α. Then $\mathcal{X} := \{f_\alpha : \alpha < \kappa\}$ is an orthogonal family of self-adjoint elements. In particular, $\mathcal{X}$ is an irredundant set (by Lemma 2.6).

With the above remark and assuming PFA, the proof of the consistency of the fact that every nonseparable scattered commutative C*-algebra admits an uncountable irredundant set can be done as follows:

Theorem 2.20. *Assume PFA. Then every nonseparable scattered commutative C*-algebra admits an uncountable irredundant set.*

Proof. Let $\mathcal{A} = C_0(X)$ be a nonseparable scattered commutative C*-algebra. We can assume that X has weight ω_1. Consider $(K^{(\alpha)})_{\alpha<\kappa}$ the Cantor-Bendixson composition series[5] for X. If $\kappa < \omega_1$, then there is $\alpha < \kappa$ such that $K^{(\alpha)}$ has an uncountable discrete set. In particular $C_0(K^{(\alpha)})$ has an uncountable irredundant set. Since $C_0(K^{(\alpha)})$ is a quotient of $C_0(K)$, we conclude that $C_0(X)$ has an uncountable irredundant set. Suppose now that $\kappa \geq \omega_1$. Choose for every $\alpha < \kappa$ an element $x_\alpha \in K^{(\alpha+1)} \setminus K^{(\alpha)}$. Then $(x_\alpha)_{\alpha<\kappa}$ is an uncountable right-separated sequence. By Corollary 1.7, X has an uncountable discrete subspace and the result follows from Remark 2.19. □

In the noncommutative scenario, we have the following:

Theorem 2.21. *Assume PFA. Suppose $\mathcal{A}$ is a nonseparable scattered C*-algebra. Then $\mathbb{P}(\mathcal{A})$ admits an uncountable discrete set.*

Proof. Let $(\mathcal{I}_\alpha)_{\alpha<\kappa}$ be the Cantor-Bendixson composition series for $\mathcal{A}$.

If there is $\xi < \kappa$ such that $\mathcal{I}_{\xi+1}/\mathcal{I}_\xi$ is *-isomorphic to a nondegenerate subalgebra of $\mathcal{K}(\ell_2(\lambda))$ for some $\lambda \geq \omega_1$, then $\mathcal{I}_{\xi+1}/\mathcal{I}_\xi$ has an uncountable orthogonal family of projections. In particular, by Lemma 1.62 we can lift this family to an uncountable family of projections $(p_\alpha)_{\alpha<\omega_1}$ in $\mathcal{A}$ such that $p_\alpha p_\beta \in \mathcal{I}_\xi$ for every $\alpha \neq \beta$. For each $\alpha < \omega_1$, consider a pure state σ_α of $\mathcal{A}/\mathcal{I}_\xi$ such that $\sigma_\alpha([p_\alpha]_{\mathcal{I}_\xi}) = 1$ and define τ_α as the composition of σ_α with the quotient map from $\mathcal{A}$ onto $\mathcal{A}/\mathcal{I}_\xi$. Then $(\tau_\alpha)_{\alpha<\omega_1}$ is a family of pure states of $\mathcal{A}$ such that $\tau_\alpha(p_\alpha) = 1$ and $\tau_\alpha(p_\beta) = 0$ for every $\alpha \neq \beta$. In particular, $(\tau_\alpha)_{\alpha<\omega_1}$ is an uncountable discrete subset of $\mathbb{P}(\mathcal{A})$.

Suppose now that $wd(\mathcal{A}) = \omega$. Since $\mathcal{A}$ is nonseparable, we have that $\kappa \geq \omega_1$. Then the Cantor-Bendixson composition series $(\mathcal{I}_\alpha)_{\alpha<\kappa}$ gives us a strictly decreasing sequence of weak* closed[6] sets $(F_\alpha)_{\alpha<\kappa}$ in $\mathbb{P}(\mathcal{A})$. An argument like in the proof of Theorem 2.20 provides an uncountable discrete subset in $\mathbb{P}(\mathcal{A})$. □

To prove that it is consistent that every nonseparable scattered C*-algebra admits an uncountable irredundant set, it would be enough to prove that for every discrete set of pure states, we can find an irredundant set of the same size. One of the difficulties to prove this fact comes from the nonexistence of a suitable noncommutative Urysohn lemma. In Section 2.6, we will prove that every discrete set of pure states gives a set of the same size in the C*-algebra with a property which is weaker than that irredundance.

We will consider now the relation between irredundant sets and representations. In this situation we have the following theorem:

Theorem 2.22. *Let $\mathcal{A}$ be a C*-algebra. Suppose $\mathcal{X}$ is a subset of $\mathcal{A}$. Then $\mathcal{X}$ is an irredundant set if and only if for every element $a \in \mathcal{A}$, there are two representations (π_a^1, H_a) and (π_a^2, H_a) such that $\pi_a^1(a) \neq \pi_a^2(a)$ and $\pi_a^1(b) = \pi_a^2(b)$ for every $b \in \mathcal{X} \setminus \{a\}$.*

[5] See Chapter 17 of [22] for details about scattered compact spaces.

[6] See Theorem 5.4.10 of [28].

Proof. If $\mathcal{X}$ is an irredundant set, the existence of the representations (π_a^1, H_a) and (π_a^2, H_a) for each $a \in \mathcal{X}$ is a consequence of Theorem 1.38.

The other part of the proof follows from the fact that for all representations (π^1, H_a) and (π^2, H_a) we have that $\{a : \pi^1(a) = \pi^2(a)\}$ is a C*-subalgebra. □

Before proving a relation between irredundant sets and discrete sets in the spectrum we need the following noncommutative version of the Urysohn lemma:

Theorem 2.23 (S. Takahasi [37])**.** *Let $\mathcal{A}$ be a C*-algebra, and let S_1, S_2 be two nonempty closed subsets of $\hat{\mathcal{A}}$. Then the following two statements are equivalent:*

- $S_1 \cap S_2 = \emptyset$.
- *For any positive operator $a \in \mathcal{A}$ there exists a positive operator $x \in \mathcal{A}$ such that $0 \leq x \leq a$, $\pi(x) = 0$ for all $\pi \in S_1$, and $\pi(x) = \pi(a)$ for all $\pi \in S_2$.*

Theorem 2.24. *Let $\mathcal{A}$ be a C*-algebra such that $\hat{\mathcal{A}}$ is T_1, i.e., every singleton in $\hat{\mathcal{A}}$ is closed. Then for every discrete set in $\hat{\mathcal{A}}$, there is an irredundant set in $\mathcal{A}$ of the same size.*

Proof. Consider $(\pi_\alpha)_{\alpha<\kappa}$ a discrete set in $\hat{\mathcal{A}}$. Then for every $\alpha < \kappa$ we have that

$$\overline{\{\pi_\alpha\}} \cap \overline{\{\pi_\beta : \beta \neq \alpha\}} = \emptyset.$$

Consider $a \in \mathcal{A}_+$ such that $\pi_\alpha(a) = 1$. Then by Theorem 2.23, there is a positive operator a_α such that $\pi_\beta(a_\alpha) = 0$ for $\beta \neq \alpha$ and $\pi_\alpha(a_\alpha) = \pi_\alpha(a) = 1$. Then $(a_\alpha)_{\alpha<\kappa}$ is an irredundant set. □

Remark 2.25. *As examples of C*-algebras with spectrum T_1, we have the class of CCR algebras (see Proposition 13.2.7 of [26]). In the case of scattered C*-algebras, we have that a scattered C*-algebra has spectrum T_1 if and only it is CCR (see Theorem 4 of [17]).*

We conclude this section with a version of Mackenzie's theorem[7] for scattered C*-algebras:

Theorem 2.26. *If $\mathcal{A}$ is a scattered C*-algebra, then*

$$d(\mathcal{A}) \leq 2^{irr(\mathcal{A})}.$$

Proof. Let κ be the minimal cardinal such that $\mathcal{I}^{At}(\mathcal{A})$ is a subalgebra of the algebra of all compact operators on $\ell_2(\kappa)$. By the characterization of subalgebras of the algebra of compact operators (see Proposition 13.2.4 of [26]), $\mathcal{A}$ must contain a pairwise orthogonal set of cardinality κ which is irredundant by Lemma 2.4. So $\kappa \leq irr(\mathcal{A})$. By the essentiality of $\mathcal{I}^{At}(\mathcal{A})$, which follows from Lemma 1.54, we can embed $\mathcal{A}$ into $\mathcal{B}(\ell_2(\kappa))$, so $d(\mathcal{A}) \leq 2^\kappa \leq 2^{irr(\mathcal{A})}$ as required. □

Observe that, if $\mathcal{A}$ is a scattered C*-algebra with density bigger than the continuum, then by Theorem 2.26 $\mathcal{A}$ has an uncountable irredundant set.

In the next section, we deal with C*-algebras of density continuum and in Chapter 4 we analyse C*-algebras of density ω_1.

[7] See Proposition 4.23 of [22].

2.5 Extracting irredundant sets

This section deals with the problem of extracting an irredundant subset from a given family of operators.

Let $\mathcal{A}$ be a C*-algebra. If $\mathcal{X} \subseteq \mathcal{A}$ is an irredundant set, then as we have seen in Lemma 2.6, $\mathcal{X}$ is norm-discrete. We are interested in extracting an uncountable irredundant set from a given uncountable norm-discrete subset of $\mathcal{A}$. The following proposition tells us that extracting an uncountable irredundant family from a discrete set of size ω_1 is not always possible.

Proposition 2.27. *There is a scattered C*-algebra $\mathcal{A}$ and an uncountable discrete family $\{a_\alpha : \alpha < \omega_1\}$ in $\mathcal{A}$ such that for every $\alpha < \beta < \omega_1$, we have that $a_\alpha \in C^*(a_\beta)$. In particular, there is no irredundant set $\mathcal{X} \subseteq \{a_\alpha : \alpha < \omega_1\}$ of cardinality bigger than 1.*

Proof. Theorem 8 of [31] states that $\mathcal{B}\otimes\mathcal{K}(\ell_2)$ is singly generated whenever $\mathcal{B}$ is a separable C*-algebra. Consider the scattered C*-algebra $\mathcal{A}$ constructed in Theorem 7.6 of [16] and its Cantor-Bendixson composition series $(\mathcal{I}_\alpha)_{\alpha<\omega_1}$. Then $\mathcal{A}$ has the property that $\mathcal{I}_{\alpha+1}$ is *-isomorphic to $\widetilde{\mathcal{I}_\alpha}\otimes\mathcal{K}(\ell_2)$ for every $\alpha < \omega_1$. In particular, since $\widetilde{\mathcal{I}_\alpha}$ is separable, there is an element $a_\alpha \in \mathcal{I}_{\alpha+1}$ which singly generated $\mathcal{I}_{\alpha+1}$. Then the family $\{a_\alpha : \alpha < \omega_1\}$ satisfies the condition in the proposition. □

We will see that the situation for a discrete family of size continuum is different.

Lemma 2.28. *Let $\mathcal{A}$ be a C*-algebra of density continuum generated by $(a_\alpha)_{\alpha<2^\omega}$. Then there is $\Gamma \subset 2^\omega$ of size continuum such that for each $\alpha \in \Gamma$, a_α is not in the C*-subalgebra generated by $\{a_\beta : \beta < \alpha, \beta \in \Gamma\}$.*

Proof. We construct Γ by induction. Suppose we have defined $(a_{\alpha_\xi})_{\xi<\beta}$ with $\beta < 2^\omega$. If there is no $\gamma < 2^\omega$ with $\gamma > \sup_{\xi<\beta}\alpha_\xi$ and such that a_γ is not in the C*-subalgebra generated by $(a_{\alpha_\xi})_{\xi<\beta}$, then we would have that $(a_{\alpha_\xi})_{\xi<\beta}$ generates $\mathcal{A}$. In particular, $d(\mathcal{A}) \leq |\beta| < 2^\omega$, a contradiction. Choose γ minimal such that a_γ does not belong to the C*-subalgebra generated by $(a_{\alpha_\xi})_{\xi<\beta}$ and define $\alpha_\beta = \gamma$. □

Corollary 2.29. *Let $\mathcal{A}$ be a C*-algebra of density continuum generated by $(a_\alpha)_{\alpha<2^\omega}$. Then there is $\Gamma \subset 2^\omega$ of size continuum and $\varepsilon > 0$ such that for each $\alpha \in \Gamma$ and each finite subset F of Γ with $max(F) < \alpha$, we have that*

$$d(a_\alpha, C^*(\{a_\beta : \beta \in F\}) := \sup\{\|a_\alpha - b\| : b \in C^*(\{a_\beta : \beta \in F\}\} > \varepsilon.$$

Proof. By Lemma 2.28, we can assume that a_α is not in $C^*(\{a_\beta : \beta < \alpha\})$. In particular, there is a rational number q_α such that $d(a_\alpha, C^*(a_\beta : \beta < \alpha)) > q_\alpha$. Passing to a subset $\Gamma \subset 2^\omega$ of size continuum, we can assume that $q_\alpha = \epsilon$ for every $\alpha \in \Gamma$ and this finishes the proof. □

Definition 2.30 ([7]). *Let $\mathbb{P}$ be a partial order. A nice name for an element of $\mathcal{B}(\ell_2)$ is a name of the form*

$$\dot{f} = \bigcup_{n,m,k\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}} \{\langle[[[\check{n},\check{m}],\check{k}],\check{q}_{n,m,k}(p)],p\rangle : p \in A_{n,m,k}\}$$

where $[\check{n},\check{m}]$ stands for the canonical name for an ordered pair whose first element is $\check{n}$ and the second element is $\check{m}$, $A_{m,n,k}$ is an antichain in $\mathbb{P}$ and $q_{n,m,k} : A_{n,m,k} \to \mathbb{Q}$ are functions.

Remark 2.31 ([7]). *The value of a nice name for an element of* $\mathcal{B}(\ell_2)$ *is a function* $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$. *This function codes the operator* $T \in \mathcal{B}(\ell_2)$ *such that*

$$\langle Te_n, e_m \rangle = \lim_{k\to\infty} f(n, m, k)$$

where $(e_n)_n$ *is the canonical basis of* ℓ_2.

Theorem 2.32. *It is relatively consistent that whenever* $(T_\alpha : \alpha < 2^\omega)$ *is a collection of operators in* $\mathcal{B}(\ell_2)$ *which generates a C*-algebra of density continuum, then there is a set* $I \subseteq 2^\omega$ *of cardinality continuum such that* $(T_\alpha : \alpha \in I)$ *is irredundant.*

Proof. To obtain the relative consistency we will use the method of forcing (see [24]). We start with the ground model V satisfying the generalized continuum hypothesis (GCH) and we will consider the generic extension $V[G]$ where G is a generic set in the forcing $\mathbb{P} = Fn(\omega_2, 2)$ for adding ω_2 Cohen reals (see Chapter VIII §2 of [24]).

Consider $(\dot{T}_\alpha)_{\alpha<\omega_2}$ a sequence of nice names for elements in $\mathcal{B}(\ell_2)$ which generates a C*-algebra of density continuum. By Corollary 2.29, we can assume that there is $\varepsilon > 0$ such that

$$\mathbb{P} \Vdash d(\dot{T}_\alpha, C^*(\{\dot{T}_{\beta_1}, \cdots, \dot{T}_{\beta_n}\})) > \varepsilon \tag{2.1}$$

for every finite sequence $\beta_1, \cdots, \beta_n < \alpha$. For each $\alpha < \omega_2$, define

$$B_\alpha = supp(\dot{T}_\alpha) = \bigcup\{dom(p) : \exists\dot{a}(\langle \dot{a}, p\rangle \in \dot{T}_\alpha)\}$$

Since $\mathbb{P}$ is c.c.c, B_α is countable for each $\alpha < \omega_2$. Since CH holds in V, by Theorem 1.17, we can assume that $(B_\alpha)_{\alpha<\omega_2}$ is a Δ-system with root Δ.

For each $\xi < \sigma$, consider the order preserving bijection $s_{\sigma,\xi}$ from B_ξ onto B_σ and let $f_\xi : \omega \to B_\xi$ be an order preserving bijection.

Claim 1: We can assume that, for every $\xi < \sigma$, the bijection $S_{\sigma,\xi} : \omega_2 \to \omega_2$ given by

$$S_{\sigma,\xi}(\alpha) = \begin{cases} s_{\sigma,\xi}(\alpha), & \text{if} \quad \alpha \in B_\xi \\ s^{-1}_{\sigma,\xi}(\alpha), & \text{if} \quad \alpha \in B_\sigma \\ \alpha, & \text{otherwise} \end{cases}$$

is such that, the induced automorphism $S^c_{\sigma,\xi} : \mathbb{P} \to \mathbb{P}$ (as in Lemma 1.13) satisfies

1. $(S^c_{\sigma,\xi})_*(\dot{T}_\xi) = \dot{T}_\sigma$
2. $(S^c_{\sigma,\xi})_*(\dot{T}_\sigma) = \dot{T}_\xi$ and
3. $(S^c_{\sigma,\xi})_*(\dot{T}_\beta) = \dot{T}_\beta$ for $\beta \notin \{\xi, \sigma\}$.

