Chern-Weil-Lecomte morphism for L_∞-algebras

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Tese apresentada
ao
Instituto de Matemática e Estatística
da
Universidade de São Paulo
para
obtenção do título
de
Doutor em Ciências

Programa: Matemática
Orientador: Prof. Dr. Cristian Ortiz

Durante o desenvolvimento deste trabalho o autor recebeu auxílio financeiro da CAPES

São Paulo, fevereiro de 2022

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Esta versão da tese contém as correções e alterações sugeridas pela Comissão Julgadora durante a defesa da versão original do trabalho, realizada em 11/03/2022. Uma cópia da versão original está disponível no Instituto de Matemática e Estatística da Universidade de São Paulo.

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- Prof. Dr. Marco Zambon - KU-Leuven

Agradecimientos

Primeramente, quiero agradecer especialmente a mi familia, a mi madre Edilma y a mis hermanos Jorge y Alejandro, por todo su apoyo incondicional, por soportar mi ausencia en estos años de doctorado. Quiero expresar mi gratitud total a mi tía Luz Marina Carmona Presiga (1939-2021) por todo su cariño y apoyo hacía a mí y a mi familia, por todas sus enseñanzas y por ser un gran ejemplo a seguir. Sin lugar a dudas, sin ellos no hubiera llegado hasta este punto de mi carrera.

Quiero agradecer a mi orientador Cristian Ortiz por haber creído en mi potencial para hacer matemáticas, por toda la atención y energía que invirtió en mí durante estos 4 años de doctorado.

A mi grande amigo Fabricio Valencia por todas las buenas conversaciones sobre matemáticas, comentarios y sugerencias que enriquecieron mi formación profundamente.

A mis amigos en Brasil por los grandes momentos compartidos durante esta estadía, por su compañía en esos difíciles momentos de cuarentena COVID-19: Juan Luis Magiña, Juan David Cabrera, Ngoc Phuc Ngo, Fabricio Valencia, Diego Lopez, André Gomes, Sebastián Daza, Daniela Bohórquez, Juan David Bayona, Natalia Penitente, Leonel Zuaznabar, Hugo Brango, Andrea Delgado, Francimario Lima, Genaro Zamudio, Denisse Mallaupoma.

Por último y no menos importante a Alejandra, y Luz por hacernos compañía en estos últimos años de doctorado para ambos y por el inicio de esta nueva historia juntos.

Quiero expresar mi agradecimiento a varios de los profesores que influyeron fuertemente en mi educación y me motivaron a seguir en este camino, al Prof. Omar Saldarriaga, al Prof. Alberto Medina, al Prof. Pedro Hernandez Rizzo.

El IME-USP por el apoyo institucional, a CAPES por el financiamiento y a todo el pueblo Brasilero por el caluroso acogimiento.

Resumo

Herrera-Carmona, J. S. **O morfismo de Chern-Weil-Lecomte para L_∞-algebras**. 2022. Tese (Doutorado) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022. Nesta tese, estendemos o morfismo de Chern-Weil-Lecomte para o contexto de extensões de álgebras L_∞ junto com uma representação a menos de homotopia. Este morfismo toma valores na cohomologia de uma álgebra L_∞ com coeficentes em um espaço vetorial graduado. Provamos que esta construção é natural e que a cohomologia é invariante por quasi-isomorfismos equivariantes de álgebras L_∞. Como aplicação obtemos um morfismo de Chern-Weil-Lecomte para 2-fibrados principais sobre grupoides de Lie que admitem uma 2-conexão.

Palavras-chave: grupoides de Lie, 2-grupos de Lie, 2-fibrados principais, álgebras L_∞, extensões de álgebras L_∞, homomorfismo de Chern-Weil.

Abstract

Herrera-Carmona, J. S. **Chern-Weil-Lecomte morphism for L_∞-algebras**. 2022. Thesis (PhD) - Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, 2022.

In this thesis we extend the Chern-Weil-Lecomte morphism to the setting of extensions of L_∞-algebras together with a representation up to homotopy. This morphism takes values in the L_∞-algebra cohomology with coefficients in a graded vector space. We prove that this construction is natural and that the L_∞-algebra cohomology is invariant by equivariant L_∞-quasi-isomorphisms. As an application we obtain a Chern-Weil-Lecomte morphism for principal 2-bundles over a Lie groupoid that admit a 2-connection form.

Keywords: Lie groupoids, Lie 2-groups, principal 2-bundles, L_∞-algebras, extensions of L_∞-algebras, Chern-Weil homomorphism.

Contents

Introduction

Lie groupoids were introduced by Charles Ehresmann in [Ehr59] and have played a central role in differential geometry as unifying objects. Examples of Lie groupoids include manifolds, Lie groups, manifolds with symmetries, fibrations, regular foliations, among others. Also, Lie groupoids have strong connections with several areas of mathematics and mathematical physics, including Lie theory [Mac05, CF03], Poisson geometry [Wei87, Wei88, BCWZ04], index theory [DS19] and noncommutative geometry [Con90, Haw08], among others

Another important fact of Lie groupoids is that they serve as smooth models for the study of singular spaces e.g. orbifolds. These are topological spaces whose local model is a quotient of an open set of $\mathbb{R}^n$ by the action of a finite linear group. Orbifolds can be viewed as Morita equivalence classes of proper and étale groupoids [MP97]. In general, a Lie groupoid can be seen as a generalized atlas for a virtual structure on an orbit space [Pra04]. This approach to the study of singular spaces was originally introduced in the framework of algebraic geometry [Gro62] with the Deligne-Mumford stacks [DM69] and it arises also in genuine problems of differential geometry under the name of differentiable stacks. Concretely, a differentiable stack is represented by a Morita equivalence class of a Lie groupoid [Blo08, BX11].

In recent years many attention has been paid to geometric structures on Lie groupoids having as main goal extending classical differential geometry to the realm of differentiable stacks. For instance, vector fields on differentiable stacks were introduced in [Hep09], the Lie 2-algebra structure on the space of the vector fields on a differentiable stack studied in [BL20, OW19], Lie algebroids over a differentiable stack developed in [Wal15], vector bundles over differentiable stacks introduced in [dHO20], Riemannian metrics on differentiable stacks as defined in [dHF19], equivariant cohomology for differentiable stacks studied in [BTN21], and principal actions of stacky Lie groupoids over stacks in [BNZ20], among others.

We are interested in the study of principal 2-bundles over Lie groupoids as models for principal bundles over differentiable stacks. In particular, these can be seen as (strict) principal actions of Lie 2-groups considered as stacky Lie groupoids over a point, in the sense of [BNZ20]. In order to determine invariants of isomorphism classes of these principal bundles over differentiable stacks, we are led to study techniques associated with Chern-Weil theory compatible with Morita equivalence of Lie groupoids.

Chern-Weil theory was developed by Shiin-Shen Chern and André Weil in [Che51]. Initially, it was motivated by the study of topological invariants of isomorphim classes of fiber bundles. Then, it became relevant due to the fact that it offers an alternative construction of the characteristic homomorphism of a principal bundle relying on purely differential geometry techniques. More concretely, for a principal G-bundle (P, π, M, G) the Chern-Weil homomorphism is a map that goes from the space of Ad-invariant polynomials on the Lie algebra $\mathfrak{g}$ of G to the de Rham cohomology of a manifold M,

$$cw : S(\mathfrak{g}^*)^G \to H_{dR}(M).$$

It allows us to construct representatives of de Rham cohomology classes which classify isomorphism classes of principal bundles over M. Those classes are usually called **characteristic classes** and have a variety of applications in differential geometry, usually arising as obstructions to the existence of geometric structures.

Due to the naturality of the Chern-Weil theory, several authors have generalized its techniques to more algebraic contexts with the aim of constructing invariants of associated geometric structures. For instance, Lecomte extended the Chern-Weil homomorphism to the framework of Lie algebras and differential graded Lie algebras in [Lec82, Lec94], Kamber and Tondeur did it for semisimplicial Weil algebras in [KT75], Connes extended it to noncommutative geometry in [Con90], Alekseev and Meinrenken to non-commutative graded algebras in [AM05], Laurent-Gengoux, Tu and Xu to G-differential simplicial algebras in [LTX07], among others.

In particular, the Lecomte approach is central in this thesis. This construction of the Chern-Weil homomorphism takes as initial data an extension of Lie algebras,

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \hat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0,$$

together with a representation of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. The outcome is a more general Chern-Weil homomorphism defined on the space of symmetric $\hat{\mathfrak{g}}$-invariant maps on $\mathfrak{n}$ with values in V into the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with values in V

$$cw : \mathrm{Sym}(\mathfrak{n}; V)^{\hat{\mathfrak{g}}} \to H_{CE}(\mathfrak{g}; V).$$

The main objective of this thesis is to generalize the Chern-Weil homomorphism via the Lecomte approach to the context of extensions of L_∞-algebras together with a representation up to homotopy. Our main result states the following.

Theorem 0.0.1. *Let us consider an extension of L_∞-algebras*

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \hat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

and ρ a representation up to homotopy of $\mathfrak{g}$ on a dg-vector space $\mathbb{V}$. There is a natural map

$$cw : \mathrm{Hom}^\bullet(\wedge^k \mathfrak{n}[1], \mathbb{V})^{\hat{\mathfrak{g}}} \to H_{CE}^{k+\bullet}(\mathfrak{g}; \mathbb{V}); \quad cw(f) = [f_h]$$

that is independent of the chosen linear section h of π.

We stress that the cohomology involved in the previous theorem is the cohomology of an L_∞-algebra with values in a dg-vector space, which has been studied before in [Pen95, Kje01, Rei19]. It turns out that this cohomology is well-behaved with respect to equivariant L_∞ quasi-isomorphisms. This is the content of our second main result

Theorem 0.0.2. *Let ρ and ρ' be two representations up to homotopy of $\mathfrak{g}$ and $\mathfrak{h}$ on the dg-vector spaces $\mathbb{V}$ and $\mathbb{W}$, respectively. If $(F, f) : \mathfrak{g} \to \mathfrak{h}$ is a (ρ, ρ')-equivariant L_∞-quasi-isomorphism along to $f : \mathbb{W} \to \mathbb{V}$, then the induced map*

$$F^* : H_{CE}(\mathfrak{h}; \mathbb{W}) \to H_{CE}(\mathfrak{g}; \mathbb{V})$$

is an isomorphism.

As an application of this result, we introduce a Chern-Weil homomorphism for principal 2-bundles over Lie groupoids that admit a 2-connection form. For that, we extend the classical Atiyah sequence of principal bundles to a short exact sequence of $\mathcal{LA}$-groupoids. Then, based on the fact that the category of multiplicative sections of an $\mathcal{LA}$-groupoid has a structure of 2-term L_∞-algebra [OW19], we obtain a natural sequence of L_∞-algebras for a principal 2-bundle over a Lie groupoid.

Part of the data in Lecomte's approach is given by a representation up to homotopy. In our case, for a given Lie groupoid we consider a canonical representation up to homotopy of the Lie 2-algebra of multiplicative vector fields on the 2-term complex of multiplicative functions. It is roughly given by the action of multiplicative vector fields by derivations of multiplicative functions. Putting all this pieces together gives rise to a Chern-Weil-Lecomte homomorphism for principal 2-bundles.

Theorem 0.0.3. *Let $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ be a principal 2-bundle over a Lie groupoid that admits a 2-connection form. Then, for each $k \geq 1$ there exists a natural morphism*

$$cw : \mathrm{Hom}^\bullet(\wedge^k \mathfrak{X}^\bullet_{mult}(\mathrm{Ad}(\mathbb{P}))[1], \mathcal{C}^\infty(\mathbb{X})) \to H^{k+\bullet}_{CE,\psi^{\mathbb{X}}}(\mathfrak{X}^\bullet_{mult}(\mathbb{X}); \mathcal{C}^\infty(\mathbb{X})),$$

that is independent of the 2-connection form.

This thesis is composed of nine chapters, in the following we will give a brief description of these. Chapter 1 is about the basic material that will be used throughout this thesis, including principal bundles, the classical construction of the Chern-Weil homomorphism for principal bundles given by Chern in [Che51], as well as the Lecomte approach to the Chern-Weil homomorphism for extensions of Lie algebras together with a linear representation [Lec82]. Also, with the aim of fixing notation we review the basics on Lie groupoids and their cohomology.

Chapter 2 is about Lie 2-groups, Lie 2-algebras and their relation with both crossed modules of Lie groups and Lie algebras, respectively. The main result of this chapter is Theorem 2.1.1 which states a one-to-one correspondence between Morita morphism of Lie 2-groups and elementary equivalences of crossed modules of Lie groups.

In chapter 3 we study the notion of principal 2-bundle over a Lie groupoid. For that, we follow closely [HOV]. Briefly, a principal 2-bundle is a categorification of the notion of principal bundle in the sense that both the base space and the fibers are categorified from a manifold to a Lie groupoid, with the suitable requirement that all the fibers will be isomorphic to a Lie 2-group. The main result in this chapter is Theorem 3.2.1 gives conditions for a morphism between principal 2-bundles over a Lie groupoid to be a Morita map.

In Chapter 4 we introduce the Atiyah sequence of a principal 2-bundle over a Lie groupoid, this is a short exact sequence of $\mathcal{LA}$-groupoids that extends the classical Atiyah sequence of a principal bundle. As main results in this chapter we have Proposition 4.1.5 and Proposition 4.2.3.

In Chapter 5 we start with the study of 2-connection forms on principal 2-bundles over Lie groupoids. These results are part of [HOV]. Here we present some examples of 2-connection forms, we show their correspondence with multiplicative splittings of the Atiyah sequence of principal 2-bundles. Also, we focus in the study of its curvature and finally, we introduce the notion of a flat up to homotopy 2-connection form. As main result of this chapter we have Theorem 5.3.1, in which we prove that the Maurer-Cartan elements in the DGLA of multiplicative forms on the total space with values in the Lie 2-algebra of the structural 2-group induce 2-connection forms that are flat up to homotopy.

In Chapter 6 we proceed to study L_∞-algebras. For that, we review some basic algebraic concepts, we introduce both the graded-symmetric and skew-symmetric algebras, and study coderivations of the symmetric algebra. Finally, we recall the notion of L_∞-algebra and present some examples.

In Chapter 7 we introduce the L_∞-cohomology of a L_∞-algebra with values in a graded vector space, already studied in [Pen95, Kje01, Rei19]. For that, we study representations up to homotopy of L_∞-algebras, we present some examples that make these notions more natural. We define the notion of equivariant L_∞-morphism with the objective of inducing a morphism at level of cohomology. Then, we study representations up to homotopy in terms of the Maurer-Cartan elements of a certain DGLA. Finally, we study the canonical spectral sequence of this cohomology and show the main result of this chapter, namely Theorem 7.6.1. Which states that the L_∞-cohomology with coefficients is invariant under equivariant quasi-isomorphisms.

In Chapter 8 we present the main theorem of this thesis, that is Theorem 8.2.1 which gives an L_∞ version of the Chern-Weil-Lecomte homomorphism. For that, we start by studying extensions of L_∞-algebras, then we state and prove Theorem 8.2.1. Furthermore, we show some results about naturality of the Chern-Weil-Lecomte morphism. Finally, we compute the Chern-Weil-Lecomte morphism for the particular case of extensions of (strict) Lie 2-algebras together with a representation up to homotopy on a 2-term vector space.

In Chapter 9 we show applications of the previous chapters. On the one hand, for a Lie groupoid we have that the category of multiplicative vector fields has structure of Lie 2-algebra [OW19, BL20]. This is the same thing that a 2-term L_∞-algebra. On the other hand, in Theorem 9.1.2 we have show a canonical representation up to homotopy of the Lie 2-algebra of multiplicative vector fields on the 2-term vector space of multiplicative functions on the Lie groupoid. Therefore, for a Lie groupoid we can associate a cohomology in the sense of Chapter 7. We refer to this cohomology as the L_∞-cohomology of multiplicative vector fields on a Lie groupoid. As a consequence we state Theorem 9.1.3 in which we show that two Morita equivalent Lie groupoids have the same L_∞-cohomology. We interpret this result as that this cohomology is an invariant of the differentiable stack represented by the Lie groupoid. Now, when we consider a principal 2-bundle over a Lie groupoid that admits a 2-connection form, we have an extension of 2-term L_∞-algebras associated to its Atiyah sequence. Thus, this extension of L_∞-algebras together with the representation up to homotopy in Theorem 9.1.2 allows us to apply Theorem 8.2.1 giving rise to a Chern-Weil-Lecomte morphism for principal 2-bundles over a Lie groupoid that admit a 2-connection form.

Chapter 1

Preliminaries

In this chapter we present the basic material that will be used throughout the thesis. Our purpose is to present some basic concepts of principal bundles, the construction of the Chern-Weil homomorphism, both the classic construction that can be found in [Che51] or in [KN69], and the construction via Lecomte's approach as in [Lec82]. We also recall the basics on Lie groupoids.

1.1 Principal bundles

Definition 1.1.1. Let P and M be smooth manifolds and G be a Lie group. A **principal G-bundle** (P, π, M, G) is a quadruple composed of a surjective submersion $\pi : P \to M$ and a right action of G on P satisfying the following conditions

i. $\pi(pg) = \pi(p)$, for all $g \in G, p \in P$;

ii. the action is transitive on the fibers, i.e., the orbits of the action are the fibers of $\pi : P \to M$,

$$\mathcal{O}_p = \pi^{-1}(\pi(p)), \quad \text{for all } p \in P;$$

iii. the action is free, that is, if $pg = p$ for $p \in P, g \in G$ then $g = e$ the identity of G.

Example 1.1.1. Let G be a Lie group and M a manifold. The **trivial G-bundle** over M is given by the canonical projection $\mathrm{pr}_1 : M \times G \to M$ together with the right action $(x, g)h := (x, gh)$.

Example 1.1.2. Given a principal G-bundle (P, π, M, G) and an embedded submanifold N of M, the fiber bundle

$$\pi|_{\pi^{-1}(N)} : \pi^{-1}(N) \to N$$

is a principal G-bundle called the **restricted** G-bundle to N and denoted by $\pi|_N : P|_N \to N$.

Example 1.1.3. For a principal G-bundle (P, π, M, G) together with $f : N \to M$ a smooth map the pullback bundle

$$\pi' : f^*P \to N, \pi'(x, p) = x,$$

is principal G-bundle with the right action given by $(x, p)g := (x, pg)$.

Definition 1.1.2. A **morphism** between the principal bundles (P, π_P, M, G) and (Q, π_Q, N, H) consists of a triple (F, f, ϕ) composed of a bundle map $F : P \to Q$ covering the map $f : M \to N$, and a Lie group homomorphism $\phi : G \to H$ such that

$$F(pg) = F(p)\phi(g), \quad \text{for all } p \in P, g \in G.$$

In these terms, we say that the map $F : P \to Q$ is a **bundle morphism along** $\phi : G \to H$ **covering the map** $f : M \to N$,

$$\begin{array}{ccc} P & \xrightarrow{F} & Q, \\ {\scriptstyle \pi_P}\downarrow & & \downarrow{\scriptstyle \pi_Q} \\ M & \xrightarrow{f} & N \end{array} \qquad \curvearrowleft \qquad (G \xrightarrow{\phi} H).$$

Principal bundles with morphisms between them constitute a category that we denote $\mathcal{PB}$.

Remark 1.1.1. It is easy to prove that if a principal G-bundle (P, π, M, G) admits a section $s : M \to P, \pi \circ s = \mathrm{Id}_M$, then it is isomorphic to the trivial G-bundle. Therefore, as $\pi : P \to M$ is a surjective submersion, the local form of a submersion implies the existence of smooth local sections of π. In other words, a principal G-bundle is locally trivial. That is, for every $x \in M$ there is an open $U \subseteq M$ such that $P|_U$ is isomorphic to $U \times G$,

$$\begin{array}{ccc} P|_U & \xrightarrow{\quad\simeq\quad} & U \times G \\ & {\scriptstyle \pi|_U}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U. & \end{array}$$

Remark 1.1.2. Given a principal bundle (P, π, M, G) one observes that the right action of G on P is proper. Indeed,

$$\varphi : P \times G \to P \times_M P, \quad (p, g) \mapsto (pg, p)$$

is a diffeomorphism. Conversely, by the slice theorem (see for example [Mic08]) given a free and proper action of G on M, the orbit space M/G admits a unique smooth structure such that $M \to M/G$ is a principal G-bundle.

For a principal bundle (P, π, M, G) we shall denote the right action of $g \in G$ on P by $R_g : P \to P$ and the fundamental vector field associated to vector $X \in \mathfrak{g}$ by $\widetilde{X}$. Also the adjoint representation of a Lie group G over its Lie algebra $\mathfrak{g}$ by Ad.

Definition 1.1.3. Let (P, π, M, G) be a principal G-bundle and $\mathfrak{g}$ be the Lie algebra of the Lie group G. A **connection 1-form** is a $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(P; \mathfrak{g})$ with the properties

i. $\theta_p(\widetilde{X}_p) = X$, for all $X \in \mathfrak{g}, p \in P$;

ii. $R_g^* \theta_p = \theta_{pg} \circ R_{g*,p} = \mathrm{Ad}_{g^{-1}} \circ \theta_p = \mathrm{Ad}_{g^{-1}} \cdot \theta_p$ for all $g \in G$.

Let (P, π_P, M, G) and (Q, π_Q, N, H) be two principal bundles and $F : P \to Q$ a morphism along the Lie group homomorphism $\phi : G \to H$ covering the map $f : M \to N$. We write $\phi_* : \mathfrak{g} \to \mathfrak{h}$ for the morphism induced on the Lie algebras. Note that given a connection 1-form θ_P on P and a connection 1-form θ_Q on Q, both the form $F^*\theta_Q$ and the form $\phi_* \cdot \theta_P$ are $\mathfrak{h}$-valued 1-forms on P,

$$F^*\theta_Q \in \Omega^1(P; \mathfrak{h}), \quad \phi_* \cdot \theta_P \in \Omega^1(P; \mathfrak{h}).$$

We introduce the category $\mathcal{PBC}$ of **principal bundles with connection** the category with objects (P, π, M, G, θ) where (P, π, M, G) is a principal bundle and θ a connection 1-form on P. A morphism between two principal bundles with connection $(P, \pi_P, M, G, \theta_P)$ and $(Q, \pi_Q, N, H, \theta_Q)$ is a morphism $(F, f, \phi) : (P, \pi_P, M, G) \to (Q, \pi_Q, N, H)$ such that $F^*\theta_Q = \phi_* \cdot \theta_P$. In these terms, we say that $F : P \to Q$ is a**bundle morphism along** $\phi : G \to H$ **covering** $f : M \to N$ **preserving the connections**.

$$F^*\theta_Q = \phi_* \cdot \theta_P, \qquad \begin{array}{ccc} P & \xrightarrow{F} & Q, \\ {\scriptstyle \pi_P}\downarrow & & \downarrow{\scriptstyle \pi_Q} \\ M & \xrightarrow{f} & N \end{array} \qquad \curvearrowleft \qquad G \xrightarrow{\phi} H.$$

Recall that every principal bundle admits a connection, which can be shown by using a partition of unity argument (see for example [KN63, Bor12]).

A connection 1-form θ on a principal bundle P determines an G-invariant splitting of the tangent bundle $TP = \mathcal{V} \oplus \mathcal{H}$, where

$$\mathcal{V} = \ker \pi_*, \quad \mathcal{H} = \ker \theta.$$

This splitting allows us to define the **horizontal lift** of any vector field $X \in \mathfrak{X}(M)$ to a vector field $X^h \in \mathfrak{X}(P)$ in such a way that $X^h_p \in H_p$ and $\pi_{*,p} X^h_p = X_{\pi(p)}$. The horizontal lift is G-invariant, that is, $R_{g_*} X^h = X^h$ for all $g \in G$. In this sense, having a connection on a principal bundle is equivalent to the choice of an G-invariant horizontal distribution $\mathcal{H} \subseteq TP$, where by horizontal we mean transverse to the fibers.

Let $h : TP \to \mathcal{H}$ be the projection onto the horizontal distribution given by the splitting determinate by θ. If V is a vector space, we consider the application induced in V-valued forms

$$h^* : \Omega^k(P; V) \to \Omega^k(P; V), \quad (h^*\varphi)(X_1, \ldots, X_k) = \varphi(h(X_1), \ldots, h(X_k)),$$

this is the projection onto the space of **horizontal differential forms**

$$\Omega_{hor}(P; V) := \left\{ \varphi \in \Omega(P; V) \,|\, \iota_{\widetilde{X}} \varphi = 0, \quad \text{for all } X \in \mathfrak{g} \right\}.$$

We define **the exterior covariant derivative** induced by θ as $D_\theta : \Omega(P; V) \to \Omega_{hor}(P; V), D_\theta = h^* \circ d$.

Definition 1.1.4. Let (P, π, M, θ) be a principal bundle with connection. The **curvature 2-form** of θ is the $\mathfrak{g}$-valued 2-form $\Omega \in \Omega^2(P; \mathfrak{g})$ defined by

$$\Omega := D_\theta \theta.$$

This is a **tensorial form of type** (G, Ad), that alternatively could be defined by the **structure equation**

$$\Omega = d\theta + \frac{1}{2} [\theta, \theta]. \tag{1.1}$$

For more details about it see [KN63, Mic08].

Remark 1.1.3. Let $(P, \pi_P, M, G, \theta_P)$ and $(Q, \pi_Q, N, H, \theta_Q)$ be two principal bundles with connection and $F : P \to Q$ be a bundle morphism along $\phi : G \to H$ covering $f : M \to N$ that preserves the connections. Then we have that F also preserves the curvatures

$$\begin{aligned} F^*\Omega_Q =& F^*(d\theta_Q + \frac{1}{2} [\theta_Q, \theta_Q]) \\ =& d(F^*\theta_Q) + \frac{1}{2} [F^*\theta_Q, F^*\theta_Q] \\ =& d(\phi_* \cdot \theta_P) + \frac{1}{2} [\phi_* \cdot \theta_P, \phi_* \cdot \theta_P] \\ =& \phi_* \cdot (d\theta_P + \frac{1}{2} [\theta_P, \theta_P]) \\ =& \phi_* \cdot \Omega_P. \end{aligned}$$

Associated to a principal G-bundle (P, π, M, G) we have a canonical short exact sequence of vector bundles,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(d\pi) & \longrightarrow & TP & \xrightarrow{d\pi} & \pi^*(TM) & \longrightarrow & 0 \\ & & & \searrow & \downarrow & \swarrow & & & \\ & & & & P. & & & & \end{array}$$

Due to the action of G on P is free and proper we have that the tangent lifting action of G on TP is a linear action that is free and proper as well. Moreover, we can induce a natural right linear action of G on $\pi^*(TM)$ in a such a way that it is a free and proper action and the projection map $d\pi$ is G-equivariant. In other words, it is a short exact sequence of G-equivariant vector bundles. We get that the induced sequence of quotient spaces is a short exact sequence of vector bundles over M. This sequence is known as the **Atiyah sequence** and we denote it by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ad}(P) & \longrightarrow & \mathrm{At}(P) & \xrightarrow{\tilde{d\pi}} & TM & \longrightarrow & 0 \\ & & & \searrow & \downarrow & \swarrow & & & \\ & & & & M, & & & & \end{array} \tag{1.2}$$

where $\mathrm{At}(P) := TP/G$ is called the **Atiyah bundle** and $\mathrm{Ad}(P) := P \times_G \mathfrak{g}$ is the associated vector bundle to P and the adjoint representation of G on $\mathfrak{g}$. The vector bundle $\mathrm{Ad}(P)$ is called the **adjoint bundle**. For more detail about the Atiyah sequence see [Ati57].

1.2 Classical Chern-Weil theory

Let (P, π, M, G, θ) be a principal bundle with connection. Let us consider the following construction: for each covector $\alpha \in \mathfrak{g}^*$ we can construct a 1-form on P in the following way

$$TP \xrightarrow{\theta} \mathfrak{g} \xrightarrow{\alpha} \mathbb{R}, \quad X_p \mapsto \alpha(\theta_p(X_p)).$$

This procedure defines the morphism:

$$c_\theta : \mathfrak{g}^* \to \Omega^1(P), \quad \alpha \mapsto \alpha \circ \theta,$$

and induces a morphism between the graded commutative algebras

$$c_\theta : \wedge(\mathfrak{g}^*) \to \Omega_{dR}(P).$$

Consider the **Chevalley-Eilenberg dga** $(\wedge \mathfrak{g}^*, d_{CE})$ where the differential is defined by

$$\wedge^n(\mathfrak{g}^*) \longrightarrow \wedge^{n+1}(\mathfrak{g}^*),\ \alpha \mapsto d_{CE}\alpha.$$

For $X_i \in \mathfrak{g}, i = 0, \ldots, n$ we have

$$d_{CE}\alpha(X_0, \ldots, X_n) = \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n).$$

In general c_θ is not a differential graded homomorphism. Indeed, for each $\alpha \in \mathfrak{g}^*$ the following holds

$$d_{dR} \circ c_\theta(\alpha) - c_\theta \circ d_{CE}(\alpha) = \alpha \circ \Omega.$$

Applying the last procedure to the curvature 2-form we get a map,

$$c_\Omega : \mathfrak{g}^* \to \Omega^2_{dR}(P), \quad \alpha \mapsto \alpha \circ \Omega,$$

that can be extended to a homomorphism of graded algebras

$$c_\Omega : S(\mathfrak{g}^*) \to \Omega_{dR}(P),$$

where $S(\mathfrak{g}^*)$ denotes the symmetric algebra of $\mathfrak{g}^*$ with even grading $S^{2k}(\mathfrak{g}^*) = \mathrm{Sym}^k(\mathfrak{g}^*)$ and $S^{2k+1}(\mathfrak{g}^*) = 0$ for all $0 \leq k$. Note that a homogeneous element of degree $2k$ in $S(\mathfrak{g}^*)$ is actually a homogeneous polynomial on $\mathfrak{g}$ of degree k, and it is mapped by c_Ω into a differential form of degree $2k$. Thus c_Ω preserves the grading. Moreover, the following equation holds,

$$d_{dR} \circ c_\theta - c_\theta \circ d_{CE} = c_\Omega,$$

yielding the next definition.

Definition 1.2.1. The Weil algebra of $\mathfrak{g}$ is the dga

$$W(\mathfrak{g}) := \wedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*),$$

with differential given by $d = d_{CE} + \mathbf{d}$, where

- $\mathbf{d}$ is the derivation of degree 1 generated by $\mathbf{d}(x \otimes 1) = 1 \otimes x, \mathbf{d}(1 \otimes x) = 0$, and
- d_{CE} is extended uniquely on generators such that $d_{CE} \circ \mathbf{d} = -\mathbf{d} \circ d_{CE}$.

We extend the morphisms c_θ and c_Ω to morphisms from $W(\mathfrak{g})$ and define the map

$$w_\theta := c_\theta \otimes c_\Omega : \wedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \longrightarrow \Omega_{dR}(P).$$

The following relation hold:

$$\begin{aligned} w_\theta \circ \mathbf{d} =& c_\Omega \circ \mathbf{d}, \\ d_{dR} \circ c_\theta - c_\theta \circ d_{CE} =& c_\Omega \circ \mathbf{d}, \\ d_{dR} \circ w_\theta =& w_\theta \circ (d_{CE} + \mathbf{d}). \end{aligned}$$

The map $w_\theta : W(\mathfrak{g}) \to \Omega_{dR}(P)$ is a dga-homomorphism. Recall that for a principal bundle P the right action of the structure group G induces a left action on the de Rham complex $\Omega_{dR}(P)$ given by pullback $R^*_{gg'} = (R_{g'} \circ R_g)^* = R^*_g \circ R^*_{g'}$. Infinitesimally, it generates a linear action of the Lie algebra $\mathfrak{g}$ on $\Omega_{dR}(P)$ by derivations of degree 0. That means, for each $X \in \mathfrak{g}$ and $\alpha, \beta \in \Omega_{dR}(P)$ we have

$$\mathcal{L}_{\widetilde{X}}\alpha = \left.\frac{d}{dt}\right|_{t=0} R^*_{\exp(tX)}\alpha,$$

which satisfies

$$\mathcal{L}_{\widetilde{X}}(\alpha \wedge \beta) = (\mathcal{L}_{\widetilde{X}}\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_{\widetilde{X}}\beta).$$

Moreover, the interior product with respect to $\widetilde{X}$ induces an action by derivations of degree -1 on $\Omega_{dR}(P)$, and these two actions interact by mean of Cartan's relations

$$\begin{aligned} \mathcal{L}_{\widetilde{X}} \circ \mathcal{L}_{\widetilde{Y}} - \mathcal{L}_{\widetilde{Y}} \circ \mathcal{L}_{\widetilde{X}} =& \mathcal{L}_{\widetilde{[X,Y]}}, \\ d \circ \iota_{\widetilde{X}} + \iota_{\widetilde{X}} \circ d =& \mathcal{L}_{\widetilde{X}}, \\ \mathcal{L}_{\widetilde{X}} \circ \iota_{\widetilde{Y}} - \iota_{\widetilde{Y}} \circ \mathcal{L}_{\widetilde{X}} =& \iota_{\widetilde{[X,Y]}}, \\ \iota_{\widetilde{X}} \circ \iota_{\widetilde{Y}} + \iota_{\widetilde{Y}} \circ \iota_{\widetilde{X}} =& 0. \end{aligned}$$

We introduce the concept of G-dga which is relevant in this work.

Definition 1.2.2. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. A ***G*-differential graded algebra** is a triple $(\mathbf{X}, \rho, \iota)$ that consists of a dga $(\mathbf{X}, \cdot, d)$ together with a left action of G by graded automorphisms $\rho : G \to \mathrm{Aut}(\mathbf{X})$, and a linear action of $\mathfrak{g}$ by derivations of degree -1 $\iota : \mathfrak{g} \to \mathrm{Der}(\mathbf{X})$

such that they satisfy the Cartan calculus equations

$$\begin{aligned} L_X \circ L_Y - L_Y \circ L_X =& L_{[X,Y]}, \\ d \circ \iota_X + \iota_X \circ d =& L_X, \\ L_X \circ \iota_Y - \iota_Y \circ L_X =& \iota_{[X,Y]}, \\ \iota_X \circ \iota_Y + \iota_Y \circ \iota_X =& 0. \end{aligned}$$

where $L_X a = \frac{d}{dt}\big|_{t=0} \rho(\exp(tX)) \cdot a$, for $X \in \mathfrak{g}, a \in \mathbf{X}$.

For more details about this concept see [AL10, GS99]. The model example of a G-dga is the de Rham complex $\Omega_{dR}(P)$ of a principal G-bundle P, so that, for analogy we call the derivation ι_A as the interior product with respect A.

Definition 1.2.3. A dga-morphism $f : (\mathbf{X}, \cdot, d) \to (\mathbf{Y}, \cdot, d)$ between the two G-dga's $\mathbf{X}$ and $\mathbf{Y}$ is a G-**dga-morphism** if it commutes with both actions, namely, the action by automorphism of G and the linear action by derivations of $\mathfrak{g}$.

Example 1.2.1. The Weil algebra $W(\mathfrak{g})$ is a G-dga with grading given by

$$W(\mathfrak{g}) = \bigoplus_{0 \leq n} W^n(\mathfrak{g}), \quad W^n(\mathfrak{g}) = \bigoplus_{\substack{2p+q=n, \\ 0 \leq p,q}} \wedge^q(\mathfrak{g}^*) \odot \mathrm{Sym}^p(\mathfrak{g}^*).$$

The coadjoint action of G on $\mathfrak{g}^*$ induces an action by automorphisms on $W(\mathfrak{g})$ and the morphism w_θ commutes with the actions by G due to the fact that $R_g^*\theta = \mathrm{Ad}_{g^{-1}} \cdot \theta$ and $R_g^*\Omega = \mathrm{Ad}_{g^{-1}} \cdot \Omega$. The interior product ι_X over $W(\mathfrak{g})$ is completely determined by the commutativity with the map w_θ. Essentially, it is constructed by dualizing the equations $\iota_{\widetilde{X}}\theta = X, \iota_{\widetilde{X}}\Omega = 0$. The Cartan calculus equations are deduced from the Lie algebra structure of $\mathfrak{g}$. Therefore, the Weil algebra $W(\mathfrak{g})$ is a G-dga and the dga-morphism

$$w_\theta : W(\mathfrak{g}) \to \Omega_{dR}(P)$$

is a G-dga morphism.

Given a principal G-bundle (P, π, M, G) is well-known that the map $\pi^* : \Omega_{dR}(M) \to \Omega_{dR}(P)$ is an injection. The image $\mathrm{im}(\pi^*)$ is better known as the space of **basic forms** on P. These forms are completely characterized for the properties of being horizontal to the fibers and invariant by the action of G. Therefore, it can be described as

$$\Omega_{basic}(P) = \left\{ \varphi \in \Omega_{dR}(P) \,|\, R_g^*\varphi = \varphi, \iota_{\widetilde{X}}\varphi = 0, \text{ for all } g \in G,\, X \in \mathfrak{g} \right\}.$$

This description motives the study of differential forms on quotient spaces as the space of basic forms.

Definition 1.2.4. Let $\mathbf{X}$ be a G-dga. The space of **basic elements** of $\mathbf{X}$ is

$$\mathbf{X}_{basic} = \{ a \in \mathbf{X} \,|\, \rho(g) \cdot a = a,\, \iota_X a = 0, \text{ for all } g \in G, X \in \mathfrak{g} \}.$$

Remark 1.2.1. Note that the Cartan calculus implies that the space of basic elements of a G-dga is a dga, and if $f : \mathbf{X} \to \mathbf{Y}$ is a morphism of G-dgca, then $f|_{\mathbf{X}_{basic}} : \mathbf{X}_{basic} \to \mathbf{Y}_{basic}$ is a well-defined dga morphism. Therefore, the G-dga morphisms are precisely these type of morphisms of G-dga that descend to the quotient space.

Example 1.2.2. For the Weil algebra $W(\mathfrak{g})$ we have that the dga of basic elements is

$$W(\mathfrak{g})_{basic} = (S(\mathfrak{g}^*)^G, d = 0),$$

In other words, the basic elements of the Weil algebra is the commutative algebra of Ad-invariant polynomials on $\mathfrak{g}$. We want to recall that all homogeneous elements have even grading.

Theorem 1.2.1. *Let (P, π, M, G) be a principal bundle. For a connection 1-form θ the homomorphism of G-dga*

$$w_\theta : W(\mathfrak{g}) \to \Omega_{dR}(P)$$

restricts to a dga-morphism

$$w_\theta : S(\mathfrak{g}^*)^G \to \Omega_{dR}(M)$$

that induces a morphism in cohomology that is ***independent of the connection 1-form***

$$cw : S(\mathfrak{g}^*)^G \to H_{dR}(M). \tag{1.3}$$

The morphism cw is called the ***Chern-Weil homomorphism****.*

Proof. The proof of this fact can be found in [KN69, pag. 295]. □

More concretely, the Chern-Weil construction says that for a given basis $\{X_1, \dots, X_n\}$ of the Lie algebra $\mathfrak{g}$ we have

$$[X_i, X_j] = \sum_{k=1}^{n} C_{ij}^k X_k,$$

where $C_{ij}^k \in \mathbb{R}$ for $1 \leq i, j \leq n$ are the structure constants relative to this basis. If we consider the dual basis $\{\alpha_1, \dots, \alpha_n\}$ and a copy of it $\{u_1, \dots, u_n\}$ one has that the Weil algebra is given by

$$W(\mathfrak{g}) = \wedge(\alpha_1, \dots, \alpha_n) \odot \mathbb{R}[u_1, \dots, u_n].$$

Actually, it is the free graded commutative algebra generated by elements α_i of degree 1 and the elements u_i of degree 2. Also, the connection 1-form θ and its curvature 2-form Ω can be expressed in this basis as

$$\theta = \sum_{i=1}^{n} \theta_i X_i, \quad \Omega = \sum_{i=1}^{n} \Omega_i X_i,$$

where $\theta_i \in \Omega^1_{dR}(P)$ and $\Omega_i \in \Omega^2_{dR}(P)$. The properties defining a connection 1-form 1.1.3 imply that these differential forms satisfy the following conditions

- $\theta_i(\widetilde{X}_j) = \delta_i^j$ (Kronecker delta: $\delta_i^j = 1$ or 0 if $i = j$ or $i \neq j$);
- $\iota_{\widetilde{X}_j} \Omega_i = 0$;
- for $1 \leq i \leq n$
$$R_g^* \theta_i = \sum_{j=1}^{n} \mathrm{Ad}_{g^{-1}}{}_j^i \cdot \theta_j, \quad R_g^* \Omega_i = \sum_{j=1}^{n} \mathrm{Ad}_{g^{-1}}{}_j^i \cdot \Omega_j.$$
- The structure equation (1.1) can be written as
$$d\theta_k = \Omega_k - \sum_{i<j} C_{ij}^k \theta_i \wedge \theta_j.$$
- Bianchi's identity can be written as
$$d\Omega_k = -\sum_{i,j} C_{ij}^k \theta_i \wedge \Omega_j.$$

The G-dga morphism $w_\theta : W(\mathfrak{g}) \to \Omega_{dR}(P)$ is defined on generators by

$$w_\theta(\alpha_i) = \theta_i, \quad w_\theta(u_i) = \Omega_i.$$

The G-dga structure in $W(\mathfrak{g})$ is defined on generators for the automorphism action of G as

$$g \cdot \alpha_i := \alpha_i \circ \mathrm{Ad}_{g^{-1}}, \quad g \cdot u_i := u_i \circ \mathrm{Ad}_{g^{-1}},$$

and for the linear action by derivations of $\mathfrak{g}$ on generators by

$$i_{X_i}\alpha_j = \delta_i^j, \quad i_{X_i}u_j = 0.$$

Thus a homogeneous elements of degree $2p+q$

$$g \cdot (\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_p} \odot u_{j_1} \odot \cdots \odot u_{j_q}) = (g \cdot \alpha_{i_1}) \wedge \cdots \wedge (g \cdot \alpha_{i_p}) \odot (g \cdot u_{j_1}) \odot \cdots \odot (g \cdot u_{j_q})$$

and

$$\begin{aligned}\iota_{X_k}(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_p} \odot u_{j_1} \odot \cdots \odot u_{j_q}) &= \sum_{s=1}^{p} (-1)^{(-1)|\alpha_{i_s}|} \alpha_{i_1} \wedge \cdots \wedge \iota_{A_k}\alpha_{i_s} \wedge \cdots \wedge \alpha_{i_p} \odot u_{j_1} \odot \cdots \odot u_{j_q} + 0 \\ &= \sum_{s=1}^{p} \alpha_{i_1} \wedge \cdots \wedge \delta_k^{i_s} \wedge \cdots \wedge \alpha_{i_p} \odot u_{j_1} \odot \cdots \odot u_{j_q}\end{aligned}$$

The differential of $W(\mathfrak{g})$ on generators is

$$d\alpha_k = u_k - \sum_{i<j} C_{ij}^k \alpha_i \wedge \alpha_j, \quad du_k = -\sum_{i,j} C_{ij}^k \alpha_i \odot u_j.$$

An Ad-invariant homogeneous polynomial f on $\mathfrak{g}$ of degree $2k$ has the form

$$f = \sum_{i_1,\dots,i_k}^{r} a_{i_1,\dots,i_k} u_{i_1} \odot \cdots \odot u_{i_k}$$

where $a_{i_1,\dots,i_k}$ is symmetric in $i_1, \dots, i_k$ and for all $g \in G$ holds that $g \cdot f = f$. Therefore, the Chern-Weil homomorphism is the map $c_\theta : S(\mathfrak{g}^*)^G \to \Omega_{dR}(M)$ that takes Ad-invariant polynomials of degree $2k$ on $\mathfrak{g}$ and maps them to the $2k$-differential forms

$$f^\Omega = \sum_{i_1,\dots,i_k}^{r} a_{i_1,\dots,i_k} \Omega_{i_1} \wedge \cdots \wedge \Omega_{i_k}.$$

Remark 1.2.2 (Relation with the classifying space)**.** It is well-known that for a topological group G, in particular for a Lie group, there is a **universal principal G-bundle**, that is, a topological principal G-bundle (EG, π, BG, G) such that EG is contractible and BG is a paracompact Hausdorff space homotopy equivalent to a CW-complex, which is universal in the sense that for any principal G-bundle (P, π_P, M, G) there is a map $f : M \to BG$ such that $P \simeq f^*EG$,

$$\begin{array}{ccc} P \simeq f^*(EG) & \longrightarrow & EG \\ \downarrow{\scriptstyle \pi_P} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\ f\ } & BG. \end{array}$$

The space BG is called the **classifying space** of G and the map $f : M \to BG$ the **classifying map**. These names are justified because isomorphism classes of principal G-bundles over M are in one-to-one correspondence with homotopy classes of maps from M to BG. Therefore, the isomorphism class of (P, π, M, G) determines the homotopy class of its classifying map $f : M \to BG$, and this in turn determines a unique morphism

$$f^* : H^*_{\text{sing}}(BG; \mathbb{R}) \to H^*_{\text{sing}}(M; \mathbb{R}).$$

The Chern-Weil Theory offers a differential geometric method to compute f^* through the Chern-Weil homomorphism. It is well-known too that in the case where G is a compact Lie group one has $H^*_{\text{sing}}(BG;\mathbb{R}) \simeq S(\mathfrak{g}^*)^G$, and that the Chern-Weil construction agree with the topological construction. For more information about this topological perspective see for example [Tu20].

1.3 Lecomte's approach

There is an algebraic approach to the Chern-Weil homomorphism due to P. B. Lecomte [Lec82]. Let us consider an extension of Lie algebras,

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \hat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0,$$

together with a representation (V,ρ) of the Lie algebra $\mathfrak{g}$ on a vector space V. We denote by $\mathrm{Sym}^k(\mathfrak{n},V)^{\hat{\mathfrak{g}}}$ the space of linear k-symmetric and $\hat{\mathfrak{g}}$-invariant maps on $\mathfrak{n}$ with values in V. That means, if $f \in \mathrm{Sym}^k(\mathfrak{n},V)^{\hat{\mathfrak{g}}}$, then f is a k-linear map $f : \mathfrak{n}\times\cdots\times\mathfrak{n} \to V$ that is symmetric and for all $X \in \hat{\mathfrak{g}}$ and $Y_1,\dots,Y_k \in \mathfrak{n}$ the following holds

$$\rho(\pi(X))f(Y_1,\dots,Y_k) = \sum_{i=1}^{k} f(Y_1,\dots,[X,Y_i],\dots,Y_k).$$

Given a linear section $h : \mathfrak{g} \to \hat{\mathfrak{g}}$, i.e. $\pi \circ h = \mathrm{Id}_{\mathfrak{g}}$, its curvature is defined as

$$K_h : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{n}, \quad K_h(X,Y) := [hX,hY] - h[X,Y].$$

Every element $f \in \mathrm{Sym}^k(\mathfrak{n},V)^{\hat{\mathfrak{g}}}$ determines a $2k$-cocycle $f_h \in \wedge^{2k}\mathfrak{g}^* \otimes V$ defined by

$$f_h(X_1,X_2,\dots,X_{2k-1},X_{2k}) := \sum_{\sigma(2i-1)<\sigma(2i)} \mathrm{sgn}(\sigma) f(K_h(X_{\sigma(1)},X_{\sigma(2)}),\dots,K_h(X_{\sigma(2k-1)},X_{\sigma(2k)})).$$

The Chern-Weil-Lecomte homomorphism is the map

$$cw : \mathrm{Sym}^\bullet(\mathfrak{n};V)^{\hat{\mathfrak{g}}} \to H^{2\bullet}_{CE}(\mathfrak{g};V), \quad cw(f) = [f_h],$$

where $H^{2\bullet}_{CE}(\mathfrak{g};V)$ denotes the even part of the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with values in V.

Remark 1.3.1. In the particular case of a principal G-bundle (P,π,M,G) we have that its Atiyah sequence (1.2) induces a sequence of space sections. In fact, it is a short exact sequence of Lie algebras,

$$0 \longrightarrow \Gamma(\mathrm{Ad}(P)) \xrightarrow{\iota} \mathfrak{X}(P)^G \xrightarrow{\pi_*} \mathfrak{X}(M) \longrightarrow 0.$$

Thus, on one side, we have the following equivalences

i.
$$\text{Connection 1-form } \theta \in \Omega^1_{dR}(P) \quad \leftrightarrows \quad h : \mathfrak{X}(M) \to \mathfrak{X}(P)^G,\ \pi_* \circ h = \mathrm{Id}_{\mathfrak{X}(M)}.$$

where h denotes the horizontal lift of vector fields. It is worth noticing that h preserves the $C^\infty(M)$-module structure, that means, h is a $C^\infty(M)$-linear splitting.

ii.
$$\text{Curvature 2-form } \Omega \in \Omega^2_{dR}(P) \quad \leftrightarrows \quad K_h \in \Omega^2_{dR}(M,\mathrm{Ad}(P))$$

where
$$K_h : \wedge^2_{C^\infty(M)}\mathfrak{X}(M) \to \Gamma(\mathrm{Ad}(P)),\ K_h(X\wedge Y) = [X^h,Y^h] - [X,Y]^h.$$

On the other side, for G connected it holds

$$S^k(\mathfrak{g}^*)^G \simeq \mathrm{Sym}^k(\Gamma(\mathrm{Ad}(P)), C^\infty(M))^{\Gamma(\mathrm{At}(P))}.$$

For $f \in S^k(\mathfrak{g}^*)^G$, we can induce a map

$$f : \Gamma(\mathrm{Ad}(P)) \times \cdots \times \Gamma(\mathrm{Ad}(P)) \to C^\infty(M),$$

such that

$$\mathcal{L}_X(f(\xi_1, \ldots, \xi_k)) = \sum_{i=1}^{k} f(\xi_1, \ldots, [X^h, \xi_i], \ldots, \xi_k),$$

for all $X \in \mathfrak{X}(M)$ and $\xi_i \in \Gamma(\mathrm{Ad}(P))$. Therefore, the Chern-Weil-Lecomte morphism is well-defined and takes values in the Lie algebra cohomology of the vector fields with values in the space of smooth functions. Given that the horizontal lift preserves the $C^\infty(M)$-module structure it yields differential forms, that is, the Chern-Weil-Lecomte takes values in the de Rham cohomology of M,

$$\begin{array}{ccc} S(\mathfrak{g}^*)^G & \xrightarrow{cw} & H_{CE,\mathcal{L}}(\mathfrak{X}(M); C^\infty(M)) \\ & \overset{cw}{\searrow} & \uparrow \\ & & H_{dR}(M). \end{array} \tag{1.4}$$

Therefore, the Chern-Weil-Lecomte associated to a principal G-bundle agrees with the classical Chern-Weil homomorphism (1.3).

1.4 Lie groupoids

A **groupoid** is a small category in which all arrows are invertible. Concretely, a groupoid consists of two sets X_1 and X_0 that we call the **set of arrows** and the **set of objects**, respectively, together with the **structural maps**,

$$X_{1\,s}\times_t X_1 \xrightarrow{m} X_1 \underset{t}{\overset{s}{\rightrightarrows}} X_0 \xrightarrow{u} X_1 \xrightarrow{\iota} X_1,$$

satisfying the axioms of a category. We denote an arrow $g \in X_1$ as $g : y \leftarrow x$ where $s(g) = x$ and $t(g) = y$, and its inverse arrow as $\iota(g) = g^{-1} : x \leftarrow y$. For an object $x \in M$ we denote the unit at x as $u(x) = 1_x$, and for a pair of **composable arrows** $(g', g) \in X_{1\,s}\times_t X_1$ we denote the composition arrow $m(g', g)$ as the concatenation $g'g$. In general, we write a groupoid by $X_1 \rightrightarrows X_0$ or $\mathbb{X}$ when the context does not demand to specify the base.

Definition 1.4.1. A **Lie groupoid** is a groupoid $X_1 \rightrightarrows X_0$ in which X_1 and X_0 are smooth manifolds, and the source and the target maps $s, t : X_1 \to X_0$ are surjective submersions, and all the other structural maps are smooth.

Remark 1.4.1. The composition map m is a smooth map from the set of composable arrows $X_2 := X_{1\,s}\times_t X_1$ with the smooth structure of embedded submanifold of $X_1 \times X_1$. We have that the inversion map ι is a diffeomorphism of X_1 and the unit map u is a smooth bisection of the source and the target.

Let us see some examples of Lie groupoids.

Example 1.4.1 (Manifolds)**.** We can see a smooth manifold M as a Lie groupoid with only identity arrows $M \rightrightarrows M$

$$\triangle(M) = M_{\mathrm{Id}}\times_{\mathrm{Id}} M \xrightarrow{pr} M \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightrightarrows}} M \xrightarrow{\mathrm{Id}} M \xrightarrow{\mathrm{Id}} M$$

This Lie groupoid is called **unit groupoid**.

Example 1.4.2 (Lie group)**.** A **Lie group** G can be seen as a Lie groupoid with only one object $G \rightrightarrows *$

$$G \times G \xrightarrow{m} G \rightrightarrows * \xrightarrow{e} G \xrightarrow{(-)^{-1}} G.$$

Example 1.4.3 (Manifolds with symmetries)**.** If a Lie group G acts on a smooth manifold M, then the product $M \times G$ inherits a structure of Lie groupoid over M. We call this Lie groupoid the **transformation groupoid** and denote it by $G \ltimes M$,

$$(G \ltimes M)_s{\times}_t(G \ltimes M) \xrightarrow{m} G \ltimes M \underset{t}{\overset{s}{\rightrightarrows}} M \xrightarrow{i_2} G \ltimes M \xrightarrow{\iota} G \ltimes M,$$

its structural maps are given by

$$s(g,x) = x, \quad t(g,x) = g \cdot x, \quad m((h, g \cdot x), (g,x)) = ((hg) \cdot x, x)$$

for all $(g,x) \in G \times M$.

Other important examples of Lie groupoids include: the fundamental groupoid of a manifold, the holonomy groupoid of a foliation, the gauge groupoid of a principal bundle, among others. For more details about Lie groupoids see [MM03, Mac05].

Definition 1.4.2. A **Lie groupoid morphism** (F, f) from $X_1 \rightrightarrows X_0$ to $Y_1 \rightrightarrows Y_0$ is a pair of smooth maps $F : X_1 \to Y_1$ and $f : X_0 \to Y_0$ such that the following diagram commutes

$$\begin{array}{ccccccccc}
X_{1s}{\times}_t X_1 & \xrightarrow{m} & X_1 & \underset{t}{\overset{s}{\rightrightarrows}} & X_0 & \xrightarrow{u} & X_1 & \xrightarrow{\iota} & X_1 \\
\downarrow{\scriptstyle F\times F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\
Y_{1s}{\times}_t Y_1 & \xrightarrow{m} & Y_1 & \underset{t}{\overset{s}{\rightrightarrows}} & Y_0 & \xrightarrow{u} & Y_1 & \xrightarrow{\iota} & Y_1
\end{array}$$

In this case we say that $F : X_1 \to Y_1$ is a Lie groupoid morphism covering the map $f : X_0 \to Y_0$.

Remark 1.4.2. It is common to find in the literature an alternative definition of a morphism of Lie groupoids as a functor $\Phi : \mathbb{X} \to \mathbb{Y}$ composed by smooth maps in arrows and objects spaces.

Example 1.4.4 (Pullback groupoid)**.** Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a Lie groupoid and $f : Y_0 \to X_0$ be a smooth map. Assume that $t \circ \mathrm{pr}_2 : Y_{0f}{\times}_s X_1 \to X_0$ is surjective submersion then $f^!\mathbb{X} := (Y_{0f}{\times}_t X_{1s}{\times}_f Y_0 \rightrightarrows Y_0)$ is a Lie groupoid where the structural maps are

$$\begin{aligned} s^!(x,g,y) = y, \quad t^!(x,g,y) = x, \quad u^!_y = (y, u_{f(y)}, y) \quad \iota(x,g,y) = (y, g^{-1}, x) \\ m^!((z,g,y),(y,g',x)) =& (z, gg', x)). \end{aligned}$$

We call this Lie groupoid the **pullback groupoid** of $\mathbb{X}$ along f.

Definition 1.4.3. Let $X_1 \rightrightarrows X_0$ be a Lie groupoid. A $\mathcal{VB}$-groupoid over $X_1 \rightrightarrows X_0$ is a Lie groupoid $\mathcal{V} \rightrightarrows E$, where both $q_{\mathcal{V}} : \mathcal{V} \to X_1$ and $q_E : E \to X_0$ are vector bundles compatible with the groupoid structure of $\mathcal{V}$ in the sense that all the structural maps are vector bundle maps over the corresponding structural maps of $X_1 \rightrightarrows X_0$. We denote the structural maps of $\mathcal{V}$ by $\tilde{s}, \tilde{t}, \tilde{m}, \tilde{\iota}, \tilde{1}$. In diagram as

$$\begin{array}{ccc}
\mathcal{V} & \underset{\tilde{t}}{\overset{\tilde{s}}{\rightrightarrows}} & E \\
\downarrow{\scriptstyle q_{\mathcal{V}}} & & \downarrow{\scriptstyle q_E} \\
X_1 & \underset{t}{\overset{s}{\rightrightarrows}} & X_0.
\end{array}$$

Observe that since $\tilde{s}$ is a vector bundle map of maximal rank, then $\ker(\tilde{s}) \to X_1$ is a vector bundle. We call the vector bundle $C := \ker(\tilde{s})|_{X_0} \to X_0$ by the **core bundle** and the vector bundle $E \to X_0$ by the **side bundle** of the $\mathcal{VB}$-groupoid $\mathcal{V}$.

The following notion is relevant in the forthcoming work

Definition 1.4.4. A morphism of Lie groupoids $\Phi : X_1 \to Y_1$ covering $\phi : X_0 \to Y_0$ is a **Morita morphism** if it satisfies the following conditions

i. (Fully faithfull) the following diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi} & Y_1 \\ {\scriptstyle (s_X,t_X)}\downarrow & \lrcorner & \downarrow{\scriptstyle (s_Y,t_Y)} \\ X_0 \times X_0 & \xrightarrow{\phi\times\phi} & Y_0 \times Y_0 \end{array}$$

is a pullback diagram,

ii. (Essentially surjective) the map

$$t_Y \circ \mathrm{pr}_2 : X_0{}_\phi\times_{s_Y} Y_1 \to Y_0$$

is a surjective submersion.

Definition 1.4.5. Two Lie groupoids $\mathbb{X}$ and $\mathbb{Y}$ are **Morita equivalent** if there exists a third Lie groupoid $\mathbb{W}$ together with Morita morphisms $\Psi : \mathbb{W} \to \mathbb{X}$ and $\Phi : \mathbb{W} \to \mathbb{Y}$.

Intuitively, we may think that the Lie groupoid $\mathbb{W}$ determines a notion of isomorphism $\frac{\Phi}{\Psi} : \mathbb{X} \to \mathbb{Y}$ between the Lie groupoids $\mathbb{X}$ and $\mathbb{Y}$. Formally, what we are doing is localizing the category of Lie groupoids along the Morita morphisms. Thus, two Lie groupoids are Morita equivalents if and only if they are isomorphic in the localized Lie groupoid category. Morphisms in the localized category are called **generalized** morphisms between Lie groupoids [MM03, del13].

