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Linhas de Wilson como Defeitos Superconformes

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Wilson Lines as Superconformal Defects

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Abstract

This dissertation started as a review of the relation between **deformations** of straight Wilson lines in $\mathcal{N} = 4$ super Yang-Mills theory and conformal field theory (CFT) **correlators** of local operators inserted along that Wilson line, which can then be thought of as a defect defining a defect CFT. These deformations/correlators capture interesting physical content, such as the Bremsstrahlung function of the theory. The work evolved subsequently to trying to extend this relation to the case of the ABJM theory in three dimensions. I start by recalling all the necessary ingredients to understand this relation and perform the corresponding calculations, from the basics of CFTs and CFTs with defects/boundaries to the study of (supersymmetric) Wilson loops in the theories mentioned above. With this groundwork in place, I set up the analysis of 2- and 3-point functions of so-called *displacement operators* inserted along a 1/2 BPS Wilson line of ABJM theory.

Keywords:
CFT, defect CFT, deformed Wilson loops, displacement operator

Resumo

Esta dissertação começou como uma revisão da relação entre **deformações** de linhas de Wilson na teoria de $\mathcal{N} = 4$ super Yang-Mills e **funções de correlação** de teoria de campos conformes (CFT) feitas de inserções de operadores locais ao longo daquela linha de Wilson, que pode então ser pensada como um defeito definindo uma CFT com defeito. Estas deformações/funções de correlação capturam conteúdo físico interessante, como por exemplo a função Bremstrahlung da teoria. O trabalho evoluiu posteriormente a uma tentativa de estender aquela relação para o caso da teoria ABJM em três dimensões. Eu começo recordando todos os ingredientes necessários para entender essa relação e executar os cálculos correspondentes, desde o básico de CFTs e CFTs com defeitos/barreiras até o estudo de laços de Wilson (supersimétricos) nas teorias mencionadas acima. Com essas bases colocadas, eu monto a análise das funções de 2- e 3- pontos dos chamados *operadores deslocamento* inseridos ao longo da linha de Wilson 1/2 BPS da teoria ABJM.

Palavras-chave:
CFT, CFT com defeito, laços de Wilson deformados, operador deslocamento

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Introduction

Conformal field theories (CFT) play a central role in physics, from high energy physics and string theory to condensed matter theory and the description of critical phenomena (see [1–5] for an overview). Moreover, conformal symmetry was found to play a central role in the duality between gauge theory and gravity, also known as the AdS/CFT correspondence [6]. The theories relevant in this context also enjoy supersymmetry (see [7]) and are called superconformal field theories (SCFT), see [8]. The prototypical examples of such theories in 4 and 3 space-time dimensions are called $\mathcal{N} = 4$ super Yang-Mills (SYM) [9, 10] and $\mathcal{N} = 6$ super Chern-Simons-matter, or ABJM (see [11]) theory, respectively.

An important class of non-local operator that can be studied in these SCFTs is given by supersymmetric Wilson loops (see [12–17]). These are fundamental observables of a theory, capturing its global features, and knowing as much as possible about them represents something valuable and can provide powerful checks of the AdS/CFT correspondence. In this work we are primarily interested in viewing Wilson loops as 1-dimensional defects, defining a 1-dimensional CFT, along which local operators can be inserted. One can then compute the CFT data of the inserted operators (anomalous dimensions and structure constants) and extract from them physical information such as, for example, the radiated energy and Bremsstrahlung functions [18–20]. An alternative way to study such operator insertions along the defect is to consider deformations of the contour on which the Wilson operator is defined (which from a straight line become 'wavy' or 'wiggled').

This point of view on Wilson lines as superconformal defects has been studied quite extensively in recent years in $\mathcal{N} = 4$ SYM (see for example [21] and [22] for an overview), much less so for ABJM. So far, the only case that has been considered, to the best of our knowledge, is the computation of 2-point functions ([23] and [24]) of a certain operator built out of the gauge field and scalars of the theory, called *displacement operator* [46]. The aim of this dissertation is to push forward this study to the case of 3- and 4- point functions.

Outline This work is organized as follows. The first chapter is devoted to reviewing all the necessary ingredients encountered in the study of conformal field theories and is based mostly on [4, 5, 25–30, 56, 59]. It also contains a review of superconformal field theories (SCFT) which are the central framework of subsequent chapters.

In the second chapter, the reader will find a formulation of defect conformal field theories in the embedding formalism, a tool borrowed from General Relativity. Special attention is given to the introduction of the displacement operator and of its supermultiplet. The relevant literature for this chapter includes [46–48, 51].

Chapter 3 is then dedicated to the presentation, from the basics, of the extended operators used in our applications, namely the Wilson loops. We follow mostly [28,62,63].

We discuss their formulation, how to compute them, and their role in the AdS/CFT correspondence.

Finally, chapter 4 applies all the knowledge developed in the preceding ones, showing explicit calculations concerning the determination of structure constants of defect correlators as well as computations of deformed Wilson lines. Part of the content here is a review of [23], but new results for the ABJM theory are presented, following [24].

The reader might find some general comments about the Poincarè group and an overview of useful tools provided by General Relativity in the appendices.

1 Conformal Field Theory

One of the main motivations for studying CFTs resides in the fact that they are the endpoints of the renormalization group (RG) flow of quantum field theories (QFT). In the space of QFTs, as one varies the couplings, the CFTs are fixed points characterized by couplings invariant under scale variations. Different microscopic theories may have similar behaviour as they are rescaled to the macroscopic environment and become indistinguishable. This can be verified thermodynamically, by determining the set of coefficients that describes the system behaviour, known as *critical exponents*. As an example, it is well known that boiling water, uni-axial magnets (by means of the Ising Model) and the ϕ^4 theory can be put in the same box of IR equivalent theories, or in more technical language, in the same *universality class*, because the phase transitions involved are similar. This example appeals to our intuition on such phenomena due to real life notions, but of course we have other non-trivial examples in Nature.

Studying CFTs then allows us to map out the possible endpoints of RG flows in the space of QFTs, therefore leading to an understanding of such space. The goal is then to solve/determine fixed points, which means, as we will see, to compute the observables: operator dimensions and correlation functions. The existing methods to compute them nowadays include Monte-Carlo simulations or high-temperature expansion for lattice models, the so called ϵ-expansion for analytical results and also the most recently developed Conformal Bootstrap.

This chapter intends to give a review of the essential topics in CFT, starting with the basic notions on the quantum theory of fields, passing through generalities on conformal symmetries and particularities encountered in CFTs, finishing with the inclusion of more symmetries into the SCFT, of special importance in the subsequent chapters.

1.1 The essentials of QFT

We start by recapitulating some of the main ingredients of the quantum theory of fields, since CFTs can be understood as nothing more than QFTs invariant under conformal transformations. As we are going to see, the most important objects in our study are the so called n-point functions (or *correlators*, or yet *correlation functions*), so let us start defining them through the formulation of QFTs by path integrals.

1.1.1 Correlation Functions

The scheme to construct them always follows the same script: one defines the *vacuum persistence* of the theory via a path integral, then external sources are linearly coupled to the fields of the theory in a proper way, depending on the nature of each field, and finally such functions are obtained from functional derivatives with respect to such sources.

The simplest case to illustrate this procedure is the single massive scalar field ϕ. The action in this case is given by:

$$S[\phi] \equiv \int d^d x \mathcal{L}\left[\phi, \partial_\mu \phi\right] = \int d^d x \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - V(\phi)\right) \ , \tag{1}$$

where m stands for the mass of the field and V for any potential. The integral is taken over all spacetime points. From this, the so called *vacuum persistence* is just the vacuum ($|0\rangle$) normalization as given in path integral formulation:

$$Z_0 \equiv \langle 0|0\rangle = \int \mathcal{D}\phi e^{iS[\phi]} \ . \tag{2}$$

The next step then is to couple an external source. In the case of a scalar field such source is simply a spacetime function $J(x)$. The notion of *vacuum persistence* then becomes dependent of the source and we can write:

$$Z[J] \equiv \mathcal{N} \int \mathcal{D}\phi e^{iS[\phi] + i\int d^d x J(x)\phi(x)} \tag{3}$$

in which $\mathcal{N}$ is a normalization we note to be equal to $Z[0]/Z_0$, traditionally identified to unity for a pre-normalized to unity vacuum, $\langle 0|0\rangle = 1$.

The n-point functions are just expectation values of n time-ordered[1] operators inserted into the vacuum. As in quantum mechanics, such insertions are verified to be equivalent to insertions of the corresponding (space-time) position eigenvalues of those operators. In the case being treated we have then:

$$\langle 0|T\left(\widehat{\phi}(x_1)...\widehat{\phi}(x_n)\right)|0\rangle = \int \mathcal{D}\phi \ \phi(x_1)...\phi(x_n) \ e^{iS[\phi]} \ , \tag{4}$$

[1] From right to left, the operators are put in chronological increasing order.

where the hats stand for operators. We easily verify then that:

$$\boxed{\langle 0|T\left(\widehat{\phi}(x_1)...\widehat{\phi}(x_n)\right)|0\rangle = \frac{(-i)^n}{Z[0]}\frac{\delta^n Z[J]}{\delta J(x_1)...\delta J(x_n)}} \,. \tag{5}$$

For this reason $Z[J]$ is known as the *generating functional*, it gives us a way of calculating correlators between operators.

The construction we have just done can then be extended to general cases very simply: in order to include more fields, obviously we need more external sources, one for each. Naturally we will be interested in cases involving fermionic fields, in these situations the source cannot be so simple, instead must obey the extra condition of behaving as a *grassmannian* variable, as fermions do. It need to be emphasized that the introduction of external sources are just a mathematical tool, even though it can be verified that the sources are related to classical quantities.

1.1.2 Symmetries

As in any area of physics, symmetries play an important role here, then it is necessary to review the basics involving them. The most important result, due to E. Noether,[2] relates conserved quantities to continuous symmetries of a system.

To see such relation, we consider the effect of an infinitesimal transformation in the action of a given general field ϕ. The transformation can affect both the position and the field:

$$\begin{aligned} x &\to x' \\ \phi(x) &\to \phi'(x') = \mathcal{F}(\phi(x)) \end{aligned} \,, \tag{6}$$

and, in general, can be written as:

$$\begin{aligned} x'^{\mu} &= x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \\ \phi'(x') &= \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \end{aligned} \,, \tag{7}$$

where ω_a's denote a set of parameters of the transformation.

The hyphotesis of invariance of the action S under these transformations allows us to

[2]Amalie Emmy Noether: German mathematician, gave fundamental contributions to theoretical physics and abstract algebra, $\star$03/23/1882 †04/14/1935.

write:

$$
\begin{aligned}
S \to S' &= \int d^d x \mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) \\
&= \int d^d x' \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x')) \\
&= \int d^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu (\phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x))\right)
\end{aligned} .
$$

Note now that the Jacobian appearing above has the form (at first order):

$$
\begin{aligned}
\frac{\partial x'^\mu}{\partial x^\nu} &= \delta^\mu_\nu + \partial_\nu \left(\omega_a \frac{\delta x^\mu}{\delta \omega_a}\right) \\
\frac{\partial x^\nu}{\partial x'^\mu} &= \delta^\nu_\mu - \partial_\mu \left(\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right)
\end{aligned} . \tag{8}
$$

Using the property $\det(1+E) \sim 1 + \mathrm{Tr}E$, for E small, we can then write:

$$
\begin{aligned}
S' = &\int d^d x \, (1 + \partial_\mu(\omega_a \delta x^\mu / \delta \omega_a)) \\
&\times \mathcal{L}\left(\phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x), (\delta^\nu_\mu - \partial_\mu(\omega_a \delta x^\nu / \delta \omega_a)) \partial_\nu (\phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x))\right)
\end{aligned} . \tag{9}
$$

Now we just have to expand the Lagrangian density appearing inside the integral to be able to see the variation caused by the infinitesimal transformation we did. With no explicit x dependence in the Lagrangian or in $\mathcal{F}$, we have:

$$
\begin{aligned}
&\mathcal{L}' \equiv \mathcal{L}\left(\phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x), (\delta^\nu_\mu - \partial_\mu(\omega_a \delta x^\nu / \delta \omega_a)) \partial_\nu (\phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x))\right) \\
&= \mathcal{L}(\phi, \partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi} \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a} \\
&+ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \left(\partial_\mu \left(\omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}\right) - \partial_\mu \omega_a \frac{\delta x^\nu}{\delta \omega_a} \partial_\nu \left(\phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x)\right)\right) + \mathcal{O}(\omega_a^2, \partial_\mu^2 \omega_a)
\end{aligned} . \tag{10}
$$

Considering the unity in the expression of the Jacobian, the first term in the second line above will cancel out the contribution coming from the original action for the variation δS, while the second term in the second line will join with the first term (after distribution) in the third line after integration by parts to vanish in virtue of the equations of motion; the relevant contribution to the variation due to the trace term in the Jacobian expression will come from its multiplication with the original Lagrangian only. The net result is:

$$
\boxed{
\begin{aligned}
&\delta S \equiv S' - S = -\int d^d x j^\mu_a \partial_\mu \omega_a \\
&\text{with } j^\mu_a \equiv \left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}\right\} \frac{\delta x^\nu}{\delta \omega_a} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \mathcal{F}}{\delta \omega_a}
\end{aligned}
} . \tag{11}
$$

Integration by parts then yields (noticing that the parameters vanish at the endpoints):

$$\delta S = \int d^d x \partial_\mu j_a^\mu \omega_a \ . \tag{12}$$

From this it is clear the relation we were looking for: null variation of the action implies conservation of the canonical[3] currents j_a^μ, which has an associated conserved charge $Q_a \equiv \int d^{d-1}x j_a^0(x)$, where $d^{d-1}x$ stands for integration over the spatial part. It must be emphasized that this holds at classical level, since only on-shell (read equations of motion satisfied) configurations of the fields imply invariance of the action according to the action principle; in other words, what we have just done tells nothing at the quantum level, but we will see right below that it imposes constraints on the correlation functions. Moreover, since the procedure above was made for infinitesimal parameters, it can be extended to finite transformations through exponentiation as always, with the generators G_a given through the definition:

$$\phi'(x) - \phi(x) \equiv -i\omega_a G_a \phi(x) \ . \tag{13}$$

Equations (7) then gives:

$$\boxed{iG_a\phi = \frac{\delta x^\mu}{\delta\omega_a}\partial_\mu\phi - \frac{\delta\mathcal{F}}{\delta\omega_a}} \ . \tag{14}$$

1.1.3 Ward Identities

A first consequence about symmetry transformations in the quantum scenario[4] can be derived as follows. Consider the following n-point function:

$$\langle\phi(x_1')...\phi(x_n')\rangle = \int \mathcal{D}\phi \ \phi(x_1')...\phi(x_n') \ e^{iS[\phi]} \ . \tag{15}$$

Supposing the measure is invariant under such transformation of the field, if we rename the dumb variable ϕ in the integrand to ϕ' and perform a change of variable writing

$$\phi'(x') = \mathcal{F}(\phi(x)) \ ,$$

the assumption of invariant action gives:

$$\begin{aligned}\langle\phi(x_1')...\phi(x_n')\rangle &= \int \mathcal{D}\phi \ \mathcal{F}(\phi(x_1))...\mathcal{F}(\phi(x_n)) \ e^{iS[\phi]} \\ &= \langle\mathcal{F}(\phi(x_1))...\mathcal{F}(\phi(x_n))\rangle\end{aligned} \ . \tag{16}$$

[3]The so derived current carries the name canonical, but we can add the divergence of any antisymmetric tensor to it, so the definition is somehow ambiguous.

[4]After all, nothing prevents us to assume we are dealing with quantum fields if we do not use classical equations of motion.

This result shows how simple correlators between operators at transformed spacetime points are related to the correlation functions at the original spacetime points: we see that only the functional form of the transformation affecting the fields is important; in the case of spacetime translation, for example, the field is not affected (that is, $\mathcal{F}(\phi) = \phi$) and therefore only relative positions between operators are relevant in the computation of correlators. From a more ample perspective, considering the Lorentz transformations entirely, this result implies that the n-point function at the new coordinates is obtained from the same at the original points simply from the nature of the fields involved: if one has one vector field times one tensor field inside the brackets, for example, the resulting correlator will be equal to (transformation rule for vector) $\times$ (transformation rule for tensor) $\times$ (original correlation function).

Now, we have just seen that the main characters in the classical case about continuous symmetries are the conserved currents and charges. How can we relate them to n-point functions? This question has a well known answer and gives us the reflection of those symmetries on quantum quantities: the *Ward-Takahashi* identities.

According to equation (13), an infinitesimal transformation on the field reads: $\phi'(x) = \phi(x) - i\omega_a G_a \phi(x)$. Using this to change the functional integration variable on the defining equation (4) for correlators, we have[5]

$$\begin{aligned}\langle \phi(x_1)...\phi(x_n)\rangle &\equiv \int \mathcal{D}\phi \ \phi(x_1)...\phi(x_n) \ e^{iS[\phi]} \\ &= \int \mathcal{D}\phi' \ \phi'(x_1)...\phi'(x_n) \ e^{iS[\phi']} \\ &= \int \mathcal{D}\phi \prod_{\mathrm{i}=1}^{n} \left(\phi(x_{\mathrm{i}}) - i\omega_a G_a \phi(x_{\mathrm{i}})\right) e^{iS[\phi] + i\int d^d y \partial_\mu j^\mu_a \omega_a}\end{aligned} ,$$

where (12) was used to get the last line. To first order in ω_a, the product of fields inside the integral can be written as:

$$\prod_{\mathrm{i}=1}^{n} \left(\phi(x_{\mathrm{i}}) - i\omega_a G_a \phi(x_{\mathrm{i}})\right) = \phi(x_1)...\phi(x_n) - i\sum_{\mathrm{i}=1}^{n} \phi(x_1)...G_a\phi(x_{\mathrm{i}})...\phi(x_n)\omega_a(x_{\mathrm{i}}) + \mathcal{O}(\omega_a^2) \ .$$

On the other hand, the exponential can also be expanded to first order, resulting in:

$$e^{iS[\phi(x)] + \int d^d y \partial_\mu j^\mu_a \omega_a} \simeq e^{iS[\phi(x)]} + i\int d^d y \partial_\mu j^\mu_a(y) e^{iS[\phi(x)]} \omega_a(y) \ ,$$

where dependences on the variables were emphasized and the partial derivatives are taken with respect to y.

[5] Note that a time-ordering is already assumed here.

Putting all together, still at first order, we get:

$$\int d^d y \partial_\mu \langle j_a^\mu(y)\phi(x_1)...\phi(x_n)\rangle \omega_a(y) = \sum_{\mathrm{i}=1}^{n} \langle \phi(x_1)...G_a\phi(x_\mathrm{i})...\phi(x_n)\omega_a(x_\mathrm{i})\rangle \ ,$$

which gives us the desired result noticing that $\omega_a(x_\mathrm{i})$ can be written as

$$\omega_a(x_\mathrm{i}) = \int d^d y \delta^d(y - x_\mathrm{i})\omega_a(y) \ ,$$

and also that it is an arbitrary function of y:

$$\boxed{\partial_\mu \langle j_a^\mu(y)\phi(x_1)...\phi(x_n)\rangle = \sum_{\mathrm{i}=1}^{n} \langle \phi(x_1)...G_a\phi(x_\mathrm{i})...\phi(x_n))\rangle \delta^d(y - x_\mathrm{i})} \ . \tag{17}$$

Finally, what about the charges? A result involving them can be obtained from these identities as follows. Integrate the expression above over the entire $(d-1)$ spatial coordinates and over a tiny time interval (t^-, t^+) around x_1^0 (without lost of generality), in a way that this hypervolume excludes all the points $x_2...x_n$. This particular "surface" allows us to make use of the Gauss theorem in the l.h.s of that equation, which in addition to a trivial integral of a delta function in the r.h.s yields:

$$\langle Q_a(t^+)\phi(x_1)\phi(x_2)...\phi(x_n)\rangle - \langle \phi(x_1)Q_a(t^-)\phi(x_2)...\phi(x_n)\rangle = \langle G_a\phi(x_1)\phi(x_2)...\phi(x_n)\rangle \ ,$$

where it was used the definition of classical charge seen before.

In the limit where the time interval goes to zero, since the procedure is valid for any set of fields $\phi(x_2)...\phi(x_n)$, we get the following identity between operators:

$$\boxed{[Q_a, \phi] = G_a\phi} \ , \tag{18}$$

that is, in the formalism of operators, the supposedly conserved charge Q_a generates infinitesimal transformations.

1.1.4 The special current: Stress-Energy Tensor

With the conservation law well established, let us apply it to an important case. Suppose our system has translation symmetry, that is, it is invariant under the following transformations:

$$\begin{aligned} x'^\mu &= x^\mu + a^\mu \\ \phi'(x') &= \phi(x) \end{aligned} \ , \tag{19}$$

for some four-vector $a^\mu = \eta^{\mu\nu} a_\nu$. According to what we have just done, in view of the four components of the (vectorial) parameter, the conserved current will be a rank-2 tensor:

$$\begin{aligned} j^{\mu\nu}_{\text{translation}} \equiv T_c^{\mu\nu} &= \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\rho \phi - \delta^\mu_\rho \mathcal{L} \right\} \frac{\delta x^\rho}{\delta a_\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \mathcal{F}}{\delta a_\nu} \\ &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \end{aligned} . \tag{20}$$

The conserved charge is obtained from $P^\mu \equiv \int d^{d-1}x T_c^{\mu 0}(x)$ and in the case of $\mu = 0$ for example, we have:

$$\begin{aligned} P^0 &= \int d^{d-1}x T_c^{00}(x) \\ &= \int d^{d-1}x \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - \mathcal{L} \right) , \\ &= \int d^{d-1}x \left(\pi(x) \dot{\phi}(x) - \mathcal{L} \right) \end{aligned}$$

that is, it is equal to the energy (note the Legendre transform of the Lagrangian density). We note those charges are already well known by us: from equations (18) and (14) we get that $[P^\mu, \phi] = -i\partial_\mu \phi$, that is, the charges are nothing more than the momentum operator! That is why the tensor $T_c^{\mu\nu}$ is known as the (canonical) *stress-energy* tensor.

What is so special about this tensor? First of all it defines the energy and momentum of the field. Secondly, as we will see when we talk about conformal symmetries, basically it makes possible local relations between conformal correlation functions; also its trace contains important informations. At this point, however, another nice fact involving it is its use to axiomatic definition of local theories: we say a quantum field theory is local when it has a conserved stress-energy tensor.[6]

Usually we will be interested in such tensor in its symmetrized form. Although in the case illustrated above it is already symmetric in the canonical form, not always this will be true. Fortunately, due to the freedom in the definition of currents stated before, we can make it symmetric in general by means of Lorentz invariance asumption. Under this condition, a clever extra term gives a new tensor physically equivalent to the first one, but now satisfying the desired feature. Consider then the infinitesimal Lorentz transformation:

$$\begin{aligned} x'^\rho &= x^\rho + \omega_{\mu\nu} \eta^{\mu\rho} x^\nu \\ \mathcal{F}(\phi) &= L_\Lambda \phi \simeq \phi - \frac{1}{2} i \omega_{\mu\nu} \mathcal{S}^{\mu\nu} \phi \end{aligned} , \tag{21}$$

where $\mathcal{S}^{\mu\nu}$ are hermitian matrices satisfying Lorentz algebra (see appendix A) and the

[6] Actually what is usually required is a continuous unitary representation of the translation group, which englobes therefore the conservation law we are talking about; moreover, besides this, other axioms are necessary to define local field theories properly, but it is not our focus here, so we refer the interested reader to [74].

omegas are well known to be antisymmetric (the condition to preserve line elements under the transformation). We have then that:

$$\frac{\delta x^\rho}{\delta\omega_{\mu\nu}} = \frac{1}{2}(\eta^{\mu\rho}x^\nu - \eta^{\nu\rho}x^\mu) \ , \qquad \frac{\delta\mathcal{F}}{\delta\omega_{\mu\nu}} = -\frac{i}{2}\mathcal{S}^{\mu\nu}\phi \ ,$$

the second of them gives us that $\mathcal{S}^{\mu\nu} = -\mathcal{S}^{\nu\mu}$. The conserved current is obtained from equation (11):

$$\begin{aligned} j^{\sigma\mu\nu} &= \left\{\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi)}\partial_\rho\phi - \delta^\sigma_\rho\mathcal{L}\right\}\frac{\delta x^\rho}{\delta\omega_{\mu\nu}} + \frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi)}\frac{\delta\mathcal{F}}{\delta\omega_{\mu\nu}} \\ &= \frac{1}{2}(T_c^{\sigma\mu}x^\nu - T_c^{\sigma\nu}x^\mu) - \frac{i}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi)}\mathcal{S}^{\mu\nu}\phi \end{aligned} \ . \tag{22}$$

The next step is to add the divergence of an antisymmetric (in the first two indices) tensor $B^{\sigma\mu\nu}$ to the canonical stress-energy tensor to construct the symmetric one, $T_B^{\sigma\mu}$. We choose $B^{\sigma\mu\nu}$ in a manner that we can make use of the conservation law for Lorentz current. At our disposal we have the following facts:

$$\begin{aligned} &T_B^{\mu\nu} = T_c^{\mu\nu} + \partial_\sigma B^{\sigma\mu\nu} \\ &\partial_\sigma T_c^{\sigma\nu} = 0 \\ &\partial_\sigma T_B^{\sigma\nu} = 0 \\ &\partial_\sigma j^{\sigma\mu\nu} = 0 \end{aligned} \ .$$

If we require then that $j^{\sigma\mu\nu} = 1/2(T_B^{\sigma\mu}x^\nu - T_B^{\sigma\nu}x^\mu)$, $T_B^{\mu\nu}$ automatically will be symmetric and we get a way to look for it explicitly. Inserting this into equation (22), we have:

$$T_B^{\sigma\mu}x^\nu - T_B^{\sigma\nu}x^\mu = T_c^{\sigma\mu}x^\nu - T_c^{\sigma\nu}x^\mu - i\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi)}\mathcal{S}^{\mu\nu}\phi \ .$$

Therefore, it just remains to find a tensor satisfying:

$$\partial_\rho B^{\rho\sigma\mu}x^\nu - \partial_\rho B^{\rho\sigma\nu}x^\mu = -i\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi)}\mathcal{S}^{\mu\nu}\phi \ ,$$

or, differentiating with respect to x^σ:

$$\partial_\rho(B^{\rho\nu\mu} - B^{\rho\mu\nu}) = \partial_\rho\left(-i\frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\mathcal{S}^{\mu\nu}\phi\right) \ .$$

Thus, together with the condition of antisymmetry between the first two indices, this last relation induces the following simplest choice for $B^{\rho\mu\nu}$:

$$B^{\rho\mu\nu} = \frac{i}{2}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\mathcal{S}^{\mu\rho}\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\rho\phi)}\mathcal{S}^{\mu\nu}\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\mathcal{S}^{\nu\rho}\phi\right) \ . \tag{23}$$

We succeed in our mission, and the new stress-energy tensor carries the name of *Belinfante* energy-momentum tensor, as well known in the literature. From now on we are going to assume we are always dealing with such a symmetric tensor (valid on shell, note) and the notation for it will be simplified to $T^{\mu\nu}$.

The symmetric form of such tensor also allows us to relate it directly to the energy-momentum tensor obtained from the point of view in which we consider a transformation on the metric of a system instead of on the positions and fields. In fact this is an alternative definition for it, usually given in the literature concerning General Relativity, equivalent to the one we have just given. In that context, the energy-momentum tensor is defined to be the variation of the matter content of the action, S_M, with respect to the metric $g_{\mu\nu}$:

$$T^{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \ ,$$

where g stands for the determinant of the metric. Consequently, the variation of S_M can be written as:

$$\delta S_M = -\frac{1}{2} \int d^d x T^{\mu\nu} \delta g_{\mu\nu} \ . \tag{24}$$

1.1.5 Feynman Diagrams and Renormalization Group

This last subsection closes the fundamentals of quantum field theory giving a more practical way for obtaining the correlators in a given theory and also making relevant considerations about theories with interactions.

In a previous subsection we saw the definition of correlation functions in the path integral formulation. Computing them consisted in performing functional derivatives of the generating functional with respect to external sources, the choice of source with respect to which one have to derive depending on the field insertions in the correlators.

The generating functional can be put in a more practical form by making use of the 2-point correlators, called *propagators*. That is because they are Green functions of quadratic operators[7] appearing in Lagrangians defining non-interacting field theories and can be used to manipulate them; for example, in (1), integrating by parts and assuming fields vanish far from the origin, the Lagrangian can be put easily in the form:

$$S[\phi] = -\frac{1}{2} \int d^d x \phi(x) \left(\partial^2 + m^2\right) \phi(x) \ ,$$

where the potential was set to zero also, after all it is a free theory, and $\partial^2 \equiv \partial_\mu \partial^\mu$. The propagator in this case is then:

$$D(x-y) = \int \frac{d^d k}{(2\pi)^d} \frac{i e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} \ , \tag{25}$$

[7] We do not give a precise definition here, just appeal to the intuitive meaning.

where we omit derivation because it is reproduced in any text book, including [28]. Basically it consists in finding the Green function in momentum space and then integrating the result to take it back to spacetime.

Using such function, one can show that the generating functional in this case can be put in the form:

$$Z[J] = Z_0 \exp\left(-\frac{1}{2}\int d^d x d^d y J(x) D(x-y) J(y)\right) \ ,$$

notice that an odd number of fields ϕ has vanishing correlation function in this case, since J is always set to zero at the end.

Once the expression of a propagator is given, higher correlators can be found as products of them with combinatorial factors in front of them. This special result is just the famous Wick's theorem: in each case being computed, the result is the sum of all possible contractions of the fields.[8]

For theories including more fields, for each field we are going to have a quadratic operator associated and, therefore, a propagator; some of them can be not uniquely defined, which is the case of *gauge* fields, therefore demanding a choice of *gauge* to have it right. The point is that, once one has the propagators of a theory in hands, one should be able to compute any correlator by multiplying them and, as we are about to see, integrating the result when dealing with interacting theories.

A subtlety raises in interacting field theories. Basically the action now is going to have two parts: one describing the free theory, say S_0, and one containing interaction terms of the involved fields, say S_{int}. Our propagators are defined only based in S_0, so what to do? How do we separate those terms, but at the same time include the interactions on propagators? Well, usually interaction terms come up with a coupling constant in front of them to tell us the magnitude of it. Assuming then that such couplings are small, we can always expand that part of the exponential in powers of them up to the order desired. For the first order we are going to drop the fields defining the interaction and an integral, the second order drops the double of fields and two integrals, and so on. For example, a theory of two massive real scalars A and B and an interaction like $gABB$, being g the coupling constant, has the vacuum persistence to first order given by:

$$\begin{aligned} Z &= \int \mathcal{D}[A,B] e^{iS_0[A,B] + ig\int d^d x A(x)B(x)B(x)} \\ &= \int \mathcal{D}[A,B] e^{iS_0[A,B]} \left(1 + ig\int d^d x A(x)B(x)B(x) + \mathcal{O}(g^2)\right) \end{aligned} .$$

Computing any n-point function now will include the calculation of two terms, one from the free theory and one from the first order *perturbation* correction due to the interaction,

[8] A contraction between to fields is defined to be their propagator.

which at the end of the day means the integration of another $(n+3)$-point function. For example:

$$\langle A(x)\rangle = ig\int d^dy\langle A(x)A(y)B(y)B(y)\rangle + \mathcal{O}(g^2)\ .$$

Generally speaking then, perturbative calculations can be performed once one knows the propagators of the free theory and the dropped form of each interaction term. It turns out this set of tools carries a name, the so called *Feynman rules*, due to Feynman.[9] The last term, also called *vertex*, is usually represented diagramatically, together with the propagators, to make computations intuitive, easier and fun. Those diagrams consist in points from which different kind of lines comes out, depending on the kind of field involved in the interaction. Traditionally, scalars are represented by straight lines, while fermionic fields by dashed lines and gauge fields by wavy lines. For example, if one has an interaction like the previous one, but with B fermionic, the diagram associated to the dropping term would be like:

$$ig\int d^dxABB \qquad \rightarrow$$

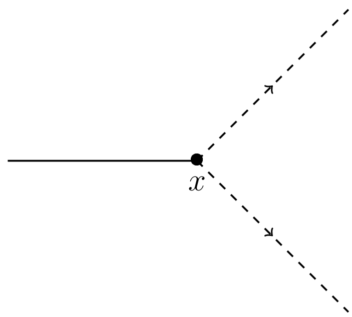

Figure 1: Example of a Feynman rule for one scalar (denoted by A, solid line) and two fermionic fields (denoted by B, dashed lines).

Notice that the fields attached to vertices are evaluated at an internal point x, which is integrated over.

All of that, however, only make sense provide the couplings are small. It is important then, to consider the fact that such couplings generally can depend on the *scale of energy* being considered, which is translated in the language of fields as predetermined *cutoffs* for their momentum frequencies, *infrared* for low frequencies (low energies) and *ultraviolet* for high frequencies (high energies). It turns out that the variation of the magnitude of those couplings with respect to the energy scale is dictated by a well known relation, which is a consequence of the so called *Callan-Symanzik* equation, a differential equation describing the evolution of n-point functions under changes on the energy scale.

Technically, choosing an energy scale is equivalent to assuming *cutoffs* for a theory, which can be interpreted also as a mass scale M or an inverse length scale $1/a$ for the theory, since in natural units momenta, mass, length^{-1} and energy have the same dimen-

[9] Richard Philips Feynman: American theoretical physicist, gave exceptional contributions to quantum electrodynamics and quantum physics, and to physics and mathematics in general, ⋆05/11/1918 †02/15/1988.

sion. Naturally then, correlators are going to depend on the energy scale and that is why they vary accordingly. Commonly, divergences appear in their computation: for a given limit of the cutoffs, the correlation function explodes. To avoid that complication, in order to maintain physically relevant quantities finite, and therefore measurable quantities, so-called *counterterms* are introduced in the action to cancel out the divergent part. This is a standard procedure in QFTs and carries the name of *Renormalization Group*; the theory that underwent such a process is said to be *renormalized*, in some *renormalization scheme* (chosen scale in which the physical couplings are defined). Counterterms are traditionally represented by δ's of the important quantities, like δm for the mass parameter, for example.

The Callan-Symanzik equation is then expressed in terms of such counterterms, for a given theory:

$$\boxed{\left[M\frac{\partial}{\partial M}+\beta(g)\frac{\partial}{\partial g}+n\gamma(g)\right]G^{(n)}\left(\{x^i\};M,g\right)=0}\,, \tag{26}$$

where the theory was assumed to depend only on a coupling g. $G^{(n)}$ is the renormalized n-point function. Two significant quantities were then defined: the beta-function and the gamma-function. They describe the variations of the coupling and of the fields with respect to the mass (or some equivalent parameter) scale, respectively:

$$\boxed{\beta(g)\equiv\frac{\delta g}{\delta\log M}\quad,\quad\gamma(g)\equiv-\frac{\delta\eta}{\delta\log M}}\,. \tag{27}$$

We have $g\to g+\delta g$ and $\phi\to(1+\delta\eta)\phi$, assuming we have only one scalar field ϕ in the theory; if there are more fields, there are more gamma functions. The Callan-Symanzik equation (26) follows directly from $dG^{(n)}=n\delta\eta G^{(n)}$.

Once one knows the beta function for a given coupling of the theory, one also knows how the coupling behaves in all scales of lengths. As we have just seen, the computation of correlation functions is usually done perturbatively, such that we also know such quantity perturbatively. Explicitly, that function is computed once the coupling counterterm δ_g and the so called *field-strength* counterterm δ_Z is known:

$$\beta(g)=M\frac{\partial}{\partial M}\left(-\delta_g+\frac{1}{2}g\delta_Z\right)\,.$$

In the case of more fields included in the theory, the second term between brackets is turned into a sum over δ_Z's. That counterterm is defined to be the one balancing a scale factor relating the renormalized field ϕ and the *bare* (non-renormalized) field ϕ_0: $\phi=Z^{1/2}\phi_0$.

Not only the appearance of the strength field counterterm, but also and mainly the existence of gamma functions, shows a feature in the behaviour of fields: they feel changes of the length scale. In the next section we are going to define the so-called *scaling dimension* of the fields more precisely, the quantity behind all this, and study it more deeply. Field scaling plays a very important role in what follows, as basically things are classified according to the sign of their scaling dimension: positive value indicates *relevant* operators, negative value *irrelevant* operators and vanishing ones *marginal* operators.

At the end of the day, rescaling the theory is a procedure to study its behaviour in infrared and ultraviolet limits. The scaling dimension then enters in the game to tell us how operators behave in those limits too, some of them losing influence over the system, while the others acquiring more influence. The name of what kind of operators plays each role is suggestive, being given conventionally by their behaviour in the infrared limit, that is, when the theory is rescaled from microscopic lengths to macroscopic lengths.

To finish the section, we point out that it is not always possible to renormalize a theory. When an infinite number of counterterms are required to do so, we say the theory is *non-renormalizable*. Of course, this is a very important problem, generally to have a physically meaningful theory we will need *renormalizability*, because it is what makes physical quantities measurable. One of the steps in this process is called *dimensional regularization* and consists in computing divergent integrals in shifted non-integer dimensions and then expanding the result around the desired dimension.

1.2 Conformal Symmetry

All the essential ingredients from quantum field theory were reviewed above and we can finally go to the case of interest: theories containing conformal symmetry. The first step to study them is obviously to get familiarized with the conformal group and the structure of the associated algebra. We start by looking at the so-called *conformal transformations*. By definition, they are the ones which preserves angles, unlike Lorentz transformations, which also preserves lengths. This is equivalent to say that such transformations leave the metric tensor unaltered, up to a scaling factor, which can depend on the position. That is, we can express their effect as:

$$g_{\mu\nu} \to \Lambda(x) g_{\mu\nu} \ ,$$

where $\Lambda(x)$ is some positive function, equaling to unity if lengths preservation is demanded. To see this more clearly, consider for example the definition of angles between two vectors. If x and y are two vectors, the angle θ between them is defined by means of the scalar product:

$$\cos\theta = \frac{x \cdot y}{|x||y|} \ ,$$

from which we see that, a scaling factor on the coordinates does not affect the left-hand side. Fig. 2 below illustrates a generic modification on the geometry of the spacetime under conformal transformations:

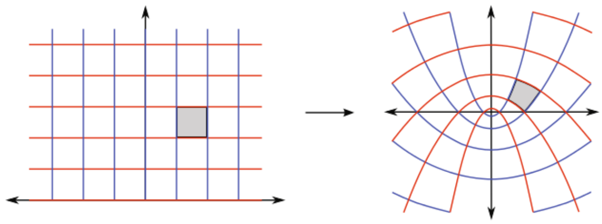

Figure 2: Flat 2-dimensional space is conformally mapped onto a curved space, while angles are preserved. Figure adapted from [30].

In the following we consider d-dimensional flat spaces with Lorentzian signature of the metric tensor, that is, $g_{\mu\nu} \equiv \eta_{\mu\nu}$. Under coordinate transformations like $x^\mu \to x'^\mu$, the metric tensor changes as $g'_{\rho\sigma} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = g_{\mu\nu}$. Identifying then $g' \equiv \Lambda(x) g$, which makes the transformation conformal, we arrive at:

$$\eta_{\rho\sigma} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = \Lambda^{-1}(x) \eta_{\mu\nu} \ . \tag{28}$$

Consider now an infinitesimal transformation on the coordinates, parametrized by ϵ^ρ in the following way: $x'^\rho = x^\rho + \epsilon^\rho(x)$. Imposing then that the equation above be satisfied, and noticing that $\epsilon_\mu = \eta_{\mu\nu}\epsilon^\nu$, we have for the l.h.s.:

$$\begin{aligned} \eta_{\rho\sigma} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} &= \eta_{\rho\sigma} \left(\delta^\rho_\mu + \partial_\mu \epsilon^\rho\right) \left(\delta^\sigma_\nu + \partial_\nu \epsilon^\sigma\right) \\ &= \left(\eta_{\mu\sigma} + \partial_\mu \epsilon_\sigma\right) \left(\delta^\sigma_\nu + \partial_\nu \epsilon^\sigma\right) \\ &= \eta_{\mu\nu} + \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) + \mathcal{O}(\epsilon^2) \end{aligned} \ .$$

We see then that the condition to satisfy such equation is, to first order in ϵ:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = F(x) \eta_{\mu\nu} \ ,$$

for some smooth function $F(x)$, which can be determined by tracing this equation with $\eta^{\mu\nu}$:

$$\begin{aligned} \eta^{\mu\nu} \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) &= \eta^{\mu\nu} F(x) \eta_{\mu\nu} \\ 2\partial^\mu \epsilon_\mu &= F(x) d \\ \therefore F(x) &= \frac{2}{d} \partial \cdot \epsilon \end{aligned} \ ,$$

where $\partial \cdot \epsilon \equiv \partial^\mu \epsilon_\mu$ stands for the d-divergence of ϵ. Thus, we can read off the scale factor $\Lambda(x)$ as $\Lambda(x) = 1 - \frac{2}{d} \partial \cdot \epsilon + \mathcal{O}(\epsilon^2)$, and the condition for the transformation to be conformal is just:

$$\boxed{\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \eta_{\mu\nu} \partial \cdot \epsilon} \ . \tag{29}$$

The next step then is to solve this equation for ϵ. Before doing so, we massage the equation above to find more useful relations. Firstly, deriving it with respect to ∂^ν and summing over it, we have:

$$\begin{aligned} \partial^\nu \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) &= \frac{2}{d}\eta_{\mu\nu}\partial^\nu \left(\partial\cdot\epsilon\right) \\ \Box\epsilon_\mu + \left(1-\frac{2}{d}\right)\partial_\mu\left(\partial\cdot\epsilon\right) = 0 & \end{aligned},$$

where the defintion $\Box \equiv \partial^\mu\partial_\mu$ was employed. Then, we take a derivative with respect to x^ν to write:

$$\begin{aligned} \partial_\nu\Box\epsilon_\mu + \left(1-\frac{2}{d}\right)\partial_\nu\partial_\mu\left(\partial\cdot\epsilon\right) &= 0 \\ \Box\partial_\nu\epsilon_\mu + \left(1-\frac{2}{d}\right)\partial_\mu\partial_\nu\left(\partial\cdot\epsilon\right) &= 0 \end{aligned},$$

where it was used the fact that partial derivatives commute to write the second line. If we now interchange the indices μ and ν in this last equation and sum both equations, using the relation (29) we can write:

$$\begin{aligned} \Box\left(\partial_\nu\epsilon_\mu + \partial_\mu\epsilon_\nu\right) + \left(2-\frac{4}{d}\right)\partial_\mu\partial_\nu\left(\partial\cdot\epsilon\right) &= 0 \\ \frac{2}{d}\eta_{\mu\nu}\Box\left(\partial\cdot\epsilon\right) + \left(2-\frac{4}{d}\right)\partial_\mu\partial_\nu\left(\partial\cdot\epsilon\right) &= 0 \\ \left[\eta_{\mu\nu}\Box + (d-2)\,\partial_\mu\partial_\nu\right]\left(\partial\cdot\epsilon\right) &= 0 \end{aligned}.$$

Finally, contracting this equation with $\eta^{\mu\nu}$, we get:

$$\boxed{(d-1)\,\Box\left(\partial\cdot\epsilon\right) = 0}\,. \tag{30}$$

In this work we are interested in two particular cases of this equation: $d \geq 3$ and $d = 1$; the 2-dimensional case is of great importance in physics, and it has been widely studied, but it will bring no applicable toolkit to what we are going to do here further. We concentrate mostly on the first case in what follows. But before doing so, let us consider the trivial case, in which equation (30) is automatically satisfied: $d = 1$.

1.2.1 Conformal group in $d = 1$

Since we do not have a notion of angle over a line, any invertible smooth mapping, say $f(x)$, is conformal in $d = 1$. The general form of such function can be understood by analytically continuing f to the complex plane. In order for f to be invertible, it can have no essential singularities nor branch points.