Proof of Claim 1. For each $\xi < \omega_2$, we have that $\dot{T}_\xi$ is a nice name in $Fn(B_\xi)$. Using the notation of Section 1.1.3 we have that $\tau_\xi := (f^c_\xi)_*(\dot{T}_\xi)$ is a nice name in $Fn(\omega)$. Since we have at most $2^\omega = \omega_1$ nice names in $Fn(\omega)$ we can assume that there is a name τ in $Fn(\omega)$ such that $\tau = \tau_\xi = (f^c_\xi)_*(\dot{T}_\xi)$ for every $\xi < \omega_2$. By Theorem 1.15, for each $\xi < \sigma$ we have that $(f^c_\xi)_* \circ (s^c_{\sigma,\xi})_* = (f^c_\sigma)_*$. In particular

$$(f^c_\xi)_*(\dot{T}_\xi) = \tau = (f^c_\sigma)_*(\dot{T}_\sigma) = ((f^c_\xi)_* \circ (s^c_{\sigma,\xi})_*)(\dot{T}_\sigma) = (f^c_\xi)_*((s^c_{\sigma,\xi})_*(\dot{T}_\sigma))$$

which implies that $(s^c_{\sigma,\xi})_*(\dot{T}_\sigma) = \dot{T}_\xi$. In the same way, we prove that $((s^{-1}_{\sigma,\xi})^c)_*(\dot{T}_\xi) = \dot{T}_\sigma$. To conclude the proof of Claim 1, we observe that

$$(S^c_{\sigma,\xi})_*(\dot{a}) = \begin{cases} (s^c_{\sigma,\xi})_*(\dot{a}), & \text{if} \quad \dot{a} \text{ is a name in } Fn(B_\xi, 2) \\ ((s^{-1}_{\sigma,\xi})^c)_*(\dot{a}), & \text{if} \quad \dot{a} \text{ is a name in } Fn(B_\sigma, 2) \\ \dot{a}, & \text{otherwise.} \end{cases}$$

□

Claim 2: There is no $p \in \mathbb{P}$ which forces some $\dot{T}_\eta$ to be in the algebra generated by $\{\dot{T}_\xi : \xi \neq \eta\}$.

Proof of Claim 2. Suppose the claim is false and let us get a contradiction. Then there is $p \in \mathbb{P}$ and $\beta_1, \cdots, \beta_n < \omega_2$ with $\beta_i \neq \eta$ such that

$$p \Vdash d(\dot{T}_\eta, C^*(\{\dot{T}_{\beta_1}, \cdots, \dot{T}_{\beta_n}\})) < \varepsilon$$

where ε satisfies 2.1.

Consider $\alpha > \max\{\beta_1, \cdots, \beta_n, \eta\}$ and the automorphism $S^c_{\alpha,\eta}$. Then by Theorem 7.13 of [24] we have that

$$S^c_{\alpha,\eta}(p) \Vdash d(\dot{T}_\alpha, C^*(\{\dot{T}_{\beta_1}, \cdots, \dot{T}_{\beta_n}\})) < \varepsilon$$

which is a contradiction with the choice of $(\dot{T}_\xi)_{\xi<\omega_2}$ because $\alpha > \beta_1, \cdots, \beta_n$. □

From Claim 2, we conclude that $(\dot{T}_\xi)_{\xi<\omega_2}$ is an irredundant set. □

The above theorem shows us that the sentence "every C*-subalgebra of $\mathcal{B}(\ell_2)$ of density continuum has an irredundant set of size continuum" is consistent with ZFC. In Section 4 we will consistently construct a thin-tall C*-algebra of density continuum ($2^\omega = \omega_1$) without irredundant sets of size continuum.

2.6 Almost irredundant sets

In this section we consider a weaker version of irredundance, namely almost irredundance.

Definition 2.33. *Let $\mathcal{A}$ be a C*-algebra. A family $(a_\alpha)_{\alpha<\kappa}$ in $\mathcal{A}$ is almost irredundant if for every $\alpha < \kappa$, the element a_α does not belong to the norm-closure of*

$$\{a = \sum_{i=1}^{n} \lambda_i \prod_{j=1}^{n_i} a_{\beta_{i,j}} : \text{ where } \beta_{i,j} \neq \alpha \text{ and} \sum |\lambda_i| \leq 1\}.$$

Clearly, every irredundant set is almost irredundant. But this two notions are not equivalent:

Remark 2.34 (An example of a set that is almost irredundant but not irredundant)**.** *Consider $\mathcal{A} = M_2(\mathbb{C})$. Let p_1, p_2 be projections onto $(1, 0)$ and onto $(0, 1)$ respectively, and Id the identity of $\mathcal{A}$. Then $\{p_1, p_2, Id\}$ is almost irredundant but it is not irredundant.*

The goal of this section is to prove that under PFA, every nonseparable scattered C*-algebra has an uncountable almost irredundant set of projections. To this end, we need some lemmas:

Lemma 2.35. *Let $\mathcal{A}$ be a unital C*-algebra. Suppose τ is a pure state and p is a projection such that $\tau(p) \geq 1-\varepsilon$ for some $\varepsilon > 0$. Then for every $a \in \mathcal{A}$ such that $\|a\| \leq 1$ we have that*

$$|\tau(a) - \tau(pap)| \leq 2\sqrt{\varepsilon}$$

Proof. By (2) of Proposition 1.34 we have that

$$|\tau(a - pa)| = |\tau((1-p)a| \leq \sqrt{\tau(1-p)\tau(a^*a)} \leq \sqrt{\varepsilon}.$$

for every $a \in \mathcal{A}$ with $\|a\| \leq 1$. Since $\|pa\| \leq 1$, we conclude that

$$|\tau(a) - \tau(pap)| = |\tau(a) - \tau(pa) + \tau(pa) - \tau(pap)| \leq |\tau(a) - \tau(pa)| + |\tau(pa) - \tau(pap)| \leq 2\sqrt{\varepsilon}.$$

□

Proposition 2.36. *Let $\mathcal{A}$ be a unital C*-algebra of real rank zero. Consider $\tau \in \mathbb{P}(\mathcal{A})$ a pure state. Then the sets*

$$U_{p,\varepsilon} := \{\sigma : \sigma(p) > 1 - \varepsilon\},$$

where $\varepsilon > 0$ and $p \in \mathcal{A}$ is a projection such that $\tau(p) = 1$, form a local neighbourhood basis of τ in $\mathbb{P}(\mathcal{A})$.

Proof. **Claim 1(Proposition 2.2 of [1]):** There is a family of projections $(p_\alpha)_{\alpha\in\Lambda}$ such that $\tau(p_\alpha) = 1$ and for every $b \in \mathcal{A}$ we have that

$$\lim_{\alpha\in\Lambda} \|p_\alpha b p_\alpha - \tau(b)p_\alpha\| = 0.$$

Proof of Claim 1. Denote by $L = \{a : \tau(a^*a) = 0\}$ the left kernel of τ and $N = L \cap L^*$. By hypothesis, $\mathcal{A}$ has real rank zero and therefore, since N is a hereditary C*-subalgebra of $\mathcal{A}$, there is an approximate unit[8] formed by projections[9] $\{q_\alpha : \alpha < \kappa\}$ for N. Define for each $\alpha < \kappa$ the projection $p_\alpha = 1 - q_\alpha$. Consider now $b \in A$. Then $\tau(b - \tau(b)) = 0$ and therefore $b - \tau(b) \in Ker(\tau)$. Since τ is a pure state, we have that[10] $Ker(\tau) = L + L^*$. Then there are $c, d \in L$ such that

$$b - \tau(b) = c + d^*.$$

But then

$$\|p_\alpha b p_\alpha - \tau(b)p_\alpha\| = \|p_\alpha(b - \tau(b))p_\alpha\| = \|p_\alpha(c + d^*)p_\alpha\| \leq$$
$$\|p_\alpha(c + d^*)\| \leq \|(1 - q_\alpha)(c + d^*)\|$$

which converges to zero because $(q_\alpha)_\alpha$ is an approximate unit in N and $c + d^* \in N$. This concludes the proof of Claim 1. □

Consider V a weak* open neighbourhood of τ. We can assume that V is of the form

$$V = \{\sigma : \|\sigma(b_i) - \tau(b_i)\| < \delta \text{ for } i = 1, \cdots, n\},$$

for some $b_i, \ldots, b_n$ in $\mathcal{A}$ with $\|b_i\| = 1$. Let $\varepsilon < \frac{\delta}{4}$ be such that $2\sqrt{\varepsilon} < \frac{\delta}{4}$. By Claim 1, let

[8] An approximate unit in a C*-algebra $\mathcal{A}$ is an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ of positive elements in the closed unit ball of $\mathcal{A}$ such that $a = \lim_\lambda au_\lambda$ for every $a \in \mathcal{A}$.

[9] See Theorem 3.2.9 of [5].

[10] See Theorem 5.3.4 of [28].

$p \in \mathcal{A}$ be a projection such that $\tau(p) = 1$ and

$$\|\tau(b_i)p - pb_ip\| < \delta/4.$$

for every $i = 1, \cdots, n$. We claim that $U_{p,\varepsilon} \subseteq V$.

Fix $\sigma \in U_{p,\varepsilon}$. Since $\|\tau(b_i)p - pb_ip\| < \delta/4$ we conclude that

$$\frac{\delta}{4} \geq \|\sigma(\tau(b_i)p - pb_ip)\| = \|\tau(b_i)\sigma(p) - \sigma(pb_ip)\|$$

On the other hand, since $\sigma \in U_{p,\varepsilon}$ we have that $\sigma(p) > 1-\varepsilon$ and by Lemma 2.35 we conclude that

$$\|\sigma(b_i) - \sigma(pb_ip)\| \leq \frac{\delta}{4}.$$

Then

$$\frac{\delta}{4} \geq \|\tau(b_i)\sigma(p) - \sigma(pb_ip)\| \geq \|\tau(b_i)\sigma(p) - \sigma(b_i) + \sigma(b_i) - \sigma(pb_ip)\| \geq$$
$$\geq \|\tau(b_i)\sigma(p) - \sigma(b_i)\| - \|\sigma(b_i) - \sigma(pb_ip)\| \geq \|\tau(b_i)\sigma(p) - \sigma(b_i)\| - \frac{\delta}{4} \geq$$
$$\geq \|\tau(b_i)\sigma(p) - \sigma(b_i) + \tau(b_i) - \tau(b_i)\| - \frac{\delta}{4} \geq \|\tau(b_i) - \sigma(b_i)\| - \|\tau(b_i)(\sigma(p) - 1)\| - \frac{\delta}{4} \geq$$
$$\geq \|\tau(b_i) - \sigma(b_i)\| - \varepsilon - \frac{\delta}{4} \geq \|\tau(b_i) - \sigma(b_i)\| - \frac{\delta}{2}.$$

This shows that $\|\tau(b_i) - \sigma(b_i)\| < \delta$ and therefore, $\sigma \in V$. □

Lemma 2.37. *Let $\mathcal{A}$ be a unital C^*-algebra. Suppose $a_1, \cdots, a_n$ are self adjoint elements such that $0 \leq a_i \leq 1$ and $\tau(a_i) < \varepsilon$ for τ a positive functional. Then*

$$\tau(a_1 \cdots a_n) \leq \sqrt{\varepsilon}.$$

Proof. Using (2) and (3) of Proposition 1.34 repeatedly, we get that:

$$|\tau(a_1 \cdots a_n)|^2 \leq \tau(a_1^2)\tau((a_2 \cdots a_n)^*(a_2 \cdots a_n)) \leq \tau((a_2 \cdots a_n)^*(a_2 \cdots a_n)) =$$
$$\tau((a_3 \cdots a_n)^* a_2^* a_2 (a_3 \cdots a_n)) \leq \|a_2^* a_2\| \tau((a_3 \cdots a_n)^*(a_3 \cdots a_n)) \leq$$
$$\leq \tau((a_3 \cdots a_n)^*(a_3 \cdots a_n)) \leq \cdots \leq \tau(a_n^* a_n) = \tau(a_n^2) \leq \tau(a_n) \leq \varepsilon$$

□

Lemma 2.38. *Let $\mathcal{A}$ be a unital C^*-algebra such that $\mathbb{P}(\mathcal{A})$ has an uncountable discrete set. Then $\mathcal{A}$ has an uncountable almost irredundant set of projections.*

Proof. Suppose $(\tau_\alpha)_{\alpha<\omega_1}$ is a discrete set in $\mathbb{P}(\mathcal{A})$. By Proposition 2.36 and by a counting argument, we can assume that there is a family $(p_\alpha)_{\alpha<\omega_1}$ of projections and a $\delta > 0$ such that

- $\tau_\alpha(p_\alpha) = 1$,
- $\tau_\alpha(p_\beta) < 1 - \delta$ for $\alpha \neq \beta$.

We claim that $(p_\alpha)_{\alpha<\omega_1}$ is almost irredundant. Suppose the claim is false and let us get a contradiction. Consider $\alpha < \omega_1$ such that p_α belongs to the closure of

$$\{a = \sum_{i=1}^{n} \lambda_i \prod_{j=1}^{n_i} p_{\beta_{i,j}} : \text{ where } \beta_{i,j} \neq \alpha \text{ and} \sum |\lambda_i| \leq 1\}.$$

Consider $\varepsilon > 0$ such that $1-\sqrt{1-\delta} > \varepsilon$ and $a = \sum_{i=1}^{n} \lambda_i \prod_{j=1}^{n_i} p_{\beta_{i,j}}$ such that $\|p_\alpha - a\| < \varepsilon$. In particular,

$$|\tau_\alpha(p_\alpha - a)| < \varepsilon.$$

On the other hand,

$$|\tau_\alpha(a)| = |\tau_\alpha(\sum_{i=1}^{n} \lambda_i \prod_{j=1}^{n_i} p_{\beta_{i,j}})| = |\sum_{i=1}^{n} \lambda_i \tau_\alpha(\prod_{j=1}^{n_i} p_{\beta_{i,j}})| \leq \sum_{i=1}^{n} |\lambda_i||\tau_\alpha(\prod_{j=1}^{n_i} p_{\beta_{i,j}})| \leq$$

$$\overset{*}{\leq} \sum_{i=1}^{n} |\lambda_i| \sqrt{(1-\delta)} \leq \sqrt{(1-\delta)}$$

where in $*$ we used Lemma 2.37. But then,

$$|\tau_\alpha(p_\alpha - a)| = |1 - \tau_\alpha(a)| \geq 1 - |\tau_\alpha(a)| > 1 - \sqrt{(1-\delta)} > \varepsilon$$

which is a contradiction. □

Theorem 2.39. *Assume PFA. Let $\mathcal{A}$ be a unital nonseparable scattered C*-algebra. Then $\mathcal{A}$ has an uncountable almost irredundant set of projections.*

Proof. By Theorem 2.21, the pure state space $\mathbb{P}(\mathcal{A})$ has an uncountable discrete set. Then by Lemma 2.38 $\mathcal{A}$ has an uncountable almost irredundant set. □

We conclude this section with a result about almost irredundant sets in the unitization of a C*-algebra. With this result, we prove that Theorem 2.39 can be generalized for non-unital scattered C*-algebras.

Proposition 2.40. *Let $\mathcal{A}$ be a C*-algebra and $\tilde{\mathcal{A}}$ its unitization. Then if $\tilde{A}$ has an almost irredundant set of projections of size $\kappa \geq \omega$, then $\mathcal{A}$ has an almost irredundant set of projections of size κ.*

Proof. Let $((a_\alpha, \lambda_\alpha))_{\alpha \leq \kappa}$ be an almost irredundant set of projections of size κ in $\tilde{A}$. Since $\kappa \geq \omega$, we can assume that $(0,1) \neq (a_\alpha, \lambda_\alpha)$ for every $\alpha < \kappa$. In particular, since $(a_\alpha, \lambda_\alpha)$ is a projection which is not the unit, it follows from Lemma 1.31 that $a_\alpha \in \mathcal{A}$ is a projection and $\lambda_\alpha = 0$. Then $(a_\alpha)_{\alpha<\kappa}$ is the required almost irredundant set of size κ in $\mathcal{A}$. □

Theorem 2.41. *Assume PFA. Let $\mathcal{A}$ be a nonseparable scattered C*-algebra. Then $\mathcal{A}$ has an uncountable almost irredundant set of projections.*

Proof. Consider $\widetilde{\mathcal{A}}$ the unitization of $\mathcal{A}$. By Proposition 1.63, $\widetilde{\mathcal{A}}$ is a scattered C*-algebra. In particular, $\widetilde{\mathcal{A}}$ is a unital nonseparable scattered C*-algebra. By Theorem 2.39, $\widetilde{\mathcal{A}}$ has an uncountable almost irredundant set of projections. Then by Proposition 2.40, $\mathcal{A}$ has an uncountable almost irredundant set of projections and this concludes the proof. □

Chapter 3

The forcing notion

In this chapter we consider a forcing notion which will be used in the next chapter to force a nonseparable and noncommutative scattered C*-algebra $\mathcal{A}$ in $\mathcal{B}(\ell_2(\omega_1 \times \mathbb{N}))$ which will be generated by a set of the form $\{A_{\xi,m,n} : \xi < \omega_1, n, m \in \mathbb{N}\}$. Our forcing notion will be formed by finite approximation to $A_{\xi,m,n}$'s.

3.1 Notation

In this chapter we consider some special C*-algebras which are subalgebras of $\mathcal{B}(\ell_2(\omega_1 \times \mathbb{N}))$. We look at $\ell_2(\omega_1 \times \mathbb{N})$ as the direct sum of columns of the form $\ell_2(\{\xi\} \times \mathbb{N})$ for $\xi < \omega_1$ and this columns will be invariant for all the operators in our algebras. To make the last statement more precise, we need some notation:

- $(e_{\xi,n} : \xi < \omega_1, n \in \mathbb{N})$ is the canonical orthonormal basis of $\ell_2(\omega_1 \times \mathbb{N})$.

  For $X \subseteq \omega_1 \times \mathbb{N}$:

- Given an operator $A \in \mathcal{B}(\ell_2(\omega_1 \times \mathbb{N}))$ and $\xi < \omega_1$, we say that $\ell_2(X \cap (\{\xi\} \times \mathbb{N}))$ is A-invariant if $Av \in \ell_2(X \cap (\{\xi\} \times \mathbb{N}))$ for every $v \in \ell_2(X \cap (\{\xi\} \times \mathbb{N}))$.

- The family of all operators A in $\mathcal{B}(\ell_2(\omega_1 \times \mathbb{N}))$ such that

  - $\ell_2(X \cap (\{\xi\} \times \mathbb{N}))$ is A-invariant for all $\xi < \omega_1$,
  - $A(e_{\xi,n}) = 0$ whenever $(\xi, n) \notin X$,

  will be denoted by $\mathcal{B}_X$.

- The unit of the C*-algebra $\mathcal{B}_X$ will be denoted by P_X.

- $1_{\xi,m,n}$ is the operator in $\mathcal{B}_{\omega_1 \times \mathbb{N}}$ satisfying

$$1_{\xi,m,n}(e_{\eta,k}) = \begin{cases} e_{\xi,m} & \text{if } k = n, \xi = \eta \\ 0 & \text{otherwise.} \end{cases}$$

- If $A \in \mathcal{B}_{\omega_1 \times \mathbb{N}}$ we define $A \restriction X = AP_X$.