Remark 1.4.3. It is common to find in the literature the notion of Morita equivalence between Lie groupoids under the name of weak equivalence of Lie groupoid, see for example [MM03, §5.4]. Another interesting approach to the notion of Morita equivalence of Lie groupoids is in terms of bi-bundles or anafunctors, for more details about this perspective see for instance [del13, §4.6] and [Wal18, §2.3] and references therein.

1.5 The de Rham cohomology of a Lie groupoid

Given a Lie groupoid $\mathbb{X} := (X_1 \rightrightarrows X_0)$ one has a simplicial manifold associated to it called **the nerve of the Lie groupoid** $X_\bullet := \{X_n\}_{0\leq n}$, whose n-simplices X_n is the manifold of strings of composable arrows of length n

$$X_n = \{(g_1, \ldots, g_n) \,|\, s(g_i) = t(g_{i+1}), 1 \leq i \leq n-1\},$$

the face maps are for $n = 1$ the source and the target maps, and for $n > 1$:

$$d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n), & \text{if} \quad i = 0; \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n), & \text{if} \quad 1 < i < n; \\ (g_1, \ldots, g_{n-1}), & \text{if} \quad i = n. \end{cases} \tag{1.5}$$

and the degeneracy maps are for $n = 0$ the unit map and for $n \geq 1$:

$$s_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i, u_{s(g_i)}, g_{i+1}, \ldots, g_n)$$

for $0 \leq i \leq n$.
Associated to this simplicial manifold $X_\bullet$ there is many information about the geometry of the Lie groupoid. On the one hand, we have the **classifying space of a Lie groupoid**, denoted by $B\mathbb{X}$, that is the geometric realization of the nerve of the Lie groupoid. The name comes from the fact the classifying space of the Lie group mentioned in Remark 1.2.2 coincides with the classifying space of the Lie group seen as Lie groupoid as in Example 1.4.2. On the other hand, we have the **Bott-Shulman-Stasheff Complex** which computes the de Rham cohomology of $\mathbb{X}$. The Bott-Shulman-Stasheff complex is the double complex

$$\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \\
{\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\
\Omega^3_{dR}(X_0) & \xrightarrow{\partial} & \Omega^3_{dR}(X_1) & \xrightarrow{\partial} & \Omega^3_{dR}(X_2) & \xrightarrow{\partial} & \Omega^3_{dR}(X_3) & \xrightarrow{\partial} & \cdots \\
{\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\
\Omega^2_{dR}(X_0) & \xrightarrow{\partial} & \Omega^2_{dR}(X_1) & \xrightarrow{\partial} & \Omega^2_{dR}(X_2) & \xrightarrow{\partial} & \Omega^2_{dR}(X_3) & \xrightarrow{\partial} & \cdots \\
{\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\
\Omega^1_{dR}(X_0) & \xrightarrow{\partial} & \Omega^1_{dR}(X_1) & \xrightarrow{\partial} & \Omega^1_{dR}(X_2) & \xrightarrow{\partial} & \Omega^1_{dR}(X_3) & \xrightarrow{\partial} & \cdots \\
{\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\
\Omega^0_{dR}(X_0) & \xrightarrow{\partial} & \Omega^0_{dR}(X_1) & \xrightarrow{\partial} & \Omega^0_{dR}(X_2) & \xrightarrow{\partial} & \Omega^0_{dR}(X_3) & \xrightarrow{\partial} & \cdots
\end{array}$$

where the vertical differential is given by the usual de Rham differentials, $d^k_p : \Omega^k(X_p) \mapsto \Omega^{k+1}(X_p)$, and the horizontal differential is given by the simplicial structure. It is defined as the alternating sum of the pull-back by the face maps of $X_\bullet$

$$\partial^q : \Omega^k(X_q) \mapsto \Omega^k(X_{q+1}), \quad \partial^q = \sum_{i=0}^{q} (-1)^i d_i^*.$$

The Bott-Shulman-Stasheff complex is denoted by $\Omega_{\mathrm{BSS}}(\mathbb{X})$. The cohomology of the total complex of this double complex is called **de Rham cohomology** of $\mathbb{X}$,

$$H^*_{dR}(\mathbb{X}) = H^* \left(\Omega_{tot}(\mathbb{X}),\, d_{tot}|_{\Omega^p_{dR}(X_q)} = \partial^p + (-1)^p d^q_p \right).$$

An important property of the de Rham cohomology of a Lie groupoid is that it is Morita invariant. The following result is a well-known fact [BSS76, Cra03, BX11], we present a proof in order to clarify the discussion in the following remark.

Theorem 1.5.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two morita equivalent Lie groupoids, then*

$$H_{dR}(\mathbb{X}) \simeq H_{dR}(\mathbb{Y})$$

Proof. Following the tom Dieck Theorem in [BSS76, pag.53] if $\mathbb{X}$ and $\mathbb{Y}$ are morita equivalents then their classifying spaces $B\mathbb{X}$ and $B\mathbb{Y}$ are homotopy equivalents, and by a multiple application of the Simplicial de Rham Theorem [BSS76, pag.51] we have

$$H_{dR}(\mathbb{X}) \simeq H_{\mathrm{sing}}(B\mathbb{X}; \mathbb{R}) \simeq H_{\mathrm{sing}}(B\mathbb{Y}; \mathbb{R}) \simeq H_{dR}(\mathbb{Y}).$$

□

Remark 1.5.1. It is worth noticing that the previous theorem does not say anything about the multiplicative structure on cochains so that it does not say anything about the ring structure of the cohomology. It only provides information about the homotopy type of the isomorphic classes of the differentiable stack represented by them. Some authors call this cohomology as the de Rham cohomology of the differentiable stack represented by X. The reader is recommended to visit the references [Beh04] and [Gin13] as well as the references therein. They provided a panoramic point of view about the relationship between Lie groupoids and differentiable stacks which we find useful.

Chapter 2

Crossed modules

In this chapter we will recall the notions of Lie 2-groups and Lie 2-algebras and their correspondence with crossed modules of both Lie groups and Lie algebras, respectively. We study that correspondence and show that elementary equivalences of crossed modules of Lie groups are in one-to-one correspondence with Morita maps of Lie 2-groups. In the second section we apply the Lie functor to a crossed module of Lie groups yielding a crossed module of Lie algebras. We also define the Lie 2-algebra of a Lie 2-group.

2.1 Lie 2-groups and crossed modules of Lie groups

Definition 2.1.1. A **Lie 2-group** is a groupoid internal to the category of Lie groups.

For a Lie 2-group $G_1 \rightrightarrows G_0$ we have that both the space of arrows and the space of objects are Lie groups, and the structural maps are homomorphisms of Lie groups. In particular, the source and target maps are surjective submersions which are at the same time homomorphisms of Lie groups. Thus, the space of composable arrows G_2 is an embedded submanifold of $G_1 \times G_1$ as well as a subgroup of the direct product. The Lie structure of the direct product restricts to it, therefore G_2 is a Lie subgroup of $G_1 \times G_1$. In general, we shall denote a Lie 2-group by $G_1 \rightrightarrows G_0$ or $\mathbb{G}$ when the context does not demand to specify the base.

Remark 2.1.1. Given that on the space of arrows G_1 there are two different products, for composable pairs $(g, g') \in G_2$ we denote by $g * g'$ its composition and by gg' its product with respect to the Lie group multiplication of G_1. These two product interact in the following way:

$$\begin{aligned}(g * g')(h * h') =& m(g, g')m(h, h')\\ =& m((g, g')(h, h'))\\ =& m(gh, g'h')\\ =& (gh) * (g'h'),\end{aligned}$$

for two composable pairs $(g, g'), (h, h') \in G_2$. The equation $(g * g')(h * h') = (gh) * (g'h')$ is known as the **exchange law**.

Definition 2.1.2. A **morphism of Lie 2-groups** $\Phi : \mathbb{G} \to \mathbb{H}$ is a Lie groupoid morphism internal to the category of Lie groups.

In other words, a morphisms of Lie 2-groups $\Phi : \mathbb{G} \to \mathbb{H}$ is a Lie groupoid morphism which is a Lie group homomorphism on objects and arrows.

Definition 2.1.3. A **crossed module of Lie groups** is a couple of homomorphisms of Lie groups $\rho : H \to G$ and $\alpha : G \to \mathrm{Aut}(H)$ satisfying the following conditions

i. (G-equivariance)
$$\rho(\alpha_g(h)) = g\rho(h)g^{-1} = c_g(\rho(h));$$

ii. (Peiffer identity)

$$\alpha_{\rho(h)}(h') = hh'h^{-1} = c_h(h').$$

for all $g, g' \in G$ and $h, h' \in H$. We denote a crossed module of Lie groups by $[H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)]$.

Remark 2.1.2. There is a one-to-one correspondence between crossed modules of Lie groups and Lie 2-groups. For a crossed module $[H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)]$ we associate the Lie 2-group given by the transformation groupoid corresponding to the action of H on G through ρ

$$H \ltimes G \rightrightarrows G$$

where the Lie group structure in the space of arrows is the semi-direct product $H \rtimes_\alpha G$. Conversely, for a Lie 2-group $G_1 \rightrightarrows G_0$ we associate the crossed module of Lie groups given by $H = \ker(s : G_1 \to G_0)$, $G = G_0$, $\rho = t|_H : H \to G$ and $\alpha_g(h) = 1_g h 1_g^{-1} = c_{1_g}(h)$, for all $g \in G, h \in H$. For more about this correspondence see for instance [BL04, §8.4].

Remark 2.1.3. Due to this correspondence, for practicality, we will abuse the notation and denote $\mathbb{G} = [H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)]$ when is more convenient to see a Lie 2-group as its associated crossed module.

Example 2.1.1 (Linear representations)**.** Let $G \to \mathrm{Aut}(V)$ be a **linear representation** of G on the vector space V. Consider V as abelian Lie group with the addition of vectors and $\rho : V \to G$ with $\rho(v) = e$ for all $v \in V$. Then for all $g \in G$ and $v \in V$ we have

- $\rho(\alpha_g(v)) = e = c_g(e) = c_g(\rho(v))$;
- $\alpha_{\rho(v)}(v') = \alpha_e(v') = v' = c_v(v')$.

Hence a linear representation of G on V can be seen as a crossed module of Lie groups whose associated Lie 2-group is the transformation groupoid $V \ltimes G \rightrightarrows G$. In fact, this Lie 2-group is the vector bundle $\pi : V \rtimes G \to G$ seen as groupoid with $s = \pi$ and $t = \pi$ where composable pairs are vectors in a same fiber and the composition is their addition in the fiber. In particular, for a Lie group G the tangent and cotangent bundles TG and T^*G can be seen as a Lie 2-groups associated to the adjoint and coadjoint representations of G, respectively.

Example 2.1.2. A **central extension of Lie group** that *splits*

$$1 \longrightarrow K \xrightarrow{i} H \underset{\sigma}{\overset{\rho}{\rightleftarrows}} G \longrightarrow 1$$

can be seen as a crossed module of Lie groups with

$$\alpha : G \to \mathrm{Aut}(H), \quad \alpha_g(h) = \sigma(g)h\sigma(g)^{-1}.$$

To see this, note that $H \xrightarrow{\rho} G$ is a principal K-bundle, then for $h \in H$ there is a unique $k \in K$ such that $\sigma(\rho(h)) = hk$. Therefore

$$\begin{aligned}
\alpha_{\rho(h)}(h') =& \sigma(\rho(h))h'\sigma(\rho(h))^{-1} \\
=& (hk)h'(k^{-1}h^{-1}) \\
=& h(kh'k^{-1})h^{-1}, \quad K \text{ is central in } H \\
=& hh'h^{-1}.
\end{aligned}$$

Example 2.1.3. Let G be a Lie group and H be a normal Lie subgroup of G. Then we have a natural crossed module of Lie groups

$$[H \overset{i}{\hookrightarrow} G \xrightarrow{c} \mathrm{Aut}(H)]$$

where the map i is the inclusion and the representation c is the conjugation. In particular, there are interesting sub examples that appear when we consider extreme cases, the first one is given by taking $H = \{e\}$, it allows us to see every Lie group G as a Lie 2-group $G \rightrightarrows G$. In this case the transformation groupoid is in fact the unit groupoid of G. The second one is given for $H = G$, then the associated Lie 2-group is the transformation groupoid $G \ltimes G \rightrightarrows G$ of the left action of G on itself, and its Lie group structure in the space of arrows is $G \rtimes_c G$.

Proposition 2.1.1. *Let* $[H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)]$ *be a crossed module of Lie groups, then*

i. $\ker(\rho)$ *is a central Lie subgroup of* H*;*

ii. $\mathrm{im}(\rho)$ *is a normal subgroup of* G*;*

iii. $\ker(\rho)$ *is* G*-invariant.*

Proof. To see (i), let $k \in \ker(\rho)$, then for every $h \in H$,

$$khk^{-1} = c_k(h) = \alpha_{\rho(k)}(h) = \alpha_e(h) = h$$

thus $hk = kh$ for all $h \in H$, hence $\ker(\rho)$ is central. To see (ii) let $h \in H$ and $g \in G$,

$$g\rho(h)g^{-1} = c_g(\rho(h)) = \rho(\alpha_g(h)) \in \mathrm{im}(\rho)$$

then $\mathrm{im}(\rho)$ is normal subgroup of G. To see (iii), let $k \in \ker(\rho)$ and $g \in G$, then

$$\rho(\alpha_g(k)) = c_g(\rho(k)) = c_g(e) = e,$$

hence $\alpha_g(\ker(\rho)) \subseteq \ker(\rho)$. □

Definition 2.1.4. A **morphism** (F, f) from the crossed module $[H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)]$ to the crossed module $[H' \xrightarrow{\rho'} G' \xrightarrow{\alpha'} \mathrm{Aut}(H')]$ is a couple of homomorphism of Lie groups $F : H \to H'$ and $f : G \to G'$ such that the following diagram commutes

$$\begin{array}{ccc} H & \xrightarrow{F} & H' \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho'} \\ G & \xrightarrow{f} & G' \end{array}$$

and the map F is $(G \xrightarrow{f} G')$-equivariant. That is,

$$F(\alpha_g(h)) = \alpha'_{f(g)}F(h) \quad \text{for all } g \in G \text{ and } h \in H.$$

Remark 2.1.4. We point out that the quotient $G/\mathrm{im}(\rho)$ is not necessarily a Lie group, since $\mathrm{im}(\rho)$ is normal but not closed normal in general. Therefore, the map $\bar{f} : G/\mathrm{im}(\rho) \to G'/\mathrm{im}(\rho')$ is only a group homomorphism.

Definition 2.1.5. A morphism of crossed modules

$$(F, f) : [H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)] \to [H' \xrightarrow{\rho'} G' \xrightarrow{\alpha'} \mathrm{Aut}(H')],$$

is said to be an **elementary equivalence** if it has the following properties

i. $\tilde{F} : \ker(\rho) \xrightarrow{\simeq} \ker(\rho')$ is an isomorphism of Lie groups;

ii. $\tilde{f} : G/\mathrm{im}(\rho) \xrightarrow{\simeq} G'/\mathrm{im}(\rho')$ is an isomorphism of groups, and

iii. the maps f and ρ' are transversal.

Remark 2.1.5. Observe that the existence of an elementary equivalence as above implies that

$$\dim(G) - \dim(H) = \dim(G') - \dim(H').$$

We say that two crossed modules are **elementary equivalent** if they are equivalent in the equivalence relation generated by elementary equivalence of morphisms of crossed modules. One sees that two crossed modules of Lie groups are equivalent if there exists a zig-zag of elementary equivalences going from one to the other (which are not necessarily going all in the same direction).

Remark 2.1.6. It is worth pointing out that there is a classification of crossed modules of groups up to elementary equivalence in terms of degree 3 cohomology classes of its cokernel group with values in its kernel group [Wei94, §6.6 Thm 13].

In the same spirit of the correspondence between crossed modules of Lie groups and Lie 2-groups as in Remark 2.1.2 we have the next proposition which upgrades the correspondence to level of morphisms, yielding an isomorphism of categories.

Proposition 2.1.2. *There is a one-to-one correspondence between morphisms of crossed modules of Lie group and morphisms of Lie 2-groups.*

Proof. It follows by Theorem 43 in [BL04]. □

Proposition 2.1.3. *Let* $[H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)]$ *be a crossed module of Lie groups. If* $\mathrm{im}(\rho)$ *is closed in* G*, then*

$$[\ker(\rho) \xrightarrow{e} G/\mathrm{im}(\rho) \xrightarrow{\bar{\alpha}} \mathrm{Aut}(\ker(\rho))]$$

is a crossed module of Lie groups. We denote by $e : \ker(\rho) \to G/\mathrm{im}(\rho)$ *the constant map that sends all elements to the identity* $[e]$.

Proof. Since $\mathrm{im}(\rho)$ is closed in G by [Bou98, §3.8, Prop.3.8] and Proposition 2.1.1 item (ii) we have $\mathrm{im}(\rho)$ is a normal Lie subgroup of G, hence by [Bou98, §1.6, Prop.11] implies that $G/\mathrm{im}(\rho)$ has a canonical Lie group structure such that the projection map $\pi : G \to G/\mathrm{im}(\rho)$ is a homomorphism of Lie groups. To see the crossed module axioms note that the map

$$\bar{\alpha} : G/\mathrm{im}(\rho) \to \mathrm{Aut}(\ker(\rho)), \quad \bar{\alpha}_{[g]} := \alpha_g,$$

is well-defined. For that, let $g, g' \in G$ and $h \in H$ such that $g = g'\rho(h)$ then

$$\begin{aligned}\alpha_g(k) =& \alpha_{g'}(\alpha_{\rho(h)}(k)) = \alpha_{g'}(c_h(k)) \\ =& \alpha_{g'}(k), \quad \text{by Prop. 2.1.1 item } (i),\end{aligned}$$

for all $k \in \ker(\rho)$, and Proposition 2.1.1 item (iii) implies that $\bar{\alpha}_{[g]} \in \mathrm{Aut}(\ker(\rho))$. Now

i. ($G/\mathrm{im}(\rho)$-equivariance) for $k \in \ker(\rho)$ and $[g] \in G/\mathrm{im}(\rho)$

$$e(\bar{\alpha}_{[g]}(k)) = [e] = c_{[g]}([e]) = c_{[g]}(e(k)).$$

ii. (Peiffer Identity) by Proposition 2.1.1 item (i)

$$\bar{\alpha}_{e(k)}(k') = \alpha_e(k') = k' = c_k(k').$$

□

The next result is motivated by a well-known fact about 2-vector spaces, namely, a 2-vector space is quasi-isomorphism to its cohomology.

Proposition 2.1.4. *Let $[H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)]$ be a crossed module of Lie groups such that $\mathrm{im}(\rho)$ is closed in G. If there exists a homomorphism of Lie groups σ which is a section of the map $\pi : G \to G/\mathrm{im}(\rho)$, then*

$$(\iota, \sigma) : [\ker(\rho) \xrightarrow{e} G/\mathrm{im}(\rho) \xrightarrow{\bar{\alpha}} \mathrm{Aut}(\ker(\rho))] \to [H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)]$$

is an elementary equivalence.

Proof. Let $[g] \in G/\mathrm{im}(\rho)$ and $k \in \ker(\rho)$ then

$$\iota(\bar{\alpha}_{[g]}(k)) = \alpha_{\sigma([g])}(k) = \alpha_{\sigma([g])}(\iota(k)).$$

It is clear that the other axioms in Definition 2.1.5 are satisfied. □

Example 2.1.4. Let $[H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)]$ be a crossed module with ρ a surjective submersion, then we have $[\ker(\rho) \xrightarrow{e} 1 \xrightarrow{\iota} \mathrm{Aut}(\ker(\rho))]$ has a unique structure crossed module and it is elementary equivalent to $[H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)]$.

Proposition 2.1.5. *Let $[H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(G)]$ be a crossed module of Lie groups and $f : G' \to G$ be a homomorphism of Lie groups such that*

i. $\pi \circ f : G' \to G/\mathrm{im}(\rho)$ is a surjective homomorphism of groups, and

ii. f and ρ are transversal,

then

$$[G' \times_G H \xrightarrow{p_1} G' \xrightarrow{\bar{\alpha}} \mathrm{Aut}(G' \times_G H)]$$

has a unique structure of crossed module of Lie groups. Moreover, the natural induced morphism of crossed modules is an elementary equivalence. We call this crossed module the ***pullback crossed module*** *of $[H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(G)]$ by f and denote it by $f^*([H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(G)])$.*

Proof. Consider the following diagram

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & \ker(\rho) & \longrightarrow & H & \xrightarrow{\rho} & G & \xrightarrow{\pi} & G/\mathrm{im}(\rho) & \longrightarrow & 1 \\
 & & \tilde{p}_2 \uparrow & & p_2 \uparrow & & f \uparrow & & \tilde{f} \uparrow & & \\
1 & \longrightarrow & \ker(p_1) & \longrightarrow & G' \times_G H & \xrightarrow{p_1} & G' & \xrightarrow{\pi'} & G'/\mathrm{im}(p_1) & \longrightarrow & 1.
\end{array}$$

The item (ii) supports the Lie group structure in $G' \times_G H$. It is easy to check that

$$\tilde{p}_2 : \ker(p_1) = \{e\} \times_G H \xrightarrow{\simeq} \ker(\rho)$$

is an isomorphism and

$$\tilde{f} : G'/\mathrm{im}(p_1) \to G/\mathrm{im}(\rho)$$

is injective. Thus the item (i) implies that $\tilde{f}$ is an isomorphism. The crossed module structure in $[G' \times_G H \xrightarrow{p_1} G' \xrightarrow{\bar{\alpha}} \mathrm{Aut}(G' \times_G H)]$ is defined as follows: for some $g \in G'$, the map $\tilde{\alpha}_g$ is defined by

$$\tilde{\alpha}_g : G' \times_G H \to G' \times_G H, \quad \tilde{\alpha}_g(g', h) := (gg'g^{-1}, \alpha_{f(g)}(h)).$$

It is well-defined, because for $g \in G$ and $h \in H$ hold that

$$f(gg'g^{-1}) = f(g)f(g')f(g)^{-1} = f(g)\rho(h)f(g)^{-1} = \rho(\alpha_{f(g)}(h)).$$

Thus $\tilde{\alpha}_g(g', h) \in G' \times_G H$. Now let us see the crossed module axioms

i. (G-equivariance)

$$p_1(\tilde{\alpha}_g(g', h)) = p_1(gg'g^{-1}, \alpha_{f(g)}(h)) = gg'g^{-1} = c_g(g') = c_g(p_1(g', h)).$$

ii. (Peiffer identity)

$$\begin{aligned}\tilde{\alpha}_{p_1(g,h)}(g',h') &= \tilde{\alpha}_g(g',h') = (gg'g^{-1}, \alpha_{f(g)}(h')) \\ &= (gg'g^{-1}, \alpha_{\rho(h)}(h')) = (gg'g^{-1}, hh'h^{-1}) \\ &= (g,h)(g',h')(g^{-1},h^{-1}) = c_{(g,h)}(g',h').\end{aligned}$$

□

Corollary 2.1.1. *If a morphism of crossed modules* $(F,f) : [H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)] \to [H' \xrightarrow{\rho'} G' \xrightarrow{\alpha'} \mathrm{Aut}(H')]$ *is an elementary equivalence, then*

$$f^*\left([H' \xrightarrow{\rho'} G' \xrightarrow{\alpha'} \mathrm{Aut}(H')]\right) = [H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)].$$

Proof. It is a straightforward computation. □

Theorem 2.1.1. *There is a one-to-one correspondence between Morita morphisms of Lie 2-groups and elementary equivalences of crossed modules.*

Proof. Let $(F,f) : [H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(G)] \to [H' \xrightarrow{\rho'} G' \xrightarrow{\alpha'} \mathrm{Aut}(H')]$ be an elementary equivalence, then one has that f and ρ' are transversal and the map $\pi' \circ f : G \to G'/\mathrm{im}(\rho')$ is surjective. Let us see that the associated groupoid morphism $\Phi : (H \ltimes G \rightrightarrows G) \to (H' \ltimes G' \rightrightarrows G')$ satisfies the axioms of a Morita morphism, these are:

i. the following diagram is a pullback, and

$$\begin{array}{ccc} H \ltimes G & \xrightarrow{\Phi} & H' \ltimes G' \\ {\scriptstyle (s,t)}\downarrow & \ulcorner & \downarrow{\scriptstyle (s',t')} \\ G \times G & \xrightarrow{f \times f} & G' \times G' \end{array}$$

ii. $\psi : G \times_{G'} (H' \ltimes G') \to G', \quad (g,h,f(g)) \mapsto \rho'(h)f(g)$, is a surjective submersion.

For the pullback diagram note that

$$(G \times G)_{f\times f}\times_{(s',t')}(H' \ltimes G') = \left\{ (g,g',h,f(g)) \middle| \, f(g') = \rho'(h)f(g) \right\},$$

If $(g,g',h,f(g)) \in (G \times G)_{f\times f}\times_{(s',t')}(H' \ltimes G')$ then $f(g'g^{-1}) = \rho'(h)$ implies that $(g'g^{-1},h) \in G_f\times_\rho H'$, since $G_f\times_\rho H' \simeq H$, there exists a unique $\tilde{h} \in H$ with $\rho(\tilde{h}) = g'g^{-1}$ and $F(\tilde{h}) = h$, hence $(\tilde{h},g) \in H \ltimes G$ is such that

$$(g, \rho(\tilde{h})g, F(\tilde{h}), f(g)) = (g,g',h,f(g)).$$

Therefore $H \ltimes G \simeq (G \times G)_{f\times f}\times_{(s',t')}(H' \times G')$. Now to see that ψ is surjective note that for $g' \in G'$, since $\pi' \circ f : G \to G'/\mathrm{im}(\rho')$ is surjective, there is $g \in G$ with $(\pi' \circ f)(g) = \pi'(g')$, so that, $g'f(g)^{-1} \in \mathrm{im}(\rho')$. Thus there is $h' \in H'$ with $\rho'(h') = g'f(g)^{-1}$, then

$$(g,h',f(g)) \in G \times_{G'} (H' \ltimes G') \quad \text{and} \quad \psi(g,h',f(g)) = \rho'(h')f(g) = g'.$$

Finally to see that $\psi : G \times_{G'} (H' \ltimes G') \to G'$ is indeed a submersion, we shall note that it is a Lie group homomorphism such that $\psi_{*,e}$ is surjective, therefore ψ submersion. Let $(g,h,f(g)),(g',h',f(g')) \in$

$G \times_{G'} (H' \ltimes G')$, then

$$\begin{aligned}\psi((g,h,f(g))(g',h',f(g'))) &= \psi(gg', h\alpha_{f(g)}(h'), f(g)f(g')) \\ &= \rho'(h\alpha_{f(g)}(h'))f(gg') = \rho'(h)\rho'(\alpha_{f(g)}(h'))f(g)f(g') \\ &= \rho'(h)f(g)\rho'(h')f(g)^{-1}f(g)f(g') = \rho'(h)f(g)\rho'(h')f(g') \\ &= \psi(g,h,f(g))\psi(g',h',f(g')).\end{aligned}$$

Therefore, ψ is a homomorphism of Lie groups. For $X \in T_eG, Y \in T_eH'$ we have

$$\psi_{*,e}(X + Y + f_{*,e}X) = \rho'_{*,e}Y + f_{*,e}X \tag{2.1}$$

as ρ' and f are transversal,

$$\mathrm{im}(\psi_{*,e}) = \mathrm{im}(\rho'_{*,e}) + \mathrm{im}(f_{*,e}) = T_eG'.$$

Hence $\psi_{*,e}$ is surjective. For the converse implication is known that a Morita morphism induces isomorphisms in the isotropy groups and a homeomorphism between the orbit space [del13]. In our case, for a Lie 2-group $H \ltimes G \rightrightarrows G$ all isotropy groups are isomorphic to $\ker(\rho)$ and the orbit space is $G/\mathrm{im}(\rho)$, then item (i) and (ii) are satisfied in Definition 2.1.5. Finally the essential surjectivity and Equation (2.1) implies that f and ρ are transversal. □

Corollary 2.1.2. *Let $\mathbb{G}$ be a Lie 2-group with associated crossed module of Lie groups $[H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(G)]$. If $f : G' \to G$ is a homomorphism of Lie groups transversal to ρ, then the pullback groupoid $f^!\mathbb{G}$ is a Lie 2-group and its associated crossed module of Lie groups is the pullback crossed module $f^*([H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(G)])$.*

Proof. It is a straightforward computation. □

2.2 Lie 2-algebras and crossed modules of Lie algebras

In this section we introduce the infinitesimal counterparts of Lie 2-groups.

Definition 2.2.1. A **Lie 2-algebra** is a groupoid internal to the category of Lie algebras.

It is worth observing that for a Lie 2-algebra $\mathfrak{g}_1 \rightrightarrows \mathfrak{g}_0$ the space of composable arrows has a Lie algebra structure given by the fiber product of Lie algebra homomorphisms. That is, the restricted Lie algebra structure of the direct product of $\mathfrak{g}_1 \oplus \mathfrak{g}_1$. We shall denote a Lie 2-algebra by $\mathfrak{g}_1 \rightrightarrows \mathfrak{g}_0$ or by $\mathfrak{g}_\bullet$ when the context does not demand to specify the base space. As in the case of Lie 2-groups, we have a one-to-one correspondence between Lie 2-algebras and crossed modules of Lie algebras.

Definition 2.2.2. A **crossed module of Lie algebras** is a couple of Lie algebra homomorphisms $\partial : \mathfrak{h} \to \mathfrak{g}$ and $\mathcal{L} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ such that satisfy the next conditions

i. ($\mathfrak{g}$-equivariance)
$$\partial(\mathcal{L}_X Y) = [X, \partial(Y)] = \mathrm{ad}_X(\partial(Y)); \tag{2.2}$$

ii. (Peiffer identity)
$$\mathcal{L}_{\partial(X)}Y = [X, Y] = \mathrm{ad}_X(Y). \tag{2.3}$$

We denote a crossed module of Lie algebras by $[\mathfrak{h} \xrightarrow{\partial} \mathfrak{g} \xrightarrow{\mathcal{L}} \mathrm{Der}(\mathfrak{h})]$.

Remark 2.2.1. The one-to-one correspondence between Lie 2-algebras and crossed modules of Lie algebras is given as follows. On the one hand, for a Lie 2-algebra $\mathfrak{g}_1 \rightrightarrows \mathfrak{g}_0$ we consider the crossed module of Lie algebras given by $\mathfrak{h} := \ker(s : \mathfrak{g}_1 \to \mathfrak{g}_0)$, $\mathfrak{g} := \mathfrak{g}_0$ and the homomorphisms $\partial := t|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{g}$, $\mathcal{L} : \mathfrak{g} \to \mathrm{Der}(h), \mathcal{L}_x := \mathrm{ad}_{u(x)}$. On the other hand, for a crossed module of Lie algebras $[\mathfrak{h} \xrightarrow{\partial} \mathfrak{g} \xrightarrow{\mathcal{L}} \mathrm{Der}(\mathfrak{h})]$ we consider the Lie 2-algebra whose underlying groupoid structure is the transformation groupoid associated to the action of $\mathfrak{h}$ on $\mathfrak{g}$ through ∂. Thus

$$(\mathfrak{h} \ltimes \mathfrak{g})_s\times_t(\mathfrak{h} \ltimes \mathfrak{g}) \xrightarrow{m} \mathfrak{h} \ltimes \mathfrak{g} \overset{s}{\underset{t}{\rightrightarrows}} \mathfrak{g} \xrightarrow{u} \mathfrak{h} \ltimes \mathfrak{g} \xrightarrow{\iota} \mathfrak{h} \ltimes \mathfrak{g},$$

with structural maps given by

$$s(x,y) = y, \quad t(x,y) = \partial(x) + y, \quad u(y) = (0,y) \quad \iota(x,y) = (-x, y + \partial(x)) \qquad (2.4)$$
$$m((x', y + \partial(x)), (x,y)) = (x + x', y).$$

The Lie algebra structure on the space of arrows is given by the semi-direct product of $\mathfrak{g}$ and $\mathfrak{h}$ through $\mathcal{L} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$.

Remark 2.2.2. Due to this correspondence, for practicality we denote $\mathfrak{g}_\bullet = [\mathfrak{h} \xrightarrow{\partial} \mathfrak{g} \xrightarrow{\mathcal{L}} \mathrm{Der}(\mathfrak{h})]$ when is more convenient to see a Lie 2-algebra as its associated crossed module of Lie algebras.

The application to the Lie functor to any Lie 2-group gives rise to a Lie 2-algebra: Let $\mathbb{G} = [H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)]$ be a Lie 2-group as in Remark 2.1.2. Then applying the Lie functor to its crossed module of Lie groups we get a crossed modules of Lie algebras with $\mathfrak{g} = \mathrm{Lie}(G), \mathfrak{h} = \mathrm{Lie}(H)$ and $\partial = \rho_{*,e}, \mathcal{L} = \mu_{*,e}$, in which $\mu : G \to \mathrm{Aut}(\mathfrak{h}), g \mapsto (\alpha_g)_{*,e}$. The Lie 2-algebra associated to this crossed module of Lie algebras is said to be the **Lie 2-algebra of** $\mathbb{G}$,

$$\mathrm{Lie}(\mathbb{G}) = \mathrm{Lie}([H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)]) = [\mathfrak{h} \xrightarrow{\partial} \mathfrak{g} \xrightarrow{\mathcal{L}} \mathrm{Der}(\mathfrak{h})].$$

For the next example we need to introduce some terminology about $\mathcal{LA}$-groupoids. Let $\mathcal{V}$ be a $\mathcal{VB}$-groupoid over the Lie groupoid $X_1 \rightrightarrows X_0$ as in Definition 1.4.3. **The category of multiplicative sections** of $\mathcal{V}$ is the subcategory $\mathrm{Sec}(\mathbb{X}, \mathcal{V})$ of $\mathrm{Hom}_{\mathrm{LieGpd}}(\mathbb{X}, \mathcal{V})$ whose objects are the multiplicative sections $(\xi, v) : \mathbb{X} \to \mathcal{V}$ of $q_\mathcal{V}$, and morphisms are the natural transformations $\tau : (\xi, v) \Rightarrow (\eta, w)$ such that $1_{q_\mathcal{V}} \bullet \tau = 1_{\mathrm{Id}_\mathbb{X}}$, where $\bullet$ denotes the horizontal composition of natural transformations.

Definition 2.2.3. An **$\mathcal{LA}$-groupoid** is a $\mathcal{VB}$-groupoid $\mathcal{V} \rightrightarrows E$ over the Lie groupoid $X_1 \rightrightarrows X_0$ where both vertical arrows $q_\mathcal{V} : \mathcal{V} \to X_1$ and $q_E : E \to X_0$ are Lie algebroids and all the structural maps are Lie algebroid morphisms.

Example 2.2.1. It was shown in [OW19] that the category of multiplicative sections of an $\mathcal{LA}$-groupoid has a canonical structure of Lie 2-algebra. If $\mathcal{V} \rightrightarrows E$ is an $\mathcal{LA}$-groupoid over $X_1 \rightrightarrows X_0$ with core bundle $C \to X_0$, then the structure of Lie 2-algebra on category of multiplicative sections is given by the crossed module of Lie algebras

$$\left[\Gamma(C) \xrightarrow{\delta} \mathfrak{X}_{mult}(\mathcal{V}) \xrightarrow{D} \mathrm{Der}(\Gamma(C)),\right]$$

where $\delta(c) = c^r - c^l$ and $D_X(c) = [X, c^r]|_{X_0}$, for all $c \in \Gamma(C)$ and $X \in \mathfrak{X}_{mult}(\mathcal{V})$.

Example 2.2.2. For every Lie groupoid $\mathbb{X} := (X_1 \rightrightarrows X_0)$ we have that the tangent groupoid $TX_1 \rightrightarrows TX_0$ is an $\mathcal{LA}$-groupoid over $X_1 \rightrightarrows X_0$ with core bundle given by its Lie algebroid $A \to X_0$. The structure of Lie 2-algebra on the category of multiplicative sections is referred as the **Lie 2-algebra of multiplicative vector fields** and is given by

$$\left[\Gamma(A) \xrightarrow{\delta} \mathfrak{X}_{mult}(\mathbb{X}) \xrightarrow{\mathcal{L}} \mathrm{Der}(\Gamma(A))\right],$$

where $\delta(a) = a^r - a^l$ and $\mathcal{L}_{(\xi,v)}a := [\xi, a^r]\big|_{X_0}$, for all $a \in \Gamma(A)$ and $(\xi, v) \in \mathfrak{X}_{mult}(\mathbb{X})$.

Chapter 3

Principal 2-bundles over Lie groupoids

In this chapter we discuss the notion of **principal 2-bundle over a Lie groupoid**, we follow closely [HOV]. In a few words, it is a categorification of the notion of principal bundle in which both the base and the fibers are categorified with the requirement that all the fibers are isomorphic.

3.1 Principal 2-bundles over a Lie groupoid

We start by introducing some terminology that shall be used throughout the whole chapter. Let $\mathbb{G} = (G_1 \rightrightarrows G_0)$ be a Lie 2-group and $\mathbb{X} = (X_1 \rightrightarrows X_0)$ be a Lie groupoid.

Definition 3.1.1. Let $\mathbb{G}$ be a Lie 2-group and $\mathbb{X}$ be a Lie groupoid. A **right 2-action** of $\mathbb{G}$ on $\mathbb{X}$ is a Lie groupoid morphism

$$\begin{array}{ccc} X_1 \times G_1 & \xrightarrow{a_1} & X_1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ X_0 \times G_0 & \xrightarrow{a_0} & X_0, \end{array}$$

where $X_1 \times G_1 \rightrightarrows X_0 \times G_0$ is the direct product Lie groupoid and both the arrows map a_1 and the objects map a_0 are usual actions of Lie groups. The left actions are defined in a similar manner.

In the forthcoming, we shall simply denote the 2-action by concatenation.

Remark 3.1.1. Given a right 2-action of $\mathbb{G}$ on $\mathbb{X}$, the structural maps of $\mathbb{X}$ are equivariant with respect to the structural maps of $\mathbb{G}$. That is, for all $g \in G_1, g_0 \in G_0$ and $x \in X_1, x_0 \in X_0$ it holds

$$s_X(xg) = s_X(x)s_G(g), \quad t_X(xg) = t_X(x)t_G(g), \quad u_X(x_0g_0) = u_X(x_0)u_G(g_0)$$

and for all $(x, y) \in X_2$ and $(g, g') \in G_2$ the following interchange law is fulfilled

$$(x * y)(g * h) = (xg) * (yh) \tag{3.1}$$

where we are denoting $m_X(x, y) = x * y$ and $m_G(g, h) = g * h$.

Let us consider the following natural examples of actions of Lie 2-groups.

Example 3.1.1. A usual action of a Lie group G on a manifold X can be seen as a 2-action of the Lie 2-group $G \rightrightarrows G$ on the unit groupoid $X \rightrightarrows X$.

Example 3.1.2. Every Lie 2-group $\mathbb{G}$ has a right 2-action on itself by right translations. More generally, for a Lie 2-subgruop $\mathbb{H}$ of $\mathbb{G}$, we have a right action of $\mathbb{H}$ on $\mathbb{G}$ by right translations.

Example 3.1.3. Any Lie 2-group $\mathbb{G}$ acts on itself on the left by conjugation at both the level of arrows and the level of objects. This 2-action induces an adjoint 2-action of $\mathbb{G}$ on its Lie 2-algebra $\mathfrak{g}_1 \rightrightarrows \mathfrak{g}_0$.

Example 3.1.4. Given a right 2-action of $\mathbb{G}$ on $\mathbb{X}$, there is an induced right 2-action of $\mathbb{G}$ on **the tangent groupoid** $T\mathbb{X} = (TX_1 \rightrightarrows TX_0)$ given by the tangent lift of the action of G_i on X_i, for $i = 0, 1$.

Definition 3.1.2. Let $\mathbb{X} = (X_1 \rightrightarrows X_0)$ and $\mathbb{E} = (E_1 \rightrightarrows E_0)$ be two Lie groupoids. A Lie groupoid morphism $\pi : \mathbb{E} \to \mathbb{X}$ is said to be a **fibration** if it satisfies the following conditions

i. the base map $\pi_0 : E_0 \to X_0$ is a surjective submersion, and

ii. the map $\tilde{\pi} : E_1 \to X_1{}_s\times_{\pi_0} E_0, e \mapsto (\pi_1(e), s_E(e))$ is a surjective submersion.

If $\pi : \mathbb{E} \to \mathbb{X}$ is a groupoid fibration, then the **fiber** over a point $x_0 \in X_0$ is defined by

$$\mathbb{E}_x := (\pi_1^{-1}(1_x) \rightrightarrows \pi_0^{-1}(x)) \subseteq \mathbb{E}.$$

Definition 3.1.3. Let $\pi : \mathbb{P} \to \mathbb{X}$ be a groupoid fibration together with a right 2-action of a Lie 2-group $\mathbb{G}$ on $\mathbb{P}$. We say that the action is **principal along** π if the groupoid morphism

$$\begin{array}{ccc} P_1 \times G_1 & \xrightarrow{\Phi} & P_1 \times_{X_1} P_1 \\ \downdownarrows & & \downdownarrows \\ P_0 \times G_0 & \xrightarrow{\phi} & P_0 \times_{X_0} P_0 \end{array} \qquad \begin{array}{c} \Phi(p_1, g_1) = (p_1, p_1 g_1) \\ \\ \phi(p_0, g_0) = (p_0, p_0 g_0), \end{array}$$

is a Lie groupoid isomorphism.

Definition 3.1.4. Let $\mathbb{P}$ and $\mathbb{X}$ be Lie groupoids and $\mathbb{G}$ be a Lie 2-group. A **principal 2-bundle** is a quadruple $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ composed of a fibration $\pi : \mathbb{P} \to \mathbb{X}$ and a principal 2-action of $\mathbb{G}$ along π.

As in the case of principal bundles, the fibers of a principal 2-bundle are isomorphic to the Lie 2-group $\mathbb{G}$. For a principal 2-bundle $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ we call $\mathbb{P}$ **the total space**, $\mathbb{X}$ **the base space** and $\mathbb{G}$ **the structural 2-group**.

Example 3.1.5. A **classical principal bundle** (P, π, M, G) can be seen as a principal 2-bundle regarding manifolds as unit groupoids.

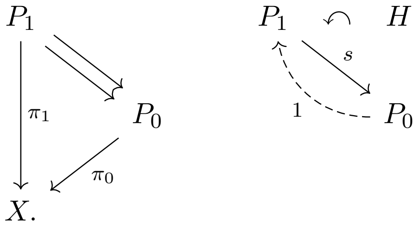
Example 3.1.6. A **principal 2-bundle over a manifold** is a principal 2-bundle $(\mathbb{P}, \pi, X, \mathbb{G})$ where the base space is the unit groupoid of a manifold X. This is a family of isomorphic groupoids that are parametrized by X. If one thinks of the structural 2-group $\mathbb{G}$ in terms of its associated crossed module of Lie groups $\left[H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)\right]$, then one can check that the total space is isomorphic to the groupoid $P_0 \times H \rightrightarrows P_0$ whose structural maps are given by

$$s(p, h) = p, \quad t(p, h) = p\rho(h), \quad 1_p = (p, e) \quad \iota(p, h) = (p\rho(h), h^{-1})$$
$$m((p\rho(h_2), h_1), (p, h_2)) = (p, h_2 h_1),$$

for all $(p, h) \in P \times H$, and $(h, g) \in H \rtimes_\alpha G$. The right action of $H \rtimes_\alpha G$ on $P \times H$ is given by

$$(p, h')(h, g) := (pg, \alpha_{g^{-1}}(h'h)).$$

Note that $s : P_1 \to P_0$ is a principal H-bundle with a section $1 : P_0 \to P_1$.



Example 3.1.7. Let $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ be a principal 2-bundle and $f : \mathbb{Y} \to \mathbb{X}$ be a Lie groupoid morphism, then the pullback by f is a principal 2-bundle with structural 2-group $\mathbb{G}$ and base $\mathbb{Y}$. In particular, the **trivial principal 2-bundle with structural 2-group $\mathbb{G}$ over $\mathbb{X}$** is the pullback of $\mathbb{G} \to \{*\}$ by the morphism $\mathbb{X} \to \{*\}$.

Example 3.1.8. Let $\pi : P \to X_0$ be a principal G-bundle with a left action of a Lie groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ along π which commutes with the right action of G. The transformation groupoid $\mathbb{P} = X_1 \ltimes P \rightrightarrows P$ is a principal 2-bundle over $\mathbb{X}$ with structural 2-group $\mathbb{G}$ seen as the unit groupoid of G. These kind of principal bundles are better known as **principal G-bundle over a Lie groupoid** $X_1 \rightrightarrows X_0$ and have been extensively studied in [LTX07].

Remark 3.1.2 ($\mathcal{PB}$-groupoid)**.** We find useful looking at a principal 2-bundle $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ as a diagram

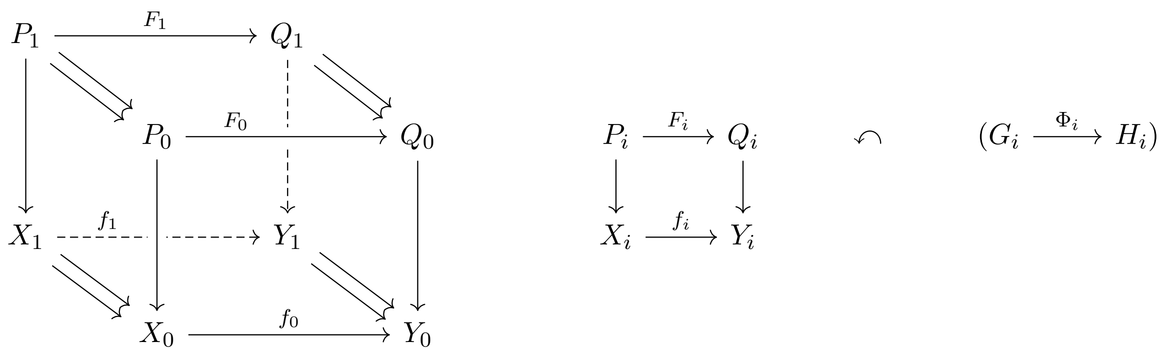
$$\begin{array}{ccc} P_1 & \overset{s_P}{\underset{t_P}{\rightrightarrows}} & P_0 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} \\ X_1 & \overset{s_X}{\underset{t_X}{\rightrightarrows}} & X_0. \end{array} \qquad \curvearrowleft \qquad (G_1 \overset{s_G}{\underset{t_G}{\rightrightarrows}} G_0) \tag{3.2}$$

The principal 2-action of $\mathbb{G}$ is principal on both objects and arrows, that is both (P_1, π_1, X_1, G_1) and (P_0, π_0, X_0, G_0) are principal bundles and the structural maps s_P, t_P are morphisms of principal bundles along of s_G, t_G covering the maps s_X, t_X. Therefore by Theorem A.1.3, the space $P_2 = P_{1s}{\times}_t P_1$ is a principal $G_2 = G_{1s_G}{\times}_{t_G} G_1$-bundle over $X_2 = X_{1s_X}{\times}_{t_X} X_1$ for which $m : P_2 \to P_1$ is a morphism of principal bundles along $m_G : G_2 \to G_1$ covering $m_X : X_2 \to X_1$. Thus a principal 2-bundle is indeed a groupoid internal to $\mathcal{PB}$ the category of principal bundles.

Example 3.1.9. Let $\mathbb{G}$ be a Lie 2-group and $\mathbb{H}$ be a Lie 2-subgroup such that $H_i \subseteq G_i$ is a closed subgruop for $i = 0, 1$. Then $\mathbb{G}$ is a principal 2-bundle over the Lie groupoid, $G_1/H_1 \rightrightarrows G_0/H_0$ with structural 2-group $\mathbb{H}$.

We introduce now the notion of morphism between principal 2-bundles.

Definition 3.1.5. Let $(\mathbb{P}, \pi_P, \mathbb{X}, \mathbb{G})$ and $(\mathbb{Q}, \pi_Q, \mathbb{Y}, \mathbb{H})$ be two principal 2-bundles. A **2-bundle morphism** (F, f, Φ) from $(\mathbb{P}, \pi_P, \mathbb{X}, \mathbb{G})$ to $(\mathbb{Q}, \pi_Q, \mathbb{Y}, \mathbb{H})$ is a triple consisting of a Lie groupoid morphism $F : \mathbb{P} \to \mathbb{Q}$ covering the morphism $f : \mathbb{X} \to \mathbb{Y}$, and a Lie 2-group morphism $\Phi : \mathbb{G} \to \mathbb{H}$ such that $F_i : P_i \to Q_i$ is a bundle map along $\Phi_i : G_i \to H_i$ covering $f_i : X_i \to Y_i$, for each $i = 0, 1$.



Proposition 3.1.1. *Let $(\mathbb{P}, \pi_P, \mathbb{X}, \mathbb{G})$ and $(\mathbb{Q}, \pi_Q, \mathbb{X}, \mathbb{G})$ be two principal 2-bundles. If $(F, \mathrm{Id}_{\mathbb{X}}, \mathrm{Id}_{\mathbb{G}}) : (\mathbb{P}, \pi_P, \mathbb{X}, \mathbb{G}) \to (\mathbb{Q}, \pi_Q, \mathbb{X}, \mathbb{G})$ is a 2-bundle morphism then the Lie groupoid morphism $F : \mathbb{P} \to \mathbb{Q}$ is an isomorphism.*

Proof. The Lie groupoid morphism F consists of bundle morphisms that cover the identity and are equivariant with respect to the identity, then by Theorem A.1.2 F is a Lie groupoid isomorphism. $\square$

3.2 Morita morphisms

In this section we will study the some conditions under which 2-bundle morphisms become Morita maps internal to the category of principal bundles.

Theorem 3.2.1. *Let* $(\mathbb{P}, \pi_P, \mathbb{X}, \mathbb{G})$ *and* $(\mathbb{Q}, \pi_Q, \mathbb{Y}, \mathbb{H})$ *be two principal 2-bundles and* $(F, f, \Phi) : (\mathbb{P}, \pi_P, \mathbb{X}, \mathbb{G}) \to (\mathbb{Q}, \pi_Q, \mathbb{Y}, \mathbb{H})$ *be a 2-bundle morphism. If both* f *and* Φ *are Morita morphisms then* F *is a Morita morphism.*

Proof. Following Definition 1.4.4 the groupoid morphism F is fully-faithful if the following diagram is a pullback

$$\begin{array}{ccc} P_1 & \xrightarrow{F} & Q_1 \\ \downarrow{\scriptstyle (s_P,t_P)} & & \downarrow{\scriptstyle (s_Q,t_Q)} \\ P_0 \times P_0 & \xrightarrow{F_0\times F_0} & Q_0 \times Q_0, \end{array}$$

meaning that the map

$$\overline{F} : P_1 \to (P_0 \times P_0)_{F_0\times F_0}\times_{(s_Q,t_Q)} Q_1, \quad \overline{F}(x) = (s_P(x), t_P(x), F(x)),$$

is a diffeomorphism. Observe that $\overline{F}$, is indeed, a bundle morphism along

$$\overline{\Phi} : G_1 \to (G_0 \times G_0)_{\phi\times\phi}\times_{(s_H,t_H)} H_1, \quad \overline{\Phi}(g) = (s_G(g), t_G(g), \Phi(g))$$

covering the map $\overline{f}$. That is

$$\begin{array}{ccc} P_1 & \xrightarrow{\overline{F}} & (P_0 \times P_0)_{F_0\times F_0}\times_{(s_Q,t_Q)} Q_1, \\ {\scriptstyle \pi_{P_1}}\downarrow & & \downarrow{\scriptstyle (\pi_{P_0}\times\pi_{P_0})\times\pi_{Q_1}} \\ X_1 & \xrightarrow{\overline{f}} & (X_0 \times X_0)_{f_0\times f_0}\times_{(s_X,t_X)} X_1. \end{array} \quad \curvearrowleft \quad (G_1 \xrightarrow{\overline{\Phi}} (G_0 \times G_0)_{\phi\times\phi}\times_{(s_H,t_H)} H_1)$$

Thus, as $\overline{\Phi}$ is a Lie group isomorphism and $\overline{f}$ is a diffeomorphism, then the commutativity of the previus diagram and the Rank Theorem imply that $\overline{F}$ is a diffeomorphism. Therefore, F is fully faithful. To see that F is essentially surjective note that $(P_0{}_{F_0}\times_{s_Q} Q_1, \pi_{P_0} \times \pi_{Q_1}, X_0{}_{f_0}\times_{s_Y} Y_1, G_0 \times_{H_0} H_1)$ is a principal bundle and the map $t_Q \circ \mathrm{pr}_2 : P_0{}_{F_0}\times_{s_Q} Q_1 \to Q_0$ is a bundle map along the homomorphism of Lie groups

$$t_H \circ \mathrm{pr}_2 : G_0{}_{\phi}\times_{s_H} H_1 \to H_0,$$

covering the map $t_X \circ \mathrm{pr}_2$. That is

$$\begin{array}{ccc} P_0 \times_{Q_0} Q_1 & \xrightarrow{t_Q\circ\mathrm{pr}_2} & Q_0, \\ {\scriptstyle \pi_{P_0}\times\pi_{Q_1}}\downarrow & & \downarrow{\scriptstyle \pi_{Q_0}} \\ X_0 \times_{Y_0} Y_1 & \xrightarrow{t_X\circ\mathrm{pr}_2} & Y_0. \end{array} \quad \curvearrowleft \quad (G_0 \times_{H_0} H_1 \xrightarrow{t_H\circ\mathrm{pr}_2} H_0)$$

Thus, as the maps $t_X \circ \mathrm{pr}_2$ and $t_H \circ \mathrm{pr}_2$ are surjective then the map $t_Q \circ \mathrm{pr}_2$ is surjective. Moreover, we get that

$$\begin{aligned} \mathrm{rank}(t_Q \circ \mathrm{pr}_2) \geq& \mathrm{rank}(t_X \circ \mathrm{pr}_2) + \mathrm{rank}(t_H \circ \mathrm{pr}_2) \\ =& \dim(Y_0) + \dim(H_0) = \dim(Q_0) \end{aligned}$$

therefore, the rank of $t_Q \circ \mathrm{pr}_2$ is maximal, then it is a submersion. □

The next example shows that the converse of the previous theorem is not true in general.

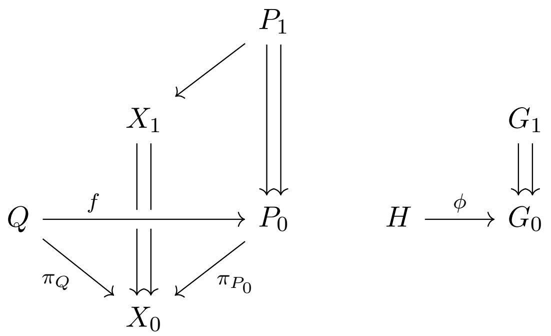
Example 3.2.1. Let us consider the trivial principal bundles $(\mathbb{R}^2 \times (\mathbb{R}, +), \mathrm{pr}_1, \mathbb{R}^2, (\mathbb{R}, +))$ and $(\mathbb{R}\times$

$(\mathbb{R}^2,+), \mathrm{pr}_1, \mathbb{R}, (\mathbb{R}^2,+))$, and the bundle morphism (Ψ,ψ,ι) where $\Psi(x,y;z) = (x;y,z), \psi(x,y) = x, \iota(z) = (0,z)$

$$\begin{array}{ccc} \mathbb{R}^2 \times (\mathbb{R},+) & \xrightarrow{\Psi} & \mathbb{R} \times (\mathbb{R}^2,+), \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathbb{R}^2 & \xrightarrow{\psi} & \mathbb{R}. \end{array} \qquad (\mathbb{R},+) \xrightarrow{\iota} (\mathbb{R}^2,+)$$

When we regard these principal bundles as principal 2-bundles as in Example 3.1.5 we get that (Ψ,ψ,ι) is a morphism of principal 2-bundles in which Ψ is a Morita map but ψ and ι are not Morita maps.

Proposition 3.2.1. *Let $(\mathbb{P},\pi_P,\mathbb{X},\mathbb{G})$ be a principal 2-bundle and $Q \to X_0$ be a principal H-bundle. If $f : Q \to P_0$ is a bundle morphism along the homomorphism $\phi : H \to G_0$ covering the identity in X_0,*



such that the pullback groupoid $f^!\mathbb{P}$ is defined, then it is a principal 2-bundle over $\mathbb{X}$ with structural 2-group $\phi^!\mathbb{G}$.

Proof. To see that $f^!\mathbb{P}$ is well-defined it suffices to assume that $f \times f$ and (s_P, t_P) are transversal. And by Theorem A.1.4, this is equivalent to saying that $\phi \times \phi$ and (s_G, t_G) are transversal. Note that if the crossed module associated to $\mathbb{G}$ is $[H \xrightarrow{\rho} G \xrightarrow{\alpha} \mathrm{Aut}(H)]$ then ϕ and ρ are transversal, and implying that $\phi \times \phi$ and (s_G, t_G) are transversal. Let us suppose that any of these conditions are satisfied, then by Theorem A.1.3 the space of arrows of $f^!\mathbb{P}$

$$Q_f{\times}_{t_P} P_{1\,s_P}{\times}_f Q = (Q \times Q)_{f\times f}{\times}_{(s_P,t_P)} P_1$$

is a principal $H_\phi{\times}_{t_G} G_{1\,s_G}{\times}_\phi H$-bundle over X_1. One can check that $f^!\mathbb{P}$ is a principal $\phi^!\mathbb{G}$-bundle over $\mathbb{X}$ and $f^! : f^!\mathbb{P} \to \mathbb{P}$ is a 2-bundle morphism along $\phi^! : \phi^!\mathbb{G} \to \mathbb{G}$ covering the identity of $\mathbb{X}$. $\square$

Chapter 4

The Atiyah sequence of a principal 2-bundle over a Lie groupoid

In this chapter we introduce the Atiyah sequence of a principal 2-bundle. Just as a usual principal bundle has an associated exact sequence of Lie algebroids, its Atiyah sequence (1.2), principal 2-bundles have an Atiyah sequence wich is exact in the category of $\mathcal{LA}$-groupoids.

4.1 The adjoint 2-bundle

In the same way that a principal bundle has an adjoint bundle, that is, a vector bundle associated to the principal bundle through the adjoin action of the structural group on its Lie algebra, for a principal 2-bundle $\mathbb{P}$ we have an **adjoint 2-bundle** $\mathrm{Ad}(\mathbb{P})$. This 2-bundle is an $\mathcal{LA}$-groupoid over $\mathbb{X}$ defined as follows. Initially, given that the vertical arrows of $\mathbb{P}$ are principal bundles, we can take their respective adjoint bundles,

$$\begin{array}{ccc} P_1 \times_{G_1} \mathfrak{g}_1 & \rightrightarrows & P_0 \times_{G_0} \mathfrak{g}_0 \\ \downarrow & & \downarrow \\ X_1 & \rightrightarrows & X_0. \end{array}$$

Since the structural maps of the principal 2-bundle are equivariant maps between the vertical principal bundles, then these induce well-defined maps between the associated bundles. Observe that the outcome of this process is a $\mathcal{VB}$-groupoid. Moreover, these two vertical vector bundles inherit the structure of Lie algebroid with null anchor, then each one is a bundle of Lie algebras, actually it is an $\mathcal{LA}$-groupoid. We denote

$$\mathrm{Ad}(\mathbb{P}) := (P_1 \times_{G_1} \mathfrak{g}_1 \rightrightarrows P_0 \times_{G_0} \mathfrak{g}_0)\,. \tag{4.1}$$

Definition 4.1.1. The **adjoint 2-bundle** of a principal 2-bundle $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ is the $\mathcal{LA}$-groupoid $(\mathrm{Ad}(\mathbb{P}), \tilde{\pi}, \mathbb{X})$ defined in (4.1).

