Representations of low copolarity and the orbifold structures of $\mathrm{Sp}(2) /\!/ \mathrm{Sp}(1)$

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A casa grande surta quando a senzala aprende a ler.

Agradecimentos

É isso aí você não pode parar
Esperar o tempo ruim vir te abraçar
Acreditar que sonhar sempre é preciso
É o que mantém os irmãos vivos
Mano Brown

Essa é a tese de doutoramento de um filho de camelôs muito orgulhoso do esforço que seus pais tiveram para que ele pudesse seguir seus sonhos.

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Resumo

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Este trabalho tem dois objetivos. Primeiramente estudamos representações de grupos de Lie compactos pela análise de seu quociente, visto como espaço métrico. Como resultado classificamos representações irredutíveis e que admitem redução não trivial de copolaridade variando entre 7 e 9. Em segundo lugar estudamos a conexão entre biquocientes e orbifolds, que ainda é um dos principais meios na busca por novos exemplos de orbifolds de curvatura positiva. Como resultado, classificamos do ponto de vista topológico os biquocientes de $\mathrm{Sp}(2)$. Dentre estes está a esfera exótica de Gromoll-Meyer, já bastante conhecida na literatura. Mas há também dois novos exemplos, dos quais um foi demonstrado que admite uma métrica de curvatura almost positive.

Palavras-chave: grupos de Lie compactos, representações de grupos de Lie, copolaridade, biquocientes, orbifolds, geometria Riemanniana.

Abstract

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The aim of this work is twofold. Firstly, we study representations of compact Lie groups from the point of view of their quotient spaces, considered as metric spaces. As result we classified irreducible representations that admit a non-trivial reduction of copolarities varying from 7 to 9. Secondly, we study the connection between biquotients and orbifolds, which still is one of the main techniques used to construct new examples of positively curved orbifolds. As result, we classified the biquotients of Sp(2) from a topological point of view. The Gromoll-Meyer sphere figures among them, which is well-known in the literature. But there is yet two new examples, of which we constructed for one of them a metric of almost-positive curvature.

Keywords: compact Lie groups, Lie groups representations, copolarity, biquotients, orbifolds, Riemannian geometry.

Contents

Chapter 1

Introduction

This work is divided in two parts.

In the first part we present a partial answer the following question.

Question 1.1. *What are the irreducible representations of low abstract copolarity ?*

First of all, let us contextualize this question.

One of the most important invariants of an orthogonal representation $\rho : G \to \mathrm{O}(V)$ is the quotient metric space V/G, since it encodes the information of horizontal geometry of V with respect to the orbits of the action. For this reason we define two representations $\rho_i : G_i \to \mathrm{O}(V_i)$, $i = 1, 2$, to be *quotient-equivalent* if V_1/G_1 is isometric to V_2/G_2. And, for defining a more restrictive (but quite useful) equivalence relation, we say that those representations are *orbit-equivalent* if there is an isometry from V_1 to V_2 mapping G_1-orbits onto G_2-orbits. It is clear from the definition that any representation is orbit-equivalent to its effectivization. Thus we will restrict our analysis to effective representations.

Many interesting properties are understood by studying the quotient space from a metric point of view. In fact, Lemma 2.1, which is shown in [GL14], states that for a given representation (G, V), all G-invariant subspace of V can be recognized metrically; in particular, reducibility of a representation is an invariant in its quotient-equivalence class.

It is then interesting to ask what are the algebraic properties that can be recovered from the metric of V/G. But answering this question is not a trivial task. For instance, $\dim V$ may not be constant among the the quotient equivalence class. But the codimension of the principal orbits of the given representation, which is called *cohomogeneity*, always agrees with the topological dimension of the quotient space and is therefore an invariant among the quotient equivalence classes.

Recently, [Goz18], irreducible representations of compact Lie groups of cohomogeneity up to 8 have been classified. And this classification will help us to fulfill our task to give an partial answer to Question 1.1. This fact, amongst others in the literature (see [GL14]), suggests that there is some relation between the algebraic invariants of quotient equivalence classes of compact Lie groups that are presented here.

Remember that a representation (G, V) is *polar* when there exists a subspace Σ of V, called *section*, that meets all G-orbits orthogonally. It is clear that the dimension of the section of a polar representation coincides with the cohomogeneity of the representation. We also know that an equivalent definition to a representation of a compact Lie group being polar is that it is quotient equivalent to a representation of a finite group. In fact we have a more accurate result: by [Dad85], we know that every polar representation of a connected compact Lie group is always orbit equivalent to a *s-representation*; that is an isotropy representation of a symmetric space and is also a representation of finite groups.

One possible and useful approach to understand these invariants is looking for the "simplest" action on each class, and we may do that by looking for the action whose group has the lowest dimension among the groups of the representations in its class. For example, by the facts exposed in the last paragraph, the polar case can be understood by the studying the representations of finite groups. For this reason we say the representation $\rho_1 : G_1 \to \mathrm{O}(V_1)$ is a *reduction* of a representation $\rho_2 : G_2 \to \mathrm{O}(V_2)$ in its quotient-equivalence class if $\dim G_1 < \dim G_2$ and we say that a representation $\rho : G \to \mathrm{O}(V)$ is a *minimal reduction* of any element of its class if it does not admit any reduction (the nomenclature may be a little confusing, since an irreducible representation may admit a reduction in the quotient-equivalence class). The $\dim G$ of a minimal reduction is an invariant of the quotient-equivalence class called *abstract copolarity.* Observe that polar representations have abstract copolarity 0.

But a representation may fail to be polar, or in other words, it may fail on having a section. For that reason, we say that a totally geodesic, complete e connected submanifold $\Sigma \subset V$ which intersects every G-orbit and such that its tangent space contains the normal spaces to the principal orbits with codimension k is called a *generalized k-section.*

Observe that the whole V is a generalized section and that the intersection of two generalized sections through (that contains) a same point is also a generalized section through that point. Thus, trough every point there is a minimal generalized section. The excess of a minimal generalized section over the cohomogeneity is called *copolarity* of the action (G, V). Furthermore, if V contains no generalized k-sections but itself we say that the action (G, V) has *trivial copolarity.* Observe that the action is polar when the copolarity is 0.

Given a generalized k-section Σ of the action (G, V), the group $\overline{N} = N/Z$, in which N (respectively, Z) is the subgroup of G that preserves Σ (respectively, fixes Σ pointwisely), acts on Σ and the actions (G, V) and $(\overline{N}, \Sigma)$ are quotient equivalent. Due to the minimality of Σ, the action of $\overline{N}$ has trivial principal isotropy groups, thus the copolarity of (G, V) is equal to $\dim \overline{N}$. So we conclude that abstract copolarity is bounded above by copolarity. This leads us to the following question.

Question 1.2. *Does a minimal reduction of a representation always come from a minimal generalized section?*

Observe that the set V^H of points in V fixed by a principal isotropy group H provides a proper generalized section and the subquotient $\overline{N} = N_G(H)/Z_G(H)$ of G acts on V^H and $V/G = V^H/\overline{N}$; here $N_G(H)$ and $Z_G(H)$ denote the normalizer and the centralizer of H in G, respectively. This reduction is known as Luna-Richardson-Straume reduction. Also, for this reason, a reduced representation has trivial principal isotropy groups.

One can often apply a LRS reduction after passing to the maximal group with same orbits. This enlarges the principal isotropy groups and shrinks the fixed point set. This naturally leads us to the following question.

Question 1.3. *Does every minimal generalized section is obtained via LRS reduction, after passing to the maximal group in the orbit-equivalence class?*

Observe that in the polar case the copolarity and the abstract copolarity coincide. This is also true when the irreducible representation is non-polar and admits a toric reduction, that is, when it is quotient equivalent to a representation of a finite extension of a torus T^k, which can be seen by the classification of such representations in [GL15]. Furthermore, Gorodski and Gozzi have shown in [GG18] that this also holds for a representation admitting a quaternion-toric reduction, that is, when it is quotient equivalent to a representation of a group whose identity component is of the form $\mathrm{Sp}(1) \times \cdots \times \mathrm{Sp}(1)$. But it is still an open question whether or not copolarity and abstract copolarity always coincide.

Also, the relations between copolarity and cohomogeneity are not fully understood yet. In order to understand their relation, it was shown in [GL14] that if an irreducible representation has copolarity $1 \leq k \leq 6$ then it admits a toric reduction and its cohomogeneity is $k + 2$. But when the representation is polar, the cohomogeneity can by arbitrary. Furthermore, Gorodski and Lytchak have given an example also in [GL14] of an irreducible representation of copolarity 7 and cohomogeneity 5, which is discribed thoroughly in section 6.1.3 below.

Our main result in this work is the classification of non-polar, non-reduced representations of abstract copolarity varying from 7 up to 9, by looking at the reduced representations of each class, which is done in Chapter 6. More specifically, we prove:

Theorem 1.4. *Let $\rho : G \to \mathrm{O}(V)$ be a non-polar, non-reduced, irreducible representation of a compact connected Lie group G on the Euclidean space V. Assume the abstract copolarity of ρ is 7, 8 or 9. Then ρ is either toric, quaternion-toric, or equivalent to $(\mathrm{U}(3) \times \mathrm{Sp}(2), \mathbb{C}^3 \otimes_{\mathbb{C}} \mathbb{C}^4)$.*

It is a remarkable fact that for every representation in this theorem, the copolarity coincides with the copolarity. And, to prove it, we analyzed what are the reduced representations that are non-trivial reductions of an irreducible representation of some connected group, which leads us to the following questions.

Question 1.5. *Which representations admit reductions? Which representations can be minimal reductions of some representation?*

Question 1.6. *If $\rho_i : G_i \to \mathrm{O}(V_i)$ for $i = 1, 2$ are minimal reductions of the same representation (resp. quotient-equivalent and reduced) is it true that $\rho_1(G_1)$ and $\rho_2(G_2)$ (or at least their identity components) must be conjugate by an isometry $V_1 \cong V_2$?*

Let (G, V) be a reduced representation that is a non-trivial reduction of an irreducible representation of a connected compact Lie group. As it is reduced, it must have trivial principal isotropy groups. Furthermore, using propositions 2.3 and 2.5, we can conclude that either the group G must be connected and V/G must have a non-trivial boundary, or G^0 must be normalized by an involution in $\mathrm{O}(V)$ that acts as a reflection on V/G^0.

The case when G^0 acts reducibly on V is well understood and, as shown in [GL14], it is equivalent to the action of a finite extension of a maximal torus of $\mathrm{SU}(k+1)$ on $\mathbb{C}^{k+1}$; in which k is the abstract copolarity of the representation (G, V).

So, to our case, when $k = 7, 8, 9$, we are left to understand the case when G^0 acts irreducibly on V. Since $\dim G = 7, 8, 9$, using the effectiveness of the representation and the classification of compact Lie groups, we can conclude that G^0 is covered by $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ (if $k = 7$), by $\mathrm{SU}(3)$ (if $k = 8$) and by $\mathrm{U}(3)$ or by $SU(2) \times SU(2) \times SU(2)$ (if $k = 9$), respectively. The case when G^0 is covered by $SU(2) \times SU(2) \times SU(2)$ is already studied in [Goz18]. In case G is connected, the presence of boundary in V/G will impose severe restrictions to the representation, leaving us to analyze but a few specific cases. In case G is disconnected, the existence of an involution in $\mathrm{O}(V)$ that normalizes G^0 and that acts as a reflection on V/G^0 will also impose severe restrictions, also leaving us to but a few other cases to analyze.

In the second part of this work we present a new family of *biquotients.*

A *biquotient* is a generalization of the concept of homogeneous space defined as follows. Fix a compact Lie group G. Then, a closed subgroup U of $G \times G$ acts on G via $(u_1, u_2) \cdot g = u_1 g u_2^{-1}$. The quotient of this action is called a *biquotient* and it is denoted by $G//U$.

It is clear that asking $G//U$ to be an orbifold is equivalent to asking that each point of G has a finite istropy group, since the orbifold group of a point's projection is the isotropy group (or the stabilizer) of this point, that is $\Gamma_{\pi(g)} = \mathrm{Stab}(g)$. And this condition is relatively easy to verify, since this isotropy group is finite exactly when its Lie algebra is trivial, and its Lie algebra is formed by $(X_1, X_2) \in \mathfrak{u}$ such that $X_1 = \mathrm{Ad}(g)X_2$. Using the fact that every element in $\mathfrak{u}$ is conjugate to an element of a fixed Cartan subalgebra and that isotropy groups occurs in conjugacy classes, we have that $G//U$ is an orbifold if, and only if, for all non-zero $(X_1, X_2) \in \mathfrak{t}_{\mathfrak{u}} \subset \mathfrak{u} \subset \mathfrak{g} \oplus \mathfrak{g}$ and for all $g \in G$, $X_1 \neq \mathrm{Ad}(g)X_2$.

Therefore, biquotients form a powerful tool to produce new examples of non-negatively curved orbifolds – since G is compact and thus admits a bi-invariant metric, so the O'Neill's formula guarantees that the biquotient does not admit a plane with negative sectional curvature anywhere.

But by deforming the bi-invariant metric of G by shrinking the vertical directions with respect to the action of a group K on G we may expect to obtain a better result than a non-negative sectional curvature in $G//U$. Such a deformation is called a *Cheeger deformation* and is discussed in Chapter 7.

Biquotients were first intruduced in [GM74], in which Gromoll and Meyer showed that an exotic 7-dimensional sphere is diffeomorphic to a biquotient $\mathrm{Sp}(2)//\,\mathrm{Sp}(1)$. This exotic sphere is known as

Gromoll-Meyer sphere in the literature. De Vito have classified simply connected biquotients that are manifolds of dimension at most 7 in [DeV11]. But there are infinitely many diffeomorphisms classes of orbifold biquotients even in low dimension. For this reason, Yeroshkin has studied biquotients arising from $\mathrm{SU}(3)$, [Yer14]. In the same direction we study here biquotients of $\mathrm{Sp}(2)$.

In Chapter 8 we study the orbifold structures of $\mathrm{Sp}(2)/\!/\,\mathrm{SU}(2)$; here $\mathrm{SU}(2)$ is embedded as a subgroup of $\mathrm{Sp}(2)\times\mathrm{Sp}(2)$. Such embeddings can be described by studying the symplectic representations of $\mathrm{SU}(2)$. The first example of such biquotient is a manifold. Actually, it is the Gromoll-Meyer sphere. The further two examples shown are orbifolds new in the literature, and we discuss its topology thoroughly here. Also, using a Cheeger deformation with respect to the action of symmetric pair $(\mathrm{Sp}(2), K)$, and improving the techniques used in [Yer14], we construct an *almost-positive* (i.e., the points that do not admit flat planes form a open dense subset) metric on the second example. Whereas there are no severe restrictions to an orbifold to have (almost-)positive sectional curvature, there are only few known examples of such.

The same technique has not been sufficient to construct any metric with almost-positive sectional curvature in the last example.

Part I

Representations of Small Copolarity

Chapter 2

Preliminaries

In this chapter we will give a summary on some preliminar topics to the full understanding of this thesis. We assume that the reader is familiarized with isometric group actions and representations of compact Lie groups.

In the first section of the present chapter we remember some basic results on isometric group actions and their stratification. In the second section we present some specific concepts to the theory of algebraic invariants on the orbit equivalence class of an action. And in the latter section we remember some basic results on the orbifold theory, which will be crucial to the understanding of the second part of this work.

2.1 Stratification and boundary points

In this section we will fix a compact Lie group G acting effectively by isometries on a connected Riemannian manifold M. As the action $\rho = (G, M)$ (which sometimes will be denoted simply by ρ or simply by (G, M)) is effective, we will identify G with its image in $\mathrm{Iso}(M)$, when convinient. We will also denote by X the orbit space M/G.

For a given point $p \in M$, we say that an embedded submanifold S of M containing p is a *slice* through p if:

1. $T_pM = d\mu_p\mathfrak{g} \oplus T_pS$ and $T_qM = d\mu_q\mathfrak{g} + T_qS$, for all $q \in S$;
2. S is invariant under G_p;
3. If $q \in S$ and $g \in G$ are such that $g \cdot q \in S$, then $g \in G_p$;

where $\mu_p : G \to M$ is the product map $g \mapsto g \cdot p$ and $G_p = \{g \in G : g \cdot p = p\}$ is the isotropy group of p.

As we are interested in actions of compact Lie groups, the Slice Theorem (see Theorem 3.49 of [AB15]) guarantees the existence of a slice through every point of M.

