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Fases geométricas em medições fracas

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Geometric phases in weak measurements

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Resumo

O formalismo de medições fracas explora a simetria temporal da teoria quântica. Dentro deste panorama, consideráveis avanços têm sidos feitos em relação à medição de fases geométricas. No entanto, o problema têm sido limitado a apenas uma medição. Neste trabalho, exploramos o problema, extendendo o formalismo para uma sequência de medições. Exploramos ainda uma extensão do formalismo de fase geométrica não-Abeliana em sequências de medições incompletas a fim de incluir estados de altas dimensões no contexto das medições fracas. Para isto, nos focamos em medições sequenciais de projetores não compatíveis e combinamos a ideia de medições fracas e fase geométrica não-Abeliana. Neste sentido, encontramos que a fase geométrica não-Abeliana é importante para ganhar informação sobre o espaço de estados e bem como a conexão entre eles, cenário favorável para reproduzir a matriz de overlap e o loop de Wilson.

Palavras-chave: fases geométricas; medições fracas; informação quântica.

Abstract

The formalism of weak measurements exploits the time-symmetric feature of quantum theory. Within this context, considerable insights have been gained concerning the geometric phase. However, the current formalism has been limited to a single measurement. We address this gap in the literature, extending the formalism to a sequence of measurements. We also explore a natural extension of the Abelian geometric phase in sequences of incomplete measurements to include high dimensional quantum states in the weak measurement scenario. To achieve the goal, we focus on sequential weak measurements of noncommuting projectors and combine the idea of weak measurements and the notion of a non-Abelian geometric phase. In this way, we find that the non-Abelian geometric phase in the weak measurement scenario can be useful to gain information about the state space and the connection between states, favorable to reproduce the overlap matrix and the Wilson Loop.

Key-words: geometric phases; weak measurements; quantum information .

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Chapter 1

Introduction

Quantum information science deals with knowledge from the most fundamental level of nature. Over the past decades, the improvement of mathematical methods and experimental techniques offer coherent control of quantum systems and their interactions [1, 2].

The intensity of interactions defines the type and amount of information to be extracted from a quantum system at different scales of distance and temperature. The concept of weak values is related to experimental setups with small coupling parameters between the system and the device. The weak measurements role provides an interesting approach to measure quantum systems with minimum disturbance compared to the strong measurements. Since its early development, weak measurements present curious phenomena as well as a huge variety of experimental applicability [3].

Within this framework, the weak measurement of quantum ensembles provides valuable insights and partial information regarding the probabilistic nature and foundations of quantum theory. Recent applications cover quantum interferometric experiments to measure the geometric phase, quantum state tomography, tests of Bell's inequalities, and diverse quantum information processing tasks [4–8].

Furthermore, the geometric properties of quantum systems draw considerable attention due to their applicability to perform quantum computation. For instance, the non-commutativity of non-Abelian geometric phases can be used to implement a universal set of quantum gates, in what was called robust all-geometric quantum computation [9, 10].

The combination of the concepts of geometric phase and weak measurements allows us to retrieve quantum information on the state space, as suggested by Sjöqvist et al. [11, 12]. To access this structure, we use the concept of geometric phase (GP) formulated by Anandan and Pines [13]. The authors proposed a natural extension of GP to the projective Hilbert space by removing the adiabatic condition. This means no external parameter space is required to describe a cyclic evolution.

The present project aims to investigate the interplay between weak measurements

and GP in both, Abelian and non-Abelian structures. After a preliminary exploration of these topics, we extend the formalism developed in Refs. [11] and [14] to a sequence of measurements and also to the spin coherent states (SCSs).

This dissertation is organized as follows: Chapter 2 gives a brief introduction to the fundamental concepts in the literature of measurements. The next chapter looks at the geometric phases. Chapter 4 introduces the main idea of the formalism of the geometric phase in the weak measurement scenario. The main findings are given in Chapter 5. A discussion of the results is given in Chapter 6. The final considerations are shown in Chapter 7.

Chapter 2

Retrieving quantum information

One of the key problems of quantum information is to retrieve information from quantum systems. To achieve this goal, one approach is called quantum measurement or test. In this chapter, we present a brief introduction to this process and its formalism. We start with the quantum mechanics postulates, and the measurement process, then introduce the two-state vector formalism. At the end of this chapter, we describe how to perform weak measurements [15–18].

2.1 Fundamental concepts

The first postulate of quantum mechanics establishes the complex vector space used to describe quantum mechanical systems and their interactions, called Hilbert space and usually denoted by $\mathcal{H}$ [15].

Postulate 1: Any isolated physical system is described by its state vector, a *ket* which belongs to the complex vector space with inner product $\mathcal{H}$, known as the *state space* of the system.

We encode the information into state vectors in $\mathcal{H}$ to deal with quantum systems. This kind of information may be a silver atom with a definite spin orientation, a polarization state of photons, etc [16]. The state vector on the space $\mathcal{H}$ is called ket and represented by $|\rangle$. It is also possible to form a linear combination of kets. Let us consider a general quantum state $|\Psi\rangle$ in the computational basis

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \tag{2.1}$$

where coefficients $\alpha, \beta \in \mathbb{C}$ are probability amplitudes. This state has a two-dimensional state space. Moreover, the kets $|0\rangle$ and $|1\rangle$ are important to quantum information processing tasks, because they form possible basis states of a *qubit*. The qubit is a mathematical object that can store information. Many interesting phenomena rise from the coherent sum of coefficients or superposition, such as destructive

interference [19, 20]

The *bra* provides the transpose conjugated of the state defined in Eq. (2.1)

$$\langle \Phi | = \delta^* \langle \Phi_1 | + \gamma^* \langle \Phi_2 | , \tag{2.2}$$

where δ^* and γ^* are the complex conjugates of the probability amplitudes $\delta, \gamma \in \mathbb{C}$, respectively.

The space $\mathcal{H}$ is endowed with the inner product between two vectors $|\Psi\rangle$ and $|\Phi\rangle$, denoted by $\langle \Phi | \Psi \rangle$ [21], which can be computed using the corresponding bra $\langle \Phi |$. The overlap (inner product) between these states

$$\langle \Phi | \Psi \rangle = \delta^* \alpha \langle \Phi_1 | 0 \rangle + \gamma^* \beta \langle \Phi_2 | 1 \rangle \tag{2.3}$$

is a complex number. Note that, although the calculations depend on the basis one chooses for $\mathcal{H}$, the inner product depends only on $|\Psi\rangle$ and $|\Phi\rangle$. If the overlap is zero, then the states are said to be orthogonal.

To manipulate information, such as those contained in kets and bras, we need operators. An operator $\mathcal{P}$ acting on any state of this system produces another state $\mathcal{P} |\Psi\rangle$ [17]. Reciprocally, the Hermitian conjugate $\mathcal{P}^\dagger$ of the operator is defined by the requirement that

$$\langle \Psi | \mathcal{P}^\dagger | \Phi \rangle = \langle \Phi | \mathcal{P} | \Psi \rangle^* . \tag{2.4}$$

Of particular interest are the Hermitian operators. This kind of operator is associated with physical observables, under the requirement that $\mathcal{P}^\dagger = \mathcal{P}$. The eigenvalues λ_n of a Hermitian operator $\mathcal{P}$ satisfy the eigenvalue equation

$$\mathcal{P} |\lambda_n\rangle = \lambda_n |\lambda_n\rangle , \tag{2.5}$$

where the $|\lambda_n\rangle$ are eigenvectors of that operator and λ_n real numbers. With this background, the next postulate establishes the relation between kets that evolve in time, using operators.

Postulate 2: The evolution of a closed quantum system is described by a unitary transformation. That is, the state (2.1) at time t_1 is related to the system at time t_2 by a unitary operator U

$$|\Psi'\rangle = U(t_2, t_1) |\Psi\rangle . \tag{2.6}$$

Following the discussion, we can define an equivalent version of *Postulate 2*, given by the evolution in continuous time. The equation that governs the time evolution of

a given quantum system is called the *Schrödinger* equation

$$i\hbar \frac{d\,|\Psi\rangle}{dt} = H\,|\Psi\rangle\,, \tag{2.7}$$

where H is a hermitian operator known as *Hamiltonian* of the system. This operator, depending on the system, can be time-dependent or time-independent. In Sections (2.3) and (2.4) we deal with Hamiltonian and time-evolution operators in the weak measurement scenario.

2.2 Measurements

A measurement is a physical process corresponding to an evolution that a quantum system can undergo. This mechanism allows us to retrieve classical information from a quantum state. This is exactly what the *Postulate 3* establishes.

Postulate 3: Quantum measurements are described by a set $\{M_k\}$ of measurement operators. These are operators acting on the state space of the system being measured.

The measurement operators satisfy the completeness equation to preserve probabilities

$$\sum_k M_k^\dagger M_k = I, \tag{2.8}$$

where I is the identity matrix.

An ideal measurement of the observable M associated with the operator M, for instance, will yield as its result one of the real eigenvalues of M

$$|\psi\rangle = \sum_n m_n\,|\lambda_n\rangle\,, \tag{2.9}$$

where m_n are eigenvalues of M. The probability of getting the outcome λ_n is then $|m_n|^2$. Note that this decomposition is exactly what Eq.(2.5) establishes.

For instance, an intuitive example of a measurement is the measurement of a qubit, Eq.(2.1), on a computational basis, $\{|0\rangle\,, |1\rangle\}$. The operators $M_0 = |0\rangle\langle 0|$ and $M_1 = |1\rangle\langle 1|$ are associated with the two possible outcomes. The probability to get the outcome 0 is given by

$$p(0) = \langle\Psi|M_0^\dagger M_0|\Psi\rangle = |\alpha|^2. \tag{2.10}$$

The state of the system after the measurement process is

$$\frac{M_0 |\Psi\rangle}{\sqrt{\langle\Psi|M_0^\dagger M_0|\Psi\rangle}} = \frac{\alpha}{|\alpha|} |0\rangle . \tag{2.11}$$

The measurement of the states (2.1) and (2.10), leads to a probabilistic result. The probabilistic rule that governs the evolution of a quantum system after the measurement is called the Born rule, See Eq.(2.10). The rule says that the state may be found in the state $|0\rangle$ with probability $|\alpha|^2$ and into the state $|1\rangle$ with probability $|\beta|^2$. The rule guarantees the reproducibility of tests, i.e., the consistency of results for many repetitions of the test.

A special class of measurements is known as projective measurements, described as observables such as $\mathcal{P}$. The observable has a spectral decomposition [15]

$$\mathcal{P} = \sum_k k \, |p_k\rangle \langle p_k| , \tag{2.12}$$

where $|p_k\rangle \langle p_k|$ is the projector onto the eigenspace of $\mathcal{P}$ with eigenvalue k. The possible outcomes of the measurement correspond to the eigenvalues, k, of the observable.

It is worth noting that the state of the system after measurement is different from the initial state. The device interacts with the physical system under study, in such a way that a property of that system affects a corresponding property of the apparatus, such as the pointers' position and momentum. We call this kind of measurement a von Neumann or strong measurements [18].

We have different tools to predict the outputs of measurements. Here, we are interested in probability distributions or expected values of observables. However, the most common way to do this is by evaluating the expectation value of the operator $\mathcal{P}_1$ concerning the state $|\Psi\rangle$

$$\langle \mathcal{P}_1 \rangle = \frac{\langle\Psi|\mathcal{P}_1|\Psi\rangle}{|\langle\Psi|\Psi\rangle|^2}, \tag{2.13}$$

where the expectation value $\langle \mathcal{P}_1 \rangle$ corresponds to the mean value of the physical observable represented by $\mathcal{P}_1$ when the system is described by the wave function Ψ. As mentioned, this property is the mean or average of numerous measurements of $\mathcal{P}_1$ concerning the state Ψ.

A different approach proposed by Ref. [14] is to read out information immediately after the measurement process. Sometimes, the Hilbert space $\mathcal{H}$ is an infinite-dimensional space, such as the set of square-integrable functions. In this case, kets are also known as wave functions and are denoted by Ψ. The measurement device

is also supposed to be described by an infinite-dimensional Hilbert space, and the characteristic that we are interested in is described by a value q, generally called the pointer position, although it can be implemented physically in many ways.