It was shown in [OW19] that the category of multiplicative sections of an $\mathcal{LA}$-groupoid has a canonical structure of Lie 2-algebra, see Example 2.2.1. Then with the aim of making explicit this structure for the adjoint 2-bundle we shall compute its core bundle and its space of multiplicative sections.

Proposition 4.1.1. *The core bundle of the adjoint 2-bundle* $\mathrm{Ad}(\mathbb{P})$ *is the Lie algebra bundle* $(P_1|_{X_0} \times_{G_1} \mathfrak{h}, \tilde{\pi}_0, X_0)$ *with the core anchor map given by*

$$\partial :\, P|_{X_1} \times_{G_1} \mathfrak{h} \to P_0 \times_{G_0} \mathfrak{g}_0, \quad \partial([q, v]) = [t_P(q), \partial(v)]\,.$$

Moreover, the core of $\mathrm{Ad}(\mathbb{P})$ *is isomorphic to the Lie algebra bundle*

$$\begin{array}{c} P_0 \times_{G_0} \mathfrak{h} \\ \downarrow \\ X_0, \end{array}$$

the associated bundle of (P_0, π_0, X_0, G_0) *through the action of* G_0 *on* $\mathfrak{h}$ *given by* $\alpha_* : G_0 \to \mathrm{Aut}(\mathfrak{h}), \alpha_{*g} := (\alpha_g)_{*,e}$.

Proof. Initially we compute directly the core bundle of $\mathrm{Ad}(\mathbb{P})$

$$\begin{aligned} 1^*(\ker(\tilde{s})) &= \left\{[q,v] \in P_1 \times_{G_1} \mathfrak{g} \,|\, s_{G*}(v) = 0,\ \tilde{\pi}_1([q,v]) = 1^X_x, \text{ for some } x \in X_0\right\} \\ &= P_1|_{X_0} \times_{G_0} \mathfrak{h} \end{aligned}$$

and the core anchor map

$$\partial([q,v]) = \tilde{t}\,[q,v] = [t_P(q), t_{G*}(v)] = [t_P(q), \partial(v)]\,.$$

To see the isomorphism $P_1|_{X_0} \times_{G_0} \mathfrak{h} \simeq P_0 \times_{G_0} \mathfrak{h}$, observe that for $[q,v] \in P_1|_{X_0} \times_{G_0} \mathfrak{h}$ there exists $x \in X_0$ with $1^X_x = \pi_1(q)$, then if we consider the point $1_{Pt_P(q)} \in P_1$ we have that $1_{Pt_P(q)}$ and q are in the same fiber,

$$\begin{aligned} \pi_1(1_{Pt_P(q)}) =& 1^X_{\pi_0(t_P(q))} = 1^X_{t^X \pi_1(q)} \\ =& 1^X_{t^X(1^X_x)} = 1^X_x \\ =& \pi_1(q). \end{aligned}$$

Thus, there exists a unique $h \in G_1$ such that $q = 1_{Pt_P(q)}h$. Note that if we take the target map it holds that

$$t_P(q) = t_P(1_{Pt_P(q)})t_G(h) = t_P(q)t_G(h)$$

so $t_G(h) = e$, then $h \in \ker(t_G)$. Thus, on the one side if we consider $G_1 = H \rtimes_\alpha G$ then $h \in \ker(t_G)$ is equivalent to $h = (z, \rho(z^{-1}))$ for some $z \in H$. Then one has that for all $a \in H$

$$c_{(z,\rho(z^{-1}))}(a,e) = (a,e),$$

hence $c_h|_H = \mathrm{Id}_H$ then for all $v \in \mathfrak{h}$ holds that

$$\mathrm{Ad}_h(v) = v.$$

Therefore, for all $[q,v]$ in $P_1|_{X_1} \times_{G_0} \mathfrak{h}$ we have

$$[q,v] = \left[1_{Pt_P(q)}, v\right].$$

On the other side, for $g_1 \in G_1$ one has that there exists $k \in \ker(t_G)$ such that $g_1 = k1_G(t_G(g_1))$, indeed $g_1 = (h,g) = (h, \rho(h^{-1}))(e, \rho(h)g)$, then

$$\begin{aligned} \left[qg_1, \mathrm{Ad}_{g_1^{-1}}v\right] &= \left[1_{Pt_P(q)}1_G(t_G(g_1)), \mathrm{Ad}_{g_1^{-1}}(v)\right] \\ &= \left[1_{Pt_P(q)}1_G(t_G(g_1)), \mathrm{Ad}_{t_G(g_1)^{-1}}\mathrm{Ad}_{k^{-1}}(v)\right] \\ &= \left[1_{Pt_P(q)}1_G(t_G(g_1)), \mathrm{Ad}_{t_G(g_1)^{-1}}(v)\right]. \end{aligned}$$

In conclusion, the map

$$P_1|_{X_0} \times_{G_1} \mathfrak{h} \to P_0 \times_{G_0} \mathfrak{h},\ [q,v] \mapsto [t_P(q), v]$$

induces an isomorphism of vector bundles. □

A Lie groupoid morphism $f:\mathbb{P}\to\mathfrak{g}_\bullet$ is Ad-equivariant if it is compose of both at arrows level, and at objects level by Ad-equivariant maps in the classical sense, that is, the maps $f_1:P_1\to\mathfrak{g}_1$ and $f_0:P_0\to\mathfrak{g}_0$ the next identities hold

$$f_1(qg)=\mathrm{Ad}^{G_1}_{g^{-1}}f_1(q),\quad f_0(pg)=\mathrm{Ad}^{G_0}_{g^{-1}}f_0(p).$$

Proposition 4.1.2. *The set of multiplicative sections of* Ad($\mathbb{P}$) *is in one-to-one correspondence with the set of* Ad*-equivariant Lie groupoid morphism from* $\mathbb{P}$ *to the Lie 2-algebra* $\mathfrak{g}_\bullet$*,*

$$\mathfrak{X}_{mult}(\mathrm{Ad}(\mathbb{P}))\simeq C^\infty_{mult}(\mathbb{P},\mathfrak{g}_\bullet)^{\mathbb{G}}.$$

Proof. To see this statement it suffices to recall that for a principal bundle (P,π,M,G) it holds that $\Gamma(\mathrm{Ad}(P))\simeq C^\infty(P;\mathfrak{g})^G$. Thus, for $s\in\Gamma(\mathrm{Ad}(\mathbb{P}))$ we have a Lie groupoid morphism composed by sections of adjoint bundles

$$\begin{array}{ccccl} X_1 & \xrightarrow{s_1} & P_1\times_{G_1}\mathfrak{g}_1, & & \tilde{\pi}_1\circ s_1=\mathrm{Id}_{X_1},\\ \downdownarrows & & \downdownarrows & & \\ X_0 & \xrightarrow{s_0} & P_0\times_{G_0}\mathfrak{g}_0, & & \tilde{\pi}\circ s_0=\mathrm{Id}_{X_0}.\end{array}$$

Then there are equivariant maps $f_1\in C^\infty(P_1,\mathfrak{g}_1)^{G_1}$ and $f_0\in C^\infty(P_0,\mathfrak{g}_0)^{G_0}$ such that

$$s_1(x)=[q,f_1(q)]\,,\quad s_0(x')=[p,f_0(p)]$$

for $q\in P_1,p\in P_0$ with $\tilde{\pi}_1(q)=x,\tilde{\pi}_0(p)=x'$. Finally, it is straightforward to see that the section s is a Lie groupoid morphism if and only if the map $f:\mathbb{P}\to\mathfrak{g}_\bullet$ is a Lie groupoid morphism. □

Proposition 4.1.3. *For* $f\in C^\infty(P_0;\mathfrak{h})^{G_0}$*, the following hold:*

i. the map

$$f^r:P_1\to\mathfrak{g}_1,\ f^r(q):=f(t_P(q))$$

is an Ad*-equivariant map which induces a right-invariant section on* Ad($\mathbb{P}$)*, and*

ii. the map

$$f^l:P_1\to\mathfrak{g}_1,\ f^l(q):=\iota_{G*}f(s_G(q))$$

is an Ad*-equivariant map that induces a left-invariant section on* Ad($\mathbb{P}$)*. These correspondences are one-to-one.*

Proof. For item (i) note that for $g\in G_1$ one has that $g^{-1}1_G(t_G(g))\in\ker(t_G)$ and $\mathrm{Ad}_{1_G(t_G(g))}f^r(q)\in\mathfrak{h}$, then

$$\begin{aligned}\mathrm{Ad}_{g^{-1}}(f^r(q))=&\mathrm{Ad}_{g^{-1}1_G(t_G(g))}\left(\mathrm{Ad}_{1_G(t_G(g^{-1}))}f^r(q)\right)\\=&\mathrm{Ad}_{1_G(t_G(g^{-1}))}f^r(q)=\mathrm{Ad}_{1_G(t_G(g^{-1}))}f(t_P(q))\\=&f(t_P(q)t_G(g))=f(t_P(qg))\\=&f^r(qg).\end{aligned}$$

Thus, f^r is an Ad-equivariant map. Let us consider the associated section $s_{f^r}\in\Gamma(P_1\times_{G_0}\mathfrak{g}_1)$, then for $x\in X_1$ and $q\in P_1$ with $\pi_1(q)=x$

$$\begin{aligned}s_{f^r}(1^X_{t^X(x)})\tilde{0}_x=&[1_P(t_P(q)),f^r(1_P(t_P(q)))]\,[q,0_e]\\=&[1_P(t_P(q))*q,m_{G*}(f(t_P(q)),0_e)]\\=&[q,f(t_P(q))]=[q,f^r(q)]\\=&s_{f^r}(x).\end{aligned}$$

Now for item (ii) one has that

$$\begin{aligned}
f^l(qg) =& \iota_{G*}f(s_P(q)s_G(g)) \\
=& \iota_{G*}\mathrm{Ad}_{1_G(s_G(g^{-1}))}f(s_P(q)) \\
=& \mathrm{Ad}_{1_G(s_G(g^{-1}))}\iota_{G*}f(s_P(q)) \\
=& \mathrm{Ad}_{1_G(s_G(g^{-1}))}f^l(q) \\
=& \mathrm{Ad}_{g^{-1}1_G(s_G(g))}\left(\mathrm{Ad}_{1_G(s_G(g^{-1}))}f^l(q)\right) \\
=& \mathrm{Ad}_{g^{-1}}f^l(q).
\end{aligned}$$

Therefore, f^l is an Ad-equivariant map. Now let us see that the associated section s_{f^l} is left-invariant

$$\begin{aligned}
\tilde{0}_x s_{f^l}(1^X_{s^X(x)}) =& [q, 0_e]\left[1_{P_{s_P(q)}}, f^l(1_{P_{s_P(q)}})\right] \\
=& [q, 0_e]\left[1_{P_{s_P(q)}}, \iota_{G*}f(s_P(q))\right] \\
=& \left[q * 1_{P_{s_P(q)}}, m_{G*}(0_e, \iota_{G*}f(s_P(q)))\right] \\
=& [q, \iota_{G*}f(s_P(q))] \\
=& \left[q, f^l(q)\right] \\
=& s_{f^l}(x).
\end{aligned}$$

□

Proposition 4.1.4. *For a map* $f \in C^\infty(P_0;\mathfrak{h})^{G_0}$ *we have* $(f^r + f^l, \partial f) \in C^\infty_{mult}(\mathbb{P}, \mathfrak{g}_\bullet)^{\mathbb{G}}$.

Proof. Let us show that

$$\begin{array}{ccc}
P_1 & \xrightarrow{f^r+f^l} & \mathfrak{g}_1 \\
\downarrow\downarrow & & \downarrow\downarrow \\
P_0 & \xrightarrow{\partial f} & \mathfrak{g}_0
\end{array}$$

is a Lie groupoid morphism. Indeed,

$$\begin{aligned}
s_{G*}(f^r + f^l) =& s_{G*}f^r + s_{G*}f^l \\
=& 0 + s_{G*}\iota_{G*}fs_P \\
=& t_{G*}(f \circ s_P) \\
=& (\partial f) \circ s_P,
\end{aligned}$$

and

$$\begin{aligned}
t_{G*}(f^r + f^l) =& t_{G*}ft_P + t_{G*}\iota_{G*}fs_P \\
=& (\partial f) \circ t_P + s_{G*}fs_P \\
=& (\partial f) \circ t_P + 0 \\
=& (\partial f) \circ t_P.
\end{aligned}$$

Finally for $(q, p) \in P_2$ we have

$$\begin{aligned}
(f^r + f^l)(q * p) =& f(t_P(q * p)) + \iota_{G*}f(s_P(q * p)) \\
=& f(t_P(q)) + \iota_{G*}f(s_P(p)) \\
=& f^r(q) + f^l(p) \\
=& m_{G*}(f^r(q) + f^l(q), f^r(p) + f^l(p)).
\end{aligned}$$

The last equality follows from the identity $m_{G*}((X,0),(-Y,\partial Y)) = (X-Y,\partial Y)$. □

Remark 4.1.1. Observe that by Equations (2.4) we obtain an explicit expression for $f^r + f^l$. Namely it is given by $t^*f - s^*f + \partial \circ f \circ s$.

Proposition 4.1.5. *The category of multiplicative sections of the adjoint 2-bundle* $(\mathrm{Ad}(\mathbb{P}), \tilde{\pi}, \mathbb{X})$ *has structure of Lie 2-algebra given by*

$$[\Gamma(P_0 \times_{G_0} \mathfrak{h}) \xrightarrow{\delta} \mathfrak{X}_{mult}(\mathrm{Ad}(\mathbb{P})) \xrightarrow{D} Der(\Gamma(P_0 \times_{G_0} \mathfrak{h}))]$$

where $\delta(f) := (t^*f - s^*f, \partial \circ f \circ s)$ *and* $D_F(f) := [F, f^r]$ *for all* $F \in \mathfrak{X}_{mult}(\mathrm{Ad}(\mathbb{P}))$*, and for all* $f \in \Gamma(P_0 \times_{G_0} \mathfrak{h})$.

Proof. It follows from the fact that $\mathrm{Ad}(\mathbb{P})$ is an $\mathcal{LA}$-groupoid and then applying the Theorem 7.1 in [OW19]. □

4.2 The Atiyah 2-bundle

From the Lie groupoid viewpoint each principal bundle (P, π, X, G) has an associated gauge groupoid $\mathcal{G}(P) = (\mathrm{Pair}(P)/G \rightrightarrows X)$. Actually, this assignment is a functor that sends principal bundles to Lie groupoids, and bundle morphisms to groupoid morphisms. It is well-known that the Lie algebroid of the gauge groupoid is the Atiyah algebroid $\mathrm{At}(P) = (TP/G, [\cdot,\cdot]^R, \tilde{\pi})$, and clearly, this assignment is again a functor, the Lie functor. Then the composition of these functors maps principal bundles into Lie algebroids, and bundle morphisms to algeborid morphisms. Thus, for a principal 2-bundle $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ we have that the next arrangement is an $\mathcal{LA}$-groupoid,

$$\begin{array}{ccc} \mathrm{At}(P_1) & \rightrightarrows & \mathrm{At}(P_0) \\ \downarrow & & \downarrow \\ X_1 & \rightrightarrows & X_0. \end{array} \tag{4.2}$$

It is worth to note that $\mathrm{At}(P_1) \rightrightarrows \mathrm{At}(P_0)$ is the quotient groupoid of the tangent groupoid of $\mathbb{P}$ by the tangent lifting of the 2-action of $\mathbb{G}$, as in Example 3.1.4. We denote the Lie groupoid $\mathrm{At}(P_1) \rightrightarrows \mathrm{At}(P_0)$ by $\mathrm{At}(\mathbb{P})$.

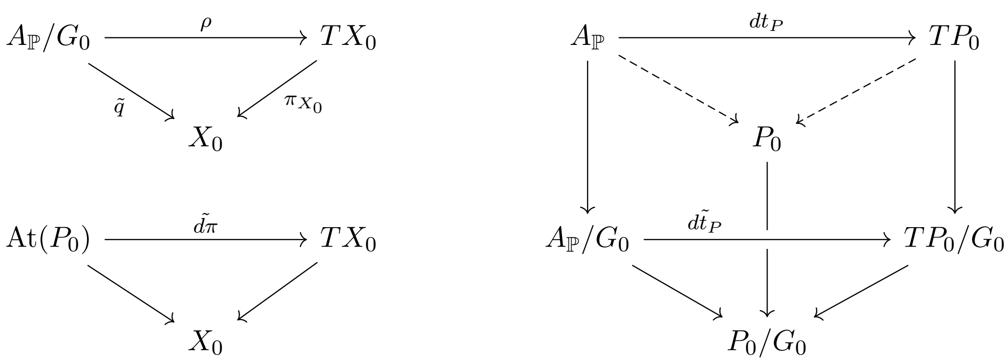
Definition 4.2.1. The **Atiyah 2-bundle** of a principal 2-bundle $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ is the $\mathcal{LA}$-groupoid $(\mathrm{At}(\mathbb{P}), \tilde{\pi}, \mathbb{X})$ as in (4.2).

As in the case of the adjoint 2-bundle we shall determine the core algebroid associated to the Atiyah 2-bundle and its space of multiplicative sections.

Proposition 4.2.1. *Let* $A_{\mathbb{P}}$ *be the Lie algebroid of the total space* $\mathbb{P}$*. Then the core algebroid of the Atiyah 2-bundle* $\mathrm{At}(\mathbb{P})$ *is isomorphic to the Lie algebroid with underlying vector bundle*

$$\begin{array}{c} A_{\mathbb{P}}/G_0 \\ \downarrow \\ X_0, \end{array}$$

The Lie algebra of sections is isomorphic to $\mathfrak{X}^R(\mathbb{P})^{G_1}$*, and the anchor map is* $\rho := \tilde{d\pi_0} \circ \tilde{dt}$*, where* $\tilde{dt_P}$ *is the core anchor map given by the quotient map of the anchor of the Lie algebroid* $A_{\mathbb{P}}$ *by the natural action of the Lie group* G_0*, and* $\tilde{d\pi}$ *is the anchor map of the algebroid* $\mathrm{At}(P_0)$.



Proof. First, note that for $\mathrm{At}(P_1) \rightrightarrows \mathrm{At}(P_0)$ one has that $\ker(\tilde{s}) = \ker(ds_P)/G_1$. Since

$$A_{\mathbb{P}} = 1_P^* \ker(ds_P) = \{(p_0, v) \in P_0 \times \ker(ds_P) \,|\, 1_P(p_0) = q_1(v)\}\,.$$

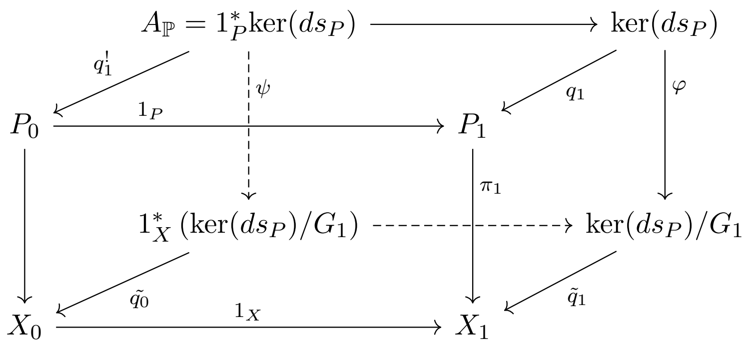
The group G_0 acts naturally on $A_{\mathbb{P}}$ via

$$A_{\mathbb{P}} \times G_0 \to A_{\mathbb{P}},\ (p_0, v) \cdot g = (p_0 g, R_{1_G(g)_*} v).$$

To see that indeed this action is well-defined note that for $p_0 \in P_0, v \in \ker(ds_P) \subseteq TP_1$ and $g \in G_0$ with $1_P(p_0) = q_1(v)$ we have

$$R_{1_G(g)_*} : T_{q_1(v)} P_1 \to T_{q_1(v) 1_G(g)} P_1,$$

then $q_1(R_{1_G(g)_*} v) = q_1(v) 1_G(g) = 1_P(p_0) 1_G(g) = 1_P(p_0 g)$. It is easy to check the action axioms. Consider now the next diagram



The top face is the pullback diagram of the maps 1_P and q_1 conforming the underlying vector bundle of the Lie algebroid $A_{\mathbb{P}}$. The face at the right hand side is given by the quotient of the G_1-equivariant vector bundle $\ker(ds_P) \xrightarrow{q_1} P_1$, recall that the action of G_1 on $\ker(ds_P)$ is the tangent lifting action on the right action of G_1 on P_1. As G_1 acts free and proper over P_1 then its tangent lifting also is free and proper, so it is a good quotient. We denote the quotient map by $\varphi : \ker(ds_P) \to \ker(ds_P)/G_0$. The bottom face is the pullback diagram of the maps 1_X and $\tilde{q}_1$ that conform the underlying vector bundle of the core of $\mathrm{At}(\mathbb{P})$. Let us now prove that the map

$$\psi : A_{\mathbb{P}} \to 1_X^* \left(\ker(ds_P)/G_0\right),\ \psi(p_0, v) = (\pi_0(p_0), \varphi(v))$$

descends to the quotient to an isomorphism of vector bundles. For $(p_0, v) \in A_{\mathbb{P}}$ and $g \in G_0$ one has that

$$\begin{aligned}\psi((p_0, v)g) =& \psi(p_0 g, R_{1_G(g)_*} v) = (\pi_0(p_0 g), \varphi(R_{1_G(g)_*} v)) \\ =& (\pi_0(p_0), \varphi(v)) = \psi(p_0, v)\end{aligned}$$

then ψ descends to $A_{\mathbb{P}}/G_0$. To see that $\tilde{\psi} : A_{\mathbb{P}}/G_0 \to 1_X^* \left(\ker(ds_P)/G_0\right)$ is a surjective map, let $(x_0, \varphi(v)) \in 1_X^* \left(\ker(ds_P)/G_0\right)$ then take some point $p_0 \in P_0$ such that $\pi_0(p_0) = x_0$ and note that

$1_P(p_0)$ and $q_1(v)$ are in the same fiber, because

$$\begin{aligned}\pi_1(1_P(p_0)) =& 1_X\pi_0(p_0) = 1_X(x_0)\\ =& \tilde{q}_1(\varphi(v)) = \pi_1(q_1(v)).\end{aligned}$$

Thus, there exists a unique $h \in G_1$ such that $1_P(p_0) = q_1(v)h$. Then the point $(p_0, R_{h*}v) \in A_{\mathbb{P}}$, to check this $1_P(p_0) = q_1(v)h = q_1(R_{h*}v)$ and

$$\begin{aligned}\psi(p_0, R_{h*}v) =& (\pi_0(p_0), \varphi(R_{h*}v))\\ =& (x_0, \varphi(v)).\end{aligned}$$

To see the injectivity, let $[p_0, v], [q_0, u] \in A_{\mathbb{P}}/G_0$ such that

$$\tilde{\psi}([p_0, v]) = (\pi_0(p_0), \varphi(v)) = (\pi_0(q_0), \varphi(u)) = \tilde{\psi}([q_0, u])$$

then $\pi_0(p_0) = \pi_0(q_0)$ and $\varphi(v) = \varphi(u)$, thus there exists a unique $g' \in G_1$ and $g \in G_0$ with $q_0 = p_0 g$ and $u = R_{g'*}v$, moreover $1_P(p_0) = q_1(v)$ and $1_P(q_0) = q_1(u)$, therefore

$$1_P(q_0) = 1_P(p_0 g) = 1_P(p_0)1_G(g) = q_1(v)1_G(g) = q_1(u),$$

so $q_1(u) = q_1(v)1_G(g)$ and $q_q(u) = q_1(R_{g'*}v) = q(v)g'$, then $1_G(g) = g'$. Hence, there exists a unique $g \in G_0$ such that

$$(p_0, v)g = (p_0 g, R_{1_G(g)_*}v) = (q_0, u).$$

In conclusion ψ is an isomorphism

$$A_{\mathbb{P}}/G_0 = (1_P^*\ker(ds_P))\,/G_0 \simeq 1_X^*\,(\ker(ds_P)/G_1) = \mathrm{Core}(\mathrm{At}(\mathbb{P})).$$

Now to see the isomorphism of Lie algebras at level of section spaces, observe that it is well-known that $\Gamma(\mathrm{At}(P_1)) \simeq \mathfrak{X}(P_1)^{G_1}$ and $\Gamma(A_{\mathbb{P}}) \simeq \mathfrak{X}^R(\mathbb{P})$, then it is straightforward to see that $\Gamma(\mathrm{At}(\mathbb{P}))^R \simeq \mathfrak{X}^R(\mathbb{P})^{G_1}$. Therefore

$$\mathfrak{X}^R(\mathbb{P})^{G_1} = \left\{X \in \mathfrak{X}(P_1) \,|\, R_{q*}X_p = X_{p*q},\ R_{g*}X_p = X_{pg},\ \forall p, q \in \mathbb{P}, g \in G_1\right\}.$$

Now we will exhibit that correspondence between $\Gamma(A_{\mathbb{P}}/G_0)$ and $\mathfrak{X}^R(\mathbb{P})^{G_1}$. For a section $X \in \Gamma(A_{\mathbb{P}})$ the pullback diagram implies that there is a map $X^! : P_0 \to \ker(ds_P)$ such that $q_1 \circ X^! = 1_P$ and $X_{p_0} = (p_0, X^!_{p_0})$ for all $p_0 \in P_0$,

$$\begin{array}{ccc} A_{\mathbb{P}} & \xrightarrow{1_P^!} & \ker(ds_P) \\ {\scriptstyle X}\uparrow\downarrow{\scriptstyle q_1^!} & \overset{X^!}{\nearrow} & \downarrow{\scriptstyle q_1} \\ P_0 & \xrightarrow{1_P} & P_1. \end{array}$$

Thus, $X \in \Gamma(A_{\mathbb{P}})^{G_0}$ if for all $g_0 \in G_0$

$$X_{pg_0} = R_{g_0*}X_p = (pg_0, R_{1_G(g_0)_*}X^!_p).$$

Recall $X \in \Gamma(A_{\mathbb{P}})$ induces $X^R \in \mathfrak{X}^R(\mathbb{P})$ where for all $q \in P_1$

$$X^R_q = R_{q*}X^!_{t_P(q)}.$$

Therefore, if $X \in \Gamma(A_{\mathbb{P}})^{G_0}$ then $X^R \in \mathfrak{X}^R(\mathbb{P})^{G_1}$, because for all $q \in P_1$ and $g \in G_1$

$$X^R_{qg} = R_{qg*}X^!_{t_P(q)t_G(g)} = R_{qg*}R_{1_G(t_G(g))_*}X^!_{t_P(q)}$$

and since the 2-action is multiplicative one has that for an arbitrary $p \in P_1$

$$\begin{aligned} R_{qg} \circ R_{1_G(t_G(g))}(p) =& R_{qg}(p1_G(t_G(g)) \\ =& (p1_G(t_G(g))) * qg \\ =& (p * q)(1_G(t_G(g)) * g) \\ =& (p * q)g \\ =& R_g(p * q) = R_g \circ R_q(p). \end{aligned}$$

Then $R_{qg} \circ R_{1_G(t_G(g))} = R_g \circ R_q$ and

$$\begin{aligned} X^R_{qg} =& (R_{qg} \circ R_{1_G(t_G(g))})_* X^!_{t_P(q)} \\ =& (R_g \circ R_q)_* (X^!_{t_P(q)}) \\ =& R_{g_*} X^R_q. \end{aligned}$$

□

Now we are ready to describe multiplicative sections of $\mathrm{At}(\mathbb{P})$.

Proposition 4.2.2. *The set of multiplicative sections of* $\mathrm{At}(\mathbb{P})$ *is in one-to-one correspondence with the set of multiplicative vector fields on* $\mathbb{P}$ *that are right invariant by the action of the structural 2-group* $\mathbb{G}$

$$\begin{aligned} \mathfrak{X}_{mult}(\mathrm{At}(\mathbb{P})) \simeq& \mathfrak{X}_{mult}(\mathbb{P})^{\mathbb{G}} \\ =& \left\{(X, e) \in \mathfrak{X}(P_1)^{G_1} \times \mathfrak{X}(P_0)^{G_0} \mid (X, e) : \mathbb{P} \to T\mathbb{P} \quad \textit{Lie groupoid morphism}\right\}. \end{aligned}$$

Proof. To see this correspondence let us recall that for a principal G-bundle (P, π, M, G) one has that $\Gamma(\mathrm{At}(P)) \simeq \mathfrak{X}(P)^G$, then if we take $(X, e) \in \mathfrak{X}_{mult}(\mathbb{P})^{\mathbb{G}}$ there exists $(\tilde{X}, \tilde{e}) \in \Gamma(\mathrm{At}(P_1)) \times \Gamma(\mathrm{At}(P_0))$ such that $\tilde{X} \circ \pi = \tilde{\pi} \circ X$, where $\tilde{\pi}$ is the quotient map of the tangent lift action of the structural 2-group $\mathbb{G}$ on $T\mathbb{P}$. Therefore, to check that $(\tilde{X}, \tilde{e})$ is a multiplicative section, let $(x, y) \in X_2$ and $(p, q) \in P_2$ such that $\pi_1(p) = x$ and $\pi_1(q) = y$, then

$$\tilde{X}_{x*y} = \tilde{\pi}_1(X_{p*q}) = \tilde{\pi}(X_p * X_q) = \tilde{\pi}_1(X_p) * \tilde{\pi}_1(X_q) = \tilde{X}_x * \tilde{X}_y.$$

Note that this definition is independent of the fiber since the action is multiplicative, that means, if $(p, q), (p', q') \in P_2$ such that $\pi_1(p) = \pi_1(p') = x$ and $\pi_1(q) = \pi_1(q') = y$ we have that there exists $g, h \in G_1$ such that $p = p'g$ and $q = q'h$, hence, $p * q = (p'g) * (q'h) = (p' * q')(g * h)$, thus

$$\tilde{\pi}_1(X_{p*q}) = \tilde{\pi}_1(X_{(p'*q')(g*h)}) = \tilde{\pi}_1\left(R_{g*h_{*,p'*q'}} X_{p'*q'}\right) = \tilde{\pi}_1(X_{p'*q'}).$$

Therefore, $(\tilde{X}, \tilde{e}) \in \mathfrak{X}_{mult}(\mathrm{At}(\mathbb{P}))$. Now for $(V, e) \in \mathfrak{X}_{mult}(\mathrm{At}(\mathbb{P}))$ we have that there exists $(\bar{V}, \bar{e}) \in \mathfrak{X}(P_1)^{G_1} \times \mathfrak{X}(P_0)^{G_0}$ such that $\tilde{\pi} \circ V = \bar{V} \circ \pi$, then one has that

$$\tilde{\pi}(\bar{V}_{p*q}) = V_{\pi(p*q)} = V_{\pi(p)*\pi(q)} = V_{\pi(p)} * V_{\pi(q)} = \tilde{\pi}(\bar{V}_p * \bar{V}_q),$$

so that there is a unique $g \in G_1$ such that

$$R_{g_{*,p*q}} V_{p*q} = V_p * V_q.$$

Hence, $p * q = (p * q)g$, thus $g = e$. In conclusion, $(\bar{V}, \bar{e}) \in \mathfrak{X}_{mult}(\mathbb{P})^{\mathbb{G}}$. □

Proposition 4.2.3. *The category of multiplicative section of the Atiyah 2-bundle* $(\mathrm{At}(\mathbb{P}), \tilde{\pi}, \mathbb{X})$ *has structure of Lie 2-algebra given by*

$$[\Gamma(A_{\mathbb{P}}/G_0) \xrightarrow{\delta} \mathfrak{X}_{mult}(\mathrm{At}(\mathbb{P})) \xrightarrow{D} Der(\Gamma(A_{\mathbb{P}}/G_0))]$$

where $\delta(\alpha) = \alpha^r - \alpha^l$ *and* $D_X(\alpha) := [X, \alpha^r]$ *for all* $\alpha \in \Gamma(A_{\mathbb{P}}/G_0)$ *and* $(X, e) \in \mathfrak{X}_{mult}(\mathrm{At}(\mathbb{P}))$.

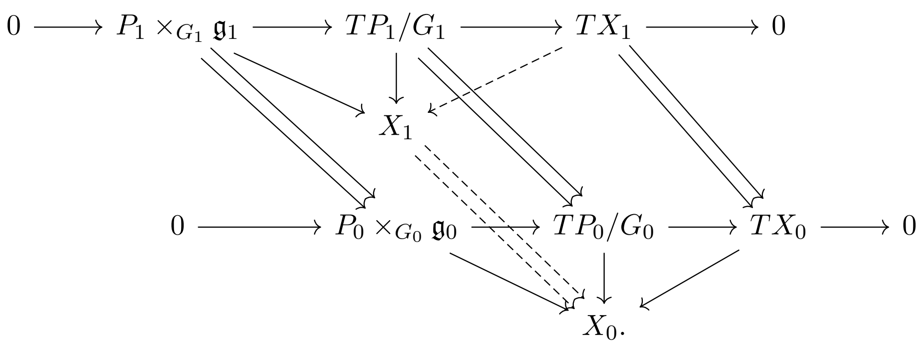
Proof. Combining the previous results with Theorem 7.1 in [OW19] we obtain the following description of the crossed module of multiplicative section of $\mathrm{At}(\mathbb{P})$. □

4.3 Atiyah sequence of a principal 2-bundle over a Lie groupoid

Associated to a principal 2-bundle over a Lie groupoid there is a short exact sequence of $\mathcal{LA}$-groupoids that generalizes the Atiyah sequence for classical principal bundles. For the construction of this short exact sequence let us consider a principal 2-bundle as in the next diagram

$$\begin{array}{ccccc} P_1 & \rightrightarrows & P_0 & \quad \curvearrowleft G_1 & \rightrightarrows G_0 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} & & \\ X_1 & \rightrightarrows & X_0. & & \end{array}$$

Recall that each vertical arrow has the structure of a classical principal bundle, these are (P_1, π, X_1, G_1) and (P_0, π_0, X_0, G_0). Then we have two Atiyah sequences corresponding to the vertical arrows, and as seen in the previous sections the structural maps of $\mathbb{P}$ induce $\mathcal{LA}$-groupoids that fit in the next arrangement



This is a short exact sequence of $\mathcal{LA}$-groupoids.

Definition 4.3.1. For a principal 2-bundle over a Lie groupoid $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ the short exact sequence of $\mathcal{LA}$-groupoid above, denoted by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ad}(\mathbb{P}) & \longrightarrow & \mathrm{At}(\mathbb{P}) & \longrightarrow & T\mathbb{X} & \longrightarrow & 0 \\ & & & \searrow & \downarrow & \swarrow & & & \\ & & & & \mathbb{X}, & & & & \end{array}$$

is called the **Atiyah sequence of the principal 2-bundle** $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$.

Now by Ortiz and Waldron in [OW19], the category of multiplicative sections of an $\mathcal{LA}$-groupoid has structure of 2-term L_∞-algebra (algebraic object to be defined in section 6.3). Thus, the Atiyah sequence of a principal 2-bundle induces a sequence 2-term L_∞-algebras given by its complex of multiplicative sections as follows

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Gamma(P_0 \times_{G_0} \mathfrak{h}) & \xrightarrow{\delta} & \mathfrak{X}_{mult}(\mathrm{Ad}(\mathbb{P})) \\
\downarrow & & \downarrow \\
\Gamma(A_{\mathbb{P}}/G_0) & \xrightarrow{\delta} & \mathfrak{X}_{mult}(\mathrm{At}(\mathbb{P})) \\
\downarrow & & \downarrow \\
\Gamma(A_{\mathbb{X}}) & \xrightarrow{\delta} & \mathfrak{X}_{mult}(T\mathbb{X}) \\
\downarrow & & \\
0 & &
\end{array} \tag{4.3}$$

Chapter 5

2-connection form

In this chapter we introduce the notion of 2-connection form on a principal 2-bundle over a Lie groupoid, for this we follow closely [HOV]. It is a notion that extends the concept of connection 1-form on principal bundles and formalizes the idea of a horizontal distribution invariant by the action of the structural group that is compatible with the groupoid structure. In Section 1 we introduce 2-connection forms and present some natural examples, and we introduce a criterion which ensures the existence of 2-connections. In section 2 we study the curvature of a 2-connection and in Section 3 we introduce the notion of flat up to homotopy 2-connection, which is a kind of 2-connection that is not flat in general, but when viewed it in cohomology it is flat.

5.1 2-connection form

Definition 5.1.1. Let $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ be a principal 2-bundle. A **2-connection form** on $\mathbb{P}$ is a $\mathcal{VB}$-map $\theta_\bullet$ from the tangent groupoid $T\mathbb{P}$ to the product groupoid $\mathbb{P} \times \mathfrak{g}_\bullet$ covering the identity on $\mathbb{P}$, where $\mathfrak{g}_\bullet$ is the Lie 2-algebra of the structural 2-group $\mathbb{G}$,

$$\begin{array}{ccc} TP_1 & \xrightarrow{\theta_1} & P_1 \times \mathfrak{g}_1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ TP_0 & \xrightarrow{\theta_0} & P_0 \times \mathfrak{g}_0. \end{array}$$

and $\theta_1 \in \Omega^1_{dR}(P_1; \mathfrak{g})$ and $\theta_0 \in \Omega^1_{dR}(P_0; \mathfrak{g}_0)$ are usual connection 1-forms.

Looking at 2-connection $\theta_\bullet$ in terms of the associated crossed module of Lie algebras, we have that as $\theta_\bullet$ is a $\mathcal{VB}$-map the following diagram is commutative

$$\begin{array}{ccccccccc} TP_2 & \xrightarrow{dm} & TP_1 & \overset{ds}{\underset{dt}{\rightrightarrows}} & TP_0 & \xrightarrow{du} & TP_1 & \xrightarrow{d\iota} & TP_1 \\ \downarrow{\scriptstyle \theta_1 \times_{P_0} \theta_1} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_1} \\ P_2 \times \mathfrak{g}_2 & \xrightarrow{m \times m} & P_1 \times \mathfrak{g}_1 & \overset{s \times s}{\underset{t \times t}{\rightrightarrows}} & P_0 \times \mathfrak{g}_0 & \xrightarrow{u \times u} & P_1 \times \mathfrak{g}_1 & \xrightarrow{\iota \times \iota} & P_1 \times \mathfrak{g}_1. \end{array}$$

Spelling out the commutativity of the diagram for $\mathfrak{g}_\bullet = (\mathfrak{h} \ltimes \mathfrak{g} \rightrightarrows \mathfrak{g})$ we get the following equations

$$\theta_1 = \omega \oplus \omega_1 \in \Omega^1_{dR}(P_1, \mathfrak{h} \oplus_{\mathcal{L}} \mathfrak{g}), \quad \theta_0 \in \Omega^1_{dR}(P_0, \mathfrak{g}_0).$$

i. The first square give us that for $\omega \in \Omega^1_{dR}(P_1; \mathfrak{h})$

$$\mathrm{pr}_1^* \omega - m^* \omega + \mathrm{pr}_2^* \omega = 0, \tag{5.1}$$

hence ω is a **multiplicative 1-form** on $\mathbb{P}$ with values in $\mathfrak{h}$;

ii. the second square tells us that

$$\omega_1 = s^*\theta_0, \quad \text{and} \quad t^*\theta_0 - s^*\theta_0 = \partial \cdot \omega. \tag{5.2}$$

Recall that $\partial \cdot \omega(X) := \partial(\omega(X))$ for all $X \in TP_1$;

iii. the third square provides us that

$$u^*\omega = 0; \tag{5.3}$$

iv. the fourth square gives us

$$\iota^*\omega = -\omega. \tag{5.4}$$

Remark 5.1.1. The equivariance of θ_1 implies that for $(h, e), (e, g) \in H \rtimes_\alpha G$ one has that

$$R^*_{(h,e)}\theta_1 = \mathrm{Ad}_{(h^{-1},e)} \cdot \theta_1, \quad \text{and} \quad R^*_{(e,g)}\theta_1 = \mathrm{Ad}_{(e,g^{-1})} \cdot \theta_1$$

then

$$R^*_{(h,e)}\omega = \mathrm{Ad}_{h^{-1}} \cdot \omega + (\tilde{\alpha}_{h^{-1}})_{*,e} \cdot s^*\theta_0, \quad \text{and} \quad R^*_{(e,g)}\omega = (\alpha_{g^{-1}})_{*,e} \cdot \omega. \tag{5.5}$$

Moreover, for $X = X_1 + X_2 \in \mathfrak{h} \oplus_{\mathcal{L}} \mathfrak{g}$ with $X_1 \in \mathfrak{h}$ and $X_2 \in \mathfrak{g}$ one has that

$$i_{\widetilde{X}}\theta_1 = \omega(\widetilde{X}) + s^*\theta_0(\widetilde{X}) = X_1 + X_2,$$

implies that $i_{\widetilde{X}}\omega = X_1$ and $i_{\widetilde{X}}s^*(\theta) = X_2$, then

$$i_{\widetilde{X_1}}\omega = X_1, \quad X_1 \in \mathfrak{h}, \quad i_{\widetilde{X_2}}\omega = 0, \quad X_2 \in \mathfrak{g}. \tag{5.6}$$

Remark 5.1.2 ($\mathcal{PBC}$-groupoid)**.** Note that having a principal 2-bundle $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ together with a 2-connection $\theta_\bullet$ is the same thing that connection 1-forms on the principal bundles (P_0, π_0, X_0, G_0), (P_1, π_1, X_1, G_1) and (P_2, π_2, X_2, G_2), see also Remark 3.1.2. Indeed, for θ_0, θ_1 and $\theta_1 \times_P \theta_1$, we have that the structural maps preserve these,

i. $m^*\theta_1 = m_{G*} \cdot \theta_1 \times_{P_0} \theta_1$ by Equation (5.1);

ii. $s^*\theta_0 = s_{G*} \cdot \theta_1, \quad$ and $\quad t^*\theta_0 = t_{G*} \cdot \theta_1$ by Equation (5.2);

iii. $1^*\theta_0 = 1_{G*} \cdot \theta_0$ by Equation (5.3);

iv. $\iota^*\theta_1 = \iota_{G*} \cdot \theta_1$ by Equation (5.4).

Therefore, the couple given by a principal 2-bundle together with a 2-connection is a groupoid internal to the category $\mathcal{PBC}$ of principal bundles with connection.

Now let us see some examples of principal 2-bundles that admit 2-connections and an example that does not admit a 2-connection.

Example 5.1.1. If we consider a classical principal bundle as a principal 2-bundle as in Example 3.1.5, then any classical connection 1-form can be seen as a 2-connection form on it.

Example 5.1.2. Consider a Lie 2-group $\mathbb{G}$ as a principal 2-bundle over a point and with itself acting on the right. Then one has that its Maurer-Cartan 1-form defines a 2-connection on $\mathbb{G}$. Explicitly, for

$$\begin{array}{ccc} \mathbb{G} = H \rtimes_\alpha G & \rightrightarrows & G \\ \downarrow & & \downarrow \\ \{*\} & \rightrightarrows & \{*\} \end{array}$$

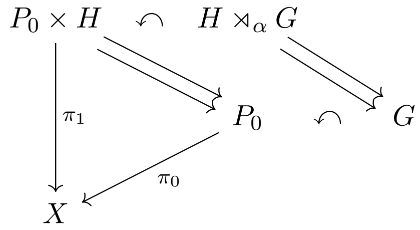
one has

$$\theta_1 = \theta_{MC}^{H \rtimes_\alpha G} = \omega + s^*\theta_0 \in \Omega^1_{dR}(H \rtimes_\alpha G; \mathfrak{h} \oplus_{\mathcal{L}} \mathfrak{g}).$$

where

$$\omega = (\alpha_{\mathrm{pr}_2^{-1}})_*(\mathrm{pr}_1^*\theta^H_{MC}) \in \Omega^1_{dR}(H \rtimes_\alpha G; \mathfrak{h}), \quad \theta_0 = \theta^G_{MC} \in \Omega^1_{dR}(G; \mathfrak{g}).$$

Example 5.1.3. Consider a principal 2-bundle over a manifold as in Example 3.1.6

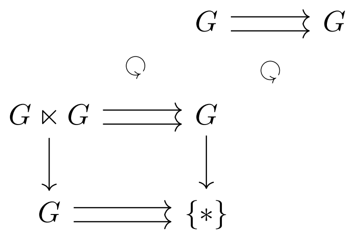


For a connection 1-form θ_0 in P_0, we can determine the differential form ω, in this case the Equations (5.5), (5.3), and (5.6)) imply that

$$\omega = (\tilde{\alpha}_{\mathrm{pr}_2^{-1}})_{*,e} \cdot s^*\theta_0 + \mathrm{pr}_2^*\theta^H_{MC},$$

for θ^H_{MC} the Maurer-Cartan form of H and $\mathrm{pr}_2 : P_0 \times H \to H$. We point out that this example also appears in [Wal18, Ex.5.1.11].

Example 5.1.4. Let G be a Lie group. Consider $G \to \{*\}$ as a principal G-bundle over the Lie groupoid $G \rightrightarrows \{*\}$ as in Example 3.1.8, where the groupoid $G \rightrightarrows \{*\}$ acts on G by left translation. That is,

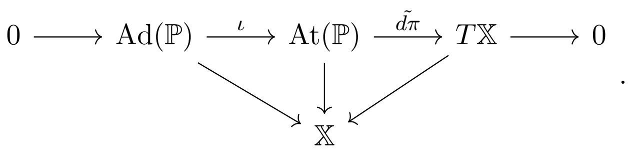


with crossed module associated to the structural 2-group given by $[* \to G]$, and the structural maps of $G \ltimes G \rightrightarrows G$ given by

$$s(g_1, g_2) = g_2, \quad t(g_1, g_2) = g_1g_2, \quad 1(g) = (e, g)$$
$$m((g_2, g_1g), (g_1, g)) = (g_2g_1, g).$$

Then we have $\omega \in \Omega^1_{dR}(G \ltimes G; 0)$ implies $\omega = 0$ and for all connection 1-form $\theta_0 \in \Omega^1_{dR}(G; \mathfrak{g})$ Equation (5.2) holds only when G is a discrete Lie group. Therefore this example shows that **in general there not exists a 2-connection on a principal 2-bundle.** This example appeared [LTX07, ex. 2.3].

Definition 5.1.2. Let us consider a principal 2-bundle $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ and its Atiyah sequence as in Section 4.3,



Then a **multiplicative horizontal lift** of $\mathrm{At}(\mathbb{P})$ is a $\mathcal{VB}$-map $h : T\mathbb{X} \to \mathrm{At}(\mathbb{P})$ such that splits the Atiyah sequence, i.e. $\tilde{d\pi} \circ h = \mathrm{Id}_{T\mathbb{X}}$.

Remark 5.1.3. Observe that to have a 2-connection form $\theta_\bullet$ on $\mathbb{P}$ is equivalent to having a $\mathcal{VB}$-map $\theta : \mathrm{At}(\mathbb{P}) \to \mathrm{Ad}(\mathbb{P})$ that covers the identity on $\mathbb{X}$ such that $\theta \circ \iota = \mathrm{Id}_{\mathrm{Ad}(\mathbb{P})}$.

Proposition 5.1.1. *Let $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ be a principal 2-bundle. There is a one-to-one correspondence between 2-connections on $\mathbb{P}$ and multiplicative horizontal lifts of* $\mathrm{At}(\mathbb{P})$.

Proof. Given h a multiplicative horizontal lift, we define the 2-connection as $\theta := \mathrm{Id}_{At(\mathbb{P})} - h \circ \tilde{d\pi}$. Let us see that $\theta : \mathrm{At}(\mathbb{P}) \to \mathrm{Ad}(\mathbb{P})$ is a $\mathcal{VB}$-map. By the definition it suffices to verify that $\theta(u*v) = \theta(u)*\theta(v)$ for all $(u,v) \in \mathrm{At}(\mathbb{P})_2$. For that let us recall the interchange law for $\mathcal{VB}$-groupoids, see e.g. [GM17, §3]. If $(\gamma_1, \gamma_3) \in \mathrm{At}(\mathbb{P})_2$ and $(\gamma_2, \gamma_4) \in \mathrm{At}(\mathbb{P})_2$ with $\tilde{\pi}(\gamma_1) = \tilde{\pi}(\gamma_2)$ and $\tilde{\pi}(\gamma_3) = \tilde{\pi}(\gamma_4)$ then $(\gamma_1 + \gamma_2) * (\gamma_3 + \gamma_4) = \gamma_1 * \gamma_3 + \gamma_2 * \gamma_4$. Thus

$$\begin{aligned}\theta(u) * \theta(v) =& (u - h\tilde{d\pi}(u)) * (v - h\tilde{d\pi}(v)) \\ =& u * v + (-h\tilde{d\pi}(u)) * (-h\tilde{d\pi}(v)).\end{aligned}$$

Given that the zero section of $\mathrm{At}(\mathbb{P})$ is multiplicative, $0_x * 0_y = 0_{x*y}$ for all $(x,y) \in X_2$, then $(-\gamma_1) * (-\gamma_3) = -(\gamma_1 * \gamma_3)$ for all $(\gamma_1, \gamma_3) \in \mathrm{At}(\mathbb{P})$. Hence

$$\begin{aligned}\theta(u) * \theta(v) =& u * v + (-h\tilde{d\pi}(u)) * (-h\tilde{d\pi}(v)) \\ =& u * v - (h\tilde{d\pi}(u) * h\tilde{d\pi}(v)) \\ =& \theta(u * v).\end{aligned}$$

Now given θ on $\mathbb{P}$ a 2-connection we define the horizontal lift as follows

$$h : T\mathbb{X} \to \mathrm{At}(\mathbb{P}), \quad h(v) = v - \iota(\theta(v))$$

for some $v \in \mathrm{At}(\mathbb{P})$ such that $\tilde{d\pi}(v) = x$. Consider $u, v \in \mathrm{At}(\mathbb{P})$ with $\tilde{d\pi}(u) = \tilde{d\pi}(v) = x$, then one has that $u - v \in \ker(\tilde{d\pi})$, thus $\iota(\theta(u-v)) = u - v$, and

$$\begin{aligned}v - \iota(\theta v) =& u + (v - u) - \iota(\theta(u - (v - u))) \\ =& u - \iota(\theta(u)) + (v - u) - (\iota(\theta(v - u))) \\ =& u - \iota(\theta(u)).\end{aligned}$$

Showing that h is well-defined, it is smooth because the horizontal distribution is smooth as well. Now let us check that h preserves products. Let $(x,y) \in T\mathbb{X}_2$ and $(u,v) \in \mathrm{At}(\mathbb{P})_2$ such that $\tilde{d\pi}_2(u,v) = (x,y)$. Then $h(x) = u - \iota(\theta(u))$ and $h(y) = v - \iota(\theta(v))$. Hence

$$\begin{aligned}h(x * y) =& u * v - \iota(\theta(u * v)) \\ =& u * v - (\iota(\theta(u)) * \iota(\theta(v)) \\ =& u * v + (-\iota(\theta(u))) * (-\iota(\theta(v))) \\ =& h(x) * h(y),\end{aligned}$$

proving that h is multiplicative. □

5.2 The curvature of a 2-connection

Let $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ be a principal 2-bundle with 2-connection form (θ_0, θ_1). Then for $\theta_1 = \omega \oplus s^*\theta_0 \in \Omega^1(P_1, \mathfrak{h} \oplus_{\mathcal{L}} \mathfrak{g})$ the curvature 2-form is given by

$$\Omega_1 = d\theta_1 + \frac{1}{2}[\theta_1, \theta_1] = \hat{\Omega}_1 + s^*\Omega_0, \tag{5.7}$$

where

$$\hat{\Omega}_1 = d\omega + \frac{1}{2}[\omega, \omega] + [s^*\theta_0, \omega]. \tag{5.8}$$

Proposition 5.2.1. *The 2-form $\hat{\Omega}_1 \in \Omega^2(P_1, \mathfrak{h})$ as in (5.8) satisfies*

$$\hat{\Omega}_1 = D_{\theta_1}\omega, \quad D_{\theta_1}\hat{\Omega}_1 = 0.$$

Proof. Given that $s : P_1 \to P_0$ is a map of principal bundles with connection, for X a horizontal vector on P_1, one has that $s_*(X)$ is a horizontal vector on P_0. It implies that for any differential form α on P_0,

$$\begin{aligned} D_{\theta_1}(s^*\alpha) =& (ds^*\alpha)(h_1) = (s^*d\alpha)(h_1) \\ =& d\alpha(s_*h_1) = d\alpha(h_0 s_*) \\ =& s^*((d\alpha)h_0) = s^*(D_{\theta_0}\alpha). \end{aligned}$$

Therefore $D_{\theta_1} \circ s^* = s^* \circ D_{\theta_0}$. On the one hand

$$\begin{aligned} \Omega_1 =& D_{\theta_1}\theta_1 = D_{\theta_1}(\omega + s^*\theta_0) \\ =& D_{\theta_1}\omega + D_{\theta_1}s^*\theta_0 \\ =& D_{\theta_1}\omega + s^*D_{\theta_0}\theta_0 \\ =& D_{\theta_1}\omega + s^*\Omega_0 \end{aligned}$$

thus $\hat{\Omega}_1 = D_{\theta_1}\omega$. On the other hand, we get

$$\begin{aligned} D_{\theta_1}\Omega_1 = 0 =& D_{\theta_1}(\hat{\Omega}_1 + s^*\Omega_0) \\ =& D_{\theta_1}\hat{\Omega}_1 + s^*D_{\theta_0}\Omega_0 \\ =& D_{\theta_1}\hat{\Omega}_1 + s^*0 \\ =& D_{\theta_1}\hat{\Omega}_1, \end{aligned}$$

hence $D_{\theta_1}\hat{\Omega}_1 = 0$. □

Proposition 5.2.2. *The following equation holds:*

$$t^*\Omega_0 - s^*\Omega_0 = \partial \cdot \hat{\Omega}_1. \tag{5.9}$$

Proof. Before checking the identity above we are going to prove two auxiliar statements that shall simplify our computations, $\partial \cdot [s^*\theta_0, \omega] = [s^*\theta_0, \partial \cdot \omega]$ and $\partial \cdot ([\omega, \omega] + 2[s^*\theta_0, \omega]) = (t^* - s^*)[\theta_0, \theta_0]$. For the first one let ξ_1, ξ_2 be arbitrary tangent vectors at some point of P_1 then

$$\begin{aligned} (\partial \cdot [s^*\theta_0, \omega])(\xi_1, \xi_2) =& \partial([s^*\theta_0, \omega](\xi_1, \xi_2)) = \partial([s^*\theta_0(\xi_1), \omega(\xi_2)] - [s^*\theta_0(\xi_2), \omega(\xi_1)]) \\ =& \partial\left(\mathcal{L}_{s^*\theta_0(\xi_1)}(\omega(\xi_2))\right) - \partial(\mathcal{L}_{s^*\theta_0(\xi_2)}(\omega(\xi_1))) \\ =& [s^*\theta_0(\xi_1), \partial(\omega(\xi_2))] - [s^*\theta_0(\xi_2), \partial(\omega(\xi_1))] \quad \text{(by Equation (2.2))} \\ =& [s^*\theta_0, \partial \cdot \omega](\xi_1, \xi_2). \end{aligned}$$

For the second one note that

$$\begin{aligned} \partial \cdot ([\omega, \omega] + 2[s^*\theta_0, \omega]) =& [\partial \cdot \omega, \partial \cdot \omega] + 2[s^*\theta_0, \partial \cdot \omega] \quad \text{(previous statement)}, \\ =& [(t^* - s^*)\theta_0, (t^* - s^*)\theta_0] + 2[s^*\theta_0, (t^* - s^*)\theta_0] \\ =& [t^*\theta_0, t^*\theta_0] - [s^*\theta_0, s^*\theta_0] = t^*([\theta_0, \theta_0]) - s^*([\theta_0, \theta_0]) \\ =& (t^* - s^*)[\theta_0, \theta_0]. \end{aligned}$$

Now we have

$$\begin{aligned}\partial \cdot \hat{\Omega}_1 =&\partial \cdot \left(d\omega + \frac{1}{2}[\omega,\omega] + [s^*\theta_0, \omega]\right)\\ =&d(\partial\cdot\omega) + \frac{1}{2}\partial\cdot([\omega,\omega]+2[s^*\theta_0,\omega])\\ =&d((t^*-s^*)\theta_0) + \frac{1}{2}(t^*-s^*)[\theta_0,\theta_0]\\ =&(t^*-s^*)(d\theta_0 + \frac{1}{2}[\theta_0,\theta_0])\\ =&(t^*-s^*)\Omega_0.\end{aligned}$$

□

Proposition 5.2.3. $\hat{\Omega}_1 \in \Omega^2(P_1;\mathfrak{h})$ *is a multiplicative 1-form.*

$$pr_1^*\hat{\Omega}_1 - m^*\hat{\Omega}_1 + pr_2^*\hat{\Omega}_1 = 0. \tag{5.10}$$

Proof. For this statement first we are going to prove two auxiliary equations,

$$(\mathrm{pr}_1^* - m^* + \ \mathrm{pr}_2^*)[\omega,\omega] = -2[\mathrm{pr}_1^*\omega, \mathrm{pr}_2^*\omega] \tag{5.11}$$

and

$$(\mathrm{pr}_1^* - m^* + \mathrm{pr}_2^*)[s^*\theta_0,\omega] = [\mathrm{pr}_1^*\omega, \mathrm{pr}_2^*\omega]. \tag{5.12}$$

For Equation (5.11)

$$\begin{aligned}(\mathrm{pr}_1^* - m^* + \mathrm{pr}_2^*)([\omega,\omega]) =&[\mathrm{pr}_1^*\omega,\mathrm{pr}_1^*\omega] - [m^*\omega, m^*\omega] + [\mathrm{pr}_2^*\omega,\mathrm{pr}_2^*\omega], \quad \text{by (5.1)}\\ =&[\mathrm{pr}_1^*\omega,\mathrm{pr}_1^*\omega] - [(\mathrm{pr}_1^*+\mathrm{pr}_2^*)\omega, (\mathrm{pr}_1^*+\mathrm{pr}_2^*)\omega] + [\mathrm{pr}_2^*\omega,\mathrm{pr}_2^*\omega]\\ =&-2[\mathrm{pr}_1^*\omega,\mathrm{pr}_2^*\omega].\end{aligned}$$

For Equation (5.12), recall that $s \circ m = s \circ \mathrm{pr}_2, s \circ \mathrm{pr}_1 = t \circ \mathrm{pr}_2$ and Equation (5.1), then

$$\begin{aligned}(\mathrm{pr}_1^* - m^* + \mathrm{pr}_2^*)([s^*\theta_0,\omega]) =&[\mathrm{pr}_1^*s^*\theta_0, \mathrm{pr}_1^*\omega] - [m^*s^*\theta_0, m^*\omega] + [\mathrm{pr}_2^*s^*\theta_0,\mathrm{pr}_2^*\omega]\\ =&[\mathrm{pr}_1^*s^*\theta_0, \mathrm{pr}_1^*\omega] - [\mathrm{pr}_2^*s^*\theta_0, \mathrm{pr}_1^*\omega] - [\mathrm{pr}_2^*s^*\theta_0,\mathrm{pr}_2^*\omega]\\ &+[\mathrm{pr}_2^*s^*\theta_0,\mathrm{pr}_2^*\omega]\\ =&[\mathrm{pr}_1^*s^*\theta_0 - \mathrm{pr}_2^*\theta_0, \mathrm{pr}_1^*\omega]\\ =&[\mathrm{pr}_2^*t^*\theta_0 - \mathrm{pr}_2^*s^*\theta_0, \mathrm{pr}_1^*\omega] = [\mathrm{pr}_2^*(t^*-s^*)\theta_0, \mathrm{pr}_1^*\omega]\\ =&[\mathrm{pr}_2^*(\partial\cdot\omega), \mathrm{pr}_1^*\omega] = [\partial\cdot\mathrm{pr}_2^*\omega, \mathrm{pr}_1^*\omega].\end{aligned}$$

Take ξ_1, ξ_2 tangent vectors at some point of P_1, by Equation (2.3)

$$\begin{aligned}[\partial\cdot\mathrm{pr}_2^*\omega, \mathrm{pr}_1^*\omega](\xi_1,\xi_2) =&[\partial(\mathrm{pr}_2^*\omega(\xi_1)), \mathrm{pr}_1^*\omega(\xi_2)] - [\partial(\mathrm{pr}_2^*\omega(\xi_2)), \mathrm{pr}_1^*\omega(\xi_1)]\\ =&\mathcal{L}_{\partial(\mathrm{pr}_2^*\omega(\xi_1))}(\mathrm{pr}_1^*\omega(\xi_2)) - \mathcal{L}_{\partial(\mathrm{pr}_2^*\omega(\xi_2))}(\mathrm{pr}_1^*\omega(\xi_1)), \quad \text{by (2.2)}\\ =&[\mathrm{pr}_2^*\omega(\xi_1), \mathrm{pr}_1^*\omega(\xi_2)] - [\mathrm{pr}_2^*\omega(\xi_2), \mathrm{pr}_1^*\omega(\xi_1)]\\ =&[\mathrm{pr}_2^*\omega, \mathrm{pr}_1^*\omega](\xi_1,\xi_2).\end{aligned}$$

Therefore

$$(\mathrm{pr}_2^* - m^* + \mathrm{pr}_2^*)([s^*\theta_0,\omega]) = [\mathrm{pr}_2^*\omega, \mathrm{pr}_1^*\omega],$$

and thus

$$
\begin{aligned}
(\mathrm{pr}_1^* - m^* + \mathrm{pr}_2^*)\hat{\Omega}_1 =& (\mathrm{pr}_1^* - m^* + \mathrm{pr}_2^*)(d\omega + \frac{1}{2}([\omega,\omega] + 2[s^*\theta_0,\omega])) \\
=& d(\mathrm{pr}_1^* - m^* + \mathrm{pr}_2^*)\omega) \\
&+ \frac{1}{2}((\mathrm{pr}_1^* - m^* + \mathrm{pr}_2^*)[\omega,\omega] + 2(\mathrm{pr}_1^* - m^* + \mathrm{pr}_2^*)[s^*\theta_0,\omega]) \\
=& 0 + \frac{1}{2}\left(-2[\mathrm{pr}_1^*\omega, \mathrm{pr}_2^*\omega] + 2[\mathrm{pr}_1^*\omega, \mathrm{pr}_2^*\omega]\right) \\
=& 0.
\end{aligned}
$$

□

5.3 Flat up to homotopy 2-connections

Let us consider briefly the notion of Lie 2-algebra valued differential forms on a Lie groupoid introduced in [Wal18, §4]. Let $\mathbb{P} = (P_1 \rightrightarrows P_0)$ be a Lie groupoid and a Lie 2-algebra $\mathfrak{g}_\bullet = \left[\mathfrak{h} \xrightarrow{\partial} \mathfrak{g} \xrightarrow{\mathcal{L}} \mathrm{Der}(\mathfrak{h})\right]$ as in Section 2.2. Here we denote the simplicial differential, see Section 1.5, induced on differential forms of $\mathbb{P}$ by δ

$$\delta : \Omega^\bullet_{dR}(P_n) \to \Omega^\bullet_{dR}(P_{n+1}), \quad \delta = \sum_{i=0}^{n} (-1)^n d_i^*.$$

Definition 5.3.1. A p-form $\Psi \in \Omega^p(\mathbb{P}, \mathfrak{g}_\bullet)$ is a triple $\Psi = (\Psi^a, \Psi^b, \Psi^c)$ composed of differential forms

$$\Psi^a \in \Omega^p_{dR}(P_0, \mathfrak{g}), \quad \Psi^b \in \Omega^p_{dR}(P_1, \mathfrak{h}), \quad \text{and} \quad \Psi^c \in \Omega^{p+1}_{dR}(P_0; \mathfrak{h})$$

such that $\delta\Psi^a = \partial \cdot \Psi^b$ and $\delta\Psi^b = 0$. The differential of a p-form Ψ is a $(p+1)$-form $\mathrm{D}\Psi$ whose components are given by

$$\mathrm{D}\Psi^a = d\Psi^a - (-1)^p \partial \cdot \Psi^c, \quad \mathrm{D}\Psi^b = d\Psi^b - (-1)^p \delta\Psi^c, \quad \mathrm{D}\Psi^c = d\Psi^c.$$

By [Wal18, Lemma 4.1.2] it follows that $(\Omega(\mathbb{P};\mathfrak{g}_\bullet), \mathrm{D}, [\cdot \wedge \cdot])$ is a dgla.

