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Deformações integráveis de teoria de cordas

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On integrable deformations of string theory

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O glaube, mein Herz, o glaube:
Es geht dir nichts verloren!
Dein ist, ja dein, was du gesehnt!
Dein, was du geliebt,
Was du gestritten!

O glaube
Du wardst nicht umsonst geboren!
Hast nicht umsonst gelebt, gelitten!

Was entstanden ist
Das muss vergehen!
Was vergangen, auferstehen!
Hör' auf zu beben!
Bereite dich zu leben!

O Schmerz! Du Alldurchdringer!
Dir bin ich entrungen!
O Tod! Du Allbezwinger!
Nun bist du bezwungen!

Mit Flügeln, die ich mir errungen,
In heißem Liebesstreben,
Werd' ich entschweben
Zum Licht, zu dem kein Aug' gedrungen!

Sterben werd' ich, um zu leben!
Aufersteh'n, ja aufersteh'n
wirst du, mein Herz, in einem Nu!
Was du geschlagen
zu Gott wird es dich tragen!

-Gustav Mahler, Die 2. Sinfonie-

Resumo

O nosso objetivo é estudar as deformações integráveis da teoria de supercordas em $AdS_4 \times \mathbb{CP}^3$ formulada como um modelo sigma não-linear no supercoset $\frac{UOSp(2,2|6)}{SO(1,3)\times U(3)}$. Estudamos a deformação de Yang-Baxter deste modelo e determinamos os backgrounds deformados nos quais a supercorda se propaga para algumas escolhas da matriz r. Para isto propomos algumas matrizes r que satisfazem a equação clássica de Yang-Baxter (CYBE) e determinamos os duais gravitacionáis da teoria ABJM não comutativa, da sua deformação de dipolo com um parâmetro e o seu limite não relativístico, que corresponde ao espaço-tempo de Schrödinger.

Palavras-chave: Supercordas; Modelo sigma não-linear; Integrabilidade; Deformações integráveis; Matriz r.

Abstract

Our aim is to study integrable deformations for the superstring theory in $AdS_4 \times \mathbb{CP}^3$ formulated as a σ-model on the supercoset $\frac{UOSp(2,2|6)}{SO(1,3)\times U(3)}$. We study the Yang-Baxter deformation of this model and determine the deformed backgrounds on which the string propagates for some choices of r-matrix. To this end we propose some r-matrices that satisfy the classical Yang-Baxter equation and show the gravity duals of the non-commutative ABJM theory, its one-parameter dipole deformation and its non-relativistic limit which corresponds to the so-called Schrödinger spacetime.

Keywords: Superstrings; Nonlinear sigma model; Integrability; Integrable deformations; r-matrix.

Contents

Chapter 1

Introduction

The AdS/CFT correspondence conjectures that certain gauge theories have a dual description in terms of string theories. The first case of the AdS/CFT correspondence states that $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on a four-dimensional flat spacetime is dual to type IIB superstring theory propagating in $AdS_5 \times S^5$ [1].

Many features of the AdS/CFT correspondence have been studied along the time including its integrability properties. On the string theory side, since it is formulated as two-dimensional field theory, the notion of integrability is associated to the existence of a Lax connection which ensures the existence of an infinite number of conserved charges. In the case of $AdS_5 \times S^5$ superstrings the theory is described as a σ-model on the supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ [2]. The $\mathbb{Z}_4$-grading of the $\mathfrak{psu}(2,2|4)$ superalgebra is a fundamental ingredient to obtain the Lax connection and thus to prove its integrability [3].

Recently, techniques to deform integrable theories keeping their integrability have been developed. One of them is based on r-matrices that satisfy the Yang-Baxter equation. These deformations were proposed by Klimcik as a way to obtain an integrable deformation of the Principal Chiral Model (PCM) [4, 5]. In this case, the type of r-matrix that was considered is called Drinfeld-Jimbo r-matrix [6, 7] and satisfies the modified classical Yang-Baxter equation. These deformations were also applied for the case of a symmetric coset σ-model [8], and furthermore to the $AdS_5 \times S^5$ σ-model [9, 10]. The supercoset construction was made in [11, 12] whose background is called η-deformed $AdS_5 \times S^5$. The important feature of this deformed background is that it does not satisfy the type IIB supergravity field equations. This fact led to postulate the existence of generalized supergravity equations [13, 14]. In a recent work [15], it was shown that the stardard supergravity equations are satisfied by an η-deformed background if the Drinfeld-Jimbo r-matrix associated to this deformation is constructed in a specific form. Furthermore, Yang-Baxter deformation of the $AdS_5 \times S^5$ in the pure spinor formulation was developed recently in [16].

It is possible to consider also an r-matrix that is solution of the classical Yang-Baxter equation (CYBE). In this case, the deformation of the symmetric coset σ-model was obtained in [17]. Moreover, this was studied for superstrings in $AdS_5 \times S^5$ in [18]. The interesting property of these deformations is that they lead to several known backgrounds of type IIB supergravity [19–22]: Lunin-Maldacena-Frolov [23, 24], Hashimoto-Itzhaki-Maldacena-Russo [25, 26] and Schrödinger spacetimes [27–29], which can be also obtained via TsT transformations [30]. In these cases, the r-matrices are all abelian. These results were extended to the nonabelian case [31] and it was conjectured in [32] that deformations using solutions of the CYBE are equivalent to nonabelian T-duality transformations [33, 34].

There is another type of integrable deformation known as λ-deformation, which was first introduced by gauging a combination of a PCM and a Wess-Zumino-Witten (WZW) model [35], and extended to string theory in symmetric spaces [36] and $AdS_5 \times S^5$ [37], as well as to the pure spinor formulation in [38]. Due to a work by Klimcik [39–42], it is conjectured that the η- and λ-deformations are related by an extension of nonabelian T-duality known as Poison-Lie T-duality [43, 44].

Another well-known example of the AdS/CFT correspondence is the duality between $\mathcal{N} = 6$ superconformal Chern-Simons theory in three dimensions (ABJM theory) and type IIA superstrings in $AdS_4 \times \mathbb{CP}^3$ [45]. The string theory is partially described by a nonlinear σ-model on the supercoset $UOSp(2, 2|6)/(SO(1, 3) \times U(3))$ [46, 47]. The superalgebra $\mathfrak{uosp}(2, 2|6)$ has a $\mathbb{Z}_4$-grading which allows to show the integrability of the model [46].

Only recently Yang-Baxter deformations of the nonlinear σ-model on this supercoset were considered. In [48], a solution of the CYBE for an abelian r-matrix, in which only the $\mathbb{CP}^3$ subspace was deformed, was found. This deformation leads to a three-parameter deformation of the $AdS_4 \times \mathbb{CP}^3$ background that can be obtained also by using TsT transformation [49]. Thus, inspired by that work, our aim is to study other possible integrable deformations of this background. The r-matrices associated to this deformation are constructed in terms of combinations of generators of the superalgebra $\mathfrak{uosp}(2,2|6)$. We propose a Drinfeld-Jimbo (DJ) r-matrix which involves only the Cartan basis of the superalgebra. In this case, the DJ r-matrix has only bosonic generators of the Cartan basis, which is the first step to construct the η-deformed $AdS_4 \times \mathbb{CP}^3$ background. In addition to this, we also provide some unimodular nonabelian r-matrices based on the classification given in [50].

Besides that, we provide the r-matrices that lead to the gravity duals of noncommutative ABJM theory as well as its one-parameter dipole deformation. These backgrounds were found initially by performing TsT transformations on the $AdS_4 \times \mathbb{CP}^3$ background [49]. In addition to this, we also present the r-matrix that leads to the gravity dual of the non-relativistic limit of ABJM which corresponds to the Schrödinger spacetime. We expect this result is compatible with the one obtained via a certain class of TsT transformations called null Melvin twists [51].

This thesis is organized as follows. In Chapter 2, we review the Green-Schwarz formalism for superstrings. We start with the case of flat space and present the symmetries of the theory. Then, we generalize it to curved backgrounds and present the Green-Schwarz action as a nonlinear σ-model. We focus on the case of superstrings in $AdS_5 \times S^5$ and examine in detail its $\mathfrak{psu}(2,2|4)$ superalgebra. We derive the corresponding Lax pair to show that the type IIB superstring theory in $AdS_5 \times S^5$ is integrable. In the Chapter 3 we review Yang-Baxter deformations and explain the relation between r-matrices and integrability. We present the Lagrangian of the Yang-Baxter deformed σ-model in terms of the R operator associated to the r-matrix. For models with $\mathbb{Z}_4$-grading we compute in detail its equations of motion and symmetries. We review the construction of different r-matrices leading to integrable deformations of the $AdS_5 \times S^5$ background. In Chapter 4, we discuss the Yang-Baxter deformation of the $AdS_4 \times \mathbb{CP}^3$ σ-model. We review the supermatrix realization of the $\mathfrak{uosp}(2,2|6)$ and its $\mathbb{Z}_4$-grading. Then, we discuss a nonlinear σ-model describing superstrings in $AdS_4 \times \mathbb{CP}^3$. After that we present the Yang-Baxter deformation of $AdS_4 \times \mathbb{CP}^3$ and discuss in detail its κ-symmetry. Finally, we compute some deformed backgrounds generated by r-matrices: the gravity dual of noncommutative ABJM theory as well as the gravity dual of the one-parameter dipole deformation and the non-relativistic limit of ABJM theory. In Chapter 5 we discuss some future perspectives along the lines of our work.

Chapter 2

The σ-model description of superstrings

This chapter is dedicated to the study of superstring theory as a nonlinear σ-model. First, we review superstrings on a flat space and its generalization to curved spaces. Then, we introduce the supercoset formulation of the superspace in order to write the Green-Schwarz action as a nonlinear σ-model. In this context we study the Green-Schwarz formalism for $AdS_5 \times S^5$ along the lines of [52]. We start by describing the $AdS_5 \times S^5$ background and the Green-Schwarz-Metsaev-Tseytlin action based on the supercoset formulation of the $\mathfrak{psu}(2,2|4)$ superalgebra. Finally, we discuss its integrability properties.

2.1 Type II superstrings in flat space

In order to have a supersymmetric theory fermionic fields must be introduced, either as worldsheet fermions, giving rise to the Ramond-Neveu-Schwarz (RNS) superstring theory, or as spacetime fermions, corresponding to the Green-Schwarz (GS) superstring theory. The GS formalism is more convenient since it can be applied to any curved background so let us start describing briefly this formalism in flat spacetime.

Supersymmetry can be introduced by generalizing the bosonic string action [53, 54],

$$S_1 = -\frac{1}{2\pi}\int_{\mathcal{M}} \mathrm{d}^2\,\sigma\sqrt{-g}g^{ab}\Pi_a^m\Pi_b^n\eta_{mn}, \tag{2.1}$$

where

$$\Pi_a^m = \partial_a X^m - i\delta_{AB}\bar{\theta}^{A\alpha}\left(\gamma^m\right)_{\alpha\beta}\partial_a\theta^{B\beta}, \tag{2.2}$$

X^m are bosonic coordinates with $m = 0, 1, ..., 9$, $\left(\gamma^m\right)_{\alpha\beta}$ are the 16×16 gamma matrices and $\mathcal{M}$ is the two-dimensional worldsheet with metric g_{ab}. The Grassmann coordinates are Majorana-Weyl fermions $\theta^{A\alpha}$ with spinor indices $A, B = 1, 2$ and $\alpha, \beta = 1, ..., 16$. These fermionic coordinates may have chiralities chosen independently. If θ^1 and θ^2 have opposite chirality, the theory is called type IIA, otherwise we refer it as type IIB superstrings.

In order to have the correct number of physical fermionic degrees of freedom we need to implement a local fermionic symmetry called κ-symmetry by adding to (2.1) an extra supersymmetric term known as the Wess-Zumino (WZ) term,

$$S_2 = -\frac{1}{\pi}\int_{\mathcal{M}} \mathrm{d}^2\,\sigma\{-\epsilon^{ab}(\bar{\theta}^1\gamma^m\partial_a\theta^1)(\bar{\theta}^2\gamma_m\partial_b\theta^2) + i\epsilon^{ab}\partial_a X_m(\bar{\theta}^1\gamma^m\partial_b\theta^1 - \bar{\theta}^2\gamma^m\partial_b\theta^2)\}, \tag{2.3}$$

where ϵ^{ab} is the Levi-Civita tensor. Thus, the supersymmetric action invariant under κ-symmetry is

$$S_{GS} = S_1 + S_2. \tag{2.4}$$

The WZ term S_2 is independent of g_{ab}, therefore it does not contribute to the energy-momentum tensor.

2.2 Type II superstrings in a curved background

In this section we study the GS superstrings action as a nonlinear σ-model on a coset superspace [55–57].

2.2.1 Green-Schwarz σ-model

In a curved background, the Green-Schwarz σ-model is

$$S_{GS} = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^2\,\sigma(\sqrt{-g}g^{ab}G_{MN}(Z) + \epsilon^{ab}B_{NM}(Z))\partial_a Z^M\partial_b Z^N, \tag{2.5}$$

where $Z^M = \left\{X^\mu, \theta^\alpha, \hat{\theta}^{\hat{\alpha}}\right\}$[1] are $\mathcal{N} = 2$, $D = 10$ curved supercoordinates, with $\mu = 0, ..., 9$ and $\alpha, \hat{\alpha} = 1, ..., 16$, and G_{MN} and B_{MN} are the background superfields. The first term in (2.5) corresponds to the kinetic term and the second one is the WZ term.

[1] A little change of notation was made here with respect to the previous section, we use hats instead of bars to discriminate the chirality of the spinorial coordinates.

At each point of this curved superspace we can define the supervielbein $E_M^A(Z)$, where $A = \{m, \alpha, \hat{\alpha}\}$ are the flat indices with $m = 0, ..., 9$. Then, G_{MN} and B_{MN} can be written as

$$G_{MN}(Z) = E_M^m(Z)E_N^n(Z)\eta_{mn}, \qquad B_{MN}(Z) = E_M^A(Z)E_N^B(Z)B_{AB}(Z), \tag{2.6}$$

where η_{mn} is a flat metric on each point Z of the superspace. Also, an orthonormal basis can be defined as

$$J^A = E_M^A \,\mathrm{d}\, Z^M, \qquad J_a^A = E_M^A \partial_a Z^M, \tag{2.7}$$

where $a = \{0, 1\}$ are the indices of worldsheet coordinates $\sigma^a = (\tau, \sigma)$ such that we can write on the worldsheet,

$$J^A = J_a^A \,\mathrm{d}\, \sigma^a. \tag{2.8}$$

In this terms we write the GS action (2.5) as

$$S_{GS} = -\frac{1}{2}\int \mathrm{d}^2\,\sigma \left(\sqrt{-g}g^{ab}J_a^m J_b^n \eta_{mn} + \epsilon^{ab}B_{AB}J_a^A J_b^B\right). \tag{2.9}$$

In particular, the WZ term can be expressed as

$$S_{WZ} = -\int B_{AB}\, J^A \wedge J^B = -\int_{\mathcal{M}} B, \tag{2.10}$$

which represents the integral of a two-form.

The type II GS action in flat space (2.4) can be recovered by taking

$$J_a^m = \Pi_a^m, \quad J_a^\alpha = \partial_a\theta^\alpha, \quad J_a^{\hat{\alpha}} = \partial_a\theta^{\hat{\alpha}}, \tag{2.11}$$

$$B_{m\alpha} = (\theta\gamma_m)_\alpha, \quad B_{m\hat{\alpha}} = -(\hat{\theta}\gamma_m)_{\hat{\alpha}}, \quad B_{\alpha\hat{\alpha}} = (\theta\gamma_m)_\alpha\,(\hat{\theta}\gamma^m)_{\hat{\alpha}}. \tag{2.12}$$

2.2.2 Supercoset formulation

One of the main motivations to study the GS superstring as nonlinear σ-model on a supercoset is that it allows us to manage algebraically the symmetries and properties of the theory as well as the demonstration of its integrability.

Coset spaces

This section introduces some notions about cosets and it is based on references [58–60]. A space $\mathcal{M}$ is said to be homogeneous if it admits as an isometry the transitive action of a group G, i.e. any point of the space can be reached from any other by the group action. Thus, it is natural to label a point X on $\mathcal{M}$ by parameters describing elements of G which move X to X'. The subgroup $H \subset G$ which leaves a point X on $\mathcal{M}$ fixed is called the isometry subgroup. Hence, there exists a redundancy when labelling $\mathcal{M}$ in terms of G. In order to describe $\mathcal{M}$ correctly we must identify those elements of the group that leave a point X on $\mathcal{M}$ fixed, which means to describe $\mathcal{M}$ in terms of the coset G/H. This equivalence is defined by the right action of $G/H : g \sim gh$, with $g \in G$ but not in H, and with $h \in H$.

If G is a Lie group we say that $\mathcal{M}$ is a coset manifold, then $\mathcal{M}$ has a Riemannian structure parametrized by coordinates.

The Lie algebra $\mathfrak{g}$ of G can be split as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}, \tag{2.13}$$

where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{k} = \mathfrak{g} \,/\, \mathfrak{h}$ contains the coset generators that remain in G/H. Thus, any element $g \in G$ can be expressed in the following form,

$$g = \exp\left(y^m K_m\right) \exp\left(x^i H_i\right), \qquad H_i \in \mathfrak{h}, \quad K_m \in \mathfrak{k}, \tag{2.14}$$

where y^m are the coordinates on the coset with $m = 1, ..., \dim G - \dim H$ and x^i are parameters of H with $i = 1, ..., \dim H$.
This suggests a natural parametrization of the coset space by choosing the representative

$$\exp\left(y^m K_m\right) \in G/H, \tag{2.15}$$

which corresponds to $x^i = 0$.

The Lie algebra-valued one-form

$$J(y) = g^{-1}(y)\, \mathrm{d}\, g(y), \qquad g(y) \in G \tag{2.16}$$

can be spanned in terms of the generators of $\mathfrak{g}$,

$$J(y) = J^m(y) K_m + \omega^i(y) H_i. \tag{2.17}$$

Here $J^m(y) = J^m_a(y)\, \mathrm{d}\, \sigma^a$ is a vielbein on G/H and $\omega^i(y) = \omega^i_a(y)\, \mathrm{d}\, \sigma^a$ is the spin-connection in terms of another set of coordinates σ^a. This one-form J satisfies the zero-curvature condition,

$$\mathrm{d}\, J + J \wedge J = 0 \tag{2.18}$$

by construction.

The simplest coset is the S^2 sphere. It can be written as the coset space $\frac{SO(3)}{SO(2)}$, where $SO(3)$ is the global symmetry group and $SO(2)$ is its local isometry. In general, an n-sphere is described as

$$S^n \equiv \frac{SO(n+1)}{SO(n)}. \tag{2.19}$$

Another useful and important construction is the coset for AdS spaces,

$$AdS_n \equiv \frac{SO(2, n-1)}{SO(1, n-1)}, \tag{2.20}$$

and the n-dimensional complex projective space $\mathbb{CP}^n$

$$\mathbb{CP}^n \equiv \frac{SU(n+1)}{U(n)}. \tag{2.21}$$

Green-Schwarz supercoset σ-model

The generalization to a supercoset consists in extending the numerator of the coset G/H to a supergroup such that it contains G as its bosonic subgroup. There exists a classification of Lie superalgebras, given in [61, 62], in which we can identify the bosonic subalgebras.

The Maurer-Cartan one-form J, in the case of a supercoset G/H [2], can be written in the same way as in (2.17),

$$J = J^A K_A + J^I H_I, \tag{2.22}$$

where $K_A \in \mathfrak{g}/\mathfrak{h}$ and $H_I \in \mathfrak{h}$ with $A = 1, ..., \dim G - \dim H$ and $I = 1, ..., \dim H$. We can write (2.22) as

$$J = J^A_M \,\mathrm{d}\, Z^M K_A + J^I_M \,\mathrm{d}\, Z^M H_I. \tag{2.23}$$

By taking $\mathrm{d}\, Z^M = \partial_a Z^M \,\mathrm{d}\,\sigma^a$ where $a = \{0, 1\}$ are the indices of worldsheet coordinates $\sigma^a = (\tau, \sigma)$ such that we can write

$$J^A_a = J^A_M \partial_a Z^M, \tag{2.24}$$

Then, if the target space is a supercoset the kinetic term for the GS superstring (2.9) can be constructed.

[2] Henceforth G refers to a supergroup with superalgebra $\mathfrak{g}$.

A σ-model on a superspace contains a term that can be constructed from a closed three-form $\Omega^{(3)}$, i.e. $\mathrm{d}\,\Omega^{(3)} = 0$, whose pullback on the worldsheet is built in terms of the Maurer-Cartan one-form [55, 63],

$$\Omega^{(3)} = \text{Str } J \wedge J \wedge J = f_{ABC} J^A \wedge J^B \wedge J^C, \tag{2.25}$$

where f_{ABC} are constants. This three-form is closed by construction due to the zero-curvature condition and the Jacobi identity. We can define $\Omega^{(3)} = \mathrm{d}\,B^{(2)}$. The WZ term is the integral of $\Omega^{(3)}$ on a three-dimensional manifold whose boundary is the string worldsheet,

$$S_{WZ} = -\int_{\mathcal{M}} B, \tag{2.26}$$

which has the same form as in (2.10) for GS superstring.

It was shown that this approach reproduces the type II GS superstrings on flat spacetime as a nonlinear σ model on the $\frac{SUSY(\mathcal{N}=2)}{SO(1,9)}$ coset, being $SO(1,9)$ the Lorentz subgroup of the $\mathcal{N} = 2$ super Poincaré group ten dimensional flat space.

Lie superalgebra

A superalgebra $\mathcal{V}$ is defined as a $\mathbb{Z}_2$-graded vector space. It can be written as $\mathcal{V} = \mathcal{V}^{(0)} \oplus \mathcal{V}^{(1)}$ where $\dim(\mathcal{V}^{(0)}) = m$, and $\dim(\mathcal{V}^{(1)}) = n$, for $m, n \geqslant 0$. The subalgebra $\mathcal{V}^{(0)}$ is called even or bosonic and $\mathcal{V}^{(1)}$ is called odd or fermionic.

A $\mathbb{Z}_2$-graded superalgebra $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ is a Lie superalgebra if it is equipped with a graded commutator defined as

$$[A, B] = AB - (-1)^{[\alpha][\beta]}\, BA, \tag{2.27}$$

that satisfies the Jacobi identity,

$$(-1)^{[\alpha][\gamma]}[A, [B, C]] + (-1)^{[\alpha][\beta]}[B, [C, A]] + (-1)^{[\beta][\gamma]}[C, [A, B]] = 0, \tag{2.28}$$

where $[\alpha], [\beta], [\gamma]$ correspond to the gradings of $A, B, C \in \mathfrak{g}^{(\alpha)}$ for $\alpha = 0, 1$

$$[\alpha] = \begin{cases} 0 & \text{if } \alpha \text{ is even,} \\ 1 & \text{if } \alpha \text{ is odd.} \end{cases} \tag{2.29}$$

Symmetric coset space

A Lie superalgebra $\mathfrak{g}$ with an automorphism Ω of order two, i.e. $\Omega: \mathfrak{g} \mapsto \mathfrak{g}$ with $\Omega^2 = I$ is split in the same way as in (2.13),

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}, \tag{2.30}$$

such that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}\,. \tag{2.31}$$

From the first relation in (2.31) $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$; the second one indicates that $\mathfrak{g}$ is reductive, and the third one that it is symmetric [59]. A superalgebra $\mathfrak{g}$ with these properties defines a symmetric space with $\mathbb{Z}_2$-grading.

There are superalgebras with an automorphism of order four, such that $\Omega^4 = I$, which is induced by a $\mathbb{Z}_4$-grading instead of a $\mathbb{Z}_2$. This defines semi-symmetric spaces like $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$ cases which we will consider in the following section and in Chapter 4.

2.3 Superstrings in $AdS_5 \times S^5$

The $AdS_5 \times S^5$ background is a solution of the type IIB supergravity equations together with a constant dilaton and a F_5 flux. This background plays a crucial role in the AdS/CFT correspondence since it is dual to $\mathcal{N} = 4$ SYM theory in four-dimensions [1,64].

2.3.1 The $AdS_5 \times S^5$ background

From (2.20) and (2.19), for $n = 5$, we have

$$AdS_5 \equiv \frac{SO(2,4)}{SO(1,4)}, \qquad S^5 \equiv \frac{SO(6)}{SO(5)}. \tag{2.32}$$

Thus, $AdS_5 \times S^5$ is written as the coset

$$AdS_5 \times S^5 \equiv \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}. \tag{2.33}$$

We need to look for a supergroup having $SO(2,4) \times SO(6)$ as its bosonic subgroup in order to describe superstrings in this background. Indeed, from the Nahm classification [62] we find that this bosonic group is part of the supergroup $PSU(2,2|4)$.

2.3.2 The Green-Schwarz-Metsaev-Tseytlin action

Metsaev and Tseylin constructed the type IIB Green-Schwarz superstring in $AdS_5 \times S^5$ as a nonlinear σ-model with target space given by the supercoset [2]

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}. \tag{2.34}$$

The $\mathfrak{psu}(2,2|4)$ superalgebra

In order to introduce the supermatrix realization of the $\mathfrak{psu}(2,2|4)$ superalgebra let us consider the general linear Lie superalgebra $\mathfrak{gl}(m|n)$. It is defined as the set of supermatrices

$$M = \begin{pmatrix} X & \theta \\ \eta & Y \end{pmatrix}, \tag{2.35}$$

where X is an $m \times m$-matrix and Y is an $n \times n$-matrix, both of even grading, and θ is an $m \times n$-matrix and η is an $n \times m$-matrix, both of odd grading. The operations of supertrace and supertranspose are defined as

$$\mathrm{Str}\, M \equiv \mathrm{tr}\, X - \mathrm{tr}\, Y, \qquad M^{st} = \begin{pmatrix} X^T & -\eta^T \\ \theta^T & Y^T \end{pmatrix}. \tag{2.36}$$

The special linear Lie superalgebra $\mathfrak{sl}(m|n)$ is defined as

$$\mathfrak{sl}(m|n) = \{M \in \mathfrak{gl}(m|n); \quad \mathrm{Str}\, M = 0\}. \tag{2.37}$$

In particular, the $\mathfrak{sl}(4|4)$ superalgebra is defined by $(4|4) \times (4|4)$ supermatrices M as in (2.35) with vanishing supertrace which are constructed in terms of 4×4 blocks. Then, in order to define the $\mathfrak{su}(2,2|4)$ Lie superalgebra M must also satisfy the following condition [52],

$$MH + HM^\dagger = 0, \tag{2.38}$$

where the Hermitian matrix H is defined as

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & I_{4\times 4} \end{pmatrix} \qquad \text{with} \qquad \Sigma = \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}. \tag{2.39}$$

The condition (2.38) acts on M as follows

$$\begin{pmatrix} X\Sigma & \theta \\ \eta\Sigma & Y \end{pmatrix} = \begin{pmatrix} -\Sigma X^\dagger & -\Sigma\eta^\dagger \\ -\theta^\dagger & -Y^\dagger \end{pmatrix}, \tag{2.40}$$

which implies that

$$X^\dagger = -\Sigma X \Sigma, \qquad Y^\dagger = -Y, \qquad \eta^\dagger = -\Sigma\theta. \tag{2.41}$$

From these conditions, the matrix blocks X and Y span the unitary algebras $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$, respectively. The bosonic superalgebra of $\mathfrak{su}(2,2|4)$ is then given by

$$\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1), \tag{2.42}$$

where $\mathfrak{u}(1)$ is the center factor [3]. By definition, the projective $\mathfrak{psu}(2,2|4)$ superalgebra corresponds to the quotient algebra of $\mathfrak{su}(2,2|4)$ over $\mathfrak{u}(1)$.