- If $A \in \mathcal{B}_{\omega_1 \times \mathbb{N}}$ and $a \subseteq \omega_1$ we define $A \restriction a$ as $A \restriction (a \times \mathbb{N})$.

- $\mathcal{A} \restriction X = \{A \restriction X : A \in \mathcal{A}\}$ for $\mathcal{A} \subseteq \mathcal{B}_{\omega_1 \times \mathbb{N}}$ and $X \subseteq \omega_1 \times \mathbb{N}$.

Definition 3.1.

1. *Fix a countable field* $\mathcal{D} \subseteq \mathbb{C}$ *such that*
   - $\mathbb{Q} + i\mathbb{Q} \subseteq \mathcal{D}$.
   - $\sqrt{\lambda} \in \mathcal{D}$ *for every* $\lambda \in \mathcal{D}$.
2. *Given an operator* $A \in \mathcal{B}(\ell_2(\omega_1 \times \mathbb{N}))$, *we say that* A *has a matrix with entries in* $\mathcal{D}$ *if*
$$\langle A(e_{\alpha,n}), e_{\beta,m} \rangle \in \mathcal{D}$$
for every $\alpha, \beta < \omega_1$ *and* $n, m \in \mathbb{N}$.

3.2 The forcing notion

Definition 3.2. *Define a partial order* $\mathbb{P}_{\omega_1}(\mathcal{D})$ *consisting of elements*
$$p = \big(a_p, \{n^p_\xi : \xi \in a_p\}, \{A^p_{\xi,m,n} : \xi \in a_p, n, m \in [0, n^p_\xi)\}\big),$$
where

1. a_p *is a finite subset of* ω_1,
2. $n^p_\xi \in \mathbb{N}$ *for each* $\xi \in a_p$,
3. $A^p_{\xi,m,n} \in \mathcal{B}_{X_p}$ *for each* $\xi \in a_p$ *and* $n, m \in [0, n^p_\xi)$, *where*
$$X_p = \{(\alpha, n) : \alpha \in a_p;\ n \in [0, n^p_\alpha)\},$$
and $A^p_{\xi,m,n}$ *has a matrix with entries in* $\mathcal{D}$.
4. $A^p_{\xi,m,n} = (A^p_{\xi,m,n} \restriction \xi) + 1_{\xi,m,n}$ *for each* $\xi \in a_p$ *and* $n, m \in [0, n^p_\xi)$.
The order $\leq_{\mathbb{P}_{\omega_1}(\mathcal{D})}$ *(or simply* $\leq$*) on* $\mathbb{P}_{\omega_1}(\mathcal{D})$ *is defined by declaring* $p \leq q$ *if and only if:*
   (a) $a_p \supseteq a_q$,
   (b) $n^p_\xi \geq n^q_\xi$ *for* $\xi \in a_q$,
   (c) *there is a (nonunital) *-embedding* $i_{p,q} : \mathcal{B}_{X_q} \to \mathcal{B}_{X_p}$ *such that* $i_{p,q}(A^q_{\xi,m,n}) = A^p_{\xi,m,n}$ *for all* $\xi \in a_q$ *and* $m, n \in [0, n^q_\xi)$,
   (d) $i_{p,q}(A) \restriction X_q = A$ *for all* $A \in \mathcal{B}_{X_q}$.

Definition 3.3. *Suppose that* $p \in \mathbb{P}_{\omega_1}(\mathcal{D})$ *and* $X \subseteq X_p$. *Then the* C^**-subalgebra of* $\mathcal{B}_{X_p}$ *generated by* $\{A^p_{\xi,m,n} : (\xi, m), (\xi, n) \in X\}$ *is denoted by* $\mathcal{A}^p_X$.

Lemma 3.4. *For every* $\alpha \in \omega_1$ *and every* $p \in \mathbb{P}_{\omega_1}(\mathcal{D})$ *we have*
$$\mathcal{A}^p_{X_p \cap (\alpha \times \mathbb{N})} = \mathcal{B}_{X_p \cap (\alpha \times \mathbb{N})}.$$
In particular $\mathcal{A}^p_{X_p} = \mathcal{B}_{X_p}$.

Proof. We will prove it by induction on $|a_p \cap \alpha|$. If $a_p \cap \alpha = \emptyset$, then both of the algebras are $\{0\}$. Suppose $|a_p \cap \alpha| = n + 1$ and we have proved the lemma for every $q \in \mathbb{P}$ and $\alpha < \omega_1$ such that $|a_q \cap \alpha| = n$. Let $\xi = max(a_p \cap \alpha)$. By the definition of $\mathcal{B}_{X_p}$ we have that $\mathcal{B}_{X_p \cap (\alpha \times \mathbb{N})}$ is *-isomorphic to $\mathcal{B}_{X_p \cap (\xi \times \mathbb{N})} \oplus \mathcal{B}_{\{\xi\} \times [0, n^p_\xi)}$. More precisely, we can write every element $A \in \mathcal{B}_{X_p \cap (\alpha \times \mathbb{N})}$ in the form $A = A_1 + A_2$ where $A_1 \in \mathcal{B}_{X_p \cap (\xi \times \mathbb{N})}$ and $A_2 \in \mathcal{B}_{\{\xi\} \times [0, n^p_\xi)}$.

By the inductive hypothesis, $\mathcal{B}_{X_p \cap (\xi \times \mathbb{N})}$ is generated by $\{A^p_{\eta,m,n} : \eta \in a_p \cap \xi; m, n \in [0, n^p_\xi)\}$. But by (4) in Definition 3.2, we have that $1_{\xi,m,n} = A^p_{\xi,m,n} - A$ for some $A \in \mathcal{B}_{X_p \cap (\xi \times \mathbb{N})}$ and all $m, n \in [0, n^p_\xi)$. In particular, $\mathcal{B}_{\{\xi\} \times [0, n^p_\xi)}$ is included in the algebra generated by $\{A^p_{\eta,m,n} : \eta \in a_p \cap \alpha; m, n \in [0, n^p_\eta)\}$ which is $\mathcal{A}^p_{X_p \cap (\alpha \times \mathbb{N})}$. Then $\mathcal{B}_{X_p \cap (\alpha \times \mathbb{N})} = \mathcal{B}_{X_p \cap (\xi \times \mathbb{N})} \oplus \mathcal{B}_{\{\xi\} \times [0, n^p_\xi)} \subseteq \mathcal{A}^p_{X_p \cap (\alpha \times \mathbb{N})}$. Since $\mathcal{A}^p_{X_p \cap (\alpha \times \mathbb{N})} \subset \mathcal{B}_{X_p \cap (\alpha \times \mathbb{N})}$, we conclude that

$$\mathcal{A}^p_{X_p \cap (\alpha \times \mathbb{N})} = \mathcal{B}_{X_p \cap (\alpha \times \mathbb{N})}.$$

□

Remark 3.5. *If $p \leq q$, then by (c) in Definition 3.2 we have that $\mathcal{A}^q_X$ is *-isomorphic to $\mathcal{A}^p_X$ for every $X \subseteq X_q$, where the *-isomorphism is given by the restriction of $i_{p,q}$ to $\mathcal{A}^q_X$. In particular, $\mathcal{B}_{X_q}$ is *-isomorphic to $\mathcal{A}^p_{X_q}$.*

Lemma 3.6. *Suppose that $\alpha < \omega_1$ and $p, q \in \mathbb{P}_{\omega_1}(\mathcal{D})$ satisfy $p \leq q$ and $A = i_{p,q}(B)$, where $B \in \mathcal{A}^q_{X_q}$. Then*

$$\|A \restriction [\alpha, \omega_1)\| = \|B \restriction [\alpha, \omega_1)\|.$$

Proof. Since $B \restriction \alpha$ and $B \restriction [\alpha, \omega_1)$ are in $A^q_{X_q}$ by Lemma 3.4, we have

$$\|A \restriction [\alpha, \omega_1)\| = \|i_{p,q}(B) \restriction [\alpha, \omega_1)\| = \|i_{p,q}(B \restriction \alpha) \restriction [\alpha, \omega_1) + i_{p,q}(B \restriction [\alpha, \omega_1)) \restriction [\alpha, \omega_1)\|.$$

But $B \restriction \alpha \in \mathcal{A}^q_{X_q \cap (\alpha \times \mathbb{N})}$ by Lemma 3.4, and by Remark 3.5 $i_{p,q}(B \restriction \alpha) \in \mathcal{A}^p_{X_q \cap (\alpha \times \mathbb{N})}$. In particular $i_{p,q}(B \restriction \alpha) \restriction [\alpha, \omega_1) = 0$ and so

$$\|A \restriction [\alpha, \omega_1)\| = \|i_{p,q}(B \restriction [\alpha, \omega_1)) \restriction [\alpha, \omega_1)\| \leq \|i_{p,q}(B \restriction [\alpha, \omega_1))\|.$$

Since $i_{p,q}$ is a *-homomorphism (in particular norm-decreasing), we conclude that

$$\|A \restriction [\alpha, \omega_1)\| \leq \|B \restriction [\alpha, \omega_1)\|.$$

The other inequality follows from the fact that $A \restriction X_q = B$ by Definition 3.2 (c-d). □

3.3 Density lemmas

In this section, we define some dense sets in our partial order $\mathbb{P}_{\omega_1}(\mathcal{D})$, where by a dense set in $\mathbb{P}_{\omega_1}(\mathcal{D})$ we mean a subset $\mathbb{D} \subseteq \mathbb{P}_{\omega_1}(\mathcal{D})$ such that for every element $p \in \mathbb{P}_{\omega_1}(\mathcal{D})$, there is an element $d \in \mathbb{D}$ such that $d \leq p$.

Lemma 3.7. *Suppose that $\xi < \omega_1$. Then*

$$\mathbb{D}_\xi = \{p \in \mathbb{P}_{\omega_1}(\mathcal{D}) : \xi \in a_p\}$$

is a dense subset of $\mathbb{P}_{\omega_1}(\mathcal{D})$.

Proof. Let $q \in \mathbb{P}_{\omega_1}(\mathcal{D})$ be such that $\xi \notin a_q$. Define p as follows:

- $a_p = a_q \cup \{\xi\}$,
- $n^p_\eta = n^q_\eta$ for $\eta \in a_q$ and $n^p_\xi = 1$,
- $A^p_{\eta,m,n} = A^q_{\eta,m,n}$ for $\eta \in a_q$, $m, n \in [0, n^p_\eta)$ and $A^p_{\xi,0,0} = 1_{\xi,0,0}$.

It is clear that $p \in \mathbb{P}_{\omega_1}(\mathcal{D})$. Also $p \leq q$ as $Id_{\mathcal{B}_{X_q}} : \mathcal{B}_{X_q} \to \mathcal{B}_{X_p}$ is a *-embedding good for $i_{p,q}$ in Definition 3.2 (c). □

Lemma 3.8. *Suppose that $\xi < \omega_1$, $k \in \mathbb{N}$ and $q \in \mathbb{P}_{\omega_1}(\mathcal{D})$ is such that $\xi \in a_q$. Then there is*

$$p \in \mathbb{E}_{\xi,k} = \{p \in \mathbb{P}_{\omega_1}(\mathcal{D}) : \xi \in a_p, n^p_\xi \geq k\}$$

such that $p \leq q$ and $a_p = a_q$.

Proof.
Consider $q \in \mathbb{P}_{\omega_1}(\mathcal{D})$ such that $\xi \in a_q$ but $n^q_\xi < k$. Define p as follows:

- $a_p = a_q$,
- $n^p_\eta = n^q_\eta$ for $\eta \in a_p \setminus \{\xi\}$ and $n^p_\xi = k$,
- $A^p_{\eta,m,n} = A^q_{\eta,m,n}$ for $\eta \in a_q \setminus \{\xi\}$, $m, n \in [0, n^q_\eta)$,
- $A^p_{\xi,m,n} = A^q_{\xi,m,n}$ for $n, m \in [0, n^q_\xi)$,
- $A^p_{\xi,m,n} = 1_{\xi,m,n}$ if $n, m \in [0, k)$ and $\{n, m\} \cap [n^q_\xi, k) \neq \emptyset$.

It is clear that $p \in \mathbb{E}_{\xi,k}$. Also $p \leq q$ as $Id_{\mathcal{B}_{X_q}} : \mathcal{B}_{X_q} \to \mathcal{B}_{X_p}$ is a *-embedding good for $i_{p,q}$ in Definition 3.2 (c). □

Lemma 3.9. *Suppose that $q \in \mathbb{P}_{\omega_1}(\mathcal{D})$ and $X \subseteq X_q$ and that $\alpha \in a_q$. Then there is $p \leq q$ such that $p \in \mathbb{F}_{X,\alpha}$, where*

$$\mathbb{F}_{X,\alpha} = \{p \in \mathbb{P}_{\omega_1}(\mathcal{D}) : \alpha \in a_p,\ X \subseteq X_p,\ \text{and}\ \ \forall A \in \mathcal{A}^p_X\ \ \|A|\{\alpha\}\| \geq \|A|[\alpha, \omega_1)\|\}.$$

Moreover, $a_p = a_q$ and $n^p_\xi = n^q_\xi$ whenever $\xi \in a_p \setminus \{\alpha\}$.

Proof. Let $q \in \mathbb{P}_{\omega_1}(\mathcal{D})$. If $\alpha = \max(a_p)$, then there is nothing to prove. So let $a_q \setminus (\alpha + 1) = \{\xi_1, ..., \xi_k\}$ for some $k \in \mathbb{N}$ and put

$$l = \sum \{n^q_{\xi_i} :\ 1 \leq i \leq k\}.$$

Consider $Y = X_q \cap ((\alpha, \omega_1) \times \mathbb{N})$. Let $\phi : Y \to [n^q_\alpha, n^q_\alpha + l)$ be any bijection. We obtain a *-homomorphism $i : \mathcal{B}_{X_q} \to \mathcal{B}_{X_q \cup \left(\{\alpha\} \times [n^q_\alpha, n^q_\alpha + l)\right)}$ given by $i(A) = A + i_r(A)$ where $i_r : \mathcal{B}_{X_q} \to \mathcal{B}_{\{\alpha\} \times [n^q_\alpha, n^q_\alpha + l)}$ satisfies

$$\langle i_r(A)(e_{\alpha, n^q_\alpha + \phi(\xi_i, k)}), e_{\alpha, n^q_\alpha + \phi(\xi_{i'}, k')} \rangle = \langle A(e_{\xi_i, k}), e_{\xi_{i'}, k'} \rangle$$

for all $(\xi, k), (\xi', k') \in Y$ and every $A \in \mathcal{B}_{X_q}$. Define p in the following way

- $a_p = a_q$,
- $n^p_\xi = n^q_\xi$ if $\xi \in a_p \setminus \{\alpha\}$ and $n^p_\alpha = n^q_\alpha + l$,

- $A^p_{\xi,m,n} = i(A^q_{\xi,m,n})$ for $(\xi, m), (\xi, n) \in X_q$,
- $A^p_{\alpha,m,n} = 1_{\alpha,m,n}$ if $\{m, n\} \cap [n^q_\alpha, n^p_\alpha) \neq \emptyset$.

It is clear from the construction that $p \in \mathbb{P}_{\omega_1}(\mathcal{D})$ as condition (4) of Definition 3.2 is satisfied due to the fact that we change only $A^q_{\xi,m,n}$ for $\xi > \alpha$ on $\{\alpha\}\times\mathbb{N}$, and that (a), (b) of Definition 3.2 are satisfied.

If we put $i_{p,q} = i$, condition (c) follows from the fact that i is a *-embedding since $\{\alpha\} \times [n^q_\alpha, n^q_\alpha + l) \cap X_q = \emptyset$. We also have $i_{p,q}(A^q_{\xi,m,n}) = A^p_{\xi,m,n}$ for $(\xi, m), (\xi, n) \in X_q$. The construction yields (d) of Definition 3.2.

Finally let us check the main assertion of the lemma. Note that for any $A \in \mathcal{B}_{X_q}$ we have that

$$\|i_{pq}(A) \upharpoonright \{a\} \times [n^q_\alpha, n^q_\alpha + l)\| = \|i_r(A)\| = \|A \upharpoonright (\alpha, \omega_1)\|.$$

Then by Lemma 3.6 we have

$$\|i_{p,q}(A)|\{\alpha\}\| = \max(\|i_{p,q}(A)|\{\alpha\}\|, \|i_{p,q}(A)|\{\alpha\} \times [n^q_\alpha, n^q_\alpha + l)\|) =$$

$$= \max(\|i_{p,q}(A)|\{\alpha\}\|, \|A|(\alpha, \omega_1)\|) = \max(\|i_{p,q}(A)|\{\alpha\}\|, \|i_{p,q}(A)|(\alpha, \omega_1)\|) =$$

$$= \|i_{p,q}(A)|[\alpha, \omega_1)\|$$

for any $A \in \mathcal{B}_{X_q}$ as required since $X \subseteq X_q$. □

Lemma 3.10. *Let $X \subseteq \omega_1 \times \mathbb{N}$ be finite and $\alpha \in X$. If $q \in \mathbb{F}_{X,\alpha}$ and $p \leq q$, then $p \in \mathbb{F}_{X,\alpha}$.*

Proof. Let $A \in \mathcal{A}^p_X$. As $X \subseteq X_q$ we have that $A = i_{p,q}(B)$ for some $B \in \mathcal{A}^q_X \subset \mathcal{A}^q_{X_q}$. First note that by Lemma 3.6

$$\|A \upharpoonright [\alpha, \omega_1)\| = \|B \upharpoonright [\alpha, \omega_1)\|.$$

Now $\|B \upharpoonright [\alpha, \omega_1)\| \leq \|B \upharpoonright \{\alpha\}\|$ by the hypothesis that $q \in F_{X,\alpha}$. But $\|B \upharpoonright \{\alpha\}\| \leq \|A \upharpoonright \{\alpha\}\|$ by the fact that $A \upharpoonright X_q = B$ by Definition 3.2 (d). So $\|A \upharpoonright [\alpha, \omega_1)\| \leq \|A \upharpoonright \{\alpha\}\|$ as required. □

3.4 The forcing $\mathbb{P}_{\omega_1}(\mathcal{D})$ has the c.c.c

The following definition is essential for all amalgamations[1] in this thesis:

Definition 3.11. *We say that two elements $p, q \in \mathbb{P}_{\omega_1}(\mathcal{D})$ are in the convenient position if and only if* [2]

$$\Delta := a_p \cap a_q < a_p \setminus \Delta < a_q \setminus \Delta$$

and there is an order preserving bijection $\sigma : a_p \to a_q$ such that

- $n^p_\xi = n^q_{\sigma(\xi)}$ *for* $\xi \in a_p$,

*and the *-isomorphism of $\mathcal{B}_{X_q}$ onto $\mathcal{B}_{X_p}$ induced by σ, denoted by j_σ, which is given by*

$$\langle j_\sigma(A)(e_{\xi,k}), e_{\xi,l}\rangle = \langle A(e_{\sigma(\xi),k}), e_{\sigma(\xi),l}\rangle$$

for every $(\xi, k), (\xi, l) \in X_p$ and $A \in \mathcal{B}_{X_q}$ satisfies

[1] Given two conditions p, q in the partial order $\mathbb{P}_{\omega_1}(\mathcal{D})$, we say that a condition r is an amalgamation of p and q if $r \leq p, q$ and r is obtained by a combination of p and q.