To measure the pointer position q after a von Neumann measurement (pre/post-selection),

$$\langle q \rangle = \frac{\int q|\Psi_{\mathcal{M}_1}|^2 dq}{\int |\Psi_{\mathcal{M}_1}|^2 dq}, \tag{2.14}$$

where $\Psi_{\mathcal{M}_1}$ is the space distribution of the coupled wave function of the system and the measurement device.

This integral can be interpreted as the average value of the pointer position for a given ensemble $\Psi_{\mathcal{M}_1}$. It is worth noticing that we evaluate the deviation of the pointer distribution immediately after a strong measurement. The post-selection can retrieve information about the past of a quantum state. In the next section, we introduce the formalism that leads to the concept of weak measurements and pre- and post-selection.

2.3 Two-State Vector formalism

Here, we introduce the two-state vector formalism, a time-symmetric formulation of measurements in quantum theory. Our formalism is based on Ref.[22]. We also deal with the basic formalism introduced in Section (2.1).

The von Neumann measurement allows us to get information immediately after the measurement process. However, there are many situations in which we want to deal with the system's information at a specific time, such as in the relative past. The answer relies on the time-symmetric formulation of quantum theory known as two-state vector formalism (TSVF).

Based on the idea of "reduction of the wave packet", this scheme uses the interplay between the past and future of the strong measurement process with minimum disturbance in-between them. The TSVF combined with the notion of weak measurements provides a powerful description and tool to recover information about a quantum system at a time t. Particularly, this combination leads to interesting phenomena and various experimental applications.

The state $|\Psi_i\rangle$ which is obtained from a von Neumann measurement of an observable $\mathcal{P}$ at a time t_1 [23]

$$|\Psi_i\rangle = U(t, t_1)\,|\Psi_a\rangle = e^{-\mathrm{i}\int_{t_1}^{t} H dt}\,|\Psi_a\rangle \tag{2.15}$$

is called a pre-selected state, with $t_1 < t$ and $|\Psi_a\rangle$ as the initial state of the system

immediately before the von Neumann measurement.

The measurement for a backward evolving state arriving from the future

$$\langle \Psi_f | = \langle \Psi_b | U^{\dagger}(t, t_2) = \langle \Psi_b | e^{i \int_{t_2}^{t} H dt}, \tag{2.16}$$

where $\langle \Psi_f |$ is called post-selected state. The state $\langle \Psi_b |$ is the system's initial state at the time immediately before the measurement process.

The measurement disturbs the system, projecting its state into the eigenvectors (subspace) of the measurement operator. Then, after each strong measurement performed on the system, both states, Eqs.(2.15) and (2.16), are completely determined. The mathematical structure of quantum theory allows us to combine the system's history from both measurements.

We consider now the reduced wave packet at an intermediate time t

$$|\psi(t)\rangle = \langle \Psi_f | \, |\Psi_i\rangle \, , \tag{2.17}$$

in between the two strong measurements, where the state $|\Psi_i\rangle$ is defined by the results of (strong) von Neumann measurements performed in the relative past (pre-selection), Eq.(2.15) at the time $t_1 < t$ of a backward evolving quantum state $\langle \Psi_f |$ defined by the results of strong measurements performed on this system in the future, Eq.(2.16) at $t_2 > t$ (post-selection). Henceforth, we omitted the time in the notation.

In summary, we deal with the information obtained at an intermediate time in between the pre-selection and post-selection processes.

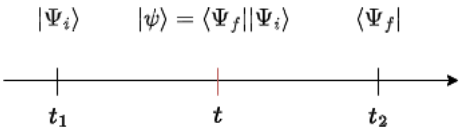

Figure 2.1: Two-state vector formalism in the weak measurement scheme, $|\psi\rangle$,$|\Psi_i\rangle$, and $\langle \Psi_f |$ are the generalized quantum state, pre-selected and post-selected states, respectively.

Fig. (2.1) shows the central idea of the reduced wave packet. It is possible to recover the expressions for prediction from time-symmetric expressions by separating the final (initial) selection procedure from the measurements under consideration, by performing strong measurements. The formalism differs from the usual scalar product between quantum states. For instance, Ref. [24] showed that weakly measuring some systems can minimize decoherence effects.

Note that the described formalism allows us to recover different steps of the history of a system by performing certain types of measurements. We focus on this special type of measurement in the next section.

2.4 Weak measurements

Revisiting Section (2.2), Eq.(2.14) shows the pointers' mean after the measurement process. Here, we describe how to get the information about the coupled system $\Psi_{\mathcal{M}_1}$ using the TSVF.

The weak value of an operator $\mathcal{P}$ is a physical property of a quantum system within the minimal disturbance scenario, in the time interval between two von Neumann measurements, using TSVF. The notion of weak value leads to fascinating and strange features. For instance, Ref. [25] showed that the outcome of a weak measurement for the spin component is 100. The result is known as the amplification effect. The reader can understand how it works in the following paragraphs.

The concept of the weak value was introduced by Aharonov, Albert, and Vaidman [25] in the context of the TVSF. Let an ensemble of particles prepared in the initial state $|\Psi_i\rangle$ and the final state $|\Psi_f\rangle$. The system and device interact during a finite time as described by the Hamiltonian

$$H(t) = -g(t)Q\mathcal{P}_B, \tag{2.18}$$

where $g(t)$ is a time-dependent coupling function, Q is an operator associated with the canonical variable of the measurement device and $\mathcal{P}_B$ acts on the system of interest. We focus on the case where the weakly measured observable $\mathcal{P}_B$ is a projector.

At a certain time in between preparation and post-selection, we switch on the interaction (2.18), in which the initial state of each measurement apparatus is $[1/\sqrt{\Delta}(2\pi)^{1/4}]$ $e^{(-q^2/4\Delta^2)}$. After the post-selection, the reduced wave packet (2.17) is given by

$$\langle\Psi_f|\, e^{-i\int g(t)q\mathcal{P}_B dt}\, |\Psi_i\rangle\, e^{-q^2/4\Delta^2} \tag{2.19}$$

with measurement strength

$$\kappa = \frac{1}{\hbar}\int g(t)dt, \tag{2.20}$$

where $2\pi\hbar$ is Planck's constant.

The measurement strength quantifies the amount of correlations between the system and the measurement device. Now, if the disturbance is sufficiently small, the

spread (Δ) is also small. Then, we expand the exponential term

$$\begin{aligned}
&\langle\Psi_f|\, e^{-i\kappa q\mathcal{P}_B}\,|\Psi_i\rangle\, e^{-q^2/4\Delta^2} \\
&\cong \langle\Psi_f|\Psi_i\rangle\left[1+\frac{iq\kappa(\langle\Psi_f|\mathcal{P}_B|\Psi_i\rangle)}{\langle\Psi_f|\Psi_i\rangle}\right] e^{-q^2/4\Delta^2} \\
&\cong \langle\Psi_f|\Psi_i\rangle\, e^{\left[\frac{iq\kappa(\langle\Psi_f|\mathcal{P}_B|\Psi_i\rangle)}{\langle\Psi_f|\Psi_i\rangle}\right]e^{-q^2/4\Delta^2}} \\
&\cong \langle\Psi_f|\Psi_i\rangle\, e^{[iq\kappa(\mathcal{P}_B)_w]e^{-q^2/4\Delta^2}}
\end{aligned} \tag{2.21}$$

for Δ such that

$$\Delta\kappa \ll \frac{|\,\langle\Psi_f|\Psi_i\rangle\,|}{(|\,\langle\Psi_f|\mathcal{P}_B|\Psi_i\rangle\,|^{1/n})}, \tag{2.22}$$

where

$$(\mathcal{P}_B)_w = \frac{\langle\Psi_f|\mathcal{P}_B|\Psi_i\rangle}{\langle\Psi_f|\Psi_i\rangle} = \frac{\langle\Psi_f|b\rangle\,\langle b|\Psi_i\rangle}{\langle\Psi_f|\Psi_i\rangle} \tag{2.23}$$

is the first order of the weak value of the operator $\mathcal{P}_B$ concerning the pre-selected state $|\Psi_i\rangle$ and the post-selected state $|\Psi_f\rangle$ [3].

The weak value defined in Eq.(2.23) is a complex number, i.e., has real and imaginary parts. In the following paragraphs, the reader may note that these real and imaginary parts are related to the deviations on the pointers.

Note from Eqs. (2.23) and (2.13) that although conceptually similar to expectation values, these weak values may show curious behavior. For instance, if the overlap between the final and initial states $\langle\Psi_f|\Psi_i\rangle$ is a small number, i.e., the states are non-orthogonal, the weak values can assume large values. This effect is called the amplification effect and is very useful in signal amplification schemes.

The resulting wave packet $\psi(q)$ of the measuring apparatus reads

$$\psi(q) = e^{iq\kappa\mathrm{Re}(\mathcal{P}_B)_w}\, e^{\left[-\frac{1}{4\Delta^2}(q+2\kappa\Delta^2\mathrm{Im}(\mathcal{P}_B)_w)^2\right]}. \tag{2.24}$$

The wave packet is related to the weak values. The meaning of 'weak' is then related to defining the strength of interaction, Eq.(2.22). However, the weak measurement scheme also depends on the relation of pre- and post-selected states, $|\Psi_i\rangle$ and $\langle\Psi_f|$, to the observable $\mathcal{P}_B$, and the relation of these states to the operator Q for an interaction such as defined as in Eq. (2.18). This fact is of summary importance for the scheme developed in Section (4.1), in which we are interested in the read-out of the pointer device.

The post-selection procedure causes a shift in the position of the pointer by a

factor

$$\delta q = -\kappa \Delta^2 \mathrm{Im}(\mathcal{P}_B)_w \tag{2.25}$$

and its momentum by

$$\delta p = \hbar \kappa \mathrm{Re}(\mathcal{P}_B)_w, \tag{2.26}$$

where the shift in the canonical variables of the measurement apparatus is related to the real and imaginary parts of the weak values [11]. In the following chapters, we present a method to measure the geometric phase by combining real and imaginary parts of the weak values. From (2.22), the uncertainty for p for each measurement apparatus is $1/2\Delta\kappa$, much bigger than the measured value. However, for sufficiently large N , then $(1/\sqrt{N})\Delta p \ll (\mathcal{P}_B)_w$ can be ascertained with arbitrary accuracy.

Chapter 3

Geometric phase

In this chapter, we briefly review the broad perspectives of the non-Abelian geometric phase. When a quantum system evolves under a cyclic evolution it may acquire an additional geometric phase (GP) in contrast to the dynamical one. The abelian and non-abelian geometric phases are defined as quantum holonomies. Sjöqvist [12] showed that these GPs appear naturally in interferometric schemes. Moreover, these holonomies arise in different interferometric experiments with photons, condensed matter systems, and cold atoms [2].

Berry demonstrated that the GP arises from the geometric structure of the state space after a cyclic evolution in the adiabatic regime. Simon connected Berry's GP with the holonomy of a closed path. Wilczek and Zee demonstrated that non-Abelian gauge structures arise in simple quantum systems [26–28] .

A natural extension of the GP for the non-adiabatic scheme was given by Anandan and Pines [13]. This generalization of the GP allows us to explore the relation between the Berry potential and the curvature of projective Hilbert spaces. Here, we would like to highlight that the holonomy can be interpreted as the indicator of confining behavior in lattice gauge theory [10, 29–32].

3.1 The adiabatic geometric phase

This section is based on Berry's article [26]. The notion of adiabaticity relies on dynamic effects when the system is submitted to slow changes. The quantum mechanical version of the adiabatic theorem describes the long-time behavior of solutions of the Schrödinger equation with the Hamiltonian slowly evolving in time.

To speak about slow changes in time one needs an intrinsic time scale to determine what slow and fast mean. In quantum mechanics, the intrinsic time scale is usually determined by the energy gaps in the spectrum. If the spectrum is non-degenerate then the gap condition ($|E_n(t) - E_m(t)| > 0$, i.e., we can order the eigenenergies)

is automatically satisfied.