For a slice S we call the set $G(S) := \{g \cdot s : g \in G,\ s \in S\}$ by *tubular neighborhood* of p in M.

Observe that the restriction of the action of G on S defines a G-equivariant diffeomorphism between $G(S)$ and the total space of the *associated fiber bundle*

$$S \to G \times_{G_p} S \to G/G_p,$$

in which $G \times_{G_p} S$ is the orbit space of the action $(G_p, G \times S)$ given by $h \cdot (g, q) = (gh^{-1}, hq)$.

Note that the isotropy group G_p acts linearly on T_pM by $g \cdot v = dg_pv$, for every $g \in G_p$ and $v \in T_pM$. Since G_p leaves the orbit $G(p)$ of p in M invariant and $G_p < \mathrm{Iso}(M)$, this action leaves $T_pG(p)$ and its orthogonal complement in T_pM, namely $\nu_pG(p)$, invariant. The action of G_p on $\nu_pG(p)$ is called *slice representation*. Observe that, by the Normal Neighborhood Theorem, there is a radius $\epsilon > 0$ such that the ball $B_\epsilon(0) \subset \nu_pG(p)$ is isometrically mapped by the exponential map onto a slice through p.

We say that two points $p, q \in M$ have the same *orbit type* if their isotropy groups G_p and G_q are conjugate in G. And we define the *stratum* of p, denoted by $\mathrm{St}(p)$, to be the connected component (that contains p) of the elements $q \in M$ with same orbit type as p. The canonical projection of $\mathrm{St}(p)$ in X is called *stratum* through $x = G \cdot p$ and is denoted by $\mathrm{St}_X(x)$.

Observe that, as the orbit types are invariant along the orbits, since $G_{gp} = gG_pg^{-1}$, the orbit types of M around p, that are precisely the orbit types of $G(S)$, are determined by the orbit types of S; which in turn is determined by the orbit types of the slice representation. Indeed, if two points $r, s \in S$ have the same G_p-orbit type, then they also have the same G-orbit type.

For sake of convenience, denote $H := G_p$. Let S be a slice through p in M and $\mathcal{I} = G(S)$ a tubular neighborhood of p. Then, a point $q = g \cdot s \in \mathcal{I}$, for $g \in G$ and $s \in S$, is fixed by an element $h \in G$ if and only $h \in gH_sg^{-1}$. So, G_q is conjugate to the subgroup H_s of H. Furthermore, $q \in \mathrm{St}(p)$ if and only if G_q is conjugate to H in G, or, equivalently, if and only if H_s is conjugate to H in G. Since $H_s \subset H$, we deduce that $H_s = H$, which implies that s is fixed H. If S^H denotes the subset of S formed by the points which are fixed by H, it follows that $\mathrm{St(p)} \cap \mathcal{I}$ can be identified with $G \times_H S^H = \frac{G}{H} \times S^H$.

Fix M_0^H the connected component of the fix point set M^H through p. Consider now the surjective map

$$\begin{aligned} \eta : \tfrac{G}{H} \times (M_0^H \cap \mathcal{I}) &\to \mathrm{St}(p) \cap \mathcal{I} \\ (gH, q) &\mapsto g \cdot q \end{aligned}$$

As the normalizer $N_G(H)$ acts on $M_0^H \cap \mathcal{I}$ we deduce that η descends to a diffeomorphism

$$\frac{G}{H} \times_{N_G(H)/H} (M_0^H \cap \mathcal{I}) \to \mathrm{St}(p) \cap \mathcal{I};$$

in which $\frac{G}{H} \times_{N_G(H)/H} (M_0^H \cap \mathcal{I})$ is the orbit space of the action $(N_G(H)/H, \frac{G}{H} \times (M_0^H \cap \mathcal{I}))$ defined by $nH \cdot (gH, q) = (gn^{-1}H, n \cdot q)$. Therefore,

$$\dim \mathrm{St}(p) = \dim G + \dim M_0^H - \dim N_G(H). \tag{2.1}$$

From the fact that $\mathrm{St}(p) \cap \mathcal{I}$ can be identified with $G \times_H S^H$, as $T_pS = \nu_pG(p)$, we deduce that the tangent space to $\mathrm{St}(p)$ is identified with $T_pG(p) \oplus (\nu_pG(p))^H$. We also deduce from this fact that the projection $\mathrm{St}(p) \to \mathrm{St}_X(x)$ gives us a bundle

$$G/H \to \mathrm{St}(p) \to \mathrm{St}_X(x)$$

and that $\mathrm{St}_X(x)$ is a totally geodesic submanifold of the space X, hence the tangent space of $\mathrm{St}_X(x)$ at x identifies with $(\nu_pG(p))^H$.

Let $(\nu_pG(p))^\dagger$ be the orthogonal complement of $(\nu_pG(p))^H$ in $\nu_pG(p)$. We say that p is *regular* if $(\nu_pG(p))^\dagger$ is trivial. It is called *exceptional* if the action of $H = G_p$ on $(\nu_pG(p))^\dagger$ has discrete orbits. If it is neither regular nor exceptional, it is called *singular*. It is an easy work to verify that regular points are exactly those with minimal isotropy groups, that is, if K isotropy group of the action (G, M) and g is a regular point, then H is conjugate in G to a subgroup of K; such a isotropy group H is called a *principal isotropy group*. Moreover, q is exceptional if and only G_q^0 is conjugate to H^0 in G and G_q has more connected components than H and q is singular if and only if $\dim G_q > \dim G_p$ for any regular point p.

As G is compact, there is an isotropy group K which has the smallest number of connected components among the isotropy groups with lowest dimension. This group K will be a principal isotropy group and the regular points are exactly those whose isotropy groups are conjugate to K in G. Therefore, the set M_{reg} of all regular points in M is non-empty. Furthermore, given an open set $U \subset M$, the points whose isotropy group has the smallest number of connected components among the isotropy groups with lowest dimension among U are regular points. Also, every point in a slice of a regular point is regular. That said we can easily conclude that M_{reg} is an open dense in M. Now, using the slice representation in a regular point and the fact that for any isometric action

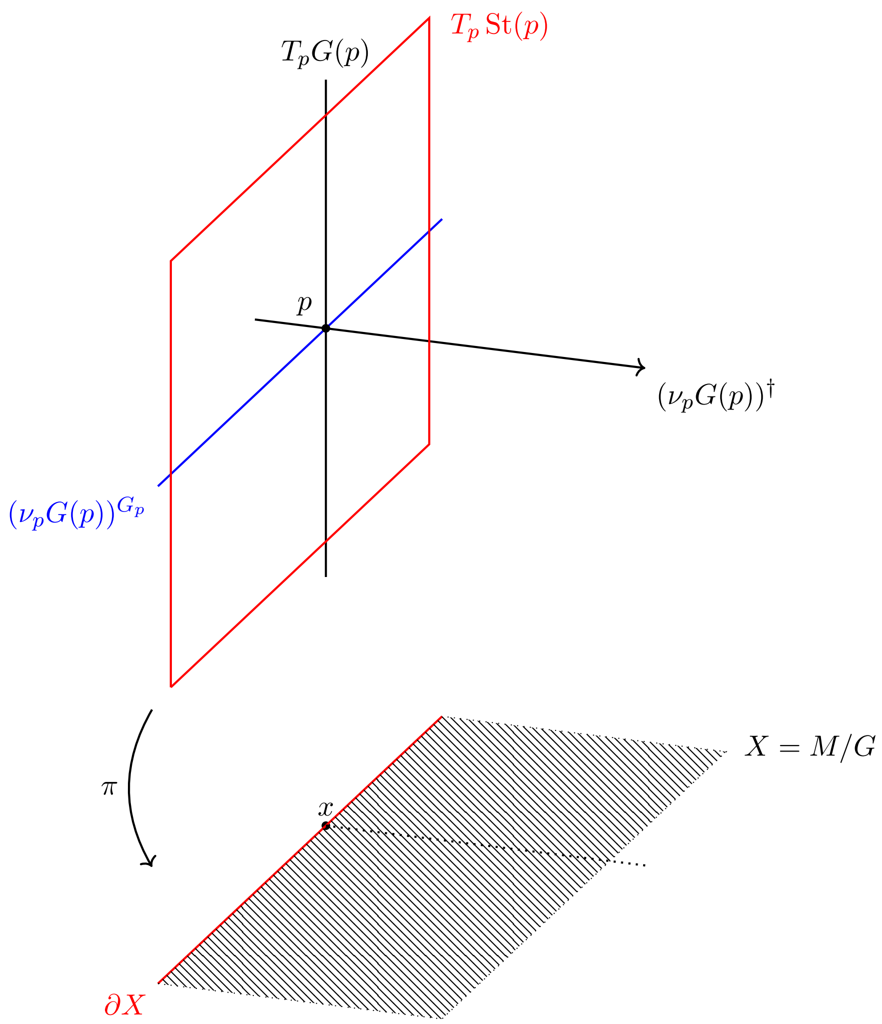

Figure 2.1: *Linear visualization of the projection at a G-important point.*

$(K, \mathbb{R}^m)$, $\mathbb{R}^m_{\mathrm{reg}}/K$is path connected, the projection $X_{\mathrm{reg}} = M_{\mathrm{reg}}/G$ is connected; X_{reg} is exactly the stratum corresponding to the unique conjugacy class of isotropy groups.

We call the codimension of a regular orbit on M by *cohomogeneity* of the action (G, M), which is the number $\mathrm{chm}(G, M)$.

Observe that

$$\mathrm{chm}(G_p, (\nu_pG(p))^\dagger) = \dim((\nu_pG(p))^\dagger/G_p).$$

So we define the *quotient codimension* of $\mathrm{St}_X(x)$ (remember that $x = G \cdot p$) as the number

$$\mathrm{qcodim}(\mathrm{St}(x)) = \mathrm{chm}(G_p, (\nu_pG(p))^\dagger).$$

The *boundary* of X, denoted by ∂X, is defined to be the closure of all strata of quotient codimension 1. A point p which is projected to a stratum of quotient codimension 1 in X is called a *G-important point*. As discussed below, a G-important point has a neighborhood that is isometric to the quotient of a Riemannian manifold by an isometric reflection on a hyperplane; and this property will be useful enough to justify the *important* in the name.

Observe that if $p\,\mathrm{im}\,M$ is a G-important point, as $\mathrm{chm}(G_p, (\nu_pG(p))^\dagger) = 1$, G_p acts transitively on the unity sphere $S((\nu_pG(p))^\dagger)$; because, as G_p is compact, its orbits are closed, and as its orbits have the same dimension as the spheres and are open, they must be exactly the spheres (see Figure 2.1).

Let G' be a normal subgroup of G such that the quotient $\Gamma = G/G'$ is finite. Then, Γ acts by isometries in $X' = M/G'$ and $X = X'/\Gamma$. Thus, $(G')^0 = G^0$ and, therefore, the orbits $G'(p)$ and

$G(p)$ have same connected components through p. This way, $\nu_p G(p) = \nu_p G'(p)$. Also, as $G_p^0 = (G_p')^0$, the orbits associated to their respective slice representations, i.e., their actions on the normal space to the tangent space to their respective orbits, have same dimension. And $(\nu_p G(p))^{G_p} \subset (\nu_p G(p))^{G_p'}$. As the orbits of their associated slice representations have same dimension, the cohomogeneity of the actions $(G_p', (\nu_p G(p))^\dagger)$ and $(G_p, (\nu_p G(p))^\dagger)$ are equal. Thus, $\dim \mathrm{St}_X(\pi(p)) \leq \dim \mathrm{St}_{X'}(\pi'(p))$. Thus, a stratum of X' is mapped to a union of strata of X under the action of Γ and, as Γ is finite, $\dim X = \dim X'$, we have that $\pi(\partial X') \subset \partial X$; in which $\pi : X' \to X$ is the canonical projection. Thus, if p is a G-important point which is not G'-important, then it must be G'-regular.

2.2 Reductions and (Abstract) Copolarity

A *generalized k-section* of the action G on M is a complete, connected and totally geodesic submanifold Σ which intersects every G-orbit and such that its tangent space contains $\nu_p G(p)$ as a k comdimensional subspace at every for every $p \in M_{\mathrm{reg}} \cap \Sigma$. If we are not interested in the number k, Σ will be simply called a *generalized section*. A generalized 0-section is just called a *section*. An action which admits a section is called *polar*.

Observe that the whole M is a generalized section. Furthermore, the intersection of two generalized sections through a given regular point is also a generalized section through this point. Thus, through any regular point p there is a smallest integer k_0 such that there is exactly one generalized k_0-section through p. In this case k_0 is called *copolarity* of the action (G, M). In addition, we say that the action has *trivial copolarity* if M contains no proper generalized sections.

Also, if Σ is a minimal generalized section of the action (G, M), then the effectivization of the action of the group $G_\Sigma := \{g \in G : g\Sigma = \Sigma\}$ on Σ, which will be denoted by $(\overline{G}_\Sigma, \Sigma)$, is such that the canonical map $\Sigma/\overline{G}_\Sigma \to M/G$ is an isometry. Observe that the copolarity of (G, M) is $\dim \Sigma - \dim(M/G)$. Due to the minimality of Σ, the action of $\overline{G}_\Sigma$ has trivial copolarity, thus the copolarity of (G, M) equals the dimension of $\overline{G}_\Sigma$.

If the action (G, M) has non-trivial principal isotropy groups, then the connected components of the set of fixed points of any principal isotropy group are generalized sections. Thus, such an action does not have trivial copolarity.

Two Riemannian actions (G, M) and (G', M') are called *orbit-equivalent* if there is an isometry $M \xrightarrow{F} M'$ such that $F(G(p)) = G'(F(p))$; that is, F maps orbits to orbits. We can identify M and M' via F and, then, view G and G' as subgroups of $\mathrm{Iso}(M)$. Thus, in this case, $G(p) = G'(p)$, which implies that (G, M) (and therefore (G', M)) is orbit equivalent to the action of the closure of the group generated by G and G' on M, and this later action have no nontrivial principal isotropy groups if G and G' are not the same subgroup of $\mathrm{Iso}(M)$. Therefore, in this case, the actions of G and G' have non-trivial copolarity.

The actions $\rho = (G, M)$ and $\rho' = (G', M')$ are said to be *quotient-equivalent* if there is an isometry between M/G and M'/G'. Furthermore, if $\dim G' < \dim G$, we say that ρ' is a *reduction* of ρ. An action which is minimal in its quotient-equivalence class is said to be *reduced* and is also called a *minimal reduction* of the any element of its class. The dimension of a minimal reduction of ρ is called as the *abstract copolarity* of ρ. The crux of the study of quotient-equivalence classes is that the orbit space somehow determines the transverse geometry of the action.

Lemma 2.1. *Let $\rho : K \to \mathrm{O}(U)$ and $\rho' : K' \to \mathrm{O}(U')$ be quotient-equivalent representations, with projections $\pi : U \to U/K$ and $\pi' : U' \to U'/K'$. Then ρ is irreducible if and only if ρ' is. More precisely, if $I : U/K \to U'/K'$ is an (origin preserving) isometry, then for any K-invariant subspace V of U the subset $\pi'^{-1}(I(\pi(V)))$ is K'-invariant subspace of U'.*

As mentioned before, if Σ is a generalized section of (G, M) then the actions $(\overline{G}_\Sigma, \Sigma)$ and (G, M) are quotient equivalents. Thus, the abstract copolarity is bounded above by the copolarity. In case this action is polar we have that $\dim \overline{G}_\Sigma = 0$. Thus, the polar actions of compact groups are quotient equivalent to actions of finite groups. The converse is also true, and a brief argument is shown in the end of section 2 of [GL14].

2.2.1 Basic observations on boundary pointss

Observe that if the action (G, M) has trivial principal isotropy groups, then so does its slices representations. Thus, if p is a G-important point, as G_p acts transitively on a sphere with trivial isotropy groups, it is diffeomorphic to this sphere. It is a well-known result in Lie group Theory that such a sphere must be S^a with $a \in \{0, 1, 3\}$. Clearly, in this case, (see Figure 2.1)

$$\dim \mathrm{St}(p) = \dim V - a - 1 = f_p + \dim(G) - \dim N_G(G_p); \tag{2.2}$$

f_p denotes the dimension of the connected component of the set of fixed points of G_p. Also, if $a \in \{1, 3\}$, then $G_p \subset G^0$ and, thus, p is also G^0-important. Conversely, if p is a G^0-important point, then it cannot lie on an exceptional orbit, due to Lemma 2.6. Therefore, the slice representation at p cannot have discrete orbits, which implies that $a \neq 0$.