In the first case, any neighborhood of such singularity sweeps the whole plane (this is the content of the *Great Picard's Theorem*, see [31] for example) with at most a single exception. As a consequence, the points outside such neighborhood surely will have coincident values with those ones, and the inverse cannot be defined. In the second case the function is not uniquely defined. For example, taking $z = re^{i\theta}$, $\log z = \log r + i\theta$, the argument θ is ill-defined due to the fact we can add multiples of 2π to it and consequently the logarithm is not uniquely defined.

Excluding this kind of functions, we are left with functions that can be expanded into a Laurent series with finite principal part, consisting of linear combinations of polynomials that can have poles. Therefore it must be a fraction of polynomials, without common zeros. Moreover, each polynomial cannot have more than one zero and in the case of zero of order greater than 1 we also have problems to define an inverse (remember, the inverse of x^2 is not uniquely defined). So, f must be a fraction of linear functions:

$$f(x) = \frac{ax+b}{cx+d} \quad \text{with} \quad ad - bc \neq 0 \ .$$

The condition between the coefficients guarantee the mapping is invertible; if $ad-bc=0$, then f is a constant. Without losing generality, selecting $ad - bc = 1$, we see that the coefficients are freely determined as for the real matrices of order 2 with determinant 1. Such matrices, equipped with the group operations of matrix products and matrix inversion, form the *(Projective) Special Linear group* of degree 2 over $\mathbb{R}$, (P)SL(2,$\mathbb{R}$).

It turns out that such group is closely related to the Lorentz group in 3 dimensions, $SO(2,1)$. To see this, notice that a general element g of the group can be parameterized as:

$$g = \begin{bmatrix} 1+a & b \\ c & \frac{1+bc}{1+a} \end{bmatrix} \ . \tag{31}$$

If the parameters are infinitesimal, we have its infinitesimal form like:

$$g = \begin{bmatrix} 1+a & b \\ c & 1-a \end{bmatrix} \ ,$$

which allows us to read off the generators as being:

$$D \equiv \frac{i}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ , \quad P \equiv i \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ \text{and} \ \ K \equiv i \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \ .$$

They generate the algebra:

$$[D,K] = -iK, \quad [K,P] = -2iD \ \ \text{and} \ \ [D,P] = iP \ . \tag{32}$$

We claim this algebra is equal to $so(2,1)$. To verify this, notice that the commutator between two generators $\mathcal{M}_{\mu\nu}$ of $SO(2,1)$ is in the algebra for the Poincarè group (201):

$$[\mathcal{M}_{\mu\nu}, \mathcal{M}_{\rho\sigma}] = i\left(\eta_{\nu\rho}\mathcal{M}_{\mu\sigma} + \eta_{\mu\sigma}\mathcal{M}_{\nu\rho} - \eta_{\mu\rho}\mathcal{M}_{\nu\sigma} - \eta_{\nu\sigma}\mathcal{M}_{\mu\rho}\right) \ ,$$

where $\mu, \nu = 0, 1, 2$ and $\eta_{\mu\nu} = \mathrm{diag}(-, +, +)$. It is just a question of redefinition of the three independent generators (remember they are antisymmetric) then: $-i(\mathcal{M}_{01}+\mathcal{M}_{12}) \equiv K$, $-i(\mathcal{M}_{12} - \mathcal{M}_{01}) \equiv P$ and $\mathcal{M}_{02} \equiv D$. In fact, from the algebra right above we have:

$$\begin{aligned}
[D, K] &= -i\left[\mathcal{M}_{02}, \mathcal{M}_{01}\right] - i\left[\mathcal{M}_{02}, \mathcal{M}_{12}\right] \\
&= (-\eta_{00}\mathcal{M}_{21}) + (-\eta_{22}\mathcal{M}_{01}) = -iK \\
[D, P] &= -i\left[\mathcal{M}_{02}, \mathcal{M}_{12}\right] + i\left[\mathcal{M}_{02}, \mathcal{M}_{01}\right] \\
&= (-\eta_{22}\mathcal{M}_{01}) + (\eta_{00}\mathcal{M}_{21}) = iP \\
[K, P] &= -\left[\mathcal{M}_{01}, \mathcal{M}_{12}\right] + \left[\mathcal{M}_{12}, \mathcal{M}_{01}\right] \\
&= (-i\eta_{11}\mathcal{M}_{02}) + (i\eta_{11}\mathcal{M}_{20}) = -2iD
\end{aligned} \ .$$

Finite transformations are found by exponentiating these generators D, P, and K. Noticing that $P^2 = K^2 = 0$ and considering the parametrization (31), exponentiation of these generators gives the following functions of x:

$$\begin{aligned}
e^{-i\alpha D} &= \begin{bmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{bmatrix} &&\rightarrow \quad f(x) = e^{\alpha}x \\
e^{-i\beta P} &= \mathbb{I} + \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} &&\rightarrow \quad f(x) = x + \beta \ , \\
e^{-i\gamma K} &= \mathbb{I} + \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix} &&\rightarrow \quad f(x) = \frac{x}{\gamma x + 1}
\end{aligned} \tag{33}$$

where α, β and γ are real numbers. It is easy to see then that D, called *dilation operator*, generates scales transformation while P generates translations, therefore being nothing more than the momentum operator. The transformation generated by K is called *Special Conformal Transformation* (SCT) and will be explored in the next section.

Now, since we are interested in theories containing fields, we need to consider field representations of this group. Firstly, we note that fields can have spin, but since it is a property independent on the position of the field, the operator $\mathcal{S}_{\mu\nu}$ associated commutes with all the others above. Considering we are acting on fields with already labeled spin, the following results are valid for each component also. Secondly, in order to do this, we define the way a general field ϕ transform under a scaling transformation like $x' = \lambda x$ on the coordinates:

$$\boxed{\phi'(\lambda x) = \lambda^{\Delta}\phi(x)} \ , \tag{34}$$

with $\lambda = e^{\alpha}$ the dilatation factor. $\Delta \in \mathbb{R}$ defines the so called *scaling dimension* of the operator (field) ϕ.

Noting that translations, as in (19), do not affect fields and considering the equation above, the use of (14) gives the following representations for D and P:

$$\mathcal{D}\phi(x) = \left(-ix\frac{d}{dx} + i\Delta\right)\phi(x), \qquad \mathcal{P}\phi(x) = -i\frac{d}{dx}\phi(x) \ . \tag{35}$$

The representation of K and $\mathcal{K}$ is found by translating its form at the origin to any value of x using Baker-Campbell-Hausdorff formula for expanding products of exponentials of operators with others operators: $e^Y X e^{-Y} = X + [X, Y] + 1/2!\,[X, [X, Y]] + \ldots$. We have:

$$\begin{aligned}
\mathcal{K}(x)\phi(x) &= e^{ix\mathcal{P}}\mathcal{K}(0)e^{-ix\mathcal{P}}\phi(x) \\
&= \mathcal{K}(0)\phi(x) + [ix\mathcal{P}, \mathcal{K}(0)]\,\phi(x) + 1/2!\,[ix\mathcal{P}, [ix\mathcal{P}, \mathcal{K}(0)]]\,\phi(x) \\
&= -2ixD(0)\phi(x) - ix^2\mathcal{P}\phi(x) \\
&= \left(-2ix\Delta + ix^2\frac{d}{dx}\right)\phi(x)
\end{aligned} \ . \tag{36}$$

We are going to explore more about conformal invariance in field theories further, after studying the symmetry group in the interesting case of dimensions greater than two. We conclude this section with an intriguing comment, which will not be discussed here: if ones take the dimension to be time-like, all we have done was conformal quantum mechanics! See, for example, [32] for details.

1.2.2 Conformal Group in $d \geq 3$

We start noticing that the equation (30), together with the equation preceding it, implies that $\partial \cdot \epsilon$ is at most linear in x^μ, thus ϵ_μ itself must be at most quadratic. We have then:

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \ , \tag{37}$$

with $|a_\mu|$, $|b_{\mu\nu}|$, $|c_{\mu\nu\rho}| \ll 1$ and $c_{\mu\nu\rho}$ symmetric in the last two indices in order to make the divergence linear in x^μ.

The equations worked out before do not depend on the position, so we can treat each of the terms appearing in the expression of ϵ_μ separately. The coefficients a_μ, $b_{\mu\nu}$ and $c_{\mu\nu\rho}$ are our set of transformation parameters (in the language of section 1.1.2) and what we have to do next is to interpret them and to find the generators associated. Note, however, that up to this moment we have not commented on the effect of conformal transformations on the fields (that is, up to here $\phi' \equiv \mathcal{F}(\phi) = \phi$), thereby the full form of the generators of the conformal symmetry for a field theory cannot be stated yet.

Restricting the discussion to the level of spacetime coordinates, let us analyze the meaning of those terms. The constant one in (37) has no constraint by the equations developed before and it is of easy interpretation: it consists of spacetime translation, $x'^{\mu} = x^{\mu} + a^{\mu}$, with (partial) generator P_{μ} given by:

$$P_{\mu} \equiv -i\frac{\delta x^{\nu}}{\delta a^{\mu}}\partial_{\nu} = -i\delta^{\nu}_{\mu}\partial_{\nu} = -i\partial_{\mu} \ . \tag{38}$$

The scale factor obviously is 1, as can be seen from the expression for it we got previously: $\Lambda(x) = 1 - (2/d)\partial \cdot \epsilon + \mathcal{O}(\epsilon^2)$.

Now we go for the second term in equation (37). Putting it into equation (29), we get:

$$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{d}b^{\rho}{}_{\rho}\eta_{\mu\nu} \ ,$$

from which it can be seen that the components of $b_{\mu\nu}$ can be split into an antisymmetric (and therefore traceless) part and a symmetric one proportional to $\eta_{\mu\nu}$. Let us take then $b_{\mu\nu} \equiv m_{\mu\nu} + \alpha\eta_{\mu\nu}$, with $m_{\mu\nu}$ antisymmetric and $\alpha \equiv b^{\rho}{}_{\rho}/d$.

We see then that the symmetric term amounts to an infinitesimal scale transformation: $x'_{\mu} = (1+\alpha)x_{\mu}$. As before, the spacetime part of the generator associated to invariance of the system with respect to this transformation is (note the parameter here is α):

$$D \equiv -i\frac{\delta x^{\nu}}{\delta \alpha}\partial_{\nu} = -ix^{\nu}\partial_{\nu} \ . \tag{39}$$

In this case, with our definitions, the scale factor reads $\Lambda(x) = 1 - 2\alpha + \mathcal{O}(\epsilon^2)$. Analogously, the (spacetime part of the) generator we find for $m_{\mu\nu}$ is:

$$\mathcal{M}_{\mu\nu} \equiv -i\eta_{\mu\rho}\eta_{\nu\sigma}\frac{\delta x^{\gamma}}{\delta m_{\rho\sigma}}\partial_{\gamma} = \frac{-i}{2}\eta_{\mu\rho}\eta_{\nu\sigma}\left(\eta^{\gamma\rho}x^{\sigma} - \eta^{\gamma\sigma}x^{\rho}\right)\partial_{\gamma} = \frac{i}{2}\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right) \ , \tag{40}$$

which we recognize as the generator of rotations in spacetime (see appendix A), despite of a constant factor $1/2$ in front of it that can be dropped; therefore $\mathcal{M}_{\mu\nu} = -i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right)$ from now on. Moreover, since $m_{\mu\nu}$ is traceless, the scale factor associated to this transformation is 1, to first order in ϵ. After all, rigid rotations do not affect lengths.

Before continuing, let us explore a little bit more about scale transformations, which play the most important role here. To consolidate our understanding about it, consider the example of the action of a free scalar field (equation (1) without potential) with scaling dimension Δ; under what condition is it invariant under scale transformation? Considering the transformations above, we have (remember $[m] = [x^{-1}]$ in natural units):

$$S' \equiv \frac{1}{2}\int d^dx'\partial'_{\mu}\phi'\partial'^{\mu}\phi' = \lambda^{d-2+2\Delta}S \ ,$$

therefore, this action is scale invariant only if $\Delta = 1 - d/2$. Notice that scaling invariance makes the mass parameter of a theory dependent on the scale also, therefore with no interpretation as a particle as usual. In fact, there is this mantra: *there is only massless* *particles in conformal field theories*[10].

But why would we like a theory to be scale invariant? It turns out that in a large class of physically relevant relativistic theories that are manifestly invariant under scale transformation, it can be verified they are also conformally invariant. To have an idea of the discussion, one can understand this in two steps: fields usually have virial[11] described by the divergence of a 2-rank tensor $\sigma^{\mu\nu}$; this particularity makes possible the addition of another divergenceless term to the energy-momentum tensor that makes its trace equals to the divergence of the current associated to scale invariance. Therefore, once such current j_D^μ is conserved, $T^{\mu\nu}$ is traceless, and from this we have conformally invariance. Calculations of the first pass are omitted here and we assume the energy-momentum tensor is traceless in what follows, but we refer the reader to [4] and [33] for a more detailed revision.

A consequence of $T^\mu_{\ \mu} = \partial_\mu j_D^\mu$ is that we can write $j_D^\mu = T^\mu_{\ \nu} x^\nu$. Now, we know that the action of our system can be written in terms of the energy-momentum tensor by means of (24); under a general coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, the change in the metric is:

$$\begin{aligned} g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \\ &= \left(\delta^\alpha_\mu - \partial_\mu \epsilon^\alpha\right)\left(\delta^\beta_\nu - \partial_\nu \epsilon^\beta\right) g_{\alpha\beta} \\ &= g_{\mu\nu} - \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) \end{aligned} .$$

If the transformation is conformal, then (30) tells us that $\delta g_{\mu\nu} = -\frac{2}{d}\eta_{\mu\nu}\partial\cdot\epsilon$. Consequently, the variation on the action S is just:

$$\delta S = \frac{1}{d}\int d^d x T^\mu_{\ \mu} \partial\cdot\epsilon \ , \tag{41}$$

which holds for any conformal transformation described by ϵ. Therefore, it can be seen that, provided the system is scale invariant and assumed that $T^\mu_{\ \mu} = \partial_\mu j_D^\mu$ holds, the energy-momentum tensor is traceless and the system has conformal invariance.

Finally, the quadratic term appearing in (37) can be studied with the help of a relation derived from (29). Differentiating it with ∂_ρ and summing the three equations obtained when permuting the indices μ, ν and ρ cyclically, we get:

$$\partial_\mu\partial_\nu\epsilon_\rho = \frac{1}{d}\left(-\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu\right)(\partial\cdot\epsilon) \ .$$

[10] Actually, this is true for flat space. if you have curvature you can turn on conformal masses.

[11] The virial of a field is defined by: $V^\mu \equiv \frac{\delta\mathcal{L}}{\delta(\partial^\sigma\phi)}\left(\eta^{\mu\sigma}\Delta + i S^{\mu\sigma}\right)\phi$, which can be written as $V^\mu = \partial_\alpha\sigma^{\alpha\mu}$ usually.

Inserting that term into this equation and using the following results:

$$\begin{aligned}\partial_\mu\partial_\nu\epsilon_\rho &= c_{\rho\sigma\gamma}\partial_\mu\left(\delta^\sigma_\nu x^\gamma + \delta^\gamma_\nu x^\sigma\right) = c_{\rho\sigma\gamma}\left(\delta^\sigma_\nu\delta^\gamma_\mu + \delta^\gamma_\nu\delta^\sigma_\mu\right) = 2c_{\rho\mu\nu} \\ \partial_\sigma\left(\partial\cdot\epsilon\right) &= c_{\mu\nu\rho}\partial_\sigma\left(\eta^{\mu\gamma}\partial_\gamma\left(x^\nu x^\rho\right)\right) = c_{\mu\nu\rho}\eta^{\mu\gamma}\left(\delta^\nu_\gamma\delta^\rho_\sigma + \delta^\rho_\gamma\delta^\nu_\sigma\right) = 2c^\mu{}_{\mu\sigma}\end{aligned} ,$$

we find that:

$$\begin{aligned}c_{\rho\mu\nu} &= \frac{1}{d}\left(-\eta_{\mu\nu}c^\gamma{}_{\gamma\rho} + \eta_{\rho\mu}c^\gamma{}_{\gamma\nu} + \eta_{\nu\rho}c^\gamma{}_{\gamma\mu}\right) \\ \therefore c_{\mu\nu\rho} &= \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu + \eta_{\rho\mu}b_\nu\end{aligned} ,$$

where we employed the definition $b_\mu \equiv c^\nu{}_{\nu\mu}/d$ to write the second line. The corresponding infinitesimal transformation is:

$$\begin{aligned}x'_\mu &= x_\mu + c_{\mu\nu\rho}x^\nu x^\rho \\ &= x_\mu + (\eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu + \eta_{\rho\mu}b_\nu)x^\nu x^\rho \ . \\ &= x_\mu + 2\left(b\cdot x\right)x_\mu - (x\cdot x)b_\mu\end{aligned} \tag{42}$$

To interpret this transformation, note the following:

$$\begin{aligned}\delta\left(\frac{x_\mu}{x^2}\right) &= \frac{\delta x_\mu}{x^2} - 2x_\mu\frac{x^\mu\cdot\delta x_\mu}{x^4} \\ &= \frac{2\left(b\cdot x\right)x_\mu}{x^2} - b_\mu - \frac{2\left(b\cdot x\right)x_\mu}{x^2} \ , \\ &= -b_\mu\end{aligned}$$

that is, under such infinitesimal transformation, the covector x_μ/x^2 experiences just a translation! The correspondent finite transformation then reads:

$$\frac{x'_\mu}{x'^2} = \frac{x_\mu}{x^2} - b_\mu \ .$$

The map $x^\mu \to x^\mu/x^2$ is known as an *inversion*, and consists of a discrete transformation; however, notice that, at the end of the day, the effect of the transformation above on x_μ is equivalent to an inversion of x_μ followed by a translation by $-b_\mu$ and again an inversion. Squaring this last equation we get:

$$\frac{1}{x'^2} = \frac{1}{x^2} + b^2 - 2\frac{b\cdot x}{x^2} = \frac{1 - 2(b\cdot x) + b^2x^2}{x^2} \ .$$

The finite transformation associated to the third term in (37) is given by:

$$x'_\mu = \frac{x_\mu - x^2 b_\mu}{1 - 2(b\cdot x) + b^2x^2} \ . \tag{43}$$

This one is known as *Special Conformal Transformation*, as we have seen before in the case $d = 1$, but now generalized for higher dimensions; its infinitesimal form is clearly given by (42), and the generator associated to it can be obtained from (14):

$$
\begin{aligned}
K_\mu \equiv -i\frac{\delta x^\nu}{\delta b^\mu}\partial_\nu &= -i\eta_{\mu\rho}\eta^{\nu\sigma}\frac{\delta x_\sigma}{\delta b_\rho}\partial_\nu = -i\eta_{\mu\rho}\eta^{\nu\sigma}\left[2x^\rho x_\sigma - x^2\delta^\rho_\sigma\right]\partial_\nu \\
&= -i\left[2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right]
\end{aligned} . \tag{44}
$$

At last, we can also derive the scale factor associated to this transformation. Its finite form follows from equation (28); using the result (43) and defining $\beta(x) \equiv 1 - 2(b\cdot x) + b^2x^2$:

$$
\begin{aligned}
\Lambda^{-1}(x) &= \frac{\eta^{\mu\nu}}{d}\eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} \\
&= \frac{\eta^{\mu\nu}}{d}\eta_{\rho\sigma}\left[\beta^{-1}(x)\left(\delta^\rho_\mu - 2x_\mu b^\rho\right) + \left(x^\rho - x^2b^\rho\right)\beta^{-2}(x)\left(2b_\mu - 2b^2x_\mu\right)\right] \\
&\times\left[\beta^{-1}(x)\left(\delta^\sigma_\nu - 2x_\nu b^\sigma\right) + \left(x^\sigma - x^2b^\sigma\right)\beta^{-2}(x)\left(2b_\nu - 2b^2x_\nu\right)\right]
\end{aligned} ,
$$

which simplifies as:

$$
\begin{aligned}
\Lambda^{-1}(x) &= \frac{1}{d}\left[\beta^{-1}(x)\left(\eta^{\rho\nu} - 2x^\nu b^\rho\right) + \left(x^\rho - x^2b^\rho\right)\beta^{-2}(x)\left(2b^\nu - 2b^2x^\nu\right)\right] \\
&\times\left[\beta^{-1}(x)\left(\eta_{\rho\nu} - 2x_\nu b_\rho\right) + \left(x_\rho - x^2b_\rho\right)\beta^{-2}(x)\left(2b_\nu - 2b^2x_\nu\right)\right] \\
&= \frac{\beta^{-2}(x)}{d}\left[\left(\eta^{\rho\nu} - 2x^\nu b^\rho\right) + \left(x^\rho - x^2b^\rho\right)\beta^{-1}(x)\left(2b^\nu - 2b^2x^\nu\right)\right]^2 \\
&= \frac{\beta^{-2}(x)}{d}\left\{\left[d - 4(b\cdot x) + 4b^2x^2\right] + \left[4b^2x^2\right] + \left[4(b\cdot x) - 8b^2x^2\right]\right\}
\end{aligned} .
$$

Therefore,

$$
\boxed{\Lambda(x) = \beta^2(x) = \left(1 - 2(b\cdot x) + b^2x^2\right)^2} . \tag{45}
$$

We have been emphasizing that the generators (38), (39), (40) and (44) we got were incomplete, and we will find their complete form when acting on fields very soon, but it turns out they constitute the so called conformal algebra in dimensions $d \geq 3$. Below we compute two of the commutators involved and summarize the results right after. The computations are straightforward to do, some of them are just reproduction of the Poincarè algebra in appendix A.

$$
\begin{aligned}
[P_\mu, K_\nu] &= \left[-i\partial_\mu, -i\left(2x_\nu x^\rho\partial_\rho - x^2\partial_\nu\right)\right] \\
&= -\left[\partial_\mu, 2x_\nu x^\rho\partial_\rho\right] + \left[\partial_\mu, x^2\partial_\nu\right] \\
&= -2(\eta_{\mu\nu}x^\rho\partial_\rho + x_\nu\partial_\mu) + (2x_\mu\partial_\nu) \\
&= -2i\left(\mathcal{M}_{\mu\nu} + \eta_{\mu\nu}D\right)
\end{aligned} ,
$$

and also:

$$\begin{aligned}
[K_\rho, \mathcal{M}_{\mu\nu}] &= \left[-i\left(2x_\rho x^\sigma \partial_\sigma - x^2\partial_\rho\right), -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)\right] \\
&= -\left[2x_\rho x^\sigma\partial_\sigma - x^2\partial_\rho, x_\mu\partial_\nu\right] + \mu \leftrightarrow \nu \\
&= -2\left(-\eta_{\nu\rho}x_\mu x^\sigma\partial_\sigma\right) + \left(\eta_{\mu\rho}x^2\partial_\nu - 2x_\mu x_\nu\partial_\rho\right) + \mu \leftrightarrow \nu \ . \\
&= -\left(-2\eta_{\nu\rho}x_\mu x^\sigma\partial_\sigma - \eta_{\mu\rho}x^2\partial_\nu + 2x_\mu x_\nu\partial_\rho\right) + \mu \leftrightarrow \nu \\
&= -i\eta_{\mu\rho}K_\nu + i\eta_{\nu\rho}K_\mu
\end{aligned}$$

In summary

$$\boxed{\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[P_\rho, \mathcal{M}_{\mu\nu}] &= -i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right) \\
[P_\mu, K_\nu] &= 2i\left(\mathcal{M}_{\mu\nu} - \eta_{\mu\nu}D\right) \\
[D, K_\mu] &= -iK_\mu \\
[K_\rho, \mathcal{M}_{\mu\nu}] &= -i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right) \\
[\mathcal{M}_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\nu\rho}\mathcal{M}_{\mu\sigma} + \eta_{\mu\sigma}\mathcal{M}_{\nu\rho} - \eta_{\mu\rho}\mathcal{M}_{\nu\sigma} - \eta_{\nu\sigma}\mathcal{M}_{\mu\rho}\right) \\
[\mathcal{M}_{\mu\nu}, D] &= [P_\mu, P_\nu] = [K_\mu, K_\nu] = [D, D] = 0
\end{aligned}} \ , \tag{46}$$

where we see that P_μ and K_μ behave as vectors under rotations, while D is scalar under them.

The reason why we obtained "incomplete" generators is that because we were assuming that the fields considered are invariant under such transformations, that is, they are in the trivial representation of the conformal group. In fact the algebra obtained above is well established and holds for more general fields, but that is the point: in our work we are going to find such non trivial fields and, then, we have to fit the form of the generators to them. This means that we need to consider non trivial representations of the conformal group in $d \geq 3$, that is, we seek matrices representation $\mathcal{T}_g$ such that a multicomponent field ϕ transforms as:

$$\phi'(x') = \left(1 - \omega_g\mathcal{T}_g\right)\phi(x) \ .$$

Poincarè generators can be generalized just by noticing that fields are scalars under translations and might have spinorial indices, which demands another generator of rotations independent on x to suplement $\mathcal{M}_{\mu\nu}$, that turns out to be just the $\mathcal{S}_{\mu\nu}$ we have seen before. Therefore, P_μ does not change, although from now on we refer to it as $\mathcal{P}_\mu$, and $\mathcal{M}_{\mu\nu}$ goes to $\mathcal{J}_{\mu\nu} = \mathcal{M}_{\mu\nu} + \mathcal{S}_{\mu\nu}$; their action on fields are then:

$$\begin{aligned}
\mathcal{P}_\mu\phi(x) &= -\partial_\mu\phi \\
\mathcal{J}_{\mu\nu}\phi(x) &= -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)\phi + \mathcal{S}_{\mu\nu}\phi
\end{aligned} \ .$$

The generalization of K_μ and D follows from considering the subalgebra formed by these operators evaluated at the origin of spacetime, and then translating them to any point x by means of Baker-Campbell-Hausdorff, similar but more complicatedly to what was done at the end of the previous section. This is done for example in [4] and results in the generalized operators $\mathcal{K}_\mu$ and $\mathcal{D}$ that acts on fields according to:

$$\begin{aligned} \mathcal{D}\phi(x) &= \left(-ix^\mu\partial_\mu + i\Delta\right)\phi(x) \\ \mathcal{K}_\mu\phi(x) &= \left(2x_\mu\Delta - x^\nu\mathcal{S}_{\mu\nu} - 2ix_\mu x^\nu\partial_\nu + ix^2\partial_\mu\right)\phi(x) \end{aligned} , \tag{47}$$

where Δ stands for the scaling dimension of the operator(field) ϕ. Note that Δ is nothing more than the eigenvalue of $\mathcal{D}$ at $x=0$ for the field ϕ.

To conclude this section, analogously to what we did at the end of the last section, we show the isomorphism between the conformal group in dimension $d \geq 3$ and the rotation group in $d+2$ dimensions. To see this, define the following generators in addition to $\mathcal{J}_{\mu\nu}$:

$$\begin{aligned} \mathcal{J}_{d+1,d+2} &= \mathcal{D} \\ \mathcal{J}_{d+1,\mu} &= \frac{1}{2}\left(\mathcal{P}_\mu - \mathcal{K}_\mu\right) . \\ \mathcal{J}_{d+2,\mu} &= \frac{1}{2}\left(\mathcal{P}_\mu + \mathcal{K}_\mu\right) \end{aligned} \tag{48}$$

Using (46), we have then:

$$\begin{aligned} \left[\mathcal{J}_{d+1,d+2}, \mathcal{J}_{d+1,\mu}\right] &= \left[\mathcal{D}, \frac{1}{2}\left(\mathcal{P}_\mu - \mathcal{K}_\mu\right)\right] = \frac{i}{2}\left(\mathcal{P}_\mu + \mathcal{K}_\mu\right) = i\mathcal{J}_{0,\mu} \\ \left[\mathcal{J}_{d+1,\mu}, \mathcal{J}_{d+2,\nu}\right] &= \left[\frac{1}{2}\left(\mathcal{P}_\mu - \mathcal{K}_\mu\right), \frac{1}{2}\left(\mathcal{P}_\nu + \mathcal{K}_\nu\right)\right] = \frac{1}{4}\left(\left[\mathcal{P}_\mu, \mathcal{K}_\nu\right] - \left[\mathcal{K}_\mu, \mathcal{P}_\nu\right]\right) \\ &= -i\eta_{\mu\nu}\mathcal{D} = -i\eta_{\mu\nu}\mathcal{J}_{d+2,d+1} \\ \left[\mathcal{J}_{d+1,\rho}, \mathcal{J}_{\mu\nu}\right] &= \left[\frac{1}{2}\left(\mathcal{P}_\rho - \mathcal{K}_\rho\right), \mathcal{J}_{\mu\nu}\right] = \frac{1}{2}\left(\left[\mathcal{P}_\rho, \mathcal{J}_{\mu\nu}\right] - \left[\mathcal{K}_\rho, \mathcal{J}_{\mu\nu}\right]\right) \\ &= \frac{1}{2}\left[i\left(\eta_{\mu\rho}\mathcal{P}_\nu - \eta_{\nu\rho}\mathcal{P}_\mu\right) - i\left(\eta_{\rho\mu}\mathcal{K}_\nu - \eta_{\rho\nu}\mathcal{K}_\mu\right)\right] \\ &= i\left(\eta_{\rho\mu}\mathcal{J}_{d+1,\nu} - \eta_{\rho\nu}\mathcal{J}_{d+1,\mu}\right) \end{aligned} ,$$

and similarly for other commutators. They obey the following algebra:

$$\left[\mathcal{J}_{ab}, \mathcal{J}_{cd}\right] = i\left(\eta_{bc}\mathcal{J}_{ad} + \eta_{ad}\mathcal{J}_{bc} - \eta_{ac}\mathcal{J}_{bd} - \eta_{bd}\mathcal{J}_{ac}\right) , \tag{49}$$

with $a, b = 0, ..., d-1, d+1, d+2$ and $\eta_{ab} = \mathrm{diag}(+, ..., +, -, +)$; remembering that $\mathcal{J}_{\mu\nu}$ already satisfies such algebra.

The algebra of these new generators is the same as for the Lorentz group in $d+2$ dimensions. This and also the result we obtained at the end of the last section are the manifestation of a very useful tool at our disposal, the so called *Embedding Space Formalism*, which relates a theory in d dimensions invariant under global conformal transformations to another one in $d+2$ dimensions invariant under rotations. Its usefullness is realized when talking about defects in conformal field theories, the subject of the next chapter, so there we come back to this with more detail. In the next sections of this chapter we explore the consequences of what we have just done, aiming to establish the main aspects of a field theory invariant under conformal transformations, including in the quantum level, concluding with an extension of the conformal group, which has been demonstrated of great significance in theoretical physics.

A last, but not least, point here is a parallel we can now establish with the renormalization group idea developed at the end of the previous section. There we saw that beta functions tell us how coupling constants of interaction terms in a theory behave under scaling. In the present section we have seen that conformal field theories are invariants under that same transformations, besides others. Therefore, CFTs are characterized as field theories having vanishing beta functions. In that case, once considered the "space" of coupling constants, one can see a CFT as a fixed point there. So that quantum field theories undergoing a scaling process will possibly end up in a CFT or, conversely, a QFT can be seen as CFT perturbed by relevant operators.

1.3 Primary and Descendant Operators

Since we are interested in theories invariant under rescaling, a natural starting point is to consider the operators which diagonalize the dilatation operator $\mathcal{D}$ at the origin. For each operator (field) $\mathcal{O}(x)$, with correspondent scaling dimension Δ, we have:

$$\begin{aligned}[D,\mathcal{O}(0)] &= \mathcal{D}\mathcal{O}(0) = i\Delta\mathcal{O}(0) \\ [D,\mathcal{O}(x)] &= \mathcal{D}\mathcal{O}(x) = \left(-ix^{\mu}\partial_{\mu} + i\Delta\right)\mathcal{O}(x)\end{aligned},$$

where it was made use of the result (18), relating charges to operators, which allows us to see the scaling dimensions as being just charges associated to each local operator $\mathcal{O}(x)$. The set of these dimensions and respective operators form the spectrum of local operators. Conformal dimensions together with the structure constants appearing in 3-point functions specify the *CFT data* of a conformal field theory.

Consider the algebra (46). At the origin we can write:

$$\begin{aligned}\mathcal{D}\mathcal{K}_{\mu}\mathcal{O}(0) &= \left([\mathcal{D},\mathcal{K}_{\mu}] + \mathcal{K}_{\mu}\mathcal{D}\right)\mathcal{O}(0) = i\left(\Delta-1\right)\mathcal{K}_{\mu}\mathcal{O}(0) \\ \mathcal{D}\mathcal{P}_{\mu}\mathcal{O}(0) &= \left([\mathcal{D},\mathcal{P}_{\mu}] + \mathcal{P}_{\mu}\mathcal{D}\right)\mathcal{O}(0) = i\left(\Delta+1\right)\mathcal{P}_{\mu}\mathcal{O}(0)\end{aligned},$$

that is, $\mathcal{K}_{\mu}$ and $\mathcal{P}_{\mu}$ act like lowering and raising operators, respectively, for the scaling dimension. This suggests a way for constructing irreducible representations of the conformal group, resembling the analogous method used in quantum mechanics for angular momentum.

Acting with $\mathcal{P}_\mu$ on $\mathcal{O}(x)$ is equivalent to differentiating with respect to x^μ. *A priori* there is no limit to its successive applications, that is, there is no upper bound limit to scaling dimensions. However, as it is going to be shown at the end of this section, we do have a lower limit to Δ for unitary CFTs. Assuming this is true, there must exist some operators annihilated by the action of $\mathcal{K}_\mu$:

$$[K_\mu, \mathcal{O}(0)] = \mathcal{K}_\mu \mathcal{O}(0) = 0 \ .$$

These operators are called *primary operators*. Other operators are obtained from them by taking derivatives directly (that is, acting with $\mathcal{P}_\mu$ successively) or by taking linear combinations of derivatives with appropriate factors of x^μ. Operators that are not primaries are called *descendants*. Summarizing, for $\mathcal{O}(0)$ primary: We call $\mathcal{P}_\mu \mathcal{O}(0)$ a first

operator	scaling dimension
.	.
.	.
.	.
$\mathcal{P}_\nu \mathcal{P}_\mu \mathcal{O}(0)$	$\Delta + 2$
$\mathcal{P}_\mu \mathcal{O}(0)$	$\Delta + 1$
$\mathcal{O}(0)$	Δ

Table 1: n aplications of $\mathcal{P}_\mu$ give an operator of dimension $\Delta + n$.

level descendant, $\mathcal{P}_\nu \mathcal{P}_\mu \mathcal{O}(0)$ a second level descendent, and so on.

Now, remember (see (46)) that dilation and rotation generators commute. We can then construct a field representation of the conformal algebra using eigenvalues of $\mathcal{D}$ and $\mathcal{M}_{\mu\nu}$ to label the operators. Starting with a primary of scaling dimension Δ and spin l, by acting with momentum operators successively we move over operators of higher dimensions; this is analogous to the construction of irreducible representations of $SU(2)$ in quantum mechanics, in our case procceding from a *lowest-weight* “state” to higher-weight ones. Moreover, notice that first level descendants carry another Lorentz index, second level descendants two more and so on. Therefore descendants live in a different space from their respective primaries, given by a simple tensor product between the vector representation and the irreducible representation of the primary. In unitary CFTs any local operator is a linear combination of primaries and descendants (as we are going to see at the end of the section).

It is customary to refer to primaries far from the origin, for example as $\mathcal{O}(x)$. This must be understood as translated primaries. In fact, the formalism above shows $\mathcal{D}$ has eigenstates only at the origin. However, translated primaries still carry information from the ones at the origin by rigorous definition: as we are about to see, the scaling dimension and the spin are taken into account on transformations of them. Looking at the infinitesimal Jacobian of a conformal transformation, we have (at first order in epsilon):

$$\begin{aligned}\frac{\partial x'^{\nu}}{\partial x^{\mu}} &= \delta_{\mu}^{\nu} + \partial_{\mu}\epsilon^{\nu} \simeq \left(\delta_{\mu}^{\nu} + \partial_{\mu}\epsilon^{\nu}\right)\left(1 - \frac{1}{d}\partial\cdot\epsilon\right)\left(1 + \frac{1}{d}\partial\cdot\epsilon\right) \\ &\simeq \left[\delta_{\mu}^{\nu} + \frac{1}{2}\left(\partial_{\mu}\epsilon^{\nu} - \partial^{\nu}\epsilon_{\mu}\right)\right]\left(1 + \frac{1}{d}\partial\cdot\epsilon\right)\end{aligned} ,$$

that is, it is equivalent to a rotation followed by a scaling operation (or vice-versa, remembering such operations commute), both dependents on the position. Exponentiating the expression for the Jacobian above, we expect then:

$$\frac{\partial x'^{\nu}}{\partial x^{\mu}} = \Lambda^{-1/2}(x) M_{\mu}{}^{\nu} ,$$

where $\Lambda(x)$ and $M_{\mu}{}^{\nu}$ stand for a finite position-dependent scaling and a finite Lorentz rotation, respectively.

Therefore, the transformation rule of a primary operator $\phi(x)$ with intrinsic spin should depend on the rotation matrix $M_{\mu}{}^{\nu}$. Without proof, we claim here that such operator, supposedly in a irreducible representation R of $SO(d)$, will transform as:

$$\boxed{\phi(x) \to \lambda^{\Delta}(x) R\left[M_{\mu}{}^{\nu}\right]\phi(x)} , \tag{50}$$

where we defined $\lambda(x) \equiv \Lambda^{-1/2}(x)$ and, as always, Δ stands for the scaling dimension of the field. For example, if $\phi(x)$ is a vector field, then $R\left[M_{\mu}{}^{\nu}\right] = M_{\mu}{}^{\nu}$. So, if $V_{\nu}(x)$ is a spin-one field, it will transform like:

$$\widehat{V}_{\mu}(x') = \lambda^{\Delta}(x) M_{\mu}{}^{\nu} V_{\nu}(x) . \tag{51}$$

Transformation rules for a given descendant can then be derived from the one above. It is not so simple as for primaries, in view of the non-commutativeness between rotations and translations, as well as between dilations and translations. But, reinforcing what was already stated, *a priori* we just need to worry about primaries behaviour.

1.4 Radial quantization

To finish this section, we talk now about some important topics which will provide some groundwork to what is about to be developed in the following sections and chapters. Basically, the aim is to close the understanding on primaries and to emphasize particularities of unitary theories.

The starting point in this way is to adopt a different, but more powerful quantization. Typically, in view of time translation invariance, one chooses to deal with states living in

Hilbert spaces supported on slices of the time direction. Then time evolution connects these slices and all the usual treatment can be applied. The quantizantion adopted thus is motivated from a given symmetry respected by the theory. For scaling invariant ones, a possibility is to foliate the spacetime with spherical shells centered at the origin, which is known as *radial quantization*. Hilbert spaces are defined on the surface of these spheres and they are connected via action of the dilation operator; therefore, we can act on this space of states by inserting operators on the surface of those spheres, for example with charges integrated all over the sphere, see Fig. 1.4. The arrow indicates scaling evolution through

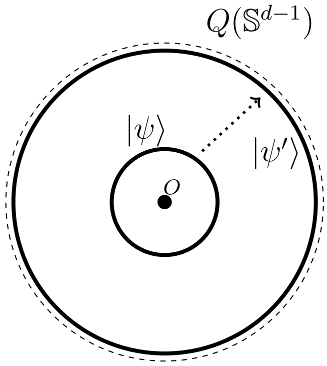

Figure 3: Radial slices of spacetime. Charge operator $Q(\mathbb{S}^{d-1})$ surrounding the sphere.

$e^{i\mathcal{D}\Delta\tau}$, with $\Delta\tau$ standing for the diference between the logarithms of the largest radius and the smallest one. In fact, making use of the logarithm of the radius characterizes a *cylinder* perspective of the quantization, in which the exponential changing in distances due to scaling is translated into simple translations of the logarithm of these distances, resembling the usual time slices:

$$r_2 = e^{\alpha} r_1 \quad \text{and} \quad \tau \equiv \log r \Rightarrow \Delta\tau = \log r_2 - \log r_1 = \alpha \ ,$$

in this case α is greater than zero, note, without losing generality.

We create states on a sphere by inserting operators inside it. The idea behind this is the following: the vacuum (on that given sphere, or space of states) is given by the path integral over the interior of the sphere, thus a general state is just the same path integral with the insertion of the operator (then said to create it) in the sphere and, therefore, inside the path integral expression. As one might already be imagining, we have an analogy of future and past here also: states in the "past" are on the sphere of null radius (or equivalently, minus infinity cylinder time τ), while states in the "future" are over the sphere of infinity radius.

The overlap of states on that Hilbert space is then equal to correlation functions of operators which create such states inserted inside the sphere. In this case, a different ordering is required than the usual time ordering in n-point functions, namely a radially ordered product of operators. The farthest from the origin come on the left; the ones at the same radius but different angles commute. For example, the configuration below is related to the two point function $\langle O(y)O(x)\rangle$ on the sphere:

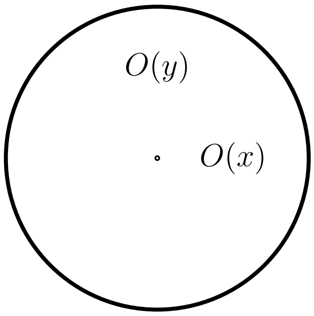

Figure 4: The overlap between $O(x)|0\rangle$ and $O(y)|0\rangle$ is just the two point function between the operators.

In this construction, notice, primary operators are going to create states that are automatically eigenstates of dilation. In fact, this is part of the so called *state-operator* correspondence: as it is clear, given a primary operator $\mathcal{O}(0)$, $\mathcal{O}(0)|0\rangle$ is automatically eigenstate of $\mathcal{D}$, in view of (47); conversely, given an eigenstate of $\mathcal{D}$ that is annihilated by $\mathcal{K}_\mu$, say $|\mathcal{O}\rangle$, then a primary $\mathcal{O}$ can be constructed by means of the correlation functions with other fields:

$$\langle 0|\phi_1\phi_2...\mathcal{O}|0\rangle \equiv \langle 0|\phi_1\phi_2...|\mathcal{O}\rangle \ , \tag{52}$$

that is because, from the right-hand side, applying rotation and dilation operators we can infer the spin and the scaling dimension of the operator $\mathcal{O}$, which defines a primary.

The state-operator correspondence tells us that all states in a conformal field theory can be created by operators which act locally at the origin (or in a small neighborhood of the origin, in the cylinder point of view). Which means that the entire Hilbert space of a CFT lives in a single point. The key is that such states evolve radially outward in a unitary way.

About the unitarity, it allows us to establish conjugation of the operators. Having the cylinder interpretation in mind and working in the euclidian time $t_E \equiv i\tau$, notice that conjugation in this case means to reflect the time t_E in view of the hermicity of D, which in turn means an inversion of the radius. Thus, unitarity in cylinder is translated to inversion in radial quantization. This operation of inversion, denoted by $\mathcal{R}$ from now on, already appeared before, when deriving special conformal transformations (see (43)); then we saw that such transformations actually were equivalent to an inversion followed by a translation and followed by another inversion. In terms of operators, this means:

$$\boxed{\mathcal{K}_\mu = \mathcal{R}\mathcal{P}_\mu\mathcal{R}} \ . \tag{53}$$

So, in radial quantization, $\mathcal{K}_\mu$ and $\mathcal{P}_\mu$ are hermitian conjugate of each other! Consequently, hermitian conjugate of descendant states are easy to write down:

$$(P_\mu|\mathcal{O}\rangle)^\dagger = \langle\mathcal{O}|\mathcal{K}_\mu \ , \tag{54}$$

where $\mathcal{O}$ is assumed to be a primary or a descendant.