The most important property of $\mathfrak{psu}(2,2|4)$ is that it has a fourth-order automorphism $\Omega : M \mapsto \Omega(M)$ defined as

$$\Omega(M) = \begin{pmatrix} JX^\dagger J & -J\theta^\dagger J \\ J\eta^\dagger J & JY^\dagger J \end{pmatrix}; \quad J = \begin{pmatrix} 0 & -I_{2\times 2} \\ I_{2\times 2} & 0 \end{pmatrix}. \tag{2.43}$$

This definition satisfies $\Omega^4(M) = M$, that is $\Omega^4 = I$. So the linear map Ω has eigenvalues $\pm 1, \pm i$. Thus, if we denote $\mathcal{A}^{(k)}$ as the eigenspace associated to the eigenvalue i^k ($k = 0, 1, 2, 3$), we can write

$$\mathcal{A}^{(k)} = \left\{ M \in \mathfrak{psu}(2,2|4),\, \Omega(M) = i^k M \right\}. \tag{2.44}$$

This automorphism allows to decompose $\mathfrak{psu}(2,2|4)$ in a direct sum of four subspaces, implying that this superalgebra has a $\mathbb{Z}_4$-grading

$$\mathfrak{psu}(2,2|4) = \mathcal{A}^{(0)} \oplus \mathcal{A}^{(1)} \oplus \mathcal{A}^{(2)} \oplus \mathcal{A}^{(3)}, \tag{2.45}$$

where the subspaces satisfy

$$\left[\mathcal{A}^{(k)}, \mathcal{A}^{(m)}\right] \subseteq \mathcal{A}^{(k+m)} \quad \text{modulo} \quad \mathbb{Z}_4. \tag{2.46}$$

This happens because

$$\Omega\left([\mathcal{A}^{(k)}, \mathcal{A}^{(m)}]\right) = i^{k+m}[\mathcal{A}^{(k)}, \mathcal{A}^{(m)}]. \tag{2.47}$$

For a supermatrix $M \in \mathfrak{psu}(2,2|4)$ its projection $M^{(k)} \in \mathcal{A}^{(k)}$ is given by

$$M^{(k)} = \frac{1}{4}(M + i^{3k}\Omega(M) + i^{2k}\Omega^2(M) + i^k\Omega^3(M)). \tag{2.48}$$

Here the projections $M^{(0)}$ and $M^{(2)}$ are even, while $M^{(1)}$ and $M^{(3)}$ are odd.

[3]The center of a group is defined as $Z(G) = \{z \in G | \forall g \in G, zg = gz\}$

Constructing the action

The Maurer-Cartan one-form is defined as $A = -g^{-1}\,\mathrm{d}\,g$, where $g(\tau,\sigma)$ is an element of the supergroup $PSU(2,2|4)$ and A takes values in $\mathfrak{psu}(2,2|4)$. Due to the $\mathbb{Z}_4$-grading, A splits as

$$A = -g^{-1}\,\mathrm{d}\,g = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}, \tag{2.49}$$

and satisfies the zero-curvature condition $\mathrm{d}\,A - A \wedge A = 0$, which is written in components as

$$\mathcal{Z} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0. \tag{2.50}$$

Also there is a local symmetry which corresponds to right-multiplication of the coset representative g by $h(\tau,\sigma) \in SO(1,4) \times SO(5)$ (2.34)

$$g \to gh, \tag{2.51}$$

under which the projections of the currents transform as

$$\begin{aligned} A^{(0)} &\to h^{-1}A^{(0)}h - h^{-1}\,\mathrm{d}\,h, \\ A^{(i)} &\to h^{-1}A^{(i)}h, \quad i = 1,2,3, \end{aligned} \tag{2.52}$$

where the component $A^{(0)}$ transforms as a gauge field so we can interpret it as the $SO(1,4) \times SO(5)$ gauge sector [4], while the components $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ transform according to the adjoint representation of $SO(1,4) \times SO(5)$. Then, any gauge invariant action in the supercoset cannot contain $A^{(0)}$, but depends exclusively on the coset elements.

The action for the sigma-model of type IIB superstrings in $AdS_5 \times S^5$ is

$$S = \int \mathrm{d}^2\,\sigma \mathcal{L}, \tag{2.53}$$

and the density Lagrangian $\mathcal{L}$ in terms of A_α is then

$$\mathcal{L} = -\frac{1}{2}\left[\gamma^{\alpha\beta}\,\mathrm{Str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) + \kappa\epsilon^{\alpha\beta}\,\mathrm{Str}\left(A^{(1)}_\alpha A^{(3)}_\beta\right)\right], \tag{2.54}$$

where $\gamma^{\alpha\beta}$ is related to worldsheet metric $g_{\alpha\beta}$ as $\gamma^{\alpha\beta} = g^{\alpha\beta}\sqrt{-g}$ such that $\det\gamma = 1$ and $\epsilon^{\alpha\beta}$ is the Levi-Civita tensor [5].

[4] Since $[A^{(0)}, A^{(0)}] \subset A^{(0)}$ in (2.46).

[5] Here α and β denote worldsheet coordinates, so you should not confuse them with spinorial indices.

The first term in (2.54) corresponds to the bosonic kinetic term and the second term is the WZ term, which has contributions from the odd components of A_α, and thus it contains the fermionic degrees of freedom of the theory. The parameter κ must be a real constant number to guarantee the reality of the Lagrangian.

Equations of motion

By taking the variation of the Lagrangian (2.54) we obtain

$$\delta\mathcal{L} = -\frac{1}{2}\left[2\gamma^{\alpha\beta}\,\mathrm{Str}\left(\delta A^{(2)}_\alpha A^{(2)}_\beta\right) + \kappa\epsilon^{\alpha\beta}\,\mathrm{Str}\left(\delta A^{(1)}_\alpha A^{(3)}_\beta + A^{(1)}_\alpha \delta A^{(3)}_\beta\right)\right]. \tag{2.55}$$

From equation (2.48) and the following property

$$\mathrm{Str}(\Omega^k(M_1)M_2) = \mathrm{Str}(M_1\Omega^{4-k}(M_2)), \tag{2.56}$$

we can write the first term of (2.55) as follows

$$\begin{aligned}\mathrm{Str}(\delta A^{(2)}_\alpha A^{(2)}_\beta) &= \frac{1}{4}\,\mathrm{Str}\left(\delta A_\alpha A^{(2)}_\beta - \Omega(\delta A_\alpha)A^{(2)}_\beta + \Omega^2(\delta A_\alpha)A^{(2)}_\beta - \Omega^3(\delta A_\alpha)A^{(2)}_\beta\right)\\ &= \mathrm{Str}\left(\delta A_\alpha A^{(2)}_\beta\right).\end{aligned} \tag{2.57}$$

In a similar way the second term is

$$\epsilon^{\alpha\beta}\,\mathrm{Str}\left(\delta A^{(1)}_\alpha A^{(3)}_\beta + A^{(1)}_\alpha\delta A^{(3)}_\beta\right) = \frac{\epsilon^{\alpha\beta}}{4}\,\mathrm{Str}\left(\delta A_\alpha\left(A^{(1)}_\beta - A^{(3)}_\beta\right)\right). \tag{2.58}$$

Thus, we write the variation of the Lagrangian (2.54) as follows

$$\delta\mathcal{L} = -\,\mathrm{Str}\left(\delta A_\alpha\Lambda^\alpha\right), \tag{2.59}$$

where

$$\Lambda^\alpha = \gamma^{\alpha\beta}A^{(2)}_\beta - \frac{1}{2}\kappa\epsilon^{\alpha\beta}\left(A^{(1)}_\beta - A^{(3)}_\beta\right). \tag{2.60}$$

If we consider the variation of A_α,

$$\delta A_\alpha = -\delta\left(g^{-1}\partial_\alpha g\right) = -g^{-1}\delta g A_\alpha - g^{-1}\partial_\alpha(\delta g), \tag{2.61}$$

we can write the variation for the Lagrangian in (2.59) as

$$\delta\mathcal{L} = -\,\mathrm{Str}\left[g^{-1}\delta g\left(\partial_\alpha\Lambda^\alpha - [A_\alpha,\Lambda^\alpha]\right)\right], \tag{2.62}$$

such that the equations of motions are

$$\mathcal{E} \equiv \partial_\alpha \Lambda^\alpha - [A_\alpha, \Lambda^\alpha] = 0. \tag{2.63}$$

The grading 2 component of (2.63) is

$$\mathcal{E}^{(2)} \equiv \partial_\alpha \left(\gamma^{\alpha\beta} A^{(2)}_\beta\right) - \gamma^{\alpha\beta}\left[A^{(0)}_\alpha, A^{(2)}_\beta\right] + \frac{1}{2}\kappa\epsilon^{\alpha\beta}\left(\left[A^{(1)}_\alpha, A^{(1)}_\beta\right] - \left[A^{(3)}_\alpha, A^{(3)}_\beta\right]\right) = 0, \tag{2.64}$$

while the grading 1 and grading 3 components of (2.63) are given, respectively, by

$$\mathcal{E}^{(1)} \equiv \gamma^{\alpha\beta}\left[A^{(3)}_\alpha, A^{(2)}_\beta\right] + \kappa\epsilon^{\alpha\beta}\left[A^{(2)}_\alpha, A^{(3)}_\beta\right] = 0, \tag{2.65}$$

$$\mathcal{E}^{(3)} \equiv \gamma^{\alpha\beta}\left[A^{(1)}_\alpha, A^{(2)}_\beta\right] - \kappa\epsilon^{\alpha\beta}\left[A^{(2)}_\alpha, A^{(1)}_\beta\right] = 0. \tag{2.66}$$

Let us define

$$P^{\alpha\beta}_\pm = \frac{1}{2}\left(\gamma^{\alpha\beta} \pm \kappa\epsilon^{\alpha\beta}\right), \tag{2.67}$$

such that the equations of motion (2.65) and (2.66) can be written as

$$\mathcal{E}^{(1)} \equiv P^{\alpha\beta}_-\left[A^{(2)}_\alpha, A^{(3)}_\beta\right] = 0, \tag{2.68}$$

$$\mathcal{E}^{(3)} \equiv P^{\alpha\beta}_+\left[A^{(2)}_\alpha, A^{(1)}_\beta\right] = 0. \tag{2.69}$$

By varying the Lagrangian (2.54) with respect to $\gamma^{\alpha\beta}$ gives rise to the Virasoro constraint

$$\mathrm{Str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\delta}\,\mathrm{Str}\left(A^{(2)}_\rho A^{(2)}_\delta\right) = 0. \tag{2.70}$$

These constraints represent the reparameterization invariance of the string action with respect to worldsheet diffeomorphisms.

κ-symmetry

Since the global symmetry acts from the left we construct the κ-symmetry as a transformation from the right on g [65]

$$gG, \qquad G = \exp\epsilon, \tag{2.71}$$

where $\epsilon = \epsilon(\tau, \sigma)$ is a local fermionic infinitesimal parameter taking values in $\mathfrak{psu}(2,2|4)$. Unlike the global symmetry case, the string action is not invariant for an arbitrary form of the ϵ parameter. Then, we have to find the conditions on ϵ which guarantee the invariance of the action.

The transformation of the Maurer-Cartan one-form under the κ-symmetry transformations (2.71) is given by

$$\begin{aligned} A = -g^{-1}\,\mathrm{d}\,g \quad &\rightarrow \quad -(ge^{\epsilon})^{-1}\,\mathrm{d}(ge^{\epsilon}) \\ &= \quad e^{-\epsilon}Ae^{\epsilon} - \mathrm{d}\,\epsilon, \end{aligned} \tag{2.72}$$

so we write the transformation of A as [6]

$$\delta_\epsilon A = -\,\mathrm{d}\,\epsilon + [A, \epsilon]\,. \tag{2.73}$$

Let us consider $\epsilon = \epsilon^{(1)} + \epsilon^{(3)}$, in the fermionic sector, such that the above transformation is written as

$$\delta_\epsilon A^{(0)} = \left[A^{(3)}, \epsilon^{(1)}\right] + \left[A^{(1)}, \epsilon^{(3)}\right], \tag{2.74}$$

$$\delta_\epsilon A^{(2)} = \left[A^{(1)}, \epsilon^{(1)}\right] + \left[A^{(3)}, \epsilon^{(3)}\right], \tag{2.75}$$

$$\delta_\epsilon A^{(1)} = -\,\mathrm{d}\,\epsilon^{(1)} + \left[A^{(0)}, \epsilon^{(1)}\right] + \left[A^{(2)}, \epsilon^{(3)}\right], \tag{2.76}$$

$$\delta_\epsilon A^{(3)} = -\,\mathrm{d}\,\epsilon^{(3)} + \left[A^{(2)}, \epsilon^{(1)}\right] + \left[A^{(0)}, \epsilon^{(3)}\right]. \tag{2.77}$$

The variation of the Lagrangian (2.54) with respect to ϵ reads

$$\begin{aligned} \delta_\epsilon \mathcal{L} =& \delta_\epsilon \gamma^{\alpha\beta}\,\mathrm{Str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) + 2\gamma^{\alpha\beta}\,\mathrm{Str}\left(\delta_\epsilon A^{(2)}_\alpha A^{(2)}_\beta\right) \\ &+ \kappa\epsilon^{\alpha\beta}\,\mathrm{Str}\left(\delta_\epsilon A^{(1)}_\alpha A^{(3)}_\beta - \delta_\epsilon A^{(3)}_\alpha A^{(1)}_\beta\right). \end{aligned} \tag{2.78}$$

By using (2.75), we express the second term in (2.78) as

$$\begin{aligned} \mathrm{Str}\left(\delta_\epsilon A^{(2)}_\alpha A^{(2)}_\beta\right) &= \mathrm{Str}\left(\left[A^{(1)}_\alpha, \epsilon^{(1)}\right] A^{(2)}_\beta + \left[A^{(3)}_\alpha, \epsilon^{(3)}\right] A^{(2)}_\beta\right) \\ &= \mathrm{Str}\left(\left[A^{(2)}_\beta, A^{(1)}_\alpha\right] \epsilon^{(1)} + \left[A^{(2)}_\beta, A^{(3)}_\beta\right] \epsilon^{(3)}\right). \end{aligned} \tag{2.79}$$

Similarly, by using (2.76) and (2.77), the third and fourth terms become

$$\mathrm{Str}\left(\delta_\epsilon A^{(1)}_\alpha A^{(3)}_\beta\right) = \mathrm{Str}\left(-\partial_\alpha \epsilon^{(1)} A^{(3)}_\beta + \left[A^{(3)}_\beta, A^{(0)}_\alpha\right] \epsilon^{(1)} + \left[A^{(3)}_\beta, A^{(2)}_\alpha\right] \epsilon^{(3)}\right), \tag{2.80}$$

$$\mathrm{Str}\left(\delta_\epsilon A^{(3)}_\alpha A^{(1)}_\beta\right) = \mathrm{Str}\left(-\partial_\alpha \epsilon^{(3)} A^{(1)}_\beta + \left[A^{(1)}_\beta, A^{(2)}_\alpha\right] \epsilon^{(1)} + \left[A^{(1)}_\beta, A^{(0)}_\alpha\right] \epsilon^{(3)}\right). \tag{2.81}$$

[6] We use here the Baker-Hausdorff formula

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2}\,[X, [X, Y]] + ...$$

Putting (2.79), (2.80) and (2.81) into (2.78) we find the variation of the Lagrangian (2.54) under ϵ

$$\begin{aligned}
\delta_\epsilon \mathcal{L} =& -\frac{1}{2}\Big[\delta_\epsilon \gamma^{\alpha\beta}\operatorname{Str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) + 2\gamma^{\alpha\beta}\operatorname{Str}\left(\delta_\epsilon A^{(2)}_\alpha A^{(2)}_\beta\right) \\
&+ \kappa\epsilon^{\alpha\beta}\operatorname{Str}\left(\delta_\epsilon A^{(1)}_\alpha A^{(3)}_\beta - \delta_\epsilon A^{(3)}_\alpha A^{(1)}_\beta\right)\Big] \\
=& -\frac{1}{2}\Big[\delta_\epsilon \gamma^{\alpha\beta}\operatorname{Str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) - 2\gamma^{\alpha\beta}\operatorname{Str}\left(\left[A^{(1)}_\alpha, A^{(2)}_\beta\right]\epsilon^{(1)} + \left[A^{(3)}_\alpha, A^{(2)}_\beta\right]\epsilon^{(3)}\right) \\
&+ \kappa\epsilon^{\alpha\beta}\operatorname{Str}\Big(\partial_\alpha A^{(3)}_\beta \epsilon^{(1)} - \partial_\alpha A^{(1)}_\beta \epsilon^{(3)} + \left[A^{(3)}_\beta, A^{(0)}_\alpha\right]\epsilon^{(1)} + \left[A^{(3)}_\beta, A^{(2)}_\alpha\right]\epsilon^{(3)} \\
&+ \left[A^{(1)}_\beta, A^{(2)}_\alpha\right]\epsilon^{(1)} + \left[A^{(1)}_\beta, A^{(0)}_\alpha\right]\epsilon^{(3)}\Big)\Big]. \qquad (2.82)
\end{aligned}$$

In order to reduce this expression we use the zero-curvature condition (2.50) and the $\mathbb{Z}_4$ decomposition to compare the terms of grading 1 and 3 which gives

$$\epsilon^{\alpha\beta}\partial_\alpha A^{(1)}_\beta = \epsilon^{\alpha\beta}\left[A^{(0)}_\alpha, A^{(1)}_\beta\right] + \epsilon^{\alpha\beta}\left[A^{(2)}_\alpha, A^{(3)}_\beta\right], \qquad (2.83)$$

$$\epsilon^{\alpha\beta}\partial_\alpha A^{(3)}_\beta = \epsilon^{\alpha\beta}\left[A^{(0)}_\alpha, A^{(3)}_\beta\right] + \epsilon^{\alpha\beta}\left[A^{(1)}_\alpha, A^{(2)}_\beta\right]. \qquad (2.84)$$

By using these identities, the variation of the Lagrangian (2.82) is

$$\delta_\epsilon \mathcal{L} = -\frac{1}{2}\delta_\epsilon \gamma^{\alpha\beta}\operatorname{Str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) + 2\operatorname{Str}\left(P^{\alpha\beta}_+\left[A^{(1)}_\beta, A^{(2)}_\alpha\right]\epsilon^{(1)} + P^{\alpha\beta}_-\left[A^{(3)}_\beta, A^{(2)}_\alpha\right]\epsilon^{(3)}\right), \qquad (2.85)$$

where $P^{\alpha\beta}_\pm$ are defined in (2.67).

If we consider the Virasoro constraint (2.70), the first term in (2.85) is zero, whereas the last two terms vanish due to the equations of motion (2.68) and (2.69). This is an on-shell cancellation. In order to (2.85) be zero off-shell, and then represent a symmetry of (2.54), we need to find an appropriate form for $\delta_\epsilon\gamma^{\alpha\beta}$.

The orthogonality of $P^{\alpha\beta}_\pm$ implies that $\kappa = \pm 1$, satisfying the following relations

$$P^{\alpha\beta}_+ + P^{\alpha\beta}_- = \gamma^{\alpha\beta}, \qquad P^{\alpha\delta}_\pm P^{\beta}_{\pm\delta} = P^{\alpha\beta}_\pm, \qquad P^{\alpha\delta}_\pm P^{\beta}_{\mp\delta} = 0. \qquad (2.86)$$

By defining the projection $A^\alpha_\pm$ of any vector A^α as $A^\alpha_\pm = P^{\alpha\beta}_\pm A_\beta$, the variation in (2.85) can be written as

$$\delta_\epsilon \mathcal{L} = -\frac{1}{2}\delta_\epsilon \gamma^{\alpha\beta}\operatorname{Str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) + 2\operatorname{Str}\left(\left[A^{(1)\alpha}_+, A^{(2)}_{\alpha-}\right]\epsilon^{(1)} + \left[A^{(3)\alpha}_-, A^{(2)}_{\alpha+}\right]\epsilon^{(3)}\right). \qquad (2.87)$$

Now, we consider the following ansatz for the components of the fermionic parameter ϵ

$$\epsilon^{(1)} = A^{(2)}_{-\alpha}\kappa^{(1)\alpha}_+ + \kappa^{(1)\alpha}_+ A^{(2)}_{-\alpha}, \qquad (2.88)$$

$$\epsilon^{(3)} = A^{(2)}_{+\alpha}\kappa^{(3)\alpha}_- + \kappa^{(3)\alpha}_- A^{(2)}_{+\alpha}, \qquad (2.89)$$

where $\kappa_{\pm}^{(k)\alpha}$ are new independent parameters of the κ-symmetry transformations. The even traceless component $A^{(2)}$ can be expressed as a supermatrix

$$A^{(2)} = \begin{pmatrix} m^i\gamma^i & 0 \\ 0 & n^i\gamma^i \end{pmatrix}, \tag{2.90}$$

in terms of the $SO(5)$ Dirac matrices γ^i. The coefficients n^i are chosen to be purely imaginary, while m^i are real for $i = 1, ..., 4$ and imaginary for $i = 5$. After using the projector, we write the following product,

$$A^{(2)}_{\alpha\pm}A^{(2)}_{\beta\pm} = \begin{pmatrix} m^i_{\alpha\pm}m^j_{\beta\pm}\frac{1}{2}\{\gamma^i,\gamma^j\} & 0 \\ 0 & n^i_{\alpha\pm}n^j_{\beta\pm}\frac{1}{2}\{\gamma^i,\gamma^j\} \end{pmatrix}. \tag{2.91}$$

Since $P^{\alpha\beta}_{\pm}A_{\beta\mp} = 0$, $A_{\tau\pm}$ and $A_{\sigma\pm}$ must be proportional to each other. This allows us to write

$$\begin{aligned} A^{(2)}_{\alpha\pm}A^{(2)}_{\beta\pm} = &\begin{pmatrix} m^i_{\alpha\pm}m^i_{\beta\pm} & 0 \\ 0 & n^i_{\alpha\pm}n^i_{\beta\pm} \end{pmatrix} \\ =&\frac{1}{8}\Upsilon\,\text{Str}\left(A^{(2)}_{\alpha\pm}A^{(2)}_{\beta\pm}\right) + \frac{1}{2}\left(m^i_{\alpha\pm}m^i_{\beta\pm} + n^i_{\alpha\pm}n^i_{\beta\pm}\right) I_8, \end{aligned} \tag{2.92}$$

where I_8 is the identity matrix and Υ is the diagonal matrix defined as $\Upsilon = \text{diag}(I_4, -I_4)$. The product (2.92) appears in (2.87) after substituting the ansatz given in (2.88) and (2.89). It allows us to write the variation (2.87) as

$$\begin{aligned} \delta_\epsilon\mathcal{L} = &-\frac{1}{2}\delta_\epsilon\gamma^{\alpha\beta}\,\text{Str}\left(A^{(2)}_{\alpha}A^{(2)}_{\beta}\right) + \frac{1}{4}\,\text{Str}\left(A^{(2)}_{\alpha-}A^{(2)}_{\beta-}\right)\text{Str}\left(\Upsilon\left[\kappa^{(1)\beta}_{+}, A^{(1)\alpha}_{+}\right]\right) \\ &+ \frac{1}{4}\,\text{Str}\left(A^{(2)}_{\alpha+}A^{(2)}_{\beta+}\right)\text{Str}\left(\Upsilon\left[\kappa^{(3)\beta}_{-}, A^{(3)\alpha}_{-}\right]\right). \end{aligned} \tag{2.93}$$

Then, it is possible to deduce the transformation of the worldsheet metric $\gamma^{\alpha\beta}$ under the κ-symmetry,

$$\begin{aligned} \delta_\epsilon\gamma^{\alpha\beta} =&\frac{1}{4}\,\text{Str}\Big(\Upsilon\Big(\left[\kappa^{(1)\alpha}_{+}, A^{(1)\beta}_{+}\right] + \left[\kappa^{(1)\beta}_{+}, A^{(1)\alpha}_{+}\right] \\ &+ \left[\kappa^{(3)\alpha}_{-}, A^{(3)\beta}_{-}\right] + \left[\kappa^{(3)\beta}_{-}, A^{(3)\alpha}_{-}\right]\Big)\Big). \end{aligned} \tag{2.94}$$

By using the fact that the supertrace become a regular trace with an insertion of Υ and the identity $P^{\alpha\gamma}_{\pm}P^{\beta\delta}_{\pm} = P^{\beta\gamma}_{\pm}P^{\alpha\delta}_{\pm}$, we finally obtain

$$\delta_\epsilon\gamma^{\alpha\beta} = \frac{1}{2}\,\text{tr}\left(\left[\kappa^{(1)\alpha}_{+}, A^{(1)\beta}_{+}\right] + \left[\kappa^{(3)\alpha}_{-}, A^{(3)\beta}_{-}\right]\right). \tag{2.95}$$

2.3.3 Integrability of $AdS_5 \times S^5$ superstrings

In this section we review some notions on integrability before introducing it for superstrings in $AdS_5 \times S^5$.

Lax pair and conserved quantities

The central object to study integrable systems is the Lax pair defined by a pair of matrices L and M, built in such a way that

$$\frac{dL}{dt} = [M, L]\,, \tag{2.96}$$

is equivalent to the equations of motion of the system. The equation (2.96) is called the Lax equation and $[M, L]$ denotes the commutator of the matrices M and L. The importance of the Lax representation is that, once found, it allows us to construct the set of conserved quantities of the system [66],

$$I_k = \mathrm{tr} L^k, \qquad k = 1, ..., n\,, \tag{2.97}$$

which are in involution, i.e. their Poisson bracket vanishes, $\{I_k, I_j\} = 0$.

Integrability of two-dimensional field theory

A field theory has an infinite number of degrees of freedom, thus in order to guarantee its integrability we need to have an infinite number of conserved quantities which leads to an infinite set of powers (2.97) of the Lax matrix L. To this end we introduce a spectral parameter, λ. It allows us to have families of matrices $M(\tau, \lambda)$ and $L(\tau, \lambda)$, satisfying

$$\partial_\tau L(\tau, \lambda) = [M(\tau, \lambda), L(\tau, \lambda)]\,. \tag{2.98}$$

In this case the invariants are

$$I_k(\lambda) = \mathrm{tr}\, L^k(\tau, \lambda), \qquad k \in \mathbb{Z}\,. \tag{2.99}$$

In two dimensions, with coordinates (τ, σ), the equations of motion of $\Psi = \Psi(\tau, \sigma, \lambda)$ for an integrable field theory can be written as

$$\begin{aligned} \partial_\tau \Psi &= L_\tau(\tau, \sigma, \lambda)\Psi, \\ \partial_\sigma \Psi &= L_\sigma(\tau, \sigma, \lambda)\Psi, \end{aligned} \tag{2.100}$$

which implies, for consistency, the zero-curvature condition,

$$\partial_\sigma L_\tau - \partial_\tau L_\sigma + [L_\sigma, L_\tau] = 0 \tag{2.101}$$

for all values of the spectral parameter λ. A matrix L_α with $\alpha = \{\tau, \sigma\}$ satisfying (2.101) is called Lax connection. This allows to define the monodromy matrix,

$$T(\tau, \lambda) = \overleftarrow{\exp} \int_0^{2\pi} \mathrm{d}\,\sigma L_\sigma(\tau, \sigma, \lambda), \tag{2.102}$$

where $\overleftarrow{\exp}$ denotes path-ordered exponentiation in which it was assumed that the fields are periodic in σ with period 2π. Then, the invariants can be constructed as the trace of powers of the monodromy matrix,

$$I_k(\lambda) = \mathrm{tr}\, T^k(\tau, \lambda), \tag{2.103}$$

since it satisfies

$$\partial_\tau T(\tau, \lambda) = [L_\tau(\tau, 0, \lambda), T(\tau, \lambda)]\,. \tag{2.104}$$

This equation has a structure similar to (2.98). Expanding in λ we obtain an infinite set of conserved quantities, as required for integrability in field theory. Then, it is the monodromy matrix which plays the role of the Lax matrix in the field theory framework.