This way, p is G-important and not G^0-important if and only if $G_p = S^0$. Thus, there is exactly one element $w \in G_p$ which is not the identity. This element w is an involution in $G \setminus G^0$ which normalizes G^0 and acts as a reflection on M/G^0.

Assume now that $\Gamma = G/G^0$ acts as a reflection group on M/G^0, i.e., Γ is generated by reflections on M/G^0. Since the action has trivial principal isotropy groups, the action $(\Gamma, M/G^0)$ is effective. Indeed, suppose that g is in the innefective kernel of the action of Γ on M/G^0. Then, for any regular point p, there is a $h \in G^0$ such that $g \cdot p = h \cdot p$, which implies that $h^{-1}g \cdot p = p$, or, equivalently, that $h^{-1}g \in G_p = \{1\}$. This means that $g \in G^0$ and that g is the unit in Γ.

Now, take $\eta \in G/G^0$ a reflection and let $x \in V/G^0$ be a principal fixed by η point and let $p \in M$ be a preimage of x. Thus, by the previous discussion, p is G^0-regular and G-important. Thus, G_p has only one non-trivial point w, which is equal to $g_0\eta$, for some $g_0 \in G^0$. This element w is an involution (i.e., $w^2 = 1$) and comparing (2.1) with (2.2) for $\dim \mathrm{St}(p)$ we have:

$$\dim M - \dim M^w = \dim G - \dim Z_G(w) + 1. \tag{2.3}$$

We call the involutions which can be constructed as w was by *nice involutions*. Furthermore, since w and η are equivalent modulo G^0, we conclude that nice involutions generate Γ.

The above discussion can be summarized by the following Theorem.

Theorem 2.2. *Let (G, M) be a faithful representation of a compact Lie group G with trivial principal isotropy groups. Assume moreover that G/G^0 is generated by reflections on M/G^0. Then G/G^0 admits a set of generators whose elements are projections of nice involutions of G.*

This is the opportune moment to appreciate the strengh of the next proposition, found in [GL14].

Proposition 2.3. *Let $\rho_i : G_i \to \mathrm{O}(V_i)$, $i = 1, 2$, be two quotient-equivalent representations. Assume that G_1 is connected. Then the action of the finite group G_2/G_2^0 of connected components of G_2 on V_2/G_2^0 is generated by reflections at subspaces of codimension 1 in V_2/G_2^0.*

By next proposition, a representation which admits a non-trivial reduction also admits non trivial boundary, and therefore G-important points.

Proposition 2.4. *Let $\rho : G \to O(V)$ be an effective representation. If $X_0 = V/G^0$ has an empty boundary, then the representation ρ is reduced.*

This implies directly the next proposition and also suggests that the abstract copolarity of the representation and that of its identity component induced representation may always coincide. Its proof can be found in [GL14], Proposition 5.2. And it is a basic fact to our problem.

Proposition 2.5. *Let $\rho_i : G_i \to \mathrm{O}(V_i)$, $i = 1, 2$ be two quotient-equivalent representations. If the quotient space V_i/G_i have no boundary (in the Alexandrov sense), then*

$$\dim V_1 = \dim V_2.$$

Moreover, if we replace the hypothesis with weaker ones, precisely: if the quotient space V_1/G_1^0 has no boundary, then the inequality $\dim V_1 \leq \dim V_2$ holds true.

2.3 A Brief Introduction to Riemannian Orbifolds

As aforementioned, this chapter is a brief synthesis on basic Orbifold theory. We will not prove the facts here enunciated.

The notion of an orbifold appeard for the first time in Satake's works under the name of V-manifold.Later, Thurston defined the notions of coverings and fundamental groups of orbifolds and showed that these notions worked in this setting just as the usual theory does for manifolds. Nevertheless, the main definitions here are due to [Lan20], which has given a rather elegant definition to Riemannian Orbifolds, that is equivalent to the traditional one, and some are also due to Thurston's works. For a more profound approach we recommend the reading of [Lan20] and [Dav11].

A *Riemannian Orbifold* of dimension n is a length space $\mathcal{O}$ such that for each point $x \in \mathcal{O}$ there exist an open neighborhood U of x in $\mathcal{O}$ and a connected Riemannian manifold M of dimension n together with a finite group G of isometries of M such that U and M/G are isometric with respect to the induced length metrics. Here $M/$G is endowed with the quotient metric, i.e. the distance between two points is defined as the distance between their respective orbits in M.

A *covering orbifold* of a Riemannian orbifold $\mathcal{O}$ is a Riemannian orbifold $\mathcal{O}'$ together with a surjective map $p : \mathcal{O}' \to \mathcal{O}$ such that for each point $x \in \mathcal{O}$ there is a neighborhood $U \subset \mathcal{O}$ isometric to some orbit space associated to the action of a finite group of isometries G of M, namely M/G, for which each connected component U_i of $p^{-1}(U)$ is isometric to M/G_i for some $G_i < G$ such that the following diagrams commute

$$\begin{tikzcd} U_i \arrow[r, "\sim"] \arrow[d, "p"] & M/G_i \arrow[d] \\ U \arrow[r, "\sim"] & M/G \end{tikzcd}$$

The map p is called as *orbifold covering.* The group of isometries of $\mathcal{O}'$ which leave the fibers of p invariant is called the *deck transformation group* of the covering p. Also, if the deck transformation group acts transitively on the fibers of p we call p a *Galois covering*, and, in this case, $\mathcal{O}$ coincides with the metric quotient of the action of the deck transformation group of p on $\mathcal{O}'$. Hereafter every covering considered is a Galois covering, unless stated otherwise.

An orbifold $\mathcal{O}$ is said *simply connected* if it is connected and it does not admit a non-trivial cover. That is, if $p : \mathcal{O}' \to \mathcal{O}$ is an orbifold covering with $\mathcal{O}'$ connected, then p is a homeomorphism.

Any connected orbifold $\mathcal{O}$ admits a simply connected cover, $\mathcal{O}'$, which covers every other covering of $\mathcal{O}$. Such a cover is called *universal covering.* The group of deck transformations of the universal cover is called the *orbifold fundamental group* of $\mathcal{O}$ and is denoted by $\pi_1^{\text{orb}}(\mathcal{O})$. This group acts as a group of discrete isometries of $\mathcal{O}'$ and is such that $\mathcal{O}'/\pi_1^{\text{orb}}(\mathcal{O}) = \mathcal{O}$.

Observe that, $\pi_1^{\text{orb}}(\mathcal{O})$ is trivial if and only if $\mathcal{O} = \mathcal{O}'$. In this case $\mathcal{O}$ is a simply connected topological space and has no boundary. And if $\pi_1^{\text{orb}}(\mathcal{O} \setminus \partial\mathcal{O}) = 1$, then $\pi_1^{\text{orb}}(\mathcal{O})$ is a Coxeter group. Here, $\partial\mathcal{O}$ is the *boundary* of $\mathcal{O}$, which is the closure of the strata of codimension 1.

A *reflection* on an orbifold is an orbifold-isometry $f : \mathcal{O} \to \mathcal{O}$ whose restriction to the regular stratum $\mathcal{O}_{\text{reg}}$ fixes a submanifold of dimension 1. A rather intuitive result, whose proof can be found at Lemma 3.5 of [GL14], is that any reflection group (that is, a group generated by reflections) on a simply connected Riemannian orbifold is a Coxeter group.

It has been shown in [LT07] that, given a isometric action (G, M), the points in M whose slice representation is polar are exactly those whose projection on $X = M/G$ have a neighborhood isometric to a Riemannian. For obvious reasons, these points in X are called *orbifold points.* The set of orbifold points of X is denoted by X_{orb} and is a connected open subset and is a union of strata containing all strata that have codimension at most 2 in X, in particular, all G-important points. Also, X_{orb} has non-empty boundary if and only if X has non-empty boundary.

The following lemma is a well-known result whose proof can be found in [Lyt10].

Lemma 2.6. *Let M be a simply connected complete Riemannian manifold. Let G be a connected compact group of isometries of M. Let X be the quotient M/G. Let X_{orb} be the set of orbifold points* *in X and set $X_0 = X_{\mathrm{orb}} \setminus \partial X_{\mathrm{orb}}$. Then X_0 is exactly the set of non-singular G-orbits. Moreover, X_0 has trivial orbifold fundamental group.*

Chapter 3

Irreducible Representations of Small Copolarity

With the aim of understanding the irreducible representations of small copolarity, Gorodski and Lytchak have shown in [GL14] that representatios of connected compact groups with abstract copolarity up to 6 have minimal reductions with a toric indentity component. Furthermore, in this case, if the representation is non polar and non reduced, then its cohomogeneity is $k+2$; in which k denotes the abstract copolarity of the representation. But also in that paper they have shown an example of a non polar irreducible representation of copolarity 7 and cohomogeneity 5. Thus, the relations between cohomogeneity and copolarity are not fully understood yet, even for small values of copolarity.

In order to extend the just mentioned results we will consider an irreducible, non-reduced and faithful representation $\tau : H \to \mathrm{O}(W)$, with H connected, and a minimal reduction $\rho : G \to \mathrm{O}(V)$ with $\dim G = k$.

As mentioned *en passant* in the introduction, observe that if G is connected, by Proposition 2.5, V/G must have a non-trivial boundary. Otherwise, if G is disconnected, Proposition 2.3 says that G/G^0 is generated by reflections on V/G^0, which implies, by Theorem 2.2, that G/G^0 admits a set of generators whose elements are projections of nice involutions. So, in the latter case, G^0 is normalized by a nice involution that acts as a reflection on V/G^0.

By Lemma 2.1, in the connected case G^0 must act irreducibly on V. In the disconnected case, if the action of G^0 is reducible, we can apply the following theorem – proved in [GL14].

Theorem 3.1. *Let $\rho : H \to O(W)$ and $\rho' : H' \to O(W')$ be quotient-equivalent representations. Assume that the action of the identity component H^0 on W is irreducible and that of $(H')^0$ on W' is reducible. Then there is precisely one effective representation $\tau : G \to O(V)$ in the quotient class of ρ and ρ' which has trivial copolarity. If this quotient-equivalence class is non-polar, then the identity component of G is a torus T^k and its action on V can be identified with that of a maximal torus of $SU(k+1)$ on $\mathbb{C}^{k+1}$.*

In this case, we deduce that the representation admits toric reduction and such representations are already classified in [GL15].

So we are left to analyze the possible cases when the action of G^0 is irreducible. By assumption, the action of G is effective, hence the abelian summand of the Lie algebra of G^0 is at most one-dimensional, by the Schur's Lemma. Which restricts the possible covers of G^0 by compact Lie groups.

When the group G is connected, the existence of a nontrivial boundary implies the existence of a G-important point p. As $G = G^0$, p must also be G^0-important, trivially. Thus, by the discussion in section 5, G_p is either S^1 or $\mathrm{SU}(2)$. We must then analyze the possible S^1 and $\mathrm{SU}(2)$ subgroups of G^0 according to its cover.

If $G_p = \mathrm{SU}(2)$ we must make a case study. But, in case $G_p = S^1$, we fix a maximal torus T on G^0 containing G_p. Therefore, the fixed set point V^{G_p} is T-invariant and, then, a sum of weight spaces. As the action of G_p on V^{G_p} is trivial, every weights appearing in the decomposition of V^{G_p} vanishes on the Lie algebra of G_p, which is one dimensional. This way, they can be associated to elements in the dual space of the Lie algebra $\mathfrak{t}$ of T which lies on a hyperplane. Observe that if the representation is of real type, we can use the same line of thought to its complexification. Therefore we conclude that if the rank of G is r, there are at most $r-1$ linearly independent weight spaces appearing on V^{G_p}, which may help us to majorate $\dim V^{G_p}$. Furthermore, by Equation (2.1),

$$\dim V \leq \dim G + \dim V^{G_p} - r + 2; \tag{3.1}$$

since $T \subset N(G_p)$. So, as we majorated $\dim V$, and the principal isotropy groups are trivial, we also majorated the cohomogeneity of the action. Furthermore, irreducible representations with cohomogeneity up to 8 are classified in [Goz18].

Now, if G is disconnected, according to the possible covering groups of G^0 one can obtain further information about the nice involutions w that normalize G^0, such as information about their centralizer and fixed-point set, in view of using equation (2.3) to get further information about the cohomogeneity of the action, limiting its possibilities and so doing a case study.

We highlight the fact that as we are concerned about the effectivization of $\rho : G \to \mathrm{O}(V)$, we will make the abuse of identifying $G = \rho(G) \in \mathrm{O}(V)$.

3.1 Abstract copolarity 7

Since $\dim G = 7$ and the abelian summand of its Lie algebra is at most one dimensional, the group G^0 is covered by $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ – by the classification of compact groups.

Let us then analyze the connected and the disconnected cases separately.

3.1.1 Connected case

$G = G^0$ and the representation ρ is irreducible, which implies by Proposition 2.4 that $\partial(V/G) = \partial(W/H) \neq \varnothing$. Then there is some G-important point p. As discussed in section 5, $G_p = S^a$ with $a \in \{1, 3\}$. Observe that, as G is covered by $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$, then the $\mathrm{SU}(2)$-subgroups of G can only be the projections of the $\mathrm{SU}(2)$-factors or the projection of the diagonal in the last two factors. Then, any $\mathrm{SU}(2)$-subgroup contained in G has an unique involution that is central in G, namely the projection of the element corresponding to $-\mathrm{Id}$ in this group. Since our representation is irreducible, such a central involution cannot have fixed points. Therefore, $a \neq 3$. This way, $G_p = S^1$.

Then, by Equation (3.1), observing that $\dim N(G_p) \geq 3$, since a maximal torus centralizes the circle G_p, we conclude that

$$\dim V \leq 6 + \dim V^{G_p}.$$

We are left then to get an upper bound to $\dim V^{G_p}$.

Fix a maximal torus T in G containing the circle G_p. As mentioned earlier in this chapter, V^{G_p} is a sum of weight spaces and all these weights vanish on the Lie algebra of the circle G_p. Hence, the weight spaces appearing in V^{G_p} are associated to hyperplanes on the dual space of the Lie algebra $\mathfrak{t}$ of T. As $\dim T = 3$, all hyperplanes are 2 dimensional, thus there are at most two linearly independent weight spaces appearing in V^{G_p}.

As G^0 is covered by $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$, then all G^0-irreducible representations are complex, since the central $\mathrm{U}(1)$ induces a complex structure, by the Schur's lemma. Therefore, the restriction of each weight to the central $\mathrm{U}(1)$ – that is the projection of $\mathrm{U}(1) \times \{\mathrm{Id}\} \times \{\mathrm{Id}\}$ – is independent of the weight, and thus there are no different linearly dependent weights. As each weight space is complex one dimensional, one concludes that $\dim V^{G_p} \leq 4$, since it is a sum of weight spaces and there are at most 2 linearly independent weights appearing in this decomposition, like mentioned above. This way, $\dim V \leq 10$.

As $G_p \cong S^1$, the principal isotropy groups must be discrete. This implies that the cohomgeneity of the action is at most 3. As the representation is complex and it must be even dimensional, its cohomogeneity is 1 or 3. When the cohomogeneity is 1, we have that the representation is polar, while when the cohomogeneity is 3, then the copolarity is one. So we exclude this case from our analysis.

3.1.2 Disconnected case

In this case G/G^0 acts on V/G^0 by reflections and is generated by nice involutions. Fix $w \in G/G^0$ a nice involution. Then, evoking Equation (2.3),

$$\dim V = 8 - \dim Z_G(w) + \dim V^w. \tag{3.2}$$

As discussed in the previous case, the representation must be of complex type. Thus, the representation space V must be $\mathbb{C}^n$. Here, again, the central circle in G^0, which is the connected component of identity of the center, induces the a complex structure. As the nice involution w normalizes G^0 it also normalizes this central circle. Thus w is either complex linear or complex antilinear. In the first case $\dim V^w$ is even, while in the second it is n.

Also, the conjugation by w can act on G^0 either as an inner or as an outer autormorphism. Let us deal with these cases separately.

Outer automorphism:

Let us analyze the case $G^0 = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$, once G^0 is covered by it. And let $\pi = (G^0, \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^m)$ be the representation induced by ρ.

Remember that $\mathrm{Out}(\mathrm{SU}(2))$ is trivial. So, if the $\mathrm{SU}(2)$ factors of G^0 are fixed by w, as it is an outer automorphism, it must act as the complex conjugation on the central circle $\mathrm{U}(1)$. Of course the automorphism that interchanges the $\mathrm{SU}(2)$ factors is also an outer automorphism.

