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# Não Markovianidade em modelos colisionais Gaussianos

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# Non-Markovianity in Gaussian Collisional Models

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*Ernest Hemingway once wrote: "The world is a fine place and worth fighting for." I agree with the second part.*

— Se7en (1995)

*A mi abuelo Arnulfo Ramirez.*

# *Abstract*

Memory effects in the dynamics of open systems have been the subject of significant interest in the last decades. The methods involved in quantifying this effect, however, are often difficult to compute and may lack analytical insight. With this in mind, we study collisional models where non-Markovianity is introduced by means of additional interactions between neighboring environmental units. By focusing on continuous-variables Gaussian dynamics, we are able to analytically study models of arbitrary size. We show that the dynamics can be cast in terms of a Markovian Embedding of the covariance matrix, which yields closed form expressions for the memory kernel that governs the dynamics, a quantity that can seldom be computed analytically. The same is also possible for a divisibility monotone, based on the complete positivity of intermediate maps. We analyze in detail two types of interactions between ancillas, a beam-splitter implementing a partial SWAP and a two-mode squeezing, which entangles them and, at the same time, feeds excitations into the system. By analyzing the memory kernel and divisibility for these two representative scenarios, our results help to shed light on the intricate mechanisms behind memory effects in the quantum domain.

**Keywords:** Open Quantum Systems; Collisional Model; Non-Markovianity; Memory Kernel; Map Divisibility.

## *Resumo*

Os efeitos da memória na dinâmica dos sistemas abertos têm sido objeto de grande interesse nas últimas décadas. Os métodos envolvidos na quantificação desse efeito, no entanto, muitas vezes são difíceis de calcular e podem carecer de uma visão analítica. Com isso em mente, estudamos modelos colisionais em que a não markovianidade é introduzida por meio de interações adicionais entre unidades ambientais vizinhas. Mostramos que a dinâmica pode ser moldada em termos de um Embedding Markoviano da matriz de covariância, que produz expressões de forma fechada para o kernel de memória que governa a dinâmica, uma quantidade que raramente pode ser calculada analiticamente. O mesmo também é possível para uma divisibilidade monótona, baseada na positividade completa dos mapas intermediários. Ao focarmos na dinâmica gaussiana de variáveis contínuas, podemos estudar analiticamente modelos de tamanho arbitrário. Analisamos em detalhes dois tipos de interações, um beam splitter implementando um SWAP parcial e uma compressão de dois modos, que emaranha as ancillas e, ao mesmo tempo, alimenta excitações no sistema. Ao analisar o kernel de memória e a divisibilidade para esses dois cenários representativos, nossos resultados ajudam a lançar luz sobre os intrincados mecanismos por trás dos efeitos da memória no domínio quântico.

**Keywords:** Sistemas quânticos abertos; Modelos colisionais; Não-Markovianidade; Kernels de memória; CP-Divisibilidade.

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# Chapter 1

## Introduction

The theory of open quantum systems is the backbone of modern research in quantum mechanics and its applications. Some examples include the decay of unstable states in nuclei, transport of electrons through quantum dots, molecular networks, nanophotonic structures, among others [1–4]. This is because, in real life phenomena, the systems are never isolated, but are constantly interacting with its surrounding environment. In some cases, when the effects of the environment are negligible, the closed system description is accurate, described by the well-known Liouvillian equation. Nevertheless, more often than not, it is not possible. Obtaining the most general microscopic description of the open system dynamics is a very difficult problem, one that is yet to be solved analytically.

One instance we can get a microscopic derivation is when we assume that the system-environment interaction is weak but also that the bath correlation functions decay quickly. This is known as the Born-Markov approximation [5–7]. Typically, when the system interacts with the environment, information about the former is transferred to the latter. When the environment is very large and complex, this information may never return. In this case the dynamics is called Markovian. This is characterized by the Lindblad equation given by [8]:

$$\frac{d}{dt}\rho_S = -\frac{i}{\hbar}[H, \rho_S] + \sum_k g_k \left( L_k \rho_S L_k^\dagger - \frac{1}{2} \{ L_k^\dagger L_k, \rho_S \} \right), \quad (1.1)$$

where  $\rho_S$  is the system's density matrix,  $H$  is the Hamiltonian, and  $L_k$  are arbitrary operators. In general, however, there may be a partial backflow of information which charac-

terizes the non-Markovian evolution [9]. The dynamics of the open quantum systems in the non-Markovian regime can generally be written as [10, 11]:

$$\frac{d\rho_S}{dt} = -\frac{i}{\hbar}[H_S, \rho_S] + \int_0^t \mathcal{K}_{t-t'}[\rho_S(t')] dt', \quad (1.2)$$

where  $\mathcal{K}_{t-t'}$ , called the memory kernel (MK), is a linear superoperator condensing all the information on how the evolution of  $\rho_S$  at time  $t$  depends on its past values. We see that the Markovian case (1.1) is recovered when  $\mathcal{K}_{t-t'} \propto \delta(t - t')$ . The MK has been intensively studied in recent years, as it provides clear insights onto the inner workings of non-Markovianity [12–17]. From the point of view of causality, this backflow quantifies the ability of the dynamics to communicate past information to the future [18]. Non-Markovianity therefore touches at the core of information processing where the notion of information flow is fundamental.

Considerable attention was given in recent years on how to characterize and quantify non-Markovianity in the quantum domain [19, 20]. Due to the richness involved, however, there is no single approach capable of capturing its full essence. The most important notion is that of map divisibility: non-Markovianity requires that the underlying dynamical map should not be CP-divisible [21, 22]. The notion of information flow, on the other hand, relies on information-theoretic quantifiers and is thus not uniquely defined. The most widely used measures involve the trace distance [21–24] between different initial states or entanglement [25] between the system and an ancilla. Several other quantifiers have also been explored [26–34]. However, analyzing non-Markovianity for general environments is in general an extremely difficult task. First, the calculations quickly become impractical when the size of the bath is large. And second, realistic baths often have many additional features which tend to mask the effects one is interested in. This motivates the search for controllable models, where the degree of non-Markovianity can be finely tuned.

One way to accomplish this, which has seen an enormous surge in popularity in recent years, are through the so-called collisional models [35–46]. Usually, a common configuration to study non-Markovianity is to consider the system  $S$  coupled to  $N$  environmental units  $E_n$  all at the same time (e.g. the Caldeira-Legett model [47]). In the collisional models setup, instead, only a small part of the environment is interacting with the sys-

tem at each instant of time. Afterwards, this part leaves and returns to the bath where it will collide with the other bath units and thermalize, forgetting all information about the system. We model this by replacing the open dynamics of a system by a series of sequential interactions between the system  $S$  and small environmental units  $E_1, E_2, E_3 \dots$  (henceforth referred to as ancillas). All ancillas are prepared in the same state and each interaction only lasts for a fixed time, after which they never interact again. This therefore leads to a stroboscopic dynamics for the system.

Although seemingly artificial, collisional models actually faithfully describe some experimentally relevant situations, for instance when a system is coupled to a 1D waveguide [48–51]. The dispersion relation of 1D waveguides allows one to discretize the field operator into time bins, so that the interaction at each time interval only involves one bin operator. The picture that emerges is then exactly that of a collisional model. Moreover, this model is in general non-Markovian by construction, which depends on the input state of the electromagnetic field, as well as on the nature of the interaction. The specific conditions determining whether the ensuing dynamics will be Markovian or not are discussed in detail in a recent review on the subject [52].

The advantage of collisional models is that non-Markovianity can be introduced in a fully controllable manner. There are two main ways to do so. The first is to consider that the ancillas already start correlated [53–57]. The other one is to assume information is transmitted between the ancillas during the process [58–66]. In this dissertation, we shall focus on the second case. That is, we consider a scenario where neighboring ancillas  $E_n E_{n+1}$  interact with each other in between the interactions  $S E_n$  and  $S E_{n+1}$ . (see Fig. 1.1(a)). This additional interaction *signals* information from the past to the future, so that when the  $S E_{n+1}$  interaction arrives, the ancilla  $E_{n+1}$  will already contain some information about the system.

In particular, we focus on continuous-variable collisional models, undergoing Gaussian-preserving dynamics [67–75]. The advantages that come with the Gaussian toolbox allows us to construct a complete framework for the study of non-Markovianity, which: (i) encompass a broad range of scenarios; (ii) can be easily use to compute the mutual information; (iii) allows for the explicit construction and computation of the memory kernel and (iv) provides access to a CP-divisibility monotone, which can be directly compared

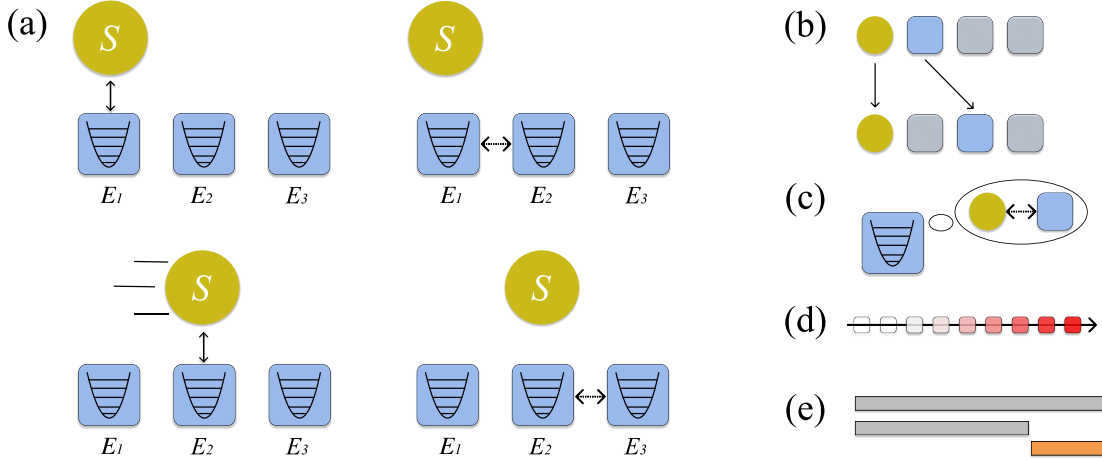


Figure 1.1: Non-Markovian collisional models. (a) First few steps of the dynamics. The system-ancilla interactions  $SE_n$  are interspersed by ancilla-ancilla interactions  $E_n E_{n+1}$ , which propagate information to the future, making the dynamics non-Markovian in a fully controllable way. (b) Basic structure of the Markovian embedding dynamics, which is a map from the Hilbert space of  $SE_n$  to that of  $SE_{n+1}$ . (c) Mutual information. Previous interactions  $SE_n$  and  $E_n E_{n+1}$  correlate the system  $S$  and the ancilla  $E_{n+1}$  even before explicitly interacting. (d) The memory kernel quantifies how different instants of the past affect the evolution at present times. (e) CP-divisibility. The maps in gray, from time 0 to  $t_n$  or  $t_m$  are, by construction, CPTP. But the intermediate map from  $t_n$  to  $t_m > t_n$  may not necessarily be.

with the memory kernel. The framework is also amenable to analytical calculations and extremely efficient from a numerical perspective. Thus, despite being restricted to Gaussian interactions, it offers multiple advantages over more general maps. We also provide a complete numerical library for efficiently simulating Gaussian collisional models in Python [76]. All plots were generated with this code.

The dissertation is divided as follows. We begin in Chapter 2 by introducing important concepts of open quantum systems and review the different methods to describe the open system's dynamics. Particularly, we consider a qubit interacting with a bosonic environment to exemplify the approximations needed to obtain a microscopic derivation. Next, we study the continuous variable formalism in Chapter 3. Specifically, we examine Gaussian states and maps as they will become cornerstone for this work. The following chapters are the most important of the dissertation. They represent the original contribution from the authors to the field of non-Markovianity in collisional models. The general framework is developed in Chapter 4. We show that the full non-Markovian dynamics

can be cast as a Markovian embedding, involving a Markovian map at a higher dimension (Fig. 1.1(b)). We then specialize this to the case of Gaussian models, where the embedding is written as a set of matrix-difference equations with clear physical interpretation. Throughout this chapter our exposition will be example-oriented, with a focus on two specific types of interactions. The framework, however, is general and we will specify, in each part, how to properly make this generalization. Following, in Chapter 5, we review non-Markovianity in the quantum domain as well as the quantifiers we will use for our collisional model. Armed with all these results, in Chapter 6 we provide a full characterization of the memory effects using the mutual information (Fig. 1.1(c)), the memory kernel (Fig. 1.1(d)) and the map divisibility (Fig. 1.1(e)). Finally, in Chapter 7 we draw our conclusions and our perspectives on future works which can be done using this robust framework. Throughout the dissertation we use natural units  $\hbar = k_B = 1$ .



# Chapter 2

## Open Quantum Systems

### 2.1 States and Dynamics

Quantum mechanics is the fundamental theory that describes the behavior of nature at the micro scale [77–82]. However, it is often assumed that the quantum system is isolated, overlooking the effects of the interaction with its surroundings. Such effects can lead to dramatic changes in the system’s dynamics such as the dissipation of the quantum correlations, making them necessary for an accurate depiction of real phenomena. The field that studies this kind of phenomenon is called open quantum systems [5–7].