This property then enables us to derive what is known as *unitarity bounds*. Basically they are restrictions on the spectrum of the theory. Imposing unitarity we get the allowed values of scaling dimensions. In terms of states, unitarity means their norm must be non-negative. From this, supposing $\mathcal{O}$ is a scalar primary of dimension Δ, we have for example:

$$\langle\mathcal{O}|\mathcal{K}_0\mathcal{P}_0|\mathcal{O}\rangle \geq 0 \Leftrightarrow 2\Delta\langle\mathcal{O}|\mathcal{O}\rangle \geq 0 \Rightarrow \Delta \geq 0 \ , \tag{55}$$

where it was used (46). Suppose we consider the norm of a second level descendant obtained from the action of $\mathcal{P}^\mu\mathcal{P}_\mu$ on $|\mathcal{O}\rangle$. We have:

$$\begin{aligned}
\langle\mathcal{O}|\mathcal{K}^\nu\mathcal{K}_\nu\mathcal{P}_\mu\mathcal{P}^\mu|\mathcal{O}\rangle &= \langle\mathcal{O}|\mathcal{K}^\nu\left([\mathcal{K}_\nu,\mathcal{P}_\mu]+\mathcal{P}_\mu\mathcal{K}_\nu\right)\mathcal{P}^\mu|\mathcal{O}\rangle \\
&= \langle\mathcal{O}|\mathcal{K}^\nu\left(\left[[\mathcal{K}_\nu,\mathcal{P}_\mu],\mathcal{P}^\mu\right]+\mathcal{P}^\mu\left[\mathcal{K}_\nu,\mathcal{P}_\mu\right]\right)|\mathcal{O}\rangle \ , \\
&+ \langle\mathcal{O}|\left[\mathcal{K}^\nu,\mathcal{P}_\mu\right]\mathcal{K}_\nu\mathcal{P}^\mu|\mathcal{O}\rangle
\end{aligned}$$

and therefore:

$$\begin{aligned}
\langle\mathcal{O}|\mathcal{K}^\nu\mathcal{K}_\nu\mathcal{P}_\mu\mathcal{P}^\mu|\mathcal{O}\rangle &= \langle\mathcal{O}|\left[\mathcal{K}^\nu,\left[[\mathcal{K}_\nu,\mathcal{P}_\mu],\mathcal{P}^\mu\right]\right]|\mathcal{O}\rangle \\
&+ \langle\mathcal{O}|\left[\mathcal{K}^\nu,\mathcal{P}^\mu\right]\left[\mathcal{K}_\nu,\mathcal{P}_\mu\right]|\mathcal{O}\rangle + \langle\mathcal{O}|\left[\mathcal{K}^\nu,\mathcal{P}_\mu\right]\left[\mathcal{K}_\nu,\mathcal{P}^\mu\right]|\mathcal{O}\rangle \ , \\
&= \left[8d\Delta^2 - 4\left(d-2\right)d\Delta\right]\langle\mathcal{O}|\mathcal{O}\rangle
\end{aligned}$$

where from the second line on we used the fact that primaries are annihilated by $\mathcal{K}_\mu$. To write the last line we just apply (46) again. From this we see that a stronger constraint on the values of Δ in this case due to imposition of unitarity is given by:

$$\boxed{\Delta = 0 \quad \text{or} \quad \Delta \geq \frac{d-2}{2}} \ . \tag{56}$$

The same exercise for a primary $\mathcal{O}^a$ of dimension Δ in a nontrivial irreducible representation $\mathcal{R}_\mathcal{O}$ of the $SO(d)$, where a stands for spin (which we assume to equals l here) indices, leads to (using (46) again):

$$\langle\mathcal{O}_b|\mathcal{O}^a\rangle \equiv \delta^a_b \Rightarrow \langle\mathcal{O}_b|\mathcal{K}_\mu\mathcal{P}_\nu|\mathcal{O}^a\rangle \geq 0 \Leftrightarrow \delta_{\mu\nu}\delta^a_b\Delta - \left(\mathcal{S}_{\mu\nu}\right)^a_b \geq 0 \ .$$

Our state lives in the space $\mathcal{V}\times\mathcal{R}_\mathcal{O}$, where $\mathcal{V}$ is the vector representation. However, the inequality in the last step must be understood as a requirement of positive-definition of the matrix depending only on $\mathcal{S}$ acting on the representation space of primaries itself. Positiveness in this case is equivalent to demand the greatest eigenvalue of $\left(\mathcal{S}_{\mu\nu}\right)_a^b$ to be minor or equal than Δ. To obtain such eigenvalue, define $\left(L^{\alpha\beta}\right)_{\mu\nu} \equiv \delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu$, and notice that $(\mathcal{S}_{\mu\nu})_a^{\ b}$ can then be written as:

$$\frac{1}{2}\left(L^{\alpha\beta}\right)_{\mu\nu}(\mathcal{S}_{\alpha\beta})_a^{\ b} = \frac{1}{2}\left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu\right)(\mathcal{S}_{\alpha\beta})_a^{\ b} = \frac{1}{2}\left[(\mathcal{S}_{\mu\nu})_a^{\ b} - (\mathcal{S}_{\nu\mu})_a^{\ b}\right] = (\mathcal{S}_{\mu\nu})_a^{\ b} \ ,$$

where the antisymmetry of rotation matrices were used in the last step.

Not coincidently, $L^{\alpha\beta}$ satisfies Lorentz algebra and, in fact, is the traditional vector representation of the group (notice it acts on the greek indices), while $\mathcal{S}_{\mu\nu}$ acts on the roman indices. We have then a kind of inner product between L and $\mathcal{S}$, which recalls spin-orbit interaction from quantum mechanics problems. That actually is how we are going to obtain such eigenvalue. We know that:

$$L \cdot \mathcal{S} = \frac{1}{2}\left[(L+\mathcal{S})^2 - \mathcal{L}^2 - \mathcal{S}^2\right] \ .$$

Moreover, the operators appearing on the right-hand side of this expression are Casimir operators, used to label primaries. So the eigenvalue of the one on the left-hand side when apllied to a primary is easy to obtain, noting that the orbital angular momentum has defined value equals 1 (after all it refers to a vector representation). Besides, the eigenvalue of the quadratic Casimir operator of the group $SO(d)$ when applied to a primary of spin l is known (see [57]) to be given by $l(l+d-2)$ (result valid for symmetric traceless representations of the group, which is the case for primaries, we are commenting more about it further). The maximum eigenvalue is then the one minimizing the positive term in that expression,[12] that is, the one related to the $l-1$ subspace:

$$\begin{aligned} \text{max. eigenvalue}[(\mathcal{S}_{\mu\nu})_b^{\ a}] &= \frac{1}{4}\left[-(l-1)(l-1+d-2) + 1(1+d-2) + l(l+d-2)\right] \\ &= \frac{l+d-2}{2} \end{aligned} .$$

So, we arrived in the following bound for the scaling dimension of primaries depending on their spin:

$$\boxed{\Delta \geq \frac{l+d-2}{2}} \ , \qquad (57)$$

which, together with (56), is known to be the best constraints we can have for general conformal field theories.

Requiring unitarity of the theory then brings up two imediate consequences: first, it tells us that primaries have scaling dimensions bounded from below, and second, the emergence of the state-operator correspondence guarantees that any operator of the theory can be written as a linear combination of primaries and descendants.

[12] The quadratic Casimir in this case carries a minus sign in view of the fact that the generators are antihermitian, that is why we minimize such term instead of maximizing it.

Formally, a linear combination stands for a finite summation, so that mathematically, the claim above is proved as follows: assume the partition function of the theory is finite, which means that the probability of being in a given subspace spanned by eigenvectors of $\mathcal{D}$ with the same also given eigenvalue is finite too. Notice that the subspace associated to each eigenvalue is finite too, in view of unitarity bounds. Under these assumptions, consider for simplicity a generic state which is eigenstate of $\mathcal{D}$. Now, subtract from such state its projection onto the given subspace and all other subspaces with associated eigenvalues less than that one. The resulting state can then only be zero, because, otherwise, by succesive application of $\mathcal{K}_\mu$ we would encounter a primary which should be in one of those subspaces, a contradiction. The finiteness of those subspaces allows us then to assume the existence of the linear combination of primaries and descendants.

A last comment on the unitarity bounds. Notice that, according to the procedure for obtaining such bounds, for a non-trivial scalar field $\mathcal{O}(x)$ that saturates the condition, that is, a scalar field with dimension $\Delta = (d-2)/2$, there is a state with null norm[13], namely $\mathcal{P}^2|\mathcal{O}\rangle$, which in operator language means that $\mathcal{O}(x)$ satisfies the Klein-Gordon equation $\partial^2\mathcal{O}(x) = 0$, which in turn means that such field is free and therefore decouples from the others. Analogously, for a spin-l primary, say $\mathcal{O}^{\mu\mu_2\cdots\mu_l}(x)$, with dimension $\Delta = (d+l-2)/2$, a state with norm equals to zero is given by $\mathcal{P}_\mu|\mathcal{O}^{\mu\mu_2\cdots\mu_l}\rangle$ [14]; again, in terms of operators, this is equivalent to the conservation equation $\partial_\mu\mathcal{O}^{\mu\mu_2\cdots\mu_l}(x) = 0$. The inverse implication also works once one relates ∂_μ to $\mathcal{P}_\mu$. We conclude then that a given spin-l operator is a conserved current if and only if its dimension equals $\Delta = (d+l-2)/2$. For example, the stress tensor in a CFT has $l = 2$ and $\Delta = d$.

1.5 Conformal Correlators

Besides determining primaries, we also need to know how their dynamics works, that is, how they interact with each other. This is established once we know their correlators. Conformal symmetry brings up some particularities. Since there is more symmetry than in Poincarè group we are going to have more contraints and, in fact, as we will see, correlators in CFTs are well determined, except for some constants. This makes the determination of a CFT be simply given by the CFT data.

Let us start with one-point functions. Obviously it is a function of one variable, say x. In view of invariance under translations due to conformal symmetry, it must be a constant. However, in order to be compatible with scale transformation, this constant must be zero! Otherwise it would not be scaling invariant. Therefore:

$$\langle\mathcal{O}(x)\rangle = 0 \ , \tag{58}$$

[13]Usually called in the literature as *null states*.

[14]Remember we saw that the saturation occurs for the $l-1$ subspace, that is why $\mathcal{P}_\mu$ appears contracted.

where $\mathcal{O}(x)$ is a primary different from identity (in this case such correlator would be just the vacuum norm).

Two-point functions are more interesting. For simplicity, take two scalar primaries $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$, with respective scaling dimensions Δ_1 and Δ_2. Invariance under translations and rotations says then that this function, f, depends only on the absolute value of the difference between both positions:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = f(|x_1 - x_2|) \ .$$

Requiring covariance of the correlator (remember: a correlation function transforms as in (16)) with respect to the scaling transformation (34), we get that the only possibility for its form is:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{C}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \ .$$

Finally, to have a function consistent with the special conformal transformations, which tells us that distances transform under them like (one can show it using (43) directly on the left-hand side of the equation below):

$$|x'_i - x'_j| = \beta^{-1/2}(x_i)\beta^{-1/2}(x_j)|x_i - x_j| \ , \tag{59}$$

we impose, in view of (45) and from the fact that $\lambda(x) = \beta(x)$ in this case:

$$\frac{C}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \lambda(x_1)^{-\Delta_1}\lambda(x_2)^{-\Delta_2}\frac{C\,(\lambda(x_1)\lambda(x_2))^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \Leftrightarrow \Delta_1 = \Delta_2 \ .$$

Therefore,

$$\boxed{\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \begin{cases} \dfrac{C}{|x_1 - x_2|^{2\Delta}} & , \text{ if } \quad \Delta_1 = \Delta_2 = \Delta \\ \\ 0 & , \text{ otherwise} \end{cases}} \ , \tag{60}$$

that is, 2-point functions are zero for operators with different scaling dimensions and are fixed up to a normalization constant C, which is usually taken to be equals to 1.

3-point functions are also fixed by conformal symmetry. Considering another scalar primary $\mathcal{O}_3(x_3)$ with scaling dimension Δ_3, the previous arguments lead us to conclude that this correlator will depend on $x_{12} \equiv |x_1 - x_2|$, $x_{13} \equiv |x_1 - x_3|$ and $x_{23} \equiv |x_2 - x_3|$ in a similar way. All this quantities should appear in the same term, to be consistent with (34) again. Matching the powers with the scaling dimensions through (59), we see there is only one possibility too:

$$\boxed{\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\lambda_{123}}{{x_{12}}^{\Delta_1+\Delta_2-\Delta_3}{x_{13}}^{\Delta_1+\Delta_3-\Delta_2}{x_{23}}^{\Delta_2+\Delta_3-\Delta_1}}} \ , \tag{61}$$

where λ_{123} is a constant.

Now, naturally we want to see the implications of conformal invariance on 4-point functions or higher. Well, these symmetries do not fix them, but restrict their dependence very nicely. The discussion above taught us that a given correlator will depend only on the quantities x_{ij}, the distances between the points, and also that scaling dimensions should appear in the power degree of them. Moreover, (59) tells us that conformally invariant quantities can not be constructed with only two or three points[15]. Four-point functions and higher then will be basically functions of these invariant quantities multiplied by factors of x_{ij} with determined powers.

For example, four points x_1, x_2, x_3 and x_4 can be arranged in a way to build two invariants:

$$u \equiv \frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|} \quad \text{and} \quad v \equiv \frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_4||x_2 - x_3|} \ .$$

Noticing then that each x_i appear in three of the x_{ij}, and inspired on the result for the three-points correlator obtained before, we conclude that 4-point function between scalar primaries has the form:

$$\boxed{\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = g(u,v)\prod_{i<j}^{4} x_{ij}^{\Delta/3-\Delta_i-\Delta_j}} \ , \tag{62}$$

where $\Delta \equiv \sum_{i=1}^{4}\Delta_i$ and g an arbitrary function of u and v.

We could go on and find generic expressions for correlators with more than four points, but actually it is not necessary. Moreover it would be a more difficult work due to the presence of more cross-ratios. That is because, as we are going to see below, conformal invariance in unitary theories allows us to obtain such correlation functions in a recursive way, so that they can be expressed, at the end of the day, in terms of 2-point functions only, which are fixed. To close this subject we are also expliciting what are the Ward identities satisfied by those correlators, which could be used as check of validity for any n-point function one could come across.

1.5.1 Conformal Ward Identities

Before exploring the technology which allows the obtention of general correlators in terms of two point functions, let us quickly present how conformal symmetry reflects on a generic quantum quantity like those. As we have seen, these restrictions from the symmetries are dictated by the so called Ward identities. So, basically, what we are going to do here is to explicit them by just applying (17) to each generator we have in the group.

[15]Notice: in the case of two and three-point functions, they are covariant quantities with respect to conformal tranformations, and not invariant!

Starting with translations, the generator and the conserved quantity is well known:

$$\mathcal{P}_\nu = -i\partial_\nu \quad \rightarrow \quad T^{\mu\nu} \ ,$$

where the energy-momentum tensor is symmetric, remember. In this case, the Ward identity associated is obtained directly:

$$\boxed{\partial_\mu \langle T^\mu{}_\nu(y)\mathcal{X}\rangle = -i\sum_{\mathrm{i}=1}^{n} \delta^d(y-x_\mathrm{i})\frac{\partial}{\partial x_\mathrm{i}^\nu}\langle\mathcal{X}\rangle} \ , \tag{63}$$

where $\mathcal{X}$ stands for $\mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)$, the product of local operators.

Consider now Lorentz rotations. Once $T^{\mu\nu}$ above is symmetrized, (22) tells us that the current associated to the generator $\mathcal{J}_{\mu\nu}(x) = -i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \mathcal{S}_{\mu\nu}$ of rotations is given by:

$$j^{\sigma\mu\nu} = \frac{1}{2}\left(T^{\sigma\mu}x^\nu - T^{\sigma\nu}x^\mu\right) \ .$$

Consequently, the identity we get come from:

$$\frac{1}{2}\partial_\sigma\langle\left(T^{\sigma\mu}(y)y^\nu - T^{\sigma\nu}(y)y^\mu\right)\mathcal{X}\rangle = -i\sum_{\mathrm{i}=1}^{n}\delta^d(y-x_\mathrm{i})\left(x_\mathrm{i}^\mu\frac{\partial}{\partial x_{\mathrm{i}\nu}} - x_\mathrm{i}^\nu\frac{\partial}{\partial x_{\mathrm{i}\mu}} + i\mathcal{S}_\mathrm{i}^{\mu\nu}\right)\langle\mathcal{X}\rangle \ .$$

Using conservation of the energy-momentum tensor, the left-hand side of the equation above simplifies. Notice, however, that the right-hand side can also be simplified by making use of (63) to keep only the spin part. Therefore:

$$\boxed{\frac{1}{2}\langle\left(T^{\nu\mu}(y) - T^{\mu\nu}(y)\right)\mathcal{X}\rangle = \sum_{\mathrm{i}=1}^{n}\delta^d(y-x_\mathrm{i})\mathcal{S}_\mathrm{i}^{\mu\nu}\langle\mathcal{X}\rangle} \ , \tag{64}$$

that is, within a correlator, the energy-momentum tensor is symmetric only at points not coincident to any of the ones at which the local operators are.

Now, moving our attention to dilations, according to section 1.2.2 we do not need to derive the explicit form of the current j_D^μ associated to scaling invariance here, instead we can use its relation to the energy-momentum tensor due to invariance under the entire group of conformal symmetry, that is, the relation $j_D^\mu = T^\mu{}_\nu x^\nu$ we have seen there. The generator of dilations is $\mathcal{D} = -ix^\mu\partial_\mu + i\Delta$, so the Ward identity in this case reads, after similar manipulations:

$$\boxed{\langle T^\mu{}_\mu(y)\mathcal{X}\rangle = i\sum_{\mathrm{i}=1}^{n}\delta^d(y-x_\mathrm{i})\Delta_\mathrm{i}\langle\mathcal{X}\rangle} \ , \tag{65}$$

as we can see, within a correlator, the energy-momentum tensor is traceless only at points different from those of the local operators.

This completes the set of Conformal Ward identities. Special conformal transformation brings up nothing new, the resulting Ward identity is dependent on those ones (see for example [58]) .

1.5.2 Operator Product Expansion

The last important point concerning conformal field theories has to do with singularities. As might be clear at this point, correlation functions are of great importance in physics, specially in particle physics, after all they are closely related to scattering problems and so on. Briefly speaking, correlators are the building blocks for quantities with physical meaning and consequently their behaviour under some limits reflects directly in the physics. In particular, from operators very close to each other we can infer the high energy panorama of a given interacting theory.

Understand this problem is not a recent worry, instead it started long ago. By that time, Wilson[16] hypothesized that the singular part of the product between two operators, say $\phi(x)$ and $\psi(y)$, when $x \to y$, is given by a sum of other local (here considered renormalized) operators $\mathcal{O}$ like in:

$$\phi(x)\psi(y) = \sum_{\mathcal{O}} F_{\mathcal{O}}^{\phi\psi}(x-y)\mathcal{O}(y) \ ,$$

where $F_{\mathcal{O}}^{\phi\psi}(x-y)$ stand for singular functions.

This hypothesis is supported by the fact that, by dimensional analysis, the function $F_{\mathcal{O}}^{\phi\psi}(x-y)$ behaves for $x \to y$ like the power $\Delta_{\mathcal{O}} - \Delta_{\phi} - \Delta_{\psi}$ of the difference $x - y$. So, since the addition of more fields and/or more derivatives to $\mathcal{O}$ increases its complexity, therefore raising $\Delta_{\mathcal{O}}$, as we consider more complicated fields in the expansion, their contribution will be less important, because they will also become less singular.

In fact, it was proved (see [59] for example) that such expansion is valid asymptotically, where the operators $\mathcal{O}$ appearing in there are local ones within a region supposed to surround $\phi(x)$ and $\psi(y)$ apart from other fields. We ommit the proof here for pedagogical reasons, but we do not forget to emphasize a powerful detail: it is a relation valid between operators and, therefore, need not to be inside correlation functions to be used.

Here is the point where conformal symmetry enters. As we have seen, in a unitary theory any operator can be written as linear combination of primaries and descendants. Moreover, we know that descendants are obtained from primaries by applications of the momentum operator. So, we conclude that any operator in a unitary theory can be written actually as a linear combination of only primaries, once the coefficients of the expansion are considered to depend on partial derivatives too.

[16] Kenneth Geddes Wilson: american theoretical physicist, gave great contributions to the understanding of critical phenomena, $\star$06/08/1936 †06/15/2013.

In the present discussion, the last paragraph says then that, in conformal field theories, the product between two operators can be expanded exactly as the sum over primaries. The coefficients in the expansion will depend on the positions and also on derivatives with respect to the position of these primaries. That is because the resulting operator from the product of the two operators creates a state at the boundary of the surrounding area, which can be written as combination of primary and descendant states on the inside, in view of the state-operator correspondence. Therefore, in CFTs the previous result is exact! Moreover, it holds for any distance the operators involved be apart from each other, provided there is no other field within the surrounding region traced. This exact result carries a special name, we call it *Operator Product Expansion* (OPE from now on):

$$\boxed{\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C_{ijk}(x_{12}, \partial_{x_2})\mathcal{O}_k(x_2)} \ , \tag{66}$$

where was adopted the notation from [5] and k stands for primary labels, such that the contribution of each multiplet on the expansion is packaged into the function $C_{ijk}(x_{12}, \partial_{x_2})$, in accordance to what we just said.

There is no need to primaries appearing on the right-hand side of the equation above to be in the same position as one of the operators in the product. That is because radial quantization can be performed around another origin. The consequence of this freedom is that we can also write the equation above like:

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k C_{ijk}(x_{13}, x_{23}, \partial_{x_3})\mathcal{O}_k(x_3) \ ,$$

that is, just adopting a different point of reference.

Another important particularity we need to stress is again the fact that it is an operator relation, therefore it can be manipulated freely even within correlators. Consider for example the product between three ordered scalars $\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)$. We could expand their product firstly surrounding ϕ_2 and ϕ_3, and secondly taking the operators inside the resulting sum and surrounding them together with ϕ_1, producing a final expansion with double sum; alternatively, we could do the same process beginning with ϕ_1 and ϕ_2, an the result would be the same! After all we have an equality in (66). This is a special feature in CFTs called *crossing symmetry*[17], intuitively represented in the figure below.

Conformal invariance fixes 2- and 3-point functions, as we have seen. Now, using OPE we can reduce any n-point function to the sum of only 2-point functions, which are fixed! Therefore, the knowledge of all two and three point correlators automatically allows one to express any other correlator. Of course it is necessary to know the coefficients $C(x, \partial_x)$ of the expansion. It turns out it is also fixed in view of conformal invariance

[17] This name makes sense if one imagines the operators inserted in a correlation function, which allows particle process interpretation and etc.

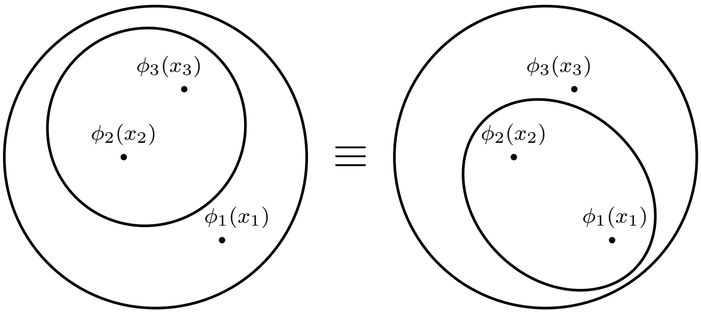

Figure 5: Crossing symmetry: associativeness of the expansions involving products of more than two operators.

of the correlation functions; they can be determined using a simple way: considering for example three scalar primaries, the fact that correlators of two and three points are fixed allows us to write:

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \sum_{k'} C_{ijk'}(x_{12}, \partial_{x_2})\langle \mathcal{O}_{k'}(x_2)\mathcal{O}_k(x_3)\rangle \ .$$

Using then (60) and (61), and normalizing $\langle \mathcal{O}_{k'}(x_2)\mathcal{O}_k(x_3)\rangle = \delta_{kk'} x_{23}^{-2\Delta_k}$, we have:

$$\frac{\lambda_{ijk}}{{x_{12}}^{\Delta_i+\Delta_j-\Delta_k}{x_{13}}^{\Delta_i+\Delta_k-\Delta_j}{x_{23}}^{\Delta_j+\Delta_k-\Delta_i}} = C_{ijk}(x_{12}, \partial_{x_2})\left(\frac{1}{x_{23}^{2\Delta_k}}\right) \ , \tag{67}$$

which tells us that the coefficient can be found by matching an expansion on x_{12}/x_{23} on both sides, being just a differential operator proportional to λ_{ijk}.

So, once one has the scaling dimensions of the fields and also the so called structure constants λ_{ijk} (and similar for spinful operators), one is able to compute any correlator in the theory. These informations together with anomalous dimensions compose what we cited as CFT data before and are the aim of several studies nowadays, specially *Conformal Bootstrap* (particularly stimulated after the seminal work by Riccardo Rattazzi [60] in 2008), and also the review being presented here.

We finish this section studying the nature of the primaries appearing within the OPE (66). We want to prove that they must be in traceless symmetric tensor representations of $SO(d)$. The proof goes as follows: consider the product between two identical scalar primaries, for simplicity, say ϕ, then we have:

$$\phi(x_1)\phi(x_2) = \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}} C_a(x_2, \partial_{x_2})\mathcal{O}^a(x_2) \ ,$$

where a stands for spin indices and $\lambda_{\phi\phi\mathcal{O}}$ is the proportionality constant we encountered previously, being actually the three-point structure constant.

Now, we know that (see for example [61] and [75]) any tensor of a given representation of $SO(d)$ can be decomposed onto a part of non-null trace components and another orthogonal traceless part. Therefore, any irreducible representation ("irreps") of $SO(d)$ must be traceless. Among these tensor irrep of $SO(d)$ we can still separate it onto parts according to the symmetries of the rotation indices, that is, we have a totally symmetric block, partially symmetric ones, and of course totally antisymmetric too. This occurs because rotations do not mix them, specially totally symmetric tensor with others.

Consider then the matrix element between the primaries in the OPE and the operator being expanded, that is $\langle \mathcal{O}^a(x_2)\phi(x_1)\phi(x_2)\rangle$. The most general form for this function can be constructed by using the vectors x_1^μ and x_2^ν and also the metric $\eta^{\mu\nu}$ that we have at our disposal, like:

$$\begin{aligned}\langle \mathcal{O}^a(x_2)\phi(x_1)\phi(x_2)\rangle = {} & f_1(|x_1-x_2|)x_1^{\mu_1}x_1^{\mu_2}...x_1^{\mu_l} + f_2(|x_1-x_2|)x_1^{\mu_1}x_2^{\mu_2}...x_1^{\mu_l} \\ & + f_3(|x_1-x_2|)\eta^{\mu_1\mu_2}x_1^{\mu_3}...x_2^{\mu_l} + ...\end{aligned},$$

from which its clear that $a = \mu_1\mu_2...\mu_l$. Notice that derivatives of a function depending on the absolute value $|x_1 - x_2|$ with respect to any of the variables still have the form of one of those terms above. Note also that $\eta^{\mu\nu}$ is the unique tensor invariant under rotations we have, and that contractions between it and the vectors are also included in those terms.

In this way, clearly irrep of $SO(d)$ can not contain terms proportional to the metric, since it has non-vanishing trace. Therefore the matrix element in question can only have symmetric traceless terms, because positions commute.

Moreover, notice that once $\langle \mathcal{O}^a(x_2)\phi(x_1)\phi(x_2)\rangle = 0$, all the descendants have vanishing matrix element with $\phi(x_1)\phi(x_2)|0\rangle$ too:

$$\begin{aligned}\langle \mathcal{O}^a(x_2)|\mathcal{K}^{\mu_1}...\mathcal{K}^{\mu_n}\phi(x_1)\phi(x_2)\rangle &= \langle \mathcal{O}^a(x_2)|\mathcal{K}^{\mu_1}...\mathcal{K}^{\mu_{(n-1)}}\phi(x_1)\mathcal{K}^{\mu_n}|\phi(x_2)\rangle \\ &= 0\end{aligned},$$

where was used (54) and the fact that $|\phi\rangle \equiv \phi|0\rangle$ is primary.

Therefore in OPE only primaries in symmetric traceless tensor representatios of the group of rotations appear. The indices of more complicated fields on the left hand-side of (66) are incorporated into the coefficient indices, not entering into the primaries labels, and for this reason not affecting the argument above.

The rest of this chapter is devoted to conformal field theories containing extra symmetries, the so called supersymmetries. There are also such theories of physical interest, in fact we focus on some of them in the present work, and at the end of the chapter we present two examples, which are going to be explored later.

1.6 Superconformal Field Theory

Up to now we have been talking only about transformations composing a Lie Group of symmetries in a system. The main characteristic of this kind of group is that the bilinear operation appearing in its defining algebra is the commutator. In appendix A we explored the Poincarè group and above we extended it to the Conformal group. Both of them describes spacetime symmetries.

It turns out, however, that is the most we can do considering Lie Groups of transformations in spacetime. In fact, there is a theorem concerning it, the so called *No-go*[18] *Coleman-Mandula theorem*; it says that any QFT under the three reasonable assumptions below can only present a Lie Group of symmetry consisting in the direct product between the Poincarè group at most and an internal group (usually direct products of $U(1)$) if the theory has massive particles (see for example [34]). Therefore, can not have symmetries mixing different spins.

- • For any given mass M, there are only a finite number of particle types with mass less than it;

- •• Any two-particles state undergoes some reaction at almost all energies (except perhaps an isolated set);

- • • • The amplitudes for elastic two body scattering are analytic functions of the scattering angle at almost all energies and angles,

as in the last reference.[19] In that reference also, one can find that if there is no mass gap in the theory, the result holds for the Conformal group at most, which is an extension of the Poincarè as we said.

Such theorem is from 1967 and, after then, ways to bypass it have emerged. Specially motivated by the idea of having bigger multiplets of symmetry groups containing particles of different spins. The most successful alternative is the systematic study of *Supersymmetry*, traditionally taken to have started with the work [35] by Wess and Zumino. The main idea for its construction is to put bosons and fermions at the same level, in the sense that one can be carried onto the other by means of a symmetry transformation.

1.6.1 Supersymmetry

Bosons and fermions are the building blocks of theoretical particle physics. Their basic difference is statistics: fermions are subjected to the Pauli exclusion priciple and can not be encountered in arbitrary number at a given state of energy, bosons can. The *Spin-Statistics theorem* then connects that to statistical behaviour: bosons have integer spins and fermions semi-integers. In more technical terms, bosonic operators (the ones that create bosons) obey commutation relations, while fermionic operators obey anticommutation relations, represented as always by $\{,\}$.

[18] In Physics, a no-go theorem stands for a statement that a particular situation is not physically possible.

[19] There are some peculiarities in gauge theories and we really encourage the interest reader to see [34].

Supersymmetry (SUSY) then enters with the intention of introducing one or more charges $Q's$ that are responsible to perform transformations of boson type states into fermionic ones and vice-versa; and since spins are connected to spatial rotations behaviour, supersymmetry then is in some sense a spacetime transformation. Here we are going to present some general ideas without going into too much details, keeping only the necessary to make the reader able to understand with some comfort the structure of the theories that are going to be explored in this work. Their construction are not the focus here. The brief discussion to be presented in this section is mainly based in [7], a very good reference for the interested reader, with some support from [37].

Therefore, the way supersymmetry comes to circumvent the limitations of the no-go Coleman-Mandula theorem is by bringing up fermionic charges to the group of symmetry. In fact, that theorem was generalized in this way by Haag, Łopuszański and Sohnius in [36]; they found spacetime and internal symmetries could be related only through fermionic operators of spin 1/2. Besides that kind of relation, the possibility of having in the same multiplet states of different statistics showed up, in accordance with the basic premise.

In this last paragraph the main feature of a *supersymmetric* charge was already put: it must be a fermionic operator of spin 1/2, that is, a spinorial charge. To have an idea, notice that such a charge, say Q, performs the following transformation:

$$Q|\text{boson}\rangle = |\text{fermion}\rangle \quad \text{and} \quad Q|\text{fermion}\rangle = |\text{boson}\rangle \ ,$$

which depdends on the model been studied, including how many $Q's$ there are, the number of which is usually refered to as $\mathcal{N}$.

Now, under an unitary transformation U representing a spatial rotation by 2π around some axis we have:

$$\begin{aligned} UQ|\text{boson}\rangle &= UQU^{\dagger}U|\text{boson}\rangle = U|\text{fermion}\rangle \\ UQ|\text{fermion}\rangle &= UQU^{\dagger}U|\text{fermion}\rangle = U|\text{boson}\rangle \end{aligned} \ .$$

Remembering that fermion states pick up a sign under such a rotation, while bosons do not, that is:

$$U|\text{boson}\rangle = |\text{boson}\rangle \quad \text{and} \quad U|\text{fermion}\rangle = -|\text{fermion}\rangle \ .$$

We conclude that:

$$UQU^{\dagger} = -Q \ , \tag{68}$$

therefore, Q picks up a sign under a 2π spatial rotation, like spinorial operators do. Of course the above argument is not a proof that the operator has spin 1/2, the original paper should contain a more detailed analysis and we refer the reader to it. The important point here is to have in mind, from now on, that supersymmetric charges appear in pairs (after all we have left and right handed spinors) and with spinorial indices: Q_α and $\bar{Q}_{\dot{\alpha}}$. So, for example, in four dimensions, a theory with one supersymmetric charge (therefore $\mathcal{N}=1$, or *simple* SUSY) is going to have four fermionic operators, since we have two Weyl spinors of two entries each.

Technically, these new operators relate to other bosonic symmetries of the system through the commutation rules between them. The new set of commutation and anti-commutation relations of the (super) symmetry group of transformations will then form the *superalgebra* of operators to be explored in the subsequent construction of multiplets.

More precisely, instead of having a Lie algebra, we are going to have what is called *graded* Lie algebra. Let $\mathcal{O}_a$ and $\mathcal{O}_b$ be two operators of the algebra; we associate to them *grades* η_a and η_b, respectively, 0 if it refers to a bosonic operator and 1 if to a fermionic one. The algebra then is generically expressed in terms of structure constants ${C^e}_{ab}$ as:

$$[\mathcal{O}_a, \mathcal{O}_b\} \equiv \mathcal{O}_a\mathcal{O}_b - (-1)^{\eta_a\eta_b}\mathcal{O}_b\mathcal{O}_a = i{C^e}_{ab}\mathcal{O}_e \ , \tag{69}$$

which is an anticommutation relation only for two fermionic insertions, notice. And also with a graded Jacobi identity:

$$[[\mathcal{O}_a, \mathcal{O}_b\}, \mathcal{O}_c\} + [[\mathcal{O}_b, \mathcal{O}_c\}, \mathcal{O}_a\} + [[\mathcal{O}_c, \mathcal{O}_a\}, \mathcal{O}_b\} = 0 \ . \tag{70}$$

Let us work out the case in which we have relativistic theory with only one supersymmetric charge. We start by rewriting the known Poincarè algebra:

$$\begin{aligned} [\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] &= i\left(\eta^{\nu\rho}\mathcal{J}^{\mu\sigma} + \eta^{\mu\sigma}\mathcal{J}^{\nu\rho} - \eta^{\mu\rho}\mathcal{J}^{\nu\sigma} - \eta^{\nu\sigma}\mathcal{J}^{\mu\rho}\right) \\ [\mathcal{P}^{\rho}, \mathcal{J}^{\mu\nu}] &= -i\left(\eta^{\mu\rho}\mathcal{P}^{\nu} - \eta^{\nu\rho}\mathcal{P}^{\mu}\right) \\ [\mathcal{P}^{\mu}, \mathcal{P}^{\nu}] &= 0 \end{aligned} \quad .$$

The inclusion of one poincarè *supercharge* Q_α (and $\bar{Q}_{\dot{\alpha}}$ of course), the additional relations we have to find are:

$$[Q_\alpha, \mathcal{J}^{\mu\nu}] \quad , \quad [Q_\alpha, \mathcal{P}^{\mu}] \quad , \quad \left\{Q_\alpha, Q^\beta\right\} \quad , \quad \left\{Q_\alpha, \bar{Q}^{\dot{\alpha}}\right\} \quad ,$$

together with commutation relations with some internal symmetry generators T^i (in the case they are present), that is, $[Q_\alpha, T^i]$. Similarly for the adjoint $\bar{Q}^{\dot{\beta}}$.

The first of them just reflects the fact that it is a spinor, and since a left-handed (right-handed) one transforms with $\sigma^{\mu\nu}$ ($\bar{\sigma}^{\mu\nu}$), we have:

$$[Q_\alpha, \mathcal{J}^{\mu\nu}] = {(\sigma^{\mu\nu})_\alpha}^\beta Q_\beta \quad \text{and} \quad \left[\bar{Q}^{\dot{\alpha}}, \mathcal{J}^{\mu\nu}\right] = {(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}}_{\dot{\beta}}\bar{Q}^{\dot{\beta}} \ ,$$

valid for 2-dimensional spinors, left (right)-handed spinors in higher dimensions transforms according to generalizations of Dirac gamma matrices, so that we must change $\sigma^\mu \to \gamma^\mu$, and similarly for "bar" quantities. We keep the 2-dimensional notation until the end of this section to avoid confusion.

The second of the list is a consequence of the Jacobi identity for $\mathcal{P}^\mu$, $\mathcal{P}^\nu$ and Q_α. Noticing that Q_α is in $(1/2, 0)$ representation of the Lorentz group, while the momenta is in $(1/2, 1/2)$, the vectorial one, a product of them can only be in $(0, 1/2)$ or $(1, 1/2)$, and since none element of the algebra is in this last representation, we must have $[Q_\alpha, \mathcal{P}^\mu] = c(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}}$ (and also $[\bar{Q}^{\dot{\alpha}}, \mathcal{P}^\mu] = c^*(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}Q_\alpha$), for some constant c. Consequently:

$$\begin{aligned}
[\mathcal{P}^\mu, [\mathcal{P}^\nu, Q_\alpha]] + [\mathcal{P}^\nu, [Q_\alpha, \mathcal{P}^\mu]] + [Q_\alpha, [\mathcal{P}^\mu, \mathcal{P}^\nu]] &= 0 \\
-c(\sigma^\nu)_{\alpha\dot{\alpha}}\left[\mathcal{P}^\mu, \bar{Q}^{\dot{\alpha}}\right] + c(\sigma^\mu)_{\alpha\dot{\alpha}}\left[\mathcal{P}^\nu, \bar{Q}^{\dot{\alpha}}\right] &= 0 \\
|c|^2(\sigma^\nu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}Q_\beta - |c|^2(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}^\nu)^{\dot{\alpha}\beta}Q_\beta &= 0 \ , \\
-4i|c|^2(\sigma^{\nu\mu})_\alpha^\beta Q_\beta &= 0 \\
\therefore c &= 0
\end{aligned}$$

where was used the Poincarè algebra for writing the second line and the definitions from appendix A to write the last result. We see then that the supercharge commutes with the momenta.

Index structure allows only the following candidate for the third commutator in the list:

$$\left[Q_\alpha, Q^\beta\right] = k(\sigma^{\mu\nu})_\alpha^\beta \mathcal{J}_{\mu\nu} \ ,$$

for some constant k. This must be zero because the left-hand side of the equation above commutes with $\mathcal{P}^\mu$, while the right-hand side does not.

For the last case, appealing again to representation arguments, $Q_\alpha\bar{Q}^{\dot{\alpha}}$ must be in $(1/2, 1/2)$, so that the anticommutator is proportional to the only compatible generator of the algebra, $\mathcal{P}^\mu$:

$$\left\{Q_\alpha, \bar{Q}^{\dot{\alpha}}\right\} = 2(\sigma^\mu)_\alpha^{\ \dot{\alpha}}\mathcal{P}_\mu \ ,$$

where the factor 2 is adopted by convention. Notice index structure also implies such form.

Lastly, internal symmetries usually commutes with supercharges, except for automorphisms of the supersymmetry charges, known as *R symmetry*. For $\mathcal{N} = 1$ case, notice the set of relations above is not altered under the following transformations:

$$Q_\alpha \to e^{i\lambda}Q_\alpha \quad , \quad \bar{Q}_{\dot{\alpha}} \to e^{-i\lambda}\bar{Q}_{\dot{\alpha}} \ ,$$

for any real λ.

So let R be a $U(1)$ generator, then we have:

$$[Q_\alpha, R] = Q_\alpha \quad \text{and} \quad \left[\bar{Q}_{\dot\alpha}, R\right] = -\bar{Q}_{\dot\alpha} \ .$$

Putting all together, the superalgebra of generators of a $\mathcal{N}=1$ supersymmetric relativistic theory is given by:

$$\boxed{\begin{array}{l} [\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i\left(\eta^{\nu\rho}\mathcal{J}^{\mu\sigma} + \eta^{\mu\sigma}\mathcal{J}^{\nu\rho} - \eta^{\mu\rho}\mathcal{J}^{\nu\sigma} - \eta^{\nu\sigma}\mathcal{J}^{\mu\rho}\right) \\ [\mathcal{P}^{\rho}, \mathcal{J}^{\mu\nu}] = -i\left(\eta^{\mu\rho}\mathcal{P}^{\nu} - \eta^{\nu\rho}\mathcal{P}^{\mu}\right) \\ [\mathcal{P}^{\mu}, \mathcal{P}^{\nu}] = 0 \\ [Q_\alpha, \mathcal{J}^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta \ \ , \quad \left[\bar{Q}^{\dot\alpha}, \mathcal{J}^{\mu\nu}\right] = (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\bar{Q}^{\dot\beta} \\ [Q_\alpha, \mathcal{P}^{\mu}] = 0 \ \ , \quad \left[\bar{Q}^{\dot\alpha}, \mathcal{P}^{\mu}\right] = 0 \\ \left\{Q_\alpha, \bar{Q}^{\dot\alpha}\right\} = 2(\sigma^\mu)_\alpha^{\ \dot\alpha}\mathcal{P}_\mu \ \ , \quad [Q_\alpha, R] = Q_\alpha \ \text{ and } \ \left[\bar{Q}_{\dot\alpha}, R\right] = -\bar{Q}_{\dot\alpha} \end{array}} \ . \tag{71}$$

Specialising the fourth line to $\mu\nu = 12$ (that is, to the spin projection onto z direction, or the *helicity*), we have that $\sigma^{12} = \frac{1}{2}\sigma^3$ and $\bar\sigma^{12} = -\frac{1}{2}\sigma^3$, consequently $[Q_\alpha, \mathcal{J}^{12}] = \frac{1}{2}(\sigma^3)_\alpha^{\ \beta}Q_\beta$ and $\left[\bar{Q}^{\dot\alpha}, \mathcal{J}^{12}\right] = -\frac{1}{2}(\sigma^3)^{\dot\alpha}_{\ \dot\beta}\bar{Q}^{\dot\beta}$. We see then that while Q_1 and $\bar{Q}^{\dot 2}$ lower the helicity by 1/2, Q_2 and $\bar{Q}^{\dot 1}$ raise it. The commutation relation with the momenta tells us that it does not matter what operation one takes first: supercharge or translation, and in addition, the anticommutation relation of the supercharges Q and $\bar{Q}$ says that if one tries to recover the original state by changing its spin twice, what one gets is in fact the same state, but translated in spacetime.

To extend that superalgebra to the case of more supersymmetries is easy. One just has to notice that the anticommutation relation between two $Q's$ might not be trivial, that is, there can be a *central charge*, say Z, a quantity commuting with all other elements of the algebra. Despite of this detail, it is just a matter of inserting kronecker deltas in the other relations as below:

$$\begin{array}{r} \left\{Q^A_\alpha, \bar{Q}^{B\dot\alpha}\right\} = 2\delta_A^{\ B}(\sigma^\mu)_\alpha^{\ \dot\alpha}\mathcal{P}_\mu \\ \left\{Q^A_\alpha, Q^B_\beta\right\} = \epsilon_{\alpha\beta}Z^{AB} \end{array} \ , \tag{72}$$

where $\epsilon_{\alpha\beta}$ stands for the totally antisymmetric symbol in two dimensions, which makes Z^{AB} also antisymmetric, note. A and B go from 1 to $\mathcal{N}$, the number of supersymmetries. In this case we say we have an *extended* supersymmetry.