Remark 5.3.1. In these terms, for a principal 2-bundle $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ in which $\mathfrak{g}_\bullet$ denotes the Lie 2-algebra of $\mathbb{G}$, any 2-connection (θ_1, θ_0) can be seen as a differential form on $\mathbb{P}$ taking values in the Lie 2-algebra $\mathfrak{g}_\bullet$,

$$\theta_\bullet = (\theta_0, \omega, 0) \in \Omega^1(\mathbb{P}; \mathfrak{g}_\bullet).$$

Note that by Equations (5.2) and (5.1)

$$\delta(\theta_0) = (t^* - s^*)\theta_0 = \partial \cdot \omega, \quad \text{and} \quad \delta(\omega) = (\mathrm{pr}_1^* - m^* + \mathrm{pr}_2^*)\omega = 0.$$

Let us consider a principal 2-bundle $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ and its Atiyah sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ad}(\mathbb{P}) & \xrightarrow{\iota} & \mathrm{At}(\mathbb{P}) & \xrightarrow{\tilde{d\pi}} & T\mathbb{X} & \longrightarrow & 0 \\
 & & & \searrow & \downarrow & \swarrow & & & \\
 & & & & \mathbb{X} & & & &
\end{array}
$$

Then a multiplicative horizontal lift h induces a lift of multiplicative vector fields

$$h_m : \mathfrak{X}_m(\mathbb{X}) \to \mathfrak{X}_m(\mathrm{At}(\mathbb{P})), \quad h_m(\xi, u) := (h \circ \xi, h_0 \circ u).$$

It induces a linear splitting of the sequence of Lie 2-algebras of multiplicative sections of the Atiyah sequence as follows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(P_0\times_{G_0}\mathfrak{h}) & \xrightarrow{\iota_c} & \Gamma(A_{\mathbb{P}}/G_0) & \underset{h_c}{\overset{\pi_c}{\rightleftarrows}} & \Gamma(A_{\mathbb{X}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{X}_m(\mathrm{Ad}(\mathbb{P})) & \xrightarrow{\iota_m} & \mathfrak{X}_m(\mathrm{At}(\mathbb{P})) & \underset{h_m}{\overset{\pi_m}{\rightleftarrows}} & \mathfrak{X}_m(\mathbb{X}) & \longrightarrow & 0 \end{array} \tag{5.13}$$

where the map h_c is the core map induced by the $\mathcal{VB}$-map h.

Definition 5.3.2. Let $(\mathbb{P},\pi,\mathbb{X},\mathbb{G})$ be a principal 2-bundle and $h: T\mathbb{X}\to \mathrm{At}(\mathbb{P})$ be a multiplicative horizontal lift. Then we say that h **flat up to homotopy** if its linear splitting induced on multiplicative sections is a weak morphism of Lie 2-algebras. In other words, there exists a 2-form $\Omega_m: \wedge^2\mathfrak{X}_m(\mathbb{X})\to\Gamma(P_0\times_{G_0}\mathfrak{h})$ satisfying the following conditions

i. $\delta\circ h_c = h_m\circ\delta$;

ii. $h_m[\xi,\eta]-[h_m(\xi),h_m(\eta)]=\delta\Omega_m(\xi,\eta)$;

iii. $h_c(D_\xi\alpha)-D_{h_m(\xi)}(h_c(\alpha))=\Omega_m(\xi,\delta(\alpha))$;

iv. $\sum_{\circlearrowleft\xi,\eta,\gamma}\Omega_m([\xi,\eta],\gamma)-\sum_{\circlearrowleft\xi,\eta,\gamma}D_{h_m(\xi)}(\Omega_m(\eta,\gamma))=0$.

$$\begin{array}{ccc} \Gamma(A_{\mathbb{X}}) & \xrightarrow{h_c} & \Gamma(A_{\mathbb{P}}/G_0) \\ \downarrow\delta & & \downarrow\delta \\ \mathfrak{X}_m(\mathbb{X}) & \xrightarrow{h_m} & \mathfrak{X}_m(\mathrm{At}(\mathbb{P})), \end{array} \qquad \wedge^2\mathfrak{X}_m(\mathbb{X}) \xrightarrow{\Omega_m} \Gamma(P_0\times_{G_0}\mathfrak{h}).$$

Remark 5.3.2. A multiplicative horizontal lift is said to be flat up to homotopy because it induces a map that is flat in cohomology. To see this, note that on bi-invariant sections of the Lie algebroid of $\mathbb{X}$ the map h_c preserves brackets. Let $\alpha,\beta\in\Gamma(A_{\mathbb{X}})$ such that $\delta(\alpha)=0,\delta(\beta)=0$ then by the properties above and the Peiffer identity for differential crossed modules it follows that

$$\begin{aligned} h_c([\beta,\alpha])-[h_c(\beta),h_c(\alpha)] =& h_c(D_{\delta\beta}\alpha)-D_{\delta(h_c(\beta))}(h_c(\alpha)) \\ =& h_c(D_{\delta\beta}\alpha)-D_{h_m(\delta\beta)}(h_c(\alpha)) \\ =& \Omega_m(\delta(\beta),\delta(\alpha)) \\ =& \Omega_m(0,0)=0. \end{aligned}$$

Observe that h_m preserves the brackets up to an exact element determined by Ω_m.

Theorem 5.3.1. *Let $\theta_\bullet$ be a 2-connection form on $\mathbb{P}$ and $\Omega_c\in\Omega^2_{dR}(P_0;\mathfrak{h})$ an equivariant 2-form, $R_g^*\Omega_c=(\alpha_{g^{-1}})_*\cdot\Omega_c$ for all $g\in G$, such that the 1-form $\Psi=(\theta_0,\omega,\Omega_c)\in\Omega^1(\mathbb{P},\mathfrak{g}_\bullet)$ is a Maurer-Cartan element in the dgla $(\Omega(\mathbb{P},\mathfrak{g}_\bullet),D,[\cdot\wedge\cdot])$, that is*

$$D\Psi+\frac{1}{2}[\Psi\wedge\Psi]=0.$$

Then the multiplicative horizontal lift induced by the 2-connection $\theta_\bullet$ is flat up to homotopy.

Proof. Let us consider the linear 2-form $\Omega_m: \bigwedge^2\mathfrak{X}_m(\mathbb{X})\to\Gamma(P_0\times_G\mathfrak{h})$ defined by

$$\Omega_m(\xi,\eta):=(-\Omega^c)(u^h,v^h),$$

where u^h and v^h are the horizontal lifts of the vector fields that cover the multiplicative vector fields ξ and η. Then we will see that Equations (ii) and (iv) of Definition 5.3.2 hold. Thus, on the one hand, by Theorem 4.1.5, we have that the Lie 2-algebra of multiplicative sections of the adjoint 2-bundle is given by

$$[\Gamma(P_0\times_G\mathfrak{h})\xrightarrow{\delta}\mathfrak{X}_m(\mathrm{Ad}(\mathbb{P}))\xrightarrow{D}\mathrm{Der}(\Gamma(P_o\times_G\mathfrak{h}))],$$

where $\delta(f)=(t^*f-s^*f,\partial\circ f\circ s)\in\mathfrak{X}_m(\mathrm{Ad}(\mathbb{P}))$, for all $f\in\Gamma(P_0\times_G\mathfrak{h})$. On the other hand, in our notation, the Maurer-Cartan equation $\mathrm{D}\Psi+\frac{1}{2}[\Psi\wedge\Psi]=0$ is equivalent to the equations

$$d\theta_0+\frac{1}{2}[\theta_0,\theta_0]+\partial\cdot\Omega^c=0; \tag{5.14}$$

$$(t^*-s^*)\Omega^c+d\omega+\frac{1}{2}[\omega,\omega]+[s^*\theta_0,\omega]=0; \tag{5.15}$$

$$d\Omega^c+[\theta_0,\Omega^c]=0. \tag{5.16}$$

Hence, considering the curvature 2-form of the connection $\theta_1\in\Omega^1_{dR}(P_1;\mathfrak{h}\oplus_{\mathcal{L}}\mathfrak{g})$ one has that

$$\begin{aligned}\Omega_1(\xi^h,\eta^h)=&(d\theta_1+\frac{1}{2}[\theta_1,\theta_1])(\xi^h,\eta^h)=\xi^h\theta_1(\eta^h)-\eta^h\theta_1(\xi^h)-\theta_1([\xi^h,\eta^h])\\=&-\theta_1([\xi^h,\eta^h])\\=&-([\xi^h,\eta^h]-[\xi,\eta]^h)\\=&h_m[\xi,\eta]-[h_m\xi,h_m\eta].\end{aligned}$$

Besides for the decomposition $\theta_1=\omega+s^*\theta_0$ it holds that

$$\Omega_1=(d\omega+\frac{1}{2}[\omega,\omega]+[s^*\theta_0,\omega])+s^*\Omega_0$$

where $\Omega_0=d\theta_0+\frac{1}{2}[\theta_0,\theta_0]$ is the curvature 2-form of $\theta_0\in\Omega^1_{dR}(P_0,\mathfrak{g})$; see Section 5.2. Therefore, by Equations (5.14) and (5.15) it follows

$$\begin{aligned}h_m[\xi,\eta]-[h_m\xi,h_m\eta]=&\Omega_1(\xi^h,\eta^h)\\=&(d\omega+\frac{1}{2}[\omega,\omega]+[s^*\theta_0,\omega])(\xi^h,\eta^h)+s^*\Omega_0(\xi^h,\eta^h)\\=&(t^*-s^*)(-\Omega^c)(\xi^h,\eta^h)+s^*(\partial\cdot(-\Omega^c))(\xi^h,\eta^h)\\=&(-\Omega^c)(u^h,v^h)\circ t-(-\Omega^c)(u^h,v^h)\circ s+\partial(-\Omega^c(u^h,v^h))\circ s\\=&t^*(\Omega_m(\xi,\eta))-s^*(\Omega_m(\xi,\eta))+\partial(\Omega_m(\xi,\eta))\circ s\\=&\delta(\Omega_m(\xi,\eta)).\end{aligned}$$

So Equation (ii) in Definition 5.3.2 holds. For seeing Equation (iii), let us consider $\alpha\in\Gamma(A_{\mathbb{X}})$ and $\xi\in\mathfrak{X}_m(\mathbb{X})$ and note that

$$\begin{aligned}h_c(D_\xi\alpha)-D_{h_m(\xi)}(h_c(\alpha))=&h\circ[\xi,\alpha^r]\circ u-[\xi^h,h_c(\alpha)^r]\circ u\\=&([\xi,\alpha^r]^h-[\xi^h,h_c(\alpha)^r])\circ u\\=&-\theta_1(\xi^h,h_c(\alpha)^r)\circ u\\=&\Omega_1(\xi^h,h_c(\alpha)^r)\circ u\\=&(d\omega+\frac{1}{2}[\omega,\omega]+[s^*\theta_0,\omega])(\xi^h,h_c(\alpha)^r)\circ u+s^*\Omega_0(\xi^h,h_c(\alpha)^r)\circ u\\=&(t^*-s^*)(-\Omega^c)(\xi^h,h_c(\alpha)^r)\circ u+\Omega_0(u^h,0)\circ u\\=&(-\Omega^c)(u^h,\rho(h_c(\alpha)))\circ t\circ u\\=&(-\Omega^c)(u^h,\rho(h_c(\alpha))).\end{aligned}$$

Observe that by Equation (i) in Definition 5.3.2 we have $h_m(\delta(\alpha)) = \delta(h_c(\alpha))$ and it covers the vector field $\rho(h_c(\alpha))$, hence

$$h_c(D_\xi \alpha) - D_{h_m\xi} h_c(\alpha) = \Omega_m(\xi, \delta(\alpha)).$$

Finally, to see Equation (iv), recall that for a horizontal right invariant vector field $\xi^h \in \mathfrak{X}(P_1)^{G_1}$ and a vertical vector field $\iota_c(f) \in \mathfrak{X}^v(P_1)$ we have $[\xi^h, \iota_c(f)] = \iota_c(\xi^h(f))$. Thus, from Equation (5.16) and $(\xi, u), (\eta, v), (\gamma, w) \in \mathfrak{X}_m(\mathbb{X})$ one has that

$$\begin{aligned}
0 =& d\Omega^c + [\theta_0, \Omega^c](u^h, v^u, w^h) \\
=& \sum_{\circlearrowleft u^h, v^h, w^h} u^h(\Omega^c(v^h, w^h)) - \sum_{\circlearrowleft u^h, v^h, w^h} \Omega^c([u^h, v^h], w^h) + \mathcal{L}(\theta_0 \wedge \Omega^c)(u^h, v^h, w^h) \\
=& \sum_{\circlearrowleft u^h, v^h, w^h} (-\Omega^c)([u^h, v^h], w^h) - \sum_{\circlearrowleft u^h, v^h, w^h} \xi^h(\Omega_m(\eta, \gamma)) \circ u + 0 \\
=& \sum_{\circlearrowleft \xi, \eta, \gamma} \Omega_m([\xi, \eta], \gamma) - \sum_{\circlearrowleft \xi, \eta, \gamma} [\xi^h, \Omega_m(\eta, \gamma)] \circ u \\
=& \sum_{\circlearrowleft \xi, \eta, \gamma} \Omega_m([\xi, \eta], \gamma) - \sum_{\circlearrowleft \xi, \eta, \gamma} D_{\xi^h}(\Omega_m(\eta, \gamma)).
\end{aligned}$$

Hence, Equation (iv) in Definition 5.3.2 holds, and therefore the morphism (h_c, h_m, Ω_m) is a weak morphism of Lie 2-algebras. □

Chapter 6

L_∞-algebras

In this chapter we will establish some algebraic features relevant for the last chapters of the thesis. In section 1, we review some concepts about graded vector spaces, in particular, we discuss the construction of the graded-symmetric algebra and the exterior algebra of a graded vector space. In section 2 we study coderivations of the coalgebra structure of the graded-symmetric algebra. In section 3, we introduce the notion of L_∞-algebra. The L_∞-algebras are extensions of the notions of Lie algebras and differential graded Lie algebras that have became relevant by their strong connection with mathematical physics and string theory [LS93, BC04]. Recently, they have appeared in several novel approaches to problems in higher differential geometry as in [BC04, Rog09, Zam12, OW19] among others. One of the main features that makes L_∞-algebras remarkable is that they are a kind of algebraic structure that can be preserved by homotopic transformations.

6.1 Basic concepts

A graded vector space $\mathbb{V}$ is a real vector space equipped with a $\mathbb{Z}$-grading, $\mathbb{V} = \bigoplus_{n\in\mathbb{Z}} V_n$. An element $x \in V_n$ is said to be homogeneous of degree n and we denote its degree by $|x|$. For two graded vector spaces $\mathbb{V}$ and $\mathbb{W}$ the set of linear maps from $\mathbb{V}$ to $\mathbb{W}$ is denoted by $\mathrm{Hom}(\mathbb{V}, \mathbb{W})$. We say that a linear map $f : \mathbb{V} \to \mathbb{W}$ has degree p, denoted by $|f| = p$, if for all $k \in \mathbb{Z}$ we have $f(V_k) \subseteq W_{k+p}$. The space of homogeneous linear maps of degree p is denoted by $\mathrm{Hom}^p(\mathbb{V}, \mathbb{W})$. A map of degree zero is called a linear degree-preserving map. The tensor product $\mathbb{V} \otimes \mathbb{W}$ is a graded vector space with grading given by

$$(\mathbb{V} \otimes \mathbb{W})_n := \bigoplus_{i+j=n} V_i \otimes W_j,$$

for all $n \in \mathbb{Z}$. Given linear maps $f \in \mathrm{Hom}^p(\mathbb{V}, \mathbb{W})$ and $g \in \mathrm{Hom}^q(\mathbb{V}', \mathbb{W}')$ their tensor product belongs to $\mathrm{Hom}^{p+q}(\mathbb{V} \otimes \mathbb{V}', \mathbb{W} \otimes \mathbb{W}')$ and for homogeneous elements $x \in \mathbb{V}$ and $y \in \mathbb{V}'$ it is defined by

$$(f \otimes g)(x \otimes y) := (-1)^{|g||x|} f(x) \otimes g(y).$$

The tensor product of composable linear maps is composable, that is, for f_1, g_1 and f_2, g_2 two couples of composable homogeneous linear maps, one has the following composition rule

$$(f_1 \otimes f_2) \circ (g_1 \otimes g_2) = (-1)^{|f_2||g_1|} (f_1 \circ g_1) \otimes (f_2 \circ g_2).$$

It is well-known that the group S_n of permutations of n-elements induces two linear right actions on the vector space $\mathbb{V}^{\otimes n}$. The first action, defined on the set of generators, is given by the translations $\sigma_i = (i, i+1)$ for $1 \le i \le n-1$ as

$$\hat{\epsilon}(\sigma_i) := 1^{\otimes i-1} \otimes T \otimes 1^{\otimes n-i-1} : \mathbb{V}^{\otimes n} \to \mathbb{V}^{\otimes n},$$

where T is the **twisting map**. The twisting map $T : \mathbb{V}^{\otimes 2} \to \mathbb{V}^{\otimes 2}$ is a degree-preserving linear map defined on homogeneous elements $x, y \in \mathbb{V}$ by $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$. Thus, explicitly for homogeneous elements $x_i \in \mathbb{V}$ with $1 \leq i \leq n$,

$$\hat{\epsilon}(\sigma_i)(x_1 \otimes \cdots \otimes x_n) = (-1)^{|x_i||x_{i+1}|} x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_n.$$

For an arbitrary permutation $\sigma \in S_n$, one has that

$$\hat{\epsilon}(\sigma)(x_1 \otimes \cdots \otimes x_n) = \epsilon(\sigma; x_1, \ldots, x_n) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},$$

where $\epsilon(\sigma; x_1, \ldots, x_n) = \pm 1$. This last symbol is known as the **Koszul sign**. We denote the set of invariant elements by this action by $(\mathbb{V}^{\otimes n})^{S_n}_{\epsilon}$. The other linear action of S_n on $\mathbb{V}^{\otimes n}$ is defined on the same set of generators as

$$\hat{\chi}(\sigma_i) := -\hat{\epsilon}(\sigma_i).$$

Thus for an arbitrary permutation $\sigma \in S_n$

$$\hat{\chi}(\sigma)(x_1 \otimes \cdots \otimes x_n) = \chi(\sigma; x_1, \ldots, x_n) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},$$

where $\chi(\sigma; x_1, \ldots, x_n) = \operatorname{sgn}(\sigma)\epsilon(\sigma; x_1, \ldots, x_n)$. Here $\operatorname{sgn}(g)$ denotes the sign of the permutation σ. We denote the set of invariant elements by this action by $(\mathbb{V}^{\otimes n})^{S_n}_{\chi}$.The **graded-symmetric algebra** $\operatorname{Sym}(\mathbb{V})$ is the free unital graded-commutative algebra on $\mathbb{V}$ given by

$$\operatorname{Sym}(\mathbb{V}) = \bigoplus_{0 \leq n} \operatorname{Sym}^n(\mathbb{V}) := \bigoplus_{0 \leq n} (\mathbb{V}^{\otimes n})^{S_n}_{\epsilon},$$

We denote the multiplication map on $\operatorname{Sym}(\mathbb{V})$ by μ_s. A homogeneous element in $\operatorname{Sym}^n(\mathbb{V})$ is denoted by $x = x_1 \vee \cdots \vee x_n$, and its **weight** is defined to be the integer n. The **degree** of x is defined as $|x| = |x_1| + \cdots + |x_n|$. In the same way, as we have defined the symmetric algebra, we define the exterior algebra of $\mathbb{V}$ by

$$\wedge \mathbb{V} = \bigoplus_{0 \leq n} \wedge^n(\mathbb{V}) := \bigoplus_{0 \leq n} (\mathbb{V}^{\otimes n})^{S_n}_{\chi}.$$

It is well-known that for an n-linear map $f : \mathbb{V}^n \to \mathbb{W}$ one has a linear one from the tensor product of $\mathbb{V}$ denoted by the same letter $f : \mathbb{V}^{\otimes n} \to \mathbb{W}$. We say that the linear map f is **symmetric** if for all $\sigma \in S_n$ the following equation holds

$$f \circ \hat{\epsilon}(\sigma) = f.$$

Explicitly, taking $x_i \in \mathbb{V}$ homogeneous for $1 \leq i \leq n$,

$$\epsilon(\sigma; x_1, \ldots, x_n) f(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}) = f(x_1 \otimes \cdots \otimes x_n).$$

The set of n-linear maps from $\mathbb{V}$ to $\mathbb{W}$ that are symmetric is denoted by $\operatorname{Sym}^n(\mathbb{V}; \mathbb{W})$. Analogously, we say that a linear map f is **skew-symmetric** if for all $\sigma \in S_n$ hold the equation $f \circ \hat{\chi}(\sigma) = f$. The set of n-linear skew-symmetric maps from $\mathbb{V}$ to $\mathbb{W}$ is denoted by $\operatorname{Skew}^n(\mathbb{V}; \mathbb{W})$. We point out that $\operatorname{Sym}^n(\mathbb{V}; \mathbb{W}) \simeq \operatorname{Hom}(\operatorname{Sym}^n(\mathbb{V}); \mathbb{W})$ and $\operatorname{Skew}^n(\mathbb{V}; \mathbb{W}) \simeq \operatorname{Hom}(\operatorname{Skew}^n(\mathbb{V}); \mathbb{W})$.

From now on we will extensively use the coalgebra structure of the graded-symmetric algebra. So, we briefly introduce the main tools that will be necessary. We say that a permutation $\sigma \in S_n$ is a $(k, n-k)$-**unshuffle**, $1 \leq k \leq n$, if it satisfies $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(n)$. The set of $(k, n-k)$-unshuffles is denoted by $\operatorname{Sh}^{-1}_{k,n-k}$.

The symmetric algebra of a graded vector space $\mathbb{V}$ is a **coaugmentated coalgebra**,

$$\operatorname{Sym}(\mathbb{V}) = \mathbb{R}1 \oplus \overline{\operatorname{Sym}}(\mathbb{V}) := \mathbb{R}1 \oplus \bigoplus_{1 \leq k} \operatorname{Sym}^k(\mathbb{V}),$$

where its coproduct is given by $\Delta_s = 1 \otimes \mathrm{Id} + \bar{\Delta}_s + \mathrm{Id} \otimes 1$, and its **reduced coproduct** is

$$\bar{\Delta}_s(x_1 \vee \cdots \vee x_n) = \sum_{k=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}_{k,n-k}^{-1}} \epsilon(\sigma)(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(k)}) \otimes (x_{\sigma(k+1)} \vee \cdots \vee x_{\sigma(n)}). \tag{6.1}$$

In particular, this coproduct is coassociative and cocommutative

$$(\Delta_s \otimes \mathrm{Id}_s) \circ \Delta_s = (\mathrm{Id}_s \otimes \Delta_s) \circ \Delta_s, \quad T \circ \Delta_s = \Delta_s,$$

where T is the twisting map.

6.2 Coderivation on $\mathrm{Sym}(\mathbb{V})$

Definition 6.2.1. (Coderivation) Let F and G be two coalgebra morphisms $F, G : (C, \Delta) \to (C', \Delta)$. An (F, G)-coderivation is a linear map $D : C \to D'$ which satisfies the coleibniz identity with respect to F and G, that is

$$\Delta_s \circ D = (F \otimes D + D \otimes G) \circ \Delta_s.$$

From a simple computation we obtain the following lemma.

Lemma 6.2.1. *The following equation holds*

$$\Delta_s^n \circ D = \sum_{i+j=n} (F^{\otimes i} \otimes D \otimes G^{\otimes j}) \circ \Delta_s^n.$$

Furthermore

Proposition 6.2.1. *There is a one-to-one correspondence between coderivations* $D : \mathrm{Sym}(\mathbb{V}) \to \mathrm{Sym}(\mathbb{V})$ *with* $D(1) = 0$ *and linear maps* $D : \mathrm{Sym}(\mathbb{V}) \to \mathbb{V}$, $D \mapsto D_1 = pr^1_{\mathrm{Sym}(\mathbb{V})} \circ D$.

Proof. Consider the next identity

$$\mathrm{pr}_{\mathrm{Sym}^n(\mathbb{V})} = \pi_n \circ \mathrm{pr}_{\mathbb{V}}^{\otimes n} \circ \Delta_s^{n-1}, \tag{6.2}$$

where $\pi_n : T^n(\mathbb{V}) \to \mathrm{Sym}^n(\mathbb{V})$ is given by $\pi_n(v_1 \otimes \cdots \otimes v_n) = \frac{1}{n!} v_1 \vee \cdots \vee v_n$. This identity can be found in [Rei19, pag.8]. Let D be a coderivation, and decompose it as $D = \sum_{n>0} D_n$, with $D_0 = 0$ and $D_n := \mathrm{pr}_{\mathrm{Sym}^n(\mathbb{V})} \circ D$. By both the identity (6.2), and the Lemma 6.2.1 one has that

$$\begin{aligned}
D_n =& \mathrm{pr}_{\mathrm{Sym}^n(\mathbb{V})} \circ D \\
=& \pi_n \circ \mathrm{pr}_{\mathbb{V}}^{\otimes n} \circ \Delta_s^{n-1} \circ D \\
=& \pi_n \circ \mathrm{pr}_{\mathbb{V}}^{\otimes n} \circ \left(\sum_{i+j=n-1} (F^{\otimes i} \otimes D \otimes G^{\otimes j}) \circ \Delta_s^{n-1} \right) \\
=& \pi_n \left(\sum_{i+j=n-1} (\mathrm{pr}_{\mathbb{V}} \circ F)^{\otimes i} \otimes (\mathrm{pr}_{\mathbb{V}} \circ D) \otimes (\mathrm{pr}_{\mathbb{V}} \circ G)^{\otimes j} \circ \Delta_s^{n-1} \right) \\
=& \pi_n \left(\sum_{i+j=n-1} F_1^{\otimes i} \otimes D_1 \otimes G_1^{\otimes j} \circ \Delta_s^{n-1} \right) \\
=& \frac{1}{n!} \sum_{i+j=n-1} F_1^{\vee i} \vee D_1 \vee G_1^{\vee j} \circ \Delta_s^{n-1}.
\end{aligned}$$

Thus

$$D = \sum_{n>0} \frac{1}{n!} \sum_{i+j=n-1} F_1^{\vee i} \vee D_1 \vee G_1^{\vee j} \circ \Delta_s^{n-1}.$$

□

6.3 L_∞-algebras

In this section we will introduce L_∞-algebras and their morphisms. We will exhibit three different, but equivalent formulations of the notion of L_∞-algebra. We warn that, for practicality, we will choose to work with a particular formulation depending on which of them is more suitable in each situation. For more details see [LS93, LM95, Man04].

Definition 6.3.1. Let $\mathfrak{g}$ be a $\mathbb{Z}_{\leq 0}$-graded vector space. An L_∞-**structure** on $\mathfrak{g}$ is a collection of maps $[\cdot] = \{[\cdot]^k\}_{1\leq k}$ called **higher brackets** which satisfy the next conditions

i. for all $1 \leq k < \infty$

$$[\cdot]^k : \wedge^k \mathfrak{g} \to \mathfrak{g},$$

is a linear map of degree $\mid [\cdot]^k \mid = 2 - k$. In other words, the k-th higher bracket $[\cdot]^k$ comes from a linear map of degree $2 - k$

$$[\cdot]^k : \mathfrak{g}^{\otimes k} \to \mathfrak{g},$$

such that for all $\sigma \in S_n$

$$[\cdot]^k \circ \hat{\chi}(\sigma) = [\cdot]^k .$$

For homogeneous elements $x_i \in \mathfrak{g}$ with $1 \leq i \leq k$

$$[x_i \otimes \cdots \otimes x_k]^k = \chi(\sigma; x_1, \ldots, x_k) \left[x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}\right]^k ;$$

ii. for all $1 \leq n$ the equation below holds

$$\sum_{\substack{i+j=n+1,\\ 1\leq i}} \sum_{\sigma\in \mathrm{Sh}_{i,n-i}^{-1}} (-1)^{i(j-1)} \chi(\sigma) \left[\left[x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(i)}\right]^i \wedge x_{\sigma(i+1)} \wedge \cdots \wedge x_{\sigma(n)}\right]^j = 0. \tag{6.3}$$

This is called **the generalized Jacobi identity**.

Now we define the notion of L_∞-algebra.

Definition 6.3.2. An L_∞-**algebra** is a pair $(\mathfrak{g}, [\cdot])$ consisting of a $\mathbb{Z}_{\leq 0}$-graded vector space $\mathfrak{g} = \bigoplus_{n\leq 0} \mathfrak{g}_n$ and an L_∞-structure $[\cdot] = \{[\cdot]^k\}_{1\leq k}$.

Let us consider the following examples of L_∞-algebras.

Example 6.3.1. Let us consider an L_∞-algebra in which the unique non-zero bracket is the first one. That is, the collection of brackets $[\cdot] = \left\{[\cdot]^k\right\}_{1\leq k}$ is given by

$$[\cdot]^k : \mathfrak{g}^{\otimes k} \to \mathfrak{g} = \begin{cases} [\cdot]^1 := \partial & k = 1; \\ [\cdot]^k := 0, & 2 \leq k. \end{cases}$$

Note that $\mid [\cdot]^1 \mid = |\partial| = 2 - 1 = 1$. In this case, we have that the generalized Jacobi identity (6.3) is equivalent to $\partial^2 = 0$. Thus we can conclude that the L_∞-algebras in which the only non-zero bracket is $[\cdot]^1$ are nothing but chain complexes.

Example 6.3.2. Let us consider an L_∞-algebra concentrated in degree zero. That is, $\mathfrak{g} = \bigoplus_{0\leq k} \mathfrak{g}_k$ where $\mathfrak{g}_k = 0$ for all $k \neq 0$. Notice that for degree reasons the only non-zero bracket is $[\cdot]^2$. The generalized Jacobi identity 6.3 in this case is the classical Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

We conclude that a Lie algebra is the same thing that an L_∞-algebra concentrated in degree zero.

Example 6.3.3. Let us consider an L_∞-algebra $(\mathfrak{g}, [\cdot])$ in which the unique non-zero brackets are the first two brackets. That is,

$$[\cdot]^k = \begin{cases} [\cdot]^1 = \partial; \\ [\cdot]^2 = [\cdot, \cdot]; \\ [\cdot]^k = 0, \quad 3 \leq k. \end{cases}$$

The generalized Jacobi identity 6.3 is equivalent the next identities

i. $\partial^2 = 0$;

ii. $\partial([x, y]) = [\partial x, y] + (-1)^{|x|}[x, \partial y]$;

iii. $[[u, v], w] + (-1)^{|u|(|v|+|w|)}[[v, w], u] + (-1)^{|v|(|u|+|w|)}[[w, u], v] = 0$.

In other words, the L_∞-algebra $(\mathfrak{g}, [\cdot])$ is the same thing that a differential graded Lie algebra.

Example 6.3.4. Let us consider an L_∞-algebra $(\mathfrak{g}, [\cdot])$ whose non-zero elements are in the first two grades. That is, $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ and $\mathfrak{g}_k = 0$ for all $k \leq -2$. Observe that the unique non-necessary zero brackets are

$$[\cdot]^k : (\mathfrak{g}_{-1} \oplus \mathfrak{g}_0)^{\otimes k} \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 = \begin{cases} [\cdot]^1 : \mathfrak{g}_{-1} \to \mathfrak{g}_0, & |\,[\cdot]^1\,| = 1; \\ [\cdot]^2 : \mathfrak{g}_{-1} \otimes \mathfrak{g}_0 \to \mathfrak{g}_{-1}, \quad \text{and} \quad [\cdot]^2 : \mathfrak{g}_0^{\otimes 2} \to \mathfrak{g}_0, & |\,[\cdot]^2\,| = 0; \\ [\cdot]^3 : \mathfrak{g}_0^{\otimes 3} \to \mathfrak{g}_{-1}, & |\,[\cdot]^3\,| = -1; \\ [\cdot]^k = 0, & 3 \leq k. \end{cases}$$

The set of identities that compose the generalized Jacobi identity can be found in [BC04, §4.3, Lemma 33]. A well-known fact is that a differential crossed module of Lie algebras, as in section 2.2, can be seen as a particular example of a 2-term L_∞-algebra in which $[\cdot]^3 = 0$. Moreover, in [BC04] it was shown that the 2-category of semistrict Lie 2-algebras is equivalent to the 2-category of 2-term L_∞-algebras.

Now let us comment two equivalent formulations of the notion of L_∞-algebra. The first formulation says that we can see an L_∞-algebra $\mathfrak{g}$ as a pair $(\mathfrak{g}[1], \lambda)$ where the brackets are the same that a map

$$\lambda = \sum_{1\leq k} \lambda_k \in \mathrm{Hom}^1(\overline{\mathrm{Sym}}(\mathfrak{g}[1]), \mathfrak{g}[1])$$

such that

i. for each $1 \leq k$ the map

$$\lambda_k : \mathrm{Sym}^k(\mathfrak{g}[1]) \to \mathfrak{g}[1],$$

is a linear map of degree 1, i.e. the map λ_k comes from a linear map of degree 1

$$\lambda_k : \mathfrak{g}[1]^{\otimes k} \to \mathfrak{g}[1],$$

such that for every $\sigma \in S_k$

$$\lambda_k \circ \hat{\epsilon}(\sigma) = \lambda_k.$$

For homogeneous elements $x_i \in \mathfrak{g}$ with $1 \leq i \leq k$

$$\lambda_k(x_1 \otimes \cdots \otimes x_k) = \epsilon(\sigma; x_1, \ldots, x_k)\lambda_k(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}).$$

ii. for all $n \geq 1$ the generalized Jacobi identity holds

$$\sum_{i=1}^{n} \sum_{\sigma \in \mathrm{Sh}^{-1}_{i,n-i}} \epsilon(\sigma)\lambda_{n-i+1}(\lambda_i(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(i)}) \vee x_{\sigma(i+1)} \vee \cdots \vee x_{\sigma(n)}) = 0.$$

The second formulation says that the collection of degree 1 maps

$$\{\lambda_n : \mathrm{Sym}^n(\mathfrak{g}\,[1]) \to \mathfrak{g}\,[1]\}_{1 \leq n},$$

defines a degree 1 coderivation $d_\lambda := \mu_s \circ (\lambda \otimes \mathrm{Id}_s) \circ \Delta_s$ of $(\mathrm{Sym}(\mathfrak{g}\,[1]), \Delta_s)$ and the generalized Jacobi identity is encoded in the property $d_\lambda^2 = 0$. So an L_∞-algebra is the same thing that the pair $(\mathrm{Sym}(\mathfrak{g}\,[1]), d)$ where d is a degree 1 coderivation of square zero. For more details about these equivalences see for example [Deh11].

Example 6.3.5. Let $\mathfrak{g}$ be an L_∞-algebra concentrated in degree zero, as in Example 6.3.2. Observe that in this particular case the symmetric algebra on $\mathfrak{g}\,[1]$ is the free commutative algebra generated by elements of degree -1,

$$\mathrm{Sym}(\mathfrak{g}[1]) = \bigwedge \mathfrak{g}.$$

The coderivation induced by the L_∞-structure is given on homogeneous elements $x_i \in \mathfrak{g}, 1 \leq i \leq n$ by

$$d(x_1 \wedge \cdots \wedge x_n) = \sum_{1 \leq i < j \leq n} (-1)^{i+j}[x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge \hat{x_j} \wedge \cdots \wedge x_n.$$

In fact, the chain complex $(\wedge\mathfrak{g}, d)$ determines the Lie algebra homology of $\mathfrak{g}$ with trivial coefficients. The bracket satisfies the Jacobi identity if and only if the derivation d squares to zero.

Remark 6.3.1. In the forthcoming, for the cases where is not strictly necessary to specify the L_∞-structure we shall denote an L_∞-algebra simply by $\mathfrak{g}$. For the cases in which is necessary to specify the L_∞-structure we denote it by $(\mathfrak{g}, [\cdot]), (\mathfrak{g}\,[1], \lambda)$ or $(\mathrm{Sym}(\mathfrak{g}\,[1]), d)$.

Definition 6.3.3. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two L_∞-algebras. An **L_∞-morphism** $F : \mathfrak{g} \to \mathfrak{h}$ is a coalgebra morphism $F : \mathrm{Sym}(\mathfrak{g}\,[1]) \to \mathrm{Sym}(\mathfrak{h}\,[1])$ that intertwines the differentials induced by the L_∞-structures. That is, F is a linear degree-preserving map that satisfy the next identities

$$F \otimes F \circ \Delta_s = \Delta_s \circ F, \quad F \circ d_{\lambda^{\mathfrak{g}}} = d_{\lambda^{\mathfrak{h}}} \circ F.$$

Remark 6.3.2. A coalgebra morphism $F : \mathrm{Sym}(\mathfrak{g}\,[1]) \to \mathrm{Sym}(\mathfrak{h}\,[1])$ is determined by its projection onto $\mathfrak{h}\,[1]$ and it is the same that a collection of linear maps of degree 0

$$\left\{F_n^1 : \mathrm{Sym}^n(\mathfrak{g}\,[1]) \to \mathfrak{h}\,[1]\right\}_{1 \leq n}.$$

In the particular case of 2-term L_∞-algebras, the set of constraint equations that should satisfy a coalgebras morphism for being an L_∞-morphism are presented in [BC04, §4.3, Def.34].

A **strict L_∞-morphism** is an L_∞-morphism F in which $F_n^1 = 0$ for $2 \leq n$, that is, $F = \mathcal{S}(F_1^1)$ where for $x_i \in \mathfrak{g}\,[1]$, with $1 \leq i \leq n$ the map $\mathcal{S}(F_1^1)$ is given by

$$\mathcal{S}(F_1^1)(x_1 \vee \cdots \vee x_n) = F_1^1(x_1) \vee \cdots \vee F_1^1(x_n),$$

and the map $F_1^1 : \mathfrak{g}\,[1] \to \mathfrak{h}\,[1]$ has degree zero and preserves the higher brackets. For all $k \geq 1$ we have

$$F_1^1 \circ \lambda_k^{\mathfrak{g}} = \lambda_k^{\mathfrak{h}} \circ {F_1^1}^{\otimes k}.$$

An L_∞ **quasi-isomorphism** is an L_∞-morphism F for which the map $H(F_1^1) : H(\mathfrak{g}, [\cdot]^1) \to H(\mathfrak{h}, [\cdot]^1)$ is an isomorphism of vector spaces. Now with the aim of presenting a non-strict L_∞-morphism let us see the next example.

Example 6.3.6. Let us consider an extension of Lie algebras

$$0 \longrightarrow \mathfrak{n} \longrightarrow \hat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0,$$

and σ a linear section of $\pi : \hat{\mathfrak{g}} \to \mathfrak{g}$, $\pi \circ \sigma = \mathrm{Id}_{\mathfrak{g}}$. Denote by $K_\sigma : \wedge^2\mathfrak{g} \to \mathfrak{n}$ its curvature $K_\sigma(x, y) = [\sigma(x), \sigma(y)] - \sigma([x, y])$. Now consider the Lie 2-algebra of derivations of $\mathfrak{n}$. This is a 2-term L_∞-algebra as in Example 6.3.4. It is defined by the crossed module of Lie algebras $\left[\mathfrak{n} \xrightarrow{\mathrm{ad}} \mathrm{Der}(\mathfrak{n}) \xrightarrow{\mathrm{Id}} \mathrm{Der}(\mathfrak{n})\right]$. Then the following collection of maps $\{F_1^1, F_2^1\}$ determines an L_∞-morphism between the Lie algebra $\mathfrak{g}$ and the Lie 2-algebra of derivations.

$$F_1^1 : 0 \to \mathfrak{n}, F_1^1 := 0, \quad F_1^1 : \mathfrak{g} \to \mathrm{Der}(\mathfrak{n}), F_1^1(x) := \mathrm{ad}_{\sigma(x)}, \quad F_2^1 : \wedge^2\mathfrak{g} \to \mathfrak{n}, F_2^1 := K_\sigma.$$

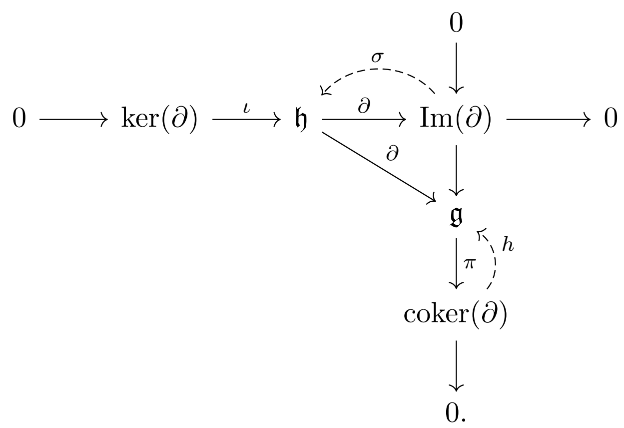
We summarize all this information in the following diagram

$$\begin{array}{ccc} 0 & \xrightarrow{F_1^1} & \mathfrak{n} \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mathrm{ad}} \\ \mathfrak{g} & \xrightarrow{F_1^1} & \mathrm{Der}(\mathfrak{n}) \end{array} \qquad\qquad \begin{array}{ccc} & & \mathfrak{n}. \\ & \overset{F_2^1}{\nearrow} & \\ \wedge^2\mathfrak{g} & & \end{array}$$

We do not check here the constraint equations that defines an L_∞-morphism, but we point out that these are obtained from Bianchi's identity of K_σ. This construction appears in [MZ12, §6, Ex.6.4] and [BZ21, §5,Prop.5.3] as examples of L_∞-action in the context of Lie algebras and Lie algebroids.

Another interesting example of non-strict L_∞-morphism arises when we study the minimal model of a Lie 2-algebra.

Example 6.3.7. Let $\left[\mathfrak{h} \xrightarrow{\partial} \mathfrak{g} \xrightarrow{\mathcal{L}} \mathrm{Der}(\mathfrak{h})\right]$ be a Lie 2-algebra. For the construction of the example let us consider the associated Lie algebra cohomology class to it. For that consider the next diagram and the next steps



i. By Proposition 2.1.1 we have $\ker(\partial)$ is an abelian subalgebra of $\mathfrak{h}$ and $\mathrm{im}(\partial)$ is an ideal of $\mathfrak{g}$.

ii. Let $h : \mathrm{coker}(\partial) \to \mathfrak{g}$ be a linear section of π, and $K_h : \wedge^2\mathrm{coker}(\partial) \to \mathrm{im}(\partial)$ its curvature $K_h(X, Y) = h([X, Y]) - [h(X), h(Y)]$, then for $S := \mathrm{ad}_h|_{\mathrm{im}(\partial)} : \mathrm{coker}(\partial) \to \mathrm{Der}(\mathrm{im}(\partial))$ one has $d_S K_h = 0$ by Bianchi's identity.

iii. Let $\sigma : \mathrm{im}(\partial) \to \mathfrak{h}$ be a linear section of ∂. Then $\omega_h := \sigma \circ K_h : \wedge^2\mathrm{coker}(\partial) \to \mathfrak{h}$ and $L := \mathcal{L} \circ h : \mathrm{coker}(\partial) \to \mathrm{Der}(\mathfrak{h})$ and the equivariance of ∂ implies (S, L)-equivariance.

iv. For the covariant derivative d_L one has that $d_L\omega_h : \wedge^3\text{coker}(\partial) \to \mathfrak{h}$ is such that $\partial(d_L\omega_h) = d_S(\partial\omega_h) = d_S K_h = 0$. Thus
$$d_L\omega_h : \wedge^3\text{coker}(\partial) \to \ker(\partial),$$
and the map $L' := L|_{\ker(\partial)} : \text{coker}(\partial) \to \text{Der}(\ker(\partial))$ is a representation of Lie algebras such that
$$d_{L'}(d_L\omega_h) = 0.$$

Therefore, we have a cohomology class $[d_L\omega_h] \in H^3(\text{coker}(\partial); \text{im}(\partial))$. This cohomology class is better known as **the characteristic class of the crossed module of Lie algebras**. See [Nee06, Wag06] for further discussion. We claim that

$$\left(\ker(\partial) \oplus \text{coker}(\partial), [\cdot] = \{[\cdot]^1 := 0, [\cdot]^2 := L', [\cdot]^3 := d_L\omega_h\}\right),$$

is a 2-term L_∞-algebra (semi-strict Lie 2-algebra) and that the collection of maps $F^1 := \{F^1_1, F^1_2\}$ defined by

$$F^1_1 : \ker(\partial) \to \mathfrak{h}, F^1_1 := \iota \quad F^1_1 : \text{coker}(\partial) \to \mathfrak{g}, F^1_1 := h \quad F^1_2 : \wedge^2\text{coker}(\partial) \to \mathfrak{h}, F^1_2 := \omega_h,$$

determines an L_∞-morphism quasi-isomorphism.

$$\begin{array}{ccc} \ker(\partial) & \xrightarrow{\iota} & \mathfrak{h} \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} \\ \text{coker}(\partial) & \xrightarrow{h} & \mathfrak{g} \end{array} \qquad \wedge^2\text{coker}(\partial) \xrightarrow{\omega_h} \mathfrak{h}.$$

Remark 6.3.3. According to the homotopy classification of L_∞-algebras due to Kontsevich in [Kon03, Chap.4.5.1], every L_∞-algebra is isomorphic to the direct sum of a minimal and of a linear contractible L_∞-algebra. The minimal L_∞-algebra involved in the decomposition of an L_∞-algebra is quasi-isomorphic to this and unique up to L_∞-isomorphism. This minimal algebra is called the **minimal model** of the L_∞-algebra.

Chapter 7

L_∞-algebra cohomology with values in a graded vector space

In this chapter we treat the cohomology of an L_∞-algebra with values in a graded vector space. Such a cohomology has been studied in [Pen95, Kje01, Rei19] extending the Chevalley-Eilenberg cohomology of a Lie algebra. Accordingly, representations up to homotopy are considered instead of strict representations. In Section 1 we deal with representations up to homotopy and define the L_∞-algebra cohomology with values in a graded vector space, as well as equivariant morphisms. In Sections 2 and 3 we focus on the tensor product of two representations up to homotopy and on the wedge product of cochains of L_∞-algebra cohomology. In Sections 4 we start with the study of representations up to homotopy in terms of Maurer-Cartan elements of a particular dgla. Finally, in Sections 5 and 6 we study the associated canonical spectral sequence to the L_∞-algebra cohomology. As a main result, we show that the L_∞-algebra cohomology is invariant with respect to equivariant L_∞-quasi-isomorphisms.

7.1 Representations up to homotopy

Definition 7.1.1. Let $\mathfrak{g}$ be an L_∞-algebra and $\mathbb{V}$ be a graded vector space. A **representation up to homotopy** of $\mathfrak{g}$ on $\mathbb{V}$ is a linear map

$$\rho : \mathrm{Sym}(\mathfrak{g}\,[1]) \otimes \mathbb{V} \to \mathbb{V},$$

of degree 1 which satisfies the equation

$$\rho \circ (d \otimes \mathrm{Id}_v) + \rho \circ (\mathrm{Id}_s \otimes \rho) \circ (\Delta_s \otimes \mathrm{Id}_v) = 0 \tag{7.1}$$

where d is the coderivation induced by the L_∞-structure on $\mathfrak{g}$.

Let us see some examples.

Example 7.1.1. Let $(\mathfrak{g}\,[1]\,, \lambda)$ be an L_∞-algebra. There exists a natural representation given by the higher brackets

$$\mathrm{ad} : \mathrm{Sym}(\mathfrak{g}\,[1]) \otimes \mathfrak{g}\,[1] \to \mathfrak{g}\,[1]\,, \quad \mathrm{ad}(x \otimes y) := \lambda(x \vee y).$$

This representation is referred to the **adjoint representation** of $\mathfrak{g}$.

Example 7.1.2. Let $\mathfrak{g}$ be an L_∞-algebra and $\mathbb{V}$ be a graded vector space both concentrated at degree zero. Let ρ be a representation up to homotopy of $\mathfrak{g}$ on $\mathbb{V}$. Note that since $|\rho| = 1$, by degree reasons the unique non-zero component of ρ is

$$\rho : \mathfrak{g}[1] \otimes V_0 \to V_0.$$

By Example 6.3.5, the Equation 7.1 says that for all $x, y \in \mathfrak{g}$ and $v \in V_0$ one has that

$$\rho([x,y]\otimes v) - \rho(x \otimes \rho(y\otimes v)) + \rho(y\otimes\rho(x\otimes v)) = 0.$$

Therefore, a representation up to homotopy of a Lie algebra on a vector space is the same thing that a usual linear representation.

The particular case of representations up to homotopy of 2-term L_∞-algebras over 2-term graded vector spaces has been studied before by several authors, for instance [SZ12, BSZ13, LSZ14]. In order to introduce this case, let us consider a 2-term L_∞-algebra $\mathfrak{g} = (\mathfrak{g}_{-1}\oplus\mathfrak{g}_0, [\cdot])$ and a 2-term graded vector space $\mathbb{V} = V_0 \oplus V_1$, and consider a linear map $\rho : \mathrm{Sym}(\mathfrak{g}[1]) \otimes \mathbb{V} \to \mathbb{V}$ of degree 1 as in Definition 7.1.1. Observe that the unique non-zero components of ρ are

$$\rho = \begin{cases} \rho^1 : \mathrm{Sym}^0(\mathfrak{g}[1]) \otimes V_0 \to V_1, & \rho^1 : V_0 \to V_1; \\ \rho^2 : \mathrm{Sym}^1(\mathfrak{g}[1]) \otimes \mathbb{V} \to \mathbb{V}, & \begin{cases} \rho^2_1 : \mathfrak{g}_{-1}\otimes V_1 \to V_0 \\ \rho^2_{0,1} : \mathfrak{g}_0 \otimes V_1 \to V_1 \\ \rho^2_{0,0} : \mathfrak{g}_0 \otimes V_0 \to V_0; \end{cases} \\ \rho^3 : \mathrm{Sym}^2(\mathfrak{g}[1]) \otimes \mathbb{V} \to \mathbb{V}, & \rho^3 : \wedge^2\mathfrak{g}_0 \otimes V_0 \to V_0. \end{cases}$$

The map ρ is a representation up to homotopy if it satisfies the constraint Equation in 7.1.1. We present an interpretation of this equation in terms of L_∞-morphisms. This is a particular case of [LM95, Thm 5.4] and can be found in [SZ12, BSZ13]. For that, let us introduce the Lie 2-algebra of endomorphisms of a 2-term dg vector space. Let $\mathbb{V} = V_0 \xrightarrow{\partial} V_1$ be a 2-term dg-vector space and $\mathrm{End}(\mathbb{V})$ be its dgla of endomorphisms

$$\mathrm{End}(\mathbb{V}) : \underbrace{\mathrm{Hom}(V_1, V_0)}_{-1} \xrightarrow{\delta_{-1}} \underbrace{\mathrm{End}(V_0)\oplus\mathrm{End}(V_1)}_{0} \xrightarrow{\delta_0} \underbrace{\mathrm{Hom}(V_0, V_1)}_{1}.$$

Consider the truncated 2-term subcomplex defined by

$$\underbrace{\mathrm{Hom}(V_1, V_0)}_{-1} \xrightarrow{\delta_{-1}} \underbrace{\ker(\delta_0)}_{0}.$$

This is a 2-term dgla, given that $\mathrm{End}_0(\mathbb{V})$ is a Lie algebra and δ a derivation. The following crossed module of Lie algebras determines the Lie 2-algebra of truncated endomorphisms

$$\left[\mathrm{End}^{-1}_\partial(\mathbb{V}) \xrightarrow{\partial^{end}} \mathrm{End}^0_\partial(\mathbb{V}) \xrightarrow{\mathcal{L}^{end}} \mathrm{Der}(\mathrm{End}^{-1}_\partial(\mathbb{V}))\right],$$

where $\mathrm{End}^{-1}_\partial(\mathbb{V}) = \mathrm{Hom}(V_1, V_0)$, $\quad \mathrm{End}^0_\partial(\mathbb{V}) = \{(S,T) \in \mathrm{End}(V_0)\times\mathrm{End}(V_1) \,|\, T\circ\partial = \partial\circ S\}$, and

$$\mathcal{L}^{end}_{(S,T)}(D) = D\circ T - S\circ D, \quad \partial^{end} D = (D\circ\partial, \partial\circ D).$$

We denote the above crossed module of Lie algebras by $\mathrm{End}_\partial(\mathbb{V})$. For more details see [SZ12]. Now we state the next definition that sum up the previous discussion.