The description of open quantum systems starts by defining what the object of study is: the states. In quantum mechanics, we were interested in kets  $|\psi_i\rangle$  which live in a Hilbert space  $\mathcal{H}$ . Analogously, open quantum systems is interested in the density matrix  $\rho = \sum c_i |\psi_i\rangle\langle\psi_i|$  which lives in  $\mathcal{L}(\mathcal{H})$ , the state space of linear operators from  $\mathcal{H}$  to  $\mathcal{H}$ . This density matrix  $\rho$  satisfies the following properties:

$$\text{Hermicity: } \rho = \rho^\dagger,$$

$$\text{Positivity: } \rho \geq 0,$$

$$\text{Normalization: } \text{Tr}(\rho) = 1.$$

Besides these properties, we also want to know how a state  $\rho$  evolves in time. For that purpose, we look at the evolution of isolated systems which evolve through Schrödinger’s

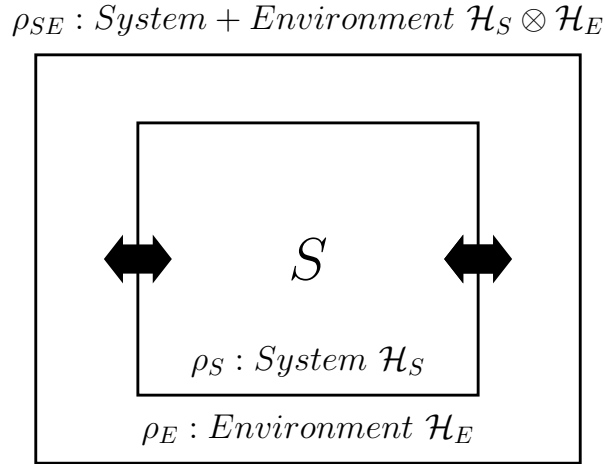


Figure 2.1: Schematics of an open quantum system. The system  $S$  is interacting with the environment while the whole bipartite system  $\rho_{SE}$  is considered as closed.

equation:

$$\partial_t |\psi(t)\rangle = -iH(t) |\psi(t)\rangle. \quad (2.1)$$

The analogue evolution in the density matrix formalism is known as von Neumann's equation:

$$\frac{d\rho(t)}{dt} = -i[H(t), \rho(t)]. \quad (2.2)$$

This result holds only for closed systems. The next step is to obtain a description of the open system's dynamics. One way to achieve this is to consider a closed bipartite system  $\rho_{SE}$  which contemplates both the system  $S$  and an environment  $E$ , as a closed system evolving under von Neumann's equation, as depicted in Fig. 2.1. Tracing out the environment  $E$  will then lead to an effective description of the system's density matrix  $\rho_S$  dynamics. Unfortunately, this is a complicated task, often relying on multiple approximation schemes to obtain analytical results. We study this procedure in detail in the next sections.

## 2.2 Memory Kernel

We follow the seminal works of Nakajima and Zwanzig throughout this subsection [10, 11]. We consider the complete density matrix  $\rho_{SE}$  of a system  $S$  plus an environment

$E$  initially uncorrelated  $\rho_{SE}(t_0) = \rho_S(t_0) \otimes \rho_E(t_0)$  evolving under a time-independent Hamiltonian  $H_{SE} = H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + V$ :

$$\frac{d\rho_{SE}(t)}{dt} = -i[H_{SE}, \rho_{SE}(t)]. \quad (2.3)$$

It is convenient to formulate (2.3) in the interaction picture  $\rho_{SE}^I = e^{iH_0 t} \rho_{SE} e^{-iH_0 t}$  with respect to  $H_0 = H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E$ :

$$\frac{d\rho_{SE}^I(t)}{dt} = -i[V_I(t), \rho_{SE}^I(t)] := \mathcal{L}_t \rho_{SE}^I(t), \quad V_I = e^{iH_0 t} V e^{-iH_0 t}, \quad (2.4)$$

where we defined the superoperator  $\mathcal{L}_t$ . From here on, we will loose the superscript  $I$  in  $\rho_{SE}^I$  for notational convenience.

Our goal is to obtain the master equation for the system  $\rho_S(t) = \text{Tr}_E(\rho_{SE}(t))$ . To that end, we define a pair of orthogonal projection operators  $\mathcal{P}$  and  $\mathcal{Q}$

$$\mathcal{P}\rho_{SE} = \text{Tr}_E(\rho_{SE}) \otimes \rho_E^*, \quad \mathcal{Q}\rho_{SE} = \rho_{SE} - \text{Tr}_E(\rho_{SE}) \otimes \rho_E^*, \quad (2.5)$$

where  $\rho_E^*$  is an arbitrary environmental density matrix, not necessarily related to the state of the environment. Being orthogonal projection operators, they satisfy  $\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = 0$  and  $\mathcal{P} + \mathcal{Q} = \mathcal{I}$  which follow from the definition. Applying  $\mathcal{P}$  to Eq. (2.4) we obtain

$$\frac{d}{dt} \mathcal{P}\rho_{SE}(t) = \mathcal{P}\mathcal{L}_t \rho_{SE}(t). \quad (2.6)$$

We can then introduce the identity  $\mathcal{I} = \mathcal{P} + \mathcal{Q}$  to get a differential equation for  $\mathcal{P}\rho_{SE}(t)$

$$\frac{d}{dt} \mathcal{P}\rho_{SE}(t) = \mathcal{P}\mathcal{L}_t(\mathcal{P} + \mathcal{Q})\rho_{SE}(t) = \mathcal{P}\mathcal{L}_t\mathcal{P}\rho_{SE}(t) + \mathcal{P}\mathcal{L}_t\mathcal{Q}\rho_{SE}(t). \quad (2.7)$$

Similarly, we can obtain a differential equation for  $\mathcal{Q}\rho_{SE}(t)$  applying  $\mathcal{Q}$  to Eq. (2.3)

$$\frac{d}{dt} \mathcal{Q}\rho_{SE}(t) = \mathcal{Q}\mathcal{L}_t\mathcal{Q}\rho_{SE}(t) + \mathcal{Q}\mathcal{L}_t\mathcal{P}\rho_{SE}(t). \quad (2.8)$$

We now have two coupled differential equations. We solve Eq. (2.8) for  $\mathcal{Q}\rho_{SE}(t)$ ,

$$\mathcal{Q}\rho_{SE}(t) = G(t, t_0)\mathcal{Q}\rho_{SE}(t_0) + \int_{t_0}^t dt' G(t, t')\mathcal{Q}\mathcal{L}_{t'}\mathcal{P}\rho_{SE}(t'), \quad (2.9)$$

where  $G(t, t')$  is the Green's function  $G(t, t_0) = \mathcal{T} \exp(\int_{t_0}^t dt' \mathcal{Q}\mathcal{L}_{t'})$ . Inserting (2.9) in (2.7) leads to the Nakajima-Zwanzig master equation

$$\frac{d}{dt}\mathcal{P}\rho_{SE}(t) = \mathcal{P}\mathcal{L}_t(\mathcal{P}\rho_{SE}(t) + G(t, t_0)\mathcal{Q}\rho_{SE}(t_0) + \int_{t_0}^t dt' G(t, t')\mathcal{Q}\mathcal{L}_{t'}\mathcal{P}\rho_{SE}(t')). \quad (2.10)$$

This is a reduced equation for  $\mathcal{P}\rho_{SE}(t)$  after integrating out the environment. Most importantly, this equation is exact. We can further reduce this expression by fixing the arbitrary  $\rho_E^*$  to be the environment's initial state  $\rho_E(t_0)$ . This selection implies that  $\mathcal{Q}\rho_{SE}(t_0) = 0$ , leading to

$$\frac{d}{dt}\rho_S(t) \otimes \rho_E(t_0) = \mathcal{P}\mathcal{L}_t\mathcal{P}\rho_{SE}(t) + \int_{t_0}^t dt' \text{Tr}_E(\mathcal{L}_t G(t, t')\mathcal{Q}\mathcal{L}_{t'}(\rho_S(t') \otimes \rho_E(t_0))) \otimes \rho_E(t_0). \quad (2.11)$$

Finally, if  $[\rho_E(t_0), H_E] = 0$  we can redefine the interaction such that  $\mathcal{P}\mathcal{L}_t\mathcal{P}\rho_{SE}(t) = 0$ , thus arriving to:

$$\frac{d}{dt}\rho_S(t) = \int_{t_0}^t dt' K_{t,t'}\rho_S(t'), \quad K_{t,t'} = \text{Tr}_E(\mathcal{L}_t G(t, t')\mathcal{Q}\mathcal{L}_{t'}(\dots \otimes \rho_E(t_0))), \quad (2.12)$$

where  $K_{t,t'}$  is the memory kernel, a linear superoperator condensing all the information on how the evolution of  $\rho_S(t)$  depends on its past values.

### 2.3 Microscopic derivation:

Eq. (2.12) still involves a superoperator, which is quite generally difficult to compute analytically. Approximations are then needed to obtain a more manageable equation, one which will be more useful under certain conditions. We start by re-scaling the interaction term  $V_I$  to  $\alpha V_I$  by a dimensionless small parameter  $\alpha$ . This assumption is usually called

the Born approximation. Expanding the memory kernel to second order, we get

$$\mathcal{K}_{t,t'} = \alpha^2 \text{Tr}_E (\mathcal{L}_t \mathcal{Q} \mathcal{L}_{t'} \mathcal{P}) + \mathcal{O}(\alpha^3), \quad (2.13)$$

which leads to an equation of motion of second order for  $\rho_S(t)$

$$\frac{d}{dt} \rho_S(t) = \alpha^2 \int_{t_0}^t dt' \text{Tr}_E (\mathcal{L}_t \mathcal{L}_{t'} \rho_S(t') \otimes \rho_E(t_0)). \quad (2.14)$$

Putting the explicit expressions for the superoperator  $\mathcal{L}_t$  we get

$$\frac{d}{dt} \rho_S(t) = - \int_{t_0}^t dt' \text{Tr}_E [V_I(t), [V_I(t'), \rho_S(t') \otimes \rho_E(t_0)]]. \quad (2.15)$$

Next, we proceed with the Markovian approximation. We assume that the environment's correlations decays much faster than the system's so that the state of the system at time  $t$  only depends on the present state  $\rho_S(t)$ . To that end, we make the previous equation time local by replacing  $\rho_S(t')$  by  $\rho_S(t)$ :

$$\frac{d}{dt} \rho_S(t) = - \int_{t_0}^t dt' \text{Tr}_E [V_I(t), [V_I(t'), \rho_S(t) \otimes \rho_E(t_0)]]. \quad (2.16)$$

This is called the Redfield equation. Still this equation is not yet Markovian since the time evolution depends upon the initial time  $t_0$ . We can make the substitution  $t'$  by  $t - t'$  in Eq. (2.16) and integrate up to infinity. This is possible as we are assuming that the time scale over which the state of the system varies is large compared to the time scale of the decay of the environment's correlations. Without loss of generality, we can set  $t_0 = 0$  and obtain

$$\frac{d}{dt} \rho_S(t) = - \int_0^\infty dt' \text{Tr}_E [V_I(t), [V_I(t - t'), \rho_S(t) \otimes \rho_E(0)]]. \quad (2.17)$$

The result is an integro-equation for  $\rho_S(t)$  which we can calculate by plugging in the interaction potential in the interaction picture. Even so, we will often need to add more approximations to obtain a differential equation for the  $\rho_S(t)$ .

## 2.4 Qubit interacting with bosonic environment

One interesting model which simulates the effects an environment can have on the system is a two-level system interacting with a bosonic bath. For example, this describes an atom interacting with the many modes of a radiation field. The Hamiltonian of this model is:

$$H = \frac{1}{2}\omega_0\sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k g_k (\sigma_+ b_k + \sigma_- b_k^\dagger), \quad (2.18)$$

where we are modeling the environment as infinite harmonic oscillators, all with different frequencies  $\omega_k$ . The interaction between our qubit system and the environment is of the form of a typical exchange: creates an excitation on the system/bath by destroying one on the bath/system with strength  $g_k$ .

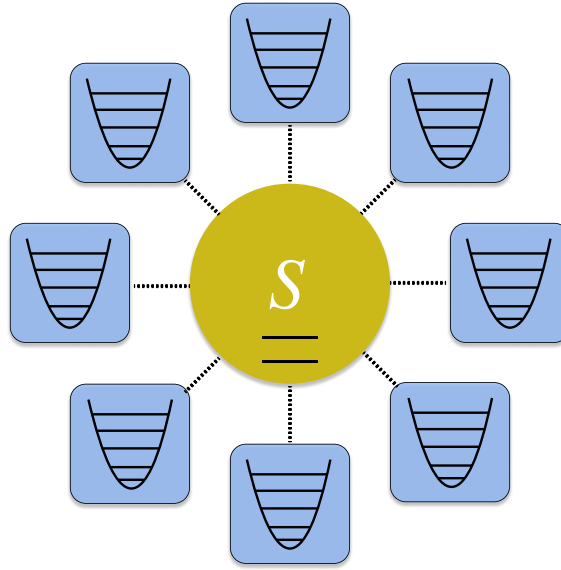


Figure 2.2: Qubit system  $S$  interchanging excitations with infinite bosonic environments. The environmental units are modeled as harmonic oscillators with frequency  $\omega_k$  interacting only with the system proportional to the coupling strength  $g_k$ .

We move to the interaction picture to get  $V_I(t)$ :

$$V_I(t) = e^{iH_0 t} V e^{-iH_0 t} = \sum_k g_k (e^{i\Delta_k t} \sigma_+ b_k + e^{-i\Delta_k t} \sigma_- b_k^\dagger), \quad (2.19)$$

where  $\Delta_k = \omega_0 - \omega_k$ , and compute the commutator term inside the trace of Eq. (2.17).

$$\frac{d}{dt}\rho_S(t) = - \int_0^\infty dt' \text{Tr}_E (V_I(t)\rho_S(t)\rho_E V_I(t-t') - \rho_S(t)\rho_E V_I(t-t')V(t)) + h.c. \quad (2.20)$$

where  $h.c.$  stands for hermitian conjugate. We focus on the first term inside the integral  $\text{Tr}_E V_I(t)\rho_S(t)\rho_E V_I(t-t')$ :

$$\sum_{k,q} g_k g_q \left( e^{i(\Delta_k + \Delta_q)t - i\Delta_q t'} \sigma_+ \rho_S \sigma_+ \langle b_q b_k \rangle + e^{i(\Delta_k - \Delta_q)t + i\Delta_q t'} \sigma_+ \rho_S \sigma_- \langle b_q^\dagger b_k \rangle \right. \\ \left. e^{-i(\Delta_k - \Delta_q)t - i\Delta_q t'} \sigma_- \rho_S \sigma_+ \langle b_q b_k^\dagger \rangle + e^{-i(\Delta_k + \Delta_q)t + i\Delta_q t'} \sigma_- \rho_S \sigma_- \langle b_q^\dagger b_k^\dagger \rangle \right).$$

To compute the correlations, we need to know the state of the environment. We assume that the bath is in a thermal state, meaning that:

$$\langle b_q b_k \rangle = \langle b_q^\dagger b_k^\dagger \rangle = 0, \quad \langle b_q^\dagger b_k \rangle = \langle b_k b_q^\dagger \rangle - \delta_{kq} = \delta_{kq} n_{\omega_k}, \quad (2.21)$$

where  $n_{\omega_k}$  is the bosonic occupation distribution  $1/(e^{\beta\omega_k} - 1)$ . This leads to:

$$\text{Tr}_E V_I(t)\rho_S\rho_E V_I(t-t') = \sum_k g_k^2 (e^{i\Delta_k t'} n_{\omega_k} \sigma_+ \rho_S \sigma_- + e^{-i\Delta_k t'} (n_{\omega_k} + 1) \sigma_- \rho_S \sigma_+). \quad (2.22)$$

It is useful to define a quantity called the spectral density of the bath:

$$J(\omega) = 2\pi \sum_k g_k^2 \delta(\omega - \omega_k), \quad (2.23)$$

such that the first term in Eq. (2.20) now looks like:

$$\int_0^\infty dt' \int_0^\infty \frac{d\omega}{2\pi} J(\omega) (e^{i(\omega_0 - \omega)t'} n_\omega \sigma_+ \rho_S \sigma_- + e^{-i(\omega_0 - \omega)t'} (n_\omega + 1) \sigma_- \rho_S \sigma_+). \quad (2.24)$$

To compute the integral we use the known identity:

$$\int_0^\infty \frac{dt'}{2\pi} e^{i(\omega_0 - \omega)t'} = \frac{1}{2} \delta(\omega_0 - \omega) - \frac{i}{2} P \left( \frac{1}{\omega_0 - \omega} \right), \quad (2.25)$$

where  $P$  is the Cauchy principal value. This term only accounts to a rescaling of the frequency  $\omega_0$  to  $\omega_0 + \Delta$  in the unitary dynamics. This net effect is called the Lamb shift.