One could be wondering if there is a maximum value for $\mathcal{N}$ for a given theory. It turns out there is, and it depends on the dimension and on the renormalizability of the Lagrangian terms. Without going into too much details, for example in four dimensions, theories without gravity (flat spacetime) are limitied to contain only fields with at most spin 1; since each application of supercharge raises (or lower) the helicity by 1/2, from -1 we can apply four different charges to reach 1, therefore the maximal value of $\mathcal{N}$ for those theories is 4. In the case we have gravity (non-flat spacetime), as another mere example, it is known that it is not possible to accomodate spin 5/2 or greater in the theory, therefore limiting the helicity λ to the range $-2 \leq \lambda \leq 2$, and consequently $\mathcal{N} = 8$.

In the following section we include the other symmetries $\mathcal{K}_\mu$ and $\mathcal{D}$ to complete the conformal superalgebra. This is in fact the algebra we are interested in and which we are going to explore in the rest of this chapter to understand a little bit more about the contruction of supersymmetric conformal field theories.

1.6.2 Conformal Field Theories with Supersymmetry

The intention of this section is to finally close the generalities of supersymmetry applied to conformal field theories. Basically we want now to expose the so called *superconformal* algebra and go into its field represention. Again, rigorous and detailed mathematical treatment is not the focus here and we refer the reader to [7] for deeper investigation.

It is not necessary an entire reconstruction of the previous argument in order to include conformal generators into the algebra of supersymmetric relativistic theories. Actually, and following [38], notice that we have to include only $\mathcal{K}_\mu$ and $\mathcal{D}$. It turns out, however, to do that, we are going to need more supercharges to have a closed algebra. The new supercharges S_α^A and $\bar{S}^{A\dot{\alpha}}$ are defined by means of commutation relations of Q's with the special conformal transformation generator:

$$\left[Q_\alpha^A, \mathcal{K}_\mu\right] = \gamma_\mu S_\alpha^A \quad , \quad \left[\bar{Q}^{A\dot{\alpha}}, \mathcal{K}_\mu\right] = \bar{\gamma}_\mu \bar{S}^{A\dot{\alpha}} \ , \tag{73}$$

notice now the presence of γ^μ's instead of σ^μ's.

With this definition, all the commutators and anticommutators of the superconformal algebra can then be determined once the Poincarè superalgebra and the conformal algebra are given, using also the Jacobi identities. Considering for simplicity $\mathcal{N} = 1$, we are led then to the following (considering non-vanishing relations only):

$$\boxed{\begin{array}{cc} [Q_\alpha, \mathcal{J}_{\mu\nu}] = \sigma_{\mu\nu} Q_\alpha & [S_\alpha, \mathcal{J}_{\mu\nu}] = \sigma_{\mu\nu} S_\alpha \\ [Q_\alpha, \mathcal{D}] = -\dfrac{1}{2} i Q_\alpha & [S_\alpha, \mathcal{D}] = \dfrac{1}{2} i S_\alpha \\ [Q_\alpha, \mathcal{P}_\mu] = 0 & [S_\alpha, \mathcal{P}_\mu] = \gamma_\mu Q_\alpha \\ [Q_\alpha, \mathcal{K}_\mu] = \gamma_\mu S_\alpha & [S_\alpha, \mathcal{K}_\mu] = 0 \\ [Q_\alpha, \mathcal{R}] = i\gamma_{(d+1)} Q_\alpha & [S_\alpha, \mathcal{R}] = -i\gamma_{(d+1)} S_\alpha \\ \left\{Q_\alpha, \bar{Q}^{\dot{\alpha}}\right\} = 2\gamma^\mu \mathcal{P}_\mu & \left\{S_\alpha, \bar{S}^{\dot{\alpha}}\right\} = 2\gamma^\mu \mathcal{K}_\mu \\ \multicolumn{2}{l}{\left\{S_\alpha, \bar{Q}^{\dot{\alpha}}\right\} = 2i\mathcal{D} + \sigma^{\mu\nu}\mathcal{J}_{\mu\nu} + 3i\gamma_{(d+1)}\mathcal{R}} \end{array}} \ , \tag{74}$$

where the last of these equations can be seen as the one defining $\mathcal{R}$, the internal symmetry between supercharges. The S's therefore play a role with the $\mathcal{K}_\mu$ similar to the Q's with $\mathcal{P}_\mu$. A very interesting relation appears, note: supercharges also raise and lower dilation eigenvalues! $\gamma_{(d+1)}$ of course is γ_5 for $d = 4$, being just a generalization for higher dimensions.

It is traditional also to organize conformal spinors differently, instead of keeping Q_α and S_α separate, one adopts the following object [39]:

$$\Sigma \equiv \begin{bmatrix} Q_\alpha \\ \bar{S}^{\dot{\alpha}} \end{bmatrix} , \tag{75}$$

the conformal spinor. That is because it simplifies the superconformal algebra in a way to involve only a higher dimensional rotation group plus supersymmetric charges and R-symmetry. There is no need to explicit this massaged algebra here, we just point out that the trick behind it is the *embedding space* formalism, to be talked about in the next chapter.

To finish this section we finally go for field representations of that superconformal algebra, which at the end of the day is what we are really interested in to comprehend the theories composing our framework in this thesis. The main ideas are explored for simplicity for a single supersymmetric charge, that is, $\mathcal{N} = 1$.

A proper construction of field representations of superalgebras involves the definitions of *superspaces* and *superfields* and etc., however we keep the discussion as simple as possible here, therefore avoiding that complicated terminology, but, again, the interested reader is refered to [7].

Usually, a field representation of the superconformal algebra (74) is built up following two steps basically: defining fields via the algebra relations (imposing some conditions) and then defining their infinitesimal supersymmetric variations by means of the so called *anticommuting spinor parameters* ζ^α (for Q_α) and ξ^α (for S_α); they are defined to anti-commute with every fermionic quantity and to commute with every bosonic quantity.

The simplest case, for example, is obtained when one starts assuming the ground state to be a scalar field $A(x)$ and to satisfy the so called *chirality* condition:

$$\left[A, \bar{Q}^{\dot{\alpha}}\right] = 0 \ .$$

From this, the Jacobi identity involving A, Q and $\bar{Q}$ then gives:

$$\left\{[A, Q]\,, \bar{Q}\right\} + \left\{\left[A, \bar{Q}\right], Q\right\} = \left[A, \left\{Q, \bar{Q}\right\}\right] = 2i\gamma^\mu \partial_\mu A \ ,$$

which tells us that $A(x)$ must be complex in order to not be constant.

Other fields $\psi_\alpha(x)$, $F_{\alpha\beta}(x)$ and $X_{\alpha\dot{\beta}}(x)$ can then be defined to explicit the commutation between A and Q:

$$[A, Q_\alpha] \equiv 2i\psi_\alpha \quad , \quad \{\psi_\alpha, Q_\beta\} \equiv -iF_{\alpha\beta} \quad , \quad \left\{\psi_\alpha, \bar{Q}_{\dot{\beta}}\right\} \equiv X_{\alpha\dot{\beta}} \; . \tag{76}$$

Enforcing the algebra on A and ψ we finish the construction. Firstly with A, the previous Jacobi identity gives:

$$2i\gamma^\mu \partial_\mu A = 2i\left\{\psi_\alpha, \bar{Q}_{\dot{\beta}}\right\} = 2iX_{\alpha\dot{\beta}} \; .$$

A similar identity involving A, Q and another Q gives:

$$F_{\alpha\beta} = \epsilon_{\alpha\beta}F \; ,$$

for a complex scalar field $F(x)$. Analogously, from the Jacobi identities involving ψ, Q, $\bar{Q}$ and ψ, Q, Q, the restrictions we arrive enforcing the algebra on ψ are:

$$-i\epsilon_{\alpha\beta}\bar{\chi}_{\dot{\beta}} + 2i(\gamma^\mu)_{\alpha\dot{\beta}}\partial_\mu\psi_\beta = 2i(\gamma^\mu)_{\beta\dot{\beta}}\partial_\mu\psi_\alpha \quad , \quad \epsilon_{\alpha\beta}\lambda_\gamma + \epsilon_{\alpha\gamma}\lambda_\beta = 0 \; ,$$

where $\lambda_\alpha \equiv [F, Q_\alpha]$ and $\bar{\chi}_{\dot{\alpha}} \equiv \left[F, \bar{Q}_{\dot{\alpha}}\right]$. The solution follows from contraction with $\epsilon^{\alpha\beta}$:

$$\bar{\chi}_{\dot{\alpha}} = 2\partial_\mu\psi^\beta(\gamma^\mu)_{\beta\dot{\alpha}} \quad , \quad \lambda_\alpha = 0 \; . \tag{77}$$

The remaining relations can also be verified:

$$\left[\psi, \left\{\bar{Q}, \bar{Q}\right\}\right] = [F, \{Q, Q\}] = \left[F, \left\{\bar{Q}, \bar{Q}\right\}\right] = 0 \quad , \quad \left[F, \left\{Q, \bar{Q}\right\}\right] = 2i\gamma^\mu\partial_\mu F \; .$$

We constructed then a field representation of the $\mathcal{N} = 1$ superalgebra on a multiplet $\phi \equiv (A; \psi; F)$ of fields in terms of the commutators and anticommutators. We have four bosonic degrees of freedom from the real and imaginary parts of the complex scalar fields $A(x)$ and $F(x)$ and four fermionic degrees of freedom composed also by the real and imaginary parts of the two complex spinor components of ψ_α. This representation is usually called *chiral*; by starting with the condition $[A, Q] = 0$ instead of $\left[A, \bar{Q}\right] = 0$ we get the *anti-chiral* multiplet $\bar{\phi} \equiv (A^\dagger, \bar{\psi}, F^\dagger)$. Their supersymmetric variations are defined in view of equation (18):

$$\delta\phi \equiv -i\left[\;\phi, \zeta Q + \bar{Q}\bar{\zeta}\;\right] \Rightarrow \begin{cases} \delta A & = 2\zeta\psi \\ \delta\psi & = -\zeta F - i\partial_\mu A\gamma^\mu\bar{\zeta} \\ \delta F & = -2i\partial_\mu\psi\gamma^\mu\bar{\zeta} \end{cases} , \tag{78}$$

which has an important consequence valid for any supersymmetric field theory, to be commented at the end of this section.

That simple example therefore contains the essential technology behind the construction of any field representation of a superalgebra. Assuming a more complicated ground state (like a vector or something else) would lead us to an also more complicated multiplet, as well as taking a condition different from the chirality; it is expected, however, to get reducible representations in that first case, in analogy to the construction of general representantions of the Lorentz group via tensor products of spinors.

Relaxing the condition over the ground state, actually imposing none, leads us to a general form of that multiplet. Definitions of fields would then have to be more embracing and generic and the algebra enforcements too. Technically, the reproduction of the procedure to this case here would brings nothing new, so we omit it. It is important, nevertheless, to have in mind the whole appearance of such a general multiplet. Let us call it V, we would have then:

$$V = (C; \chi; M; N; A_\mu; \lambda; D) \ , \tag{79}$$

where C, N and D are complex pseudoscalars, M is a complex scalar and A_μ a complex vector, while λ and χ are Dirac spinors, therefore 8 + 8 field components. Their transformation rules are:

$$\begin{aligned}
\delta C &= \bar{\zeta}\gamma_{d+1}\chi \\
\delta\chi &= \left(M + \gamma_{d+1}N\right)\zeta - i\gamma^\mu\left(A_\mu + \gamma_{d+1}\partial_\mu C\right)\zeta \\
\delta M &= \bar{\zeta}\left(\lambda - i\partial\!\!\!/\chi\right) \\
\delta N &= \bar{\zeta}\gamma_{d+1}\left(\lambda - i\partial\!\!\!/\chi\right) \\
\delta A_\mu &= i\bar{\zeta}\gamma_\mu\lambda + \bar{\zeta}\partial_\mu\chi \\
\delta\lambda &= -i\sigma^{\mu\nu}\zeta\partial_\mu A_\nu - \gamma_{d+1}\zeta D \\
\delta D &= -i\bar{\zeta}\partial\!\!\!/\gamma_{d+1}\lambda
\end{aligned} \quad . \tag{80}$$

With the general multiplet in hands one could then impose conditions over it, in contrast to the original example, where the condition is a starting point. For example, imposing the reality condition $V = V^\dagger$ would give the so called *real general multiplet*, in which all components are real or Majorana.

It is important to remember that, up to now, we have been considering only the Poincarè superalgebra. However, we are interested in the superconformal algebra, so we must know how to include conformal generators and also S-supercharges in the procedure above. It turns out that it is sufficient to assume transformation rules for C (now allowed to have Lorentz indices) under dilations, Lorentz and R-transformations:

$$\begin{aligned}[C, D] &= ix \cdot \partial C + i\Delta C \\ [C, J_{\mu\nu}] &= i(x_\mu \partial_\nu - x_\nu \partial_\mu)C + \mathcal{S}_{\mu\nu} C \ , \\ [C, R] &= nC\end{aligned}$$

where Δ is the scaling dimension of C, $\mathcal{S}_{\mu\nu}$ the spin matrix acting on C and n is the R quantum number of C.

A detailed analysis then shows that the general multiplet will be the same with only different transformation rules for a combined variation $\delta V = -i \left[\ V, \bar{\zeta} Q + \bar{\xi} S \ \right]$:

$$\begin{aligned}\delta C &= \bar{\eta}\gamma_{d+1}\chi \\ \delta\chi &= (M + \gamma_{d+1}N)\,\eta - i\gamma^\mu\,(A_\mu + \gamma_{d+1}\partial_\mu C)\,\eta + 2\mathring{X}^+\gamma_{d+1}\xi C \\ \delta M &= \bar{\eta}\left(\lambda - i\partial\!\!\!/\chi\right) + \bar{\xi}\mathring{X}^-\chi - 2\bar{\xi}\chi \\ \delta N &= \bar{\eta}\gamma_{d+1}\left(\lambda - i\partial\!\!\!/\chi\right) - \bar{\xi}\gamma_{d+1}\mathring{X}^-\chi + 2\bar{\xi}\gamma_{d+1}\chi \\ \delta A_\mu &= i\bar{\eta}\gamma_\mu\lambda + \partial_\mu\,(\bar{\eta}\chi) + i\bar{\xi}\mathring{X}^-\gamma_\mu\chi \\ \delta\lambda &= -i\sigma^{\mu\nu}\eta\partial_\mu A_\nu - \gamma_{d+1}\eta D - \mathring{X}^+\,(M - \gamma_{d+1}N)\,\xi + i\gamma^\mu\mathring{X}^+\,(A_\mu + \gamma_{d+1}\partial_\mu C)\,\xi \\ \delta D &= -i\bar{\eta}\partial\!\!\!/\gamma_{d+1}\lambda + 2\bar{xi}\gamma_{d+1}\mathring{X}^-\left(\lambda - \frac{1}{2}i\partial\!\!\!/\chi\right)\end{aligned} \quad , \qquad (81)$$

where we made use of two definitions: $\eta \equiv \zeta - ix^\mu\gamma_\mu\xi$ and $\mathring{X}^\pm \equiv \Delta - \dfrac{3in}{2}\gamma_{d+1} \pm \dfrac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}$.

Building field representations of the superconformal algebra then is resumed in finding fields that transform between under all supersymmetry transformations of the algebra by means of the algebra itself; that is, we put in the same box all fields that are related to each other via application of supercharges. From another point of view, given an operator of the theory, we know that it is going to transform in combinations of the other operators composing the multiplet under any application of supercharges. The other charges of the algebra then will be responsible to connect those different multiplets!

Fields will then be organized accordingly to the quadratic Casimir operators of the algebra and the set of commuting operators formed from them, as always. The conformal algebra (46) tells us that $\mathcal{D}$ and $\mathcal{J}_{\mu\nu}$ commutes, therefore being very nice candidates for labeling operators, in fact this is done when performing radial quantisation as we have seen before. Naturally then $\mathcal{J}^2$ will be a quadratic Casimir, so that we can use it together with its projection in some direction, say $\mathcal{J}_{12}$ (we choose this because in three dimensions it corresponds to z direction), and the dilation operator $\mathcal{D}$ to organize our operators; the corresponding labels will be j, l and Δ, respectively. The special conformal generator and the momentum generator lowers and raises, respectively, the scaling dimension of the operator by 1, as well as they change the Lorentz structure of a given operator, thereby altering also the vector space in which the operator lives, so the spin. On the other hand, as we have just seen, supercharges also promote movements of the operators inside the superconformal multiplet they belongs. Moreover, notice R also gives a good labeling.

Let us now take a closer look on the effect of other generators of the algebra and how multiplets are connected. A given field will carry labels of spin projection and scaling dimension only, like $\mathcal{O}_{l;\Delta}$, that is because j will be implicit and the R quantum number is associated to internal symmetry. We have then, for any local operator constructed from the conformal theory:

$$[D, \mathcal{O}_{l;\Delta}(0)] = i\Delta\mathcal{O}_{l;\Delta}(0) \quad , \quad [J_{12}, \mathcal{O}_{l;\Delta}(0)] = l\mathcal{O}_{l;\Delta}(0) \ . \tag{82}$$

Following the direction given by radial quantisation, there will be special operators from which we can derive others. These operators are called *superconformal primary operators* and are defined to be the ones satisfying:

$$[S_\alpha, \mathcal{O}_{l;\Delta}\} = 0 \quad , \quad [\bar{S}_{\dot{\alpha}}, \mathcal{O}_{l;\Delta}\} = 0 \ , \tag{83}$$

where the brackets depend on the nature of the operator $\mathcal{O}_{l;\Delta}$. This conditions are considered in view of the fact that S's as well as $\bar{S}$'s lower the scaling dimension of a given operator by $1/2$, see (74). So, these superconformal primaries are the operators of lowest scaling dimension on a superconformal multiplet. Moreover, note that, since $\{S, \bar{S}\} \sim \mathcal{K}$, those operators are also conformal primaries, the inverse although is not truth.

Consequently, descendants can be obtained from them. There are two ways of doing so, however. The first one is by applying momentum operators, in which case we yield a new operator of conformal dimension increased by 1, that is, just conformal descendants as befor; an infinite number of them, therefore. The second type of descendants are *superdescendants* obtained from succesive applications of Q's or $\bar{Q}$'s, after all they also increase scaling dimensions according to (74); this time, notice, that eigenvalue is increased by $1/2$ for each application, moreover, the number of successive applications can be limited due to the fact that two Q's anticommute as well as two $\bar{Q}$'s.

Superdescendants are special because they are conformal primaries. They are defined by:

$$\mathcal{O}' \equiv [Q, \mathcal{O}\} \ . \tag{84}$$

The Jacobi identity involving $\mathcal{K}_\mu$, Q and $\mathcal{O}$ together with $[Q_\alpha, \mathcal{K}_\mu] = \gamma_\mu S_\alpha$ shows that:

$$\begin{aligned}
[\mathcal{K}_\mu, [Q, \mathcal{O}\}\} + [Q, [\mathcal{K}_\mu, \mathcal{O}]\} + [\mathcal{O}, [\mathcal{K}_\mu, Q]\} &= 0 \\
[\mathcal{K}_\mu, \mathcal{O}'\} &= \gamma_\mu[\mathcal{O}, S\} \ , \\
\therefore [\mathcal{K}_\mu, \mathcal{O}'\} &= 0
\end{aligned}$$

where in the second line we used the fact that $\mathcal{O}$ is a superconformal primary.

We saw then that every operator of the theory can be constructed from the special ones, in particular that each superdescendant of them generates a conformal multiplet. The supercharges then not only surround operators organizing them onto superconformal multiplets, but also are very useful in the radial quantisation procedure in superconformal field theories.

To finish this subsection, we make some important comments. If one consider a theory with more supersymmetries, that is, an extended one, what changes we would have in the development above? The algebra of course would change, specially because of the presence of kronecker's deltas, central charges and also a more complicated internal symmetry. In consequence of that, the construction of superconformal multiplets would also be more complicated, leading then of course to different supersymmetric transformations inside a general multiplet. However, despite of transformations and representations, the role of each operator would not change and therefore quantisation can be done in the same way; the novelty is that each supercharge S^a (and also $\bar{S}^a$), $a = 1, ..., \mathcal{N}$, would also be used to define superconformal primaries, while Q^a and $\bar{Q}^a$ would expand the number of superdescendants. From now on then, we have enough intuition to comprehend the construction of multiplets of a superconformal field theory as well as the organization of the fields in a given theory. So we make considerations for any $\mathcal{N}$.

An important kind of operator also emerges in what was done previously. In the case Q^a does not create a superdescendant, that is, $[Q, \mathcal{O}_{l\Delta}\} = 0$, we have one more supersymmetry preserved (after all they commute/anticommute) by the operator, this time a Poincarè supercharge. These operators satisfying such conditions carry a special name depending on the number of Poincarè supercharges preserved, they are called *chiral primaries* or also $1/2^k$ *BPS operators*, where $k \in \{1, ..., \mathcal{N}\}$ is the number of those preserved charges. More generally, it is customary to extend the terminology for any operator that commutes/preserves those Q^a and, actually, as we will see, such operators play an important role in the context of the correspondence AdS/CFT. The name BPS stands for *Bogomolnyi–Prasad–Sommerfield*, due to their contribution into the 1-state representations of the superconformal algebra, see [38].

Chiral primaries are very special because, in view of the Jacobi identity involving Q, $\bar{S}$ and $\mathcal{O}$, their scaling dimension are protected from quantum corrections, after all they will be given in terms of spin and R-symmetry eigenvalues:

$$[\{\bar{S}, Q\}, \mathcal{O}_{l;\Delta}(0)] = 0 \Rightarrow [-2i\mathcal{D} + \bar{\sigma}^{\mu\nu}\mathcal{J}_{\mu\nu} - 3i\gamma_{d+1}\mathcal{R}, \mathcal{O}_{l;\Delta}(0)] = 0 \ , \tag{85}$$

where the superconformal algebra was used.

In this last subsection, we finish the chapter with important examples. The usefulness of these will be clear along this work and their construction are not explored here. Nonetheless, as a glimpse, one should have in mind, as can be seen more easily from (78) (take two variations, $[\delta_1, \delta_2]$), that invariant quantities under supersymmetric transformations can only be constants, demanding then the Lagrangian densisty of a given theory to be at most a divergent.

1.6.3 $\mathcal{N} = 4$ SYM and ABJM

We present here the field content of two of very important SCFTs in the context of gauge/gravity duality. Generally speaking, the so called AdS/CFT correspondence, discovered by Maldacena in [8], relates conformal field theories to gravity theories on asympotically Anti-de-Sitter spacetimes. Having two such different theories physically equivalent is a powerful way of making them computationally tractable and also clearer in concepts, which is the reason of that be one of the most exciting discoveries in the last two decades.

The most important example is the widely explored maximally supersymmetric Yang-Mills theory in $4d$, the $\mathcal{N} = 4$ Super Yang-Mills (SYM) (see for example [9]) with gauge group[20] $U(N)$ and Yang-Mills coupling constant g_{YM}, dynamically equivalent to a type IIB superstring theory with string length $l_s = \sqrt{\alpha'}$ and coupling constant g_s on $AdS_5 \times S^5$ with radius of curvature L and N units of $F_{(5)}$ flux on S^5, by means of:

$$g_{YM}^2 = 2\pi g_s \quad \text{and} \quad 2g_{YM}^2 N = L^4/\alpha'^2 \ . \tag{86}$$

A very usual formulation of that SCFT is by making use of a technique called *dimensional reduction*, which we do not explore here, but basically gives the desired theory from a $\mathcal{N} = 1$ SYM theory in ten dimensions. For this reason, it is traditional and simpler to organize the fields into 10-dimensional pieces, which also makes things concise and practical; we refer the reader to [38] as guideline. As in [40], the action for it reads:

$$S_{\mathcal{N}=4SYM} = \frac{1}{g_{YM}} \int d^4x \mathrm{Tr} \left[-\frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi_I)^2 + \frac{1}{2}[\Phi_I, \Phi_J]^2 + i\bar{\Psi}\gamma^\mu D_\mu \Psi + \bar{\Psi}\gamma^I [\Phi_I, \Psi] \right] \ , \tag{87}$$

where we have four gauge potentials A_μ, six scalars Φ_I, $I = 1, ..., 6$ indices of R-symmetry, and four Majorana fermions $\Psi_{\alpha A}$, $\alpha = 1, 2$ spinor indices and $A = 1, 2, 3, 4$ R-symmetry indices, all in the adjoint representation of the gauge group; D_μ stands for covariant derivatives, to be explained in the last chapter. The scalars are in the vector representation of the $SO(6)$ R-symmetry group, while the fermions in the spinor representation of that group. The Dirac matrices $\Gamma^M = (\gamma^\mu, \gamma^I)$ form the Clifford algebra in ten dimensions and fermions satisfy the conditions: $\gamma^{11}\Psi = \Psi$ and $\bar{\Psi} = \Psi^{\mathrm{T}} C$, with the chirality matrix γ^{11} and the charge-conjugation matrix C in ten dimensions; it is possible to choose $\gamma^I = \gamma^5 \Gamma^I$, where Γ^I are the Dirac matrices in six dimensions. The gauge fields and the scalars

[20] We are going to talk more about this in the last chapter, for now it is sufficient to see it as possible extra symmetry in a theory.

combine into a ten dimensional vector potential and the fermions into a single Majorana-Weyl spinor in $10d$.

Besides the conformal symmetry group in $4d$ and the gauge group $U(N)$ we have also the $SO(6)$ R-symmetry group in the theory, and of course the supersymmetries. Together, conformal and R-symmetry groups form the supergroup of symmetries $PSU(2,2|4)$ of the theory; since dilations and rotations can be splitted, operators are labelled according to the cartesian product $SO(3,1)\times U(1)\times SO(6)_{\mathrm{R}}$, with the respective quantum numbers: spin s, scaling dimension Δ and the set of those ones associated to R, known by *Dynkin* labels[21].

The supersymmetry transformations under which such action is invariant are:

$$\boxed{\delta_Q\Psi=\frac{i}{2}F^{MN}\Gamma_{MN}\xi \quad , \quad \delta_Q A_M=-i\bar{\xi}\Gamma_M\Psi \quad , \quad \Gamma_{MN}\equiv\frac{i}{2}(\Gamma_M\Gamma_N-\Gamma_N\Gamma_M)}\ , \tag{88}$$

for the Poincarè supercharges, where ξ is a constant spinor. And

$$\boxed{\delta_S\Psi=\frac{i}{2}F^{MN}\Gamma_{MN}x^\mu\gamma_\mu\eta \quad , \quad \delta_S A_M=-i\bar{\eta}x^\mu\gamma_\mu\Gamma_M\Psi}\ , \tag{89}$$

for the superconformal charges, where again η is a constant spinor.

Another important SCFT goes by the name of $\mathcal{N}=6$ ABJM in three dimensions, with gauge groups $U(N)\times U(N)$ and Chern-Simons levels k and $-k$ (see [41] and [42]). Here it plays the role of the dual or correspondent of a gravity theory: M-theory on $AdS_4\times S^7/\mathbb{Z}_k$ with four-form flux $F^{(4)}\sim N$ through AdS_4. Couplings are connected depending on some limits of treatment, we are going to talk more about this in the last chapter.

The field content now is: four matter scalars $(C_I)_{\hat{\mathrm{i}}}^{\mathrm{i}}$, $I=1,2,3,4$ R-symmetry indices and the others are gauge indices, and correspondent fermions (Weyl-spinors) $(\bar{\psi}^I)_{\hat{j}}^{j}$ in the bifundamental representation of the gauge groups and in the fundamental representation of the R-symmetry group, and conjugate fields $(\bar{C}^I)_{\mathrm{i}}^{\hat{\mathrm{i}}}$ and $(\psi_I)_{j}^{\hat{j}}$, respectively, in the antibifundamental; two Chern-Simons gauge fields A_μ and $\hat{A}_\mu$ for the two gauge groups. As in [43], using also [44], the euclidian action[22] for it reads:

$$S_{\mathrm{ABJM}}=S_{\mathrm{CS}}+S_{\mathrm{matter}}+S_{\mathrm{gf}}\ ,$$

[21] They are just labels for high or lowest weight states in a given representation, similar to quantum mechanics too.

[22] For future purposes we present it in euclidian space;

where:

$$\begin{aligned}
S_{CS} &= -i\frac{k}{4\pi}\int d^3x\epsilon^{\mu\nu\rho}\left[\mathrm{Tr}(A_\mu\partial_\nu A_\rho+\frac{2}{3}iA_\mu A_\nu A_\rho)-\mathrm{Tr}(\hat{A}_\mu\partial_\nu\hat{A}_\rho+\frac{2}{3}i\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho)\right]\\
S_{gf} &= \frac{k}{4\pi}\int d^3x\mathrm{Tr}\left[1/\xi\left(\partial_\mu A_\mu\right)^2+\partial_\mu\bar{c}D^\mu c-1/\xi\left(\partial_\mu\hat{A}^\mu\right)^2-\partial_\mu\bar{\hat{c}}D^\mu\hat{c}\right] \qquad ,\\
S_{\text{matter}} &= \int d^3x\mathrm{Tr}\left[D_\mu\bar{C}_I D^\mu C^I+i\bar{\psi}^I\gamma^\mu D_\mu\psi_I\right]\\
&\quad+\lambda_4(\bar{\psi}\psi)(\bar{C}C)+\lambda_4'(\bar{\psi}C)(\bar{C}\psi)+\lambda_4''\left[(\bar{\psi}C)(\bar{\psi}C)+(\bar{C}\psi)(\bar{C}\psi)\right]+\lambda_6(\bar{C}C)^3
\end{aligned} \tag{90}$$

with S_{gf} the action of auxiliary fields c and $\bar{c}$ for quantisation, the so called *ghosts*, to be understood in the last section, although they will not play an important role in this work. The couplings λ_4, λ_4', λ_4'' and λ_6 control the interaction terms and depend on the Chern-Simons level k. ξ is a parameter to fix the gauge.

In this case the group of symmetry besides the gauges $U(N)_k\times U(N)_{-k}$ is formed by the conformal group in three dimensions, $SO(4,1)$, the R-symmetry group $SO(6)$ and the supersymmetries. The global symmetry composed by the cartesian product of the conformal and R-symmetry groups is denoted by $OSp(6|4)$[23] and, again, we know it is possible to split the conformal part onto dilation and rotation pieces, so that the global group can be written as $SO(3)\times U(1)\times SO(6)_{\text{R}}$, and operators are labelled in accordance.

Finally, supersymmetry transformations for this theory read (see [45]):

$$\boxed{\begin{aligned}
\delta C_I &= -\theta_{IJ}\bar{\psi}^J\\
\delta\bar{C}^I &= -\bar{\theta}^{IJ}\psi_J\\
\delta\psi_I^\alpha &= -2\theta_{IJ}^\beta(\gamma^\mu)_\beta{}^\alpha D_\mu\bar{C}^J+\frac{4\pi}{k}\theta_{IJ}^\alpha\left(\bar{C}^J C_K\bar{C}^K-\bar{C}^K C_K\bar{C}^J\right)+\frac{8\pi}{k}\theta_{KL}^\alpha\bar{C}^K C_I\bar{C}^L\\
\delta\bar{\psi}_\alpha^I &= -2\bar{\theta}^{IJ\beta}(\gamma^\mu)_{\beta\alpha}D_\mu C_J+\frac{4\pi}{k}\bar{\theta}_\alpha^{IJ}\left(C_K\bar{C}^K C_J-C_J\bar{C}^K C_K\right)+\frac{8\pi}{k}\bar{\theta}_\alpha^{KL}C_L\bar{C}^I C_K\\
\delta A_\mu &= -\frac{2\pi i}{k}\left(\bar{\theta}^{IJ}\gamma_\mu C_I\psi_J+\theta_{IJ}\gamma_\mu\bar{\psi}^I\bar{C}^J\right)\\
\delta\hat{A}_\mu &= -\frac{2\pi i}{k}\left(\bar{\theta}^{IJ}\gamma_\mu\psi_J C_I+\theta_{IJ}\gamma_\mu\bar{C}^J\bar{\psi}^I\right)
\end{aligned}} \; , \tag{91}$$

where θ_{IJ}^α and its conjugate is the (spinor) parameter associated to the supercharge Q_α^{IJ}, whose application is obtained trough:

$$Q_\alpha^{IJ}=\frac{\partial}{\partial\theta_{IJ}^\alpha} \; ,$$

and we have the reality condition: $\bar{\theta}^{IJ}=1/2\epsilon^{IJKL}\theta_{KL}$, with $\epsilon^{1234}=1$.

The transformations are similar for superconformal charges, which we do not expose here, except for the spinor parameter $x^\mu\gamma_\mu\eta_{IJ}$.

[23]This is the orthosymplectic group, it is not necessary going into details of it here;

2 CFT with defects/boundaries

Restrictions imposed by conformal invariance were already explored in the previous sections; correlators in this case have closed form, and they are entirely determined via OPE once we know the CFT data. Moreover, in spite of CFTs be more symmetric (and weaker, one would expect) theories, we also saw that the machinery of renormalization group makes them very strong. Breaking some of those symmetries, however, has been shown of great usefulness in describing phenomenological and theoretical situations in Physics. More specifically, the so called conformal defects are the objects that can be used for causing such a break. They should be viewed as structures preserving some of the original conformal symmetries and possibly dividing our theory onto different regions.

The approach for this adopted here is a consequence of the fact that generators of special conformal transformations act non-linearly on the fields. The way we "linearize" such generators is by relating them to Lorentz group generators, using for this the so called Embedding Space formalism; remember our previous treatment of the conformal group in $d = 1$ and $d \geq 3$. The conformal defects will come then as objects preserving part of the starting group of rotations, possibly breaking it into the direct sum of two smaller groups of rotations.

Given the necessary toolbox for analyzing a generic Defect CFT, we work on the particular example of spherical defects, preparing the reader to the next chapters. There we explore the very special case of inserting Wilson loops in 3D and 4D (super) conformal field theories.

2.1 Embedding Space Formalism

Firstly, let us understand the big idea due to Dirac behind this formalism: the natural habitat of the conformal group in d dimensions is the *embedding space* $\mathbb{M}^{d+2}$ of linear isometries, which can be realized as the usual Minkowski $\mathbb{R}^{d+1,1}$ spacetime if we restrict to proper and orthocronus transformations. We refer the conformal group then to the $SO(d+1,1)$ group. Therefore, somehow the original d dimensional spacetime is put within the $d+2$ dimensional spacetime.

Technically speaking, the original spacetime is lifted into a higher dimensional one, via the *push-forward* apparatus from general relativity (see appendix B). The former ends up in a section of a *null-cone* of the latter. Light-cone coordinates turn out to be very usefull to understand this.

Using capital letters to denote coordinates on the embedding space, like X^A with $A = 1, 2, ..., d+2$, we define then two new coordinates: $X^+ \equiv X^{d+1} + X^{d+2}$ and $X^- \equiv$

$X^{d+1} - X^{d+2}$, so that the line element dS^2 is just:

$$dS^2 = \eta_{AB} dX^A dX^B = \sum_{n=1}^{\mathrm{d}} dX^n dX^n - dX^- dX^+ \ ,$$

where η_{AB} is the mostly plus minkowskian signature (timelike coordinate being X^{d+1}) and the light-cone coordinates are then just $X^A = (X^+, X^-, X^a)$, with $a = 1, ..., d$. The d-uple labeled with a refers to the coordinates of the d-dimensional spacetime if we Wick-rotate X^d and identify $x^0 = iX^d$.

Now we reduce the dimension of the manifold in two in order to obtain the physical spacetime. To do this we impose two constraints on the coordinates of the embedding space. Making the *embedded space* be on the null-cone X^2 (note it is invariant under Lorentz transformations) and identifying points on it up to a rescaling: $X \sim \Omega X, \Omega \in \mathbb{R}^+$, we only need to choose a section of the cone to get $\mathbb{R}^{d-1,1}$; this means that to each physical point x^μ corresponds a line on the light-cone (see Figure 6). We select the so called Poincarè section, namely $X^+(x^\mu) = 1$. For future interests, notice that our section is parametrized as $X^M_x = (1, x^2, x^\mu)$.

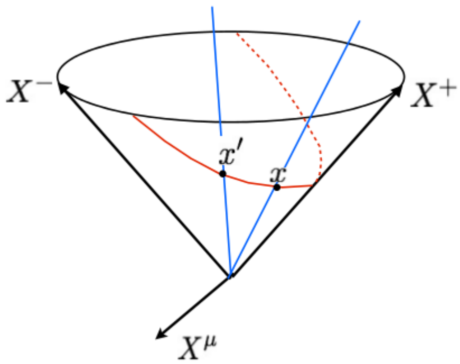

Figure 6: Embedding into the null-cone. Figure adapted from [25].

Speaking in terms of vectors, acting with elements of the $SO(d+1,1)$ group on a given point of the section must reflect as the action of a conformal transformation on x^μ. This means that the induced metric from $\mathbb{R}^{d+1,1}$ in $\mathbb{R}^{d-1,1}$ (or simply $\mathbb{R}^d$) is conformal. We investigate this studying infinitesimal changes at points $X(x^\mu)$ and $X' = \Omega(x^\mu)\Lambda X(x^\mu)$ on the section under rotations, where Λ stands for the rotation of X while $\Omega(x^\mu)$ brings the rotated point back into the section rescaling it. If the initial claim is true, the induced metric should transform like $ds'^2 = c(x)ds^2$, for positive $c(x)$. In fact:

$$ds'^2 \equiv dS'^2\big|_{X'^+=1, X'^-=x'^2/X'^2} = \left[d\left(\Omega(x)\Lambda X\right)\big|_{X^+=1, X^-=x^2/X^2} \right]^2 \ .$$

So,

$$\begin{aligned} ds'^2 &= \left[\Omega(x) d\left(\Lambda X\right)\big|_{X^+=1, X^-=x^2/X^2} + \Lambda X \left(\boldsymbol{\nabla}\Omega \cdot dX\right)\big|_{X^+=1, X^-=x^2/X^2}\right]^2 \\ &= \Omega^2(x)\left(d\Lambda X\big|_{X^+=1, X^-=x^2/X^2}\right)^2 \\ &= \Omega^2(x)(dX)^2\big|_{X^+=1, X^-=x^2/X^2} = \Omega^2(x) ds^2 \end{aligned} ,$$

where to write the second equality it was used the fact that on the null-cone we have $X^2 = 0$ and therefore $X \cdot dX = 0$, while the invariance of the line element under rotation was used to identify the last line.

We see then that $c(x) = \Omega^2(x)$ and therefore the conformal group in d dimensions is in fact embedded into minkowskian spacetime in $d+2$ dimensions, thus conformal calculations can be translated into the language of linear transformations in higher dimensions, making easier and simpler all computations.

As before, the conformal generators can be easily identified. Compare with equation (48):

$$\boxed{\begin{array}{ll} \mathcal{J}_{\mu\nu} = \mathcal{M}_{\mu\nu} \quad , & \mathcal{J}_{\mu +} = \mathcal{P}_\mu \\ \mathcal{J}_{\mu -} = \mathcal{K}_\mu \quad , & \mathcal{J}_{+-} = \mathcal{D} \end{array}} , \tag{92}$$

where $\mu, \nu = 0, ..., d-1$ and antisymmetry under $\mu \leftrightarrow \nu$ is implicit.

2.1.1 Tensors and their encoding by polynomials

Naturally we are interested in extending the formalism above to fields, that is, in formalizing how fields defined in the physical spacetime are embedded. This is a straightforward procedure and goes as follows. Let $F_{A_1, A_2, ..., A_l}(X)$ be a field on $\mathbb{R}^{d+1,1}$, therefore a tensor of $SO(d+1,1)$, with the following properties:

i. It is defined on the cone $X^2 = 0$;

ii. Homogeneous of degree $-\Delta$: $F_{A_1, A_2, ..., A_l}(\lambda X) = \lambda^{-\Delta} F_{A_1, A_2, ..., A_l}(X)$, $\lambda > 0$;

iii. Transverse in all indices: $(X \cdot F)_{A_1, ..., A_{k-1}, A_{k+1}, ..., A_l} \equiv X^{A_k} F_{A_1, ..., A_k, ..., A_l} = 0$, $k = 1, ..., l$.

The first condition is obvious, since we want a well defined physical theory. The requirement of homogeneity guarantees the field is known on the entire light-cone once it is known on the Poincarè section; by means of a *pull-back* of the tensor field (see appendix B), we project it onto the section, defining a field on $\mathbb{R}^{d-1,1}$:

$$f_{a_1, ..., a_l}(x) \equiv \frac{\partial X^{A_1}}{\partial x^{a_1}} \cdots \frac{\partial X^{A_l}}{\partial x^{a_l}} F_{A_1, A_2, ..., A_l}(X_x) \ , \tag{93}$$

this new operator is identified to the physical one. Note that tensors differing by an amount proportional to any of the X^A project onto the same physical operator. This is a gauge freedom; fields proportional to X^A project to zero, we refer to them as *pure gauge*.

For a scalar field Δ is easily interpreted as the scaling dimension, explaining the requirement; note that in this case $f(x)$ is just the restriction of $F(X)$ to the section. This interpretation holds for more complicated fields, it will become clearer below when we derive the transformation rule for the embedded field $f_{a_1,\dots,a_l}(x)$. In this way, transversality condition **iii.** ensures such transformation coincides with general conformal transformation rules for primary tensors. In fact, for a coordinate transformation like $X'^A = \Lambda^A_B X^B_x$, we have:

$$f_{a_1,\dots,a_l}(x) \to \widehat{f}_{a_1,\dots,a_l}(x') = \frac{\partial X^{A_1}(X')}{\partial x^{a_1}} \dots \frac{\partial X^{A_l}(X')}{\partial x^{a_l}} \Lambda^{B_1}_{\ A_1} \dots \Lambda^{B_l}_{\ A_l} F_{B_1,\dots,B_l}(X') \ .$$

Noticing then that, in view of $X'^B dX'_B = 0$ on the cone:

$$\frac{\partial X'^B}{\partial x^a} dX'_B = \Lambda^B_{\ A} \frac{\partial X^A}{\partial x^a} dX'_B \Rightarrow \frac{\partial X'^B}{\partial x^a} - \Lambda^B_{\ A} \frac{\partial X^A}{\partial x^a} \propto X'^B \ .$$

Transversality condition allows us to write:

$$\widehat{f}_{a_1,\dots,a_l}(x') = \frac{\partial X'^{B_1}(X')}{\partial x^{a_1}} \dots \frac{\partial X'^{B_l}(X')}{\partial x^{a_l}} F_{B_1,\dots,B_l}(X') \ ,$$

which is just a massaged form for the starting expression for $\widehat{f}_{a_1,\dots a_l}$.

Remember now that a second step in embedding space formalism is the scaling of the points back to the Poincarè section, which contains the physical information. In other words, the transformation above carries the original field to a point not necessarily on the physical section of the null-cone, we need then to project the result on it. This is done by means of the addition of a position dependent scaling factor $\lambda(Y_y)$ into the mapping, where Y_y is the final point at the section again, thus understanding X' as related to Y_y by $X' = \lambda(y) Y_y$.

In virtue of the homogeneity satisfied by $F_{A_1,\dots,A_l}$, when projected it is going to yield an overall factor of $\lambda^{-\Delta}(y)$ (as if it was calculated in $\lambda(y)Y_y$). The partial derivatives, on the other hand, can be manipulated as below:

$$\begin{aligned} \frac{\partial X'^B}{\partial x^a} &= \frac{\partial x'^b}{\partial x^a} \frac{\partial X'^B}{\partial x'^b} \\ &= \frac{\partial x'^b}{\partial x^a} \left[\frac{\partial Y^B_y}{\partial y^b} + Y^B_y \lambda'(y) \right] \end{aligned} ,$$

the second term above vanishes when contracted with $F_{\dots,B,\dots}$ in view of transversality. Therefore, as expected (see (50)), $f_{a_1,\dots,a_l}$ transforms as a primary tensor field under a general conformal transformation denoted by $x \to y$:

$$\begin{aligned} f_{a_1,\ldots,a_l}(x) \to \widehat{f}_{a_1,\ldots,a_l}(y) &= \lambda^{-\Delta}(y)\frac{\partial x'^{b_1}}{\partial x^{a_1}}\ldots\frac{\partial x'^{b_l}}{\partial x^{a_l}}f_{b_1,\ldots,b_l}(y) \\ &= \lambda^{-\Delta}(y)M_{a_1}{}^{b_1}\ldots M_{a_l}{}^{b_l}f_{b_1,\ldots,b_l}(y) \end{aligned} , \tag{94}$$

from which it is clear the meaning of Δ as stated before.