During a cyclic quantum evolution, the slow changing of external parameters gives rise to an adiabatic GP. Let the Hamiltonian H be slowly changed by varying external parameters $\mathbf{R} = (r_1, r_2, ...)$, such that $H(\mathbf{R})$. These parameters can be viewed as points of a manifold M, such that $\mathbf{R} = (r_1, r_2, ...) \in M$, then M is mapped to $H(\mathbf{R})$. We require that the spectrum of H be non-degenerate. Otherwise, the eigenstates will not provide a unique definition of the GP. The time evolution is governed by the Schrödinger equation [10]

$$H(\mathbf{R}(t))\,|\psi(t)\rangle = i\hbar\frac{d}{dt}\,|\psi(t)\rangle\,. \tag{3.1}$$

The eigenstates $|n(\mathbf{R})\rangle$ of $H(\mathbf{R})$ satisfy

$$H(\mathbf{R})\,|n(\mathbf{R})\rangle = E_n\,|n(\mathbf{R})\rangle \tag{3.2}$$

with eigenergies $E_n(\mathbf{R})$. The vector $|\Psi(0)\rangle = |n(\mathbf{R}(0))\rangle$ is the solution of the Schrödinger equation of the lowest energy of H. In the adiabatic limit, the vector $|\Psi(0)\rangle$ evolves at t into

$$|\psi(t)\rangle = \mathrm{e}^{\frac{-i}{\hbar}\int_0^t dt' E_n(\mathbf{R}(t))}\mathrm{e}^{i\gamma_n(t)}\,|n(\mathbf{R}(t)\rangle = \mathrm{e}^{i\phi}\mathrm{e}^{i\gamma_n(t)}\,|n(\mathbf{R}(t)\rangle\,, \tag{3.3}$$

where the quantity ϕ is the dynamical phase. However, the term $\gamma_n(t)$ is non-integrable, i.e., $\gamma_n(t)$ cannot be written as a function of $\mathbf{R}$ and may assume different values for each t.

We can evaluate $\gamma_n(t)$ making the requirement that $|\psi(t)\rangle$ must satisfy the Schrödinger equation. The direct substitution of (3.3) into (3.1) leads to

$$\frac{d}{dt}\gamma_n(t) = i\,\langle n(\mathbf{R}(t))|\nabla_R|n(\mathbf{R}(t)\rangle\cdot\frac{d}{dt}\mathbf{R}(t) \tag{3.4}$$

The total phase shift of (3.3) around the closed path $\mathcal{C}$ is given by

$$|\psi(t)\rangle = e^{i\gamma_n(\mathcal{C})}e^{\frac{-i}{\hbar}\int_0^t dt' E_n(\mathbf{R}(t))}\,|\psi(0)\rangle\,, \tag{3.5}$$

where we define the GP

$$\gamma_n(\mathcal{C}) = i\oint_{\mathcal{C}}\langle n(\mathbf{R})|\nabla_R|n(\mathbf{R})\rangle\cdot d\mathbf{R} \tag{3.6}$$

as the Berry phase. Note that the closed integral in parameter space is dependent on the traversed loop.

We may rewrite (3.6) as,

$$\gamma_n(\mathcal{C}) = i \oint_{\mathcal{C}} \mathcal{A}^n(\mathbf{R}) \cdot d\mathbf{R}, \tag{3.7}$$

where the quantity $\mathcal{A}^n(\mathbf{R})$ is called Berry connection coupled to the slow degrees of freedom. The Berry connection is a vector potential as in classical electrodynamics and its integral around $\mathcal{C}$ is analog to the magnetic flux [28].

Non-Abelian geometric phase

Simon [27] was the first to conciliate the concept of holonomy and the GP. The main idea concerns the parallel transport along loops in a fiber bundle. This means that in the geometric view, a loop corresponds to a cyclic path traced by a tangent vector at $\mathbf{R}$ of a manifold M endowed with a linear combination, the idea of parallel transportation along a closed curve [33].

In general, when a vector is parallel transported along a loop, the resulting vector is different from the original one. The difference can be related to the curvature of the connection between them. The geometric measure of the failure of the resulting vector to coincide with the initial one is known as a holonomy. This scheme was used to describe the effects of the two-dimensional electron gas in a uniform magnetic field, known as the quantum Hall effect [33, 34].

Consider the same loop $\mathcal{C}(t)$ and a choice of basis $|n(\mathbf{R}(0))\rangle$ of the previous section. If the vector $|n(\mathbf{R}(0))\rangle$ is parallel transported along this curve, the single-valued GP (3.6) is equivalent to the holonomy associated with this connection. Wilczek and Zee [28] proposed an extension of GP to the non-Abelian structure, which is a matrix-valued holonomy.

Let the Hamiltonian $H(\mathbf{R})$ depending on the set of external parameters. For an arbitrary set of basis $|\psi_a(t)\rangle$ we can set

$$H(\mathbf{R})\,|\psi_a(t)\rangle = 0 \tag{3.8}$$

where this choice can be made locally. The GP is a global feature of the quantum evolution, in which the external parameters are picked up locally along the path in state space [10].

In the adiabatic limit, we consider the solutions of the Schrödinger equation (3.1) such that $|\eta_a(0)\rangle = |\psi_a(0)\rangle$. Naturally, we can decompose a general state at t

$$|\eta_a(t)\rangle = U_{ab}(t)\,|\psi_b(t)\rangle\,, \tag{3.9}$$

and where the eigenstate $|\eta_a(t)\rangle$ is normalized .

The goal is to define the holonomy $U(t)$ using the previous assumption (3.9). The normalization condition requires that

$$\left\langle \eta_b | \frac{d}{dt} \eta_a \right\rangle = \left\langle \eta_b | \frac{d}{dt} U_{ac} | \eta_c \right\rangle + \left\langle \eta_b | U_{ac} | \frac{d}{dt} \eta_c \right\rangle = 0, \tag{3.10}$$

where the vector potential

$$\left\langle \psi_b | \frac{d}{dt} \psi_a \right\rangle = \left\langle \eta_b | U^{-1} \frac{d}{dt} U | \eta_a \right\rangle = A_{ab}, \tag{3.11}$$

where A_{ab} is matrix-valued and depends on the geometry of the space of degenerate levels. We can write the above equation in terms of a path-ordered integral

$$U(t) = \mathcal{P} e^{\int_0^t A(\tau) d\tau}. \tag{3.12}$$

Note that the ordered integral depends only on the path and not on its parametrization. In particular, for a closed path, the integral is the Wilson Loop (WL), which is gauge invariant.

If we set a different set of basis

$$|\psi'(t)\rangle = \Omega(t) |\psi(t)\rangle, \tag{3.13}$$

where $\Omega(t)$ is a function of t, i.e., $\Omega(t)$ is unchanged under local phase transformations of $|\psi(t)\rangle$.

The A field transforms as

$$A'(t) = \frac{d}{dt} \Omega \Omega^{-1} + \Omega A \Omega^{-1}, \tag{3.14}$$

as a proper gauge potential.

3.2 Non-adiabatic geometric phase

In this section, we present the mathematical formalism developed in Refs. [13, 35, 36] concerning the non-adiabatic GP. First, we explain the formalism introduced in Refs.[35, 36] and then, we explore the formalism of the non-adiabatic GP, introduced by Anandan and Pines [13].

The projective Hilbert space $\mathcal{P}_H$ represents the set of all possible physical states of the system. Each state represents a point in that space and the evolution of the system can be described by a curve in $\mathcal{P}_H$. For a finite-dimensional Hilbert space $\mathcal{H}(n)$, the space states $\mathcal{P}_H$ is a complex projective space $(n-1)-$ dimensional.

Interference patterns can occur during the evolution of the system, in which state

vectors interfere constructively and destructively. Hence, the probability amplitude of transition during a measurement is obtained from the overlap (Eq.(2.3)) between these states on $\mathcal{H}$. Therefore, the path of any state vector in $\mathcal{H}$ can be obtained from a knowledge of the motions of a complete set of state vectors.

Under a cyclic evolution of states, the state space is a curve $\mathcal{C}$ in $\mathcal{H}$, which projects to a curve $\mathcal{C}'$ in $\mathcal{P}_H$ with the projection map: $\Pi : \mathcal{H} \to \mathcal{P}_H$, that maps any state in $\mathcal{H}$ into the ray that contains it. For closed curves $\mathcal{C}'$, the process is called cyclic and the state that undergoes this evolution will be called a cyclic state. During such an evolution, the state vector acquires a phase factor, as shown in Ref. [35]. Now, let us give some mathematical background about the process.

Consider the time-dependent Hamiltonian $H = V_n(t) \oplus V_m(t)$, resulting from the decomposition of a $(n+m)-$ dimensional Hilbert space into two subspaces of dimensions n and m, respectively.

If the subspace V_n undergoes cyclic evolution, we can write $V_n(\tau) = V_n(0)$. Here, we are interested in the states and the connection between them in that subspace. Then, consider a set of orthonormal basis $\left\{ |\tilde{\psi}_a\rangle , a = 1, \dots n \right\}$ of $V_n(t)$, with $|\tilde{\psi}_a(t)\rangle = |\tilde{\psi}_a(0)\rangle$ for every a. Another possible choice is the orthonormal basis $\{|\psi_a(t)\rangle , a = 1, \dots n\}$ which evolves in time according to the Schrödinger equation, Eq.(3.1)

$$i\hbar \frac{d}{dt} |\psi_a(t)\rangle = H |\psi_a(t)\rangle \tag{3.15}$$

with $|\psi_a(0)\rangle = |\tilde{\psi}_a(0)\rangle$. Then we can associate a unitary evolution (2.6)

$$|\psi_a(t)\rangle = \sum_{b=1}^{n} U_{ba} |\tilde{\psi}_b(t)\rangle , \tag{3.16}$$

where U_{ba} is the connection between these states. Substituting Eq.(3.16) into Eq. (3.15), we obtain the path-ordering integral

$$U(t) = P \exp\left(\int_0^t i(A_{ab} - K_{ab}) \right) , \tag{3.17}$$

where $A_{ab} = \mathrm{i} \langle \tilde{\psi}_a | d/dt | \tilde{\psi}_b \rangle$ and $K_{ab} = (1/\hbar) \langle \tilde{\psi}_a | H | \tilde{\psi}_b \rangle$ are hermitian matrices and P represents the path ordering linked to this transformation. The geometric quantity A_{ab} depends on the space states. A similar development was given for the adiabatic case in Eqs.(3.11) and (3.12).

A natural choice of basis is $|\tilde{\psi}'_a(t)\rangle = \sum_{b=1}^{n} \Omega_{ba}(t) |\tilde{\psi}_b(t)\rangle$, with Ω_{ba} being a

unitary operator. The transformation of A_{ab} and K_{ab} is given by

$$A' = \mathrm{i}\Omega^{\dagger}\dot{\Omega} + \Omega^{\dagger}A\Omega,$$
$$K' = \Omega^{\dagger}K\Omega. \tag{3.18}$$

The next step is to relate a basis manifold for the subspaces and then, set a connection between them. We identify the Grassmann manifold $G_{n,m}$ as the set of all $n-$dimensional subspaces of $\mathcal{H}$. A natural option is to choose an open covering $\{U_\alpha\}$ of $G_{n,m}$ such that on each U_α, $\left\{|\tilde{\psi}_a\rangle\,, a = 1, \ldots n\right\}$ are smooths fields, with their values forming an orthonormal basis of V_n.

Therefore, the matrix $\mathrm{i}\,\langle\tilde{\psi}_a|d|\tilde{\psi}_b\rangle$ is linked to the coefficients of a non-Abelian connection or gauge field on the space $G_{n,m}$. If a state vector $|\psi\rangle \in V_n(t)$ is parallel transported with respect to this connection, then

$$\langle\tilde{\psi}_a|d|\psi(t)\rangle = 0, \tag{3.19}$$

with $a = 1, 2, \ldots n$.

In terms of differential geometry, this connection is in a vector bundle $E_{n,m}$ over $G_{n,m}$ such that the fiber of $E_{n,m}$ over $V_n \in G_{n,m}$ is V_n itself. Since the subgroup of the unitary group $U(n+m)$, that acts on $\mathcal{H}$, that leaves $V_n(t)$ invariant is $U(n) \times U(m)$,$G_{m,n}$ may also be taken to be $U(n,m)/U(n) \times U(m)$.

Now, consider a set of n-orthogonal vectors $\{|b_i\rangle\}$ of $V_n \in G_{m,n}$ with associated projection operator

$$\mathcal{P} = \sum_{i=1}^{N} |b_i\rangle\,\langle b_i|\,, \tag{3.20}$$

where the operator $\mathcal{P}$ is independent of the chosen orthonormal basis and therefore invariant under the unitary group $U(n)$ between the orthonormal basis of V_n.