On the other hand, the first term will lead to new terms in the evolution of  $\rho_S$  which are of dissipative nature. Focusing on this term we then get:

$$\int_0^{\infty} \frac{d\omega}{2} J(\omega) \delta(\omega_0 - \omega) (n_{\omega} \sigma_+ \rho_S \sigma_- + (n_{\omega} + 1) \sigma_- \rho_S \sigma_+) = \frac{J(\omega_0)}{2} (n_{\omega_0} \sigma_+ \rho_S \sigma_- + (n_{\omega_0} + 1) \sigma_- \rho_S \sigma_+).$$

A similar procedure can be done to the other terms in Eq. (2.20), resulting in equal terms up to some permutations of the operators. Plugging them back, we get:

$$\frac{d}{dt} \rho_S(t) = \gamma \bar{n} (\sigma_+ \rho_S \sigma_- - \frac{1}{2} \{\sigma_- \sigma_+, \rho_S\}) + \gamma (\bar{n} + 1) (\sigma_- \rho_S \sigma_+ - \frac{1}{2} \{\sigma_+ \sigma_-, \rho_S\}). \quad (2.26)$$

Going back to the Schrödinger picture simply reintroduces the unitary contribution and we thus finally arrive at

$$\frac{d}{dt} \rho_S(t) = -i \left[ \frac{\omega_0 + \Delta}{2} \sigma_z, \rho_S(t) \right] + \mathcal{D}(\rho_S(t)), \quad (2.27)$$

where the first term is the familiar unitary term and  $\mathcal{D}(\rho_S(t))$  is the dissipative contribution found in Eq. (2.26). By comparison with the closed system's evolution, the effects of tracing out the environment results in new additional terms. Still, several approximations and assumptions were needed to arrive to this equation, most of which do not hold in real experiments.

## 2.5 Lindblad Equation

Suppose the evolution of the system is time local,

$$\frac{d}{dt} \rho_S(t) = \mathcal{R}_t \rho_S(t), \quad (2.28)$$

for some superoperator  $\mathcal{R}_t$ . We then ask: What is the general structure of  $\mathcal{R}_t$  required to ensure that starting with a density matrix  $\rho(0)$ , the evolved state  $\rho(t)$  is always a genuine density matrix for all later times  $t$ ? Let us start with a simpler question. Suppose the



solution is given by a linear map of the form:

$$\rho_S(t) = \mathcal{V}_t(\rho_S(0)), \quad (2.29)$$

for some superoperator  $\mathcal{V}_t$ . The question now becomes: What conditions shall  $\mathcal{V}_t$  fulfill in order for  $\rho_S(t)$  be a density matrix? The answer is that the map should be a quantum operation, one that is completely positive trace preserving (CPTP) [83]. This can be represented by the operator-sum representation

$$\mathcal{V}_t(\rho_S(0)) = \sum_k M_k \rho_S(0) M_k^\dagger, \quad \sum_k M_k^\dagger M_k = 1, \quad (2.30)$$

where the set of  $M_k$  are called the Kraus operators [84].

Returning now to the original question, if (2.29) is to be the solution of (2.28) then  $\mathcal{V}_t$  must satisfy the semi-group property

$$\mathcal{V}_{t_2} \mathcal{V}_{t_1} = \mathcal{V}_{t_2+t_1}. \quad (2.31)$$

The semi-group property, together with the restriction that  $\mathcal{V}_t$  must be a quantum operation, completely determines the basic structure of  $\mathcal{R}_t$ . The generator of any quantum operation satisfying the semigroup property must have the form [8]

$$\frac{d}{dt} \rho_S = -i[H, \rho_S] + \sum_k g_k \left( L_k \rho_S L_k^\dagger - \frac{1}{2} \{ L_k^\dagger L_k, \rho_S \} \right), \quad (2.32)$$

where  $H$  is an Hermitian operator,  $L_k$  are arbitrary operators and  $g_k \geq 0$ . This is called Lindblad's theorem and master equations having this structure are called Lindblad equations. We can see that our microscopic derivation of the system's density evolution Eq. (2.27) already has this structure.

We proceed to proof Lindblad's theorem. The semigroup property ensures that the evolution of the density matrix for an infinitesimal  $\Delta t$  is

$$\rho_S(t + \Delta t) = \sum_k M_k(\Delta t) \rho_S(t) M_k^\dagger(\Delta t), \quad (2.33)$$

and expanding  $\rho_S(t + \Delta t)$  to first order in  $\Delta t$ , we find

$$\rho_S(t + \Delta t) = \rho_S(t) + \Delta t \frac{d}{dt} \rho_S(t) + \mathcal{O}(\Delta t^2). \quad (2.34)$$

This fixes the correct scaling of the  $M_k$  on  $\Delta t$ . We choose the  $k = 0$  term to be proportional to the identity which will guarantee that in the limit of  $\Delta t \rightarrow 0$  we get the correct result. Similarly, the other terms are proportional to  $\sqrt{\Delta t}$  to ensure that products  $M_k(\dots)M_k^\dagger$  are of order  $\Delta t$ . That is, we take

$$M_0 = I + G\Delta t, \quad M_k = \sqrt{\Delta t}L_k, \quad k \neq 0, \quad (2.35)$$

where  $G$  and  $L_k$  are arbitrary operators. Normalization of the Kraus operators leads to

$$\sum_k M_k^\dagger M_k = (I + G^\dagger \Delta t)(I + G\Delta t) + \Delta t \sum_{k \neq 0} L_k^\dagger L_k, \quad (2.36)$$

$$= I + (G + G^\dagger)\Delta t + \Delta t \sum_{k \neq 0} L_k^\dagger L_k + \mathcal{O}(\Delta t^2). \quad (2.37)$$

We can parametrize  $G = K - iH$  where  $K$  and  $H$  are both Hermitian. It follows from the normalization condition

$$K = -\frac{1}{2} \sum_{k \neq 0} L_k^\dagger L_k. \quad (2.38)$$

We now substitute our results into Eq. (2.33) to finally get:

$$\rho_S(t + \Delta t) = (I + G\Delta t)\rho_S(I + G^\dagger \Delta t) + \Delta t \sum_{k \neq 0} L_k \rho_S L_k^\dagger, \quad (2.39)$$

$$= \rho_S(t) + \Delta t(G\rho_S + \rho_S G^\dagger) + \Delta t \sum_{k \neq 0} L_k \rho_S L_k^\dagger, \quad (2.40)$$

$$= \rho_S(t) - i\Delta t[H, \rho_S] + \Delta t \sum_{k \neq 0} \left( L_k \rho_S L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho_S\} \right). \quad (2.41)$$

Rearranging and taking the limit  $\Delta t \rightarrow 0$  we get:

$$\frac{\rho_S(t + \Delta t) - \rho_S(t)}{\Delta t} = \frac{d}{dt} \rho_S = -i[H, \rho_S] + \sum_{k \neq 0} \left( L_k \rho_S L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho_S\} \right), \quad (2.42)$$

which is exactly Eq. (2.32).

# Chapter 3

## Continuous Variable Formalism

### 3.1 Coherent States

A useful set of physical states are the coherent states. This basis allows the description of several experiments, most prominently in quantum optics where the electromagnetic field is described by quantized harmonic oscillators [6, 85]; it allows for the description of quantum many-body phenomena by means of the coherent path integral [86–90]; and, specially in quantum information, they represent a cornerstone in all facets of information processing [4, 67, 83, 91].

The coherent states  $|\alpha\rangle$  are defined as the eigenstates of the annihilation operator  $a$ :

$$a |\alpha\rangle = \alpha |\alpha\rangle, \quad |\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a} |0\rangle, \quad (3.1)$$

where we will henceforth refer to  $|0\rangle$  as the vacuum state. Although not crucial, it will be useful to state some properties of the coherent states [6, 77, 86]:

- Taking the hermitian conjugate of Eq. (3.1), we get the bra  $\langle\alpha|$  which is a left eigenstate of the creation operator  $a^\dagger$ ,

$$\langle\alpha| a^\dagger = \langle\alpha| \alpha^*, \quad (3.2)$$

where  $\alpha^*$  is the complex conjugate of  $\alpha$ .

- The action of the creation operator  $a$  on the coherent state yields

$$a^\dagger |\alpha\rangle = \partial_\alpha |\alpha\rangle. \quad (3.3)$$

- The overlap between two coherent states is given by

$$\langle\mu|\alpha\rangle = e^{\mu^*\alpha - \frac{1}{2}|\mu|^2 - \frac{1}{2}|\alpha|^2}. \quad (3.4)$$

In particular, the norm of a coherent state is  $|\langle\alpha|\alpha\rangle|^2 = 1$ .

- Most importantly, the coherent states form an overcomplete set of states:

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1, \quad (3.5)$$

where  $d^2\alpha = d\text{Re } \alpha d\text{Im } \alpha$ .

- The trace of a density matrix  $\rho$  in the coherent basis is given by

$$\text{Tr}(\rho) = \int \frac{d^2\alpha}{\pi} \langle\alpha|\rho|\alpha\rangle. \quad (3.6)$$

- The expectation value of any operator  $\mathcal{O}$  in the coherent basis is

$$\langle\mathcal{O}\rangle = \text{Tr}(\rho\mathcal{O}) = \int \frac{d^2\alpha}{\pi} \langle\alpha|\rho\mathcal{O}|\alpha\rangle. \quad (3.7)$$

The coherent basis is particularly useful when representing the density matrix in quantum phase space, the quantum analogue of classical phase space. Among the different representations, we choose the Husimi-Q function [92]. This is defined as the expectation value of the density matrix  $\rho$  in the coherent basis  $|\alpha\rangle$

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \langle\alpha|\rho|\alpha\rangle. \quad (3.8)$$

As an example, let us take  $\rho$  to be the thermal Gibbs state

$$\rho = \frac{e^{-\beta\omega a^\dagger a}}{\text{Tr}(e^{-\beta\omega a^\dagger a})}. \quad (3.9)$$

Its Husimi function will be

$$Q(\alpha, \alpha^*) = \frac{1 - e^{-\beta\omega}}{\pi} \sum_{n=0}^{\infty} e^{-\beta\omega n} \langle \alpha | n \rangle \langle n | \alpha \rangle = \frac{1}{\pi(\bar{n} + 1)} e^{-\frac{\alpha\alpha^*}{\bar{n}+1}}, \quad (3.10)$$

where  $\bar{n} = (e^{\beta\omega} - 1)^{-1}$  is the bosonic occupation number. We plot in Fig. 3.1 the Gibbs state with  $\bar{n} = 20$ .

We can also use the Husimi function to calculate the expectation value of anti-normally ordered operators. That is operators of the form

$$\langle a^k (a^\dagger)^l \rangle = \int d^2\alpha \, a^k (\alpha^*)^l Q(\alpha, \alpha^*), \quad (3.11)$$

which reduces to an integral in the complex plane of  $\alpha$ .

## 3.2 Gaussian States and Maps

The Gibbs state is an example of a Gaussian state. These are characterized by their representation in quantum phase space where its Husimi-Q function is a Gaussian function of the coherent variables

$$Q(\alpha, \alpha^*) = \frac{1}{\pi\sqrt{|\Theta|}} \exp\left(-\frac{1}{2}\vec{\alpha}^\dagger \Theta^{-1} \vec{\alpha}\right), \quad \vec{\alpha} = (\alpha, \alpha^*), \quad (3.12)$$

where  $\Theta$  is called the covariance matrix and  $|\Theta|$  is the determinant of such matrix.

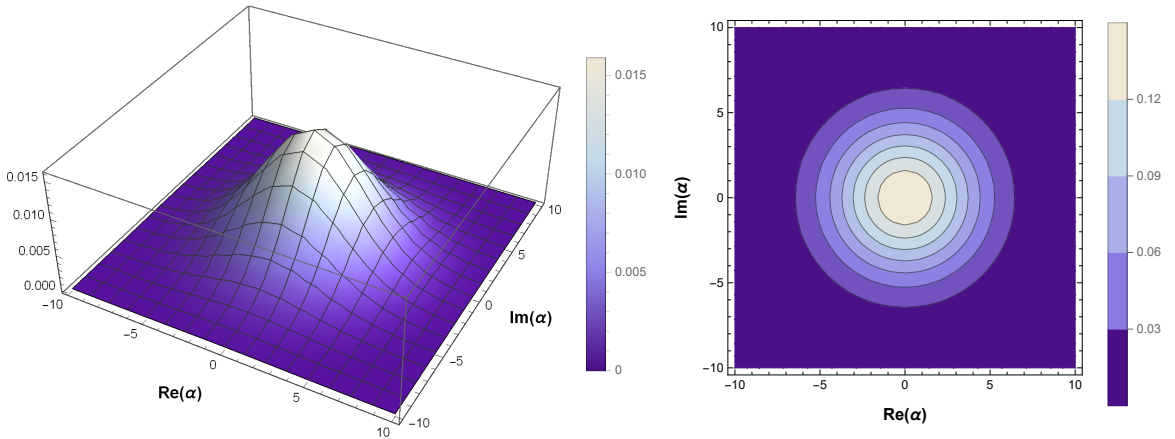


Figure 3.1: Husimi-Q function of the thermal Gibbs state with occupation number  $\bar{n} = 20$ .