Classical integrability of $AdS_5 \times S^5$ superstrings

In 2003, Bena, Polchinski and Roiban showed that the Green-Schwarz superstring on $AdS_5 \times S^5$, described by a nonlinear σ-model on $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ supercoset, is classically integrable [3]. Due to the $\mathbb{Z}_4$-grading of this supercoset the structure of the Lax connection in terms of the components of the Maurer-Cartan one-form (2.49) can be assumed to have the following form

$$L_\alpha = \ell_0 A_\alpha^{(0)} + \ell_1 A_\alpha^{(2)} + \ell_2 \gamma_{\alpha\beta}\epsilon^{\beta\rho} A_\rho^{(2)} + \ell_3 A_\alpha^{(1)} + \ell_4 A_\alpha^{(3)}, \tag{2.105}$$

where ℓ_i are constants to be determined by requiring that (2.105) satisfies (2.101).

Let us write the zero-curvature condition of L_α for each grading. First, for $L_\alpha^{(0)}$,

$$2\ell_0 \epsilon^{\alpha\beta}\partial_\alpha A_\beta^{(0)} - \epsilon^{\alpha\beta}\left(\ell_0^2\left[A_\alpha^{(0)}, A_\beta^{(0)}\right] + (\ell_1^2 - \ell_2^2)\left[A_\alpha^{(2)}, A_\beta^{(2)}\right] + 2\ell_3\ell_4\left[A_\alpha^{(1)}, A_\beta^{(3)}\right]\right) = 0. \tag{2.106}$$

This condition gives

$$\ell_0 = 1, \qquad \ell_1^2 - \ell_2^2, = 1 \qquad \ell_3 \ell_4 = 1. \tag{2.107}$$

For $L_\alpha^{(2)}$ we have

$$\begin{aligned} &\ell_1 \epsilon^{\alpha\beta} \partial_\alpha A_\beta^{(2)} + \ell_2 \partial_\alpha (\gamma^{\alpha\beta} A_\beta^{(2)}) \\ &-(\epsilon^{\alpha\beta} \ell_0 \ell_1 + \gamma^{\alpha\beta} \ell_0 \ell_2) \left[A_\alpha^{(0)}, A_\beta^{(2)}\right] - \frac{1}{2} \epsilon^{\alpha\beta} \ell_3^2 \left[A_\alpha^{(1)}, A_\beta^{(1)}\right] - \frac{1}{2} \epsilon^{\alpha\beta} \ell_4^2 \left[A_\alpha^{(3)}, A_\beta^{(3)}\right] = 0. \end{aligned} \tag{2.108}$$

The equation (2.108) can be put into the form of the equations of motion (2.64) if the following relations hold

$$\frac{\ell_3^2 - \ell_1}{\ell_2} = -\kappa, \qquad \frac{\ell_4^2 - \ell_1}{\ell_2} = \kappa. \tag{2.109}$$

For $L_\alpha^{(1)}$ and $L_\alpha^{(3)}$ we have

$$\begin{aligned} &\ell_3 \epsilon^{\alpha\beta} \partial_\alpha A_\beta^{(1)} - \epsilon^{\alpha\beta} \ell_0 \ell_3 \left[A_\alpha^{(0)}, A_\beta^{(1)}\right] - \epsilon^{\alpha\beta} \ell_1 \ell_4 \left[A_\alpha^{(2)}, A_\beta^{(3)}\right] + \gamma^{\alpha\beta} \ell_2 \ell_4 \left[A_\alpha^{(2)}, A_\beta^{(3)}\right] = 0, \\ &\ell_4 \epsilon^{\alpha\beta} \partial_\alpha A_\beta^{(3)} - \epsilon^{\alpha\beta} \ell_0 \ell_4 \left[A_\alpha^{(0)}, A_\beta^{(3)}\right] - \epsilon^{\alpha\beta} \ell_1 \ell_3 \left[A_\alpha^{(2)}, A_\beta^{(1)}\right] + \gamma^{\alpha\beta} \ell_2 \ell_3 \left[A_\alpha^{(2)}, A_\beta^{(1)}\right] = 0. \end{aligned} \tag{2.110}$$

The zero-curvature condition (2.50) of $A^{(1)}$ and $A^{(3)}$ allows to write (2.110) as

$$\left(\gamma^{\alpha\beta} - \frac{\ell_1 \ell_4 - \ell_3}{\ell_2 \ell_4} \epsilon^{\alpha\beta}\right) \left[A_\alpha^{(2)}, A_\beta^{(3)}\right] = 0, \tag{2.111}$$

$$\left(\gamma^{\alpha\beta} + \frac{\ell_4 - \ell_1 \ell_3}{\ell_2 \ell_3} \epsilon^{\alpha\beta}\right) \left[A_\alpha^{(2)}, A_\beta^{(1)}\right] = 0. \tag{2.112}$$

These equations will be equal to the equations of motion for $A^{(1)}$ (2.65) and $A^{(3)}$ (2.66) provided

$$\frac{\ell_1 \ell_4 - \ell_3}{\ell_2 \ell_4} = \kappa, \qquad \frac{\ell_4 - \ell_1 \ell_3}{\ell_2 \ell_3} = \kappa. \tag{2.113}$$

From (2.109) we find

$$2\ell_1 = \ell_3^2 + \ell_4^2. \tag{2.114}$$

This equation also follows from (2.113) if $\ell_3 \ell_4 = 1$. From (2.107), (2.109) and (2.113), we get

$$\kappa^2 = 1, \tag{2.115}$$

which is the condition for κ-symmetry. This is an important result because the integrability of the equations of motion implies κ-symmetry.

Finally, we can write the coefficients ℓ_i in terms of a spectral parameter z as

$$\ell_0 = 1, \quad \ell_1 = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right), \quad \ell_2 = -\frac{1}{2\kappa}\left(z^2 - \frac{1}{z^2}\right), \quad \ell_3 = z, \quad \ell_4 = \frac{1}{z}, \tag{2.116}$$

which allows write the Lax connection (2.105) in terms of z.

Chapter 3

Yang-Baxter deformations of semisymmetric σ-models

In this chapter we study a family of integrable deformations known as Yang-Baxter deformations. The main characteristic of this type of deformations is that integrability is preserved from the outset. This is due to the algebraic procedure which relies on the possibility of defining a r-matrix.

To start we introduce some brief comments on how these deformations emerged. Then, we discuss the connection between the r-matrix and the integrability of the Yang-Baxter deformed models and give a short classification of r-matrices for the case of $AdS_5 \times S^5$ background. After that, we present the Lagrangian for an Yang-Baxter deformed σ-model with $\mathbb{Z}_4$-grading and discuss its main properties. Finally, we show some examples of deformed backgrounds obtained by using particular choices of r-matrices which are equivalent to those computed via TsT transformation of the $AdS_5 \times S^5$ solution.

The first evidence for integrable deformations was found in the $SU(2)$ Principal Chiral Model (PCM) [67] which is known to be integrable. Klimcik generalized this for any compact Lie group in [4] and showed its integrability in [5]. This model is called Yang-Baxter σ-model. Delduc, Magro and Vicedo considered the extension to a symmetric coset space in [8] and to a semisymmetric coset in [9, 10].

3.1 The r-matrix and integrability

Along the lines of [68–72] let us define a Poisson-Lie group as a Lie group G equipped with a Poisson structure. A Lie algebra $\mathfrak{g}$ is defined by the operation $[\cdot,\cdot] : \mathfrak{g}\otimes\mathfrak{g} \mapsto \mathfrak{g}$ known as the Lie bracket where $\otimes$ denotes the tensor product. The Poisson structure, i.e. the Poisson bracket $\{\cdot,\cdot\}$, can be defined on the dual $\mathfrak{g}^*$, but it is not necessarily a Lie algebra. If a map $r : \mathfrak{g} \mapsto \mathfrak{g}\otimes\mathfrak{g}$ is defined, an algebra with this map is called bialgebra, and it is a Lie bialgebra if also $\mathfrak{g}^*$ is a Lie algebra. This is because ${}^t r : \mathfrak{g}^*\otimes\mathfrak{g}^* \mapsto \mathfrak{g}^*$ plays the role as the Lie bracket on $\mathfrak{g}^*$, which in turn allows us to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of its scalar product. So, given r, it is possible to define a Lie bialgebra $(\mathfrak{g}, r)$ with a Poisson structure. The requirement that this r-map connects with the Poisson bracket leads to [71]

$$\{L \overset{\otimes}{,} L\}^r \equiv [L\otimes 1 + 1\otimes L, r] \in \mathfrak{g}\otimes\mathfrak{g}, \tag{3.1}$$

where r is the r-matrix. Let us consider the following expression

$$\{L \overset{\otimes}{,} L\}^r = \{L_1, L_2\}, \tag{3.2}$$

where $L_1 = L\otimes 1$ and $L_2 = 1\otimes L$. Since

$$\mathrm{tr}\,\{L_1, L_2\} = \{\mathrm{tr}\,L, \mathrm{tr}\,L\}, \tag{3.3}$$

it is possible to write

$$\mathrm{tr}\left\{L_1^k, L_2^\ell\right\} = \left\{\mathrm{tr}\,L^k, \mathrm{tr}\,L^\ell\right\}, \qquad k,\ell\in\mathbb{Z}_+. \tag{3.4}$$

Then we find

$$\begin{aligned}\left\{\mathrm{tr}\,L^k, \mathrm{tr}\,L^\ell\right\} =& \mathrm{tr}\left\{L_1^k, L_2^\ell\right\} = k\ell\,\mathrm{tr}\left(L_1^{k-1}L_2^{\ell-1}\{L_1, L_2\}\right)\\ =& k\ell\,\mathrm{tr}\left(L_1^{k-1}L_2^{\ell-1}[L_1+L_2, r]\right) = 0.\end{aligned} \tag{3.5}$$

The latter result allows us to identify L as the Lax matrix and to establish that all the invariant we can construct from it are in involution.

r-matrix

The r-matrix is defined in terms of wedge product of $T_i\in\mathfrak{g}$ as

$$r = \frac{1}{2}r^{ij}T_i\wedge T_j, \tag{3.6}$$

which belongs to $\mathfrak{g}\otimes\mathfrak{g}$. The wedge operation is defined as

$$T_i \wedge T_j \equiv T_i \otimes T_j - T_j \otimes T_i \quad \in \mathfrak{g}\otimes\mathfrak{g}\,. \tag{3.7}$$

Due to its definition (3.6), the r-matrix is skew-symmetric.

A new operator $R:\ \mathfrak{g} \mapsto \mathfrak{g}$ exists due to the presence of the r-map and can be understood as $R:\ \mathfrak{g} \overset{r}{\mapsto} \mathfrak{g}\otimes\mathfrak{g} \overset{{}^t r}{\mapsto} \mathfrak{g}$. It is defined as

$$R\,(M) \equiv \mathrm{tr}_2\,(r\,(1 \otimes M))\,, \qquad M \in \mathfrak{g}, \tag{3.8}$$

where tr_2 is the trace on the second subspace.

The existence of the r-matrix in a Poisson-Lie bialgebra allows to define a Lie bracket on $\mathfrak{g}$ in terms of the R operator,

$$[M,N]_R \equiv [R(M),N] + [M,R(N)]\,, \qquad M,N \in \mathfrak{g}, \tag{3.9}$$

which must satisfy the Jacobi identity to be well-defined. The Jacobi identity for (3.9) leads to the so-called Yang-Baxter equation (YBE) [68]

$$[R(M),R(N)] - R\,([R(M),N] + [M,R(N)]) = c\,[M,N]\,, \begin{cases} c = 0 & \text{CYBE} \\ c = \pm 1 & \text{mCYBE} \end{cases} \tag{3.10}$$

where $M,N \in \mathfrak{g}$. In (3.10), CYBE refers to classical Yang-Baxter equation and mCYBE to modified classical Yang-Baxter equation.

3.2 r-matrices of the Yang-Baxter equation

Let $\mathfrak{g}$ be any Lie algebra, then $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where $\mathfrak{n}_\pm$ are maximal nilpotent subalgebras of $\mathfrak{g}$ and $\mathfrak{h}$ is a Cartan subalgebra. The subalgebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are generated by the positive and negative root vectors. The subalgebra of $\mathfrak{g}$ is $\mathfrak{b}_+ := \mathfrak{h} \oplus \mathfrak{n}_+$ is called Borel subalgebra. To identify the roots of $\mathfrak{g}$ we introduce a Cartan-Weyl basis, composed of the Cartan generators $h_i \in \mathfrak{h}$, positive $e_j \in \mathfrak{n}_+$ and negative $f_j \in \mathfrak{n}_-$ roots. The Cartan generators and the simple roots satisfy the defining relations

$$[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_j, \tag{3.11}$$

where a_{ij} are the elements that compose the Cartan matrix (a_{ij}). We now define the non-simple roots such that if e_a and e_b are two positive roots commuting to give a further positive root e_c, then

$$[e_a, e_b] = N_{abc} e_c, \quad [f_a, f_b] = -N_{abc} f_c, \tag{3.12}$$

where N_{abc} are constants.

A typical solution of the mCYBE is the Drinfeld-Jimbo type solution [68],

$$r_{\text{DJ}} = i\, e_j \wedge f_j. \tag{3.13}$$

The associated linear R-operator is defined by its action on the Cartan generators and the positive and negative roots

$$R\,(e_a) = i e_a, \quad R\,(f_a) = -i f_a, \quad R\,(h_i) = 0, \tag{3.14}$$

and satisfies the mCYBE (3.10) with $c = 1$.

On the other hand, some solutions of the CYBE are

- abelian r-matrices

$$r_{\text{Ab}} = h_i \wedge h_j, \quad [h_i, h_j] = 0, \tag{3.15}$$

  otherwise r-matrices construced by non-commuting generators are called nonabelian.

- Jordanian r-matrices

$$r_{\text{Jor}} = h_i \wedge e_j, \quad r^3_{\text{Jor}} = 0, \tag{3.16}$$

- abelian Jordanian r-matrices

$$r_{\text{AJ}} = e_i \wedge e_j, \quad r^2_{\text{AJ}} = 0. \tag{3.17}$$

3.3 Yang-Baxter deformed σ-model

The action of the Yang-Baxter deformed σ-model with $\mathbb{Z}_4$-grading is [9, 18, 50],

$$S = -\frac{(1+c\eta^2)^2}{2(1-c\eta^2)} \int \mathrm{d}^2\,\sigma P_-^{\alpha\beta}\,\mathrm{Str}\left(A_\alpha\, d \circ \frac{1}{1-\eta R_g \circ d} A_\beta\right), \tag{3.18}$$

where $P_\pm^{\alpha\beta} = \frac{1}{2}(\gamma^{\alpha\beta} \pm \kappa\epsilon^{\alpha\beta})$, $\gamma^{\alpha\beta}$ is the worldsheet metric with det$\gamma = -1$, $\epsilon^{01} = 1$, c is the constant in the YBE (3.10) and $A_\alpha = g^{-1}\partial_\alpha g$ are the components of the Maurer-Cartan one-form which take values on $\mathfrak{g}$. The operators d and $\tilde{d}$ correspond to projections on the $\mathbb{Z}_4$ grading, given by

$$d = P_1 + 2\hat{\eta}^{-2}P_2 - P_3, \qquad \tilde{d} = -P_1 + 2\hat{\eta}^{-2}P_2 + P_3, \tag{3.19}$$

with the property Str$(M\,dN) =$ Str$(\tilde{d}M\,N)$ where $\hat{\eta} = \sqrt{1 - c\eta^2}$ and η is the deformation parameter. In equation (3.18), $\circ$ denotes the function composition [1] and R_g is defined as

$$R_g(M) \equiv \mathrm{Ad}_g^{-1} \circ R \circ \mathrm{Ad}_g(M) = g^{-1}R(gMg^{-1})g, \tag{3.20}$$

with

$$\begin{aligned} R(M) =& \mathrm{Str}_2(r(1 \otimes M)) = r^{ij}\,\mathrm{Str}_2\left(T_i \otimes T_j M - (-1)^{[\alpha][\beta]} T_j \otimes T_i M\right) \\ =& r^{ij}\left(T_i\,\mathrm{Str}\,(T_j M) - (-1)^{[i][j]} T_j\,\mathrm{Str}\,(T_i M)\right), \end{aligned} \tag{3.21}$$

where $[i]$ and $[j]$ represent the gradings of T_i and T_j, respectively. The R operator satisfies

$$\mathrm{Str}(M\,R(N)) = -\,\mathrm{Str}(R(M)\,N). \tag{3.22}$$

It is also convenient to define the following currents:

$$\begin{aligned} J_\alpha &\equiv \frac{1}{1 - \eta R_g \circ d} A_\alpha \equiv \mathcal{O}^{-1} A_\alpha, \quad J^\alpha_\pm \equiv P^{\alpha\beta}_\pm J_\beta, \\ \tilde{J}_\alpha &\equiv \frac{1}{1 + \eta R_g \circ \tilde{d}} A_\alpha \equiv \tilde{\mathcal{O}}^{-1} A_\alpha, \quad \tilde{J}^\alpha_\pm \equiv P^{\alpha\beta}_\pm \tilde{J}_\beta. \end{aligned} \tag{3.23}$$

where the operators $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are

$$\mathcal{O} = 1 - \eta R_g \circ d, \qquad \tilde{\mathcal{O}} = 1 + \eta R_g \circ \tilde{d}. \tag{3.24}$$

Undeformed action

The undeformed action corresponds to taking $\eta = 0$ in (3.18). Indeed, when η vanishes, the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2} P^{\alpha\beta}_- \,\mathrm{Str}\left(A_\alpha\, d|_{\eta=0} \circ A_\beta\right), \tag{3.25}$$

with $d|_{\eta=0} = P_1 + 2P_2 - P_3$. When applying each projector on A_β and taking into account the $\mathbb{Z}_4$-grading we get

$$\mathcal{L} = -\frac{1}{2}\left[\gamma^{\alpha\beta}\,\mathrm{Str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) + \kappa\epsilon^{\alpha\beta}\,\mathrm{Str}\left(A^{(1)}_\alpha A^{(3)}_\beta\right)\right], \tag{3.26}$$

which is precisely the Lagrangian (2.54).

[1] The function composition is defined as $f \circ g(x) = f(g(x))$

Derivation of the equations of motion

In order to find the equations of motion let us vary the Lagrangian of the action (3.18)

$$\delta\mathcal{L} = -\frac{1}{2}\frac{(1+c\eta^2)^2}{(1-c\eta^2)}P_-^{\alpha\beta}\left\{\text{Str}\left(\delta A_\alpha, d\circ J_\beta\right) + \text{Str}\left(A_\alpha, d\circ \delta J_\beta\right)\right\}. \tag{3.27}$$

The variation of δA_α is

$$\begin{aligned}\delta A_\alpha &= -g^{-1}\delta g\tilde{J}_\alpha + g^{-1}\delta g(\eta R_g\circ\tilde{d}\tilde{J}_\alpha) + g^{-1}\partial_\alpha(gg^{-1}\delta g),\\ &= -g^{-1}\delta g\tilde{J}_\alpha + g^{-1}\delta g(\eta R_g\circ\tilde{d}\tilde{J}_\alpha) + \tilde{J}_\alpha(g^{-1}\delta g) - (\eta R_g\circ\tilde{d}\tilde{J}_\alpha)g^{-1}\delta g + \partial_\alpha(g^{-1}\delta g),\\ &= \left[\tilde{J}_\alpha, g^{-1}\delta g\right] - \left[\eta R_g\circ\tilde{d}\tilde{J}_\alpha, g^{-1}\delta g\right] + \partial_\alpha(g^{-1}\delta g).\end{aligned} \tag{3.28}$$

In order to calculate the δJ_β in (3.27), we can use the properties of the R operator, so by using (3.20) the variation of R_g is

$$\delta R_g(M) = R_g(\delta M) + \left[R_g(M), g^{-1}\delta g\right] - R_g\left(\left[M, g^{-1}\delta g\right]\right). \tag{3.29}$$

For $M = \tilde{d}A_\beta$ in (3.29) we get

$$\delta\left(\left(R_g\circ\tilde{d}\right)(A_\beta)\right) = \left(R_g\circ\tilde{d}\right)(\delta A_\beta) + \left[\left(R_g\circ\tilde{d}\right)A_\beta, g^{-1}\delta g\right] - R_g\left(\left[\tilde{d}A_\beta, g^{-1}\delta g\right]\right), \tag{3.30}$$

and using this relation repeatedly we get

$$\begin{aligned}\delta\left(\left(R_g\circ\tilde{d}\right)^n(A_\beta)\right) &= \left(R_g\circ\tilde{d}\right)^n(\delta A_\beta) + \sum_{k=0}^{n-1}\left(R_g\circ\tilde{d}\right)^k\left[\left(R_g\circ\tilde{d}\right)^{n-k}(A_\beta), g^{-1}\delta g\right]\\ &\quad - \sum_{k=0}^{n-1}\left(R_g\circ\tilde{d}\right)^k R_g\left(\left[\tilde{d}\circ(R_g\circ\tilde{d})^{n-1-k}(A_\beta), g^{-1}\delta g\right]\right),\end{aligned} \tag{3.31}$$

for $n \geqslant 0$. By multiplying by η^n on both sides of the above equation and summing in n from 0 to ∞, we obtain

$$\begin{aligned}&\delta\left(\frac{1}{1-\eta R_g\circ\tilde{d}}A_\beta\right) =\\ &= \frac{1}{1-\eta R_g\circ\tilde{d}}\left(\delta A_\beta + \left[\frac{\eta R_g\circ\tilde{d}}{1-\eta R_g\circ\tilde{d}}A_\beta, g^{-1}\delta g\right] - \eta R_g\left[\tilde{d}\frac{1}{1-\eta R_g\circ\tilde{d}}A_\beta, g^{-1}\delta g\right]\right),\end{aligned} \tag{3.32}$$

then we find the variation of J_β,

$$\delta J_\beta = \frac{1}{1-\eta R_g\circ\tilde{d}}\left(\delta A_\beta + \left[\eta R_g\circ\tilde{d}J_\beta, g^{-1}\delta g\right] - \eta R_g\circ\left[\tilde{d}J_\beta, g^{-1}\delta g\right]\right). \tag{3.33}$$

By replacing (3.28) into (3.33) we obtain

$$\delta J_\beta = \frac{1}{1-\eta R_g \circ \tilde{d}} \left(\partial_\beta(g^{-1}\delta g) + \left[J_\beta, g^{-1}\delta g\right] - \eta R_g \circ \left[\tilde{d} J_\beta, g^{-1}\delta g\right]\right). \tag{3.34}$$

Then, by plugging (3.28) and (3.34) into (3.27), we obtain

$$\begin{aligned}\delta\mathcal{L} = &-\frac{1}{2}\frac{(1+c\eta^2)^2}{(1-c\eta^2)}\Big\{P_-^{\alpha\beta}\,\mathrm{Str}\left(\partial_\alpha(g^{-1}\delta g)\,, dJ_\beta\right) + P_-^{\alpha\beta}\,\mathrm{Str}\left(\left[\tilde{J}_\alpha + \eta R_g \circ \tilde{d}\tilde{J}_\alpha, g^{-1}\delta g\right], dJ_\beta\right)\\ &+ P_-^{\alpha\beta}\,\mathrm{Str}\left(\tilde{d}\tilde{J}_\alpha, \partial_\beta(g^{-1}\delta g) + \left[J_\beta, g^{-1}\delta g\right] - \eta R_g \circ \left[dJ_\beta, g^{-1}\delta g\right]\right)\Big\}.\end{aligned} \tag{3.35}$$

Now, by using the properties of the projectors $P_-^{\alpha\beta} J_\beta = J_-^\alpha$ in the first term and using (2.86) in the second and third terms, and neglecting the total derivative terms, we have

$$\begin{aligned}\delta\mathcal{L} \;=\; &-\frac{1}{4}\frac{(1+c\eta^2)^2}{(1-c\eta^2)}\Big\{-\mathrm{Str}\left(g^{-1}\delta g, d\partial_\alpha J_-^\alpha\right) + \mathrm{Str}\left(\left[\tilde{J}_+^\sigma, g^{-1}\delta g\right], dJ_{\sigma-}\right)\\ &+\mathrm{Str}\left(\left[dJ_{\sigma-}, \eta R_g \circ \tilde{d}\tilde{J}_+^\sigma\right], g^{-1}\delta g\right) - \mathrm{Str}\left(\tilde{d}\partial_\alpha \tilde{J}_+^\alpha, g^{-1}\delta g\right)\\ &+\mathrm{Str}\left(\tilde{d}\tilde{J}_+^\sigma, \left[J_{\sigma-}, g^{-1}\delta g\right]\right) + \mathrm{Str}\left(\eta R_g \tilde{d}\tilde{J}_+^\sigma, \left[dJ_{\sigma-}, g^{-1}\delta g\right]\right)\Big\},\end{aligned} \tag{3.36}$$

where in the last term we used the skew-symmetry of the R_g operator (3.22). Then after some simplifications we get

$$\delta\mathcal{L} \sim \Big\{\mathrm{Str}\left(g^{-1}\delta g\left(d\left(\partial_\alpha J_-^\alpha\right) + \tilde{d}\left(\partial_\alpha \tilde{J}_+^\alpha\right) + \left[\tilde{J}_{+\alpha}, d\left(J_-^\alpha\right)\right] + \left[J_{-\alpha}, \tilde{d}\left(\tilde{J}_+^\alpha\right)\right]\right)\right)\Big\} \tag{3.37}$$

The equation of motion is then given by

$$\mathcal{E} \equiv d\left(\partial_\alpha J_-^\alpha\right) + \tilde{d}\left(\partial_\alpha \tilde{J}_+^\alpha\right) + \left[\tilde{J}_{+\alpha}, d\left(J_-^\alpha\right)\right] + \left[J_{-\alpha}, \tilde{d}\left(\tilde{J}_+^\alpha\right)\right] = 0. \tag{3.38}$$

Zero-curvature condition

By definition the left-invariant one-form $A = g^{-1}\,\mathrm{d}\,g$ satisfies the zero-curvature condition $\mathcal{Z}$ which, in components, is

$$\mathcal{Z} \equiv \frac{1}{2}\epsilon^{\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]) = 0. \tag{3.39}$$

Plugging the relation (3.23) into the above expression one can recast it into the following form,

$$\begin{aligned}\mathcal{Z} \;\equiv\; &P_-^{\alpha\beta}\Big\{\partial_\alpha J_\beta - \partial_\beta \tilde{J}_\alpha - \eta R_g \partial_\alpha dJ_\beta - \eta R_g \partial_\beta \tilde{d}\tilde{J}_\alpha + \left[\tilde{J}_\alpha, J_\beta\right]\\ &-\left[\tilde{J}_\alpha, \eta R_g dJ_\beta\right] + \left[\eta R_g \tilde{d}\tilde{J}_\alpha, J_\beta\right] - \left[\eta R_g \tilde{d}\tilde{J}_\alpha, \eta R_g dJ_\beta\right]\Big\},\\ =\; &\partial_\alpha J_-^\alpha - \partial_\alpha \tilde{J}_+^\alpha - \eta R_g \partial_\alpha dJ_-^\alpha - \eta R_g \partial_\alpha \tilde{d}\tilde{J}_+^\alpha + \left[\tilde{J}_+^\alpha, J_{-\alpha}\right]\\ &-\left[\tilde{J}_{+\alpha}, \eta R_g dJ_-^\alpha\right] - \left[J_{-\alpha}, \eta R_g \tilde{d}\tilde{J}_+^\alpha\right] - \left[\eta R_g \tilde{d}\tilde{J}_{+\alpha}, \eta R_g dJ_-^\alpha\right] = 0,\end{aligned} \tag{3.40}$$

and by using (3.10) for the operator R_g, because it is also a skew-symmetric solution of the YBE, and (3.38) the zero-curvature condition becomes

$$\mathcal{Z} \equiv \partial_\alpha \tilde{J}^\alpha_+ - \partial_\alpha J^\alpha_- + \left[J_{-\alpha}, \tilde{J}^\alpha_+\right] + \eta R_g(\mathcal{E}) - \eta^2 c \left[dJ^\alpha_-, \tilde{d}\tilde{J}_{+\alpha}\right] = 0. \tag{3.41}$$

Let us remark that the field equations on the odd sector $\mathcal{E}^{(1)} = 0$ and $\mathcal{E}^{(3)} = 0$ simplify if we consider the following combinations:

$$\begin{aligned} P_1 \circ (1 - \eta R_g)(\mathcal{E}) + P_1(\mathcal{Z}) &= -4\left[\tilde{J}^{(2)}_{+\alpha}, J^{\alpha(3)}_-\right], \\ P_3 \circ (1 + \eta R_g)(\mathcal{E}) - P_3(\mathcal{Z}) &= -4\left[J^{(2)}_{-\alpha}, \tilde{J}^{\alpha(1)}_+\right]. \end{aligned} \tag{3.42}$$

And, as a consequence, one can write the field equations in the odd sector as

$$\mathcal{E}^{(1)} \equiv \left[\tilde{J}^{(2)}_{+\alpha}, J^{\alpha(3)}_-\right] = 0, \qquad \mathcal{E}^{(3)} \equiv \left[J^{(2)}_{-\alpha}, \tilde{J}^{\alpha(1)}_+\right] = 0, \tag{3.43}$$

which have the same form as those of the undeformed model written in terms of undeformed currents (2.63).