Then the manifold $G_{m,n}$ can be connected to the set of rank-n projection operators P uniquely associated with the subspaces V_n. The set of $(n+m)$-frames can be identified with the group $U(n+m)$, and V_n with the equivalence class of $(n+m)$-frames each consisting of n vectors in V_n and m vectors in the orthogonal complement V_m of V_n in $\mathcal{H}$. We can also identify the Grassmann manifold as $G_{m,n} = U(n+m)/U(n)\times U(m)$.

The Stiefel manifold $S_{m,n}$ is the set of n-frames which $S_{m,n} = SU(n+m)/SU(m)$. The idea is to interpret the holonomy in terms of the connection of Stiefel-bundle over a Grassmann manifold. Then, $U(m+n)$ is a $U(m)$-principal fiber bundle over $S_{m,n}$ with projection map Φ, while $S_{m,n}$ is $U(n)$-principal fiber bundle over $G_{m,n}$ with $U(n) \times U(m)$ as the structure group and projection map $\chi = \Pi\Phi$.

There is a connection in the bundle $S_{m,n}$ over $G_{m,n}$ whose connection one-form with

respect to a field of n-frames $\{|b_i\rangle\}$ on $G_{m,n}$ is $B_{ij} = i\langle\tilde{\Psi}_i|d\tilde{\Psi}_j\rangle$. The orthonormality of $\{|b_i\rangle\}$ implies that B_{ij} is a Hermitian matrix, i.e., is in the Lie algebra of $U(n)$.

Anandan [36, 37] produced extensive works showing that this connection gives the non-Abelian GP in the cyclic evolution in a closed curve $\mathcal{C}$ in $G_{m,n}$. The formalism can be extended to n successive incomplete measurements. Two parallel bases in subspaces corresponding to P and $P' \in G_{n,m}$ are related by parallel transport along the geodesic C joining P and P'. For more technical details, see Ref. [13] .

Chapter 4

Geometric phase in weak measurements

Weak measurements are performed in experimental setups with precise control over the interaction between the system and the measuring device. For instance, it can be realized in interferometric experiments with photons encoding different polarization states [38]. We present different approaches to measure the GP using weak measurements. First, we implement the formalism of sequential measurements in Refs. [11, 14] for a single measurement. After that, we focus on the non-Abelian geometric phase, using the formalism formulated in Ref. [12].

4.1 Abelian geometric phase

Ref. [14] showed that the deflections on the pointers devices distributions are related to the real and imaginary parts of the weak values. These results draw our attention to focus on the mixed products of positions and momenta to measure the GP.

Let an ensemble of particles $\mathcal{S}$ prepared in the initial state

$$\Psi_{\mathcal{S},\mathcal{M}_1} = e^{-igp_1\mathcal{P}_1} U |\Psi_i\rangle_{\mathcal{S}} \phi(q_1), \tag{4.1}$$

where the system and device are coupled by g and evolves under U from $|\Psi_i\rangle$. The measurement of the observable $\mathcal{P}_i$ causes a shift in the pointer momentum p_1 and in the initial distribution $\phi(q_1)$.

The system and measuring device interact during a finite time interval according to the Hamiltonian

$$H(t) = g(t)P_1 \otimes \mathcal{P}_1, \tag{4.2}$$

where $\mathcal{P}_1$ is a one-dimensional projector , the time-dependent coupling parameter g turns on and off the interaction between the measurement device and the measured system and P_1 is the pointer momentum operator. Conditioned on the post selection of the state $|\Psi_f\rangle$, we get

$$\psi_{\mathcal{M}_1} = \langle\Psi_f| W e^{-igp_1\mathcal{P}_1} U |\Psi_i\rangle_{\mathcal{S}} \phi(q_1), \tag{4.3}$$

where the system evolves under U from $|\Psi_i\rangle$ to the point where $\mathcal{P}_1$ is measured, and finally under W to $|\Psi_f\rangle$. For simplicity, from now on we take $U, W = 1$.

Under the assumption of a small coupling parameter g, we expand the exponential term as

$$\psi_{\mathcal{M}_1} = \langle\Psi_f| \left(1 - gp_1\mathcal{P}_1 - \frac{g^2}{2}p_1^2\mathcal{P}_1^2 + \dots\right) \times e^{-igp_1\mathcal{P}_1} |\Psi_i\rangle \phi(q_1). \tag{4.4}$$

Using $p = -i\partial/\partial q$ ($\hbar = 1$), we write

$$\psi_{\mathcal{M}_1} = \langle\Psi_f|\Psi_i\rangle [\phi(q_1) - g(\mathcal{P}_1)_w\phi^{'}(q_1) + \frac{g^2}{2}(\mathcal{P}_1)_w^2 \times\phi^{''}(q_1) + O(g^3)], \tag{4.5}$$

where

$$(\mathcal{P}_1)_w = \frac{\langle\Psi_f|\mathcal{P}_1|\Psi_i\rangle}{\langle\Psi_f|\Psi_i\rangle} \tag{4.6}$$

is the weak value of the operator $\mathcal{P}_1$ with respect to the pre and post-selected states $|\Psi_i\rangle$ and $|\Psi_f\rangle$, respectively.

As mentioned, the post-selection causes a shift in the pointer device position. To analyze the distribution, we consider the expectation value of q_1 (2.14)

$$\langle q_1\rangle = \frac{\int q_1|\psi_{\mathcal{M}_1}|^2 dq_1}{\int |\psi_{\mathcal{M}_1}|^2 dq_1}. \tag{4.7}$$

To evaluate the deviation, we assume that the initial pointer distribution ϕ is real and its mean is zero, i.e., $\int q\phi^*(q)\phi(q)dq = 0$. Also, we assume that the pointer normal distribution is normalized, i.e., $\int \phi^2(q)dq = 1$. These assumptions leads to keep only the first order in g, and we get

$$\langle q_1\rangle = -g[(P_1)_w + (\overline{P_1})_w] \left(\int q_1\phi(q_1)\phi^{'}(q_1)dq_1\right). \tag{4.8}$$

Integrating (by parts) over the position space,

$$\int q_1\phi(q_1)\phi^{'}(q_1)dq_1 = \int q_1\phi^2(q_1)dq_1 - \int q_1\phi(q_1)\phi^{'}(q_1)dq_1 - \int \phi^2(q_1)dq_1, \tag{4.9}$$

where $\int q_1\phi^2(q_1)dq_1 = 0$ and $\int \phi^2(q_1)dq_1 = 1$ and then

$$\int q_1\phi(q_1)\phi^{'}(q_1)dq_1 = -\frac{1}{2}. \tag{4.10}$$

Also, using the properties of complex numbers [39, 40]

$$\begin{aligned} z + \overline{z} &= 2\mathrm{Re}(z) \\ i(z - \overline{z}) &= -2\mathrm{Im}(z). \end{aligned} \tag{4.11}$$

Then, we get

$$\langle q_1\rangle = g\mathrm{Re}[(\mathcal{P}_1)_w]. \tag{4.12}$$

Reciprocally, writing $\langle q_1\rangle$ in the momentum basis representation, $\phi(q_1) = \tilde{\phi}(p_1)$ and $\phi^{'}(q_1) = -ip_1\tilde{\phi}(p_1)$, its momentum is given by

$$\langle p_1\rangle = 2gv\mathrm{Im}[(\mathcal{P}_1)_w], \tag{4.13}$$

where $v = \int p^2\tilde{\phi}(p)dp$ is the velocity distribution. As mentioned, the deviations in the pointer device are related to the real and imaginary parts of the weak values [11]. Then, we may write the GP

$$\Delta_p = \arg[(\mathcal{P}_1)_w] \tag{4.14}$$

as the argument of the weak values.

Finally, with the results in Eqs. (4.12) and (4.13), we get

$$\Delta_p = \arctan\left(\frac{\langle p_1\rangle}{2v\langle q_1\rangle}\right). \tag{4.15}$$

Notice that by measuring the shift in position and momentum of the measuring device immediately after the weak interaction, we determine the Pancharatnam or Abelian geometric phase, as shown in Ref. [41]. Similar results obtained in Ref. [11] address the measure of the geometric phase in the weak measurement scenario. However, the formalism has been restricted to a single measurement. As a result, we extend the formalism to two measurements.

4.2 Formalism of holonomic scheme

Let us begin the discussion by introducing the concept of discrete holonomy. Following Ref. [12], consider a cyclic sequence $\mathcal{C}$ of $q+1$ projections (q of which are distinct). We can construct the geometric quantity $\Gamma_{\mathcal{C}}$

$$\Gamma_{\mathcal{C}} = \mathcal{P}_m \ldots \mathcal{P}_1, \tag{4.16}$$

which can be viewed as a sequence of projective filtering measurements.

In the N-dimensional Hilbert space, a projection onto a K-dimensional subspace p_b, spanned by a (nonunique) frame $\mathcal{F}_b = \{|b_k\rangle\}_{k=1}^{K}$ can be realised with a projector $\mathcal{P}_b$. The set of frames constitutes a Stiefel manifold, which is a fiber bundle with the Grassmanian as the base manifold and the set of K-dimensional unitary matrices as fibers. The overlap matrix is defined as

$$(\mathcal{F}_c|\mathcal{F}_b)_{kl} = \langle c_k|b_l\rangle\,, \tag{4.17}$$

the connection between the subspaces.

The product of overlap matrices of a sequence of projective measurements

$$D = (\mathcal{F}_1|\mathcal{F}_q)(\mathcal{F}_q|\mathcal{F}_{q-1})\ldots(\mathcal{F}_2|\mathcal{F}_1) \tag{4.18}$$

defines the holonomy (non-Abelian GP) of the system

$$U = |D|^{-1}D \tag{4.19}$$

where $|D| = \sqrt{DD^\dagger}$ is the positive part of D.

4.2.1 Holonomic scheme and weak measurements

Arecchi et al. [42] introduced the concept of atomic coherent states of two-level atoms. The spin coherent states (SCSs) are defined by the angular momentum operator in the Hilbert space as irreducible representations of some symmetry Lie group. Making the spin as a degree of freedom, we can characterize the states by a vector of spin direction. Here, we apply the concept of direct holonomy and the three-spin-$\frac{1}{2}$ setting formalism to the general spin j developed in Refs. [9, 43].

Let us introduce the main idea. Consider a measuring device described by the unit vector $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ in spherical coordinates. In the weak measurements scenario, the filtering measurements select the maximal angular momentum quantum numbers, $m = \pm j$. The selection corresponds to the 2-rank operators $\mathcal{P}_{\mathbf{n}_\alpha} = |j;\mathbf{n}_\alpha\rangle\langle j;\mathbf{n}_\alpha| + |j;-\mathbf{n}_\alpha\rangle|j;-\mathbf{n}_\alpha\rangle = \sum_{k=\pm}|j;k\mathbf{n}_\alpha\rangle\langle j;k\mathbf{n}_\alpha|$. The use of SCSs

simplifies the subsequent calculations since $|j;\mathbf{n}\rangle$ can be viewed as a product of $2j$ copies of spin-$\frac{1}{2}$ state $|\frac{1}{2};\mathbf{n}\rangle$ and $|-j;\mathbf{n}\rangle$ similarly as $2j$ copies of $|-\frac{1}{2};\mathbf{n}\rangle$ [9, 44].