Related to Gaussian states is the concept of Gaussian maps [71–73, 75, 93]. Gaussian maps are transformations that preserve Gaussianity. That is, that take Gaussian states into Gaussian states. Dynamically, this is achieved by imposing certain restrictions in the Lindblad equation (2.32):

- The Hamiltonian  $H$  must be at most quadratic in  $a$  and  $a^\dagger$ . Thus, the most general Gaussian preserving Hamiltonian has the form

$$H = \omega a^\dagger a + \gamma a^\dagger a^\dagger + \gamma^* a a + f a^\dagger + f^* a. \quad (3.13)$$

- The Lindblad generators  $L_k$  must be linear on  $a$  and  $a^\dagger$ . As an example the thermal bath generator

$$\mathcal{D}(\rho) = \gamma \bar{n} (a^\dagger \rho a - \frac{1}{2} \{a a^\dagger, \rho\}) + \gamma (\bar{n} + 1) (a \rho a^\dagger - \frac{1}{2} \{a^\dagger a, \rho\}), \quad (3.14)$$

is a Gaussian preserving map.

Gaussian states and Gaussian maps are important tools since they simplify dramatically a potentially unsolvable problem. While, in general, continuous variable systems require an infinite dimensional Hilbert space, Gaussian states are fully determined by the first and second moments. Moreover, using Gaussian maps allows the dynamics to be described by a finite set of parameters. The reason is that for a Gaussian map, the equations for the first and second moments are closed, therefore reducing the complexity of the problem which otherwise would be intractable.

Even though we could, in principle, tackle the problem numerically, this approach becomes obsolete as we start considering multi-mode states. Gaussianity becomes essential to obtain analytical solutions.

### 3.3 Multi-mode States

We consider  $N$  bosonic modes with algebra given by

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad (3.15)$$

and quadratures by

$$q_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger), \quad p_i = \frac{i}{\sqrt{2}}(a_i^\dagger - a_i). \quad (3.16)$$

We would like to merge all these relations into a single condition. For this reason, we define two  $2N$  dimensional vectors

$$\vec{X} = (a_1, a_1^\dagger, a_2, a_2^\dagger, \dots, a_N, a_N^\dagger), \quad \vec{Y} = (q_1, p_1, q_2, p_2, \dots, q_N, p_N). \quad (3.17)$$

These two vectors are equivalent representations of the  $N$  bosonic modes. It will be convenient to write down the basic results for both of them. The connection between these two vectors is given by

$$\vec{Y} = \Lambda \vec{X}, \quad \Lambda = \bigoplus_{i=1}^N \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}, \quad (3.18)$$

where  $\Lambda$  is a unitary transformation analogue to (3.16). Additionally, the multi-mode commutation relations (3.15) are now condensed in the vector commutators

$$[\vec{X}, \vec{X}^T] = \bigoplus_{i=1}^N \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [\vec{Y}, \vec{Y}^T] = \bigoplus_{i=1}^N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (3.19)$$

where the vector commutator is defined as  $[\vec{A}, \vec{B}^T] = \vec{A} \otimes \vec{B}^T - (\vec{B}^T \otimes \vec{A})^T$ . Equations (3.17)-(3.19) establishes the algebra of the group of operators.

As mentioned in the previous section, we are interested in the first and second moments. We define the first moments  $\vec{x}$  and  $\vec{y}$  related to  $\vec{X}$  and  $\vec{Y}$  as simply  $x_i = \langle X_i \rangle$  and  $y_i = \langle Y_i \rangle$ . More interesting, we define the covariance matrices (CM) as

$$\Theta_{ij} = \frac{1}{2} \langle \{X_i, X_j^\dagger\} \rangle - \langle X_i \rangle \langle X_j^\dagger \rangle, \quad \sigma_{ij} = \frac{1}{2} \langle \{Y_i, Y_j\} \rangle - \langle Y_i \rangle \langle Y_j \rangle, \quad (3.20)$$

where by construction  $\Theta$  is Hermitian, and  $\sigma$  is real and symmetric. For example, for two

modes the covariance matrices would look like:

$$\sigma = \begin{pmatrix} \langle q_1^2 \rangle & \frac{1}{2}\langle \{q_1, p_1\} \rangle & \langle q_1 q_2 \rangle & \langle q_1 p_2 \rangle \\ \frac{1}{2}\langle \{q_1, p_1\} \rangle & \langle p_1^2 \rangle & \langle p_1 q_2 \rangle & \langle p_1 p_2 \rangle \\ \langle q_2 q_1 \rangle & \langle q_2 p_1 \rangle & \langle q_2^2 \rangle & \frac{1}{2}\langle \{q_2, p_2\} \rangle \\ \langle p_2 q_1 \rangle & \langle p_2 p_1 \rangle & \frac{1}{2}\langle \{q_2, p_2\} \rangle & \langle p_2^2 \rangle \end{pmatrix}, \quad (3.21)$$

and

$$\Theta = \begin{pmatrix} \langle a_1^\dagger a_1 \rangle + \frac{1}{2} & \langle a_1 a_1 \rangle & \langle a_1 a_2^\dagger \rangle & \langle a_1 a_2 \rangle \\ \langle a_1^\dagger a_1^\dagger \rangle & \langle a_1^\dagger a_1 \rangle + \frac{1}{2} & \langle a_1^\dagger a_2^\dagger \rangle & \langle a_1^\dagger a_2 \rangle \\ \langle a_1^\dagger a_2 \rangle & \langle a_1 a_2 \rangle & \langle a_2^\dagger a_2 \rangle + \frac{1}{2} & \langle a_2 a_2 \rangle \\ \langle a_1^\dagger a_2^\dagger \rangle & \langle a_1 a_2^\dagger \rangle & \langle a_2^\dagger a_2^\dagger \rangle & \langle a_2^\dagger a_2 \rangle + \frac{1}{2} \end{pmatrix}, \quad (3.22)$$

where is implicitly assumed that  $\langle a_i \rangle = 0$ . Notice that the covariance matrix is structured in  $2 \times 2$  blocks where the diagonal blocks are the CMs of modes 1 and 2, while the off-diagonal blocks represent their correlations. These two matrices are also related via the unitary transformation (3.18), namely

$$\sigma = \Lambda \Theta \Lambda^\dagger. \quad (3.23)$$

### 3.4 Gaussian Dynamics

We now focus on the dynamical evolution of Gaussian states subject to the Lindblad equation:

$$\frac{d}{dt}\rho = -i[H, \rho] + \mathcal{D}(\rho), \quad (3.24)$$

where  $H$  is a Gaussian Hamiltonian and  $\mathcal{D}$  is a Gaussian preserving dissipator. Given the master equation, we can obtain the evolution of the expectation value of any operator:

$$\frac{d}{dt}\langle \mathcal{O} \rangle = i\langle [H, \mathcal{O}] \rangle + \text{Tr}(\mathcal{O}\mathcal{D}(\rho)). \quad (3.25)$$



Using the cyclic property of the trace, we can write

$$\text{Tr} \left( \mathcal{O} \left( L \rho L^\dagger - \frac{1}{2} \{ L^\dagger L, \rho \} \right) \right) = \langle L^\dagger \mathcal{O} L - \frac{1}{2} \{ L^\dagger L, \mathcal{O} \} \rangle := \langle \tilde{\mathcal{D}}(\mathcal{O}) \rangle. \quad (3.26)$$

Hence, we can rewrite Eq. (3.25) as:

$$\frac{d}{dt} \langle \mathcal{O} \rangle = i \langle [H, \mathcal{O}] \rangle + \langle \tilde{\mathcal{D}}(\mathcal{O}) \rangle. \quad (3.27)$$

So far we haven't use the fact that the evolution is described by Gaussian maps. Imposing Gaussianity results in the equations to close for the first and second moments. That means that the expected values of operators like  $\langle a_i \rangle$  or  $\langle a_i^\dagger a_j \rangle - \langle a_i^\dagger \rangle \langle a_j \rangle$  will only depend on the elements of the vector  $\vec{x}$  and the covariance matrix  $\Theta$ . In contrast, in the non-Gaussian scenario the equations for these moments will depend also on the higher moments, leading to an infinite hierarchy of coupled equations. Thus, the Gaussian dynamics will be encapsulated in the evolution of the vector of averages and the covariance matrix.

Taking the evolution of the vector of averages  $\vec{x}$  and  $\vec{y}$  into Eq. (3.27) we get

$$\frac{d}{dt} \vec{x} = W \vec{x} - \vec{f}, \quad \frac{d}{dt} \vec{y} = M \vec{y} - \vec{g}, \quad (3.28)$$

where  $W$  and  $M$  depend on the choice of Hamiltonian  $H$  while  $\vec{f}$  and  $\vec{g}$  on the choice of dissipator  $\mathcal{D}$ . Similarly, we get for the covariance matrices:

$$\frac{d}{dt} \Theta = W \Theta + \Theta W^\dagger + F, \quad \frac{d}{dt} \sigma = M \sigma + \sigma M^\dagger + G, \quad (3.29)$$

where the matrices  $F$  and  $G$  depend only on the dissipator  $\mathcal{D}$  whereas the matrices  $W$  and  $M$  are the same as Eq. (3.28).

Let us illustrate the procedure with two simple examples that we will further explore in Chapter 4. Suppose we have two bosonic modes  $a_1$  and  $a_2$  interacting via a beam-splitter Hamiltonian

$$H_{BS} = ig(a_1^\dagger a_2 - a_2^\dagger a_1). \quad (3.30)$$

This will create/destroy a particle in one mode and destroy/create a particle in the other

mode, conserving always the total number of particles  $N = a_1^\dagger a_1 + a_2^\dagger a_2$ . We use Eq. (3.27) to get the first moments evolution

$$\frac{d\langle a_1 \rangle}{dt} = \langle i[H_{BS}, a_1] \rangle = g\langle a_2 \rangle, \quad (3.31)$$

$$\frac{d\langle a_1^\dagger \rangle}{dt} = \langle i[H_{BS}, a_1^\dagger] \rangle = g\langle a_2^\dagger \rangle, \quad (3.32)$$

$$\frac{d\langle a_2 \rangle}{dt} = \langle i[H_{BS}, a_2] \rangle = -g\langle a_1 \rangle, \quad (3.33)$$

$$\frac{d\langle a_2^\dagger \rangle}{dt} = \langle i[H_{BS}, a_2^\dagger] \rangle = -g\langle a_1^\dagger \rangle. \quad (3.34)$$

The matrix  $W$  associated to this beam-splitter Hamiltonian is given by

$$W_{BS} = \begin{pmatrix} 0 & 0 & g & 0 \\ 0 & 0 & 0 & g \\ -g & 0 & 0 & 0 \\ 0 & -g & 0 & 0 \end{pmatrix}. \quad (3.35)$$

The solution for the first moments vector  $\vec{x}(t)$  is thus

$$\vec{x}(t) = V_{BS}(t) \vec{x}_0, \quad (3.36)$$

where the  $V_{BS}$  matrix introduced is

$$V_{BS}(t) := e^{W_{BS}t} = \begin{pmatrix} \cos(gt) & 0 & \sin(gt) & 0 \\ 0 & \cos(gt) & 0 & \sin(gt) \\ -\sin(gt) & 0 & \cos(gt) & 0 \\ 0 & -\sin(gt) & 0 & \cos(gt) \end{pmatrix}. \quad (3.37)$$

It will be useful to re-state this matrix in terms of  $2 \times 2$  blocks

$$V_{BS}(t) = \begin{pmatrix} \cos(gt) & \sin(gt) \\ -\sin(gt) & \cos(gt) \end{pmatrix} \otimes \mathbb{I}, \quad (3.38)$$

where  $\mathbb{I}$  is the  $2 \times 2$  identity matrix.

We can also obtain the  $\vec{y}(t)$  solution by the interdependence relation (3.18),

$$\vec{y}(t) = S_{BS}(t) \vec{y}_0, \quad S_{BS}(t) = \Lambda^\dagger V_{BS}(t) \Lambda = \begin{pmatrix} \cos(gt) & \sin(gt) \\ -\sin(gt) & \cos(gt) \end{pmatrix} \otimes \mathbb{I}. \quad (3.39)$$

Likewise, the solution for the covariance matrices are:

$$\Theta(t) = V_{BS}(t) \Theta_0 V_{BS}^\dagger(t), \quad \sigma(t) = S_{BS}(t) \sigma_0 S_{BS}^\dagger(t). \quad (3.40)$$

Particularly, we will be interested in obtaining the solution of one of the modes (the system) by tracing out the other modes (environment). This is achieved by looking at the system's block term in Eq. (3.40). As an example, let us take the first mode as the system and the second mode as the environment. Tracing out the second mode, we get:

$$\sigma_1(t) = \cos^2(gt) \sigma_1(0) + \sin^2(gt) \sigma_2(0), \quad (3.41)$$

where  $\sigma_1(0)$  and  $\sigma_2(0)$  are the initial states of the covariance matrices for the first and second mode correspondingly. This equation can be also expressed as:

$$\sigma_1(t) = X \sigma_1(0) X^\dagger + Y, \quad (3.42)$$

where  $X = \cos(gt) \mathbb{I}$ , and  $Y = \sin^2(gt) \sigma_2(0)$ . This is the Gaussian analogue of Eq. (2.30) and it is known as a Gaussian CP-map [67].

Another Gaussian Hamiltonian we will be interested in is the two-mode squeezing

$$H_{SQ} = i\mu(a_1^\dagger a_2^\dagger - a_2 a_1). \quad (3.43)$$

In opposition to the beam-splitter, this Hamiltonian does not conserve the total number of particles  $N$ : it creates/destroys a particle in both modes with strength  $\mu$ .