Let $i, \epsilon \in \mathrm{O}(V)$ be elements such that $i(u \otimes v) = v \otimes u$ and $\epsilon(u \otimes v) = \overline{u} \otimes \overline{v}$; note that such i can only exist when $m = n$. And define the automorphism $\varphi_i : G^0 \to G^0$ via $(z, g, h) \mapsto (z, h, g)$ and the automorphism $\varphi_\epsilon : G^0 \to G^0$ via $(z, g, h) \mapsto (\overline{z}, \overline{g}, \overline{h})$. So $\mathrm{Aut}(G^0)/\mathrm{Inn}(G^0) = \langle \varphi_i, \varphi_\epsilon \rangle$. Thus, for each $g \in G^0$, $wgw^{-1} = h\varphi(g)h^{-1}$, with $h \in G^0$ and $\varphi \in \langle \varphi_i, \varphi_\epsilon \rangle$.

Since the $\mathrm{U}(1)$ factor of G^0 is central we may assume that $h = (1, h_1, h_2)$. So that the conjugation by $h^{-1} \cdot w$ is the intertwining operator between π and $\pi \circ \varphi$, meaning that these representations are equivalent.

As $w^2 = \mathrm{Id}$,

$$g = w^2 g w^{-2} = wh\varphi(g)h^{-1}w^{-1} = h\varphi(h\varphi(g)h^{-1})h^{-1} = h\varphi(h)g\varphi(h)^{-1}h^{-1};$$

which means that $h\varphi(h)$ is in the centralizer of G^0, that coincides with the center of G^0. Thus, $\varphi(h) = \gamma h^{-1}$ for some $\gamma \in \mathrm{U}(1)$. As $\varphi \in \langle \varphi_i, \varphi_\epsilon \rangle$ we conclude that $\varphi(h) = h^{-1}$, since the $\mathrm{U}(1)$-entry of $\varphi(h)$ must be 1, $\varphi_i(1, h_1, h_2) = (1, h_2, h_1)$ and $\varphi_\epsilon(1, h_1, h_2) = (1, \overline{h_1}, \overline{h_2})$.

Suppose that φ interchanges the $\mathrm{SU}(2)$-factors of G^0, that is, that φ is either φ_i or $\varphi_i \varphi_\epsilon$. Then, $m = n$, meaning that the action π is of the form $(G^0, \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^n)$.

Observe that when $\varphi = \varphi_i$, as $\varphi(h) = h^{-1}$, $h_2 = h_1^{-1}$. In this case, conjugate w by $(1, \mathrm{Id}, \pm h_1)$. Then, its conjugation on an element $(z, g_1, g_2) \in G^0$ is given by:

$$(1, \mathrm{Id}, h_1)w(1, \mathrm{Id}, h_1^{-1})(z, g_1, g_2)(1, \mathrm{Id}, h_1)w^{-1}(1, \mathrm{Id}, h_1^{-1}) =$$

$$(1, \mathrm{Id}, h_1)w(z, g_1, h_1^{-1}g_2h_1)w^{-1}(1, \mathrm{Id}, h_1^{-1}) =$$

$$(1, \mathrm{Id}, h_1)(1, h_1, h_1^{-1})(z, h_1^{-1}g_2h_1, g_1)(1, h_1^{-1}, h_1)(1, \mathrm{Id}, h_1^{-1}) =$$

$$(1, \mathrm{Id}, h_1)(z, g_2, h_1^{-1}g_1h_1)(1, \mathrm{Id}, h_1^{-1}) = (z, g_2, g_1).$$

Thus, if $\varphi = \varphi_i$, by conjugating w with an element in G^0 we may assume that w acts on G^0 by $(z, g_1, g_2) \mapsto (z, g_2, g_1)$. And then $i \cdot w$ commutes with G^0, which means by Schur's Lemma that $w = \lambda i$, for $\lambda \in \mathrm{U}(1)$. As $w^2 = \mathrm{Id}$ and $\mathrm{U}(1)$ centralizes $\langle i, \epsilon \rangle$, we conclude that $\lambda = \pm 1$, that is, that $w = \pm i$.

Analogously, we have that if $\varphi = \varphi_i\varphi_\epsilon$, from $\varphi(h) = h^{-1}$, which means that $h_2 = \overline{h_1}^{-1}$. Then, by conjugating w by $(1, \mathrm{Id}, \overline{h_1}^{-1})$ we may assume that w a acts on G^0 by $(z, g_1, g_2) \mapsto (\overline{z}, \overline{g_2}, \overline{g_1})$. In this case, as $(i\epsilon) \cdot w$ commutes with G^0 we conclude that $w = \pm i\epsilon$.

Then, let us analyze separately the cases when $w = \pm i$ or when $w = \pm(i\epsilon)$.

In case $w = \pm i$, the connected component of the centralizer of w is the product of the central circle and the diagonal of the last two factors. So $\dim Z_G(w) = 4$. From the dimension formula we conclude that $\dim V - \dim V^w = 4$.

In case $w = i$, V^w is formed by the symmetric tensors. Thus, $\dim V^w = n(n+1)$. Since $\dim V = 2n^2$, the equation $\dim V - \dim V^w = 4$, that is $n^2 - n = 4$, has no integer solution for n, and we exclude this case.

In case $w = -i$, V^w is the set of antisymetric tensors. Thus, $\dim V^w = n(n-1)$. Again the equation $\dim V - \dim V^w = 4$ has no integer solution for n, and we also exclude this case.

In case $w = \pm(i\epsilon)$, then $Z_{G^0} = \{\pm 1\} \times \{(g, \overline{g}) : g \in \mathrm{SU}(2)\}$. Thus, $\dim Z_G = 3$, which implies that $\dim V - \dim V^w = 5$.

Observe that $\sum_{p,q=1}^{n} T_{p,q} e_p \otimes e_q$ is in $V^{i\epsilon}$ if and only if the matrix $(T_{p,q})_{p,q}$ is Hermetian and it is in $V^{-i\epsilon}$ if and only if the matrix $(T_{p,q})_{p,q}$ is anti-Hermetian. Thus, in both cases $\dim V^w = n^2$ and from the dimension formula we get $n^2 = 5$, which also does not have integer solution.

Therefore φ_i cannot be a factor of φ. So $\varphi = \varphi_\epsilon$. But remember that $\varphi(h) = h^{-1}$. That is, $\varphi(1, h_1, h_2) = (1, h_1^*, h_2^*)$, which implies that $\overline{h}_k = h_k^*$; in which $k = 1, 2$ and x^* denotes the conjugate transpose of x. This means that h_k is symmetric for $k = 1, 2$. Let $h_k = X_k + iY_k$ with X_k, Y_k being real symmetric matrices. Observe that $\mathrm{Id} = h_k h_k^* = X_k^2 + Y_k^2 + i(X_kY_k - Y_kX_k)$. This way, we conclude that $X_kY_k = Y_kX_k$, so there is a matrix x_k' in the special orthogonal group that simultaneously diagonalizes X_k and Y_k. This means that $x_k' h_k (x_k')^t = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{-i(\theta_1+\theta_2)})$. Observe then that if $x_k = \mathrm{diag}(e^{-i\theta_1/2}, e^{-i\theta_2/2}, e^{i(\theta_1+\theta_2)/2}) \cdot x_k'$ we have that $x_k h_k x_k^t = \mathrm{Id}$.[1]

Observe that for $x = (1, x_1, x_2)$,

$$xwx^{-1}gxw^{-1}x^{-1} = xh\varphi_\epsilon(x^{-1}gx)h^{-1}x^{-1} = xhx^t\varphi_\epsilon(g)\varphi_\epsilon(xhx^t) = \varphi(g).$$

So, by conjugating w by an element of G^0 we may assume that w acts by conjugation og G^0 via $g \mapsto \varphi(g)$. In this case $\epsilon \cdot w$ commutes with G^0. As $w^2 = 1$, by Schur's Lemma, $w = \pm\epsilon$. Therefore, $Z_{G^0}(w)^0 = \{1\} \otimes \mathrm{SO}(2) \otimes \mathrm{SO}(2)$, which is two dimensional. So $\dim V - \dim V^w = 6$.

Now the representation π is $(G^0, \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^m)$. As the action of $\mathrm{U}(1) \times \mathrm{SU}(2)$ on $\mathbb{C}^n$ is given by $(z, A) \cdot u = zAu$ and the map $\mathrm{U}(1) \times \mathrm{SU}(2) \mapsto \mathrm{U}(2)$ defined by $(z, A) \mapsto U(2)$ is a covering map, this representation is equivalent to $(\mathrm{U}(2), \mathbb{C}^n)$. Thus, π is equivalent to $(\mathrm{U}(2) \times \mathrm{SU}(2), \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^m)$.

In both cases $(w = \pm\epsilon)$, $\dim V^w = m \cdot n$, since V^w is either the real tensors or the pure imaginary ones. Therefore $m \cdot n = 6$. Then, the representation of G^0 is $(\mathrm{U}(2) \times \mathrm{SU}(2), \mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^3)$.

Inner automorphism:

Observe that $w = qj$ for some q that centralizes G^0 and some $j \in G^0$. As the centralizer of G^0 is contained in G^0 we conclude that $w \in G^0$, which is a direct contradiction.

3.1.3 $(\mathrm{O}(3) \times \mathrm{U}(2), \mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^4)$ as a reduction of $(\mathrm{U}(3) \times \mathrm{Sp}(2), \mathbb{C}^3 \otimes_{\mathbb{C}} \mathbb{C}^4)$

In this section we will explicit the example of a faithful, non-reduced, non-polar representation of abstract copolarity 7 and cohomogeneity 5, whose the induced representation of its connected component is covered by $(\mathrm{U}(2) \times \mathrm{SU}(2), \mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^3)$, that is presented in[GL14].

This will be the case of the effectivization of $(\mathrm{U}(3) \times \mathrm{Sp}(2), \mathbb{C}^3 \otimes_{\mathbb{C}} \mathbb{C}^4)$ – whose kernel is generated by $(-\mathrm{Id}, -\mathrm{Id})$ – which reduces to $(\mathrm{O}(3) \times \mathrm{U}(2), \mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^4)$, and which is covered by $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ and has a nice involution that acts by conjugation on its connected component as an outer isomorphism, just as previously discussed.

[1] This procedure is an application of the Autonne–Takagi factorization (see [HJ13] page 263), which guarantees that for every complex symmetric matrix A there is a unitary matrix U such that UAU^t is a real diagonal matrix with non-negative entries.

One easily sees that the representation of $\mathrm{U}(3) \times \mathrm{Sp}(2)$ on $\mathbb{C}^3 \otimes_{\mathbb{C}} \mathbb{C}^4$ is equivalent to its representation on $M(3 \times 4, \mathbb{C})$ given by $(A, B) \cdot X = AXB^{-1}$.

We know that for a given representation (G, V) we have that the principal isotropy groups of its slice representation at a given point p, namely $(G_p, \nu_p G(p))$, is a principal isotropy group of (G, V). We can iterate this process, by looking at the slice representation of $(G_p, \nu_p G(p))$. By dimensional reasons this algorithm stops when the slice represetantion is trivial.

Take

$$p_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have that $V_2 := \nu_{p_1} G(p_1)$ consists of the matrices

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix};$$

with $a \in \mathbb{R}$. So every G-orbit intersects a point like that. By multiplying on the left by $\mathrm{diag}(-1, 1, 1)$ we may assume that $a \geq 0$. Take $G_2 = G_{p_1}$ whose elements are of the form

$$\left(\begin{bmatrix} z & \\ & A \end{bmatrix}, \begin{bmatrix} \overline{z} & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & z & 0 \\ 0 & -\overline{b} & 0 & \overline{\alpha} \end{bmatrix} \right);$$

with $z \in \mathrm{U}(1)$, $A \in \mathrm{U}(2)$ and

$$\begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \in \mathrm{SU}(2).$$

Observe that G_2 acts in V_2 by fixing the radial direction $\mathbb{R}p_1$.

Now, let us analyze (G_2, V_2). Take

$$p_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have that by looking at $V_3 := \nu_{p_2} G_2(p_2)$ we see that every orbit intersects an element of form

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & * & * \\ 0 & 0 & * & * \end{bmatrix};$$

with $b \geq 0$, multiplying on the left by $\mathrm{diag}(1, -1, 1)$ if necessary. Now $G_3 := (G_2)_{p_2} = \{(\mathrm{diag}(\omega, \gamma, \theta), \mathrm{diag}(\overline{\omega}, \overline{\gamma}, \omega, \gamma)$
$\omega, \gamma, \theta \in \mathrm{U}(1)\}$.

Take

$$p_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

By looking at $V_4 := \nu_{p_3} G_3(p_3)$ we see that every orbit intersects an element of form

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & * & 0 \\ 0 & 0 & c & * \end{bmatrix};$$

with $c \geq 0$, multiplying on the left by $\mathrm{diag}(1, 1, -1)$ if necessary.

Let $G_4 := (G_3)_{p_3} = \{(\mathrm{diag}(\omega, \gamma, \overline{\omega}), \mathrm{diag}(\overline{\omega}, \overline{\gamma}, \omega, \gamma)) : \omega, \gamma \in \mathrm{U}(1)\}$.

Take

$$p_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By looking at $V_5 := \nu_{p_4} G_4(p_4)$ we see that every orbit intersects an element of form

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & d & 0 \\ 0 & 0 & c & * \end{bmatrix};$$

with $d \geq 0$, multiplying on the left by $\mathrm{diag}(-1, 1, -1)$ and on the right by $\mathrm{diag}(-1, 1, -1, 1)$, if necessary.

Let now $G_5 := (G_4)_{p_4} = \{(\mathrm{diag}(\omega, \overline{\omega}, \overline{\omega}), \mathrm{diag}(\overline{\omega}, \omega, \overline{\omega}, \omega)) : \omega \in \mathrm{U}(1)\}$.

Finally, take

$$p_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By looking at $V_6 := \nu_{p_5} G_5(p_5)$ we see that every orbit intersects an element of form

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & d & 0 \\ 0 & 0 & c & e \end{bmatrix};$$

with $e \geq 0$, multiplying on the left by $\mathrm{diag}(1, -1, 1)$ and on the right by $\mathrm{diag}(1, -1, 1, -1)$, and maybe even taking $d \leq 0$ earlier in this process, if necessary. So such matrices represents the orbit spaces of the G-action.

Furthermore, we have that $G_6 := (G_5)_{p_5}$ is trivial.

Therefore every matrix in $M(3 \times 4, \mathbb{C})$ is conjugated to a real matrix and that our representation has trivial principal isotropy groups. So it follows that the group obtained by adjoining to G the complex conjugation of matrices, ϵ, still has the same orbits as G. Also, the principal isotropy group is the one generated by ϵ. We shall then apply a LRS reduction.

Let us calculate then the normalizer $N(\epsilon)$. Take $(A, B) \in N(\epsilon)$. Then, for every $X \in M(3 \times 4, \mathbb{C})$,

$$\overline{X} = (A, B)\epsilon(A, B)^{-1} \cdot X = AA^t \overline{X} (BB^t)^{-1}.$$

Then, for every $X \in M(3 \times 4, \mathbb{C})$,

$$AA^t X = XBB^t. \tag{3.3}$$

Name $AA^t = C$ and $BB^t = D$. Observe then that the equation (3.3) can be rewritten as

$$\sum_k c_{ik} x_{kl} = \sum_l x_{il} d_{lj}, \tag{3.4}$$

for each pair (i, j).

Fix a index i and let $j \neq i$ be another index. Let X be the null matrix except by $x_{j1} = 1$. Then, $\sum_k c_{ik} x_{k1} = \sum_l x_{il} d_{l1}$ is equivalent to $c_{ij} = 0$. By taking X to be null except for $x_{i1} = 1$, we have that $\sum_k c_{ik} x_{k1} = \sum_l x_{il} d_{l1}$ is equivalent to $c_{ii} = d_{11}$, which means that C is scalar. Using $x_{ii} = 1$ as the only non null entry of X we have that $\sum_k c_{ik} x_{ki} = \sum_l x_{il} d_{li}$ is equivalent to $c_{ii} = d_{ii}$ and for $j \neq i$, $\sum_k c_{ik} x_{kj} = \sum_l x_{il} d_{lj}$ is equivalent to $d_{ij} = 0$, which means that $C = D = \lambda\,\mathrm{Id}$ with $\lambda \in \mathbb{C}$. As C is unitary and D is symplectic we have that $\lambda = \pm 1$. That is, we have seen that $(AA^t, BB^t) = \pm(\mathrm{Id}, \mathrm{Id})$. This means that $N(\epsilon)/\epsilon$ is generated by $\mathrm{O}(3) \times \mathrm{U}(2)$ and $\theta = (\mathrm{diag}(i, i, i), \mathrm{diag}(i, i, -i, -i))$.