The overlap matrix of an ensemble of particles with the pre- and post-selected states, $\mathcal{F}_b = \text{span}\left\{|\mathrm{b}_{k'}\rangle\right\}_{\mathrm{k}=1}^{\mathrm{K}}$ and $\mathcal{F}_c = \text{span}\left\{|\mathrm{c}_{k''}\rangle\right\}_{\mathrm{k}=1}^{\mathrm{K}}$ is defined as

$$(\mathcal{F}_c|\mathcal{F}_b)_{k'',k'} = \langle c_{k''}|b_{k'}\rangle\,, \tag{4.20}$$

which is the connection between the subspaces. For the SCCs, the overlap matrix takes the form [44]

$$(\mathcal{F}(\theta_c,\phi_c)|\mathcal{F}(\theta_b,\phi_b)) = \begin{pmatrix} R(c,b) & S(c,b) \\ (-1)^{2j}S(c,b)^* & R(c,b)^* \end{pmatrix}, \tag{4.21}$$

where

$$\begin{aligned} R(c,b) = \Bigg[&\cos\left(\frac{\theta_\mathrm{c}-\theta_\mathrm{b}}{2}\right)\cos\left(\frac{\phi_\mathrm{c}-\phi_\mathrm{b}}{2}\right) \\ &+\mathrm{i}\cos\left(\frac{\theta_\mathrm{c}+\theta_\mathrm{b}}{2}\right)\sin\left(\frac{\phi_\mathrm{c}-\phi_\mathrm{b}}{2}\right)\Bigg]^{2j}, \end{aligned}$$

$$\begin{aligned} S(c,b) = \Bigg[&\sin\left(\frac{\theta_\mathrm{c}-\theta_\mathrm{b}}{2}\right)\cos\left(\frac{\phi_\mathrm{c}-\phi_\mathrm{b}}{2}\right) - \\ &\mathrm{i}\sin\left(\frac{\theta_\mathrm{c}+\theta_\mathrm{b}}{2}\right)\sin\left(\frac{\phi_\mathrm{c}-\phi_\mathrm{b}}{2}\right)\Bigg]^{2j}, \end{aligned} \tag{4.22}$$

are rotations by the angles (θ,ϕ).

For any $j \in \frac{1}{2}\mathbb{N}$, we can find an irreductible representation of $SU(2)$. If j is a half-odd integer, then a projective representation of $SO(3)$ exists. This means that we describe each overleap as

$$(\mathcal{F}(\theta_c,\phi_c)|\mathcal{F}(\theta_b,\phi_b)) = \kappa U_{c,b}, \tag{4.23}$$

where $U_{c,b}$ is the relative phase associated to the transformation. In interferometric setups, the relative phase can be measured as oscillations caused by local manipulations of the internal states of the interfering particles.

Also κ is defined by

$$\kappa = \sqrt{|R(c,b)|^2 + |S(c,b)|^2} \tag{4.24}$$

the transition probability associated with this transformation.

Moreover, the number κ has an interesting interpretation in interferometric schemes and can be viewed as the visibility of such experimental devices. In order to associate

the non-adiabatic GP to the product of overlap matrices, we require that the overlap matrices are unitary up to a multiplicative factor. On the contrary, it is only possible to get the holonomy by performing unitary operations on the system [9, 12].

Chapter 5

Results

Here, we formulate three different forms of writing the weak values. The first consists of the TSVF in the vectorial form. In this context, we are interested in the direction of the pointer's device, i.e., the angles of the pre- and post-selected states as well as the projection operator of the weak measurement as degrees of freedom. The second represents the expectation values of the position and momentum of the pointers devices, and lastly, the formalism in terms of the components or pre- and post-selected states. The formalism can be extended to high-dimensional quantum systems. For instance, it can be applied when a photon carries many degrees of freedom.

5.1 Weak measurements and values

There is a non-trivial relation between the Abelian GP and the weak values. Here, we explore the geometric properties of SCSs to measure the GP. In this section, we write the pre- and post-selected SCSs, and the projection operator in terms of vectors in spherical coordinates, $|\Psi_i\rangle = |j; k'\mathbf{n}\rangle$, $\langle\Psi_f| = \langle j; k''\mathbf{m}|$ and $\mathcal{P}_{\mathbf{n}_\alpha} = \sum_{k=\pm} |j; k\mathbf{n}_\alpha\rangle \langle j; k\mathbf{n}_\alpha|$, with the indices k, k', $k'' = \pm$ corresponding to different combinations of signs.

The overlap between two SCSs [18, 45]

$$\langle j; k''\mathbf{m}|j; k'\mathbf{n}\rangle = e^{ij\Phi(\mathbf{n},\mathbf{m})} \left(\frac{1 + k''k'\mathbf{n}\cdot\mathbf{m}}{2}\right)^j, \tag{5.1}$$

where $\mathbf{n}\cdot\mathbf{m}$ is the scalar product between the vectors and $\Phi(\mathbf{n},\mathbf{m})$ is a real number. Then, the weak values become

$$(\mathcal{P}_{\mathbf{n}_\alpha})_w = \frac{\langle j; k''\mathbf{m}|j; k\mathbf{n}_\alpha\rangle \langle j; k\mathbf{n}_\alpha|j; k'\mathbf{n}\rangle}{\langle j; k''\mathbf{m}|j; k'\mathbf{n}\rangle}, \tag{5.2}$$

the sum of all possible combinations of signs.

As we are interested in a cyclic process, upon multiplication and division by $\langle j; k''\mathbf{m}|j; k'\mathbf{n}\rangle$, the above equation can be written

$$(\mathcal{P}_{\mathbf{n}_\alpha})_w = \frac{\langle j; k''\mathbf{m}|j; k\mathbf{n}_\alpha\rangle \langle j; k\mathbf{n}_\alpha|j; k'\mathbf{n}\rangle \langle j; k'\mathbf{n}|j; k''\mathbf{m}\rangle}{|\langle j; k''\mathbf{m}|j; k'\mathbf{n}\rangle|^2}$$
$$= \frac{e^{ij\Phi(k\mathbf{n}_\alpha, k''\mathbf{m})} e^{ij\Phi(k'\mathbf{n}, k\mathbf{n}_\alpha)} e^{ij\Phi(k''\mathbf{m}, k'\mathbf{n})} \left(\frac{1+kk''\mathbf{n}_\alpha\cdot\mathbf{m}}{2}\right)^j \left(\frac{1+kk'\mathbf{n}\cdot\mathbf{n}_\alpha}{2}\right)^j}{\left(\frac{1+k'k''\mathbf{m}\cdot\mathbf{n}}{2}\right)^j}, \tag{5.3}$$

which is defined in terms of the vectors $\mathbf{n}, \mathbf{m}, \mathbf{n}_\alpha$.

Our idea is based on the lattice approach. The vertices of the triangle represent the connection and thus, the overlap between the states. If the spherical triangle $\{\mathbf{n}, \mathbf{m}, \mathbf{n}_\alpha\}$ is an Euler triangle, then we get the Abelian GP

$$(\mathcal{P}_{\mathbf{n}_\alpha})_w = \frac{e^{ij\Delta_P(k''\mathbf{m}, k\mathbf{n}_\alpha, k'\mathbf{n})} \left(\frac{1+kk''\mathbf{n}_\alpha\cdot\mathbf{m}}{2}\right)^j \left(\frac{1+kk'\mathbf{n}\cdot\mathbf{n}_\alpha}{2}\right)^j}{\left(\frac{1+k'k''\mathbf{m}\cdot\mathbf{n}}{2}\right)^j}, \tag{5.4}$$

where the quantities $e^{ij\Delta_P(k''\mathbf{m}, k\mathbf{n}_\alpha, k'\mathbf{n})}$ are exactly the Pancharatnam GP [46].

Eq.(5.4) shows that the Abelian or Pancharatnam GP is the phase accumulated during the cyclic evolution. Note that, depending on the interplay between the sign of pre- and post-selected states and the measuring device, the probability amplitudes can interfere constructively or destructively.

The geometric properties are determined by different paths in the state space. The accumulation of phases is thus, related to the phase acquired by overlapping states. For SCSs, the relationship shows the interference between states. The connection arises due to the coherent behavior of SCSs and the geometric properties of the parameter space. The interference of different spin coherent states gives rise to a relative phase, which contributes to the geometric phase acquired during the cyclic evolution.

When SCSs with different orientations are superposed, their interference results in a relative phase that depends uniquely on the angle between the spin directions. This fact is fascinating because, for SCSs, the parameter space corresponds to the orientation of the apparatus device. By varying the angle of the pointers device, for instance, the system undergoes a cyclic evolution in the parameter space.

5.2 Sequence of measurements

In this section, we investigate a sequence of measurements. The advantage of performing weak measurements of such sequences concerns the possibility of combining incompatible observables. We based our formalism mostly on Ref.[14] to get an

extension of Ref.[11].

Our main task is to obtain the GP by combining different mixed products of momenta and position deviations. The advantage of performing such a sequence is to combine the results of two incompatible observables, by combining different degrees of freedom of the pointers devices, such as position and momentum. Refs.[5, 6] demonstrated different implementations of weak measurements to establish bounds of uncertainty and sequential measurements.

The idea is to read the pointer's position and momentum deviations immediately after the measurement, as highlighted in Eqs.(2.25) and (2.26). The observables $\mathcal{P}_1$ and $\mathcal{P}_2$, are projector operators measured at different times on a system $\mathcal{S}$.

Let us define the initial state of the coupled system in terms of the momentum-space wave functions

$$\Psi_{\mathcal{S},\mathcal{M}_1\mathcal{M}_2} = e^{-ig(p_2\mathcal{P}_2+p_1\mathcal{P}_1)}\,|\Psi_i\rangle_{\mathcal{S}}\,\phi(q_1)\phi(q_2), \tag{5.5}$$

where the subscripts $\mathcal{M}_1$ and $\mathcal{M}_2$ indicate the distributions of the pointers devices.

The system and measuring devices interact during a finite time interval according to the Hamiltonian

$$H(t) = g(t)(p_2 \otimes \mathcal{P}_2 + p_1 \otimes \mathcal{P}_1), \tag{5.6}$$

where $\mathcal{P}_2, \mathcal{P}_1$ are observables and the time-dependent coupling parameter g turns on and off the interaction between the measurement device and the measured system.

Conditioned on the post selection of the state $|\Phi_f\rangle$, we get

$$\Psi_{\mathcal{M}_1\mathcal{M}_2} = \langle\Psi_f|\,e^{-ig(p_2\mathcal{P}_2+p_1\mathcal{P}_1)}\,|\Psi_i\rangle\,\phi(q_1)\phi(q_2). \tag{5.7}$$

Now, if the disturbance of the measurement process on the system is sufficiently small, the terms in the order of $(gp_i)^n$, for any $n \geq 2$, can be neglected. Under this assumption, we expand the exponential term

$$\begin{aligned}\Psi_{\mathcal{M}_1\mathcal{M}_2} = \langle\Psi_f|\left(1 - gp_2\mathcal{P}_2 - \frac{g^2}{2}p_2^2\mathcal{P}_2^2 + \ldots\right)\\ \times\left(1 - gp_1\mathcal{P}_1 - \frac{g^2}{2}p_1^2\mathcal{P}_1^2 + \ldots\right)|\Psi_i\rangle\,\phi(q_1)\phi(q_2).\end{aligned} \tag{5.8}$$

Using $p = -i\partial/\partial q$, we get

$$\begin{aligned}\Psi_{\mathcal{M}_1\mathcal{M}_2} = \langle\Psi_f|\Psi_i\rangle\,[&\phi(q_1)\phi(q_2) - g(\mathcal{P}_1)_w\phi^{'}(q_1)\phi(q_2)\\ &-g(\mathcal{P}_2)_w\phi(q_1)\phi^{'}(q_2) + \frac{g^2}{2}(\mathcal{P}_1^2)_w\phi^{''}(q_1)\phi(q_2)\\ &+\frac{g^2}{2}(\mathcal{P}_1^2)_w\phi(q_1)\phi^{''}(q_2)\\ &+g^2(\mathcal{P}_2,\mathcal{P}_1)_w\phi^{'}(q_1)\phi^{'}(q_2)\\ &+O(g^3)],\end{aligned} \tag{5.9}$$

where

$$(\mathcal{P}_2,\mathcal{P}_1)_w = \frac{\langle\Psi_f|\mathcal{P}_2\mathcal{P}_1|\Psi_i\rangle}{\langle\Psi_f|\Psi_i\rangle}. \tag{5.10}$$

is the weak value of the sequential weak measurements.