The equations of motion for the first moments are:

$$\frac{d\langle a_1 \rangle}{dt} = \langle i[H_{SQ}, a_1] \rangle = \mu \langle a_2^\dagger \rangle, \quad (3.44)$$

$$\frac{d\langle a_1^\dagger \rangle}{dt} = \langle i[H_{SQ}, a_1^\dagger] \rangle = \mu \langle a_2 \rangle, \quad (3.45)$$

$$\frac{d\langle a_2 \rangle}{dt} = \langle i[H_{SQ}, a_2] \rangle = \mu \langle a_1^\dagger \rangle, \quad (3.46)$$

$$\frac{d\langle a_2^\dagger \rangle}{dt} = \langle i[H_{SQ}, a_2^\dagger] \rangle = \mu \langle a_1 \rangle. \quad (3.47)$$

The matrix  $W$  associated with this Hamiltonian is

$$W_{SQ} = \begin{pmatrix} 0 & 0 & 0 & \mu \\ 0 & 0 & \mu & 0 \\ 0 & \mu & 0 & 0 \\ \mu & 0 & 0 & 0 \end{pmatrix}. \quad (3.48)$$

Similar solutions (3.36)-(3.40) are obtained for the two-mode squeezing provided we change  $V_{BS}$  with the corresponding  $V_{SQ}$  matrix

$$V_{SQ}(t) := e^{W_{SQ}t} = \begin{pmatrix} \cosh(\mu t)\mathbb{I} & \sinh(\mu t)\sigma_x \\ \sinh(\mu t)\sigma_x & \cosh(\mu t)\mathbb{I} \end{pmatrix}, \quad (3.49)$$

and change  $S_{BS}$  with  $S_{SQ}$

$$S_{SQ}(t) = \Lambda^\dagger V_{SQ}(t) \Lambda = \begin{pmatrix} \cosh(\mu t)\mathbb{I} & \sinh(\mu t)\sigma_z \\ \sinh(\mu t)\sigma_z & \cosh(\mu t)\mathbb{I} \end{pmatrix}, \quad (3.50)$$

where  $\sigma_x$  and  $\sigma_z$  are the familiar  $2 \times 2$  Pauli Matrices.

Finally, we can get the Gaussian CP-map (3.42) for the two-mode squeezing:

$$\sigma_1(t) = \cosh^2(\mu t) \sigma_1(0) + \sinh^2(\mu t) \sigma_z \sigma_2(0) \sigma_z, \quad (3.51)$$

where  $X = \cosh(\mu t)\mathbb{I}$ , and  $Y = \sinh^2(\mu t) \sigma_z \sigma_2(0) \sigma_z$ .

# Chapter 4

## Collisional Models

This chapter contains one of the main original results developed in this dissertation. It presents the theoretical model to be studied in these last chapters. The exposition here follows closely Sec. II of the submitted arXiv preprint [94].

### 4.1 Motivation

As we have seen previously, obtaining a microscopic description of the open quantum system's dynamics is a really complicated task, often relying on approximation schemes to get analytical results. We particularly analyzed the case of a qubit interacting with a bosonic environment and realized how many approximations were made to obtain the system's evolution. This motivates the search for alternative models, where analytic equations can be obtained without resorting to such severe approximations. One way to accomplish this, which has seen an enormous surge in popularity in recent years, are through the so-called collisional models [35–46, 94]. The basic idea is to replace the open dynamics of a system by a series of sequential interactions between the system  $S$  and small environmental units  $E_1, E_2, E_3 \dots$  (also referred to as ancillas). All interactions lasts for a fixed time after which they never interact again. This therefore leads to a stroboscopic dynamics for the system.

The collisional model we are interested in this work is the one depicted in Fig. 4.1. That is, we consider a scenario where neighboring ancillas  $E_n E_{n+1}$  interact with each other in between the interactions  $SE_n$  and  $SE_{n+1}$ . This additional interaction *signals*

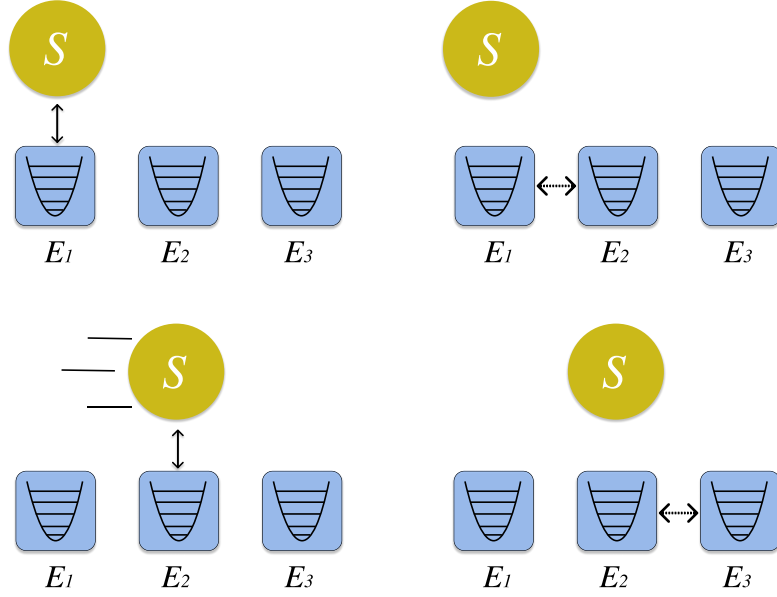


Figure 4.1: First few steps of the collisional model dynamics. The system-ancilla interactions  $SE_n$  are interspersed by ancilla-ancilla interactions  $E_n E_{n+1}$ , which propagate information forward, making the dynamics non-Markovian in a fully controllable way.

information from the past to the future, so that when the  $SE_{n+1}$  interaction arrives, the ancilla  $E_{n+1}$  will already contain some information about the system.

## 4.2 Formal Framework

Here we consider the collisional model scenario presented in Fig. 4.1. A system  $S$  is put to interact sequentially with an arbitrary number of environment ancillas  $E_1, E_2, E_3 \dots$ . The ancillas are independent and identically prepared, each with the initial density matrix  $\rho_E$ . The interaction between  $S$  and  $E_n$  is described by a unitary  $U_n$ . After this,  $S$  and  $E_n$  never interact again. After the collision  $SE_n$  but before the  $SE_{n+1}$ , we put ancillas  $E_n$  and  $E_{n+1}$  to interact with each other by means of another unitary  $V_{n,n+1}$ . Since  $E_n$  already interacted with  $S$ , it contains some information about each other, obtained from  $E_n$ . Past information about  $S$  can thus *backflow* at  $SE_{n+1}$ .

Let  $\rho^0 = \rho_S \otimes \rho_{E_1} \otimes \rho_{E_2} \otimes \dots$  denote the initial state of the composite system  $SE_1 E_2 \dots$ . We count time in integer steps, such that at time  $n$  the collisions  $SE_n$  and  $E_n E_{n+1}$  already took place. That is, at time  $n$  the system has already interacted with its corresponding

ancilla  $E_n$  and this ancilla has already passed down its information to the next one. The map taking the composite system  $SE_1E_2 \dots$  from  $n - 1$  to  $n$  reads

$$\rho^n = V_{n,n+1}U_n \rho^{n-1} U_n^\dagger V_{n,n+1}^\dagger. \quad (4.1)$$

To avoid confusion we henceforth use superscripts to denote time so that  $\rho^n$  refers to the global state of  $SE_1E_2 \dots$  at time  $n$ . The map (4.1) involves only  $SE_nE_{n+1}$ . All ancillas  $E_m$  with  $m \geq n + 2$  did not yet participate in the process and therefore remain in a product state with everything else. In addition, the ancillas with  $m < n$  will never participate again and hence can be traced out (discarded). The process (4.1) can thus be equivalently written as

$$\rho_{SE_nE_{n+1}}^n = V_{n,n+1}U_n(\rho_{SE_n}^{n-1} \otimes \rho_{E_{n+1}})U_n^\dagger V_{n,n+1}^\dagger, \quad (4.2)$$

where  $\rho_{SE_n}^{n-1}$  is the state of  $SE_n$  at time  $n - 1$  and  $\rho_{E_{n+1}} = \rho_E$  refers to the initial state of  $E_{n+1}$ . This also holds for the first step, provided one recalls that  $\rho_{SE_1}^0 = \rho_S^0 \otimes \rho_{E_1}$ . After the interaction (4.2), one may trace out  $E_n$ , leading to

$$\rho_{SE_{n+1}}^n = \text{Tr}_{E_n} \left( V_{n,n+1}U_n(\rho_{SE_n}^{n-1} \otimes \rho_{E_{n+1}})U_n^\dagger V_{n,n+1}^\dagger \right) := \Phi(\rho_{SE_n}^{n-1}). \quad (4.3)$$

This can now be fed again to Eq. (4.2) to evolve to the next step. This equation also defines the quantum channel  $\Phi(\cdot)$ , which is a map from the Hilbert space of  $SE_n$  to that of  $SE_{n+1}$ . Moreover, since we are assuming that the unitaries  $U_n$  and  $V_{n,n+1}$  are the same for all collisions, the map  $\Phi$  itself is actually independent of  $n$ ; the only  $n$  dependence is in the input  $\rho_{SE_n}^{n-1}$ .

Crucially, we see that the map  $\Phi(\cdot)$  is both time-local and completely positive since it is written as a Stinespring dilation [95]. Hence, it represents an entirely Markovian evolution. Eq. (4.3) is known as a Markovian embedding of the dynamics [63]: it expresses a non-Markovian evolution as a Markovian one at the expense of working with maps that act between different Hilbert spaces and also have a higher dimension.

It is convenient to define the more compact notation  $\varrho^n = \rho_{SE_{n+1}}^n$  for the joint state of  $SE_{n+1}$  at time  $n$ . The entire dynamics can then be captured by the stroboscopic,

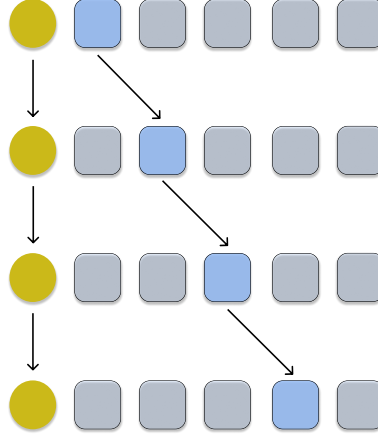


Figure 4.2: Basic structure of the Markovian embedding dynamics (4.20), which is a map from the Hilbert space of  $SE_n$  to that of  $SE_{n+1}$

Markovian, CPTP evolution:

$$\varrho^{n+1} = \Phi(\varrho^n), \quad (4.4)$$

and the corresponding sequence of states  $\varrho^0, \varrho^1, \varrho^2, \dots$  that it generates. At each step, the reduced state of the system is always available as  $\rho_S^n = \text{Tr}_{E_{n+1}} \varrho^n$ . The Markovian embedding (4.4) will be central to this work.

Since initially the system is uncorrelated from all ancillas, it is possible to define a CPTP map taking  $\rho_S^0 \rightarrow \rho_S^n$ :

$$\rho_S^n = \mathcal{E}_n(\rho_S^0) = \text{Tr}_{E_{n+1}} \Phi^n(\rho_S^0 \otimes \rho_{E_1}). \quad (4.5)$$

Even though this map is CPTP, the map taking  $\rho_S^m \rightarrow \rho_S^n$  will not generally be CPTP due to the non-Markovian dynamics.

### 4.3 Gaussian Collisional Model

We are interested in obtaining analytical results. To accomplish this, we therefore specialize now to the case of continuous-variable systems undergoing Gaussian-preserving dynamics reviewed in Chapter 3. Our exposition, in what follows, will be example oriented. However, the final results will be quite general [Eqs. (4.19), (4.20) and (4.22)].



We assume the system  $S$  is described by bosonic annihilation operator  $a$  and corresponding quadrature  $Q = (a + a^\dagger)/\sqrt{2}$  and  $P = i(a^\dagger - a)/\sqrt{2}$ . Similarly, the ancillas are described by bosonic annihilation operators  $b_1, b_2, \dots$  with corresponding quadratures  $q_n, p_n$ . The generalization to multimode system is straightforward. We take the system-ancilla interaction  $U_n$  in Eq. (4.2) to be a simple beam-splitter-type unitary

$$U_n = e^{\lambda_s(a^\dagger b_n - b_n^\dagger a)}, \quad (4.6)$$

described by a parameter  $\lambda_s$ . One can view (4.6) as an interaction with a Hamiltonian  $ig(a^\dagger b_n - b_n^\dagger a)$  that lasts for a time  $\tau$  such that  $g\tau = \lambda_s$ . Since we are only interested in the stroboscopic dynamics, we can omit the internal details for simplicity. As for the  $E_n E_{n+1}$  collision unitary  $V_{n,n+1}$ , we shall explore two possibilities. The first is again a beam-splitter map

$$V_{n,n+1} = e^{\lambda_e(b_n^\dagger b_{n+1} - b_{n+1}^\dagger b_n)}, \quad (4.7)$$

with interaction strength  $\lambda_e$ . We shall henceforth refer to this as the BS dynamics. In addition, we shall also look at a two-mode squeezing interaction (TMS),

$$\tilde{V}_{n,n+1} = e^{\nu_e(b_n^\dagger b_{n+1}^\dagger - b_{n+1} b_n)}, \quad (4.8)$$

with strength  $\nu_e$  and view (4.8) as coming from a Hamiltonian  $ig(b_n^\dagger b_{n+1}^\dagger - b_{n+1} b_n)$  for a time  $\tau$  such that  $g\tau = \nu_e$ . The reason behind this choice is related to the fact that two-mode squeezing interactions generate stronger forms of correlations (e.g. entanglement) between the ancillas.

The unitaries (4.6)-(4.8) are Gaussian preserving. If we assume that the initial state is Gaussian, the dynamics will then be completely characterized by the first and second moments. We assume for simplicity that the first moments are initially zero so that they will remain so throughout. The covariance matrix of the initial state is block-diagonal, of the form

$$\sigma^0 = \text{diag}(\theta^0, \epsilon, \epsilon, \dots), \quad (4.9)$$

where each block is  $2 \times 2$ :  $\theta^0$  is the arbitrary initial CM of the system and  $\epsilon$  is the initial CM of the ancillas (which are all the same, since we are assuming the ancillas are iid).

