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Tópicos da correspondência fluido/gravidade em espaços planos

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Topics of fluid/gravity correspondence in flat spaces

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To my parents

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Abstract

In this dissertation we study some aspects of the fluid/gravity correspondence applied to flat space in ingoing Rindler coordinates. Our main goal is to study the effect of Ehlers transformations and symmetries of the Einstein equations in the context of fluid/gravity correspondence.

To do so, we review the main aspects of General Relativity and Hydrodynamics which will be employed throughout the text. We devote significant attention to a method that allows us to find solutions to the Einstein equations that by performing a derivative expansion, which will be utilized afterwards to generate our seed solution, upon which we later apply the Ehlers transformations.

We show that the metric of flat spacetime in ingoing Rindler coordinates is related to a Taub spacetime by an Ehlers transformation and we utilize an approach in which we solve the Killing equation perturbatively in the ε-expansion. The results obtained by using this approach are not entirely conclusive, and further investigation is still required.

Resumo

Nesta dissertação estudamos alguns aspectos da correspondência fluido/gravitação aplicada ao espaço plano em coordenadas de Rindler *ingoing*. Nosso principal objetivo é estudar o efeito de transformações de Ehlers e simetrias das equações de Einstein no contexto da correspondência fluido/gravitação.

Para isso, fazemos uma revisão dos aspectos principais da Relatividade Geral e da Hidrodinâmica, os quais serão empregados ao longo do texto. Damos bastante atenção ao desenvolvimento de um método que permite encontrar soluções da equações de Einstein por meio de uma expansão em derivadas, o qual sera utilizado posteriormente para gerar uma solução-base sobre a qual aplicaremos transformações de Ehlers.

Nós mostramos que a métrica de um espaçotempo plano em coordenadas de Rindler *ingoing* está relacionada a um espaçotempo de Taub por meio de uma transformação de Ehlers e nós utilizamos um método em que nós resolvemos a equação de Killing perturbativamente na expansão no parâmetro ε. Os resultados obtidos com este método não são inteiramente conclusivos, de modo que faz-se necessária uma futura investigação.

Contents

Notation and Conventions

Unfortunately, there is no universal consensus about what notation to use in General Relativity, either when it comes to metric signature or what alphabet to use index labeling. Still, we have tried to adopt a mostly self-consistent notation. Any deviations from our standard notations will be explicitly stated.

- The metric signature for flat spacetime is $(-,+,+,+)$.
- Greek indices $(\mu,\nu,\ldots)$ represent temporal and spatial coordinates, while latin indices $(a,b,i,j,k,\ldots)$ represent only spatial coordinates.
- Partial derivatives will be written either as ∂_μ or $\frac{\partial}{\partial x^\mu}$. Covariant derivatives are denoted by ∇_μ. The "comma and semicolon" notation is not used.
- Unless otherwise stated, we set $c=\hbar=k=G=1$.
- Throughout this dissertation we made extensive use of the Mathematica RGTC package available at http://www.inp.demokritos.gr/~sbonano/RGTC/.

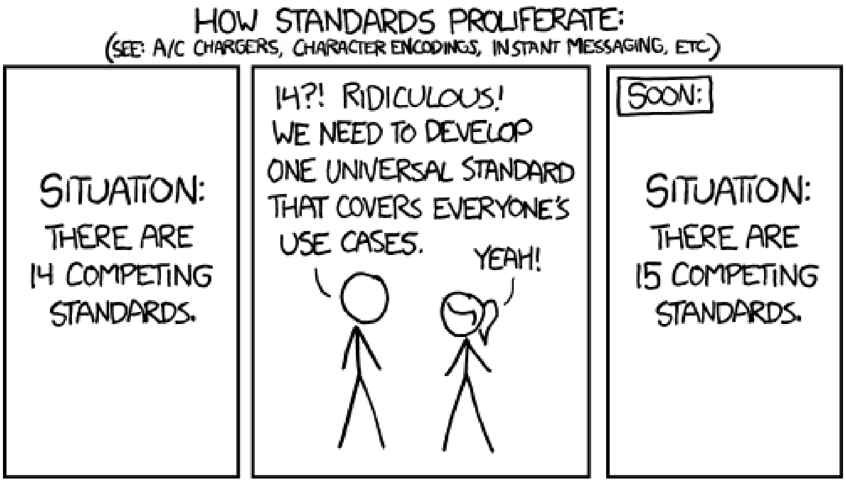

Figure 1: Also valid for notation conventions in Physics. Source: https://xkcd.com/927/

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Introduction

The Einstein equations of General Relativity, which describe gravity as a result of the interaction between matter, energy and spacetime, and the Navier-Stokes equation of fluid dynamics, which describes the motion of viscous fluids, have been since their discovery a source of continuous research and interest, not only because they can be applied to a broad set of physical phenomena and to many problems in engineering (turbulence, the Big Bang, black holes, weather modelling, GPS, gravitational lensing, aerodynamics and many others), but also because they possess a very rich, yet not fully understood, non-linear structure, which makes it difficult to find exact solutions to these equations and has attracted interest in itself from a purely mathematical point of view, especially in the case of the Navier-Stokes equation.

Surprisingly, there are hints suggesting that these two equations are not totally unrelated. It was shown by Damour [1] that the dynamics of the fluctuations around the event horizon of a black hole is very similar to that of a viscous fluid, explained by a 2-dimensional Navier-Stokes like equation for the event horizon (known as the Damour-Navier-Stokes equation [2]). The use of fluid dynamics to describe the black hole horizon, including the works by Damour, eventually led to the membrane paradigm [4], an approach in which the 4-dimensional black hole horizon is treated as a 2-dimensional membrane in a 3-dimensional space.

Shortly before the work of Damour, there appeared other indications from black hole thermodynamics that would become the basis of what would later be known as the holographic principle. When studying the notion of black hole entropy, Bekenstein suggested [5–7] that the black hole entropy is proportional to its area. This was confirmed by Hawking [8, 9], who showed that $S_{BH} = \frac{A_{BH}}{4}$, where S_{BH} and A_{BH} are the black hole entropy and area, respectively. It was also shown by Damour that the ratio between the viscosity and the entropy is given by $\frac{\eta}{s} = \frac{1}{4\pi}$. These works by Bekenstein and Hawking were done in the much larger context of black hole information. In particular, there was (and there still is) a great interest in understanding what happens to the information of something that falls inside a black hole. Nevertheless, as we said earlier, these ideas would later resurface as an inspiration for the holographic principle.

Indeed, motivated in part by these works on black hole thermodynamics, 't Hooft proposed in 1993 [10] that the information contained within some region of spacetime could be regarded as a hologram, that is, it could be entirely described by a theory living on the boundary of this region, i.e. if this region of spacetime contains $d+1$ dimensions, it could be described by a theory living on its d-dimensional boundary. This proposal was later expanded by Susskind [11], who formulated the holographic principle in the context of string theory. For a review on the holographic principle, including the motivations arising from black hole thermodynamics in the works of Bekenstein and Hawking cited above, see [12].

This idea of gravitational theories in $d+1$ dimensions being equivalent to field theories in one dimension less, living on the boundary of this higher dimensional spacetime, gained a substantial boost in activity after Maldacena proposed the AdS/CFT correspondence [13], which may be thought as an implementation of holography, in the sense that it relates quantum gravity (as in string theory) living in anti-de Sitter space to conformal field theories living at the boundary of AdS. The most famous example of AdS/CFT correspondence relates string theory on $AdS_5 \times S^5$ to a $N = 4$ supersymmetric Yang-Mills theory living on the four-dimensional boundary of $AdS_5 \times S^5$. For reviews of AdS/CFT correspondence, see [14, 15].

Working in the context of AdS/CFT correspondence, Policastro, Starinets and Son [16, 17] found that a black hole dissipating in AdS space behaves just like a viscous fluid, that it, its behavior may be explained using hydrodynamics. Moreover, they found that the ratio between entropy and shear viscosity is a constant, namely $\frac{\eta}{s} = \frac{1}{4\pi}$, which is precisely the same value obtained decades earlier by Damour. It should be pointed out that the calculation by Policastro, Son and Starinets is done at spatial infinity, unlike that of Damour, which is performed at the black hole horizon. However, there is a relation between these results.

Also within the context of AdS/CFT correspondence, Bhattacharyya et al. [18] (see also [19–21] for reviews and the references therein for developments of the original idea of Bhattacharyya et al.) found a relation between fluid dynamics in d dimensions and General Relativity with negative cosmological constant in $d+1$ dimensions, in the sense that for an arbitrary solution in fluid dynamics, they constructed an asymptotically AdS black hole spacetime where the evolution of the horizon is the same as that of a fluid flow.

Up to this point, the works on fluid/gravity correspondence were done with Maldacena's conjecture in mind, but a question that arises is whether it is possible to do a fluid/gravity correspondence in other spacetimes, that is, can we move away from anti-de Sitter spaces? It turns out that it is indeed possible to move away from AdS spaces. This was first addressed in [23]. Upon introduction of a cutoff surface Σ_c at a fixed radius $r = r_c$ outside the horizon of a metric given by $\mathrm{d}s^2 = -h(r)\mathrm{d}\tau^2 + 2\mathrm{d}\tau\mathrm{d}r + e^{2t(r)}\mathrm{d}x_i\mathrm{d}x^i$, they imposed boundary conditions to fix the induced metric on Σ_c, but these conditions do not specify a solution. By taking the cutoff to the horizon r_h, the authors of [23] found that the geometry is Rindler space, and they also found that $\frac{\eta}{s} = \frac{1}{4\pi}$. All of this was done without assuming an asymptotically AdS region.

Building up on [23], it was later shown [24] that it is possible to map solutions of the incompressible Navier-Stokes equation into solutions of the Einstein equation, which was achieved by considering gravitational fluctuations around a background solution $\mathrm{d}s^2 = -r\mathrm{d}t^2 + 2\mathrm{d}t\mathrm{d}r + \mathrm{d}x_i\mathrm{d}x^i$, with a dual fluid living on a cutoff surface Σ_c, constrained to be flat. It was found that the deformed geometry at the cutoff surface does satisfy the Einstein equations. As in the previous case [23], the authors of [24] did not assume an asymptotically AdS spacetime.

The dual metric of [24] was later extended to arbitrarily high orders of a class of Ricci-flat metrics by Compère et al. [25], by means of a procedure similar to the one previously employed in [18], although in [18] it was done in the context of AdS/CFT correspondence. Nevertheless, the algorithm developed in [25] allowed one not only to adopt a systematic approach to this and similar problems (see below), but also showed that the incompressibility condition and the Navier-Stokes equation must receive corrections at higher orders so that the metric remains Ricci-flat.

Further developments of this fluid/gravity correspondence include a procedure similar to the one developed in [25], that applies the reasoning of ε-expansion in the relativistic case [26–28], the discussion of Petrov types [29], the possibility of a similar correspondence for magnetohydrodynamics [30, 31], as well as the AdS/Ricci-flat correspondence [32, 33], which relates a class of asymptotically anti-de Sitter spacetimes with another class of Ricci-flat spacetimes and therefore may provide a bridge between the AdS cases and the Rindler ones.

Finally, as we said in the beginning, the Einstein and Navier-Stokes equations are of great interest from a mathematical point of view. They have long been studied with focus on their properties, such as exact solutions of the Einstein equations [34], as well as the problem of existence and smoothness of solutions of the Navier-Stokes equations. The mathematical interest in these equations has led to noticeable developments, such as transformation groups mapping exact solutions of the Einstein equations into other exact solutions. One such example is known as the Ehlers group [35–39], which has first been applied in the context of fluid/gravity correspondence in 2012 [40]. On the other hand, there is a known symmetry group of solutions of the Navier-Stokes equations [41], which might also be of interest whenever someone mentions a duality between gravitational and fluid dynamical systems.

Our focus in this dissertation will be on fluid/gravity correspondence in flat (Rindler) spacetimes and the possibility of using Ehlers transformations to generate new solutions to which the correspondence may be applied.

Outline of this dissertation

We have tried to keep this dissertation didactic and as self-contained as possible. In some cases, a full treatment of a certain topic would require many prior developments well outside the scope of this dissertation. In such cases, we only give a short review of said topic, focusing on its direct usefulness and applications to this dissertation, and refer the reader to the references containing the full details.

- Chapter 1 contains a brief review of the main aspects of General Relativity that will be used thoroughly throughout this text, such as a discussion on the Einstein equations as well as the concepts of hypersurfaces, Killing vectors and Lie derivatives.

- Chapter 2 discusses general aspects of relativistic fluid dynamics, its status as a long-wavelength limit of quantum field theories and the Navier-Stokes equation as a non-relativistic limit of the equations of motion for a relativistic fluid at first order in the ε-expansion which we discuss in the text.

- Chapter 3 applies the ideas of fluid/gravity correspondence to the case of a flat spacetime. We review an algorithmic procedure developed in [25], which allows us to solve the Einstein equations for the flat spacetime in Rindler coordinates order by order, and explicitly solve the equations at third order, a result which will be used afterwards.

- Chapter 4 contains a discussion on the Ehlers transformations, which allows us to find solutions of the Einstein equations based on existing solutions. Following the ideas of [40], we employ a similar reasoning to the flat metric discussed in chapter 3, which is done by solving the Killing equations perturbatively in the ε-expansion. We also found that Ehlers transformations may relate the Rindler and Taub spacetimes.

- Chapter 5 contains our conclusions and possible future developments.

Chapter 1

General Relativity

In order to make this work as self-contained as possible, we have chosen to dedicate this first chapter to aspects of General Relativity. Given the immense breadth [42, 43] of this subject, however, a full treatment is completely out of hand, which already indicates somewhat of a failure in making this work properly self-contained, but the content we provide here should be enough to understand the aspects of General Relativity that will appear in the following chapters. Some of the main references for a more in-depth coverage of General Relativity should be, among many others, [42–45], with the latter being particularly useful for some technical details.

1.1 Spacetime and the Einstein equations

General Relativity is the theory that describes space, time and gravity, as well as how these things intertwine, especially in the presence of very massive bodies. Formulated by Einstein in 1915, it is built upon Special Relativity, proposed by Einstein ten years earlier, which redefines our notions of space and time by treating them as two sides of the same coin and introducing the concept of spacetime.

Einstein's general theory of relativity can be regarded as a generalization of Newton's theory of gravitation, by describing gravity as a geometric property of spacetime. Indeed, quoting [44], we may say that "*spacetime is a manifold $\mathcal{M}$ on which there is defined a Lorentz metric $g_{\mu\nu}$. The curvature of $g_{\mu\nu}$ is related to the matter distribution in spacetime* *by Einstein's equation*". In mathematical terms, this means that[1]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \tag{1.1}$$

where the left-hand side represents curvature and the right-hand side represents the matter content. Before explaining the Einstein equations eq. (1.1) in more detail we should digress a little bit into the metric tensor.

The metric tensor (or simply "the metric") is what defines the geometry of spacetime. It is essential in the definitions of spatial concepts such as distance, curvature and angles, as well as the causal structure of spacetime. In other words, the metric is also responsible for the notions of past and future in a given spacetime.

The simplest case of a spacetime is the Minkowski space, also known as flat space. In this case, we denote the metric by $\eta_{\mu\nu}$, with the indices representing the spacetime coordinates. For more general spacetimes, we use the notation $g_{\mu\nu}$ for the metric. For our 4-dimensional flat spacetime, the indices are time and the three spatial coordinates: (t,x,y,z). The *interval* defined by this metric is

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2, \tag{1.2}$$

where $\eta_{tt} = -\eta_{xx} = -\eta_{yy} = -\eta_{zz} = -1$, all the other $\eta_{\mu\nu}$ being zero. In the above equation we have introduced the Einstein summation convention: repeated indices (one upper, one lower) are summed. It should be noted that eq. (1.2) is not the only way in which we can write the interval for a flat space.

We may, for example, use the following metric

$$\mathrm{d}s^2 = -r\mathrm{d}\tau^2 + 2\mathrm{d}\tau\mathrm{d}r + \mathrm{d}x_i\mathrm{d}x^i \tag{1.3}$$

to describe flat space in the so-called ingoing Rindler coordinates. Indeed, if we set $\tau = 2\log(x+t)$ and $4r = x^2 - t^2$, we recover $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$. The metric described in eq. (1.3) will be used thoroughly in the rest of this work.

If we perform a coordinate transformation $x \to x'$, then the metric will transform according to

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}(x), \tag{1.4}$$

which is simply a more generalized transformation law for vectors.

[1]In this work, we ignore the presence of an additional term $\Lambda g_{\mu\nu}$ representing the cosmological constant on the left-hand side of the Einstein equations.

As two final comments on the metric tensor, we first note that it is symmetric upon the interchange between μ and ν, that is, $g_{\mu\nu} = g_{\nu\mu}$. Also, we note that the metric admits the tensor $g^{\mu\nu}$, which is called the *inverse metric* and is such that

$$g_{\mu\rho}g^{\rho\nu} = \delta_{\mu}^{\nu}, \tag{1.5}$$

with δ_{μ}^{ν} being the Kronecker delta, equating 1 if $\mu = \nu$ or zero if $\mu \neq \nu$.

With the above considerations we may proceed to the other terms appearing in the left-hand side of the Einstein equations eq. (1.1). To do this, we first consider free particles on a flat spacetime. The paths describing their movement can be parametrized by λ and may be written as $x^{\mu}(\lambda)$, and they obey the following equation:

$$\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\lambda^2} = 0. \tag{1.6}$$

This is simply a straight line, which means that free particles on a flat spacetime move along straight lines. However, if we consider other spacetimes, now displaying some kind of curvature, the free particles will no longer move along straight lines. Instead, they will move along the "straightest possible lines", which are called *geodesics*, and eq. (1.6) must be replaced by the *geodesic equation*

$$\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\lambda^2} + \Gamma^{\mu}_{\rho\sigma}\frac{\mathrm{d}x^{\rho}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\sigma}}{\mathrm{d}\lambda} = 0. \tag{1.7}$$

The $\Gamma^{\mu}_{\rho\sigma}$ are known as Christoffel symbols[2] and arise out of the fact that partial derivatives do not remain the same if we change our system of coordinates; we would like a more general derivative operator, one that follows the transformation law for tensors, that is, one which is independent of the coordinates. For a vector V^{μ}, the expression

$$\nabla_{\mu}V^{\nu} = \partial_{\mu}V^{\nu} + \Gamma^{\nu}_{\mu\sigma}V^{\sigma} \tag{1.8}$$

satisfies our needs, and this is the definition of the *covariant derivative* for a contravariant vector. For a covariant vector, it is simply

$$\nabla_{\mu}V_{\nu} = \partial_{\mu}V_{\nu} - \Gamma^{\lambda}_{\mu\nu}V_{\lambda}. \tag{1.9}$$

[2]To be rigorous, we should use either the term "connection coefficients" or "affine connection". This is because the Christoffel symbols are a special case of a more general derivative operator $\nabla_{\mu}V^{\nu} = \tilde{\nabla}_{\mu}V^{\nu} + C^{\nu}_{\mu\sigma}V^{\sigma}$, which arises out of the discussion on the uniqueness of derivative operators, as two different derivative operators ∇_{μ} and $\tilde{\nabla}_{\mu}$ do not necessarily act on vectors and tensors in exactly the same way. See Section 3.1 of [44] for a more rigorous discussion. If $\tilde{\nabla}_{\mu} = \partial_{\mu}$, that is, the usual partial derivative operator, then we write the $C^{\mu}_{\rho\sigma}$ as $\Gamma^{\mu}_{\rho\sigma}$ and call them Christoffel symbols. Since, in our case $\tilde{\nabla}_{\mu} = \partial_{\mu}$, we are not making an abuse of nomenclature by adopting the name Christoffel symbols from the start.

It can be shown (see references cited above) that the Christoffel symbols may be calculated directly from the metric by[3]

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}). \tag{1.10}$$

The Christoffel symbols are symmetric[4] under $\mu \leftrightarrow \nu$. With them we may calculate the *Riemann curvature tensor* ("Riemann tensor" for short) $R^{\rho}_{\sigma\mu\nu}$:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{1.11}$$

By lowering ρ in the above equation, one may show that the Riemann tensor satisfies the Bianchi identities

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0. \tag{1.12}$$

A useful quantity derived from the Riemann tensor is the Kretschmann scalar

$$K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}. \tag{1.13}$$

The Kretschmann scalar is a curvature invariant, and is useful to identify singularities of the metric. From the Riemann tensor, we may get the *Ricci tensor* $R_{\mu\nu}$ upon index contraction:

$$R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu}. \tag{1.14}$$

The Ricci tensor too is symmetric under $\mu \leftrightarrow \nu$. Also, a metric is said to be *Ricci flat* if $R_{\mu\nu} = 0$. Finally, if we contract the Ricci tensor with the metric, we arrive at the *Ricci scalar*

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{1.15}$$

We see from eqs. (1.10), (1.11), (1.14) and (1.15) that the entire left side of eq. (1.1) can be given once the metric is known.

The $T_{\mu\nu}$ on the right-hand side of eq. (1.1) is called the *energy-momentum tensor* and represents the matter content within a region characterized by the metric $g_{\mu\nu}$. In many cases, we would like that $T_{\mu\nu}$ takes the form of a perfect fluid, with energy density ρ and

[3]Despite the look of this equation, the Christoffel symbols are not tensors, as they do not transform according to a equation like eq. (1.4). Instead, it is the covariant derivative that follows such a transformation law.

[4]The affine connection is only symmetric upon this interchange if it is torsion free, which is the case in General Relativity. Again, see [44] for a more general discussion.

pressure p:

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{1.16}$$

In chapter 2 we describe fluid dynamics in some detail.