Moreover, condition **iii.** has other important consequences: traceless and symmetry nature in the indices of embedding operators are carried over to the physical spacetime. The second correspondence is obvious, the first deserves some lines of calculations:

$$\begin{aligned} \eta^{a_1a_2}f_{a_1,a_2,\ldots,a_l}(x) &= \eta^{a_1a_2}\frac{\partial X^{A_1}}{\partial x^{a_1}}\frac{\partial X^{A_2}}{\partial x^{a_2}}\ldots\frac{\partial X^{A_l}}{\partial x^{a_l}}F_{A_1,A_2,\ldots,A_l}(X_x) \\ &= \left(\eta^{A_1A_2} + X_x^{A_1}K^{A_2} + X_x^{A_2}K^{A_1}\right)\frac{\partial X^{A_3}}{\partial x^{a_3}}\ldots\frac{\partial X^{A_l}}{\partial x^{a_l}}F_{A_1,A_2,\ldots,A_l}(X_x) \ , \\ &= \frac{\partial X^{A_3}}{\partial x^{a_3}}\ldots\frac{\partial X^{A_l}}{\partial x^{a_l}}\eta^{A_1A_2}F_{A_1,A_2,\ldots,A_l}(X_x) \end{aligned}$$

where it was chosen the first two entries without loss of generality. To write the second line we used $K^A \equiv (0,2,0)^A$ in light-cone coordinates and the identity:

$$\eta^{a_1a_2}\frac{\partial X^{A_1}}{\partial x^{a_1}}\frac{\partial X^{A_2}}{\partial x^{a_2}}(X_x) = \eta^{A_1A_2} + X_x^{A_1}K^{A_2} + X_x^{A_2}K^{A_1} \ ,$$

that can be checked explicitly without difficulty. Last line is a consequence of transversality.

So, $d+2$ tensors in irreducible representations of $SO(d+1,1)$ correspond to irreducible ones of $SO(d-1,1)$, since in such representations generators of rotations are traceless. Moreover, remember that traceless energy-momentum tensor guarantees conformal invariance of the system. The most pleasant consequence, however, is that any conformally invariant quantity in $\mathbb{R}^{d-1,1}$ is lifted to a $SO(d+1,1)$-invariant in the embedding space, in particular the correlation functions. This makes kinematics of conformal field theories as simples as kinematics of Lorentz-invariant field theories. We can do computations with tensor fields in $\mathbb{R}^{d+1,1}$ and project the result to the physical spacetime, conformal invariance then will be automatic.

Before exploring correlators and defects, a last tool turns out to be very useful in that study, with which we end this subsection. It is basically a clever glimpse of the fact that primary operators are represented by symmetric traceless tensor, as stated in the previous section. We use this fact to compile the tensor into a polynomial, making computations even simpler. Detailed mathematical discussion is not of our interest here, so we refer the reader to [46] and [48] for a guidance in that direction.

The basic idea is that a symmetric traceless tensor $f_{\mu_1 \ldots \mu_l}(x)$ can be encoded into a polynomial $f(x,z)$ restricted to the manifold $z^2 = 0$ by using an auxiliary vector z^μ:

$$f_{\mu_1 \ldots \mu_l}(x) \to f_l(x,z) \equiv z^{\mu_1} \ldots z^{\mu_l} f_{\mu_1 \ldots \mu_l}(x), \quad z^2 = 0 \ , \tag{95}$$

the condition in the end is to enforce tracelessness.

The correspondence is one to one, this can be seen directly by expanding $f(x,z)$ on z, the initial tensor is recovered as the coefficient of one of those terms in the expansion. On the other hand, a more pratical way of recovering the index structure is via the *Todorov* differential operator [49]:

$$D_\mu = \left(\frac{d-2}{2} + z \cdot \frac{\partial}{\partial z} \right) \frac{\partial}{\partial z^\mu} - \frac{1}{2} z_\mu \frac{\partial^2}{\partial z \cdot \partial z} \ . \tag{96}$$

For example, one index is made free by applying the equation above once:

$$f_{\mu_1 \mu_2 \ldots \mu_l} z^{\mu_2} \ldots z^{\mu_l} = \frac{D_{\mu_1} f_l(x,z)}{l \, (d/2 + l - 2)} \ .$$

And of course, all of them are released by applying such operator l times:

$$f_{\mu_1 \ldots \mu_l} = \frac{D_{\mu_1} \ldots D_{\mu_l} f_l(x,z)}{l! \left(\frac{d-2}{2} \right)_l} \ ,$$

where $(a)_l = \Gamma(a+l)/\Gamma(a)$ is the *Pochhammer* symbol.

This is a way of encoding primary fields, therefore embedded ones. Remembering symmetry structure is preserved in projections, the embedding fields $F_{A_1 \ldots A_l(X)}$ from which the primaries come can be encoded into $F_l(X)$ in the same way presented above, except for the extra condition concerning their need of being transverse, namely $Z \cdot X = 0$.

A close relation between both encodings should be expected then, and in fact we have it. Using the explicit form of $\partial X_x / \partial x$, the encoding procedure agrees with (93) through:

$$f_l(x,z) = F_l \left(X_x, Z_{z,x} \right) \ , \tag{97}$$

where $Z_{z,x} \equiv (0, 2x \cdot z, z)$ is a consequence of contracting the partial derivatives with the auxiliary vectors, with the already expected properties $Z_{z,x} \cdot X_x = 0$ and $Z_{z,x}^2 = z^2$. In resume, diagramatically we have:

$$\begin{tikzcd} F_{a_1 \ldots a_l}(X) \arrow[r] \arrow[d, dashed] & F_l(X,Z) \arrow[d, dashed] \\ f_{a_1 \ldots a_l}(x) \arrow[r] & f_l(x,z) \end{tikzcd} \ ,$$

where dashed lines indicate projections, and the others, encoding.

2.1.2 Correlation Functions

In this section we are going to explore technical consequences of the powerful formalism developed previously. We saw the conformal group in d dimensions is equivalent to $SO(d+1,1)$, which means invariant quantities under the former are taken to invariant quantities of the later.

We are going to show then how conformal correlators can be obtained by projecting Lorentz-invariant equivalent quantities from the embedding space. The standard procedure will be to start from the most general form possible for these quantities and *pull-back* it; a given embedding correlation function will depend on a set of spacetime points X_N and a set of auxiliary vectors Z_N that equals to $(0, 2x_n \cdot z_n, z_n^\mu)$ when projected, if one chooses to deal with polynomlials. In order to do so, we derive the most useful rules for projecting it down to physical space:

$$\begin{aligned}
-2X_N \cdot X_M &= -2x_n \cdot x_m + 2X_n^{d+1}X_m^{d+1} - 2X_n^{d+2}X_m^{d+2} \\
&\to -2x_n \cdot x_m + x_n^2 + x_m^2 = (x_n - y_m)^2 \\
Z_N \cdot Z_M &= z_n \cdot z_m - Z_n^{d+1}Z_m^{d+1} + Z_n^{d+2}Z_m^{d+2} \to z_n \cdot z_m \\
X_N \cdot Z_M &= x_n \cdot z_m - X_n^{d+1}Z_m^{d+1} + X_n^{d+2}Z_m^{d+2} \to x_n \cdot z_m - x_m \cdot z_m = (x_n - x_m) \cdot z_m
\end{aligned} .$$

Summarizing:

$$\boxed{-2X_N \cdot X_M = x_{nm}^2 \quad , \quad Z_N \cdot Z_M = z_n \cdot z_m \quad , \quad X_N \cdot Z_M = x_{nm} \cdot z_m} \, , \tag{98}$$

where it was defined $x_{nm}^\mu \equiv (x_n - x_m)^\mu$.

we are ready now to get physical correlators from embedding ones. Some of the results can be checked in [50]. Working with polynomials instead of tensors turns out to be worthful when considering spinning correlators. We are going to start simple, considering a 2-point function between two scalar fields $\phi(X)$ and $\phi(Y)$.

At our disposal, the only Lorentz invariant we have is $X \cdot Y$. By inspection, one can verify that the only structure possible to this starting 2-point function then is:

$$\langle \phi(X)\phi(Y) \rangle = \frac{c}{(X \cdot Y)^\Delta} \, ,$$

where c is some constant and Δ is the common degree of the scalars. Notice this is in agreement with homogeneity condition, and also that quadratic or higher power terms on X or Y are not allowed in view of the cone condition. Moreover, from our construction, this expression is already conformally invariant and refers to 2-point function between primaries in the physical spacetime.

Using (98) then, we arrive at the known result directly:

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{(x-y)^{2\Delta}} ,$$

where the overall factor was taken to be 1.

To fix ideas, let us also obtain 3-point functions between scalars $\phi_1(X_1)$, $\phi_2(X_2)$ and $\phi_3(X_3)$, with respective scaling dimensions (more precisely, homogeneity degree): Δ_1, Δ_2 and Δ_3. This time we have three Lorentz invariants obviously: $X_1 \cdot X_2$, $X_2 \cdot X_3$ and $X_1 \cdot X_3$. Inspired from the previous case, we found that the most general form for this correlator is[24]:

$$\langle \phi_1(X_1)\phi_2(X_2)\phi_3(X_3)\rangle = \frac{c_{123}}{(X_1\cdot X_2)^{\alpha_{123}}(X_1\cdot X_3)^{\alpha_{132}}(X_2\cdot X_3)^{\alpha_{231}}} , \tag{99}$$

where α_{123}, α_{132} and α_{231} are constants that must satisfy:

$$\begin{aligned} \alpha_{123} + \alpha_{132} &= \Delta_1 \\ \alpha_{123} + \alpha_{231} &= \Delta_2 \; . \\ \alpha_{132} + \alpha_{231} &= \Delta_3 \end{aligned}$$

Solving it, we get:

$$\alpha_{ijk} = \frac{\Delta_i + \Delta_j - \Delta_k}{2} .$$

Projecting it, we have:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{|x_{12}|^{2\alpha_{123}}|x_{13}|^{2\alpha_{132}}|x_{23}|^{2\alpha_{231}}} .$$

Now an example following the same reasoning, but with more complicated fields. Consider the 2-point function between two vector operators $V_M(X)$ and $V_N(Y)$. We have only one Lorentz invariant $X\cdot Y$ to infer a homogeneous quantity, but besides we now have to worry about transversality. Non-linear terms on X and Y are prohibited as before. Linear terms on X_M or Y_N are pure gauge and need not to be considered. In this way, one checks that the resulting possibility is:

$$\langle V_M(X)V_N(Y)\rangle = \frac{c}{(X\cdot Y)^{\Delta}}\left(\eta_{MN} + \alpha\frac{Y_M X_N}{X\cdot Y}\right) , \tag{100}$$

where, again, c is an overall constant and Δ stands for the degree of the operator.

Tranversality condition then enforces $\alpha = -1$. Projecting the result and taking again

[24] We reproduce some steps from [25]

the overall constant to be equal to 1, we have:

$$\boxed{\langle v_\mu(x) v_\nu(y)\rangle = \frac{I_{\mu\nu}(x-y)}{(x-y)^{2\Delta}} \quad , \quad \text{with} \quad I_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}} \ . \tag{101}$$

At this point, we should make use of the tensorial encoding developed previously. We are going to apply it with the same purpose, but it works in a little bit different way. The encoding absorbs the indices from the tensor, so the approach goes in the way of starting with a polynomial which carries the desired conditions for the field, then projecting it using the rules involving auxiliary vectors and finally obtaining a physical field via Todorov operator if necessary.

We saw that tranversality of the embedding fields guarantees they are projected onto primaries, besides homogeneity of course. Tranversality is going to play the most important role now, so we begin reaffirming it in the encoding language. Suppose $F_l(X,Z)$ is a polynomial encoding such a field, then looking at (95), the condition is translated to:

$$X \cdot \frac{\partial F_l(X,Z)}{\partial Z} = 0 \quad \equiv \quad F_l(X, Z + \alpha X) = F_l(X,Z) \quad , \quad \forall \alpha \ . \tag{102}$$

In general, a transverse field $F(x)$ may contain pure gauge terms, such that $X \cdot F = 0$ is valid only once $X^2 = 0$ is considered (they are referred to as tensors transverse *modulo* X^2). It turns out we can simplify and actually make the condition stronger, without using $X^2 = 0$. For this, we define the so called *identically transverse* tensors, which are tensors that are transverse in each index, but not only in the cone, that is, $X \cdot F = 0$ is satisfied identically. These tensors are easier to characterize, and as seen before they project onto the same field, because differ by pure gauge terms only. We are going to use their encoding as building blocks for the encoded correlators, exploring all possible contractions.

Usually we are going to work with fields constructed from metrics and components of $\mathcal{M}^{d+1,1}$ vectors. For such tensors there is a "canonical rule" for obtaining the identically transverse encoding polynomials: dropping $Z \cdot X$ and Z^2 terms, the pure gauge ones. This is a consequence of the following fact: given a tensor $F_{A_1 \dots A_l(X)}$ like those that is also transverse modulo X^2, dropping any terms within it proportional to X^2, $\eta_{A_i A_j}$ or X_{A_i}, the resultant tensor will be identically transverse. The proof is clarifying: separate F in two parts $\widehat{F}$ and $\tilde{F}$, the first one containing all terms to be dropped, so that $X \cdot \widehat{F}$ contain only terms proportional to X_{A_i} and/or X^2. Then $X \cdot \tilde{F}$ is going to contain only terms proportional to the other possible vectors Q_{B_i}, with coefficients proportional to $Q \cdot X$ or $Q_{B_i} \cdot Q_j$ (if more than one extra vector); those terms will not mix up, and, therefore, will not be possible to cancel them out, so if $X \cdot F$ is to vanish when $X^2 = 0$, then $X \cdot \tilde{F}$ must vanish identically, and of course $\tilde{F}$ is the identically transverse tensor. Notice also that $\tilde{F}_l(X,Z)$ satisfies (102) identically, and so we extend the definition and refer to it as *identically tranverse polynomial.*

All the discussion in the last paragraphs finally provides the recipe for obtaining encoded correlators from embedding space. The whole problem resides now in constructing the most general polynomial that encodes the n-point function between identically transverse tensors. However, this is not a difficult task, and this suffices because we know such polynomial is going to produce the right projection. Let us then convince ourselves that a polynomial $\tilde{F}_l(X,Z)$ is identically transverse if and only if the variable Z_A appears in it only by means of the tensor:

$$\boxed{C_{AB} = Z_A X_B - Z_B X_A} \ . \tag{103}$$

On one hand, if Z_A appear on it only through this expression, clearly (102) is identically satisfied. On the other hand, if we have such an identically transverse polynomial, it can only have terms linear on Z_A, but no terms like $Z \cdot X$. That is, it should appear contracted with different X's or Z's coming from encoding other tensors. Therefore Z_A must appear besides X_B only, but in a way that (102) is satisfied identically, this is done antisymmetrizing the indices with (103).

This recipe should be consistent with what was done before, so we start using it to recalculate the correlator between two vector fields. We have in this case two spacetime points X_1 and X_2, and two auxiliary vectors Z_1 and Z_2, respectively, each of which appears only once since vectors are $(1,0)$ tensors. So two C's appear:

$$\begin{aligned} C_{1AB} &= Z_{1A}X_{1B} - Z_{1B}X_{1A} \\ C_{2CD} &= Z_{2C}X_{2D} - Z_{2D}X_{2C} \end{aligned} \ .$$

One verifies we have only one possible non-vanishing contraction:

$$C_{1AB}{C_2}^{AB} = 2\left[(X_1 \cdot X_2)(Z_1 \cdot Z_2) - (X_1 \cdot Z_2)(X_2 \cdot Z_1)\right] \ .$$

That is because C_{AB} is traceless and also because a string made of alternating C's reduces to powers of $C_1 \cdot C_2$ multiplying one of the C's, so that the most general solution is a function of $C_1 \cdot C_2$:

$$\begin{aligned} {C_{1A}}^B {C_{2B}}^C C_{1CD} &= (Z_{1A}{X_1}^B - {Z_1}^B X_{1A})(Z_{2B}{X_2}^C - {Z_2}^C X_{2B})(Z_{1C}X_{1D} - Z_{1D}X_{1C}) \\ &= [Z_{1A}(X_1 \cdot Z_2){X_2}^C - Z_{1A}(X_1 \cdot X_2){Z_2}^C \\ &- X_{1A}(Z_1 \cdot Z_2){X_2}^C + X_{1A}(Z_1 \cdot X_2){Z_2}^C](Z_{1C}X_{1D} - Z_{1D}X_{1C}) \\ &= -\frac{1}{2}(C_1 \cdot C_2)C_{1AD} \end{aligned} \ .$$

In order to the 2-point function $G(X_1, X_2; Z_1, Z_2)$ obtained using the contractions be homogeneous of degree Δ, we multiply each possible term by a power of $X_1 \cdot X_2$ consistent with homogeneity condition; notice these powers do not affect the transversality of the polynomial and also that it is made of a Lorentz-invariant quantity linear on the positions. In this case, we just have to divide by $(X_1 \cdot X_2)^{\Delta+1}$ and get:

$$G(X_1, X_2; Z_1, Z_2) = 2\frac{(X_1 \cdot X_2)(Z_1 \cdot Z_2) - (X_1 \cdot Z_2)(X_2 \cdot Z_1)}{(X_1 \cdot X_2)^{\Delta+1}} \ . \tag{104}$$

Projecting it onto the physical space using (98), we get the following encoded correlator:

$$\begin{aligned} g(x_1, x_2; z_1, z_2) &= 2\frac{-\frac{1}{2}x_{12}^2(z_1 \cdot z_2) - (z_2 \cdot x_{12})(z_1 \cdot x_{21})}{\left(-\frac{1}{2}{x_{12}}^2\right)^{\Delta+1}} \\ &= \frac{1}{{x_{12}}^{2\Delta}}\left[z_1 \cdot z_2 - 2\frac{(z_1 \cdot x_{12})(z_2 \cdot x_{12})}{{x_{12}}^2}\right] \end{aligned} , \tag{105}$$

where an overall constant was taken to be 1 in the last line.

The indices can be recovered using Todorov operators, that is, deriving the expression above with respect to z_1 and/or z_2, but it is clearly in agreement with (101).

Notice this result can be easily generalized to 2-point function between operators with any spin. Firstly, it is clear the operators must have the same scaling dimension, Δ say. Secondly, they must be tensors of same rank, say l indices each; this is not so trivial but it can be seen from the fact that two equal C's contracted are not allowed since it yields pure gauge terms. Therefore, the resultant 2-point function $G_l(X_1, X_2; Z_1, Z_2)$ between two primary tensor fields of rank l (therefore with nontrivial spin) is quickly read to be:

$$\boxed{G_l(X_1, X_2; Z_1, Z_2) = \text{const.}\frac{\left[(X_1 \cdot X_2)(Z_1 \cdot Z_2) - (X_1 \cdot Z_2)(X_2 \cdot Z_1)\right]^l}{(X_1 \cdot X_2)^{\Delta+l}}} \ . \tag{106}$$

We could keep following the recipe to obtain other correlators, but we are going to do it as we need (if so) along this work. Nevertheless, for completeness, we finish this section sketching the general prescription one should follows to get 3-point functions between arbitrary operators. Adopting the short notation $\chi \equiv [l, \Delta]$ for referring to the set of eigenvalues labelling a given operator (traceless, symetric and transverse, of course), the generalized encoded 3-point function $G_{\chi_1\chi_2\chi_3}(X_i; Z_i)$ should have the form below, in comparison with (99) and according to our discussion just above:

$$G_{\chi_1\chi_2\chi_3}(X_i; Z_i) = \frac{Q_{\chi_1\chi_2\chi_3}(X_i; Z_i)}{(X_1 \cdot X_2)^{\frac{\tau_1+\tau_2-\tau_3}{2}}(X_1 \cdot X_3)^{\frac{\tau_1+\tau_3-\tau_2}{2}}(X_2 \cdot X_3)^{\frac{\tau_2+\tau_3-\tau_1}{2}}} \ , \tag{107}$$

where we have defined $\tau_i \equiv \Delta_i + l_i$ and $Q_{\chi_1\chi_2\chi_3}(X_i; Z_i)$ is responsible for carrying the tranversality information, that is, it is an identically transverse polynomial of degree l_i in each Z_i, with coefficients dependent on X_i, and also homogeneous of degree l_i in each X_i.

So that the whole function is homogeneous of degree Δ. Thus:

$$Q_{\chi_1\chi_2\chi_3}(\lambda_i X_i; \alpha_i Z_i + \beta_i X_i) = Q_{\chi_1\chi_2\chi_3}(X_i; Z_i) \prod_i (\lambda_i \alpha_i)^{l_i} \ . \tag{108}$$

Moreover, generically, identically transverse polynomials must be built from contractions of the tensors C_{iAB}, as we have seen. But not all contractions are useful, notice $C_i \cdot C_i$, $C_i \cdot X_i$ and $C_i \cdot Z_i$ produce terms proportional to Z_i^2 and $Z_i \cdot X_i$. Therefore, nontrivial building blocks are going to be given by contractions using different points, as in $C_1 \cdot C_2$. In fact, besides $C_i \cdot C_j$ there is just one more possible contraction given by $X_j \cdot C_i \cdot X_k$ and adjusted by dividing it by $X_j \cdot X_k$, explicitly:

$$\boxed{\begin{aligned} H_{ij} &\equiv C_i \cdot C_j = 2\left[(X_i \cdot X_j)(Z_i \cdot Z_j) - (X_i \cdot Z_j)(X_j \cdot Z_i)\right] \\ V_{i,jk} &\equiv \frac{X_j \cdot C_i \cdot X_k}{X_j \cdot X_k} = \frac{(Z_i \cdot X_j)(X_i \cdot X_k) - (Z_i \cdot X_k)(X_i \cdot X_j)}{(X_j \cdot X_k)} \end{aligned}} \ , \tag{109}$$

where H_{ij} satisfies the scaling condition $l_i = l_j = 1$ and $l_k = 0$ and $V_{i,jk}$ satisfies the conditions $l_i = 1$, $l_j = l_k = 0$.

At the end of the day, note, this null-cone formalism is just another way, sometimes simpler, to compute conformal constraints. In fact it should be used together with the point of view of the physical space. Encoding technology enhances the power of such formalism and also makes the calculations more elegant.

Along the rest of this chapter we are going to use the fundamentals developed up to here to understand particular theories that do not manifest conformal invariance entirely due to the introduction of the so called defects on it. It is going to be of our interest to charaterize correlation functions in those theories as well as studying how physics is modified on them.

2.2 Defects

In this section our aim is to present all the essential for analyzing a generic Defect CFT, and then explore some specific cases that are known to be very useful nowadays. In fact these defects appear in several situations, in both phenomenological and theoretical problems.

An ordinary CFT has the vacuum invariant under general conformal transformations.An interesting way of breaking such symmetries is by introducing interfaces or non-trivial boundaries on it, that is, defects in the spacetime. If we do so such that we preserve part of the conformal symmetry, then we get new theories that we could study just adapting the formalism we developed previously, that is why we refer to them as Defect CFTs.

As we will see, the breaking pattern is not the only thing characterizing a defect CFT, besides we need the CFT data. But now the spacetime has different regions, namely the *bulk* and the defect, and, therefore, new ways of fusing operators appear. Consequently, new difficulties and particularities arise in the computation of correlation functions.

2.2.1 Types of defects

Technically we are going to understand such defects as non-local operators parallel to its characterizing geometric object. In this sense, the presence of it is felt computing correlators with the insertion of these *extended operators* within them. Notice we also use the term defect to refer to the world-volume of the object itself.

As mentioned, we want to introduce structures that preserves a subgroup of the conformal symmetry of the homogeneous vacuum. Since rotations are part of the conformal transformations, intuitively the most obvious objects we could choose are spheres. However, we know that planes and spheres are connected via stereographic projections, therefore, we are going to study defects consisting of extended operators on spheres or planes.

If the original spacetime has dimension d, our defect placed in $\mathbb{R}^{d-1,1}$ is going to have dimension $p \in \{1, .., d-1\}$, or, alternatively, codimension q, such that $p + q = d$. As just said, a sphere $\mathbb{S}^p$ can be mapped into a hyperplane. $SO(p+1, 1)$ then is the group of conformal symmetry preserved by this hyperplane. Besides, $SO(q)$ composes the possible rotations around such plane. Thus the subgroup of symmetries preserved in the presence of those defects is just $SO(p+1, 1) \times SO(q)$, varying according to the values of p and q.

Defects of codimension 1 carry the special name of boundaries, as one should expect. It is widely studied and a large set of applications can be encountered, see for example [52] to get into it and [53] for a review and also for future perspectives concerning boundary conformal field theories (BCFTs). An interesting application is its involvement with the Heisenberg spinchain, as reviewed in [54]. Here we are not going to be so interested in CFTs with domain containing boundaries.

For p between 1 and $d-1$ one should interpret the surface of the defect as an interface between two regions (maybe more) of the domain, therefore possibly diving the original theory between two connected CFTs for example. That is, like *domain walls*. Through this interface some physical modes can propagate while others can not. In this sense, it is even possible to talk about in optics terms, reflection and transmission coefficients for the defect, as in [55].

The last possibility, and actually the most important in what follows, is the case in which p equals 1. It does not carry a special name, but it is very special when treating superconformal field theories, as we will see. In this category we will have the so called *Wilson loops*, non-local observables in the presence of which we are going measure our correlation functions.

2.2.2 Correlation functions in a Defect CFT

In this section we establish the rules to play the game and explore them to point out the most important features in such teories. We are going to follow the approach by polynomial encoding and the aim of this part is to make us able to deal with correlators in defect CFTs, including OPE applications.

Now we have two kind of operators: bulk ones, which depend on the entire set of coordinates x and defect ones, which are functions only of coordinates related to the defect itself, from now on x^a. These two kinds of operators bring new possibilities of combinations and, thus, new information. In the following, bulk operators are going to be represented as usual, while defect fields will have a hat over them; generically, $O_{\Delta,l}$ for bulk ones and $\widehat{O}_{\widehat{\Delta},j,s}$ for defect ones. Correlation functions between them will be measured in the presence of the defect $\mathcal{O}_{\mathbb{D}}$ properly saying, which has its expectation value in the vacuum of the original CFT divided out; for n bulk insertions and m defect insertions, we have:

$$\langle\langle O_1(x_1)...O_n(x_n)\widehat{O}_1(x_1^a)...\widehat{O}_m(x_m^a)\rangle\rangle \equiv \frac{1}{\langle O_{\mathbb{D}}\rangle}\langle O_1(x_1)...O_n(x_n)\widehat{O}_1(x_1^a)...\widehat{O}_m(x_m^a)O_{\mathbb{D}}\rangle \ , \tag{110}$$

where eigenvalues were omitted.

Bulk operators are encoded according to the formalism developed on section 2.1.1 previously. Defect ones have something different: since they are in a representation of the subgroup $SO(p+1,1)\times SO(q)$, they are going to have quantum numbers refering to both $SO(p)$ and $SO(q)$, j and s, respectively; we call them parallel and transverse spin, respectively. To encode such operators two auxiliary variables are required, z^a and ω^i, the first for parallel indices and the second for transverse spin indices.

As before, we are going to work with primary fields only, so that the requirement of symmetric and traceless representations of $SO(p)$ and $SO(q)$ holds, therefore imposing $\omega^i\omega_i = 0$ and $z^a z_a = 0$. This said, if necessary to recover indices from polynomials of course two different Todorov operators are going to be needed:

$$\begin{aligned} D_a &= \left(\frac{p-2}{2} + z^b\frac{\partial}{\partial z^b}\right)\frac{\partial}{\partial z^a} - \frac{1}{2}z_a\frac{\partial^2}{\partial z^b\partial z_b} \\ D_i &= \left(\frac{q-2}{2} + \omega^j\frac{\partial}{\partial \omega^j}\right)\frac{\partial}{\partial \omega^i} - \frac{1}{2}\omega_i\frac{\partial^2}{\partial \omega^j\partial \omega_j} \end{aligned} .$$

From the point of view of the defect, transverse spins are kind of charges under some internal symmetry, but they also come from the same original symmetry structure. So at the end of the day, tensor structures coupling both transverse and parallel spin to bulk indices indeed occur.

In the embedding space we also split the coordinates into two sets: letters from the beginning of the alphabet (like $A, B, ...$) refer to parallel directions while letters from the middle (as $I, J, ...$) correspond to transverse ones. Bulk quantities are still constrained by the condtions $X^2 = 0$, $Z^2 = 0$ and $Z \cdot X = 0$, consequently, since symmetry realization in the embedding space is still linear, scalar quantities are going to be built from two scalar products instead of one, but only a subset of scalar products are independent:

$$X \bullet Y \equiv X^A \eta_{AB} Y^B \quad , \quad X \circ Y \equiv X^I \delta_{IJ} Y^J \ , \tag{111}$$

where η_{AB} is the remanescent minkowskian metric on the defect and δ_{IJ} the euclidian metric related to the other coordinates. And using the conditions for bulk quantities:

$$X \bullet X = -X \circ X \quad , \quad X \bullet Z = -X \circ Z \quad , \quad Z \bullet Z = -Z \circ Z \ . \tag{112}$$

Remember we have used the building block tensor (103) to construct the correlation functions in this approach. With such splitting on the coordinates, it is also expected that tensor to break into different pieces. In fact, notice we can have three parts: C^{AB}, C^{AI} and C^{IJ}. It turns out, however, that only the second of them is necessary when considering bulk quantities. That is because the others can be written as linear combinations of it, see below.

$$\begin{aligned}
C_{AB}Q^A R^B &= (X \bullet Q)(Z \bullet R)(X \bullet R)(Z \bullet Q) - (X \bullet Q)(Z \bullet R) \\
\div (X \circ G) \rightarrow &= -\frac{(X \bullet Q)}{(X \circ G)}(Z \bullet R)(X \circ G) + \frac{(X \bullet R)}{(X \circ G)}(Z \bullet Q)(X \circ G) \\
&= -\frac{(X \bullet Q)}{(X \circ G)}\left[(X \bullet R)(Z \circ G) - C_{AI}R^A G^I\right] \\
&+ \frac{(X \bullet R)}{(X \circ G)}\left[(X \bullet Q)(Z \circ G) - C_{AI}Q^A G^I\right] \\
&= \left(-\frac{(X \bullet R)}{(X \circ G)}Q^A + \frac{(X \bullet Q)}{(X \circ G)}R^A\right) G^I C_{AI}
\end{aligned} \quad ,$$

where Q, R and G are generic vectors (in particular, note, we can take $G = X$). To write the third line it was used the following identity: $C_{AI}R^A G^I = -(X \bullet R)(Z \circ G) + (X \circ G)(Z \bullet R)$. Analogously,

$$\begin{aligned}
C_{IJ}Q^I R^J &= -(X \circ Q)(Z \circ R) + (X \circ R)(Z \circ Q) \\
&= -\frac{(X \circ Q)}{(X \bullet G)}(Z \circ R)(X \bullet G) + \frac{(X \circ R)}{(X \bullet G)}(Z \circ Q)(X \bullet G) \\
&= -\frac{(X \circ Q)}{(X \bullet G)}\left[(X \circ R)(Z \bullet G) + C_{AI}G^A R^I\right] \\
&+ \frac{(X \circ R)}{(X \bullet G)}\left[(X \circ Q)(Z \bullet G) + C_{AI}G^A Q^I\right] \\
&= \left(-\frac{(X \circ Q)}{(X \bullet G)}R^I + \frac{(X \circ R)}{(X \bullet G)}Q^I\right) G^A C_{AI}
\end{aligned} \ .$$

Moreover, the fact that more than two C's concatenated is unnecessary still holds. Observe:

$$C^{AI}C_{BI}C^{BJ} = \left[(X\circ X)Z^AZ_B + (Z\circ Z)X^AX_B - (X\circ Z)(X^AZ_B + X_BZ^A)\right]C^{BJ} \ ,$$

which opens to:

$$\begin{aligned}
C^{AI}C_{BI}C^{BJ} &= -(X\circ X)Z^A\left[(X\bullet Z)Z^J - (Z\bullet Z)X^J\right] \\
&- (Z\circ Z)X^A\left[(X\bullet X)Z^J - (X\bullet Z)X^J\right] \\
&+ (X\circ Z)[X^A(X\bullet Z)Z^J - (Z\bullet Z)X^AX^J \\
&- Z^A(X\bullet X)Z^J - Z^A(X\bullet Z)X^J] \\
&= \left[(X\bullet X)(Z\circ Z) - (X\circ Z)(X\bullet Z)\right]C^{AJ} \\
&= \frac{1}{2}(C^{BI}C_{BI})C^{AJ}
\end{aligned} \ .$$

Defects live in a $(p+1)$-dimensional light-cone section inside the total ambient space. Primaries whithin it clearly will be lifted to fields under null-cone, homogeneity and transversality conditions like before. In this way, embedding operators are going to be encoded by means of two auxiliary variables Z^A and W^I, the first a p-vector and the second a q-vector. Translation of those conditions in this language is straightforward.

Before computing some correlators, a last point to understand is the projection of expressions containing both scalar products onto physical space, besides the rules expressed in (98). In order to do this, we assume our extended defect lays on a flat sub-manifold $\mathbb{D}$, which is embedded into the Poincarè section as:

$$X_x^M \in \mathbb{D}: \quad X^A = (1, x^2, x^a) \ , \quad X^I = 0 \ . \tag{113}$$

Demanding similar consistency condition as in (97), we see that Z's and W's, for a given X^M, must be of the form: $Z^A = (0, 2x^a z_a, z^a)$ and $W^I = \omega^{\mathrm{i}}$. Projection rules then are easy to obtain. Orthogonal scalar products projections are trivial, the metric is euclidian and the components assumed to be spacelike only. However, for generic vectors X_M, X_N and their respective auxiliaries Z_M and Z_N, we have:

$$\begin{aligned}
-2(X_M\bullet X_N) &= -2\left[(X_M\cdot X_N) - (X_M\circ X_N)\right] \\
&= -2\left[x_m\cdot x_n - \left(\frac{1+x_m^2}{2}\right)\left(\frac{1+x_n^2}{2}\right) + \left(\frac{1-x_m^2}{2}\right)\left(\frac{1-x_n^2}{2}\right) - x_m^{\mathrm{i}}x_{n\mathrm{i}}\right], \\
&= x_m^2 + x_n^2 - 2x_m^a x_{na} = (x_{mn}^a)^2 + (x_m^{\mathrm{i}})^2 + (x_n^{\mathrm{i}})^2
\end{aligned}$$

and

$$
\begin{aligned}
(X_M \bullet Z_n) &= (X_M \cdot Z_N) - (X_M \circ Z_N) \\
&= x_m \cdot z_n - \left(\frac{1+x_m^2}{2}\right)(x_n \cdot z_n) + \left(\frac{1-x_m^2}{2}\right)(-x_n \cdot z_n) - x_m^{\mathrm{i}} z_{n\mathrm{i}} \\
&= x_m^a z_{na} - x_n \cdot z_n \\
&= x_{mn}^a z_{na} - x_n^{\mathrm{i}} z_{n\mathrm{i}}
\end{aligned} .
$$

That is, two more practical rules are necessary in a defect theory:

$$
\boxed{-2(X_M \bullet X_N) = (x_{mn}^a)^2 + (x_m^{\mathrm{i}})^2 + (x_n^{\mathrm{i}})^2 \quad , \text{ and } \quad (X_M \bullet Z_N) = x_{mn}^a z_{na} - x_n^{\mathrm{i}} z_{n\mathrm{i}}} \; . \tag{114}
$$

With this in hands, we are now able to construct correlation functions. We restrict ourselves to correlators fixed up to numerical coefficients, OPE then takes care of the rest. Firstly, notice that if only defect insertions are used, the n-point functions are constrained by ordinary CFT in p dimensions, having conformal group as global symmetry. As an example, take the 2-point function between scalar primaries:

$$
\langle\langle \widehat{O}_{\widehat{\Delta},0,s}(X_1, W_1) \widehat{O}_{\widehat{\Delta},0,s}(X_2, W_2) \rangle\rangle = \frac{(W_1 \circ W_2)^s}{(X_1 \bullet X_2)^{\widehat{\Delta}}} \; , \tag{115}
$$

where an overall constant was taken to be 1. Notice terms contracting W's with X's automatically vanish in view of (113).

Consider then 2-point functions between a defect and a bulk operator, which is one of the particularities in such theories. These correlators will depend basically on 5 variables:

$$
\langle\langle O_{\Delta,J}(X_1, Z_1) \widehat{O}_{\widehat{\Delta},j,s}(X_2, Z_2, W_2) \rangle\rangle \; .
$$

The auxiliary Z_2^A is subject to the same conditions as Z's in ordinary CFT and, therefore, should appear in correlations only through the equivalent of (103), that is, by means of the tensor:

$$
C_2^{AB} = X_2^B Z_2^A - X_2^A Z_2^B \; .
$$

Moreover, W_2 should appear only contracted to Z_1 or X_1, since it is by definition orthogonal to X_2 and it is also subjected to null-cone condition. Summed to this we have the fact commented previously that, when considering bulk insertions, only the tensor C_1^{AI} is suficient for contractions. Let us explore the possibilities and construct the more general form for such functions then.

Contraction between C_2's produce pure gauge terms only, remember. So factors containing Z_2 can appear only in one way:

$$Q^0_{\text{b-d}} \equiv \frac{C_1^{AB}C_{2AB}}{2(X_1 \bullet X_2)} = \frac{[(X_1 \bullet X_2)(Z_1 \bullet Z_2) - (X_1 \bullet Z_2)(X_2 \bullet Z_1)]}{(X_1 \bullet X_2)} \,, \tag{116}$$

where the denominator was chosen conveniently. Notice that degree j in Z_2 enforces this term to appear with the power of j in the final form.

The more general structure then comes from multiplying the expression above by terms constructed from contractions between C_1, W_2, X_1 and X_2. Below we present a set of independent ones, the derivation is straightforward by performing the proposed contractions. The idea to form them is simple: we can have two C_1's contracted between themselves, or only a single C_1^{AI} contracted with either $X_{2A}X_{1I}$, $X_{1A}W_{2I}$ or $X_{2A}W_{2I}$, or yet none C_1.

$$\boxed{\begin{aligned}
Q^1_{b-d} &\equiv \frac{C_1^{AI}C_{1AI}}{2(X_1 \circ X_1)} = \frac{[(X_1 \bullet X_1)(Z_1 \circ Z_1) - (X_1 \bullet Z_1)(X_1 \circ Z_1)]}{(X_1 \circ X_1)} \\
Q^2_{b-d} &\equiv \frac{C_1^{AI}X_{2A}X_{1I}}{(X_1 \circ X_1)^{1/2}(X_1 \bullet X_2)} = \frac{[(X_1 \circ X_1)(X_2 \bullet Z_1) - (X_1 \bullet X_2)(X_1 \circ Z_1)]}{(X_1 \circ X_1)^{1/2}(X_1 \bullet X_2)} \\
Q^3_{b-d} &\equiv \frac{C_1^{AI}X_{1A}W_{2I}}{(X_1 \bullet X_1)} = \frac{[(X_1 \circ W_2)(X_1 \bullet Z_1)]}{(X_1 \bullet X_1)} - (Z_1 \circ W_2) \\
Q^4_{b-d} &\equiv \frac{X_1 \circ W_2}{(X_1 \circ X_1)^{1/2}}
\end{aligned}} \,, \tag{117}$$

the term formed by $C_1^{AI}X_{2A}W_{2I}$ was omitted because, although not obvious, it is dependent on the others above, explicitly it equals $Q^3_{b-d} + Q^2_{b-d}Q^4_{b-d}$.

Taking into account homogeneity in Z_1, Z_2 and W_2, a generic bulk-to-defect 2-point function is thus given by:

$$\boxed{\langle\langle O_{\Delta,J}(X_1, Z_1)\widehat{O}_{\widehat{\Delta},j,s}(X_2, Z_2, W_2)\rangle\rangle = \left(Q^0_{\text{b-d}}\right)^j \sum_{\{n_i\}} b_{n_1,\ldots,n_4} \frac{\prod_{k=1}^4 (Q^k_{b-d})^{n_k}}{(X_1 \bullet X_2)^{\widehat{\Delta}}(X_1 \circ X_1)^{(\Delta-\widehat{\Delta})/2}}} \,, \tag{118}$$

where the sum is over values of $\{n_i\}$ such that $2n_1 + n_2 + n_3 = J - j$ and $n_3 + n_4 = s$.

In particular, if one takes the defect operator to be the identity, which means to exclude defect quantities from the expression above, only Q^1 is relevant to construct the correlator and we have:

$$\boxed{\langle\langle O_{\Delta,J}(X_1, Z_1)\rangle\rangle = \frac{a_O}{(X_1 \circ X_1)^{\Delta/2}} \left[(Z_1 \circ Z_1) - \frac{(X_1 \circ Z_1)^2}{(X_1 \circ X_1)}\right]^J} \tag{119}$$

where a_O is some constant to be given, being actually part of the CFT data.

This should be interpreted as a non-vanishing 1-point function of bulk insertions and it is in fact other distinguishing feature of defect field theories; notice 1-point function of defect insertions still vanishes. Some symmetries were broken and the consequence is that the remaining ones are not enough to constrain correlators as usual.

We have seen then that correlation functions of defect insertions are just like in ordinary CFT, the conformal data related to this part of the theory is thus well known. The non-vanishing aspect of 1-point functions with bulk insertions demands the addition of more data to establish the theory. It turns out those data are all we need, that is because, as we are about to see, even 2-point functions involving only bulk insertions depend on cross-ratios, bringing up the OPE topic again, which completes the necessary toolkit to be able to compute any n-point function.

2.2.3 OPE and 2-point Functions of Bulk Primaries

Above we arrived at the conclusion that inside the defect we have a conformal field theory. In view of that, invoking the state-operator correspondence again, it can be said that any state created at a surface not necessarily entirely contained in the defect can be written as a sum of states of defect primaries and descendants. Consequently, any bulk operator associated to a state created from such a surface can be written as an expansion over primaries centered at a point in the defect, that is, we are going to have OPE of bulk primaries in terms of defect primaries. For a bulk primary $\mathcal{O}_\Delta(x)$, for example, it would be like:

$$\mathcal{O}_\Delta(x) = \sum_k \frac{b_{\mathcal{O}\hat{\mathcal{O}}_k}}{|x^i|^{\Delta-\hat{\Delta}}} C_{\hat{\mathcal{O}}_k}(|x^i|, \partial_a)\hat{\mathcal{O}}_k(x^a) \ , \tag{120}$$

where was chosen the centering to be the part of x in the defect, and the partial derivatives are with respect to that coordinates also, after all they are associated to descendants of the defect primaries $\hat{\mathcal{O}}_k$ located on it. Therefore the expansions can depend only on coordinates out of the defect, that is why $|x^i|$, which should be understood here as the distance of the operator from the center, $b_{\mathcal{O}\hat{\mathcal{O}}_k}$ is a constant isolated for convenience.

On the other hand, it is also possible to take a surface that does not contain the defect at all. In this case, it is like there was no defect, and therefore OPE would work as in (66). We call this the OPE *bulk channel* in contrast to the *defect channel* just above.