Virasoro constraints

The action (3.18) in terms of the deformed currents (3.23) gives

$$S = -\frac{1}{4}\frac{(1+c\eta^2)^2}{(1-c\eta^2)}\int \mathrm{d}\,\sigma^2(\gamma^{\alpha\beta} - \kappa\epsilon^{\alpha\beta})\{\mathrm{Str}(J_\alpha, d\circ J_\beta)) - \mathrm{Str}\left(\eta R_g \circ dJ_\alpha, d\circ J_\beta)\right\}.$$

By using (3.19) the part of the action proportional to the metric takes the form,

$$S_\gamma = -\frac{1}{2}\frac{(1+c\eta^2)^2}{(1-c\eta^2)}\frac{1}{\hat{\eta}^2}\int \mathrm{d}\,\sigma^2\gamma^{\alpha\beta}\,\mathrm{Str}\left(J^{(2)}_\alpha J^{(2)}_\beta\right). \tag{3.44}$$

The Virasoro constraint is

$$T_{\alpha\beta} = -2\frac{\delta S}{\delta\gamma^{\alpha\beta}} = 0, \tag{3.45}$$

where $T_{\alpha\beta}$ is the energy-momentum tensor, implies that

$$\mathrm{Str}(J^{(2)}_\alpha J^{(2)}_\beta) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\gamma\delta}\,\mathrm{Str}(J^{(2)}_\gamma J^{(2)}_\delta) = 0. \tag{3.46}$$

Finally, taking the projector on (3.46) we get

$$\mathrm{Str}(J^{\alpha(2)}_- J^{\beta(2)}_-) = 0, \qquad \mathrm{Str}(\tilde{J}^{\alpha(2)}_+ \tilde{J}^{\beta(2)}_+) = 0. \tag{3.47}$$

The Lax connection

From (3.38) and (3.41) we find the deformed Lax pair

$$L_{+}^{\alpha} = \tilde{J}_{+}^{(0)\alpha} + \lambda\sqrt{1+c\eta^2}\tilde{J}_{+}^{\alpha(1)} + \lambda^{-2}\frac{1+c\eta^2}{1-c\eta^2}\tilde{J}_{+}^{\alpha(2)} + \lambda^{-1}\sqrt{1+c\eta^2}\tilde{J}_{+}^{\alpha(3)} \quad (3.48)$$

$$M_{-}^{\alpha} = J_{-}^{(0)\alpha} + \lambda\sqrt{1+c\eta^2}J_{-}^{\alpha(1)} + \lambda^{2}\frac{1+c\eta^2}{1-c\eta^2}J_{-}^{\alpha(2)} + \lambda^{-1}\sqrt{1+c\eta^2}J_{-}^{\alpha(3)} \quad (3.49)$$

where λ is the spectral parameter. So the Lax connection is constructed by the linear combination of these equations,

$$\mathcal{L}_{\alpha} = L_{+\alpha} + M_{-\alpha}, \tag{3.50}$$

and it satisfies the zero-curvature condition.

The results above are valid for any σ-model with $\mathbb{Z}_4$-grading.

3.4 Yang-Baxter deformations of $AdS_5 \times S^5$

The deformed action to describe type IIB superstring theory in $AdS_5 \times S^5$ is (3.18) since it possesses a $\mathbb{Z}_4$-grading; thus, we apply the same formulae we computed in Section 3.3. The equations of motion arising from the Lagrangian of the action (3.18) are given by (3.38), the zero curvature condition $\mathcal{Z} = 0$ in terms of $\tilde{J}_{+}^{\alpha}$ and J_{-}^{α} is (3.41), the Virasoro constraint is (3.47) and the Lax connection is (3.50).

κ-symmetry

In this case, let us consider the following infinitesimal right translation of the coset repesentative g,

$$\delta g = g\left[(1-\eta R_g)\,\epsilon^{(1)} + (1+\eta R_g)\,\epsilon^{(3)}\right]. \tag{3.51}$$

The variation of the action (3.18) it, is given by

$$\begin{aligned}\delta_g S =& \frac{(1+c\eta^2)^2}{2(1-c\eta^2)}\int d\sigma^2\,\mathrm{Str}\left(\epsilon\mathcal{E}\right),\\ =& \frac{(1+c\eta^2)^2}{2(1-c\eta^2)}\int d\sigma^2\,\mathrm{Str}\left(\epsilon^{(1)}P_3\circ(1+\eta R_g)\mathcal{E} + \epsilon^{(3)}P_1\circ(1-\eta R_g)\,\mathcal{E}\right).\end{aligned} \tag{3.52}$$

Using the identities (3.42) and the zero curvature condition in terms of the deformed current $P_1(\mathcal{Z}) = P_3(\mathcal{Z}) = 0$, the variation of the action can be written as follows

$$\delta_g S = -2\frac{(1+c\eta^2)^2}{(1-c\eta^2)}\int d\sigma^2\,\mathrm{Str}\left(\epsilon^{(1)}\left[J^{(2)}_{-\alpha}, \tilde{J}^{\alpha(1)}_{+}\right] + \epsilon^{(3)}\left[\tilde{J}^{(2)}_{+\alpha}, J^{\alpha(3)}_{-}\right]\right). \tag{3.53}$$

In a very similar to the undeformed case (2.88) and (2.89), an ansatz for the transformation parameters $\epsilon^{(1)}$ and $\epsilon^{(3)}$ can be taken as,

$$\epsilon^{(1)} = \left(J^{(2)\alpha}_{-}\kappa^{(1)}_{+\alpha} + \kappa^{(1)}_{+\alpha}J^{(2)\alpha}_{-}\right), \tag{3.54}$$

$$\epsilon^{(3)} = \left(\tilde{J}^{(2)\alpha}_{+}\kappa^{(3)}_{-\alpha} + \kappa^{(3)}_{-\alpha}\tilde{J}^{(2)\alpha}_{+}\right), \tag{3.55}$$

where $\kappa^{(1)}_{+}$ and $\kappa^{(3)}_{-}$ are the vectors corresponding to 1 and 3 grading. From now on, the computation is totally parallel to the procedure leading to equation (2.92) and (2.93). We obtain

$$\begin{aligned}\delta_g S =& \frac{(1+c\eta^2)^2}{4(1-c\eta^2)}\int d\sigma^2\Big\{\,\mathrm{Str}\left(J^{\alpha(2)}_{-}J^{\beta(2)}_{-}\right)\mathrm{Str}\left(\Upsilon\left[\tilde{J}^{(1)}_{\alpha+}, i\kappa^{(1)}_{+\beta}\right]\right)\\ &+ \mathrm{Str}\left(\tilde{J}^{\alpha(2)}_{+}\tilde{J}^{\beta(2)}_{+}\right)\mathrm{Str}\left(\Upsilon\left[J^{(3)}_{-\alpha}, i\kappa^{(3)}_{-\beta}\right]\right)\Big\},\end{aligned} \tag{3.56}$$

where Υ is the diagonal matrix defined as $\Upsilon = \mathrm{diag}(I_4, -I_4)$. The vanishing of the total variation of the action off-shell, with respect to g and $\gamma^{\alpha\beta}$ gives the following condition:

$$\delta\gamma^{\alpha\beta} = \frac{1-c\eta^2}{2}\mathrm{Str}\left(\Upsilon\left[\kappa^{(1)\alpha}_{+}, \tilde{J}^{(1)\beta}_{+}\right] + \Upsilon\left[\kappa^{(3)\alpha}_{-}, J^{(3)\beta}_{-}\right]\right), \tag{3.57}$$

for the transformation of the metric $\gamma^{\alpha\beta}$ in order to ensure κ-symmetry, .

3.4.1 r-matrices for $AdS_5 \times S^5$

The construction of the r-matrices can be done in terms of a basis of $\mathfrak{gl}(4|4)$ since it is possible to write the generators of $\mathfrak{psu}(2,2|4)$ in terms of combinations of the generators of $\mathfrak{gl}(4|4)$. This is developed in Appendix A.

Drinfeld-Jimbo r-matrix

The DJ r-matrix (3.13) can be written as

$$r_{\mathrm{DJ}} = i\sum_{1\leqslant i<j\leqslant 8} E_{ij}\wedge E_{ji}\,(-1)^{[i][j]}\,, \tag{3.58}$$

where $[i]$ is the grading associated to the index i, such that $[i] = 0$ for $i = 1, ..., 4$ and $[i] = 1$ for $i = 5, ..., 8$. In this basis is easy to see that the positive and negative roots are E_{ij} with $i < j$ and E_{ji} with $i > j$, respectively, and the R operator associated to this r-matrix in (3.14) is

$$R(E_{ij}) = \begin{cases} +iE_{ij} & \text{if} \quad i < j \\ 0 & \text{if} \quad i = j \\ -iE_{ij} & \text{if} \quad i > j \end{cases} \cdot \tag{3.59}$$

Abelian r-matrix

An abelian r-matrix (3.15) can be written in terms of diagonal elements

$$r_{\text{Ab}} = E_{ii} \wedge E_{jj} \quad \text{with} \quad i \neq j, \tag{3.60}$$

such that any linear combination of them also satisfies the CYBE. In terms of the associated linear R-operator it only acts on the Cartan generators

$$R_{\text{Ab}}(E_{ii}) = -E_{jj}, \qquad R_{\text{Ab}}(E_{jj}) = E_{ii}. \tag{3.61}$$

Jordanian r-matrix

The Jordanian r-matrix (3.16) can be written in general as [18]

$$r_{\text{Jor}} = E_{ij} \wedge (\alpha E_{ii} - \beta E_{jj}) - \gamma \sum_{i<k<j} E_{ik} \wedge E_{kj}, \tag{3.62}$$

for $1 \leqslant i < j \leqslant m$ with $\gamma = \alpha + \beta$ or $\gamma = 0$. The action of the linear R-operator is

$$R_{\text{Jor}}(E_{ji}) = -\alpha E_{ii} + \beta E_{jj}, \quad R_{\text{Jor}}(E_{jk}) = -\gamma E_{ik}, \tag{3.63}$$

$$R_{\text{Jor}}(E_{kk}) = (\alpha\delta_{ki} - \beta\delta_{kj})E_{ij}, \quad R_{\text{Jor}}(E_{ki}) = \gamma E_{kj}, \tag{3.64}$$

where $i < j < k$ and the nilpotency is $(R_{\text{Jor}})^n = 0$ for $n \geqslant 3$.

When $\alpha = \beta = c$ and $\gamma = 2c$, the Jordanian bosonic r-matrix is obtained from the Drinfeld-Jimbo r-matrix (3.13) through a twisting [18, 73]. Moreover, unimodular Jordanian r-matrices including fermionic generators were obtained recently in [74].

Abelian-Jordanian r-matrix

Another type of r-matrix can be constructed from bosonic commutating positive or negative roots generators. These matrices are referred as abelian-Jordanian type (3.17)

$$r_{AJ} = E_{ij} \wedge E_{kl} \qquad \text{with} \qquad i < j,\ k < l,\ j \neq k,\ i \neq l. \tag{3.65}$$

The associated linear R-operator is given by

$$R_{AJ}(E_{lk}) = E_{ij}, \quad R_{AJ}(E_{ji}) = -E_{kl}. \tag{3.66}$$

In this case, the nilpotency is $(R_{AJ})^n = 0$ for $n \geqslant 2$.

Unimodular nonabelian r-matrix

It is possible to construct several nonabelian r-matrices but, as was remarked in [50], based on a classification in [75] only a subset of the possible r-matrices satisfy the unimodularity condition,

$$r^{ij}\,[T_i, T_j] = 0, \qquad T_i \in \mathfrak{g}\,. \tag{3.67}$$

Notice that this condition is trivially satisfied for abelian r-matrices. For the nonabelian case, however, unimodularity is a more subtle issue. In [50] it is given a list of the unimodular nonabelian bosonic r-matrices that solve the CYBE. These r-matrices have the form

$$r = a \wedge b + c \wedge d \tag{3.68}$$

and are called rank four r-matrices. In (3.68) a, b, c and d are linear combinations of the generators T_i such that $[a, b] = [c, d] = 0$. The list of r-matrices in terms of the $\mathfrak{so}(2, 4)$ generators p_μ, k_μ, $m_{\mu\nu}$ and D with $\mu, \nu = 0, ..., 3$ given in Appendix B is

$$r_1 = p_1 \wedge p_2 + (p_0 + p_3) \wedge (m_{01} - m_{13}),$$

$$r_2 = p_1 \wedge p_2 + (p_0 + p_3) \wedge (p_3 + m_{01} - m_{13}),$$

$$r_3 = p_1 \wedge (m_{02} - m_{23}) + (p_0 + p_3) \wedge (p_2 + m_{01} - m_{13}),$$

$$r_4 = (p_1 - m_{02} + m_{23}) \wedge (k_0 + k_3 + 2p_3 - 2m_{12}) + 2(p_0 + p_3) \wedge (p_2 + m_{01} - m_{13}),$$

$$r_5 = p_1 \wedge (m_{02} - m_{23}) + (p_0 + p_3) \wedge (D + m_{03}),$$

$$r_6 = p_1 \wedge m_{03} + 2p_0 \wedge p_3,$$

$$
\begin{aligned}
r_7 &= m_{03} \wedge m_{12} + 2p_0 \wedge p_3, \\
r_8 &= p_1 \wedge p_2 + (p_0 + p_3) \wedge m_{12}, \\
r_9 &= p_1 \wedge p_2 + (p_0 + p_3) \wedge (p_3 + m_{12}), \\
r_{10} &= p_1 \wedge p_2 + p_3 \wedge (p_0 + m_{12}), \\
r_{11} &= p_1 \wedge p_2 + p_3 \wedge m_{12}, \\
r_{12} &= p_1 \wedge p_2 + p_0 \wedge (p_3 + m_{12}), \\
r_{13} &= p_1 \wedge p_2 + p_0 \wedge m_{12}, \\
r_{14} &= p_1 \wedge p_2 + m_{12} \wedge m_{03}, \\
r_{15} &= p_1 \wedge p_3 + (m_{01} - m_{13}) \wedge (p_0 + p_3), \\
r_{16} &= p_1 \wedge p_3 + (p_2 + m_{01} - m_{13}) \wedge (p_0 + p_3), \\
r_{17} &= p_1 \wedge (p_3 + m_{02} - m_{23}) + (p_0 + p_3) \wedge (p_2 + m_{01} - m_{13}). \qquad (3.69)
\end{aligned}
$$

However, unimodular nonabelian r-matrices cannot be constructed for the compact algebra $\mathfrak{su}(4)$ of S^5, thus we cannot define this type of r-matrix in this subspace [76].

3.4.2 Deformed Backgrounds generated by r-matrices

Here we list some backgrounds that can be obtained by deforming $AdS_5 \times S^5$. They result from the choice of different types of r-matrices we presented above.

Yang-Baxter deformation of $AdS_5 \times S^5$

For $c = 0$ and switching off the fermionic degrees of freedom, so that $d = 2P_2$, the deformed Yang-Baxter σ-model Lagrangian of the action (3.18) can be written as

$$
\mathcal{L} = -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \,\mathrm{Str}\left(A_\alpha P_2 \circ \mathcal{O}^{-1} A_\beta\right), \qquad (3.70)
$$

where the operator $\mathcal{O}^{-1}$ depending on the deformation parameter η is given by

$$
\mathcal{O}^{-1} = \frac{1}{1 - 2\eta R_g \circ P_2}, \qquad (3.71)
$$

and R_g was defined in (3.20). In order to extract the background fields from the Lagrangian (3.70) we need to define a basis for the coset (2.33),

$$
\frac{\mathfrak{so}(2,4) \oplus \mathfrak{so}(6)}{\mathfrak{so}(1,4) \oplus \mathfrak{so}(5)} = \mathrm{span}_{\mathbb{R}} \{K_m\}, \qquad m = 1, ..., 10, \qquad (3.72)
$$

With this, the action of the projector P_2 on the components of the Maurer-Cartan one-form is

$$P_2 (A_\alpha) = E_\alpha{}^m K_m, \tag{3.73}$$

and

$$P_2 (R_g(K_m)) = \Lambda_m{}^n K_n. \tag{3.74}$$

Moreover, the projected action of the operator (3.71) can be computed in a similar way

$$P_2 \left(\mathcal{O}^{-1}(K_m)\right) = C_m{}^n K_n, \tag{3.75}$$

where the projector on the coset is

$$P_2 (X) = \sum_m \frac{\text{Str}\,(K_m X)}{\text{Str}\,(K_m K_m)}. \tag{3.76}$$

Combining equations (3.71), (3.74) and (3.75) we can find the relation between the coefficients $\Lambda_m{}^n$ and $C_m{}^n$

$$K_m = (C_m{}^n K_n - 2\eta\, C_m{}^n \Lambda_n{}^p K_p)\,, \tag{3.77}$$

or in matrix notation,

$$\mathbf{C} = (\mathbf{I} - 2\eta \mathbf{\Lambda})^{-1}\,, \tag{3.78}$$

which can be solved for $\mathbf{C}$.

We can rewrite the deformed Lagrangian (3.70) as

$$\mathcal{L} = -\frac{1}{2} \left(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}\right) E_\alpha{}^m E_\beta{}^n C_n{}^p\, \text{Str}\,(K_m K_p)\,, \tag{3.79}$$

or

$$\mathcal{L} = -\frac{N}{2} \left(\gamma^{\alpha\beta} C_{(mn)} E_\alpha{}^m E_\beta{}^n - \epsilon^{\alpha\beta} C_{[mn]} E_\alpha{}^m E_\beta{}^n\right)\,, \tag{3.80}$$

where N is a constant due to the supertrace in (3.79), the coefficients $C_{(mn)}$ and $C_{[mn]}$ are the symmetric and antisymmetric parts of the matrix (3.78) and E_α^m represent the coefficients in front of each of generators in (3.73).

Gravity dual of SYM on non-commutative spacetime

To evaluate the Lagrangian (3.70), let us take the following coset parametrization,

$$g = \exp\left[p_0 x^0 + p_1 x^1 + p_2 x^2 + p_3 x^3\right] \exp\left[-D \log z\right] \;\in SO(2,4)/SO(1,4). \tag{3.81}$$

The abelian Jordanian r-matrix [20]

$$r_{\mathrm{AJ}} = \mu p_2 \wedge p_3 + \nu p_0 \wedge p_1, \tag{3.82}$$

where p_μ is defined in Appendix B and μ, ν are constant parameters.

The procedure to compute the deformed background is the following. First, we compute the Maurer-Cartan one-form $A = g^{-1}\,\mathrm{d}\,g$ and the coefficients $E_\alpha^{\ m}$ from (3.73) by using the projector P_2 on the components of A. Then, we compute the coefficients $\Lambda_m^{\ n}$ by applying the projector P_2 on the action of the operator R_g (3.20) on the generators of the coset K_m. The coefficients $C_{(mn)}$ and $C_{[mn]}$ can be identified from the matrix $\mathbf{C}$ in (3.78). Finally, we can compute the deformed Lagrangian (3.80) explicitly and extract the metric from the symmetric part and B-field from the antisymmetric part.

In this case, the metric and B-field are

$$\mathrm{d}\,s^2 = \frac{z^2}{z^4 + a'^4}(\mathrm{d}\,x_0^2 + \mathrm{d}\,x_1^2) + \frac{z^2}{z^4 + a^4}(\mathrm{d}\,x_2^2 + \mathrm{d}\,x_3^2) + \frac{\mathrm{d}\,z^2}{z^2} + \mathrm{d}\,\Omega_5^2, \tag{3.83}$$

$$B = \frac{a'^2}{z^4 + a'^4}\,\mathrm{d}\,x_0 \wedge \mathrm{d}\,x_1 + \frac{a^2}{z^4 + a^4}\,\mathrm{d}\,x_2 \wedge \mathrm{d}\,x_3, \tag{3.84}$$

with $2\eta\mu = a^2$, $2\eta\nu = a'^2$. This background was first obtained via TsT transformations of $AdS_5 \times S^5$ [49].

In the following, the procedure to compute the metric and B-field will be the same only with different choices of coset parametrization and r-matrix.

γ-deformed $AdS_5 \times S^5$ with three parameters

To evaluate the deformed Lagrangian (3.70), let us adopt the following coset parametrization

$$g = \Lambda(\phi_1, \phi_2, \phi_3)\Xi(\zeta)\check{g}_r(r) \ \in \ SU(4)/SO(5). \tag{3.85}$$

where the matrices Λ, Ξ and $\check{g}_\rho$ are defined as

$$\Lambda(\phi_1, \phi_2, \phi_3) \equiv \exp\left[\frac{i}{2}\left(\phi_1 h_1 + \phi_2 h_2 + \phi_3 h_3\right)\right], \tag{3.86}$$

$$\Xi(\zeta) \equiv \begin{pmatrix} \cos\frac{\zeta}{2} & \sin\frac{\zeta}{2} & 0 & 0 \\ -\sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\zeta}{2} & -\sin\frac{\zeta}{2} \\ 0 & 0 & \sin\frac{\zeta}{2} & \cos\frac{\zeta}{2} \end{pmatrix}, \quad \breve{g}_r(r) \equiv \begin{pmatrix} \cosh\frac{r}{2} & 0 & 0 & \sinh\frac{r}{2} \\ 0 & \cosh\frac{r}{2} & -\sinh\frac{r}{2} & 0 \\ 0 & -\sinh\frac{r}{2} & \cosh\frac{r}{2} & 0 \\ \sinh\frac{r}{2} & 0 & 0 & \cosh\frac{r}{2} \end{pmatrix}. \tag{3.87}$$

The abelian r-matrix [19]

$$r_{\text{Ab}} = \mu_3 h_1 \wedge h_2 + \mu_1 h_2 \wedge h_3 + \mu_2 h_3 \wedge h_1, \tag{3.88}$$

where μ_i are constant parameters, and h_i $(i = 1, 2, 3)$ are the Cartan generators of $\mathfrak{su}(4)$.

With the following coordinate transformation

$$\rho_1 = \sin r \cos\zeta, \qquad \rho_2 = \sin r \sin\zeta, \qquad \rho_3 = \cos r, \tag{3.89}$$

the metric and B-field are

$$ds^2 = ds^2_{AdS_5} + \sum_{i=1}^{3}(\mathrm{d}\,\rho_i^2 + \mathcal{M}\rho_i^2\,\mathrm{d}\,\phi_i^2) + \mathcal{M}\rho_1^2\rho_2^2\rho_3^2\left(\sum_{i=1}^{3}\hat{\gamma}_i\,\mathrm{d}\,\phi_i\right)^2, \tag{3.90}$$

$$B = \mathcal{M}(\hat{\gamma}_3\rho_1^2\rho_2^2\,\mathrm{d}\,\phi_1 \wedge \mathrm{d}\,\phi_2 + \hat{\gamma}_1\rho_2^2\rho_3^2\,\mathrm{d}\,\phi_2 \wedge \mathrm{d}\,\phi_3 + \hat{\gamma}_2\rho_3^2\rho_1^2\,\mathrm{d}\,\phi_3 \wedge \mathrm{d}\,\phi_1), \tag{3.91}$$

where

$$\mathcal{M}^{-1} \equiv 1 + \hat{\gamma}_3^2\rho_1^2\rho_2^2 + \hat{\gamma}_1^2\rho_2^2\rho_3^2 + \hat{\gamma}_2^2\rho_3^2\rho_1^2, \tag{3.92}$$

and the relation between parameters is: $8\eta\mu_1 = \hat{\gamma}_1$, $8\eta\mu_2 = \hat{\gamma}_2$ and $8\eta\mu_3 = \hat{\gamma}_3$.

If we consider $\hat{\gamma}_1 = \hat{\gamma}_2 = \hat{\gamma}_3 \equiv \hat{\gamma}$ in (3.90) and (3.91) we get the Lunin-Maldacena background. This background was first obtained via TsT transformations of $AdS_5 \times S^5$ [49]

Gravity dual of the non-relativistic limit of SYM: Schrödinger spacetime

We are now ready to parametrize bosonic group elements of $\mathrm{PSU}(2,2|4)$. The group elements of $SO(2,4)$ and $SO(6)$ are parametrized as

$$\begin{aligned} g_a &= \exp\left(x^1 p_1 + x^2 p_2 + x^3 p_3 + x^0 p_0\right)\exp\left(-D\log z\right) \\ &= \exp\left(x^1 p_1 + x^2 p_2 + x^+ p_+ + x^- p_-\right)\exp(-D\log z) \ \in SO(2,4), \end{aligned} \tag{3.93}$$

$$g_s = \exp\left(\phi_1 h_4 + \phi_2 h_5 + \phi_3 h_6\right) \exp\left(-\zeta n_{13}\right) \exp\left(-\frac{i}{2} r \gamma_1^s\right) \in SO(6). \tag{3.94}$$

The light-cone coordinates and the associated generators are given by

$$x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}}, \quad p_{\pm} = \frac{p_0 \pm p_3}{\sqrt{2}}. \tag{3.95}$$

Thus, a bosonic element g of $\mathrm{PSU}(2,2|4)$ is represented by

$$g = g_a g_s \quad \in SO(2,4) \times SO(6) \subset \mathrm{PSU}(2,2|4). \tag{3.96}$$

The abelian r-matrix [21]

$$r_{\text{Ab}} = -\frac{i\beta}{4\eta} p_- \wedge (h_4 + h_5 + h_6), \tag{3.97}$$

where $p_- = p_0 - p_3$ is defined for a particular representation of $\mathfrak{so}(2,4)$ and β is a constant parameter.

In order to write the metric and B-field in a convenient form we make the following change of coordinates

$$\phi_1 = \chi + \frac{1}{2}\left(\psi + \phi\right), \qquad r = \mu, \tag{3.98}$$

$$\phi_2 = \chi + \frac{1}{2}\left(\psi - \phi\right), \qquad \zeta = \frac{1}{2}\theta, \tag{3.99}$$

$$\phi_3 = \chi, \tag{3.100}$$

and then we get

$$\mathrm{d}\, s^2 = \frac{-2\,\mathrm{d}\, x^+\,\mathrm{d}\, x^- + (dx^1)^2 + (dx^2)^2 + \mathrm{d}\, z^2}{z^2} - \beta^2 \frac{(\mathrm{d}\, x^+)^2}{z^4} + \mathrm{d}\, s^2_{S^5}, \tag{3.101}$$

$$B = \frac{\beta}{z^2}\,\mathrm{d}\, x^+ \wedge (\mathrm{d}\,\chi + w). \tag{3.102}$$

Here, the S^5 space is written in terms of the coordinates $(\chi, \mu, \psi, \theta, \phi)$

$$\begin{aligned} \mathrm{d}\, s^2_{S^5} &= (\mathrm{d}\,\chi + \omega)^2 + \mathrm{d}\, s^2_{\mathbb{CP}^2}, \\ \mathrm{d}\, s^2_{\mathbb{CP}^2} &= \mathrm{d}\,\mu^2 + \sin^2\mu \left(\sigma_1^2 + \sigma_2^2 + \cos^2\mu\,\sigma_3^2\right), \end{aligned} \tag{3.103}$$

where

$$\sigma_1 = \frac{1}{2}\left(\cos\psi\,\mathrm{d}\,\theta + \sin\psi \sin\theta\,\mathrm{d}\,\phi\right),$$

$$\sigma_2 = \frac{1}{2}\left(\cos\psi\,\mathrm{d}\,\theta - \sin\psi\sin\theta\,\mathrm{d}\,\phi\right),$$
$$\sigma_3 = \frac{1}{2}\left(\mathrm{d}\,\psi + \cos\theta\,\mathrm{d}\,\phi\right), \qquad \omega = \sin^2\mu\,\sigma_3. \tag{3.104}$$

This background was first obtained via a TsT transformations called null Melvin twist of $AdS_5 \times S^5$ [51].