1.2 Symmetries, Killing vectors and the Lie derivative

1.2.1 Symmetries of manifolds

A manifold $\mathcal{M}$ is said to be symmetric if its geometry is invariant under some kind of transformation (diffeomorphism) mapping $\mathcal{M}$ to itself. In the case of a metric, such diffeomorphism is called an *isometry*[5]. In component notation, an isometry is such that it satisfies the following equation:

$$g_{\mu\nu}(x) = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g_{\alpha\beta}(x'). \tag{1.17}$$

In other words, an isometry preserves the metric. Bringing this discussion to spacetime, we may naturally ask ourselves if we can find such isometries for a given spacetime and, in case this is true, we may also ask ourselves how many isometries we may find for a given metric. In order to answer these questions, we must first introduce the idea of Killing vectors. To do this, we consider the transformation

$$x_{\mu} \to x'_{\mu} = x_{\mu} + \varepsilon \xi_{\mu}, \qquad |\varepsilon| \ll 1. \tag{1.18}$$

The term $\varepsilon\xi_{\mu}$ represents an infinitesimal displacement. If this transformation is an isometry, then we may put eq. (1.18) in eq. (1.17). If we expand this to first order in ε and lower the ξ^{μ} coordinates that will appear, then eq. (1.17) may be rewritten as

$$\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0. \tag{1.19}$$

This equation is known as the *Killing equation*, and a vector ξ_{μ} that satisfies the Killing equation is known as a Killing vector. Therefore, if we want to find the isometries of a given spacetime, we must find its Killing vectors. We should point out that a linear

[5]See [44] or [46] for more complete discussions.

combination of Killing vectors is itself a Killing vector, which follows from the linearity of the covariant derivative.

As a last comment on Killing vectors and isometries, it can be shown that a d-dimensional Minkowski spacetime ($d \geq 2$) there are $\frac{d(d+1)}{2}$ Killing vectors, which generate translations, boosts and space rotations. The spacetimes admitting $\frac{d(d+1)}{2}$ Killing vectors are called *maximally symmetric spaces*.

1.2.2 Lie derivative

The Lie derivative measures how a tensor varies along a certain vector field. For a general tensor $T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l}$ the Lie derivative along V may be written as

$$\begin{aligned}\mathcal{L}_V T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} &= V^\sigma \nabla_\sigma T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} \\ &\quad - (\nabla_\lambda V^{\mu_1}) T^{\lambda\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} - (\nabla_\lambda V^{\mu_2}) T^{\mu_2\lambda\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} - \ldots \\ &\quad + (\nabla_{\nu_1} V^\lambda) T^{\mu_1\mu_2\ldots\mu_k}_{\lambda\nu_2\ldots\nu_l} + (\nabla_{\nu_2} V^\lambda) T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\lambda\ldots\nu_l} + \ldots \end{aligned} \tag{1.20}$$

For the metric $g_{\mu\nu}$, the above equation becomes

$$\begin{aligned}\mathcal{L}_V g_{\mu\nu} &= V^\sigma \nabla_\sigma g_{\mu\nu} + (\nabla_\mu V^\lambda) g_{\lambda\nu} + (\nabla_\nu V^\lambda) g_{\mu\lambda} \\ &= \nabla_\mu V_\nu + \nabla_\nu V_\mu, \end{aligned} \tag{1.21}$$

upon lowering the indices. In particular, if V_μ is a Killing vector, the Killing equation eq. (1.19) tells us that

$$\mathcal{L}_V g_{\mu\nu} = 0. \tag{1.22}$$

Hence, we may say that a vector V^μ is a Killing vector if the Lie derivative of the metric in the direction of V^μ is zero.

1.3 Hypersurfaces and spacetime foliations

1.3.1 Hypersurfaces

So far, we have been talking about manifolds, but we have said nothing about what happens when we take a subset of a given manifold, namely a *submanifold*, and study it. The ideas presented here are going to be very useful in what follows, since in the rest of this work we will be studying fluids living on the boundary of given spacetimes, that is, fluids living on a submanifold of a spacetime manifold.

We are particularly interested in a class of submanifolds called *hypersurfaces*. For a d-dimensional manifold, a hypersurface Σ is a $(d-1)$-dimensional submanifold (we say that Σ is of codimension 1).

So, let Σ be a hypersurface of a larger manifold $\mathcal{M}$. If $\mathcal{M}$ is described by the coordinates x^α and Σ is equipped with the coordinates y^a, we may describe the hypersurface by means of an embedding $\Phi : x^\alpha = x^\alpha(y^a)$, which are simply parametric equations. The Jacobian of this embedding is given by

$$E_a^\alpha = \frac{\partial x^\alpha}{\partial y^a}, \tag{1.23}$$

and we may define the normal vectors to Σ by noting that they obey the following equation:

$$E_a^\alpha \xi_\alpha = 0, \tag{1.24}$$

where ξ_α are the normal vectors to Σ. According to the norm of ξ_α, the hypersurface may be defined as either spacelike, timelike or null, as follows:

$$\Sigma \text{ is called } \begin{cases} \text{spacelike} & \text{if } \xi^\alpha \xi_\alpha < 0, \\ \text{timelike} & \text{if } \xi^\alpha \xi_\alpha > 0, \\ \text{null} & \text{if } \xi^\alpha \xi_\alpha = 0. \end{cases} \tag{1.25}$$

If Σ is not a null hypersurface, then we may define the normalized normal vector n^α as

$$n^\alpha = \frac{\xi^\alpha}{\sqrt{|\xi^\alpha \xi_\alpha|}}, \tag{1.26}$$

so that

$$n^\alpha n_\alpha = \varepsilon = \begin{cases} -1 & \text{if } \Sigma \text{ is spacelike}, \\ +1 & \text{if } \Sigma \text{ is timelike}. \end{cases} \tag{1.27}$$

We may also define the *induced metric* h_{ab} on Σ, which is the metric induced on Σ by the metric $g_{\alpha\beta}$ on $\mathcal{M}$:

$$\begin{aligned} \mathrm{d}s^2|_\Sigma &= g_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta|_\Sigma \\ &= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^a} \frac{\partial x^\beta}{\partial y^b} \mathrm{d}y^a \mathrm{d}y^b \\ &\equiv h_{ab} \mathrm{d}y^a \mathrm{d}y^b, \end{aligned} \tag{1.28}$$

where

$$h_{ab} \equiv g_{\alpha\beta} E_a^\alpha E_b^\beta \tag{1.29}$$

is the induced metric on Σ. At this point we introduce the notation a, b for the coordinates of Σ, while keeping α, β for the coordinates on the larger spacetime.

Finally, we may define the *extrinsic curvature* of a hypersurface Σ with a metric h_{ab} and vector n normal to Σ by

$$K_{ab} \equiv \frac{1}{2}\mathcal{L}_n h_{ab}. \tag{1.30}$$

The extrinsic curvature describes how the space is embedded in some larger space. In this case, it describes how the hypersurface Σ is embedded in the manifold $\mathcal{M}$, that is, how it bends in $\mathcal{M}$. Considering that $\mathcal{M}$ is not embedded in any larger manifold, it obviously only makes sense to speak of an extrinsic curvature on Σ. The *intrinsic curvature*, on the other hand, is measured by the Riemann tensor, and it describes how the space is curved. Both $\mathcal{M}$ and Σ have their own intrisic properties, and it is possible to relate these quantities via the Gauss-Codazzi equations (see, for example, [44]).

1.3.2 Spacetime foliations

It is possible to foliate a given spacetime $(\mathcal{M}, g)$ into a family of nonintersecting spacelike surfaces, each denoted by Σ_t, one for each "instant of time". We may parametrize the Σ_t by a scalar field $t(x^\alpha)$, which is a completely arbitrary single-valued function of the $g_{\alpha\beta}$ coordinates x^α, sometimes called the "global time", or "time function". One requirement to $t(x^\alpha)$ is that the unit normal to Σ_t, which we denote by n^α as before, be a future-oriented timelike vector field.

We would like to work on a coordinate system defined by $x^\alpha = x^\alpha(t, y^a)$ so that we may define

$$t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a}, \tag{1.31a}$$

$$E^\alpha_a = \left(\frac{\partial x^\alpha}{\partial y^a}\right)_t. \tag{1.31b}$$

At first, it is not possible to assume that t^α is orthogonal to Σ_t. To see this, suppose a curve γ connecting points at different hypersurfaces (for example P_1 at Σ_{t_1} and P_2 at Σ_{t_2}) such that these points have the same spatial coordinates: $y^a(P_1) = y^a(P_2)$. This curve then provides the notion of "time evolution" from Σ_{t_1} to Σ_{t_2}, but they are not necessarily normal to the Σ_t hypersurfaces because these are not necessarily parallel to each other; we only require the Σ_t to be nonintersecting to each other. Thus, there is no requirement that these curves enter the Σ_t orthogonally. Therefore, we must decompose t^α into its normal and tangent components

$$t^\alpha = Nn^\alpha + N^a E_a^\alpha, \tag{1.32}$$

where N is called the *lapse function*, which measures how the proper time of a co-moving observer differs from the coordinate time, that is, it measures the rate of flow of proper time with respect to t. We define N by

$$N = -t^\alpha n_\alpha. \tag{1.33}$$

The term N^a in eq. (1.32) is the *shift vector*, which may be defined as

$$N^a = h_b^a t^b. \tag{1.34}$$

It measures the movement parallel to Σ_t. In summary, the lapse function and the shift vector measure the non-orthogonality of t with respect to Σ_t.

We can now use the coordinate transformation $x^\alpha(t, y^a)$ along with eqs. (1.31a), (1.31b) and (1.32) to rewrite the line element $\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$. By noting that

$$\begin{aligned} \mathrm{d}x^\alpha &= \frac{\partial x^\alpha}{\partial t}\mathrm{d}t + \frac{\partial x^\alpha}{\partial y^a}\mathrm{d}y^a = t^\alpha \mathrm{d}t + E_a^\alpha \mathrm{d}y^a = (Nn^\alpha + N^a E_a^\alpha)\mathrm{d}t + E_a^\alpha \mathrm{d}y^a \\ &= (N\mathrm{d}t)n^\alpha + (\mathrm{d}y^a + N^a \mathrm{d}t)E_a^\alpha, \end{aligned} \tag{1.35}$$

we may write the line element as

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = -N^2 \mathrm{d}t^2 + h_{ab}(\mathrm{d}y^a + N^a \mathrm{d}t)(\mathrm{d}y^b + N^b \mathrm{d}t), \tag{1.36}$$

where we used the relations $g_{\alpha\beta}n^\alpha n^\beta = -1$ and $n_\beta E_b^\beta = 0$ and where, as usual,

$$h_{ab} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^a}\frac{\partial x^\beta}{\partial y^b} \tag{1.37}$$

is the induced metric on Σ_t.

Using eq. (1.21) and then eq. (1.32), the extrinsic curvature may be rewritten in terms of these variables as[6]

$$\begin{aligned}
K_{ab} &= \frac{1}{2}\mathcal{L}_n h_{ab} = \frac{1}{2}[n^c\nabla_c h_{ab} + h_{ac}\nabla_b n^c + h_{bc}\nabla_a n^c] \\
&= \frac{1}{2N}[Nn^c\nabla_c h_{ab} + h_{ac}\nabla_b(Nn^c) + h_{bc}\nabla_a(Nn^c)] \\
&= \frac{1}{2N}[(t^c - N^c)\nabla_c h_{ab} + h_{ac}\nabla_b(t^c - N^c) + h_{bc}\nabla_a(t^c - N^c)] \\
&= \frac{1}{2N}(\mathcal{L}_t h_{ab} - \mathcal{L}_N h_{ab}). \qquad (1.38)
\end{aligned}$$

In the above example, we foliated spacetime according to the time coordinate, but in principle we may do an analogous procedure to other coordinates as well.

1.3.3 The Brown-York tensor

In the Hamiltonian formulation of Classical Mechanics, we have the Hamiltonian function

$$H(p,q) = p\dot{q} - L, \qquad (1.39)$$

where p is the canonical momentum, q the generalized coordinates and L the Lagrangian function. By writing $L = p\dot{q} - H$, we may write the action S as

$$S = \int_{t_1}^{t_2} L\mathrm{d}t = \int_{t_1}^{t_2} (p\dot{q} - H)\mathrm{d}t, \qquad (1.40)$$

Upon varying the action, while requiring that $\delta q = 0$ at the endpoints t_1 and t_2, we are eventually led to the Hamilton equations and, further along, to the momentum and the Hamilton-Jacobi equations

$$p = \frac{\partial S}{\partial x}, \qquad (1.41a)$$

$$H = -\frac{\partial S}{\partial t}. \qquad (1.41b)$$

Now, we would like to consider a region of spacetime foliated by two spacelike surfaces Σ_{t_1} and Σ_{t_2}, at the "bottom" and "top", respectively, with time flowing upwards, both bounded by closed two-surfaces S_t, and we denote the union of S_t by $\mathcal{B}$. The idea originally proposed by Brown and York [47] is to apply an analogous procedure to the

[6]To be rigorous, we should have followed [45] and done this for the general case in α coordinates, and then projected the result onto the hypersurface, but here we are assuming from start that we are already on Σ_t.

one described for the Classical Mechanical case, but this time for an action given by[7]

$$\begin{aligned} S = &\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}R \\ &+2\int_{\Sigma_{t_2}} \mathrm{d}^3x\sqrt{h}K - 2\int_{\Sigma_{t_1}} \mathrm{d}^3x\sqrt{h}K - 2\int_{\mathcal{B}} \mathrm{d}^3x\sqrt{-\gamma}\Theta \\ &+ S^m, \end{aligned} \tag{1.42}$$

where the term in the first line is the usual Einstein-Hilbert action term, K is the extrinsic curvature scalar in the hypersurfaces Σ_t, h is the metric in Σ_t, Θ and γ are, respectively, the extrinsic curvature scalar and the metric in $\mathcal{B}$. The reason why we have a minus sign before the second integral in the second line is that the normal vectors at Σ_{t_1} and Σ_{t_2} point at opposite directions (the normal to Σ_{t_1} must be future-directed, and therefore must point inward). The term in the third line corresponds to the contribution of matter terms. Collectively, the terms in the second line are the boundary terms, and they are known as the Gibbons-Hawking-York term [48–50].

Upon varying eq. (1.42), it can be shown [45, 47] that

$$\begin{aligned} \delta S = &\,(\text{equations of motion}) + \delta S^m \\ &+ \int_{\Sigma_{t_2}} \mathrm{d}^3x P^{ij}\delta h_{ij} - \int_{\Sigma_{t_1}} \mathrm{d}^3x P^{ij}\delta h_{ij} + \int_{\mathcal{B}} \mathrm{d}^3x \pi^{ij}\delta\gamma_{ij}. \end{aligned} \tag{1.43}$$

The equations of motion are just the Einstein equations given as the Euler-Lagrange equations. In eq. (1.43) we have defined the conjugated momenta to h_{ij} and γ_{ij}, respectively, as

$$P^{ij} = \sqrt{h}(Kh^{ij} - K^{ij}), \tag{1.44a}$$

$$\pi^{ij} = -\sqrt{-\gamma}(\Theta\gamma^{ij} - \Theta^{ij}). \tag{1.44b}$$

From eqs. (1.44a) and (1.44b) we may define in analogy to the Classical Mechanical case the momentum

$$P^{ij} = \frac{\delta S}{\delta h_{ij}}. \tag{1.45}$$

Now, for the analogous of the Hamilton-Jacobi equation, the notion of energy will be generalized to a stress-energy tensor, namely,

$$T^{ij}_{BY} \equiv \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma_{ij}}, \tag{1.46}$$

[7] In units where $\frac{1}{16\pi G} = 1$.

known as the *Brown-York* tensor[8] [47].

We must note here that the Brown-York tensor appears exclusively on the boundary of our system. It is useful to note this now, as we are later going to apply the so-called fluid/gravity duality to a metric at the bulk so as to study the fluids appearing on the boundary. The stress-energy tensor for these fluids will be a Brown-York tensor.

[8]We may add or subtract terms to the action and still leave the dynamics of the system unaltered. This "ambiguity" is resolved in [47] by adding a term S_0 to the action. This new term will then generate a "momentum" π_0^{ij}. Therefore, the action S in eq. (1.46) should be $\frac{\delta S}{\delta \gamma_{ij}} = \pi^{ij} - \pi_0^{ij}$. For simplicity, we are setting $S_0 = \pi_0^{ij} = 0$.

Chapter 2

Hydrodynamics

In this chapter we will introduce some features and basic results of relativistic hydrodynamics. We also dedicate a small section to the non-relativistic limit, where we follow [51] to show that the Navier-Stokes equations appear naturally as a non-relativistic limit of the relativistic hydrodynamical equations at first order in the ε expansion.

We start with an overview of some of the most fundamental aspects of the theory, such as its status as a long-wavelength limit of interacting quantum field theories, the equations of motion for hydrodynamics in covariant form and the Landau frame. Then, we discuss the perfect fluid and use it as a starting point for a discussion on the derivative expansion. We end this chapter by remarking on the non-relativistic limit. The main standard reference for the basics of fluid dynamics should be [52]. Parts of this chapter will follow closely [20]. Other useful references are [53–55], as well as the references cited along the text.

2.1 Overview of hydrodynamics

2.1.1 Hydrodynamics as a long-wavelength limit of QFTs

When we speak of quantum field theories (QFTs), we are particularly interested in their action, from which we may get the stress-energy tensor $T_{\mu\nu}$ and the conserved currents J_μ. Another useful feature of a QFT is its mean free path, or correlation length ℓ_{corr}. This is the length scale at which the interactions of this QFT occur. In order to treat hydrodynamics as a long-wavelength limit of QFTs, we must first of all explain what we mean by this and how this is related to what we just said. To do this, we shift for a moment to the statistical description of fluid dynamics, which is intrinsical to it, since we are dealing with a very large number of microscopic constituents.

Following [20], let us consider a system initially in global thermal equilibrium with conserved currents. Then, we must describe it statistically in terms of the grand canonical ensemble, whose chemical potentials will be associated with the currents. If we choose to perturb this system so that it moves away from equilibrium as the thermodynamic variables fluctuate, it is possible to do this in such a way that these fluctuations will be so slow that, *locally*, the thermodynamic quantities such as temperature, pressure and chemical potential will *not* change. This is possible to achieve if the scales L at which these fluctuations occur are much larger than the scales of the interactions between the constituents of the system, that is, if $L \gg \ell_{corr}$.

Now, fluid dynamics is used exactly to describe the systems in which these fluctuations occur at long wavelengths when compared to the scale set by ℓ_{corr}. This is why we may regard hydrodynamics as the long-wavelength limit of an interacting field theory at a finite temperature. We express this assertion mathematically by defining the Knudsen number

$$K_n = \frac{\ell_{corr}}{L}, \tag{2.1}$$

Therefore, when we speak of a long-wavelength limit, we speak of a small Knudsen number. This is what we call the *hydrodynamic limit*.

2.1.2 Equations of motion

Generally, ideal fluids are described by the Euler and the continuity equations [52]

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla p}{\rho}, \tag{2.2a}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{2.2b}$$

where $\mathbf{v}$ is the fluid velocity, ρ the energy density and p the fluid pressure. The Euler equation comes from Newton's law and describes the fluid flow, while the continuity equation states that the rates at which mass enters and leaves the system are the same. If the fluid is viscous, then it is generally described by the incompressible Navier-Stokes equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{v}, \tag{2.3a}$$

$$\vec{\nabla} \cdot \vec{v} = 0, \tag{2.3b}$$

where ν is fluid viscosity.

However, in our case it is far more useful to write the equations of motion for fluids in a covariant manner, such that the characteristics of the fluid will now be contained in the stress-energy tensor $T_{\mu\nu}$. By doing so, the equations we have shown above are reduced to

$$\nabla^{\mu} T_{\mu\nu} = 0. \tag{2.4}$$

In the above equation, T_{00} is the energy density, also denoted by ρ, T_{ii} is the pressure in the x_i direction, T_{0i} the momentum density and T_{ij} the shear stress parallel to the ij surface.

The non-diagonal terms appearing on the general stress-energy tensor $T_{\mu\nu}$ defined above are related to the dissipation of the fluid, that is, to derivatives of the fluid velocity, which we now denote by u^{μ}. The dissipative corrections to the stress-energy tensor may be constructed in a derivative expansion (also called the hydrodynamic expansion) of the velocity field and the thermodynamic variables.

2.2 The hydrodynamic expansion

The main goal of this section is to arrive at the scaling limit that shows that we can obtain the non-relativistic Navier-Stokes equations from the first-order relativistic equations of motion for a fluid. With this in mind, and also in order to make this presentation somewhat didactic, we need in first place to discuss the perfect fluid as well as dissipative terms added to the equations of ideal fluids, upon which we will develop the hydrodynamic expansion that will eventually lead to our desired scaling limit.

The basic assumption that allows us to perform the hydrodynamic expansion is that the velocity field $u^{\mu}(x^{\nu})$ varies slowly. This means that we may expand the derivatives of u^{μ} and solve our equations perturbatively. To do so, we must rescale our coordinates by multiplying them by factors of our scale parameter ε. It is possible to apply a hydrodynamic expansion to both relativistic and non-relativistic fluids, according to scalings we show below. However, our focus in this work will be on the non-relativistic case. In the relativistic case, we have $\tau \sim x^i$. Therefore, we may write [26]

$$u(\tau, \vec{x}) \to u(\varepsilon\tau, \varepsilon\vec{x}) \tag{2.5}$$

and leave the equations of motion invariant under this scaling. In the non-relativistic regime the velocity is much smaller, so $\tau \sim x^i$ no longer applies. Instead, the rescaling should be [51]

$$u_i(\tau,\vec{x}) = \varepsilon u_i(\varepsilon^2\tau,\varepsilon\vec{x}) \text{ and} \tag{2.6a}$$

$$P(\tau,\vec{x}) = \varepsilon^2 P(\varepsilon^2\tau,\varepsilon\vec{x}) \tag{2.6b}$$

(we use the notation u_i for the non-relativistic fluid velocity at the spatial x_i coordinate), as we are going to see below. In any case, this is the prescription for the hydrodynamic expansion. With this in mind, we proceed to talk about the perfect fluid, whose stress-energy tensor will be the basis for the hydrodynamic expansion in the relativistic case.