[2] Given two set of ordinals A, B we define $A < B$ if and only if $\alpha < \beta$ for every $\alpha \in A$ and $\beta \in B$.

- $j_\sigma(A^q_{\sigma(\xi),n,m}) = A^p_{\xi,n,m}$ *for every* $\xi \in a_p$, $n, m \in [0, n^p_\xi)$.

Remark 3.12. *The *-isomorphism* j_σ *is such that the matrices of* A *and* $j_\sigma(A)$ *are the same. More precisely, define the matrix of* $A \in \mathcal{B}_{X_q}$ *as the sum* $[A] = \oplus_{\alpha\in a_q}[A]_\alpha$ *where for each* $\alpha \in a_q$, $[A]_\alpha = (a^\alpha_{i,j})_{i,j\in[0,n^q_\alpha)} \in M_{n^q_\alpha}(\mathbb{C})$ *is such that*

$$a^\alpha_{i,j} = \langle Ae_{\alpha,i}, e_{\alpha,j}\rangle$$

Then $[A]_\alpha = [j_\sigma(A)]_{\sigma^{-1}(\alpha)}$ *for each* $\alpha \in a_q$.

Lemma 3.13. *Suppose that two elements* $p, q \in \mathbb{P}_{\omega_1}(\mathcal{D})$ *are in the convenient position as witnessed by* $\sigma : a_p \to a_q$ *and that* $\xi \in \Delta = a_p \cap a_q$. *Then* $A^q_{\xi,n,m} = A^p_{\xi,n,m}$ *for every* $n, m \in [0, n^p_\xi) = [0, n^q_\xi)$.

Proof. Note that in Definition 3.11 the bijection σ must be the identity on Δ because it is order-preserving and Δ is the initial fragment of both a_p and a_q and so any $\xi \in \Delta$ must have the same position in both a_p and a_q. So $j_\sigma(A^q_{\xi,n,m}) = A^p_{\xi,n,m}$ and it is enough to prove that $j_\sigma(A^q_{\xi,n,m}) = A^q_{\xi,n,m}$.

For $\eta \in a_p \setminus \Delta$ we have

$$\langle j_\sigma(A^q_{\xi,n,m})(e_{\eta,k}), e_{\eta,l}\rangle = \langle A^q_{\xi,n,m}(e_{\sigma(\eta),k}), e_{\sigma(\eta),l}\rangle = 0$$

for every $k, l \in \mathbb{N}$ such that $(\eta, k), (\eta, l) \in X_p$ as $\sigma(\eta) \in a_q \setminus \Delta$.

On the other hand for $\eta \in \Delta$ we have $\sigma(\eta) = \eta$ and so

$$\langle j_\sigma(A^q_{\xi,n,m})(e_{\eta,k}), e_{\eta,l}\rangle = \langle A^q_{\xi,n,m}(e_{\eta,k}), e_{\eta,l}\rangle$$

for every $k, l \in \mathbb{N}$ such that $(\eta, k), (\eta, l) \in X_p$. This proves the required $A^q_{\xi,n,m} = j_\sigma(A^q_{\xi,n,m}) = A^p_{\xi,n,m}$. □

The interesting thing about two conditions p and q in the convenient position relies on the fact that we can construct amalgamations for them. As a first example, we have the following:

Lemma 3.14. *Suppose that* $p, q \in \mathbb{P}_{\omega_1}(\mathcal{D})$ *are in the convenient position as witnessed by* $\sigma : a_p \to a_q$. *Then there is* $r \le p, q$ *such that*

- $a_r = a_p \cup a_q$,
- $n^r_\xi = n^p_\xi$ *if* $\xi \in a_p$ *and* $n^r_\xi = n^q_\xi$ *if* $\xi \in a_q$,
- $i_{r,p} = Id_{\mathcal{B}_{X_p}}$, $i_{r,q} = Id_{\mathcal{B}_{X_q}}$.

In particular,

- $A^r_{\xi,m,n} = A^p_{\xi,m,n}$ *for each* $\xi \in a_p$ *and* $n, m \in [0, n^r_\xi)$,
- $A^r_{\xi,m,n} = A^q_{\xi,m,n}$ *for each* $\xi \in a_q$ *and* $n, m \in [0, n^r_\xi)$.

The element r *will be called the disjoint amalgamation of* p *and* q.

Proof. Define r as in the lemma. Observe that by Lemma 3.13, r is well defined. As $p, q \in \mathbb{P}_{\omega_1}(\mathcal{D})$, we get that $r \in \mathbb{P}_{\omega_1}(\mathcal{D})$. To see that $r \le p, q$ note that $Id_{\mathcal{B}_{X_p}}$ and $Id_{\mathcal{B}_{X_q}}$ are *-embeddings into $\mathcal{B}_{X_r}$. □

Lemma 3.15. *For every uncountable family $(p_\alpha : \alpha < \omega_1)$ of elements in $\mathbb{P}_{\omega_1}(\mathcal{D})$, there is an uncountable set $\Gamma \subseteq \omega_1$ such that $(p_\alpha : \alpha \in \Gamma)$ are pairwise in the convenient position.*

Proof. We can assume that $(a_{p_\alpha} : \alpha < \omega_1)$ is a Δ-system with root Δ such that $\Delta < a_{p_\alpha} \setminus \Delta < a_{p_\beta} \setminus \Delta$ for every $\alpha < \beta$ and $a_{p_\alpha} = (\xi_1(\alpha), \cdots, \xi_k(\alpha))$, where $\xi_i(\alpha) < \xi_{i+1}(\alpha)$ for every $\alpha < \omega_1$ and $i = 1, \cdots, k-1$.

By a counting argument, we can assume that there are natural numbers $n_1, \cdots, n_k$ such that $n^{p_\alpha}_{\xi_i(\alpha)} = n_i$ for every $\alpha < \omega_1$ and $i = 1, \cdots, k$. For $\alpha < \beta$, define $\sigma_{\beta,\alpha} : a_{p_\alpha} \to a_{p_\beta}$ by $\sigma_{\beta,\alpha}(\xi_i(\alpha)) = \xi_i(\beta)$.

Fix $i = 1, \cdots, k$ and $u, v \in [0, n_i)$. Then for each $\alpha < \omega_1$, there is a matrix $(q^\alpha_{(j,l),(m,n)})$ with entries in $\mathcal{D}$ for[3] $j, m = 1, \cdots, i$ and $l, n \in \bigcup_{s=1,\cdots,i}[0, n_s)$ such that

$$q^\alpha_{(j,l),(m,n)} = \langle A^{p_\alpha}_{\xi_i(\alpha),u,v}(e_{\xi_j(\alpha),l}), e_{\xi_m(\alpha),n}\rangle.$$

As $\mathcal{D}$ is countable, we can assume that for every $i = 1, \cdots, k$ and $u, v \in [0, n_i)$, there is a matrix $(q_{(j,l),(m,n)})$ with entries in $\mathcal{D}$ such that for every $\alpha < \omega_1$

$$(q^\alpha_{(j,l),(m,n)}) = (q_{(j,l),(m,n)}).$$

In particular, for every $\alpha < \beta$ $(q^\alpha_{(j,l),(m,n)}) = (q^\beta_{(j,l),(m,n)})$, i.e.,

$$\langle A^{p_\alpha}_{\xi_i(\alpha),u,v}(e_{\xi_j(\alpha),l}), e_{\xi_m(\alpha),n}\rangle = \langle A^{p_\beta}_{\xi_i(\beta),u,v}(e_{\xi_j(\beta),l}), e_{\xi_m(\beta),n}\rangle =$$

$$= \langle A^{p_\beta}_{\sigma_{\beta,\alpha}(\xi_i(\alpha)),u,v}(e_{\sigma_{\beta,\alpha}(\xi_j(\alpha)),l}), e_{\sigma_{\beta,\alpha}(\xi_m(\alpha)),n}\rangle.$$

Then for every $\alpha < \beta$

$$\langle A^{p_\alpha}_{\eta,u,v}(e_{\xi,l}), e_{\xi,n}\rangle = \langle A_{\sigma_{\beta,\alpha}(\eta),u,v}(e_{\sigma_{\beta,\alpha}(\xi),l}), e_{\sigma_{\beta,\alpha}(\xi),n}\rangle.$$

This shows that $(p_\alpha, \alpha \in \Gamma)$ is a family in the convenient position. □

Lemma 3.16. $\mathbb{P}_{\omega_1}(\mathcal{D})$ *has the c.c.c.*

Proof. Consider an uncountable family $(p_\alpha : \alpha < \omega_1)$ of elements in $\mathbb{P}_{\omega_1}(\mathcal{D})$. By Lemma 3.15, there are $\alpha < \beta < \omega_1$ such that p_α and p_β are in the convenient position. Then use Lemma 3.14 to get an element $r \in \mathbb{P}_{\omega_1}(\mathcal{D})$ such that $r \leq p_\alpha, p_\beta$. This shows that p_α and p_β are compatible and therefore, $(p_\alpha : \alpha < \omega_1)$ is not an antichain. □

[3] $\mathcal{D}$ is the countable complex field in Definition 3.2

Chapter 4

A scattered C*-algebra without uncountable irredundant sets

In this chapter, we prove our main result in this thesis:

Theorem 4.1. *It is consistent that there is a nonseparable scattered fully noncommutative C*-algebra $\mathcal{A}$ with the following properties:*

1. *There is a directed family of finite-dimensional algebras whose union $\mathcal{B}$ is dense in $\mathcal{A}$ such that whenever $(P_\xi : \xi < \omega_1) \subseteq \mathcal{B}$ is a family of projections which generate a nonseparable subalgebra of $\mathcal{A}$, then for every $\varepsilon > 0$*
   - *(a) there are $\xi_1 < \xi_2 < \xi_3 < \omega_1$ such that $\|P_{\xi_1} - P_{\xi_2}P_{\xi_3}\| < \varepsilon$,*
   - *(b) there are $\xi_1 < \xi_2 < \omega_1$ such that $\|[P_{\xi_1}, P_{\xi_2}]\| < \varepsilon$,*
   - *(c) there are $\xi_1 < \xi_2 < \omega_1$ such that $\|[P_{\xi_1}, P_{\xi_2}]\| > 1/2 - \varepsilon$.*
2. *$\mathcal{A}$ has no uncountable almost irredundant set of projections (and therefore, no uncountable irredundant sets).*
3. *$\mathcal{A}$ has no nonseparable abelian subalgebra.*

The proof is based on the forcing notion defined in Chapter 3. Throughout this chapter, M is a fixed countable transitive model of ZFC (i.e., some finite fragment of ZFC) and $\mathcal{D} \in M$ a countable field of complex numbers as in Definition 3.1. Define in M the partial order $\mathbb{P} := \mathbb{P}_{\omega_1}(\mathcal{D})^M$, i.e., the partial order $\mathbb{P}_{\omega_1}(\mathcal{D})$ relativized[1] to M . We say that $\mathbb{G}$ is $\mathbb{P}$-generic over M (or simply that $\mathbb{G}$ is a generic filter in $\mathbb{P}$) if and only if $\mathbb{G}$ is a filter on $\mathbb{P}$ and $\mathbb{G} \cap D \neq \emptyset$ for every dense set $D \subseteq \mathbb{P}$ such that $D \in M$.

Fix a generic filter $\mathbb{G}$ in $\mathbb{P}$. We will prove that there is a C*-algebra $\mathcal{A}^{\mathbb{G}}$ in $M[\mathbb{G}]$ satisfying all the conditions in Theorem 4.1.

In Section 4.2, we prove that there is a thin-tall fully noncommutative scattered C*-algebra $\mathcal{A}^{\mathbb{G}}$ in $M[\mathbb{G}]$. Section 4.3 is devoted to prove some technical lemmas that will be used to prove 1(a) and 1(c) and, finally, in Section 4.4 we prove Theorem 4.1.

We conclude this chapter with Section 4.5, where we summarize some of the independence results obtained in this thesis and we prove a new result about commutators in $\mathcal{B}(\ell_2)$.

[1] Formally, in Chapter 3 we defined a family of partial orders $\mathbb{P}_{\omega_1} = \{\mathbb{P}_{\omega_1}(D) : D \text{ is a countable field as in Definition 3.1}\}$. Our partial order $\mathbb{P}$ is the partial order in the family $\mathbb{P}_{\omega_1}^M$ which is indexed by the countable field $\mathcal{D}$.

4.1 $\mathbb{P}$ and $\overline{\mathbb{P}}$

In this section, we consider a partial order $\overline{\mathbb{P}}$ in $M[\mathbb{G}]$ such that $\mathbb{P}$ is a sub-order of $\overline{\mathbb{P}}$. This will help us to translate properties in $\mathbb{P}$ (in M) to properties in $\overline{\mathbb{P}}$ (in $M[\mathbb{G}]$).

By Proposition 3.16, $\mathbb{P}$ is c.c.c and therefore, it preserves cardinalities. In particular[2], $\omega_1^M = \omega_1^{M[\mathbb{G}]}$. Observe that, the countable field $\mathcal{D}$ of $\mathbb{C}$ in the ground model M remains as a countable field of $\mathbb{C}$ in $M[\mathbb{G}]$ and still satisfies all the conditions in Definition 3.1 (1). This allows us to make the following definition:

Definition 4.2. *In $M[\mathbb{G}]$, define the partial order $\overline{\mathbb{P}} := \mathbb{P}_{\omega_1}(\mathcal{D})^{M[\mathbb{G}]}$.*

Observe that the partial order $\overline{\mathbb{P}}$ is defined using the same countable field $\mathcal{D}$ of $\mathbb{C}$ in the ground model M. This will allow us to construct (in $M[\mathbb{G}]$) an order-preserving map $p \to \overline{p}$ from $\mathbb{P}$ into $\overline{\mathbb{P}}$.

First, let us define a map from $\mathbb{P}$ into $\overline{\mathbb{P}}$:

Definition 4.3. *Fix $p = \big(a_p, \{n_\xi^p : \xi \in a_p\}, \{A^p_{\xi,m,n} : \xi \in a_p, n, m \in [0, n_\xi^p)\}\big)$ in $\mathbb{P}$. For every operator $A^p_{\xi,m,n}$ we associate an operator $\overline{A^p_{\xi,m,n}}$ in $M[\mathbb{G}]$ in the following way: In $M[\mathbb{G}]$ consider the canonical basis $\{e_{\alpha,n}^{M[\mathbb{G}]} : \alpha < \omega_1, n \in \mathbb{N}\}$ of $\ell_2(\omega_1 \times \mathbb{N})^{M[\mathbb{G}]}$ and in M, let $\{e^M_{\alpha,n} : \alpha < \omega_1, n \in \mathbb{N}\}$ the canonical basis of $\ell_2(\omega_1 \times \mathbb{N})^M$.*

Define in $M[\mathbb{G}]$ the operator $\overline{A^p_{\xi,m,n}}$ given by

$$\langle \overline{A^p_{\xi,m,n}}(e^{M[\mathbb{G}]}_{\alpha,m}), e^{M[\mathbb{G}]}_{\beta,n}\rangle = \langle A_{\xi,m,n}(e^M_{\alpha,m}), e^M_{\beta,n}\rangle \in \mathcal{D}$$

for every $\alpha, \beta \in \omega_1$ and $m, n \in \mathbb{N}$.

Essentially, the operators $\overline{A^p_{\xi,m,n}}$ and $A^p_{\xi,m,n}$ have the same matrices with entries in $\mathcal{D}$. But formally, $\overline{A^p_{\xi,m,n}}$ is an operator in $\mathcal{B}(\ell_2(\omega_1 \times \mathbb{N}))$ of $M[\mathbb{G}]$ while $A^p_{\xi,m,n}$ is an operator in $\mathcal{B}(\ell_2(\omega_1 \times \mathbb{N}))$ of M.

Define $\overline{p} := \big(a_p, \{n^p_\xi : \xi \in a_p\}, \{\overline{A^p_{\xi,m,n}} : \xi \in a_p, n, m \in [0, n^p_\xi)\}\big)$. Then $\overline{p}$ is an element in $\overline{\mathbb{P}}$.

In the same spirit, for every operator $A \in \mathcal{B}_{X_p}$ in M, we consider its extension $\overline{A} \in \mathcal{B}_{X_p}$ in $M[\mathbb{G}]$ as the operator such that $\overline{A}$ has the same matrix of A. Observe that A and $\overline{A}$ have the same norm, since the norm of these operators depend on the matrices which define them.

In $M[\mathbb{G}]$, for every $X \subseteq X_p = X_{\overline{p}}$, we will denote by $\overline{\mathcal{A}^p_X}$ the C*-algebra $\mathcal{A}^{\overline{p}}_X$ i.e., the C*-algebra generated by $\{\overline{A^p_{\xi,m,n}} : (\xi, n), (\xi, m) \in X\}$. In particular, we will use the notation $\overline{\mathcal{B}_{X_p}}$ for $\mathcal{B}_{X_{\overline{p}}}$.