Until now we have obtained backgrounds by using only abelian r-matrices which satisfy trivially the unimodularity condition, and then the backgrounds derived above are standard supergravity backgrounds since they coincide with those obtained via TsT transformation.

On the other hand, nonabelian r-matrices do not lead, in general, to standard supergravity backgrounds [50]. The classification of nonabelian r-matrices given in (3.69), which satisfy the unimodularity condition, leads to standard supergravity backgrounds, some of them were computed in [50]. In [76], non-unimodular r-matrices were considered, leading to deformed backgrounds which are solutions to the generalized supergravity and some of them reduce to the original $AdS_5 \times S^5$ background after performing a generalized TsT transformation.

The Drinfeld-Jimbo r-matrix including fermionic roots can be constructed by using Dynkin diagrams associated to a Cartan-Weyl basis [10]. Choosing different configurations of Dynkin diagrams lead to different deformed backgrounds . For some particular example of DJ r-matrix, the metric and B-field were constructed in [11] and the Ramond-Ramond fluxes in [12]. These background fields do not solve the standard supergravity equations but a set of generalized type supergravity equations [13, 14]. For the $AdS_5 \times S^5$ and $AdS_2 \times S^2 \times T^6$, Seilbold and Hoare found in [15] that the unimodularity condition is satisfied if and only if all the simple roots are fermionic.

Chapter 4

Yang-Baxter deformations of the $AdS_4 \times \mathbb{CP}^3$ σ-model

In this chapter we consider deformations of superstrings in the $AdS_4 \times \mathbb{CP}^3$ background. The σ-model admits a supercoset description with $\mathbb{Z}_4$-grading and it is integrable. Then we can apply the same procedures described in Chapter 3 to study its integrable deformations.

We start describing the $AdS_4 \times \mathbb{CP}^3$ background as a supercoset and the $\mathbb{Z}_4$-grading of the superalgebra $\mathfrak{uosp}(2,2|6)$. Then, we review the supercoset description of type IIA superstrings in $AdS_4 \times \mathbb{CP}^3$. In the following, we discuss this in the context of the Yang-Baxter deformation. We propose some r-matrices for $AdS_4 \times \mathbb{CP}^3$ which include the bosonic DJ r-matrix as well as one abelian Jordanian r-matrix, two mixed r-matrices and some examples of nonabelian r-matrices based on the classification in [50]. Finally, we calculate some deformed backgrounds associated to the gravity dual of ABJM theory.

4.1 Superstrings in $AdS_4 \times \mathbb{CP}^3$

The $AdS_4 \times \mathbb{CP}^3$ background is a solution of Type IIA supergravity equations of motion together with a constant dilaton, an F_2 and an F_4 flux. Due to the AdS/CFT correspondence this background is dual to $\mathcal{N} = 6$ $SU(N) \times SU(N)$ Chern-Simons theory in three dimensions with $N, k \to \infty$ and N/k large, where N is the rank of the gauge group $SU(N)$ and k is the level of the Chern-Simons action [45].

4.1.1 The $AdS_4 \times \mathbb{CP}^3$ background

For this background, from (2.20), for $n = 4$, and (2.21), for $n = 3$, we get

$$AdS_4 \equiv \frac{SO(2,3)}{SO(1,3)} \simeq \frac{Sp(4)}{SO(1,3)}, \qquad \mathbb{CP}^3 \equiv \frac{SU(4)}{U(3)} \simeq \frac{SO(6)}{U(3)}, \tag{4.1}$$

so the bosonic background is

$$AdS_4 \times \mathbb{CP}^3 \equiv \frac{Sp(4) \times SO(6)}{SO(1,3) \times U(3)}. \tag{4.2}$$

When adding fermions we need to extend the numerator of the coset (4.2) to a supergroup that contains it as its bosonic subgroup. In this case, due to the isomorphism $Sp(4) \simeq USp(2,2)$, the supergroup $UOSp(2,2|6)$ allows us to write the supercoset as [77]

$$\frac{UOSp(2,2|6)}{SO(1,3) \times U(3)}. \tag{4.3}$$

4.1.2 The Arutyunov-Frolov-Stefanski action

Arutyunov and Frolov [46] and, in parallel, Stefanski [47] proposed a way to investigate the dynamics of type IIA superstrings in $AdS_4 \times \mathbb{CP}^3$. The main idea was to follow the Green-Schwarz-Metsaev-Tseytlin approach for type IIB superstrings in $AdS_5 \times S^5$, where the supercoset σ-model formulation gives an alternative to the GS formalism. Thus, type IIA superstring theory in $AdS_4 \times \mathbb{CP}^3$ can be described as a σ-model on the supercoset $UOSp(2,2|6)/(SO(1,3) \times U(3))$. In ten dimensions, superstring theory requires spinors with 32 components in total. This supercoset description lacks 8 fermionic components and then does not describe the full superstring. It was shown that these components have been gauged away in this σ-model so the coset description has partially fixed κ-symmetry [46].

The $\mathfrak{uosp}(2,2|6)$ superalgebra and its $\mathbb{Z}_4$-grading

The $\mathfrak{osp}(4|6)$ superalgebra can be realized in terms of 10×10 supermatrices. Such supermatrices are of the form

$$M = \begin{pmatrix} X & \theta \\ \eta & Y \end{pmatrix}, \tag{4.4}$$

where X and Y are 4×4 and 6×6 bosonic matrices, and η and θ are 6×4 and 4×6 fermionic matrices. The supermatrices M belong to $\mathfrak{uosp}(2,2|6)$ if they obey the following conditions [77]

$$M^{st}\begin{pmatrix} C_4 & 0 \\ 0 & I_{6\times 6}\end{pmatrix} + \begin{pmatrix} C_4 & 0 \\ 0 & I_{6\times 6}\end{pmatrix} M = 0,$$
$$M^{\dagger}\begin{pmatrix} \gamma_0 & 0 \\ 0 & -I_{6\times 6}\end{pmatrix} + \begin{pmatrix} \gamma_0 & 0 \\ 0 & -I_{6\times 6}\end{pmatrix} M = 0, \tag{4.5}$$

where the supertranspose of a matrix M is defined as

$$M^{st} = \begin{pmatrix} X^t & -\eta^t \\ \theta^t & Y^t \end{pmatrix}. \tag{4.6}$$

In condition (4.5), C_4 and γ_0 are

$$C_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{4.7}$$

C_4 denotes the real skew-symmetric matrix satisfying $C_4^2 = -I_{4\times 4}$ while γ_0 is part of the $SO(1,3)$ Clifford algebra. The first condition in (4.5) defines the algebra $\mathfrak{osp}\,(4|6)$, whereas the second gives a real section of $\mathfrak{osp}\,(4|6)$ denoted by $\mathfrak{uosp}(2,2|6)$. From the first condition in (4.5) we get,

$$X^t = -C_4XC_4^{-1}, \quad Y^t = -Y, \quad \theta^t = -\eta C_4^{-1}, \quad \eta^t = C_4\theta, \tag{4.8}$$

and from the second one,

$$X^* = (\gamma^0C_4)X(\gamma^0C_4)^{-1}, \quad Y^* = Y, \quad \theta^* = (\gamma^0C_4)\theta, \quad \eta^* = -\eta(\gamma^0C_4). \tag{4.9}$$

The conditions on the bosonic matrix X show that it belongs to $\mathfrak{usp}(2,2)$, the unitary form of $\mathfrak{sp}(4)$. For Y, these are the conditions for the $\mathfrak{so}(6)$ algebra. We also notice that the generic matrix M contains 96 real fermionic components but the condition on θ and η reduce this number to 24.

The superalgebra $\mathfrak{uosp}(2,2|6)$ admits a fourth order automorphism [46] with stationary subalgebra $\mathfrak{so}(1,3) \oplus \mathfrak{u}(3)$. This automorphism can be used to define a $\mathbb{Z}_4$-grading such that $\mathfrak{uosp}(2,2|6)$ decomposes as a direct sum of four subalgebras,

$$\mathfrak{uosp}(2,2|6) = \mathcal{A}^{(0)} \oplus \mathcal{A}^{(1)} \oplus \mathcal{A}^{(2)} \oplus \mathcal{A}^{(3)}, \tag{4.10}$$

where each subspace is an eigenspace of Ω

$$\Omega\left(\mathcal{A}^{(k)}\right) = i^k \mathcal{A}^{(k)}, \qquad \left[\mathcal{A}^{(k)}, \mathcal{A}^{(m)}\right] \subseteq \mathcal{A}^{(k+m)} \bmod \mathbb{Z}_4, \tag{4.11}$$

similar to the $AdS_5 \times S^5$ case.

Constructing the action

Let g be an element of the supercoset (4.3) belonging to the supergroup $UOSp(2,2|6)$. We use g to build the Maurer-Cartan one-form A defined as follows,

$$A = -g^{-1}\,\mathrm{d}\,g = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}, \qquad A \in \mathfrak{uosp}(2,2|6), \qquad A^{(k)} \in \mathcal{A}^{(k)} \tag{4.12}$$

By construction A satisfy the zero-curvature condition

$$\mathcal{Z} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0. \tag{4.13}$$

The supercoset description of type IIA superstring theory in $AdS_4 \times \mathbb{CP}^3$ has the action [46]

$$S = \int \mathrm{d}^2\,\sigma \mathcal{L}, \tag{4.14}$$

with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\left[\gamma^{\alpha\beta}\,\mathrm{Str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) + \kappa\epsilon^{\alpha\beta}\,\mathrm{Str}\left(A^{(1)}_\alpha A^{(3)}_\beta\right)\right], \tag{4.15}$$

where $\gamma^{\alpha\beta}$ is related to worldsheet metric $g_{\alpha\beta}$ as $\gamma^{\alpha\beta} = g^{\alpha\beta}\sqrt{-g}$ such that $\det\gamma = 1$. The first term of (4.15) corresponds to the kinetic term. The second term, proportional to the parameter κ, is the WZ term and has contributions only from the odd components of A_α and thus it contains the fermionic degrees of freedom of the theory.

The equations of motion derived from this Lagrangian are

$$\mathcal{E} \equiv \partial_\alpha \Lambda^\alpha - [A_\alpha, \Lambda^\alpha] = 0, \tag{4.16}$$

where

$$\Lambda^\alpha = \gamma^{\alpha\beta} A^{(2)}_\beta - \frac{1}{2}\kappa\epsilon^{\alpha\beta}\left(A^{(1)}_\beta - A^{(3)}_\beta\right). \tag{4.17}$$

The equations of motion $\mathcal{E}$ in (4.16) can be projected on the subspaces of the $\mathbb{Z}_4$ automorphism. The grading 2 component of (4.16) is

$$\mathcal{E}^{(2)} \equiv \partial_\alpha\left(\gamma^{\alpha\beta}A^{(2)}_\beta\right) - \gamma^{\alpha\beta}\left[A^{(0)}_\alpha, A^{(2)}_\beta\right] + \frac{1}{2}\kappa\epsilon^{\alpha\beta}\left(\left[A^{(1)}_\alpha, A^{(1)}_\beta\right] - \left[A^{(3)}_\alpha, A^{(3)}_\beta\right]\right) = 0, \tag{4.18}$$

while the grading 1 and grading 3 components of (4.16) are given, respectively, by

$$\mathcal{E}^{(1)} \equiv \gamma^{\alpha\beta}\left[A^{(3)}_{\alpha}, A^{(2)}_{\beta}\right] + \kappa\epsilon^{\alpha\beta}\left[A^{(2)}_{\alpha}, A^{(3)}_{\beta}\right] = 0, \tag{4.19}$$

$$\mathcal{E}^{(3)} \equiv \gamma^{\alpha\beta}\left[A^{(1)}_{\alpha}, A^{(2)}_{\beta}\right] - \kappa\epsilon^{\alpha\beta}\left[A^{(2)}_{\alpha}, A^{(1)}_{\beta}\right] = 0. \tag{4.20}$$

Using the tensor $P^{\alpha\beta}_{\pm}$ defined in (2.67), the equations of motion (4.19) and (4.20) can be written as

$$\mathcal{E}^{(1)} \equiv P^{\alpha\beta}_{-}\left[A^{(2)}_{\alpha}, A^{(3)}_{\beta}\right] = 0, \tag{4.21}$$

$$\mathcal{E}^{(3)} \equiv P^{\alpha\beta}_{+}\left[A^{(2)}_{\alpha}, A^{(1)}_{\beta}\right] = 0. \tag{4.22}$$

By varying the Lagrangian (4.15) with respect to $\gamma^{\alpha\beta}$ gives rise to the Virasoro constraint

$$\mathrm{Str}\left(A^{(2)}_{\alpha}A^{(2)}_{\beta}\right) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\delta}\,\mathrm{Str}\left(A^{(2)}_{\rho}A^{(2)}_{\delta}\right) = 0. \tag{4.23}$$

So the construction of the coset sigma model goes along similar lines as for the $AdS_5 \times S^5$ superstrings Section 2.3.2. The next step is to study the κ-symmetry in the action (4.14).

κ symmetry

The κ- symmetry transformations can be understood as the infinitesimal right local action of a element $G = \exp\epsilon$ from $UOSp(2,2|6)$ on a coset representative g,

$$\delta g = g\left(\epsilon^{(1)} + \epsilon^{(3)}\right), \quad \epsilon = \epsilon^{(1)} + \epsilon^{(3)}. \tag{4.24}$$

The variation of the action (4.14) it, is given by

$$\delta_g S = \int \mathrm{d}\,\sigma^2\,\mathrm{Str}\left(\epsilon\mathcal{E}\right),$$

Using the identities (4.21), (4.22) and the zero curvature condition $P_1(\mathcal{Z}) = P_3(\mathcal{Z}) = 0$, the variation of the action can be written as follows

$$\delta_g S = \int \mathrm{d}\,\sigma^2\Big\{\delta_g\gamma^{\alpha\beta}\,\mathrm{Str}\left(A^{(2)}_{\alpha}A^{(2)}_{\beta}\right) - 4\,\mathrm{Str}\left(\epsilon^{(1)}\left[A^{(1)\alpha}_{+}, A^{(2)}_{\alpha-}\right] + \epsilon^{(3)}\left[A^{(3)\alpha}_{-}, A^{(2)}_{\alpha+}\right]\right)\Big\}. \tag{4.25}$$

Under κ-symmetry transformations the action should remain invariant off-shell. The crucial point of this construction is the ansatz for the transformation parameters $\epsilon^{(1)}$ and $\epsilon^{(3)}$ [46],

$$\epsilon^{(1)} = A^{(2)}_{\alpha-}A^{(2)}_{\beta-}\kappa^{\alpha\beta}_{++} + \kappa^{\alpha\beta}_{++}A^{(2)}_{\alpha-}A^{(2)}_{\beta-} + A^{(2)}_{\alpha-}\kappa^{\alpha\beta}_{++}A^{(2)}_{\beta-} - \frac{1}{8}\,\mathrm{Str}\left(\Sigma A^{(2)}_{\alpha-}A^{(2)}_{\beta-}\right)\kappa^{\alpha\beta}_{++}, \tag{4.26}$$

$$\epsilon^{(3)} = A^{(2)}_{\alpha +} A^{(2)}_{\beta +} \kappa^{\alpha\beta}_{--} + \kappa^{\alpha\beta}_{--} A^{(2)}_{\alpha +} A^{(2)}_{\beta +} + A^{(2)}_{\alpha +} \kappa^{\alpha\beta}_{--} A^{(2)}_{\beta +} - \frac{1}{8} \text{Str} \left(\Sigma A^{(2)}_{\alpha +} A^{(2)}_{\beta +} \right) \kappa^{\alpha\beta}_{--}, \quad (4.27)$$

where $\Sigma = \text{diag}(I_4, -I_6)$ and $\kappa^{(1)}_{+}$, $\kappa^{(3)}_{-}$ are the vectors corresponding to 1 and 3 grading. From now on, the computation is similar to the procedure leading to equation (2.92) and (2.93). The vanishing of the total variation off-shell of the action , with respect to g and $\gamma^{\alpha\beta}$, gives the transformation of the metric $\gamma^{\alpha\beta}$ in order to ensure κ-symmetry,

$$\delta\gamma^{\alpha\beta} = \frac{1}{2} \text{Str} \left(\Sigma A^{(2)}_{\delta -} \left[\kappa^{\alpha\beta}_{++}, A^{(1)\delta}_{+} \right] \right) + \frac{1}{2} \text{Str} \left(\Sigma A^{(2)}_{\delta +} \left[\kappa^{\alpha\beta}_{--}, A^{(3)\delta}_{-} \right] \right). \quad (4.28)$$

It is worth to mention that for the derivation of the κ-symmetry we used the fact that $P^{\alpha\beta}_{\pm}$ is an orthogonal projector, and then, the realization of this symmetry required that $\kappa = \pm 1$.

4.1.3 Integrability of $AdS_4 \times \mathbb{CP}^3$ superstring

The equations of motion (4.16) admit the Lax connection,

$$L_\alpha = \ell_0 A^{(0)}_\alpha + \ell_1 A^{(2)}_\alpha + \ell_2 \gamma_{\alpha\beta} \epsilon^{\beta\rho} A^{(2)}_\rho + \ell_3 A^{(1)}_\alpha + \ell_4 A^{(3)}_\alpha, \quad (4.29)$$

where ℓ_i are constants to be determined by requiring that (4.29) satisfies (2.101). The procedure to find the Lax connection explicitly in terms of a spectral parameter z is completely analogous to the one discussed for the superstring in $AdS_5 \times S^5$ (2.105) and requires that $\kappa = \pm 1$. Thus, the Lax connection is

$$L_\alpha = A^{(0)}_\alpha + \frac{1}{2} \left(z^2 + \frac{1}{z^2} \right) A^{(2)}_\alpha - \frac{1}{2\kappa} \left(z^2 - \frac{1}{z^2} \right) \gamma_{\alpha\beta} \epsilon^{\beta\rho} A^{(2)}_\rho + z A^{(1)}_\alpha + \frac{1}{z} A^{(3)}_\alpha. \quad (4.30)$$

However, as we mentioned before, the σ-model for $AdS_4 \times \mathbb{CP}^3$ does not contemplate all the fermions of type IIA superstrings. The analysis of the Lax pair for the complete theory, which includes the non-coset fermions, i.e. fermions that are not in the supercoset σ-model, was done in [78–81].

4.2 Yang-Baxter deformations of $AdS_4 \times \mathbb{CP}^3$

All the results presented in Section 3.3 can be applied to $AdS_4 \times \mathbb{CP}^3$. The Lagrangian density of the Yang-Baxter deformed σ-model with $\mathbb{Z}_4$-grading (3.18) gives the action

$$S = -\frac{(1 + c\eta^2)^2}{2(1 - c\eta^2)} \int \text{d}^2\sigma \, P^{\alpha\beta}_{-} \text{Str} \left(A_\alpha \, d \circ \frac{1}{1 - \eta R_g \circ d} A_\beta \right), \quad (4.31)$$

where $P_{\pm}^{\alpha\beta} = \frac{1}{2}(\gamma^{\alpha\beta} \pm \kappa\epsilon^{\alpha\beta})$ and c is the constant in the YBE (3.10). The operators d and $\tilde{d}$ are defined by the combination of projectors

$$d = P_1 + 2\hat{\eta}^{-2}P_2 - P_3, \quad \tilde{d} = -P_1 + 2\hat{\eta}^{-2}P_2 + P_3, \tag{4.32}$$

satisfying Str$(M, dN) =$ Str$(\tilde{d}M, N)$ where $\hat{\eta} = \sqrt{1 - c\eta^2}$ and η is a deformation parameter. We define the currents as

$$\begin{aligned} J_\alpha &\equiv \frac{1}{1 - \eta R_g \circ d} A_\alpha \equiv \mathcal{O}^{-1} A_\alpha, \qquad J_\pm^\alpha \equiv P_\pm^{\alpha\beta} J_\beta, \\ \tilde{J}_\alpha &\equiv \frac{1}{1 + \eta R_g \circ \tilde{d}} A_\alpha \equiv \tilde{\mathcal{O}}^{-1} A_\alpha, \qquad \tilde{J}_\pm^\alpha \equiv P_\pm^{\alpha\beta} \tilde{J}_\beta, \end{aligned} \tag{4.33}$$

where the operators $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are

$$\mathcal{O} = 1 - \eta R_g \circ d, \qquad \tilde{\mathcal{O}} = 1 + \eta R_g \circ \tilde{d}. \tag{4.34}$$

Thus, the equation of motion becomes

$$\mathcal{E} \equiv d\left(\partial_\alpha J_-^\alpha\right) + \tilde{d}\left(\partial_\alpha \tilde{J}_+^\alpha\right) + \left[\tilde{J}_{\alpha+}, d\left(J_-^\alpha\right)\right] + \left[J_{\alpha-}, \tilde{d}\left(\tilde{J}_+^\alpha\right)\right] = 0, \tag{4.35}$$

and the zero curvature condition is now

$$\mathcal{Z} \equiv \partial_\alpha \tilde{J}_+^\alpha - \partial_\alpha J_-^\alpha + \left[J_{\alpha-}, \tilde{J}_+^\alpha\right] + \eta R_g(\mathcal{E}) - \eta^2 c\left[dJ_-^\alpha, \tilde{d}\tilde{J}_{\alpha+}\right] = 0, \tag{4.36}$$

while the Virasoro constraint is

$$\text{Str}(J_-^{\alpha(2)} J_-^{\beta(2)}) = 0, \qquad \text{Str}(\tilde{J}_+^{\alpha(2)} \tilde{J}_+^{\beta(2)}) = 0. \tag{4.37}$$

These results reduce to the ones for the undeformed case [46] when $\eta = 0$.

Due to the $\mathbb{Z}_4$-grading of this σ-model the Lax pair can be constructed in the same way as was done for the deformed $AdS_5 \times S^5$ (3.48).

κ-symmetry

To show that the deformed action (4.31) is invariant under κ-symmetry let us consider an infinitesimal right translation, $\delta g = g\epsilon$, with

$$\epsilon = (1 - \eta R_g)\epsilon^{(1)} + (1 + \eta R_g)\epsilon^{(3)}. \tag{4.38}$$

$\epsilon^{(1)}$ and $\epsilon^{(3)}$, whose expressions will be determined below, take values in $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(3)}$ respectively. Then, the variation of the action (4.31) with respect to g reads

$$\delta_g S = \frac{(1+c\eta^2)^2}{2(1-c\eta^2)} \int \mathrm{d}^2\sigma \left[\mathrm{Str}\left((1-\eta R_g)\epsilon^{(1)}, \mathcal{E}^{(3)}\right) + \mathrm{Str}\left((1+\eta R_g)\epsilon^{(3)}, \mathcal{E}^{(1)}\right)\right]. \quad (4.39)$$

By using the following property of the R operator $\mathrm{Str}(M, R(N)) = -\,\mathrm{Str}(R(M), N)$, we can write (4.39) as

$$\delta_g S = \frac{(1+c\eta^2)^2}{2(1-c\eta^2)} \int \mathrm{d}^2\sigma \left[\mathrm{Str}\left(\epsilon^{(1)}, P_3 \circ (1+\eta R_g)\mathcal{E}\right) + \mathrm{Str}\left(\epsilon^{(3)}, P_1 \circ (1-\eta R_g)\mathcal{E}\right)\right], \quad (4.40)$$

where $\mathcal{E}$ is given in (4.35). By considering the combinations given in (3.42)

$$\begin{aligned} P_1 \circ (1-\eta R_g)(\mathcal{E}) + P_1(\mathcal{Z}) &= -4\left[\tilde{J}^{(2)}_{\alpha+}, J^{\alpha(3)}_{-}\right], \\ P_3 \circ (1+\eta R_g)(\mathcal{E}) - P_3(\mathcal{Z}) &= -4\left[J^{(2)}_{\alpha-}, \tilde{J}^{\alpha(1)}_{+}\right], \end{aligned} \quad (4.41)$$

and the zero curvature condition $P_1(\mathcal{Z}) = P_3(\mathcal{Z}) = 0$, we get

$$\begin{aligned} \delta_g S &= -2\frac{(1+c\eta^2)^2}{(1-c\eta^2)} \int \mathrm{d}^2\sigma\, \mathrm{Str}\left(\epsilon^{(1)}\left[J^{(2)}_{\alpha-}, \tilde{J}^{\alpha(1)}_{+}\right]\right) + \mathrm{Str}\left(\epsilon^{(3)}, \left[\tilde{J}^{(2)}_{\alpha+}, J^{\alpha(3)}_{-}\right]\right), \\ &= 2\frac{(1+c\eta^2)^2}{(1-c\eta^2)} \int \mathrm{d}^2\sigma \left\{\mathrm{Str}\left(\left[J^{(2)}_{\alpha-}, \epsilon^{(1)}\right], \tilde{J}^{\alpha(1)}_{+}\right) + \mathrm{Str}\left(\left[J^{(2)}_{\alpha+}, \epsilon^{(3)}\right], J^{\alpha(3)}_{-}\right)\right\}. \end{aligned} \quad (4.42)$$

Now, we propose the following ansatz for $\epsilon^{(1)}$ and $\epsilon^{(3)}$, inspired by the form they take in the undeformed case (4.26) and (4.27),

$$\begin{aligned} \epsilon^{(1)} &= J^{(2)}_{\alpha-}J^{(2)}_{\beta-}\kappa^{\alpha\beta}_{++} + \kappa^{\alpha\beta}_{++}J^{(2)}_{\alpha-}J^{(2)}_{\beta-} + J^{(2)}_{\alpha-}\kappa^{\alpha\beta}_{++}J^{(2)}_{\beta-} - \frac{1}{8}\,\mathrm{Str}\left(\Sigma J^{(2)}_{\alpha-}J^{(2)}_{\beta-}\right)\kappa^{\alpha\beta}_{++}, \\ \epsilon^{(3)} &= \tilde{J}^{(2)}_{\alpha+}\tilde{J}^{(2)}_{\beta+}\kappa^{\alpha\beta}_{--} + \kappa^{\alpha\beta}_{--}\tilde{J}^{(2)}_{\alpha+}\tilde{J}^{(2)}_{\beta+} + \tilde{J}^{(2)}_{\alpha+}\kappa^{\alpha\beta}_{--}\tilde{J}^{(2)}_{\beta+} - \frac{1}{8}\,\mathrm{Str}\left(\Sigma\tilde{J}^{(2)}_{\alpha+}\tilde{J}^{(2)}_{\beta+}\right)\kappa^{\alpha\beta}_{--}, \end{aligned} \quad (4.43)$$

where Σ is a diagonal matrix $\Sigma = \mathrm{diag}(I_4, -I_6)$. The $\kappa^{\alpha\beta}_{++}$ and $\kappa^{\alpha\beta}_{--}$ are the κ-symmetry parameters which are assumed to be independent of the dynamical fields of the model. The automorphism Ω_4 acts on $\epsilon^{(1)}$ and $\epsilon^{(3)}$ above, and in order to $\epsilon^{(1)} \in \mathcal{A}^{(1)}$ and $\epsilon^{(3)} \in \mathcal{A}^{(3)}$ we need $\kappa^{\alpha\beta}_{++} \in \mathcal{A}^{(1)}$ and $\kappa^{\alpha\beta}_{--} \in \mathcal{A}^{(3)}$. Then, the commutators in (4.42) can be written as

$$\begin{aligned} \left[J^{(2)}_{\alpha-}, \epsilon^{(1)}\right] &= \left[J^{(2)}_{\alpha-}J^{(2)}_{\beta-}J^{(2)}_{\delta-} - \frac{1}{8}\,\mathrm{Str}\left(\Sigma J^{(2)}_{\beta-}J^{(2)}_{\delta-}\right)J^{(2)}_{\alpha-}, \kappa^{\beta\delta}_{++}\right], \\ \left[J^{(2)}_{\alpha+}, \epsilon^{(3)}\right] &= \left[J^{(2)}_{\alpha+}\tilde{J}^{(2)}_{\beta+}\tilde{J}^{(2)}_{\delta+} - \frac{1}{8}\,\mathrm{Str}\left(\Sigma\tilde{J}^{(2)}_{\beta+}\tilde{J}^{(2)}_{\delta+}\right)J^{(2)}_{\alpha+}, \kappa^{\beta\delta}_{--}\right]. \end{aligned} \quad (4.44)$$