2.2.1 The perfect fluid

The simplest case of a fluid is that of the perfect (or ideal) fluid. The stress-energy tensor for this fluid does not contain dissipative terms, that is, its non-diagonal terms are zero. Therefore, it is not necessary to perform the derivative expansion in this case. However, as we stated above, the expansion will be built upon the stress-energy tensor for the perfect fluid, thus it is only natural that we discuss this simplest case now.

The stress-energy tensor for a perfect fluid is given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}. \tag{2.7}$$

It is customary and often useful to express the stress-energy tensor above in terms of

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \tag{2.8}$$

which is a projector onto spatial directions perpendicular to u^μ. It is immediate to verify that $P_{\mu\nu}u^\mu = 0$. Using eq. (2.8), we can rewrite eq. (2.7) as

$$T^{\mu\nu} = \rho u^\mu u^\nu + p P^{\mu\nu}. \tag{2.9}$$

Another feature of a perfect fluid is that the conserved currents J_I^μ, where I indicates the corresponding conserved charge, may be expressed in the local rest frame as

$$J_I^\mu = q_I u^\mu, \tag{2.10}$$

where q_I are the conserved charges. Since the perfect fluid lacks derivatives of the fluid velocity, the dynamical equation for the current is

$$\nabla_\mu J_I^\mu = 0. \tag{2.11}$$

One such current is the entropy current J_s^μ, which measures how the local entropy density varies in the fluid. For the perfect fluid entropy density in we have

$$(J_s^\mu)_{(0)} = su^\mu, \tag{2.12a}$$

$$\nabla_\mu (J_s^\mu)_{(0)} = 0. \tag{2.12b}$$

The second equation means that the fluid flow involves no entropy production.

2.2.2 Landau and Eckart frames

Before we proceed, there is a subtlety that might remain unnoticed if we do not think carefully about how to define the velocity field. This subtlety arises essentially from the fact that, in the relativistic regime, the distinction between mass and energy becomes somewhat shady.

In the case of a non-relativistic fluid, the energy current and the particle current are the same[1], and it is possible to define the velocity in terms of the mass flux density without further complications. However, if we try to apply a similar definition to a relativistic fluid, it makes no sense.

The current and momentum densities do not necessarily have to coincide with each other, and the mass-energy equivalence does not help in making this easy. In fact, the notion of "rest frame" becomes ambiguous, because in a rest frame $u^i = 0$, yet we don't know in principle if we are going to define the velocity according to current or momentum flow.

To fix this, we choose the velocity field so that in the local rest frame of a fluid element the stress-energy tensor components which are longitudinal to the velocity will give the local energy density of the fluid. That is, u^μ is chosen to represent the energy flux (momentum density). A key point here is that we wish the same choice to be applied to the dissipative terms. In other words, we wish the dissipative terms to be orthogonal to the fluid velocity, that is

$$\Pi^{\mu\nu} u_\mu = 0, \tag{2.13a}$$

$$\Upsilon^\mu u_\mu = 0. \tag{2.13b}$$

[1]Rigorously, the four-velocity defining the rest-mass density current is also an eigenvector of $T^{\mu\nu}$. See the discussion in [55] for more details.

This choice in which the fluid velocity is defined as going along the energy current is known as the *Landau frame* [52] (also known as the energy frame).

It is also possible to work on the Eckart frame [56] (also known as the particle frame), in which the local rest frame is that of the particle current, so that the fluid velocity is related to the charge transport: the velocity field is set to be proportional to the charge current. In this frame, there is no charge flow (particle diffusion) in the rest frame.

Throughout this dissertation, we will work on the Landau frame.

2.2.3 Dissipative terms

If the fluid dissipates, eqs. (2.7) and (2.9) are no longer valid. In order to take into account the dissipative terms, we should have

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + \Pi^{\mu\nu}, \tag{2.14}$$

where $T^{\mu\nu}_{(0)}$ is the stress-energy tensor for the perfect fluid eqs. (2.7) and (2.9) and $\Pi^{\mu\nu}$ represents the dissipative terms at all orders

$$\Pi^{\mu\nu} = \Pi^{\mu\nu}_{(1)} + \Pi^{\mu\nu}_{(2)} + \cdots, \tag{2.15}$$

where the subscripts indicate the order of our corrections. To make this clearer, we do as follows: by inspecting the zeroth order terms (perfect fluid), we notice that they do not contain derivatives of the fluid variables. Therefore, it is reasonable to assume that the dissipative terms appearing in $\Pi^{\mu\nu}$ will be described in terms of higher-order derivatives of the fluid variables. In fact, we will be working with higher order derivatives of u^μ. Thus, the order at the subscripts of the $\Pi^{\mu\nu}_{(n)}$ is just the order of the derivatives in u^μ.

The discussion above leads us to the conclusion that to study the dissipative terms, we must analyze the different combinations of derivatives of the velocity field. In order to achieve this goal, we decompose the fluid velocity derivatives into different terms. Moreover, we can give these different terms an actual physical meaning. A spacetime tensor can be covariantly split (see, for example, [53]) as a sum of a vector, a nonzero trace part and two traceless parts, one symmetric and one antisymmetric. This irreducible representation decomposes the tensor into components that are either parallel or orthogonal to the vector. We may apply this to the derivatives of the fluid velocity and write

$$\nabla^\mu u^\nu = -a^\mu u^\nu + \sigma^{\mu\nu} + \omega^{\mu\nu} + \frac{1}{d-1}\theta P^{\mu\nu}, \tag{2.16}$$

where we define, respectively, the divergence θ, the acceleration a^μ, shear $\sigma^{\mu\nu}$ and the vorticity $\omega^{\mu\nu}$ as

$$\theta = \nabla_\mu u^\mu = P^{\mu\nu}\nabla_\mu u_\nu, \tag{2.17a}$$

$$a^\mu = u^\nu \nabla_\nu u^\mu, \tag{2.17b}$$

$$\sigma^{\mu\nu} = \nabla^{(\mu}u^{\nu)} + u^{(\mu}a^{\nu)} - \frac{1}{d-1}\theta P^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\nabla_{(\alpha}u_{\beta)} - \frac{1}{d-1}\theta P^{\mu\nu}, \tag{2.17c}$$

$$\omega^{\mu\nu} = \nabla^{[\mu}u^{\nu]} + u^{[\mu}a^{\nu]}. \tag{2.17d}$$

The usefulness of this decomposition is that it will directly allow us to write the dissipative parts of the stress tensor in the derivative expansion. We thus investigate some properties of the elements defined above and take into account the Landau frame condition, which states that the dissipative contributions must be orthogonal to the fluid velocity:

$$u_\mu \Pi^{\mu\nu} = 0. \tag{2.18}$$

This already indicates that the acceleration must not contribute to $\Pi^{\mu\nu}_{(1)}$, since the acceleration term is parallel with u^μ. Other properties of these elements are

$$\sigma^{\mu\nu}u_\mu = 0 \qquad \omega^{\mu\nu}u_\mu = 0 \qquad \sigma^{\mu\rho}P_{\rho\nu} = \sigma^\mu_\nu \qquad \omega^{\mu\rho}P_{\rho\nu} = \omega^\mu_\nu$$
$$\sigma^\mu_\mu = 0 \qquad \omega^\mu_\mu = 0 \qquad a_\mu u^\mu = 0.$$

The other useful property that will lead us to the form of $\Pi^{\mu\nu}_{(1)}$ is the fact that we want $\Pi^{\mu\nu}$ to remain symmetric. This means that the vorticity $\omega^{\mu\nu}$ does not contribute. Therefore, we may write

$$\Pi^{\mu\nu}_{(1)} = -2\eta\sigma^{\mu\nu} - \zeta\theta P^{\mu\nu}, \tag{2.20}$$

where the parameters η and ζ are, respectively, the *shear viscosity* and the *bulk viscosity*.

The currents must also be modified to include the terms Υ^μ_I. In the Landau frame, these dissipative contributions must be orthogonal to the fluid velocity, which for the current means

$$\Upsilon^\mu_I u_\mu = 0. \tag{2.21}$$

2.2.4 Scaling limit

With the discussion on dissipative terms in the previous section, we may write the equations of relativistic hydrodynamics at first order as

$$\nabla_\mu T^{\mu\nu} = 0, \tag{2.22a}$$

$$T^{\mu\nu} = \rho u^\mu u^\nu + P P^{\mu\nu} - 2\eta\sigma^{\mu\nu} - \zeta\theta P^{\mu\nu}, \tag{2.22b}$$

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \tag{2.22c}$$

$$\sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha\right) - \frac{1}{d-1}\theta P^{\mu\nu}, \tag{2.22d}$$

$$\theta = \nabla^\beta u_\beta. \tag{2.22e}$$

These are the equations for a fluid propagating on a space with metric $g_{\mu\nu}$. Now, following [51], we wish to study the motion of a fluid propagating in a space with metric given by $G_{\mu\nu} = g_{\mu\nu} + H_{\mu\nu}$, where the background metric has the form

$$g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + g_{ij}\mathrm{d}x^i \mathrm{d}x^j, \tag{2.23}$$

and $H_{\mu\nu}$ is an arbitrary small fluctuation. The goal of this section is to show that the perturbations $H_{\mu\nu}$ will be related to forcing terms in fluid dynamics. Moreover, we are going to show that the equation $\nabla_\mu T^{\mu\nu} = 0$ will be reduced to the Navier-Stokes equation on $g_{\mu\nu}$, while the contributions from $H_{\mu\nu}$ will be related to a function $A_i(t, x_i)$ which will later be identified as the electromagnetic potential, so that the forcing terms will be the force applied on a charged fluid by this background electromagnetic potential. Therefore, when we write $G_{\mu\nu} = g_{\mu\nu} + H_{\mu\nu}$, we wish to see the fluid flow on $G_{\mu\nu}$ as a forced flow on the space $g_{\mu\nu}$.

We denote by $\tilde{u}^{\mu\nu}$ the fluid velocity in $G_{\mu\nu}$, while u^μ refers to the fluid velocity referred to $g_{\mu\nu}$. We assume that

$$\tilde{u}^\mu = \frac{1}{\sqrt{V^2}}(1,\vec{V}) = \frac{1}{\sqrt{G_{\alpha\beta}V^\alpha V^\beta}}(1,\vec{V}), \tag{2.24}$$

in which V^α has components $(1,\vec{V})$ in d dimensions, while V has components V^i in $d-1$ dimensions. Now, we expand $\tilde{u}^\mu$ to first order in $H_{\alpha\beta}$

$$\begin{aligned}\tilde{u}^\mu = u^\mu + \delta u^\mu &= \frac{1}{\sqrt{1-g_{ij}V^iV^j}}(1,\vec{V}) - u^\mu\frac{u^\alpha u^\beta H_{\alpha\beta}}{2} \\ &= u^\mu\left(1-\frac{u^\alpha u^\beta H_{\alpha\beta}}{2}\right),\end{aligned} \tag{2.25}$$

where

$$u^\mu = \frac{1}{\sqrt{1-g_{ij}V^iV^j}}(1,\vec{V}), \tag{2.26}$$

$$\delta u^\mu = -u^\mu \frac{u^\alpha u^\beta H_{\alpha\beta}}{2}. \tag{2.27}$$

We now wish to treat fluctuations about the uniform fluid at rest, with pressure P_0 and energy density ρ_0, by setting

$$H_{tt} = \varepsilon^2 h_{tt}(\varepsilon^2 t, \varepsilon\vec{x}), \tag{2.28a}$$

$$H_{ti} = \varepsilon A_i(\varepsilon^2 t, \varepsilon\vec{x}), \tag{2.28b}$$

$$H_{ij} = \varepsilon^2 h_{ij}(\varepsilon^2 t, \varepsilon\vec{x}), \tag{2.28c}$$

$$V^i = \varepsilon v^i(\varepsilon^2 t, \varepsilon\vec{x}), \tag{2.28d}$$

$$\frac{P-P_0}{\rho_0+P_0} = \varepsilon^2 p(\varepsilon^2 t, \varepsilon\vec{x}), \tag{2.28e}$$

where $\rho_e \equiv \rho_0 + P_0$ was chosen for further convenience. Also, ε is taken to be arbitrarily small. With the rescaling defined above, we look at the equations of motion $\nabla_\mu T^{\mu\nu} = 0$ under the scaling we have just described. In particular, we want to calculate $\nabla_\mu T^{\mu t}$ and $\nabla_\mu T^{\mu i}$.

To do this, we use eq. (2.25) in eqs. (2.22a) to (2.22e), then apply the rescalings defined in eqs. (2.28a) to (2.28e) and collect the terms order by order in ε. This calculation is straightforward, but very long and tedious, and we just quote the results here,

$$\nabla_\mu T^{\mu t} = \varepsilon^2[\rho_e(\nabla_i v^i)] + O(\varepsilon^4), \tag{2.29}$$

where $\rho_e = \rho_0 + P_0$. Analogously, for $T^{\mu i}$ we have

$$\nabla_\mu T^{\mu i} = \varepsilon^3 \left[\rho_e \nabla^i p + \rho_e \nabla_\mu(v^i v^\mu) - 2\eta\nabla_j\left(\frac{\nabla^i v^j + \nabla^j v^i}{2} - g^{ij}\frac{\vec{\nabla}\cdot\vec{v}}{d-1}\right) - \zeta\nabla_i\vec{\nabla}\cdot\vec{v} - f^i\right] + O(\varepsilon^5), \tag{2.30}$$

where

$$f^i = \rho_e\left(\frac{\partial_i h_{tt}}{2} - \partial_t A_i - \frac{\partial_j(\sqrt{g}A_i v^j)}{\sqrt{g}} + v^j(\partial^i A_j)\right). \tag{2.31}$$

Equation (2.29) reduces to $\nabla_i v^i = 0$ for small ε, while eq. (2.30) can be simplified by using the well-known relation (see, for example, [44])

$$\nabla_i v^i = \frac{\partial_i(\sqrt{g}v^i)}{\sqrt{g}} = 0, \tag{2.32}$$

so that

$$\frac{\partial_j(\sqrt{g}A_i v^j)}{\sqrt{g}} = A_i\left[\frac{\partial_i(\sqrt{g}v^i)}{\sqrt{g}}\right] + v^j(\partial_i A_j) = v^j(\partial_i A_j) \tag{2.33}$$

and

$$f^i = \rho_e\left(\frac{\partial_i h_{tt}}{2} - \partial_t A_i + v^j(\partial^i A_j - \partial_j A_i)\right). \tag{2.34}$$

It can be shown [51] that eq. (2.30) with eq. (2.34) may be written as

$$\partial_t v^i + (\vec{v}\cdot\nabla)v^i - \nu(\nabla^2 v^i + R^i_j v^j) + \nabla^i p = \frac{\partial^i h_{tt}}{2} - \partial_t A^i + F^i_j v^j. \tag{2.35}$$

To proceed, we make some redefinitions of our variables. First, we write

$$A_i = a_i + \nabla_i \chi, \tag{2.36}$$

where χ is chosen so as to have $\nabla_i a^i = 0$. Now, we define the "effective" pressure as

$$p_e = p - \frac{1}{2}h_{tt} + \dot{\chi}. \tag{2.37}$$

With these redefinitions, eq. (2.35) becomes

$$\partial_t v_i + (\vec{v}\cdot\nabla)v_i - \nu(\nabla^2 v_i + R_{ij}v^j) + \nabla_i p_e = -\partial_t a^i - v^j F_{ji}, \tag{2.38}$$

which are the Navier-Stokes equations with forcing terms which are generated by an electromagnetic field (on the right-hand side) as well as the curvature term given by the Ricci tensor. We note that in the absence of these terms, eq. (2.38) reduces to

$$\partial_t v_i + (v^j\partial_j)v_i - \nu\partial^2 v_i + \partial_i p_e = 0, \tag{2.39}$$

which is the "usual" Navier-Stokes equation and the one we are going to use throughout this work. This equation, together with eq. (2.29), shows that the scalings eqs. (2.28a) to (2.28e) allow us to treat the incompressible Navier-Stokes equations as a non-relativistic limit of the relativistic fluid equations of motion at first order when considering dissipative terms in the stress-energy tensor.

2.2.5 Non-relativistic hydrodynamics

In the rest of this dissertation, we intend to work on the non-relativistic limit, and we are not going to consider the forcing terms on the right-hand side of the "full" Navier-Stokes equation eq. (2.38). For convenience, we rewrite the incompressible Navier-Stokes equations below,

$$\partial_t v_i + v^j \partial_j v_i - \eta \partial^2 v_i + \partial_i p = 0, \tag{2.40a}$$

$$\partial_i v^i = 0. \tag{2.40b}$$

Now, we apply to eqs. (2.40a) and (2.40b) the very same scaling which we used to derive the Navier-Stokes equation as a non-relativistic limit of the hydrodynamical equations. Namely, we write

$$v_i^{(\varepsilon)}(t,\vec{x}) = \varepsilon v_i(\varepsilon^2 t, \varepsilon \vec{x}) \text{ and} \tag{2.41a}$$

$$p^{(\varepsilon)}(t,\vec{x}) = \varepsilon^2 p(\varepsilon^2 t, \varepsilon \vec{x}), \tag{2.41b}$$

so that

$$\begin{aligned}
\partial_t v_i^{(\varepsilon)} &= \varepsilon \partial_t v_i(\varepsilon^2 t, \varepsilon \vec{x}) = \varepsilon^3 \partial_t v_i, \\
\partial_i v_i^{(\varepsilon)} &= \varepsilon \partial_i v_i(\varepsilon^2 t, \varepsilon \vec{x}) = \varepsilon^2 \partial_i v_i, \\
v^{j(\varepsilon)} \partial_j v_i^{(\varepsilon)} &= \varepsilon v^j(\varepsilon^2 t, \varepsilon \vec{x}) \varepsilon^2 \partial_j v_i = \varepsilon^3 v^j \partial_j v_i, \\
\partial^2 v_i^{(\varepsilon)} &= \partial_i [\varepsilon \partial_i v_i(\varepsilon^2 t, \varepsilon \vec{x})] = \varepsilon^3 \partial^2 v_i, \\
\partial_i p^{(\varepsilon)} &= \varepsilon^2 \partial_i p(\varepsilon^2 t, \varepsilon \vec{x}) = \varepsilon^3 \partial_i p.
\end{aligned}$$

These, when inserted in eqs. (2.40a) and (2.40b), result in

$$\varepsilon^3 (\partial_t v_i + v^j \partial_j v_i - \eta \partial^2 v_i + \partial_i p) = 0, \tag{2.42a}$$

$$\varepsilon^2 (\partial_i v^i) = 0. \tag{2.42b}$$

What this means is that the scaling operation shown in the previous section is also a symmetry of the unforced incompressible Navier-Stokes equations. A fluid obeying these equations and the derivative expansion in ε will be an integral part of our discussion of fluid/gravity correspondence in the context of a Rindler spacetime, starting in chapter 3.

Chapter 3

Fluid/gravity Correspondence on a Flat Spacetime

In [24], the authors found a relation between incompressible non-relativistic fluids in $d+1$ dimensions satisfying the Navier-Stokes equations and Ricci-flat metrics in $d+2$ dimensions. Working on flat space, they found that a Ricci-flat metric exists if the Brown-York tensor on a hypersurface Σ_c at the boundary of this space satisfies the equations for an incompressible Navier-Stokes fluid. This relation was found by studying the effect of perturbations of the extrinsic curvature of Σ_c and analyzing them in a hydrodynamic limit, which enabled them to construct the metric up to third order in the hydrodynamic expansion.

Instead of reviewing this work, we opt to follow closely [25], which develops an algorithmic procedure that generalizes the result of [24] and calculates the solutions order by order in the ε-expansion. As discussed in [25], in order for this procedure to be valid for all orders in the expansion, it will be necessary to add corrections to the Navier-Stokes at higher orders. However, since we will focus on the expansion up to order ε^3, these corrections will not be necessary.

One of the main merits of the works of [24, 25], is that they provide us with an extension of a procedure which was initially done within the context of the AdS/CFT correspondence. This hopefully sheds light on the possibility of new insights about holography.

3.1 Algorithmic procedure

We start with the metric of flat space written in ingoing Rindler coordinates (see chapter 1), which we rewrite here for convenience:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -r d\tau^2 + 2 d\tau dr + dx_i dx^i. \tag{3.1}$$

We recall that the ingoing Rindler coordinates $x^\mu = (\tau, r, x^i)$ are related to the Cartesian chart (t, z, x^i) by $z^2 - t^2 = 4r$ and $z + t = e^{\tau/2}$. This metric satisfies three features:

1. They admit a hypersurface Σ_c defined by a constant $r = r_c$, where r is the coordinate into the bulk. The induced metric at Σ_c is given by

$$\gamma_{ab}dx^a dx^b = -r_c d\tau^2 + dx_i dx^i, \tag{3.2}$$

where $x^a = (\tau, x^i)$ and $\sqrt{r_c}$ is the speed of light;

2. The Brown-York tensor [47] (see also chapter 1) on Σ_c is that of a perfect fluid

$$T_{ab} = 2(K\gamma_{ab} - K_{ab}), \tag{3.3}$$

where K_{ab} is the extrinsic curvature of Σ_c and $K = \gamma^{ab}K_{ab}$;

3. Diffeomorphisms applied to eq. (3.1) return metrics which are stationary in τ and homogeneous in the x^i directions.

As shown in the appendix B of [25], the set of diffeomorphisms that can be applied to eq. (3.1) while preserving the three features described above is a boost, followed by a shift and a rescaling of the coordinates.