Now, we prove that the map $p \to \overline{p}$ is an order-preserving map:

Definition 4.4. *Consider now $p, q \in \mathbb{P}$ such that $p \leq q$ and $i_{p,q}$ the *-embedding (in M) as in Definition 3.2 (c). We will define a *-embedding $\overline{i_{p,q}}$ in $M[\mathbb{G}]$ from $\overline{\mathcal{A}^q_{X_q}}$ into $\overline{\mathcal{A}^p_{X_p}}$ in the following way:*

First, consider for each $q \in \mathbb{P}$, the set $\mathbb{B}_q$ formed by all elements of the form $\Psi(\overline{A^q_{\xi,m,n}} : (\xi, m), (\xi, n) \in X)$ where Ψ is a C- polynomial[3] (with rational coefficients) and $X \subseteq X_q$. Observe that $\Psi(\overline{A^q_{\xi,m,n}} : (\xi, m), (\xi, n) \in X) = \overline{\Psi(A^q_{\xi,m,n} : (\xi, m), (\xi, n) \in X)}$ for each C*-polynomial Ψ with rational coefficients. In particular*

$$\|\Psi(\overline{A^q_{\xi,m,n}} : (\xi, m), (\xi, n) \in X)\| = \|\Psi(A^q_{\xi,m,n} : (\xi, m), (\xi, n) \in X)\|.$$

[2]Whenever it is convenient, we will also drop the superscript and we simply write $\alpha = \alpha^M = \alpha^{M[\mathbb{G}]}$.

[3]By a C*-polynomial Ψ we mean a expression $\Psi(x_1, \cdots, x_n)$ which is a linear combination of elements of the form $\prod_{i=1}^j y_i$ where $y_i \in \{x_1, \cdots x_n, x_1^*, \cdots, x_n^*\}$.

Now, define a map $(\overline{i_{p,q}})_0 : \mathbb{B}_q \to \overline{\mathcal{A}^p_{X_p}}$ *by*

$$(\overline{i_{p,q}})_0(\Psi(\overline{A^q_{\xi,m,n}} : (\xi,m),(\xi,n) \in X))) = \overline{i_{p,q}(\Psi(A^q_{\xi,m,n} : (\xi,m),(\xi,n) \in X)))}$$

Then $(\overline{i_{p,q}})_0$ *is a map which preserves the* C^**-operations and is defined in a dense set of* $\overline{\mathcal{A}^q_{X_q}}$. *Also, it preserves the norm (because* $(\overline{i_{p,q}})_0$ *is a map which sends an operator* A *to an operator* $(\overline{i_{p,q}})_0(A)$ *which has the same matrix of* A*). In particular we can extend* $(\overline{i_{p,q}})_0$ *to a* $*$*-homomorphism* $\overline{i_{p,q}}$ *from* $\overline{\mathcal{A}^q_{X_q}}$ *into* $\overline{\mathcal{A}^p_{X_p}}$. *Moreover,* $\overline{i_{p,q}}$ *is a* $*$*-embedding. In fact, suppose there is a* $0 \neq a \in \overline{\mathcal{A}^q_{X_q}}$ *such that* $\overline{i_{p,q}}(a) = 0$. *Then there is a Cauchy sequence* $(\overline{a_n})_n \in \mathbb{B}_q$ *which converges to* a. *But then we would have a Cauchy sequence* $(a_n)_n$ *in* $\mathcal{A}^q_{X_q}$ *converging to an element* $0 \neq b$ *such that* $\overline{i_{p,q}(b)} = \overline{i_{p,q}}(a) = 0$, *and therefore* $i_{p,q}(b) = 0$, *which is a contradiction with the fact that* $i_{p,q}$ *is a* $*$*-embedding. This concludes the proof that* $\overline{i_{p,q}}$ *is a* $*$*-embedding. Moreover,* $\overline{i_{p,q}}$ *witnesses the fact that* $\overline{p} \leq \overline{q}$ *in* $\overline{\mathbb{P}}$.

Also, if p *and* q *are in the convenient position in* $\mathbb{P}$ *and* j_σ *is the* $*$*-isomorphism of* $\mathcal{B}_{X_q}$ *onto* $\mathcal{B}_{X_p}$ *induced by the order preserving bijection* $\sigma : a_p \to a_q$, *then* $\overline{p}$ *and* $\overline{q}$ *are in the convenient position in* $\overline{P}$ *witnessed by the same order preserving bijection* $\sigma : a_p \to a_q$ *and by the* $*$*-isomorphism* $\overline{j_\sigma}$ *in* $M[\mathbb{G}]$ *induced by* $\sigma : a_p \to a_q$.

We conclude this section with the following lemma which will be used in the proof of Theorem 4.1.

Lemma 4.5. *In* M*, suppose that* p_1, p_2, p_3 *are distinct elements in* $\mathbb{P}$ *which are pairwise in the convenient position as witnessed by* $\sigma_{j,i} : a_{p_i} \to a_{p_j}$ *for* $1 \leq i < j \leq 3$ *and* $r \leq p_1, p_2, p_3$.

1. *Suppose that*

$$i_{r,p_3}(A) i_{r,p_2}(j_{\sigma_{3,2}}(A)) = i_{r,p_1}(j_{\sigma_{3,1}}(A))^2$$

for every $A \in \mathcal{B}_{X_{p_3}}$.

Then in $M[\mathbb{G}]$ *we have that*

$$\overline{i_{r,p_3}}(A)\overline{i_{r,p_2}}(\overline{j_{\sigma_{3,2}}}(A)) = \overline{i_{r,p_1}}(\overline{j_{\sigma_{3,1}}}(A))^2$$

for every $A \in \overline{\mathcal{B}_{X_{p_3}}}$.

2. *In* M*, suppose that*

$$\|[i_{rp_1}(A), i_{rp_2}(j_{\sigma_{2,1}}(A))]\| = 1/2$$

for every nonexpanding $A \in \mathcal{B}_{X_{p_1}}$ *such that there is* $\xi \in a_{p_2} \backslash a_{p_1}$ *with* $A(\sum_{k<n} v^1_k e^M_{\xi,k}) = \sum_{k<n} v^1_k e^M_{\xi,k}$ *and* $A(\sum_{k<n} v^2_k e^M_{\xi,k}) = 0$ *where* $(v^1_1, \cdots, v^1_{n-1}), (v^2_1, \cdots, v^2_{n-1})$ *are two orthogonal vectors with rational coordinates. Then in* $M[\mathbb{G}]$ *we have that*

$$\|[\overline{i_{rp_1}}(A), \overline{i_{rp_2}}(j_{\sigma_{2,1}}(A))]\| = 1/2$$

for every nonexpanding $A \in \overline{\mathcal{B}_{X_{p_1}}}$ *such that* A *has a matrix with rational entries and such that there is* $\xi \in a_{p_2} \backslash a_{p_1}$ *with* $A(\sum_{k<n} v^1_k e^{M[\mathbb{G}]}_{\xi,k}) = \sum_{k<n} v^1_k e^{M[\mathbb{G}]}_{\xi,k}$ *and* $A(\sum_{k<n} v^2_k e^{M[\mathbb{G}]}_{\xi,k}) = 0$.

Proof. 1. First, observe that the result is true for every element $\overline{B} \in \mathbb{B}_{p_3}$. In fact,

$$\overline{i_{r,p_3}}(\overline{B})\overline{i_{r,p_2}}(\overline{j_{\sigma_{3,2}}}(\overline{B})) = \overline{i_{r,p_3}(B) i_{r,p_2}(j_{\sigma_{3,2}}(B))} =$$

$$= \overline{i_{r,p_1}(j_{\sigma_{3,1}}(B))^2} = \overline{i_{r,p_1}}(\overline{j_{\sigma_{3,1}}}(\overline{B}))^2.$$

To conclude, we observe that $\mathbb{B}_{p_3}$ is a dense set in $\overline{\mathcal{B}_{X_{p_3}}}$.

2. The proof follows from the fact that if $A \in \overline{\mathcal{B}_{X_{p_1}}}$ has a matrix with rational entries, then there is $B \in \mathcal{B}_{X_{p_1}}$ such that $\overline{B} = A$.

□

4.2 A generic scattered C*-algebra

Definition 4.6. *In $M[\mathbb{G}]$, define $A^{\mathbb{G}}_{\xi,n,m} \in \mathcal{B}_{\omega_1 \times \mathbb{N}}$ by*

$$\langle A^{\mathbb{G}}_{\xi,n,m}(e^{M[\mathbb{G}]}_{\eta,k}), e^{M[\mathbb{G}]}_{\eta,l}\rangle = \langle \overline{A^{p}_{\xi,n,m}}(e^{M[\mathbb{G}]}_{\eta,k}), e^{M[\mathbb{G}]}_{\eta,l}\rangle$$

for any (all) $p \in \mathbb{G}$ such $(\eta,k),(\eta,l),(\xi,n),(\xi,m) \in X_p$.

Observe that $A^{\mathbb{G}}_{\xi,n,m}$ is well defined. In fact, suppose $p_1, p_2 \in \mathbb{G}$ are such that $(\eta,k),(\eta,l),(\xi,n),(\xi,m) \in X_{p_1} \cap X_{p_2}$. Since $\mathbb{G}$ is a filter, there is $q \in \mathbb{G}$ such that $q \leq p_1, p_2$. By (c) and (d) of Definition 3.2, we have that $A^{p_i}_{\xi,n,m} = A^{q}_{\xi,n,m} \restriction X_{p_i}$ for $i = 1, 2$. In particular, $A^{p_1}_{\xi,n,m} \restriction X_{p_1} \cap X_{p_2} = A^{p_2}_{\xi,n,m} \restriction X_{p_1} \cap X_{p_2}$ and therefore

$$\langle A^{p_1}_{\xi,n,m}(e^{M}_{\eta,n}), e^{M}_{\eta,m}\rangle = \langle A^{p_2}_{\xi,n,m}(e^{M}_{\eta,n}), e^{M}_{\eta,m}\rangle$$

for every $(\eta,k),(\eta,l),(\xi,n),(\xi,m) \in X_{p_1} \cap X_{p_2}$. Then

$$\langle A^{\mathbb{G}}_{\xi,n,m}(e^{M[\mathbb{G}]}_{\eta,k}), e^{M[\mathbb{G}]}_{\eta,l}\rangle = \langle \overline{A^{p_1}_{\xi,n,m}}(e^{M[\mathbb{G}]}_{\eta,n}), e^{M[\mathbb{G}]}_{\eta,m}\rangle = \langle \overline{A^{p_2}_{\xi,n,m}}(e^{M[\mathbb{G}]}_{\eta,n}), e^{M[\mathbb{G}]}_{\eta,m}\rangle.$$

for every $(\eta,k),(\eta,l),(\xi,n),(\xi,m) \in X_{p_1} \cap X_{p_2}$.

By abuse of notation, we will drop the superscript $e^{M}_{\alpha,n}, e^{M[\mathbb{G}]}_{\alpha,n}$ and simply write $e_{\alpha,n}$ when no confusion may arise, i.e., when the model (M or $M[\mathbb{G}]$) in which we are working is clear.

Definition 4.7. *In $M[\mathbb{G}]$, define $\mathcal{A}^{\mathbb{G}}$ as the subalgebra of $\mathcal{B}_{\omega_1 \times \mathbb{N}}$ generated by the operators $A^{\mathbb{G}}_{\xi,m,n}$ for all $\xi \in \omega_1$ and $m, n \in \mathbb{N}$. Let X be a subset of $\omega_1 \times \mathbb{N}$. We define $\mathcal{A}^{\mathbb{G}}_X$ to be the C^*-subalgebra of $\mathcal{A}^{\mathbb{G}}$ generated by $(A^{\mathbb{G}}_{\xi,m,n} : (\xi,n),(\xi,m) \in X)$. In particular, for every $\alpha < \omega_1$, by $\mathcal{A}^{\mathbb{G}}_\alpha$ we mean the C^*-subalgebra of $\mathcal{A}^{\mathbb{G}}$ generated by $\{A^{\mathbb{G}}_{\xi,m,n} : \xi < \alpha, m, n \in \mathbb{N}\}$.*

Lemma 4.8. *Let p be an element in $\mathbb{G}$. Then in $M[\mathbb{G}]$ there is a *-embedding $i_{\mathbb{G},p} : \overline{\mathcal{A}^{p}_{X_p}} \to \mathcal{A}^{\mathbb{G}}_{X_p}$ such that*

1. *$i_{\mathbb{G},p}(\overline{A^{p}_{\xi,m,n}}) = A^{\mathbb{G}}_{\xi,m,n}$ and*
2. *$i_{\mathbb{G},p}(\overline{A^{p}_{\xi,m,n}}) \restriction X_p = \overline{A^{p}_{\xi,m,n}}$*

for every ξ, n, m such that $(\xi,n),(\xi,m) \in X$.

Proof. Consider i_0 defined by

$$i_0(\Psi(\overline{A^{p}_{\xi,m,n}} : (\xi,m),(\xi,n) \in X)) = \Psi(A^{\mathbb{G}}_{\xi,m,n} : (\xi,m),(\xi,n) \in X)$$

for every C*-polynomial Ψ and for every $X \subseteq X_p$. Since for every $q \leq p$ there is a *-embedding $\overline{i_{q,p}}$ such that $\overline{i_{q,p}}(\overline{A^{p}_{\xi,m,n}}) = \overline{A^{q}_{\xi,m,n}}$ for every $\xi \in a_p$, we conclude that

$$\|i_0(\Psi(\overline{A^{p}_{\xi,m,n}} : (\xi,m),(\xi,n) \in X))\| = \|\Psi(\overline{A^{p}_{\xi,m,n}} : (\xi,m),(\xi,n) \in X)\|.$$

In particular, i_0 can be extended to a *-homomorphism $i_{\mathbb{G},p} : \overline{\mathcal{A}^p_{X_p}} \to \mathcal{A}^{\mathbb{G}}_{X_p}$. Its kernel must be null as the kernels of $\overline{i_{q,p}}$ for $q \leq p$ are null.

To prove the second part of the lemma, use the first part and Definition 4.6. □

Consider now $\mathcal{A} = lim_{p\in\mathbb{G}}\overline{\mathcal{A}^p_{X_p}}$ as the inductive limit of the inductive system $(\overline{\mathcal{A}^p_{X_p}}, \{\overline{i_{q,p}}, q \leq p\})_{p\in\mathbb{G}}$, together with the maps $\overline{i_p} : \overline{\mathcal{A}^p_{X_p}} \to \mathcal{A}$ for $p \in \mathbb{G}$ as in Theorem 1.44.

Lemma 4.9. *In $M[\mathbb{G}]$ there is a *-isomorphism j from $lim_{p\in\mathbb{G}}\overline{\mathcal{A}^p_{X_p}}$ onto $\mathcal{A}^{\mathbb{G}}$ such that*

$$j \circ \overline{i_p} = i_{\mathbb{G},p}$$

for every $p \in \mathbb{G}$.

Proof. It is enough to prove (2) of Theorem 1.45, i.e., we should prove that $i_{\mathbb{G},q} = i_{\mathbb{G},p}\overline{i_{p,q}}$ for every $p, q \in \mathbb{G}$ such that $p \leq q$. This follows from the fact that by Definition 4.6 we have $i_{\mathbb{G},p}(\overline{i_{p,q}}(\overline{A^q_{\xi,n,m}})) = A^{\mathbb{G}}_{\xi,n,m} = i_{\mathbb{G},q}(\overline{A^q_{\xi,n,m}})$ for ξ, m, n such that $(\xi, m), (\xi, n) \in X_q$. But these elements generate $\overline{\mathcal{A}^q_{X_q}}$. □

Lemma 4.10. *Let p be an element in $\mathbb{G}$. In $M[\mathbb{G}]$, if $B \in \overline{\mathcal{A}^p_{X_p}}$, then for every $\alpha < \omega_1$*

$$\|i_{\mathbb{G},p}(B) \restriction [\alpha, \omega_1)\| = \|i_{\mathbb{G},p}(B \restriction [\alpha, \omega_1))\| = \|B \restriction [\alpha, \omega_1)\|$$

Proof. Since $B \restriction \alpha \in \overline{\mathcal{A}^p_\alpha}$ we have that $i_{\mathbb{G},p}(B \restriction \alpha) \in \mathcal{A}^{\mathbb{G}}_\alpha$ and therefore $i_{\mathbb{G},p}(B \restriction \alpha) \restriction [\alpha, \omega_1) \equiv 0$. In particular

$$\|i_{\mathbb{G},p}(B) \restriction [\alpha, \omega_1)\| = \|i_{\mathbb{G},p}(B \restriction \alpha) \restriction [\alpha, \omega_1) + i_{\mathbb{G},p}(B \restriction [\alpha, \omega_1)) \restriction [\alpha, \omega_1)\| =$$

$$= \|i_{\mathbb{G},p}(B \restriction [\alpha, \omega_1)) \restriction [\alpha, \omega_1)\| \leq \|i_{\mathbb{G},p}(B \restriction [\alpha, \omega_1))\| \leq \|B \restriction [\alpha, \omega_1)\|.$$

On the other hand, since $i_{\mathbb{G},p}(B) \restriction X_p = B$, it follows that

$$\|B \restriction [\alpha, \omega_1)\| \leq \|i_{\mathbb{G},p}(B) \restriction [\alpha, \omega_1)\|$$

□

Lemma 4.11. *In $M[\mathbb{G}]$, for every $\alpha < \omega_1$ the following hold:*

1. *$\mathcal{A}^{\mathbb{G}}_\alpha$ is equal to $\{A \in \mathcal{A}^{\mathbb{G}} : A \restriction [\alpha, \omega_1) = 0\}$ (and therefore, an ideal of $\mathcal{A}^{\mathbb{G}}$),*
2. *there is a *-isomorphism $j_\alpha : \mathcal{A}^{\mathbb{G}}/\mathcal{A}^{\mathbb{G}}_\alpha \to \mathcal{A}^{\mathbb{G}} \restriction [\alpha, \omega_1)$,*
3. *the representation $\pi_\alpha : \mathcal{A}^{\mathbb{G}} \restriction [\alpha, \omega_1) \to \mathcal{A}^{\mathbb{G}} \restriction \{\alpha\}$ given by $\pi_\alpha(A) = A \restriction \{\alpha\}$ is faithful.*

Proof. As $\ell_2(\{\xi\} \times \mathbb{N})$ is $\mathcal{A}^{\mathbb{G}}$-invariant for every $\xi < \omega_1$, it is clear that the map sending $A \in \mathcal{A}^{\mathbb{G}}$ to $A \restriction [\alpha, \omega_1)$ is a *-homomorphism. So for (1) and (2) we are left with proving that its kernel is equal to $\mathcal{A}^{\mathbb{G}}_\alpha$.