Here some terms cancel each other due to the cyclicity of the indices α, β and δ. Using the following identity

$$A^3 = \frac{1}{8}\,\mathrm{Str}(\Sigma A^2)A + \frac{1}{8}\,\mathrm{Str}(A^2)\Sigma A, \quad (4.45)$$

we can write

$$J^{(2)}_{\alpha-}J^{(2)}_{\beta-}J^{(2)}_{\delta-} - \frac{1}{8}\,\mathrm{Str}\left(\Sigma J^{(2)}_{\beta-}J^{(2)}_{\delta-}\right)J^{(2)}_{\alpha-} = \frac{1}{8}\,\mathrm{Str}\left(J^{(2)}_{\beta-}J^{(2)}_{\delta-}\right)\Sigma J^{(2)}_{\alpha-},$$
$$J^{(2)}_{\alpha+}\tilde{J}^{(2)}_{\beta+}\tilde{J}^{(2)}_{\delta+} - \frac{1}{8}\,\mathrm{Str}\left(\Sigma \tilde{J}^{(2)}_{\beta+}\tilde{J}^{(2)}_{\delta-}\right)J^{(2)}_{\alpha+} = \frac{1}{8}\,\mathrm{Str}\left(\tilde{J}^{(2)}_{\beta+}\tilde{J}^{(2)}_{\delta+}\right)\Sigma J^{(2)}_{\alpha+}. \tag{4.46}$$

so the commutators (4.44) reduce to

$$\left[J^{(2)}_{\alpha-},\epsilon^{(1)}\right] = \left[\frac{1}{8}\,\mathrm{Str}(J^{(2)}_{\beta-}J^{(2)}_{\delta-})\Sigma J^{(2)}_{\alpha-},\kappa^{\beta\delta}_{++}\right] = \frac{1}{8}\,\mathrm{Str}(J^{(2)}_{\beta-}J^{(2)}_{\delta-})\left[\Sigma J^{(2)}_{\alpha-},\kappa^{\beta\delta}_{++}\right],$$
$$\left[J^{(2)}_{\alpha+},\epsilon^{(3)}\right] = \frac{1}{8}\,\mathrm{Str}(\tilde{J}^{(2)}_{\beta+}\tilde{J}^{(2)}_{\delta+})\left[\Sigma J^{(2)}_{\alpha+},\kappa^{\beta\delta}_{--}\right]. \tag{4.47}$$

Then, the variation of the action (4.42) under κ-symmetry becomes

$$\begin{aligned}
\delta_g S &= -2\frac{(1+c\eta^2)^2}{(1-c\eta^2)}\int \mathrm{d}^2\sigma\Big\{-\frac{1}{8}\,\mathrm{Str}\left(J^{(2)}_{\beta-}J^{(2)}_{\delta-}\right)\times\mathrm{Str}\left(\left[\Sigma J^{(2)}_{\alpha-},\kappa^{\beta\delta}_{++}\right]\tilde{J}^{\alpha(1)}_{+}\right)\\
&\quad -\frac{1}{8}\,\mathrm{Str}\left(\tilde{J}^{(2)}_{\beta+}\tilde{J}^{(2)}_{\delta+}\right)\times\mathrm{Str}\left(\left[\Sigma J^{(2)}_{\alpha+},\kappa^{\beta\delta}_{--}\right]J^{\alpha(3)}_{-}\right)\Big\},\\
&= \frac{(1+c\eta^2)^2}{4(1-c\eta^2)}\int \mathrm{d}^2\sigma\Big\{\mathrm{Str}\left(\Sigma J^{(2)}_{\alpha-}\left[\kappa^{\beta\delta}_{++},\tilde{J}^{\alpha(1)}_{+}\right]\right)\mathrm{Str}\left(J^{(2)}_{\beta-}J^{(2)}_{\delta-}\right)\\
&\quad +\mathrm{Str}\left(\Sigma J^{(2)}_{\alpha+}\left[\kappa^{\beta\delta}_{--},J^{\alpha(3)}_{-}\right]\right)\mathrm{Str}\left(\tilde{J}^{(2)}_{\beta+}\tilde{J}^{(2)}_{\delta+}\right)\Big\}.
\end{aligned} \tag{4.48}$$

On the other hand, the variation of the action (4.31) with respect to $\gamma^{\alpha\beta}$ is

$$\delta_\gamma S = -\frac{1}{2}\frac{(1+c\eta^2)^2}{(1-c\eta^2)^2}\int \mathrm{d}^2\sigma\,\delta\gamma^{\alpha\beta}\Big\{\mathrm{Str}\left(J^{(2)}_{\alpha}J^{(2)}_{\beta}\right)+\mathrm{Str}\left(\tilde{J}^{(2)}_{\alpha}\tilde{J}^{(2)}_{\beta}\right)\Big\}. \tag{4.49}$$

By adding (4.48) and (4.49) and requiring that the total variation of the deformed action (4.31) vanishes off-shell we can deduce the transformation of the worldsheet metric $\gamma^{\alpha\beta}$ under κ-symmetry,

$$\delta\gamma^{\alpha\beta} = \frac{1}{2}(1-c\eta^2)\,\mathrm{Str}\left(\Sigma J^{(2)}_{\delta-}\left[\kappa^{\alpha\beta}_{++},\tilde{J}^{\delta(1)}_{+}\right]+\Sigma J^{(2)}_{\delta+}\left[\kappa^{\alpha\beta}_{--},J^{\delta(3)}_{-}\right]\right). \tag{4.50}$$

4.3 r-matrices for $AdS_4 \times \mathbb{CP}^3$

Based on the classification shown in Section 3.2 we present here some solutions of the YBE for $AdS_4 \times \mathbb{CP}^3$.

Drinfeld-Jimbo r-matrix

A typical solution of the mCYBE is the Drinfeld-Jimbo type solution which is constructed in terms of the positive and negative roots of the superalgebra. The DJ r-matrix of $\mathfrak{uosp}(2,2|6)$

at the bosonic level is just the sum of the $\mathfrak{usp}(2,2)$ and $\mathfrak{so}(6)$ DJ r-matrices, respectively. The DJ r-matrix associated to $\mathfrak{usp}(2,2)$ is

$$r_{DJ} = e_1 \wedge f_1 + e_2 \wedge f_2, \tag{4.51}$$

and for $\mathfrak{so}(6)$ is

$$\tilde{r}_{DJ} = \tilde{e}_1 \wedge \tilde{f}_1 + \tilde{e}_2 \wedge \tilde{f}_2 + \tilde{e}_3 \wedge \tilde{f}_3, \tag{4.52}$$

where e_i, f_i, $\tilde{e}_i$ and $\tilde{f}_i$ are the bosonic roots of $\mathfrak{usp}(2,2)$ and $\mathfrak{so}(6)$ defined in Appendix D. The R operator corresponding to this DJ r-matrix is defined by its action on the Cartan generators, the positive and the negative roots,

$$Re_i = ie_i, \qquad Rf_i = -if_i, \qquad Rh_i = 0, \tag{4.53}$$

$$R\tilde{e}_i = i\tilde{e}_i, \qquad R\tilde{f}_i = -i\tilde{f}_i, \qquad R\tilde{h}_i = 0, \tag{4.54}$$

which satisfies the mCYBE with $c = 1$.

Abelian r-matrix

An r-matrix of this type is constructed by commuting generators and satisfy the unimodularity condition trivially. It can be built in terms of the Cartan generators L, L_3 and M_3 of $\mathbb{CP}^3$ [48]

$$r = \mu_1 L \wedge M_3 + \mu_2 L_3 \wedge M_3 + \mu_3 L_3 \wedge L, \tag{4.55}$$

where the μ_i's are constant parameters.

Abelian-Jordanian and mixed r-matrix

Abelian-Jordanian r-matrices are abelian and nilpotent and one example for AdS_4 is

$$r = P_1 \wedge P_2, \tag{4.56}$$

where P_1 and P_2 are the AdS_4 generators given in Appendix E.

Two examples of r-matrices which are composed by generators of isometries of both subspaces, AdS_4 and $\mathbb{CP}^3$, are

$$r = \frac{\alpha}{\eta} P_2 \wedge M_3, \tag{4.57}$$

$$r = \frac{1}{\eta} P_- \wedge (\beta_1 L + \beta_2 L_3 + \beta_3 M_3) , \tag{4.58}$$

where α, β_i are constants and P_2 and $P_- = P_0 - P_2$ are AdS_4 generators given in Appendix E and L, L_3, M_3 are Cartan generators of $\mathbb{CP}^3$.

Unimodular nonabelian r-matrix

A list of unimodular nonabelian r-matrices for $\mathfrak{so}(2,4)$ is given in (3.69). From it, r_6, r_{13} and r_{15} are written in terms of a subset of generators that can be embedded in $\mathfrak{so}(2,3)$ and thus be used to construct deformations of AdS_4, [1],

$$r_6 = P_2 \wedge M_{01} + 2P_0 \wedge P_1, \tag{4.59}$$

$$r_{13} = P_1 \wedge P_2 + P_0 \wedge M_{12}, \tag{4.60}$$

$$r_{15} = P_1 \wedge P_2 + (M_{01} - M_{12}) \wedge (P_0 + P_2) . \tag{4.61}$$

Also, we could consider generators of isometries of $\mathbb{CP}^3$, which obviously commute with those of $\mathfrak{so}(2,4)$, in order to construct unimodular nonabelian deformations of $AdS_4 \times \mathbb{CP}^3$,

$$r_1 = P_1 \wedge S_A + (P_0 + P_2) \wedge (M_{01} - M_{12}) , \tag{4.62}$$

$$r_2 = P_1 \wedge S_A + (P_0 + P_2) \wedge (P_2 + M_{01} - M_{12}) \tag{4.63}$$

$$r_7 = M_{02} \wedge S_A + 2P_0 \wedge P_2, \tag{4.64}$$

where S_A is any standard generator of isometries of $\mathbb{CP}^3$.

The complete classification of this type of r-matrices has not been done yet. The r-matrices we considered above are some examples of those that can be used to deform AdS_4.

4.4 Deformed Backgrounds generated by r-matrices

In this section we will compute the backgrounds of some of the r-matrices we proposed.

[1] As was mentioned at the end of Section 3.4.1, it is not possible to construct unimodular nonabelian r-matrices for compact Lie algebras [76]

4.4.1 Gravity dual of ABJM on non-commutative spacetime

The non-commutative deformation of the ABJM three-dimensional theory corresponds, on the string theory side, to a deformation of the AdS_4 space.

Coset construction of AdS_4

The corresponding coset for AdS_4 is

$$AdS_4 \equiv \frac{USP(2,2)}{SO(1,3)} \simeq \frac{SO(2,3)}{SO(1,3)}. \tag{4.65}$$

In terms of its algebra the coset for AdS_4 can be written as

$$\mathfrak{so}(2,3) = \frac{\mathfrak{so}(2,3)}{\mathfrak{so}(1,3)} \oplus \mathfrak{so}(1,3), \tag{4.66}$$

where $\mathfrak{so}(1,3)$ corresponds to the local isometries. In order to write an appropriate coset parametrization of (4.65), we consider the following basis

$$\frac{\mathfrak{so}(2,3)}{\mathfrak{so}(1,3)} = \text{span}_{\mathbb{R}} \{K_m\}, \qquad m = 1, ..., 4 \tag{4.67}$$

where we have renaming the generators using the notation in Appendix E:

$$K_1 = M_{04}, \qquad K_2 = M_{14}, \qquad K_3 = M_{24}, \qquad K_4 = iM_{34}. \tag{4.68}$$

The $\mathfrak{so}(1,3)$ generators are $\{H_a\} = \{M_{01}, M_{02}, M_{03}, M_{12}, M_{13}, M_{23}\}$. We will use (4.68) to parametrize AdS_4. An appropriate representative coset, which will allow us to get the desired AdS_4 metric, is [82]

$$g = \exp\left[x_0 P_0 + x_1 P_1 + x_2 P_2\right] \exp\left[-\log z D\right] \in SO(2,3)/SO(1,3), \tag{4.69}$$

where P_a and D are the translation and dilation generators respectively, defined in Appendix E. Then, we can write a Lie algebra element of $\mathfrak{so}(2,3)$ by using the Maurer-Cartan one-form.

Any $X \in \mathfrak{so}(2,3)$ can be written as

$$X = \sum_m \frac{\text{tr}\,(K_m X)}{\text{tr}\,(K_m K_m)} K_m + \sum_a \frac{\text{tr}\,(H_a X)}{\text{tr}\,(H_a H_a)} H_a, \tag{4.70}$$

with $K_m \in \mathfrak{so}(2,3)/\mathfrak{so}(1,3)$ and $H_a \in \mathfrak{so}(1,3)$. The projector into the coset can be defined as

$$P\,(X) = X - \sum_a \frac{\text{tr}\,(H_a X)}{\text{tr}\,(H_a H_a)} H_a = \sum_m \frac{\text{tr}\,(K_m X)}{\text{tr}\,(K_m K_m)} K_m. \tag{4.71}$$

The Maurer-Cartan one-form A has generators that are out of the coset so we need to project them back into those of the coset by using (4.71)

$$\begin{aligned} P(A) &= A - \frac{\text{tr}(M_{01}A)}{\text{tr}(M_{01}M_{01})}M_{01} - \frac{\text{tr}(M_{02}A)}{\text{tr}(M_{02}M_{02})}M_{02} - \frac{\text{tr}(M_{03}A)}{\text{tr}(M_{03}M_{03})}M_{03} \\ &\quad - \frac{\text{tr}(M_{12}A)}{\text{tr}(M_{12}M_{12})}M_{12} - \frac{\text{tr}(M_{13}A)}{\text{tr}(M_{13}M_{13})}M_{13} - \frac{\text{tr}(M_{23}A)}{\text{tr}(M_{23}M_{23})}M_{23} \\ &= \frac{dx_0}{z}K_1 + \frac{dx_1}{z}K_2 + \frac{dx_2}{z}K_3 - \frac{dz}{z}K_4 = E^m K_m. \end{aligned} \tag{4.72}$$

The metric for AdS_4 is obtained from the symmetric part of the Lagrangian (4.15) with fermionic degrees of freedom switched off,

$$\mathcal{L} = -\frac{1}{2}\gamma^{\alpha\beta}\,\text{tr}\left[A_\alpha P(A_\beta)\right], \tag{4.73}$$

which gives

$$ds^2_{AdS_4} = \frac{1}{z^2}\left(dz^2 + dx_0^2 - dx_1^2 - dx_2^2\right), \tag{4.74}$$

where $z > 0$ and x_a ($a = 0, 1, 2$) parametrize a three-dimensional Minkowski space with signature $(+--)$.

Yang-Baxter deformation of AdS_4

The deformed Yang-Baxter σ-model Lagrangian of the action (4.31), with $c = 0$, $\kappa = 1$ and the fermionic degrees of freedom switched-off, can be written as

$$\mathcal{L} = -\frac{1}{2}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta})\,\text{tr}\left(A_\alpha P \circ \mathcal{O}^{-1} A_\beta\right), \tag{4.75}$$

where the operator $\mathcal{O}$ depending on the deformation parameter η is given by

$$\mathcal{O}^{-1} = \frac{1}{1 - 2\eta R_g \circ P}. \tag{4.76}$$

In order to extract the background fields from the Lagrangian (4.75), we need to find the projected action of R_g on each generator of the basis (4.67), such that

$$P\left(R_g(K_m)\right) = \Lambda_m{}^n K_n, \tag{4.77}$$

where K_m are defined in (4.68). Moreover, the projected action of the operator (4.76) can be computed in a similar way

$$P\left(\mathcal{O}^{-1}(K_m)\right) = C_m{}^n K_n. \tag{4.78}$$

Combining equations (4.76), (4.77) and (4.78) we can find the relation between the coefficients $\Lambda_m{}^n$ and $C_m{}^n$

$$K_m = \left(C_m{}^n K_n - 2\eta\, C_m{}^n \Lambda_n{}^p K_p\right), \tag{4.79}$$

or in matrix notation,

$$\mathbf{C} = \left(\mathbf{I} - 2\eta\mathbf{\Lambda}\right)^{-1}, \tag{4.80}$$

which can be solved for $\mathbf{C}$.

We can rewrite the deformed Lagrangian (4.75) as

$$\mathcal{L} = -\frac{N}{2}\left(\gamma^{\alpha\beta}C_{(mn)}E^m_\alpha E^n_\beta - \epsilon^{\alpha\beta}C_{[mn]}E^m_\alpha E^n_\beta\right), \tag{4.81}$$

where N is a constant, the coefficients $C_{(mn)}$ and $C_{[mn]}$ are the symmetric and antisymmetric parts of the matrix (4.80) and E^m_α represent the coefficients in front of each of generators in (4.72).

The r-matrix to be used is built by taking abelian generators as in (4.56),

$$r = P_1 \wedge P_2. \tag{4.82}$$

This matrix is of abelian-Jordanian type. From the coset parametrization (4.69) together with the definitions (3.20) and (3.21) for the R operator, we find that the only non-vanishing components (4.77) of the matrix $\mathbf{\Lambda}$ are

$$\Lambda_2{}^3 = -\Lambda_3{}^2 = \frac{1}{z^2}, \tag{4.83}$$

and those of $\mathbf{C}$ (4.80) are

$$C_1{}^1 = C_4{}^4 = 1, \quad C_2{}^2 = C_3{}^3 = \frac{z^4}{z^4+4\eta^2}, \quad C_2{}^3 = -C_3{}^2 = \frac{z^4}{z^4+4\eta^2}. \tag{4.84}$$

Then, we can identify the symmetric and antisymmetric part of the deformed Lagrangian as

$$\mathcal{L}_{\text{sym}} = \frac{1}{4}\gamma^{\alpha\beta}\left(\frac{1}{z^2}\left(\partial_\alpha x_0\partial_\beta x_0 + \partial_\alpha z\partial_\beta z\right) + \mathcal{M}\left(\partial_\alpha x_1\partial_\beta x_1 + \partial_\alpha x_2\partial_\beta x_2\right)\right), \tag{4.85}$$

$$\mathcal{L}_{\text{antisym}} = -\frac{1}{4}\epsilon^{\alpha\beta}\mathcal{M}\left(\partial_\alpha x_2\partial_\beta x_1 - \partial_\alpha x_1\partial_\beta x_2\right), \tag{4.86}$$

where

$$\mathcal{M}^{-1} = z^2 + \frac{4\eta^2}{z^2}. \tag{4.87}$$

The metric and the antisymmetric field can easily be read off

$$ds^2_{AdS_4} = \frac{1}{z^2}\left(dx_0^2 + dz^2\right) + \mathcal{M}\left(dx_1^2 + dx_2^2\right), \tag{4.88}$$

$$B = \frac{\eta}{z^2}\mathcal{M}\, dx_1 \wedge dx_2. \tag{4.89}$$

This background agrees with the gravity dual of non-commutative ABJM theory presented in [49] up to a Wick rotation $x_0 \to ix_0$. This result shows that the gravity dual of a non-commutative ABJM theory is an integrable deformation of $AdS_4 \times \mathbb{CP}^3$ string theory.

4.4.2 Gravity dual of one-parameter dipole deformation of ABJM

In this section, we construct the deformation for both spaces AdS_4 and $\mathbb{CP}^3$. We will choose an r-matrix with one parameter constructed in terms of one generator of AdS_4 and one of the three Cartan generators of the $\mathbb{CP}^3$.

Coset construction of $AdS_4 \times \mathbb{CP}^3$

In [48] it was used an extension of the coset describing the $\mathbb{CP}^3$ space in order to get the standard Fubini-Study metric. To obtain the full $AdS_4 \times \mathbb{CP}^3$ metric we consider the extended coset

$$AdS_4 \times \mathbb{CP}^3 \equiv \frac{SO(2,3)}{SO(1,3)} \times \frac{SU(4) \times SU(2)}{U(3) \times SU(2)}. \tag{4.90}$$

To this end we need to choose a suitable supermatrix realization for the generators of this extended supercoset. As explained in [83] we can write a algebra valued $(6|4) \times (6|4)$ supermatrix as

$$M = \left(\begin{array}{cc|c} \mathfrak{so}(2,3) & 0 & \bar{Q} \\ 0 & \mathfrak{su}(2) & 0 \\ \hline Q & 0 & \mathfrak{su}(4) \end{array}\right), \tag{4.91}$$

where we extended $\mathfrak{so}(2,3) \oplus \mathfrak{su}(4)$ to $\mathfrak{so}(2,3) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(4)$ and $Q, \bar{Q}$ are fermionic generators. In terms of algebras we now have

$$\mathfrak{so}(2,3) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(4) = \left(\frac{\mathfrak{so}(2,3) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(4)}{\mathfrak{so}(1,3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(3)}\right) \oplus \mathfrak{so}(1,3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(3). \tag{4.92}$$

Notice that in (4.91), in the form of (4.4), we put the extra $\mathfrak{su}(2)$ in the X-block of the supermatrix and the $\mathfrak{su}(4)$ in the 4×4 Y-block, such that the supertrace is

$$M^{\rm st} = \mathrm{tr}\, X_1 + (\mathrm{tr}\, X_2 - \mathrm{tr}\, Y)\,, \tag{4.93}$$

where $X_1 \in \mathfrak{so}(2,3)$, $X_2 \in \mathfrak{su}(2)$ and $Y \in \mathfrak{su}(4)$ and the fermionic blocks are preserved.

The basis of $\mathfrak{so}(2,3) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(4)$ that we will consider is composed of $\mathfrak{so}(2,3)$ generators denoted by F_A ($A = 1, ..., 10$), $\mathfrak{su}(2)$ generators denoted by M_a ($a = 1,2,3$) and $\mathfrak{su}(4)$ generators denoted by L_m ($m = 1, ...15$),

$$F_A = \left(\begin{array}{cc|c} f_A & & \\ & 0 & \\ \hline & & 0 \end{array}\right), \quad M_a = -\frac{i}{2}\left(\begin{array}{cc|c} 0 & & \\ & \sigma_a & \\ \hline & & 0 \end{array}\right), \quad L_m = -\frac{i}{2}\left(\begin{array}{cc|c} 0 & & \\ & 0 & \\ \hline & & \lambda_m \end{array}\right), \tag{4.94}$$

where f_A are the 4×4 matrices representing the generators of isometries of AdS_4, σ_a and λ_m are the conventional 2×2 Pauli and 4×4 Gell-Mann matrices of $\mathfrak{su}(2)$ and $\mathfrak{su}(4)$, respectively. All these generators are defined in Appendix E.

The commutation relations and the supertraces are [2]

$$[L_m, L_n] = f_{mn}{}^{p} L_p, \qquad [M_a, M_b] = \epsilon_{ab}{}^{c} M_c, \tag{4.95}$$

$$\mathrm{Str}\,(L_m L_n) = \frac{1}{2}\delta_{mn}, \qquad \mathrm{Str}\,(M_a M_b) = -\frac{1}{2}\delta_{ab}. \tag{4.96}$$

The Cartan generators of $\mathfrak{su}(2) \oplus \mathfrak{su}(4)$ are given by L_3, L_8, L_{15} and M_3. The following combination of generators will be useful,

$$T_1 = L_6 - L_9, \qquad T_2 = L_6 + L_9 + 2M_1. \tag{4.97}$$

The basis for $\mathfrak{su}(2) \oplus \mathfrak{su}(4)$ can be chosen as

$$\mathfrak{su}(2) \oplus \mathfrak{su}(4) = \mathrm{span}_{\mathbb{R}}\, \{L_{m'}, M_2, M_3, T_1, T_2, H\}\,, \tag{4.98}$$

where $L_{m'}$ is the set of generators of $\mathfrak{su}(4)$ except for L_6, L_9 and

$$H = L_6 + L_9 + M_1, \tag{4.99}$$

[2]We have denoted the 10 generators of $\mathfrak{so}(2,3)$ as $f_A = \{D, M_{01}, M_{02}, M_{12}, P_0, P_1, P_2, K_0, K_1, K_2\}$. F_A is simply the extension to a supermatrix of these generators and the algebra they satisfy is given in (E.10) of Appendix E.

satisfying

$$\text{Str}\left(HH\right) = \frac{1}{2}. \tag{4.100}$$

The basis for the coset is this,

$$\frac{\mathfrak{su}(2) \oplus \mathfrak{su}(4)}{\mathfrak{u}(3) \oplus \mathfrak{su}(2)} = \text{span}_{\mathbb{R}} \left\{K_m\right\}, \qquad m = 1, ..., 6, \tag{4.101}$$

where

$$\begin{aligned} &K_1 = L_{11}, && K_2 = L_{12}, && K_3 = L_{13} \\ &K_4 = L_{14}, && K_5 = H, && K_6 = L_{10}, \end{aligned} \tag{4.102}$$

with

$$\text{Str}\left(K_m K_n\right) = \frac{1}{2}. \tag{4.103}$$

The generators of $\mathfrak{u}(3) \oplus \mathfrak{su}(2)$ are

$$\mathfrak{u}(3) \oplus \mathfrak{su}(2) = \text{span}_{\mathbb{R}} \left\{H_a\right\} = \text{span}_{\mathbb{R}} \left\{L_1, L_2, L_3, L_4, L_5, L_7, L_8, L_{15}, T_1, T_2, M_2, M_3\right\}. \tag{4.104}$$

An appropriate coset representative which will allows us to get the desired $AdS_4 \times \mathbb{CP}^3$ is

$$g = g_{AdS_4} g_{\mathbb{CP}^3}, \tag{4.105}$$

where

$$g_{AdS_4} = \exp\left[x_0 P_0 + x_1 P_1 + x_2 P_2\right] \exp\left[\log r D\right], \tag{4.106}$$

$$\begin{aligned} g_{\mathbb{CP}^3} &= \exp\left[\left(\varphi_1 L_3 + \varphi_2 L - \psi M_3\right)\right] \exp\left[\left(\theta_1 L_2 + \left(\theta_2 + \pi\right) L_{14}\right)\right] \times \\ &\times \exp\left[\left(2\xi + \pi\right)\left(L_{10} + M_2\right)\right], \end{aligned} \tag{4.107}$$

where

$$L = -\frac{1}{\sqrt{3}} L_8 + \sqrt{\frac{2}{3}} L_{15}. \tag{4.108}$$

We can get the $AdS_4 \times \mathbb{CP}^3$ background by following the same steps as in the previous section. The projection $P\left(A\right)$ allows us to define,

$$P\left(A\right) = E^m K_m, \qquad m = 0, ..., 9, \tag{4.109}$$

in terms of all the coset generators

$$\left\{K_m\right\} = \left\{P_0, P_1, P_2, D, L_{11}, L_{12}, L_{13}, L_{14}, H, L_{10}\right\}. \tag{4.110}$$

In this case, the projector is defined as in (4.71) but extended to supermatrices,

$$P(A) = A - \sum_a \frac{\text{Str}(H_a A)}{\text{Str}(H_a H_a)} H_a = \sum_m \frac{\text{Str}(K_m A)}{\text{Str}(K_m K_m)} K_m. \tag{4.111}$$

Then we find

$$\begin{gathered} E^1 = r\,\mathrm{d}\,x_0, \quad E^2 = r\,\mathrm{d}\,x_1, \quad E^3 = r\,\mathrm{d}\,x_2, \quad E^4 = \frac{\mathrm{d}\,r}{r}, \\ E^5 = \cos\xi \sin\theta_1\,\mathrm{d}\,\varphi_1, \quad E^6 = \cos\xi\,\mathrm{d}\,\theta_1, \quad E^7 = -\sin\theta_2 \sin\xi\,\mathrm{d}\,\varphi_2 \\ E^8 = -\sin\xi\,\mathrm{d}\,\theta_2, \quad E^9 = \frac{1}{2}(\cos\theta_1\,\mathrm{d}\,\varphi_1 - \cos\theta_2\,\mathrm{d}\,\varphi_2 + 2\,\mathrm{d}\,\psi)\sin 2\xi, \quad E^{10} = 2\,\mathrm{d}\,\xi. \end{gathered} \tag{4.112}$$

The metric can be computed from

$$\mathcal{L} = -\frac{1}{2}\gamma^{\alpha\beta}\,\text{Str}\left[A_\alpha P(A_\beta)\right], \tag{4.113}$$

where $\mathrm{d}\,s^2_{AdS_4}$ was given in (4.74) and $\mathrm{d}\,s^2_{\mathbb{CP}^3}$ is

$$\begin{aligned} \mathrm{d}\,s^2_{\mathbb{CP}^3} = &\,\mathrm{d}\,\xi^2 + \frac{1}{4}\cos^2\xi\left(\mathrm{d}\,\theta_1^2 + \sin^2\theta_1\,\mathrm{d}\,\varphi_1^2\right) + \frac{1}{4}\sin^2\xi\left(\mathrm{d}\,\theta_2^2 + \sin^2\theta_2\,\mathrm{d}\,\varphi_2^2\right) \\ &+ \cos^2\xi\sin^2\xi\left(\frac{1}{2}\cos\theta_1\,\mathrm{d}\,\theta_1 - \frac{1}{2}\cos\theta_2\,\mathrm{d}\,\theta_2 + \mathrm{d}\,\psi\right)^2, \end{aligned} \tag{4.114}$$

where (θ_1, φ_1) and (θ_2, φ_2) parametrize two spheres, the angle $0 \leqslant \xi \leqslant \pi/2$ determines their radii and $0 \leqslant \psi \leqslant 2\pi$.