The boost is given by

$$\sqrt{r_c}\tau \to \gamma\sqrt{r_c}\tau - \gamma\beta_i x^i, \tag{3.4a}$$

$$x^i \to x^i - \gamma\beta^i\sqrt{r_c}\tau + (\gamma - 1)\frac{\beta^i\beta_j}{\beta^2}x^j, \tag{3.4b}$$

where $\gamma = (1-\beta^2)^{-1/2}$ and $\beta_i = r_c^{-1/2}v_i$. Note that β_i is constant. The second transformation corresponds to a translation in r and a subsequent rescaling in τ:

$$r \to r - r_h, \tag{3.5a}$$

$$\tau \to \frac{\tau}{\sqrt{1 - r_h/r_c}}, \tag{3.5b}$$

which shifts the horizon from $r=0$ to $r=r_h$. Since these are linear transformations, it is possible to apply them to eq. (3.1) in any order. Applying the eqs. (3.5a) and (3.5b), we have

$$\begin{aligned} \mathrm{d}s^2 = & \frac{1}{1-\frac{v^2}{r_c}}\left(v^2-\frac{r-r_h}{1-\frac{r_h}{r_c}}\right)\mathrm{d}\tau^2+\frac{2\gamma}{\sqrt{1-\frac{r_h}{r_c}}}\mathrm{d}\tau\mathrm{d}r-\frac{2\gamma v_i}{r_c\sqrt{1-\frac{r_h}{r_c}}}\mathrm{d}x^i\mathrm{d}r \\ & +\frac{2v_i}{1-\frac{v^2}{r_c}}\left(\frac{r-r_c}{r_c-r_h}\right)\mathrm{d}x^i\mathrm{d}\tau+\left[\delta_{ij}-\frac{v_iv_j}{r_c^2\left(1-\frac{v^2}{r_c}\right)}\left(\frac{r-r_c}{1-\frac{r_h}{r_c}}\right)\right]\mathrm{d}x^i\mathrm{d}x^j. \end{aligned} \tag{3.6}$$

This equation is just the eq. (3.1) written in a more complicated way. Indeed, the Brown-York tensor has the perfect fluid form with the identifications

$$\rho=0, \qquad p=\frac{1}{\sqrt{r_c-r_h}}, \qquad u^a=\frac{1}{\sqrt{r_c-v^2}}(1,v_i). \tag{3.7}$$

Before proceeding, we note that if we apply the Hamiltonian constraint [44] $R_{\mu\nu}n^\mu n^\nu = 0$, where $R_{\mu\nu}$ is the Ricci tensor on the bulk and n^μ is the vector normal to the hypersurface Σ_c, we have[1]

$$dT_{ab}T^{ab}=T^2. \tag{3.8}$$

This is done by writing [44]

$$G_{\mu\nu}n^\mu n^\nu=\frac{1}{2}(\mathcal{R}-K_{ab}K^{ab}+K^2)=0, \tag{3.9}$$

where $\mathcal{R}$ is the Ricci scalar on Σ_c. Since we want Σ_c to be flat, $\mathcal{R}$ vanishes and we arrive at eq. (3.8). Inserting the perfect fluid form eq. (3.3) into eq. (3.8), we get

$$\rho=0, \tag{3.10a}$$

$$\rho=\frac{-2d}{d-1}p. \tag{3.10b}$$

The first is the case we have been working so far, while the second is associated with the Taub geometry [27,57],

$$\mathrm{d}s^2=-\frac{\mathcal{A}}{r^{d-1}}\mathrm{d}t^2+2\mathrm{d}t\mathrm{d}r+r^2\mathrm{d}x_i\mathrm{d}x^i. \tag{3.11}$$

We will show in the next chapter that these two results are actually related by an Ehlers transformation.

Now, going back to our previous discussion, we wish to investigate the hydrodynamic system which is dual to the metric given by eq. (3.6). We want to do this in the context of

[1] The d here refers to dimension. It is not a differential operator.

the ε-expansions, that is, we need to consider the metric perturbations within the hydrodynamic limit. This will be achieved by promoting the velocity v_i and the pressure p to functions depending on space and time. We write [51]

$$v_i^{(\varepsilon)}(\tau,\vec{x}) = \varepsilon v_i(\varepsilon^2\tau, \varepsilon\vec{x}), \qquad P^{(\varepsilon)}(\tau,\vec{x}) = \varepsilon^2 P(\varepsilon^2\tau, \varepsilon\vec{x}), \tag{3.12}$$

where P is the non-relativistical pressure, which is related to r_h by $r_h = 2P^{(\varepsilon)} + O(\varepsilon^4)$. Recall from chapter 2 that the scalings defined above are such that the Navier-Stokes equations still hold for v_i and P. Also, note that we are scaling to larger times and distances while scaling down the amplitudes, so that eq. (3.12) corresponds to small perturbations in the hydrodynamic limit, as discussed before. Finally, by promoting the parameters to functions depending on the coordinates, we may no longer guarantee that our results will satisfy the Einstein equations.

If we treat $v_i = v_i^{(\varepsilon)}$ and $p = r_c^{-1/2} + r_c^{-3/2} P^{(\varepsilon)}$, where $v_i^{(\varepsilon)}$ and $P^{(\varepsilon)}$ are described by the scaling above, as small fluctuations, upon expansion we obtain

$$\begin{aligned} ds^2 &= -r d\tau^2 + 2 d\tau dr + dx_i dx^i \\ &\quad - 2\left(1 - \frac{r}{r_c}\right) v_i^{(\varepsilon)} dx^i d\tau - \frac{2 v_i^{(\varepsilon)}}{r_c} dx^i dr \\ &\quad + \left(1 - \frac{r}{r_c}\right)\left[(v^{2(\varepsilon)} + 2P^{(\varepsilon)}) d\tau^2 + \frac{v_i^{(\varepsilon)} v_j^{(\varepsilon)}}{r_c} dx^i dx^j\right] + \left(\frac{v^{2(\varepsilon)} + 2P^{(\varepsilon)}}{r_c}\right) d\tau dr \\ &\quad + O(\varepsilon^3), \end{aligned} \tag{3.13}$$

where each line corresponds to an order in the ε expansion, with the first line being at order zero. This metric eq. (3.13) preserves the γ_{ab} induced in Σ_c. To see this, it is only necessary to do $r = r_c$ and $dr = 0$ in the above equation. This is the seed metric obtained in [24], which does satisfy the Einstein equations. The Brown-York tensor on Σ_c constructed from this metric is[2], up to order ε^2,

$$T_{ab} dx^a dx^b = \frac{d\vec{x}^2}{\sqrt{r_c}} - \frac{2 v_i}{\sqrt{r_c}} dx^i d\tau + \frac{v^2}{\sqrt{r_c}} d\tau^2 + r_c^{-3/2}[P\delta_{ij} + v_i v_j - r_c \partial_i v_j] dx^i dx^j + O(\varepsilon^3). \tag{3.14}$$

If we now attempt to apply this procedure and expand the velocity and pressure to orders > 2, we will not be able to arrive at a Ricci flat solution. Indeed, at order ε^3, the terms $R_{\tau i}^{(3)}$ and $R_{ri}^{(3)}$ will not vanish, as we would expect. Thus, if we wish to construct a

[2] We note that our result does not have the factor of 2 multiplying $r_c \partial_i v_j$ in the spatial term.

general procedure to find metrics that satisfy the Einstein equations at any order, we need to add corrections to our results, so that these corrections will cancel out the extra terms.

3.1.1 Higher-order corrections

To fix the problem discussed above and guarantee that the Einstein equations will be satisfied at orders larger than 2 in ε, it is necessary to add corrections to the seed metric eq. (3.13), which is accomplished with an algorithmic procedure developed in [25].

We begin by assuming that the bulk metric is known at order ε^{n-1}, so that the first nonzero terms of the Ricci tensor appear at order ε^n. Next, we add new terms $g^{(n)}_{\mu\nu}$ to the metric at order ε^n, so that the Ricci tensor at order ε^n will be written as

$$R^{(n)}_{\mu\nu} = \hat{R}^{(n)}_{\mu\nu} + \delta R^{(n)}_{\mu\nu}, \tag{3.15}$$

where $\hat{R}^{(n)}_{\mu\nu}$ are the contributions at order ε^n arising from the metric up to order ε^{n-1} and $\delta R^{(n)}_{\mu\nu}$ are the contributions from the corrections to the metric $g^{(n)}_{\mu\nu}$. The idea is that, in order for the Einstein equations to be satisfied, we must have

$$R^{(n)}_{\mu\nu} = \hat{R}^{(n)}_{\mu\nu} + \delta R^{(n)}_{\mu\nu} = 0. \tag{3.16}$$

Before proceeding, we note that since

$$\partial_r \sim \varepsilon^0 \qquad \partial_i \sim \varepsilon^1 \qquad \partial_\tau \sim \varepsilon^2,$$

only derivatives of $g^{(n)}_{\mu\nu}$ in r will appear on $\delta R^{(n)}_{\mu\nu}$, since $g^{(n)}_{\mu\nu}$ is of order ε^n by construction.

We add n-th order terms $g^{(n)}_{\mu\nu}$ to the original metric, these new terms will produce the following corrections to the Ricci tensor:

$$\delta R^{(n)}_{rr} = -\frac{1}{2}\partial_r^2 g^{(n)}_{ii} \tag{3.17a}$$

$$\delta R^{(n)}_{ij} = -\frac{1}{2}\partial_r(r\partial_r g^{(n)}_{ij}) \tag{3.17b}$$

$$\delta R^{(n)}_{\tau i} = -r\delta R^{(n)}_{ri} = -\frac{r}{2}\partial_r^2 g^{(n)}_{\tau i} \tag{3.17c}$$

$$\delta R^{(n)}_{\tau\tau} = -r\delta R^{(n)}_{\tau\tau} = -\frac{r}{4}[\partial_r(r g^{(n)}_{rr}) + 2\partial_r g^{(n)}_{\tau\tau} - \partial_r g^{(n)}_{ii} + 2\partial_r^2 g^{(n)}_{\tau\tau}] \tag{3.17d}$$

A particular solution $\tilde{g}^{(n)}_{\mu\nu}$ that satisfies $R^{(n)}_{\mu\nu} = \hat{R}^{(n)}_{\mu\nu} + \delta R^{(n)}_{\mu\nu} = 0$ is

$$\tilde{g}^{(n)}_{r\mu} = 0, \tag{3.18a}$$

$$\tilde{g}^{(n)}_{\tau\tau} = \beta^{(n)}_1(\tau,\vec{x}) + (1 - r/r_c)\beta^{(n)}_2(\tau,\vec{x}) + \int_r^{r_c} \mathrm{d}r' \int_{r'}^{r_c} \mathrm{d}r'' (\hat{R}^{(n)}_{ii} - r\hat{R}^{(n)}_{rr} - 2\hat{R}^{(n)}_{r\tau}), \tag{3.18b}$$

$$\tilde{g}^{(n)}_{\tau i} = \beta^{(n)}_{3i}(\tau,\vec{x}) + (1 - r/r_c)\beta^{(n)}_{4i}(\tau,\vec{x}) - 2\int_r^{r_c} \mathrm{d}r' \int_{r'}^{r_c} \mathrm{d}r'' \hat{R}^{(n)}_{ri}, \tag{3.18c}$$

$$\tilde{g}^{(n)}_{ij} = \beta^{(n)}_{5ij}(\tau,\vec{x}) + \ln(r/r_c)\beta^{(n)}_{6ij}(\tau,\vec{x}) - 2\int_r^{r_c} \mathrm{d}r' \frac{1}{r'} \int_r^{r_c} \mathrm{d}r'' \hat{R}^{(n)}_{ij}. \tag{3.18d}$$

The above solution is not unique. Recalling that $g^{(n)}_{\mu\nu} \to g^{(n)}_{\mu\nu} + \nabla_\mu \xi^{(n)}_\nu + \nabla_\nu \xi^{(n)}_\mu$, we may allow for gauge transformations $\xi^{(n)\mu}(r,\tau,\vec{x})$ at order ε^n, while keeping our result unchanged, that is, while still allowing our results to satisfy eq. (3.16).

As an example,

$$g^{(n)}_{ri} = \tilde{g}^{(n)}_{ri} + \partial_r \xi^{(n)}_i + \partial_i \xi^{(n)}_r - 2\Gamma^\lambda_{ri}\xi^{(n)}_\lambda = \partial_r \xi^{(n)}_i. \tag{3.19}$$

The first term vanishes because $\tilde{g}^{(n)}_{r\mu} = 0$, the third because the derivative in x_i raises the order in ε and the last term vanishes because the only non-vanishing Christoffel symbols are $\Gamma^\tau_{\tau\tau} = \frac{1}{2}, \Gamma^r_{\tau\tau} = \frac{r}{2}$ and $\Gamma^r_{\tau r} = \Gamma^r_{r\tau} = -\frac{1}{2}$.

We must also take into account field redefinitions $\delta v^{(n)}_i(\tau,\vec{x})$ and $\delta P^{(n)}(\tau,\vec{x})$ of velocity and pressure at order ε^n. These terms are obtained by looking how v_i and P appear linearly in the seed solution, and we must guarantee that they indeed remain of order ε^n. For our example eq. (3.19), we will have

$$g^{(n)}_{ri} = \partial_r \xi^{(n)}_i - \frac{1}{r_c}\delta v^{(n)}_i. \tag{3.20}$$

With these considerations, we may apply this reasoning to the remaining $g^{(n)}_{\mu\nu}$. The result is

$$g^{(n)}_{rr} = 2\partial_r \xi^{(n)\tau}, \tag{3.21a}$$

$$g^{(n)}_{r\tau} = -r\partial_r \xi^{(n)\tau} + \partial_r \xi^{(n)r} + \frac{1}{r_c}\delta P^{(n)}, \tag{3.21b}$$

$$g^{(n)}_{ri} = \partial_r \xi^{(n)}_i - \frac{1}{r_c}\delta v^{(n)}_i, \tag{3.21c}$$

$$g^{(n)}_{\tau\tau} = \tilde{g}^{(n)}_{\tau\tau} - \xi^{(n)r} + (1 - r/r_c)2\delta P^{(n)}, \tag{3.21d}$$

$$g^{(n)}_{\tau i} = \tilde{g}^{(n)}_{\tau i} - (1 - r/r_c)\delta v^{(n)}_i, \tag{3.21e}$$

$$g^{(n)}_{ij} = \tilde{g}^{(n)}_{ij}. \tag{3.21f}$$

Since our redefinitions $\delta v_i^{(n)}$ and $\delta P^{(n)}$ and gauge transformations do not contribute to $\delta R_{\mu\nu}^{(n)}$, the $g_{\mu\nu}^{(n)}$ above still solve the Einstein equations.

We can now choose a gauge in which $g_{r\mu}^{(n)} = 0$ for all $n > 2$, which is a useful choice because we don't have boundary conditions at Σ_c for $g_{r\mu}^{(n)}$. This choice implies that the only non-vanishing $g_{r\mu}$ of the full metric are those already known from our seed metric, namely

$$g_{rr} = 0, \qquad g_{r\tau} = 1 + \frac{v^2}{2r_c} + \frac{P}{r_c}, \qquad g_{ri} = -\frac{v_i}{r_c}. \tag{3.22}$$

For the above gauge to be valid, we are forced to fix

$$\xi^{(n)r} = (1 - r/r_c)\delta P^{(n)} + \tilde{\xi}^{(n)r}(\tau, \vec{x}), \tag{3.23a}$$

$$\xi^{(n)\tau} = \tilde{\xi}^{(n)\tau}(\tau, \vec{x}), \tag{3.23b}$$

$$\xi_i^{(n)} = -(1 - r/r_c)\delta v_i^{(n)} + \tilde{\xi}_i^{(n)}(\tau, \vec{x}). \tag{3.23c}$$

Our solution then becomes

$$g_{r\mu}^{(n)} = 0, \tag{3.24a}$$

$$\begin{aligned} g_{\tau\tau}^{(n)} &= \beta_1^{(n)}(\tau, \vec{x}) - \tilde{\xi}^{(n)r} + (1 - r/r_c)(\beta_2^{(n)}(\tau, \vec{x}) + \delta P^{(n)}) \\ &\quad + \int_r^{r_c} \mathrm{d}r' \int_{r'}^{r_c} \mathrm{d}r'' (\hat{R}_{ii}^{(n)} - r\hat{R}_{rr}^{(n)} - 2\hat{R}_{r\tau}^{(n)}), \end{aligned} \tag{3.24b}$$

$$g_{\tau i}^{(n)} = \beta_{3i}^{(n)}(\tau, \vec{x}) + (1 - r/r_c)(\beta_{4i}^{(n)}(\tau, \vec{x}) - 2\delta v_i^{(n)}) - 2\int_r^{r_c} \mathrm{d}r' \int_{r'}^{r_c} \mathrm{d}r'' \hat{R}_{ri}^{(n)}, \tag{3.24c}$$

$$g_{ij}^{(n)} = \beta_{5ij}^{(n)}(\tau, \vec{x}) + \ln(r/r_c)\beta_{6ij}^{(n)}(\tau, \vec{x}) - 2\int_r^{r_c} \mathrm{d}r' \frac{1}{r'} \int_{r^*}^{r_c} \mathrm{d}r'' \hat{R}_{ij}^{(n)}. \tag{3.24d}$$

To treat the remaining β functions, we need to impose the boundary condition $g_{ab}^{(n)} = 0$, so that the induced metric on Σ_c remains fixed. This implies that

$$\beta_1^{(n)}(\tau, \vec{x}) = \tilde{\xi}^{(n)r}, \qquad \beta_{3i}^{(n)}(\tau, \vec{x}) = \beta_{5ij}^{(n)}(\tau, \vec{x}) = 0. \tag{3.25}$$

In order to fix the term $\beta_{6ij}^{(n)}(\tau, \vec{x})$, we first note that it is useful to choose the lower limit in the $\hat{R}_{ij}^{(n)}$ integral to be zero, because $\hat{R}_{ij}^{(n)}$ is a polynomial in r, and this choice guarantees that no other logarithmic term other than the one associated with $\beta_{6ij}^{(n)}$ will appear when we do the $1/r'$ integral. However, the presence of this logarithmic term is problematic, because we wish the metric to remain regular, at least order by order, at $r = 0$. Therefore, if we want to satisfy these conditions, we must set

$$\beta_{6ij}^{(n)}(\tau, \vec{x}) = 0. \tag{3.26}$$

We note that if $\hat{R}^{(n)}_{\mu\nu}$ is regular at $r = 0$, then $g^{(n)}_{\mu\nu}$ will itself be regular at $r = 0$, as it depends on $\hat{R}^{(n)}_{\mu\nu}$. Now, since $\hat{R}^{(n+1)}_{\mu\nu}$ arises from $g^{(n)}_{\mu\nu}$, we may guarantee that $g^{(n)}_{\mu\nu}$ will be regular at every n, because, as we will show below, $\hat{R}^{(3)}_{\mu\nu}$ is regular.

Now that the $\beta^{(n)}(\tau,\vec{x})$ have been fixed, we may finally write the general solution for the new part of the bulk metric at order n as

$$g^{(n)}_{r\mu} = 0, \tag{3.27a}$$

$$g^{(n)}_{\tau\tau} = (1 - r/r_c)F^{(n)}_{\tau}(\tau,\vec{x}) + \int_r^{r_c} \mathrm{d}r' \int_{r'}^{r_c} \mathrm{d}r'' (\hat{R}^{(n)}_{ii} - r\hat{R}^{(n)}_{rr} - 2\hat{R}^{(n)}_{r\tau}), \tag{3.27b}$$

$$g^{(n)}_{\tau i} = (1 - r/r_c)F^{(n)}_{i}(\tau,\vec{x}) - 2\int_r^{r_c} \mathrm{d}r' \int_{r'}^{r_c} \mathrm{d}r'' \hat{R}^{(n)}_{ri}, \tag{3.27c}$$

$$g^{(n)}_{ij} = -2\int_r^{r_c} \mathrm{d}r' \frac{1}{r'} \int_r^{r_c} \mathrm{d}r'' \hat{R}^{(n)}_{ij}, \tag{3.27d}$$

with

$$F^{(n)}_{\tau}(\tau,\vec{x}) = \beta^{(n)}_{2}(\tau,\vec{x}) + \delta P^{(n)}(\tau,\vec{x}), \quad F^{(n)}_{i}(\tau,\vec{x}) = \beta^{(n)}_{4i}(\tau,\vec{x}) - 2\delta v^{(n)}_{i}(\tau,\vec{x}). \tag{3.28}$$

At this point it is extremely useful to note an interesting feature of this procedure, which greatly simplifies our future calculations. First of all, we recall that

$$v_i \sim \varepsilon, \quad P \sim \varepsilon^2, \quad \partial_r \sim \varepsilon^0, \quad \partial_i \sim \varepsilon, \quad \partial_\tau \sim \varepsilon^2, \tag{3.29}$$

which in turn means that

$$\partial_\tau P \sim \varepsilon^4, \quad \partial_\tau v_i \sim \varepsilon^3, \quad \partial_i P \sim \varepsilon^3, \quad \partial_i v_i \sim \varepsilon^2, \quad \partial_r P \sim \varepsilon^2, \quad \partial_r v_i \sim \varepsilon. \tag{3.30}$$

From eq. (3.30) it is not difficult to convince ourselves that any vector ($\partial_\tau v_i$, $\partial_i P$, $\partial_r v_i$ and so on) constructed from P, v_i and their derivatives will *necessarily* be of *odd* order ε, while any scalar or rank-two tensor ($\partial_\tau P$, $\partial_i v_i$, $\partial_r P$ and so on) constructed from them will be of *even* order in ε. In turn, this implies that, at orders ε^n for odd n, the components $\hat{R}^{(n)}_{rr}$, $\hat{R}^{(n)}_{\tau\tau}$ and $\hat{R}^{(n)}_{ij}$ of the Ricci tensor must be zero, just like the arbitrary function $F^{(n)}_{\tau}$, because these terms are either scalar or tensors. Therefore, only $g^{(n)}_{\tau i}$ will survive the integration scheme at odd orders in the ε expansion (see eqs. (3.27a) to (3.27d)). This same reasoning tells us that at *even* orders in ε^n, $\hat{R}^{(n)}_{\tau i}$ and $F^{(n)}_{i}$ will be zero, since they are vectors and only scalars and rank-two tensors survive at even orders. Thus, the only nonzero metric components at even orders will be $g^{(n)}_{\tau\tau}$ and $g^{(n)}_{ij}$.