The LRS reduction is then $(\overline{N}, M(3\times 4,\mathbb{C}))$, with $\overline{N} = N(\sigma)/\sigma$. Observe that the identity component here, that is $\overline{N}^0 = \mathrm{SO}(3)\times \mathrm{U}(2)$ is covered by $\mathrm{SU}(2)\times U(2)$, as expected.

Furthermore, θ acts on $M(3\times 4,\mathbb{R})$ as $(\mathrm{diag}(1,1,1), \mathrm{diag}(1,1,-1,-1))$. This way $w = (i\,\mathrm{diag}(-1,1,1), i\,\mathrm{diag}(-1,$ is an element of $\overline{N}/\overline{N}^0$ such that $w^2 = (-\mathrm{Id}, -\mathrm{Id})$, which lies on the kernel of the representation. Thus, w is in fact an involution.

Furthermore, $M(3\times 4,\mathbb{R})^w$ is formed by the matrices of the form

$$\begin{bmatrix} 0 & 0 & x & y \\ a & b & 0 & 0 \\ c & d & 0 & 0 \end{bmatrix} \in M(3\times 4,\mathbb{R}).$$

Thus, $\dim M(3\times 4,\mathbb{R})^w = 6$. Also, $Z^0_{\mathrm{SO}(3)\times \mathrm{U}^2}(w) = \mathrm{SO}(2)\times\mathrm{SO}(2)$, which means that $\dim Z_{\overline{N}}(w) = 2$. Therefore w satisfies the equation (2.3) and is a nice involution that also satisfies equation (3.2).

3.2 Abstract copolarity 8

Since $\dim G = 8$ and the abelian summand of its Lie algebra is at most one dimensional, by the classification of compact groups, the group G^0 is covered by $\mathrm{SU}(3)$.

Let us then analyze the connected and the disconnected cases separately.

3.2.1 Connected case

Our first aim is showing that the representation (G, V) does not have $\mathrm{SU}(2)$-boundary. So let $V^{\mathbb{C}} = W$, and remember that $\mathrm{SU}(3)^{\mathbb{C}} = \mathrm{SL}(3,\mathbb{C})$. Observe that, by Proposition (5.8) of [Sch80], $(\mathrm{SU}(3), V)$ has $\mathrm{SU}(2)$-boundary if and only if $(\mathrm{SL}(3), W)$ also does. But first of all, we need some preliminary language.

Let $\pi = (K, W)$ be a complex representation and let trace_V be the trace function on $\mathrm{Hom}_{\mathbb{C}}(W, W)$. Then $X \mapsto \mathrm{trace}_V(X^2)$ is an $\mathfrak{k}$-invariant bilinear form on $\mathfrak{k}$ and hence a multiple of the Cartan-Killing form $X \mapsto \mathrm{trace}_{\mathfrak{k}}(X^2)$. The multiplication factor is called *index* of W (or of (K, W)), denoted by $\mathrm{ind}_K V$ or $\mathrm{ind}_K \pi$. Clearly $\mathrm{ind}_K(\pi_1 \oplus \pi_2) = \mathrm{ind}_K \pi_1 + \mathrm{ind}_K \pi_2$. Next proposition, proven in [Sch80] (prop. (13.1)), will come in handy in this case analysis.

Proposition 3.2. *Let $G = G_1 \times \cdots \times G_s$ be a product of simple algebraic groups. Let V be an orthogonal representation space for G, H a principal isotropy group. Suppose that $j \in \mathbb{Z}^+$, $\mathrm{ind}_{G_i}(V) = 1$ for $1 \le i \le j$, and $\mathrm{ind}_{G_i}(V) > 1$ for $j < i \le s$. Then $H^0 \subset G_1 \times \cdots G_j$ ($H^0 = \{1\}$ if $j = 0$), and H^0 is a torus. Moreover, if H is finite, then (G, V) has no S^3 strata.*

Returning to our analysis of $(\mathrm{SU}(3)), V)$, tautologically we have two possibilities: either V is the realification U^r of some complex irreducible representation U or it is not.

In case $V = U^r$, we have that $(\mathrm{SL}(3,\mathbb{C}), W) = (\mathrm{SL}(3,\mathbb{C}), U \oplus \overline{U})$, wich means that $\mathrm{ind}_{\mathrm{SL}(3,\mathbb{C})} W = 2\,\mathrm{ind}_{\mathrm{SL}(3,\mathbb{C})} U$. By Proposition (3.2) we have that if $(\mathrm{SL}(3,\mathbb{C}), W)$ has $\mathrm{SU}(2)$ strata, then $\mathrm{ind}_{\mathrm{SL}(3,\mathbb{C})} W < 1$, which means that $\mathrm{ind}_{\mathrm{SL}(3,\mathbb{C})} U < \frac{1}{2}$. By Table 1 of [AVÉ67] we have then that $(\mathrm{SL}(3,\mathbb{C}), W) = (\mathrm{SL}(3,\mathbb{C}), \mathbb{C}^3)$, so that $(G, V) = (\mathrm{SU}(3), \mathbb{C}^3)$, that is polar.

In case V is not a realification of some complex irreducible representation we have that $W = V^{\mathbb{C}}$ is irreducible. Also by Proposition (3.2) we have that if $(\mathrm{SL}(3,\mathbb{C}), W)$ has $\mathrm{SU}(2)$ strata, then $\mathrm{ind}_{\mathrm{SL}(3,\mathbb{C})} W < 1$. Again, by Table 1 of [AVÉ67] we have then that $(\mathrm{SL}(3,\mathbb{C}), W)$ is either $(\mathrm{SL}(3,\mathbb{C}), \mathbb{C}^3)$ or $(\mathrm{SL}(3,\mathbb{C}), S^2\mathbb{C}^3)$. But neither of these representations is the complexification of a real irreducible representation.

Therefore ρ has no $\mathrm{SU}(2)$-boundary.

Let then p be a G-important point. From the recently discussed, $G_p = S^1$. Then, by Equation (3.1), as $N(G_p) \ge 2$, since a maximal torus centralizes G_p, we conclude that

$$\dim V \le 8 + \dim V^{G_p}.$$

Let us then majorate $\dim V^{G_p}$.

Fix a maximal torus T in G containing G_p. As V^{G_p} is a sum of weight spaces that are associated to hyperplanes on the dual space of the Lie algebra $\mathfrak{t}$ of T, which is 2-dimensional, there are no linearly independent weight spaces appearing in V^{G_p}.

Let θ_1 and θ_2 be the fundamental weights of SU(3). This way, each irreducible representation has a maximal weight $a\theta_1 + b\theta_2$ with a and b integers. Let this representation, that is unique up to isomorphism, be denoted by $\Gamma_{a,b}$.

Assuming without loss of generality that $a \geq b$, it is also known in the theory (see [FH13] section 13.2.) that the weight diagram of $\Gamma_{a,b}$ is formed by concentric hexagons H_i with vertices at the points $(a-i)\theta_1 - (b-i)\theta_2$, for $i = 0, 1, \cdots, b-1$, and a sequence of triangles T_j with vertices at points $(a-b-3j)\theta_1$ for $j = 0, 1, \cdots, ((a-b)/3)$. Furthermore, the representation $\Gamma_{a,b}$ has multiplicity $2(i+1)$ on H_i and $2b$ on T_j, when $a > b$, and it has multiplicity $i+1$ on H_i and a on T_j, when $a = b$. Also, see [Hal13],

$$\dim \Gamma_{a,b} = \begin{cases} (a+1)(b+1)(a+b+2), & \text{when } a > b \\ (a+1)^3, & \text{when } a = b. \end{cases}$$

Henceforth we are assuming that $V = \Gamma_{a,b}$. It is clear then that as the weights appearing in V^{G_p} lie all in a same line through the origin, that there are at most 2 weights of each H_i and of each T_j appearing in V^{G_p}. Thus, using the multiplicities mentioned on the previous paragraph,

$$\dim V^{G_p} \leq \begin{cases} b(b+1) + 2b\left(\frac{a-b+3}{3}\right), & \text{if } a > b \\ \frac{1}{2}a(a+1), & \text{if } a = b. \end{cases}$$

Then, by Equation (3.1), when $a > b$,

$$(a+1)(b+1)(a+b+2) \leq 8 + b(b+1) + 2b\left(\frac{a-b+3}{3}\right).$$

As a and b are integers it is not hard to verify that, with $a \geq b$, the only possibilities to this inequality are $(a,b) \in \{(0,0),(1,0)\}$. We may obviously exclude the trivial representation $\Gamma_{0,0}$. And, as $\Gamma_{1,0} = \mathbb{C}^3$ is polar, it can also be excluded from our analysis.

Also by Equation (3.1), when $a = b$,

$$(a+1)^3 \leq 8 + \frac{1}{2}a(a+1),$$

which is only possible when $a = 1$. But the representation $\Gamma_{1,1}$, that is the adjoint representation, is polar.

3.2.2 Disconnected case

Fix $w \in G/G^0$ a nice involution. Then, evoking Equation (2.3), we have the fundamental formula

$$\dim V = 9 - \dim Z_G(w) + \dim V^w. \tag{3.5}$$

As in the disconnected case of abstract copolarity 7, we shall deal with w acting by conjugation on G^0 as an inner or as an outer automorphism

Furthermore, as G^0 is covered by SU(3), and the only non-trivial normal group of SU(3) is $Z(\mathrm{SU}(3)) \cong \mathbb{Z}^3$, then G^0 is either SU(3) or SU(3)/$\mathbb{Z}^3$. Assume that $G^0 = \mathrm{SU}(3)/\mathbb{Z}^3$, take φ an automorphism of G^0 and let $p : \mathrm{SU}(3) \to G^0$ be the canonical projection. From the fact that SU(3) is simply connected, we have that $\varphi \circ p$ lifts to an isomorphism $\tilde{\varphi} : \mathrm{SU}(3) \to \mathrm{SU}(3)$. This clearly provides an monomorphism $\mathrm{Aut}(\mathrm{SU}(3)) \to \mathrm{Aut}(G^0)$. In fact, using the fact that $\mathbb{Z}^3$ is a characteristic subgroup, we have that this map is in fact an isomorphism. Furthermore, if φ is inner, that is, if $\varphi = \mathrm{Inn}_g$ for $g \in G^0$, then Inn_{g_0} provides a lift for any g_0 lifting g. Because $\ker p$

is central in SU(3), we have that $\mathrm{Inn}(\mathrm{SU}(3)) \to \mathrm{Inn}(G^0)$ is an isomorphism. This way we also have an isomorphism $\mathrm{Out}(\mathrm{SU}(3)) \to \mathrm{Out}(G^0)$. Therefore, for this analysis, we may assume without loss of generality that $G^0 = \mathrm{SU}(3)$.

Outer automorphism:

As $\mathrm{Out}(\mathrm{SU}(3))$ is generated by the complex conjugation, see [Wol72] p.290, we have that there is $h \in G^0$ such that for every $g \in G^0$

$$wgw^{-1} = h\overline{g}h^{-1}.$$

As $w^2 = 1$, we have that for every $g \in G^0$, $g = w^2gw^{-2} = h\overline{h}g\overline{h}^{-1}h^{-1}$, which means that $h\overline{h} \in Z(G^0)$. That is, there is $\omega \in \mathrm{U}(1)$ such that $\omega^3 = 1$ and $h\overline{h} = \omega\,\mathrm{Id}$. Thus, $h = \omega h^t$, which implies that $\omega = 1$. So, $h\overline{h} = 1$ and, consequently, h is symmetric. Again by the Autonne-Takagi factorization there is $x \in G^0$ such that $xhx^t = \mathrm{Id}$. Thus,

$$xwx^{-1}gxw^{-1}x^{-1} = \overline{g}.$$

Thus, by conjugating w by an element of G^0 we may assume that w acts by conjugation on G^0 via $g \mapsto \overline{g}$. So $\dim Z_G(w) = 3$ and thus $\dim V - \dim V^w = 7$.

$\rho^0 = (G^0, V)$ is absolutely irreducible

Assume ρ^0 is absolutely irreducible. Then its complexification $\pi = (\rho^0)^c : G^0 \to U(V^c)$ is unitary and irreducible. Denote ϵ the complex conjugation of V^c over V and also denote by σ the complex conjugation on G^0. Then $\epsilon \circ \pi \circ \epsilon$ and $\pi \circ \sigma$ are equivalent representations. So, there is $A \in \mathrm{U}(V^c)$ such that

$$A\epsilon\pi(g)\epsilon A^{-1} = \pi(\sigma(g)) = w\pi(g)w^{-1}$$

for all $g \in G$, in which we have considered the complex linear extension of w to V^c. This says that $w(\epsilon A)^{-1}$ centralizes $\rho(G^0)$. Owing to Schur's lemma, $w = \lambda\epsilon A$ for some $\lambda \in S^1$; we note a contradiction by referencing to the fact that the left hand-side of this equation is complex linear, whereas the right hand-side is conjugate linear.

ρ^0 is not absolutely irreducible

Assume ρ^0 is not absolutely irreducible. Then it is the realification of a complex representation $\pi : G^0 \to U(W)$, where $V = W^r$. Let ϵ be the conjugate linear involution of W over a real form. Then $\epsilon \circ \pi \circ \epsilon$ and $\pi \circ \sigma$ are equivalent representations. So, there is an $A \in U(W)$ such that

$$A\epsilon\pi(g)\epsilon A^{-1} = w\pi(g)w^{-1}$$

for all $g \in G$, as above. Now $w = \lambda\epsilon A$ for some $\lambda \in S^1$, due to Schur's lemma. Since A is complex linear and ϵ is conjugate linear, we have $\lambda^{-1/2}Iw(\lambda^{-1/2}I)^{-1} = \epsilon A$, so we may assume $w = \epsilon A$. Now $w^2 = \mathrm{Id}$ implies

$$A\bar{A} = A(\epsilon A\epsilon) = A(\epsilon A)^2A^{-1} = \mathrm{Id},$$

that is, A is unitary and symmetric. Write $A = BB^t$, where B is unitary and symmetric. Then $B^tw(B^t)^{-1} = \epsilon$, so we may assume $w = \epsilon$. Finally, $Z_{G^0}(w) = SO(3)$, $\dim V^w = \frac{1}{2}\dim V$ and the fundamental formula yields $\dim V = 12$. This implies that the cohomogeneity of ρ is 4, but all non-reduced, irreducible representations of cohomogeneity 4 of compact connected Lie groups have abstract copolarity 2 (see [GL14]).

Inner automorphism:

Observe that $w = hz$ for some z that centralizes G^0 and some $h \in G^0$. Using the fact that $w^2 = \mathrm{Id}$ we have that $z^2 = h^2$ lies in the center $\{\mathrm{Id}, e^{i2\pi/3}\,\mathrm{Id}, e^{i4\pi/3}\,\mathrm{Id}\}$ of SU(3). As $z, h \in \mathrm{O}(V)$ we have that they are diagonalizable. Moreover, as the eigenspaces of z are G^0-invariant and the G^0-action is irreducible, we have that z is scalar, so that $z = \lambda\,\mathrm{Id}$ with $\lambda^6 = 1$ ($\lambda \neq 1$, as $w \notin G^0$). Now we have that $h^2 = \overline{\lambda}^2\,\mathrm{Id}$, implying that the eigenvalues of h are of the form $\pm\overline{\lambda}$ (remember that we are confusing G with $\rho(G)$, thus these eigenvalues are of h seen as an element of $\mathrm{O}(V)$).

Furthermore, $Z_{G^0}(w) = Z_{G^0}(h)$, and $\dim Z_{G^0}(h)$ is even, since the centralizer of any element in

SU(3) is even dimensional. This means that $\dim Z_{G^0}(w) \in \{2,4,8\}$, since SU(3) has no subgroup of dimension 6 and the maximal torus containing h lies in this centralizer. So, by Equation (3.5), we have that ρ cannot be of complex type, since otherwise, $\dim V$ and $\dim V^w$ would be both even, which would imply a direct contradiction with the fundamental formula. This way, by the Schur's Lemma, as $w \notin G^0$, $z = -1$, which implies that $w = -h$. Furthermore, the fundamental formula implies that

$$\dim V^{-w} = \dim V - \dim V^w = 9 - \dim Z_G(w) \in \{1,5,7\}. \tag{3.6}$$

The complexification $\rho^c = \pi_{a,a}$ is a complex irreducible representation of highest weight $a\theta_1 + a\theta_2$, with θ_1 and θ_2 being the fundamental weights. We have that $e_1^a \otimes e_1'^a$ is the highest weight of $\pi_{a,a}$ for e_1, e_2, e_3 and e_1', e_2', e_3' being the canonical basis of $\mathbb{C}^3$ and its dual space $(\mathbb{C}^3)^*$, respectively. Of course the elements of the Weyl orbit of the highest weight, namely the vectors of the form $e_j^a \otimes e_j'^a$, are fixed by $-w$ for a even.