Definition 7.1.2. (2-term representation up to homotopy) A representation up to homotopy of a 2-term L_∞-algebra $\mathfrak{g} = (\mathfrak{g}_{-1}\oplus\mathfrak{g}_0, [\cdot] = \{d_\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, l_3\})$ on a 2-term dg-vector space $\mathbb{V} : V_0 \xrightarrow{\partial} V_1$ is an L_∞-morphisms $\rho : \mathfrak{g} \to End_\partial(\mathbb{V})$. That means, $\rho = (\rho_{-1}, \rho_0, \rho_2)$

$$\begin{array}{ccc} \mathfrak{g}_{-1} & \xrightarrow{\rho_{-1}} & End^{-1}_\partial(\mathbb{V}) \\ \downarrow{\scriptstyle d_\mathfrak{g}} & & \downarrow{\scriptstyle \partial^{end}} \\ \mathfrak{g}_0 & \xrightarrow{\rho_0} & End^0_\partial(\mathbb{V}), \end{array} \qquad \wedge^2\mathfrak{g}_0 \xrightarrow{\rho_2} End^{-1}_\partial(\mathbb{V})$$

where are satisfied the next identities

i. $\rho_0 \circ d_{\mathfrak{g}} = \partial^{end} \circ \rho_{-1}$;

ii. $\rho_0([x,y]) - [\rho_0(x), \rho_0(y)] = \partial^{end}\rho_2(x,y)$;

iii. $\rho_{-1}(\mathcal{L}^{\mathfrak{g}}_x a) - \mathcal{L}^{end}_{\rho_0(x)}\rho_{-1}(a) = \rho_2(x, d_{\mathfrak{g}} a)$;

iv.
$$\sum_{\circlearrowleft x,y,z} \mathcal{L}^{end}_{\rho_0(x)}\rho_2(y,z) - \sum_{\circlearrowleft x,y,z} \rho_2(x,[y,z]) = \rho_{-1}(l_3(x,y,z)).$$

Example 7.1.3. Let $\mathfrak{h}$ and $\mathfrak{g}$ be two Lie algebras and $\partial : \mathfrak{h} \to \mathfrak{g}$ be a homomorphism of Lie algebras. For a linear map $\nabla : \mathfrak{g} \to \mathrm{End}(\mathfrak{h})$ we have a representation up to homotopy of the Lie algebra $\mathfrak{h}$ on the 2-term dg vector space $\mathbb{V} = \mathfrak{h} \xrightarrow{\partial} \mathfrak{g}$ given by

$$\begin{array}{ccc} 0 & \longrightarrow & End^{-1}_{\partial}(\mathbb{V}) \\ \downarrow & & \downarrow \\ \mathfrak{h} & \xrightarrow[\rho_0]{} & End^{0}_{\partial}(\mathbb{V}), \end{array} \qquad \wedge^2\mathfrak{h} \xrightarrow{\rho_2} End^{-1}_{\partial}(\mathbb{V})$$

where

$$\begin{aligned} \rho_0^1 : \mathfrak{h} \to End(\mathfrak{h}), &\quad \rho^1_{0,\alpha}(\beta) := \nabla_{\partial\beta}\alpha + [\alpha,\beta]\,; \\ \rho_0^2 : \mathfrak{h} \to End(\mathfrak{g}), &\quad \rho^2_{0,\alpha}(X) = \partial(\nabla_X\alpha) + [\partial\alpha, X]\,; \\ \rho_2 : \wedge^2\mathfrak{h} \to \mathrm{Hom}(\mathfrak{g},\mathfrak{h}), &\quad \rho_2(\alpha,\beta)(X) = \nabla_X[\alpha,\beta] - [\nabla_X\alpha,\beta] - [\alpha,\nabla_X\beta] - \nabla_{\rho^2_{0,\beta}X}\alpha + \nabla_{\rho^2_{0,\alpha}X}\beta. \end{aligned}$$

More details can be found in [AC12, Prop.2.11].

Now we will define the notion of L_∞-algebra cohomology with values in a graded vector space $\mathbb{V}$.

Definition 7.1.3. Let $\mathfrak{g}$ be an L_∞-algebra, and ρ be a representation up to homotopy of $\mathfrak{g}$ on the graded vector spce $\mathbb{V}$. **The L_∞-algebra cohomology of $\mathfrak{g}$ with values in** $\mathbb{V}$ is defined as the cohomology of the cochain complex

$$C_\rho(\mathfrak{g},\mathbb{V}) := (\mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}\,[1]),\mathbb{V}), D_\rho)\,,$$

where the boundary operator D_ρ is given for a homogeneous element $\alpha : \mathrm{Sym}(\mathfrak{g}\,[1]) \to \mathbb{V}$ of degree p, by

$$D_\rho\alpha := \rho \circ (\mathrm{Id}_s \otimes \alpha) \circ \Delta_s - (-1)^p \alpha \circ d. \tag{7.2}$$

We denote the L_∞-cohomology of $\mathfrak{g}$ with values in $\mathbb{V}$ by $H_{CE,\rho}(\mathfrak{g};\mathbb{V})$. For more details about this cohomology see [Deh11] and [Rei19]. For illustrate the nature of the formula in Equation (7.2) consider the next example.

Example 7.1.4. Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation. For $\alpha \in \mathrm{Hom}^n(\wedge\mathfrak{g}, V)$ and $x_i \in \mathfrak{g}$ for $i = 1,\dots,n+1$, we will computate

$$D_\rho\alpha(x_1 \wedge \cdots \wedge x_{n+1}) = (\rho \circ (\mathrm{Id}_s \otimes \alpha) \circ \Delta_s - (-1)^n \alpha \circ d)(x_1 \wedge \cdots \wedge x_{n+1}) = A$$

using the Example 6.3.5 and the explicit expression for the coproduct in 6.1 we get

$$\begin{aligned}
A =&\rho\circ(\mathrm{Id}_s\otimes\alpha)\left(\sum_{k=1}^{n}\sum_{\sigma\in\mathrm{Sh}^{-1}_{k,n+1-k}}\epsilon(\sigma)(x_{\sigma(1)}\wedge\cdots\wedge x_{\sigma(k)})\otimes(x_{\sigma(k+1)}\wedge\cdots\wedge x_{\sigma(n+1)})\right)\\
&-(-1)^n\alpha\left(\sum_{1\le i<j\le n+1}(-1)^{i+j}[x_i,x_j]\wedge x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_{n+1}\right)\\
=&\rho\left(\sum_{k=1}^{n}(-1)^{k+n}x_k\otimes\alpha(x_1\wedge\cdots\wedge\hat{x}_k\wedge\cdots\wedge x_{n+1})\right)\\
&-\sum_{1\le i<j\le n+1}(-1)^{i+j+n}\alpha([x_i,x_j]\wedge x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_{n+1})\\
=&(-1)^n\left(\sum_{k=1}^{n}(-1)^k\rho_{x_k}(\alpha(x_1\wedge\cdots\wedge\hat{x}_k\wedge\ldots x_{n+1}))\right.\\
&\left.-\sum_{1\le i<j\le n+1}(-1)^{i+j}\alpha([x_i,x_j]\wedge x_1\wedge\ldots\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_{n+1})\right)
\end{aligned}$$

Observe that D_ρ is (up to a sign) the Chevalley-Eilenberg coboundary operator. In particular, we have that $H_{CE,\rho}(\mathfrak{g};V)$ is the Lie algebra cohomology with values in V.

Definition 7.1.4. Let ρ and ρ' be two representations up to homotopy of $\mathfrak{g}$ on $\mathbb{V}$ and $\mathfrak{h}$ on $\mathbb{W}$, respectively. A (ρ,ρ')-**equivariant map** from $\mathfrak{g}$ to $\mathfrak{h}$ is a couple of maps (F,f) where $F:\mathfrak{g}\to\mathfrak{h}$ is an L_∞-morphism and $f:\mathbb{W}\to\mathbb{V}$ is a degree-preserving map satisfying

$$f\circ\rho'\circ(F\otimes\mathrm{Id}_v)=\rho\circ(\mathrm{Id}_s\otimes f). \tag{7.3}$$

In the particular case that $\mathfrak{g}=\mathfrak{h}$ and $F=\mathrm{Id}$ we simply say that the map $f:\mathbb{W}\to\mathbb{V}$ is (ρ,ρ')-equivariant. A (ρ,ρ')-equivariant map $(F,f):\mathfrak{g}\to\mathfrak{h}$ is said to be a (ρ,ρ')-**equivariant L_∞-quasi-isomorphism along to** $f:\mathbb{W}\to\mathbb{V}$, if F is an L_∞-quasi-isomorphism and f is a quasi-isomorphism of graded vector spaces.

One can compose equivariant maps yielding a new equivariant map. This is the content of the next proposition.

Proposition 7.1.1. *Let ρ,ρ' and ρ'' be three representations up to homotopy of $\mathfrak{g}$ on $\mathbb{V}$, $\mathfrak{g}'$ on $\mathbb{U}$ and $\mathfrak{h}$ on $\mathbb{W}$, respectively. Then for $(F,f):\mathfrak{g}\to\mathfrak{g}'$ a (ρ,ρ')-equivariant map and $(F',f'):\mathfrak{g}'\to\mathfrak{h}$ a (ρ',ρ'')-equivariant map we have*

$$(F'\circ F,f\circ f'):\mathfrak{g}\to\mathfrak{h},$$

is a (ρ,ρ'')-equivariant map.

Proof. To see that in fact it is a (ρ,ρ'')-equivariant map we shall check the equation (7.3). Indeed,

$$\begin{aligned}
(f\circ f')\circ\rho''\circ((F'\circ F)\otimes\mathrm{Id}_w)=&f\circ(f'\circ\rho''\circ(F'\otimes\mathrm{Id}_w))\circ(F\otimes\mathrm{Id}_w)\\
=&f\circ(\rho'\circ(\mathrm{Id}_s\otimes f'))\circ(F\otimes\mathrm{Id}_w)\\
=&(f\circ\rho'\circ(F\otimes\mathrm{Id}_u))\circ(\mathrm{Id}_s\otimes f')\\
=&\rho\circ(\mathrm{Id}_s\otimes f)\circ(\mathrm{Id}_s\otimes f')\\
=&\rho\circ(\mathrm{Id}_s\otimes(f\circ f')).
\end{aligned}$$

□

Therefore, for (F, f) a (ρ, ρ')-equivariant map and (F', f') a (ρ', ρ'')-equivariant map as in Proposition 7.1.1 we define their composition as the (ρ, ρ'')-equivariant map

$$(F, f) \circ (F', f') := (F' \circ F, f \circ f').$$

Remark 7.1.1. Given $F : \mathfrak{g} \to \mathfrak{h}$ an L_∞-morphism and ρ' a representation up to homotopy of $\mathfrak{h}$ on $\mathbb{V}$, we can induce a representation of $\mathfrak{g}$ on $\mathbb{V}$ called the **pullback representation** and is defined by

$$F^*\rho' : \mathrm{Sym}(\mathfrak{g}\,[1]) \otimes \mathbb{V} \xrightarrow{F\otimes \mathrm{Id}_v} \mathrm{Sym}(\mathfrak{h}\,[1]) \otimes \mathbb{V} \xrightarrow{\rho'} \mathbb{V}, \quad F^*\rho' := \rho' \circ (F \otimes \mathrm{Id}_v).$$

The map $(F, \mathrm{Id}_v) : \mathfrak{g} \to \mathfrak{h}$ is $(F^*\rho', \rho')$-equivariant and it is clear that $\mathrm{Id}_v \circ \rho' \circ (F \otimes \mathrm{Id}_v) = F^*\rho' \circ (\mathrm{Id}_s \otimes \mathrm{Id}_v)$ holds. Actually, a general (ρ, ρ')-equivariant map $(F, f) : \mathfrak{g} \to \mathfrak{h}$, can always be factored it in a canonical way as the composition of $(\rho, F^*\rho'), (F^*\rho', \rho')$-equivariant maps. Indeed, the map (Id_s, f) is $(\rho, F^*\rho')$-equivariant, since

$$f \circ F^*\rho' \circ (\mathrm{Id}_s \otimes \mathrm{Id}_v) = \rho \circ (\mathrm{Id}_s \otimes f),$$

holds by equation (7.3), and $(F, f) = (\mathrm{Id}_s, f) \circ (F, \mathrm{Id}_v)$.

Proposition 7.1.2. *Let ρ and ρ' be two representations up to homotopy of $\mathfrak{g}$ on $\mathbb{V}$ and $\mathfrak{h}$ on $\mathbb{W}$, respectively. A (ρ, ρ')-equivariant morphism from $\mathfrak{g}$ to $\mathfrak{h}$ induces a morphism of cochain complexes given by*

$$F^* : C_{\rho'}(\mathfrak{h}, \mathbb{W}) \to C_\rho(\mathfrak{g}, \mathbb{V}), \quad F^*\alpha := f \circ \alpha \circ F.$$

Proof. It is clear that the morphism F^* is well-defined. Let us see that $F^*D_{\rho'} = D_\rho F^*$. For this let α be a homogeneous element of degree p in $C_{\rho'}(\mathfrak{h}, \mathbb{W})$, then

$$\begin{aligned}
F^*(D_{\rho'}\alpha) =& f\rho'(\mathrm{Id}_s \otimes \alpha)\Delta_s F - (-1)^p f\alpha dF \\
=& f\rho'(\mathrm{Id}_s \otimes \alpha)(F \otimes F)\Delta_s - (-1)^p f\alpha F d \\
=& f\rho'(F \otimes \mathrm{Id}_v)(\mathrm{Id}_s \otimes \alpha)(\mathrm{Id}_s \otimes F)\Delta_s - (-1)^p (F^*\alpha) d \\
=& \rho(\mathrm{Id}_s \otimes f)(\mathrm{Id}_s \otimes \alpha)(\mathrm{Id}_s \otimes F)\Delta_s - (-1)^p (F^*\alpha) d, \quad \text{by equation (7.3)} \\
=& \rho(\mathrm{Id}_s \otimes (F^*\alpha))\Delta_s - (-1)^p (F^*\alpha) d \\
=& D_\rho(F^*\alpha).
\end{aligned}$$

□

7.2 Tensor product of representations up to homotopy

In this section we define the notion of tensor product of representations up to homotopy.

Proposition 7.2.1. *Let $\mathfrak{g}$ be an L_∞-algebra and $\rho^1 : \mathrm{Sym}(\mathfrak{g}\,[1]) \otimes \mathbb{V} \to \mathbb{V}$ and $\rho^2 : \mathrm{Sym}(\mathfrak{g}\,[1]) \otimes \mathbb{W} \to \mathbb{W}$ two representations up to homotopy. The following defines a representation up to homotopy of $\mathfrak{g}$ on the graded vector space $\mathbb{V} \otimes \mathbb{W}$*

$$\rho : \mathrm{Sym}(\mathfrak{g}\,[1]) \otimes (\mathbb{V} \otimes \mathbb{W}) \to \mathbb{V} \otimes \mathbb{W}, \quad \rho := \rho^1 \otimes \mathrm{Id}_w + (\mathrm{Id}_v \otimes \rho^2) \circ (T \otimes \mathrm{Id}_w)$$

where $T : \mathrm{Sym}(\mathfrak{g}\,[1]) \otimes \mathbb{V} \to \mathbb{V} \otimes \mathrm{Sym}(\mathfrak{g}\,[1])$ is the twisting map. Explicitly, for homogeneous elements $x \in \mathrm{Sym}(\mathfrak{g}\,[1]), v \in \mathbb{V}$ and $w \in \mathbb{W}$ one has that

$$\rho(x \otimes v \otimes w) = \rho^1(x \otimes v) \otimes w + (-1)^{(|x|+1)|v|} v \otimes \rho^2(x \otimes w).$$

*This representation is said to be the **tensor product representation** of ρ^1 and ρ^2, and it is denoted by $\rho := \rho^1 \otimes \rho^2$.*

Proof. To see that ρ is a representation up to homotopy we shall verify that the next equation holds

$$\rho \circ (d \otimes \mathrm{Id}_{v\otimes w}) + \rho \circ (\mathrm{Id}_s \otimes \rho) \circ (\Delta_s \otimes \mathrm{Id}_{v\otimes w}) = 0.$$

We will analyse this equation by parts. First consider the expression

$$\begin{aligned}\rho \circ (\mathrm{Id}_s \circ \rho) =&(\rho^1 \otimes \mathrm{Id}_w + (\mathrm{Id}_v \otimes \rho^2) \circ (T \otimes \mathrm{Id}_w))(\mathrm{Id}_s \otimes \rho^1 \otimes \mathrm{Id}_w + \mathrm{Id}_s \otimes ((\mathrm{Id}_w \otimes \rho^2) \circ (T \otimes \mathrm{Id}_w)))\\ =&\rho^1 \circ (\mathrm{Id}_s \otimes \rho^1) \otimes \mathrm{Id}_w + \underbrace{(\rho^1 \otimes \mathrm{Id}_w) \circ \mathrm{Id}_s \otimes ((\mathrm{Id}_v \otimes \rho^2) \circ (T \otimes \mathrm{Id}_w))}_{A} +\\ &+ \underbrace{(\mathrm{Id}_v \otimes \rho^2) \circ (T \otimes \mathrm{Id}_w) \circ \mathrm{Id}_s \otimes \rho^1 \otimes \mathrm{Id}_w}_{B}\\ &+(\mathrm{Id}_v \otimes \rho^2) \circ \underbrace{(T \otimes \mathrm{Id}_w) \circ \mathrm{Id}_s \otimes ((\mathrm{Id}_v \otimes \rho^2) \circ (T \otimes \mathrm{Id}_w))}_{C}.\end{aligned}$$

Let us check that: $B \circ (T \otimes \mathrm{Id}_v \otimes \mathrm{Id}_w) = -A$. To see this we take homogeneous elements $x, y \in \mathrm{Sym}(\mathfrak{g}[1]), v \in \mathbb{V}, w \in \mathbb{W}$, and check this equality.

$$\begin{aligned}A(x\otimes y\otimes v\otimes w) =&(\rho^1 \otimes \mathrm{Id}_w) \circ \mathrm{Id}_s \otimes ((\mathrm{Id}_v \otimes \rho^2) \circ (T \otimes \mathrm{Id}_w))(x\otimes y\otimes v\otimes w)\\ =&\rho^1 \otimes \mathrm{Id}_w((-1)^{|x|}x \otimes (\mathrm{Id}_v \otimes \rho^2) \circ (T \otimes \mathrm{Id}_w)(y\otimes v\otimes w))\\ =&\rho^1 \otimes \mathrm{Id}_w((-1)^{|x|}x \otimes (\mathrm{Id}_v \otimes \rho^2)(-1)^{|y||v|}v\otimes y\otimes w)\\ =&(-1)^{|x|+|y||v|}\rho^1 \otimes \mathrm{Id}_w(x \otimes (-1)^{|v|}v \otimes \rho^2(y\otimes w))\\ =&(-1)^{|x|+|y||v|+|v|}\rho^1 \otimes \mathrm{Id}_w(x \otimes v \otimes \rho^2(y\otimes w))\\ =&(-1)^{|x|+|y||v|+|v|}\rho^1(x\otimes v) \otimes \rho^2(y\otimes w).\end{aligned}$$

$$\begin{aligned}B(x\otimes y\otimes v\otimes w) =&(\mathrm{Id}_v \otimes \rho^2) \circ (T \otimes \mathrm{Id}_w) \circ (\mathrm{Id}_s \otimes \rho^1 \otimes \mathrm{Id}_w)(x\otimes y\otimes v\otimes w)\\ =&(\mathrm{Id}_w \otimes \rho^2) \circ (T \otimes \mathrm{Id}_w)((-1)^{|x|}x \otimes (\rho^1 \otimes \mathrm{Id}_w)(y\otimes v\otimes w))\\ =&(-1)^{|x|}(\mathrm{Id}_v \otimes \rho^2) \circ (T \otimes \mathrm{Id}_w)(x \otimes \rho^1(y\otimes v) \otimes w)\\ =&(-1)^{|x|}(\mathrm{Id}_v \otimes \rho^2)((-1)^{(|y|+|v|+1)|x|}\rho^1(y\otimes v) \otimes x \otimes w)\\ =&(-1)^{|y||x|+|v||x|+|x|+|x|}(-1)^{|y|+|v|+1}\rho^1(y\otimes v) \otimes \rho^2(x\otimes w)\\ =&(-1)^{|y||x|+|v||x|+|y|+|v|+1}\rho^1(y\otimes v) \otimes \rho^2(x\otimes w),\end{aligned}$$

then

$$\begin{aligned}B \circ (T \otimes \mathrm{Id}_v \otimes \mathrm{Id}_w)(x\otimes y\otimes v\otimes w) =&B((-1)^{|x||y|}y\otimes x\otimes v\otimes w)\\ =&(-1)^{|x||y|+|y||x|+|v||y|+|x|+|v|+1}\rho^1(x\otimes v) \otimes \rho^2(y\otimes w)\\ =&-A(x\otimes y\otimes v\otimes w).\end{aligned}$$

Let us check that: $C = (\mathrm{Id}_v \otimes \mathrm{Id}_s \otimes \rho^2) \circ (T \otimes \mathrm{Id}_{v\otimes w}) \circ (\mathrm{Id}_s \otimes T \otimes \mathrm{Id}_w)$.

$$\begin{aligned}C(x\otimes y\otimes v\otimes w) =&(T \otimes \mathrm{Id}_w) \circ (\mathrm{Id}_s \otimes ((\mathrm{Id}_v \otimes \rho^2) \circ (T \otimes \mathrm{Id}_{v\otimes w})))(x\otimes y\otimes v\otimes w)\\ =&(T \otimes \mathrm{Id}_w)((-1)^{|x|}x \otimes ((\mathrm{Id}_v \otimes \rho^2) \circ (T \otimes \mathrm{Id}_w))(y\otimes v\otimes w))\\ =&(-1)^{|x|}(T \otimes \mathrm{Id}_w)(x \otimes (\mathrm{Id}_v \otimes \rho^2)((-1)^{|y||v|}v\otimes y\otimes w))\\ =&(-1)^{|x|+|y||v|}(T \otimes \mathrm{Id}_w)(x \otimes (-1)^{|v|}v \otimes \rho^2(y\otimes w))\\ =&(-1)^{|x|+|y||v|+|v|}(-1)^{|x||v|}v \otimes x \otimes \rho^2(y\otimes w)\\ =&(-1)^{|y||v|+|x||v|+|x|+|v|}v \otimes x \otimes \rho^2(y\otimes w),\end{aligned}$$

$$\begin{aligned}(\mathrm{Id}_v\otimes\mathrm{Id}_s\otimes\rho^2)\circ&(T\otimes\mathrm{Id}_{v\otimes w})\circ(\mathrm{Id}_s\otimes T\otimes\mathrm{Id}_w)(x\otimes y\otimes v\otimes w)=\\
&=(\mathrm{Id}_v\otimes\mathrm{Id}_s\otimes\rho^2)\circ(T\otimes\mathrm{Id}_{v\otimes w})(x\otimes(T\otimes\mathrm{Id}_w)(y\otimes v\otimes w))\\
&=(\mathrm{Id}_v\otimes\mathrm{Id}_s\otimes\rho^2)\circ(T\otimes\mathrm{Id}_{v\otimes w})(x\otimes(-1)^{|y||v|}(v\otimes y\otimes w))\\
&=(-1)^{|y||v|}(\mathrm{Id}_v\otimes\mathrm{Id}_s\otimes\rho^2)((-1)^{|x||v|}v\otimes x\otimes y\otimes w)\\
&=(-1)^{|y||v|+|x||v|}(-1)^{|v|}v\otimes(\mathrm{Id}_s\otimes\rho^2)(x\otimes y\otimes w)\\
&=(-1)^{|y||v|+|x||v|+|v|}v\otimes(-1)^{|x|}x\otimes\rho^2(y\otimes w)\\
&=(-1)^{|y||v|+|x||v|+|v|+|x|}v\otimes x\otimes\rho^2(y\otimes w)\\
&=C(x\otimes y\otimes v\otimes w).\end{aligned}$$

and let us check that:

$$(T\otimes\mathrm{Id}_{v\otimes w})\circ(\mathrm{Id}_s\otimes T\otimes\mathrm{Id}_w)\circ(\Delta_s\otimes\mathrm{Id}_{v\otimes w})=(\mathrm{Id}_v\otimes\Delta_s\otimes\mathrm{Id}_w)\circ(T\otimes\mathrm{Id}_w).$$

Let $v\in\mathbb{V},w\in\mathbb{W},x_i\in\mathrm{Sym}(\mathfrak{g}\,[1]),i=1,\ldots,n$. On the one side

$$\begin{aligned}&(T\otimes\mathrm{Id}_{v\otimes w})\circ(\mathrm{Id}_s\otimes T\otimes\mathrm{Id}_w)\circ(\Delta_s\otimes\mathrm{Id}_{v\otimes w})(x_1\vee\cdots\vee x_n\otimes v\otimes w)\\
&=(T\otimes\mathrm{Id}_{v\otimes w})\circ(\mathrm{Id}_s\otimes T\otimes\mathrm{Id}_w)\sum_{k=1}^{n}\sum_{\sigma\in\mathrm{Sh}^{-1}_{k,n-k}}\epsilon(\sigma)x_{\sigma(1)}\vee\cdots\vee x_{\sigma(k)}\otimes x_{\sigma(k+1)}\vee\cdots\\
&\quad\cdots\vee x_{\sigma(n)}\otimes v\otimes w\\
&=(T\otimes\mathrm{Id}_{v\otimes w})\sum_{k=1}^{n}\sum_{\sigma\in\mathrm{Sh}^{-1}_{k,n-k}}\epsilon(\sigma)x_{\sigma(1)}\vee\cdots\vee x_{\sigma(k)}\otimes(-1)^{(\sum_{k\leq i}|x_{\sigma(i)}|)|v|}v\otimes x_{\sigma(k+1)}\vee\cdots\\
&\quad\cdots\vee x_{\sigma(n)}\otimes w\\
&=\sum_{k=1}^{n}\sum_{\sigma\in\mathrm{Sh}^{-1}_{k,n-k}}\epsilon(\sigma)(-1)^{(\sum_{i\leq k}|x_{\sigma(i)}|)|v|+(\sum_{k\leq i}|x_{\sigma(i)}|)|v|}v\otimes x_{\sigma(1)}\vee\cdots\vee x_{\sigma(k)}\otimes x_{\sigma(k+1)}\vee\cdots\\
&\quad\cdots\vee x_{\sigma(n)}\otimes w\\
&=(-1)^{|v|(\sum|x_i|)}\sum_{k=1}^{n}\sum_{\sigma\in\mathrm{Sh}^{-1}_{k,n-k}}\epsilon(\sigma)v\otimes x_{\sigma(1)}\vee\cdots\vee x_{\sigma(k)}\otimes x_{\sigma(k+1)}\vee\cdots\vee x_{\sigma(n)}\otimes w.\end{aligned}$$

On the other side

$$\begin{aligned}(\mathrm{Id}_s\otimes\Delta_s\otimes\mathrm{Id}_w)&\circ(T\otimes\mathrm{Id}_w)(x_1\vee\cdots\vee x_n\otimes v\otimes w)\\
&=(\mathrm{Id}_s\otimes\Delta_s\otimes\mathrm{Id}_w)(-1)^{(\sum|x_i|)|v|}v\otimes x_1\vee\cdots\vee x_n\otimes w\\
&=(-1)^{(\sum|x_i|)|v|}v\otimes\sum_{k=1}^{n}\sum_{\sigma\in\mathrm{Sh}^{-1}_{k,n-k}}\epsilon(\sigma)x_{\sigma(1)}\vee\cdots\vee x_{\sigma(k)}\otimes x_{\sigma(k+1)}\vee\cdots\vee x_{\sigma(n)}\otimes w\\
&=(-1)^{|v|(\sum|x_i|)}\sum_{k=1}^{n}\sum_{\sigma\in\mathrm{Sh}^{-1}_{k,n-k}}\epsilon(\sigma)v\otimes x_{\sigma(1)}\vee\cdots\vee x_{\sigma(k)}\otimes x_{\sigma(k+1)}\vee\cdots\vee x_{\sigma(n)}\otimes w.\end{aligned}$$

Then, $(T\otimes\mathrm{Id}_{v\otimes w})\circ(\mathrm{Id}_s\otimes T\otimes\mathrm{Id}_w)\circ(\Delta_s\otimes\mathrm{Id}_{v\otimes w})=(\mathrm{Id}_v\otimes\Delta_s\otimes\mathrm{Id}_w)\circ(T\otimes\mathrm{Id}_w)$. Finally, consider the main equation

$$\begin{aligned}
&\rho\circ(d\otimes \mathrm{Id}_{v\otimes w})+\rho\circ(\mathrm{Id}_s\otimes\rho)\circ(\Delta_s\otimes \mathrm{Id}_{v\otimes w})=\\
&\quad(\rho^1\otimes \mathrm{Id}_w)(d\otimes \mathrm{Id}_{v\otimes w})+(\mathrm{Id}_v\otimes\rho^2)\circ\underbrace{(T\otimes \mathrm{Id}_w)(d\otimes \mathrm{Id}_{v\otimes w})}_{=(\mathrm{Id}_v\otimes d\otimes \mathrm{Id}_w)\circ(T\otimes \mathrm{Id}_w)}+((\rho^1\circ(\mathrm{Id}_s\otimes\rho^1))\otimes\\
&\quad\otimes \mathrm{Id}_w)\circ(\Delta_s\otimes \mathrm{Id}_{v\otimes w})+A\circ(\Delta_s\otimes \mathrm{Id}_{v\otimes w})+B\circ(\Delta_s\otimes \mathrm{Id}_{v\otimes w})+(\mathrm{Id}_v\otimes\rho^2)\circ C\circ(\Delta_s\otimes \mathrm{Id}_{v\otimes w})\\
&=\rho^1\circ(d\otimes \mathrm{Id}_v)\otimes \mathrm{Id}_w+(\mathrm{Id}_v\otimes\rho^2)\circ(\mathrm{Id}_v\otimes d\otimes \mathrm{Id}_w)\circ(T\otimes \mathrm{Id}_w)+(\rho^1\circ(\mathrm{Id}_s\otimes\rho^1))\circ\\
&\quad\circ(\Delta_s\otimes \mathrm{Id}_v)\otimes \mathrm{Id}_w+A\circ(\Delta_s\otimes \mathrm{Id}_{v\otimes w})+\underbrace{B\circ(T\otimes \mathrm{Id}_{v\otimes w})}_{=-A}\circ(\Delta_s\otimes \mathrm{Id}_{v\otimes w})+\\
&+(\mathrm{Id}_v\otimes\rho^2)\circ(\mathrm{Id}_v\otimes \mathrm{Id}_s\otimes\rho^2)\circ\underbrace{(T\otimes \mathrm{Id}_{v\otimes w})\circ(\mathrm{Id}_s\otimes T\otimes \mathrm{Id}_w)\circ(\Delta_s\otimes \mathrm{Id}_{v\otimes w})}_{=(\mathrm{Id}_v\otimes\Delta_s\otimes \mathrm{Id}_w)\circ(T\otimes \mathrm{Id}_w)}\\
&=\underbrace{(\rho^1\circ(d\otimes \mathrm{Id}_v)+\rho^1\circ(\mathrm{Id}_s\otimes\rho^1)\circ(\Delta_s\otimes \mathrm{Id}_v))}_{=0,\ \rho^1\text{ representation}}\otimes \mathrm{Id}_w+\mathrm{Id}_v\otimes(\rho^2\circ(d\otimes \mathrm{Id}_w))\circ(T\otimes \mathrm{Id}_w)\\
&\quad A\circ(\Delta_s\otimes \mathrm{Id}_{v\otimes w})-A\circ(\Delta_s\otimes \mathrm{Id}_{v\otimes w})+\mathrm{Id}_v\otimes(\rho^2\circ(\mathrm{Id}_s\otimes\rho^2))\circ(\mathrm{Id}_v\otimes\Delta_s\otimes \mathrm{Id}_w)\circ(T\otimes \mathrm{Id}_w)\\
&=0\otimes \mathrm{Id}_v+\mathrm{Id}_w\otimes(\rho^2\circ(d\otimes \mathrm{Id}_w)+\mathrm{Id}_v\otimes(\rho^2\circ(\mathrm{Id}_s\otimes\rho^2)\circ(\Delta_s\otimes \mathrm{Id}_w)))\circ(T\otimes \mathrm{Id}_w)\\
&=\mathrm{Id}_w\circ\underbrace{(\rho^2\circ(d\otimes \mathrm{Id}_w)+\rho^2\circ(\mathrm{Id}_s\otimes\rho^2)\circ(\Delta_s\otimes \mathrm{Id}_w))}_{=0,\ \rho^2\text{ representation}}\circ(T\otimes \mathrm{Id}_w)=0.
\end{aligned}$$

Therefore ρ is a representation up to homotopy of $\mathfrak{g}$ on $\mathbb{V}\otimes\mathbb{W}$. □

Let us consider the right shift of the first element on $\mathbb{V}$. That is, the degree-preserving map $T^k:\mathbb{V}^{\otimes n}\to\mathbb{V}^{\otimes n}$ for $k\geq 2$,

$$T^k(v_1\otimes\cdots\otimes v_n)=(-1)^{|v_1|(\sum_{j=2}^k|v_j|)}v_2\otimes\cdots\otimes v_k\otimes v_1\otimes v_{k-1}\otimes\cdots\otimes v_n,$$

for $v_i\in\mathbb{V}, i=1,\dots,n$. It is interesting to note that it can be introduced in the following way. The k-shift at right is defined by

$$T^k:=(\mathrm{Id}^{\otimes k-2}\otimes T\otimes \mathrm{Id}^{\otimes n-k})\circ T^{k-1},$$

where $T:\mathbb{V}\otimes\mathbb{V}\to\mathbb{V}\otimes\mathbb{V}$ is the twisting map.

Now consider a representation of an L_∞-algebra $\mathfrak{g}$ on the vector space $\mathbb{V}$

$$\rho:\mathrm{Sym}(\mathfrak{g}\,[1])\otimes\mathbb{V}\to\mathbb{V}.$$

The n^{th} tensor product representation up to homotopy $\rho^{\otimes n}$ of $\mathfrak{g}$ on $\mathbb{V}^{\otimes n}$ is defined by

$$\rho^{\otimes n}:=\rho\otimes \mathrm{Id}^{\otimes n-1}+\sum_{k=2}^n(\mathrm{Id}^{\otimes k-1}\otimes\rho\otimes \mathrm{Id}^{\otimes n-k})\circ T^k.$$

Explicitly, for $x\in\mathrm{Sym}(\mathfrak{g}\,[1])$ and $v_k\in\mathbb{V}, k=1,\dots,n$ we have

$$\begin{aligned}\rho^{\otimes n}(x\otimes v_1\otimes\cdots\otimes v_n)&=\rho(x\otimes v_1)\otimes v_2\otimes\cdots\otimes v_n+\\
&+\sum_{k=2}^n(-1)^{(|x|+1)(\sum_{j=1}^{k-1}|v_j|)}v_1\otimes\cdots v_{k-1}\otimes\rho(x\otimes v_k)\otimes v_{k+1}\otimes\cdots\otimes v_n.\end{aligned}$$

7.3 Cochain products

Let $\mathfrak{g}$ be an L_∞-algebra, $m : \mathbb{V}_1 \otimes \mathbb{V}_2 \to \mathbb{V}_3$ be a linear degree-preserving map and

$$\rho^i : \mathrm{Sym}(\mathfrak{g}\,[1]) \otimes \mathbb{V}_i \to \mathbb{V}_i,$$

be a representation up to homotopy for $i = 1, 2, 3$. Suppose that m is $(\rho^1 \otimes \rho^2, \rho^3)$-equivariant, that is, the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Sym}(\mathfrak{g}\,[1]) \otimes \mathbb{V}_1 \otimes \mathbb{V}_2 & \xrightarrow{\rho^1\otimes\rho^2} & \mathbb{V}_1 \otimes \mathbb{V}_2 \\ \downarrow{\scriptstyle \mathrm{Id}_s\otimes m} & & \downarrow{\scriptstyle m} \\ \mathrm{Sym}(\mathfrak{g}\,[1]) \otimes \mathbb{V}_3 & \xrightarrow{\rho^3} & \mathbb{V}_3. \end{array}$$

Definition 7.3.1. Let $\alpha \in \mathrm{Hom}^p(\mathrm{Sym}(\mathfrak{g}\,[1]), \mathbb{V}_1)$ and $\beta \in \mathrm{Hom}^q(\mathrm{Sym}(\mathfrak{g}\,[1]), \mathbb{V}_2)$. The **cochain product** between α and β along m is defined as

$$\alpha \wedge_m \beta := m \circ \alpha \otimes \beta \circ \Delta_s \in \mathrm{Hom}^{p+q}(\mathrm{Sym}(\mathfrak{g}\,[1]), \mathbb{V}_3).$$

The product between cochains descends to a well-defined product in cohomology. In order to see this, we first exhibit some expected properties that the product of cochains should have. We say that m is **symmetric** if $m = m \circ T$, and that m is **skew-symmetric** if $m = -m \circ T$.

Proposition 7.3.1. *Let $\mathfrak{g}$ be an L_∞-algebra and $m : \mathbb{V}_1 \otimes \mathbb{V}_2 \to \mathbb{V}_3$ as above. Then for $\alpha, \alpha_1, \alpha_2 \in \mathrm{Hom}^\bullet(\mathrm{Sym}(\mathfrak{g}\,[1]), \mathbb{V}_1)$ and $\beta, \beta_1, \beta_2 \in \mathrm{Hom}^\bullet(\mathrm{Sym}(\mathfrak{g}\,[1]), \mathbb{V}_2)$,*

i. The product $\wedge_m$ is $\mathbb{R}$-bilinear.

$$(\alpha_1 + \alpha_2) \wedge_m (\beta_1 + \beta_2) = \alpha_1 \wedge_m \beta_1 + \alpha_1 \wedge_m \beta_2 + \alpha_2 \wedge_m \beta_1 + \alpha_2 \wedge_m \beta_2.$$

ii. If m is symmetric, then

$$\alpha \wedge_m \beta = (-1)^{|\alpha||\beta|} \beta \wedge_m \alpha,$$

and if m is skew-symmetric, then

$$\alpha \wedge_m \beta = -(-1)^{|\alpha||\beta|} \beta \wedge_m \alpha.$$

iii. (Leibniz rule)

$$D_{\rho^3}(\alpha \wedge_m \beta) = (D_{\rho^1}\alpha) \wedge_m \beta + (-1)^{|\alpha|} \alpha \wedge (D_{\rho^2}\beta).$$

Proof. Item (i) follows directly from the linearity of the tensor product. To see item (ii), note that $T \circ \alpha \otimes \beta \circ T = (-1)^{|\alpha||\beta|} \beta \otimes \alpha$ holds. Thus if $x, y \in \mathrm{Sym}(\mathfrak{g}\,[1])$ are homogeneous elements, then

$$\begin{aligned} (T \circ \alpha \otimes \beta \circ T)(x \otimes y) =& (-1)^{|x||y|}(T \circ \alpha \otimes \beta)(y \otimes x) \\ =& (-1)^{|x||y|+|\beta||y|} T(\alpha(y) \otimes \beta(x)) \\ =& (-1)^{|y|(|x|+|\beta|)}(-1)^{(|y|+|\alpha|)(|x|+|\beta|)} \beta(x) \otimes \alpha(y) \\ =& (-1)^{|\alpha|(|x|+|\beta|)} \beta(x) \otimes \alpha(y) \\ =& (-1)^{|\alpha||\beta|}(-1)^{|\alpha||x|} \beta(x) \otimes \alpha(y) \\ =& (-1)^{|\alpha||\beta|}(\beta \otimes \alpha)(x \otimes y). \end{aligned}$$

Hence,

$$T \circ \alpha \otimes \beta \circ T = (-1)^{|\alpha||\beta|} \beta \otimes \alpha.$$

Suppose now that the map m is symmetric, then we have

$$\begin{aligned}\alpha\wedge_m\beta =& m\circ\alpha\otimes\beta\circ\Delta_s\\ =& m\circ T\circ\alpha\otimes\beta\circ T\circ\Delta_s\\ =& (-1)^{|\alpha||\beta|}m\circ\beta\otimes\alpha\circ\Delta_s\\ =& (-1)^{|\alpha||\beta|}\beta\wedge_m\alpha.\end{aligned}$$

It is worth noticing that in the third equality it is used that the coproduct Δ_s is cocommutative, i.e. $T\circ\Delta_s=\Delta_s$. The skew symmetric case is analogous. Finally to see item *(iii)* we are going to check the Leibniz rule by a direct computation

$$\begin{aligned}D_{\rho^3}(\alpha\wedge_m\beta) =& \rho^3\circ(\mathrm{Id}_s\otimes\alpha\wedge_m\beta)\circ\Delta_s-(-1)^{|\alpha|+|\beta|}\alpha\wedge_m\beta\circ d\\ =& \rho^3\circ(\mathrm{Id}_s\otimes(m\circ\alpha\otimes\beta\circ\Delta_s))\circ\Delta_s-(-1)^{|\alpha|+|\beta|}m\circ\alpha\otimes\beta\circ\Delta_s\circ d\\ =& \rho^3\circ(\mathrm{Id}_s\otimes m)\circ(\mathrm{Id}_s\otimes\alpha\otimes\beta)\circ(\mathrm{Id}_s\otimes\Delta_s)\circ\Delta_s-\\ & (-1)^{|\alpha|+|\beta|}m\circ\alpha\otimes\beta\circ(\mathrm{Id}_s\otimes d+d\otimes\mathrm{Id}_s)\circ\Delta_s\\ =& m\circ\rho^1\otimes\rho^2\circ(\mathrm{Id}_s\otimes\alpha\otimes\beta)\circ\Delta_s^2-(-1)^{|\alpha|+|\beta|}m\circ(\alpha\otimes(\beta\circ d))\circ\Delta_s+\\ & -(-1)^{|\alpha|+|\beta|}m\circ(-1)^{|\beta|}((\alpha\circ d)\otimes\beta)\circ\Delta_s)\\ =& \underbrace{m\circ((\rho^1\circ(\mathrm{Id}_s\otimes\alpha))\otimes\beta+(\mathrm{Id}_s\otimes\rho^2)\circ(T\otimes\mathrm{Id}_s)\circ(\mathrm{Id}_s\otimes\alpha\otimes\beta))\circ\Delta_s^2}_{A}+\\ & \underbrace{-(-1)^{|\alpha|+|\beta|}m\circ(\alpha\otimes(\beta\circ d))\circ\Delta_s-(-1)^{|\alpha|}m\circ((\alpha\circ d)\otimes\beta)\circ\Delta_s)}_{B}.\end{aligned}$$

On the one side, since the coproduct is coassociative and cocommutative $\Delta_s^2=(\Delta_s\otimes\mathrm{Id}_s)\circ\Delta_s=(\mathrm{Id}_s\otimes\Delta_s)\circ\Delta_s$ and $(T\otimes\mathrm{Id}_s)\circ\Delta_s^2=\Delta_s^2$, one gets

$$\begin{aligned}A =& m\circ((\rho^1\circ(\mathrm{Id}_s\otimes\alpha))\otimes\beta\circ\Delta_s^2+(\mathrm{Id}_s\otimes\rho^2)\circ(T\otimes\mathrm{Id}_s)\circ(\mathrm{Id}_s\otimes\alpha\otimes\beta)\circ(T\otimes\mathrm{Id}_s)\circ\Delta_s^2)\\ =& m\circ((\rho^1\circ(\mathrm{Id}_s\otimes\alpha))\otimes\beta\circ(\Delta_s\otimes\mathrm{Id}_s)\circ\Delta_s+(\mathrm{Id}_s\otimes\rho^2)\circ(\alpha\otimes\mathrm{Id}_s\otimes\beta)\circ(\mathrm{Id}_s\otimes\Delta_s)\circ\Delta_s)\\ =& m\circ(((\rho^1\circ(\mathrm{Id}_s\otimes\alpha)\circ\Delta_s)\otimes\beta)\circ\Delta_s+(-1)^{|\alpha|}(\alpha\otimes(\rho^2\circ(\mathrm{Id}_s\otimes\beta)\circ\Delta_s))\circ\Delta_s)\\ =& m\circ(((\rho^1\circ(\mathrm{Id}_s\otimes\alpha)\circ\Delta_s)\otimes\beta)+(-1)^{|\alpha|}(\alpha\otimes(\rho^2\circ(\mathrm{Id}_s\otimes\beta)\circ\Delta_s)))\circ\Delta_s.\end{aligned}$$

On the other side, one has that

$$B=m\circ(-(-1)^{|\alpha|+|\beta|}\alpha\otimes(\beta\circ d)-(-1)^{|\alpha|}(\alpha\circ d)\otimes\beta)\circ\Delta_s.$$

Therefore, combining the expressions A and B yields

$$\begin{aligned}D_{\rho^3}(\alpha\wedge_m\beta) =& m\circ(((\rho^1\circ(\mathrm{Id}_s\otimes\alpha)\circ\Delta_s)\otimes\beta)+(-1)^{|\alpha|}(\alpha\otimes(\rho^2\circ(\mathrm{Id}_s\otimes\beta)\circ\Delta_s)))\circ\Delta_s\\ & +m\circ(-(-1)^{|\alpha|+|\beta|}\alpha\otimes(\beta\circ d)-(-1)^{|\alpha|}(\alpha\circ d)\otimes\beta)\circ\Delta_s\\ =& m\circ((\rho^1\circ(\mathrm{Id}_s\otimes\alpha)\circ\Delta_s-(-1)^{|\alpha|}(\alpha\circ d))\otimes\beta)\circ\Delta_s\\ & +(-1)^{|\alpha|}m\circ(\alpha\otimes(\rho^2\circ(\mathrm{Id}_s\otimes\beta)\circ\Delta_s-(-1)^{|\beta|}(\beta\circ d))\circ\Delta_s\\ =& m\circ(D_{\rho^1}\alpha\otimes\beta)\circ\Delta_s+(-1)^{|\alpha|}m\circ(\alpha\otimes D_{\rho^2}\beta)\circ\Delta_s\\ =& D_{\rho^1}\alpha\wedge_m\beta+(-1)^{|\alpha|}\alpha\wedge_m D_{\rho^2}\beta.\end{aligned}$$

In conclusion, the Leibniz rule holds

$$D_{\rho^3}(\alpha\wedge_m\beta)=D_{\rho^1}\alpha\wedge_m\beta+(-1)^{|\alpha|}\alpha\wedge_m D_{\rho^2}\beta.$$

□

Remark 7.3.1. We point out the equivariance hypothesis on m only is required in the proof of item (iii). Moreover, the Leibniz rule is totally independent if the map m is symmetric or skew-symmetric.

Corollary 7.3.1. *Let $\mathfrak{g}$ be a L_∞-algebra, and $\rho^i : \mathrm{Sym}(\mathfrak{g}[1]) \otimes \mathbb{V}_i \to \mathbb{V}_i$ be representations up to homotopy, for $i = 1, 2, 3$, and $m : \mathbb{V}_1 \otimes \mathbb{V}_2 \to \mathbb{V}_3$ be a $(\rho_1 \otimes \rho_2, \rho_3)$-equivariant map as in Definition 7.1.4. Then the cochain product $\wedge_m$ is well-defined on cohomology. That is,*

$$H_{CE,\rho^1}(\mathfrak{g}; \mathbb{V}_1) \otimes H_{CE,\rho^2}(\mathfrak{g}; \mathbb{V}_2) \to H_{CE,\rho^3}(\mathfrak{g}; \mathbb{V}_3), \quad [\alpha] \otimes [\beta] \mapsto [\alpha \wedge_m \beta].$$

Proof. It suffices to verify that if α and β are cocycles, then $\alpha \wedge_m \beta$ is so, and if α is a cocycle and β is a coboundary then $\alpha \wedge_m \beta$ is a coboundary. These implications follow from item (i) and (iii) stated in Proposition 7.3.1. □

Example 7.3.1. Let $\rho : \mathrm{Sym}(\mathfrak{g}[1]) \otimes \mathbb{V} \to \mathbb{V}$ be a representation up to homotopy of $\mathfrak{g}$ on $\mathbb{V}$, as $\mathrm{Id}_{\mathbb{V}\otimes\mathbb{V}} : \mathbb{V}\otimes\mathbb{V} \to \mathbb{V}\otimes\mathbb{V}$ is clearly $(\rho\otimes\rho, \rho\otimes\rho)$-equivariant, it defines a canonical product of cochains. For $\alpha, \beta \in \mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}[1]), \mathbb{V})$ we denote their product as

$$\alpha \wedge_{\mathrm{Id}_{\mathbb{V}\otimes\mathbb{V}}} \beta := \alpha \wedge_\otimes \beta \in \mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}[1]), \mathbb{V} \otimes \mathbb{V}).$$

Here is relevant to observe that this product is not symmetric neither skew-symmetric.

Let us recall the definition of a **graded Lie algebra**, gla for short. A **graded Lie algebra** is a pair $(L, [\cdot,\cdot])$ consisting of a graded vector space $L = \bigoplus_{n\in\mathbb{Z}} L_n$ and a skew-symmetric linear map $[\cdot,\cdot] : L \otimes L \to L$ of degree zero which satisfies the Jacobi identity:

$$[\cdot,\cdot] \circ (\mathrm{Id} \otimes [\cdot,\cdot]) = [\cdot,\cdot] \circ ([\cdot,\cdot] \otimes \mathrm{Id}) + [\cdot,\cdot] \circ (\mathrm{Id} \otimes [\cdot,\cdot]) \circ (T \otimes \mathrm{Id}). \tag{7.4}$$

Spelling out the previous definition yields:

- linearity: $x \in L_r, y \in L_s$ implies $[x, y] \in L_{r+s}$;
- skew-symmetric: $[x, y] = -(-1)^{rs} [y, x]$;
- Jacobi identity: $[x, [y, z]] = [y, [x, z]] + (-1)^{rs} [y, [x, z]]$.

A **differential graded Lie algebra** is a gla $(L, [\cdot,\cdot])$ together with a degree $+1$ map d which is a derivation of the bracket and $d^2 = 0$. That is,

$$d \circ [\cdot,\cdot] = [\cdot,\cdot] \circ (d \circ \mathrm{Id} + \mathrm{Id} \otimes d), \quad d^2 = 0, \tag{7.5}$$

For $x, y \in L$ homogeneous, one has

$$d([x, y]) = [dx, y] + (-1)^{|x|} [x, dy].$$

Corollary 7.3.2. *Let $\mathfrak{g}$ be an L_∞-algebra and $(L, [\cdot,\cdot])$ be a graded Lie algebra, then*

$$(\mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}[1]), L), \wedge_{[\cdot,\cdot]}),$$

is a graded Lie algebra with the cochain product $\wedge_{[\cdot,\cdot]}$. Moreover, if $(L, [\cdot,\cdot], d)$ is a differential graded Lie algebra, then $\mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}[1], L), \wedge_{[\cdot,\cdot]}, \boldsymbol{d})$ is differential graded Lie algebra with differential given by

$$\boldsymbol{d}(\alpha) := d \circ \alpha - (-1)^{|\alpha|} \alpha \circ d_\lambda,$$

for every $\alpha \in \mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}[1]), L)$ homogeneous.

Proof. Let us consider homogeneous elements $\alpha, \beta, \gamma \in \mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}[1]), L)$, then by Theorem (7.3.1) we have that the product $\wedge_{[\cdot,\cdot]}$ is a skew-symmetric product. Let us see that it satisfies the Jacobi identity.

$$\begin{aligned}\alpha \wedge_{[\cdot,\cdot]} (\beta \wedge_{[\cdot,\cdot]} \gamma) &= [\cdot,\cdot] \circ (\alpha \otimes \beta \wedge_{[\cdot,\cdot]} \gamma) \circ \Delta_s = [\cdot,\cdot] \circ (\alpha \otimes ([\cdot,\cdot] \circ \beta \otimes \gamma \circ \Delta_s)) \circ \Delta_s \\ &= [\cdot,\cdot] \circ (\mathrm{Id} \otimes [\cdot,\cdot]) \circ \alpha \otimes \beta \otimes \gamma \circ (\mathrm{Id} \otimes \Delta_s) \circ \Delta_s, \quad \text{by Equation (7.4)} \\ &= [\cdot,\cdot] \circ ([\cdot,\cdot] \otimes \mathrm{Id}) \circ \alpha \otimes \beta \otimes \gamma \circ \Delta_s^2 + [\cdot,\cdot] \circ (\mathrm{Id} \otimes [\cdot,\cdot]) \circ (T \otimes \mathrm{Id}) \circ \alpha \otimes \beta \otimes \gamma \circ \Delta_s^2 \\ &= (\alpha \wedge_{[\cdot,\cdot]} \beta) \wedge_{[\cdot,\cdot]} \gamma + (-1)^{|\alpha||\beta|} \beta \wedge_{[\cdot,\cdot]} (\alpha \wedge_{[\cdot,\cdot]} \gamma).\end{aligned}$$

Now we assume that $(L, [\cdot,\cdot], d)$ is **dgla** and let us see that the map $\mathbf{d}$ is derivation of $\wedge_{[\cdot,\cdot]}$ with $\mathbf{d}^2 = 0$. Verifying that $\mathbf{d}^2 = 0$ is straightforward and equivalent to $d^2 = 0$ and $d_\lambda^2 = 0$. To see that $\mathbf{d}$ is a derivation let us compute it directly

$$\begin{aligned}\mathbf{d}(\alpha \wedge_{[\cdot,\cdot]} \beta) &= d \circ \alpha \wedge_{[\cdot,\cdot]} \beta - (-1)^{|\alpha|+|\beta|} \alpha \wedge_{[\cdot,\cdot]} \beta \circ d_\lambda \\ &= d \circ [\cdot,\cdot] \circ \alpha \otimes \beta \circ \Delta_s - (-1)^{|\alpha|+|\beta|} [\cdot,\cdot] \circ \alpha \otimes \beta \circ \Delta_s \circ d_\lambda \\ &= [\cdot,\cdot] \circ (d \otimes \mathrm{Id} + \mathrm{Id} \otimes d) \circ \alpha \otimes \beta \circ \Delta_s - (-1)^{|\alpha|+|\beta|} [\cdot,\cdot] \circ \alpha \otimes \beta \circ (d_\lambda \otimes \mathrm{Id} + \mathrm{Id} \otimes d_\lambda) \circ \Delta_s \\ &= [\cdot,\cdot] \circ (d \circ \alpha) \otimes \beta \circ \Delta_s + [\cdot,\cdot] \circ (-1)^{|\alpha|} \alpha \otimes (d \circ \beta) \circ \Delta_s - (-1)^{|\alpha|} [\cdot,\cdot] \circ (\alpha \circ d_\lambda) \otimes \beta \circ \Delta_s \\ &\quad - (-1)^{|\alpha|+|\beta|} [\cdot,\cdot] \circ \alpha \otimes (\beta \circ d_\lambda) \circ \Delta_s \\ &= [\cdot,\cdot] \circ (d \circ \alpha - (-1)^{|\alpha|} \alpha \circ d_\lambda) \otimes \beta \circ \Delta_s + (-1)^{|\alpha|} [\cdot,\cdot] \circ \alpha \otimes (d \circ \beta - (-1)^{|\beta|} \beta \circ d_\lambda) \circ \Delta_s \\ &= \mathbf{d}\alpha \wedge_{[\cdot,\cdot]} \beta + (-1)^{|\alpha|} \alpha \wedge_{[\cdot,\cdot]} \mathbf{d}\beta.\end{aligned}$$

□

Let us consider the action of the symmetric group S_{n+1} on the vector space $\mathrm{Sym}(\mathfrak{g}[1])^{\otimes(n+1)}$. One observes that since the coproduct Δ_s is cocommutative and coassociative, for a translation $\sigma_i \in S_{n+1}$ the following holds

$$\begin{aligned}\hat{\epsilon}(\sigma_i) \circ \Delta_s^n &= (1^{\otimes i-1} \otimes T \otimes 1^{\otimes n-i}) \circ (1^{\otimes i-1} \otimes \Delta_s \otimes 1^{\otimes n-i}) \circ \Delta_s^{n-1} \\ &= (1^{\otimes i-1} \otimes (T \circ \Delta_s) \otimes 1^{\otimes n-i}) \circ \Delta_s^{n-1} \\ &= (1^{\otimes i-1} \otimes \Delta_s \otimes 1^{\otimes n-i}) \circ \Delta_s^{n-1} = \Delta_s^n.\end{aligned}$$

Therefore, as this action is generated by translations, for every permutation $\sigma \in S_{n+1}$ we have

$$\hat{\epsilon}(\sigma) \circ \Delta_s^n = \Delta_s^n. \tag{7.6}$$

Proposition 7.3.2. *Let ρ be a representation up to homotopy of $\mathfrak{g}$ on $\mathbb{V}$ and $\alpha_i \in \mathrm{Hom}^{|\alpha_i|}(\mathrm{Sym}(\mathfrak{g}[1]), \mathbb{V})$, for $1 \leq i \leq k$. Given a k-linear map $f \in \mathrm{Hom}(\mathbb{V}^{\otimes k}, \mathbb{W})$, consider*

$$f_{\alpha_1,\dots,\alpha_k} := f \circ \alpha_1 \otimes \cdots \otimes \alpha_k \circ \Delta_s^{k-1} : \mathrm{Sym}(\mathfrak{g}[1]) \to \mathbb{W}.$$

Then for every $\sigma \in S_k$ the following hold:

i. if $f \in \mathrm{Sym}^k(\mathbb{V}, \mathbb{W})$, then

$$f_{\alpha_1,\dots,\alpha_k} = \epsilon(\sigma; \alpha_1, \dots, \alpha_k) f_{\alpha_{\sigma(1)},\dots,\alpha_{\sigma(k)}};$$

ii. if $f \in \mathrm{Skew}^k(\mathbb{V}, \mathbb{W})$, then

$$f_{\alpha_1,\dots,\alpha_k} = \chi(\sigma; \alpha_1, \dots, \alpha_k) f_{\alpha_{\sigma(1)},\dots,\alpha_{\sigma(k)}}.$$

Proof. Note that for any $\sigma \in S_n$ one has

$$\hat{\epsilon}(\sigma) \circ \alpha_1 \otimes \cdots \otimes \alpha_n \circ \hat{\epsilon}(\sigma^{-1}) = \epsilon(\sigma; \alpha_1, \ldots, \alpha_n)\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)}. \tag{7.7}$$

In order to check (7.7) it suffices to show it for generators $\sigma_i = (i, i+1) \in S_n, 1 \leq i \leq n-1$. Thus for homogeneous elements $x_i \in \mathbb{V}, 1 \leq i \leq n$,

$$\begin{aligned}
&\hat{\epsilon}(\sigma_i) \circ \alpha_1 \otimes \cdots \otimes \alpha_n \circ \hat{\epsilon}(\sigma_i)(x_1 \otimes \cdots \otimes x_n) = (-1)^{|x_i||x_{i+1}|}(\hat{\epsilon}(\sigma_i) \circ \alpha_1 \otimes \cdots \\
&\quad \cdots \otimes \alpha_n)(x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_n) \\
&\quad =(-1)^{|x_i||x_{i+1}|}\hat{\epsilon}(\sigma)((-1)^{\sum_{k=1,k\neq i,i+1}^n |x_k|(\sum_{j=k+1}^n |\alpha_j|)+|x_{i+1}|(\sum_{j=i+1}^n |\alpha_j|)+|x_i|(\sum_{j=i+2}^n |\alpha_j|)} \\
&\quad\ \hat{\epsilon}(\sigma_i)\alpha_1(x_1) \otimes \cdots \otimes \alpha_i(x_{i+1}) \otimes \alpha_{i+1}(x_i) \otimes \cdots \otimes \alpha_n(x_n) \\
&\quad =(-1)^{|x_i||x_{i+1}|+\sum_{k=1,k\neq i,i+1}^n |x_k|(\sum_{j=k+1}^n |\alpha_j|)+|x_{i+1}|(\sum_{j=i+2}^n |\alpha_j|)+(|\alpha_i|+|x_{i+1}|)(|\alpha_{i+1}|+|x_i|)} \\
&\quad\ \alpha_1(x_1) \otimes \cdots\cdots \otimes \alpha_{i+1}(x_i) \otimes \alpha_i(x_{i+1}) \otimes \cdots \otimes \alpha_n(x_n) \\
&\quad =(-1)^{|x_i||x_{i+1}|+\sum_{k=1,k\neq i,i+1}^n |x_k|(\sum_{j=k+1}^n |\alpha_j|)+|x_{i+1}|(\sum_{j=i+1}^n |\alpha_j|)+|x_i|(\sum_{j=i+2}^n |\alpha_j|)} \\
&\quad\ (-1)^{|\alpha_i||\alpha_{i+1}|+|x_{i+1}||\alpha_{i+1}|+|x_{i+1}||x_i|}(-1)^{\sum_{k=1,k\neq i,i+1}^n |x_k|(\sum_{j=k+1}^n |\alpha_j|)+|x_i|(\sum_{j=i}^n |\alpha_j|-|\alpha_{i+1}|)} \\
&\quad\ (-1)^{|x_{i+1}|(\sum_{j=i+2}^n |\alpha_j|)}(\alpha_1 \otimes \cdots \otimes \alpha_{i+1} \otimes \alpha_i \otimes \cdots \otimes \alpha_n)(x_1 \otimes \cdots \otimes x_n) \\
&\quad =(-1)^{|\alpha_i||\alpha_{i+1}|}(\alpha_1 \otimes \cdots \otimes \alpha_{i+1} \otimes \alpha_i \otimes \cdots \alpha_n)(x_1 \otimes \cdots \otimes x_n).
\end{aligned}$$

Hence

$$\hat{\epsilon}(\sigma_i) \circ \alpha_1 \otimes \cdots \otimes \alpha_n \circ \hat{\epsilon}(\sigma_i) = (-1)^{|\alpha_i||\alpha_{i+1}|}\alpha_1 \otimes \cdots \otimes \alpha_{i+1} \otimes \alpha_i \otimes \cdots \otimes \alpha_n.$$

To see item (i), let $f \in \mathrm{Sym}^k(\mathbb{V}, \mathbb{W})$, then

$$\begin{aligned}
f_{\alpha_1,\ldots,\alpha_n} &= f \circ \alpha_1 \otimes \cdots \otimes \alpha_n \circ \Delta_s^{n-1} \\
&= f \circ \hat{\epsilon}(\sigma) \circ \alpha_1 \otimes \cdots \otimes \alpha_n \circ \underbrace{\hat{\epsilon}(\sigma^{-1}) \circ \Delta_s^{n-1}}_{=\Delta_s^{n-1}, \text{ by equation (7.6)}} \\
&= f \circ \epsilon(\sigma; \alpha_1, \ldots, \alpha_n)\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)} \circ \Delta_s^{n-1} \\
&= \epsilon(\sigma; \alpha_1, \ldots, \alpha_n) f_{\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(n)}}.
\end{aligned}$$

Regarding item (ii) let $f \in \mathrm{Skew}^k(\mathbb{V}, \mathbb{W})$, then

$$\begin{aligned}
f_{\alpha_1,\ldots,\alpha_n} &= f \circ \alpha_1 \otimes \cdots \otimes \alpha_n \circ \Delta_s^{n-1} \\
&= f \circ \hat{\chi}(\sigma) \circ \alpha_1 \otimes \cdots \otimes \alpha_n \circ \hat{\epsilon}(\sigma^{-1}) \circ \Delta_s^{n-1} \\
&= f \circ \mathrm{sgn}(\sigma)\hat{\epsilon}(\sigma) \circ \alpha_1 \otimes \cdots \otimes \alpha_n \circ \hat{\epsilon}(\sigma^{-1}) \circ \Delta_s^{n-1} \\
&= \mathrm{sgn}(\sigma) f \circ \epsilon(\sigma; \alpha_1, \ldots, \alpha_n)\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)} \circ \Delta_s^{n-1} \\
&= \mathrm{sgn}(\sigma)\epsilon(\sigma; \alpha_1, \ldots, \alpha_n) f_{\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(n)}} \\
&= \chi(\sigma; \alpha_1, \ldots, \alpha_n) f_{\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(n)}}.
\end{aligned}$$

□

7.4 Representations up to homotopy in terms of Maurer-Cartan elements

Let us recall the relation between L_∞-modules and Maurer-Cartan elements in a suitable dgla. For that we follow [LM95]. Let us consider a differential (non-negative) graded vector space (dg-vector space for short) $\mathbb{V} = \bigoplus_{0 \leq k} V_k$, with a differential $\partial \in \mathrm{Hom}^1(\mathbb{V}, \mathbb{V})$, $\quad \partial^2 = 0$,

$$\mathbb{V}: \qquad \cdots \longrightarrow 0 \xrightarrow{\partial} V_0 \xrightarrow{\partial} V_1 \xrightarrow{\partial} V_2 \xrightarrow{\partial} \cdots \xrightarrow{\partial} V_n \xrightarrow{\partial} \cdots .$$

The space of endomorphisms $\mathrm{End}(\mathbb{V}) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}^k(\mathbb{V},\mathbb{V})$ is a graded associative algebra with product given by the composition of morphisms. In particular, it inherits a graded Lie algebra with the commutator bracket. That is,

$$[S,T] = S\circ T - (-1)^{|S||T|} T\circ S.$$

for $S,T\in \mathrm{End}(\mathbb{V})$ homogeneous elements. Moreover, the differential operator ∂ of $\mathbb{V}$ induces a degree 1 derivation δ of $(\mathrm{End}(\mathbb{V}),[\cdot,\cdot])$ defined as follows, for $T\in\mathrm{End}(\mathbb{V})$

$$\delta(T) := [\partial, T] = \partial\circ T - (-1)^{|T|} T\circ\partial.$$

Thus $(\mathrm{End}(\mathbb{V}), [\cdot,\cdot], \delta)$ is **dgla** which we denote by $\mathfrak{gl}(\mathbb{V})$. The notion of representations up to homotopy was originally introduced in [LM95] with the name of L_∞-modules, and in this same work, was shown that an L_∞-module of an L_∞-algebra $(\mathfrak{g},[\cdot])$ over the dg-vector space $(\mathbb{V},\partial)$ is equivalent to an L_∞-morphism ρ from $\mathfrak{g}$ to $\mathfrak{gl}(\mathbb{V})$, where $\mathfrak{gl}(\mathbb{V})$ is considered as an L_∞-algebra, see [LM95, Thm 5.4]. In [Rei19, Lem. 46] was proved that the map ρ is a L_∞-morphism if and only if the degree 1 map $\bar\rho : \overline{\mathrm{Sym}}(\mathfrak{g}[1]) \to \mathfrak{gl}(\mathbb{V})$ satisfies the equation

$$[\partial, -]\circ\bar\rho + \bar\rho\circ d_\lambda + \frac{1}{2}[\cdot,\cdot]\circ\bar\rho\otimes\bar\rho\circ\bar\Delta_s = 0. \tag{7.8}$$

Therefore, Corollary (7.3.2) implies that $(\mathrm{Hom}(\overline{\mathrm{Sym}}(\mathfrak{g}[1]),\mathfrak{gl}(\mathbb{V})), \wedge_{[\cdot,\cdot]}, \mathbf{d})$ is a **dgla** and the Equation (7.8) allows us to interpret a representation up to homotopy as a Maurer-Cartan element in this **dgla**. By this we mean, the map $\bar\rho$ is a degree 1 element in $\mathrm{Hom}(\overline{\mathrm{Sym}}(\mathfrak{g}[1]),\mathfrak{gl}(\mathbb{V}))$ which satisfies the equation

$$\mathbf{d}\bar\rho + \frac{1}{2}\bar\rho\wedge_{[\cdot,\cdot]}\bar\rho = 0.$$

We denote the set of Maurer-Cartan elements (or representations up to homotopy) of the **dgla** $(\mathrm{Hom}(\overline{\mathrm{Sym}}(\mathfrak{g}[1]),\mathfrak{gl}(\mathbb{V})), \wedge_{[\cdot,\cdot]}, \mathbf{d})$ by $\mathcal{R}ep^\infty_{\mathfrak{g}}(\mathbb{V},\partial)$.