In analogy to the theory developed in subsection 1.4.2, the way for determining the coefficients $C_{\hat{\mathcal{O}}_k}$ is by appealing to the fixed point functions we have. In the previous subsection we saw that not only two-point functions of defect primaries are fixed, but also bulk-to-defect two-point functions too. In this simple case of bulk scalars we would have:

$$\langle \mathcal{O}_\Delta(x)\hat{\mathcal{O}}_{\hat{\Delta},j,s}(y)\rangle = \sum_k \frac{b_{\mathcal{O}\hat{\mathcal{O}}_k}}{|x^i|^{\Delta-\hat{\Delta}}} C_{\hat{\mathcal{O}}_k}(|x^i|, \partial_a)\langle\hat{\mathcal{O}}_k(x^a)\hat{\mathcal{O}}_{\hat{\Delta},j,s}(y)\rangle \ .$$

Then by means of (118) on the left hand side and of (60) on the right hand side, those coefficients should be found by matching expansions too.

To finish this short subsection, we present another subtlety of defect CFTs that is nothing more than a contextualization of those OPE prescriptions. In comparison to ordinary CFTs, conformal blocks appear already in two-point functions involving bulk primaries. To have that clear, notice that, given two bulk operators localized at X_1 and X_2, there are two cross-ratios invariants under the remaining symmetries:

$$\xi \equiv \frac{-2X_1 \cdot X_2}{(X_1 \circ X_1)^{1/2} (X_2 \circ X_2)^{1/2}} \quad , \quad \cos\phi \equiv \frac{X_1 \circ X_2}{(X_1 \circ X_1)^{1/2} (X_2 \circ X_2)^{1/2}} \tag{121}$$

where the normalization on the first of them is chosen in view of (98). Moreover, notice that ϕ is the angle between the projections of the operators onto the orthogonal space to the defect; in fact it involves only the $\circ$ type of scalar product.

The two point-function of bulk primaries will then be constructed from functions of those cross-ratios multiplying allowed structures for the correlator itself, obtained similarly to what was done to get (116) and (117) before. In the present case we have to contract C_1^{AI} and C_2^{AI} with X_1 and X_2 in all possible forms, the resulting building blocks are thus easy to obtain:

$$\boxed{\begin{array}{llll} Q^1_{b-b} = \dfrac{C_1^{AI} X_{1A} X_{2I}}{(X_1 \circ X_1)(X_2 \circ X_2)^{1/2}} & , & Q^2_{b-b} = \dfrac{C_1^{AI} X_{2A} X_{2I}}{(X_2 \circ X_2)(X_1 \circ X_1)^{1/2}} \\ Q^3_{b-b} = \dfrac{C_2^{AI} X_{1A} X_{2I}}{(X_2 \circ X_2)(X_1 \circ X_1)^{1/2}} & , & Q^4_{b-b} = \dfrac{C_2^{AI} X_{1A} X_{1I}}{(X_2 \circ X_2)^{1/2}(X_1 \circ X_1)} \\ Q^5_{b-b} = \dfrac{C_1^{AI} C_2^{BI} X_{1A} X_{2B}}{(X_2 \circ X_2)(X_1 \circ X_1)} & , & Q^6_{b-b} = \dfrac{C_1^{AI} C_2^{AJ} X_{2I} X_{2J}}{(X_2 \circ X_2)^{3/2}(X_1 \circ X_1)^{1/2}} \\ Q^7_{b-b} = \dfrac{C_1^{AI} C_{1AI}}{2X_1 \circ X_1} & , & Q^8_{b-b} = \dfrac{C_2^{AI} C_{2AI}}{2X_2 \circ X_2} \end{array}} \; . \tag{122}$$

And finally the bulk-to-bulk correlator reads:

$$\langle \mathcal{O}_{\Delta_1;J_1}(X_1, Z_1) \mathcal{O}_{\Delta_2;J_2}(X_2, Z_2) \rangle = \sum_{\{n_i\}} \frac{\prod_{k=1}^{8} \left(Q^k_{b-b}\right)^{n_k} f_{n_1,\dots,n_8}(\xi, \phi)}{(X_1 \circ X_1)^{\Delta_1/2} (X_2 \circ X_2)^{\Delta_2/2}} \, , \tag{123}$$

in analogy to (118). $f_{n_1,\dots,n_8}(\xi, \phi)$ are the functions of the cross-ratios.

Of course the OPE can be used to determine this correlation function too. The interesting fact is that it can be done in two ways as we saw above, such that structure constants present in the theory with no defect are connected to structure constants of the conformal field theory in the defect.

This subsection thus completes the technical ingredients for dealing with these kind of theories. In resume, we started looking for the preserved symmetries due to presence of a conformal defect, then we saw how correlation functions can be obtained by means of the embedding space formalism and the encoding of the correlator by polynomials. Finally we established an algorithm to obtain any correlation function in a Defect CFT using OPE and ended up finding that structure constants of non-defectual and defectual theories are connected.

Next and last subsection, actually ending the chpater too, is devoted to the presentation of a very important operator to the work to be developed at the end of this dissertation. It is an operator that emerges in Defect CFTs and is closely related to the conservation of the stress-energy tensor.

2.2.4 Displacement Operator and Conservation

It should be clear at this point that the introduction of a defect within a theory translates in a changing of the spacetime structure itself, because the defect is present in part of it only. Particular choices of them still keep part of the conformal symmetries, more specifically, we chose to work with defects conformally invariant.

As pointed out in the beginning, we will focus on flat defects of dimension $p = 1$, that is, lines in the following chapters. The conformal group in $d = 1$ was studied in a previous subsection of section 1.2, and we saw that only one translation symmetry was present, the one along the extended operator introduced.

Going a little bit more in the past, remembers that invariance under translations was connected to the stress-tensor. More specifically, once translation in the x^ν direction was preserved, we had:

$$\partial_\mu T^{\mu\nu}(x) = 0 \ .$$

In the present case then, this equation should hold only along the line over which the operator is supported. Somehow the equation above should also adapt to carry the information of broken translation symmetries on orthogonal directions. We can easily write these ideas by introducing a new operator $\mathbb{D}$ as following:

$$\partial^\mu T_{\mu\nu}(x) = \delta^{d-1}(x_\perp)\mathbb{D}_\nu(x_\parallel) \ ,$$

where $x_\parallel$ denotes the direction of the defect and $x_\perp$ orthogonal directions to it. This new operator is called *displacement operator* and has non-vanishing components for $x^\nu \neq x_\parallel$ only. The reason of the name will be clear in the last chapter.

The equation above can be seen as a Ward Identity in Defect CFT and has a very important consequence, which we are going to explore in the last chapter too. For now, notice that it gives us the amount of energy necessary to move outward the defect.

In what follows we will be interested in applying this Ward Identity to two specific cases: the $4d$ $\mathcal{N} = 4$ SYM and the $3d$ $\mathcal{N} = 6$ ABJM superconformal field theories. In the first case, we will use the usual notation $x^\mu = (x^0, x^{\mathrm{i}})$, i=1,2,3, for four dimension spacetime coordinates, while in the last, we adopt $x^\mu = (x^1, x^m)$, m=2,3, because we work in euclidian space for simplicity, x^0 and x^1 are the directions of the defect, respectively. Moreover, traditionally the displacement operator in the theory mentioned firstly is denoted by $\mathbb{F}_{\mathrm{i}}$, so that its defining equation reads:

$$\boxed{\partial^\mu T_{\mu \mathrm{i}} = \delta^3(x_\perp)\mathbb{F}_{\mathrm{i}}(x^0)}\ , \tag{124}$$

while for the ABJM:

$$\boxed{\partial^\mu T_{\mu m} = \delta^2(x_\perp)\mathbb{D}_m(x^1)}\ . \tag{125}$$

Notice, however, that the explicit form of such operators has not been given. In fact, a generic treatment of this for any defect with arbitrary dimension can be found in [46], to which we refer the interested reader. There, by making use of general relativity environment, it is derived the Ward Identity itself and the general form of the displacement operator in terms of characteristic quantities of the spacetime coupled to a metric, like the curvature tensor and etc. Here we choose to use a different approach for obtaining their form.

By assuming the validity of the Ward Identity, we will see that the displacement operator can be determined directly from the explicit form of the defect. For this reason, their complete forms are postponed to be presented in the last chapter. Actually, there is a connection between the quantities defining the defect and some quantities in the general theory of Differential Geometry and, therefore, General Relativity too. We avoid the entire derivation in view of unnecessary complications, which would demand content out of the scope of this work.

3 Wilson loops

Until the last chapter we have been worried in describing conformal field theories and, in particular, those theories containing what we called *defects*. We saw what novelties the introduction of such objects brings up and also developed all the necessary toolkit to deal with those new difficulties.

This chapter is devoted to the introduction of Wilson loops, potential defects. We focus on building these extended operators and on understanding their physical relevance. The final chapter of this dissertation will explore their usefulness in the context of defect CFTs.

3.1 Wilson loops in gauge theory

Remember the well-known *Aharonov-Bohm* effect in electromagnetism: a charged particle acquires a phase factor $e^{i\alpha(x)}$ when moving over a (closed) path in a background electromagnetic field ($\alpha(x)$ is a function which depends on the path); this is a consequence of the coupling between that wavefunction and the electromagnetic potential A_μ. This phase is essentially the holonomy of the background gauge field and the definition of Wilson loop. Before giving more details, let us take a step back and review briefly gauge transformations and gauge invariance.

Once we identify (or, conversely, impose) invariance of the theory under a gauge transformation, we need to know how to construct gauge invariant quantities, which are the physical relevant objects, once they are independent on the "frame" of description. Mathematically, a finite local phase transformation of a complex valued field $\psi(x)$ is given by $e^{i\omega(x)}\psi(x)$; its conjugate $\psi^*(x)$ then transforms like $e^{-i\omega(x)}\psi^*(x)$. Clearly then, gauge invariants can be constructed from prodcuts of ψ^* and ψ, in which exponentials cancel out. However, we know for example that propagators come from derivatives in the Lagrangian, so in order to be able to build all kind of invariants, we should treat these possibilities too.

It turns out a simple partial derivative of a field does not behave appropriately under a gauge transformation. Instead, one has to consider the so called *covariant* derivative. It is constructed from the *comparator* of the theory, a scalar quantity $U(y,x)$ defined to compensates for the difference in phase transformations between the two points y and x. Assumed to be a pure phase $U(y,x) = e^{i\phi(y,x)}$ in general, transforming like:

$$U(y,x) \to e^{i\omega(y)}U(y,x)e^{-i\omega(x)} , \tag{126}$$

which makes $\psi(y)$ and $U(y,x)\psi(x)$ transform similarly. The covariant derivative D_μ then is obtained once one assumes continuity of the comparator and considers its infinitesimal form:

$$U(x+\epsilon\eta, x) = 1 - ie\epsilon\eta^\mu A_\mu(x) + O(\epsilon^2) \ , \tag{127}$$

where e is a constant extracted for convenience, ϵ is an infinitesimal parameter and η^μ is the unitary vector dictating the direction of variation from the point x^μ to the point $y^\mu \equiv x^\mu + \epsilon\eta^\mu$; $A_\mu(x)$ are the variations. It comes from:

$$\begin{aligned} \eta^\mu D_\mu\psi(x) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\psi(x+\epsilon\eta) - U(x+\epsilon\eta, x)\psi(x)\right] \\ &= \eta^\mu\left(\partial_\mu + ieA_\mu\psi\right)\psi(x) \\ \therefore D_\mu &\equiv \partial_\mu + ieA_\mu \end{aligned} \quad , \tag{128}$$

We see then that the gauge field arises naturally as a vector in the definition of the covariant derivative, and that covariant derivatives of fields naturally yield interactions between them and gauge fields.

From equation (127) inserted in the transformation law for the comparator, one concludes that the gauge field $A_\mu(x)$ transforms like:

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\omega(x) \ , \tag{129}$$

and that, consequently:

$$D_\mu\psi(x) \to e^{i\omega(x)}D_\mu\psi(x) \ . \tag{130}$$

Gauge invariant Lagrangians can be constructed from combinations of fields and covariant derivatives of the fields. Moreover, we can also consider covariant derivatives of covariant derivatives. In fact, considering the commutator between them, we have that:

$$\left[D_\mu, D_\nu\right]\psi(x) \to e^{i\omega(x)}\left[D_\mu, D_\nu\right]\psi(x) \ .$$

Since $\psi(x)$ carries all the resulting transformation law by itself, the commutator must be a gauge invariant, and we have the well known strength tensor $F_{\mu\nu}$:

$$\left[D_\mu, D_\nu\right] \equiv ieF_{\mu\nu} = ie(\partial_\mu A_\nu - \partial_\nu A_\mu) \ . \tag{131}$$

In this way, we build gauge invariant Lagrangians from ψ, $D_\mu\psi$, $F_{\mu\nu}$ and its derivatives, keeping up all renormalizable terms that also respect other symmetries of the system one wants to impose, like parity, translation and charge conjugation, discrete ones.

A natural study in this context then comes up from the particular role played by the comparator. Firstly, it is not uniquely defined. In fact, given two points z and y, from the transformation rule for the gauge field (129) we see that the expression below is in accordance with (127) and (126):

$$U_P(z,y) = \exp\left(-ie\int_P dx^\mu A_\mu(x)\right) \ , \tag{132}$$

where P denotes any path from y to z. This is called the *Wilson line*. Taking a path that is closed, that is, starting and finishing at the same point y, we have natural nonlocal operators that are gauge invariants by construction, the so called *Wilson loops*:

$$U_C(y,y) = \exp\left(-ie\oint_C dx^\mu A_\mu(x)\right) \ . \tag{133}$$

The fact that different paths give different Wilson loops, which are nontrivial functions of the gauge fields, supports the claim that any locally gauge invariant can be constructed from combinations of them, considering particular paths. This is a result derived in differential geometry. Its proof is not the focus here, however, the basic idea behind it is that the gauge connection is defined from its *holonomies*, in this case the Wilson loops. They can be seen, by means of the application of Stokes theorem, as a function of the strength tensor $F_{\mu\nu}$, which plays similar role as curvature tensors in that context. The interested reader can find more about this for example in [64].

A last point we need to talk about gauge theory is concerning its quantisation. In the path integral formulation of quantum field theory, a pure gauge theory would be quantised by means of a functional integral as always:

$$\int \mathcal{D}A \ e^{iS[A]} \ .$$

Nonetheless, as we have just seen, gauge fields transform under gauge transformation with the addition of a term which keeps the measure $\mathcal{D}A$ unaltered, since it is a derivative that vanishes when integrated (once $\omega(x)$ is assumed to go to zero at the infinity). The action is assumed to be gauge invariant, such that the transformation rule for the gauge field makes the functional integral to contain redundancy, that is, integrations over physically equivalent configurations of the gauge fields. A well defined quantisation is thus one that removes such redundancy.

Non-redundant configurations are selected using the Faddeev-Popov procedure. As it is well known on QFT literature (see [28] for example), this is done by inserting a delta function of the linear gauge-fixing function $G(A)$, that is $\delta\left(G(A)\right)$, together with the determinant of the functional derivative $\delta G(A^\omega/\delta\omega)$, for a gauge transformation function $\omega(x)$, into the path integral, through the identity:

$$1 = \int \mathcal{D}\omega(x)\delta\left(G(A)\right)\det\left(\frac{\delta G(A^\omega)}{\delta\omega}\right) \ .$$

The delta function ensures no redundancy. The extra determinant factor has a simple form if $G(A)$ is linear in A, that is because in this case, as we can see in (129), the functional derivative will be independent on $\omega(x)$. So the integral over $\omega(x)$ then add just a negligible normalization factor. The determinant will then, as always, be expressed as a path integral for non-physical fields called *ghosts*. To clarify, consider fo example the generalized Lorentz gauge condition $G(A) = \partial^\mu A_\mu - \sigma(x)$, sigma a gaussian weight. A particular choice of $\sigma(x)$ then fixes the gauge and the determinant can be expressed through the well known result (see [28] for example):

$$\det\left(-\frac{1}{e}\partial^\mu\partial_\mu\right) = \int \mathcal{D}c\mathcal{D}\bar{c}\exp\left[i\int d^d x\bar{c}\left(\partial^\mu\partial_\mu\right)c\right] \ , \tag{134}$$

where c and $\bar{c}$ are anticommuting fields, that is fermionic fields. The constant e were absorbed in their definition. The resulting consistent path integral for pure gauge theory then englobes also the ghost fields. Generically it will have the form:

$$\int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\ e^{iS[A,c,\bar{c}]} \ , \tag{135}$$

the Faddeev-Popov path integral.

Feynman rules will depend on the theory considered of course, specially on the interactions. However, as usual we can already establish the notation for gauge and ghosts legs and propagators. Traditionally, gauge propagators are represented by wavy lines, while ghost ones are represented by dotted lines, as in the figure below:

a) b)

Figure 7: Notation for gauge fields and ghosts in Feynman diagrams, respectively.

3.2 Non-Abelian extension

In the previous reasoning, implicitly we chose to work on the simplest case of gauge transformation, consisting in a unitary transformation acting on spinless operators and, therefore, simply complex valued functions; in that case then we were dealing with the $U(1)$ gauge group. However, we are able to generalize that procedure in two ways: considering spinfull operators and after that allowing non-commuting gauge transformations.

Firtsly, once we allow $\psi(x)$ to have more than one entry, like a column vector, gauge transformations must act on them like matrices. With this simple changing, one must now interpret the gauge transformation appearing before as being proportional to the identity matrix, that is:

$$e^{i\omega(x)\mathbb{I}} \ ,$$

and of course conjugate fields will transform with the hermitian conjugate of this matrix. Notice in this case we would have a reducible representation of $U(1)$.

Suppose now we put different $\omega's$ at each site of the identity matrix. Clearly, in that case we would have independent $U(1)$ transformations parameterized for each ω, in other words, an irreducible representation of $U(1) \times U(1) \times ...U(1)$, as many $U(1)$ as entries on $\psi(x)$.

To generalize that, since identity matrices clearly commute between themselves, we consider gauge transformations not proportional to identity matrices, but instead being generated by possibly non-commuting matrices. In this case we say we have a non-abelian gauge theory, and the group of transformations are said to be a non-abelian gauge group, in contrast to the abelian previous case.

Usually, a finite gauge transformation will have the form:

$$V(x) = e^{i\omega^a(x)t_a} \quad , \quad V^\dagger(x) = e^{-i\omega^a(x)t_a} \ ,$$

where the t_a are the generators of the group of transformations considered, assumed to be hermitian for convention. The group algebra is specified through the structure constants f_{abc} as below:[25]

$$[t_a, t_b] = i f_{abc} t_c \ . \tag{136}$$

With that in place, we now generalize the procedure done before in order to be able to construct gauge invariant quantities and all that. We must then define a covariant derivative and encounter the strength tensor of the theory. To do that, we first notice that the comparator must now be a matrix transforming like:

$$U(y,x) \to V(y)U(y,x)V^\dagger(x) \ . \tag{137}$$

Assuming continuity, the comparator will have the following infinitesimal form:

$$U(x+\epsilon\eta, x) = 1 + ig\epsilon\eta^\mu A^a_\mu(x) t_a + \mathcal{O}(\epsilon^2) \ , \tag{138}$$

the constant g extracted for convenience as usual. Notice now we have a gauge field for each group generator t_a. It follows then that the covariant derivative is given by:

$$D_\mu = \partial_\mu - igA^a_\mu t_a \ , \tag{139}$$

note the representation in which the gauge group is will depend on the field its act. The

[25]Do not confuse with structure constants of 3-point correlation functions in CFT.

generators will be square matrices. The transformation law for the gauge fields follows from insertion of (138) into (137) and using the fact that

$$V(x+\epsilon\eta)V^{\dagger}(x) = 1 + \epsilon\eta^{\mu}V(x)\left(-\partial_{\mu}V^{\dagger}(x)\right) + \mathcal{O}(\epsilon^2)\ .$$

We have:

$$\begin{aligned} 1 + ig\epsilon\eta^{\mu}A^a_{\mu}(x)t_a + \mathcal{O}(\epsilon^2) &\to V(x+\epsilon\eta)\left(1 + ig\epsilon\eta^{\mu}A^a_{\mu}(x)t_a + \mathcal{O}(\epsilon^2)\right)V^{\dagger}(x) \\ &= 1 + ig\epsilon\eta^{\mu}V(x)\left(A^a_{\mu}t_a + \frac{i}{g}\partial_{\mu}\right)V^{\dagger}(x) + \mathcal{O}(\epsilon^2) \\ \therefore A^a_{\mu}t_a &\to V(x)\left(A^a_{\mu}t_a + \frac{i}{g}\partial_{\mu}\right)V^{\dagger}(x) \end{aligned} \quad . \tag{140}$$

Assuming the transformation is small and using (136), this last result acquires the infinitesimal familiar form:

$$\begin{aligned} A^a_{\mu}t_a &\to A^a_{\mu}t_a + \frac{1}{g}\partial_{\mu}\omega^a t_a + i\omega^b A^c_{\mu}\left[t_b, t_c\right] \\ &= A^a_{\mu}t_a + \frac{1}{g}\partial_{\mu}\omega^a t_a + i f^a_{\ bc}\omega^b A^c_{\mu}t_a \end{aligned} \quad . \tag{141}$$

From the definition of the covariant derivative (128) and the transformation rule for the field $\psi(x)$: $\psi(x) \to V(x)\psi(x)$ and for the comparator (137) we show that:

$$\begin{aligned} \eta^{\mu}D_{\mu}\psi(x) &\to \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[V(x+\epsilon\eta)\psi(x+\epsilon\eta) - V(x+\epsilon\eta)U(x+\epsilon\eta, x)V^{\dagger}(x)V(x)\psi(x)\right] \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}V(x+\epsilon\eta)\left[\psi(x+\epsilon\eta) - U(x+\epsilon\eta, x)\psi(x)\right] \\ &= \eta^{\mu}V(x)D_{\mu}\psi(x) \end{aligned} \quad , \tag{142}$$

that is, the covariant derivative of a field in fact transform like the field itself, even for finite transformations.

Finally, as before we can define the strength tensor, it will be given by:

$$\begin{aligned} F^a_{\mu\nu}t_a = \frac{i}{g}\left[D_{\mu}, D_{\nu}\right] &= \left(\partial_{\mu}A^a_{\nu} - \partial_{\nu}A^a_{\mu}\right)t_a - igA^b_{\mu}A^c_{\nu}\left[t_b, t_c\right] \\ &= \left(\partial_{\mu}A^a_{\nu} - \partial_{\nu}A^a_{\mu} + gf^{abc}A^b_{\mu}A^c_{\nu}\right)t_a \end{aligned} \quad . \tag{143}$$

Notice however that this time it is not gauge invariant. In fact, its transformation law now follows from the previous result and we have:

$$F^a_{\mu\nu}t_a \to V(x)F^a_{\mu\nu}t_aV^{\dagger}(x)\ .$$

Instead, its trace will be a gauge invariant:

$$\mathrm{Tr}\left(F^a_{\mu\nu}t_a\right) \to \mathrm{Tr}\left(F^a_{\mu\nu}t_a\right) \ .$$

We now generalize the Wilson loop to non-Abelian gauge groups. Remember the Wilson Line (132). A reasonable candidate to the non-abelian case should consider the fact that we have non-commuting matrices; we suppose the Wilson Line to be then the Taylor series expansion of the one in the abelian case with the substitution $A_\mu \to A^a_\mu t_a$, such that, in each term of the expansion, matrices localized at higher values of the parameter s of the curve are placed on the left; this is called *path-ordering* prescription, and is represented by $\mathcal{P}\{\}$. Therefore our candidate is:

$$U_P(z,y) = \mathcal{P}\left\{\exp\left(ig\int_P ds\frac{dx^\mu}{ds}A^a_\mu(x(s))t_a\right)\right\} \ . \tag{144}$$

It is not hard to see that it satisfies the following equation:

$$\frac{d}{ds}U_P\left(x(s),y\right) = \left(ig\frac{dx^\mu}{ds}A^a_\mu(x(s))t_a\right)U_P\left(x(s),y\right) \ ,$$

which has unique solution under the condition $U_P = 1$ for degenerate curve[26] and can be put easily in the form:

$$\frac{dx^\mu}{ds}D_\mu U_P\left(x(s),y\right) = 0 \ .$$

We have to show it transforms correctly to be our comparator between any two points, that is, we need to prove that:

$$U_P(z,y,A^V) = V(z)U_P(z,y,A)V^\dagger(y) \ ,$$

where A^V indicates transformed gauge fields. In equation (142) we proved the relation $D_\mu(A^V)V = VD_\mu(A)$. So, once $U_P(z,y)$ is defined to be the solution of the first-order differential equation above with fixed boundary condition, the proof is straighforward:

$$\begin{aligned}\frac{dz^\mu}{ds}D_\mu(A^V)U_P(z,y,A^V) &= \frac{dz^\mu}{ds}D_\mu(A^V)V(z)U_P(z,y,A)V^\dagger(y)\\ &= V(z)\frac{dz^\mu}{ds}D_\mu(A)U_P(z,y,A)V^\dagger(y)\\ &= 0\end{aligned} \ .$$

Therefore we succeed in finding a non-abelian generalization of the Wilson line. It turns out, however, that simply taking a closed curve C does not yield a gauge invariant

[26] The initial and final parameter values of the curve are equal.

object, instead, as we saw it will transform like:

$$U_C(y,y) \to V(y)U_C(y,y)V^\dagger(y)\ .$$

But as for the strength tensor, a well defined Wilson loop comes then from the trace of it:

$$\mathrm{Tr}\left(U_C(y,y)\right) = \mathrm{Tr}\left(\mathcal{P}\left\{\exp\left(ig\int_C ds\frac{dx^\mu}{ds}A^a_\mu(x(s))t_a\right)\right\}\right)\ . \tag{145}$$

Usually the non-Abelian gauge groups involved in applications are $O(N)$ (group of orthogonal transformations of N-dimensional vectors), $SO(N)$ (group of rotations on N-dimensional vectors), $Sp(N)$ (symplectic group), $SU(N)$ (special unitary transformations on N-dimensional vectors) and $U(N)$ (unitary transformations on N-dimensional vectors). Here we will be working on the last one only.

3.3 Supersymmetric Wilson loops

In this section we introduce the main objects of study for the following chapter. We want to generalize the construction of Wilson loops to theories containing more spacetime symmetries, as supersymmetries and conformal symmetries.

They are going to depend on the closed path chosen, as before. In fact, choosing paths now will be of crucial importance, because particular paths are going to preserve desired spacetime symmetries. This is important in view of the AdS/CFT correspondence mentioned at the end of the first chapter: some SCFTs are dual to some gravity theories and, therefore, matching quantities between both theories allows then check the validity of such correspondence principle.

In particular, it is known that 1/2 BPS operators, that is, operators preserving half of the supersymmetries (and superconformal symmetries), play a central role in this way. The study of 1/2 BPS Wilson loops can be viewed then as an aspect of that duality.

We are going to work over two SCFTs, those cited in the first chapter: $\mathcal{N}=4$ SYM in four dimensions and $\mathcal{N}=6$ ABJM in three dimensions. Basically, by taking an ansatz we check its validity finding preserved supersymmetries.

Starting with $\mathcal{N}=4$ SYM, the well established ([65]) WL is defined by:

$$\mathcal{W}_{N=4SYM}(C,\mathbf{n}) = \frac{1}{N}\mathrm{Tr}\ \left(\mathcal{P}\left\{\exp\left[\oint_C ds(i\dot{x}^\mu A_\mu + |\dot{x}|n^I\Phi_I)\right]\right\}\right)\ , \tag{146}$$

where n^I is a six-dimensional unitary vector controlling the coupling with the scalar fields and the trace is taken with respect to the fundamental representation of the gauge group. The WL is characterized by the contour $C=\{x^\mu(s)|s\in(0,2\pi)\}$ and is clearly gauge invariant, once it is just the non-abelian WL (145) with an extra term whose trace is gauge invariant:

$$\Phi_I \to U\Phi_I U^\dagger \ .$$

The 1/2 BPS WL's come from varying the above expression with respect to the supersymmetries transformations (88). One has:

$$\delta_\xi \mathcal{W} = \frac{1}{N}\mathrm{Tr}\ \left(\mathcal{P}\left\{\oint_C ds\bar{\xi}\left(\dot{x}^\mu\gamma_\mu - i|\dot{x}|\gamma^5 n^I\Gamma_I\right)\Psi \exp\left[\oint_C ds(i\dot{x}^\mu A_\mu + |\dot{x}|n^I\Phi_I)\right]\right\}\right) \ ,$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is the fifth Dirac matrix in four dimensions and Γ_I's are six-dimensional Dirac matrices. In order to be invariant we must then have:

$$\bar{\xi}\left(\dot{x}^\mu\gamma_\mu - i|\dot{x}|\gamma^5 n^I\Gamma_I\right)\Psi = 0 \ .$$

This equation can also be written in terms of the projectors

$$\mathcal{P}^\pm = 1 \pm i\frac{\dot{x}^\mu}{|\dot{x}|}\gamma_\mu\gamma^5 n^I\Gamma_I \ ,$$

giving:

$$\bar{\xi}\mathcal{P}^- \dot{x}^\mu\gamma_\mu\Psi = 0 \ .$$

A particular solution for this equation is found by taking the infinite straight line as path, that is, $x^\mu = (s,0,0,0)$ for example. In this case the parameter ξ must be a constant spinor orthogonal to $\mathcal{P}^-\gamma_0\Psi$, for any Ψ, that is, we must use parameters orthogonal to half the projections of such spinor. In other words, ξ need to be a $10d$ spinor with half of the degrees of freedom only, therefore preserving half of the Poincarè supercharges. A similar procedure shows that we are going to have also half of the superconformal charges preserved, resulting then in a 1/2 BPS operator.

For simplicity, usually it is fixed $n^I = (1,0,0,0,0,0)$. This definition then gives our desired object, the 1/2 BPS infinite straight Wilson Line:

$$\boxed{\mathcal{W}^{1/2}_{N=4SYM} = \frac{1}{N}\mathrm{Tr}\ \left(\mathcal{P}\left\{\exp\left[\int_{-\infty}^{\infty} ds(iA_0(s) + \Phi_1(s))\right]\right\}\right)} \ . \tag{147}$$

More recently, we had the discover of 1/2 BPS Wilson loops in ABJM too. The initial efforts in this way considered only bosonic fields appearing in the connection defining the operator, resulting in at most 1/6 BPS operators (see for example [66]), but the successful approach was given in [67], in which Drukker and Trancanelli introduced a supermatrix model for the connection of a Wilson loop, that is, a *super-connection*. A supermatrix M

consists in a matrix with well defined entries:

$$M = \begin{bmatrix} A & \Theta \\ \eta & B \end{bmatrix} ,$$

where A and B are Grassmann even and Θ and η are Grassmann odd. A simple practical way of having a matrix respecting those conditions is by putting bosonic quantities in the diagonal and fermionic quantities in the off-diagonal. Two very important properties arise:

$$\begin{aligned} \mathrm{STr}M &= \mathrm{Tr}A - \mathrm{Tr}B \\ \mathrm{Tr}M &= \mathrm{Tr}A + \mathrm{Tr}B \end{aligned} , \tag{148}$$

where was defined the trace and the *supertrace*, STr, of a supermatrix, Tr is just the ordinary trace with respect to some representation of the objects.

In that paper, the authors proposed embed the natural $U(N)\times U(N)$ gauge connection into a supergroup $U(N|N)$[27], therefore augmenting the previous connection to (as in [24]):

$$L \equiv \begin{bmatrix} A_\mu \dot{x}^\mu - \frac{2\pi i}{k} M^I_J C_I \bar{C}^J & -i\sqrt{\frac{2\pi}{k}}|\dot{x}|\eta^\alpha_I \bar{\psi}^I_\alpha \\ \sqrt{\frac{2\pi}{k}}|\dot{x}|\psi^\alpha_I \bar{\eta}^I_\alpha & \hat{A}_\mu \dot{x}^\mu - \frac{2\pi i}{k}\hat{M}^I_J \bar{C}^J C_I \end{bmatrix} , \tag{149}$$

where M^I_J, $\hat{M}^I_J$ and η^α_I are free parameters and x^μ parameterizes the curve supporting the loop.

The ansatz for the Wilson loop then will be defined with the help of another supermatrix $\mathcal{T}$, responsible for closing the loop after a supersymmetry transformation. The presence of such quantity is necessary because contours now must satisfy a weaker condition for the super-connection under a supersymmetry transformation, namely:

$$\delta_{\mathrm{susy}} L = \mathfrak{D}_\tau \mathcal{G} = \partial_\tau \mathcal{G} + i[L, \mathcal{G}] ,$$

where $\mathcal{G}$ is a supermatrix of $U(N|N)$ with some periodicity τ_0 and the *super-covariant* derivative $\mathfrak{D}$ is taken with respect to the argument τ parameterizing the curve $x^\mu(\tau)$, in contrast to $\delta_{\mathrm{susy}} L = 0$. That requeriment is in order to not have only trivial solutions, see [68] for more details. The equation defining $\mathcal{T}$ then is:

$$\mathcal{T}\mathcal{G}(\tau_0) = \mathcal{G}(0)\mathcal{T} . \tag{150}$$

The point is that if its possible to find $\mathcal{G}$ such that the condition above for the variation of the super-connection is obeyed and in accordance with (91), thus we have a solution, that is, a Wilson loop that is not only gauge invariant (by construction), but

[27] Actually that work is more general, dealing with ABJ theories, for which we have different gauge groups $U(N) \times U(M)$;

also supersymmetric.

We do not show the derivation, just present the well known choice of parameters that gives the 1/2 BPS infinite straight Wilson Line version for ABJM. Taking the parameterization $x^\mu = (s, 0, 0)$ for the line, the particular choice of M_I^J, $\hat{M}_I^J$ and η_I^α that results in that operator is (from [24]):

$$M_I^J = \hat{M}_I^J = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad \eta_I^\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_I \begin{bmatrix} 1 & 0 \end{bmatrix}^\alpha \quad , \quad \bar{\eta}_\alpha^I = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^I \begin{bmatrix} 1 \\ 0 \end{bmatrix}_\alpha . \tag{151}$$

The super-connection for it then reads:

$$L^{1/2}_{\text{ABJM}} = \begin{bmatrix} A_1 - \frac{2\pi i}{k} M_J^I C_I \bar{C}^J & -i\sqrt{\frac{2\pi}{k}}\bar{\psi}_1^1 \\ \sqrt{\frac{2\pi}{k}}\psi_1^1 & \hat{A}_1 - \frac{2\pi i}{k}\hat{M}_J^I \bar{C}^J C_I \end{bmatrix} . \tag{152}$$

Finally, the infinite straight Wilson Line in this case will be given by inserting this expression into the exponent, as always. A subtlety, nevertheless, arises now: the condition for invariance of it under supersymmetric transformations depend on the periodicity of the curve; basically, we can have $\mathcal{G}(\tau_0) = -\mathcal{G}(0)$ or $\mathcal{G}(\tau_0) = \mathcal{G}(0)$ in (150) and, depending on these conditions, the appropriate requirement for building the operator can be taking the trace or the supertrace. For the case in question, as can be checked in the original paper [67], we must have the trace, and therefore:

$$\boxed{\mathcal{W}^{1/2}_{\text{ABJM}} = \frac{1}{2N}\text{Tr}\left(\mathcal{P}\left\{\exp\left(-i\int_\infty^\infty ds \begin{bmatrix} A_1 - \frac{2\pi i}{k} M_J^I C_I \bar{C}^J & -i\sqrt{\frac{2\pi}{k}}\bar{\psi}_1^1 \\ \sqrt{\frac{2\pi}{k}}\psi_1^1 & \hat{A}_1 - \frac{2\pi i}{k}\hat{M}_J^I \bar{C}^J C_I \end{bmatrix}\right)\right\}\right)} , \tag{153}$$

where the dependence on s is on the fields.

3.4 't Hooft Limit

Along this chapter it has been repeated the importance of knowing as many Wilson loops in a theory as possible. Computing their vacuum expectation values (VEV) is not an easy job in general, unfortunately. However, within a specific limit, the so called *t'Hooft* limit, it is possible to find them perturbatively.

This tool was introduced by Gerard t'Hooft in [69] to simplify non-abelian gauge theories with large N (the gauge group label). Roughly speaking, he proposed to consider the theory in the limit $N \to \infty$ keeping $\lambda \equiv g_{YM}^2 N$, the t'Hooft parameter, fixed, where g_{YM} is the coupling constant of the Yang-Mills theory. In this situation feynman diagrams simplify a lot, that is, only some of them survives[28].

Not only there is such simplification, but also it was shown that, by rearranging the remanscent diagrams, the theory would correspond to a string theory (in its proper limit to). This was the first step in what culminates in the establishment of AdS/CFT correspondence. Moreover, this was extended to superconformal field theories by Maldacena in his seminal paper : [6], which embasis the dualities mentioned here. In the case of $\mathcal{N} = 4$ SYM the connection is more evident through (86), but in ABJM we need some more words.

Firstly, as pointed out in the first chapter, both theories (ABJM and its dual) depend on only two parameters: the Chern-Simons level k and also N. Secondly, in ABJM, the first of these works as a coupling constant and fields can be rescaled such that interactions are supressed by powers of $1/k$, which for large k means weak coupling. So, the t'Hooft limit in this case is then taken in the following way:

$$k, N \to \infty \quad , \quad \lambda \equiv \frac{N}{k} = \text{const.} \ , \tag{154}$$

with λ the t'Hooft parameter. On the gravity side, the string coupling constant is given by:

$$g_s \sim \left(\frac{N}{k^5}\right)^{1/4} .$$

Moreover, in this limit the couplings appearing in the interaction term of the ABJM action (90) reads: $\lambda_4 \sim 1/k$, $\lambda_4' \sim 1/k$, $\lambda_4'' \sim 1/k$ and $\lambda_6 \sim 1/k^2$; see [44].

In the t'Hooft limit then, both theories are weakly coupled. In fact, see for example [11], in the planar limit $\mathcal{N} = 6$ ABJM is dual to a string theory in $AdS_4 \times CP^3$, a type IIA superstring to be more specific. The lesson here is thus that under such limit, the general duality between ABJM and the M-theory mentioned before specializes.

Finally, it remains to see how the computation of VEV of Wilson loops is possible within the limit. The idea actually is simple. Remembering that path ordering organizes the integrals appearing in the expansion of the exponential defining the WL, taking expectation values of the Wilson loops itself resumes to sum up integrals of expectation values of each of the integrands we have along the expansion, that is:

$$\langle \mathcal{W} \rangle = \frac{\mathrm{Tr}_{\mathcal{R}}}{\dim(\mathcal{R})} \left(\mathbb{I} + \int_C d\tau_1 \langle L(\tau_1) \rangle + \frac{1}{2!} \int_C d\tau_1 d\tau_2 \langle L(\tau_1) L(\tau_2) \rangle + ... \right) ,$$

and so on, where $\mathbb{I}$ stands for the identity matrix in the representation $\mathcal{R}$, while $\dim(\mathcal{R})$ is the relative dimension of it. We chose a generic Wilson loop $\mathcal{W}$ with connection L, which can be a super-connection.

[28]The surviving diagrams have the common particularity of being able to be drawn on a plane without *crossing legs*, in the language of the area, for this reason called *planar* diagrams, which is also the reason of such limit be sometimes called *Planar limit*;

So, computing VEV of Wilson loops involves VEV of connection insertions, which in turn means expectation values of fields in the theory, as gauges, bosons and fermions. Therefore, by making use of feynman rules one can compute the desired quantity. Moreover, those rules will depend on the parameters of the theory, such that their expecation values in the limit above can also be expressed in terms of powers of the t'Hooft parameter λ. Assuming it is small, all we have to do then is consider diagrams contributing to the calculation up to the order required, which can be easily seen from the rules; order λ results are referred to as 1-loop calculation, λ^2 as 2-loop calculation and so on.

On the other hand, as one should be wondering, it should be possible also having information about correlation functions of the theory by knowing the VEV of Wilson loops. In fact, both point of views are the topic of the final chapter in this work, in which such relation is well explained in the context of theories with defects.

4 Wilson loop Defect CFT

We saw then that operators over lines preserve the conformal group in $d = 1$, namely $PSL(2, \mathbb{R})$. At this point it should be clear that straight Wilson Lines are operators like those, after all there is exactly an infinite line supporting its definition. Since along the line the conformal symmetry holds, operators inserted on it and measured in the presence of the WL are expected to give rise to a CFT.

This chapter is devoted to the exploration of such emergent CFTs. In one hand we have Wilson loops, whose determination is of great importance, on the other hand we have a CFT, whose data carries very significance also. It would be fantastic then if we could join both of them. In fact, this link is done by the Ward identity we saw in the end of section 2.

4.1 Defect Correlators and Deformed Wilson loops

The displacement operator was introduced previously by means of the Ward Identity:

$$\partial^\mu T_{\mu\nu}(x) = \delta^{d-1}(x_\perp)\mathbb{D}_\nu(x_\parallel) \ , \tag{155}$$

valid within any correlation function.

The approach to relate it to Wilson loops is to consider *Deformed* Wilson loops, that is, WL supported on curves slightly deviated from known ones. Generically, we can write a deformed WL $\mathcal{W}[C]$ defined over a curve C given by $x^\mu(s)+\delta x^\mu(s)$, with s the parameter and $\delta x^\mu(s)$ the profile of the small deformation introduced, as below:

$$\mathcal{W}[C] = \mathcal{W}[C_0] + \sum_{n=1}^{\infty} \frac{1}{n!}\delta^n \mathcal{W}[C_0] \ , \tag{156}$$

where C_0 indicates the undeformed loop and we wrote a functional Taylor expansion for $\mathcal{W}[C]$.

Now, remembers from (14) that charges acting on a given operator is equivalent to the variation on such operator due to them. Consider then (155) with the insertion of $\mathcal{W}[C_0]$ integrated over the whole volume with currents projected onto the deformation δx^μ:

$$\int d^d x \langle \partial^\mu T_{\mu\nu}(x)\delta x^\nu \mathcal{W}[C_0]\rangle = \int d^d x \langle \delta^{d-1}(x_\perp)\mathbb{D}_\nu(x_\parallel)\mathcal{W}[C_0]\delta x^\nu\rangle \ .$$

The right-hand side of this equation is easily computed due to the delta function, giving:

$$\int d^d x \langle \partial^\mu T_{\mu\nu}(x)\delta x^\nu \mathcal{W}[C_0]\rangle = \int dx_\parallel \langle \mathbb{D}_\nu(x_\parallel)\mathcal{W}[C_0]\delta x^\nu(x_\parallel)\rangle \ .$$

The left-hand side, noticing that $\delta x^\nu = 0$ for the parallel coordinate, conventionally taken to be the time direction for minkowskian spacetimes and x^1 for euclidian spaces, gives the expectation value of charges projected along the deformation acting on $\mathcal{W}[C_0]$, which means the first order variation of $\mathcal{W}[C_0]$ along it. We have then:

$$\langle \delta \mathcal{W}[C_0] \rangle = \int dx_\parallel \langle \mathbb{D}_\nu(x_\parallel) \mathcal{W}[C_0] \rangle \delta x^\nu(x_\parallel) \ . \tag{157}$$

Notice also that the procedure also applies to any insertion X (not localized at x) besides $\mathcal{W}[C_0]$. So the equation above holds more generically as:

$$\langle X \delta \mathcal{W}[C_0] \rangle = \int dx_\parallel \langle X \mathbb{D}_\nu(x_\parallel) \mathcal{W}[C_0] \rangle \delta x^\nu(x_\parallel) \ . \tag{158}$$

Repeating the procedure now inserting $\delta\mathcal{W}[C_0]$ instead of $\mathcal{W}[C_0]$, it follows that:

$$\langle \delta^2 \mathcal{W}[C_0] \rangle = \int dx_\parallel \langle \mathbb{D}_\mu(x_\parallel) \delta \mathcal{W}[C_0] \rangle \delta x^\mu(x_\parallel) \ ,$$

so using the previous result we get:

$$\langle \delta^2 \mathcal{W}[C_0] \rangle = \int dx_\parallel dx'_\parallel \langle \mathbb{D}_\mu(x_\parallel) \mathbb{D}_\nu(x'_\parallel) \mathcal{W}[C_0] \rangle \delta x^\mu(x_\parallel) \delta x^\nu(x'_\parallel) \ ,$$

and so on for any term on (156).