The global dynamics of  $SE_1E_2\dots$  is unitary. As a consequence, the map (4.1) is translated into a symplectic evolution for the CM:

$$\sigma^n = S_{n,n+1}S_n \sigma^{n-1} S_n^T S_{n,n+1}^T, \quad (4.10)$$

where  $S_n$  and  $S_{n,n+1}$  are the symplectic matrices associated with the unitaries  $U_n$  and  $V_{n,n+1}$ . The symplectic matrix associated to the beam-splitter interaction (4.6) is remarkably simple because all entries are proportional to the  $2 \times 2$  identity (3.39) [this is partially because of the choice of phase in the exponent of (4.6)]. For instance, the interaction  $S_2$  between the  $S$  and  $E_2$  reads

$$S_2 = \begin{pmatrix} x & 0 & y & 0 & \dots \\ 0 & \mathbb{I} & 0 & 0 & \dots \\ -y & 0 & x & 0 & \dots \\ 0 & 0 & 0 & \mathbb{I} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (4.11)$$

where each entry is a  $2 \times 2$  matrix, with  $x = \cos(\lambda_s)\mathbb{I}$  and  $y = \sin(\lambda_s)\mathbb{I}$ . The same structure also holds for the BS unitary  $V_{n,n+1}$  between  $E_nE_{n+1}$  [Eq. (4.7)], except that now the position of the non-zero entries changes. For instance,

$$S_{1,2} = \begin{pmatrix} \mathbb{I} & 0 & 0 & 0 & \dots \\ 0 & z & w & 0 & \dots \\ 0 & -w & z & 0 & \dots \\ 0 & 0 & 0 & \mathbb{I} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (4.12)$$

where  $z = \cos(\lambda_e)\mathbb{I}$  and  $w = \sin(\lambda_e)\mathbb{I}$ . The TMS interaction (4.8) is slightly more complicated since some entries are proportional to the identity, while others are proportional

to the Pauli matrix  $\sigma_z$  as seen in (3.50). For instance,

$$\tilde{S}_{1,2} = \begin{pmatrix} \mathbb{I} & 0 & 0 & 0 & \dots \\ 0 & \tilde{z} & \tilde{w}\sigma_z & 0 & \dots \\ 0 & \tilde{w}\sigma_z & \tilde{z} & 0 & \dots \\ 0 & 0 & 0 & \mathbb{I} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (4.13)$$

with  $\tilde{z} = \cosh(\nu_e)\mathbb{I}$  and  $\tilde{w} = \sinh(\nu_e)$ .

The BS dynamics is completely characterized by the pair  $(\lambda_s, \lambda_e)$ , while the TMS dynamics is characterized by  $(\lambda_s, \nu_e)$ . On top of that, one also has the choice of ancilla initial state  $\epsilon$ . More general Gaussian maps will continue to have a similar structure. The symplectic  $S_n$  will have the form

$$S_2 = \begin{pmatrix} A & 0 & B & 0 & \dots \\ 0 & \mathbb{I} & 0 & 0 & \dots \\ C & 0 & D & 0 & \dots \\ 0 & 0 & 0 & \mathbb{I} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (4.14)$$

for block matrices  $A, B, C, D$ . The matrices  $S_n$  for other values of  $n$  are obtained by simply placing  $A, B, C, D$  at the correct positions. Note also that the condition that  $S$  must be symplectic imposes constraints on  $A, B, C, D$  which, however, are not particularly illuminating. Similarly, the  $E_n E_{n+1}$  interaction reads

$$\tilde{S}_{1,2} = \begin{pmatrix} \mathbb{I} & 0 & 0 & 0 & \dots \\ 0 & E & F & 0 & \dots \\ 0 & G & J & 0 & \dots \\ 0 & 0 & 0 & \mathbb{I} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (4.15)$$

for block matrices  $E, F, G, J$ . Note that these two expressions also naturally contemplate the case where either the system or each ancilla are, individually, composed of multiple

modes (which would simply affect the size of the matrices  $A, \dots, J$ ).

## 4.4 Markovian Embedding

The biggest advantage of Gaussian collisional models, as we will now show, is that the full evolution can be converted into a simple system of matrix difference equations for only a handful of entries of the full CM  $\sigma^n$ . As already discussed below Eq. (4.2), the step from  $\sigma^{n-1}$  to  $\sigma^n$  involves only  $S$ ,  $E_n$  and  $E_{n+1}$ . At time  $n - 1$  the ancilla  $E_{n+1}$  is still uncorrelated from the rest, whereas  $S$  and  $E_n$  are already correlated because of the previous step. Thus, the tripartite CM of  $SE_nE_{n+1}$ , at time  $n - 1$ , will have the block structure

$$\sigma_{SE_nE_{n+1}}^{n-1} = \begin{pmatrix} \theta^{n-1} & \xi_n^{n-1} & 0 \\ \xi_n^{n-1,T} & \epsilon_n^{n-1} & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \quad (4.16)$$

where  $\epsilon_n^{n-1}$  is the state of ancilla  $E_n$  at time  $n - 1$ , which is no longer the original value  $\epsilon$  because it already interacted with  $E_{n-1}$  in the previous step. Moreover,  $\xi_n^{n-1}$  are the correlations between  $SE_n$  that were developed in the previous step. We then apply the map (4.10) to Eq. (4.16), using the matrices in Eqs. (4.11)-(4.13)

$$\sigma_{SE_nE_{n+1}}^n = S_{n,n+1} S_n \left( \sigma_{SE_nE_{n+1}}^{n-1} \right) S_n^T S_{n,n+1}^T. \quad (4.17)$$

This will lead to a matrix  $\sigma^n$  with many non-zero entries. However, as far as the dynamics of  $S$  is concerned, only three entries are needed: the state of the system  $\theta^n$ , the state  $\epsilon_{n+1}^n$  of ancilla  $E_{n+1}$  and the correlations  $\xi_{n+1}^n$  between  $S$  and  $E_{n+1}$ .

To gain intuition, let us first analyze the BS case, which is simple since all blocks in Eq. (4.12) are proportional to the identity. Using Eqs. (4.11) and (4.12) in (4.17), one

finds the following system of matrix difference equations:

$$\begin{aligned}
 \theta^n &= x^2\theta^{n-1} + y^2\epsilon_n^{n-1} + xy(\xi_n^{n-1} + \xi_n^{n-1,T}), \\
 \epsilon_{n+1}^n &= z^2\epsilon + w^2\left[x^2\epsilon_n^{n-1} + y^2\theta^{n-1} - xy(\xi_n^{n-1} + \xi_n^{n-1,T})\right], \\
 \xi_{n+1}^n &= w\left[xy(\theta^{n-1} - \epsilon_n^{n-1}) + y^2\xi_n^{n-1,T} - x^2\xi_n^{n-1}\right].
 \end{aligned} \tag{4.18}$$

This provides a neat illustration of the map  $\Phi(\cdot)$  in Eq. (4.3): the quantities on the left-hand and right-hand side refer to different ancillas: for instance,  $\epsilon_{n+1}^n$  is the state of ancilla  $E_{n+1}$  at time  $n$ , whereas  $\epsilon_n^{n-1}$  is the state of  $E_n$  at time  $n - 1$ . Of course, one could also compute  $\epsilon_n^n$ , but this is not necessary for describing the dynamics of  $S$ .

The system of matrix difference equations (4.18) contains the minimum amount of information required to fully account for the dynamics of  $S$ . These equations can also be recast in a more compact form in terms of the Markovian embedding (4.4). We define the reduced CM of  $SE_{n+1}$  at time  $n$  as

$$\gamma_{n+1}^n \equiv \gamma^n = \begin{pmatrix} \theta^n & \xi_{n+1}^n \\ \xi_{n+1}^{n,T} & \epsilon_{n+1}^n \end{pmatrix}, \tag{4.19}$$

where the notation  $\gamma^n$  will be used to simplify the expressions. Eq. (4.18) can then be written compactly as

$$\gamma^{n+1} = X\gamma^n X^T + Y, \tag{4.20}$$

where the time index was shifted by 1. Here  $X$  and  $Y$  are  $4 \times 4$  matrices with block form

$$X = \begin{pmatrix} x & y \\ yw & -wx \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & z^2\epsilon \end{pmatrix}, \tag{4.21}$$

where, again, each block is proportional to the identity.

Eq. (4.20) beautifully illustrates the notion of Markovian embedding. It has the structure of a typical Gaussian CPTP map (3.40), being Markovian (time-local) by construction. However, this Markovian dynamics takes place at the larger space of the system plus one ancilla (which one, in specific, changes at each collision). Thus, we have embedded

the non-Markovian dynamics into a Markovian dynamics at a larger space. Notice how the size of the space is directly related to the fact that we chose  $E_n$  to only interact with its nearest neighbor  $E_{n+1}$ . That is, we fixed the memory length to be 1, which defines the size of the minimal space required for the embedding [63].

The matrices (4.21) refer to the beam-splitter unitary (4.7). The generalization to the arbitrary Gaussian interactions (4.14) and (4.15) is similar, albeit more cumbersome. The result is

$$X = \begin{pmatrix} A & B \\ GC & GD \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & J_\epsilon J^T \end{pmatrix}. \quad (4.22)$$

For instance, in the case of the TMS interaction, Eq. (4.13), one has  $G = \tilde{w}\sigma_z$  and  $J = \tilde{z}$ , in addition to  $A = D = x$ ,  $B = y$  and  $C = -y$  (which come from  $S_n$  in (4.11)). One then finds that

$$X = \begin{pmatrix} x & y \\ -y\tilde{w}\sigma_z & \tilde{w}x\sigma_z \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{z}^2\epsilon \end{pmatrix}. \quad (4.23)$$

The blocks in  $X$  are therefore no longer proportional to the identity, but some are proportional to  $\sigma_z$ .

To summarize, the general non-Markovian dynamics will be described by the embedding (4.20), with  $\gamma^n$  defined in (4.19), and with  $X$  and  $Y$  given by (4.22). This framework therefore provides a quite general platform, enabling one to study a broad range of situations.

## 4.5 Example Dynamics

Eqs (4.20)-(4.23) are the first main results of this work. They provide a compact and efficient way of describing the non-Markovian dynamics of a bosonic mode in terms of a simple matrix difference equation for the augmented CM  $\gamma^n$ . The reduced state of the system is always readily accessible from the first  $2 \times 2$  block [Eq. (4.19)]. Before proceeding to quantify the non-Markovianity of the process, we first illustrate the typical behavior of the BS and TMS maps, by plotting the average system occupation  $\langle a^\dagger a \rangle$  as a function of time for different values of the  $E_n E_{n+1}$  interaction strength  $\lambda_e$  (for the BS case) or  $\nu_e$  (for the TMS case). We choose the system to start in a thermal state with

occupation number  $\langle a^\dagger a \rangle^0 = 20$ , while the ancillas start in the vacuum,  $\epsilon = \mathbb{I}_2/2$ . The results are summarized in Fig. 4.3 for the BS evolution and in Fig. 4.4 for the TMS evolution.

The BS dynamics is sensitive to the relative signs between  $\lambda_s$  and  $\lambda_e$  (and, consequently, of  $y = \sin(\lambda_s)$  and  $w = \sin(\lambda_e)$ ). This is an interference effect, which occurs due to the fact we are combining two beam-splitters [Eqs. (4.6) and (4.7)]. A similar effect was also observed in Refs [65, 66]. We emphasize this in Fig. 4.3 by comparing  $\lambda_e > 0$  and  $\lambda_e < 0$ , with  $\lambda_s > 0$ . In both cases we see that for small  $\lambda_e$  the system's excitations [Fig. 4.3(a,b)] tend to decay monotonically, which is what one would expect of a Markovian BS interaction with a vacuum bath. For larger  $\lambda_e$ , on the other hand, the occupations present oscillations. Since the interaction conserves the number of quanta, these revivals in excitations must necessarily be due to a backflow caused by the non-Markovian behavior. That is, some of the excitations that leave the system towards  $E_n$  are transferred from  $E_n$  to  $E_{n+1}$  and then make it back into the system in the  $SE_{n+1}$  interaction. The nature of these oscillations, however, is different whether  $\lambda_e > 0$  or  $\lambda_e < 0$ , being fast in the former and slow in the latter. Irrespective of the value of  $\lambda_e$ , however, after an infinite time the system will always thermalize to the ancilla's state, which in this case means  $\langle a^\dagger a \rangle^\infty = 0$  [the only exception is at  $\lambda_e = \pm\pi/2$ , which is somewhat pathological]. Despite its resemblance with the system's behavior, the ancilla's excitations [Fig. 4.3(c,d)] will have some unique features. First, the excitations spike at the first steps and then tend to decay monotonically. There are no lost excitations yet in the dropped ancillas so the ancilla gets the maximum amount from the system. And second, the number of excitations oscillates in the opposite manner to the system's oscillations. This is due to the conservation of total number of quanta: the amount that the system loses, the ancilla gains it. Finally, we can also look at the correlations build between the system and the  $n^{\text{th}}$  ancilla [Fig. 4.3(e,f)]. The modulus of the correlations will be influenced by the number of total excitations. This is related to the fact that, in order to correlate the system with the ancilla, there needs to be intermediary ancilla's excitations acting as messengers for the parties.

Results for the TMS interaction are shown in Fig. 4.4. In this case the relative signs are immaterial, but the dynamics becomes more sensitive on the magnitude of  $\nu_e$ , since  $\tilde{z}$  and  $\tilde{w}$  are hyperbolic functions. The TMS interaction entangles  $E_n E_{n+1}$ , even if both

are initially in the vacuum. As a consequence, it also spontaneously create excitations, so that the number of quanta is not preserved. At each  $E_n E_{n+1}$  collision the net number of excitations therefore increases. Part of these excitations are lost when the ancillas are discarded and part flow to the system. As a consequence, depending on the rate at which excitations are created, the dynamics can be either stable or unstable. This occurs at the critical point  $\nu_e^{\text{crit}} = \sinh^{-1}(1) \simeq 0.8813$ , which is when  $\tilde{w} = 1$ , thus marking the situation where the total number of excitations grow unboundedly [c.f. Eq. (4.23)]. When  $\nu_e < \nu_e^{\text{crit}}$  the dynamics will be stable and both, the system and the ancilla, will converge to an uncorrelated steady-state value  $\langle a^\dagger a \rangle = \sinh^2 \nu_e (1 - \sinh^2 \nu_e)^{-1}$  independently of  $\lambda_s$  [Fig. 4.4(a,c,e)]. Conversely, for  $\nu_e \geq \nu_e^{\text{crit}}$ , the dynamics becomes unstable. Therefore, the number of excitations and correlations diverge [Fig. 4.4(b,d,f)]. These asymptotic values can be understood from arguments of stability theory, as shown in Appendix A.