Now, it only remains to give a precise meaning to the functions $F^{(n)}_{\tau}(\tau,\vec{x})$ and $F^{(n)}_{i}(\tau,\vec{x})$. In order to do so, we first need to discuss the changes to the Brown-York tensor.

We proceed exactly as in the Ricci tensor case, that is, we wish to write the Brown-York tensor at order ε^n as

$$T_{ab}^{(n)} = \hat{T}_{ab}^{(n)} + \delta T_{ab}^{(n)}, \tag{3.31}$$

where $\hat{T}_{ab}^{(n)}$ are the contributions at order ε^n arising from the metric at order ε^{n-1} and $\delta T_{ab}^{(n)}$ the contributions due to the correction terms $g_{\mu\nu}^{(n)}$. The former are assumed to be known, whereas the latter may be calculated from the change in the extrinsic curvature of Σ_c due to $g_{\mu\nu}^{(n)}$:

$$\delta K_{ab}^{(n)} = \frac{1}{2}\mathcal{L}_N g_{ab}^{(n)} = \frac{1}{2}N^r\partial_r g_{ab}^{(n)} = \frac{1}{2}\sqrt{r_c}\partial_r g_{ab}^{(n)}|_{\Sigma_c}, \tag{3.32}$$

so that

$$\delta K_{\tau\tau}^{(n)} = -\frac{F_\tau^{(n)}(\tau,\vec{x})}{2\sqrt{r_c}}, \tag{3.33a}$$

$$\delta K_{\tau i}^{(n)} = -\frac{F_i^{(n)}(\tau,\vec{x})}{2\sqrt{r_c}}, \tag{3.33b}$$

$$\delta K_{ij}^{(n)} = \frac{1}{\sqrt{r_c}}\int_0^{r_c} \mathrm{d}r' \hat{R}_{ij}^{(n)}. \tag{3.33c}$$

The $\delta T_{ab}^{(n)}$ terms due to corrections $g_{\mu\nu}^{(n)}$ will then be

$$\delta T_{\tau\tau}^{(n)} = -2\sqrt{r_c}\int_0^{r_c} \mathrm{d}r' \hat{R}_{ij}^{(n)}, \tag{3.34a}$$

$$\delta T_{\tau i}^{(n)} = \frac{F_i^{(n)}(\tau,\vec{x})}{\sqrt{r_c}}, \tag{3.34b}$$

$$\delta T_{ij}^{(n)} = \frac{F_\tau^{(n)}(\tau,\vec{x})}{r_c^{3/2}}\delta_{ij} + \frac{2}{\sqrt{r_c}}\int_0^{r_c} \mathrm{d}r'(\delta_{ij}\hat{R}_{kk}^{(n)} - \hat{R}_{ij}^{(n)}), \tag{3.34c}$$

so that the full Brown-York tensor on Σ_c at order ε^n will be given by eq. (3.31).

By inspecting eq. (3.28), we note that the arbitrary $F_i^{(n)}(\tau,\vec{x})$ are related to redefinitions of the velocity field. We may fix the arbitrariness by defining the fluid velocity as the boost from the local frame to the lab frame, obeying

$$0 = h_a^b T_{bc} u^c, \qquad h_a^b = \delta_a^b + u^b u_a. \tag{3.35}$$

Working on the above equation yields (noting that scalar and rank-two tensors vanish at even orders while vectors vanish at odd orders)

$$0 = T_{i\tau}^{(n)} + T_{ij}^{(n-1)} v_j + \rho^{(n-1)} v_i, \tag{3.36}$$

where $\rho = T_{ab}u^a u^b$ is the energy density in the local rest frame. Using eq. (3.34b), we arrive at

$$\frac{F_i^{(n)}(\tau,\vec{x})}{\sqrt{r_c}} + \hat{T}_{i\tau}^{(n)} + T_{ij}^{(n-1)} v_j + \rho^{(n-1)} v_i = 0, \tag{3.37}$$

which allows us to calculate $F_i^{(n)}$ in terms of previously known quantities.

To fix $F_\tau^{(n)}(\tau,\vec{x})$, which is related to our redefinition of the pressure fluctuation, we define the pressure fluctuation P in such a way that the isotropic part of the stress-energy tensor T_{ij} is fixed to be

$$T_{ij}^{\text{isotropic}} = \left(\frac{1}{\sqrt{r_c}} + \frac{P}{r_c^{3/2}} \right) \delta_{ij}. \tag{3.38}$$

In [25] it has been shown that it is possible to make other gauge choices and therefore define the pressure in a different way. However, this discussion is not very important in our case, because we are going to work at third order in ε, which means that, according to our discussion surrounding eq. (3.30), $F_\tau^{(3)} = 0$.

3.1.2 Example: third-order calculations

We are going to apply the procedure developed above to the metric at third order in ε. By adding $g_{\mu\nu}^{(3)}$ to our seed metric eq. (3.13), the Ricci tensor $R_{\mu\nu}^{(3)}$ will now be

$$R_{\tau\tau}^{(3)} = -\frac{r}{4}\partial_r(rg_{rr}^{(3)} - 2g_{\tau r}^{(3)} + 2\partial_r g_{\tau\tau}^{(3)} + g_{xx}^{(3)} + g_{yy}^{(3)} + g_{zz}^{(3)}), \tag{3.39a}$$

$$R_{\tau r}^{(3)} = \frac{1}{4}\partial_r(rg_{rr}^{(3)} - 2g_{\tau r}^{(3)} + 2\partial_r g_{\tau\tau}^{(3)} + g_{xx}^{(3)} + g_{yy}^{(3)} + g_{zz}^{(3)}), \tag{3.39b}$$

$$\begin{aligned} R_{\tau i}^{(3)} = &-\frac{1}{2r_c}\{(\vec{v}\cdot\vec{\nabla})v_i + v_i(\vec{\nabla}\cdot\vec{v}) + \partial_i P + \partial_\tau v_i \\ &+ (r-r_c)[\partial_y^2 v_x + \partial_z^2 v_x - \partial_x(\partial_y v_y + \partial_z v_z)]\} - \frac{r}{2}\partial_r^2 g_{\tau i}^{(3)}, \end{aligned} \tag{3.39c}$$

$$R_{rr}^{(3)} = -\frac{1}{2}\partial_r^2(g_{xx}^{(3)} + g_{yy}^{(3)} + g_{zz}^{(3)}), \tag{3.39d}$$

$$R_{ri}^{(3)} = \frac{1}{2}\partial_r^2 g_{\tau i}^{(3)}, \tag{3.39e}$$

$$R_{ij}^{(3)} = -\frac{1}{2}\partial_r(r\partial_r g_{ij}^{(3)}). \tag{3.39f}$$

Upon comparison with eqs. (3.17a) to (3.17d) and recalling that $R^{(3)}_{\mu\nu} = \hat{R}^{(3)}_{\mu\nu} + \delta R^{(3)}_{\mu\nu}$, we immediately see that the only surviving $\hat{R}^{(3)}_{\mu\nu}$ is

$$\begin{aligned}\hat{R}^{(3)}_{\tau i} &= -\frac{1}{2r_c}\{(\vec{v}\cdot\vec{\nabla})v_i + v_i(\vec{\nabla}\cdot\vec{v}) + \partial_i P + \partial_\tau v_i + (r-r_c)[\partial_y^2 v_x + \partial_z^2 v_x - \partial_x(\partial_y v_y + \partial_z v_z)]\}\\ &= -\frac{1}{2r_c}\{(\vec{v}\cdot\vec{\nabla})v_i + v_i(\vec{\nabla}\cdot\vec{v}) + \partial_i P + \partial_\tau v_i + (r-r_c)[\partial_y^2 v_x + \partial_z^2 v_x - \partial_x(\vec{\nabla}\cdot\vec{v} - \partial_x v_i)]\}\\ &= -\frac{1}{2r_c}[\partial_\tau v_i + (\vec{v}\cdot\vec{\nabla})v_i + (r-r_c)\nabla^2 v_i + \partial_i P], \end{aligned} \tag{3.40}$$

where we used the incompressibility condition $\vec{\nabla}\cdot\vec{v} = 0$ when going from the second to the third line. We note that this is very similar to the i-th component of a Navier-Stokes equation with viscosity $r - r_c$. However, it doesn't make much sense to speak of a viscosity involving the coordinate r, but as we are going to see below, this term will be canceled out. All the other terms in eqs. (3.39a) to (3.39f) are the contributions from $\delta R^{(3)}_{\mu\nu}$. These results greatly simplify eqs. (3.27a) to (3.27d) to

$$g^{(3)}_{r\mu} = 0, \tag{3.41a}$$

$$g^{(3)}_{\tau\tau} = (1 - r/r_c)F^{(3)}_\tau(\tau,\vec{x}), \tag{3.41b}$$

$$g^{(3)}_{\tau i} = (1 - r/r_c)F^{(3)}_i(\tau,\vec{x}), \tag{3.41c}$$

$$g^{(3)}_{ij} = 0, \tag{3.41d}$$

so we only need to focus our attention on $F^{(3)}_\tau$ and $F^{(3)}_i$. Now, since we are working at order ε^3, that is, at an odd order in ε, our discussion in section 3.1.1 implies that $F^{(3)}_\tau$ must be set to zero. Therefore, it only remains to find the $F^{(3)}_i$ and, as stated in section 3.1.1, only $g^{(3)}_{\tau i}$ survives.

To calculate the $F^{(3)}_i$, we need the Brown-York tensor at second and third orders. At second order, the nonzero components of the full Brown-York tensor are

$$T^{(2)}_{\tau\tau} = r_c^{-1/2}[v^2 - 2r_c\vec{\nabla}\cdot\vec{v}], \tag{3.42a}$$

$$T^{(2)}_{xx} = r_c^{-3/2}[P + v_x^2 + 2r_c(\partial_y v_y + \partial_z v_z)], \tag{3.42b}$$

$$T^{(2)}_{yy} = r_c^{-3/2}[P + v_y^2 + 2r_c(\partial_x v_x + \partial_z v_z)], \tag{3.42c}$$

$$T^{(2)}_{zz} = r_c^{-3/2}[P + v_z^2 + 2r_c(\partial_y v_x + \partial_z v_y)], \tag{3.42d}$$

$$T^{(2)}_{ij} = r_c^{-3/2}[v_i v_j - r_c(\partial_i v_j + \partial_j v_i)] \qquad (i \neq j), \tag{3.42e}$$

where we note that $\vec{\nabla}\cdot\vec{v} = 0$. At third order, the full non-vanishing components of the full Brown-York tensor are

$$T^{(3)}_{\tau i} = \frac{1}{r_c^{3/2}} r_c F^{(3)}_i + P v_i + r_c(\partial_i P + v_x\partial_i v_x + v_y\partial_i v_y + v_z\partial_i v_z - \partial_\tau v_i). \tag{3.43}$$

We must then put eqs. (3.42a) to (3.42e) and (3.43) into eq. (3.37) to find $F_i^{(3)}$ and put the result into eq. (3.41c). The correction to the metric at order ε^3, $g_{\tau i}^{(3)}$, is found to be

$$g_{\tau i}^{(3)} = \frac{(r-r_c)}{2r_c}\left[(v^2+2P)\frac{2v_i}{r_c}+4\partial_i P-(r+r_c)\partial^2 v_i\right], \tag{3.44}$$

so that the full metric at order ε^3 becomes

$$\begin{aligned} \mathrm{d}s^2 &= -r\mathrm{d}\tau^2 + 2\mathrm{d}\tau\mathrm{d}r + \mathrm{d}x_i\mathrm{d}x^i \\ &\quad -2\left(1-\frac{r}{r_c}\right)v_i^{(\varepsilon)}\mathrm{d}x^i\mathrm{d}\tau - \frac{2v_i^{(\varepsilon)}}{r_c}\mathrm{d}x^i\mathrm{d}r \\ &\quad +\left(1-\frac{r}{r_c}\right)\left[(v^{2(\varepsilon)}+2P^{(\varepsilon)})\mathrm{d}\tau^2+\frac{v_i^{(\varepsilon)}v_j^{(\varepsilon)}}{r_c}\mathrm{d}x^i\mathrm{d}x^j\right]+\left(\frac{v^{2(\varepsilon)}+2P^{(\varepsilon)}}{r_c}\right)\mathrm{d}\tau\mathrm{d}r \\ &\quad +\frac{(r-r_c)}{2r_c}\left[(v^2+2P)\frac{2v_i}{r_c}+4\partial_i P-(r+r_c)\partial^2 v_i\right]\mathrm{d}\tau\mathrm{d}x^i \\ &\quad +O(\varepsilon^4), \end{aligned} \tag{3.45}$$

Now, going back to eq. (3.39c), we must take the r-derivative of eq. (3.44) twice, which gives

$$\partial_r^2 g_{\tau i}^{(3)} = -\frac{\partial^2 v_i}{r_c}. \tag{3.46}$$

Putting together eqs. (3.39c), (3.40) and (3.46), we can finally arrive at the only remaining term of the Ricci tensor at order ε^3:

$$R_{ri}^{(3)} = -\frac{1}{2r_c}[\partial_\tau v_i+(v^j\partial_j)v_i-r_c\partial^2 v_i+\partial_i P], \tag{3.47}$$

with all the other components being zero at this order. We immediately notice that the terms inside the square brackets in eq. (3.47) are precisely the Navier-Stokes equations, which should vanish according to eq. (2.42a). What eq. (3.47) then implies is that the Ricci tensor for the metric eq. (3.45) vanishes, that is, the metric at order ε^3 is Ricci-flat, as required.

Proceeding with our calculations, the Brown-York tensor at order ε^3 gets corrected by

$$T_{ab}^{(3)}\mathrm{d}x^a\mathrm{d}x^b = 2r_c^{-3/2}[r_c\sigma_{ik}v_k-(v^2+P)v_i]\mathrm{d}x^i\mathrm{d}\tau, \tag{3.48}$$

where

$$\sigma_{ij} \equiv \partial_i v_j + \partial_j v_i \tag{3.49}$$

is the shear. It is clear then, that at orders ε^n for $n>2$, the Brown-York tensor no longer satisfies our previous requirement that it should take the form of a perfect fluid.

3.1.3 Comments on the expansion at higher orders

Reference [25] uses this algorithmic procedure to calculate the metric and Brown-York corrections up to order $O(\varepsilon^6)$. Up to order ε^3, as we have just seen, we only need to add corrections to the metric in order to have a vanishing Ricci tensor. However, once one goes to orders higher than ε^3, it becomes necessary to add corrections to either the Navier-Stokes equation or the incompressibility condition. In particular, at even orders of ε^n, the incompressibility condition $\partial_i v_i = 0$ gets corrected, while at odd orders, it is the Navier-Stokes equations that must be modified so that the Ricci tensor vanishes.

These results at higher orders are not relevant to our work, since we will go no further than third order in our expansion.

Chapter 4

Solution-generating Symmetries

Despite their power and usefulness in describing some properties of spacetime, the Einstein equations remain a difficult subject of study even a century after their discovery, in particular because finding their exact solutions is by no means an easy task. Therefore, it is only natural that one starts to look for alternative methods to find exact solutions to the Einstein equations. One such way of achieving this was developed by Geroch [38], in a work that generalizes the previous works of Ehlers [35], Buchdahl [36] and Harrison [37]. In short, this method generates explicit, exact and source-free solutions of the Einstein equations by associating to an exact solution which contains a Killing vector a family of new solutions, each with a Killing vector.

The method developed by Geroch, sometimes called the *projection formalism*, is very useful when applied in strictly stationary spacetimes, i.e., spacetimes admitting an everywhere timelike Killing vector, as well as when applied in spacetimes in which the Killing vector is everywhere spacelike. In the context of this projection formalism, the Ehlers group was discovered as a set of group transformations that maps solutions of the Einstein vacuum field equations for stationary spacetimes into other solutions of stationary spacetimes.

One of the main caveats of this projection formalism is that stationary spacetimes may develop ergospheres and horizons, at which the Killing vectors become null, which in turn means that the projection formalism is no longer applicable. Since the Ehlers group is developed within this formalism, it becomes necessary to adopt a different approach to this problem, or at least define the Ehlers group in such a way that they include the case of null Killing vectors. As stated on [39], there were examples suggesting that it would be possible to extend the Ehlers group to encompass null Killing vectors, and it is shown in [39] that this is indeed possible, if one no longer works in the projection formalism and instead works in a spacetime setting, i.e., using only spacetime objects. Moreover, it is shown that the Ehlers group can be included within an infinite-dimensional group of transformations mapping Lorentzian metrics into Lorentzian metrics. In their most general form, these transformations are given by

$$g_{\mu\nu} \to h_{\mu\nu}(\xi, W, g) = \Omega^2 g_{\mu\nu} - \xi_\mu W_\nu - \xi_\nu W_\mu - \frac{\lambda}{\Omega^2} W_\mu W_\nu, \tag{4.1}$$

where $\xi = \xi^\mu \partial_\mu$ is a vector field (particularly for our case, a Killing vector field) and $W = W_\mu dx^\mu$ a one-form, both defined on a manifold $\mathcal{M}$ with a metric $g_{\mu\nu}$. Here, $\lambda = -\xi^\mu \xi_\mu$ and W_μ is constrained to satisfy $\Omega^2 \equiv 1 + \xi^\mu W_\mu > 0$. We are soon going to describe eq. (4.1) in somewhat more detail, but the full treatment in which they are developed lies well outside the scope of our work, so we refer the reader to [39] for all details.

For convenience, we are going to adopt the nomenclature used in [39, 40]: we shall refer to the general transformations described by eq. (4.1) as the *generalized Ehlers group*, while their subset which specifically maps vacuum solutions into vacuum solutions shall be called the *spacetime Ehlers group*. This group in particular will be very useful for our work, since the solutions they map include the metric of flat spacetime in ingoing Rindler coordinates which we have studied in the previous chapter.

In a certain sense, we are going to build upon the work of [40], in that we will apply the Ehlers transformations in the context of the fluid/gravity duality. Their idea was to apply the Ehlers group to the Rindler metric at order ε^3 in the derivative expansion we described in the previous chapter. That is, they imposed a Killing symmetry in a spacetime admitting a metric ansatz corresponding to the Navier-Stokes equation, applied the generalized Ehlers transformations preserving the ansatz and determined the induced transformation of the dual fluid's parameters, such as velocity, pressure and viscosity. In doing so, the authors of [40] applied some constraints on the new metrics as well as the isometries. An interesting result of [40] is that they found that the transformations they find are not part of the spacetime Ehlers group, but still produce solution-generating transformations for the dual fluid's fields. Here, we would like to do a similar approach, but without *a priori* constraining the isometries or making assumptions about what the new metric should look like.

As a final remark, we note that the Ehlers transformations only apply to spacetimes of $(3+1)$ dimensions. Therefore, when we treat the Rindler spacetime, our coordinates will be $x^\mu = (\tau, r, x, y)$.

4.1 Ehlers transformations

Following [39], we will start with some definitions of objects that will be used throughout this text. As stated before, a full treatment lies outside the scope of this work, so we are only going to list the tools which are necessary for further development. Details can be found in [39, 46].

We denote our spacetime by $(\mathcal{M}, g)$, and we assume it admits a Killing vector field $\vec{\xi}$. We define the norm λ and the twist ω_α of $\vec{\xi}$ respectively as

$$\lambda = -\xi^\alpha \xi_\alpha, \tag{4.2}$$

$$\omega_\alpha = \eta_{\alpha\beta\gamma\delta}\xi^\beta \nabla^\gamma \xi^\delta, \tag{4.3}$$

where $\eta_{\alpha\beta\gamma\delta}$ is the metric volume form of $(\mathcal{M}, g)$. Following the notation of [39], we denote p-forms in boldface characters, while their components are denoted by non-boldface characters. We define the 2-form

$$F_{\alpha\beta} \equiv \nabla_\alpha \xi_\beta \tag{4.4}$$

and its self-dual associate

$$\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + iF^\star_{\alpha\beta}, \tag{4.5}$$

where $F^\star_{\alpha\beta}$ is the Hodge-dual of $F_{\alpha\beta}$ and is given by

$$F^\star_{\alpha\beta} = \frac{\sqrt{\det g}}{2} \sum_{\mu,\nu} F^{\mu\nu} \varepsilon_{\mu\nu\alpha\beta} \tag{4.6}$$

In the above, $\varepsilon_{\mu\nu\alpha\beta} = +1$ if $\mu\nu\alpha\beta$ is an even permutation of the coordinates, while it is -1 if $\mu\nu\alpha\beta$ is an odd permutation of the coordinates, and zero otherwise.

We call the 2-form $\boldsymbol{\mathcal{F}} \equiv \frac{1}{2}\mathcal{F}_{\alpha\beta}\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta$ the *Killing form*. It may be written as $\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + iF^\star_{\alpha\beta}$. Also, we define the *Ernst 1-form* $\sigma = \sigma_\mu \mathrm{d}x^\mu$ associated with $\vec{\xi}$ by

$$\sigma_\mu \equiv 2\xi^\alpha \mathcal{F}_{\alpha\mu} = \nabla_\mu \lambda - i\omega_\mu. \tag{4.7}$$

The action of the Ehlers group was worked out by Geroch [38] and is defined by transforming σ according to a Möbius map[1]

$$\sigma' = \frac{\alpha\sigma + i\beta}{i\gamma\sigma + \delta}, \tag{4.8}$$

[1]Geroch's work extends the results of Ehlers in that it shows that the set of transformations discovered by Ehlers is an element of $SL(2, \mathbb{R})$, that is, the Ehlers group is isomorphic to $SL(2, \mathbb{R})$.