Theorem 7.4.1. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two L_∞-algebras. An L_∞-morphism $F:\mathfrak{g}\to\mathfrak{h}$ induces a **dgla**-morphism*

$$F^* : \mathrm{Hom}(\overline{\mathrm{Sym}}(\mathfrak{h}[1]),\mathfrak{gl}(\mathbb{V})) \to \mathrm{Hom}(\overline{\mathrm{Sym}}(\mathfrak{g}[1]),\mathfrak{gl}(\mathbb{V})), \quad F^*\alpha := \alpha\circ F.$$

*Moreover, if F is an L_∞-quasi-isomorphism then F^* is a quasi-isomorphism of **dgla's**.*

Proof. Let $\alpha,\beta\in\mathrm{Hom}(\overline{\mathrm{Sym}}(\mathfrak{h}[1]),\mathfrak{gl}(\mathbb{V}))$ be two homogeneous elements, then

$$\begin{aligned} F^*(\alpha\wedge_{[\cdot,\cdot]}\beta) &= [\cdot,\cdot]\circ\alpha\otimes\beta\circ\Delta_s\circ F \\ &= [\cdot,\cdot]\circ\alpha\otimes\beta\circ F\otimes F\circ\Delta_s \\ &= [\cdot,\cdot]\circ(\alpha\circ F)\otimes(\beta\circ F)\circ\Delta_s \\ &= F^*\alpha\wedge_{[\cdot,\cdot]} F^*\beta, \end{aligned}$$

and

$$\begin{aligned} F^*(\mathbf{d}\alpha) &= ([\partial,\cdot]\circ\alpha - (-1)^{|\alpha|}\alpha\circ d_{\lambda\mathfrak{h}})\circ F \\ &= [\partial,\cdot]\circ(\alpha\circ F) - (-1)^{|\alpha|}(\alpha\circ F)\circ d_{\lambda\mathfrak{g}} \\ &= [\partial,\cdot]\circ F^*\alpha - (-1)^{|\alpha|}F^*\alpha\circ d_{\lambda\mathfrak{g}} \\ &= \mathbf{d}(F^*\alpha). \end{aligned}$$

Hence, F^* is a **dgla**-morphism. Now, let us suppose that F is an L_∞-quasi-isomorphism then it is well-known that there exists an inverse L_∞-quasi-isomorphism $G:\mathfrak{h}\to\mathfrak{g}$, see for example [Kon03, Mar04, ADM02, Mar06]. Furthermore, there exists linear maps $H\in\mathrm{Hom}^{-1}(\mathrm{Sym}(\mathfrak{g}[1]),\mathrm{Sym}(\mathfrak{g}[1]))$

and $H' \in \mathrm{Hom}^{-1}(\mathrm{Sym}(\mathfrak{h}\,[1]), \mathrm{Sym}(\mathfrak{h}\,[1]))$ such that

$$G \circ F - \mathrm{Id}_{\mathfrak{g}} = d_{\lambda^{\mathfrak{g}}} \circ H + H \circ d_{\lambda^{\mathfrak{g}}}, \quad \text{and} \quad F \circ G - \mathrm{Id}_{\mathfrak{h}} = d_{\lambda^{\mathfrak{h}}} \circ H' + H' \circ d_{\lambda^{\mathfrak{h}}}.$$

One proof for this fact can be found in [Kaj07, Thm 7.5]. Thus, for some $\omega \in \mathrm{Hom}(\overline{\mathrm{Sym}}(\mathfrak{h}\,[1]), \mathfrak{gl}(\mathbb{V}))$ homogeneous element we have

$$\begin{aligned}
(G^* \circ F^* - \mathrm{Id})(\omega) =& G^*(\omega \circ F) - \omega \\
=& \omega \circ F \circ G - \omega = \omega \circ (F \circ G - \mathrm{Id}_{\mathfrak{h}}) \\
=& \omega \circ (H' \circ d_{\lambda^{\mathfrak{h}}} + d_{\lambda^{\mathfrak{h}}} \circ H') \\
=& -(-1)^{|\omega|}\,[\partial, \cdot] \circ \omega \circ H' + \omega \circ d_{\lambda^{\mathfrak{h}}} \circ H' + (-1)^{|\omega|}\,[\partial, \cdot] \circ \omega \circ H' + \omega \circ H' \circ d_{\lambda^{\mathfrak{h}}} \\
=& -(-1)^{|\omega|} H'^*(\mathbf{d}\omega) + (-1)^{|\omega|}([\partial, \cdot] \circ H'^*\omega - (-1)^{|\omega|-1} H'^*\omega \circ d_{\lambda^{\mathfrak{h}}}) \\
=& (-1)^{|\omega|}(\mathbf{d}(H'^*\omega) - H'^*(\mathbf{d}\omega)),
\end{aligned}$$

then

$$(G^* \circ F^* - \mathrm{Id})(\omega) = (-1)^{|\omega|}(\mathbf{d}(H'^*\omega) - H'^*(\mathbf{d}\omega)).$$

Therefore in cohomology one has that G^* is left inverse of F^*, and analogously we can see that in cohomology G is right inverse. Therefore F^* is an isomorphism in cohomology, and so, a quasi-isomorphism of **dgla's**. □

The next theorem allows us to give an interpretation of the word *up to homotopy* in the sense that a representation up to homotopy of $\mathfrak{g}$ on a dg-vector space $\mathbb{V}$, actually, represents $\mathfrak{g}$ in the homological homotopy class of $(\mathbb{V}, \partial)$.

Theorem 7.4.2. *Let ρ be a representation up to homotopy of the L_∞-algebra $\mathfrak{g}$ on the dg-vector space $\mathbb{V}$. Then there is an induced representation up to homotopy on the dg-vector space $H(\mathbb{V}, \partial)$.*

Proof. First observe that $H(\mathbb{V}, \partial)$, or simple $H(\mathbb{V})$, is a dg-vector space with null-differential and it is well-known that the dg-vector space $\mathbb{V}$ is a deformation retract of $H(\mathbb{V})$, that means, there are linear maps

$$H \circlearrowright (\mathbb{V}, \partial) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (H(\mathbb{V}, \partial), 0)$$

such that $\pi \circ \iota = \mathrm{Id}_{H(\mathbb{V})}$ and $\mathrm{Id}_{\mathbb{V}} - \iota \circ \pi = \partial \circ H + H \circ \partial$. This deformation allows us to construct a strict L_∞-quasi-isomorphism between the **dgla's**,

$$(\cdot)^* : \mathfrak{gl}(H(\mathbb{V})) \to \mathfrak{gl}(\mathbb{V}), \quad f^* := \iota \circ f \circ \pi.$$

This assignment is a linear dg-associative algebras homomorphism, for f and g in $\mathfrak{gl}(H(\mathbb{V}))$ one has that

$$(f \circ g)^* = \iota \circ (f \circ g) \circ \pi = (\iota \circ f \circ \pi) \circ (\iota \circ g \circ \pi) = f^* \circ g^*.$$

Moreover, if $F \in \mathfrak{gl}(\mathbb{V})$ is a class representative for a homogeneous element in $H(\mathfrak{gl}(\mathbb{V}), \delta)$, then we have $\partial \circ F = (-1)^{|F|} F \circ \partial$, so that for $H(F) \in \mathfrak{gl}(H(\mathbb{V}))$ the contraction above implies that

$$\begin{aligned}
F - F \circ \iota \circ \pi =& F \circ \partial \circ H + F \circ H \circ \partial \\
=& (-1)^{|F|} \partial \circ (FH) + (FH) \circ \partial \\
=& \partial \circ ((-1)^{|F|} FH) - (-1)^{|F|-1}((-1)^{|F|} FH) \circ \partial.
\end{aligned}$$

Hence $F - F \circ \iota \circ \pi = \left[\partial, (-1)^{|F|} FH\right]$ and $F \circ \iota \circ \pi = H(F)^*$. Therefore $[F] = [H(F)^*]$ and so $(\cdot)^*$ is an L_∞-quasi-isomorphism. Now according to [Kon03] there exists an inverse L_∞-quasi-isomorphism $P : \mathfrak{gl}(\mathbb{V}) \to \mathfrak{gl}(H(\mathbb{V}))$. Therefore the representation induced on $H(\mathbb{V})$ is given by

$$\rho' : \mathfrak{g} \xrightarrow{\rho} \mathfrak{gl}(\mathbb{V}) \xrightarrow{P} \mathfrak{gl}(H(\mathbb{V})), \quad \rho' := P \circ \rho.$$

□

7.5 The reduced cohomology

Throughout this section let $\mathfrak{g}$ be an L_∞-algebra and ρ be a representation up to homotopy of the L_∞-algebra $\mathfrak{g}$ on the dg-vector space $\mathbb{V}$.

Proposition 7.5.1. *Let $\alpha : \mathrm{Sym}(\mathfrak{g}[1]) \to \mathbb{V}$ be a homogeneous linear map with $\alpha(1) = 0$, then $D_\rho\alpha$ satisfies $D_\rho\alpha(1) = 0$ and*

$$D_\rho\alpha = \overline{\rho} \wedge_{ev} \alpha + \partial \circ \alpha - (-1)^{|\alpha|}\alpha \circ d_\lambda.$$

Proof. Note that

$$\begin{aligned} D_\rho\alpha(1) =& \rho \circ (\mathrm{Id}_s \otimes \alpha) \circ \Delta_s(1) - (-1)^{|\alpha|}\alpha \circ d(1) \\ =& \rho \circ (\mathrm{Id}_s \otimes \alpha)(1 \otimes 1) - 0 \\ =& \rho(1 \otimes \alpha(1)) = \partial(\alpha(1)), \end{aligned}$$

thus $\alpha(1) = 0$ implies $D_\rho\alpha(1) = 0$. Now for $x \in \overline{\mathrm{Sym}}(\mathfrak{g}[1])$ we have

$$\begin{aligned} D_\rho\alpha(x) =& \rho(\mathrm{Id}_s \otimes \alpha)(\overline{\Delta}(x) + 1 \otimes x + x \otimes 1) - (-1)^{|\alpha|}\alpha(d(x)) \\ =& \rho(\mathrm{Id}_s \otimes \alpha)\overline{\Delta}(x) + \rho(\mathrm{Id}_s \otimes \alpha)(1 \otimes x) + \rho(\mathrm{Id}_s \otimes \alpha)(x \otimes 1) - (-1)^{|\alpha|}\alpha(d(x)) \\ =& \overline{\rho}(\mathrm{Id}_s \otimes \alpha)\overline{\Delta}(x) + \partial(\alpha(x)) + (-1)^{|\alpha||x|}\overline{\rho}_x(\alpha(1)) - (-1)^{|\alpha|}\alpha(d(x)), \end{aligned}$$

and since $\overline{\rho}(1 \otimes \alpha)\overline{\Delta} = \overline{\rho} \wedge_{ev} \alpha$, then

$$D_\rho\alpha = \overline{\rho} \wedge_{ev} \alpha + \partial \circ \alpha - (-1)^{|\alpha|}\alpha \circ d.$$

□

The above proposition allows us to reduce the coboundary operator D_ρ to the subcomplex

$$\mathrm{Hom}(\overline{\mathrm{Sym}}(\mathfrak{g}[1]), \mathbb{V}) \subseteq \mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}[1]), \mathbb{V}),$$

that is

$$D_{\overline{\rho}} : \mathrm{Hom}^\bullet(\overline{\mathrm{Sym}}(\mathfrak{g}[1]), \mathbb{V}) \to \mathrm{Hom}^{\bullet+1}(\overline{\mathrm{Sym}}(\mathfrak{g}[1]), \mathbb{V}), \quad D_{\overline{\rho}}\alpha = \overline{\rho} \wedge_{ev} \alpha + \partial \circ \alpha - (-1)^{|\alpha|}\alpha \circ d.$$

The next formula tells us that the operator $D_{\overline{\rho}}$ is a coboundary operator if and only if $\bar{\rho}$ is a representation up to homotopy as in Equation (7.8).

Proposition 7.5.2. *The next formula holds*

$$D^2_{\overline{\rho}}(\cdot) = \left(\frac{1}{2}\overline{\rho} \wedge_{[\cdot,\cdot]} \overline{\rho} + \overline{\rho} \circ d + [\partial, \cdot] \circ \overline{\rho}\right) \wedge_{ev} \cdot$$

Proof. Let us explicitly check the above formula in an arbitrary homogeneous linear map $\alpha : \overline{\mathrm{Sym}}(\mathfrak{g}[1]) \to \mathbb{V}$

$$\begin{aligned} D_{\overline{\rho}}(D_{\overline{\rho}}\alpha) =& \overline{\rho} \wedge_{ev} (D_{\overline{\rho}}\alpha) + \partial(D_{\overline{\rho}}\alpha) - (-1)^{|\alpha|+1}D_{\overline{\rho}}\alpha \circ d \\ =& \overline{\rho} \wedge_{ev} (\overline{\rho} \wedge_{ev} \alpha) + \overline{\rho} \wedge_{ev} (\partial \circ \alpha) + \overline{\rho} \wedge_{ev} (-(-1)^{|\alpha|}\alpha \circ d) + \partial \circ \overline{\rho} \wedge_{ev} \alpha + \partial \circ \partial \circ \alpha + \\ & - (-1)^{|\alpha|}\partial \circ \alpha \circ d - (-1)^{|\alpha|+1}\overline{\rho} \wedge_{ev} \alpha \circ d - (-1)^{|\alpha|+1}\partial \circ \alpha \circ d - (-1)^{|\alpha|+1}((-1)^{|\alpha|+1}\alpha \circ d \circ d) \\ =& \underbrace{\overline{\rho} \wedge_{ev} (\overline{\rho} \wedge_{ev} \alpha)}_{A} + \underbrace{\overline{\rho} \wedge_{ev} (\partial \circ \alpha) + \partial \circ (\overline{\rho} \wedge_{ev} \alpha)}_{B} \underbrace{- (-1)^{|\alpha|}\,\overline{\rho} \wedge_{ev} (\alpha \circ d) + (-1)^{|\alpha|}(\overline{\rho} \wedge_{ev} \alpha) \circ d}_{C}. \end{aligned}$$

Then let us analyse expressions A, B and C, separately

$$\begin{aligned}
A = (\overline{\rho} \wedge_{ev} \alpha) \circ d =& ev \circ \overline{\rho} \otimes \alpha \circ \Delta_s \circ d \\
=& ev \circ \overline{\rho} \otimes \alpha \circ (\mathrm{Id}_s \otimes d + d \otimes \mathrm{Id}_s) \circ \Delta_s \\
=& \overline{\rho} \wedge_{ev} (\alpha \circ d) + (-1)^{|\alpha|} (\overline{\rho} \circ d) \wedge_{ev} \alpha. \\
B = \overline{\rho} \wedge_{ev} (\overline{\rho} \wedge_{ev} \alpha) =& ev \circ \overline{\rho} \otimes (ev \circ \overline{\rho} \otimes \alpha \circ \Delta_s) \circ \Delta_s \\
=& ev \circ (\mathrm{Id}_s \otimes ev) \circ (\overline{\rho} \otimes \overline{\rho} \otimes \alpha) \circ (\mathrm{Id} \otimes \Delta_s) \circ \Delta_s \\
=& (C \otimes ev) \circ (\overline{\rho} \otimes \overline{\rho} \otimes \alpha) \circ \Delta_s^2 \\
=& (\overline{\rho} \wedge_C \overline{\rho}) \wedge_{ev} \alpha \\
=& \frac{1}{2} (\overline{\rho} \wedge_{[\cdot,\cdot]} \overline{\rho}) \wedge_{ev} \alpha.
\end{aligned}$$

Regarding expression C, let x be in $\mathrm{Sym}(\mathfrak{g}[1])$ then

$$\begin{aligned}
([\partial, \cdot] \circ \overline{\rho}) \wedge_{ev} \alpha(x) =& (ev \circ ([\partial, \cdot] \circ \overline{\rho}) \otimes \alpha) \left(\sum x_{(1)} \otimes x_{(2)} \right) \\
=& \sum (-1)^{|x_{(1)}||\alpha|} [\partial, \overline{\rho}_{x_{(1)}}](\alpha(x_{(2)}) \\
=& \sum (-1)^{|x_{(1)}||\alpha|} \left(\partial \overline{\rho}_{x_{(1)}}(\alpha(x_{(2)})) - (-1)^{|x_{(1)}|+1} \overline{\rho}_{x_{(1)}} \partial(\alpha(x_{(2)})) \right) \\
=& \sum (-1)^{|x_{(1)}||\alpha|} \partial \overline{\rho}_{x_{(1)}}(\alpha(x_{(2)})) + (-1)^{(|x_{(1)}|+1)|\alpha|} \overline{\rho}_{x_{(1)}} \partial(\alpha(x_{(2)})) \\
=& \partial \circ (\overline{\rho} \wedge_{ev} \alpha)(x) + \overline{\rho} \wedge_{ev} (\partial \alpha)(x).
\end{aligned}$$

Putting all these observations together gives rise to

$$\begin{aligned}
D_\rho^2 \alpha =& (\frac{1}{2} \overline{\rho} \wedge_{[\cdot,\cdot]} \overline{\rho}) \wedge_{ev} \alpha + ([\partial, \cdot] \circ \overline{\rho}) \wedge_{ev} \alpha + (\overline{\rho} \circ d) \wedge_{ev} \alpha \\
=& \left(\frac{1}{2} \overline{\rho} \wedge_{[\cdot,\cdot]} \overline{\rho} + [\partial, \cdot] \circ \overline{\rho} + \overline{\rho} \circ d \right) \wedge_{ev} \alpha.
\end{aligned}$$

□

Definition 7.5.1. Let $\mathfrak{g}$ be an L_∞-algebra, and ρ be a representation up to homotopy of $\mathfrak{g}$ on the dg-vector space $(\mathbb{V}, \partial)$. The cohomology of the subcomplex

$$\left(\mathrm{Hom}(\overline{\mathrm{Sym}}(\mathfrak{g}[1]), \mathbb{V}), D_{\overline{\rho}} \right)$$

is said to be the L_∞ **reduced cohomology of $\mathfrak{g}$ with values in** $\mathbb{V}$. We denote this complex by $C_{red}(\mathfrak{g}, \mathbb{V})$.

7.6 The canonical spectral sequence

In this section we study the canonical spectral sequence associated to the L_∞-algebra cohomology with values in a dg-vector space. In particular, as the main result we show that this cohomology is invariant by equivariant quasi-isomorphisms.

Let $\mathfrak{g}$ be an L_∞ algebra and ρ be a representation up to homotopy of $\mathfrak{g}$ on $(\mathbb{V}, \partial)$. Let us consider the primitive filtration of the coalgebra $(\mathrm{Sym}(\mathfrak{g}[1]), \Delta_s, \epsilon, \eta)$. That is,

$$\mathrm{Sym}_{[0]}(\mathfrak{g}[1]) = 0, \quad \mathrm{Sym}_{[1]}(\mathfrak{g}[1]) = \mathbb{R}, \quad \mathrm{Sym}_{[p]}(\mathfrak{g}[1]) = \bigoplus_{n<p} \mathrm{Sym}^n(\mathfrak{g}[1]).$$

Abstractly $\mathrm{Sym}_{[p]}(\mathfrak{g}[1]) = \mathbb{R} \oplus \ker(\bar{\Delta}_s^{(p)})$, for all $p > 1$. It is well-known that the primitive filtration is an ascending filtration that is exhaustive, Hausdorff, and bounded below.

$$\bigcup_{n=0}^{\infty} \mathrm{Sym}_{[n]}(\mathfrak{g}\,[1]) = \mathrm{Sym}(\mathfrak{g}\,[1]), \quad \bigcap_{n=0}^{\infty} \mathrm{Sym}_{[n]}(\mathfrak{g}\,[1]) = 0, \quad \text{and} \quad \mathrm{Sym}_{[-1]}(\mathfrak{g}\,[1]) = 0.$$

Moreover, the coderivation d induced by the L_∞ structure preserves the filtration because of Lemma (6.2.1). The primitive filtration on $\mathrm{Sym}(\mathfrak{g}\,[1])$ induces a descending filtration on the cochain complex $C_\rho(\mathfrak{g}, \mathbb{V})$. For all p integer it is given by

$$F^p C_\rho(\mathfrak{g}, \mathbb{V}) := \left\{ \alpha : \mathrm{Sym}(\mathfrak{g}\,[1]) \to \mathbb{V} \,|\, \alpha(\mathrm{Sym}_{[p]}(\mathfrak{g}\,[1])) = 0 \right\}.$$

Observe that $F^0 C_\rho(\mathfrak{g}, \mathbb{V}) = C_\rho(\mathfrak{g}, \mathbb{V})$ and $F^1 C_\rho(\mathfrak{g}, \mathbb{V}) = C_{red}(\mathfrak{g}, \mathbb{V})$ then

$$\cdots F^{p+1} C_\rho(\mathfrak{g}, \mathbb{V}) \subseteq F^p C_\rho(\mathfrak{g}, \mathbb{V}) \subseteq \cdots \subseteq C_{red}(\mathfrak{g}, \mathbb{V}) \subseteq C_\rho(\mathfrak{g}, \mathbb{V}).$$

This filtration is clearly exhaustive, $\bigcup_{n=0}^{\infty} F^n C_\rho(\mathfrak{g}, \mathbb{V}) = C_\rho(\mathfrak{g}, \mathbb{V})$, and complete, since it is straightforward to see that

$$C_\rho(\mathfrak{g}, \mathbb{V}) / F^n C_\rho(\mathfrak{g}, \mathbb{V}) \simeq \mathrm{Hom}(\mathrm{Sym}_{[n]}(\mathfrak{g}\,[1]), \mathbb{V}).$$

Then one has that

$$\varprojlim \mathrm{Hom}(\mathrm{Sym}_{[n]}(\mathfrak{g}\,[1]), \mathbb{V}) \simeq \mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}\,[1]), \mathbb{V}) = C_\rho(\mathfrak{g}, \mathbb{V}).$$

Moreover, we have

$$D_\rho(F^p C^\bullet_\rho(\mathfrak{g}, \mathbb{V})) \subseteq F^p C^{\bullet+1}_\rho(\mathfrak{g}, \mathbb{V}).$$

Since $\alpha \in C_\rho(\mathfrak{g}, \mathbb{V})$ satisfies $\alpha(\mathrm{Sym}_{[p]}(\mathfrak{g}\,[1])) = 0$, the following hold

- $\partial\alpha(\mathrm{Sym}_{[p]}(\mathfrak{g}\,[1])) \subseteq \partial 0 = 0$;
- $\alpha d(\mathrm{Sym}_{[p]}(\mathfrak{g}\,[1])) \subseteq \alpha(\mathrm{Sym}_{[p]}(\mathfrak{g}\,[1])) = 0$;
- $\bar{\rho} \wedge_{ev} \alpha(\mathrm{Sym}_{[p]}(\mathfrak{g}\,[1])) \subseteq (\mathrm{ev} \circ (\bar{\rho} \otimes \alpha))\,(\mathrm{Sym}_{[p]}(\mathfrak{g}\,[1])^{\otimes 2}) \subseteq \mathrm{ev}(\mathfrak{gl}(\mathbb{V}) \otimes 0) = 0$.

Then $D_\rho \alpha(\mathrm{Sym}_{[p]}(\mathfrak{g}\,[1])) = 0$. As a result we have the following proposition.

Proposition 7.6.1. *The associated spectral sequence to the filtration above weakly converges*

$$E_1^{p,q}(\mathfrak{g}; \mathbb{V}) \Rightarrow H^{p+q}_{CE,\rho}(\mathfrak{g}; \mathbb{V}).$$

Proof. The descending filtration above is exhaustive and complete then the spectral sequence weakly converges by Theorem 10 Section 5.5 in [Wei94]. □

The associated spectral sequence to the filtration of the complex $C(\mathfrak{g}; \mathbb{V})$ starts with

$$E_0^{p,q} = F^p C^{p+q}(\mathfrak{g}, \mathbb{V}) / F^{p+1} C^{p+q}(\mathfrak{g}, \mathbb{V}) \simeq \mathrm{Hom}^{p+q}(\mathrm{Sym}^p(\mathfrak{g}\,[1]), \mathbb{V})$$

and $d_0 : E_0^{p,q} \to E_0^{p,q+1}$ is given by $\bar{D}_\rho$. For $\alpha \in \mathrm{Hom}^{p+q}(\mathrm{Sym}^p(\mathfrak{g}\,[1]), \mathbb{V})$ we have

$$d_0 \alpha = \partial\alpha - (-1)^{p+q} \alpha d_1^{1\otimes}, \tag{7.9}$$

where $d_1^{1\otimes}$ is the coderivation induced by λ_1. Explicitly for $x_i \in \mathfrak{g}\,[1]\,, 1 \le i \le p$ it holds that

$$d_0\alpha(x_1 \vee \cdots \vee x_p) = \partial(\alpha(x_1 \vee \cdots \vee x_p)) - \sum_{\sigma \in \mathrm{Sh}^{-1}_{1,p-1}} (-1)^{p+q} \epsilon(\sigma) \alpha(\lambda_1(x_{\sigma(1)}) \vee x_{\sigma(2)} \cdots \vee x_{\sigma(p)}).$$

We are ready to state the main result of this chapter which establishes that the L_∞-cohomology is invariant by L_∞-quasi-isomorphisms

Theorem 7.6.1. *Let ρ and ρ' be two representations up to homotopy of $\mathfrak{g}$ and $\mathfrak{h}$ on the dg-vector spaces $\mathbb{V}$ and $\mathbb{W}$, respectively. If $(F,f):\mathfrak{g}\to\mathfrak{h}$ is a (ρ,ρ')-equivariant L_∞-quasi-isomorphism along to $f:\mathbb{W}\to\mathbb{V}$, then the induced map*

$$F^*: H_{CE,\rho'}(\mathfrak{h};\mathbb{W})\to H_{CE,\rho}(\mathfrak{g};\mathbb{V}),$$

is an isomorphism.

Proof. Initially we observe that as F is a coalgebra morphism it preserves the primitive filtration on $\mathrm{Sym}(\mathfrak{g}[1])$ and $\mathrm{Sym}(\mathfrak{h}[1])$ then F^* preserves the filtration on the cochain complexes $C_{\rho'}(\mathfrak{h},\mathbb{W})$ and $C_\rho(\mathfrak{g},\mathbb{V})$. Thus, F^* induces a morphism between spectral sequences. Now we shall see that

$$F_1^*: E_1^{p,q}(\mathfrak{h};\mathbb{W})\to E_1^{p,q}(\mathfrak{g};\mathbb{V}),$$

is an isomorphism. For this we take an L_∞-homotopy inverse T for F, it exists by [Kaj07, Thm.7.5]. Note that by Equation (6.2.1) a coderivation preserves the primitive filtration, then in particular for a homotopy coderivation H of $\mathrm{Sym}(\mathfrak{h}[1])$ the equation

$$F\circ T-\mathrm{Id}=Hd+dH,$$

holds on $\mathrm{Sym}^p(\mathfrak{h}[1])$ for all $1\le p$. On the other hand, one has that for some $p,q>0$ the map $F_0^*: E_0^{p,q}(\mathfrak{h};\mathbb{W})\to E_0^{p,q}(\mathfrak{g};\mathbb{V})$ depends only on F_1^1 and f, indeed, for $x_i\in\mathfrak{g}[1], 1\le i\le p$ and $\alpha\in E_0^{p,q}(\mathfrak{h};\mathbb{W})$

$$F_0^*\alpha(x_1\vee\cdots\vee x_p)=f\alpha(F_1^1(x_1)\vee F_1^1(x_2)\vee\cdots\vee F_1^1(x_p)).$$

Then if t is a homotopy inverse for f, then the maps $(T,t):\mathfrak{h}\to\mathfrak{g}$, although they are not a (ρ',ρ)-equivariant map, they determine a well-defined map on

$$T_0^*: E_0^{p,q}(\mathfrak{g};\mathbb{V})\to E_0^{p,q}(\mathfrak{h};\mathbb{W}),$$

for all $p,q>0$. Now it is straightforward to verify that T_0^* is a homotopy left inverse for F_0^*,

$$\begin{aligned}T_0^*F_0^*\alpha=&(\mathrm{Id}+h\partial+\partial h)\alpha(\mathrm{Id}+Hd_1^{1\otimes}+d_1^{1\otimes}H)\\ =&\alpha+d_0\left(h\alpha+h\alpha Hd_1^{1\otimes}+(-1)^{|\alpha|}(\alpha H+h\partial\alpha H)\right).\end{aligned}$$

In a similar way we can see that T_0^* is a homotopy right inverse of F_0^*. Hence, F_1^* is an isomorphism, and therefore by the Eilenberg-Moore Comparison Theorem [Wei94, Thm 5.5.11] we have

$$H(F^*): H_{CE,\rho'}(\mathfrak{h};\mathbb{W})\to H_{CE,\rho}(\mathfrak{g};\mathbb{V}),$$

is an isomorphism. □

In some sense, the next result tells us that the L_∞-cohomology depends strongly on the homotopy type of both the L_∞-algebra and the coefficient space.

Theorem 7.6.2.

$$E_1^{p,q}(\mathfrak{g};\mathbb{V})\simeq \mathrm{Hom}^{p+q}(\mathrm{Sym}^p(H(\mathfrak{g})[1]),H(\mathbb{V})).$$

Proof. Let us consider the minimal model of $\mathfrak{g}$, that is, an L_∞-structure on $H(\mathfrak{g}):=H(\mathfrak{g},[\cdot]^1)$ together with an L_∞-quasi-isomorphism $q:H(\mathfrak{g})\to\mathfrak{g}$. Then $\rho^*:=q^*\rho$ determines a representation up to homotopy of $H(\mathfrak{g})$ on $\mathbb{V}$ such that

$$(q,\mathrm{Id}):H(\mathfrak{g})\to\mathfrak{g},$$

is a (ρ^*,ρ)-equivariant map, thus by Theorem 7.6.1 one has

$$E_1^{p,q}(\mathfrak{g};\mathbb{V})\simeq E_1^{p,q}(H(\mathfrak{g});\mathbb{V}),$$

and as
$$E_1^{p,q}(H(\mathfrak{g});\mathbb{V}) = H(E_0^{p,q}(H(\mathfrak{g});\mathbb{V}), d_0)$$
where d_0 is induced by ∂, by Equation (7.9). Hence,
$$\begin{aligned} H(E_0^{p,q}(H(\mathfrak{g});\mathbb{V}), d_0) &\simeq H(\mathrm{Hom}^{p+q}(\mathrm{Sym}^p(H(\mathfrak{g})\,[1]), \mathbb{V}), d_0) \\ &\simeq \mathrm{Hom}^{p+q}(\mathrm{Sym}^p(H(\mathfrak{g})\,[1]), H(\mathbb{V})). \end{aligned}$$

□

Chapter 8

The Chern-Weil-Lecomte morphism

This is the main chapter of this thesis. We present the extension of the Chern-Weil-Lecomte homomorphism to the of extensions of L_∞-algebras together with representations up to homotopy. This construction extends the algebraic approach to the Chern-Weil homomorphism made by P. B. Lecomte in [Lec82, Lec94]. For that, in Section 1 we study extensions of L_∞-algebras and present some useful results about splittings, in particular we get a Bianchi identity in this context. In Section 2 we state and prove the main result of this work, namely, Theorem 8.2.1, which is a L_∞-version of the Chern-Weil-Lecomte morphism. In Section 3 we will see some results about the naturality of this construction, and finally in Section 4 we study the particular case of extensions of (strict) Lie 2-algebras together with a 2-term representation up to homotopy.

8.1 Extensions of L_∞-algebras

We start by reviewing extensions of L_∞-algebras. For a broader discussion see [MZ12, Chap.6], [CL13, Chap.3.1], and [Rei19, Chap. 5.3].

Definition 8.1.1. An **extension of L_∞-algebras** is a short exact sequence of L_∞-algebras

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \hat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0,$$

in which the L_∞-morphism π and ι are strict L_∞-morphisms. In that case we say that $\hat{\mathfrak{g}}$ it is an **extension** of $\mathfrak{g}$ by $\mathfrak{n}$.

Remark 8.1.1. As π is a strict L_∞-morphism it is determined by π_1^1, $\pi = \mathcal{S}(\pi_1^1)$. Then a linear section $h_1^1 : \mathfrak{g}\,[1] \to \hat{\mathfrak{g}}\,[1]$ of π_1^1 induces a coalgebra morphism $h = \mathcal{S}(h_1^1)$ which is a linear section of π,

$$\pi \circ h = \mathcal{S}(\pi_1^1) \circ \mathcal{S}(h_1^1) = \mathcal{S}(\pi_1^1 \circ h_1^1) = \mathcal{S}(\mathrm{Id}) = \mathrm{Id}.$$

Therefore, in the sequel we shall denote a linear section of π_1^1 simply by h and say that it is a **linear section** of π.

Proposition 8.1.1. *Let*

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \hat{\mathfrak{g}} \underset{h}{\overset{\pi}{\rightleftarrows}} \mathfrak{g} \longrightarrow 0,$$

be an extension of L_∞-algebras and h a linear section of π. Then for each $k \geq 1$ the linear map

$$K_h^k := \hat{\lambda}_k \circ h^{\otimes k} - h \circ \lambda_k : \mathfrak{g}\,[1]^{\otimes k} \to \hat{\mathfrak{g}}\,[1]\,,$$

has the following properties:

i. K_h^k is a linear symmetric map of degree 1, and *ii.* $\operatorname{im}(K_h^k) \subseteq \mathfrak{n}\,[1]$.

In particular, there is a well-defined linear map $K_h^k : \operatorname{Sym}^k(\mathfrak{g}\,[1]) \to \mathfrak{n}\,[1]$, *for all* $k \in \mathbb{N}$ *and it defines a linear map*

$$K_h := \sum_{k\geq 1} K_h^k : \operatorname{Sym}(\mathfrak{g}\,[1]) \to \mathfrak{n}\,[1]\,,$$

of degree one referred to as the ***curvature of*** h.

Proof. For item (i) if $x_i \in \mathfrak{g}$ for $1 \leq i \leq k$, then

$$\begin{aligned}\hat{\lambda}_k \circ h^{\otimes k}(x_1 \otimes \cdots \otimes x_k) =& \hat{\lambda}_k(h(x_1) \otimes \cdots \otimes h(x_k))\\ =&\epsilon(\sigma; h(x_1), \ldots, h(x_n))\hat{\lambda}_k(h(x_{\sigma(1)}), \otimes \cdots \otimes h(x_{\sigma(k)}))\\ =&\epsilon(\sigma; x_1, \ldots, x_n)(\hat{\lambda}_k \circ h^{\otimes k})(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}),\end{aligned}$$

since, h is degree-preserving map $\epsilon(\sigma; h(x_1), \ldots, h(x_n)) = \epsilon(\sigma; x_1, \ldots, x_n)$. So,

$$\begin{aligned}K_h^k(x_1 \otimes \cdots \otimes x_n) =&\epsilon(\sigma; x_1, \ldots, x_n)(\hat{\lambda}_k \circ h^{\otimes k})(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)})\\ &- h(\epsilon(\sigma; x_1, \ldots, x_n)\lambda_k(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}))\\ =&\epsilon(\sigma; x_1, \ldots, x_n)(\hat{\lambda}_k \circ h^{\otimes k} - h \circ \lambda_k)(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}).\end{aligned}$$

Therefore, $K_h^k(x_1 \otimes \cdots \otimes x_n) = \epsilon(\sigma; x_1, \ldots, x_k)K_h^k(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)})$ is symmetric, moreover as the degree of $\lambda_k, \hat{\lambda}_k$ is 1 and the degree of h is 0, then K_h^k has degree 1. For item (ii) note that $\pi : \hat{\mathfrak{g}} \to \mathfrak{g}$ is a strict morphism between L_∞-algebras, then one has that $\pi_1^1 \circ \hat{\lambda}_k = \lambda_k \circ (\pi_1^1)^{\otimes k}$. Thus

$$\begin{aligned}\pi_1^1(K_h^k) =&\pi_1^1 \circ \hat{\lambda}_k \circ h^{\otimes k} - \pi_1^1 \circ h \circ \lambda_k\\ =&\lambda_k \circ (\pi_1^1)^{\otimes k} \circ h^{\otimes k} - \pi_1^1 \circ h \circ \lambda_k\\ =&\lambda_k \circ (\pi_1^1 \circ h)^{\otimes k} - (\pi_1^1 \circ h) \circ \lambda_k\\ =&\lambda_k - \lambda_k\\ =&0.\end{aligned}$$

Then, by exactness of the sequence, one has that

$$\operatorname{Im}(K_h^k) \subseteq \ker(\pi_1^1) = \operatorname{Im}(\iota) = \mathfrak{n}\,[1]\,.$$

□

Remark 8.1.2. We can interpret the curvature K_h as the failure of h being an L_∞-morphism. Indeed,

$$K_h = \hat{\lambda} \circ \mathcal{S}(h) - h \circ \lambda = \operatorname{Pr}_{\mathfrak{g}[1]} \circ (\hat{d} \circ \mathcal{S}(h) - \mathcal{S}(h) \circ d).$$

Definition 8.1.2. Let $(\mathfrak{g}\,[1]\,, \lambda)$ be an L_∞-algebra. A graded subspace $\mathfrak{n}\,[1]$ is an **ideal** if for every $x \in \mathfrak{n}\,[1]$ and $y \in \operatorname{Sym}(\mathfrak{g}\,[1])$ we have $\lambda(x \vee y) \in \mathfrak{n}\,[1]$.

Example 8.1.1. For an extension of L_∞-algebras as in (8.1.1) the space $\ker(\pi_1^1) = \mathfrak{n}\,[1]$ is an ideal of $\hat{\mathfrak{g}}\,[1]$, since for $x \in \mathfrak{n}\,[1]$ and $y \in \operatorname{Sym}(\hat{\mathfrak{g}}\,[1])$ we have

$$\begin{aligned}\pi_1^1(\hat{\lambda}(x \vee y)) =&\lambda(\pi_1^1(x) \vee \pi(y))\\ =&\lambda(0 \vee \pi(y))\\ =&0,\end{aligned}$$

hence, $\hat{\lambda}(x \vee y) \in \mathfrak{n}\,[1]$.

Remark 8.1.3. For an extension of L_∞-algebras, we can restrict the adjoint representation, see Example 7.1.1, to the ideal $\mathfrak{n}[1]$. That is,

$$\mathrm{ad}_{\mathfrak{n}} : \mathrm{Sym}(\hat{\mathfrak{g}}[1]) \otimes \mathfrak{n}[1] \to \mathfrak{n}[1], \quad \mathrm{ad}_{\mathfrak{n}}(x \otimes y) := \hat{\lambda}(x \vee y).$$

This is a representation up to homotopy of $\hat{\mathfrak{g}}$ on $\mathfrak{n}[1]$. Consider the linear map

$$S : \mathrm{Sym}(\mathfrak{g}[1]) \otimes \mathfrak{n}[1] \to \mathfrak{n}[1], \quad S(x \otimes y) := \hat{\lambda}(\mathcal{S}(h)(x) \vee y).$$

Equivalently,

$$S = \hat{\lambda} \circ \hat{\mu} \circ (\mathcal{S}(h) \otimes \mathrm{Id}_n),$$

with $\hat{\mu}$ being the multiplication map on the algebra $\mathrm{Sym}(\hat{\mathfrak{g}}[1])$. A priori, if the map h is not an L_∞ morphism, there is no reason for S being a representation of $\mathfrak{g}$. Now for a homogeneous map $f \in \mathrm{Hom}(\mathrm{Sym}(\mathfrak{g}[1]), \mathfrak{n}[1])$ of degree p we define the map

$$d_S f := S \circ (\mathrm{Id}_s \otimes f) \circ \Delta_s - (-1)^p f \circ d \in \mathrm{Hom}^{p+1}(\mathrm{Sym}(\mathfrak{g}[1]), \mathfrak{n}[1]).$$

Theorem 8.1.1 (Bianchi's Identity)**.** *Let $K_h \in \mathrm{Hom}^1(\mathrm{Sym}(\mathfrak{g}[1]), \mathfrak{n}[1])$ be the curvature of a splitting h of π. Then*

$$d_S K_h = 0.$$

Proof. This follows from a direct computation. Indeed,

$$\begin{aligned} d_S K_h =& S \circ (\mathrm{Id}_s \otimes K_h) \circ \Delta_s - (-1)^1 K_h \circ d \\ =& S \circ (\mathrm{Id}_s \otimes (\hat{\lambda} \circ \mathcal{S}(h)) \circ \Delta_s - S \circ (\mathrm{Id}_s \otimes h \circ \lambda) \circ \Delta_s \\ &+ (\hat{\lambda} \circ \mathcal{S}(h)) \circ d - \underbrace{(h \circ \lambda) \circ d}_{\lambda \circ d = 0} \\ =& \underbrace{S \circ (\mathrm{Id}_s \otimes (\hat{\lambda} \circ \mathcal{S}(h))) \circ \Delta_s}_{A} - \underbrace{S \circ (\mathrm{Id}_s \otimes (h \circ \lambda)) \circ \Delta_s}_{B} + \underbrace{\hat{\lambda} \circ \mathcal{S}(h) \circ d}_{C} \end{aligned}$$

Computing the term A:

$$\begin{aligned} A =& \hat{\lambda} \circ \hat{\mu} \circ (\mathcal{S}(h) \otimes \mathrm{Id}_n) \circ (\mathrm{Id}_s \otimes (\hat{\lambda} \circ \mathcal{S}(h))) \circ \Delta_s \\ =& \hat{\lambda} \circ \hat{\mu} \circ (\mathcal{S}(h) \otimes (\hat{\lambda} \circ \mathcal{S}(h)) \circ \Delta_s \\ =& \hat{\lambda} \circ \underbrace{\hat{\mu}}_{=\hat{\mu} \circ T} \circ (\mathrm{Id}_s \otimes \hat{\lambda}) \circ \underbrace{(\mathcal{S}(h) \otimes \mathcal{S}(h)) \circ \Delta_s}_{=\Delta_s \circ \mathcal{S}(h),\, T \circ \Delta_s = \Delta_s} \\ =& \hat{\lambda} \circ \hat{\mu} \circ \underbrace{T \circ (\mathrm{Id}_s \otimes \hat{\lambda}) \circ T}_{=\hat{\lambda} \otimes \mathrm{Id}_s} \circ \Delta_s \otimes \mathcal{S}(h) \\ =& \hat{\lambda} \circ \underbrace{\hat{\mu} \circ (\hat{\lambda} \otimes \mathrm{Id}_s) \circ \Delta_s}_{=d} \circ \mathcal{S}(h) \\ =& \underbrace{\hat{\lambda} \circ \hat{d}}_{=0} \circ \mathcal{S}(h) \\ =& 0. \end{aligned}$$

Computing the term B:

$$\begin{aligned} B =& -\hat{\lambda} \circ \hat{\mu} \circ (\mathcal{S}(h) \otimes \mathrm{Id}_n) \circ (\mathrm{Id}_s \otimes (h \circ \lambda)) \circ \Delta_s \\ =& -\hat{\lambda} \circ \hat{\mu} \circ (\mathcal{S}(h) \otimes (h \circ \lambda)) \circ \Delta_s. \end{aligned}$$

Computing the term C:

$$\begin{aligned} C =&\hat{\lambda} \circ \underbrace{\mathcal{S}(h) \circ \mu}_{=\hat{\mu}\circ(\mathcal{S}(h)\otimes\mathcal{S}(h))} \circ (\lambda \otimes \mathrm{Id}_s) \circ \Delta_s \\ =&\hat{\lambda} \circ \hat{\mu} \circ (\mathcal{S}(h) \otimes \mathcal{S}(h)) \circ (\lambda \otimes \mathrm{Id}_s) \circ \Delta_s \\ =&\hat{\lambda} \circ \underbrace{\hat{\mu}}_{=\hat{\mu}\circ T} \circ ((\mathcal{S}(h) \circ \lambda) \otimes \mathcal{S}(h)) \circ \underbrace{\Delta_s}_{=T\circ\Delta_s} \\ =&\hat{\lambda} \circ \hat{\mu} \circ \underbrace{T \circ ((h \circ \lambda) \otimes \mathcal{S}(h)) \circ T}_{=\mathcal{S}(h)\otimes(h\circ\lambda)} \circ \Delta_s \\ =&\hat{\lambda} \circ \hat{\mu} \circ (\mathcal{S}(h) \otimes (h \circ \lambda)) \circ \Delta_s. \end{aligned}$$

Hence,

$$\begin{aligned} d_S K_h =& A + B + C \\ =&0. \end{aligned}$$

□

8.2 The Chern-Weil-Lecomte morphism

In [Lec82, Lec94] P.B. Lecomte gave a construction of the Chern-Weil map for an extension of Lie algebras together with a representation of Lie algebras. See section 1.3 and also [Nee10] for more details. In this section we extend the Lecomte's construction to the setting of extensions of L_∞-algebras and representations up to homotopy. For that, let us consider an extension of L_∞-algebras

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \hat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0,$$

and ρ a representation up to homotopy of $\mathfrak{g}$ on the dg-vector space $\mathbb{V}$. A space of great interest for us is the set of k-linear skew-symmetric maps between $\mathfrak{n}[1]$ and $\mathbb{V}$ which are $\hat{\mathfrak{g}}$-equivariant, that is,

$$\mathrm{Hom}^\bullet(\wedge^k \mathfrak{n}[1], \mathbb{V})^{\hat{\mathfrak{g}}} = \left\{ f \in \mathrm{Hom}^\bullet(\wedge^k \mathfrak{n}[1], \mathbb{V}) \mid f \quad (\pi^*\rho, \mathrm{ad}_{\mathfrak{n}}^{\otimes k})\text{-equivariant} \right\}.$$

Note that given a linear section $h : \mathfrak{g}[1] \to \hat{\mathfrak{g}}[1]$ of π, one has that $K_h \in \mathrm{Hom}^1(\mathrm{Sym}(\mathfrak{g}[1]), \mathfrak{n}[1])$, then

$$K_h^{\wedge_\otimes k} := K_h \underbrace{\wedge_\otimes \cdots \wedge_\otimes}_{k-\text{times}} K_h \in \mathrm{Hom}^k(\mathrm{Sym}(\mathfrak{g}[1]), \mathfrak{n}[1]^{\otimes k}),$$

so that, for each $f \in \mathrm{Hom}^\bullet(\wedge^k \mathfrak{n}[1], \mathbb{V})$ we define

$$f_h := f \circ K_h^{\wedge_\otimes k} \in \mathrm{Hom}^{k+\bullet}(\mathrm{Sym}(\mathfrak{g}[1]), \mathbb{V}).$$

Theorem 8.2.1. *There is a natural map*

$$cw : \mathrm{Hom}^\bullet(\wedge^k \mathfrak{n}[1], \mathbb{V})^{\hat{g}} \to H_{CE}^{k+\bullet}(\mathfrak{g}; \mathbb{V}); \quad f \mapsto [f_h] \tag{8.1}$$

that is independent of the chosen linear section h of π.

The map cw in (8.1) is called the L_∞**- Chern-Weil-Lecomte morphism**. Before providing a proof of this theorem we shall present some auxiliary propositions.

Proposition 8.2.1. *If for all $k \geq 1$ the map $f : \mathfrak{n}[1]^{\otimes k} \to \mathbb{V}$ is $(\pi^*\rho, \mathrm{ad}_{\mathfrak{n}}^{\otimes k})$-equivariant and $h : \mathfrak{g}[1] \to \hat{\mathfrak{g}}[1]$ is a section of π, then f is $(\rho, \mathrm{ad}_{\mathfrak{n}}^{\otimes k} \circ (\mathcal{S}(h) \otimes \mathrm{Id}_v))$-equivariant.*

Proof. We shall see the equivariance statement. Let us denote by $S^{\otimes k} = \mathrm{ad}_{\mathfrak{n}}^{\otimes k} \circ (\mathcal{S}(h) \otimes \mathrm{Id}_v)$

$$\begin{array}{ccc} \mathrm{Sym}(\hat{\mathfrak{g}}\,[1]) \otimes \mathfrak{n}\,[1]^{\otimes k} & \xrightarrow{\mathrm{ad}_{\mathfrak{n}}^{\otimes k}} & \mathfrak{n}\,[1]^{\otimes k} \\ \big\downarrow{\scriptstyle \mathrm{Id}_{\hat{s}}\otimes f} & & \big\downarrow{\scriptstyle f} \\ \mathrm{Sym}(\hat{\mathfrak{g}}\,[1]) \otimes \mathbb{V} & \xrightarrow{\pi^*\rho} & \mathbb{V}, \end{array} \qquad \begin{array}{ccc} \mathrm{Sym}(\mathfrak{g}\,[1]) \otimes \mathfrak{n}\,[1]^{\otimes k} & \xrightarrow{S^{\otimes k}} & \mathfrak{n}\,[1]^{\otimes k} \\ \big\downarrow{\scriptstyle \mathrm{Id}_{s}\otimes f} & & \big\downarrow{\scriptstyle f} \\ \mathrm{Sym}(\mathfrak{g}\,[1]) \otimes \mathbb{V} & \xrightarrow{\rho} & \mathbb{V}. \end{array}$$

Then

$$\begin{aligned} f \circ S^{\otimes k} =& f \circ \mathrm{ad}_{\mathfrak{n}}^{\otimes k} \circ (\mathcal{S}(h) \otimes \mathrm{Id}_v) \\ =& \hat{\rho} \circ (\mathrm{Id}_{\hat{s}} \otimes f) \circ (\mathcal{S}(h) \otimes \mathrm{Id}_v) \\ =& \rho \circ (\mathcal{S}(\pi) \otimes \mathrm{Id}_v) \circ (\mathcal{S}(h) \otimes \mathrm{Id}_v) \circ (\mathrm{Id}_s \otimes f) \\ =& \rho \circ (\underbrace{(\mathcal{S}(\pi) \circ \mathcal{S}(h))}_{=\mathcal{S}(\pi\circ h)} \otimes \mathrm{Id}_v) \circ (\mathrm{Id}_s \otimes f) \\ =& \rho \circ (\mathrm{Id}_s \otimes \mathrm{Id}_v) \circ (\mathrm{Id}_s \otimes f) \\ =& \rho \circ (\mathrm{Id}_s \otimes f). \end{aligned}$$

□

Proposition 8.2.2. *Let $h_0, h_1 : \mathfrak{g}\,[1] \to \hat{\mathfrak{g}}\,[1]$ be two sections of π and consider*

$$h_t := h_0 + t(h_1 - h_0) = h_0 + t\bar{\alpha}, \quad 0 \leq t \leq 1$$

with $\bar{\alpha} = h_1 - h_0 : \mathfrak{g}\,[1] \to \mathfrak{n}\,[1]$. For $\alpha = \bar{\alpha} \circ Pr_{\mathfrak{g}[1]} : \mathrm{Sym}(\mathfrak{g}\,[1]) \to \mathfrak{n}\,[1]$ one has that

$$\frac{d}{dt} K_{h_t} = d_{S_t} \alpha,$$

where $S_t := \hat{\lambda} \circ \hat{\mu} \circ (\mathcal{S}(h_t) \otimes \mathrm{Id}_s) : \mathrm{Sym}(\mathfrak{g}\,[1]) \otimes \hat{\mathfrak{g}}\,[1] \to \hat{\mathfrak{g}}\,[1]$.