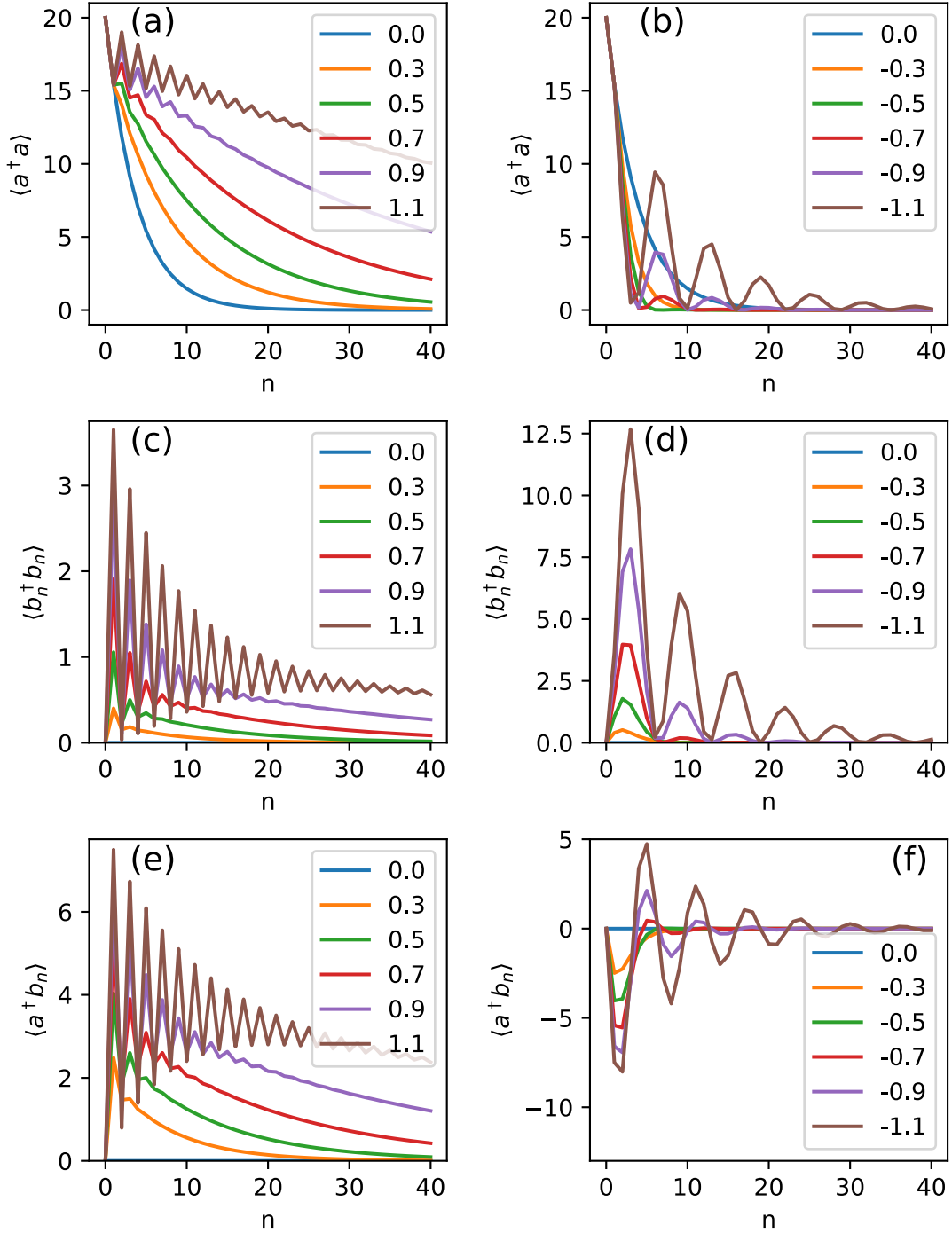


Figure 4.3: Second moments as a function of time, computed from Eq. (4.20) for the BS dynamics (4.21) with  $\lambda_s = 0.5$  and different values of  $\lambda_e$  (with  $\lambda_e > 0$  on the left column and  $\lambda_e < 0$  on the right column). The ancillas are assumed to start in the vacuum, and the system in a thermal state with  $\langle a^\dagger a \rangle^0 = 20$ . (a,b) Number of excitations in the system  $\langle a^\dagger a \rangle$ . (c,d) Number of excitation in the  $n^{\text{th}}$  ancilla  $\langle b_n^\dagger b_n \rangle$ . (e,f) Correlation function between the system and the  $n^{\text{th}}$  ancilla  $\langle a^\dagger b_n \rangle$ .

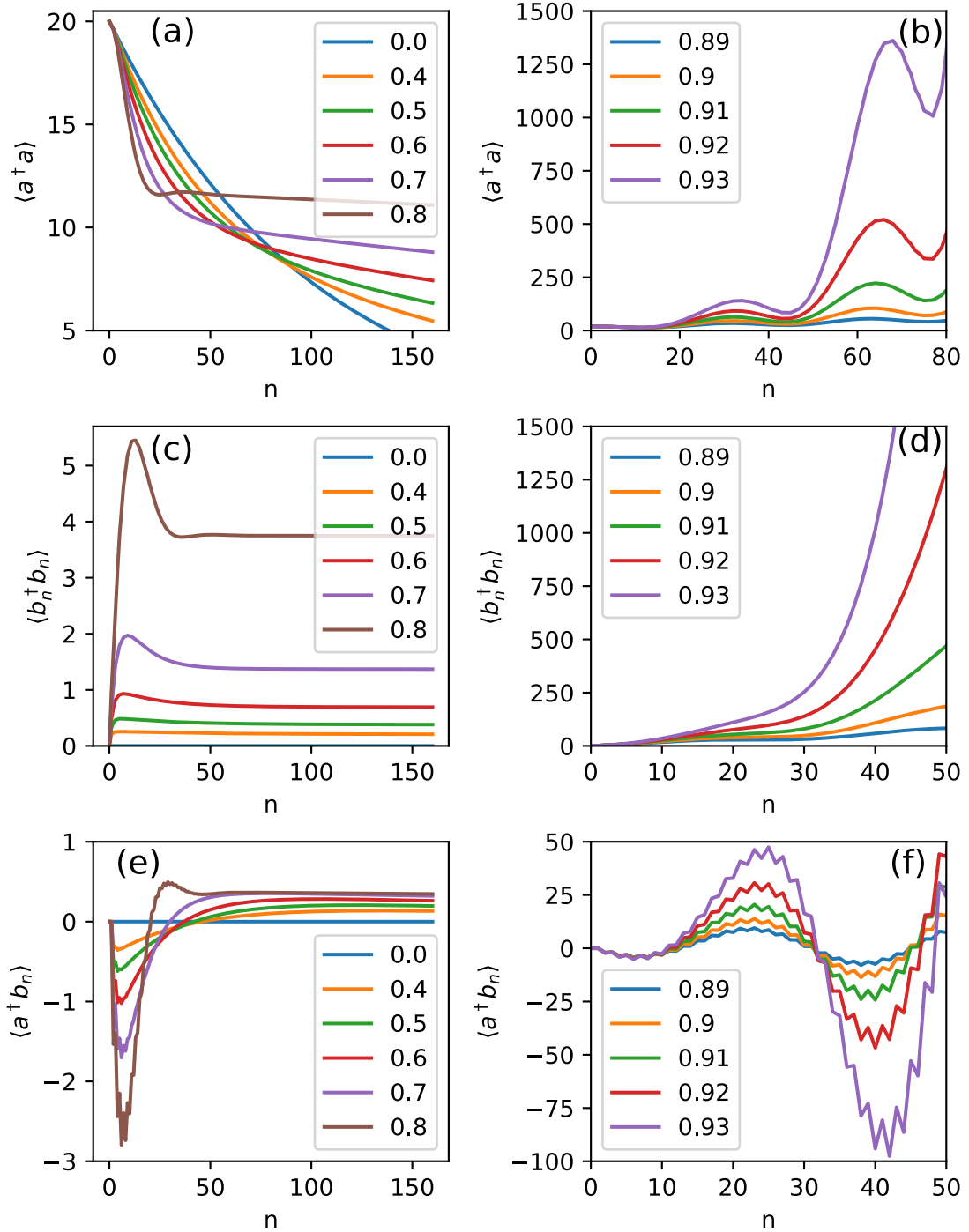


Figure 4.4: Second moments as a function of time, computed from Eq. (4.20) for the TMS dynamics (4.23) with  $\lambda_s = 0.1$  and different values of  $\nu_e$  (with  $\nu_e < \nu_e^{\text{crit}}$  on the left column and  $\nu_e \geq \nu_e^{\text{crit}}$  on the right column, where  $\nu_e^{\text{crit}} = \sinh^{-1}(1)$ ). The ancillas are assumed to start in the vacuum, and the system in a thermal state with  $\langle a^\dagger a \rangle^0 = 20$ . (a,b) Number of excitations in the system  $\langle a^\dagger a \rangle$ . (c,d) Number of excitations in the  $n^{\text{th}}$  ancilla  $\langle b_n^\dagger b_n \rangle$ . (e,f) Correlation function between the system and the  $n^{\text{th}}$  ancilla  $\langle a^\dagger b_n \rangle$ .

# Chapter 5

## Non-Markovianity

### 5.1 Quantum non-Markovianity

In Chapter 2, we reviewed the dynamics of open quantum systems, in which we describe the system's evolution by a Lindblad master equation (2.32). We found out in Sec. 2.3 that a microscopic derivation was possible provided we resort to some approximation schemes, for example the Born-Markov approximation. However, these approximations cannot be employed to many processes in open quantum systems and thus, the system's dynamics cannot be described by a divisible dynamical map. Typically, this is because the environment's correlation time scale is not small compared to the system's decoherence time, rendering the Markov approximation impossible. Examples are the cases of strongly correlated systems, low temperatures, finite reservoirs, or large initial correlations for which some experimental demonstrations can be found in [96–100]. These processes are called non-Markovian.

Non-Markovianity first arose in the theory of classical stochastic processes [101–104]. Its classical definition is based on classical probability theory where we say that a process is non-Markovian if the conditional probability of the future states of the process depends on a sequence of events that preceded it. Quantum theory, however, cannot be formulated as a statistical theory on classical probability space [5, 105]. Therefore, the concept of quantum non-Markovianity cannot be based on the classical notions only.

On the one hand, there's the notion of information flow between the system and the surrounding environment. In the quantum realm, information leaks are much more effi-

cient, so that when a system interacts with an environment, information about the former is inevitably transferred to the latter. When the environment is very large and complex, this information may never return. In general, however, there may be a partial backflow of information, which characterizes a non-Markovian evolution. From the point of view of causality, this backflow quantifies the ability of the dynamics to communicate past information to the future. On the other hand, there's the notion of map divisibility. For instance, considering the case of collisional models studied in Chapter 4, the map from  $\rho_S^0 \rightarrow \rho_S^n$  in Eq. (4.5) is always CPTP by construction. One may also define more general maps  $\mathcal{E}_{m \rightarrow n}$  taking the density matrix  $\rho_S^m$  to  $\rho_S^n$  ( $m < n$ ). Assuming that the inverse  $\mathcal{E}_n^{-1}$  exists, they are defined as  $\mathcal{E}_n \circ \mathcal{E}_m^{-1}$ . Albeit mathematically well defined, these maps are in general not CP. Markovian maps, conversely, are CP-divisible by construction. This defines the notion of CP-divisibility, which provides a widely used criteria for characterization of non-Markovianity: a map is CP-divisible when the intermediate maps  $\mathcal{E}_{m \rightarrow n}$  are CP.

Considerable attention was given in recent years on how to characterize and quantify non-Markovianity in the quantum domain (for recent reviews, see [19, 20]). Due to the richness involved, however, there is no single approach capable of capturing its full essence. Several quantifiers have been studied: dynamical divisibility [26, 106–108], trace distance [21, 23, 24], mutual information [27], relative entropy [28, 29], memory kernel [12–14], among others [30–34]. In this work, we will be particularly interested on the ones best suited for the collisional model setup described in Chapter 4. We will explore them in detail in the following sections.

## 5.2 Mutual Information

The first approach to quantum non-Markovianity to be discussed is based on the idea that memory effects in the open quantum system's dynamics are linked to the exchange of information between the system and its environment. While in a Markovian process the open system continuously loses information to the environment, a non-Markovian process is characterized by a flow of information from the environment back into the system. In this way, quantum non-Markovianity is associated with a notion of quantum

memory, namely, information which has been transferred to the environment, in the form of system-environment correlations or changes in the environmental states, and is later recovered by the system.

Non-Markovianity and backflow of information must be related to correlations that develop between system and bath. However, when a system interacts with multiple modes of the bath at the same time, there are many different correlations one may consider, between the system and all possible parts of the bath. And it is not clear which of these correlations are relevant for the non-Markovian evolution. For instance, the correlation with a part of the bath the system will never interact again is irrelevant, as far as non-Markovianity is concerned. But in the standard scenario, it is in general not possible to identify which are the relevant correlations.

In a collisional model picture, on the other hand, this is unambiguous: the relevant correlations are those between  $S$  and ancilla  $E_{n+1}$  at time  $n$  (see Fig. 4.1). These are the correlations that ancilla  $E_n$  transferred to  $E_{n+1}$  after its interaction. Hence, they represent the only possible source of information backflow at each collision. Conveniently, this is also exactly what the Markovian embedding (4.4) offers. A useful measure of correlations, for instance, is the quantum mutual information (MI), defined as

$$\mathcal{I}^n(SE_{n+1}) = S(\rho_S^n) + S(\rho_{E_{n+1}}^n) - S(\varrho^n), \quad (5.1)$$

where  $S(\rho) = -\text{tr}(\rho \ln \rho)$  is the von Neumann entropy. The states  $\rho_S^n$  and  $\rho_{E_{n+1}}^n$  are both computed from  $\varrho^n$  by taking the appropriate partial trace. Thus, by monitoring  $\varrho^n$  as a function of time, one has direct access to the relevant measure of correlation. Of course,  $\mathcal{I}^n(SE_{n+1})$  is not the only relevant measure of correlation. Different choices, from two-point functions, to quantifiers of entanglement and quantum discord, may also be of interest. The relevant point is that any such measure can be evaluated from  $\varrho^n$ .