(we use the notation of [39], where α, β, γ and δ are real constants satisfying $\alpha\delta + \beta\gamma = 1$) and the general form of the transformations, proposed in [39], is given by eq. (4.1)

$$T(\xi, W, g)_{\mu\nu} = \Omega^2 g_{\mu\nu} - \xi_\mu W_\nu - \xi_\nu W_\mu - \frac{\lambda}{\Omega^2} W_\mu W_\nu, \tag{4.9}$$

which is shown to be smooth and Lorentzian in [39]. The specific case of maps of vacuum metrics into vacuum metrics is covered by the following theorem, which presents two necessary conditions for an Ehlers transformation (the subgroup of eq. (4.9) called the "spacetime Ehlers transformation") to generate a vacuum metric:

Theorem 1. *Let $(\mathcal{M}, g)$ be a smooth spacetime admitting a Killing vector $\vec{\xi}$ and satisfying the Einstein vacuum field equations. Let δ, $\gamma \in \mathbb{R}$ satisfy $\delta^2 + \gamma^2 \neq 0$. Define λ, $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\sigma}$ as the squared norm, the Killing form and the Ernst one-form associated to $\vec{\xi}$. If the two following conditions are satisfied*

1. *The Ernst one-form is exact, i.e. there exists a complex smooth function $\sigma \equiv \lambda - i\omega$ such that $\boldsymbol{\sigma} = \mathbf{d}\sigma$;*

2. *The closed two-form $\Re(-4\gamma(\gamma\bar{\sigma} + i\delta)\boldsymbol{\mathcal{F}})$ is exact and the equation $\boldsymbol{dW} = \Re(-4\gamma(\gamma\bar{\sigma} + i\delta)\boldsymbol{\mathcal{F}})$ admits a solution satisfying $W_\alpha \xi^\alpha + 1 = (i\gamma\sigma + \delta)(-i\gamma\bar{\sigma} + \delta) \equiv \Omega^2$;*

then the symmetric tensor $T_W^{\vec{\xi}}(g) \equiv \Omega^2 g - \boldsymbol{\xi} \otimes \boldsymbol{W} - \boldsymbol{W} \otimes \boldsymbol{\xi} - \frac{\lambda}{\Omega^2} \boldsymbol{W} \otimes \boldsymbol{W}$ defines a smooth vacuum metric on the spacetime $\tilde{\mathcal{M}} = \{p \in \mathcal{M}; \lambda|_p \neq 0 \text{ or } (\gamma\omega + \delta) \neq 0\}$.

The proof of this theorem can be found in [39]. To be concise, then, there are two conditions that W must satisfy for the new metric to be a vacuum metric, namely

$$\nabla_{[\mu} W_{\nu]} = -2\gamma\Re[(\gamma\sigma + i\delta)\mathcal{F}_{\mu\nu}] \Rightarrow \nabla_\mu W_\nu - \nabla_\nu W_\mu = -4\gamma\Re[(\gamma\bar{\sigma} + i\delta)\mathcal{F}_{\mu\nu}], \tag{4.10a}$$

$$\Omega^2 \equiv \xi^\mu W_\mu + 1 = (i\gamma\sigma + \delta)(-i\gamma\bar{\sigma} + \delta), \tag{4.10b}$$

and the new metric $h_{\mu\nu}$ will be given by eq. (4.9).

4.2 Ehlers transformations in a Rindler fluid/gravity scenario

The authors of [40] employed the formalism of Ehlers transformation to the "seed metric" plus corrections at order ε^3 (see eq. (3.45)), which we rewrite here for convenience:

$$
\begin{aligned}
ds^2 &= -r d\tau^2 + 2 d\tau dr + dx_i dx^i \\
&- 2\left(1-\frac{r}{r_c}\right) v_i^{(\varepsilon)} dx^i d\tau - \frac{2v_i^{(\varepsilon)}}{r_c} dx^i dr \\
&+ \left(1-\frac{r}{r_c}\right)\left[(v^{2(\varepsilon)} + 2P^{(\varepsilon)}) d\tau^2 + \frac{v_i^{(\varepsilon)} v_j^{(\varepsilon)}}{r_c} dx^i dx^j\right] + \left(\frac{v^{2(\varepsilon)} + 2P^{(\varepsilon)}}{r_c}\right) d\tau dr \\
&+ \frac{(r-r_c)}{2r_c}\left[(v^2+2P)\frac{2v_i}{r_c} + 4\partial_i P - (r+r_c)\partial^2 v_i\right] d\tau dx^i.
\end{aligned} \tag{4.11}
$$

However, when doing this they forced the transformed metrics $h(\xi, W, g)$ to preserve the functional form of g, which is not the most general case. By making this assumption, they defined the transformed parameters $\tilde{v}_i$, $\tilde{P}$ and $\tilde{r}_c$, so that the transformed metric will yield the incompressible Navier-Stokes for these parameters, that is,

$$\partial_\tau \tilde{v}_i + \partial_i \tilde{P} + \tilde{v}_k \partial_k \tilde{v}_i - \tilde{r}_c \partial^2 \tilde{v}_i = 0, \tag{4.12a}$$

$$\partial_i \tilde{v}_i = 0. \tag{4.12b}$$

Therefore, these new parameters are such that they will represent a new set of solutions. By imposing that the new metric preserves the form of the old metric, we have

$$g_{\mu\nu}(r_c, v_i, P) \to h_{\mu\nu}(\xi, W, g) = \tilde{g}_{\mu\nu} = g_{\mu\nu}(\tilde{r}_c, \tilde{v}_i, \tilde{P}), \tag{4.13}$$

where

$$\tilde{g}_{\tau r} = 1 + \frac{\tilde{v}(x^a)^2 + 2\tilde{P}(x^a)}{2\tilde{r}_c}, \tag{4.14a}$$

$$\tilde{g}_{ir} = -\frac{\tilde{v}_i(x^a)}{\tilde{r}_c}, \tag{4.14b}$$

$$\tilde{g}_{rr} = 0, \tag{4.14c}$$

$$\tilde{g}_{ab}|_{\tilde{r}_c} = \tilde{\gamma}_{ab}, \tag{4.14d}$$

$$\tilde{\gamma}_{\tau\tau} = -\tilde{r}_c, \tag{4.14e}$$

$$\tilde{\gamma}_{ai} = \gamma_{ai}. \tag{4.14f}$$

From these, it is immediate to find, upon using eq. (4.9),

$$W_r = -2\alpha \xi_r \frac{\Omega^2}{\lambda}, \tag{4.15}$$

where $\alpha = 0, 1$. Also, upon contracting eq. (4.9) with boundary indices $a, b, \ldots$ of the Killing vector, we have

$$W_a = \frac{\Omega^2 \xi^r \left(g_{ar} + \frac{\xi_a \xi_r}{\lambda} \right) + \xi^b \tilde{g}_{ab}}{\frac{\lambda}{\Omega^2} + (1 - 2\alpha) \xi^r \xi_r} - \xi_a \frac{\Omega^2}{\lambda}. \tag{4.16}$$

To illustrate, it can be shown that we may write [40]

$$\tilde{g}_{ar} = g_{ar} - 2\xi_r \left(\frac{\xi^b \gamma_{ab} - \xi^r g_{ar}}{\xi^c \xi^d \gamma_{cd} - \xi^r \xi_r} \right). \tag{4.17}$$

With these, it can be shown [40] that with $\xi = c_k \partial_k$ (c_k being constants obeying $\Sigma_k c_k^2 = 1$), the isometries $c_k \partial_k v_i = c_k \partial_k P = 0$ imply that

$$\tilde{v}_i = v_i - 2 c_i c_k v_k, \tag{4.18a}$$

$$\tilde{P} = P. \tag{4.18b}$$

As we are going to explain in section 4.4, we are going to use a different procedure to search for symmetries of eq. (4.11), in which we will not assume a priori constraints on the form of the new metric.

4.3 Example: Rindler metric at order zero

Before we apply the Ehlers transformations to the Rindler metric at order ε^3, we employ the same reasoning for a simpler case, namely the Rindler metric at order zero, in four dimensions ($d = 2$), with $x^\mu = (\tau, r, x, y)$,

$$g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -r \mathrm{d}\tau^2 + 2 \mathrm{d}\tau \mathrm{d}r + \mathrm{d}x_i \mathrm{d}x^i, \tag{4.19}$$

whose inverse, in matrix form, is

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.20}$$

For this metric, we may obtain the Killing vectors by solving the Killing equation eq. (1.19). We note that, since this is flat space, it is maximally symmetric, which means that, according to our discussion immediately after eq. (1.19), there must be 10 linearly independent Killing vectors $\xi_A^{(0)\mu}$. They are found to be

$$\xi_1^{(0)\mu}(t,r,x,y) = (1,0,0,0), \tag{4.21a}$$
$$\xi_2^{(0)\mu}(t,r,x,y) = (0,0,1,0), \tag{4.21b}$$
$$\xi_3^{(0)\mu}(t,r,x,y) = (0,0,0,1), \tag{4.21c}$$
$$\xi_4^{(0)\mu}(t,r,x,y) = (0,e^{t/2},0,0), \tag{4.21d}$$
$$\xi_5^{(0)\mu}(t,r,x,y) = (0,0,y,-x), \tag{4.21e}$$
$$\xi_6^{(0)\mu}(t,r,x,y) = (2e^{-t/2},re^{-t/2},0,0), \tag{4.21f}$$
$$\xi_7^{(0)\mu}(t,r,x,y) = (0,xe^{t/2},-2e^{t/2},0), \tag{4.21g}$$
$$\xi_8^{(0)\mu}(t,r,x,y) = (0,ye^{t/2},0,-2e^{t/2}), \tag{4.21h}$$
$$\xi_9^{(0)\mu}(t,r,x,y) = (2xe^{-t/2},xre^{-t/2},-2re^{-t/2},0), \tag{4.21i}$$
$$\xi_{10}^{(0)\mu}(t,r,x,y) = (2ye^{-t/2},yre^{-t/2},0,-2re^{-t/2}), \tag{4.21j}$$

where, in anticipation of a future development, we use the superscript (0) to denote the Killing vectors at order ε^0, while the subscript A labels the different Killing vectors. In section 4.4 we are going to explain how to find the Killing vectors order by order.

4.3.1 Temporal translation

In particular, we are going to work with the vector representing the temporal translation

$$\xi_\mu = (-r,1,0,0), \tag{4.22}$$

that is, $\xi_\tau = -r$ and $\xi_r = 1$, such that $\xi^\tau = g^{\tau\mu}\xi_\mu = 1$ and $\xi^r = g^{r\mu}\xi_\mu = 0$.

For eqs. (4.19) and (4.20), the only non-zero Christoffel symbols are

$$\Gamma^\tau_{\tau\tau} = \frac{1}{2}, \qquad \Gamma^r_{\tau\tau} = \frac{r}{2}, \qquad \Gamma^r_{\tau r} = \Gamma^r_{r\tau} = -\frac{1}{2}. \tag{4.23}$$

Using $F_{\alpha\beta} = \nabla_\alpha \xi_\beta$, the only non-vanishing components turn out to be

$$F_{\tau r} = -F_{r\tau} = \frac{1}{2}. \tag{4.24}$$

Hence, we may write the $F_{\alpha\beta}$ matrix for this Killing vector:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.25}$$

The Hodge dual $F^{\star}_{\alpha\beta}$ of the above is

$$F^{\star}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \tag{4.26}$$

so that $\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + iF^{\star}_{\alpha\beta}$ is

$$\mathcal{F}_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}. \tag{4.27}$$

With $\mathcal{F}_{\alpha\beta}$ we may calculate $\sigma_\mu = 2\xi^\alpha \mathcal{F}_{\alpha\mu}$. The only non-vanishing term is

$$\sigma_r = 2\xi^\alpha \mathcal{F}_{\alpha r} = 2\xi^\tau \mathcal{F}_{\tau r} = 2\frac{1}{2} = 1. \tag{4.28}$$

Thus,

$$\sigma_\mu = (0,1,0,0) \Rightarrow \sigma = r. \tag{4.29}$$

With the results obtained so far we can calculate the 1-forms W_μ using the conditions given by eqs. (4.10a) and (4.10b).

$$\begin{aligned} \nabla_{[\mu}W_{\nu]} &= -2\gamma\Re[(\sigma\gamma + i\delta)\mathcal{F}_{\mu\nu}] \\ &= -2\gamma\Re\left[(r\gamma + i\delta)\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}\right] \\ &= -\gamma\Re\begin{pmatrix} 0 & r\gamma + i\delta & 0 & 0 \\ -(r\gamma + i\delta) & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta - ir\gamma \\ 0 & 0 & -\delta + ir\gamma & 0 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 0 & -r\gamma^2 & 0 & 0 \\ r\gamma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta\gamma \\ 0 & 0 & \delta\gamma & 0 \end{pmatrix} \tag{4.30}$$

In component notation, this is

$$(\tau, r) : \nabla_\tau W_r - \nabla_r W_\tau = -2r\gamma^2, \tag{4.31a}$$

$$(x, y) : \nabla_x W_y - \nabla_y W_x = -2\delta\gamma, \tag{4.31b}$$

$$(\tau, x) : \nabla_\tau W_x - \nabla_x W_\tau = 0, \tag{4.31c}$$

$$(\tau, y) : \nabla_\tau W_y - \nabla_y W_\tau = 0, \tag{4.31d}$$

$$(r, x) : \nabla_r W_x - \nabla_x W_r = 0, \tag{4.31e}$$

$$(r, y) : \nabla_r W_y - \nabla_y W_r = 0. \tag{4.31f}$$

Before proceeding, we note that since the only surviving ξ^μ is ξ^τ, we may easily get an expression for W_τ:

$$\xi^\alpha W_\alpha + 1 = \gamma^2 r^2 + \delta^2 \Rightarrow W_\tau = \gamma^2 r^2 + \delta^2 - 1. \tag{4.32}$$

Plugging this in the previous equations we may get the remaining W_μ:

$$W_\tau = \gamma^2 r^2 + \delta^2 - 1 \tag{4.33a}$$

$$W_r = 0 \tag{4.33b}$$

$$W_x = \gamma\delta(x - y) \tag{4.33c}$$

$$W_y = \gamma\delta(x - y). \tag{4.33d}$$

Note that these are *not* the only possible solutions. In particular, we wish to maintain a symmetry in the W_x and W_y functions, which in principle is not necessary.

Finally, we note that $\lambda = -\xi^\alpha \xi_\alpha = r$. We may then find the metric generated by the Ehlers transformation for the Killing vector representing a translation in τ. Putting the results derived in this section into eq. (4.9), we arrive at the most general form of the new metric $h_{\mu\nu}$:

$$h_{\mu\nu} = \begin{pmatrix} -\frac{r}{\gamma^2 r^2+\delta^2} & 1 & \frac{r(x-y)\gamma\delta}{\gamma^2 r^2+\delta^2} & \frac{r(x-y)\gamma\delta}{\gamma^2 r^2+\delta^2} \\ 1 & 0 & -(x-y)\gamma\delta & -(x-y)\gamma\delta \\ \frac{r(x-y)\gamma\delta}{\gamma^2 r^2+\delta^2} & -(x-y)\gamma\delta & \gamma^2 r^2 + \delta^2 - \frac{r(x-y)^2\gamma^2\delta^2}{\gamma^2 r^2+\delta^2} & -\frac{r(x-y)^2\gamma^2\delta^2}{\gamma^2 r^2+\delta^2} \\ \frac{r(x-y)\gamma\delta}{\gamma^2 r^2+\delta^2} & -(x-y)\gamma\delta & -\frac{r(x-y)^2\gamma^2\delta^2}{\gamma^2 r^2+\delta^2} & \gamma^2 r^2 + \delta^2 - \frac{r(x-y)^2\gamma^2\delta^2}{\gamma^2 r^2+\delta^2} \end{pmatrix} \tag{4.34}$$

This metric is Ricci-flat[2], but in its most general form it does not seem to admit a foliation in flat space. This might be due to an intrinsic property of the metric, but it may be possible to avoid this by choosing a suitable coordinate transformation.

Working on eq. (4.9), if we set $\gamma = 0$ while leaving δ untouched, we'll have

$$h_{\mu\nu} = \begin{pmatrix} -\frac{r}{\delta^2} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \delta^2 & 0 \\ 0 & 0 & 0 & \delta^2 \end{pmatrix}. \tag{4.35}$$

Note that setting $\delta = 1$ in this case gives a Rindler metric, so we conclude that the Ehlers transformation in this case simply maps a Rindler spacetime into another Rindler spacetime.

If we instead set $\delta = 0$, the metric takes the following form:

$$h_{\mu\nu} = \begin{pmatrix} -\frac{1}{\gamma^2 r} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \gamma^2 r^2 & 0 \\ 0 & 0 & 0 & \gamma^2 r^2 \end{pmatrix}. \tag{4.36}$$

Setting $\gamma = 1$, we see that the metric will be precisely the one discovered by Taub [27,57] (see eq. (3.11)), with $\mathcal{A} = 1$. The Taub metric describes a vacuum solution exterior to an infinite plane-symmetric object with uniform mass density. It should be noted [27] that the energy density computed from Brown-York tensor always negative.

Hence, it is possible to use an Ehlers transformation to map a metric of flat spacetime in ingoing Rindler coordinates into the Taub metric, which also satisfies the equation of state

$$dT_{ab}T^{ab} = T^2 \tag{4.37}$$

(see eq. (3.8) in the previous chapter) for a perfect fluid whose energy density is given by

$$\rho = -\frac{2d}{d-1}p. \tag{4.38}$$

4.4 Solving the Killing equation perturbatively

In the previous section we applied the Ehlers transformation to a very simple case, the flat spacetime in Rindler coordinates, and we showed that there is a relation between the

[2] Verified using Mathematica.

two possible solutions for the equation

$$dT_{ab}T^{ab} = T^2,$$

namely the Rindler spacetime and the Taub spacetime. In this section, we would like to do a similar analysis, but this time considering the expanded metric up to order ε^3. In other words, we would like to apply the Ehlers transformation to our "seed metric" plus corrections at order ε^3 (see eq. (4.11)).

As we said in section 4.2, a similar procedure was employed by [40], but they forced the new metric to have the same "form" as the initial one. Also, they forced some symmetries to appear, both because they wanted to preserve the form of the metric and also because it is not immediately clear that the Killing vectors for eq. (4.11) can be easily found.

Our idea here is to find approximate isometries of eq. (4.11), that is, we would like to solve the Killing equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{4.39}$$

perturbatively in the ε expansion up to order ε^3, by expanding the Killing vectors ξ^μ as

$$\xi^\mu = \xi^{(0)\mu} + \xi^{(1)\mu} + \xi^{(2)\mu} + \xi^{(3)\mu} + \cdots, \tag{4.40}$$

so that each term at order n is proportional to ε^n.

The ten Killing vectors $\xi^{(0)\mu}$ at order ε^0 have been found previously in eqs. (4.21a) to (4.21j), and since any linear combination of them is itself a Killing vector, we may write the ten Killing vectors at order ε^0 in a general form as

$$\xi^{(0)}(t,r,x,y) = \sum_{A=1}^{10} c_A \xi_A^{(0)\mu}(t,r,x,y), \tag{4.41}$$

where the c_A's are constant coefficients. Equation (4.41) is also a solution of the Killing equation at order ε^0.

The method used to solve the Killing equations at order ε^0 as well as in higher orders is explained in more detail in Appendix B.

4.4.1 Order ε^1

Now we solve the Killing equation at order ε^1, that is

$$\partial_\mu \xi_\nu^{(1)} + \partial_\nu \xi_\mu^{(1)} - \Gamma^{(0)\rho}{}_{\mu\nu} \xi_\rho^{(1)} - \Gamma^{(1)\rho}{}_{\mu\nu} \xi_\rho^{(0)} = 0, \tag{4.42}$$

where $\xi^{(0)}_{\mu}$ is obtained from eq. (4.41). The form of eq. (4.42) indicates the general procedure which will be employed in future orders: at higher orders, we will have products of the form $\Gamma^{(m)\rho}_{\mu\nu}\xi^{(n)}_{\rho}$, where the sum $m+n$ will be of the desired order; in this case, $m+n=1$, hence the products $\Gamma^{(0)\rho}_{\mu\nu}\xi^{(1)}_{\rho}$ and $\Gamma^{(1)\rho}_{\mu\nu}\xi^{(0)}_{\rho}$. Therefore, the procedure is merely iterative: once we have the Killing vectors at order ε^0, we may calculate $\xi^{(1)\mu}$, and with these, we may proceed to order ε^2 and so on.

The nonvanishing Christoffel symbols at order ε^1 are

$$\Gamma^{(1)t}_{\ \ ti} = -\frac{v_i}{2r_c}, \tag{4.43a}$$

$$\Gamma^{(1)r}_{\ \ ti} = -\frac{rv_i}{2r_c}, \tag{4.43b}$$

$$\Gamma^{(1)r}_{\ \ ri} = \frac{v_i}{2r_c}, \tag{4.43c}$$

$$\Gamma^{(1)i}_{\ \ tt} = \frac{v_i}{2}. \tag{4.43d}$$

The Killing vectors at order ε^1 are then

$$\xi_1^{(1)\mu}(t,r,x,y) = (1,0,0,0), \tag{4.44a}$$

$$\xi_2^{(1)\mu}(t,r,x,y) = (0,0,1,0), \tag{4.44b}$$

$$\xi_3^{(1)\mu}(t,r,x,y) = (0,0,0,1), \tag{4.44c}$$

$$\xi_4^{(1)\mu}(t,r,x,y) = (F_t^{(1)}(t,x,y),0,0,0), \tag{4.44d}$$

$$\xi_5^{(1)\mu}(t,r,x,y) = (0,0,F_x^{(1)}(t,x,y),0), \tag{4.44e}$$

$$\xi_6^{(1)\mu}(t,r,x,y) = (0,0,0,F_y^{(1)}(t,x,y)), \tag{4.44f}$$

where the $F_I^{(1)}(t,x,y)$ are arbitrary functions of t,x and y at order ε^1. The general form of these Killing vectors is

$$\xi^{(1)}(t,r,x,y) = \sum_{A=1}^{6} c_A \xi_A^{(1)\mu}(t,r,x,y), \tag{4.45}$$

where the c_A's are arbitrary constants.