The expectation value of $\mathcal{W}[C]$ can then be written as:

$$\langle \mathcal{W}[C] \rangle = \langle \mathcal{W}[C_0] \rangle + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_\parallel^{(1)} ... dx_\parallel^{(n)} \langle \mathbb{D}_{\mu_1}(x_\parallel^{(1)}) ... \mathbb{D}_{\mu_n}(x_\parallel^{(n)}) \mathcal{W}[C_0] \rangle \delta x^{\mu_1} ... \delta x^{\mu_n} \ .$$

Dividing this expression by $\langle \mathcal{W}[C_0] \rangle$ and adopting the shorthand notation:

$$\frac{1}{n!} \int dx_\parallel^{(1)} ... dx_\parallel^{(n)} \equiv \int dx_\parallel^{(1)} > dx_\parallel^{(2)} > ... > dx_\parallel^{(n)} \ ,$$

we can also write:

$$\langle \delta \log \mathcal{W} \rangle = \sum_{n=1}^{\infty} \int dx_\parallel^{(1)} > dx_\parallel^{(2)} > ... > dx_\parallel^{(n)} \langle\langle \mathbb{D}_{\mu_1}(x_\parallel^{(1)}) ... \mathbb{D}_{\mu_n}(x_\parallel^{(n)}) \rangle\rangle \delta x^{\mu_1} ... \delta x^{\mu_n} \ ,$$

where, in accordance with (110), the double notation stands for correlation functions in the presence of the defect $\mathcal{W}[C_0]$.

But we are considering conformal defects only, so correlation functions in the presence of it obey the usual constrictions, in particular one-point functions vanish, so the sum above should start from 2 and therefore:

$$\boxed{\langle \delta \log \mathcal{W} \rangle = \sum_{n=2}^{\infty} \int \; dx_{\parallel}^{(1)} > dx_{\parallel}^{(2)} > ... > dx_{\parallel}^{(n)} \langle\langle \mathbb{D}_{\mu_1}(x_{\parallel}^{(1)}) ... \mathbb{D}_{\mu_n}(x_{\parallel}^{(n)}) \rangle\rangle \delta x^{\mu_1} ... \delta x^{\mu_n}} \,. \tag{159}$$

The displacement operator then is responsible to *displace* the Wilson loop out of its defining direction, justifying the name. Its explicit form then is not unique, and each one deforms a given Wilson loop. The point is that, once the WL is given, the explicit form of the displacement can be obtained from the steps above through:

$$\boxed{\mathbb{D}_{\mu}(s) = \frac{\delta \log \mathcal{W}[C_0]}{\delta x^{\mu}}(s) \propto \frac{\delta L(x)}{\delta x^{\mu}(s)}} \,, \tag{160}$$

where $L(x)$ stands for the connection or super-connection in the definition of $\mathcal{W}[C_0]$.

The previous equation then connects correlators on the defect with variations of the deformed Wilson loop. Thus once one knows some information at some level about one side, one also knows at the same level informations about the other side; more specifically, the t'Hooft parameter will be the level we are referring to here and the connected informations will be structure constants of two and three-points functions on the defects and VEV of Deformed Wilson loops. The two last sections work on both of these implications by making use of the Wilson loops constructed previously.

4.2 Structure Constants from Wavy Line in $\mathcal{N} = 4$ SYM

This section is reserved to the review of some of the calculations found in [23], the paper which motivated this writing. By considering known expressions for a wavy wilson line in $\mathcal{N} = 4$ SYM at order λ and λ^2, the authors found structure constants for two and three insertions of displacement operators onto the Straight Wilson Line (147).

Before going into the computations, we start adapting the discussion above to this case. The Straight Wilson line is described by $x^{\mu}(s) = (s, 0, 0, 0)$, so that $x_{\parallel} = x^0 = s$; coordinates perpendicular to it will be indexed by i. We will have then:

$$\delta x^{\mu}(s) = \big(0, \epsilon^{\mathrm{i}}(s)\big) \,,$$

with $\mathrm{i} = 1, 2, 3$, $\epsilon = |\epsilon| \ll 1$ serves as our expansion parameter, and it is also made the assumption $\dfrac{d^n \epsilon(s)}{ds^n} \sim \epsilon(s)$, that is, higher order variations of the parameter should be considered in the computations as well.

Moreover, the displacement operator can be obtained directly from (146) and (147) according to the last section:

$$\mathbb{F}_{\mathrm{i}}(s) = iF_{\mathrm{i}0} + D_{\mathrm{i}}\Phi_1(s) \,, \tag{161}$$

where the stress-tensor above appears from manipulations of the variation of the gauge field term in the connection. Notice also that the displacement can not be written as the derivative of a field, so it is a conformal primary.

Equation (159) then tells us that to second order variations of the curve, the expectation value of the wavy line reads:

$$\langle \delta \log \mathcal{W} \rangle = \int ds_1 > ds_2 \langle\langle \mathbb{F}_{\mathrm{i}0}(s_1)\mathbb{F}_{j0}(s_2)\rangle\rangle \epsilon^{\mathrm{i}}(s_1)\epsilon^{j}(s_2) + \mathcal{O}(\epsilon^3) \ .$$

This expression simplifies a little bit more using the fact that the VEV of the Straight Wilson Line is equal to 1, according to [70], so:

$$\langle \delta \mathcal{W} \rangle = \int ds_1 > ds_2 \langle\langle \mathbb{F}_{\mathrm{i}0}(s_1)\mathbb{F}_{j0}(s_2)\rangle\rangle \epsilon^{\mathrm{i}}(s_1)\epsilon^{j}(s_2) + \mathcal{O}(\epsilon^3) \ .$$

The correlation function appearing in the integrand has fixed form due to conformal symmetry. Its form can be obtained using the formalism developed in the second chapter, but can also be guessed from the index structure to be:

$$\langle\langle \mathbb{F}_{\mathrm{i}0}(s_1)\mathbb{F}_{j0}(s_2)\rangle\rangle = \frac{a_{\mathbb{F}}(\lambda)\delta_{\mathrm{i}j}}{(s_1 - s_2)^{2\Delta_{\mathbb{F}}}} \ , \tag{162}$$

where the dependence of $a_{\mathbb{F}}$ on the t'Hooft parameter λ was made clear and $\Delta_{\mathbb{F}}$ is the scaling dimension of the displacement.

Clearly then, the structure constant appearing above can be read off once known the left-hand side of the previous relation. It is needed then the expression of $\delta\mathcal{W}$ for the wavy line to ϵ^2 order. As given in [23], we have at 1-loop:

$$\langle \mathcal{W} \rangle^{(\text{1-loop})} = -\frac{\lambda}{16\pi^2} \int ds_1 ds_2 \frac{\dot{x}_1 \cdot \dot{x}_2 + |\dot{x}_1||\dot{x}_2|}{x_{12}^2} \ , \tag{163}$$

where x_{i}, $\mathrm{i} = 1, 2$, stands for $x(s_{\mathrm{i}})$, and $x_{\mathrm{i}j} = x_{\mathrm{i}} - x_j$. The two integrals indicate that at 1-loop the VEV of the wavy line is sensible exactly to two insertions of displacements only; to match the desired constant with the expression above we will have to massage the integral.

The wavy line is parameterized by $x^\mu = (s, \epsilon^{\mathrm{i}}(s))$. Inserting this into the expression above, we get:

$$\langle \mathcal{W} \rangle^{(\text{1-loop})} = -\frac{\lambda}{32\pi^2} \int ds_1 ds_2 \frac{(\dot{\epsilon}(s_1) - \dot{\epsilon}(s_2))^2}{(s_1 - s_2)^2} + \mathcal{O}(\epsilon^3) \ .$$

Opening the integrand, we will have three contributing terms:

$$\begin{aligned}\langle \mathcal{W} \rangle^{(\text{1-loop})} = & -\frac{\lambda}{32\pi^2}\Bigg[\int ds_1 ds_2 \frac{\dot{\epsilon}^2(s_1)}{(s_1-s_2)^2} + \int ds_1 ds_2 \frac{\dot{\epsilon}^2(s_2)}{(s_1-s_2)^2} \\ & - 2\int ds_1 ds_2 \frac{\dot{\epsilon}^{\mathrm{i}}(s_1)\dot{\epsilon}_{\mathrm{i}}(s_2)}{(s_1-s_2)^2}\Bigg]\end{aligned} .$$

Their contributions will be found once a regularization procedure is followed in order to computed the integrals, that is because separatedly they diverge. A cutoff μ is introduced in order to keep the range of integration controlled by $s_1 - \mu > s_2$, however, this brings up terms not relevant here, that is, regularizationn dependent terms with no physical meaning. Hopefully, such terms appear in single integrals only, which can be neglected since we want those terms with double integrals. Moreover, the λ^0 order contribution to the expectation value of the wavy line is carried with those terms, after all is just a normalization taken to be equal to 1.

So, the first and second terms in the equation above clearly do not give relevant contributions, because they can be easily integrated in s_2 and s_1 (exchange $s_1 \leftrightarrow s_2$ to see), respectively. The third term can be integrated by parts twice, giving:

$$\begin{aligned}\int ds_1 ds_2 \frac{\dot{\epsilon}^{\mathrm{i}}(s_1)\dot{\epsilon}_{\mathrm{i}}(s_2)}{(s_1-s_2)^2} &= -2\int_{-\infty}^{\infty} ds_1 \dot{\epsilon}^{\mathrm{i}}(s_1)\int_{\infty}^{s_1-\mu} ds_2 \frac{\epsilon_{\mathrm{i}}(s_2)}{(s_1-s_2)^3} + (\text{single integral term}) \\ &= -6\int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1-\mu} ds_2 \frac{\epsilon^{\mathrm{i}}(s_1)\epsilon_{\mathrm{i}}(s_2)}{(s_1-s_2)^4} + (\text{single integral terms})\end{aligned} ,$$

where to write the second line we used the Leibniz's rule for derivation under the integral sign.

The variation of the wavy line at λ order then reads:

$$\langle \delta\mathcal{W} \rangle^{(\text{1-loop})} = -\frac{3\lambda}{4\pi^2}\int ds_1 > ds_2 \frac{\epsilon^{\mathrm{i}}(s_1)\epsilon^{\mathrm{j}}(s_2)\delta_{ij}}{(s_1-s_2)^4} , \tag{164}$$

from which we see that, at order λ we also have:

$$\boxed{a_{\mathbb{F}}(\lambda) = -\frac{3\lambda}{4\pi^2} \quad , \quad \Delta_{\mathbb{F}} = 2} , \tag{165}$$

which, as mentioned in [23], is in accordance with the literature, see for example results of [71]. As a imediate check, however, notice that the scaling dimnension of the displacement is the same of the stress-tensor, which in turn is equal to the mass dimension of the gauge fields minus one; from the action we see gauge fields have classical scaling dimension 1 and, therefore, (classical) $\Delta_{\mathbb{F}} = 2$.

The same procedure could be done using higher orders in λ for the VEV of the wavy line, and this is done in [23] to λ^2 actually. Similar calculations are going to appear, the regularization scheme will require more detail and more cutoffs, possibly new difficulties arise, but the script is the same as above.

On the other hand, one could also try a different path and go onto higher order variations of $\mathcal{W}$, that is, consider terms with ϵ^3 and so on. This now instead of giving information about the correlation function of two displacement insertions, would give information about the structure constant of the three-point function of displacements, also fixed by conformal symmetry. Notice, however, that such operators are composed by bosonic fields only, so that they commmute between themselves; the three-point function would then have three indices totally symmetric. The contribution given by such quantity to the third variation of the wavy line would then be found once such indices were contracted with a tensor, which turns out to be the totally antisymmetric Levi-Civita tensor, the only one at our disposition, resulting, therefore, in zero.

4.3 Deformed WL from Fixed Defect Correlators in ABJM

Conversely to the procedure above, as shown previously, the computation of deformed Wilson loops from defect correlators is also a imminent consequence of the relation (159). Working on this side of the relation then encloses our topic and, therefore, this last section is dedicated to it.

Nonetheless, we choose a different background now, the $\mathcal{N} = 6$ ABJM superconformal field theory. In this case, as we saw before, the 1/2 BPS Straight Wilson Line is given by (153). We must look then to correlation functions of insertions of displacements into this line. To compute such correlators, we go to their definition, given by (110), which adapted to this case reads:

$$\langle\langle \mathbb{D}_{m_1}(s_1)...\mathbb{D}_{m_\mathrm{i}}(s_\mathrm{i})\rangle\rangle = \frac{\frac{1}{2N}\langle \mathrm{Tr}\ \mathcal{P}\{\mathbb{D}_{m_1}(s_1)...\mathbb{D}_{m_\mathrm{i}}(s_\mathrm{i})\mathcal{W}^{1/2}_{\text{ABJM}}\}\rangle}{\langle \mathcal{W}^{1/2}_{\text{ABJM}}\rangle}\ , \tag{166}$$

where we introduced the necessary normalization factor $1/2N$ and $\mathcal{P}$ stands for path-ordering, so that the numerator on the right-hand side is to be understood as:

$$\begin{aligned}\langle\frac{1}{2N}\mathrm{Tr}\ \mathcal{P}\{\mathbb{D}_{m_1}(s_1)...\mathbb{D}_{m_\mathrm{i}}(s_\mathrm{i})\mathcal{W}^{1/2}_{\text{ABJM}}\}\rangle = \frac{1}{2N}\langle \mathrm{Tr}\ &\mathcal{W}(\infty, s_1)\mathbb{D}_{m_1}(s_1)\mathcal{W}(s_1, s_2)... \\ &\times \mathcal{W}(s_{\mathrm{i}-1}, s_\mathrm{i})\mathbb{D}_{m_\mathrm{i}}(s_\mathrm{i})\mathcal{W}(s_\mathrm{i}, -\infty)\rangle\end{aligned}\ , \tag{167}$$

for $s_1 > s_2 > ... > s_\mathrm{i}$, supposedly. Note that in the expression above it was used a new definition, the partial wilson line in ABJM:

$$\mathcal{W}(s_1, s_2) = \mathcal{P}\{\exp\left(-i\int_{s_1}^{s_2} d\tau\ L^{1/2}_{\text{ABJM}}(\tau)\right)\}\ .$$

The explicit form of the displacement operator in this case can be derived using again the relation (160) in the expression above. Splitting the super-connection as:

$$L^{1/2}_{\text{ABJM}} = \mathcal{A} + \mathcal{L}_B + \mathcal{L}_F \ , \tag{168}$$

with:

$$\mathcal{A} = \begin{bmatrix} A_1 & 0 \\ 0 & \hat{A}_1 \end{bmatrix} \ , \ \mathcal{L}_B = \frac{2\pi i}{k}\begin{bmatrix} C\bar{C} & 0 \\ 0 & \bar{C}C \end{bmatrix} \ , \ \mathcal{L}_F = \sqrt{\frac{2\pi}{k}}\begin{bmatrix} 0 & -i\bar{\psi}_+ \\ \psi^+ & 0 \end{bmatrix} \ , \tag{169}$$

we have:

$$\mathbb{D}_m = \mathcal{F}_{m1} + \mathcal{D}_m\left(\mathcal{L}_B + \mathcal{L}_F\right) \ , \tag{170}$$

where:

$$\mathcal{D}_m\mathcal{O} = \partial_m\mathcal{O} + i\left[\mathcal{A}_m, \mathcal{O}\right] \ , \ \mathcal{F}_{m1} = \partial_m\mathcal{A}_1 - \partial_1\mathcal{A}_m + i\left[\mathcal{A}_m, \mathcal{A}_1\right] \ , \ \mathcal{A}_\mu = \begin{bmatrix} A_\mu & 0 \\ 0 & \hat{A}_\mu \end{bmatrix} \ , \tag{171}$$

and remember: $m = 2, 3$, the straight line is along x^1, the space is euclidian. Moreover, in the definition of $\mathcal{L}_F$ above we already set the Dirac matrices to be $\gamma^\mu = (\sigma^3, \sigma^2, -\sigma^1)$, σ's the Pauli matrices, so that ψ_1^1 in (152) stands for the positive eigenstate of γ^1, namely ψ^+.

Equation (167) tells us that displacements are linked to partial wilson lines. To compute correlators as in (166) we will have then to expand those partial lines up to the necessary order, resulting in VEV values of the trace of terms coming from the product between localized displacements and super-connections dropped together with integrals when expanding the exponentials. Moreover, the VEV of the 1/2 BPS WL is taken to be equal to 1 (see [43]).

For example, let us see the defect 2-point function of displacements:

$$\langle\langle \mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\rangle\rangle = \frac{1}{2N}\langle \text{Tr}\ \mathcal{P}\ \mathcal{W}(\infty, s_1)\mathbb{D}_m(s_1)\mathcal{W}(s_1, s_2)\mathbb{D}_n(s_2)\mathcal{W}(s_2, -\infty)\rangle \ , \tag{172}$$

where $s_1 > s_2$ supposedly.

We stop the expansion of the partial lines at the term for which the resulting diagrams will be of the the desired order in power of the t'Hooft parameter (154). In order to get such correlator with structure constant at first order in λ, for example, we consider partial lines only as:

$$\mathcal{W}(s_1, s_2) \simeq 1 - i\int_{s_2}^{s_1} d\tau \mathcal{L}(\tau) \ .$$

Inserting this into the previous equation we get:

$$\begin{aligned}\langle\langle \mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\rangle\rangle \simeq \frac{1}{2N}\Big\langle \mathrm{Tr}\Big[&\left(1-i\int_{s_1}^{\infty} d\tau\mathcal{L}(\tau)\right)\mathbb{D}_m(s_1)\left(1-i\int_{s_2}^{s_1} d\tau\mathcal{L}(\tau)\right) \\ &\times \mathbb{D}_n(s_2)\left(1-i\int_{-\infty}^{s_2} d\tau\mathcal{L}(\tau)\right)\Big]\Big\rangle\end{aligned}, \quad (173)$$

which, keeping only terms with at most one integral, resumes to:

$$\begin{aligned}\langle\langle \mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\rangle\rangle \simeq &\frac{1}{2N}\langle \mathrm{Tr}\ [\mathbb{D}_m(s_1)\mathbb{D}_n(s_2)]\rangle \\ &-\frac{i}{2N}\bigg(\int_{s_1}^{\infty} d\tau \langle \mathrm{Tr}\ [\mathcal{L}(\tau)\mathbb{D}_m(s_1)\mathbb{D}_n(s_2)]\rangle \\ &+\int_{s_2}^{s_1} d\tau \langle \mathrm{Tr}\ [\mathbb{D}_m(s_1)\mathcal{L}(\tau)\mathbb{D}_n(s_2)]\rangle \\ &+\int_{-\infty}^{s_2} d\tau \langle \mathrm{Tr}\ [\mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\mathcal{L}(\tau)]\rangle\bigg)\end{aligned}. \quad (174)$$

Basically then, we have two structures to study:

1) $\langle \mathrm{Tr}\ [\mathbb{D}_m(s_1)\mathbb{D}_n(s_2)]\rangle$

2) $\langle \mathrm{Tr}\ [\mathcal{L}(\tau)\mathbb{D}_m(s_1)\mathbb{D}_n(s_2)]\rangle$

the second of them is common to all terms between parenthesis.

In this study we will need then the feynman rules of the theory. Below we summarize them, following [43]. Covariant derivatives act on the fields like:

$$\begin{aligned} D_\mu C_I &= \partial_\mu C_I + iA_\mu C_I - iC_I\hat{A}_\mu \\ D_\mu \bar{C}^I &= \partial_\mu \bar{C}^I - i\bar{C}^I A_\mu + i\hat{A}_\mu \bar{C}^I \\ D_\mu \bar{\psi}^I &= \partial_\mu \bar{\psi}^I + iA_\mu \bar{\psi}^I - i\bar{\psi}^I \hat{A}_\mu \\ D_\mu \psi_I &= \partial_\mu \psi_I - i\psi_I A_\mu + i\hat{A}_\mu \psi_I \end{aligned}. \quad (175)$$

Propagators at three-level reads:

$$\begin{aligned} \langle A^a_\mu(x)A^b_\nu(y)\rangle^{(0)} &= \delta^{ab}\left(\frac{2\pi i}{k}\right)\frac{\Gamma\left(\frac{3}{2}-\epsilon\right)}{2\pi^{\frac{3}{2}-\epsilon}}\epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^{3-2\epsilon}} \\ \langle \hat{A}^a_\mu(x)\hat{A}^b_\nu(y)\rangle^{(0)} &= -\delta^{ab}\left(\frac{2\pi i}{k}\right)\frac{\Gamma\left(\frac{3}{2}-\epsilon\right)}{2\pi^{\frac{3}{2}-\epsilon}}\epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^{3-2\epsilon}} \\ \langle (C_I)_{\hat{\imath}}^{\hat{j}}(x)\left(\bar{C}^J\right)_{\hat{k}}^{l}(y)\rangle^{(0)} &= \delta_I^J\delta_{\hat{\imath}}^{l}\delta_{\hat{k}}^{\hat{j}}\frac{\Gamma\left(\frac{1}{2}-\epsilon\right)}{4\pi^{\frac{3}{2}-\epsilon}}\frac{1}{|x-y|^{1-2\epsilon}} \\ \langle (\psi_I^\alpha)_{\hat{\imath}}^{j}(x)\left(\bar{\psi}^J_\beta\right)_{k}^{\hat{l}}(y)\rangle^{(0)} &= -i\delta_I^J\delta_{\hat{\imath}}^{\hat{l}}\delta_k^{j}\frac{\Gamma\left(\frac{3}{2}-\epsilon\right)}{2\pi^{\frac{3}{2}-\epsilon}}\frac{\left(\gamma^\mu\right)^\alpha{}_\beta (x-y)_\mu}{|x-y|^{3-2\epsilon}} \end{aligned}, \quad (176)$$

where we have them already regularized by means of ϵ.

And the interaction vertices are:

$$\begin{aligned} \text{gauge cubic} \quad &\to \quad -i\frac{k}{12\pi}\epsilon^{\mu\nu\rho}\int d^3x f^{abc}A^a_\mu A^b_\nu A^c_\rho \\ \text{gauge-fermion cubic} \quad &\to \quad -\int d^3x \mathrm{Tr}\left[\bar{\psi}^I\gamma^\mu\psi_I A_\mu - \bar{\psi}^I\gamma^\mu\hat{A}_\mu\psi_I\right] \end{aligned} . \tag{177}$$

Moreover, our colour conventions are:

$$\begin{aligned} \mathrm{Tr}(T^aT^b) = \delta^{ab} \quad &, \quad \mathrm{Tr}(\hat{T}^{\hat{a}}\hat{T}^{\hat{b}}) = \delta^{\hat{a}\hat{b}} \\ \sum_{a=1}^{N^2}(T^a)_{ij}(T^a)_{kl} = \delta_{il}\delta_{jk} \quad &, \quad \sum_{\hat{a}=1}^{N^2}(\hat{T}^{\hat{a}})_{ij}(\hat{T}^{\hat{a}})_{kl} \quad = \delta_{il}\delta_{jk} \ , \\ f^{abc}f^{abc} = 2N^3 \quad &, \quad f^{\hat{a}\hat{b}\hat{c}}f^{\hat{a}\hat{b}\hat{c}} = 2N^3 \end{aligned} \tag{178}$$

which need extra conventions for generators of bifundamental fields like fermions and scalars:

$$T^{a\hat{a}} = \frac{i}{\sqrt{2}}T^a \times T^{\hat{a}} \ , \tag{179}$$

where we have a cartesian product, and the presence of i on the right-hand side is due to the fact that the generators T^a and $T^{\hat{a}}$ are hermitian, in this way we have consistency with non-real matter fields.

To work on that two structures, we explicit the product between two displacements, we have:

$$\begin{aligned} \mathbb{D}_m\mathbb{D}_n = \mathcal{F}_{m1}\mathcal{F}_{n1} + \mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_B + \mathcal{D}_m\mathcal{L}_B\mathcal{F}_{n1} + \mathcal{D}_m\mathcal{L}_B\mathcal{D}_n\mathcal{L}_B + \mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_F \quad &\to\equiv X_{mn} \\ + \mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_F + \mathcal{D}_m\mathcal{L}_F\mathcal{F}_{n1} + \mathcal{D}_m\mathcal{B}_F\mathcal{D}_n\mathcal{L}_F + \mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_B \quad &\to\equiv Y_{mn} \end{aligned} , \tag{180}$$

that is, the first line was defined as X_{mn} while the second line as Y_{mn}. Note, X_{mn} contains terms having nonvanishing components only in the diagonal, while Y_{mn} contains off diagonal non-vanishing terms only.

Count the associated power in λ for each diagram is equivalent to count powers of $1/k$ factors in front of them. In order to do this then we make some important comments, valid generically:

- Gauge propagators contribute with $1/k$ factors;
- Scalar and fermion propagators contribute with k^0;
- Whenever a gauge-cubic vertex is necessary, if we do not want an internal divergent bubble, we need three gauge propagators, one for each gauge in the vertex, which, together with the factor of k in front of it, gives a contribution of $1/k^2$ for such vertex;
- Gauge-fermion cubic vertex contributes with a factor of $1/k$ for the same reason;

- Gauge-scalar vertices will contribute with $1/k$ or $1/k^2$;

- Scalar-fermion interaction terms have couplings dependent inversely on k, see [44];

- $\mathcal{L}_B$ carries a $1/k$ factor and $\mathcal{L}_F$ carries a $1/\sqrt{k}$ factor.

Starting with the diagrams contributing from **1)**, we have:

$$\begin{aligned}\langle \mathrm{Tr}\ [\mathbb{D}_m(s_1)\mathbb{D}_n(s_2)]\rangle &= \langle \mathrm{Tr}\ X_{mn}(s_1,s_2)\rangle \\ &= \langle \mathrm{Tr}\ \mathcal{F}_{m1}\mathcal{F}_{n1}\rangle + \langle \mathrm{Tr}\ \mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_B\rangle + \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_B\mathcal{F}_{n1}\rangle \ . \\ &+ \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_B\mathcal{D}_n\mathcal{L}_B\rangle + \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_F\rangle\end{aligned}$$

We see that only the first and the last of these terms can give $1/k$ contributions; the second and third have at least one gauge and a factor of $1/k$ from $\mathcal{L}_B$, therefore starting from $1/k^2$, while the fourth has a factor $1/k^2$ coming from the $\mathcal{L}_B$'s. The relevant expectation values and associated diagrams are thus:

i $\langle \mathrm{Tr}\ (\partial_m A_1 - \partial_1 A_m)(\partial_n A_1 - \partial_1 A_n)\rangle$, $1/k$ contribution;

ii $\langle \mathrm{Tr}\ \left(\partial_m \hat{A}_1 - \partial_1 \hat{A}_m\right)\left(\partial_n \hat{A}_1 - \partial_1 \hat{A}_n\right)\rangle$, $1/k$ contribution;

iii $\langle \mathrm{Tr}\ \partial_m\psi_+\partial_n\bar{\psi}^+\rangle$, k^0 contribution, but there is an overall $1/k$ factor from $\mathcal{L}_F$'s;

iv $\langle \mathrm{Tr}\ \partial_m\bar{\psi}^+\partial_n\psi_+\rangle$, k^0 contribution, but there is an overall $1/k$ factor from $\mathcal{L}_F$'s.

the left one refering to i and ii and the right one to iii and iv. Just propagators.

Now the contributions from **2)**. This case is a little bit more complicated because involves products of three matrices. However we can use the previous result (180) together with (169) to keep directly nonvanishing terms from the trace procedure:

$$\begin{aligned}\langle \mathrm{Tr}\ [\mathcal{L}(\tau)\mathbb{D}_m(s_1)\mathbb{D}_n(s_2)]\rangle &= \langle \mathrm{Tr}\ \mathcal{A}(\tau)\times X_{mn}(s_1,s_2)\rangle \\ &+ \langle \mathrm{Tr}\ \mathcal{L}_B(\tau)\times X_{mn}(s_1,s_2)\rangle\ , \\ &+ \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\times Y_{mn}(s_1,s_2)\rangle\end{aligned} \tag{181}$$

notice, however, that we already have a $1/k$ factor coming from $\mathcal{L}_B$ in the second term and also a $1/\sqrt{k}$ factor from $\mathcal{L}_F$ in the third term above. Counting relevant diagrams then will be simpler than it seems. Focusing on the first of these terms, expliciting it we have:

$$\begin{aligned}\langle \mathrm{Tr}\ \mathcal{A}(\tau)\times X_{mn}(s_1,s_2)\rangle &= \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{F}_{m1}\mathcal{F}_{n1}\rangle + \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_B\rangle \\ &+ \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{D}_m\mathcal{L}_B\mathcal{F}_{n1}\rangle + \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{D}_m\mathcal{L}_B\mathcal{D}_n\mathcal{L}_B\rangle \ . \\ &+ \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_F\rangle\end{aligned} \tag{182}$$

All of them has at least one gauge field, for this reason at least a $1/k$ contribution. The ones containing $\mathcal{L}_B$'s and/or $\mathcal{L}_F$'s get another factor from the definitions of those quantities, so they can be dispensed. The first of the terms appearing above has at least three gauge fields within it, making impossible to construct diagrams proportional to $1/k$ without considering bubbles. Therefore the overall contribution to λ^1 is zero.

The second term in (181) has the same problems. The contribution coming from the double $\mathcal{F}$'s term in X_{mn} despites of having one less gauge-field, has an extra $1/k$ from $\mathcal{L}_B$ which makes everything proportional to $1/k^2$. Finally the last term in (181). Let us explicit it:

$$\begin{aligned}\langle \text{Tr } \mathcal{L}_F(\tau) \times Y_{mn}(s_1, s_2)\rangle = \langle \text{Tr } \mathcal{L}_F(\tau)\mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_F\rangle + \langle \text{Tr } \mathcal{L}_F(\tau)\mathcal{D}_m\mathcal{L}_F\mathcal{F}_{n1}\rangle \\ + \langle \text{Tr } \mathcal{L}_F(\tau)\mathcal{D}_m\mathcal{L}_B\mathcal{D}_n\mathcal{L}_F\rangle + \langle \text{Tr } \mathcal{L}_F(\tau)\mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_B\rangle\end{aligned} . \tag{183}$$

Clearly the last two of them does not give λ^1 contributions. The ones on the first line at leading order will be the result of considering gauge-fermion interaction, giving $1/k$ from 1 propagator and $1/k$ from the two $\mathcal{L}_F$, therefore a $1/k^2$ contribution.

So, in the case of two insertions of displacement operators in the line, only the propagators drawn before will contribute to λ^1. Considering more terms on wilson loops expansions in (174) will give more $\mathcal{L}_B$'s and $\mathcal{L}_F$'s, raising the order of the result. Computing the expectation values of i and ii will not be necessary, actually, because they differ by a minus sign and the trace of the supermatrix makes them sum up to zero. The relevant quantities here will be then iii and iv, we have:

$$\begin{aligned}\langle \text{Tr } \mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_F\rangle &= -\frac{2\pi i}{k}\left\langle \text{Tr } \begin{bmatrix} \partial_m\bar{\psi}_+\partial_n\psi^+ & 0 \\ 0 & \partial_m\psi^+\partial_n\bar{\psi}_+ \end{bmatrix}\right\rangle \\ &= -\frac{2\pi i}{k}\left(\langle \text{Tr } \partial_m\bar{\psi}_+\partial_n\psi^+\rangle + \langle \text{Tr } \partial_m\psi^+\partial_n\bar{\psi}_+\rangle\right) , \\ &= -\frac{4\pi i}{k}\langle \text{Tr } \partial_m\psi^+\partial_n\bar{\psi}_+\rangle\end{aligned}$$

where to write the last line we used the fact that fermions anticommute and that exchanging arguments of ψ and $\bar{\psi}$ also gives a minus sign.

Using (176) and (178), we can write:

$$\begin{aligned}\langle \text{Tr } \mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_F\rangle &= -\frac{4\pi i}{k}\frac{-1}{2}\text{Tr}(T^aT^b \times T^{\hat{a}}T^{\hat{b}})\frac{\partial^2}{\partial x^m\partial y^n}\langle(\psi^+)^a_{\hat{a}}(x)(\bar{\psi}_+)^{\hat{b}}_b(y)\rangle\Big|_{\text{points}} \\ &= \frac{2\pi i}{k}\text{Tr}(T^aT^b \times T^{\hat{a}}T^{\hat{b}})\frac{\partial^2}{\partial x^m\partial y^n}\langle(\psi^+)^a_{\hat{a}}(x)(\bar{\psi}_+)^{\hat{b}}_b(y)\rangle\Big|_{\text{points}} \\ &= \frac{2\pi i}{k}\text{Tr}(T^aT^b \times T^{\hat{a}}T^{\hat{b}})\left(-i\delta_{ab}\delta_{\hat{a}\hat{b}}\frac{\Gamma(3/2-\epsilon)}{2\pi^{3/2-\epsilon}}\right)\frac{\partial^2}{\partial x^m\partial y^n}\frac{(x-y)_1}{|x-y|^{3-2\epsilon}}\Big|_{\text{points}}\end{aligned}$$

, where "points" stands for $x = (s_1, 0, 0)$ and $y = (s_2, 0, 0)$.

Summing up traces and calculating the derivatives, noticing that $\delta_{aa} = \delta_{\hat{a}\hat{a}} = N^2$, we get:

$$\langle \text{Tr } \mathcal{D}_m \mathcal{L}_F \mathcal{D}_n \mathcal{L}_F \rangle = 3\frac{N^2}{k}\delta_{mn}\frac{1}{(s_1 - s_2)^4} \ ,$$

and, therefore, now returning to (172):

$$\boxed{\langle\langle \mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\rangle\rangle = \delta_{mn}\frac{\frac{3}{2}\lambda}{(s_1 - s_2)^4} + \mathcal{O}(\lambda^2)} \ , \tag{184}$$

that is, the structure constant of two-point function of displacements at first order in λ is equal to $\frac{3}{2}\lambda$.

In order to compare this result with the Wilson loop side of the relation (159), instead of integrating it we use a more clever approach: the second variation of a deformed Wilson loop is well known in those theories and is proportional to a quantity that goes by the name of *Bremstrahlung function*, see [45] for more details. In consequence of this, such function for a derformed wilson line is directly related to the correlator between two displacements inserted on the line. More precisely, we have the universal relation (see [24]):

$$C_{\mathbb{D}} = 12B^{1/2}_{\text{ABJM}}(\lambda) \ , \tag{185}$$

where $C_{\mathbb{D}}$ is the structure constant in question and $B^{1/2}_{\text{ABJM}}(\lambda)$ the Bremstrahlung function.

As found in equation 6.19 of [45] for example, up to λ^5 we have:

$$B^{1/2}_{\text{ABJM}}(\lambda) = \frac{\lambda}{8} - \frac{\pi^2\lambda^3}{48} + \mathcal{O}(\lambda^5) \ , \tag{186}$$

which comproves the result (184).

A little new contribution in this way we give with this work is to use the same reasoning above to compute also the structure constant of three insertions of displacements in the 1/2 BPS Infinite Straight Wilson Line and study its consequence to the evaluation of third order variations of deformed lines. The quantity we are interested in now is:

$$\begin{aligned}\langle \mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\mathbb{D}_p(s_3)\rangle_{\mathcal{W}} = \langle &\text{Tr } \mathcal{P} \ \mathcal{W}(\infty, s_1)\mathbb{D}_m(s_1)\mathcal{W}(s_1, s_2) \\ &\times \mathbb{D}_n(s_2)\mathcal{W}(s_2, s_3)\mathbb{D}_p(s_3)\mathcal{W}(s_3, -\infty)\rangle\end{aligned} \ , \tag{187}$$

where $s_1 > s_2 > s_3$ supposedly.

Expanding the $\mathcal{W}$'s and keeping only terms with one integral at most we get:

$$\begin{aligned}\langle \mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\mathbb{D}_p(s_3)\rangle_{\mathcal{W}} \simeq &\langle \text{Tr } \mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\mathbb{D}_p(s_3)\rangle \\ &- i\Big[\int_{s_1}^{\infty} d\tau \langle \text{Tr } \mathcal{L}(\tau)\mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\mathbb{D}_p(s_3)\rangle \\ &+ \int_{s_2}^{s_1} d\tau \langle \text{Tr } \mathbb{D}_m(s_1)\mathcal{L}(\tau)\mathbb{D}_n(s_2)\mathbb{D}_p(s_3)\rangle \\ &+ \int_{s_3}^{s_2} d\tau \langle \text{Tr } \mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\mathcal{L}(\tau)\mathbb{D}_p(s_3)\rangle \\ &+ \int_{-\infty}^{s_3} d\tau \langle \text{Tr } \mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\mathbb{D}_p(s_3)\mathcal{L}(\tau)\rangle\Big]\end{aligned} .$$

And again we have two important structures to study only:

1') $\langle \text{Tr } [\mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\mathbb{D}_p(s_3)]\rangle$;

2') $\langle \text{Tr } [\mathcal{L}(\tau)\mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\mathbb{D}_p(s_3)]\rangle$,

which we explore below at different orders.

4.3.1 Three-point of displacements to λ

Using the notation from two-insertions calculation, **1')** above can be splitted onto three parts:

$$\begin{aligned}\langle \text{Tr } [\mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\mathbb{D}_p(s_3)]\rangle = &\langle \text{Tr } \mathcal{F}_{m1}(s_1)X_{np}(s_2,s_3)\rangle + \langle \text{Tr } \mathcal{D}_m\mathcal{L}_B(s_1)X_{np}(s_2,s_3)\rangle \\ &+ \langle \text{Tr } \mathcal{D}_m\mathcal{L}_F(s_1)Y_{np}(s_2,s_3)\rangle\end{aligned} . \tag{188}$$

Let us treat each one separatedly. The first opens to:

$$\begin{aligned}\langle \text{Tr } \mathcal{F}_{m1}(s_1)X_{np}(s_2,s_3)\rangle = &\langle \text{Tr } \mathcal{F}_{m1}\mathcal{F}_{n1}\mathcal{F}_{p1}\rangle + \langle \text{Tr } \mathcal{F}_{m1}\mathcal{F}_{n1}\mathcal{D}_p\mathcal{L}_B\rangle \\ &+ \langle \text{Tr } \mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_B\mathcal{F}_{p1}\rangle + \langle \text{Tr } \mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_B\mathcal{D}_p\mathcal{L}_B\rangle \\ &+ \langle \text{Tr } \mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_F\mathcal{D}_p\mathcal{L}_F\rangle\end{aligned} . \tag{189}$$

The first term gives no contribution at order $1/k$ because involves at least two gauge propagators. The second and third ones do not contribute also because involve at least one gauge propagator and a $1/k$ from $\mathcal{L}_B$. The fourth is also irrelevant because of two $1/k$ factors from $\mathcal{L}_B$'s. The last one involves at least one gauge propagator and a $1/k$ from the two $\mathcal{L}_F$'s, therefore not contributing too.

The second term in (188) is similar to the previous one, except for the overall $1/k$ factor from $\mathcal{L}_B$, making leading order be $1/k^2$.

The last term in (188) is a litte bit different:

$$\begin{aligned}\langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_F(s_1)Y_{np}(s_2,s_3) &= \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_F\mathcal{F}_{n1}\mathcal{D}_p\mathcal{L}_F\rangle + \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_F\mathcal{F}_{p1}\rangle \\ &+ \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_B\mathcal{D}_p\mathcal{L}_F\rangle + \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_F\mathcal{D}_p\mathcal{L}_B\rangle\end{aligned}\ , \tag{190}$$

but again gives no relevant contribution: the last two have $1/k^2$ factors coming from the $\mathcal{L}_B$'s and $\mathcal{L}_F$'s; the first two, in spite of having only one $1/k$ factor coming from the $\mathcal{L}_F$'s, still have at least one gauge propagator, and therefore $1/k^2$ net result.

Contributions from **2')** are more interesting, but as one could expect due to the presence of more fields inside the expectation values, will not give relevant diagrams at first order in λ. We choose to see in detail here those terms though, for pedagogical reasons. Before exploring them, however, we define useful quantities in order to keep calculations organized; considering the product between three displacement operators and using (180), we have:

$$\begin{aligned}\mathbb{D}_m\mathbb{D}_n\mathbb{D}_p &= (\mathcal{F}_{m1} + \mathcal{D}_m\mathcal{L}_B + \mathcal{D}_m\mathcal{L}_F)\,(X_{np} + Y_{np}) \\ &= (\mathcal{F}_{m1} + \mathcal{D}_m\mathcal{L}_B)\,X_{np} + \mathcal{D}_m\mathcal{L}_F Y_{np} \quad \to \equiv X_{mnp}\ , \\ &+ (\mathcal{F}_{m1} + \mathcal{D}_m\mathcal{L}_B)\,Y_{np} + \mathcal{D}_m\mathcal{L}_F X_{np} \quad \to \equiv Y_{mnp}\end{aligned} \tag{191}$$

where again X_{mnp} contains non-vanishing components in the principal diagonal only, while Y_{mnp} in the secondary diagonal.

Taking into account the presence of the trace, the surviving terms contributing in **2')** are then:

$$\begin{aligned}\langle \mathrm{Tr}\ [\mathcal{L}(\tau)\mathbb{D}_m(s_1)\mathbb{D}_n(s_2)\mathbb{D}_p(s_3)]\rangle &= \langle \mathrm{Tr}\ \mathcal{A}(\tau)X_{mnp}(s_1,s_2,s_3)\rangle \quad \to \mathbf{a} \\ &+ \langle \mathrm{Tr}\ \mathcal{L}_B(\tau)X_{mnp}(s_1,s_2,s_3)\rangle \quad \to \mathbf{b}\ . \\ &+ \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)Y_{mnp}(s_1,s_2,s_3)\rangle \quad \to \mathbf{c}\end{aligned} \tag{192}$$

Expliciting **a**, we have:

$$\begin{aligned}\langle \mathrm{Tr}\ \mathcal{A}(\tau)X_{mnp}(s_1,s_2,s_3)\rangle &= \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{F}_{m1}\mathcal{F}_{n1}\mathcal{F}_{p1}\rangle + \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{F}_{m1}\mathcal{F}_{n1}\mathcal{D}_p\mathcal{L}_B\rangle \\ &+ \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_B\mathcal{F}_{p1}\rangle + \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_B\mathcal{D}_p\mathcal{L}_B\rangle \\ &+ \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_F\mathcal{D}_p\mathcal{L}_F\rangle + \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{D}_m\mathcal{L}_B\mathcal{F}_{n1}\mathcal{F}_{p1}\rangle \\ &+ \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{D}_m\mathcal{L}_B\mathcal{F}_{n1}\mathcal{D}_p\mathcal{L}_B\rangle + \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{D}_m\mathcal{L}_B\mathcal{D}_n\mathcal{L}_B\mathcal{F}_{p1}\rangle \\ &\langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{D}_m\mathcal{L}_B\mathcal{D}_n\mathcal{L}_B\mathcal{D}_p\mathcal{L}_B\rangle + \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{D}_m\mathcal{L}_B\mathcal{D}_n\mathcal{L}_F\mathcal{D}_p\mathcal{L}_F\rangle \\ &+ \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{D}_m\mathcal{L}_F\mathcal{F}_{n1}\mathcal{D}_p\mathcal{L}_F\rangle + \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_F\mathcal{F}_{p1}\rangle \\ &+ \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_B\mathcal{D}_p\mathcal{L}_F\rangle \\ &+ \langle \mathrm{Tr}\ \mathcal{A}(\tau)\mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_F\mathcal{D}_p\mathcal{L}_B\rangle\end{aligned} \tag{193}$$

from which we see we are not having first order contributions: the first red term contains at least two gauge propagators, the second and third red terms have at least a gauge propagator and another factor of $1/k$ from $\mathcal{L}_B$, the fourth contains two factos of $1/k$ from the two $\mathcal{L}_B$'s and the last red term has at least a gauge propagator and a $1/k$ factor resulting from the $\mathcal{L}_F$'s; the blue terms clearly do not contribute to first order because they have at least one factor of $1/k$ from $\mathcal{L}_B$ and one gauge propagator; similarly, green terms do not contribute because of factors from at least two $\mathcal{L}_F$'s besides at least one gauge propagator.