First note that the kernel contains every generator $A^{\mathbb{G}}_{\xi,n,m}$ for $\xi < \alpha$ and $m, n \in \mathbb{N}$ of $\mathcal{A}^{\mathbb{G}}_\alpha$ and so includes $\mathcal{A}^{\mathbb{G}}_\alpha$. This is true by Definition 3.2 (4).

For the other inclusion let $A \in \mathcal{A}^{\mathbb{G}}$ be such that $A \restriction [\alpha, \omega_1) = 0$. Since $\mathcal{A}^{\mathbb{G}}$ is the inductive limit of $\overline{\mathcal{A}^p_{X_p}}$s for $p \in \mathbb{G}$ by Lemma 4.9, for every $\varepsilon > 0$ there is $p \in \mathbb{G}$ and $B \in \overline{\mathcal{A}^p_{X_p}}$ such that $\|i_{\mathbb{G},p}(B) - A\| < \varepsilon$ and so $\|i_{\mathbb{G},p}(B) \restriction [\alpha, \omega_1)\| < \varepsilon$. By Lemma 3.4, $B \restriction \alpha \in \overline{\mathcal{A}^p_{X_p \cap (\alpha \times \mathbb{N})}} \subseteq \overline{\mathcal{A}^p_{X_p}}$ and $B \restriction [\alpha, \omega_1) \in \overline{\mathcal{A}^p_{X_p}}$, so we can apply $i_{\mathbb{G},p}$ to them. By Lemma 4.10 we have that $\|i_{\mathbb{G},p}(B) \restriction [\alpha, \omega_1)\| = \|i_{\mathbb{G},p}(B \restriction [\alpha, \omega_1))\|$.

So we have

$$\|A - i_{\mathbb{G},p}(B \restriction \alpha)\| = \|A - i_{\mathbb{G},p}(B) + i_{\mathbb{G},p}(B \restriction [\alpha, \omega_1))\| \leq$$

$$\leq \|A - i_{\mathbb{G},p}(B)\| + \|i_{\mathbb{G},p}(B) \restriction [\alpha, \omega_1)\| \leq 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary and $i_{\mathbb{G},p}(B \restriction \alpha) \in \mathcal{A}^{\mathbb{G}}_\alpha$, we conclude that $A \in \mathcal{A}^{\mathbb{G}}_\alpha$, completing the proof of (1) and (2).

To prove (3) first note that since $\ell_2(\{\alpha\} \times \omega_1)$ is $\mathcal{A}^{\mathbb{G}}$-invariant, it is clear that π_α is a representation of $\mathcal{A}^{\mathbb{G}} \restriction [\alpha, \omega_1)$. Now suppose that $A \in \mathcal{A}^{\mathbb{G}}_{X_q}$ for $q \in \mathbb{G}$. By Lemma 4.8 there is $B \in \overline{\mathcal{A}^q_{X_q}}$ such that $i_{\mathbb{G},q}(B) = A$. Since $\mathbb{G}$ is a generic filter in $\mathbb{P}$, by Lemmas 3.9 and 3.10 there is $p \in \mathbb{F}_{X_q,\alpha} \cap \mathbb{G}$ such that $p \leq q$. By Lemma 3.6 and Definition 4.6 we have

$$\|A \restriction [\alpha, \omega_1)\| = \|i_{\mathbb{G},q}(B) \restriction [\alpha, \omega_1)\| = \|B \restriction [\alpha, \omega_1)\| = \|\overline{i_{p,q}}(B) \restriction [\alpha, \omega_1)\|.$$

By the fact that $p \in \mathbb{F}_{X_q,\alpha}$ we have that

$$\|A \restriction \{\alpha\}\| \geq \|\overline{i_{p,q}}(B) \restriction \{\alpha\}\| \geq \|\overline{i_{p,q}}(B) \restriction [\alpha, \omega_1)\| = \|A \restriction [\alpha, \omega_1)\|.$$

This shows that π_α is an isometry when restricted to $\bigcup_{q \in \mathbb{G}} \mathcal{A}^{\mathbb{G}}_{X_q} \restriction [\alpha, \omega_1)$ which is dense in $\mathcal{A}^{\mathbb{G}} \restriction [\alpha, \omega_1)$ by Lemma 4.9, and so the representation is faithful. □

Proposition 4.12. *$\mathcal{A}^{\mathbb{G}}$ is a scattered thin-tall fully noncommutative C*-algebra in $M[\mathbb{G}]$ such that*

1. *$\mathcal{I}^{At}_\alpha(\mathcal{A}^{\mathbb{G}}) = \mathcal{A}^{\mathbb{G}}_\alpha$,*
2. *there is a *-isomorphism $j_\alpha : \mathcal{A}^{\mathbb{G}}/\mathcal{I}^{At}_\alpha(\mathcal{A}^{\mathbb{G}}) \to \mathcal{A}^{\mathbb{G}} \restriction [\alpha, \omega_1)$ satisfying*

$$j_\alpha([A]_{\mathcal{I}^{At}_\alpha(\mathcal{A})}) = A \restriction [\alpha, \omega_1),$$

3. *the collection $\{[A_{\alpha,m,n}]_{\mathcal{I}^{At}_\alpha(\mathcal{A})} : n, m \in \mathbb{N}\}$ satisfies the matrix unit relations and generates the essential ideal $At(\mathcal{A}^{\mathbb{G}}/\mathcal{I}^{At}_\alpha(\mathcal{A}^{\mathbb{G}}))$.*

Proof. By Theorem 1.4 of [16] it is enough to prove (1) - (3) to conclude that $\mathcal{A}$ is a scattered thin-tall fully noncommutative C*-algebra.

The proof of (1) - (3) is by induction on $\alpha < \omega_1$. For $\alpha = 0$ we have that $\mathcal{I}_\alpha = \{0\}$ and so (1) and (2) are trivial. Also $A^{\mathbb{G}}_{0,n,m} = 1_{0,n,m}$ by Definition 3.2, so these elements satisfy the matrix unit relations. Moreover they generate the algebra of all compact operators on $\ell_2(\{0\} \times \mathbb{N})$ which is an essential ideal in $\mathcal{B}_{\{0\}\times\mathbb{N}}$. Since π_0 from Lemma 4.11 is faithfull, the collection $\{A_{0,m,n} : n, m \in \mathbb{N}\}$ generates an essential ideal *-isomorphic to the algebra of all compact operators on a Hilbert space, so by Theorem 1.2 (4) of [16] this ideal is $\mathcal{I}^{At}(A^{\mathbb{G}})$ as required.

Now suppose we are done for $\beta < \alpha < \omega_1$.

(1) If α is a limit ordinal, then by (2) of Theorem 1.55 and the inductive hypothesis we have

$$\mathcal{I}^{At}_\alpha(\mathcal{A}) = \overline{\bigcup_{\beta<\alpha} \mathcal{I}^{At}_\beta(\mathcal{A})} = \overline{\bigcup_{\beta<\alpha} \mathcal{A}_\beta} = \mathcal{A}_\alpha.$$

If $\alpha = \beta + 1$, then (3) of the inductive hypothesis implies (1).

(2) follows from Lemma 4.11.

(3) analogously to the case $\alpha = 0$. □

4.3 Special amalgamation

In this section we work in the ground model M to prove some technical lemmas which will be used in the next section.

Lemma 4.13. *Suppose that p, q are two elements of $\mathbb{P}$ in the convenient position as witnessed by $\sigma_{q,p} : a_p \to a_q$. Let $U \in \mathcal{B}_{X_p \cup X_q}$ be such that U has a matrix with entries in $\mathcal{D}$ and such that U is a partial isometry satisfying $UU^* = U^*U = P_{X_p \setminus X_q}$, where $P_{X_p \setminus X_q}$ is the projection on the space spanned by $\{e_{\xi,k} : (\xi, k) \in X_p \setminus X_q\}$. Then there is $r_U = r \leq p, q$ such that*

- $a_r = a_p \cup a_q$,
- $n^r_\xi = n^p_\xi$ *if* $\xi \in a_p$, $n^r_\xi = n^q_\xi$ *if* $\xi \in a_q$,
- $i_{r,p} = Id_{\mathcal{B}_{X_p}}$,
- $i_{r,q}(A) = A + Uj_{\sigma_{q,p}}(A)U^*$ *for all* $A \in \mathcal{B}_{X_q}$,

in particular,

- $A^r_{\xi,m,n} = A^p_{\xi,m,n}$ *for* $\xi \in a_p$ *and* $m, n \in [0, n^p_\xi)$,
- $A^r_{\xi,m,n} = A^q_{\xi,m,n} + UA^p_{\sigma^{-1}_{q,p}(\xi),m,n}U^*$ *for* $\xi \in a_q \setminus a_p$ *and* $m, n \in [0, n^q_\xi)$.

The element r_U will be called the U-including amalgamation of p and q; if $U = P_{X_p \setminus X_q}$, then r_U is called the including amalgamation.

Proof. Define r_U as in the lemma. It is clear by Definition 3.2 applied to p and q that $r \in \mathbb{P}_{\omega_1}$. $r_U \leq p$ because $Id_{\mathcal{B}_{X_p}} : \mathcal{B}_{X_p} \to \mathcal{B}_{X_r}$ is a *-embedding. For $r_U \leq q$ we note that $A^r_{\xi,m,n} \upharpoonright X_q = A^q_{\xi,m,n}$ as $\left(UA^p_{\sigma^{-1}_{q,p}(\xi),m,n}U^*\right) \upharpoonright X_q = 0$ since $UU^* = U^*U = P_{X_p \setminus X_q}$ and that the formula $i_{r,q}(A) = A + Uj_\sigma(A)U^*$ for all $A \in \mathcal{B}_{X_q}$ defines a *-embedding from $\mathcal{B}_{X_q}$ to $\mathcal{B}_{X_r}$. This follows from the fact that sending A to $Uj_\sigma(A)U^*$ is a *-homomorphism since $\mathcal{B}_{X_p \setminus X_q}$ is $A^p_{X_p}$-invariant, so $i_{r,q}$ is a *-homomorphism. But its kernel is null since $Uj_\sigma(A)U^* = (Uj_\sigma(A)U^*) \upharpoonright (X_p \setminus X_q)$ for all $A \in \mathcal{B}_{X_q}$. □

The next lemma will be used in the proof of 1 (a) from Theorem 4.1.

Lemma 4.14. *Suppose that p_1, p_2, p_3 are distinct elements in $\mathbb{P}$ which are pairwise in the convenient position as witnessed by $\sigma_{j,i} : a_{p_i} \to a_{p_j}$ for $1 \leq i < j \leq 3$ such that $\Delta < a_{p_1} \setminus \Delta < a_{p_2} \setminus \Delta < a_{p_3} \setminus \Delta$. Then there is $r \leq p_1, p_2, p_3$ satisfying*

- $a_r = a_{p_1} \cup a_{p_2} \cup a_{p_3}$,
- $n^r_\xi = n^{p_i}_\xi$ *if* $\xi \in a_{p_i}$ *for* $1 \leq i \leq 3$,
- $i_{r,p_1} = Id_{\mathcal{B}_{X_{p_1}}}$,
- $i_{r,p_2}(A) = A + j_{\sigma_{2,1}}(A) \upharpoonright (X_{p_1} \setminus X_{p_2})$ *for all* $A \in \mathcal{B}_{X_{p_2}}$,
- $i_{r,p_3}(A) = A + j_{\sigma_{3,1}}(A) \upharpoonright (X_{p_1} \setminus X_{p_3})$ *for all* $A \in \mathcal{B}_{X_{p_3}}$.

In particular

$$i_{r,p_3}(A)i_{r,p_2}(j_{\sigma_{3,2}}(A)) = i_{r,p_1}(j_{\sigma_{3,1}}(A))^2$$

for every $A \in \mathcal{B}_{X_{p_3}}$. The element r is called the amalgamation of p_1, p_2, p_3 of type 1.

Proof. First consider $s_2 \leq p_1, p_2$ and $s_3 \leq p_1, p_3$ which are the including amalgamations of p_1, p_2 and p_1, p_3 as in Lemma 4.13. It is clear that s_1 and s_2 are in the convenient position as witnessed by $Id_{a_{p_1}} \cup \sigma_{3,2} : a_{p_1} \cup a_{p_2} \to a_{p_1} \cup a_{p_3}$. Now let r be the disjoint amalgamation of s_1 and s_2 as in Lemma 3.14. The properties of r follow from Lemma 4.13 and Definition 3.2.

To prove the last statement of the lemma note that

$$\begin{aligned} i_{r,p_3}(A)i_{r,p_2}(j_{\sigma_{3,2}}(A)) &= \big(A + j_{\sigma_{3,1}}(A) \restriction (X_{p_1} \setminus X_{p_3})\big)\big(j_{\sigma_{3,2}}(A) + j_{\sigma_{3,1}}(A) \restriction (X_{p_1} \setminus X_{p_3})\big) \\ &= (j_{\sigma_{3,1}}(A))^2 \\ &= i_{r,p_1}(j_{\sigma_{3,1}}(A))^2 \end{aligned}$$

□

Now we prove some lemmas which will be used in the proof of 1 (c) from Theorem 4.1.

Lemma 4.15. *Given $n > 1$, suppose that v_1, v_2 are two orthogonal unit vectors of $\mathbb{C}^n$ with rational coordinates. Then there is a unitary $U \in M_n$ such that*

$$\|[UAU^*, A]\| = 1/2$$

for every nonexpanding[4] *linear $A \in M_n$ satisfying $A(v_1) = v_1$ and $A(v_2) = 0$.*

Proof. Choose an orthonormal basis $v_1, \dots v_n$ of $\mathbb{C}^n$ starting with v_1, v_2 and consider the orthogonal projection $P \in M_n$ onto the space spanned by v_1, so in particular we have $P(v_1) = v_1$ and $P(v_2) = 0$. Let $U = V \oplus I_{n-2}$, $U^* = V^* \oplus I_{n-2}$, where

$$V = V^* = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

So we obtain that

$$UPU^* = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \oplus 0_{n-2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \oplus 0_{n-2}.$$

Hence

$$[UPU^*, P] = UPU^*P - PUPU^* =$$
$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0_{n-2} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \oplus 0_{n-2} =$$
$$= \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} \oplus 0_{n-2} - \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \oplus 0_{n-2} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \oplus 0_{n-2}.$$

And so $\|[UPU^*, P]\| = 1/2$ and in particular

- $[UPU^*, P](v_1) = -(1/2)v_2$ and
- $[UPU^*, P](v_2) = (1/2)v_1$.

Since P equals A on the space spanned by v_1 and v_2, and U, U^* leave this space invariant, we have the same equalities for A instead of P, hence $\|[UAU^*, A]\| \geq 1/2$. The other inequality

[4]We say that $A \in M_n$ is nonexpanding if $\|Av\| \leq \|v\|$ for every $v \in \mathbb{C}^n$.

follows from the fact that $\|[B,C]\| \leq 1/2$ for any two B, C satisfying $0 \leq B, C \leq 1$ by a result of Stampfli (Corollary 2 of [36]). □

Lemma 4.16. *Suppose that p, q are two elements of $\mathbb{P}$ in the convenient position as witnessed by $\sigma : a_p \to a_q$ such that $\Delta < a_p \setminus \Delta < a_q \setminus \Delta$. Suppose that $n^q_\xi = n > 1$ for every $\xi \in a_q \setminus a_p$ and that $v_1 = (v^1_0, ...v^1_{n-1}), v_2 = (v^2_0, ...v^2_{n-1})$ are two orthogonal unit vectors of $\mathbb{C}^n$ with rational coordinates. Then there is $r \leq p, q$ such that*

- $a_r = a_p \cup a_q$,
- $n^r_\xi = n^p_\xi$ *if* $\xi \in a_p$, $n^r_\xi = n^q_\xi$ *if* $\xi \in a_q$,

and

- $\|[i_{rq}(A), i_{rp}(j_\sigma(A))]\| = 1/2$

for every nonexpanding $A \in \mathcal{B}_{X_q}$ such that there is $\xi \in a_q \setminus a_p$ with $A(\sum_{k<n} v^1_k e_{\xi,k}) = \sum_{k<n} v^1_k e_{\xi,k}$ and $A(\sum_{k<n} v^2_k e_{\xi,k}) = 0$. We call r the (v_1, v_2)-anticommuting amalgamation of p and q.

Proof. By Lemma 4.15 for each $\xi \in a_q \setminus a_p$ and each $\eta_\xi = \sigma^{-1}(\xi)$ there is a unitary $U_\xi \in \mathcal{B}_{\{\eta_\xi\}\times[0,n)}$ such that

$$(*) \qquad \|[U_\xi(j_\sigma(A) \restriction \{\eta_\xi\})U^*_\xi, j_\sigma(A) \restriction \{\eta_\xi\}]\| = 1/2$$

whenever $A \in \mathcal{B}_{X_q}$ is nonexpanding such that

$$A(\sum_{k<n} v^1_k e_{\xi,k}) = \sum_{k<n} v^1_k e_{\xi,k} \text{ and } A(\sum_{k<n} v^2_k e_{\xi,k}) = 0.$$

Let $U \in \mathcal{B}_{X_p \cup X_q}$ be a partial isometry such that $U \restriction (\{\eta_\xi\} \times [0,n)) = U_\xi$ and U is zero on the columns not in $X_p \setminus X_q$, and $UU^* = P_{X_p \setminus X_q}$. Consider the U-including amalgamation $r_U \leq p, q$ as in Lemma 4.13.