Yang-Baxter deformation of $AdS_4 \times \mathbb{CP}^3$

Let us consider the r-matrix in (4.57),

$$r = \frac{\alpha}{\eta} P_2 \wedge M_3. \tag{4.115}$$

From (3.21), the R operator associated to this r-matrix is

$$R(X) = \frac{\alpha}{\eta}\left(P_2\,\text{Str}(X M_3) - M_3\,\text{Str}(X P_2)\right). \tag{4.116}$$

The projected action of R_g and $\mathcal{O}^{-1}$ on the supercoset bosonic generators K_m, (4.77) and (4.78), give the matrices $\mathbf{\Lambda}$ and $\mathbf{C}$, whose non-vanishing terms are

$$\Lambda_4^{\ 10} = -\Lambda_{10}^{\ 4} = -\frac{\alpha r \sin 2\xi}{2\eta}, \tag{4.117}$$

and for (4.80)

$$\begin{aligned} C_1{}^1 = C_2{}^2 = C_4{}^4 = C_5{}^5 = C_6{}^6 = C_7{}^7 = C_8{}^8 = C_{10}{}^{10} = 1, \\ C_3{}^3 = C_9{}^9 = \frac{1}{1+\alpha^2 r^2 \sin^2 2\xi}, \quad C_3{}^9 = -C_9{}^3 = -\frac{\alpha r \sin 2\xi}{1+\alpha^2 r^2 \sin^2 2\xi}. \end{aligned} \tag{4.118}$$

The metric and the antisymmetric field for the deformed case are

$$\begin{aligned} ds^2 =& \frac{1}{4}\left(r^2\left(\mathrm{d}\,x_0^2 + \mathrm{d}\,x_1^2 + \mathcal{M}\,\mathrm{d}\,x_2^2\right) + \frac{\mathrm{d}\,r^2}{r^2}\right) \\ & + \mathrm{d}\,\xi^2 + \mathcal{M}\cos^2\xi\sin^2\xi\left(\frac{1}{2}\cos\theta_1\,\mathrm{d}\,\varphi_1 - \frac{1}{2}\cos\theta_2\,\mathrm{d}\,\varphi_2 + \mathrm{d}\,\psi\right)^2 \\ & + \frac{1}{4}\cos^2\xi\left(\mathrm{d}\,\theta_1^2 + \sin^2\theta_1\,\mathrm{d}\,\varphi_1^2\right) + \frac{1}{4}\sin^2\xi\left(\mathrm{d}\,\theta_2^2 + \sin^2\theta_2\,\mathrm{d}\,\varphi_2^2\right), \end{aligned} \tag{4.119}$$

$$B = -\alpha\mathcal{M}r^2\sin^2 2\xi\,\mathrm{d}\,x_2 \wedge \left(\frac{1}{2}\cos\theta_1\,\mathrm{d}\,\varphi_1 - \frac{1}{2}\cos\theta_2\,\mathrm{d}\,\varphi_2 + \mathrm{d}\,\psi\right), \tag{4.120}$$

where

$$\mathcal{M}^{-1} = 1 + \alpha^2 r^2 \sin^2 2\xi. \tag{4.121}$$

The deformed background (4.119) and (4.120) agree with the gravity dual of the one-parameter dipole deformation of ABJM theory found in [49]. We then have an integrable deformation of superstring in $AdS_4 \times \mathbb{CP}^3$.

4.4.3 Gravity dual of the non-relativistic limit of ABJM: Schrödinger spacetime

As in the last section, we will use the extended supercoset with supermatrices (4.94) and $AdS_4 \times \mathbb{CP}^3$ parametrized by the coset representatives (4.106) and (4.107).

Yang-Baxter deformation of $AdS_4 \times \mathbb{CP}^3$

We consider the r-matrix in (4.58),

$$r = \frac{1}{\eta}P_- \wedge \left(\beta_1 L + \beta_2 L_3 + \beta_3 M_3\right). \tag{4.122}$$

For simplicity we will choose $\beta_1 = \beta_3 = 0$. Then, the non-vanishing elements (4.77) and (4.78) of the $\mathbf{\Lambda}$ and $\mathbf{C}$ matrices, respectively, are

$$\begin{aligned} \Lambda_1{}^5 = -\Lambda_5{}^1 = \Lambda_3{}^5 = \Lambda_5{}^3 = -\frac{\beta_2 r\cos\xi\sin\theta_1}{2\sqrt{2}}, \\ \Lambda_1{}^9 = \Lambda_3{}^9 = -\Lambda_9{}^1 = \Lambda_9{}^3 = -\frac{\beta_2 r\sin 2\xi\cos\theta_1}{4\sqrt{2}}. \end{aligned} \tag{4.123}$$

and

$$
\begin{aligned}
C_2{}^2 &= C_4{}^4 = C_5{}^5 = C_6{}^6 = C_7{}^7 = C_8{}^8 = C_9{}^9 = C_{10}^{10}, \\
C_1{}^1 &= 1 - \frac{1}{8}\beta_2^2 \left(4\cos^2\xi \sin^2\theta_1 + \sin^2 2\xi \cos^2\theta_1\right), \\
C_3{}^3 &= 1 + \frac{1}{8}\beta_2^2 \left(4\cos^2\xi \sin^2\theta_1 + \sin^2 2\xi \cos^2\theta_1\right), \\
C_1{}^3 &= -C_3{}^1 = \frac{1}{8}\beta_2^2 \left(4\cos^2\xi \sin^2\theta_1 + \sin^2 2\xi \cos^2\theta_1\right), \\
C_1{}^5 = -C_5{}^1 &= -\frac{\beta_2 r \cos\xi \sin\theta_1}{\sqrt{2}}, \quad C_1{}^9 = -C_9{}^1 = -\frac{\beta_2 r \sin 2\xi \sin\theta_1}{2\sqrt{2}}, \\
C_3{}^5 = C_5{}^3 &= -\frac{\beta_2 r \cos\xi \sin\theta_1}{\sqrt{2}}, \quad C_3{}^9 = C_9{}^3 = -\frac{\beta_2 r \sin 2\xi \sin\theta_1}{2\sqrt{2}}.
\end{aligned}
\tag{4.124}
$$

From (4.111) we find that the components proportional to the coset generators are

$$
\begin{aligned}
E^1 &= \frac{r}{\sqrt{2}}\left(\mathrm{d}\,x_+ + \mathrm{d}\,x_-\right), \quad E^2 = r\,\mathrm{d}\,x_1, \quad E^3 = \frac{r}{\sqrt{2}}\left(\mathrm{d}\,x_+ - \mathrm{d}\,x_-\right), \quad E^4 = \frac{\mathrm{d}\,r}{r}, \\
E^5 &= \cos\xi \sin\theta_1\,\mathrm{d}\,\varphi_1, \quad E^6 = \cos\xi\,\mathrm{d}\,\theta_1, \quad E^7 = -\sin\theta_2 \sin\xi\,\mathrm{d}\,\varphi_2 \\
E^8 &= -\sin\xi\,\mathrm{d}\,\theta_2, \quad E^9 = \frac{1}{2}\left(\cos\theta_1\,\mathrm{d}\,\varphi_1 - \cos\theta_2\,\mathrm{d}\,\varphi_2 + 2\,\mathrm{d}\,\psi\right)\sin 2\xi, \quad E^{10} = 2\,\mathrm{d}\,\xi.
\end{aligned}
\tag{4.125}
$$

From the symmetric part of $\mathbf{C}$ in (4.124) we can compute the deformed metric

$$
\begin{aligned}
\mathrm{d}\,s^2 =& \frac{\mathrm{d}\,r^2}{r^2} + r^2\,\mathrm{d}\,x_1^2 + 2r^2\beta_2\mathcal{M}\,\mathrm{d}\,x_+\,\mathrm{d}\,x_- + r^2\left(\mathrm{d}\,x_+^2 + \mathrm{d}\,x_-^2\right) \\
&+ \left(2\mathcal{M}\,\mathrm{d}\,\phi_1 - \frac{1}{4}\beta_2 r^2 \sin^2 2\xi \cos\theta_1\left(-\cos\theta_2\,\mathrm{d}\,\phi_2 + 2\,\mathrm{d}\,\psi\right)\right)\left(\mathrm{d}\,x_+ - \mathrm{d}\,x_-\right) \\
&+ 4ds^2_{\mathbb{CP}^3},
\end{aligned}
\tag{4.126}
$$

and from the antisymmetric part of $\mathbf{C}$ in (4.124) we can compute of the antisymmetric field

$$
\begin{aligned}
B =& \frac{1}{2}\mathcal{M}\beta_2 r^2\,\mathrm{d}\,x_+ \wedge \mathrm{d}\,x_- - \frac{\beta_2 r^2 \cos^2\xi \sin^2\theta_1}{4}\left(\mathrm{d}\,x_+ + \mathrm{d}\,x_-\right)\wedge \mathrm{d}\,\varphi_1 \\
&- \frac{\beta_2 r^2 \sin^2 2\xi \cos\theta_1}{8}\left(\mathrm{d}\,x_+ + \mathrm{d}\,x_-\right)\wedge\left(\cos\theta_1\,\mathrm{d}\,\varphi_1 - \cos\theta_2\,\mathrm{d}\,\varphi_2 + 2\,\mathrm{d}\,\psi\right),
\end{aligned}
\tag{4.127}
$$

with

$$
\mathcal{M} = -\frac{1}{8}\beta_2 r^2\left(4\cos^2\xi \sin^2\theta_1 + \sin^2 2\xi \cos^2\theta_1\right). \tag{4.128}
$$

This space should be obtained via a certain class of TsT transformations called null Melvin twist [21, 51].

Chapter 5

Concluding remarks

In this thesis we studied Yang-Baxter integrable deformations of the nonlinear σ-model describing string theories in $AdS_4 \times \mathbb{CP}^3$.

We presented some solutions of the YBE for $AdS_4 \times \mathbb{CP}^3$, like a DJ r-matrix in terms of only bosonic roots of the $\mathfrak{uosp}(2, 2|6)$. An abelian Jordanian and two mixed r-matrices were discussed. Also, some unimodular nonabelian r-matrices were provided.

We computed explicitly the backgrounds generated by some of the r-matrices we proposed. By considering an abelian Jordanian r-matrix in terms of the generators of AdS_4, we computed the metric and B-field of the gravity dual of the non-commutative ABJM theory. This deformation involved only the AdS_4 part of the spacetime, and thus we only needed to consider the usual parametrization of AdS spaces. On the other hand, in order to reproduce the full undeformed $AdS_4 \times \mathbb{CP}^3$ Fubini-Study metric, we enlarged the supercoset as was done in [48] in a consistent way. We took a coset representative that included the generators of AdS_4 and those that give the correct form of $\mathbb{CP}^3$. Along this line we computed the deformed metric and B-field corresponding to the gravity dual of the one-parameter dipole deformed ABJM theory by using an r-matrix in terms of mixed generators, one of AdS_4 and one Cartan generator of $\mathbb{CP}^3$. These backgrounds coincide with those obtained via TsT transformations in [49]. We also considered an r-matrix constructed in terms of another combination of mixed generators, one of AdS_4 and one of the three Cartan generators of $\mathbb{CP}^3$. The deformed background we obtained in this case corresponds to the gravity dual of the non-relativistic limit of ABJM theory which is known as the Schrödinger spacetime. This background is expected to be obtained by an appropriate null Melvin twist [51]. These deformations of $AdS_4 \times \mathbb{CP}^3$ can be regarded as an evidence of the relation between TsT transformations and solutions of the CYBE which is called the gravity/CYBE correspondence [19].

Let us now discuss some open problems that arise from this work. First, the DJ r-matrix we wrote was in terms of only bosonic generators. In Appendix D, it was shown the algebra of the fermionic generators of $\mathfrak{uosp}(2,2|4)$, from which it is possible to identify the full Cartan decomposition, and thus to write the full DJ r-matrix including also fermionic simple roots. This would allows to construct the η-deformed $AdS_4 \times \mathbb{CP}^3$ background and to investigate the generalized supergravity in this framework. Along this route, we expect that, as in $AdS_5 \times S^5$ and $AdS_2 \times S^2 \times T^6$, a DJ r-matrix constructed in terms of only fermionic simple roots leads to a standard supergravity backgrounds [15].

Furthermore, as conjectured by Klimcik [42], it would be interesting to pursue the relation between the η- and λ-deformed $AdS_4 \times \mathbb{CP}^3$ by Poisson-Lie T-duality [84, 85]. In order to obtain the Poisson-Lie T-dual η-deformed we need to construct the Drinfeld double which is the complexified superalgebra $\mathfrak{uosp}(2,2|6)^{\mathbb{C}}$. This superalgebra can be split into $\mathfrak{uosp}(2,2|6)$ and the Borel subsuperalgebra $\mathfrak{pb}(2,2|6)$. This is possible to do by taking our Cartan decomposition. Once we have identified the Drinfeld double, it is possible to construct the action of the Poisson-Lie σ-model.

An immediate step to complete our result of the gravity dual of the non-relativistic limit of ABJM is to compute the corresponding TsT null Melvin twist of the undeformed $AdS_4 \times \mathbb{CP}^3$ background. This should give the same Yang-Baxter deformed background we calculated. A direct generalization of our results for the cases of the gravity duals of the dipole deformed ABJM theory and Schrödinger spacetime from one to three parameters can be done if we now consider r-matrices with three constant parameters.

Finally, we can consider the unimodular nonabelian r-matrices we found in Section 4.3 and compute the corresponding deformed backgrounds which would be standard supergravity solutions.

Appendix A

Supermatrix realization of $\mathfrak{su}(2,2|4)$

The $\mathfrak{gl}(m|n)$ superalgebra can be generated by the following basis

$$(E_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}, \qquad i,j,k,\ell = 1,...,m+n, \tag{A.1}$$

satisfying

$$[E_{ij}, E_{k\ell}] = \delta_{jk}E_{i\ell} - (-1)^{[i+j][k+\ell]}\,\delta_{i\ell}E_{jk}, \tag{A.2}$$

where $[\cdot]$ is the grading of each generator. This basis can be expressed by $(m|n)\times(m|n)$ supermatrices,

$$M = \left(\begin{array}{c|c} X_{m\times m} & \theta_{m\times n} \\ \hline \eta_{n\times m} & Y_{n\times n}\end{array}\right). \tag{A.3}$$

The special linear Lie superalgebra $\mathfrak{sl}(m|n)$ is defined as

$$\mathfrak{sl}(m|n) = \{M \in \mathfrak{gl}(m|n),\, \mathrm{Str}\, M = \mathrm{tr}\, X - \mathrm{tr}\, Y = 0\}\,. \tag{A.4}$$

By using this condition for the generators of $\mathfrak{gl}(4|4)$, we obtain the following bosonic generators of $\mathfrak{sl}(4|4)$,

$$\begin{gathered} E_{11}-E_{22}, \quad E_{22}-E_{33}, \quad E_{33}-E_{44}, \quad E_{12}, \quad E_{13}, \quad E_{14}, \quad E_{23}, \\ E_{24}, \quad E_{34}, \quad E_{21}, \quad E_{32}, \quad E_{43}, \quad E_{31}, \quad E_{42}, \quad E_{41}, \end{gathered} \tag{A.5}$$

$$\begin{gathered} E_{55}-E_{66}, \quad E_{66}-E_{77}, \quad E_{77}-E_{88}, \quad E_{56}, \quad E_{57}, \quad E_{58}, \quad E_{67}, \\ E_{68}, \quad E_{78}, \quad E_{65}, \quad E_{76}, \quad E_{87}, \quad E_{75}, \quad E_{86}, \quad E_{85}. \end{gathered} \tag{A.6}$$

In addition to these generators, we also have

$$E_{44}+E_{55}, \tag{A.7}$$

which represents the $\mathfrak{u}(1)$ algebra. Then, by using the condition

$$MH + HM^\dagger = 0, \tag{A.8}$$

where the Hermitian matrix H is defined as

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & I_{4\times4} \end{pmatrix}, \qquad \text{with} \qquad \Sigma = \begin{pmatrix} I_{2\times2} & 0 \\ 0 & -I_{2\times2} \end{pmatrix}, \tag{A.9}$$

we obtain the real form of $\mathfrak{su}(2,2|4)$.

By imposing (A.8) on (A.5) and (A.6) we get the 15 generators of $\mathfrak{su}(2,2)$

$$\begin{gathered} E_{13}+E_{31}, \quad i(E_{13}-E_{31}), \quad E_{14}+E_{41}, \quad i(E_{14}-E_{41}) \\ E_{23}+E_{32}, \quad i(E_{23}-E_{32}), \quad E_{24}+E_{42}, \quad i(E_{24}-E_{42}) \\ i(E_{12}+E_{21}), \quad E_{12}-E_{21}, \quad i(E_{34}+E_{43}), \quad E_{34}-E_{43} \\ i(E_{11}-E_{22}), \quad i(E_{22}-E_{33}), \quad i(E_{33}-E_{44}), \end{gathered} \tag{A.10}$$

and the 15 generators of $\mathfrak{su}(4)$,

$$\begin{gathered} E_{56}-E_{65}, \quad i(E_{56}+E_{65}), \quad E_{57}-E_{75}, \quad i(E_{57}+E_{75}) \\ E_{58}-E_{85}, \quad i(E_{58}+E_{85}), \quad E_{67}-E_{76}, \quad i(E_{67}+E_{76}) \\ E_{68}-E_{86}, \quad i(E_{68}+E_{86}), \quad E_{78}-E_{87}, \quad i(E_{78}+E_{87}) \\ i(E_{55}-E_{66}), \quad i(E_{66}-E_{77}), \quad i(E_{77}-E_{88}). \end{gathered} \tag{A.11}$$

Appendix B

$\mathfrak{so}(2,4)$ and $\mathfrak{so}(6)$ algebras

We summarize here the notation and conventions of the $\mathfrak{so}(2,4)$ and $\mathfrak{so}(6)$ generators.

The gamma matrices

We use the gamma matrices represented by

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{B.1}$$

$$\gamma_0 = i\gamma_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{B.2}$$

To describe the $\mathfrak{so}(2,4)$ and $\mathfrak{so}(6)$ subalgebras of the $\mathfrak{psu}(2,2|4)$ superalgebra, it is necessary to introduce the following 8×8 gamma matrices

$$\gamma^a_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix}, \quad \gamma^a_5 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad \mu = 0,1,2,3, \tag{B.3}$$

$$\gamma^s_i = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_i \end{pmatrix}, \quad \gamma^s_5 = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_5 \end{pmatrix} \quad \text{with} \quad i = 1,2,3,4. \tag{B.4}$$

Here each block of the matrices is a 4×4 matrix.

The bosonic generators

Then, the Lie algebras $\mathfrak{so}(2,4)$ and $\mathfrak{so}(6)$ are spanned by the bases:

$$\mathfrak{so}(2,4) = \mathrm{span}_{\mathbb{R}} \left\{ \gamma^a_\mu, \gamma^a_5, m_{\mu\nu} = \frac{1}{4}\left[\gamma^a_\mu, \gamma^a_\nu\right], m_{\mu 5} = \frac{1}{4}\left[\gamma^a_\mu, \gamma^a_5\right] \middle|\ \mu, \nu = 0,1,2,3 \right\}, \tag{B.5}$$

$$\mathfrak{so}(6) = \mathrm{span}_{\mathbb{R}} \left\{ \gamma^s_i, \gamma^s_5, n_{ij} = \frac{1}{4}\left[\gamma^s_i, \gamma^s_j\right], n_{i5} = \frac{1}{4}\left[\gamma^s_i, \gamma^s_5\right] \middle|\ i, j = 1,2,3,4 \right\}. \tag{B.6}$$

The subalgebras $\mathfrak{so}(1,4)$ and $\mathfrak{so}(5)$ are generated by

$$\mathfrak{so}(1,4) = \mathrm{span}_{\mathbb{R}} \left\{ m_{\mu\nu}, m_{\mu 5} \middle|\ \mu, \nu = 0,1,2,3 \right\}, \tag{B.7}$$

$$\mathfrak{so}(5) = \mathrm{span}_{\mathbb{R}} \left\{ n_{ij}, n_{i5} \middle|\ i, j = 1,2,3,4 \right\}. \tag{B.8}$$

For the coset construction of AdS_5 with the Poincare coordinates, the following basis of $\mathfrak{so}(2,4)$ is convenient;

$$\mathfrak{so}(2,4) = \mathrm{span}_{\mathbb{R}} \left\{ p_\mu, k_\mu, h_1, h_2, h_3, m_{13}, m_{10}, m_{23}, m_{20} \middle|\ \mu = 0,1,2,3 \right\}, \tag{B.9}$$

where the Cartan generators h_1, h_2, h_3 and p_μ, k_μ are given by

$$\begin{aligned} h_1 &= 2i m_{12} = \mathrm{diag}(-1,1,-1,1,0,0,0,0), & p_\mu &= \frac{1}{2}\gamma^a_\mu - m_{\mu 5}, \\ h_2 &= 2 m_{30} = \mathrm{diag}(-1,1,1,-1,0,0,0,0), & k_\mu &= \frac{1}{2}\gamma^a_\mu + m_{\mu 5}, \\ h_3 &= \gamma^a_5 = \mathrm{diag}(1,1,-1,-1,0,0,0,0), & D &= \frac{\gamma^a_5}{2}. \end{aligned} \tag{B.10}$$

Notice that the generators p_μ and k_μ commute each other,

$$[p_\mu, p_\nu] = [k_\mu, k_\nu] = [p_\mu, k_\nu] = 0 \quad \text{for} \quad \mu, \nu = 0,1,2,3. \tag{B.11}$$

On the other hand, the Cartan generators of $\mathfrak{so}(6)$ read

$$\begin{aligned} h_4 &= 2i n_{12} = \mathrm{diag}(0,0,0,0,-1,1,-1,1), \\ h_5 &= 2i n_{34} = \mathrm{diag}(0,0,0,0,-1,1,1,-1), \\ h_6 &= \gamma^s_5 = \mathrm{diag}(0,0,0,0,1,1,-1,-1). \end{aligned} \tag{B.12}$$

Appendix C

Supermatrix realization of $\mathfrak{uosp}(2,2|6)$

We can consider the $\mathfrak{gl}(4|6)$ superalgebra generated by the basis defined in (A.1) in terms of $(4|6)\times(4|6)$ supermatrices M (A.3). Let us define the conditions for M

$$M^{st}\begin{pmatrix} C_4 & 0 \\ 0 & I_{6\times 6}\end{pmatrix} + \begin{pmatrix} C_4 & 0 \\ 0 & I_{6\times 6}\end{pmatrix} M = 0, \tag{C.1}$$

$$M^{\dagger}\begin{pmatrix} \gamma_0 & 0 \\ 0 & -I_{6\times 6}\end{pmatrix} + \begin{pmatrix} \gamma_0 & 0 \\ 0 & -I_{6\times 6}\end{pmatrix} M = 0, \tag{C.2}$$

where

$$C_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{C.3}$$

The supertranspose of M is defined as

$$M^{st} = \begin{pmatrix} X^t & -\eta^t \\ \theta^t & Y^t \end{pmatrix}. \tag{C.4}$$

From (C.1) we get the 10 generators of $\mathfrak{sp}(4)$

$$\begin{gathered} E_{14}, \quad E_{41}, \quad E_{23}, \quad E_{32}, \quad E_{12}+E_{34}, \quad E_{21}+E_{43}, \\ E_{11}-E_{44}, \quad E_{13}-E_{24}, \quad E_{22}-E_{33}, \quad E_{31}-E_{42}, \end{gathered} \tag{C.5}$$

and the 15 generators of $\mathfrak{so}(6)$,

$$E_{\bar{\imath}\bar{\jmath}} - E_{\bar{\jmath}\bar{\imath}}, \quad \bar{\imath},\bar{\jmath} = \bar{1},...,\bar{6}. \tag{C.6}$$

For the fermionic generators we have the following relations (4.8),

$$\theta_{i\bar{\jmath}} = -(C_4\eta^t)_{i\bar{\jmath}} = -\,(C_4)_{ik}\,(\eta^t)_{k\bar{\jmath}}, \tag{C.7}$$

so we can define θ in terms of η. Then, let us fix η as

$$\eta_{\bar{\imath}j} = E_{\bar{\imath}j}, \tag{C.8}$$

where $i, j = 1, ..., 4$ and $\bar{\imath}, \bar{\jmath} = \bar{1}, ..., \bar{6}$.

By using (C.2) we obtain the real form of $\mathfrak{osp}(4|6)$ denoted by $\mathfrak{uosp}(2,2|6)$. We can thus compute the 10 generators of the $\mathfrak{usp}(2,2)$,

$$\begin{aligned}
X_1 &= E_{14} + E_{41}, & X_5 &= E_{13} - E_{24} + (E_{31} - E_{42}),\\
X_2 &= i(E_{14} - E_{41}), & X_6 &= i(E_{13} - E_{24} - (E_{31} - E_{42})),\\
X_3 &= E_{23} + E_{32}, & X_7 &= E_{12} + E_{34} - (E_{21} + E_{43}),\\
X_4 &= i(E_{23} - E_{32}), & X_8 &= i(E_{12} + E_{34} + (E_{21} + E_{43})),\\
X_9 &= i(E_{11} - E_{44}), & X_{10} &= i(E_{22} - E_{33}).
\end{aligned} \tag{C.9}$$

The commutation relations of these generators is given in Table C.1.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1	0	$-2X_9$	0	0	$-X_7$	X_8	$-X_5$	X_6	$-2X_2$	0
X_2	$2X_9$	0	0	0	$-X_8$	$-X_7$	$-X_6$	$-X_5$	$2X_1$	0
X_3	0	0	0	$-2X_{10}$	$-X_7$	$-X_8$	$-X_5$	$-X_6$	0	$-2X_4$
X_4	0	0	$2X_{10}$	0	X_8	$-X_7$	$-X_6$	X_5	0	$2X_3$
X_5	X_7	X_8	X_7	$-X_8$	0	$-2(X_9+X_{10})$	$2(X_1+X_3)$	$2(X_2-X_4)$	$-X_6$	$-X_6$
X_6	$-X_8$	X_7	X_8	X_7	$2(X_9+X_{10})$	0	$2(X_2+X_4)$	$-2(X_1-X_3)$	X_5	X_5
X_7	X_5	X_6	X_5	X_6	$-2(X_1+X_3)$	$-2(X_2+X_4)$	0	$2(X_9-X_{10})$	$-X_8$	X_8
X_8	$-X_6$	X_5	X_6	$-X_5$	$-2(X_2-X_4)$	$2(X_1-X_3)$	$-2(X_9-X_{10})$	0	X_7	$-X_7$
X_9	$2X_2$	$-2X_1$	0	0	X_6	$-X_5$	X_8	$-X_7$	0	0
X_{10}	0	0	$2X_4$	$-2X_3$	X_6	$-X_5$	$-X_8$	X_7	0	0

Table C.1: Algebra $\mathfrak{usp}(2,2)$.