Proof. On the one hand we have

$$\begin{aligned} K_{h_{t+t_0}} =& \hat{\lambda} \circ \mathcal{S}(h_{t+t_0}) - h_{t+t_0} \circ \lambda \\ =& \hat{\lambda} \circ \mathcal{S}(h_t + t_0\bar{\alpha}) - (h_t + t_0\bar{\alpha}) \circ \lambda \\ =& \hat{\lambda} \circ (\mathcal{S}(h_t) + \mathcal{S}(t_0\bar{\alpha}) + K_{h_t, t_0\bar{\alpha}}) - h_t \circ \lambda - t_0\bar{\alpha} \circ \lambda. \end{aligned}$$

For $K_{h_t,t_0\bar{\alpha}} := \mathcal{S}(h_t + t_0\bar{\alpha}) - (\mathcal{S}(h_t) + \mathcal{S}(t_0\bar{\alpha}))$,

$$K_{h_{t+t_0}} = (\hat{\lambda} \circ \mathcal{S}(h_t) - h_t \circ \lambda) + (\hat{\lambda} \circ \mathcal{S}(t_0\bar{\alpha}) - t_0\bar{\alpha} \circ \lambda) + \hat{\lambda} \circ K_{h_t,t_0\bar{\alpha}}.$$

So that,

$$\frac{d}{dt} K_{h_t} = \frac{d}{dt}\bigg|_{t_0=0} K_{h_{t+t_0}} = \frac{d}{dt}\bigg|_{t_0=0} K_{h_t} + \frac{d}{dt}\bigg|_{t_0=0} K_{t_0\bar{\alpha}} + \frac{d}{dt}\bigg|_{t_0=0} \hat{\lambda} \circ K_{h_t,t_0\bar{\alpha}}.$$

Then, $\frac{d}{dt}\big|_{t_0=0} K_{h_t} = 0$, since it is independent of t_0,

$$\begin{aligned} \frac{d}{dt}\bigg|_{t_0=0} K_{t_0\bar{\alpha}} =& \frac{d}{dt}\bigg|_{t_0=0} (\hat{\lambda} \circ \mathcal{S}(t_0\bar{\alpha}) - t_0\bar{\alpha} \circ \lambda) \\ =& \sum_k \frac{d}{dt}\bigg|_{t_0=0} \hat{\lambda}_k \circ t_0^k \mathcal{S}^k(\bar{\alpha}) - \frac{d}{dt}\bigg|_{t_0=0} t_0\bar{\alpha} \circ \lambda \\ =& \hat{\lambda}_1 \circ \mathcal{S}^1(\bar{\alpha}) \circ Pr_{\mathfrak{g}[1]} - \bar{\alpha} \circ \lambda, \\ \frac{d}{dt}\bigg|_{t_0=0} K_{t_0\alpha} =& \hat{\lambda}_1 \circ \alpha - \bar{\alpha} \circ \lambda, \end{aligned}$$

and

$$\left.\frac{d}{dt}\right|_{t_0=0} \hat{\lambda} \circ K_{h_t, t_0\bar{\alpha}} = \sum_k \hat{\lambda}_k \left(\left.\frac{d}{dt}\right|_{t_0=0} K^k_{h_t, t_0\bar{\alpha}} \right).$$

On the other hand,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t_0=0} K^k_{h_t, t_0\bar{\alpha}} &= \left.\frac{d}{dt}\right|_{t_0=0} (\mathcal{S}^k(h_t + t_0\bar{\alpha}) - \underbrace{\mathcal{S}^k(h_t)}_{=0} - \mathcal{S}^k(t_0\bar{\alpha})) \\ &= \left.\frac{d}{dt}\right|_{t_0=0} \mathcal{S}^k(h_t + t_0\bar{\alpha}) - \begin{cases} \mathcal{S}^1(\bar{\alpha}), & k = 1, \\ 0, & k \neq 1, \end{cases} \end{aligned}$$

and for $x_i \in \mathfrak{g}[1], 1 \leq i \leq k$

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t_0=0} \mathcal{S}^k(h_t + t_0\bar{\alpha})(x_1 \vee \cdots \vee x_k) &= \left.\frac{d}{dt}\right|_{t_0=0} (h_t + t_0\bar{\alpha})x_1 \vee \cdots \vee (h_t + t_0\bar{\alpha})x_k \\ &= \sum_{i=1}^k h_t(x_1) \vee \cdots \vee h_t(x_{i-1}) \vee \bar{\alpha}(x_i) \vee h_t(x_{i+1}) \vee \cdots \vee h_t(x_k), \end{aligned}$$

hence

$$\left.\frac{d}{dt}\right|_{t_0=0} \mathcal{S}^k(h_t + t_0\bar{\alpha}) = \sum_{i=1}^k h_t \vee \cdots \vee \bar{\alpha} \vee \cdots \vee h_t.$$

Then,

$$\left.\frac{d}{dt}\right|_{t_0=0} \hat{\lambda} \circ K_{h_t, t_0\bar{\alpha}} = \sum_k \hat{\lambda}_k \left(\sum_{i=1}^k h_t \vee \cdots \vee \bar{\alpha} \vee \cdots \vee h_t \right) - \hat{\lambda}_1 \circ \bar{\alpha} \circ Pr_{\mathfrak{g}[1]}.$$

Therefore,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t_0=0} K_{h_t + t_0\alpha} =& \hat{\lambda}_1 \circ \alpha - \bar{\alpha} \circ \lambda + \sum_k \hat{\lambda}_k \left(\sum_{i=1}^k h_t \vee \cdots \vee \bar{\alpha} \vee \cdots \vee h_t \right) \\ & - \hat{\lambda}_1 \circ \bar{\alpha} \circ Pr_{\mathfrak{g}[1]} \\ =& \sum_k \left(\sum_{i=1}^k \hat{\lambda}_k (h_t \vee \cdots \vee \bar{\alpha} \vee \cdots \vee h_t) \right) - \alpha \circ \lambda, \end{aligned}$$

and observe that

$$\sum_k \left(\sum_{i=1}^k \hat{\lambda}_k (h_t \vee \cdots \vee \bar{\alpha} \vee \cdots \vee h_t) \right) = \hat{\lambda} \circ \hat{\mu} \circ (\mathcal{S}(h_t) \otimes \mathrm{Id}_s) \circ (\mathrm{Id}_s \otimes \alpha) \circ \Delta_s.$$

To check this last expression let us do the direct computation. For this recall that $\alpha = \bar{\alpha} \circ Pr_{\mathfrak{g}[1]}$

$$\begin{aligned}\hat{\lambda} \circ \hat{\mu} \circ (\mathcal{S}(h_t) \otimes \mathrm{Id}_s) \circ (\mathrm{Id}_s \otimes \alpha) \circ \Delta_s(x_1 \vee \cdots \vee x_k) = \hat{\lambda} \circ \hat{\mu} \circ (\mathcal{S}(h_t) \otimes \mathrm{Id}_s) \\ \circ \left(\sum_{\sigma \in \mathrm{Sh}^{-1}_{k-1,1}} \epsilon(\sigma) x_{\sigma(1)} \vee \cdots \vee x_{\sigma(k-1)} \otimes \bar{\alpha}(x_{\sigma(k)}) \right) \\ = \hat{\lambda} \circ \hat{\mu} \left(\sum_{\sigma \in \mathrm{Sh}^{-1}_{k-1,1}} \epsilon(\sigma) h_t(x_{\sigma(1)}) \vee \cdots \vee h_t(x_{\sigma(k-1)}) \otimes \bar{\alpha}(x_{\sigma(k)}) \right) \\ = \hat{\lambda}_k \left(\sum_{\sigma \in \mathrm{Sh}^{-1}_{k-1,1}} \epsilon(\sigma) h_t(x_{\sigma(1)}) \vee \cdots \vee h_t(x_{\sigma(k-1)}) \vee \bar{\alpha}(x_{\sigma(k)}) \right).\end{aligned}$$

Note that

$$|\mathrm{Sh}^{-1}_{k-1,1}| = |\{\sigma \in S_k \mid \sigma(1) < \sigma(2) < \cdots < \sigma(k-1)\}| = k,$$

then

$$\begin{aligned}\hat{\lambda} \circ \hat{\mu} \circ (\mathcal{S}(h_t) \otimes \mathrm{Id}_s) \circ (\mathrm{Id}_s \otimes \alpha) \circ \Delta_s(x_1 \vee \cdots \vee x_k) = \\ = \hat{\lambda}_k \left(\sum_{i=1}^{k} \epsilon(\sigma) h_t(x_1) \vee \cdots \vee \bar{\alpha}(x_i) \vee \cdots h_t(x_k) \right) \\ = \hat{\lambda}_k \left(\sum_{i=1}^{k} h_t \vee \cdots \vee \bar{\alpha} \vee \cdots \vee h_t \right) (x_1 \vee \cdots \vee x_k).\end{aligned}$$

Hence,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t_0=0} K_{h_{t+t_0}} &= \sum_k \left(\sum_{i=1}^{k} \hat{\lambda}_k (h_t \vee \cdots \vee \bar{\alpha} \vee \cdots \vee h_t) \right) - \bar{\alpha} \circ \lambda \\ &= S_t \circ (\mathrm{Id}_s \otimes \alpha) \circ \Delta_s - (-1)^{|\alpha|} \alpha \circ d \\ &= d_{S_t} \alpha.\end{aligned}$$

□

Now we shall prove Theorem 8.2.1.

Proof. Let $h : \mathfrak{g}[1] \to \hat{\mathfrak{g}}[1]$ be a linear section of π, then $K_h \in \mathrm{Hom}^1(\mathrm{Sym}(\mathfrak{g}[1]), \mathfrak{n}[1])$ by Proposition 8.1.1. Thus, for $f \in \mathrm{Hom}^\bullet(\wedge^k \mathfrak{n}[1], \mathbb{V})^{\hat{\mathfrak{g}}}$ we have

$$f_h = f \circ K_h^{\wedge \otimes k} \in \mathrm{Hom}^{k+\bullet}(\mathrm{Sym}(\mathfrak{g}[1]), \mathbb{V}).$$

Now we shall see that f_h is a cocycle and that its cohomology class is independent of the chosen linear section h of π. Let us see that f_h is a cocycle.
Indeed:

$$\begin{aligned}
D_\rho f_h =& D_\rho(f\circ K_h\wedge_\otimes\cdots\wedge_\otimes K_h)\\
=& f\circ d_{S^{\otimes k}}(K_h\wedge_\otimes\cdots\wedge_\otimes K_h), && \text{(by Proposition 7.1.2)}\\
=& f\circ\left(\sum_{i=1}^k(-1)^{i-1}K_h\wedge_\otimes\cdots\wedge_\otimes d_SK_h\wedge_\otimes\cdots\wedge_\otimes K_h\right), && \text{(by Leibniz Rule)}\\
=& f\circ\left(\sum_{i=1}^k(-1)^{i-1}K_h\wedge_\otimes\cdots\wedge_\otimes 0\wedge_\otimes\cdots\wedge_\otimes K_h\right), && \text{(by Bianchi's Identity)}\\
=& f\circ 0\\
=& 0.
\end{aligned}$$

Let us check now that $[f_h]$ does not depend on h. Let h_0, h_1 be two linear degree-preserving sections of π. Now consider $\alpha = (h_1 - h_0)\circ Pr_{\mathfrak{g}[1]} : \mathrm{Sym}(\mathfrak{g}[1]) \to \mathfrak{n}[1]$, and $h_t := h_0 + t(h_1 - h_0)$ a one-parameter family of sections of π, for $0 \le t \le 1$, then

$$f_{\alpha,K_{h_t}} := f\circ\alpha\wedge_\otimes K_{h_t}^{\wedge_\otimes k-1} \in \mathrm{Hom}^{k-1+\bullet}(\mathrm{Sym}(\mathfrak{g}[1]),\mathbb{V}),$$

and note that

$$\begin{aligned}
D_\rho f_{\alpha,K_{h_t}} =& D_\rho(f\circ\alpha\wedge_\otimes K_{h_t}^{\wedge_\otimes k-1})\\
=& f\circ d_{S_t^{\otimes k}}(\alpha\wedge_\otimes K_{h_t}^{\wedge_\otimes k-1}) && \text{(by Propositions 8.2.1 and 7.1.2)}\\
=& f\circ(d_{S_t}\alpha\wedge_\otimes K_{h_t}^{\wedge_\otimes k-1}+\alpha\wedge_\otimes(d_{S_t^{\otimes k-1}}K_{h_t}^{\wedge_\otimes k-1})) && \text{(Leibniz rule)}\\
=& f\circ(d_{S_t}\alpha\wedge_\otimes K_{h_t}^{\wedge_\otimes k-1})
\end{aligned}$$

so that,

$$D_\rho f_{\alpha,K_{h_t}} = f\circ(d_{S_t}\alpha\wedge_\otimes K_{h_t}^{\wedge_\otimes k-1}) := f_{d_{S_t}\alpha,K_{h_t}}.$$

On the other hand, for f_{h_t} we have

$$\begin{aligned}
\frac{d}{dt}f_{h_t} =& \left.\frac{d}{dt}\right|_{t_0=0} f_{h_{t+t_0}}\\
=& f\circ\left(\left.\frac{d}{dt}\right|_{t_0=0}K_{h_t}^{\wedge_\otimes k}\right) && \text{(Linearity)}\\
=& f\circ\left(\sum_{i=1}^k K_{h_{t+0}}\wedge_\otimes\cdots\wedge_\otimes\left.\frac{d}{dt}\right|_{t_0=0}K_{h_{t+t_0}}\wedge_\otimes\cdots\wedge_\otimes K_{h_{t+0}}\right), && \text{(Theorem 7.3.1)}\\
=& f\circ\left(\sum_{i=1}^k K_{h_t}\wedge_\otimes\cdots\wedge_\otimes d_{S_t}\alpha\wedge_\otimes\cdots\wedge_\otimes K_{h_t}\right). && \text{(Proposition 8.2.2)}
\end{aligned}$$

As f is skew-symmetric, by Proposition (7.3.2) we have

$$\begin{aligned}
&f\circ K_{h_t}\wedge_\otimes\cdots\wedge_\otimes d_{S_t}\alpha\wedge_\otimes\cdots\wedge_\otimes K_{h_t} =\\
&\quad = \chi(\sigma; K_{h_t},\ldots,K_{h_t},d_{S_t}\alpha,K_{h_t},\ldots,K_{h_t})f\circ d_{S_t}\alpha\wedge_\otimes K_{h_t}\wedge_\otimes\cdots\wedge_\otimes K_{h_t}
\end{aligned}$$

for σ the permutation given by $\sigma(i) = 1, \sigma(1) = i, \sigma(j) = j, j \neq 1, i$, i.e., $\sigma = (1, i)$. Since $|K_{h_t}| = 1$, and $|d_{S_t}\alpha| = 1$, then

$$\chi(\sigma; K_{h_t},\ldots,K_{h_t},d_{S_t}\alpha,K_{h_t},\cdots,K_{h_t}) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma) = 1.$$

Therefore,

$$f \circ K_{h_t} \wedge_{\otimes} \cdots \wedge_{\otimes} d_{S_t}\alpha \wedge_{\otimes} \cdots \wedge_{\otimes} K_{h_t} = f \circ d_{S_t}\alpha \wedge_{\otimes} K_{h_t} \wedge_{\otimes} \cdots \wedge_{\otimes} K_{h_t},$$

so that,

$$\begin{aligned} \frac{d}{dt} f_{h_t} =& \sum_{i=1}^{k} f \circ (d_{S_t}\alpha \wedge_{\otimes} K_{h_t} \wedge_{\otimes} \cdots \wedge_{\otimes} K_{h_t}) \\ =& k f \circ d_{S_t}\alpha \wedge_{\otimes} K_{h_t}^{\wedge_{\otimes} k-1} \\ =& k f_{d_{S_t}\alpha, K_{h_t}} \\ =& k D_{\rho} f_{\alpha, K_{h_t}}. \end{aligned}$$

Thus,

$$\begin{aligned} f_{h_1} - f_{h_0} =& \int_0^1 \frac{d}{dt} f_{h_t} dt \\ =& \int_0^1 k D_{\rho} f_{\alpha, K_{h_t}} dt \\ =& D_{\rho} \left(k \int_0^1 f_{\alpha, K_{h_t}} dt \right) \in \mathrm{Hom}^{k+\bullet}(\mathrm{Sym}(\mathfrak{g}\,[1]), \mathbb{V}) \end{aligned}$$

That is, f_{h_0} and f_{h_1} are cohomologous. □

8.3 Naturality

In this section we shall see the naturality of the L_∞-Chern-Weil-Lecomte morphism and consequently, the induced characteristic classes. Let us consider an extension of L_∞-algebras

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \hat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0,$$

and $T : \mathfrak{h} \to \mathfrak{g}$ be an L_∞-morphism. Since π is an L_∞-epimorphism and the category of L_∞-algebras has pullbacks, see [Rog20, Thm 5.9] and [Val20, Thm 2.1], the pullback of T and π give rise to an extension of L_∞-algebras of $\mathfrak{h}$ by $\mathfrak{n}$ such that the next diagram of L_∞-algebras is commutative

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathfrak{n} & \xrightarrow{\mathrm{Id}} & \mathfrak{n} \\ \downarrow{\scriptstyle \iota_\bullet} & & \downarrow{\scriptstyle \iota} \\ \hat{\mathfrak{h}} & \xrightarrow{T_\bullet} & \hat{\mathfrak{g}} \\ \downarrow{\scriptstyle \pi_\bullet} & & \downarrow{\scriptstyle \pi} \\ \mathfrak{h} & \xrightarrow{T} & \mathfrak{g} \\ \downarrow & & \downarrow \\ 0 & & 0. \end{array}$$

Now if we consider a linear section $h : \mathfrak{g}\,[1] \to \hat{\mathfrak{g}}\,[1]$ of π and the induced coalgebra morphism $\mathcal{S}(h) : \mathrm{Sym}(\mathfrak{g}\,[1]) \to \mathrm{Sym}(\hat{\mathfrak{g}}\,[1])$, the pullback property in the category of coalgebras implies the existence of a coalgebra morphism $\overline{h} : \mathrm{Sym}(\mathfrak{h}\,[1]) \to \mathrm{Sym}(\hat{\mathfrak{h}}\,[1])$ such that $\pi_\bullet \circ \overline{h} = \mathrm{Id}$ and $T_\bullet \circ \overline{h} = \mathcal{S}(h) \circ T$. We point out that due to fact that $\pi_\bullet$ is a strict L_∞-morphism it preserves the weight degree, thus $\overline{h}$ has to preserve the weight degree as well. Hence, $\overline{h}$ is a strict coalgebra morphism and therefore it is determined by $\overline{h}_1^1 : \mathfrak{h}\,[1] \to \hat{\mathfrak{h}}\,[1]$ as $\overline{h} = \mathcal{S}(\overline{h}_1^1)$. We have the following result.

Proposition 8.3.1. *Let*

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \hat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0,$$

be an extension of L_∞-algebras and ρ and ρ' be representations up to homotopy of $\mathfrak{h}$ on $\mathbb{W}$ and $\mathfrak{g}$ on $\mathbb{V}$, respectively. Suppose that $(T,t) : \mathfrak{h} \to \mathfrak{g}$ is a (ρ,ρ')-equivariant map and consider the pullback extension of $\mathfrak{h}$ by $\mathfrak{n}$ through T. Then the next diagram is commutative

$$\begin{array}{ccc} \mathrm{Hom}^\bullet(\wedge^n \mathfrak{n}\,[1]\,,\mathbb{V})^{\hat{\mathfrak{g}}} & \xrightarrow{cw} & H^{n+\bullet}_{CE,\rho'}(\mathfrak{g};\mathbb{V}) \\ \Big\downarrow{t^\sharp} & & \Big\downarrow{T^\sharp} \\ \mathrm{Hom}^\bullet(\wedge^n \mathfrak{n}\,[1]\,,\mathbb{W})^{\hat{\mathfrak{h}}} & \xrightarrow{cw} & H^{n+\bullet}_{CE,\rho}(\mathfrak{h};\mathbb{W}), \end{array} \tag{8.2}$$

where the map $t^\sharp : \mathrm{Hom}^\bullet(\wedge^n \mathfrak{n}\,[1]\,,\mathbb{V})^{\hat{\mathfrak{g}}} \to \mathrm{Hom}^\bullet(\wedge^n \mathfrak{n}\,[1]\,,\mathbb{W})^{\hat{\mathfrak{h}}}$ is define by $t^\sharp(f) := t \circ f$.

Proof. First we shall see that the map $t^\sharp$ is well-defined. Since $T_\bullet$ is an L_∞-morphism and $T_\bullet|_{\mathfrak{n}[1]} = \mathrm{Id}$ then $(T_\bullet, \mathrm{Id}_{\mathfrak{n}[1]})$ is a $(\mathrm{ad}_{\hat{h},n}, \mathrm{ad}_{\hat{g},n})$-equivariant map and it is straightforward to see that $(T_\bullet, \mathrm{Id}_{\mathfrak{n}[1]^{\otimes k}})$ is a $(\mathrm{ad}^{\otimes k}_{\hat{h},n}, \mathrm{ad}^{\otimes k}_{\hat{g},n})$-equivariant, hence we have $T^*_\bullet \mathrm{ad}^{\otimes k}_{\hat{g},n} = \mathrm{ad}^{\otimes k}_{\hat{h},n}$ for all $1 \le k$. On the other hand, as (T,t) is a (ρ,ρ')-equivariant then $(T_\bullet, t)$ is $(\pi^*_\bullet\rho, \pi^*\rho')$-equivariant map, thus for $f \in \mathrm{Hom}^\bullet(\wedge^n \mathfrak{n}\,[1]\,,\mathbb{V})$ a $(\pi^*\rho', \mathrm{ad}^{\otimes k}_{\hat{g},n})$-equivariant map we have

$$(T_\bullet, t) \circ (\mathrm{Id}_{\hat{g}}, f) = (\mathrm{Id}_{\hat{g}} \circ T_\bullet, t \circ f)$$

is $(\pi^*_\bullet\rho, \mathrm{ad}^{\otimes k}_{\hat{g},n})$-equivariant. By the decomposition seen in the Remark 7.1.1

$$(T_\bullet, t \circ f) = (\mathrm{Id}_{\hat{h}}, t \circ f) \circ (T_\bullet, \mathrm{Id}_{\mathfrak{n}[1]^{\otimes k}})$$

where the map $(\mathrm{Id}_{\hat{h}}, t \circ f)$ is $(\pi^*_\bullet\rho, T^*_\bullet \mathrm{ad}^{\otimes k}_{\hat{g},n})$-equivariant, therefore $t^\sharp(f) = t \circ f \in \mathrm{Hom}^\bullet(\wedge^n \mathfrak{n}\,[1]\,,\mathbb{W})$ is $(\pi^*_\bullet\rho, \mathrm{ad}^{\otimes k}_{\hat{h},n})$-equivariant. So $t^\sharp$ is well-defined. Now to see that Diagram (8.2) is commutative let us consider $h : \mathfrak{g}\,[1] \to \hat{\mathfrak{g}}\,[1]$ a linear section of π, and its curvature $K_h = \mathrm{Pr}_{\mathfrak{g}[1]}(\hat{d} \circ \mathcal{S}(h) - \mathcal{S}(h) \circ d) : \mathrm{Sym}(\mathfrak{g}\,[1]) \to \mathfrak{n}\,[1]$. As we discussed above there is a linear section $\overline{h} : \mathfrak{h}\,[1] \to \hat{\mathfrak{h}}\,[1]$ of $\pi_\bullet$ induced by the pullback diagram such that $T_\bullet \circ \mathcal{S}(\overline{h}) = \mathcal{S}(h) \circ T$, then one has that

$$\begin{aligned} T^*K_h =& K_h \circ T = \mathrm{Pr}_{\mathfrak{g}[1]} \circ (\hat{d} \circ \mathcal{S}(h) - \mathcal{S}(h) \circ d) \circ T \\ =& \mathrm{Pr}_{\mathfrak{g}[1]} \circ (\hat{d} \circ \mathcal{S}(h) \circ T - \mathcal{S}(h) \circ T \circ d) \\ =& \mathrm{Pr}_{\mathfrak{g}[1]} \circ (\hat{d} \circ T_\bullet \circ \mathcal{S}(\overline{h}) - T_\bullet \circ \mathcal{S}(\overline{h}) \circ d), && \text{(by } T_\bullet \circ \mathcal{S}(\overline{h}) = \mathcal{S}(h) \circ T) \\ =& \mathrm{Pr}_{\mathfrak{g}[1]} \circ T_\bullet \circ (\hat{d} \circ \mathcal{S}(\overline{h}) - \mathcal{S}(\overline{h}) \circ d), && \text{(by } T_\bullet|_{\mathfrak{n}[1]} = \mathrm{Id}_{\mathfrak{n}[1]}) \\ =& \mathrm{Pr}_{\mathfrak{h}[1]} \circ (\hat{d} \circ \mathcal{S}(\overline{h}) - \mathcal{S}(\overline{h}) \circ d) \\ =& K_{\overline{h}}, \end{aligned}$$

and for $f \in \mathrm{Hom}^\bullet(\wedge^n \mathfrak{n}\,[1]\,,\mathbb{V})^{\hat{\mathfrak{g}}}$,

$$\begin{aligned} T^*f_h =& f \circ K_h^{\wedge\otimes k} \circ T \\ =& f \circ K_h \otimes \cdots \otimes K_h \circ \Delta^{k-1} \circ T \\ =& f \circ K_h \otimes \cdots \otimes K_h \circ T^{\otimes k} \circ \Delta^{k-1}, && \text{(by } T \text{ being coalgebra morphism)} \\ =& f \circ (K_h \circ T) \otimes \cdots (K_h \circ T) \circ \Delta^{k-1} \\ =& f \circ (T^*K_h)^{\wedge\otimes k} \\ =& f \circ K_{\overline{h}}^{\wedge\otimes k}, && \text{(by } T^*K_h = K_{\overline{h}}) \\ =& f_{\overline{h}}. \end{aligned}$$

Therefore,

$$\begin{aligned}\left((T,t)^{\sharp}\circ cw\right)(f) =&[(T,t)^{*}(f_h)] = [t\circ f_h\circ T]\\ =&[t\circ T^{*}f_h] = \left[t\circ f_{\overline{h}}\right] = \left[t\circ f\circ K_{\overline{h}}^{\wedge\otimes k}\right]\\ =&\left[t^{\sharp}(f)\circ K_{\overline{h}}^{\wedge\otimes k}\right] = cw(t^{\sharp}(f))\\ =&(cw\circ t^{\sharp})(f).\end{aligned}$$

□

Proposition 8.3.2. *Let us consider two extensions of L_∞-algebras and two L_∞-isomorphisms as in the following commutative arrangement of L_∞-morphisms*

$$\begin{array}{ccccc} 0 & & & & 0\\ \downarrow & & & & \downarrow\\ \mathfrak{n} & & \xrightarrow[\simeq]{\psi} & & \mathfrak{n}'\\ \downarrow & & & & \downarrow\\ \hat{\mathfrak{g}} & & \xrightarrow[\simeq]{\varphi} & & \hat{\mathfrak{g}}'\\ & {\scriptstyle\pi}\searrow & & \swarrow{\scriptstyle\pi'} & \\ & & \mathfrak{g} & & \\ & & \downarrow & & \\ & & 0 & & \end{array} \tag{8.3}$$

and a representation up to homotopy of $\mathfrak{g}$ on the dg-vector space $\mathbb{V}$. Then the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Hom}^{\bullet}(\wedge^{n}\mathfrak{n}\,[1]\,,\mathbb{V})^{\hat{\mathfrak{g}}} & \xrightarrow{cw} & H_{CE}^{n+\bullet}(\mathfrak{g};\mathbb{V})\\ \downarrow{\scriptstyle\psi^{*}} & \nearrow{\scriptstyle cw} & \\ \mathrm{Hom}^{\bullet}(\wedge^{n}\mathfrak{n}'\,[1]\,,\mathbb{V})^{\hat{\mathfrak{g}}'} & & \end{array} \tag{8.4}$$

where the map $\psi^{*}:\mathrm{Hom}^{\bullet}(\wedge^{n}\mathfrak{n}\,[1]\,,\mathbb{V})^{\hat{\mathfrak{g}}}\to\mathrm{Hom}^{\bullet}(\wedge^{n}\mathfrak{n}\,[1]\,,\mathbb{V})^{\hat{\mathfrak{g}'}}$ is define by $\psi^{*}f := f\circ\left({\psi_1^1}^{-1}\right)^{\otimes k}$.

Proof. We have to observe that the commutativity of Diagram (8.3) implies that the maps φ and ψ preserve the weight degree, then these are strict L_∞-isomorphisms. Now to see that Diagram (8.4) is commutative let us take a linear section $h:\mathfrak{g}\,[1]\to\hat{\mathfrak{g}}\,[1]$ of π, then $\varphi_{*}h := \varphi_1^1\circ h:\mathfrak{g}\,[1]\to\hat{\mathfrak{g}}'\,[1]$ is a linear section of π',

$$\pi'\circ\mathcal{S}(\varphi_{*}h) = \pi'\circ\varphi\circ\mathcal{S}(h) = \pi\circ\mathcal{S}(h) = \mathrm{Id}_{\mathfrak{g}[1]}.$$

Note that as $\pi(d'\circ\mathcal{S}(h)-\mathcal{S}(h)\circ d) = 0$ and $\varphi|_{\ker(\pi)} = \psi$ one has that

$$\begin{aligned}K_{\varphi_{*}h} =&\mathrm{Pr}_{\mathfrak{g}[1]}\circ(d'\circ\mathcal{S}(\varphi_1^1\circ h)-\mathcal{S}(\varphi_1^1\circ h)\circ d) = \mathrm{Pr}_{\mathfrak{g}[1]}\circ(d'\circ\varphi\circ\mathcal{S}(h)-\varphi\circ\mathcal{S}(h)\circ d)\\ =&\mathrm{Pr}_{\mathfrak{g}[1]}\circ(\varphi\circ(d'\circ\mathcal{S}(h)-\mathcal{S}(h)\circ d)) = \mathrm{Pr}_{\mathfrak{g}[1]}\circ(\psi\circ(d'\circ\mathcal{S}(h)-\mathcal{S}(h)\circ d))\\ =&\psi_1^1\circ\mathrm{Pr}_{\mathfrak{g}[1]}\circ(d'\circ\mathcal{S}(h)-\mathcal{S}(h)\circ d) = \psi_1^1\circ K_h.\end{aligned}$$

Thus $K_{\varphi_* h} = \psi_1^1 \circ K_h$, and therefore

$$\begin{aligned} K_{\varphi_* h} \underbrace{\wedge_\otimes \cdots \wedge_\otimes}_{k-\text{times}} K_{\varphi_* h} =& (\psi_1^1 \circ K_h) \wedge_\otimes \cdots \wedge_\otimes (\psi_1^1 \circ K_h) \\ =& (\psi_1^1 \circ K_h) \otimes \cdots \otimes (\psi_1^1 \circ K_h) \circ \Delta_s^{k-1} \\ =& (\psi_1^{1^{\otimes k}} \circ K_h \otimes \cdots \otimes K_h) \circ \Delta_s^{k-1} \\ =& \psi_1^{1^{\otimes k}} \circ K_h \wedge_\otimes \cdots \wedge_\otimes K_h, \end{aligned}$$

so for all $1 \leq k$

$$K_{\varphi_* h}^{\wedge_\otimes k} = \psi_1^{1^{\otimes k}} \circ K_h^{\wedge_\otimes k}.$$

On the other hand, by the commutativity of Diagram (8.3) we have $(\varphi, (\psi_1^1)^{-1})$ is $(\mathrm{ad}_n, \mathrm{ad}_{n'})$-equivariant then $(\varphi, ((\psi_1^1)^{-1})^{\otimes k})$ is $(\mathrm{ad}_n^{\otimes k}, \mathrm{ad}_{n'}^{\otimes k})$-equivariant. Now if we take a linear map $f \in \mathrm{Hom}^\bullet(\wedge^n \mathfrak{n}[1], \mathbb{V})^{\hat{g}}$ then

$$(\mathrm{Id}_n, f) \circ (\varphi, (\psi_1^1)^{-1^{\otimes k}}) = (\mathrm{Id}_n \circ \varphi, f \circ (\psi_1^1)^{-1^{\otimes k}})$$

is a $(\pi^* \rho, \mathrm{ad}_{n'}^{\otimes k})$-equivariant map and the decomposition of the Remark (7.1.1) implies that $(\mathrm{Id}_n, f \circ (\psi_1^1)^{-1^{\otimes k}})$ is $(\pi^* \rho, \varphi^* \mathrm{ad}_{n'}^{\otimes k})$-equivariant map. The equation $\pi = \varphi \circ \pi'$ imply that $(\mathrm{Id}_n, f \circ (\psi_1^1)^{-1^{\otimes k}})$ is $(\varphi^* \pi'^* \rho, \varphi^* \mathrm{ad}_{n'}^{\otimes k})$-equivariant map, and given that the map φ is L_∞-isomorphisms we have $(\mathrm{Id}_n, f \circ (\psi_1^1)^{-1^{\otimes k}})$ is $(\pi'^* \rho, \mathrm{ad}_{n'}^{\otimes k})$-equivariant map. Therefore $\psi^* f := f \circ (\psi_1^{1^{-1}})^{\otimes k} \in \mathrm{Hom}^\bullet(\wedge^n \mathfrak{n}'[1], \mathbb{V})^{\hat{\mathfrak{g}}}$ is well-defined. Finally, observe that

$$(\psi^* f)_{\varphi_* h} = \psi^* f \circ K_{\varphi^* h}^{\wedge_\otimes k} = (f \circ (\psi_1^1)^{-1^{\otimes k}}) \circ (\psi_1^{1^{\otimes k}} \circ K_h^{\wedge_\otimes k}) = f_h.$$

Thus by the independence of the connection in Chern-Weil-Lecomte morphism Theorem (8.2.1) one has that

$$cw(f) = [f_h] = [(\psi^* f)_{\varphi_* h}] = cw(\psi^* f) = (cw \circ \psi^*)(f).$$

That is, the diagram (8.4) is commutative. □

Remark 8.3.1. Observe that the particular case in which $\mathfrak{n}[1] = \mathfrak{n}'[1]$ and the map $\psi = \mathrm{Id}_{\mathfrak{n}[1]}$, then

$$\mathrm{Hom}(\wedge^n \mathfrak{n}[1], \mathbb{V})^{\hat{g}} = \mathrm{Hom}(\wedge^n \mathfrak{n}[1], \mathbb{V})^{\hat{g}'},$$

and *equivalent extensions* of $\mathfrak{g}[1]$ by $\mathfrak{n}[1]$ have the same Chern-Weil-Lecomte morphism.

8.4 The Chern-Weil-Lecomte morphism for Lie 2-algebras

Let us consider an extension of Lie 2-algebras,

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \hat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0,$$

together with a representation up to homotopy ρ of $\mathfrak{g}$ on the 2-term vector space $\mathbb{V} = V_0 \to V_1$. Now, to construct the Chern-Weil morphism, let us choose a linear section $h : \mathfrak{g} \to \hat{\mathfrak{g}}$ of π. Since its curvature is a degree 1 linear map, $K_h : \mathrm{Sym}(\mathfrak{g}[1]) \to \mathfrak{n}[1]$, and

$$\mathrm{Sym}(\mathfrak{g}[1]) = \bigoplus_{-n, 0 \leq n} \bigoplus_{2p+q=n} \bigodot^p \mathfrak{g}_{-1} \odot \bigwedge^q \mathfrak{g}_0,$$

the components of K_h that are possible non-zero are given by

$$K_h^{-3} : (\mathfrak{g}_{-1} \odot \mathfrak{g}_0) \oplus \wedge^3 \mathfrak{g}_0 \to \mathfrak{n}_{-1}, \quad K_h^{-2} : \mathfrak{g}_{-1} \oplus \wedge^2 \mathfrak{g}_0 \to \mathfrak{n}_0,$$

in our case these are

i. $K_h : \mathfrak{g}_{-1} \odot \mathfrak{g}_0 \to \mathfrak{n}_{-1}, \quad K_h(x,a) := h_{-1}(\mathcal{L}_x a) - \hat{\mathcal{L}}_{h_0 x} h_{-1} a;$

ii. $K_h : \wedge^3 \mathfrak{g}_0 \to \mathfrak{n}_{-1}, \quad K_h(x,y,z) := 0;$

iii. $K_h : \mathfrak{g}_{-1} \to \mathfrak{n}_0, \quad K_h(a) := h_0(d(a)) - \hat{d}(h_{-1}a);$

iv. $K_h : \wedge^2 \mathfrak{g}_0 \to \mathfrak{n}_0, \quad K_h(x,y) := h_0 [x,y] - [h_0 x, h_0 y] .$

Remark 8.4.1. Note that in fact the curvature measure when the section h preserves the L_∞-structure

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathfrak{n}_{-1} & \longrightarrow & \mathfrak{n}_0 \\ \downarrow & & \downarrow \\ \hat{\mathfrak{g}}_{-1} & \longrightarrow & \hat{\mathfrak{g}}_0 \\ \downarrow \nwarrow h_{-1} & & \downarrow \nwarrow h_0 \\ \mathfrak{g}_{-1} & \longrightarrow & \mathfrak{g}_0 \\ \downarrow & & \downarrow \\ 0 & & 0. \end{array}$$

The item (i) measure the lack of equivariance of h, item (ii) is zero given that the Lie 2-algebras are strict. The item (iii) measure when h makes the diagram commutes, and item (iv) when h_0 is a Lie algebras morphism. That means, K_h contain the lack of h to being a Lie 2-algebra morphism.

Now let us compute the set of invariants $\mathrm{Hom}(\wedge^n \mathfrak{n}[1], \mathbb{V})^{\hat{\mathfrak{g}}}$. Initially, note that

$$\wedge^n \mathfrak{n}[1] = \bigoplus_{k=0}^{n} \bigwedge^{k} \mathfrak{n}_{-1} \wedge \bigodot^{n-k} \mathfrak{n}_0,$$

so the degrees of homogeneous elements in $\wedge^n \mathfrak{n}[1]$ are between $-2n$ and $-n$, indeed

$$\deg\left(\bigwedge^{k} \mathfrak{n}_{-1} \wedge \bigodot^{n-k} \mathfrak{n}_0 \right) = -k - n.$$

Then the degrees in $\mathrm{Hom}(\wedge^n \mathfrak{n}[1], \mathbb{V})$ with non-zero elements are between n and $2n+1$. Thus for $0 < p < n+1$

$$\mathrm{Hom}^{n+p}(\wedge^n \mathfrak{n}[1], \mathbb{V}) = \mathrm{Hom}\left(\bigwedge^{p} \mathfrak{n}_{-1} \wedge \bigodot^{n-p} \mathfrak{n}_0, V_0 \right) \bigoplus \mathrm{Hom}\left(\bigwedge^{p-1} \mathfrak{n}_{-1} \wedge \bigodot^{n-p+1} \mathfrak{n}_0, V_1 \right),$$

and for $p = 0$

$$\mathrm{Hom}^{n}(\wedge^n \mathfrak{n}[1], \mathbb{V}) = \mathrm{Hom}\left(\bigodot^{n} \mathfrak{n}_0, V_0 \right),$$

and for $p = n+1$

$$\mathrm{Hom}^{2n+1}(\wedge^n \mathfrak{n}[1], \mathbb{V}) = \mathrm{Hom}\left(\bigwedge^{n} \mathfrak{n}_{-1}, V_1 \right).$$

We will denote a map in $\mathrm{Hom}(\bigwedge^p \mathfrak{n}_{-1} \wedge \bigodot^{n-p} \mathfrak{n}_0, V_i)$ by $f_i^{p,n-p}$. Then

$$f_{n+p} = f_0^{p,n-p} + f_1^{p-1,n-p+1} \in \mathrm{Hom}^{n+p}(\wedge^n \mathfrak{n}[1], \mathbb{V})^{\hat{\mathfrak{g}}},$$

is such that for all $a \in \hat{\mathfrak{g}}_{-1}$ and $x \in \hat{\mathfrak{g}}_0$, the next equations hold

$$f_{n+p} \circ \mathrm{ad}_a^{\otimes n} = \rho_{\pi_{-1}(a)} \circ f_{n+p}, \quad f_{n+p} \circ \mathrm{ad}_x^{\otimes n} = \rho_{\pi_0(x)} \circ f_{n+p}.$$

These are equivalent to

- $f_0^{p,n-p} \circ \mathrm{ad}_a^{\otimes n} = \rho_{\pi_{-1}(a)} \circ f_1^{p-1,n-p+1}$;
- $f_0^{p,n-p} \circ \mathrm{ad}_x^{\otimes n} = \rho_{\pi_0(x)} \circ f_0^{p,n-p}$;
- $f_1^{p-1,n-p+1} \circ \mathrm{ad}_x^{\otimes n} = \rho_{\pi_0(x)} \circ f_1^{p-1,n-p+1}$.

Finally, to make explicit the above equations we will compute

$$\mathrm{ad}_a^{\otimes n} : \wedge^n \mathfrak{n}[1] \to \wedge^n \mathfrak{n}[1], \quad \mathrm{ad}_x^{\otimes n} : \wedge^n \mathfrak{n}[1] \to \wedge^n \mathfrak{n}[1].$$

So let $x_i \in \mathfrak{g}_0$ and $a_j \in \mathfrak{g}_{-1}$ for $1 \leq i \leq p$ and $1 \leq j \leq q$, then

$$\begin{aligned}\mathrm{ad}_x^{\otimes n}(a_1 \wedge \cdots \wedge a_p \wedge x_1 \cdots x_q) = &\sum_{k=1}^{p} a_1 \wedge \cdots \wedge (-1)^{(|x|+1)(\sum_{i=1}^{k-1}|a_i|)} \hat{\mathcal{L}}_x a_k \wedge \cdots \wedge a_p \wedge x_1 \cdots x_q + \\ &+ \sum_{k=1}^{q} a_1 \wedge \cdots \wedge a_p \wedge x_1 \cdots (-1)^{(|x|+1)(\sum_{i=1}^{k-1}|x_i|)} [x, x_i] \cdots x_q,\end{aligned}$$

then

$$\begin{aligned}\mathrm{ad}_x^{\otimes n}(a_1 \wedge \cdots \wedge a_p \wedge x_1 \cdots x_q) = &\sum_{k=1}^{p} (-1)^{k-1} \hat{\mathcal{L}}_x a_k \wedge a_1 \wedge \cdots \wedge \hat{a}_k \wedge \cdots \wedge a_p \wedge x_1 \cdots x_q \\ &+ \sum_{k=1}^{q} a_1 \wedge \cdots \wedge a_p \wedge [x, x_k] \, x_1 \cdots \hat{x}_k \cdots x_q,\end{aligned} \tag{8.5}$$

and

$$\begin{aligned}\mathrm{ad}_a^{\otimes n}(a_1 \wedge \cdots \wedge a_p \wedge x_1 \cdots x_q) = &\sum_{k=1}^{p} a_1 \wedge \cdots \wedge (-1)^{(|a|+1)(\sum_{i=1}^{k-1}|a_i|)} [a, a_k] \wedge \cdots \wedge a_p \wedge x_1 \cdots x_q \\ &+ \sum_{k=1}^{q} a_1 \wedge \cdots \wedge a_p \wedge x_1 \cdots \wedge (-1)^{(|a|+1)(\sum_{i=1}^{k-1}|a_i|)+1} \hat{\mathcal{L}}_{x_k} a \wedge \cdots x_q.\end{aligned}$$

hence as $[a, a_k] = 0$

$$\mathrm{ad}_a^{\otimes n}(a_1 \wedge \cdots \wedge a_p \wedge x_1 \cdots x_q) = -\sum_{k=1}^{q} a_1 \wedge \cdots \wedge a_p \wedge \hat{\mathcal{L}}_{x_k} a \wedge x_1 \cdots \hat{x}_k \cdots x_q.$$

Therefore for $0 \leq n$ we have

$$\mathrm{Hom}^n(\mathrm{Sym}(\mathfrak{g}[1]), \mathbb{V}) = \mathrm{Hom}\left(\bigoplus_{2p+q=n} \bigodot^{p} \mathfrak{g}_{-1} \odot \bigwedge^{q} \mathfrak{g}_0, V_0 \right) \oplus \mathrm{Hom}\left(\bigoplus_{2r+s=n-1} \bigodot^{r} \mathfrak{g}_{-1} \odot \bigwedge^{s} \mathfrak{g}_0, V_1 \right).$$

and the Chern-Weil-Lecomte morphism is given by

$$\begin{aligned}\mathrm{Hom}^{n+p}(\wedge^n \mathfrak{n}[1], \mathbb{V})^{\hat{\mathfrak{g}}} &\to H_{CE}^{2n+p}(\mathfrak{g}; \mathbb{V}), \\ f_{n+p} = f_0^{p,n-p} + f_1^{p-1,n-p+1} &\mapsto \left[f_0^{p,n-p}{}_{,h} + f_1^{p-1,n-p+1}{}_{,h} \right].\end{aligned}$$

Chapter 9

An application

In this chapter we introduce the L_∞-cohomology of multiplicative vector fields of a Lie groupoid and extend the Chern-Weil homomorphism to the context of principal 2-bundle over a Lie groupoid that admit a 2-connection form. In section 1 we define the L_∞-cohomology of multiplicative vector fields, this is the L_∞-cohomology of the 2-term L_∞-algebra of multiplicative vector fields of a Lie groupoid with values in its 2-vector space of multiplicative functions. The main result in this section establishes that the L_∞-cohomology of multiplicative vector fields is invariant up to Morita equivalences of Lie groupoids. In Section 2 we present a Lecomte's approach that gives us a morphism that takes values in the L_∞-cohomology of multiplicative vector fields of a Lie groupoid. In Section 3 we present a simplicial approach that gives us a morphism with values in the de Rham cohomology of a Lie groupoid.

9.1 The L_∞-cohomology of multiplicative vector fields over a Lie groupoid.

Let $\mathbb{X} := (X_1 \rightrightarrows X_0)$ be a Lie groupoid and $A \to X_0$ its Lie algebroid. Given that the tangent groupoid of $\mathbb{X}$ is an $\mathcal{LA}$-groupoid the category of multiplicative vector fields has a natural structure of 2-term L_∞-algebra, see Example 2.2.1. This L_∞-structure is actually a Lie 2-algebra structure and it is given by the next crossed module of Lie algebras

$$\left[\mathfrak{X}^R(\mathbb{X}) \xrightarrow{\delta} \mathfrak{X}_{mult}(\mathbb{X}) \xrightarrow{\mathcal{L}} \mathrm{Der}(\mathfrak{X}^R(\mathbb{X}))\right].$$

We denote by $\mathfrak{X}^R(\mathbb{X})$ the space of right invariant vector fields on $\mathbb{X}$ and by $\mathfrak{X}_{mult}(\mathbb{X})$ the space of multiplicative vector fields on $\mathbb{X}$. For any $X, Z \in \mathfrak{X}^R(\mathbb{X})$ and $(\xi, v) \in \mathfrak{X}_{mult}(\mathbb{X})$

$$\delta(X) = (X + \iota_* X, t_* X), \quad \mathcal{L}_{(\xi,v)} Z = [\xi, Z].$$

We write this L_∞-algebra by $\mathfrak{X}^\bullet_{mult}(\mathbb{X})$. It is worth noting that its symmetric algebra is given by

$$\mathrm{Sym}(\mathfrak{X}^\bullet_{mult}(\mathbb{X})\,[1]) = \bigoplus_{-n, 0 \leq n} \bigoplus_{2k+l=n} S^k(\mathfrak{X}^R(\mathbb{X})) \odot \wedge^l \mathfrak{X}_{mult}(\mathbb{X}),$$

its degree 1 coderivation induced by the L_∞-structure is

$$\begin{aligned} d(X_1\cdots X_k\odot\xi_1\wedge\cdots\wedge\xi_l) =& \sum_{i=1}^{k}\delta(X_i)\odot X_1\cdots\hat{X}_i\cdots X_k\odot\xi_1\wedge\cdots\wedge\xi_l \\ &+\sum_{\substack{1\le i\le k\\ 1\le j\le l}}(-1)^j\left(\mathcal{L}_{\xi_j}X_i\right)\cdot X_1\cdots\hat{X}_i\cdots X_k\odot\xi_1\wedge\cdots\wedge\hat{\xi}_j\wedge\cdots\wedge\xi_l \\ &-\sum_{1\le i<j\le k}(-1)^{i+j}\left[\xi_i,\xi_j\right]\odot X_1\cdots X_k\odot\xi_1\wedge\cdots\wedge\hat{\xi}_i\wedge\cdots\wedge\hat{\xi}_j\wedge\cdots\wedge\xi_k, \end{aligned}$$

where $X_i\in\mathfrak{X}^R(\mathbb{X}),\xi_j\in\mathfrak{X}_{mult}(\mathbb{X})$ for $1\le i\le k, 1\le j\le l$.

Theorem 9.1.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Morita equivalent Lie groupoids, then*

$$H(\mathrm{Sym}(\mathfrak{X}^\bullet_{mult}(\mathbb{X})\,[1]),d)\simeq H(\mathrm{Sym}(\mathfrak{X}^\bullet_{mult}(\mathbb{Y})\,[1]),d).$$

Proof. By Theorem 7.6 in [OW19] the L_∞-algebras $\mathfrak{X}^\bullet_{mult}(\mathbb{X})$ and $\mathfrak{X}^\bullet_{mult}(\mathbb{Y})$ are L_∞ quasi-isomorphic. Therefore using a spectral sequence argument similar to the one used in Theorem 7.6.1 the result follows. □

Remark 9.1.1. Since the cohomology $H(\mathrm{Sym}(\mathfrak{X}^\bullet_{mult}(\mathbb{X})[1]),d)$ is invariant by Morita equivalences of the Lie groupoid $\mathbb{X}$, it is an object associated to the differentiable stack represented by $\mathbb{X}$.

Now let us consider the line $\mathcal{VB}$-groupoid over $\mathbb{X}$,

$$\begin{array}{ccc} \mathbb{X}\times\mathbb{R} & \rightrightarrows & M\times\{*\} \\ \downarrow & & \downarrow \\ \mathbb{X} & \rightrightarrows & M. \end{array}$$

Again, by [OW19] one has that the category of multiplicative sections of this $\mathcal{VB}$-groupoid has structure of 2-vector space. Actually, it is the 2-vector space of multiplicative functions

$$\mathcal{C}^\infty(\mathbb{X}):\quad \underbrace{C^\infty_R(\mathbb{X})}_{\text{degree}\quad 0}\xrightarrow{\partial}\underbrace{C^\infty_{mult}(\mathbb{X})}_{\text{degree}\quad 1},\quad \partial(\psi)=\psi-\iota^*\psi,$$

where $C^\infty_R(\mathbb{X})$ denotes the space of right invariant functions

$$C^\infty_R(\mathbb{X})=\left\{f\in C^\infty(X_1)\,|\,f\circ R_g=f,\forall g\in X_1\right\},$$

and $C^\infty_{mult}(\mathbb{X})$ denotes the space of multiplicative functions

$$C^\infty_{mult}(\mathbb{X})=\left\{f\in C^\infty(X_1)\,|\,f(gh)=f(g)+f(h),\forall g,h\in X_1\right\}.$$

The next theorem generalizes the classical result about smooth manifolds, that says that the Lie algebra of vector fields acts by derivations in the space of smooth functions. A detailed study of multiplicative vector fields and derivations will appear in [HOW].

Theorem 9.1.2. *Let $\mathbb{X}$ be a Lie groupoid. There exists a natural representation up to homotopy $\psi^{\mathbb{X}}$ of the 2-term L_∞-algebra of multiplicative vector fields $\mathfrak{X}^\bullet_{mult}(\mathbb{X})$ on its 2-vector space of multiplicative functions $\mathcal{C}^\infty(\mathbb{X})$. It is given by*

$$\begin{array}{ccc} \mathfrak{X}^R(\mathbb{X}) & \xrightarrow{\psi^{\mathbb{X}}_{-1}} & \mathrm{End}^{-1}_\partial(\mathcal{C}^\infty(\mathbb{X})) \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\partial^{end}} \\ \mathfrak{X}_{mult}(\mathbb{X}) & \xrightarrow{\psi^{\mathbb{X}}_0} & \mathrm{End}^0_\partial(\mathcal{C}^\infty(\mathbb{X})), \end{array}\tag{9.1}$$

where for all $X \in \mathfrak{X}^R(\mathbb{X})$

$$\psi_{-1}^{\mathbb{X}}(X) : C^\infty_{mult}(\mathbb{X}) \to C^\infty_R(\mathbb{X}), \quad \psi_{-1}^{\mathbb{X}}(X)(f) := X(f),$$

and for all $(\xi, v) \in \mathfrak{X}_{mult}(\mathbb{X})$

$$\psi_0^{\mathbb{X}}(\xi, v) := (\xi, t^* \circ v \circ 1^*).$$

Proof. We will see that $\psi^{\mathbb{X}}$ is well-defined and satisfies the identities in Definition 7.1.2 in which we have $\rho = (\psi_{-1}^{\mathbb{X}}, \psi_0^{\mathbb{X}}, 0)$. Note that if $X \in \mathfrak{X}^R(\mathbb{X})$ and $f \in C^\infty_{mult}(\mathbb{X})$ then $X(f)$ is a right invariant function. In order to see that $\psi_0^{\mathbb{X}}$ is well-defined it is enough verifying that the next equation holds

$$\xi \circ (\mathrm{Id}^* - \iota^*) = (\mathrm{Id}^* - \iota^*) \circ (t^* \circ v \circ 1^*).$$

Observe that if $f \in C^\infty_R(\mathbb{X})$ then $f = f \circ (1 \circ t)$ and if $X = (\xi, v) : \mathbb{X} \to T\mathbb{X}$ is multiplicative then $dt \circ \xi = v \circ t$ and $ds \circ \xi = v \circ s$. Now these two properties imply that $\xi(f \circ t) = v(f) \circ t$ and $\xi(f \circ s) = v(f) \circ s$. Thus, on the one hand, we get

$$\begin{aligned} \xi \circ (\mathrm{Id}^* - \iota^*)(f) =& \xi(f) - \xi(f \circ \iota) \\ =& \xi(f \circ (1 \circ t)) - \xi(f \circ (1 \circ t) \circ \iota) \\ =& \xi((f \circ 1) \circ t) - \xi((f \circ 1) \circ s), \quad \text{by } s = t \circ \iota \\ =& v(f \circ 1) \circ t - v(f \circ 1) \circ s, \end{aligned}$$

and on the other hand, one has that

$$(\mathrm{Id}^* - \iota^*) \circ (t^* \circ v \circ 1^*)(f) = v(f \circ 1) \circ t - v(f \circ 1) \circ s.$$

Hence,

$$\xi \circ (\mathrm{Id}^* - \iota^*) = (\mathrm{Id}^* - \iota^*) \circ (t^* \circ v \circ 1^*).$$

Now to check that the diagram (9.1) is commutative, let $X \in \mathfrak{X}^R(\mathbb{X})$, then

$$(\partial^{end} \circ \psi_{-1}^{\mathbb{X}})(X) = (\partial \circ X, X \circ \partial), \quad (\psi_0^{\mathbb{X}} \circ \delta)(X) = (X + \iota_* X, t^* \circ t_* X \circ 1^*).$$

On the one side, if $f \in C^\infty_{mult}(\mathbb{X})$ and $x \in X_1$ then we have

$$0 = f(x\iota(x)) = f(x) + \iota^* f(x).$$

Hence, $\iota^* f = -f$, and

$$\begin{aligned} (\iota_* X)(f)(x) =& \iota_{*,\iota(x)}(X_{\iota(x)})(f) = X_{\iota(x)}(f \circ \iota) = -X_{\iota(x)}(f) \\ =& -X(f)(\iota(x)). \end{aligned}$$

That is, $\iota_* X(f) = -\iota^*(X(f))$. Now

$$\begin{aligned} (\partial \circ X)(f) =& \partial(X(f)) = X(f) - \iota^*(X(f)) \\ =& X(f) + \iota_* X(f) \\ =& (X + \iota_* X)(f). \end{aligned}$$

On the other side, for $f \in C^\infty_R(\mathbb{X})$ one has $f = f \circ (1 \circ t)$, then

$$(X \circ \partial)(f) = X(f - \iota^* f) = X(f) - X(\iota^* f),$$

and for all $x \in X_1$, given that $s_{*,x}X_x = 0$ it follows

$$X(\iota^* f)(x) = X_x(f \circ \iota) = X_x(f \circ (1 \circ t) \circ \iota) = X_x((f \circ 1) \circ (t \circ \iota)) \\ = X_x((f \circ 1) \circ s) = s_{*,x}X_x(f \circ 1) = 0.$$

Thus, $X(i^* f) = 0$, then $(X \circ \partial)(f) = X(f)$, and

$$(t^* \circ t_* X \circ 1^*)(f) = t^* \circ (t_* X(f \circ 1)).$$

Now for all $x_0 \in X_0$ one has

$$t_* X(f \circ 1)(x_0) = (t_* X)_{x_0}(f \circ 1) = (t_{*,1_{x_0}} X_{1_{x_0}})(f \circ 1) \\ = X_{1_{x_0}}((f \circ 1) \circ t) = X_{1_{x_0}}(f), \quad \text{by } f \circ (1 \circ t) = f, \\ = X(f)(1_{x_0}) = X(f) \circ 1_{x_0},$$

hence, as $X(f) \in \mathfrak{X}^R(\mathbb{X})$

$$t^* \circ (t_* X(f) \circ 1)) = X(f) \circ (1 \circ t) = X(f).$$

Thus, $X \circ \partial = t^* \circ t_* X \circ 1^*$, and the diagram is commutative. Now for checking that

$$\psi_0^{\mathbb{X}}([(\xi, v), (\eta, w)]) = \left[\psi_0^{\mathbb{X}}(\xi, v), \psi_0^{\mathbb{X}}(\eta, w)\right]$$

for all $(\xi, v), (\eta, w) \in \mathfrak{X}_{mult}(\mathbb{X})$, consider the following expression

$$\psi_0^{\mathbb{X}}([(\xi, v), (\eta, w)]) = \psi_0^{\mathbb{X}}([\xi, \eta], [v, w]) \\ = ([\xi, \eta], t^* \circ [v, w] \circ 1^*) \\ = ([\xi, \eta], t^* \circ (v \circ w - w \circ v) \circ 1^*) \\ = ([\xi, \eta], (t^* \circ v \circ \underbrace{1^*) \circ (t^*}_{\mathrm{Id}^*} \circ w \circ 1^*) - (t^* \circ w \circ \underbrace{1^*) \circ (t^*}_{\mathrm{Id}^*} \circ v \circ 1^*)) \\ = ([\xi, \eta], [t^* \circ v \circ 1^*, t^* \circ w \circ 1^*]) \\ = \left[\psi_0^{\mathbb{X}}(\xi, v), \psi_0^{\mathbb{X}}(\eta, w)\right].$$

Finally, to see that

$$\psi_{-1}^{\mathbb{X}}(\mathcal{L}_{(\xi,v)}^{\mathbb{X}} Z) = \mathcal{L}_{\psi_0^{\mathbb{X}}(\xi,v)}^{end} \psi_{-1}^{\mathbb{X}}(Z),$$

let us consider $f \in C_{mult}^\infty(\mathbb{X})$ and $Z \in \mathfrak{X}^R(\mathbb{X})$. Note that $Z(f) \in C_R^\infty(\mathbb{X})$ so $Z(f) = Z(f) \circ (1 \circ t)$ and therefore

$$\psi_{-1}^{\mathbb{X}}(\mathcal{L}_{(\xi,v)}^{\mathbb{X}} Z)(f) = \psi_{-1}^{\mathbb{X}}([\xi, Z])(f) \\ = [\xi, Z](f) \\ = \xi(Z(f)) - Z(\xi(f)) \\ = \xi(Z(f) \circ (1 \circ t)) - Z(\xi(f)) \\ = v(1^*(Z(f))) \circ t - Z(\xi(f)) \\ = (t^* \circ v \circ 1^*)(Z(f)) - Z(\xi(f)) \\ = ((t^* \circ v \circ 1^*) \circ Z - Z \circ \xi)(f) \\ = (\mathcal{L}_{\psi_0^{\mathbb{X}}(\xi,v)}^{end} \psi_{-1}^{\mathbb{X}}(Z))(f).$$

□

In order to investigate the behavior of this natural representation up to homotopy with respect to Morita morphisms of Lie groupoids, let us consider two Lie groupoids $\mathbb{X}$ and $\mathbb{Y}$ together with a Morita morphism $\Phi : \mathbb{X} \to \mathbb{Y}$ that is surjective on objects. On the one hand, by Theorem 7.3 [OW19], the elements involved in the next diagram are 2-term L_∞-algebras and L_∞-quasi-isomorphisms

$$\begin{array}{ccc} & \mathfrak{X}^\bullet_{mult}(\mathbb{X})^\Phi & \\ {\scriptstyle \iota}\swarrow & & \searrow{\scriptstyle \Phi_*} \\ \mathfrak{X}^\bullet_{mult}(\mathbb{X}) & & \mathfrak{X}^\bullet_{mult}(\mathbb{Y}), \end{array}$$

where $\mathfrak{X}^\bullet_{mult}(\mathbb{X})^\Phi$ is the subcomplex of projectable vector fields of $\mathfrak{X}^\bullet_{mult}(\mathbb{X})$. The subcomplex $\mathfrak{X}_{mult}(\mathbb{X})^\Phi$ is defined as follows

$$\Gamma(A_\mathbb{X})^\Phi \xrightarrow{\delta} \mathfrak{X}_{mult}(\mathbb{X})^\Phi,$$

where

$$\Gamma(A_\mathbb{X})^\Phi = \left\{ a \in \Gamma(A_\mathbb{X}) \,|\, \exists_{a' \in \Gamma(A_\mathbb{Y})} \mathrm{Lie}(\Phi) \circ a = a' \circ \phi \right\}$$

and

$$\mathfrak{X}_{mult}(\mathbb{X})^\Phi = \left\{ (\xi, v) \in \mathfrak{X}_{mult}(\mathbb{X}) \,|\, \exists_{(\eta,w) \in \mathfrak{X}_{mult}(\mathbb{Y})} d\Phi \circ \xi = \eta \circ \Phi \right\}.$$

If we consider the pullback representation of $\mathfrak{X}^\bullet_{mult}(\mathbb{X})^\Phi$ by ι

$$\begin{array}{ccc} \mathfrak{X}^\bullet_{mult}(\mathbb{X}) & \xrightarrow{\psi^\mathbb{X}} & \mathrm{End}(\mathcal{C}^\infty(\mathbb{X})) \\ {\scriptstyle \iota}\uparrow & \nearrow{\scriptstyle \iota^*\psi^\mathbb{X}} & \\ \mathfrak{X}^\bullet_{mult}(\mathbb{X})^\Phi, & & \end{array}$$

then we have that (ι, Id) is a $(\iota^*\psi^\mathbb{X}, \psi^\mathbb{X})$-equivariant L_∞ quasi-isomorphism. Hence by Theorem 7.6.1

$$H_{CE,\psi^\mathbb{X}}(\mathfrak{X}^\bullet_{mult}(\mathbb{X}); \mathcal{C}^\infty(\mathbb{X})) \simeq H_{CE,\iota^*\psi^\mathbb{X}}(\mathfrak{X}^\bullet_{mult}(\mathbb{X})^\Phi; \mathcal{C}^\infty(\mathbb{X})).$$

On the other hand, one has that the canonical pullback of smooth functions

$$\Phi^* : \mathcal{C}^\infty(\mathbb{Y}) \to \mathcal{C}^\infty(\mathbb{X}),$$

is a morphism of dg-vector spaces. We need the following lemma.