The case **b** is similar to the one above changing $\mathcal{A}(\tau)$ by $\mathcal{L}_B$ everywhere. The starting extra $1/k$ factor from $\mathcal{L}_B$ in all terms makes every contribution be at least of second order, because at least another gauge propagator would be needed, as it is clear from the expression (193).

Expression **c** is a little bit diffrent:

$$
\begin{aligned}
\langle \mathrm{Tr}\ \mathcal{L}_F(\tau) Y_{mnp}(s_1,s_2,s_3)\rangle = & \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\mathcal{F}_{m1}\mathcal{F}_{n1}\mathcal{D}_p\mathcal{L}_F\rangle + \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_F\mathcal{F}_{p1}\rangle \\
& + \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_B\mathcal{D}_p\mathcal{L}_F\rangle + \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_F\mathcal{D}_p\mathcal{L}_B\rangle \\
& + \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\mathcal{D}_m\mathcal{L}_B\mathcal{F}_{n1}\mathcal{D}_p\mathcal{L}_F\rangle + \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\mathcal{D}_m\mathcal{L}_B\mathcal{D}_n\mathcal{L}_F\mathcal{F}_{p1}\rangle \\
& + \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\mathcal{D}_m\mathcal{L}_B\mathcal{D}_n\mathcal{L}_B\mathcal{D}_p\mathcal{L}_F\rangle \\
& + \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\mathcal{D}_m\mathcal{L}_B\mathcal{D}_n\mathcal{L}_F\mathcal{D}_p\mathcal{L}_B\rangle \\
& \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\mathcal{D}_m\mathcal{L}_F\mathcal{F}_{n1}\mathcal{F}_{p1}\rangle + \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\mathcal{D}_m\mathcal{L}_F\mathcal{F}_{n1}\mathcal{D}_p\mathcal{L}_B\rangle \\
& + \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_B\mathcal{F}_{p1}\rangle \\
& + \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_B\mathcal{D}_p\mathcal{L}_B\rangle \\
& + \langle \mathrm{Tr}\ \mathcal{L}_F(\tau)\mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_F\mathcal{D}_p\mathcal{L}_F\rangle
\end{aligned}
\tag{194}
$$

notice that the presence of two L's and a gauge propagator, or more than two L's, makes the contribution for this term also start from $1/k^2$.

To λ, therefore, the structure constant $f_{\mathbb{DDD}}$ of three displacements is equal to zero, and we have to go beyond:

$$
f_{\mathbb{DDD}}(\lambda) = 0 + \mathcal{O}(\lambda^2)\ . \tag{195}
$$

4.3.2 Three-point of displacements to λ^2

We analyze again **1')** and **2')**. The diagrams appearing in the first of them contributing at order λ^2 are synthesized in Figure 8, where operators coming from insertions or the action are represented by bullets.

To see this, we just have to repeat the procedure in the previous section for (188); the first term reads:

$$
\begin{aligned}
\langle \mathrm{Tr}\ \mathcal{F}_{m1}(s_1) X_{np}(s_2,s_3)\rangle = & \langle \mathrm{Tr}\ \mathcal{F}_{m1}\mathcal{F}_{n1}\mathcal{F}_{p1}\rangle + \langle \mathrm{Tr}\ \mathcal{F}_{m1}\mathcal{F}_{n1}\mathcal{D}_p\mathcal{L}_B\rangle \\
& + \langle \mathrm{Tr}\ \mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_B\mathcal{F}_{p1}\rangle + \langle \mathrm{Tr}\ \mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_B\mathcal{D}_p\mathcal{L}_B\rangle\ , \\
& + \langle \mathrm{Tr}\ \mathcal{F}_{m1}\mathcal{D}_n\mathcal{L}_F\mathcal{D}_p\mathcal{L}_F\rangle
\end{aligned}
\tag{196}
$$

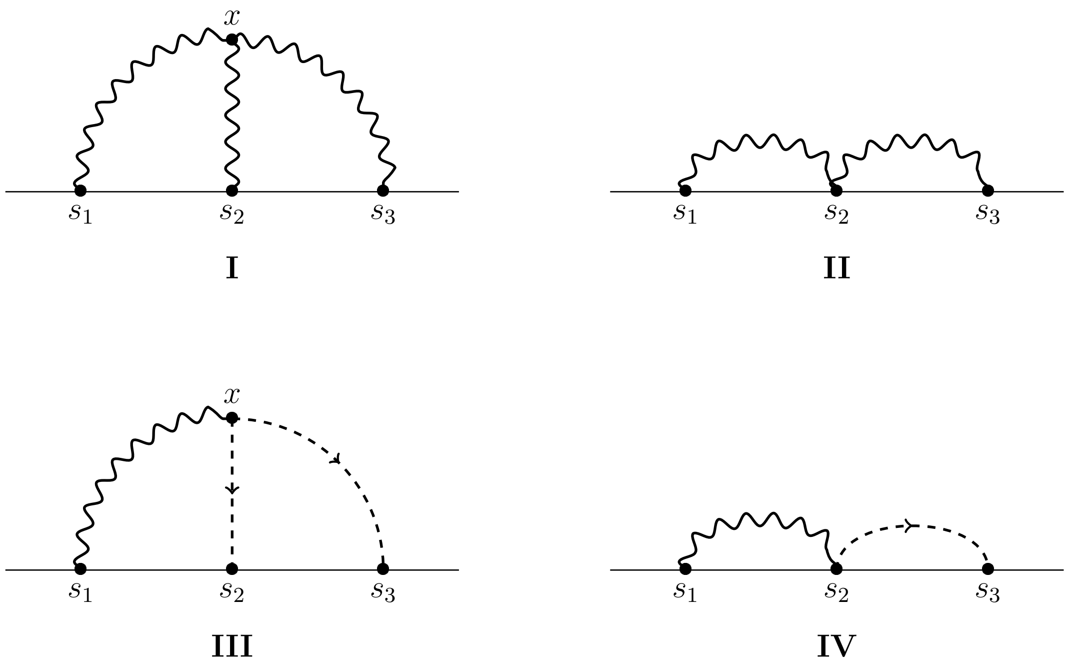

Figure 8: Resulting diagrams for none insertion from loop expansion.

resulting in four important diagrams, **I** and **II** in fig. 8 associated to the first term above and **III** and **IV** in the same figure associated to the last term in there. Scalars appear just as bubbles or higher order terms due to factors from $\mathcal{L}_B$'s or to the presence of more than one gauge propagator from gauge-scalar interaction term.

The second stays:

$$\begin{aligned}\langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_B(s_1)X_{np}(s_2,s_3)\rangle &= \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_B\mathcal{F}_{n1}\mathcal{F}_{p1}\rangle + \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_B\mathcal{F}_{n1}\mathcal{D}_p\mathcal{L}_B\rangle \\ &+ \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_B\mathcal{D}_n\mathcal{L}_B\mathcal{F}_{p1}\rangle + \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_B\mathcal{D}_n\mathcal{L}_B\mathcal{D}_p\mathcal{L}_B\rangle\ , \\ &+ \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_B\mathcal{D}_n\mathcal{L}_F\mathcal{D}_p\mathcal{L}_F\rangle\end{aligned} \tag{197}$$

no relevant contribution due to several $\mathcal{L}$'s.

The third term reads:

$$\begin{aligned}\langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_F(s_1)Y_{np}(s_2,s_3) &= \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_F\mathcal{F}_{n1}\mathcal{D}_p\mathcal{L}_F\rangle + \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_F\mathcal{F}_{p1}\rangle \\ &+ \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_B\mathcal{D}_p\mathcal{L}_F\rangle + \langle \mathrm{Tr}\ \mathcal{D}_m\mathcal{L}_F\mathcal{D}_n\mathcal{L}_F\mathcal{D}_p\mathcal{L}_B\rangle\end{aligned}\ ; \tag{198}$$

the first two terms here are similar to the last one in (196), two repeated diagrams appearing then: **III** and **IV**. The last two terms here do not contribute for the same reason as the previous case, they also appear there.

For **2')** we just have to look again at (193) and (194), **b** in (192) clearly contributes with $1/k^3$ terms only. In the following diagrams, $\times$ represents operators coming from expasions of partial lines.

In (193), we have two diagrams from the reds: first and last term, **V** and **VI** in fig. 9, respectively; scalar bubbles and/or excessive gauge propagators make second and third reds not relevant; the fourth red has $1/k^2$ factor from $\mathcal{L}_B$'s and at least one gauge propagator, therefore $1/k^3$ contribution. No contributions from blues for similar reasons. **VI** repeated diagram for the two first greens; the remaining green terms have too much $\mathcal{L}$'s and at least one gauge propagator required, making them $1/k^3$ relevants only.

Turning to (194), the first two reds contribute according to **VI** too; other reds are alike dispensed terms before. None contribution from the blues. First green term contributes according to **VI** again, the others are of at least $1/k^3$ order.

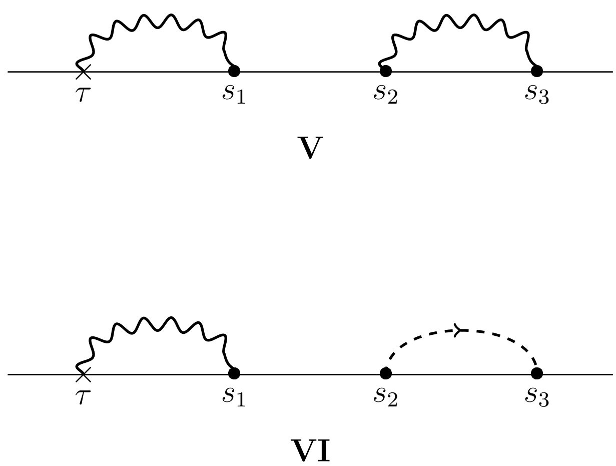

Figure 9: Resulting diagrams for one insertion from loop expansion.

Finally we go to the calculations. The way is short, however. Diagrams **I** and **III** are the least trivial, and as can be seen from (196), we also have three derivatives involved in the expectation value. The computation of the VEV in these cases requires the use of a computer program, which can be simplified using known expressions in ABJM literature, as in [72], for example. These diagrams result in zero, individually.

Diagrams **II**, **IV**, **V** and **VI** do not involve vertices, they are just products of propagators, but again there are derivatives acting on these products and evaluated at the points $x = (s_1, 0, 0)$, $y = (s_2, 0, 0)$ and $z = (s_3, 0, 0)$. It turns out they vanish too, now algebraically. In the case of **VI**, for example, it is easier to see; take for example its appearence in the first green term of (194), in that case the following VEV occurs:

$$\langle(\bar{\psi}_+)_a^{\hat{a}}(\tau)\partial_m(\psi^+)_{\hat{b}}^{b}(x)\rangle\langle F_{n1}(s_2)F_{p1}(s_3)\rangle$$

Equations (176) then tell us that the quantity above will be proportional to:

$$\frac{\partial}{\partial x^m}\frac{(w-x)_1}{|w-x|^{3-2\epsilon}}\bigg|_{\text{points}},$$

with "points" standing for $w = (\tau, 0, 0)$ and $x = (s_1, 0, 0)$.

Clearly we see that the partial derivative is going to drop coordinates in the numerator orthogonal to the line, such that when evaluated in the "points" they are going to result in zero.

We conclude then that, even to λ^2, the structure constant of three insertions of displacements on the line is zero. Updating our previous result then, we have:

$$\boxed{f_{\mathbb{DDD}} = 0 + \mathcal{O}(\lambda^3)}\ . \tag{199}$$

This result shows that third variations of deformed WL from the 1/2 BPS Wilson Line in ABJM are zero up to λ^3. It is in accordance with the fact that such variations for deformed WL from 1/2 BPS Wilson Line in $\mathcal{N} = 4$ SYM (a more supersymmetric theory) are also zero, as pointed out at the end of section 4.2. There, however, we concluded it was true for any order in λ. For this reason, it is a strong clue that our result should also hold for any order in λ. Another important check can be made by considering the two-point function of displacements at λ^2; in this case, diagrams **I** and **II** apper in a similar way in the first term of (182), as well as diagrams **III** and **IV** in the last term in there[29]; they vanish, in accordance with the fact that the Bremstrahlung function (186) has only terms proportional to λ and λ^3.

In fact, from an analogous argument as for the $4d$ case, it is possible to conclude the same. The correlation function composed of three insertions of displacements have three free indices: m, n and p, which are allowed to take the values 2 and 3 only. In order to compute the third variation on the deformed wilson line (remember (159) once more), we have to contract those indices again with the unique invariant tensor at our disposition: the totally antisymmetric Levi-Civita symbol ϵ^{ijk}. When contracting though, we certainly will have at least two equal entries in that tensor, therefore resulting in zero always, independent on the order in λ for $f_{\mathbb{DDD}}$.

4.3.3 Discussion

We studied two applications of (159) in this last chapter. In the first of them we have reproduced calculations of a recent paper, finding the structure constant for an specific two-point function in the presence of the 1/2 BPS Straight Wilson Line in $\mathcal{N} = 4$ SYM at first order in the t'Hooft parameter; that paper go beyond what we have presented here and try to compute the structure constant for the three-point functions also, but not so securely about the results found, as commented in there ([23]).

In the second application we tried to compute the structure constant of a specific three-point function in the presence now of the 1/2 BPS Straight Wilson Line in $\mathcal{N} = 6$ ABJM. We found that it vanishes up to λ^3, so that it may be nonzero only considering λ^3 or higher.

[29] More diagrams appear besides these ones actually, some of them vanishes trivially, others not.

We checked the result by making two considerations: third order variations of deformed wilson lines vanish in $\mathcal{N} = 4$ SYM, which is a theory with more symmetries, as argumented also in [23], so the same should be expected here and our result, therefore, does not violate it; moreover, the diagrams appearing in our calculations also appear in the computation of the structure constant of the two-point function of the same operator insertion and in a similar way. In that case, the term proportional to λ^2 in the structure constant must vanish alone in view of the Bremstrahlung function (186) used for comparation.

Our result is thus another step forward, but does not give any information about the content of the structure constant $f_{\mathbb{DDD}}$ for λ^3, although one could be wondering if it is true for all orders in λ too. The fact is that it does not necessarily have to be zero to guarantee the third variation of the deformed wilson line is also zero, this last already holds in view of symmetry of the indices involved, as pointed out before.

5 Closing Remarks

This thesis has been concerned primarily with the defect CFTs defined by (supersymmetric) Wilson lines in 4 and 3 dimensions. In Chapter 1 some effort has been made to present all the necessary ingredients of quantum field theory and conformal field theory, in a concise way. I have tried to provide all the particularities encountered in CFTs, in particular the fact that 2- and 3-point functions are fixed by the conformal symmetries, thus defining the CFT data given by anomalous dimensions and structure constants.

Chapter 2 has been dedicated to introducing defect CFTs, namely CFT defined along a defect in space-time. The introduction of an extended operator breaks up the original conformal group of a theory into a smaller (yet still conformal) group, giving rise to a different CFT. In that chapter we learnt how to construct the main objects of study in a defect CFT using the embedding space formalism. We then concluded with the presentation of the displacement operator, an important actor of the final part of this dissertation.

In Chapter 3 another actor of great importance has been introduced, namely the supersymmetric Wilson loop which is the operator we used to define the defect CFT. We started from the very fundamental definition of Wilson operator in a gauge theory and then extended it to some superconformal field theories of interest in holography: $\mathcal{N} = 4$ SYM in 4 dimensions and $\mathcal{N} = 6$ ABJM theory in 3 dimensions.

Finally, in the last Chapter 4, it has been shown how Wilson loops can be used to define defect CFTs. An interesting relation between Wilson loops defined along 'wavy' contours, deformations of the straight line or the circle, and correlators of local operators inserted along the contour has been discussed. This relation was derived and summarized in equation (159). The rest of the chapter has been thus reserved to explore such terrific connection, with the hope of having given a useful contribution in this sense. In particular, I have setup various computations to compute, at the perturbative level, correlation functions of displacement operators inserted along the 1/2 BPS Wilson line of ABJM theory, which to the best of my knowledge has been done only for 2-point functions, see [24].

Outlook for the future

In recent years a significant amount of attention has been given to this topic but there are still some very interesting open questions. Computing these correlators along the defect, and the structure constants in particular, is by itself very important and interesting, since this essentially represents solving the theory. Moreover, one can hope to compare the results obtained here from a perturbative computation with results obtained using bootstrap methods (at generic couplings) and/or holography (for the strong coupling limit).

The calculations were performed here in some detail in order to show possible difficulties one could face, related in particular to regularization. We found the structure constant of three insertions of displacement operators into the 1/2 BPS straight Wilson line in ABJM to order λ^2. It is possible to go on and find values for higher orders in λ, although it may be very hard to do. Maybe one could come up with some trick along the lines of [73] to avoid having to perform explicit integrals.

One could apply the same techniques to determine 4-point functions or higher n-point functions. In particular, the fourth (or higher) order deformation of a Wilson line can be found by determining the 4-point function of displacements, which in turn can be expressed in terms of 2- and/or 3-point functions of another primary operators. It can be useful then to find structure constants for other defect 2- and 3-point functions of primary insertions in the theory. This is in fact something we intend to do as a follow-up work.

A Poincarè Group

This section contains the basics about the Poicaré group and is based entirely on [75,76].

The Poincarè group is composed of translations and proper orthochronous Lorentz transformations. In section 1.1.4 we worked out the consequences of a system invariant under translations and we got that momentum operators are the generators of such symmetries. The explicit form of these generators in the field representation were also given there ($\mathcal{P}_\mu = -i\partial_\mu$) and it can be verified they satisfy:

$$[\mathcal{P}_\mu, \mathcal{P}_\nu] = 0$$

The Lorentz group is by definition the linear homogeneous transformations on the coordinates that preserves the Minkowski metric $\eta_{\mu\nu}$ in d-dimensional flat spacetime (d-1 spacelike directions and 1 timelike direction). That is, they are of the form:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \text{with} \;\; dx'^2 = dx^2$$

The second condition above tells us that $\Lambda^\mu{}_\nu$ must satisfy $\eta = \Lambda^{\mathrm{T}}\eta\Lambda$, with $\det\Lambda = \pm 1$. The subgroup of these transformations with $\det\Lambda = 1$ and with the timelike diagonal component greater or equal to 1 (in the mostly plus signature) is the one we want. We denote such group by $SO(d-1,1)$. Infinitesimally, that is, with $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu_\nu$ and small omegas, that condition says that the parameters $\omega^\mu{}_\nu$ are antisymmetric in their indices. Thus, conveniently, it is adopted the convetion that the generators associated are represented by antisymmetric objects $\mathcal{J}^{\mu\nu}$, such that a general and an infinitesimal Lorentz transformation on a field ϕ (in a possible representation of the group) are given respectively by:

$$\phi' \equiv L_\Lambda \phi = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{S}^{\mu\nu}\right)\phi$$
$$\phi'(x') \simeq \phi - \frac{i}{2}\omega_{\mu\nu}\mathcal{S}^{\mu\nu}\phi$$

where these $\mathcal{S}^{\mu\nu}$ denote the particularization of the object $\mathcal{J}^{\mu\nu}$ to the intrinsic behaviour of the field ϕ under such transformations. Therefore $\mathcal{J}^{\mu\nu}$ incorporates both $\mathcal{S}^{\mu\nu}$ and also the contribution coming from the effect of the symmetry on the coordinates, $\mathcal{M}^{\mu\nu}$. In section 1.1.4 we also calculated the conserved current for a system with Lorentz symmetry. Following then the reasoning of finding the conserved charges integrating (22) and from them obtaining the generators associated by means of (18), one arrives at:

$$
\begin{aligned}
J^{\mu\nu} &\equiv \int d^{d-1}x j^{0\mu\nu} = M^{\mu\nu} + S^{\mu\nu} \\
M^{\mu\nu} &\equiv \int d^{d-1}x \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \left(x^\mu \partial^\nu - x^\nu \partial^\mu \right) \phi \\
S^{\mu\nu} &\equiv \int d^{d-1}x \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \mathcal{S}^{\mu\nu} \phi
\end{aligned}
$$

with associated generators:

$$
\begin{aligned}
\mathcal{J}^{\mu\nu} &\equiv \mathcal{M}^{\mu\nu} + \mathcal{S}^{\mu\nu} \\
\mathcal{M}^{\mu\nu} &\equiv -i \left(x^\mu \partial^\nu - x^\nu \partial^\mu \right)
\end{aligned}
\tag{200}
$$

The expressions for conserverd charges in this case show that one component of the cross product between position and momentum does enter in the integrand of $M^{\mu\nu}$, so that it can be interpreted as the angular momentum operator and the generators $\mathcal{M}^{\mu\nu}$ as the ones which yields spacetime rotations in d dimensions (boosts and spatial rotations).

Now we have all the generators associated to the Poincarè group, the last step is to show the algebra associated to it. The computation of the commutators is straightforward to do and is omitted here; the crucial point resides in the fact that $\mathcal{S}^{\mu\nu}$ is independent on the position, after all it corresponds to an intrinsic property of the fields. So we have:

$$
\begin{aligned}
[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] &= i \left(\eta^{\nu\rho} \mathcal{J}^{\mu\sigma} + \eta^{\mu\sigma} \mathcal{J}^{\nu\rho} - \eta^{\mu\rho} \mathcal{J}^{\nu\sigma} - \eta^{\nu\sigma} \mathcal{J}^{\mu\rho} \right) \\
[\mathcal{P}^{\rho}, \mathcal{J}^{\mu\nu}] &= -i \left(\eta^{\mu\rho} \mathcal{P}^{\nu} - \eta^{\nu\rho} \mathcal{P}^{\mu} \right) \\
[\mathcal{P}^{\mu}, \mathcal{P}^{\nu}] &= 0
\end{aligned}
\tag{201}
$$

Usually when we say *realtivistic* quantum field theory, it is to be understood as a theory invariant under the symmetry transformations generated by the $J^{\mu\nu}$ and P^μ, that is, a theory invariant under Poincarè group.

A.1 Fundamental Representation

Above we presented the generalities concerning field representation of the Poincarè group. A particular and, in fact, very useful representation of it is the so called *Weyl-spinor field representation*, which is the building block for other non-trivial field representations, in which fields have extra components.

The starting point is the spinor representation of the $SU(2)$ group, that one of 2-dimensional unitary matrices with determinant equals to 1. Spinors form a 2-dimensional representation of this group. All other representations of $SU(2)$ can be constructed from tensor products of spinors, and they are labeled by an index j which takes half-integer values, including zero. The algebra of this group and the solution associated to spinors are shown below:

$$\left[J^{\mathrm{i}}, J^{j}\right] = i\epsilon^{\mathrm{i}jk} J^{k}, \qquad J^{\mathrm{i}} = \sigma^{\mathrm{i}}/2$$

where $\epsilon^{\mathrm{i}jk}$ is the totally antisymmetric levi-civita symbol and σ^{i} are the Pauli matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \;, \; \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \;\text{ and }\; \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Spinors are representations with $j = 1/2$, that is, spin $1/2$. j is the label associated to the eigenvalue $j(j+1)$ of the Casimir operator J^2. $j = 0$ is the scalar (or singlet) representation. It can be shown, for example, that vectors are equivalent to the representation obtained from tensor product between two spinors, that is $1/2 \otimes 1/2$, and in general tensorial representations are obtained from tensor products of vectorials. The dimension of the representation is equal to $2j+1$ and the construction of invariant subspaces follows the usual rule for spin summation from quantum mechanics, which allows us obtaining irreducible and, therefore, physically relevant representations of the group.

With that in mind, notice that the first commutation relation in (201) contains the one above if we restrict ourselves to $d = 4$ and then rearrange the six components of $\mathcal{J}^{\mu\nu}$ into $\mathcal{J}^{\mathrm{i}} \equiv 1/2\epsilon^{\mathrm{i}jk}\mathcal{J}^{jk}$ and $\mathcal{K}^{\mathrm{i}} \equiv \mathcal{J}^{\mathrm{i}0}$:

$$\begin{aligned}
\left[\mathcal{J}^{\mathrm{i}}, \mathcal{J}^{j}\right] &= i\epsilon^{\mathrm{i}jk}\mathcal{J}^{k} \\
\left[\mathcal{K}^{\mathrm{i}}, \mathcal{K}^{j}\right] &= -i\epsilon^{\mathrm{i}jk}\mathcal{J}^{k} \\
\left[\mathcal{J}^{\mathrm{i}}, \mathcal{K}^{j}\right] &= i\epsilon^{\mathrm{i}jk}\mathcal{K}^{k}
\end{aligned}$$

where the algebra of $\mathcal{J}^{\mathrm{i}}$ is the traditional angular momentum algebra, while $\mathcal{K}^{\mathrm{i}}$ is reponsible for boost transformations, behaving like a vector operator as the last relation above shows[30]. Moreover, if one defines $\mathcal{J}^{\pm,\mathrm{i}} \equiv (\mathcal{J}^{\mathrm{i}} \pm i\mathcal{K}^{\mathrm{i}})/2$, the relations above reduces onto two independent algebras:

$$\begin{aligned}
\left[\mathcal{J}^{\pm,\mathrm{i}}, \mathcal{J}^{\pm,j}\right] &= i\epsilon^{\mathrm{i}jk}\mathcal{J}^{\pm,k} \\
\left[\mathcal{J}^{\pm,\mathrm{i}}, \mathcal{J}^{\mp,j}\right] &= 0
\end{aligned}$$

We see then that, in four dimensions, the Lorentz group can be decomposed into two $SU(2)$ groups, so that the representations of it can be constructed as tensorial products of representations of this group, thus labeled by two half-integers (j^-, j^+). Emerges then the question of which is taken as the fundamental representation of Lorentz group: $1/2 \otimes 0$ or $0 \otimes 1/2$; it turns out both representations are of equal importance and actually motivate the definition of *Weyl-spinors* in four dimensions: *left-handed* Weyl-spinors, ψ_L, are in the (1/2,0) representation of Lorentz group, therefore with $\mathcal{J}^{-,\mathrm{i}} = \sigma^{\mathrm{i}}/2$ (consequently

[30] Therefore it must not be confused with the conformal generator of special conformal transformations.

$\mathcal{J}^i = \sigma^i/2$ and $\mathcal{K}^{\mathrm{i}} = i\sigma^{\mathrm{i}}/2$), and transforms under a Lorentz transformation Λ_L like:

$$\psi_L \to \Lambda_L \psi_L \equiv \exp\left[(-i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}^{\mathrm{i}}}{2}\right] \psi_L$$

where it was defined: $\theta^i \equiv 1/2\epsilon^{ijk}\omega^{jk}$, $\eta^i \equiv \omega^{i0}$ and $\boldsymbol{\sigma} \equiv (\sigma^1, \sigma^2, \sigma^3)$, with the ω's standing for the general parameters of a Lorentz transformation as before. *Right-handed* Weyl-spinors, ψ_R, are defined analogously, leading to:

$$\psi_L \to \Lambda_R \psi_R \equiv \exp\left[(-i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}^{\mathrm{i}}}{2}\right] \psi_R$$

The notion of Weyl-spinors can be extended to higher dimensions. The crucial point resides in the fact that rotations are generated by antisymmetric generators, and for this reason traceless, operators. Therefore, Weyl-spinors do exist only in even-dimensional spacetime. The generalization goes with the help of the so called *Dirac gamma matrices*, γ_μ, and their *Clifford* algebra:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu \equiv \{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}\mathbb{I} \tag{202}$$

Notice that $\gamma_0^2 = \mathbb{I}$ and $\gamma_k^2 = -\mathbb{I}$, for $k \in \{1, 2, ..., d-1\}$. Therefore, the eigenvalues of γ_0 are 1 and -1, and it can be chosen to be hermitian, while γ_k are antihermitian; in this way we have $\gamma_0^\dagger = \gamma_0$ and $\gamma_k^\dagger = \gamma_0\gamma_k\gamma_0$, so that they are unitary matrices. A representation then is constructed using the generators defined as below:

$$\mathcal{J}^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$$

In fact:

$$\begin{aligned}
[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] &= \frac{i}{2}\frac{i}{2}\left[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu, \gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho\right] \\
&= \frac{i}{2}\frac{i}{2}\left\{[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] - [\gamma^\nu\gamma^\mu, \gamma^\rho\gamma^\sigma] - [\gamma^\mu\gamma^\nu, \gamma^\sigma\gamma^\rho] + [\gamma^\nu\gamma^\mu, \gamma^\sigma\gamma^\rho]\right\} \\
&= \frac{i}{2}(\gamma^\mu\mathcal{J}^{\nu\rho}\gamma^\sigma + \mathcal{J}^{\mu\rho}\gamma^\nu\gamma^\sigma + \mathcal{J}^{\rho\mu}\gamma^\nu\gamma^\sigma + \gamma^\rho\mathcal{J}^{\mu\sigma}\gamma^\nu \\
&- \gamma^\nu\mathcal{J}^{\mu\rho}\gamma^\sigma - \mathcal{J}^{\nu\rho}\gamma^\mu\gamma^\sigma - \mathcal{J}^{\rho\nu}\gamma^\mu\gamma^\sigma - \gamma^\rho\mathcal{J}^{\nu\sigma}\gamma^\mu \\
&- \gamma^\mu\mathcal{J}^{\nu\sigma}\gamma^\rho - \mathcal{J}^{\mu\sigma}\gamma^\nu\gamma^\rho - \mathcal{J}^{\sigma\mu}\gamma^\nu\gamma^\rho - \gamma^\sigma\mathcal{J}^{\mu\rho}\gamma^\nu \\
&+ \gamma^\nu\mathcal{J}^{\mu\sigma}\gamma^\rho + \mathcal{J}^{\nu\sigma}\gamma^\mu\gamma^\rho + \mathcal{J}^{\sigma\nu}\gamma^\mu\gamma^\rho + \gamma^\sigma\mathcal{J}^{\nu\rho}\gamma^\mu) \\
&= \frac{i}{2}(\gamma^\mu\mathcal{J}^{\nu\rho}\gamma^\sigma + \gamma^\rho\mathcal{J}^{\mu\sigma}\gamma^\nu - \gamma^\nu\mathcal{J}^{\mu\rho}\gamma^\sigma - \gamma^\rho\mathcal{J}^{\nu\sigma}\gamma^\mu \\
&- \gamma^\mu\mathcal{J}^{\nu\sigma}\gamma^\rho - \gamma^\sigma\mathcal{J}^{\mu\rho}\gamma^\nu + \gamma^\nu\mathcal{J}^{\mu\sigma}\gamma^\rho + \gamma^\sigma\mathcal{J}^{\nu\rho}\gamma^\mu) \\
&= i(-\eta^{\mu\sigma}\mathcal{J}^{\nu\rho} - \eta^{\nu\rho}\mathcal{J}^{\mu\sigma} + \eta^{\nu\sigma}\mathcal{J}^{\mu\rho} + \eta^{\rho\mu}\mathcal{J}^{\nu\sigma})
\end{aligned}$$

where it was used the identity $[AB, CD] = A[B, C]D + [A, C]BD + CA[B, D] + C[A, D]B$ to write the third equality and the antisymmetry of the generators to write the fourth equality. The final result follows from Clifford algebra. Left and right-handed Weyl-spinors are then defined by means of the chirality $\gamma_{d+1} \equiv i\gamma_0\gamma_1...\gamma_{d-1}$, which can be verified to satisfy:

$$\{\gamma_{d+1}, \gamma_\mu\} = 0 \quad , \quad \gamma_{d+1}^2 = \mathbb{I} \quad \text{and} \quad \gamma_{d+1}^\dagger = \gamma_{d+1}$$

Multiplying the first equation above by $\gamma_\mu^\dagger$ allows us to show that γ_{d+1} is traceless, in view of unitarity of the gammas. Then, the second relation tells us that the eigenvalues of it are 1 and -1 and, since d is even, we conclude that we can made $\mathcal{J}^{\mu\nu}$ block-diagonal, in which one block corresponds to the eigenvalue -1 of γ_{d+1} and the other to 1. To these two ortoghonal subspaces we associate the left and right-handed Weyl-spinors, respectively.

Four-dimensional spacetime case is of particular interest. A more practical toolkit can be obtained for it from this generalized formalism above in comparison with the pedagocial approach adopted initially. By means of the definitions $\sigma^\mu \equiv (\mathbb{I}, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu \equiv (\mathbb{I}, -\boldsymbol{\sigma})$, where $\boldsymbol{\sigma}$ are the Pauli matrices, taking the generators as following is a traditional way of separating that subspaces and define $4d$ Weyl-spinors explicitly:

$$\mathcal{J}^{\mu\nu} = \begin{bmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{bmatrix}$$

where the elements are 2×2 matrices and we defined $\sigma^{\mu\nu} \equiv i/4(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$ and $\bar{\sigma}^{\mu\nu} \equiv i/4(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$. Notice this is in agreement with what was done at the beginning, it is just a question of redefinitions; so it automatically satisfies the algebra mentioned.

Weyl-spinors are then 2-components objects (from now on undotted indices like α for left-ones and dotted indices like $\dot{\alpha}$ for right-ones) that, under Lorentz transformations, change as follows:

$$\begin{aligned} \psi_\alpha &\to (\Lambda_L)_\alpha^{\ \beta}\psi_\beta \ \text{ with } \ \Lambda_L \equiv \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right) \\ \psi_{\dot{\alpha}} &\to (\Lambda_R)_{\dot{\alpha}}^{\ \dot{\beta}}\psi_{\dot{\beta}} \ \text{ with } \ \Lambda_R \equiv \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right) \end{aligned} \tag{203}$$

Now, since we are interested in theory of fields, we make use of the ideas just developed to define Weyl-spinor fields. These fields are operators that, under a general Lorentz transformation, behave like:

$$\begin{aligned} \psi_L(x) &\to \psi'_L(x') \equiv \Lambda_L\psi_L(x) \\ \psi_R(x) &\to \psi'_R(x') \equiv \Lambda_R\psi_R(x) \end{aligned}$$

Working infinitesimally we are able to find the generators of Lorentz symmetries in this representation. For left-handed spinors for example:

$$\begin{aligned}\delta\psi_L(x) &\equiv \psi'_L(x) - \psi_L(x) = \psi'_L(x' - \delta x) - \psi_L(x) \\ &= \psi'_L(x') - \psi_L(x) - \delta x^\rho \partial_\rho \psi_L(x)\end{aligned}$$

Noting that the contribution coming from the last term above is the symmetry effect on the coordinates only, that is, $-\frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu} = \omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)$, we can write:

$$\begin{aligned}\mathcal{J}^{\mu\nu} &= \mathcal{M}^{\mu\nu} + \sigma^{\mu\nu} \\ (\mathcal{J}^{\mu\nu})_\alpha^{\ \beta}\,\psi_\beta(x) &= \omega_{\mu\nu}\left[\delta_\alpha^\beta(x^\mu\partial^\nu - x^\nu\partial^\mu) - i/2\sigma^{\mu\nu}\right]_\alpha^{\ \beta}\psi_\beta(x)\end{aligned}$$

We see then that $\sigma^{\mu\nu}$ and $\bar{\sigma}^{\mu\nu}$ are just the spin part of the generators. General irreducible representations of Lorentz group can be constructed from this one by means of tensorial products as we saw. The inclusion of translations in order to have field representations of the Poincarè properly saying is straightforward and follows (201). Therefore, the representation explored here allows us to construct any Poincarè invariant Lagrangians.

B General Relativity toolkit

This appendix is intended to be a brief exposition of the essential tools in General Relativity which turn out to be relevant in CFTs. It is based on the fantastic set of lectures given by Fredric Schuller [77]. For more details we refer the reader to that playlist, here the presentation is kept simple, without rigorous definitions and proves.

General Relativity relies on the concept of *smooth manifolds*. Roughly speaking, a manifold is a topological space[31] that can be *charted* in an atlas; if the chart is contained in $\mathbb{R}^d$, we say we have a d-dimensional topological manifold. The importance of such atlas is to translate real object properties onto a representative and tractable way. Smoothness, for example, enters in the game once a smooth-compatible atlas is considered, where compatibility is a technical term which should be understood as a shared characteristic by any two charts through a transition map.

Naturally then, vector spaces emerge in the discussion and the description in terms of tensors become indispensable. From a vector space $(V, +, \cdot)$ we define its dual vector space $(V^*, \oplus, \odot)$ of linear maps from V to $\mathbb{R}$. A (r, s)-tensor T over V thus can be defined as a multi-linear map:

$$T : V^* \times ... \times V^* \times V \times ... \times V \to \mathbb{R}\ , \tag{204}$$

for r V^* factors and s V factors, r and s in $\mathbb{N}_0$. And the components of a tensor can also

[31] A set equipped with a topology.

be defined in terms of the basis elements $\{\epsilon^{\mathrm{i}_r}\}$ of V^* and $\{\epsilon_{j_s}\}$ of V:

$$T^{\mathrm{i}_1,\dots,\mathrm{i}_r}{}_{j_1,\dots,j_s} \equiv T(\epsilon^{\mathrm{i}_1},\dots,\epsilon^{\mathrm{i}_r},\epsilon_{j_1},\dots,\epsilon_{j_s}) \ .$$

In particular, the so called tangent spaces play the central role. They can be constructed by means of the concept of velocity of a smooth curve $\gamma : \mathbb{R} \to \mathcal{M}$ at a point p in the smooth manifold $\mathcal{M}$. The velocity of γ at p is the linear map:

$$v_{\gamma,p} : C^\infty(\mathcal{M}) \to \mathbb{R} \ ,$$

such that

$$f \longmapsto v_{\gamma,p}(f) \equiv (f \circ \gamma)'(\lambda_0) \ ,$$

where $C^\infty(\mathcal{M})$ stands for the vector space of smooth maps f and $\gamma(\lambda_0) = p$. Intuitively:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend right, "f \circ \gamma"'] & \mathcal{M} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

For each point $p \in \mathcal{M}$ we define then the set $T_p\mathcal{M} \equiv \{v_{\gamma,p}/\gamma \text{ smooth curves}\}$, the vector space tangent to $\mathcal{M}$ at p. A natural basis for it is the induced one from a chart (U, x) in the atlas, U an element of the topology:

$$\epsilon_{\mathrm{i}} = \left(\frac{\partial}{\partial x^{\mathrm{i}}}\right)_p \ .$$

For example, taking $\lambda_0 = 0$, the velocity can be written in terms of them like:

$$v_{\gamma,p}(f) = \dot{\gamma}_x(0) \left(\frac{\partial f}{\partial x^{\mathrm{i}}}\right)_p \quad \forall f \in C^\infty(\mathcal{M}) \ .$$

For a given $X \in T_p\mathcal{M}$, its components change under a change of charts $(U, x) \to (V, y)$ as:

$$X^j_{(y)} = \left(\frac{\partial y^j}{\partial x^{\mathrm{i}}}\right)_p X^{\mathrm{i}}_{(x)} \ .$$

We also have the definition of the cotangent space $(T_p\mathcal{M})^*$ of linear maps from $T_p\mathcal{M}$ to $\mathbb{R}$. A typical element of it is the gradient of f at p, $(df)_p$:

$$(df)_p(X) \equiv Xf \ ,$$

where $X \in T_p\mathcal{M}$. Naturally, the set $\{(dx^{\mathrm{i}})_p\}$ form a basis for $(T_p\mathcal{M})^*$, and a covector ω

in that space change components according to:

$$\omega_{(y),\mathrm{i}} = \left(\frac{\partial x^j}{\partial y^{\mathrm{i}}}\right)_p \omega_{(x),j} \ .$$

More generally, we also define the *total* tangent space to $\mathcal{M}$, $T\mathcal{M}$, and the *total* cotangent space associated as:

$$T\mathcal{M} \equiv \dot{\bigcup_{p\in\mathcal{M}}} T_p\mathcal{M} \ \text{ and } \ T^*\mathcal{M} \equiv \dot{\bigcup_{p\in\mathcal{M}}} (Tp\mathcal{M})^* \ .$$

They are smooth-manifolds, but the construction of the atlas is omitted.

The point in making such definitions is that now we can finally introduce the concept of a *bundle*, which supports the definition of quantities of physical interest, like fields, and also allows the connection between two smooth manifolds to be made more clearly. Essentially, a bundle is a triple $E \xrightarrow{\pi} \mathcal{M}$ of a smooth manifold E which we call *total space*, a smooth surjective map π known by *projection map* and another smooth manifold $\mathcal{M}$ which we use as *base space*; for example, $E = \text{cylinder}, \mathcal{M} = \text{circle}, \pi : E \to \mathcal{M}$. Applied to our context, by making use of the surjective map $\pi : T\mathcal{M} \to \mathcal{M}$, with $X \mapsto p$ the unique point $p \in \mathcal{M}$ such that $X \in T_p\mathcal{M}$, we construct the *tangent bundle* composed by $T\mathcal{M}$ as total space and $\mathcal{M}$ as base space.

To see the link between two different manifolds, which is of great usefulness in this work and in succeeding studies in General Relativity, we consider two smooth manifolds $\mathcal{M}$ and $\mathcal{N}$ and use a smooth function ϕ to inject the further into the later:

$$\mathcal{M} \xrightarrow[\text{injective}]{\phi} \mathcal{N} \ . \tag{205}$$

By making use then of the following directive diagram

$$\begin{tikzcd} T\mathcal{M} \arrow[r, "\phi_*"] \arrow[d, "\pi_{TM}"'] & T\mathcal{N} \arrow[d, "\pi_{TN}"] & \\ \mathcal{M} \arrow[r, "\phi"] & \mathcal{N} \arrow[r, "f"] & \mathbb{R} \end{tikzcd} \ ,$$

where π's stand for surjectives maps, we define two important maps: the *push-forward* ϕ_* and the *pull-back* ϕ^*. The first of them is defined by $X \mapsto \phi_*(X)$, with $\phi_*(X)f \equiv X(f\circ\phi)$, for any $f \in C^\infty(\mathcal{N})$; note that $\phi_*(T_p\mathcal{M}) \subseteq T\phi(p)\mathcal{N}$, so "vectors are pushed forward". The second is in analogy given by $\phi^* : T^*\mathcal{N} \to T^*\mathcal{M}$, with $\omega \mapsto \phi^*(\omega)(X) \equiv \omega\left(\phi_*(X)\right)$; now "covectors are pulled back". In components the following holds: $\phi^*_{\mathrm{i}} = \phi_{*,\mathrm{i}}$.

These new maps can the be used for obtaining tensors in one manifold from tensor in the other. A $(0,s)$-tensor in $\mathcal{M}$, for example, T_{Ms}, follows from an also $(0,s)$-tensor in

$\mathcal{N}$, T_{Ns}, through:

$$T_{Ms}(X_1, ..., X_s) \equiv T_{Ns}(\phi_*(X_1), ..., \phi_*(X_s)) \ , \tag{206}$$

which, by considering charts (U, x) of $\mathcal{M}$ and (V, y) of $\mathcal{N}$, in components reads:

$$\boxed{(T_M)_{\mathrm{i}_1,...,\mathrm{i}_s;p} = (T_N)_{a_1,...,a_s;\phi(p)} \left(\frac{\partial\hat{\phi}^{a_1}}{\partial x^{\mathrm{i}_1}}\right)_p ... \left(\frac{\partial\hat{\phi}^{a_s}}{\partial x^{\mathrm{i}_s}}\right)_p \ , \quad \text{with} \quad \hat{\phi}^a = (y \circ \phi)^a} \ . \tag{207}$$

The partial derivatives above form an induced basis of $T_p\mathcal{M}$ from the chart (U, x).

Finally, for completeness, we point out how quantities appearing in the theory developed in this work do situate in this context. A vector field in this language is just a smooth map χ satisfying $\pi \circ \chi = \mathbb{I}_{\mathcal{M}}$ in the diagram below: Tensor fields then emerge by

$$\begin{tikzcd} T\mathcal{M} \arrow[r, "\pi"] & \mathcal{M} \arrow[l, bend left, "\chi"] \end{tikzcd} \quad .$$

expanding this idea by means of a new structure called C^∞*-module* $\Gamma(T\mathcal{M})$, which we do not explore here.

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