Then the expected value of the combination of positions is

$$\langle q_1 q_2\rangle = \frac{\int\int q_1 q_2|\Psi_{\mathcal{M}_1,\mathcal{M}_2}|^2 dq_1 dq_2}{\int\int|\Psi_{\mathcal{M}_1,\mathcal{M}_2}|^2 dq_1 dq_2}. \tag{5.11}$$

The above equation is a double integral, concerning each distribution over the position spaces. To evaluate the integral, we make the following assumptions. The initial pointer distributions ϕ are real-valued [14] and its mean is zero, i.e.,

$$\int q\phi^*(q)\phi(q)dq = \int q\phi^2(q)dq = 0. \tag{5.12}$$

Also, for simplicity, we assume that the pointer normal distribution is normalized, i.e., $\int\phi^2(q)dq = 1$. With these assumptions, note that now the terms of order 0 and 1 in g in Eq. (5.11) vanish. Then we get

$$\begin{aligned}\langle q_1 q_2\rangle = g^2[&(\mathcal{P}_2,\mathcal{P}_1)_w + (\overline{\mathcal{P}_2,\mathcal{P}_1})_w + (\overline{\mathcal{P}_1})_w(\mathcal{P}_2)_w\\ &+(\mathcal{P}_1)_w(\overline{\mathcal{P}_2})_w]\left(\int q\phi(q)\phi^{'}(q)dq\right)^2.\end{aligned} \tag{5.13}$$

The integral $\int q\phi(q)\phi^{'}(q)dq = -1/2$. Using the properties in Eq. (4.11)

$$\langle q_1 q_2\rangle = \frac{g^2}{2}\mathrm{Re}[(\mathcal{P}_2,\mathcal{P}_1)_w + (\mathcal{P}_1)_w(\overline{\mathcal{P}_2})_w], \tag{5.14}$$

where $(\overline{\mathcal{P}_2})_w$ is the complex conjugate of the weak value of the operator $\mathcal{P}_2$.

This result was obtained in Ref. [14] and it is interesting to note that the product of pointers positions contains a supplementary term, indicated by $(\mathcal{P}_1)_w(\overline{\mathcal{P}_2})_w$.

The first step is to get the real and imaginary parts of the weak values by com-

bining the positions and momenta of the pointer devices. Then, we combine $\langle q_1, q_2\rangle$, $\langle p_1, p_2\rangle$, which are related to the real part of the weak values and $\langle q_1, p_2\rangle$ and $\langle p_1, q_2\rangle$, related to the imaginary part.

Similarly, we evaluate the product of momenta using Eq.(5.9) and switching the basis representation. The transformation from the q-representation to the p-representation [16]

$$\phi(q) = \int e^{ipq}\tilde{\phi}(p)dp. \tag{5.15}$$

And also,

$$\int e^{-ipq}\phi^{'}(q)dq = -ip\int e^{-ipq}\phi(q)dq \tag{5.16}$$

where $\tilde{\phi(p)} = \int e^{-ipq}\phi(q)dq$. Then, we write the expectation value

$$\langle p_1 p_2\rangle = \frac{\int\int p_1 p_2 |\Psi_{\mathcal{M}_1,\mathcal{M}_2}|^2 dp_1 dp_2}{\int\int |\Psi_{\mathcal{M}_1,\mathcal{M}_2}|^2 dp_1 dp_2}. \tag{5.17}$$

Again, note that the above equation is a double integral over the momentum space of each pointer distribution. Keeping the second order in g, we get

$$\begin{aligned}\langle p_1 p_2\rangle = g^2[-(\mathcal{P}_2,\mathcal{P}_1)_w - (\overline{\mathcal{P}_2,\mathcal{P}_1})_w + (\overline{\mathcal{P}_1})_w(\mathcal{P}_2)_w \\ +(\mathcal{P}_1)_w(\overline{\mathcal{P}_2})_w]\left(\int p\tilde{\phi}^2(p)dp\right)^2,\end{aligned} \tag{5.18}$$

where we assume that the pointer devices have the same distribution over the position and momentum spaces.

The expectation value of the momenta product

$$\langle p_1 p_2\rangle = 2g^2v^2\mathrm{Re}[-(\mathcal{P}_2,\mathcal{P}_1)_w + (\mathcal{P}_1)_w(\overline{\mathcal{P}_2})_w], \tag{5.19}$$

where $v = \int p^2\tilde{\phi}(p)dp$ is the velocity distribution.

Now, we evaluate the mixed products of position and momentum and relate them with the imaginary part of the weak values. Likewise, following the previous procedure and switching the basis representation, we get

$$\langle q_1 p_2\rangle = \frac{\int\int q_1 p_2 |\Psi_{\mathcal{M}_1,\mathcal{M}_2}|^2 dq_1 dp_2}{\int\int |\Psi_{\mathcal{M}_1,\mathcal{M}_2}|^2 dq_1 dp_2}. \tag{5.20}$$

Note, again that we have a double integral over the position and momentum spaces.

The expectation value

$$\langle q_1 p_2 \rangle = -g^2 v \mathrm{Im}[(\mathcal{P}_2, \mathcal{P}_1)_w + (\overline{\mathcal{P}_1})_w (\mathcal{P}_2)_w]. \tag{5.21}$$

Similarly, for other combinations of momentum and position

$$\langle p_1 q_2 \rangle = \frac{\int\int p_1 q_2 |\Psi_{\mathcal{M}_1,\mathcal{M}_2}|^2 dp_1 dq_2}{\int\int |\Psi_{\mathcal{M}_1,\mathcal{M}_2}|^2 dp_1 dq_2}. \tag{5.22}$$

Resulting into

$$\langle p_1 q_2 \rangle = -g^2 v \mathrm{Im}[(\mathcal{P}_2, \mathcal{P}_1)_w - (\overline{\mathcal{P}_1})_w (\mathcal{P}_2)_w]. \tag{5.23}$$

Once the results of weak measurements with the shift in the pointers device, we address the geometric phase by combining them. The idea is to evaluate the GP using the real and imaginary parts of the weak values. Then,

$$\langle q_1 q_2 \rangle - \frac{1}{(2v)^2} \langle p_1 p_2 \rangle = g^2 \mathrm{Re}[(\mathcal{P}_2, \mathcal{P}_1)_w]. \tag{5.24}$$

Similarly, for the second mixed product,

$$\frac{1}{2v} \langle q_1 p_2 \rangle + \frac{1}{2v} \langle p_1 q_2 \rangle = -g^2 \mathrm{Im}[(\mathcal{P}_2, \mathcal{P}_1)_w]. \tag{5.25}$$

Now, rewriting Eq.(5.10) upon multiplication and division by the overlap $\langle \Psi_f | \Psi_i \rangle$, we get our main result

$$\Delta_p = \arg(\mathcal{P}_2, \mathcal{P}_1)_w, \tag{5.26}$$

where Δ_p is the Pancharatnam GP. Note that the deduction does not depend on the pre-and post-selected states.

Thus, by measuring the shifts in position and momentum and combining them, we obtain the geometric phase. The pointer device is typically used to read out the measurement results in weak measurements, and variations in its behavior can provide valuable information. After performing a sequence of weak measurements, one can analyze and combine the deviations on the pointer devices to measure the geometric phase. For further studies, we intend to generalize this scheme for longer sequences of N measurements.

5.3 Reconstruction scheme

The knowledge of a quantum process is an important task in quantum information. For instance, weak measurements can be applied to perform quantum tomography SCSs. By weakly measuring spin observables along different axes, we can recover partial information about the state space. This kind of information is convenient to reconstruct the resulting overlap matrix.

Consider the special case in which we perform a weak measurement of the projection operator of rank-2. The resulting product of the overlap matrices (A.1) takes the form

$$\begin{aligned}\mathcal{F}_{c,b}\mathcal{F}_{b,a}\mathcal{F}_{a,c} = &\begin{pmatrix} \langle j;\mathbf{m}|j;\mathbf{n}_\alpha\rangle & \langle j;\mathbf{m}|j;-\mathbf{n}_\alpha\rangle \\ \langle j;-\mathbf{m}|j;\mathbf{n}_\alpha\rangle & \langle j;-\mathbf{m}|j;-\mathbf{n}_\alpha\rangle \end{pmatrix} \\ &\times \begin{pmatrix} \langle j;\mathbf{n}_\alpha|j;\mathbf{n}\rangle & \langle j;\mathbf{n}_\alpha|j;-\mathbf{n}\rangle \\ \langle j;-\mathbf{n}_\alpha|j;\mathbf{n}\rangle & \langle j;-\mathbf{n}_\alpha|j;-\mathbf{n}\rangle \end{pmatrix} \\ &\times \begin{pmatrix} \langle j;\mathbf{n}|j;\mathbf{m}\rangle & \langle j;\mathbf{n}|j;-\mathbf{m}\rangle \\ \langle j;-\mathbf{n}|j;\mathbf{m}\rangle & \langle j;-\mathbf{n}|j;-\mathbf{m}\rangle \end{pmatrix}. \end{aligned} \tag{5.27}$$

Writing in the vectorial form, we get the coefficients

$$A = |\langle j;\mathbf{m}|j;\mathbf{n}\rangle|^2(\mathcal{P}_{\mathbf{n}_\alpha+,+})_w + |\langle j;\mathbf{m}|j;-\mathbf{n}\rangle|^2(\mathcal{P}_{\mathbf{n}_\alpha+,-})_w, \tag{5.28}$$

$$B = |\langle j;-\mathbf{m}|j;\mathbf{n}\rangle|^2(\mathcal{P}_{\mathbf{n}_\alpha-,+})_w + |\langle j;-\mathbf{m}|j;-\mathbf{n}\rangle|^2(\mathcal{P}_{\mathbf{n}_\alpha-,-})_w, \tag{5.29}$$

$$C = p(\mathcal{P}_{\mathbf{n}_\alpha+,+})_w + q(\mathcal{P}_{\mathbf{n}_\alpha+,-})_w, \tag{5.30}$$

$$D = r(\mathcal{P}_{\mathbf{n}_\alpha-,+})_w + s(\mathcal{P}_{\mathbf{n}_\alpha-,-})_w, \tag{5.31}$$

where p, q, r and s are defined in Appendix A and hence, we get $p = r^*$ and $q = s^*$. Writing in a compact form

$$\mathcal{F}_{c,b}\mathcal{F}_{b,a}\mathcal{F}_{a,c} = \begin{pmatrix} A & C \\ D & B \end{pmatrix}. \tag{5.32}$$

Note that the weak measurements provide an interesting tool to extract partial information about the quantum state without a significant disturbance. By recon-

structing the overlap matrix, which characterizes the overlaps between different states, one can gain insights into the coherence properties and relative weights of different components in the quantum state. This information is valuable for understanding the system's behavior, studying quantum correlations, and exploring quantum interference phenomena.

However, it should be noted that, while the coefficients A and B are composed of transition probabilities, the other C and D coefficients have probability amplitudes. A possible interpretation is given in the next chapter.

Also, the reconstruction of the overlap matrix is useful for experimental verification in weak measurement scenarios. By comparing the experimental results with the theoretical values or reference states, one can assess the accuracy and fidelity of the weak measurements and measurement devices. This verification process helps ensure the reliability and validity of the obtained results and strengthens the experimental evidence supporting the weak measurement approach.

The reconstruction of the overlap matrix in the weak measurement scenario offers valuable insights into the coherence properties, entanglement, and superposition in quantum systems. It enables partial state characterization, and experimental verification, and plays a role in quantum tomography and state engineering, contributing to our understanding and manipulation of quantum phenomena.

5.4 Holonomy

In this section, our main purpose is to establish the relationship between the weak measurements and the non-Abelian GP, i.e., the holonomy. The notion of holonomy around a loop provides an interesting geometric view of the state space and the connections between the states. The key idea is to show that we can exploit different combinations of the pre-and post-selected states and the weak measurement device components to measure the GP [47–51].

Here, we follow the concept of holonomy, and its trace given by Wilczek-Zee [52] and the WL [53], respectively. One of the major limitations to adopting our method for obtaining geometric information is the applicability to direct holonomies [11]. For $j \in \frac{1}{2}\mathbb{N}$, the holonomy $U_{c,b}U_{b,a}U_{a,c}$ is given by (A.1)

$$\mathcal{F}_{c,b}\mathcal{F}_{b,a}\mathcal{F}_{a,c} = \left[\left(\kappa_{c,b}U_{c,b}\right)\left(\kappa_{b,a}U_{b,a}\right)\left(\kappa_{a,c}U_{a,c}\right)\right], \tag{5.33}$$

We can define the WL as a gauge invariant

$$\begin{aligned}\mathrm{Tr}[\mathcal{F}_{c,b}\mathcal{F}_{b,a}\mathcal{F}_{a,c}] \\ = \kappa_{c,b}\kappa_{b,a}\kappa_{a,c}\mathrm{Tr}[U_{c,b}U_{b,a}U_{a,c}] \\ = \kappa W_L. \end{aligned} \tag{5.34}$$

Here, κ is defined by

$$\kappa = 2^{-\frac{3}{2}}\left\{\sum_{k'',k',k} |\langle j;k''\mathbf{m}|j;k\mathbf{n}_\alpha\rangle|^2|\langle j;k\mathbf{n}_\alpha|j;k'\mathbf{n}\rangle|^2|\langle j;k'\mathbf{n}|j;k''\mathbf{m}\rangle|^2\right\}^{\frac{1}{2}}, \tag{5.35}$$

where the sum extends over all transition probabilities.