We claim that $r = r_U$ satisfies the lemma we are proving. Let $A \in \mathcal{B}_{X_q}$ be nonexpanding and $\xi \in a_q \setminus a_p$ be such that $A(\sum_{k<n} v^1_k e_{\xi,k}) = \sum_{k<n} v^1_k e_{\xi,k}$ and $A(\sum_{k<n} v^2_k e_{\xi,k}) = 0$. Since $\ell_2(\{\xi\} \times [0, n^q_\xi))$ for $\xi \in a_q \setminus a_p$ are invariant for $\mathcal{B}_{X_q}$ the operator $A \restriction \{\xi\}$ is nonexpanding as well and so is $j_\sigma(A) \restriction \{\eta_\xi\}$. By Lemma 4.13 we have $i_{r,q}(A) = A + Uj_\sigma(A)U^*$ and $i_{r,p}(j_\sigma(A)) = j_\sigma(A)$, so for $\eta_\xi = \sigma^{-1}(\xi)$ we have

$$[i_{r,q}(A), i_{r,p}(j_\sigma(A))] \restriction (\{\eta_\xi\} \times [0,n)) = [U_\xi(j_\sigma(A) \restriction \{\eta_\xi\})U^*_\xi, j_\sigma(A) \restriction \{\eta_\xi\}],$$

So by (*) we have $\|[i_{r,q}(A), i_{r,p}(j_\sigma(A))]\| \geq 1/2$. The other inequality follows from the maximality of 1/2 (Corollary 2 of [36]). □

Lemma 4.17. *Suppose that p_1, p_2, p_3 are distinct elements in $\mathbb{P}$ which are pairwise in the convenient position as witnessed by $\sigma_{j,i} : a_{p_i} \to a_{p_j}$ for $1 \leq i < j \leq 3$ such that $\Delta < a_{p_1} \setminus \Delta < a_{p_2} \setminus \Delta < a_{p_3} \setminus \Delta$ and $n^{p_i}_\xi = n$ for some $n > 1$ and each $i \in \{1,2,3\}$ and that $v_1 = (v^1_0, ...v^1_{n-1}), v_2 = (v^2_0, ...v^2_{n-1})$ are two orthogonal unit vectors of $\mathbb{C}^n$ with rational coordinates. Then there is $r \leq p_1, p_2, p_3$ satisfying*

- $a_r = a_{p_1} \cup a_{p_2} \cup a_{p_3}$,
- $n^r_\xi = n^{p_i}_\xi = n$ *if* $\xi \in a_{p_i}$ *for* $1 \leq i \leq 3$,

- *for $m = 2, 3$ we have that*

$$\|[i_{r,p_m}(A), i_{r,p_1}(j_{\sigma_{m,1}}(A))]\| = 1/2,$$

for every nonexpanding $A \in \mathcal{B}_{X_{p_m}}$ such that there is $\xi \in a_m \setminus a_{p_1}$ with $A(\sum_{k<n} v_k^1 e_{\xi,k}) = \sum_{k<n} v_k^1 e_{\xi,k}$ and $A(\sum_{k<n} v_k^2 e_{\xi,k}) = 0$.

The element r is called the amalgamation of p_1, p_2, p_3 of type 2 for vectors v_1 and v_2.

Proof. First consider $s_2 \leq p_1, p_2$ and $s_3 \leq p_1, p_3$ which are the (v_1, v_2)-anti-commuting amalgamations of p_1, p_2 and p_1, p_3 as in Lemma 4.16. It is clear that s_1 and s_2 are in the convenient position as witnessed by $Id_{a_{p_1}} \cup \sigma_{3,2} : a_{p_1} \cup a_{p_2} \to a_{p_1} \cup a_{p_3}$. Now let r be the disjoint amalgamation of s_1 and s_2 as in Lemma 3.14. The properties of s_1 and s_2 from the (v_1, v_2)-anti-commuting amalgamations s_1 and s_2 pass to r by Definition 3.2 (d). □

4.4 Proof of the main theorem

Let $\mathcal{A}^{\mathbb{G}}$ be the C*-algebra in $M[\mathbb{G}]$ constructed in Section 4.2. By Proposition 4.12 we have that $\mathcal{A}^{\mathbb{G}}$ is a thin-tall fully noncommutative scattered C*-algebra in $M[\mathbb{G}]$. We will prove that $\mathcal{A}^{\mathbb{G}}$ satisfies all the condition in Theorem 4.1. Before that, we need some lemmas:

Lemma 4.18. *In $M[\mathbb{G}]$, suppose $(P_\xi)_{\xi<\omega_1}$ is a family of elements in $\mathcal{B} := \bigcup\{\mathcal{A}^{\mathbb{G}}_{X_p} : p \in \mathbb{G}\}$ such that $(P_\xi)_{\xi<\omega_1}$ generates a nonseparable C*-subalgebra of $\mathcal{A}^{\mathbb{G}}$. Then there is an uncountable $\Gamma \subseteq \omega_1$ and distinct $\alpha_\xi \in \omega_1$ for $\xi \in \Gamma$ such that $P_\xi \restriction \{\alpha_\xi\} \times \mathbb{N} \neq 0$.*

Proof. Suppose the lemma is false and let us get a contradiction. Then there is $\alpha < \omega_1$ such that $P_\xi \restriction [\alpha, \omega_1) \equiv 0$ for every $\xi < \omega_1$. In particular $(P_\xi)_{\xi<\omega_1} \subseteq \{A \in \mathcal{A}^{\mathbb{G}} : A \restriction [\alpha, \omega_1) \equiv 0\}$. By Lemma 4.11 and Proposition 4.12, we have that $\{A \in \mathcal{A}^{\mathbb{G}} : A \restriction [\alpha, \omega_1) \equiv 0\}$ is equal to $\mathcal{I}^{At}_\alpha(\mathcal{A}^{\mathbb{G}})$, the α-th ideal of the Cantor-Bendixson composition series for $\mathcal{A}^{\mathbb{G}}$. In particular, $(P_\xi)_{\xi<\omega_1}$ would be contained in a separable C*-subalgebra of $\mathcal{A}^{\mathbb{G}}$, which is a contradiction with the fact that $(P_\xi)_{\xi<\omega_1}$ generates a nonseparable C*-subalgebra. □

Observe that the direct family of finite dimensional subalgebras $(\mathcal{A}^{\mathbb{G}}_{X_p} : p \in \mathbb{G})$ witness the fact that $\mathcal{A}^{\mathbb{G}}$ is an AF algebra (Definition 1.46)[5]. In particular the following lemma can be applied to our algebra $\mathcal{A}^{\mathbb{G}}$:

Lemma 4.19. *Suppose that $\mathcal{A}$ is an AF C*-algebra where $\{\mathcal{A}_D : D \in \mathcal{E}\}$ is a directed family of finite-dimensional subalgebras with dense union. Let $P \in \mathcal{A}$ be a projection. Then for every $0 < \varepsilon < 1$ there is $E \in \mathcal{E}$ and a projection $Q \in \mathcal{A}_E$ such that $\|Q - P\| < \varepsilon$.*

Proof. Let $E \in \mathcal{E}$ be such that there is $A \in \mathcal{A}_E$ satisfying $\|A - P\| < \varepsilon/6$. By considering $(A + A^*)/2$ instead of A we may assume that A is self-adjoint and $\|A - P\| < \varepsilon/6$. As $\mathcal{A}_E$ is finite-dimensional, it is *-isomorphic to the direct sum of full matrix algebras (see Theorem 6.3.6 of [28]). Let π be the *-isomorphism. The matrix $\pi(A)$ is self-adjoint, so it can be diagonalized. As $\|A - P\| < \varepsilon/6$ we have that $\|A^2 - A\| < \varepsilon/2$ and so the distance of each entry on the diagonal of the diagonalized $\pi(A)$ to 0 or 1 cannot be bigger than $\varepsilon/2$, so there is a projection $Q \in \mathcal{A}_E$ such that $\|\pi(Q) - \pi(A)\| < \varepsilon/2$ and hence $\|Q - A\| < \varepsilon/2$ and $\|Q - P\| < \varepsilon$ as required. □

[5]Another way to conclude that $\mathcal{A}^{\mathbb{G}}$ is an AF algebra follows from the fact that $\mathcal{A}^{\mathbb{G}}$ is scattered and therefore it is an LF algebra (see Lemma 5.2 of [27]). Since $\mathcal{A}^{\mathbb{G}}$ has density ω_1, by Theorem 1.47 we conclude again that $\mathcal{A}^{\mathbb{G}}$ is an AF algebra.

Now, we are in the position of proving the main result in this thesis:

Theorem 4.1. *It is consistent that there is a nonseparable scattered fully noncommutative C*-algebra $\mathcal{A}$ with the following properties:*

1. *There is a directed family of finite-dimensional algebras whose union $\mathcal{B}$ is dense in $\mathcal{A}$ such that whenever $(P_\xi : \xi < \omega_1) \subseteq \mathcal{B}$ is a family of projections which generate a nonseparable subalgebra of $\mathcal{A}$, then for every $\varepsilon > 0$*

   (a) there are $\xi_1 < \xi_2 < \xi_3 < \omega_1$ such that $\|P_{\xi_1} - P_{\xi_2}P_{\xi_3}\| < \varepsilon$,

   (b) there are $\xi_1 < \xi_2 < \omega_1$ such that $\|[P_{\xi_1}, P_{\xi_2}]\| < \varepsilon$,

   (c) there are $\xi_1 < \xi_2 < \omega_1$ such that $\|[P_{\xi_1}, P_{\xi_2}]\| > 1/2 - \varepsilon$.

2. *$\mathcal{A}$ has no uncountable almost irredundant set of projections (and therefore, no uncountable irredundant sets).*

3. *$\mathcal{A}$ has no nonseparable abelian subalgebra.*

Proof. We will prove that $\mathcal{A}^{\mathbb{G}}$ as in Definition 4.7 satisfies the conditions in the theorem.

To prove (1) the directed family of finite-dimensional subalgebras of $\mathcal{A}^{\mathbb{G}}$ is $\{\mathcal{A}^{\mathbb{G}}_{X_p} : p \in \mathbb{G}\}$ as in Definition 4.7. By Lemma 4.8 the algebras $\mathcal{A}^{\mathbb{G}}_{X_p}$ are *-isomorphic to the algebras $\overline{\mathcal{A}^{p}_{X_p}}$ and they are finite-dimensional since they are equal to $\overline{\mathcal{B}_{X_p}}$ by Lemma 3.4.

In $M[\mathbb{G}]$, suppose $(P_\xi)_{\xi<\omega_1}$ is a sequence of projections in $\mathcal{B}$ such that $(P_\xi)_{\xi<\omega_1}$ generates a nonseparable C*-subalgebra of $\mathcal{A}^{\mathbb{G}}$. So, by Lemma 4.18 there must be distinct elements α_ξ of ω_1 such that $P_\xi \restriction (\{\alpha_\xi\} \times \mathbb{N}) \neq 0$. Since $\mathcal{B}_{\alpha_\xi \times \mathbb{N}}$ is invariant for $\mathcal{A}^{\mathbb{G}}$ it follows that $P_\xi \restriction (\{\alpha_\xi\} \times \mathbb{N})$ is a non-zero projection. Moreover it is not the unit of $\mathcal{B}_{\alpha_\xi \times \mathbb{N}}$ because such a unit would produce a unit of $\mathcal{A}^{\mathbb{G}}/\mathcal{I}^{At}_{\alpha_\xi}(\mathcal{A}^{\mathbb{G}})$ by Lemma 4.11 and Theorem 4.12, which is impossible since $\mathcal{A}^{\mathbb{G}}$ is the union of proper ideals $\mathcal{I}^{At}_{\alpha}(\mathcal{A}^{\mathbb{G}})$ for $\alpha < \omega_1$. In M, consider $p \in \mathbb{P}$, a sequence of $\mathbb{P}$-names $(\dot{P}_\xi)_{\xi<\omega_1}$ for $(P_\xi)_{\xi<\omega_1}$ and a sequence of $\mathbb{P}$-names $(\dot{\alpha}_\xi)_{\xi<\omega_1}$ for $(\alpha_\xi)_{\xi<\omega_1}$ such that p forces that $(\dot{P}_\xi)_{\xi<\omega_1}$ and $(\dot{\alpha}_\xi)_{\xi<\omega_1}$ have all the aforementioned properties.

For every $\xi < \omega_1$, let $p_\xi \in \mathbb{P}$ and $\alpha_\xi < \omega_1$ be such that

- $p_\xi \leq p$,
- $\alpha_\xi \in a_{p_\xi}$,
- $p_\xi \Vdash \dot{P}_\xi \in \mathcal{A}^{\dot{\mathbb{G}}}_{X_{p_\xi}}$,
- $p_\xi \Vdash \check{\alpha}_\xi = \dot{\alpha}_\xi$,
- p_ξ forces that $\dot{P}_\xi \restriction (\{\dot{\alpha}_\xi\} \times [0, n^{p_\xi}_{\dot{\alpha}_\xi}))$ is a nonzero projection which is not the unit of $\dot{\mathcal{B}}_{\{\dot{\alpha}_\xi\}\times[0,n^{p_\xi}_{\dot{\alpha}_\xi})}$

In $M[\mathbb{G}]$, suppose $p \in \mathbb{G}$. For every $\xi < \omega_1$, let $Q_\xi \in \overline{\mathcal{A}^{p_\xi}_{X_{p_\xi}}}$ be such that $i_{\mathbb{G},p_\xi}(Q_\xi) = P_\xi$. Note that by Lemma 4.8, Q_ξ's are projections and $Q_\xi \restriction (\{\alpha_\xi\} \times [0, n^{p_\xi}_{\alpha_\xi}))$ is a nonzero projection which is not the unit of $\mathcal{B}_{\{\alpha_\xi\}\times[0,n^{p_\xi}_{\alpha_\xi})}$ for each $\xi < \omega_1$. Assume that $p_\xi \Vdash i_{\mathbb{G},p_\xi}(\dot{Q}_\xi) = \dot{P}_\xi$. In M, by passing to an uncountable subset, we may assume that $(p_\xi)_{\xi<\omega_1}$ are pairwise in the convenient position and that $T = \{a_{p_\xi} : \xi < \omega_1\}$ forms an increasing[6] Δ-system of elements such that $n^{p_\xi}_{\alpha_\xi} = l$ for a fixed $l \in \mathbb{N}$.

[6] $\Delta < a_{p_\alpha} \setminus \Delta < a_{p_\beta} \setminus \Delta$ for every $\alpha < \beta < \omega_1$, where Δ is the root.

In $M[\mathbb{G}]$, by a counting argument, assume that

$$|\langle Q_\xi(e_{\eta,l'}), e_{\eta,l''}\rangle - \langle Q_{\xi'}(e_{\sigma_{p_{\xi'},p_\xi}(\eta),l'}), e_{\sigma_{p_{\xi'},p_\xi}(\eta),l''}\rangle| < \varepsilon/2l$$

for every $(\eta, l'), (\eta, l'') \in X_{p_\xi}$ and every $\xi < \xi' < \omega_1$. This guarantees that

$$(+) \qquad \|\overline{j_{\sigma_{p_{\xi'},p_\xi}}}(Q_\xi) - Q_{\xi'}\| < \varepsilon/2$$

for every $\xi < \xi' < \omega_1$.

Now let us prove item (a) of (1).

In M, fix $\xi_1 < \xi_2 < \xi_3 < \omega_1$ and consider r as the amalgamation of $p_{\xi_1}, p_{\xi_2}, p_{\xi_3}$ of type 1 as in Lemma 4.14. Then

$$i_{r,p_{\xi_3}}(A) i_{r,p_{\xi_2}}(j_{\sigma_{p_{\xi_3},p_{\xi_2}}}(A)) = i_{r,p_{\xi_1}}(j_{\sigma_{p_{\xi_3},p_{\xi_1}}}(A))^2,$$

for every $A \in \mathcal{A}^{p_{\xi_3}}_{X_{p_{\xi_3}}}$. Then by (1) of Lemma 4.5 we have that

$$\overline{i_{r,p_{\xi_3}}}(A)\overline{i_{r,p_{\xi_2}}}(\overline{j_{\sigma_{p_{\xi_3},p_{\xi_2}}}}(A)) = \overline{i_{r,p_{\xi_1}}}(\overline{j_{\sigma_{p_{\xi_3},p_{\xi_1}}}}(A))^2,$$

for every $A \in \overline{\mathcal{A}^{p_{\xi_3}}_{X_{p_{\xi_3}}}}$ and so

$$\|\overline{i_{r,p_{\xi_3}}}(Q_{\xi_3})\overline{i_{r,p_{\xi_2}}}(Q_{\xi_2}) - \overline{i_{r,p_{\xi_1}}}(Q_{\xi_1})^2\| < \varepsilon$$

and hence $\|P_{\xi_3}P_{\xi_2} - P_{\xi_1}\| < \varepsilon$ since

$$i_{\mathbb{G},r} \circ \overline{i_{r,p_{\xi_i}}}(Q_{\xi_i}) = i_{\mathbb{G},p_{\xi_i}}(Q_{\xi_i}) = P_{\xi_i}$$

by Definition 4.6 and Lemma 4.8. This completes the proof of (a) of (1).