Recalling that

$$v_i \sim \varepsilon, \quad P \sim \varepsilon^2, \quad \partial_t \sim \varepsilon^2, \quad \partial_i \sim \varepsilon,$$

we see that the $F_I^{(i)}$ above will be of the form

$$F_t^{(1)}(t,x,y) = a_i v_i(t,x,y), \tag{4.46}$$

where the a_I are arbitrary constants. $F_x^{(1)}$ and $F_y^{(1)}$ are entirely analogous.

Before proceeding to order ε^2, we must point out that, when compared to the Killing vectors at order ε^0, the translations in t, x and y still remain, that is, they are symmetries also at order ε^1. To see why this happens, suppose we have infinitesimal transformations. In particular, suppose we have an infinitesimal translation in x. Then, we may write

$$P(t,x+\delta x,y) = P(t,x,y) + \delta x \partial_x P(t,x,y) + \cdots = P(t,x,y) + O(\varepsilon^3),$$
$$v_i(t,x+\delta x,y) = v_i(t,x,y) + \delta x \partial_x v_i(t,x,y) + \cdots = v_i(t,x,y) + O(\varepsilon^2).$$

The term $\partial_x P$ is of order ε^3 and $\partial_x v_i$ of order ε^2. Hence they can be left out of the expansion at this order. The same reasoning is valid for t and y. Indeed, using this reasoning, we can see that the spatial translations will no longer be a symmetry at order ε^2.

4.4.2 Order ε^2

Now we apply our iterative process to

$$\partial_\mu \xi_\nu^{(2)} + \partial_\nu \xi_\mu^{(2)} - \Gamma^{(0)\rho}{}_{\mu\nu} \xi_\rho^{(2)} - \Gamma^{(2)\rho}{}_{\mu\nu} \xi_\rho^{(0)} - 2\Gamma^{(1)\rho}{}_{\mu\nu} \xi_\rho^{(1)} = 0. \tag{4.48}$$

The Christoffel symbols at order ε^2 that do not immediately vanish are

$$\Gamma^{(2)t}{}_{tt} = \frac{1}{2r_c}\left(P + \frac{3}{2}v^2\right), \tag{4.49a}$$
$$\Gamma^{(2)t}{}_{ij} = \frac{1}{2r_c}\left[v_i v_j - r_c(\partial_i v_j + \partial_j v_i)\right], \tag{4.49b}$$
$$\Gamma^{(2)r}{}_{tt} = \frac{1}{2r_c}\left[2(r-r_c)P + rv^2\right], \tag{4.49c}$$
$$\Gamma^{(2)r}{}_{tr} = \frac{1}{4r_c}\left(2P + v^2\right), \tag{4.49d}$$
$$\Gamma^{(2)r}{}_{ij} = \frac{r v_i v_j}{2r_c^2} - \frac{1}{2}(\partial_i v_j + \partial_j v_i), \tag{4.49e}$$
$$\Gamma^{(2)i}{}_{tj} = -\frac{1}{2r_c}\left[v_i v_j + (r-r_c)(\partial_i v_j - \partial_j v_i)\right], \tag{4.49f}$$
$$\Gamma^{(2)i}{}_{rj} = -\frac{1}{2r_c}\left(\partial_i v_j - \partial_j v_i\right). \tag{4.49g}$$

Using the procedure described earlier, we find that the Killing vectors at order ε^2 are given by

$$\xi_1^{(2)\mu}(t,r,x,y) = (1,0,0,0), \tag{4.50a}$$
$$\xi_2^{(2)\mu}(t,r,x,y) = (G_t^{(2)}(t,x,y),0,0,0), \tag{4.50b}$$
$$\xi_3^{(2)\mu}(t,r,x,y) = (0,0,G_x^{(2)}(t,x,y),0), \tag{4.50c}$$
$$\xi_4^{(2)\mu}(t,r,x,y) = (0,0,0,G_y^{(2)}(t,x,y)), \tag{4.50d}$$

where the functions $G_I^{(2)}(t,x,y)$ are functions of order ε^2 entirely analogous to the $F_I^{(1)}$ in the previous case. We note that terms of order ε^2 can be constructed from P, $v_i v_j$ and $\partial_i v_j$, so that the most general form of the $G_I^{(2)}$ is (similar for $G_x^{(2)}$ and $G_y^{(2)}$)

$$G_t^{(2)}(t,x,y) = aP(t,x,y) + b_{ij}v_i(t,x,y)v_j(t,x,y) + c_{ij}\partial_i v_j(t,x,y), \tag{4.51}$$

where a, b_{ij} and c_{ij} are arbitrary constants.

As predicted before, the spatial translations are no longer symmetries at order ε^2. Still, the temporal translation remains a symmetry because upon an infinitesimal transformation in t we have

$$P(t+\delta t,x,y) = P(t,x,y) + \delta t\partial_t P(t,x,y) + \cdots = P(t,x,y) + O(\varepsilon^4),$$
$$v_i(t+\delta t,x,y) = v_i(t,x,y) + \delta t\partial_t v_i(t,x,y) + \cdots = v_i(t,x,y) + O(\varepsilon^3).$$

This already indicates that, at order ε^3, the temporal translation will no longer be a symmetry.

4.4.3 Order ε^3

At order ε^3, the Killing equation becomes

$$\partial_\mu\xi_\nu^{(3)} + \partial_\nu\xi_\mu^{(3)} - \Gamma^{(0)\rho}_{\mu\nu}\xi_\rho^{(3)} - \Gamma^{(3)\rho}_{\mu\nu}\xi_\rho^{(0)} - \Gamma^{(1)\rho}_{\mu\nu}\xi_\rho^{(2)} - \Gamma^{(2)\rho}_{\mu\nu}\xi_\rho^{(1)} = 0. \tag{4.53}$$

The nonvanishing Christoffel symbols are too numerous and their expressions are large, so we refer the reader to the Appendix for them.

The Killing vectors at order ε^3 are

$$\xi_1^{(3)\mu}(t,r,x,y) = (H_t^{(3)}(t,x,y),0,0,0), \tag{4.54a}$$
$$\xi_2^{(3)\mu}(t,r,x,y) = (0,0,H_x^{(3)}(t,x,y),0), \tag{4.54b}$$
$$\xi_3^{(3)\mu}(t,r,x,y) = (0,0,0,H_y^{(3)}(t,x,y)), \tag{4.54c}$$

where, as before, the $H_I^{(3)}(t,x,y)$ are just like the $F_I^{(1)}$ and $G_I^{(2)}$ in lower orders. The general form of $H_I^{(3)}$ is found by noting that the only objects at order ε^3 that can be constructed from P, v_i and their derivatives are $\partial_i P$, Pv_i, $v_i v_j v_k$, $v_i \partial_j v_k$ and $\partial_t v_i$. Hence,

$$H_t^{(3)}(t,x,y) = a_i\partial_i P(t,x,y) + b_i P(t,x,y)v_i(t,x,y) + c_{ijk}v_i(t,x,y)v_j(t,x,y)v_k(t,x,y) + d_{ijk}v_i(t,x,y)\partial_j v_k(t,x,y) + e_i\partial_t v_i(t,x,y), \tag{4.55}$$

with arbitraty constants a_i, b_i, c_{ijk}, d_{ijk} and e_i, and similarly for $H_x^{(3)}$ and $H_y^{(3)}$.

We now wish to apply the Ehlers transformation to eq. (4.55).

4.5 Ehlers transformation on the ε^3 Killing vector

We write a general Killing vector at order ε^3 as

$$\xi^{(3)\mu} = (H_t^{(3)}(t,x,y), 0, H_x^{(3)}(t,x,y), H_y^{(3)}(t,x,y)), \tag{4.56}$$

where the H_i are the general functions given above. To lower the indices, we use only the metric terms at order ε^0, because we wish to remain at order ε^3. We arrive at

$$\xi_\mu^{(3)} = (-rH_t, H_t, H_x, H_y). \tag{4.57}$$

From eqs. (4.56) and (4.57) we see that

$$\lambda = -\xi^{(3)\alpha}\xi_\alpha^{(3)} = O(\varepsilon^6), \tag{4.58}$$

which already simplifies the general form of the Ehlers transformation eq. (4.9) to

$$h_{\mu\nu} = \Omega^2 g_{\mu\nu} - \xi_\mu^{(3)} W_\nu - \xi_\nu^{(3)} W_\mu. \tag{4.59}$$

This, in turn, suggests that the W_μ should be either constants or depend only on r.

Now, $\nabla_\alpha \xi_\beta^{(3)} = \partial_\alpha \xi_\beta^{(3)} - \Gamma^{(0)\mu}{}_{\alpha\beta}\xi_\mu^{(3)}$ gives the only non-zero terms

$$F_{\tau r} = -F_{r\tau} = \frac{H_t(t,x,y)}{2}. \tag{4.60}$$

We note from this equation that the H_x and H_y terms no longer appear, which was already expected by looking at the equation $\nabla_\alpha \xi_\beta^{(3)} = \partial_\alpha \xi_\beta^{(3)} - \Gamma^{(0)\mu}{}_{\alpha\beta}\xi_\mu^{(3)}$ and noting that the partial derivatives would all vanish since they would either be a derivative with respect to r (upon which H_x and H_y do not depend) or would increase the order of our terms. Also, none of the Christoffel symbols at order ε^0 contain the coordinates x or y.

The Hodge dual of eq. (4.60) is

$$F^\star_{xy} = -F^\star_{yx} = -\frac{H_t}{2}, \tag{4.61}$$

so that

$$\mathcal{F}_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 0 & H_t & 0 & 0 \\ -H_t & 0 & 0 & 0 \\ 0 & 0 & 0 & -iH_t \\ 0 & 0 & iH_t & 0 \end{pmatrix}. \tag{4.62}$$

This implies that

$$\sigma_\mu = (0, O(\varepsilon^6), O(\varepsilon^6), O(\varepsilon^6)). \tag{4.63}$$

Thus, σ is either a constant or a function $f^{(3)}(t,x,y)$. We treat these two cases separately.

4.5.1 Constant σ

If σ is a constant (for simplicity, we assume σ is real), then we find

$$\nabla_{[\mu}W_{\nu]} = \begin{pmatrix} 0 & -H_t\sigma\gamma^2 & 0 & 0 \\ H_t\sigma\gamma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_t\gamma\delta \\ 0 & 0 & H_t\gamma\delta & 0 \end{pmatrix}. \tag{4.64}$$

The equation $\Omega^2 = \xi^{(3)\mu}W_\mu + 1 = \gamma^2\sigma^2 + \delta^2$ becomes

$$H_t^{(3)}W_t + H_x^{(3)}W_x + H_y^{(3)}W_y + 1 = \gamma^2\sigma^2 + \delta^2, \tag{4.65}$$

implying that W_t, W_x and W_y should be constant, otherwise we would go to even higher orders. It is easier to separate eq. (4.65) order by order,

$$H_t^{(3)}W_t + H_x^{(3)}W_x + H_y^{(3)}W_y = 0, \tag{4.66}$$

$$\gamma^2\sigma^2 + \delta^2 = 1. \tag{4.67}$$

It is also important to notice that the left-hand side of eq. (4.64) is of order ε^0, while the right-hand side is of order ε^3, so they must vanish separately. This means that there are now two cases to be solved, namely one in which $\gamma = 0$ while $H_t^{(3)}, \sigma, \delta$ are left untouched, and another in which $H_t^{(3)} = 0$ with γ, σ, δ arbitrary real constants.

If $\gamma = 0$ then eq. (4.67) tells us that $\delta^2 = \Omega^2 = 1$ (recall the definition of Ω^2 from eq. (4.10b)). Now, we wouldn't like to make a priori assumptions on the nature and form of the W_t, W_x, W_y terms, that is, we wouldn't like to impose any a priori symmetries on (for example) W_x and W_y, although we note that the redefinitions $t \to \varepsilon^2 t$, $x_i \to \varepsilon x_i$ imply that the W_μ must not depend on t, x, y if they are to remain at order zero. We also wouldn't like to make a priori assumptions on the $H^{(3)}$ terms, apart from their general form eq. (4.55). Therefore, we must set $W_t = W_x = W_y = 0$ to validate eq. (4.66). With the simplifications described above, we are then led to conclude upon inspection of eq. (4.64) that W_r depends only on the r coordinate, that is, $W_r = W_r(r)$.

In this case, the metric generated by eq. (4.59) will be

$$h_{\mu\nu} = g^{(3)}_{\mu\nu} + 2rW_r(r)H^{(3)}_t \mathrm{d}t\mathrm{d}r - 2W_r(r)H^{(3)}_t \mathrm{d}r^2 - 2W_r(r)H^{(3)}_x \mathrm{d}r\mathrm{d}x - 2W_r(r)H^{(3)}_y \mathrm{d}r\mathrm{d}y, \tag{4.68}$$

where $g^{(3)}_{\mu\nu}$ is given by eq. (4.11) (with $\tau \to t$). We will discuss this result after covering the case where $H^{(3)}_t = 0$ below.

The second case, where $H^{(3)}_t = 0$, implies that if we do not want to make a priori assumptions on the nature and form of W_x and W_y, we must set them to zero. We note here that this is due to the fact that $H^{(3)}_t = 0$ implies that eq. (4.66) becomes $H^{(3)}_x W_x + H^{(3)}_y W_y = 0$. Therefore, only the terms $W_t(r)$ and $W_r(r)$ remain to be found (recall from our discussion above that they do not depend on t, x, y.) Now, one of the equations we get from eq. (4.64) is $\nabla_t W_r - \nabla_r W_t = 0$, which implies that $\nabla_r W_t = 0$. Hence, we conclude that W_t is a constant.

By applying eq. (4.59) in this case, we arrive at a metric given by

$$h_{\mu\nu} = \Omega^2 g^{(3)}_{\mu\nu} - W_t H^{(3)}_x \mathrm{d}t\mathrm{d}x - W_t H^{(3)}_y \mathrm{d}t\mathrm{d}y - W_r(r)H^{(3)}_x \mathrm{d}r\mathrm{d}x - W_r(r)H^{(3)}_y \mathrm{d}r\mathrm{d}y. \tag{4.69}$$

Upon comparing eqs. (4.68) and (4.69) with our initial metric eq. (4.11), we note that the application of Ehlers transformations to our "general" Killing vector at order ε^3 eq. (4.55) resulted in new metrics which differ from the old metric by means of additional terms that not necessarily cancel any of the "old" terms. These new terms, however have a very general form (perhaps *too* general), so it would be interesting to check some particular cases of the $H^{(3)}_i$ functions in order to give a proper physical meaning to them. This is currently being done.

We recall that we still have a nontrivial Killing vector at second order, but we can't recover the Navier-Stokes equations at that order, only the incompressibility condition. However, we have verified that eqs. (4.68) and (4.69) do produce a Navier-Stokes equation in the Einstein tensor, at third order, just like eq. (4.11) did.

4.5.2 $\sigma = f^{(3)}(t,x,y)$

If σ is a function of the form $f^{(3)}(t,x,y)$, then we find

$$\nabla_{[\mu}W_{\nu]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_t\gamma\delta \\ 0 & 0 & H_t\gamma\delta & 0 \end{pmatrix}. \tag{4.70}$$

The equation $\Omega^2 = \xi^{(3)\mu}W_\mu + 1 = \gamma^2\sigma^2 + \delta^2$ becomes

$$H_t^{(3)}W_t + H_x^{(3)}W_x + H_y^{(3)}W_y + 1 = \gamma^2(f^{(3)})^2 + \delta^2, \tag{4.71}$$

implying that W_t, W_x and W_y should be not only constant but zero. We also note that the $(f^{(3)})^2$ will be of sixth order, so we must set it to zero. The only difference between this case and the one where σ is a constant is that, here, $\delta^2 = 1$ always. By dividing in two cases ($\gamma = 0$ or $H_t^{(3)} = 0$), as before, we arrive at the same metrics as eqs. (4.68) and (4.69), with $\Omega^2 = 1$ in eq. (4.69). Thus, the discussion on the new metrics remains valid for this case as well.

4.5.3 Discussion and interpretation of results

The functions $F_i^{(1)}$, $G_i^{(2)}$ and $H_i^{(3)}$ that appeared in our expansion process to determine the Killing vectors order by order require some explanation as to what they actually mean. We recall that they appeared as general functions of order ε^1, ε^2 and ε^3, respectively, in the sense that they are combinations of all possible terms at their order. However, this implies that any combination of v_i, P and their derivatives is a symmetry of the metric at a certain order, which does not make sense. In fact, if we apply this same reasoning to increasingly higher orders, the metric will have even more symmetries. This strongly suggests that the functions $F_i^{(1)}$, $G_i^{(2)}$ and $H_i^{(3)}$ (as well as higher order analogues) should be set to zero, that is, the constants multiplying the parameters and their derivatives should all be zero. However, it still remains to see why our method produces these functions.

What we may conclude from the above discussion is that, with the functions $F_i^{(1)}$, $G_i^{(2)}$ and $H_i^{(3)}$ being zero, the metric will have less symmetries as we go to higher orders in the perturbation, until order ε^2, where only the temporal translation remains. From order ε^3 and beyond, the metric no longer has any symmetry.

Chapter 5

Conclusions

In this work we have studied the fluid/gravity correspondence in a flat (ingoing Rindler) spacetime. To do so, we started with a review of some of the main topics necessary to understand the aspects of both gravitational and fluid dynamical sides of this correspondence and also reviewed in detail an algorithmic procedure to expand our metric order by order so that at each order the metric would satisfy the Einstein equations. Particular attention was given to the case at order ε^3, which serves as our starting point to a discussion on the application of Ehlers transformations in the context of fluid/gravity correspondence, particularly in the case of a flat spacetime in ingoing Rindler coordinates.

We have shown in chapter 4 that the metric of a flat spacetime in ingoing Rindler coordinates is related to that of a Taub spacetime by an Ehlers transformation, which suggests it is useful to apply the Ehlers transformations to other Killing vectors.

We have also solved the Killing equation perturbatively in the ε-expansion, hoping to find symmetries of our seed metric at third order. With the fluid/gravity correspondence in flat spacetimes in mind, the questions we set out to study and hopefully find an answer were related to the solutions of the Einstein and Navier-Stokes equations. Given the relations between solutions of these equations, especially those that have been found since 2008 [18], it is only natural to ask ourselves whether some properties and symmetries of their solutions may tell us something about this correspondence. Obviously, the literature in these two subjects is extremely vast, such that many choices have to be made in order to do a first approach to these questions. It is open to discussion which choices are to be made in order to do an optimal approach.

It is nevertheless interesting to see what the solutions and symmetries of either the Einstein equations or Navier-Stokes equation may tell us in the context of fluid/gravity correspondence, as these might be useful to help us test the limits of this correspondence and hopefully provide us with useful hints as to what questions to make when studying these dual systems, with the correspondence then providing a bridge to facilitate our calculations and/or to help us make the right questions. It should be noted that Ehlers transformations are just a small part of a broad range of transformations relating solutions of the Einstein equations [34], so although our approach still requires further developments, there might be other interesting and perhaps more suitable approaches in order to extend the relations between Fluid Dynamics and General Relativity. It might still be useful to investigate the transformations produced by the Killing vectors at lower orders, and see whether they give us interesting results that might be treated in the context of fluid/gravity correspondence. Our focus on the third order case was due to the direct relation between our seed metric eq. (4.11) and the Navier-Stokes equation (see also the discussion on the Navier-Stokes equation appearing at third order in the expansion in ε in chapter 2), and we have indeed verified that the Navier-Stokes equations appear at third order, but as our previous result for one of the Killing vectors at order zero suggests, it is still possible that we arrive at interesting results when considering the other Killing vectors.

A possible caveat in the approach employed in this work is that the Ehlers transformations do not work at dimensions higher than four. Given the fact that fluid/gravity correspondence has its roots in AdS/CFT correspondence and string theory, where higher dimensions are required, it could perhaps be useful to find analogous or extensions of these transformations to higher dimensional spacetimes, in order to study this subject in a larger context.

Another possibility is to study the symmetries of the Einstein tensor in this context, hoping to find a relation between these and the symmetries of the Navier-Stokes equation.