Now, we can plug Eqs. (5.28) and (5.29) into (5.34) yielding

$$\begin{aligned} W_L = \kappa^{-1}(A+B) = \kappa^{-1}|\langle j;\mathbf{m}|j;\mathbf{n}\rangle|^2|(\mathcal{P}_{\mathbf{n}_\alpha+,+})_w + \kappa^{-1}\langle j;\mathbf{m}|j;-\mathbf{n}\rangle|^2 \\ (\mathcal{P}_{\mathbf{n}_\alpha+,-})_w\kappa^{-1}|\langle j;-\mathbf{m}|j;\mathbf{n}\rangle|^2|(\mathcal{P}_{\mathbf{n}_\alpha-,+})_w + \\ \kappa^{-1}\langle j;-\mathbf{m}|j;-\mathbf{n}\rangle|^2(\mathcal{P}_{\mathbf{n}_\alpha-,-})_w, \end{aligned} \tag{5.36}$$

where the W_L is s gauge invariant defined on a closed contour. This result shows that we can recover geometric information about the state space through the non-Abelian GP [28]. However, as highlighted in Ref.[44] the above relation holds only for the case in which the holonomy is a unitary matrix up to a real number, as defined by Eq.(4.23).

Note that this approach allows us to capture the topological nature of the system. It seems reasonable that strong measurements can destroy some kind of information and thus, some topological features of the system. The Wilson loop provides an insightful view of the gauge fields and connections in the same view as Ref. [52]. By performing cyclic evolutions on the system, one can retrieve information about the gauge structure and thus, topological invariants associated with the path.

Chapter 6

Discussion

One of the main goals of this project was to attempt to find the relation between the weak measurements of higher rank operators and the non-Abelian GP. Wilson [53] proposed an invariant quantity to quantize gauge field theory on closed paths. Wilczek and Zee realized that the Wilson Loop (WL) can be an indicator of non-Abelian holonomies [52]. Based on these ideas, the WL could be useful to evaluate the non-Abelian GP and also, other related quantities such as the Chern number, and topological invariants.

Our approach relies on the lattice behavior which relates the connection between states and the measurements performed on a quantum system. Thus, after a sequence of measurements that correspond to a finite sequence of loops, Eq.(5.34), we define the WL to give geometric information about the state space.

In the weak measurement scenario, the post-selection allows us to get information about the state space. This information is accessible through the non-Abelian GP for the $j \in \frac{1}{2}\mathbb{N}$ SCSs case, where the transition probabilities are simply numbers and equal in magnitude for each step on the Grassmann manifold. This suggests some relation to the geometric structure of the system.

Notice that the operator in Eq.(5.2) corresponds to a superposition of the north and south poles of the Bloch sphere. Thus, it is natural that the solid angle composed by the vectors $\{k''\mathbf{m}, k\mathbf{n}_\alpha, k'\mathbf{n}\}$, for different combinations o signs $k'', k', k = \pm$, can be inferred by a weak measurement of spin in the $\mathbf{n}_\alpha$ direction. In this way, the closed paths acquire a specific GP associated with specific trajectories, i.e., different combinations of the angles imply different readouts of the device [54].

Note from the denominator of Eq. (5.3) that the weak values may assume large values as the post-selected being nearly orthogonal to the pre-selected state. This property is called weak value amplification (WVA) and can provide interesting and advantageous experimental effects [55]. This amplification effect has no classical analog and can be described in terms of quantum interference. For this reason, it is possible to interpret the overlap between the states in terms of transition amplitudes,

as follows.

For instance, Ref. [55] highlighted the use of WVA when the device is saturated with a high number of particles or in the case in which the detector cannot differentiate between two signals. For an interesting discussion of this topic, see [25], where it is shown that by weakly measuring improbable or rare events, it is possible to produce a reciprocal improbable outcome. For instance, the spin of an electron may acquire the value 100 [25]. In Ref. [56], the authors demonstrated the protection of SCSs from decoherence using the weak measurements scheme, which can be improved using rotations around arbitrary axes on the Bloch sphere. For instance, in the case of SCSs, the amplification effect becomes important in optical settings.

In contrast, the weak values diverge when $\mathbf{m} \to -\mathbf{n}$. This behavior can be interpreted as a limit for which a pre-selected state at the point $\mathbf{n}$ has a vanishing probability for it to pass a post-selection of $-\mathbf{n}$. The antipodal points $\pm\mathbf{n}$ correspond to orthogonal states in which the GP is undefined [11].

The transition probability quantifies the probability of getting a weak value given the pre- and post-selected states. On the other hand, the transition amplitude quantifies the overlap between those states. In Eqs. (5.30) and (5.31), the coefficients p, q, r, and s constitute transition amplitudes, in contrast to the transition probabilities in Eqs. (5.28) and (5.29).

Fig.(6.1) presents a schematic view of our interpretation

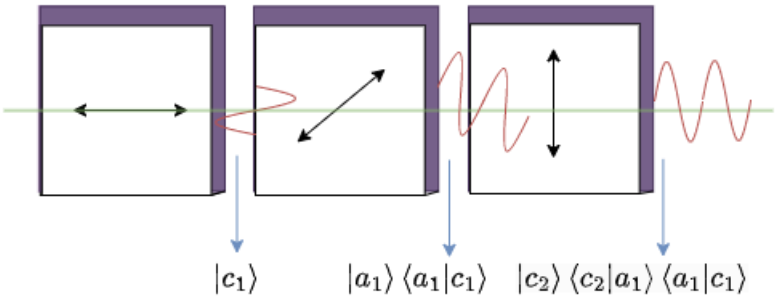

Figure 6.1: Conceptual representation of measurement back-action. The post-selected state $|c_1\rangle$ is weakly measured by sequential filtering measurements. After that, the final state is $|c_2\rangle\langle c_2|a_1\rangle\langle a_1|a_1\rangle$.

where the transition corresponds to the connection between the pre-and post-selected states.

For instance, Aharonov et al. [57] showed that quantum random walks can be described in terms of probability amplitudes. In this context, the amplitude represents the decision on whether a particle takes a given path, depending on the outcome of the measurement. In this sense, it can be interpreted as the partial information remaining after the measurement process.

The post-selection plays an important role in the scheme. The presence of transition amplitudes can be interpreted as the remaining or partial information about the past of the post-selected state. This is an important feature because it shows the nature of the weak measurement: the post-selected state carries some information about the interactions at a given time $t < t_2$. Moreover, due to this fact, it is possible to get information about the state space using the weak measurement scheme.

However, as highlighted in Ref. [44] the above relation holds only for the case in which the holonomy is a unitary matrix up to a real number, as defined by Eq.(4.23).

The significance of our findings provides considerable insight into searching for topological and geometric invariants. The deduction of the Wilson Loop in terms of weak values offers a compelling interpretation of invariants and the role of overlap between pre-selected and post-selected states. They can be considered a coming together of topology and geometry.

In summary, we have presented different approaches to measure and extract the geometric phase in the weak measurements scenario. Also, one possible method is to compare the outcomes of different measures along different paths and combine them to get the geometric phase. By analyzing the deviations or distributions of the pointer devices' position and momentum, one might observe patterns that correspond to the Abelian geometric phase. Future work will concentrate on these sequences.

Chapter 7

Conclusion

We have explored the extension of the Abelian GP in a sequence of incomplete measurements to include more than three states by performing sequential weak measurements. This thesis also has investigated the notion of non-Abelian GP in the weak measurement scenario. The results show that we can recover information about the state space by performing weak measurements. The formalism can be applied to longer sequences of operators. The advantage of performing a sequence of weak measurements concerns the possibility of combining incompatible observables.

The present study has only investigated the case in which the holonomy is a unitary matrix up to a real number. Our approach could be applied to the SCSs case ($j \in \frac{1}{2}\mathbb{N}$), where the transition probabilities are simply numbers and equal in magnitude for each step on the Grassmann manifold. This suggests some relation to the geometric structure of the system.

Our studies can be extended to a topological view in order to search for invariants in quantum materials, such as the Chern number, Bargmann invariant, or the Pancharatnam phase. Future work will investigate the possible implementations of interferometric, polarimetric schemes where κ has a physical interpretation. The motivations are to develop new experimental techniques and improve the amount of information that can be extracted from the quantum systems.

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Appendix A

Spin Coherent States (SCSs)

A.1 Product of the overlap matrices

In this section, we evaluate the product of the overlap matrices Here, we consider a special case of a given rank-2 projector operator.

$$\mathcal{F}_{c,b}\mathcal{F}_{b,a}\mathcal{F}_{a,c} = \begin{pmatrix} \langle c_1|b_1\rangle & \langle c_1|b_2\rangle \\ \langle c_2|b_1\rangle & \langle c_2|b_2\rangle \end{pmatrix} \begin{pmatrix} \langle b_1|a_1\rangle & \langle b_1|a_2\rangle \\ \langle b_2|a_1\rangle & \langle b_2|a_2\rangle \end{pmatrix} \begin{pmatrix} \langle a_1|c_1\rangle & \langle a_1|c_2\rangle \\ \langle a_2|c_1\rangle & \langle a_2|c_2\rangle \end{pmatrix} = \begin{pmatrix} A & C \\ D & B \end{pmatrix}, \quad \text{(A.1)}$$

where each matrix element represents the resulting overlap element. The formalism can be extended to any number of components of the pre and post-selected states, and the projection operator within the TSVF. The formalism can be extended to high-dimensional quantum systems and also to other degrees of freedom.

If we consider the weak value of a projector operator 2-rank

$$(\mathcal{P}^{\mathbf{n}}_{c_1,a_1})_w = \frac{\langle c_1|\mathcal{P}^{\mathbf{n}}|a_1\rangle}{\langle c_1|a_1\rangle} = \frac{\langle c_1|b_1\rangle\langle b_1|a_1\rangle}{\langle c_1|a_1\rangle} + \frac{\langle c_1|b_2\rangle\langle b_2|a_1\rangle}{\langle c_1|a_1\rangle}, \quad \text{(A.2)}$$

then we can write the matrix elements A, B, C and D as

$$A = |\langle c_1|a_1\rangle|^2(\mathcal{P}^{\mathbf{n}}_{c_1,a_1})_w + |\langle c_1|a_2\rangle|^2(\mathcal{P}^{\mathbf{n}}_{c_1,a_2})_w, \quad \text{(A.3)}$$

$$B = |\langle c_2|a_1\rangle|^2(\mathcal{P}^{\mathbf{n}}_{c_2,a_1})_w + |\langle c_2|a_2\rangle|^2(\mathcal{P}^{\mathbf{n}}_{c_2,a_2})_w, \quad \text{(A.4)}$$

$$C = \langle c_1|a_1\rangle\langle a_1|c_2\rangle(\mathcal{P}^{\mathbf{n}}_{c_1,a_1})_w + \langle c_1|a_2\rangle\langle a_2|c_2\rangle(\mathcal{P}^{\mathbf{n}}_{c_1,a_2})_w, \quad \text{(A.5)}$$

$$D = \langle c_2|a_1\rangle \langle a_1|c_1\rangle (\mathcal{P}^{\mathbf{n}}_{c_2,a_1})_w + \langle c_2|a_2\rangle \langle a_2|c_1\rangle (\mathcal{P}^{\mathbf{n}}_{c_2,a_2})_w. \tag{A.6}$$

Now, in order to get the relation between the matrix elements, we write

$$\begin{aligned} p &= \langle c_1|a_1\rangle \langle a_1|c_2\rangle , \\ q &= \langle c_1|a_2\rangle \langle a_2|c_2\rangle . \end{aligned} \tag{A.7}$$

Then, the matrix elements C and D become

$$C = p(\mathcal{P}^{\mathbf{n}}_{c_1,a_1})_w + q(\mathcal{P}^{\mathbf{n}}_{c_1,a_2})_w, \tag{A.8}$$

$$D = p^*(\mathcal{P}^{\mathbf{n}}_{c_2,a_1})_w + q^*(\mathcal{P}^{\mathbf{n}}_{c_2,a_2})_w, \tag{A.9}$$

where the coefficients C and D are related by the factors p and q .