Let us prove now item (b) of (1). By (a) of (1), there are $\xi_1 < \xi_2 < \xi_3 < \omega_1$ such that $\|P_{\xi_1} - P_{\xi_2}P_{\xi_3}\| < \varepsilon/2$. Taking the adjoints, we also get that $\|P_{\xi_1} - P_{\xi_3}P_{\xi_2}\| < \varepsilon/2$. Then

$$\begin{aligned}\|[P_{\xi_2}, P_{\xi_3}]\| &= \|P_{\xi_2}P_{\xi_3} - P_{\xi_3}P_{\xi_2}\| \\ &\leq \|P_{\xi_2}P_{\xi_3} - P_{\xi_1}\| + \|P_{\xi_1} - P_{\xi_3}P_{\xi_2}\| \\ &\leq \varepsilon.\end{aligned}$$

Now let us prove item (c) of (1). In $M[\mathbb{G}]$, for $\xi < \omega_1$ let $Q'_\xi \in \overline{\mathcal{B}_{X_{p_\xi}}}$ be projections such that $\|Q_\xi - Q'_\xi\| < \varepsilon/8$ and there is an orthonormal basis in $\mathcal{B}_{\{\alpha_\xi\}\times[0,l)}$ of eigenvectors for Q'_ξ consisting only of vectors with all rational coordinates with respect to our canonical basis $(e_{\alpha_\xi,j} : 0 \leq j < l)$. Note that by Lemma 3.4 we have that $Q'_\xi \in \overline{\mathcal{A}^{p_\xi}_{X_{p_\xi}}}$. Since $Q_\xi \restriction \big(\{\alpha_\xi\}\times[0,l)\big)$ is a nonzero projection which is not the unit of $\mathcal{B}_{\{\alpha_\xi\}\times[0,l)}$ for each $\xi < \omega_1$, Q'_ξ may be assumed to have the same rank as Q_ξ and so there are orthogonal unit vectors $v^\xi, w^\xi \in \mathbb{C}^l$ with all rational coordinates such that

$$Q'_\xi(\sum_{j<l} v^\xi_j e_{\alpha_\xi,j}) = \sum_{j<l} v^\xi_j e_{\alpha_\xi,j}, \quad Q'_\xi(\sum_{j<l} w^\xi_j e_{\alpha_\xi,j}) = 0.$$

As there are only countably many such vectors we may assume that all of them are equal to a pair (v, w) and moreover that

$$(++) \qquad \|\overline{j_{\sigma_{p_{\xi'},p_\xi}}}(Q'_\xi) - Q'_{\xi'}\| < \varepsilon/4$$

for every $\xi < \xi' < \omega_1$.

In M, fix $\xi_1 < \xi_2 < \xi_3 < \omega_1$ and consider r as the amalgamation of $p_{\xi_1}, p_{\xi_2}, p_{\xi_3}$ of type 2 for vector v and w, as in Lemma 4.17. Then

$$\|[i_{r,p_{\xi_1}}(A), i_{r,p_{\xi_2}}(j_{\sigma_{p_{\xi_2}},p_{\xi_1}}(A))]\| = 1/2$$

for every nonexpanding $A \in \mathcal{B}_{X_{p_1}}$ such that there is $\xi \in a_{p_2} \setminus a_{p_1}$ with $A(\sum_{k<l} v_k e_{\xi,k}) = \sum_{k<n} v_k e_{\xi,k}$ and $A(\sum_{k<n} w_k e_{\xi,k}) = 0$. Then by (2) of Lemma 4.5 we have that

$$\|[\overline{i_{r,p_{\xi_1}}}(Q'_{\xi_1}), \overline{i_{r,p_{\xi_2}}}(\overline{j_{\sigma_{p_{\xi_2}},p_{\xi_1}}}(Q'_{\xi_1}))]\| = 1/2$$

and so

$$\|[\overline{i_{r,p_{\xi_1}}}(Q'_{\xi_1}), \overline{i_{r,p_{\xi_2}}}(Q'_{\xi_2})]\| \geq 1/2 - \varepsilon/2$$

and hence

$$\|[\overline{i_{r,p_{\xi_1}}}(Q_{\xi_1}), \overline{i_{r,p_{\xi_2}}}(Q_{\xi_2})]\| \geq 1/2 - \varepsilon$$

as $\|Q_\xi - Q'_{\xi'}\| < \varepsilon/8$ for each $\xi < \xi' < \omega_1$, and finally

$$\|[P_{\xi_1}, P_{\xi_2}]\| \geq 1/2 - \varepsilon$$

since

$$i_{\mathbb{G},r} \circ \overline{i_{r,p_{\xi_i}}}(Q_{\xi_i}) = i_{\mathbb{G},p_{\xi_i}}(Q_{\xi_i}) = P_{\xi_i}$$

by Definition 4.6 and Lemma 4.8. This completes the proof of (c) of (1).

The proof of (2) will be based on (1) (a) and Lemma 4.19. Suppose that $\mathcal{A}^{\mathbb{G}}$ contains an uncountable almost irredundant set $\{Q_\xi : \xi < \omega_1\}$ of projections.

For each ξ let $F_{\omega_1\setminus\{\xi\}}$ be the closure of

$$\{\sum_{i=1}^{n} \lambda_i \prod_{j=1}^{n_i} Q_{\beta_{i,j}} : \text{ where } b_{i,j} \in \omega_1 \setminus \{\xi\} \text{ and} \sum |\lambda_i| = 1\}$$

By passing to an uncountable subset we may assume that there is $\varepsilon > 0$ such that for each $\xi < \omega_1$ we have $\|A - Q_\xi\| \geq \varepsilon$ for each $A \in F_{\omega_1\setminus\{\xi\}}$. Let $P_\xi \in \mathcal{B}$ be a projection satisfying $\|P_\xi - Q_\xi\| < \varepsilon/4$ which is obtained using Lemma 4.19. By (1) (a) there are $\xi_1 < \xi_2 < \xi_3 < \omega_1$ such that $\|P_{\xi_1} - P_{\xi_2} P_{\xi_3}\| < \varepsilon/4$. This implies that $\|Q_{\xi_1} - Q_{\xi_2} Q_{\xi_3}\| < \varepsilon$ which contradicts the defining property of ε, since $Q_{\xi_2} Q_{\xi_3} \in F_{\omega_1\setminus\{\xi_1\}}$.

Moreover, since for every irredundant set in $\mathcal{A}^{\mathbb{G}}$ there is an irredundant set of the same size and formed by projections (by Lemma 2.9) and since every irredundant set is in particular almost irredundant, we also conclude that there is no uncountable irredundant set in $\mathcal{A}^{\mathbb{G}}$ and this completes the proof of (2).

The proof of (3) will be based on (1) (c) and Lemma 4.19. Suppose that $\mathcal{A}^{\mathbb{G}}$ contains a nonseparable abelian subalgebra. As subalgebras of scattered algebras are scattered, and scattered locally compact spaces are totally disconnected, it follows that $\mathcal{A}^{\mathbb{G}}$ contains an uncountable Boolean algebra of (commuting) projections $\{Q_\xi : \xi < \omega_1\}$. In particular $\|Q_\xi - Q_{\xi'}\| = 1$ for all $\xi < \xi' < \omega_1$.

Let $P_\xi \in \mathcal{B}$ for $\xi < \omega_1$ be projections satisfying $\|P_\xi - Q_\xi\| < 1/10$ for each $\xi < \omega_1$ which is obtained using Lemma 4.19. In particular $\|P_\xi - P_{\xi'}\| \geq 8/10$ for all $\xi < \xi' < \omega_1$ and so they generate a nonseparable C*-algebra.

We have $\|P_{\xi_1} P_{\xi_2} - Q_{\xi_1} Q_{\xi_2}\| < 1/5$ and $\|P_{\xi_2} P_{\xi_1} - Q_{\xi_2} Q_{\xi_1}\| < 1/5$ for each $\xi_1 < \xi_2 < \omega_1$, so $[P_{\xi_1}, P_{\xi_2}] < 2/5$ for each $\xi_1 < \xi_2 < \omega_1$. But by (1) (c) there are $\xi_1 < \xi_2 < \omega_1$ such that

$\|[P_{\xi_1}, P_{\xi_2}]\| \geq 2/5$, a contradiction. □

4.5 Independence results

In this section, we summarize some of the independence results obtained in this thesis and we include a new result about commutators.

Theorem 4.20. *It is independent from ZFC whether there is a C*-subalgebra of $\mathcal{B}(\ell_2)$ of density continuum without irredundant sets of size continuum.*

Proof. Theorem 2.32 tells us that it is consistent with ZFC that every C*-subalgebra of $\mathcal{B}(\ell_2)$ of density continuum admits an irredundant set of size continuum. On the other hand, if M is a model of ZFC+CH, then Theorem 4.1 states that if $\mathbb{G}$ is a generic filter for $(\mathbb{P}_{\omega_1})^M$, then in $M[\mathbb{G}]$, the C*-algebra $\mathcal{A}^{\mathbb{G}}$ is a thin-tall C*-algebra of density continuum without irredundant sets of size continuum. Since every thin-tall C*-algebra can be embedded in $\mathcal{B}(\ell_2)$ (See Lemma 1.61), we conclude that in $M[\mathbb{G}]$, there is a C*-subalgebra of $\mathcal{B}(\ell_2)$ of density continuum without irredundant sets of size continuum. □

For almost irredundant sets, we have the following:

Theorem 4.21.

1. *Under PFA, every nonseparable scattered C*-algebra admits an uncountable almost irredundant set of projections (Theorem 2.41).*
2. *The C*-algebra $\mathcal{A}^{\mathbb{G}}$ constructed in Theorem 4.1 is a consistent example of a nonseparable scattered C*-algebra without uncountable almost irredundant sets of projections.*

The above theorem can be interpreted as a result of weak independence. This follows from the fact that the consistency of PFA requires large cardinals.

4.5.1 Commutators under OCA

In this section we prove the following result:

Theorem 4.22. *It is independent from ZFC whether there is a scattered nonseparable C*-algebra $\mathcal{A}$ in $\mathcal{B}(\ell_2)$ such that for every $0 < \varepsilon < 1/2$ among any discrete sequence of projections $(P_\xi : \xi < \omega_1)$ in $\mathcal{A}$*

- *there are $\xi_1 < \xi_2 < \omega_1$ such that $\|[P_{\xi_1}, P_{\xi_2}]\| > 1/2 - \varepsilon$,*
- *there are $\xi_1 < \xi_2 < \omega_1$ such that $\|[P_{\xi_1}, P_{\xi_2}]\| < \varepsilon$.*

The first part of Theorem 4.22 follows from Theorem 4.1. The second part will be proved assuming OCA.

Let us start with a review of the strong operator topology on $\mathcal{B}(H)$. We will follow the approach from the book [9] of K. Davidson.

Definition 4.23. *Let H be a Hilbert space. The strong operator topology (SOT) on $\mathcal{B}(H)$ is defined as the weakest topology such that the sets*

$$S(a, x) := \{b \in \mathcal{B}(H) : ||(b - a)(x)|| < 1\}$$

are open for each $a \in \mathcal{B}(H)$ and $x \in H$. In particular, a net $(T_\alpha)_\alpha$ converges to T if and only if $\lim_\alpha T_\alpha(x) = T(x)$ for every $x \in H$. We denote by $(\mathcal{B}(H), \tau_{SOT})$ and $(\mathcal{B}(H)_1, \tau_{SOT})$ respectively the space $\mathcal{B}(H)$ and the closed unit ball of $\mathcal{B}(H)$ with the strong operator topology.

Proposition 4.24. *If H is a separable Hilbert space, then $(\mathcal{B}(H)_1, \tau_{SOT})$ is metrizable and separable in the strong operator topology.*

Proof. For metrizability see [9], Proposition I.6.3. For the separability, first we observe that the operators of the form[7] $\sum_{i=1}^n x_i \otimes y_i$ are dense in $(\mathcal{B}(H)_1, \tau_{SOT})$. In fact, consider $S(a, x_1, \cdots, x_n, \varepsilon) = \{b : \|(b-a)(x_i)\| < \varepsilon, i = 1, \ldots, n\}$ an open neighbourhood of $a \in \mathcal{B}(H)$, where $x_1, \cdots, x_n$ are linearly independent. Then $b = \frac{b'}{\|b'\|}$ where $b' = \sum_{i=1}^n a(x_i) \otimes x_i$ is an element in $S(a, x_1, \cdots, x_n, \varepsilon)$.

Now we approximate every operator of the form $\sum_{i=1}^n x_i \otimes y_i$ by an element of some countable set $\mathbb{D}$.

Let D be a dense set in H and define $\mathbb{D}$ as the rational linear combinations of operators of the form $u \otimes v$ where $u, v \in D$. Given an operator of the form $b = \sum_{i=1}^n x_i \otimes y_i$, we have that

$$\|(\sum_{i=1}^n x_i \otimes y_i - \sum_{i=1}^n u_i \otimes v_i)(z)\| \leq (\sum_{i=1}^n (\|x_i - u_i\|\|v_i\| + \|y_i - v_i\|\|u_i\|))\|z\|$$

for every $u_i, v_i \in D$ and $z \in H$.

In particular, given an open neighbourhood $S(b, z_1, \cdots, z_n, \varepsilon)$ of b, consider $u_i, v_i \in D$ such that

$$(\sum_{i=1}^n (\|x_i - u_i\|\|v_i\| + \|y_i - v_i\|\|u_i\|))\|z_j\|) < \varepsilon$$

for every $j = 1, \cdots, n$. Then $b' = \sum_{i=1}^n u_i \otimes v_i \in S(b, z_1, \cdots, z_n, \varepsilon) \cap \mathbb{D}$. □

Following the remarks on page 16 and 17 of [9] we have the following:

Lemma 4.25. *The multiplication in $\mathcal{B}(H)_1$ is jointly continuous in the SOT topology and so every polynomial*[8] *is SOT continuous on $\mathcal{B}(H)_1$.*

Proof. Suppose $(S_\alpha)_\alpha$ and $(T_\alpha)_\alpha$ are nets in $\mathcal{B}(H)_1$ such that $\lim_\alpha S_\alpha = S$ and $\lim_\alpha T_\alpha = T$. For every $x \in H$ we have that

$$\|(TS - T_\alpha S_\alpha)(x)\| = \|((T - T_\alpha)S + T_\alpha(S - S_\alpha))(x)\| \leq$$
$$\leq \|(T - T_\alpha)(S(x))\| + \|T_\alpha\|\|(S - S_\alpha)(x)\|.$$

In particular, $\lim_\alpha T_\alpha S_\alpha = TS$. This shows us that the multiplication is jointly continuous in the SOT topology. The last part of the proof follows from the fact that addition and multiplication by scalar is also continuous with respect to the SOT topology. □

Lemma 4.26. *The norm $\|\cdot\|$ on $\mathcal{B}(H)$ is lower semi-continuous with respect to the SOT topology. In particular, for every $\varepsilon > 0$ the set*

$$\{B \in \mathcal{B}(H) : \|B\| > \varepsilon\}$$

is open in the SOT topology.

Proof. Consider the set $V = \{B \in \mathcal{B}(H) : \|B\| > \varepsilon\}$ and let $A \in V$. Then there is $x \in H$ of norm one such that $\|A(x)\| > \varepsilon + \delta$ for some $\delta > 0$. We claim that $S(A, x, \delta) := \{B : \|(B - A)(x)\| < \delta\} \subseteq V$. In fact, consider $B \in S(A, x, \delta)$. Then

$$\|B\| \geq \|B(x)\| = \|(B - A)(x) - A(x)\| \geq \|A(x)\| - \|(B - A)(x)\| > (\varepsilon + \delta) - \delta = \varepsilon$$

[7] Given $x, y \in H$ we define the operator $x \otimes y \in \mathcal{B}(H)$ by setting $x \otimes y(z) = \langle x, z\rangle y$ for every $z \in H$.

[8] By a polynomial $P(x, y)$ we mean a expression in the form $P(x, y) = \sum_i a_i x^i + \sum_i b_i y^i + \sum_{i,j} c_{i,j} x^i y^j + \sum_{i,j} d_{i,j} y^i x^j + e_0$.

which shows that $B \in V$. □

Theorem 4.27. *Assume OCA. Let $(A_\alpha)_{\alpha<\omega_1}$ be an uncountable family in $\mathcal{B}(\ell_2)$ and $P(x,y)$ be a polynomial satisfying $\|P(A,B)\| = \|P(B,A)\|$ for all $A, B \in \mathcal{B}(\ell_2)$. Then given $\varepsilon > 0$, either there is an uncountable $\Gamma_0 \subset \omega_1$ such that $||P(A_\alpha, A_\beta)|| > \varepsilon$ for every distinct $\alpha, \beta \in \Gamma_0$ or else there is an uncountable $\Gamma_1 \subset \omega_1$ such that $||P(A_\alpha, A_\beta)|| \leq \varepsilon$ for every distinct $\alpha, \beta \in \Gamma_1$.*

Proof. Since $(A_\alpha)_{\alpha<\omega_1}$ is uncountable, by passing to an uncountable subset, we may assume that there is $M > 0$ such that $\|A_\alpha\| \leq M$ for all $\alpha < \omega_1$. Let $X = \{A_\alpha : \alpha < \omega_1\} \subseteq M\mathcal{B}(\ell_2)_1$ and note that $M\mathcal{B}(\ell_2)_1$ is metric and separable by Proposition 4.24. Define

$$K_0 = \{\{A, B\} \in [X]^2 : ||P(A, B)|| > \varepsilon\}$$

and $K_1 = [X]^2 \setminus K_0$.

First note that the separability is hereditary for metric spaces, so X is metric separable as a subspace of $(M\mathcal{B}(l_2)_1, \tau_{SOT})$.

Now note that K_0 is open. This follows from Lemma 4.25 and Lemma 4.26. So we are in the position of applying the OCA. From OCA (Definition 1.10) either there is:

- An uncountable $\Gamma_0 \subset \omega_1$ such that $\{A_\alpha : \alpha \in \Gamma_0\}$ is 0-homogeneous set (and therefore, $||P(A_\alpha, A_\beta)|| > \varepsilon$ for every $\alpha, \beta \in \Gamma_0$) or
- we can write $X = \bigcup_n X_n$ where each X_n is 1-homogeneous (and therefore, there is $n \in \mathbb{N}$ and an uncountable $\Gamma_1 \subset \omega_1$ such that $X_n = \{A_\alpha : \alpha \in \Gamma_1\}$ and $||P(A_\alpha, A_\beta)|| \leq \varepsilon$ for every $\alpha, \beta \in \Gamma_1$).

□

Finally, the second part of Theorem 4.22 is a consequence of the following corollary:

Corollary 4.28. *It is consistent that for every $0 < \varepsilon < 1/2$ among any sequence of operators $(A_\xi : \xi < \omega_1)$ in $\mathcal{B}(\ell_2)$, there is an uncountable $\Gamma \subseteq \omega_1$ such that*

- *for every distinct $\xi_1, \xi_2 \in \Gamma$ we have $\|[A_{\xi_1}, A_{\xi_2}]\| > 1/2 - \varepsilon$ or else*
- *for every distinct $\xi_1, \xi_2 \in \Gamma$ we have $\|[A_{\xi_1}, A_{\xi_2}]\| < \varepsilon$.*

Proof. Consider $P(x, y) = xy - yx$ and apply Theorem 4.27. □

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