Lemma 9.1.1. *For the representations up to homotopy $\iota^*\psi^\mathbb{X}$ of $\mathfrak{X}^\bullet_{mult}(\mathbb{X})^\Phi$ on $\mathcal{C}^\infty(\mathbb{X})$ and $\psi^\mathbb{Y}$ of $\mathfrak{X}^\bullet_{mult}(\mathbb{Y})$ on $\mathcal{C}^\infty(\mathbb{Y})$, the map*

$$(\Phi_*, \Phi^*) : \mathfrak{X}^\bullet_{mult}(\mathbb{X})^\Phi \to \mathfrak{X}^\bullet_{mult}(\mathbb{Y})$$

is a $(\iota^*\psi^\mathbb{X}, \psi^\mathbb{Y})$-equivariant L_∞-quasi-isomorphism along Φ^*.

Proof. To see the $(\iota^*\psi^\mathbb{X}, \psi^\mathbb{Y})$-equivariance we have to check that the next equation holds

$$\Phi^* \circ \psi^\mathbb{Y} \circ (\Phi_* \otimes \mathrm{Id}) = \psi^\mathbb{X} \circ (\mathrm{Id} \otimes \Phi^*). \tag{9.2}$$

For this it suffices to verify Equation (9.2) for $X \in \mathfrak{X}(\mathbb{X})^\Phi$ and $f \in C^\infty(\mathbb{Y})$. This equation follows from the X is Ψ-projectable:

$$X(f \circ \Phi) = \Phi_*(X)(f) \circ \Phi.$$

Finally, by Theorem 7.3 in [OW19] we have that Φ_* is an L_∞ quasi-isomorphism and Φ^* is a dg-vector space quasi-isomorphism, therefore (Φ_*, Φ^*) is a $(\iota^*\psi^\mathbb{X}, \psi^\mathbb{Y})$-equivariant L_∞ quasi-isomorphism map along Φ^*. □

As a consequence, we have the next result which establishes the Morita invariance of L_∞-cohomology of $\mathfrak{X}^\bullet_{mult}(\mathbb{X})$ with values in $\mathcal{C}^\infty(\mathbb{X})$.

Theorem 9.1.3. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Lie groupoids. If $\mathbb{X}$ and $\mathbb{Y}$ are Morita equivalent then*

$$H_{CE,\psi^{\mathbb{X}}}(\mathfrak{X}^\bullet_{mult}(\mathbb{X});\mathcal{C}^\infty(\mathbb{X})) \simeq H_{CE,\psi^{\mathbb{Y}}}(\mathfrak{X}^\bullet_{mult}(\mathbb{Y});\mathcal{C}^\infty(\mathbb{Y})).$$

Proof. By Proposition 7.3 in [OW19] there exists a Lie groupoid $\mathbb{W}$ and Morita maps $\Psi : \mathbb{W} \to \mathbb{X}$ and $\Phi : \mathbb{W} \to \mathbb{Y}$ that are surjective in objects, hence there are L_∞ quasi-isomorphisms as follows

$$\begin{array}{ccccccc} & \mathfrak{X}^\bullet_{mult}(\mathbb{W})^{\Psi} & & & & \mathfrak{X}^\bullet_{mult}(\mathbb{W})^{\Phi} & \\ \swarrow^{\Psi_*} & & \searrow & & \swarrow & & \searrow^{\Phi_*} \\ \mathfrak{X}^\bullet_{mult}(\mathbb{X}) & & & \mathfrak{X}^\bullet_{mult}(\mathbb{W}) & & & \mathfrak{X}^\bullet_{mult}(\mathbb{Y}), \end{array}$$

and by Lemma 9.1.1 it holds that

$$H_{CE,\psi^{\mathbb{X}}}(\mathfrak{X}^\bullet_{mult}(\mathbb{X});\mathcal{C}^\infty(\mathbb{X})) \simeq H_{CE,\psi^{\mathbb{W}}}(\mathfrak{X}^\bullet_{mult}(\mathbb{W});\mathcal{C}^\infty(\mathbb{W})) \simeq H_{CE,\psi^{\mathbb{Y}}}(\mathfrak{X}^\bullet_{mult}(\mathbb{Y});\mathcal{C}^\infty(\mathbb{Y})).$$

□

The above Theorem motivates the next definition.

Definition 9.1.1. Let $\mathbb{X}$ be a Lie groupoid. The L_∞-cohomology of the 2-term L_∞-algebra $\mathfrak{X}^\bullet_{mult}(\mathbb{X})$ with values in the 2-vector space $\mathcal{C}^\infty(\mathbb{X})$ with respect to the representation up to homotopy $\psi^{\mathbb{X}}$,

$$H_{CE,\psi^{\mathbb{X}}}(\mathfrak{X}^\bullet_{mult}(\mathbb{X});\mathcal{C}^\infty(\mathbb{X})),$$

is called the L_∞**-cohomology of multiplicative vector fields over** $\mathbb{X}$.

9.2 Chern-Weil-Lecomte morphism for principal 2-bundles with a 2-connection form

Let us consider a principal 2-bundle over a Lie groupoid $(\mathbb{P},\pi,\mathbb{X},\mathbb{G})$ equipped with a 2-connection form. Note that by Proposition 5.1.1 a 2-connection form $\theta_\bullet = (\theta_1,\theta_0)$ is the same thing that having a multiplicative horizontal lift, that is, a $\mathcal{VB}$-map $h : T\mathbb{X} \to \mathrm{At}(\mathbb{P})$ such that $\tilde{d\pi} \circ h = \mathrm{Id}_{T\mathbb{X}}$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ad}(\mathbb{P}) & \xrightarrow{\iota} & \mathrm{At}(\mathbb{P}) & \underset{\overset{h}{\dashleftarrow}}{\xrightarrow{\tilde{d\pi}}} & T\mathbb{X} & \longrightarrow & 0 \\ & & & \searrow & \downarrow & \swarrow & & & \\ & & & & \mathbb{X}. & & & & \end{array}$$

A multiplicative horizontal lift allows us to lift multiplicative vector fields on $\mathbb{X}$ to multiplicative vector fields on $\mathbb{P}$. In particular, when we look at the sequence of 2-term L_∞-algebras induced by the Atiyah sequence, see Definition 4.3.1, we get that the existence of a multiplicative horizontal lift implies that it is an extension of 2-term L_∞-algebras,

$$
\begin{array}{cccc}
 & 0 & & 0 \\
 & \downarrow & & \downarrow \\
\mathfrak{X}^\bullet_{mult}(\mathrm{Ad}(\mathbb{P})): & \Gamma(P_0 \times_{G_0} \mathfrak{h}) & \xrightarrow{\delta} & \mathfrak{X}_{mult}(\mathrm{Ad}(\mathbb{P})) \\
\downarrow & \downarrow & & \downarrow \\
\mathfrak{X}^\bullet_{mult}(\mathrm{At}(\mathbb{P})): & \Gamma(A_{\mathbb{P}}/G_0) & \xrightarrow{\delta} & \mathfrak{X}_{mult}(\mathrm{At}(\mathbb{P})) \\
\downarrow \uparrow h & \downarrow \uparrow h_c & & \downarrow \uparrow h_m \\
\mathfrak{X}^\bullet_{mult}(\mathbb{X}): & \Gamma(A_{\mathbb{X}}) & \xrightarrow{\delta} & \mathfrak{X}_{mult}(\mathbb{X}) \\
\downarrow & \downarrow & & \downarrow \\
0: & 0 & & 0.
\end{array}
\tag{9.3}
$$

It is worth noting that this multiplicative horizontal lift induces a vector bundle morphism $h_c : A_{\mathbb{X}} \to A_{\mathbb{P}}/G_0$ at level of core bundles such that $d\pi \circ h_c = \mathrm{Id}_{A_{\mathbb{X}}}$. Moreover, Theorem 9.1.2 says that for the base groupoid there exists a representation up to homotopy of $\mathfrak{X}^\bullet_{mult}(\mathbb{X})$ on the 2-vector space $\mathcal{C}^\infty(\mathbb{X})$. Thus, for a principal 2-bundle over a Lie groupoid that admits a 2-connection form we have naturally an extension of L_∞-algebras together with a representation up to homotopy. Then Theorem 8.2.1 allows us to construct a Chern-Weil-Lecomte morphism that takes values in the L_∞-cohomology of multiplicative vector fields over the base groupoid. This is the content of the main result of this section

Theorem 9.2.1. *Let $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ be a principal 2-bundle over a Lie groupoid equipped with a 2-connection form. Then, for each $k \geq 0$ there exists a natural morphism*

$$cw : \mathrm{Hom}^\bullet(\wedge^k \mathfrak{X}^\bullet_{mult}(\mathrm{Ad}(\mathbb{P}))[1], \mathcal{C}^\infty(\mathbb{X})) \to H^{k+\bullet}_{CE,\psi^{\mathbb{X}}}(\mathfrak{X}^\bullet_{mult}(\mathbb{X}); \mathcal{C}^\infty(\mathbb{X})),$$

that is independent of the 2-connection form.

9.2.1 Simplicial approach

The main objective of this section is to present Theorem 9.2.4. It gives us a simplicial approach to the construction of the Chern-Weil homomorphism for principal 2-bundles with 2-connection form. The main feature of this approach is that it takes values in the de Rham cohomology of the base Lie groupoid, see Section 1.5. In particular, this construction motives two open questions. The first one, if it is possible to compare the de Rham cohomology of a Lie groupoid and the L_∞-cohomology of multiplicative vector field presented in Section 9.1, and the second one, if there exists a similar diagram, as in the classical case (1.4), that allows us to compare the Lecomte's approach and the simplicial approach to the extensions of the Chern-Weil homomorphism for principal 2-bundles with 2-connection form.

Let $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$ be a principal 2-bundle with a 2-connection (θ_1, θ_0). The Lie groupoid morphism $\pi : \mathbb{P} \to \mathbb{X}$ induces a morphism between simplicial manifolds $\{\pi_n : P_n \to X_n\}_{n \geq 0}$. By Theorem A.1.3 for each $n \geq 0$ one has that (P_n, π_n, X_n, G_n) is a principal bundle,

$$
\begin{array}{ccccccc}
G_\bullet: & \cdots & G_2 & \substack{\to\\ \to\\ \to} & G_1 & \rightrightarrows & G_0 \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\
P_\bullet: & \cdots & P_2 & \substack{\to\\ \to\\ \to} & P_1 & \rightrightarrows & P_0 \\
\downarrow \pi & & \downarrow \pi_2 & & \downarrow \pi_1 & & \downarrow \pi_0 \\
X_\bullet: & \cdots & X_2 & \substack{\to\\ \to\\ \to} & X_1 & \rightrightarrows & X_0.
\end{array}
$$

Moreover, by the Theorem A.2.1 the principal bundle (P_n, π_n, X_n, G_n) has a connection 1-form given by $\theta_n := \theta_1 \times_{P_0} \cdots \times_{P_0} \theta_1 \in \Omega^1_{dR}(P_n; \mathfrak{g}_n)$. Indeed, the Lie groupoid morphism $\theta : TP_1 \to P_1 \times \mathfrak{g}_1$ induces a morphism on simplicial manifolds

$$\{\theta_n : TP_n \to P_n \times \mathfrak{g}_n\}_{n \geq 0}\,.$$

Observe that $T(P_n) = T(P_1 \times_{P_0} \cdots \times_{P_0} P_1) = TP_1 \times_{TP_0} \cdots \times_{TP_0} TP_1 = (TP)_n$ and $\mathfrak{g}_n = \mathfrak{g}_1 \oplus_{\mathfrak{g}_0} \cdots \oplus_{\mathfrak{g}_0} \mathfrak{g}_1$ is the Lie algebra of the Lie group G_n. Since $\theta_\bullet : TP_\bullet \to P_\bullet \times \mathfrak{g}_\bullet$ is simplicial map, the maps $\theta_n : TP_n \to P_n \times \mathfrak{g}_n$ commute with the face maps d_i and the degeneracy maps s_i for all $i = 0, \ldots, n$, and $n \geq 0$:

$$\begin{array}{ccc} TP_n & \xrightarrow{\theta_n} & P_n \times \mathfrak{g}_n \\ {\scriptstyle d_{i*}}\downarrow & & \downarrow{\scriptstyle d_i \times d_{G*}^i} \\ TP_{n-1} & \xrightarrow{\theta_{n-1}} & P_{n-1} \times \mathfrak{g}_{n-1} \end{array} \qquad \begin{array}{ccc} TP_n & \xrightarrow{\theta_n} & P_n \times \mathfrak{g}_n \\ {\scriptstyle s_{i*}}\uparrow & & \uparrow{\scriptstyle s_i \times s_{G*}^i} \\ TP_{n-1} & \xrightarrow{\theta_{n-1}} & P_{n-1} \times \mathfrak{g}_{n-1}. \end{array}$$

Therefore, the face maps $d_i : P_n \to P_{n-1}$ are bundle morphisms along ${d_{G_n}}^i : G_n \to G_{n-1}$ covering the maps $d_{X\,i} : X_n \to X_{n-1}$ that preserve the connection forms. Analogously, the degeneracy maps are bundle morphisms preserving the connection forms. Thus by Theorem A.2.2 both face and degeneracy maps induce morphisms of dga

$$\begin{array}{ccc} W(\mathfrak{g}_n) & \xrightarrow{w_{\theta_n}} & \Omega_{dR}(P_n) \\ {\scriptstyle {d_{G_n}^i}^*}\uparrow & & \uparrow{\scriptstyle {d_i}^*} \\ W(\mathfrak{g}_{n-1}) & \xrightarrow{w_{\theta_{n-1}}} & \Omega_{dR}(P_{n-1}) \end{array} \qquad \begin{array}{ccc} W(\mathfrak{g}_n) & \xrightarrow{w_{\theta_n}} & \Omega_{dR}(P_n) \\ {\scriptstyle {s_G^i}^*}\downarrow & & \downarrow{\scriptstyle s_i^*} \\ W(\mathfrak{g}_{n-1}) & \xrightarrow{w_{\theta_{n-1}}} & \Omega_{dR}(P_{n-1}). \end{array}$$

Then, dualizing and extending the simplicial structure on $\mathfrak{g}_\bullet$, we have a cosimplicial dga $W(\mathfrak{g}_\bullet) = \{W(\mathfrak{g}_n)\}_{n \geq 0}$ such that for each $n \geq 0$ the algebra $W(\mathfrak{g}_n)$ is a G_n-dga. Moreover, the simplicial morphism $\theta_\bullet$ induces a morphism of cosimplicial differential graded commutative algebras

$$\{w_{\theta_n} : W(\mathfrak{g}_n) \to \Omega_{dR}(P_n)\}_{n \geq 0}\,.$$

The cosimplicial structure in $W(\mathfrak{g}_\bullet)$ together with the dga structure determine the following double complex

$$\begin{array}{ccccccccc} \cdots & & \cdots & & \cdots & & \cdots & & \\ {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\ W^3(\mathfrak{g}_0) & \xrightarrow{\partial} & W^3(\mathfrak{g}_1) & \xrightarrow{\partial} & W^3(\mathfrak{g}_2) & \xrightarrow{\partial} & W^3(\mathfrak{g}_3) & \xrightarrow{\partial} & \cdots \\ {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\ W^2(\mathfrak{g}_0) & \xrightarrow{\partial} & W^2(\mathfrak{g}_1) & \xrightarrow{\partial} & W^2(\mathfrak{g}_2) & \xrightarrow{\partial} & W^2(\mathfrak{g}_3) & \xrightarrow{\partial} & \cdots \\ {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\ W^1(\mathfrak{g}_0) & \xrightarrow{\partial} & W^1(\mathfrak{g}_1) & \xrightarrow{\partial} & W^1(\mathfrak{g}_2) & \xrightarrow{\partial} & W^1(\mathfrak{g}_3) & \xrightarrow{\partial} & \cdots \\ {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & {\scriptstyle d}\uparrow & & \\ W^0(\mathfrak{g}_0) & \xrightarrow{\partial} & W^0(\mathfrak{g}_1) & \xrightarrow{\partial} & W^0(\mathfrak{g}_2) & \xrightarrow{\partial} & W^0(\mathfrak{g}_3) & \xrightarrow{\partial} & \cdots \end{array}$$

where the horizontal differential at level q is the alternating sum of the face maps

$$\partial^q := \sum_{i=0}^{q} (-1)^i {d_{G_q}^i}^* : W(\mathfrak{g}_q) \to W(\mathfrak{g}_{q+1}),$$

and the vertical differential at level p is the differential of

$$d_p : W(\mathfrak{g}_p) \to W(\mathfrak{g}_p), \quad d_p^q : W^q(\mathfrak{g}_p) \to W^{q+1}(\mathfrak{g}_p).$$

for $p \geq 0$ and $q \geq 0$. We write the total complex of this complex by

$$W_{tot}(\mathfrak{g}_\bullet) := \bigoplus_{n\geq 0} W^n(\mathfrak{g}_\bullet), \quad W^n(\mathfrak{g}_\bullet) := \bigoplus_{p+q=n} W^q(\mathfrak{g}_p),$$

where the total differential is given by

$$d_{tot}|_{W^p(\mathfrak{g}_p)} = \partial^p + (-1)^p d_p^q.$$

In a diagram:

$$\begin{array}{ccc} W^{q+1}(\mathfrak{g}_p) & & \\ {\scriptstyle (-1)^p d_p^q}\uparrow & & \\ d_{tot} : W^q(\mathfrak{g}_p) & \xrightarrow{\partial^p} & W^q(\mathfrak{g}_{p+1}). \end{array}$$

Theorem 9.2.2. *The cosimplicial morphism*

$$\{w_{\theta_n} : W(\mathfrak{g}_n) \to \Omega_{dR}(P_n)\}_{n\geq 0},$$

induces a morphism on the Bott-Shulman-Stasheff complex $\Omega_{dR}(P)$

$$w_{\theta_\bullet} : W_{tot}(\mathfrak{g}_\bullet) \to \Omega_{tot}(P_\bullet).$$

Proof. It follows from the previous comments about the cosimplicial structure induced by the simplicial morphism $\theta_\bullet$. □

The following proposition suggests that the total complex $W_{tot}(\mathfrak{g}_\bullet)$ could be thought of as a candidate for a model of the de Rham complex of the universal principal 2-bundle $E(\mathbb{G})$ with structural 2-group $\mathbb{G}$.

Theorem 9.2.3. *The total complex* $W_{tot}(\mathfrak{g}_\bullet)$ *is acyclic, i.e., it has the same cohomology of the point*

$$H^k(W_{tot}(\mathfrak{g}_\bullet)) = \begin{cases} 0, & \text{if} \quad k > 0, \\ \mathbb{R}, & \text{if} \quad k = 0. \end{cases}$$

Proof. It is well-known that the Weil algebra is acyclic, therefore the columns of the double complex of $W(\mathfrak{g}_\bullet)$ are exact, then as this double complex is a first quadrant double complex, it is a bounded double complex with exact columns, then its total complex is acyclic [Wei94, pag.9]. □

Now Theorem A.2.2 also implies that restricting level by level the cosimplicial structure in $W(\mathfrak{g}_\bullet)$ to its basic elements we get a cosimplicial dga $S(\mathfrak{g}_\bullet^*)^{G_\bullet} := \{S(\mathfrak{g}_n^*)^{G_n}\}_{n\geq 0}$ that, just as $W(\mathfrak{g}_\bullet)$, it induces a double complex

$$\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
S^2(\mathfrak{g}_0^*)^{G_0} & \xrightarrow{\partial} & S^2(\mathfrak{g}_1^*)^{G_1} & \xrightarrow{\partial} & S^2(\mathfrak{g}_2^*)^{G_2} & \xrightarrow{\partial} & S^2(\mathfrak{g}_3^*)^{G_3} & \xrightarrow{\partial} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
S^1(\mathfrak{g}_0^*)^{G_0} & \xrightarrow{\partial} & S^1(\mathfrak{g}_1^*)^{G_1} & \xrightarrow{\partial} & S^1(\mathfrak{g}_2^*)^{G_2} & \xrightarrow{\partial} & S^1(\mathfrak{g}_3^*)^{G_3} & \xrightarrow{\partial} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
0 & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & 0 & \xrightarrow{\partial} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
S^0(\mathfrak{g}_0^*)^{G_0} & \xrightarrow{\partial} & S^0(\mathfrak{g}_1^*)^{G_1} & \xrightarrow{\partial} & S^0(\mathfrak{g}_2^*)^{G_2} & \xrightarrow{\partial} & S^0(\mathfrak{g}_3^*)^{G_3} & \xrightarrow{\partial} & \cdots
\end{array}$$

The morphism

$$\{w_{\theta_n} : W(\mathfrak{g}_n) \to \Omega_{dR}(P_n)\}_{n\geq 0},$$

restricts level by level to basic elements as a dga cosimplicial morphism

$$\left\{w_{\theta_n} : S(\mathfrak{g}_n^*)^{G_n} \to \Omega_{dR}(X_n)\right\}_{n\geq 0},$$

that induces a morphism on their respective double complexes

$$w_{\theta_\bullet} : S_{tot}(\mathfrak{g}_\bullet^*)^{G_\bullet} \to \Omega_{tot}(X_\bullet).$$

Therefore, a simplicial candidate for Chern-Weil homomorphism for a principal 2-bundle with a 2-connection is the previous morphism seen in cohomology. In order to strengthen our statement we are going to prove that $w_\bullet$ is independent of the 2-connection.

Theorem 9.2.4. *The morphism induced in cohomology by*

$$w_{\theta_\bullet} : S_{tot}(\mathfrak{g}_\bullet^*)^{G_\bullet} \to \Omega_{tot}(X_\bullet)$$

is independent of the 2-connection.

Proof. Initially let us note that as the double complex $S(\mathfrak{g}_\bullet)^{G_\bullet}$ has all odd rows being zero, then for $x \in S^n_{tot}(\mathfrak{g}_\bullet^*)^{G_\bullet}$ one has that

$$x = \sum_{2q+p=n} f_{2q,p}, \quad f_{2q,p} \in S^q(\mathfrak{g}_q^*)^{G_p},$$

$$\begin{aligned}
d_{tot}x =& d_{tot}\left(\sum_{2q+p=n} f_{2q,p}\right) = \sum_{2q+p=n} d_{tot}(f_{2q,p}) \\
=& \sum_{2q+p=n} \partial^p \cdot f_{2q,p} + (-1)^p d_p^{2q}(f_{2q,p}) \\
=& \sum_{2q+p=n} \partial^p \cdot f_{2q,p},
\end{aligned}$$

thus, $d_{tot}x = 0$ if and only if $\partial^p \cdot f_{2q,p} = 0$ for all $p \geq 0, q \geq 0$ with $2q + p = n$.
Now let us consider a pair of 2-connections (θ_1, θ_0) and (θ_1', θ_0') on the principal 2-bundle $(\mathbb{P}, \pi, \mathbb{X}, \mathbb{G})$, and consider for $f_{2q,p} \in S^q(\mathfrak{g}_p^*)^{G_p}$ with $\partial^p \cdot f_{2q,p} = 0$. Since $\theta_p = \theta_1 \times_{P_0} \times \cdots \times_{P_0} \theta_1$ and $\theta_p' = \theta_1' \times_{P_0} \times \cdots \times_{P_0} \theta_1'$ determine connection 1-forms on P_n, and it is well-known that for $f_{2q,p} \in S^q(\mathfrak{g}_p^*)^{G_p}$

there is a transgression form [KN69, pag. 297]

$$T(f_{2q,p}) = q\int_0^1 f_{2q,p}(\alpha,\Omega_t,\dots,\Omega_t)dt \in \Omega^{2q-1}_{dR}(P_p),$$

for $\alpha = \theta_p - \theta'_p$, and Ω_t the curvature form of the connection $\theta_t = \theta'_p + t(\theta_p - \theta'_p), 0 \leq t \leq 1$ such that

$$d_p^{2q-1}(T(f_{2q,p})) = f_{2q,p}^{\Omega_p} - f_{2q,p}^{\Omega'_p}.$$

Moreover, we have that the transgression form satisfies that

$$\begin{aligned}
\partial^p(T(f_{2q,p})) &= \left(\sum_{i=0}^p(-1)^i d_i^*\right)(q\int_0^1 f_{2q,p}(\alpha,\Omega_t,\dots,\Omega_t)dt)\\
&= \sum_{i=0}^p(-1)^i q\int_0^1 f_{2q,p}(d_i^*\alpha, d_i^*\Omega_t,\dots,d_i^*\Omega_t)dt\\
&= \sum_{i=0}^p(-1)^i q\int_0^1 f_{2q,p}(d^i_{G_p*}\cdot\alpha, d^i_{G_p*}\cdot\Omega_t,\dots,d^i_{G_p*}\cdot\Omega_t)dt\\
&= \sum_{i=0}^p(-1)^i q\int_0^1 \left({d^i_{G_p}}^* f_{2q,p}\right)(\alpha,\Omega_t,\dots,\Omega_t)dt\\
&= \int_0^1 q\left(\sum_{i=0}^p(-1)^i {d^i_{G_p}}^* f_{2q,p}\right)(\alpha,\Omega_t,\dots,\Omega_t)dt\\
&= \int_0^1 q\left(\partial^p\cdot f_{2q,p}\right)(\alpha,\Omega_t,\dots,\Omega_t)dt\\
&= \int_0^1 0dt = 0.
\end{aligned}$$

Thus, let $x = \sum_{2q+p=n} f_{2q,p} \in S^n_{tot}(\mathfrak{g}_\bullet)^{G_\bullet}$ with $d_{tot}x = 0$, then one has that

$$\begin{aligned}
w_{\theta_\bullet}(x) =& w_{\theta_\bullet}\left(\sum_{2q+p=n} f_{2q,p}\right) = \sum_{2+p=n} w_{\theta_p}(f_{2q,p})\\
&= \sum_{2q+p=n} f_{2q,p}^{\Omega_p} \in \Omega^n_{tot}(P_\bullet), \quad f_{2q,p}^{\Omega_q} \in \Omega^{2q}_{dR}(P_p),
\end{aligned}$$

Therefore for

$$y = \sum_{2q+p=n}(-1)^p T(f_{2q,p}) \in \Omega^{n-1}_{tot}(P_\bullet), \quad T(f_{2q,p}) \in \Omega^{2q-1}_{dR}(P_p),$$

one has

$$\begin{aligned}
d_{tot}y =& d_{tot}\left(\sum_{2q+p=n}(-1)^p T(f_{2q,p})\right)\\
=& \sum_{2q+p=n} d_{tot}((-1)^p T(f_{2q,p}))\\
=& \sum_{2q+p=n} (-1)^p \partial^p T(f_{2q,p}) + (-1)^{2p} d_p^{2q-1}(T(f_{2q,p})))\\
=& \sum_{2q+p=n} (-1)^p 0 + (f_{2q,p}^{\Omega_p} - f_{2q,p}^{\Omega'_p}))\\
=& \sum_{2q+p=n} f_{2q,p}^{\Omega_p} - f_{2q,p}^{\Omega'_p}\\
=& \sum_{2q+p=n} f_{2q,p}^{\Omega_p} - \sum_{2q+p=n} f_{2q,p}^{\Omega'_p}\\
=& w_{\theta_\bullet}(x) - w_{\theta'_\bullet}(x).
\end{aligned}$$

In conclusion, for every cocycle $x \in S_{tot}(\mathfrak{g}_\bullet)^{G_\bullet}$ the two images $w_{\theta_\bullet}(x)$ and $w_{\theta'_\bullet}(x)$ are cohomologous, hence these two maps are equal in cohomology. □

Appendix A

Constructions in $\mathcal{PB}$

A.1 Constructions in $\mathcal{PB}$

In this chapter we shall introduce some supporting constructions and some propositions that we find useful for a better understanding of the material included in the previous chapters involving the category $\mathcal{PB}$.

Theorem A.1.1. *Let (P, π_P, X, G) and (Q, π_Q, Y, H) be two principal bundles and $F : P \to Q$ be a bundle morphism along $\varphi : G \to H$ covering the map $f : X \to Y$,*

$$\begin{array}{ccc} P & \xrightarrow{F} & Q \\ \downarrow{\scriptstyle \pi_P} & & \downarrow{\scriptstyle \pi_Q} \\ X & \xrightarrow{f} & Y \end{array} \qquad \curvearrowleft \qquad (G \xrightarrow{\varphi} H)$$

then

$$\mathit{rank}(\varphi) + \mathit{rank}(f) \leq \mathit{rank}(F).$$

Proof. Consider a local representation of F, let $U \subset X$ and $V \subset Y$ be trivializing open sets such that

$$F : U \times G \to V \times H, \quad (x, g) \mapsto (f(x), \gamma(x)\varphi(g)),$$

for $\gamma : U \to H$, the local representation has this form because F is $G \xrightarrow{\varphi} H$-equivariant. Then it is clear that the horizontal rank is equal to $\mathrm{rank}(f)$ and the vertical rank is a mix between $\mathrm{rank}(\varphi)$ and $\mathrm{rank}(\gamma)$, therefore

$$\mathrm{rank}(f) + \mathrm{rank}(\varphi) \leq \mathrm{rank}(F).$$

□

The next example shows that it is possible to have the strict inequality.

Example A.1.1. Let us consider the trivial principal bundles $(\mathbb{R}^2 \times (\mathbb{R}, +), \mathrm{pr}_1, \mathbb{R}^2, (\mathbb{R}, +))$ and $(\mathbb{R} \times (\mathbb{R}^2, +), \mathrm{pr}_1, \mathbb{R}, (\mathbb{R}^2, +))$, and the bundle morphism (Ψ, ψ, ι) where $\Psi(x, y; z) = (x; y, z), \psi(x, y) = x, \iota(z) = (0, z)$

$$\begin{array}{ccc} \mathbb{R}^2 \times (\mathbb{R}, +) & \xrightarrow{\Psi} & \mathbb{R} \times (\mathbb{R}^2, +), \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathbb{R}^2 & \xrightarrow{\psi} & \mathbb{R}. \end{array} \qquad (\mathbb{R}, +) \xrightarrow{\iota} (\mathbb{R}^2, +)$$

Then we have that (Ψ, ψ, ι) is a morphism of principal bundles in which $\mathrm{rank}(\Psi) = 3$, $\mathrm{rank}(\psi) = 1$ and $\mathrm{rank}(\iota) = 1$, therefore

$$\mathrm{rank}(\psi) + \mathrm{rank}(\iota) < \mathrm{rank}(\Psi).$$

Theorem A.1.2. *Let (P, π_P, X, G) and (Q, π_Q, Y, H) be two principal bundles and $F : P \to Q$ be a bundle morphism along $\varphi : G \to H$ covering the map $f : X \to Y$*

$$\begin{array}{ccc} P & \xrightarrow{F} & Q \\ {\scriptstyle \pi_P}\downarrow & & \downarrow{\scriptstyle \pi_Q} \\ X & \xrightarrow{f} & Y \end{array} \qquad \curvearrowleft \qquad (G \xrightarrow{\varphi} H).$$

then $(F, f, \phi) : (P, \pi_P, X, G) \to (Q, \pi_Q, Y, H)$ is a bundle isomorphism if and only if φ is an isomorphism of Lie groups and f is a diffeomorphism.

Proof. Initially let us suppose that F is a bundle isomorphism then there is a bundle morphism $F^{-1} : Q \to P$ along $\psi : H \to G$ covering the map $g : Y \to X$ such that $F \circ F^{-1} = \mathrm{Id}_Q$ and $F^{-1} \circ F = \mathrm{Id}_P$. Thus for $z \in P$ and $g \in G$ we have that

$$zg = (F^{-1} \circ F)(zg) = F^{-1}(F(z)\varphi(g)) = z\psi(\varphi(g)),$$

and given that the action is free $g = \psi(\varphi(g))$ for all $g \in G$, it implies that $\psi \circ \varphi = \mathrm{Id}_G$, analogously $\varphi \circ \psi = \mathrm{Id}_H$. Now since π_P is surjective for $x \in X$ there is a $z \in P$ with $\pi_P(z) = x$, then

$$g(f(x)) = gf\pi_P(z) = g\pi_Q F(z) = \pi_P F^{-1} F(z) = \pi_P(z) = x.$$

Thus $g(f(x)) = x$ for all $x \in X$, then $g \circ f = \mathrm{Id}_X$, analogously $f \circ g = \mathrm{Id}_Y$. Conversely, let us suppose that φ is an isomorphism of Lie groups and f is a diffeomorphism, then we will show that F is a smooth bijection with maximal rank, then it is a diffeomorphism and finally we see that its inverse F^{-1} determines a bundle morphism $(F^{-1}, f^{-1}, \phi^{-1})$. To see that F is bijective, let $x, y \in P$ and suppose that $F(x) = F(y)$, then one has that $\pi_Q F(x) = f\pi_P(x)$ and $\pi_Q F(y) = f\pi_P(y)$, thus $f(\pi_P(x)) = f(\pi_P(y))$ and since f is injective $\pi_P(x) = \pi_P(y)$. Hence there exists a unique $g \in G$ with $xg = y$, but $F(x) = F(y) = F(xg) = F(x)\varphi(g)$ then it implies that $\varphi(g) = e$, and since φ is injective $e = g$, therefore $x = y$, and F is injective. To see that F is surjective, let $q \in Q$ and take some $z \in \pi_P^{-1}(f^{-1}(\pi_Q(q)))$, then note that

$$\pi_Q F(z) = f\pi_P(z) = ff^{-1}\pi_Q(q) = \pi_Q(q).$$

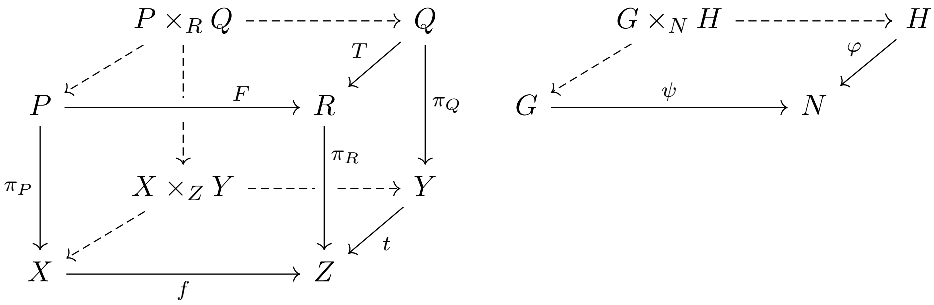
Hence $F(z)$ and q are in the same fiber then there exists a unique $h \in H$ with $F(z)h = q$, since φ is surjective there is some $g \in G$ with $\varphi(g) = h$, therefore taking $zg \in P$ one has that $F(zg) = F(z)\varphi(g) = F(z)h = q$. Thus F is a smooth bijective map, therefore by the Theorem A.1.1 we have that

$$\mathrm{rank}(F) \geq \mathrm{rank}(\varphi) + \mathrm{rank}(f) = \dim(H) + \dim(Y) = \dim(Q).$$

Then F has maximal rank, hence a diffeomorphism. Finally to see that F^{-1} is equivariant note that for some $q \in Q$ and $h \in H$ it holds that $F(F^{-1}(qh)) = qh$ and $F(F^{-1}(q)\varphi^{-1}(h)) = F(F^{-1}(q))\varphi(\varphi^{-1}(h)) = qh$, then $F(F^{-1}(qh)) = F(F^{-1}(q)\varphi^{-1}(h))$ and by the injectivity of F one has that $F^{-1}(qh) = F^{-1}(q)\varphi^{-1}(h)$.

□

Theorem A.1.3. *Let (P, π_P, X, G), (Q, π_Q, Y, H) and (R, π_R, Z, N) be principal bundles, $F : P \to R$ be a bundle morphism along $\psi : G \to N$ covering the map $f : X \to Z$, and $T : Q \to R$ a bundle* *morphism along $\varphi : H \to N$ covering the map $t : Y \to Z$,*



Then if F and T are transversal, then

$$\pi_P \times_{\pi_R} \pi_Q : P \times_R Q \to X \times_Z Y,$$

is a principal $G \times_N H$-bundle.

Proof. We shall see that

$$(\pi_P \times_{\pi_R} \pi_Q) : P \times_R Q \to X \times_Z Y,$$

is a surjective submersion and that $G \times_N H$ acts freely and transitively on the fibers, thus $P \times_R Q$ is a principal $G \times_N H$-bundle. By Theorem A.1.4 it follows that $P \times_R Q$ and $X \times_Z Y$ are smooth manifolds, and $\pi_P \times_{\pi_R} \pi_Q$ is a smooth map, in the same way $G \times_N H$ is a Lie group. Now one easily sees that for all $(p, r) \in P \times_R Q$ with $(\pi_P \times_{\pi_R} \pi_Q)(p, r) = (x, y)$ we have

$$\begin{aligned}\operatorname{im}\left((\pi_P \times_{\pi_R} \pi_Q)_{*,(p,r)}\right) &= \operatorname{im}((\pi_P)_{*,p})_{f_{*,x}} \times_{t_{*,y}} \operatorname{im}((\pi_Q)_{*,r}) \\ &= T_x Y_{f_{*,x}} \times_{t_{*,y}} T_y Z \\ &= T_{(x,y)} Y \times_X Z.\end{aligned}$$

Hence $\pi_P \times_{\pi_R} \pi_Q$ is a submersion. On the other hand, if $(x, y), (z, w) \in P \times_R Q$ with $(\pi_P \times_{\pi_R} \pi_Q)(x, y) = (\pi_P \times_{\pi_R} \pi_Q)(z, w)$ one has that $\pi_P(x) = \pi_P(z)$ then there is a unique $g \in G$ with $x = zg$, and $\pi_R(y) = \pi_R(w)$, then there is a unique $h \in H$ with $y = xh$, moreover

$$F(x) = F(zg) = F(z)\psi(g), \quad T(y) = T(xh) = T(x)\varphi(h),$$

so $F(x) = T(y), F(z) = T(w)$ implies $F(z)\psi(g) = T(w)\varphi(h)$, and then $\psi(g) = \varphi(h)$. Therefore there is a unique $(g, h) \in G \times_N H$ such that $(x, y) = (z, w)(g, h)$, thus $G \times_N H$ acts freely and transitive in the fibers. □

Theorem A.1.4. *With the same hypothesis of the last theorem, the maps f, t are transversal and ψ, φ are transversal if and only if F, T are transversal.*

Proof. Given that it is a local question, it suffices to take a local representation for F and T in trivializing open sets. Let $U \subseteq X, V \subseteq Y, W \subseteq Z$ be such that

$$F : U \times G \to W \times N, \quad (x, g) \mapsto (f(x), \gamma(x)\varphi(g))$$

and

$$T : V \times H \to W \times N, \quad (y, h) \mapsto (t(y), \delta(y)\psi(h)),$$

for some smooth maps $\gamma : U \to N$ and $\delta : V \to N$, this is because F is $(G \xrightarrow{\varphi} N)$-equivariant and T is $(H \xrightarrow{\psi} N)$-equivariant. For all $(x, g) \in U \times G, (y, h) \in V \times H$ with $F(x, g) = T(y, h)$ one has that $f(x) = t(y)$ and $\gamma(x)\varphi(g) = \delta(y)\psi(h)$. Moreover

$$\operatorname{im}(F_{*,(x,g)}) = \operatorname{im}(f_{*,x}) \oplus \widetilde{\operatorname{im}(\varphi)}_{\gamma(x)\varphi(g)},$$

where

$$\widetilde{\mathrm{im}(\varphi)}_{\gamma(x)\varphi(g)} = \left\{ \widetilde{\varphi_*(A)}_{\gamma(x)\varphi(g)} \mid A \in Lie(G) \right\}.$$

We denote the fundamental vector field in R associated to the vector $X \in Lie(N)$ by $\widetilde{X}$. Then one easily checks that

$$\mathrm{im}(F_{*,(x,g)}) + \mathrm{im}(T_{*,(y,h)}) = (\mathrm{im}(f_{*,x}) + \mathrm{im}(t_{*,y})) \oplus \left(\widetilde{\mathrm{im}(\varphi)}_{\gamma(x)\varphi(g)} + \widetilde{\mathrm{im}(\psi)}_{\delta(y)\psi(h)} \right).$$

Thus

$$\mathrm{im}(F_{*,(x,g)}) + \mathrm{im}(T_{*,(y,h)}) = T_{f(x)}M \oplus T_{\gamma(x)\varphi(g)}H,$$

if and only if

$$T_{f(x)}M = \mathrm{im}(f_{*,x}) + \mathrm{im}(t_{*,y}), \quad T_{\gamma(x)\varphi(g)}H = \widetilde{\mathrm{im}(\varphi)}_{\gamma(x)\varphi(g)} + \widetilde{\mathrm{im}(\psi)}_{\delta(y)\psi(h)}.$$

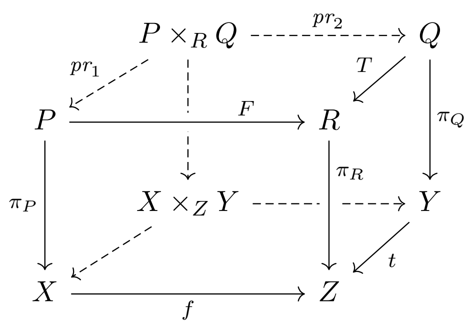
Therefore F and T are transversal if and only if f and t are transversal, and φ and ψ are transversal. □

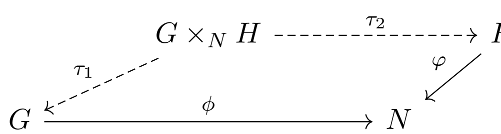
Remark A.1.1. Note that if $\psi : G \to N$ and $\varphi : H \to N$ are transversal Lie group homomorphisms then $G \times_N H$ is an embedded submanifold of $G \times H$, so that the product Lie group structure in $G \times H$ restricts to $G \times_N H$. Thus $G \times_N H$ is a Lie subgroup of $G \times H$.

A.2 Constructions in $\mathcal{PBC}$

Recall that the category of **principal bundles with connection**, denoted by $\mathcal{PBC}$, is the category whose objects are (P, π, M, G, θ) where (P, π, M, G) is a principal bundle and θ is a connection 1-form on P. A morphism between two principal bundles with connection $(P, \pi_P, M, G, \theta_P)$ and $(Q, \pi_Q, N, H, \theta_Q)$ is a morphism $(F, f, \phi) : (P, \pi_P, M, G) \to (Q, \pi_Q, N, H)$ such that $F^*\theta_Q = \phi_* \cdot \theta_P$.

Theorem A.2.1. *Let $(P, \pi_P, X, G, \theta_P), (Q, \pi_Q, Y, H, \theta_Q)$ and $(R, \pi_R, Z, N, \theta_R)$ be a principal bundles with connection and $F : P \to R$ be a bundle morphism along $\phi : G \to N$ covering $f : X \to Z$ preserving the connections and $T : Q \to R$ a bundle morphism along $\varphi : H \to N$ covering to $t : Y \to Z$ preserving the connections.*





$$T^*\theta_R = \varphi_* \cdot \theta_Q,$$

$$F^*\theta_R = \phi_* \cdot \theta_P.$$

Then if F, T are transversal, then the pullback principal bundle $(P \times_R Q, \pi_P \times_{\pi_R} \pi_Q, X \times_Z Y, G \times_N H)$ admits a connection 1-form $\theta_P \times_R \theta_Q$ such that

$$pr_1^*\theta_P = \tau_{1*} \cdot \theta_P \times_R \theta_Q, \quad pr_2^*\theta_Q = \tau_{2*} \cdot \theta_P \times_R \theta_Q.$$

Proof. By Theorem A.1.3 the pullback principal bundle is well-defined. We write by $\mathfrak{g} \oplus_{\mathfrak{n}} \mathfrak{h}$ the Lie algebra of the Lie group $G \times_N H$. Now let us see that in fact, $\theta_P \times_R \theta_Q := \mathrm{pr}_1^*\theta_P + \mathrm{pr}_2^*\theta_Q \in$

$\Omega^1_{dR}(P \times_R Q; \mathfrak{g} \oplus_{\mathfrak{n}} \mathfrak{h})$ is a connection 1-form. For this let $A + B \in \mathfrak{g} \oplus_{\mathfrak{n}} \mathfrak{h}$, then

$$\begin{aligned} i_{\widetilde{A}+\widetilde{B}}\theta_P \times_R \theta_Q =& i_{\widetilde{A}+\widetilde{B}}\mathrm{pr}_1^*\theta_P + i_{\widetilde{A}+\widetilde{B}}\mathrm{pr}_2^*\theta_Q \\ =& i_{\widetilde{A}}\theta_P + i_{\widetilde{B}}\theta_Q \\ =& A + B. \end{aligned}$$

For $(g,h) \in G \times_N H$ note that one has $\mathrm{Ad}_{(g,h)} = \mathrm{Ad}_g \oplus \mathrm{Ad}_h : \mathfrak{g} \oplus_{\mathfrak{n}} \mathfrak{h} \to \mathfrak{g} \oplus_{\mathfrak{n}} \mathfrak{h}$. Then

$$\begin{aligned} R^*_{(g,h)}(\theta_P \times_R \theta_Q) =& R^*_{(g,h)}(\mathrm{pr}_1^*\theta_P + \mathrm{pr}_2^*\theta_Q) \\ =& R^*_{(g,h)}\mathrm{pr}_1^*\theta_P + R^*_{(g,h)}\mathrm{pr}_2^*\theta_Q \\ =& \mathrm{pr}_1^*(R_g^*\theta_P) + \mathrm{pr}_2^*(R_h^*\theta_Q) \\ =& \mathrm{pr}_1^*(\mathrm{Ad}_{g^{-1}} \cdot \theta_P) + \mathrm{pr}_2^*(\mathrm{Ad}_{h^{-1}} \cdot \theta_Q) \\ =& \mathrm{Ad}_{g^{-1}} \cdot \mathrm{pr}_1^*\theta_P + \mathrm{Ad}_{h^{-1}} \cdot \mathrm{pr}_2^*\theta_Q \\ =& (\mathrm{Ad}_{g^{-1}} \oplus \mathrm{Ad}_{h^{-1}}) \cdot (\mathrm{pr}_1^*\theta_P + \mathrm{pr}_2^*\theta_Q) \\ =& \mathrm{Ad}_{(g,h)^{-1}} \cdot \theta_P \times_R \theta_Q. \end{aligned}$$

Therefore $\theta_P \times_R \theta_Q$ is a connection 1-form on $P \times_R Q$. Moreover, it is clear that $\mathrm{pr}_1^*\theta_P = \tau_{1*} \cdot \theta_P \times_R \theta_Q$ and $\mathrm{pr}_2^*\theta_Q = \tau_{2*} \cdot \theta_P \times_R \theta_Q$ from the definition. □

The following theorem relates the theory of principal bundles with connection and the theory of differential graded algebras with symmetries.

Theorem A.2.2. *Let $(P, \pi_P, M, G, \theta_P)$ and $(Q, \pi_Q, N, H, \theta_Q)$ be two principal bundles with connection and $F : P \to Q$ be a bundle morphism along $\phi : G \to H$ covering $f : M \to N$ preserving the connections, then the following diagram of dga is commutative*

$$\begin{array}{ccc} W(\mathfrak{g}) & \xrightarrow{w_{\theta_P}} & \Omega_{dR}(P) \\ {\scriptstyle \phi^*}\uparrow & & \uparrow{\scriptstyle F^*} \\ W(\mathfrak{h}) & \xrightarrow[w_{\theta_Q}]{} & \Omega_{dR}(Q), \end{array}$$

and it induces a commutative diagram in the respective basic dga's

$$\begin{array}{ccc} S(\mathfrak{g}^*)^G & \xrightarrow{w_{\theta_P}} & \Omega_{dR}(M) \\ {\scriptstyle \phi^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ S(\mathfrak{h}^*)^H & \xrightarrow[w_{\theta_Q}]{} & \Omega_{dR}(N) \end{array}$$

Proof. To see that the first diagram is commutative it is sufficient to show this in generators. Indeed, consider a basis $X_1, \ldots, X_n$ for $\mathfrak{g}$ with dual basis $\alpha^1, \ldots, \alpha^n$ and with a copy dual basis $u_1, \ldots, u_n$ and a basis $Y_1, \ldots, Y_k$ for $\mathfrak{h}$ with dual basis $\beta^1, \ldots, \beta^k$ and with a copy dual basis $v_1, \ldots, v_k$ then

$$W(\mathfrak{g}) = \wedge(\alpha^1, \ldots, \alpha^n) \otimes \mathbb{R}\left[u_1, \ldots, u_n\right], \quad W(\mathfrak{h}) = \wedge(\beta^1, \ldots, \beta^k) \otimes \mathbb{R}\left[v_1, \ldots, v_k\right].$$

Now extend the transpose application $\phi^* : \mathfrak{h}^* \to \mathfrak{g}^*$ to a differential graded map $\phi^* : W(\mathfrak{h}) \to W(\mathfrak{g})$. Note that for $\phi_* : \mathfrak{g} \to \mathfrak{h}$, $\phi_*(X_j) = \sum_{i=1}^k \phi^i_j Y_i$, we have that the transpose application $\phi^* : \mathfrak{h}^* \to \mathfrak{g}^*$ is given by $\phi^*\beta^j = \sum_{i=1}^n \phi^j_i \alpha^i$. Note that $F^*\theta_Q = \phi_* \cdot \theta_p$ implies that

$$F^*\theta_Q = F^*\left(\sum_{i=1}^k \theta_Q^i Y_i\right) = \sum_{i=1}^k F^*\theta_Q^i Y_i,$$

and

$$\begin{aligned}\phi_* \cdot \theta_P =& \phi_* \cdot \left(\sum_{j=1}^{n} \theta_P^j X_j \right) = \sum_{j=1}^{n} \theta_P^j \left(\phi_* \cdot X_j \right) \\ =& \sum_{j=1}^{n} \theta_P^j \sum_{i=1}^{k} \phi_j^i Y_i = \sum_{i=1}^{k} \left(\sum_{j=1}^{n} \phi_j^i \theta_P^j \right) Y_i,\end{aligned}$$

hence

$$F^* \theta_Q^i = \sum_{j=1}^{n} \phi_j^i \theta_P^j, \quad i = 1, \dots, k.$$

Thus

$$\begin{aligned}F^* \circ w_{\theta_Q}(\beta^i) =& F^*(\theta_Q^i) = \sum_{j=1}^{n} \phi_j^i \theta_P^j = \sum_{j=1}^{n} \phi_j^i w_{\theta_P}(\alpha^j) \\ =& w_{\theta_P} \left(\sum_{j=1}^{n} \phi_j^i \alpha^j \right) = w_{\theta_P}(\phi^*(\beta^i)) = w_{\theta_P} \circ \phi^*(\beta^i).\end{aligned}$$

Now recall that Equation (1.1.3) gives us that $F^* \Omega_Q = \phi_* \cdot \Omega_P$, and since $\phi^* : W(\mathfrak{h}) \to W(\mathfrak{g})$ is a dga morphism, then

$$\begin{aligned}\phi^*(v_j) =& \phi^*(\mathbf{d}(\beta^j)) = \phi^*(d\beta^j - d_{CE}\beta^j) = d\phi^* \beta^j - d_{CE} \phi^* \beta^j \\ =& d \left(\sum_{i=1}^{n} \phi_i^j \alpha^i \right) - d_{CE} \left(\sum_{i=1}^{n} \phi_i^j \alpha^i \right) \\ =& \sum_{i=1}^{n} \phi_i^j (d\alpha^i - d_{CE}\alpha^i) = \sum_{i=1}^{n} \phi_i^j \mathbf{d}\alpha^i = \sum_{i=1}^{n} \phi_i^j u_i.\end{aligned}$$

Hence

$$\phi^*(v_j) = \sum_{i=1}^{n} \phi_i^j u_i, \quad j = 1, \dots, k.$$

From this, it is straightforward to conclude that $F^* \circ w_{\theta_Q}(v_j) = w_{\theta_P} \circ \phi^*(v_j)$. Thus the diagram is commutative. The diagram in basic elements is commutative, because the following conditions are satisfied for $X_i \in \mathfrak{g}$ with $j = 1, \dots, n$

$$i_{\widetilde{X_j}} \circ F^* = F^* \circ i_{\widetilde{\phi_* X_j}}, \quad i_{X_j} \circ \phi^* = \phi^* \circ i_{\phi_*(X_j)},$$

and for all $g \in G$,

$$R_g^* \circ F^* = F^* \circ R_{\phi(g)}^*, \quad \mathrm{Ad}_g^* \circ \phi^* = \phi^* \circ \mathrm{Ad}_{\phi(g)}^*.$$

□

Appendix B

The adjoint representation of a semi-direct product of Lie groups

B.1 The adjoint representation of semi-direct product of Lie groups

Let G and H be two Lie groups, and $\alpha : G \to \mathrm{Aut}(H)$ be a homomorphism of Lie groups. Let us consider its semi-direct product $H \rtimes_\alpha G$. For two elements $(h, g), (x, y) \in H \rtimes_\alpha G$ their product is given by

$$(h, g)(x, y) = (h\alpha_g(x), gy).$$

The conjugation by (h, g) is given by

$$\begin{aligned} c_{(h,g)}(x, y) &= (h, g)(x, y)(h, g)^{-1} \\ &= (h\alpha_g(x), gy)(\alpha_{g^{-1}}(h^{-1}), g^{-1}), \quad \text{by} \quad (h, g)^{-1} := (\alpha_{g^{-1}}(h^{-1}), g^{-1}), \\ &= (h\alpha_g(x)\alpha_{gyg^{-1}}(h^{-1}), gyg^{-1}) \\ &= ((c_h \circ \alpha_g)(x)(\tilde{\alpha}_h \circ c_g)(y), c_g(y)), \end{aligned} \tag{B.1}$$

where for all $h \in H$ the morphism $\tilde{\alpha}_h : G \to H, g \mapsto h\alpha_g(h^{-1})$. Infinitesimally, for $X \in T_eG$ and $Y \in T_eH$ the adjoint at (h, g) is

$$\mathrm{Ad}_{(h,g)}(X + Y) = (\mathrm{Ad}_h((\alpha_g)_{*,e}(X)) + (\tilde{\alpha}_h)_{*,e}(\mathrm{Ad}_g Y), \mathrm{Ad}_g Y).$$

In particular, for $(h, e) \in H \rtimes_\alpha G$

$$\mathrm{Ad}_{(h,e)}(X + Y) = (\mathrm{Ad}_h(X) + (\tilde{\alpha}_h)_{*,e}(Y), Y),$$

and for $(e, g) \in H \rtimes_\alpha G$

$$\mathrm{Ad}_{(e,g)}(X + Y) = ((\alpha_g)_{*,e}(X), \mathrm{Ad}_g(Y)).$$

The **left invariant Maurer-Cartan 1-form** of the semi-direct Lie group $H \rtimes_\alpha G$ is

$$\theta_{MC}^{H \rtimes_\alpha G} = (\alpha_{\mathrm{pr}_2^{-1}})_*(\mathrm{pr}_1^* \theta_{MC}^H) + \mathrm{pr}_2^* \theta_{MC}^G,$$

where θ_{MC}^G, and θ_{MC}^H are the left invariant Maurer-Cartan 1-forms of G and H, respectively.

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