The 15 generators of $\mathfrak{so}(6)$ are

$$\begin{aligned}
Y_1 &= E_{\bar{1}\bar{2}} - E_{\bar{2}\bar{1}}, & Y_6 &= E_{\bar{2}\bar{3}} - E_{\bar{3}\bar{2}}, & Y_{11} &= E_{\bar{3}\bar{5}} - E_{\bar{5}\bar{3}}, \\
Y_2 &= E_{\bar{1}\bar{3}} - E_{\bar{3}\bar{1}}, & Y_7 &= E_{\bar{2}\bar{4}} - E_{\bar{4}\bar{2}}, & Y_{12} &= E_{\bar{3}\bar{6}} - E_{\bar{6}\bar{3}}, \\
Y_3 &= E_{\bar{1}\bar{4}} - E_{\bar{4}\bar{1}}, & Y_8 &= E_{\bar{2}\bar{5}} - E_{\bar{5}\bar{2}}, & Y_{13} &= E_{\bar{4}\bar{5}} - E_{\bar{5}\bar{4}}, \\
Y_4 &= E_{\bar{1}\bar{5}} - E_{\bar{5}\bar{1}}, & Y_9 &= E_{\bar{2}\bar{6}} - E_{\bar{6}\bar{2}}, & Y_{14} &= E_{\bar{4}\bar{6}} - E_{\bar{6}\bar{4}}, \\
Y_5 &= E_{\bar{1}\bar{6}} - E_{\bar{6}\bar{1}}, & Y_{10} &= E_{\bar{3}\bar{4}} - E_{\bar{4}\bar{3}}, & Y_{15} &= E_{\bar{5}\bar{6}} - E_{\bar{6}\bar{5}},
\end{aligned} \tag{C.10}$$

whose commutation relations are given in Table C.2.

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}	Y_{11}	Y_{12}	Y_{13}	Y_{14}	Y_{15}
Y_1	0	$-Y_6$	$-Y_7$	$-Y_8$	$-Y_9$	Y_2	Y_3	Y_4	Y_5	0	0	0	0	0	0
Y_2	Y_6	0	$-Y_{10}$	$-Y_{11}$	$-Y_{12}$	$-Y_1$	0	0	0	Y_3	Y_4	Y_5	0	0	0
Y_3	Y_7	Y_{10}	0	$-Y_{13}$	$-Y_{14}$	0	$-Y_1$	0	0	$-Y_2$	0	0	Y_4	Y_5	0
Y_4	Y_8	Y_{11}	Y_{13}	0	$-Y_{15}$	0	0	$-Y_1$	0	0	$-Y_2$	0	$-Y_3$	0	Y_5
Y_5	Y_9	Y_{12}	Y_{14}	Y_{15}	0	0	0	0	$-Y_1$	0	0	$-Y_2$	0	$-Y_3$	$-Y_4$
Y_6	$-Y_2$	Y_1	0	0	0	0	$-Y_{10}$	$-Y_{11}$	$-Y_{12}$	Y_7	Y_8	Y_9	0	0	0
Y_7	$-Y_3$	0	Y_1	0	0	Y_{10}	0	$-Y_{13}$	$-Y_{14}$	$-Y_6$	0	0	Y_8	Y_9	0
Y_8	$-Y_4$	0	0	Y_1	0	Y_{11}	Y_{13}	0	$-Y_{15}$	0	$-Y_6$	0	$-Y_7$	0	Y_9
Y_9	$-Y_5$	0	0	0	Y_1	Y_{12}	Y_{14}	Y_{15}	0	0	0	$-Y_6$	0	$-Y_7$	$-Y_8$
Y_{10}	0	$-Y_3$	Y_2	0	0	$-Y_7$	Y_6	0	0	0	$-Y_{13}$	$-Y_{14}$	Y_{11}	Y_{12}	0
Y_{11}	0	$-Y_4$	0	Y_2	0	$-Y_8$	0	Y_6	0	Y_{13}	0	$-Y_{15}$	$-Y_{10}$	0	Y_{12}
Y_{12}	0	$-Y_5$	0	0	Y_2	$-Y_9$	0	0	Y_6	Y_{14}	Y_{15}	0	0	$-Y_{10}$	$-Y_{11}$
Y_{13}	0	0	$-Y_4$	Y_3	0	0	$-Y_8$	Y_7	0	$-Y_{11}$	Y_{10}	0	0	$-Y_{15}$	Y_{14}
Y_{14}	0	0	$-Y_5$	0	Y_3	0	$-Y_9$	0	Y_7	$-Y_{12}$	0	Y_{10}	Y_{15}	0	$-Y_{13}$
Y_{15}	0	0	0	$-Y_5$	Y_4	0	0	$-Y_9$	Y_8	0	$-Y_{12}$	Y_{11}	$-Y_{14}$	Y_{13}	0

Table C.2: Algebra $\mathfrak{so}(6)$.

The 24 fermionic generators are

$$\begin{aligned}
Q_{1\bar{\imath}} &= E_{1\bar{\imath}} + E_{\bar{\imath}1} + E_{4\bar{\imath}} - E_{\bar{\imath}4} \\
Q_{2\bar{\imath}} &= E_{2\bar{\imath}} + E_{\bar{\imath}2} - E_{3\bar{\imath}} + E_{\bar{\imath}3} \\
Q_{3\bar{\imath}} &= i(-E_{2\bar{\imath}} + E_{\bar{\imath}2} - E_{3\bar{\imath}} - E_{\bar{\imath}3}) \\
Q_{4\bar{\imath}} &= i(-E_{1\bar{\imath}} + E_{\bar{\imath}1} + E_{4\bar{\imath}} + E_{\bar{\imath}4})
\end{aligned} \tag{C.11}$$

where $\bar{\imath} = \bar{1}, ..., \bar{6}$. The anticommutation relations of these fermionic generators are given in Table C.3 and Table C.4.

	$Q_{2\bar{1}}$	$Q_{2\bar{2}}$	$Q_{2\bar{3}}$	$Q_{2\bar{4}}$	$Q_{2\bar{5}}$	$Q_{2\bar{6}}$	$Q_{3\bar{1}}$	$Q_{3\bar{2}}$	$Q_{3\bar{3}}$	$Q_{3\bar{4}}$	$Q_{3\bar{5}}$	$Q_{3\bar{6}}$
$Q_{1\bar{1}}$	$-i(X_6+X_8)$	0	0	0	0	0	$-i(X_5-X_7)$	0	0	0	0	0
$Q_{1\bar{2}}$	0	$-i(X_6+X_8)$	0	0	0	0	0	$-i(X_5-X_7)$	0	0	0	0
$Q_{1\bar{3}}$	0	0	$-i(X_6+X_8)$	0	0	0	0	0	$-i(X_5-X_7)$	0	0	0
$Q_{1\bar{4}}$	0	0	0	$-i(X_6+X_8)$	0	0	0	0	0	$-i(X_5-X_7)$	0	0
$Q_{1\bar{5}}$	0	0	0	0	$-i(X_6+X_8)$	0	0	0	0	0	$-i(X_5-X_7)$	0
$Q_{1\bar{6}}$	0	0	0	0	0	$-i(X_6+X_8)$	0	0	0	0	0	$-i(X_5-X_7)$
$Q_{2\bar{1}}$	$-2i(X_{10}+X_4)$	0	0	0	0	0	$-2iX_3$	$-2iY_1$	$-2iY_2$	$-2iY_3$	$-2iY_4$	$-2iY_5$
$Q_{2\bar{2}}$	0	$-2i(X_{10}+X_4)$	0	0	0	0	$2iY_1$	$-2iX_3$	$-2iY_6$	$-2iY_7$	$-2iY_8$	$-2iY_9$
$Q_{2\bar{3}}$	0	0	$-2i(X_{10}+X_4)$	0	0	0	$2iY_2$	$2iY_6$	$-2iX_3$	$-2iY_{10}$	$-2iY_{11}$	$-2iY_{12}$
$Q_{2\bar{4}}$	0	0	0	$-2i(X_{10}+X_4)$	0	0	$2iY_3$	$2iY_7$	$2iY_{10}$	$-2iX_3$	$-2iY_{13}$	$-2iY_{14}$
$Q_{2\bar{5}}$	0	0	0	0	$-2i(X_{10}+X_4)$	0	$2iY_4$	$2iY_8$	$2iY_{11}$	$2iY_{13}$	$-2iX_3$	$-2iY_{15}$
$Q_{2\bar{6}}$	0	0	0	0	0	$-2i(X_{10}+X_4)$	$2iY_5$	$2iY_9$	$2iY_{12}$	$2iY_{14}$	$2iY_{15}$	$-2iX_3$
$Q_{3\bar{1}}$	$-2iX_3$	$2iY_1$	$2iY_2$	$2iY_3$	$2iY_4$	$2iY_5$	$-2i(X_{10}-X_4)$	0	0	0	0	0
$Q_{3\bar{2}}$	$-2iY_1$	$-2iX_3$	$2iY_6$	$2iY_7$	$2iY_8$	$2iY_9$	0	$-2i(X_{10}-X_4)$	0	0	0	0
$Q_{3\bar{3}}$	$-2iY_2$	$-2iY_6$	$-2iX_3$	$2iY_{10}$	$2iY_{11}$	$2iY_{12}$	0	0	$-2i(X_{10}-X_4)$	0	0	0
$Q_{3\bar{4}}$	$-2iY_3$	$-2iY_7$	$-2iY_{10}$	$-2iX_3$	$2iY_{13}$	$2iY_{14}$	0	0	0	$-2i(X_{10}-X_4)$	0	0
$Q_{3\bar{5}}$	$-2iY_4$	$-2iY_8$	$-2iY_{11}$	$-2iY_{13}$	$-2iX_3$	$2iY_{15}$	0	0	0	0	$-2i(X_{10}-X_4)$	0
$Q_{3\bar{6}}$	$-2iY_5$	$-2iY_9$	$-2iY_{12}$	$-2iY_{14}$	$-2iY_{15}$	$-2iX_3$	0	0	0	0	0	$-2i(X_{10}-X_4)$
$Q_{4\bar{1}}$	$-i(X_5+X_7)$	0	0	0	0	0	$i(X_6-X_8)$	0	0	0	0	0
$Q_{4\bar{2}}$	0	$-i(X_5+X_7)$	0	0	0	0	0	$i(X_6-X_8)$	0	0	0	0
$Q_{4\bar{3}}$	0	0	$-i(X_5+X_7)$	0	0	0	0	0	$i(X_6-X_8)$	0	0	0
$Q_{4\bar{4}}$	0	0	0	$-i(X_5+X_7)$	0	0	0	0	0	$i(X_6-X_8)$	0	0
$Q_{4\bar{5}}$	0	0	0	0	$-i(X_5+X_7)$	0	0	0	0	0	$i(X_6-X_8)$	0
$Q_{4\bar{6}}$	0	0	0	0	0	$-i(X_5+X_7)$	0	0	0	0	0	$i(X_6-X_8)$

Table C.3: Fermionic algebra. Part 1.

	$Q_{1\bar{1}}$	$Q_{1\bar{2}}$	$Q_{1\bar{3}}$	$Q_{1\bar{4}}$	$Q_{1\bar{5}}$	$Q_{1\bar{6}}$	$Q_{4\bar{1}}$	$Q_{4\bar{2}}$	$Q_{4\bar{3}}$	$Q_{4\bar{4}}$	$Q_{4\bar{5}}$	$Q_{4\bar{6}}$
$Q_{1\bar{1}}$	$-2i(X_9-X_2)$	0	0	0	0	0	$2iX_1$	$-2iY_1$	$-2iY_2$	$-2iY_3$	$-2iY_4$	$-2iY_5$
$Q_{1\bar{2}}$	0	$-2i(X_9-X_2)$	0	0	0	0	$2iY_1$	$2iX_1$	$-2iY_6$	$-2iY_7$	$-2iY_8$	$-2iY_9$
$Q_{1\bar{3}}$	0	0	$-2i(X_9-X_2)$	0	0	0	$2iY_2$	$2iY_6$	$2iX_1$	$-2iY_{10}$	$-2iY_{11}$	$-2iY_{12}$
$Q_{1\bar{4}}$	0	0	0	$-2i(X_9-X_2)$	0	0	$2iY_3$	$2iY_7$	$2iY_{10}$	$2iX_1$	$-2iY_{13}$	$-2iY_{14}$
$Q_{1\bar{5}}$	0	0	0	0	$-2i(X_9-X_2)$	0	$2iY_4$	$2iY_8$	$2iY_{11}$	$2iY_{13}$	$2iX_1$	$-2iY_{15}$
$Q_{1\bar{6}}$	0	0	0	0	0	$-2i(X_9-X_2)$	$2iY_5$	$2iY_9$	$2iY_{12}$	$2iY_{14}$	$2iY_{15}$	$2iX_1$
$Q_{2\bar{1}}$	$-i(X_6+X_8)$	0	0	0	0	0	$-i(X_5+X_7)$	0	0	0	0	0
$Q_{2\bar{2}}$	0	$-i(X_6+X_8)$	0	0	0	0	0	$-i(X_5+X_7)$	0	0	0	0
$Q_{2\bar{3}}$	0	0	$-i(X_6+X_8)$	0	0	0	0	0	$-i(X_5+X_7)$	0	0	0
$Q_{2\bar{4}}$	0	0	0	$-i(X_6+X_8)$	0	0	0	0	0	$-i(X_5+X_7)$	0	0
$Q_{2\bar{5}}$	0	0	0	0	$-i(X_6+X_8)$	0	0	0	0	0	$-i(X_5+X_7)$	0
$Q_{2\bar{6}}$	0	0	0	0	0	$-i(X_6+X_8)$	0	0	0	0	0	$-i(X_5+X_7)$
$Q_{3\bar{1}}$	$-i(X_5-X_7)$	0	0	0	0	0	$i(X_6-X_8)$	0	0	0	0	0
$Q_{3\bar{2}}$	0	$-i(X_5-X_7)$	0	0	0	0	0	$i(X_6-X_8)$	0	0	0	0
$Q_{3\bar{3}}$	0	0	$-i(X_5-X_7)$	0	0	0	0	0	$i(X_6-X_8)$	0	0	0
$Q_{3\bar{4}}$	0	0	0	$-i(X_5-X_7)$	0	0	0	0	0	$i(X_6-X_8)$	0	0
$Q_{3\bar{5}}$	0	0	0	0	$-i(X_5-X_7)$	0	0	0	0	0	$i(X_6-X_8)$	0
$Q_{3\bar{6}}$	0	0	0	0	0	$-i(X_5-X_7)$	0	0	0	0	0	$i(X_6-X_8)$
$Q_{4\bar{1}}$	$2iX_1$	$2iY_1$	$2iY_2$	$2iY_3$	$2iY_4$	$2iY_5$	$-2i(X_9+X_2)$	0	0	0	0	0
$Q_{4\bar{2}}$	$-2iY_1$	$2iX_1$	$2iY_6$	$2iY_7$	$2iY_8$	$2iY_9$	0	$-2i(X_9+X_2)$	0	0	0	0
$Q_{4\bar{3}}$	$-2iY_2$	$-2iY_6$	$2iX_1$	$2iY_{10}$	$2iY_{11}$	$2iY_{12}$	0	0	$-2i(X_9+X_2)$	0	0	0
$Q_{4\bar{4}}$	$-2iY_3$	$-2iY_7$	$-2iY_{10}$	$2iX_1$	$2iY_{13}$	$2iY_{14}$	0	0	0	$-2i(X_9+X_2)$	0	0
$Q_{4\bar{5}}$	$-2iY_4$	$-2iY_8$	$-2iY_{11}$	$-2iY_{13}$	$2iX_1$	$2iY_{15}$	0	0	0	0	$-2i(X_9+X_2)$	0
$Q_{4\bar{6}}$	$-2iY_5$	$-2iY_9$	$-2iY_{12}$	$-2iY_{14}$	$-2iY_{15}$	$2iX_1$	0	0	0	0	0	$-2i(X_9+X_2)$

Table C.4: Fermionic algebra. Part 2.

Appendix D

Cartan-Weyl basis of $\mathfrak{usp}(2,2)$ and $\mathfrak{so}(6)$

In Section 3.2 we presented a brief introduction on the Cartan-Weyl basis whose algebra was given in (3.11). In particular, for $SO(2,3) \times SO(6) \simeq Sp(2,2) \times SO(6)$, we computed the Cartan-Weyl decomposition. The Cartan matrices for $Sp(4)$ and $SO(6)$ are respectively,

$$(a_{ij})_{Sp(4)} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \qquad (a_{ij})_{SO(6)} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}. \tag{D.1}$$

whose orthogonal root systems are

$$Sp(4): \qquad \alpha_1 = (1,-1),\ \alpha_2 = (0,2), \tag{D.2}$$

$$SO(6): \qquad \alpha_1 = (1,-1,0),\ \alpha_2 = (0,1,-1),\ \alpha_3 = (0,1,1), \tag{D.3}$$

such that $a_{ij} = 2\dfrac{\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i}$ for both groups.

D.1 Cartan decomposition of $\mathfrak{usp}(2,2)$

The Cartan generators h_j with $j = 1,2$, the positive simple roots e_j, and the negative simple roots f_j are given by

$$\begin{gathered} e_1 = E_{12} + E_{34}, \quad e_2 = \frac{1}{\sqrt{2}} E_{23}, \\ f_1 = E_{21} + E_{43}, \quad f_2 = \frac{1}{\sqrt{2}} E_{32}, \\ h_1 = E_{11} - E_{44} - E_{22} + E_{33}, \quad h_2 = E_{22} - E_{33}. \end{gathered} \tag{D.4}$$

The non simple roots follow from the commutators

$$\begin{aligned} e_3 &= [e_1, e_2] = \frac{1}{\sqrt{2}} \left(E_{13} - E_{24}\right), \quad e_4 = [e_1, [e_1, e_2]] = -\frac{2}{\sqrt{2}} E_{14}, \\ f_3 &= [f_1, f_2] = -\frac{1}{\sqrt{2}} \left(E_{31} - E_{42}\right), \quad f_4 = [f_1, [f_1, f_2]] = -\frac{2}{\sqrt{2}} E_{41}. \end{aligned} \tag{D.5}$$

The basis for the Lie algebra $\mathfrak{usp}(2,2)$ is then

$$\begin{aligned} X_1 &= -\frac{1}{\sqrt{2}} \left(e_4 + f_4\right), \qquad X_2 = -\frac{i}{\sqrt{2}} \left(e_4 - f_4\right), \\ X_3 &= \frac{2}{\sqrt{2}} \left(e_2 + f_2\right), \qquad X_4 = \frac{2i}{\sqrt{2}} \left(e_2 - f_2\right), \\ X_5 &= \frac{2}{\sqrt{2}} \left(e_3 - f_3\right), \qquad X_6 = \frac{2i}{\sqrt{2}} \left(e_3 + f_3\right), \\ X_7 &= e_1 - f_1, \qquad X_8 = i \left(e_1 + f_1\right), \\ X_9 &= i \left(h_1 + h_2\right), \qquad X_{10} = i h_2. \end{aligned} \tag{D.6}$$

D.2 Cartan decomposition of $\mathfrak{so}(6)$

The roots of $SO(6)$ can be found in [86] (page 308),

$$\begin{aligned} H_i &= T_{(2i-1)(2i)}, \\ E_{ij}^{(1)} &= \frac{1}{2} \left[T_{(2i)(2j-1)} - iT_{(2i-1)(2j-1)} - iT_{(2i)(2j)} - T_{(2i-1)(2j)}\right], \\ E_{ij}^{(2)} &= \frac{1}{2} \left[T_{(2i)(2j-1)} + iT_{(2i-1)(2j-1)} + iT_{(2i)(2j)} - T_{(2i-1)(2j)}\right], \\ E_{ij}^{(3)} &= \frac{1}{2} \left[T_{(2i)(2j-1)} - iT_{(2i-1)(2j-1)} + iT_{(2i)(2j)} + T_{(2i-1)(2j)}\right], \\ E_{ij}^{(4)} &= \frac{1}{2} \left[T_{(2i)(2j-1)} + iT_{(2i-1)(2j-1)} - iT_{(2i)(2j)} - T_{(2i-1)(2j)}\right]. \end{aligned} \tag{D.7}$$

with $i, j = 1, 2, 3$ and $(T_{ij})_{km} = -i \left(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}\right)$. Here the generators labelled by $^{(1)}$ and $^{(3)}$ correspond to the positive roots and satisfy

$$[H_k, E_{\alpha_i}] = (\alpha_i)_k E_{\alpha_i}, \qquad [E_{\alpha_i}, E_{-\alpha_i}] = (\alpha_i)_k H_k. \tag{D.8}$$

Then, we find that the simple positive roots are

$$E_{\alpha_1} = E_{12}^{(1)}, \quad E_{\alpha_2} = E_{23}^{(1)}, \quad E_{\alpha_3} = E_{23}^{(3)}, \tag{D.9}$$

and the simple negative roots are

$$E_{-\alpha_1} = E_{12}^{(2)}, \quad E_{-\alpha_2} = E_{23}^{(2)}, \quad E_{-\alpha_3} = E_{23}^{(4)}. \tag{D.10}$$

Under the redefinitions

$$\tilde{e}_i = \left(\frac{2}{\alpha_i \cdot \alpha_i}\right)^{1/2} E_{\alpha_i}, \quad \tilde{f}_i = \left(\frac{2}{\alpha_i \cdot \alpha_i}\right)^{1/2} E_{-\alpha_i}, \quad \tilde{h}_i = 2\frac{\alpha_i \cdot H}{\alpha_i \cdot \alpha_i}, \tag{D.11}$$

they satisfy the Cartan-Weyl algebra. Explicitly, we have

$$\begin{gathered}
\tilde{e}_1 = E_{\alpha_1} = \frac{1}{2}\left(-iY_6 - Y_2 - Y_7 + iY_3\right), \quad \tilde{e}_2 = E_{\alpha_2} = \frac{1}{2}\left(-iY_{13} - Y_{11} - Y_{14} + iY_{12}\right), \\
\tilde{e}_3 = E_{\alpha_3} = \frac{1}{2}\left(-iY_{13} - Y_{11} + Y_{14} - iY_{12}\right) =, \\
\tilde{f}_1 = E_{-\alpha_1} = \frac{1}{2}\left(-iY_6 + Y_2 + Y_7 + iY_3\right), \quad \tilde{f}_2 = E_{-\alpha_2} = \frac{1}{2}\left(-iY_{13} + Y_{11} + Y_{14} + iY_{12}\right), \\
\tilde{f}_3 = E_{-\alpha_3} = \frac{1}{2}\left(-iY_{13} + Y_{11} - Y_{14} + iY_{12}\right), \\
\tilde{h}_1 = H_1 - H_2 = -iY_1 - Y_{10}, \quad \tilde{h}_1 = H_2 - H_3 = -iY_{10} - Y_{15}, \\
\tilde{h}_3 = H_2 + H_3 = -iY_{10} + Y_{15},
\end{gathered} \tag{D.12}$$

where Y_i with $i = 1, ..., 15$ are the generators of $\mathfrak{so}(6)$ defined in (C.10).

Appendix E

$\mathfrak{so}(2,3)$ and $\mathfrak{su}(4)$ algebras

A basis for $\mathfrak{so}(2,3)$

The 10 generators of $SO(2,3)$ can be written as

$$M_{AB} = \frac{i}{4}\left[\Gamma_A, \Gamma_B\right], \tag{E.1}$$

and satisfy

$$[M_{AB}, M_{CD}] = i\left(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{BD}M_{AC} - \eta_{AC}M_{BD}\right), \quad A = 0, ..., 4, \tag{E.2}$$

where $A, B = 0, 1, 2, 3, 4$. We choose the following representation for the $SO(2,3)$ Γ_A matrices,

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} \tag{E.3}$$

$$\Gamma_A = \begin{cases} i\gamma_5\gamma_\mu & A = \mu = 0, 1, 2, 3 \\ \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 & A = 4 \end{cases}, \tag{E.4}$$

with $\eta_{AB} = \text{diag}(+ - - - +)$, and γ_μ are the gamma matrices in a Dirac representation $SO(1,3)$ [77] (see [46] for a different choice),

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \\ \gamma_2 &= \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}. \end{aligned} \tag{E.5}$$

And,

$$\gamma_5 = \begin{pmatrix} 0 & -1_2 \\ -1_2 & 0 \end{pmatrix}. \tag{E.6}$$

From (E.1), we get

$$M_{\mu\nu} = \frac{i}{4}\left[\gamma_\mu, \gamma_\nu\right], \qquad M_{\mu 4} = \frac{1}{2}\gamma_\mu, \qquad \mu, \nu = 0, 1, 2, 3. \tag{E.7}$$

In order to explicit the conformal group, let us split the indices as

$$M_{AB} = \{M_{ab}, M_{a3}, M_{a4}, M_{34}\}, \qquad a, b = 0, 1, 2, \tag{E.8}$$

such that $\eta_{ab} = \text{diag}(+,-,-)$ [1]. Let us define [77]

$$\begin{aligned} P_a &= M_{a4} + M_{a3} \\ K_a &= M_{a4} - M_{a3} \\ D &= iM_{34} \end{aligned} \tag{E.9}$$

The conformal algebra $SO(2,3)$ is then

$$\begin{aligned} [M_{ab}, M_{cd}] &= i(\eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}) \\ [M_{ab}, D] &= 0 \\ [D, P_a] &= iP_a \\ [D, K_a] &= -iK_a \\ [K_a, P_a] &= 2i\eta_{ab}D - 2iM_{ab} \\ [M_{ab}, P_c] &= -i(\eta_{ac}P_b - \eta_{bc}P_a) \\ [M_{ab}, K_c] &= -i(\eta_{ac}K_b - \eta_{bc}K_a). \end{aligned} \tag{E.10}$$

A basis for $\mathfrak{su}(4)$

A basis for $\mathfrak{su}(4)$ can be constructed in terms of anti-Hermitian 4×4 matrices known as Gell-Mann matrices,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

[1] This is going to be the signature on the Minkowskian boundary of AdS_4.

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$

$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad \lambda_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \tag{E.11}$$

The first 8 matrices form a basis for $\mathfrak{su}(3) \subset \mathfrak{su}(4)$. Furthermore, these matrices are orthogonal and satisfy

$$\mathrm{Tr}\,(\lambda_m \lambda_n) = 2\delta_{mn}, \qquad m = 1, ..., 15, \tag{E.12}$$

and commutation relations

$$[\lambda_m, \lambda_n] = 2i f_{mn}^{p} \lambda_p. \tag{E.13}$$

A list of non-vanishing structure constants can be found in [87]. In this representation the Cartan generators are given by λ_3, λ_8 and λ_{15}.

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En el principio era el Verbo, y el Verbo era con Dios, y el Verbo era Dios. Este era en el principio con Dios. Todas las cosas por él fueron hechas, y sin él nada de lo que ha sido hecho, fue hecho.

Soli Deo gloria