Denote by ϵ the complex conjugation of the complexification V^c over V. Now $e_j^a \otimes e_j'^a + \epsilon(e_j^a \otimes e_j'^a)$ and $i(e_j^a \otimes e_j'^a - \epsilon(e_j^a \otimes e_j'^a))$ are in V^w, so that $\dim V^{-w} \geq 9$. Which is a contradiction to Equation (3.6).

3.3 Abstract copolarity 9

As stated previously, we are bound analyze the case when G^0 is covered by U(3), which implies that G^0 is covered by $\mathrm{U}(1) \times \mathrm{SU}(3)$.

Observe that as the projection of the circle $\mathrm{U}(1) \times \{1\}$ is central in G^0, every irreducible representation of G^0 is of complex type.

3.3.1 Connected case

Let p be the fixed G-important point, by Equation (3.1). Suppose that $G_p \cong \mathrm{SU}(2)$. Observe that, as $G = \mathrm{U}(1) \times \mathrm{SU}(3)$, G_p is then a subgroup of $\{1\} \times \mathrm{SU}(3)$. This means that the slice representation at p is $(G_p \cong \mathrm{SU}(2), \mathbb{C}^2)$, up to a G_p-fixed subspace of $\nu_p G(p)$.

Observe that the representation (G, V) comes from a representation $(\mathrm{SU}(3), V)$ in addition of a central circle. Assume that $(\mathrm{SU}(3), V)$ has an index non smaller than 1 and take $X \in \mathfrak{g}_p \subset \mathfrak{su}(3) \subset \mathbb{R} \oplus \mathfrak{su}(3) = \mathfrak{g}$. Then

$$\mathrm{trace}_V(X^2) = \mathrm{trace}_{\mathfrak{g}}(X^2) - \mathrm{trace}_{\mathfrak{g}_p}(X^2) + \mathrm{trace}_{\nu_p(Gp)}(X^2) < \mathrm{trace}_{\mathfrak{g}}(X^2),$$

since

$$\frac{\mathrm{trace}_{\nu_p(Gp)}(X^2)}{\mathrm{trace}_{\mathfrak{g}_p}(X^2)} = \mathrm{ind}_{\mathrm{SU}(2)}((\mathbb{C}^2)^r) = 2\frac{1}{4} = \frac{1}{2}.$$

On the other hand,

$$\frac{\mathrm{trace}_V(X^2)}{\mathrm{trace}_{\mathfrak{g}}(X^2)} = \frac{\mathrm{trace}_V(X^2)}{\mathrm{trace}_{\mathfrak{su}(3)}(X^2)} \geq 1,$$

a contradiction. Therefore, the index of the $(\mathrm{SU}(3), V)$ representation is smaller than 1. And then, the same argumentation we did to the abstract copolarity 8 holds. So we may assume that $G_p = \mathrm{U}(1)$ without loss of generality.

Observing that $\dim N(G_p) \geq 3$, we conclude then that

$$\dim V \leq 8 + \dim V^{G_p}.$$

Also following the same line of reasoning of the connected cases of abstract copolarity 7 and 8, we have that V^{G_p} is a sum of weight spaces containing at most two linearly independent weights. The restriction of each weight to the central circle is independent of the weight, thus there are no different linearly dependent weights. Assuming that $V = \Gamma_{a,b}$ we have then that, as the maximal multiplicity of a weight in this case is $2b$, as stated in the copolarity 8 case,

$$\dim V^{G_p} \leq 4b$$

Thus, Equation (3.1) is rewritten as

$$(a+1)(b+1)(a+b+2) - 4b \leq 8.$$

Therefore $(a,b) \in \{(0,0),(1,0),(1,1)\}$, which is a contradiction, since the representation is neither polar nor trivial.

3.3.2 Disconnected case

Fix $w \in G/G^0$ a nice involution. Then, evoking Equation (2.3),

$$\dim V = 10 - \dim Z_G(w) + \dim V^w. \tag{3.7}$$

We will again analyze the cases when when w acts by conjugation on G^0 as an inner or as an outer automorphism separately.

Outer automorphism:

Again, in this case we have that $w = h \cdot \epsilon$ with ϵ being the complex conjugation on V and $h \in G^0$. Also, as the U(1)-factor is central in G^0 we may assume that $h = (1, h')$, h' being either 1 or not central in SU(3).

Just as we did in the abstract copolarity 8 case, by conjugating w by an element in G^0 we may assume that w acts by conjugation on G^0 via $g \mapsto \overline{g}$. And then, by the Schur's Lemma, we have that $h\epsilon = w = \lambda\epsilon$ with $\lambda \in \mathrm{U}(1)$. Therefore h is central, which implies, by our assumptions that $h = \mathrm{Id}$. That is, $w = \epsilon$. Thus, $Z_{G^0}(w)^0 = \{1\} \times \mathrm{SO}(3)$ and $\dim V^w = \frac{1}{2}\dim V$. So Equation (3.7) is rewritten as $\dim V = 14$.

Let $V = \Gamma_{a,b}$. Then

$$(a+1)(b+1)(a+b+2) = 14,$$

when $a > b$ and

$$(a+1)^3 = 14,$$

when $a = b$. Neither equations have integer solutions.

Inner automorphism:

Observe that $w = qj$ for some q that centralizes G^0 and some $j \in G^0$. Again, the centralizer in this case is the central circle of G^0, by the Schur's Lemma, which would mean that $w \in G^0$, that is a direct contradiction.

Part II

Orbifold Structures of $\mathrm{Sp}(2)//\mathrm{SU}(2)$

Chapter 4

Biquotients and Metric

In this chapter we present the notion of an biquotient, that is closely related to the notion of an homogeneous space. And, as such, is associated to a riemannian submersion, which implies, by O'Neil's formula that it admits a non-negative curvature. Given that, we also present an important techinic in sight of constructing (almost-)positive curvatures in non-negatively curverd spaces: the Cheeger deformation.

Given a compact Lie group G then any closed subgroup U of $G \times G$ has a natural action on G given by $(u_1, u_2).g = u_1 g u_2^{-1}$. The quotient of this action is called a biquotient and denoted by $G//U$.

It is clear that to ask $G//U$ to be an orbifold is equivalent to ask that each point of G has a finite isotropy group, since the orbifold group of a point's projection is the isotropy group (or the stabilizer) of this point, that is $\Gamma_{\pi(g)} = \mathrm{Stab}(g) = \{(u_1, u_2) \in U : u_1 = g u_2 g^{-1}\}$. Furthermore, this group is finite exactly when its Lie algebra is trivial, and the Lie algebra of $\Gamma_{\pi(g)}$ is formed by $(X_1, X_2) \in \mathfrak{u}$ such that $X_1 = \mathrm{Ad}(g) X_2$. Using the fact that every element in $\mathfrak{u}$ is conjugate to an element of a fixed cartan subalgebra, $\mathfrak{t}_{\mathfrak{u}}$, and that isotropy groups occurs in conjugacy classes, we have the following lemma.

Lemma 4.1. *Let G be a Lie group and $U < G \times G$, a subgroup, with $\mathfrak{u}$ as its Lie algebra. Let $\mathfrak{t}_{\mathfrak{u}}$ be a maximal abelian subalgebra. Then, $G//U$ is an orbifold if, and only if, for all non-zero $(X_1, X_2) \in \mathfrak{t}_{\mathfrak{u}} \subset \mathfrak{u} \subset \mathfrak{g} \oplus \mathfrak{g}$ and for all $g \in G$, $X_1 \neq \mathrm{Ad}(g) X_2$. Furthermore, if $\pi : G \to G//U$ is the projection, and $g \in G$, then the orbifold group $\Gamma_{\pi(g)} \subset U$ is given by*

$$\Gamma_{\pi(g)} = \{(u_1, u_2) \in U : u_1 = g u_2 g^{-1}\}.$$

In particular, $G//U$ is a manifold if, and only if, $(u_1, u_2) \in U$, $g \in G$ with $u_1 = g u_2 g^{-1}$ implies that $u_1 = u_2 = e$.

Since biquotients are defined via a group action, they can be seen as base spaces of Riemannian submersions. So, if we impose that the canonical projection $\pi : G \to G//U$ is a Riemannian submersion, by O'Neil's formula we have

$$\sec_{G//U}(X, Y) = \sec_G(\tilde{X}, \tilde{Y}) + \frac{3}{4} |[\tilde{X}, \tilde{Y}]|^2 \geq \sec_G(\tilde{X}, \tilde{Y});$$

in which $\sec_{G//U}, \sec_G$ are the sectional curvatures of $G//U$ and of G, respectively, and $\tilde{X}, \tilde{Y}$ are the horizontal lifts of the tangent vectors X and Y.

Thus, as G is compact, and therefore admits a bi-invariant metric, which has non-negative sectional curvature, $G//U$ always admits a non-negative sectional curvature.

But non-negative curvature is not always the best we can expect. We shall rely on an important technic that may help in trying to improve an orbifold curvature: the Cheeger deformation. Eschenburg' Habilitation [Esc82] (in German) holds the most relevant information about Cheeger deformation, so does [Ker12], which is written in English. Nevertheless we shall recall it in next section.

4.1 Cheeger Deformation

Cheeger deformation is an important technic on searching for new positively curved orbifolds, since it tends to increase the sectional curvature of non-negative curved manifolds. This deformation consists of rescaling the Riemannian metric in the vertical directions of a given isometric group action, as shown next.

Let (K, M) be an isometric right-action of a compact Lie group K, endowed with a bi-invariant metric $\mathcal{Q}$, on a Riemannian manifold M with metric g_0.

For each $\lambda > 0$ we can endow $M \times K$ with the product metric $g_0 \oplus \lambda\mathcal{Q}$. Observe that K acts isometrically on $(M \times K, g_0 \oplus \lambda\mathcal{Q})$ via $h * (p, k) = (ph^{-1}, hk)$. The orbit space of this action, namely $M \times_K K$ is diffeomorphic to M. Explicitly this diffeomorphism is given by $[p, k] \mapsto pk^{-1}$. Since this just mentioned action is free and isometric, there is an unique Riemannian metric on $M = M \times_K K$ (with some abuse of language) g_1 that makes the projection $M \times K \to M$ a Riemannian submersion.

Furthermore, for a given $p \in M$ we fix an orthogonal splitting $\mathfrak{k} = \mathfrak{k}_p \oplus \mathfrak{m}_p$; with $\mathfrak{k}_p$ denoting the Lie algebra of the isotropy group K_p and identifying the orthogonal complement $\mathfrak{m}_p$ with $T_pK(p)$ via action fields. Namely, we associate each $X \in \mathfrak{m}_p$ with the action field $X^*_p = \left.\dfrac{d}{dt}p\exp(tX)\right|_{t=0}$. This also determines an orthogonal splitting $T_pM = \mathcal{V}_p \oplus \mathcal{H}_p$, in *vertical* and *horizontal* spaces, respectively,

$$\mathcal{V}_p = T_pK(p) \quad \text{and} \quad \mathcal{H}_p = \mathcal{V}_p^{\perp}.$$

A very special and useful case to our work is when M is a compact Lie group G, g_0 is a bi-invariant metric and (G, K) is a symmetric pair and the metric $\mathcal{Q}$ is the restriction $g_0|_K$. In this case it also useful to consider the orthogonal splitting with respect to g_0: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

We want now to understand the Riemannian submersion $\pi : G \times K \to G$. But, as g_1 is right-invariant, we just need to understand the submersion at (e, e). So, as for all $(X, Y) \in \mathfrak{g} \oplus \mathfrak{k}$ we have $d\pi_{(e,e)}(X, Y) = X - Y$, it follows that the vertical space at (e, e) consists of the elements of the form $(X_{\mathfrak{k}}, X_{\mathfrak{k}})$. Which implies that the horizontal vectors are of the form $\left(X_{\mathfrak{k}}, -\frac{1}{\lambda}X_{\mathfrak{k}}\right)$, where $X_{\mathfrak{k}}$ is the $\mathfrak{k}$ component of X in the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Henceforth we will also denote $X_{\mathfrak{p}}$ as the $\mathfrak{p}$ component of X.

This way, the horizontal lift of $X \in \mathfrak{g}$ is $\left(\frac{\lambda}{1+\lambda}X_{\mathfrak{k}} + X_{\mathfrak{p}}, -\frac{1}{1+\lambda}X_{\mathfrak{k}}\right)$. Therefore,

$$g_1(X, Y) = \frac{\lambda}{1 + \lambda}g_0(X_{\mathfrak{k}}, Y_{\mathfrak{k}}) + g_0(X_{\mathfrak{p}}, Y_{\mathfrak{p}}). \tag{4.1}$$

Then, the metric relating tensor ϕ, i.e., $g_0(\phi\cdot, \cdot) = g_1(\cdot, \cdot)$, is given by $\phi(X) = \dfrac{\lambda}{1 + \lambda}X_{\mathfrak{k}} + X_{\mathfrak{p}}$, which is clearly invertible with $\phi^{-1}(X) = X_{\mathfrak{p}} + \dfrac{1 + \lambda}{\lambda}X_{\mathfrak{k}}$. So, as we have seen, the horizontal lift of X is $(\phi(X), -\frac{1}{\lambda}\phi(X)_{\mathfrak{k}})$.

Remark: the definition of the tensor ϕ is quite general and can be done with respect to the action of a compact group K on a Riemannian manifold M with the hypotheses established earlier in this section, as done in [AB15]. Note also that, by Equation (4.1), $g_1(X,Y) \to g_0(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$ as $\lambda \to 0$. That is, the metric shrinks in the vertical directions and remains unchanged in the horizontal ones. An almost direct consequence is that (M, g_1) converges in the Gromov-Hausdorff sense to M/K as $\lambda \to 0$.

We want to understand when a plane in (G, g_1) has zero sectional curvature. It is obvious that a necessary condition is that a horizontal lift of the plane to $K \times G$ must have zero sectional curvature. If this plane is $\sigma = \mathrm{span}\{\phi^{-1}(X), \phi^{-1}(Y)\}$, by what was discussed above, its horizontal lift is $\tilde{\sigma} = \mathrm{span}\{(X, -\frac{1}{\lambda}X_{\mathfrak{k}}), (Y, -\frac{1}{\lambda}Y_{\mathfrak{k}})\}$. Hence, $\sec_{g_1}(\mathrm{span}\{\phi^{-1}(X), \phi^{-1}(Y)\}) = 0$ implies that $\sec_{g_0}(X, Y) = 0$ and $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$. This condition is also sufficient, as shown by [T$^+$09]. Therefore we have proven the following proposition.

Proposition 4.2. $\sec_{g_1}(\mathrm{span}\{\phi^{-1}(X), \phi^{-1}(Y)\}) = 0$ *iff* $\sec_{g_0}(X, Y) = 0$ *and* $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$.

Note that if g_0 is bi-invariant $\sec_{g_1}(\mathrm{span}\{\phi^{-1}(X), \phi^{-1}(Y)\}) = 0$ implies that $[X, Y] = 0$ and $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$. It becomes a little simpler – as the following lemma shows – if (G, K) is a symmetric pair. Also, in this case, as $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, if $[X, Y] = 0$, then $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$ is equivalent to $[X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = 0$.

Next lemma's proof is rather simple, and can be found at [DeV11]. We will omit it here for brevity.

Lemma 4.3. $\sec_{g_1}(\phi^{-1}(X), \phi^{-1}(Y)) = 0$ *iff* $\sec_{g_1}(X, Y) = 0$.

If we have a chain of of symmetric pairs $e = K_{n+1} \subset K_n \subset \cdots \subset K_1 \subset K_0 = G$, then we can iterate this process by deforming in the direction of the largest subgroup, then in the direction of the second largest, etc, to obtain metrics of non-negative sectional curvature $g_1, \cdots, g_n$ with parameters $\lambda_1, \cdots, \lambda_n$, respectively. In the case which (K_{i-1}, K_i) is a symmetric pair for each $1 \leq i \leq n$, we have, inductively:

Lemma 4.4. $\sec_{g_n}(X, Y) = 0$ *iff* $[X_{\mathfrak{k}_i}, Y_{\mathfrak{k}_i}] = 0$ *for every* i *from* 0 *to* n.