Appendix A

Christoffel Symbols at Third Order

The Christoffel symbols for eq. (4.11) at order ε^3 are given by

$$\Gamma^{(3)t}{}_{tx} = \frac{1}{4r_c^2}\{-v_x(6P+5v^2)+2rv_y(\partial_x v_y-\partial_y v_x)+2r_c[(v_x\partial_x+v_y\partial_y)v_x+\partial_x P-\partial_t v_x]\},$$

$$\Gamma^{(3)t}{}_{ty} = \frac{1}{4r_c^2}\{-v_y(6P+5v^2)+2rv_x(\partial_y v_x-\partial_x v_y)+2r_c[(v_y\partial_y+v_x\partial_x)v_y+\partial_y P-\partial_t v_y]\},$$

$$\Gamma^{(3)t}{}_{rx} = \frac{v_y(\partial_y v_x-\partial_x v_y)}{2r_c^2},$$

$$\Gamma^{(3)t}{}_{ry} = \frac{v_x(\partial_x v_y-\partial_y v_x)}{2r_c^2},$$

$$\Gamma^{(3)r}{}_{tx} = \frac{1}{2r_c^2},$$

$$\Gamma^{(3)r}{}_{ty} = \frac{1}{2r_c^2},$$

$$\Gamma^{(3)r}{}_{rx} = \frac{1}{4r_c^2}\{3v_x(2P+v^2)+2r_c[(v_x\partial_x+v_y\partial_y)v_x+\partial_x P+\partial_t v_x]\},$$

$$\Gamma^{(3)r}{}_{ry} = \frac{1}{4r_c^2}\{3v_y(2P+v^2)+2r_c[(v_x\partial_x+v_y\partial_y)v_y+\partial_y P+\partial_t v_y]\},$$

$$\Gamma^{(3)x}{}_{tt} = \frac{1}{4r_c^2}\{v_x[(2P+v^2)(-2r+3r_c)+2r_cv^2]+4r_c(r-r_c)(v_y\partial_x v_y+v_x\partial_x v_x+\partial_x P+\partial_t v_x)\},$$

$$\Gamma^{(3)x}{}_{tr} = \frac{1}{2r_c}[v_x(2P+v^2)-r_c(v_x\partial_x v_x+v_y\partial_x v_y+\partial_x P+\partial_t v_x)],$$

$$\Gamma^{(3)x}{}_{xx} = \frac{v_x^3-2r_cv_x\partial_x v_x}{2r_c^2},$$

$$\Gamma^{(3)x}{}_{xy} = \frac{v_x[v_xv_y+r(\partial_x v_y-\partial_y v_x)-2r_c\partial_x v_y]}{2r_c^2},$$

$$\Gamma^{(3)x}{}_{yy} = \frac{v_x(v_y^2-2r_c\partial_y v_y)-2(r-r_c)v_y(\partial_y v_x-\partial_x v_y)}{2r_c^2},$$

$$\Gamma^{(3)y}{}_{tt} = \frac{1}{4r_c^2}\{v_y[(2P+v^2)(-2r+3r_c)+2r_cv^2]+4r_c(r-r_c)(v_x\partial_y v_x+v_y\partial_y v_y+\partial_y P+\partial_t v_y)\},$$

$$\Gamma^{(3)y}_{\ \ tr} = \frac{1}{2r_c}[v_y(2P+v^2) - r_c(v_y\partial_y v_y + v_x\partial_y v_x + \partial_y P + \partial_t v_y)],$$
$$\Gamma^{(3)y}_{\ \ xx} = \frac{v_y(v_x^2 - 2r_c\partial_x v_x) - 2(r-r_c)v_x(\partial_x v_y - \partial_y v_x)}{2r_c^2},$$
$$\Gamma^{(3)y}_{\ \ xy} = \frac{v_y[v_x v_y + r(\partial_y v_x - \partial_x v_y) - 2r_c\partial_y v_x]}{2r_c^2},$$
$$\Gamma^{(3)y}_{\ \ yy} = \frac{v_y^3 - 2r_c v_y\partial_y v_y}{2r_c^2}.$$

Appendix B

Details of Perturbative Solutions of the Killing Equation

Here we sketch the main steps of our method to solve the Killing equation order by order. It is possible to solve the order ε^0 equations without help from numerical methods, but the equations become very complicated at higher orders, so the calculations we present here were done using the Mathematica RGTC package.

B.1 Order ε^0

We write the general form of our Killing vectors as

$$\xi^\mu(t,r,x,y) = (\xi^t(t,r,x,y), \xi^r(t,r,x,y), \xi^x(t,r,x,y), \xi^y(t,r,x,y)). \tag{B.1}$$

Using RGTC, we find that the Killing equations at order ε^0 can be written in a general form as

$$\partial_y \xi^y(t,r,x,y) = 0 \tag{B.2a}$$

$$\partial_x \xi^x(t,r,x,y) = 0 \tag{B.2b}$$

$$\partial_y \xi^x(t,r,x,y) + \partial_x \xi^y(t,r,x,y) = 0 \tag{B.2c}$$

$$\partial_r \xi^t(t,r,x,y) = 0 \tag{B.2d}$$

$$\partial_x \xi^t(t,r,x,y) + \partial_r \xi^x(t,r,x,y) = 0 \tag{B.2e}$$

$$\partial_y \xi^t(t,r,x,y) + \partial_r \xi^y(t,r,x,y) = 0 \tag{B.2f}$$

$$-r\partial_r \xi^t(t,r,x,y) + \partial_r \xi^r(t,r,x,y) + \partial_t \xi^t(t,r,x,y) = 0 \tag{B.2g}$$

$$-\xi^t(t,r,x,y) - 2r\partial_t\xi^t(t,r,x,y) + 2\partial_t\xi^r(t,r,x,y) = 0 \tag{B.2h}$$

$$-r\partial_x\xi^t(t,r,x,y) + \partial_x\xi^r(t,r,x,y) + \partial_t\xi^x(t,r,x,y) = 0 \tag{B.2i}$$

$$-r\partial_y\xi^t(t,r,x,y) + \partial_y\xi^r(t,r,x,y) + \partial_t\xi^y(t,r,x,y) = 0. \tag{B.2j}$$

We see from eqs. (B.2a), (B.2b) and (B.2d) that we can easily integrate out the y, x, and r dependences, respectively, meaning that these components do not depend on these variables. Hence,

$$\partial_y\xi^x(t,r,y) + \partial_x\xi^y(t,r,x) = 0 \tag{B.3a}$$

$$\partial_x\xi^t(t,x,y) + \partial_r\xi^x(t,r,y) = 0 \tag{B.3b}$$

$$\partial_y\xi^t(t,x,y) + \partial_r\xi^y(t,r,x) = 0 \tag{B.3c}$$

$$-r\partial_y\xi^t(t,x,y) + \partial_t\xi^y(t,r,x) + \partial_y\xi^r(t,r,x,y) = 0 \tag{B.3d}$$

$$-r\partial_x\xi^t(t,x,y) + \partial_t\xi^x(t,r,y) + \partial_x\xi^r(t,r,x,y) = 0 \tag{B.3e}$$

$$\partial_t\xi^t(t,x,y) + \partial_r\xi^r(t,r,x,y) = 0 \tag{B.3f}$$

$$-r\xi^r(t,r,x,y) - 2r\partial_t\xi^t(t,x,y) + 2\partial_t\xi^r(t,r,x,y) = 0. \tag{B.3g}$$

We may further simplify our system by noting that eqs. (B.3b), (B.3c) and (B.3f) relate derivatives of ξ^t to derivatives of the other ξ^i. The same is valid for eq. (B.3a), relating ξ^x and ξ^y. By using eq. (B.3f) into eq. (B.3g), we arrive at

$$-\xi^r(t,r,x,y) + 2r\partial_r\xi^r(t,r,x,y) + 2\partial_t\xi^r(t,r,x,y) = 0, \tag{B.4}$$

which can be solved to give

$$\xi^r(t,r,x,y) = e^{t/2}C(x,y,re^{-t}), \tag{B.5}$$

where $C(x,y,re^{-t})$ is an arbitrary function of x,y,re^{-t}. For convenience, we use the notation ∂_* to refer to derivatives of re^{-t}. Putting this into our system of equations, along with the substitutions we mentioned above, we have

$$\partial_x\xi^t(t,x,y) + \partial_r\xi^x(t,r,y) = 0 \tag{B.6a}$$

$$\partial_y\xi^t(t,x,y) + \partial_r\xi^y(t,r,x) = 0 \tag{B.6b}$$

$$e^{-t/2}\partial_*C(x,y,re^{-t}) + \partial_t\xi^t(t,x,y) = 0 \tag{B.6c}$$

$$\partial_y\xi^x(t,r,y) + \partial_x\xi^y(t,r,x) = 0 \tag{B.6d}$$

$$r\partial_r\xi^y(t,r,x) + \partial_t\xi^y(t,r,x) + e^{t/2}\partial_yC(x,y,re^{-t}) = 0 \tag{B.6e}$$

$$r\partial_r\xi^x(t,x,y) + \partial_t\xi^x(t,r,y) + e^{t/2}\partial_xC(x,y,re^{-t}) = 0. \tag{B.6f}$$

At this point, we can't find further simplifications, so we introduce the ansatz

$$\xi^t(t,x,y) = f_1(t)x + f_2(t)y + f_3(t)xy + f_4(t), \tag{B.7a}$$

$$\xi^x(t,r,y) = f_5(t)x + f_6(t)y + f_7(t)xy + f_8(t), \tag{B.7b}$$

$$\xi^y(t,r,x) = f_9(t)x + f_{10}(t)y + f_{11}(t)xy + f_{12}(t), \tag{B.7c}$$

where the $f_i(t)$ are arbitrary functions of t. We plug this ansatz into eqs. (B.6a) to (B.6f) and get, after a lengthy calculation similar to the one employed up to this point,

$$\xi^t(t,x,y) = 2e^{-t/2}(g_1x + g_2y + g_3) + g_4, \tag{B.8a}$$

$$\xi^r(t,r,x,y) = re^{-t/2}(g_1x + g_2y + g_3) + e^{t/2}(h_1x + h_2y + h_3), \tag{B.8b}$$

$$\xi^x(t,r,y) = -2g_1re^{-t/2} - 2e^{t/2}h_1 + h_4y + h_5, \tag{B.8c}$$

$$\xi^y(t,r,x) = -2g_2re^{-t/2} - 2e^{t/2}h_2 - h_4x + h_6, \tag{B.8d}$$

where the g_i and h_i are constants that appear as we integrate and separate variables when solving our system with the ansatz eqs. (B.7a) to (B.7c). By setting all except one of these 10 constants to zero, we get the 10 Killing vectors eqs. (4.21a) to (4.21j). For example, if only h_6 is non-zero, we get the y-translation eq. (4.21c).

We have verified that a linear combination of eqs. (4.21a) to (4.21j) does satisfy the Killing equation, as expected.

B.2 Orders ε^1, ε^2 and ε^3

At order ε^1, we start by writing the Killing vectors as $\xi^\mu = \xi^{(0)\mu} + \xi^{(1)\mu}$, where $\xi^{(0)\mu}$ is the general Killing vector at order ε^0, which is simply the sum of the 10 independent Killing vectors found before.

We employ a procedure entirely analogous to the previous case. Namely, we use RGTC to generate the Killing equations and solve the order ε^1 equations. As previously stated, the equations at order ε^1 and higher are too lengthy, so we will list the main steps in our calculation and to illustrate we will give only some of the equations we have obtained.

One of the equations that are generated is

$$\frac{1}{r_c}e^{-t/2}(2c_9v_x + 2c_{10}v_y + e^{t/2}r_c\partial_r\xi^t(t,r,x,y)) = 0, \tag{B.9}$$

and it is immediate to verify that the r-dependence can be integrated out, which gives an arbitrary term of order ε^1, which we call $f_1(t,x,y)$,

$$\xi^t(t,r,x,y) = -\frac{e^{-t/2}}{r_c} 2(c_9 v_x + c_{10} v_y) r + f_1(t,x,y). \tag{B.10}$$

The c_i come from the ε^0 general Killing vector. Recalling that the only ε^1 term we may have is of the form $v_i(t,x,y)$, we must then write $f_1(t,x,y) = a_1 v_x(t,x,y) + b_1 v_y(t,x,y)$, where a_1 and b_1 are constants. This procedure can be used to reduce our initial system of 10 equations to 6 equations, with the functions $f_2(t,x,y), f_3(t,x,y), f_4(t,x,y)$ all being analogous to $f_1(t,x,y)$ above, that is, they are linear combinations of the velocity components.

In this remaining set of equations, all of whom must vanish, it is possible to note that the remaining dependence on r is polynomial, that is, it may be written as $g(t,x,y) + h(t,x,y)r$. Since these equations must vanish, we must have $g(t,x,y) = h(t,x,y) = 0$. For example, one of the equations we have is

$$\begin{aligned} &\frac{1}{r_c} e^{-t/2} [(-2c_7 e^t v_x + 2c_9 r_c v_x - 2c_8 e^t v_y + 2c_{10} r_c v_y) r \\ &\qquad + (2c_7 e^t v_x - e^{t/2} a_2 v_x + 2c_8 e^t v_y - e^{t/2} b_2 v_y) r_c] = 0, \end{aligned} \tag{B.11}$$

where the a_i and b_i were explained above. It is easy to note the $g(t,x,y) + h(t,x,y)r$ form of this equation. So, if the $g(t,x,y)$ and $h(t,x,y)$ are to vanish, it is possible to show by inspecting all the remaining equations that we must set $a_2 = b_2 = c_4 = c_5 = c_6 = c_7 = c_8 = c_9 = c_{10} = 0$. We could also have imposed constraints on v_x and v_y, so that not all of the above coefficients would be set to zero (it could a priori be possible that, upon certain constraints on the velocities, none of these coefficients are zero).

Proceeding with this calculation, we are led to conclude that the general form of the Killing vectors at order ε^1 is the one given by eqs. (4.44a) to (4.44f).

The procedure to solve the Killing equations at orders ε^2 and ε^3 is entirely analogous to the one at order ε^1.

Bibliography

[1] T. Damour, *Quelques propriétés mécaniques, électromagnétiques, thermodynamiques et quantiques des trous noirs*. Thèse de Doctorat d'Etat, Université Pierre et Marie Curie, Paris VI, France (1979).

[2] Eric Gourgoulhon, *A generalized Damour-Navier-Stokes equation applied to trapping horizons*. Phys.Rev. D **72** 104007 (2005).

[3] T. Damour, *Surface effects in black hole physics*, in Proceedings of the Second Marcel Grossmann Meeting on General Relativity, Ed. R. Ruffini, North Holland (1982), p. 587.

[4] Richard H. Price, Kip S. Thorne, *Membrane viewpoint on black holes: Properties and evolution of the stretched horizon*. Phys. Rev. D **33** 915-941 (1986).

[5] J. D. Bekenstein, *Black holes and the second law*. Lett. Nuovo Cimento Soc. Ital. Fis. **4**, 737 (1972).

[6] J. D. Bekenstein, *Black holes and entropy*. Phys. Rev. D **7**, 2333 (1973).

[7] J. D. Bekestein, *Generalized second law of thermodynamics in black hole physics*. Phys. Rev. D **9**, 3292 (1974).

[8] S. W. Hawking, *Black hole explosions?* Nature (London) **248**, 30.

[9] S. W. Hawking, *Particle Creation by Black Holes*. Commun. math. Phys. **43**, 199-220 (1975).

[10] G. 't Hooft, *Dimensional reduction in Quantum Gravity*. arXiv:gr-qc/9310026.

[11] L. Susskind, *The world as a hologram*. J. Math. Phys. **36**, 6377 (1995).

[12] Raphael Bousso, *The holographic principle*. Rev. Mod. Phys. **74**, 825-874 (2002).

[13] Juan M. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*. Adv. Theor. Math. Phys. **2**, 231-252 (1998).

[14] Ofer Aharony, Steven S. Gubser, Juan Maldacena, Hirosi Ooguri, Yaron Oz, *Large N field theories, string theory and gravity*. Phys. Rept. **323**, 183-386 (2000).

[15] Makoto Natsuume, *AdS/CFT Duality User Guide*. Springer Japan, 2015. Also: arXiv:1409.3575.

[16] Giuseppe Policastro, Dam T. Son, Andrei O. Starinets, *Shear viscosity of strongly coupled N=4 supersymmetric Yang-Mills plasma*. Phys. Rev. Lett. **87**, 081601 (2001).

[17] Giuseppe Policastro, Dam T. Son, Andrei O. Starinets, *From AdS/CFT correspondence to hydrodynamics*. JHEP **09**, 043 (2002).

[18] S. Bhattacharyya, S. Minwalla, V. Hubeny, M. Rangamani, *Nonlinear fluid dynamics from gravity*. JHEP **02**, 045 (2008).

[19] V. Hubeny, *The Fluid/Gravity Correspondence: a new perspective on the Membrane Paradigm*. Class. Quantum Grav. **28**, 114007 (2011).

[20] M. Rangamani, *Gravity & Hydrodynamics: Lectures on the fluid-gravity correspondence*. Class. Quantum. Grav. **26**, 224003 (2009).

[21] N. Ambrosetti, J. Charbonneau, S. Weinfurtner, *The fluid/gravity correspondence: Lecture notes from the 2008 Summer School on Particles, Fields and Strings*. [arXiv:0810.2631v1].

[22] V. Balasubramanian, P. Kraus, *A Stress Tensor for Anti-de Sitter Gravity*. Commun. Math. Phys. **208**, 413-428 (1999).

[23] I. Bredberg, C. Keeler, V. Lysov, A. Strominger, *Wilsonian approach to fluid/gravity duality*. JHEP **03**, 141 (2011).

[24] I. Bredberg, C. Keeler, V. Lysov, A. Strominger, *From Navier-Stokes to Einstein*. [arXiv:1101.2451v2]

[25] Geoffrey Compère, Paul McFadden, Kostas Skenderis, Marika Taylor, *The holographic fluid dual to vacuum Einstein gravity*. JHEP **07**, 050 (2011).

[26] Geoffrey Compère, Paul McFadden, Kostas Skenderis, Marika Taylor, *The holographic fluid dual to vacuum Einstein gravity*. JHEP **03**, 076 (2012).

[27] C. Eling, A. Meyer, Y. Oz, *The relativistic Rindler hydrodynamics*. JHEP **05**, 116 (2012).

[28] Natalia Pinzani-Fokeeva, Marika Taylor, *Towards a general fluid/gravity correspondence*. Phys. Rev. D **91**, 044001 (2015).

[29] V. Lysov, A. Strominger, *From Petrov-Einstein to Navier-Stokes*. [arXiv:1104.5502v2]

[30] C.-Y. Zhang, Y. Ling, C. Niu, Y. Tian, X.-N. Wu, *Magnetohydrodynamics from gravity*. Phys. Rev. D **86**, 084043 (2012).

[31] V. Lysov, *On the Magnetohydrodynamics/Gravity Correspondence*. [arXiv:1310.4181 - hep-th]

[32] Marco M. Caldarelli, Joan Camps, Blaise Goutéraux, Kostas Skenderis, *AdS/Ricci-flat correspondence and the Gregory-Laflamme instability*. Phys. Rev. D **87**, 061502 (2012).

[33] Marco M. Caldarelli, Joan Camps, Blaise Goutéraux, Kostas Skenderis, *AdS/Ricci-flat correspondence*. JHEP **04**, 071 (2014).

[34] Hans Stephani, Dietrich Kramer, Malcolm MacCallum, Cornelius Hoenselaers, Eduard Herlt, *Exact solutions to Einstein's field equations*. Deutscher Verlag der Wissenschaften, Cambridge University Press, 1980, 2nd Edition 2003

[35] J. Ehlers, *Konstruktionen und Charakterisierung von Lösungen der Einsteinschen Gravitationsfeldgleichungen*. Ph.D. thesis, Hamburg, Germany (1957).

[36] H.A. Buchdahl, *Reciprocal static metrics and scalar fields in the general theory of relativity*. Phys. Rev. **115**, 1325 (1959).

[37] B. Kent Harrison, *New Solutions of the Einstein-Maxwell Equations from Old*. J. Math. Phys. **9**, 1744 (1968).

[38] R. Geroch, *A method for generating solutions of Einstein's equations*. J, Math. Phys. **12**, 918 (1971).

[39] M. Mars, *The spacetime Ehlers group: transformation law for the Weyl tensor*. Class. Quant. Grav. **18**, 719 (2001).

[40] Joel Berkeley and David S. Berman, *The Navier-Stokes equation and solution generating symmetries from holography*. JHEP **04**, 092 (2013).

[41] R. E. Boisvert, W. F. Ames, U. N. Srivastava, *Group properties and new solutions of Navier-Stokes equations*. J. Eng. Math. **17**, 203-221 (1983).

[42] C. Misner, K. Thorne, J. Wheeler, *Gravitation*. Freeman, New York (1973).

[43] Matthias Blau, *Lecture Notes on General Relativity*. Available at http://www.blau.itp.unibe.ch/GRLecturenotes.html

[44] Robert M. Wald, *General Relativity*. University of Chicago Press, First Edition (1984).

[45] Eric Poisson, *A Relativist's Toolkit*. Cambridge University Press, Cambridge (2004).

[46] M. Nakahara, *Geometry, Topology and Physics*. Taylor & Francis, Second Edition (2003).

[47] J. D. Brown, J. W. York, *Quasilocal energy and conserved charges derived from the gravitational action*. Phys. Rev. D **47**, 1407 (1993).

[48] James W. York, Jr., *Role of Conformal Three-Geometry in the Dynamics of Gravitation*. Phys. Rev. Lett. **28**, 1082 (1972).

[49] G. W. Gibbons, S. W. Hawking, *Action integrals and partition functions in quantum gravity*. Phys. Rev. D **15**, 2752 (1977).

[50] J. W. York, *Boundary terms in the action principles of general relativity*. Found. Phys. **16**, 249 (1986).

[51] S. Bhattacharyya, S, Minwalla, S.R. Wadia, *The incompressible non-relativistic Navier-Stokes equation from gravity*. JHEP **08**, 059 (2009).

[52] L. Landau, E. Lifshitz, *Fluid Mechanics: Course of Theoretical Physics, Volume 6*. Butterworth-Heinemann, Second Edition (1987).

[53] Roy Maartens, *Causal Thermodynamics in Relativity*. [arXiv:astro-ph/9609119v1].

[54] Nils Andersson, Gregory L. Comer, *Relativistic fluid dynamics: Physics for many different scales*. Living Rev. Relativity. **10**:1, (2007)

[55] Luciano Rezzolla, Olindo Zanotti, *Relativistic Hydrodynamics*. Oxford University Press 2013.

[56] C. Eckart, *The Thermodynamics of Irreversible Processes. III. Relativistic Theory of the Simple Fluid*. Phys. Rev. **58**, 919 (1940).

[57] A. H. Taub, *Empty space-times admitting a three parameter group of motions*. Ann. Math. **53**, 472 (1951).

[58] Sotirios Bonanos, *Riemannian Geometry and Tensor Calculus*. Mathematica code available at http://www.inp.demokritos.gr/~sbonano/RGTC/