A.2 Vectorial form of SCSs

Here, we deduce the expression for the overlap between two SCSs states.

Let $\mathbf{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ be a unit vector pointing in the spatial direction corresponding to a polar angle θ and azimuth ϕ, in spherical coordinates. We choose a normalized vector $|\mathbf{n}\rangle$ and for convenience, we choose $|\mathbf{n}\rangle$ to be an eigenvector of J_z. Thus $J_z|\mathbf{n}\rangle = m|\mathbf{n}\rangle$ and so we may denote $|\mathbf{n}\rangle$ by $|m;\mathbf{n}\rangle$ where $m = -j, -j+1, ..., j$. Hence, we also introduce the related expression

$$U(\theta,\phi) = e^{-i\phi J_z} e^{-i\theta J_y}, \tag{A.10}$$

and correspondingly we define,

$$|\theta,\phi\rangle = R(\theta,\phi)|m;\mathbf{n}\rangle . \tag{A.11}$$

However, we are interested only in the extreme values of m, i.e., when $m = \pm j$. The overlap becomes

$$\begin{aligned} \langle\theta',\phi'|\theta'',\phi''\rangle &= \langle m;\mathbf{n}|e^{i\theta'' J_y} e^{i(\phi'-\phi'')J_z} e^{-i\theta' J_y}|m;\mathbf{n}\rangle \\ &= \sum_{n=-j}^{j} \langle m;\mathbf{n}|e^{i\theta'' J_y}|n;\mathbf{n}\rangle \langle n;\mathbf{n}|e^{-i\theta' J_y}|m;\mathbf{n}\rangle e^{i(\phi'-\phi'')n}, \end{aligned} \tag{A.12}$$

which can be expressed in terms of the reduced Wigner coefficients of the spin-j

representation [58],

$$\mathcal{D}^{j}_{m,n} = \langle n|e^{-i\theta J_y}|m\rangle\,. \tag{A.13}$$

For the special case in which $m = j$, this expression leads to the result[18],

$$\langle\theta',\phi'|\theta'',\phi''\rangle = \left[\cos\left(\frac{\theta'}{2}\right)\cos\left(\frac{\theta''}{2}\right)e^{i(\phi'-\phi'')/2} + \sin\left(\frac{\theta'}{2}\right)\sin\left(\frac{\theta''}{2}\right)e^{-i(\phi'-\phi'')/2}\right]^{2j}. \tag{A.14}$$

Now, consider two SCSs states $\langle\mathbf{m}| = \langle(\theta_b,\phi_b)|$ and $|\mathbf{n}\rangle = |(\theta_a,\phi_a)\rangle$. The squared modulus of the overlap between them is defined by

$$\begin{aligned}|\langle\mathbf{m}|\mathbf{n}\rangle|^2 = &\left(\cos\left(\frac{\theta_b}{2}\right)\cos\left(\frac{\theta_a}{2}\right) + \cos(\phi_b-\phi_a)\sin\left(\frac{\theta_b}{2}\right)\sin\left(\frac{\theta_a}{2}\right)\right)^2 \\ &+\left(\sin(\phi_b-\phi_a)\sin\left(\frac{\theta_b}{2}\right)\sin\left(\frac{\theta_a}{2}\right)\right)^2 = \\ &= \cos^2\left(\frac{\theta_a}{2}\right)\cos^2\left(\frac{\theta_b}{2}\right) + \sin^2\left(\frac{\theta_a}{2}\right)\sin^2\left(\frac{\theta_a}{2}\right) \\ &+2\cos(\phi_a-\phi_b)\cos\left(\frac{\theta_a}{2}\right)\sin\left(\frac{\theta_a}{2}\right)\cos\left(\frac{\theta_b}{2}\right)\sin\left(\frac{\theta_b}{2}\right).\end{aligned} \tag{A.15}$$

Using the following trigonometric relation

$$\cos\left(\frac{\theta_a}{2}\right)\sin\left(\frac{\theta_a}{2}\right)\cos\left(\frac{\theta_b}{2}\right)\sin\left(\frac{\theta_b}{2}\right) = \frac{1}{2}\sin(\theta_a)\frac{1}{2}\sin(\theta_b), \tag{A.16}$$

we get,

$$\begin{aligned}|\langle\mathbf{m}|\mathbf{n}\rangle|^2 &= \frac{1}{2} + \frac{1}{2}\cos(\theta_a)\frac{1}{2}\cos(\theta_b) + \frac{1}{2}\cos(\phi_a-\phi_b)\sin(\theta_a)\sin(\theta_b) \\ &= e^{ij\phi(\mathbf{n},\mathbf{m})}\frac{1}{2}(1+\mathbf{n}\cdot\mathbf{m}).\end{aligned} \tag{A.17}$$

Expressed in terms of the vectors

$$|\langle\mathbf{m}|\mathbf{n}\rangle|^2 = e^{ij\phi(\mathbf{n},\mathbf{m})}\left(\frac{1+\mathbf{n}\cdot\mathbf{m}}{2}\right)^{j/2}, \tag{A.18}$$

where $\phi(\mathbf{n},\mathbf{m})$ is a real phase.

This leads to the result,

$$\langle\mathbf{m}|\mathbf{n}\rangle = e^{ij\Phi(\mathbf{n},\mathbf{m})}\left(\frac{1+\mathbf{n}\cdot\mathbf{m}}{2}\right)^{j}. \tag{A.19}$$

A.3 Transition Amplitudes and Probabilities

Using the concept of relative phase defined in (4.23), we can find the transition amplitude by writing

$$\kappa_{c,b} = \left[\frac{Tr(\mathcal{F}_{c,b}\mathcal{F}_{b,c})}{2}\right]^{\frac{1}{2}}$$
$$= \left[|\langle c_1|b_1\rangle|^2 + |\langle c_1|b_2\rangle|^2 + |\langle c_2|b_1\rangle|^2 + |\langle c_2|b_2\rangle|^2\right]^{\frac{1}{2}}, \tag{A.20}$$

$$\kappa_{b,a} = \left[|\langle b_1|a_1\rangle|^2 + |\langle b_1|a_2\rangle|^2 + |\langle b_2|a_1\rangle|^2 + |\langle b_2|a_2\rangle|^2\right]^{\frac{1}{2}}, \tag{A.21}$$

$$\kappa_{a,c} = \left[|\langle a_1|c_1\rangle|^2 + |\langle a_1|c_2\rangle|^2 + |\langle a_2|c_1\rangle|^2 + |\langle a_2|c_2\rangle|^2\right]^{\frac{1}{2}}. \tag{A.22}$$

In the vectorial form, we write,

$$\kappa_{c,b} = \left[\sum_{k'',k} \frac{|\langle j; k'\mathbf{m}|j; k\mathbf{n}_\alpha\rangle|^2}{2}\right]^{\frac{1}{2}}, \tag{A.23}$$

$$\kappa_{b,a} = \left[\sum_{k,k'} \frac{|\langle j; k\mathbf{n}_\alpha|j; k'\mathbf{n}\rangle|^2}{2}\right]^{\frac{1}{2}}, \tag{A.24}$$

$$\kappa_{a,c} = \left[\sum_{k',k''} \frac{|\langle j; k'\mathbf{n}|j; k''\mathbf{m}\rangle|^2}{2}\right]^{\frac{1}{2}}. \tag{A.25}$$

In this way, κ can be interpreted in terms of transition probabilities.

We can define the transition amplitudes as

$$\langle c_1|a_1\rangle = \langle j; \mathbf{m}|j; \mathbf{n}\rangle, \tag{A.26}$$

$$\langle c_1|a_2\rangle = \langle j; \mathbf{m}|j; -\mathbf{n}\rangle, \tag{A.27}$$

$$\langle c_2|a_1\rangle = \langle j; -\mathbf{m}|j; \mathbf{n}\rangle, \tag{A.28}$$

$$\langle c_2|a_2\rangle = \langle j; -\mathbf{m}|j; -\mathbf{n}\rangle. \tag{A.29}$$

And then we define the product of transition amplitudes

$$p = \langle c_1|a_1\rangle \langle a_1|c_2\rangle = \langle j;\mathbf{m}|j;\mathbf{n}\rangle \langle j;\mathbf{n}|j;-\mathbf{m}\rangle , \tag{A.30}$$

$$q = \langle c_1|a_2\rangle \langle a_2|c_2\rangle = \langle j;\mathbf{m}|j;-\mathbf{n}\rangle \langle j;-\mathbf{n}|j;-\mathbf{m}\rangle , \tag{A.31}$$

$$r = \langle c_2|a_1\rangle \langle a_1|c_1\rangle = \langle j;-\mathbf{m}|j;\mathbf{n}\rangle \langle j;\mathbf{n}|j;\mathbf{m}\rangle , \tag{A.32}$$

$$s = \langle c_2|a_2\rangle \langle a_2|c_1\rangle = \langle j;-\mathbf{m}|j;-\mathbf{n}\rangle \langle j;-\mathbf{n}|j;\mathbf{m}\rangle . \tag{A.33}$$

And similarly, for the general case, we can identify that $p = r^*$ and $q = s^*$. Rewriting the matrix elements A, B, C, and D,

$$A = |\langle j;\mathbf{m}|j;\mathbf{n}\rangle|^2(\mathcal{P}^{\mathbf{n}_\alpha}_{+,+})_w + |\langle j;\mathbf{m}|j;-\mathbf{n}\rangle|^2(\mathcal{P}^{\mathbf{n}_\alpha}_{+,-})_w, \tag{A.34}$$

$$B = |\langle j;-\mathbf{m}|j;\mathbf{n}\rangle|^2(\mathcal{P}^{\mathbf{n}_\alpha}_{-,+})_w + |\langle j;-\mathbf{m}|j;-\mathbf{n}\rangle|^2(\mathcal{P}^{\mathbf{n}_\alpha}_{-,-})_w, \tag{A.35}$$

$$C = p(\mathcal{P}^{\mathbf{n}_\alpha}_{+,+})_w + q(\mathcal{P}^{\mathbf{n}_\alpha}_{+,-})_w, \tag{A.36}$$

$$D = r(\mathcal{P}^{\mathbf{n}_\alpha}_{-,+})_w + s(\mathcal{P}^{\mathbf{n}_\alpha}_{-,-})_w, \tag{A.37}$$

where A more detailed explanation can be found in Appendix A and $p = r^*$ and $q = s^*$.

A.4 Reconstruction scheme for SCSs

For the SCSs, the coefficients A, B, C, and D are given by the elements of the overlap matrices or rotation matrices

$$A = R(c,a)R(a,c)\left(\frac{R(c,b)R(b,a) - S(c,b)S(b,a)^*}{R(c,a)}\right) + S(c,a)S(a,c)\left(\frac{R(c,b)S(b,a) + S(c,b)R(b,a)^*}{S(c,a)}\right), \tag{A.38}$$

$$B = S(c,a)^*S(a,c)^*\left(\frac{-S(c,b)^*R(b,a) - R(c,b)^*S(b,a)^*}{-S(c,a)^*}\right) + S(c,a)^*S(a,c)^*\left(\frac{-S(c,b)^*R(b,a) - R(c,b)^*S(b,a)^*}{R(c,a)^*}\right), \tag{A.39}$$

$$C = R(c,a)S(a,c)\left(\frac{R(c,b)R(b,a) - S(c,b)S(b,a)^*}{R(c,a)}\right) + S(c,a)R(a,c)^*\left(\frac{R(c,b)S(b,a) + S(c,b)R(b,a)^*}{S(c,a)}\right), \tag{A.40}$$

$$D = -S(c,a)^*R(a,c)\left(\frac{-S(c,b)^*R(b,a) - R(c,b)^*S(b,a)^*}{-S(c,a)^*}\right) - S(c,a)S(a,c)^*\left(\frac{-S(c,b)^*S(b,a) + R(c,b)^*R(b,a)^*}{R(c,a)^*}\right), \tag{A.41}$$

which we can get that $A = B^*$ and $C = -D^*$. Using (A.7), we can define the

$$D = (-1)^{2j}\{p(\mathcal{P}^{\mathbf{n}}_{c_2,a_1})_w + q(\mathcal{P}^{\mathbf{n}}_{c_2,a_2})_w\}^*. \tag{A.42}$$