If $\mathfrak{p}_i$ is defined by the splitting (orthogonal with respect to a bi-invariant metric) $\mathfrak{k}_{i-1} = \mathfrak{k}_i \oplus \mathfrak{p}_i$, then, since (K_{i-1}, K_i) is a symmetric pair, we must also have $[X_{\mathfrak{p}_i}, Y_{\mathfrak{p}_i}] = 0$ for all i.

As we are studying the curvature of $G//U$ with respect with the deformed metric g_1, we must understand its vertical and horizontal bundles.

First of all, note that at a point $g \in G$, the vertical space with respect to $G \to G//U$ is

$$V_g = \{dR_g U_1 - dL_g U_2 : (U_1, U_2) \in \mathfrak{u}\}.$$

Thus, as our metric is right-invariant, we have the vertical space translated to the identity

$$dL_{g^{-1}} V_g = \{\mathrm{Ad}_{g^{-1}} U_1 - U_2 : (U_1, U_2) \in \mathfrak{u}\}.$$

Therefore, as $g_1(\cdot, \cdot) = g_0(\phi \cdot, \cdot)$, the horizontal space translated to the identity is

$$dR_{g^{-1}} H_g := \{\phi^{-1} X : g_0(X, U_1 - Ad_g U_2) = 0 \text{ for all } (U_1, U_2) \in \mathfrak{u}\}.$$

Chapter 5

Orbifold structures of $\mathrm{Sp}(2)//\,\mathrm{SU}(2)$.

In this chapter we study the family of the possible orbifold structures of the biquotients $\mathrm{Sp}(2)//\,\mathrm{SU}(2)$, by detailing their singular sets from a topological point of view, and between them we give a new exemple of almost-positively curved orbifold.

Here we denote $\mathrm{Sp}(n)$ as the group of isometries of $\mathbb{H}^n$.

Fix g_0 via the bi-invariant metric induced by the inner product in $\mathfrak{g} = \mathfrak{sp}(n)$ given by $\langle A, B\rangle = \Re(\mathrm{trace}(AB^*))$.

Remark: If $A, B \in \mathfrak{sp}(n)$ are 2×2 quaternionic matrices and $\tilde{A}, \tilde{B}$ are its presentations as $2n \times 2n$ complex matrices, one has that $\Re(\mathrm{trace}(\tilde{A}\tilde{B}^*)) = 2\Re(\mathrm{trace}(AB^*))$, thus, when convenient, we shall see $\mathrm{Sp}(n)$ as complex matrices.

Also, in order to understand the possible orbifold structures of $\mathrm{Sp}(2)//\,\mathrm{SU}(2)$ one must understand how $\mathrm{SU}(2)$ can be seen as a subgroup of $\mathrm{Sp}(2)\times\mathrm{Sp}(2)$. Obviously this requires an understanding of the embeddings $\mathrm{SU}(2) \to \mathrm{Sp}(2)$, which are precisely the complex four dimensional symplectic representations of $\mathrm{SU}(2)$.

Let ρ_m denote the complex m-dimensional irreducible representation of $\mathrm{SU}(2)$. It is a well known fact that, for $m > 0$, ρ_m is sympletic iff m is odd. Thus, we shall only work with the representations ρ_0, ρ_1 and ρ_3. We also highlight that under the identification $\mathrm{SU}(2) = \mathrm{Sp}(1)$, $\rho_1 : \mathrm{Sp}(1) \to \mathbb{H}$ is the natural inclusion.

Therefore, the possible nontrivial morphisms $\mathrm{SU}(2) \to \mathrm{Sp}(2)$ are those who defines the representations: $\mathbb{C}^2 \oplus \mathbb{R}^4$, $\mathbb{C}^2 \oplus \mathbb{C}^2$ and $\mathbb{C}^4 = \mathbb{H}^2$; which are the morphisms $\mathrm{SU}(2) \to GL(2n;\mathbb{C})$ given by (under identifying $\mathrm{SU}(2) = \mathrm{Sp}(1)$)

$$\psi_1(g) = \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}, \quad \psi_2(g) = \mathrm{diag}(\rho_1(g), \rho_1(g)) \text{ and } \psi_3(g) = \rho_3(g);$$

in which, in complex notation, $\rho_0 \equiv \mathrm{Id}$, ρ_1 is the natural inclusion and

$$\rho_3\left(\begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}\right) = \begin{bmatrix} \alpha^3 & \sqrt{3}\alpha\overline{\beta}^2 & -\overline{\beta}^3 & -\sqrt{3}\alpha^2\overline{\beta} \\ \sqrt{3}\alpha\beta^2 & \alpha\overline{\alpha}^2 - 2\overline{\alpha}\beta\overline{\beta} & -\sqrt{3}\overline{\alpha}^2\overline{\beta} & 2\alpha\overline{\alpha}\beta - \beta^2\overline{\beta} \\ \beta^3 & \sqrt{3}\overline{\alpha}^2\beta & \overline{\alpha}^3 & \sqrt{3}\overline{\alpha}\beta^2 \\ \sqrt{3}\alpha^2\beta & \beta\overline{\beta}^2 - 2\alpha\overline{\alpha}\overline{\beta} & \sqrt{3}\overline{\alpha}\overline{\beta}^2 & \alpha^2\overline{\alpha} - 2\alpha\beta\overline{\beta} \end{bmatrix}.$$

As corollary of Lemma 4.1, the biquotientes that arises from the embeddings that maps $\mathrm{SU}(2)$ diagonally into $\mathrm{Sp}(2)$, or, equivalently, from actions of $\Delta\,\mathrm{SU}(2) \subset \mathrm{Sp}(2) \times \mathrm{Sp}(2)$, do not have an orbifold structure.

Moreover, as we are concerned about non homogeneous orbifold structures under U-actions, we will not study the cases $U = \{e\} \times \mathrm{SU}(2)$ or $U = \mathrm{SU}(2) \times \{e\}$. Which leaves us to analyze the following embeddings of $\mathrm{SU}(2)$ into $\mathrm{Sp}(2)$: $\varphi_1 = \psi_2 \times \psi_1$, $\varphi_2 = \psi_3 \times \psi_1$ and $\varphi_3 = \psi_3 \times \psi_2$. That is, we will study the biquotients $\mathrm{Sp}(2)//\,\mathrm{SU}(2)_{\varphi_i}$, with $i \in \{1, 2, 3\}$, which are those defined under the actions of $\mathrm{im}\,\varphi_i \subset \mathrm{Sp}(2) \times \mathrm{Sp}(2)$ on $\mathrm{Sp}(2)$. They all admit an orbifold structure, as shall discuss later in this section.

$T^1 = \{e^{it} : t \in \mathbb{R}\}$ is a maximal torus of $\mathrm{SU}(2) = \mathrm{Sp}(1)$ and its corresponding maximal tori and their associated Cartan subalgebra of $\mathrm{SU}(2)_{\psi_i} = \operatorname{im}\psi_i$, $i = 1, 2, 3$ are, respectively,

$$T_1 = \{\operatorname{diag}(e^{it}, 1, e^{-it}, 1) : t \in \mathbb{R}\}, \quad \mathfrak{t}_1 := \{t \operatorname{diag}(i, 0, -i, 0) : t \in \mathbb{R}\};$$

$$T_2 = \{\operatorname{diag}(e^{it}, e^{it}, e^{-it}, e^{-it}) : t \in \mathbb{R}\}, \quad \mathfrak{t}_2 := \{t \operatorname{diag}(i, i, -i, -i) : t \in \mathbb{R}\}$$

and

$$T_3 = \{\operatorname{diag}(e^{3it}, e^{it}, e^{-3it}, e^{-it}) : t \in \mathbb{R}\}, \quad \mathfrak{t}_3 := \{t \operatorname{diag}(3i, i, -3i, -i) : t \in \mathbb{R}\}.$$

It is clear that there are no pair of distinct matrices among $t\operatorname{diag}(i, 0, -i, 0)$, $t\operatorname{diag}(i, i, -i, -i)$ and $t\operatorname{diag}(3i, i, -3i, -i)$ that are conjugate, since two diagonal matrices are conjugate iff their eigenvalues are equal up to permutation. So, by Lemma 4.1, $\mathrm{Sp}(2)//\mathrm{SU}(2)_{\varphi_i}$, $i \in \{1, 2, 3\}$ are orbifolds.

Also by Lemma 4.1, to study the singular locus of $\mathrm{Sp}(2)//\mathrm{SU}(2)_{\varphi_i}$, for $i = 1, 2, 3$, one must study the elements $g \in \mathrm{Sp}(2)$ that have nontrivial stabilizer under $\psi_i(h)g\psi_j(h)^{-1}$, for $(i, j) = (1, 2), (1, 3), (2, 3)$, respectively.

5.1 $\mathrm{Sp}(2)//\mathrm{SU}(2)_{\varphi_1}$

Two elements, $\operatorname{diag}(e^{it}, 1, e^{-it}, 1) \in U_1$ and $\operatorname{diag}(e^{it}, e^{it}, e^{-it}, e^{-it}) \in U_2$ are conjugate iff $t = 0 \mod 2\pi$. So $\mathrm{Sp}(2)//\mathrm{SU}(2)_{\varphi_1}$ is a manifold, that is known as Gromoll-Meyer sphere, see [GM74].

5.2 $\mathrm{Sp}(2)//\mathrm{SU}(2)_{\varphi_2}$

5.2.1 Topology

This section is reserved to prove the following proposition.

Proposition 5.1. *The singular locus of* $\mathrm{Sp}(2)//\mathrm{SU}(2)_{\varphi_2}$ *is homeomorphic to the connected sum* $\mathbb{RP}^3\#\mathbb{RP}^3$*.*

Let $g \in \mathrm{Sp}(2)$ be a singular point. Then there is $(u_1, u_2) \in U$ such that $u_1 g u_2^{-1} = g$. Changing g to another point in its U-orbit, we may assume that there is $(t_1, t_2) \in T_U$ s.t. $t_1 g t_2^{-1} = g$, or equivalently, $g t_2 g^{-1} = t_1$. Write

$$t_1 = \operatorname{diag}(z^3, \overline{z}, \overline{z}^3, z), \quad t_2 = \operatorname{diag}(z, 1, \overline{z}, 1).$$

Comparing eigenvalues we have that either $z = 1$ or, with a little redundancy, $z^3 = 1$. The first case is degenerate, as $t_1 = t_2 = \mathrm{Id}$. In the second case $z \in \{1, e^{i2\pi/3}, e^{i4\pi/3}\}$. For sake of convenience fix $z = e^{i2\pi/3}$.

This proves that every element $u \in U$ that fixes a point in $\mathrm{Sp}(2)$ has order 3, that is, is the image of an order 3 element in $\mathrm{SU}(2)$ under φ_2.

Lemma 5.2. *Let K be a subgroup of* $\mathrm{SU}(2)$ *such that every nontrivial element has order* 3*. Then K is conjugate to $\mathbb{Z}_3 \cong \{1, z, \overline{z}\}$. Furthermore, all elements of order* 3 *in* $\mathrm{SU}(2)$ *are conjugate to each other.*

Proof. Recall that every p-group (group in which each element has a power of p, with p a prime, as its order) has nontrivial center. Additionally, take an element of order 3 in $\mathrm{SU}(2)$, then it is conjugated to an element of order 3 in the maximal circle $\{e^{i\theta}\}$. Thus, since z and $\overline{z}$ are conjugate to each other, this element is conjugate to z. This proves that all elements of order 3 in $\mathrm{SU}(2)$ are conjugate to each other. As the center is invariant under conjugation, K is abelian. Furthermore, an element in $\mathrm{SU}(2)$ can only commute with another element in the same maximal circle. Thus, $K \cong \mathbb{Z}_3$. ∎

Since conjugation does not change the order of an element, the isotropy groups in Sp(2)// SU(2) are all isomorphic – via conjugation – to $\{1, z, \overline{z}\}$.

Note that, as $\overline{z} = z^{-1}$, $\mathrm{Fix}(\overline{z}) = \mathrm{Fix}(z)$; in which $\mathrm{Fix}(x)$ denotes the fix-point set of x. As $\psi_1(z) = \mathrm{diag}(z, 1, \overline{z}, 1)$ and $\psi_3(z) = (1, \overline{z}, 1, z)$, it is straightforward to compute $\mathrm{Fix}(z)$. Which leads us to the following proposition.

Proposition 5.3. *The fix-point set of* $\mathbb{Z}_3 = \{1, z, \overline{z}\}$ *consists of matrices in* Sp(2) *of the form*

$$\begin{bmatrix} 0 & \alpha & 0 & -\overline{\beta} \\ 0 & 0 & -\overline{\omega} & 0 \\ 0 & \beta & 0 & \overline{\alpha} \\ \omega & 0 & 0 & 0 \end{bmatrix}$$

which we will conveniently parametrize by $[\omega, \alpha, \beta]$.

We have that the singular locus of $\mathrm{Sp}(2)//\mathrm{SU}(2)_{\varphi_2}$ is $\mathrm{Fix}(\mathbb{Z}_3)/N$; where $N = N(\mathbb{Z}_3) = \langle e^{i\theta}, j\rangle$ is the normalizer of $\mathbb{Z}_3$ in $\mathrm{SU}(2) = \mathrm{Sp}(1)$.

The N^0-action on $\mathrm{Fix}(\mathbb{Z}_3)$ is given by $e^{i\theta} \cdot [\omega, \alpha, \beta] = [\omega, e^{i3\theta}\alpha, e^{-i3\theta}\beta]$. Now, there is a diffeomorphism (below we identify $S^2 = \mathbb{C} \cup \infty$ via stereographic projection)

$$\mathrm{Fix}(\mathbb{Z}_3)/N^0 \to S^1 \times S^2$$

induced by

$$\mathrm{Fix}(\mathbb{Z}_3) \to S^1 \times S^2, \quad [\omega, \alpha, \beta] \mapsto \left(\omega, \frac{\overline{\alpha}}{\beta}\right).$$

The j-action on $\mathrm{Fix}(\mathbb{Z}_3)$ is given by $j \cdot [\omega, \alpha, \beta] = [-\overline{\omega}, -\beta, \alpha]$; this projects to $\left(-\overline{\omega}, -\frac{\overline{\beta}}{\alpha}\right)$. That is, the induced j-action on $S^1 \times S^2$ is given by $j \cdot (\omega, s) = (-\overline{\omega}, -\overline{s^{-1}})$.

Note that, the map $s \mapsto -\overline{s^{-1}}$ is the antipodal map in the stereographic projection. So that the image of the exceptional orbits under the projection $\mathrm{Sp}(2) \to \mathrm{Sp}(2)//\mathrm{SU}(2)_{\varphi_2}$ forms the manifold $(]0,1[\times S^2) \cup (\{0,1\} \times \mathbb{RP}^2)$, which is topologically the connected sum is $\mathbb{RP}^3 \# \mathbb{RP}^3$.

Indeed, if one takes the pole of $\mathbb{RP}^3$ – that is the mapping cone of the projection $S^2 \to \mathbb{RP}^2$ – as the base point of the connected sum $\mathbb{RP}^3 \# \mathbb{RP}^3$, we have exactly the double mapping cylinder given by $\pi : S^2 \to \mathbb{RP}^2$, which is topologically $]0,1[\times S^2 \cup \{0,1\} \times \mathbb{RP}^2$.

Metric

For sake of convenience we will see Sp(2) as quaternionic matrices in this section.

Theorem 5.4. *The orbifold* $\mathcal{O}_2 = \mathrm{Sp}(2)//\mathrm{SU}(2)_{\varphi_2}$ *equipped with the metric after the Cheeger deformation of the bi-invariant metric on* Sp(2) *with respect to the symmetric pair* $(\mathrm{Sp}(2), \mathrm{Sp}(1) \times \mathrm{Sp}(1))$ *has almost-positive curvature.*

Proof. Consider also the metric g_1 given by the deformation with respect to the symmetric pair $K \subset G$, in which

$$K = \mathrm{Sp}(1) \times \mathrm{Sp}(1) := \left\{ \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} : p, q \in \mathrm{Sp}(1) \right\}$$

Now, fix $\{i, j, k\}$ as a basis for $\mathfrak{su}(2)$.

As $\mathfrak{k} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, we fix the notation $\mathfrak{k}' = \mathfrak{sp}(1) \oplus 0$ and $\mathfrak{k}" = 0 \oplus \mathfrak{sp}(1)$. Also, following the same notation, we fix

$$K' := \left\{ \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} : p \in \mathrm{Sp}(1) \right\}, K'' := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} : p \in \mathrm{Sp}(1) \right\}.$$

The elements of K' are exactly the image of ψ_1 and they commute with the elements in K''. Thus, left translations (as well as right translations) by an element in K'' are isometries with respect to g_1.

As $\mathrm{SU}(2) = \mathrm{Sp}(1)$ has positive sectional curvature (with respect to the bi-invariant metric), a plane in $\mathfrak{k}$ has zero sectional curvature if and only if its generators are one in $\mathfrak{k}'$ and the other in $\mathfrak{k}''$. Indeed, if $A = A_1 + A_2 \in \mathfrak{k}$ and $B = B_1 + B_2 \in \mathfrak{k}$, with $A_i, B_i \in \mathfrak{k}_i$ $(i = 1, 2)$, then $[A, B] = [A_1, B_1] + [A_2, B_2]$.

Now, fix $X, Y \in \mathfrak{g}$ vectors that form a plane with zero sectional curvature at $g \in \mathrm{Sp}(2)$, that is, X, Y are horizontal vectors, or, in other words,

$$g_1(X, \mathrm{Ad}_{g^{-1}} U_1 - U_2) = g_1(Y, \mathrm{Ad}_{g^{-1}} U_1 - U_2) = 0,$$

for every $(U_1, U_2) \in \mathfrak{u}$.

As $G/K = \mathbb{S}^7$ has positive sectional curvature, $[X_\mathfrak{p}, Y_\mathfrak{p}] = 0$ implies that $X_\mathfrak{p}$ and $Y_\mathfrak{p}$ are linearly dependent, i.e., $Y_\mathfrak{p} = \lambda X_\mathfrak{p}$. So, by changing Y by $Y - \lambda X_\mathfrak{p}$ we may assume without loss of generality that $Y_\mathfrak{p} = 0$. That is, $Y = Y_\mathfrak{k} \in \mathfrak{k}$. Also, from 4.4, $0 = [X_\mathfrak{k}, Y_\mathfrak{k}] = [X_\mathfrak{k}, Y] = [X_{\mathfrak{k}'}, Y_{\mathfrak{k}'}] + [X_{\mathfrak{k}''}, Y_{\mathfrak{k}''}]$. Thus, as $\mathrm{Sp}(1)$ has positive sectional curvature, $[X_{\mathfrak{k}'}, Y_{\mathfrak{k}'}] = 0$ and $[X_{\mathfrak{k}''}, Y_{\mathfrak{k}''}] = 0$, which implies that $\{X_{\mathfrak{k}'}, Y_{\mathfrak{k}'}\}$ and $\{X_{\mathfrak{k}''}, Y_{\mathfrak{k}''}\}$ are linearly dependent.

As $Y_\mathfrak{k} \neq 0$, it follows that $Y_{\mathfrak{k}'} \neq 0$ or $Y_{\mathfrak{k}''} \neq 0$. As $\{X_{\mathfrak{k}'}, Y_{\mathfrak{k}'}\}$ and $\{X_{\mathfrak{k}''}, Y_{\mathfrak{k}''}\}$ are both linearly dependent we can assume without loss of generality that either $Y_{\mathfrak{k}'} = 0$ or $X_{\mathfrak{k}'} = 0$.

Case 1: $Y_{\mathfrak{k}'} = 0$**.**

In this case, $Y = Y_{\mathfrak{k}''} \in \mathfrak{k}''$. As Y is horizontal at g, for every $(U_1, U_2) \in \mathfrak{u}$,

$$0 = g_1(Y, \mathrm{Ad}_{g^{-1}} U_1 - U_2) = g_1(Y, \mathrm{Ad}_{g^{-1}} U_1) - g_1(Y, U_2) \tag{5.1}$$

As $U_2 \in \mathrm{span}\{d\psi_1(i), d\psi_1(j), d\psi_1(k)\}$, $U_2 \in \mathfrak{k}'$, which implies, since $Y \in \mathfrak{k}''$,

$$g_0(Y, U_2) = 0.$$

Thus,

$$g_1(Y, U_2) = \frac{\lambda}{1 + \lambda} g_0(Y, U_2) = 0.$$

This way, the Equation (5.1) implies that

$$0 = g_1(Y, \mathrm{Ad}_{g^{-1}} U_1)$$

which, by the fact that Cheeger deformation does not creates new zero curvature planes, also implies that

$$0 = g_0(Y, \mathrm{Ad}_{g^{-1}} U_1) = g_0(\mathrm{Ad}_g Y, U_1).$$

As, $Y \in \mathfrak{k}''$, there is $y = y_i i + y_j j + y_k k \in \Im(\mathbb{H})$ (here $\Im(\mathbb{H})$ denotes the set of purely imaginary quaternions) s.t.

$$Y = \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}.$$

Thus,

$$\begin{cases} \langle \mathrm{Ad}_g Y, d\psi_3(i) \rangle = 0 \\ \langle \mathrm{Ad}_g Y, d\psi_3(j) \rangle = 0 \\ \langle \mathrm{Ad}_g Y, d\psi_3(k) \rangle = 0 \end{cases}$$

is a system which can be rewritten as the real linear homogeneous system over the variables y_i, y_j, y_k. It is straightforward to verify that this system only admits the trivial solution for $g = \mathrm{Id}$. Furthermore, the points with zero curvature planes in $\mathrm{Sp}(2)/\!/\,\mathrm{SU}(2)$ are those to which the above system has nontrivial solutions, or, equivalently, such that the system's determinant is zero. But the system's determinant is a polynomial in $\mathbb{R}^{16}$ that is not zero at $\mathrm{Id} \in \mathrm{Sp}(2)$. Also, $\mathrm{Sp}(2)$ can be seen as an real algebraic variety over $\mathbb{R}^{16}$ – as the set that annihilate the polynomials that impose that its lines form an orthonormal basis for $\mathbb{H}^2$ – such that is irreducible (that is, it cannot be written as a nontrivial union of two Zariski closed sets) as it is smooth and connected. So, the preimage of 0 under the determinant is a closed set (by continuity of the determinant) with empty interior, otherwise, by the next proposition, this closed set would be the whole Sp(2), and, as we have stated, it does not contain the identity.

Case 2: $Y_{\mathfrak{k}'} \neq 0$.

Then $X_{\mathfrak{k}'} = 0$. That is, $X = X_{\mathfrak{k}''} + X_{\mathfrak{p}}$ and $Y = Y_{\mathfrak{k}'} + Y_{\mathfrak{k}''}$, with $\{X_{\mathfrak{k}''}, Y_{\mathfrak{k}''}\}$ linearly dependent. Also, as we have, by 4.4,

$$0 = [X_{\mathfrak{p}}, Y_{\mathfrak{k}'} + Y_{\mathfrak{k}''}],$$

$Y_{\mathfrak{k}''} \neq 0$. Thus, $X = \lambda Y_{\mathfrak{k}''} + X_{\mathfrak{p}}$.

We highlight that one may assume, without loss of generality, that $X_{\mathfrak{p}} \neq 0$, otherwise, also without loss of generality, one could assume that $X \in \mathfrak{k}'$ and $Y \in \mathfrak{k}''$, which is the same as the previously discussed case 1.

In this case, there are $x \in \mathbb{H}$ and $y_1, y_2 \in \Im(\mathbb{H})$ s.t.

$$X = \begin{bmatrix} 0 & x \\ -\overline{x} & \lambda y_2 \end{bmatrix} \text{ and } Y = \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}.$$

Also from 4.4 we have that

$$0 = [X, Y] = [X_{\mathfrak{k}''}, Y] = [X_{\mathfrak{p}}, Y].$$

Thus, $X_{\mathfrak{p}} Y = Y X_{\mathfrak{p}}$. Or, equivalently,

$$\begin{cases} xy_2 = y_1 x \\ -\overline{x} y_1 = -y_2 \overline{x} \end{cases}$$

Which implies that $xy_2 = y_1 x$. As $X_{\mathfrak{p}} \neq 0$, $x \neq 0$, thus,

$$x y_1 x^{-1} = y_2.$$

As conjugation preserves norm, $\|y_1\| = \|y_2\|$.

Note that if Y is horizontal at g then $\mathrm{Ad}_{k_2} Y$ is horizontal at gk_2 for any $k_2 \in K''$. So, since right translations by elements in K are isometries, by changing g to gk_2 and Y by $\mathrm{Ad}_{k_2^{-1}} Y$, we may assume that $y_1 = y_2$, which we will simply denote by y. This way, as $\langle Y, U_1 - \mathrm{Ad}_g U_2\rangle = 0$, for every $(U_1, U_2) \in \mathfrak{u}$, we have the homogeneous real system in three variables, y_i, y_j, y_k,

$$\begin{cases} \langle \mathrm{Ad}_g Y, d\psi_1(i) - d\psi_3(i)\rangle = 0 \\ \langle \mathrm{Ad}_g Y, d\psi_1(j) - d\psi_3(j)\rangle = 0 \\ \langle \mathrm{Ad}_g Y, d\psi_1(k) - d\psi_3(k)\rangle = 0 \end{cases} \tag{5.2}$$

It is also straightfoward to verify that this system only admits trivial solution when $g = \mathrm{Id}$. Thus, as discussed in the previous case, it admits no zero curvature planes on a open dense subset of Sp(2). □

5.2.2 Sp(2)// SU(2)$_{\varphi_3}$

Topology

Let g be a singular point of Sp(2) by the φ_3-action. Then, by changing g to another point in its U-orbit if necessary, there is $(t_1, t_2) \in T \setminus (1, 1)$ such that $t_1 g t_2^{-1} = g$ or, equivalently, $g t_2 g^{-1} = t_1$. Write

$$t_1 = \mathrm{diag}(z^3, \overline{z}, \overline{z}^3, z), \quad t_2 = \mathrm{diag}(z, z, \overline{z}, \overline{z}).$$

By comparing eigenvalues we get $z^3 = z$ or $z^3 = \overline{z}$. In first case we get $z = \pm 1 \in \ker \varphi_3$. In second case we get $z \in \{\pm 1, \pm i\}$, an element of order 2 or 4. Since conjugation does not change the order of an element, we have proven:

Lemma 5.5. *Let $u \in U \setminus \ker \varphi_3$ be an element fixing a point in G. Then u has order 2 or 4.*

Therefore, the lemma below characterizes the possible isotropy groups of $\mathrm{Sp}(2)//SU(2)_{\varphi_3}$.

Lemma 5.6. *Let K be a subgroup of $\mathrm{SU}(2)$ such that every non-trivial element of K has order 2 or 4. Then K is conjugate to one of:*

$$\mathbb{Z}_2 \cong \{\pm 1\}, \quad \mathbb{Z}_4 \cong \{\pm 1, \pm i\}, \quad Q_z = \{\pm 1, \pm i, \pm jz, \pm kz\},$$

in which $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

Proof. We will deal with SU(2) as Sp(1) in this proof, since they are ismorphic as Lie groups.

Let K be a 4-subgroup of Sp(1). Then it contains an element of order 2, namely -1 (this is the only element of order 2 in Sp(1).

Let $g \in K$ be an element of order 4. Then, $g \neq \pm 1$ and there is an unique maximal circle of Sp(1) containing g, which, up to conjugation, we may assume it is $\{e^{i\theta}\}$. Now $g = \pm i$ and, therefore, up to conjugation we know that K contains $\{\pm 1, \pm i\} \cong \mathbb{Z}_4$.

Note that $\mathbb{Z}_4$ is a possibility. Suppose then that K has more than 4 elements. Let $h \in K$ be an order 4 element such that $h \notin \mathbb{Z}_4$. By above, h is conjugate to i and we can write $h = qi\overline{q}$ for $q \in \mathrm{Sp}(1)$. Consider $k = iqi\overline{q} \in K$. Write $q = \alpha + j\beta$ with $\alpha, \beta \in \mathbb{C}$. Then,

$$h = qi\overline{q} = i(|\alpha|^2 - |\beta|^2) + 2j(i\overline{\alpha}\beta)$$

and

$$k = ih = -|\alpha|^2 + |\beta|^2 + 2j\overline{\alpha}\beta. \tag{5.3}$$

Note that $k \neq \pm 1$, so k has order 4. This implies that

$$k = pi\overline{p} = i(|\gamma|^2 - |\delta|^2) + 2j(i\overline{\gamma}\delta) \tag{5.4}$$

for $p = \gamma + j\delta \in \mathrm{Sp}(1)$. Comparing (5.3) with (5.4) we obtain that $|\alpha| = |\beta|$. Since $|\alpha|^2 + |\beta|^2 = 1$, we deduce that $|\alpha| = |\beta| = 1/\sqrt{2}$. Now

$$h = ke^{i\theta}.$$

We obtain the group

$$\{\pm 1, \pm i, \pm je^{i\theta}, \pm ke^{i\theta}\}.$$

(In particular, $\theta = 0$ gives the so-called *quaternion group*).

We finally prove that there can be no more elements in K. Suppose, to the contrary, there is another element of order 4. Then it is of the form $ke^{i\varphi}$. Now

$$(ke^{i\theta})(ke^{i\varphi}) = -e^{i(\varphi-\theta)}$$

must have order $1, 2$ or 4, so it must lie in $\mathbb{Z}_4 = \langle i \rangle = \{\pm 1, \pm i\}$. Hence $ke^{i\varphi}$ is already in K. ☐

To get the singular set in Sp(2) in this case we must project

$$\mathcal{B} := \mathrm{Fix}(\mathbb{Z}_4) = \left\{ \begin{pmatrix} 0 & -\overline{B} \\ B & 0 \end{pmatrix} : B \in \mathrm{U}(2) \right\} \cong \mathrm{U}(2).$$

But it contains the points that are "more singular", namely, the projection of $\mathrm{Fix}(Q_z)$.

We write

$$B = u \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix},$$

with $a, b, u \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$, $|u| = 1$. Observe that we can parametrize $B = [u, a, b] = [-u, -a, -b]$.

Also, the normalizer of $\mathbb{Z}_4$ is $N = \langle e^{i\theta}(\theta \in \mathbb{R}), j\rangle$. Observe that the map $\mathcal{B} \to \mathbb{R}\,\mathrm{P}^3$ given by $[u,a,b] \mapsto [a\overline{u}, b\overline{u}]$ induces an diffeomorphism $\mathcal{B}/N^0 \to \mathbb{R}\,\mathrm{P}^3$. Its is straightforward then to verify that the j-action on $\mathcal{B}$ induces a j-action on $\mathbb{R}\,\mathrm{P}^3$ given by $[a,b] \mapsto [\overline{a}, \overline{b}]$.

We deduce then that $\mathcal{B}/N$ is the orbifold $\mathbb{R}\,\mathrm{P}^3/\mathbb{Z}_2$; in which $\mathbb{Z}_2$ is generated by reflections on two geodesics, $\pi/2$ apart, namely, using real homogeneous coordinates the reflection is given by

$$[x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : -x_1 : x_2 : -x_3].$$

And the two geodesic (fixed point set) are

$$x_0 = x_2 = 0, \text{ and } x_1 = x_3 = 0. \tag{5.5}$$

The *"most singular"* set in Sp(2) is

$$\mathcal{B} \setminus \mathcal{B}_0 = \bigsqcup_{z \in S^1} \mathrm{Fix}(Q_z).$$

$\mathcal{B}_0$ is the interior of $\mathcal{B}$ and $\mathrm{Fix}(Q_z)$ is the disjoint union of four circles, namely,

$$\mathrm{Fix}(Q_z) = \{[\pm z, xz, yz], [\pm iz, xz, yz] : x, y \in \mathbb{R}, x^2 + y^2 = 1\}.$$

Observe that under the map $\mathcal{B} \to \mathbb{R}\,\mathrm{P}^3$ given by $[u,a,b] \mapsto [a\overline{u}, b\overline{u}]$

$$[\pm z, xz, yz] \mapsto [x, y] \in \mathbb{R}\,\mathrm{P}^3, \ [\pm iz, xz, yz] \mapsto [\pm ix, \pm iy] \in \mathbb{R}\,\mathrm{P}^3.$$

Now, $(\mathcal{B} \setminus \mathcal{B}_0)/N \subset \mathcal{B}/N$ coincides with the projections of the two closed geodesics (5.5).

$\mathcal{B}/N$ is the singular set on $\mathrm{Sp}(2)//\mathrm{SU}(2)_{\varphi_3}$ and $(\mathcal{B} \setminus \mathcal{B}_0)/N$ is its most singular set.

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