### 5.3 Memory Kernel

A much older notion of non-Markovianity is that of a memory kernel, as present already in the seminal works of Nakajima and Zwanzig [10, 11]. The basic idea is that the open

dynamics of a system's density matrix  $\rho_S$  can, quite generally, be written as

$$\frac{d\rho_S}{dt} = -i[H_S, \rho_S] + \int_0^t \mathcal{K}_{t-t'}[\rho_S(t')] dt', \quad (5.2)$$

where  $\mathcal{K}_{t-t'}$ , called the memory kernel (MK), is a linear superoperator condensing all the information on how the evolution of  $\rho_S$  at time  $t$  depends on its past values. The MK has been studied intensively in recent years [12–17], as it provides clear insights onto the inner workings of non-Markovianity. It can also be given an operational interpretation, in terms of the so-called transfer tensors [109], rendering it accessible to experiments [110]. However, being a superoperator, it is generally difficult to compute it analytically. We also mention in passing the broader notion of process tensor, which includes also all possible input and output operations performed in the system [111–113].

In the original formulation of Nakajima and Zwanzig [10, 11], the memory kernel (MK) was represented as a single-parameter continuous superoperator  $\mathcal{K}_t$  that quantifies the memory that is retained about the system's configuration a time  $t$  in the past [c.f. Eq. (5.2)]. The collisional model analog of that will be a superoperator  $\mathcal{K}_n$ , labeled by the discrete time index  $n$ . Thus, the stroboscopic analog of Eq. (5.2) should be of the form

$$\rho_S^n = \sum_{m=0}^{n-1} \mathcal{K}_{n-m}(\rho_S^m). \quad (5.3)$$

For instance,  $\rho_S^3 = \mathcal{K}_1(\rho_S^2) + \mathcal{K}_2(\rho_S^1) + \mathcal{K}_3(\rho_S^0)$ . The term  $\mathcal{K}_1(\rho_S^{n-1})$  describes the short-term memory from the very last step, while  $\mathcal{K}_n(\rho_S^0)$  describes the long-term memory all the way from the initial state. An explicit formula for the MK can be constructed using a reasoning similar to that used for transfer tensors in Ref. [109]: Starting with the reduced map  $\mathcal{E}_n$  in Eq. (4.5), we define the MK recursively from

$$\mathcal{K}_n = \mathcal{E}_n - \sum_{m=1}^{n-1} \mathcal{K}_{n-m} \mathcal{E}_m. \quad (5.4)$$

That this is indeed the correct formula can now be verified by substituting it back in Eq. (5.3). For instance,  $\mathcal{K}_1 = \mathcal{E}_1$ ,  $\mathcal{K}_2 = \mathcal{E}_2 - \mathcal{K}_1 \mathcal{E}_1$ ,  $\mathcal{K}_3 = \mathcal{E}_3 - \mathcal{K}_2 \mathcal{E}_1 - \mathcal{K}_1 \mathcal{E}_2$ , and so on. Eq. (5.4) provides an algorithmic method for computing the MK. However, this requires

heavy numerics, even in simple cases. For instance,  $\mathcal{K}_3 = \mathcal{E}_3 - \mathcal{E}_2\mathcal{E}_1 - \mathcal{E}_1\mathcal{E}_2 + \mathcal{E}_1\mathcal{E}_1\mathcal{E}_1$ , and the complexity of the formulas only grow from there. One of the main results in this work will be to show that, for the case of Gaussian collisions, it is possible to write down a closed and compact formula for  $\mathcal{K}_n$  (Sec. 6.2), which provides valuable insight into the inner workings of the memory kernel.

## 5.4 CP-Divisibility

Lastly, let us return to the idea of the divisibility of the map. Given that the inverse map  $\mathcal{E}^{-1}$  exists for all times  $t > 0$ , we can define the family of intermediate maps  $\mathcal{E}_{m \rightarrow n}$  from state  $\rho_S^m$  to state  $\rho_S^n$  ( $m < n$ )

$$\mathcal{E}_{m \rightarrow n} = \mathcal{E}_n \circ \mathcal{E}_m^{-1}. \quad (5.5)$$

The existence of the inverse is what allows the notion of divisibility. Even though  $\mathcal{E}_n$  and  $\mathcal{E}_m$  are completely-positive by construction, the intermediate map  $\mathcal{E}_{m \rightarrow n}$  will not necessarily be. This is because the inverse map  $\mathcal{E}_m^{-1}$  of the completely-positive map  $\mathcal{E}_m$  need not to be positive.

Now, if these maps (5.5) are CP for all  $m$ , the evolution is Markovian. For non-Markovianity to occur there must exist a  $m$  value to not be completely positive. Hence, by measuring how much the intermediate map  $\mathcal{E}_{m \rightarrow n}$  departs from the CP map, we are measuring the degree of non-Markovianity of the time evolution. Testing CP divisibility, however, is not always easy. For instance, it may require analyzing the distance between different pairs of initial conditions  $\rho_S^0$  [20]. According to the data processing inequality, these distances are always contractive for CP maps. Violations of contractivity are thus identified as violations of divisibility. This, however, requires a maximization over all possible initial conditions. In Sec. 6.3 we discuss the Gaussian version of this concept and show that the maximization is replaced by an alternative condition, that provides a clean and easily applicable formula for quantifying CP-divisibility.

# Chapter 6

## Memory effects in Gaussian Collisional Models

This chapter continues with the remaining original results obtained in this dissertation. The exposition here follows Sec. III and IV of the submitted arXiv preprint [94].

### 6.1 Gaussian Mutual Information

The Gaussian framework developed in Sec. 4.3 makes the mutual information (5.1) readily accessible from the CM  $\gamma^n$  in Eq. (4.19). Correlations are related to the off-diagonal blocks  $\xi_{n+1}^n$  (the MI would be zero if  $\gamma^n$  was block-diagonal). The three entropies in Eq. (5.1) can be readily computed in terms of the symplectic eigenvalues of  $\gamma^n$  [67]. Illustrative results are shown in Fig. 6.1, for the same collection of parameters as Fig. 4.3 and Fig. 4.4. As a sanity check, the MI is identically zero when  $\lambda_e = \nu_e = 0$ . It also tends to be larger for short times, tending to zero as  $n$  grows. The only exception is the unstable dynamics in Fig. 6.1(d), where the MI grows unboundedly. The oscillatory patterns in  $\langle a^\dagger a \rangle$  are also present in the MI. The mutual information will also be highly dependent on the total number of excitations, as intermediary ancilla excitations are needed for information backflow (See also Sec. 4.5).

To better understand the role of the MI in the non-Markovian dynamics we present in Fig. 6.2 a comparison between the occupation number  $\langle a^\dagger a \rangle$  of Fig. 4.3(a) and the MI of Fig. 6.1 for the BS dynamics. We focus on early times (small  $n$ ) and also compare  $\langle a^\dagger a \rangle$



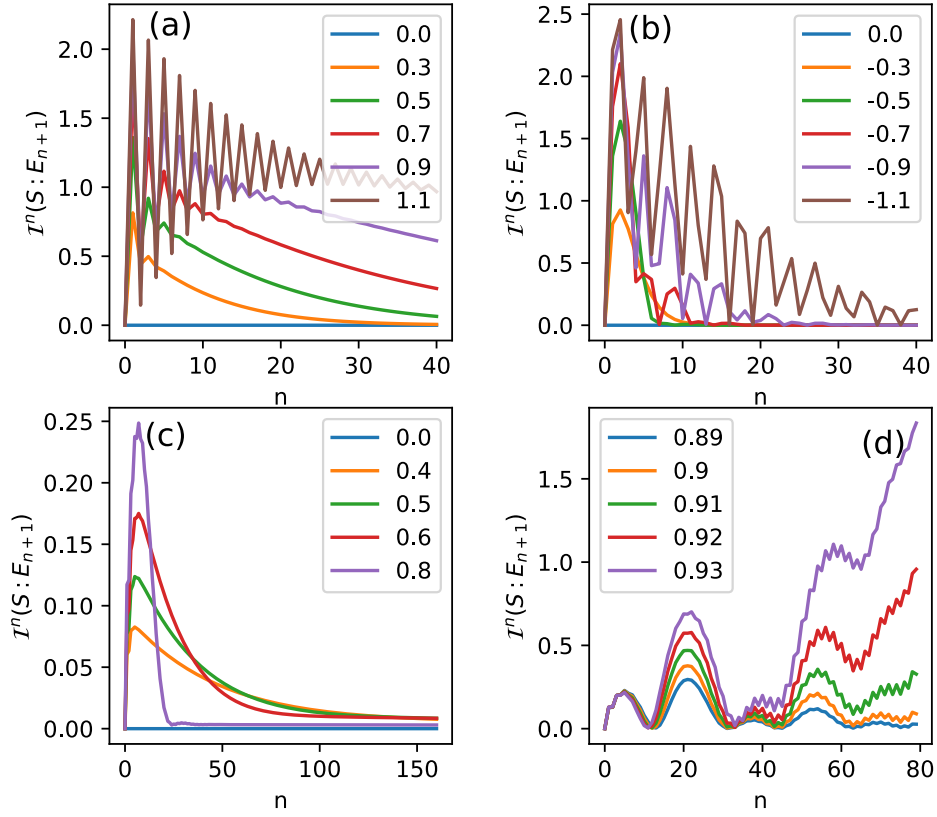


Figure 6.1: Quantum Mutual Information (5.1) for the BS (a,b) with  $\lambda_s = 0.5$  ( $\lambda_e > 0$  in (a) and  $\lambda_e < 0$  in (b)), and TMS (c,d) dynamics with  $\lambda_s = 0.1$  ( $\nu_e < \nu_e^{\text{crit}}$  in (a) and  $\nu_e \geq \nu_e^{\text{crit}}$  in (b)). Other parameters are the same as Fig. 4.3 and Fig. 4.4.

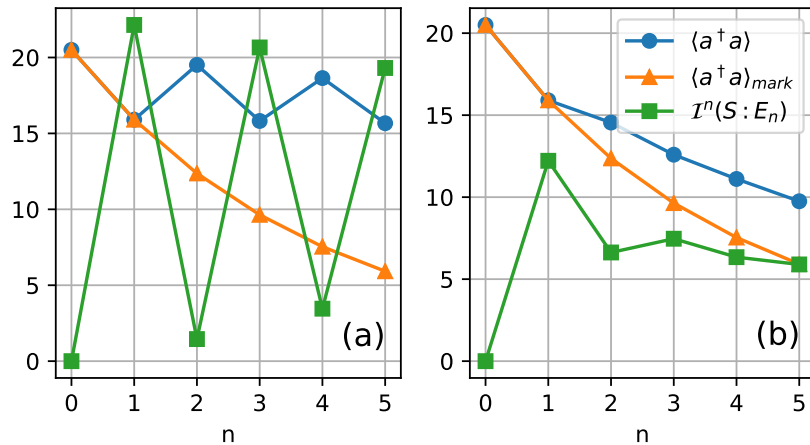


Figure 6.2: Comparison between Markovian and non-Markovian dynamics and role of the Mutual Information. In blue circles we show the early dynamics of  $\langle a^\dagger a \rangle$  vs.  $n$  for the BS dynamics with (a)  $\lambda_e = 1.1$  and (b)  $\lambda_e = 0.3$ , with fixed  $\lambda_s = 0.5$  [c.f. Fig. 4.3(a)]. The corresponding Markovian case ( $\lambda_e = 0$ ) is shown in orange triangles. These curves are to be compared with the MI (5.1), shown by green squares in the two cases [Fig. 6.1(a)]. The heights of each curve were adjusted for better visibility.

with the corresponding Markovian dynamics ( $\lambda_e = 0$ ). The difference between the non-Markovian (blue circles) and Markovian (orange triangles) dynamics reflects the extent to which the backflow of information affects the evolution. This, as can be seen in the figure, is directly correlated with the MI (green squares) of the previous step. That is, a large MI in a given step implies a large difference between the blue and orange curves in the following one. This is particularly clear in Fig. 6.2(a) and serves to illustrate how the correlations built between  $SE_{n+1}$ , at step  $n$ , affect the future interaction between  $S$  and  $E_{n+1}$  at the next step.

## 6.2 Memory Kernel

The Memory Kernel (MK) discussed in Eqs. (5.2) and (5.3), and Sec. 5.3, is perhaps the most physically transparent way of analyzing non-Markovianity. Starting from any global map between system and bath, one can always write down a differential equation for the reduced density matrix  $\rho_S$  of the system. This equation, however, will in general be time-non-local; i.e., it will be an integro-differential equation of the form (5.2), where  $\mathcal{K}_{t-t'}[\rho_S(t')]$  describes how  $d\rho_S(t)/dt$  depends on  $\rho_S(t')$  in previous times  $t' < t$ . The MK therefore contains all the information about the dynamics, with non-Markovianity being related to its overall dependence on  $t - t'$ : the slower the decay of  $\mathcal{K}_{t-t'}$  with  $t - t'$ , the longer the memory and hence the more non-Markovian is the dynamics. The Markovian case is recovered when  $\mathcal{K}_{t-t'} \propto \delta(t - t')$ .

The memory kernel  $\mathcal{K}_{t-t'}$  is a superoperator acting on the full Hilbert space of the system. Computing it is thus, in general, a very difficult task. Within our framework, however, one may equivalently formulate a memory kernel acting only in the system's CM  $\theta^n$ . This can be accomplished starting from Eq. (4.20) and writing down a difference equation for  $\theta^n$  only. As we will demonstrate below, this equation will have the form (contrast with (2.12) and (5.3)):

$$\theta^{n+1} = x^2\theta^n + \sum_{r=0}^{n-1} \mathcal{K}_{n-r-1}(\theta^r) + G_n, \quad (6.1)$$

where  $G_n$  is a contribution that depends only on the initial state of the ancillas <sup>1</sup> and  $\mathcal{K}_n$  is the memory kernel. The way we define the MK is such that  $\mathcal{K}_0$  measures how the step from  $\theta^n$  to  $\theta^{n+1}$  is affected by  $\theta^{n-1}$  and  $\mathcal{K}_{n-1}$  measures how it is affected by  $\theta^0$ .  $\mathcal{K}_n$  is still a superoperator, but one which acts on the space of  $2 \times 2$  CMs. One can write it more explicitly in terms of a Kraus operator-sum representation [4, 114]

$$\mathcal{K}_n(\theta) = \sum_{ij} \kappa_{ij}^n M_i \theta M_j^T, \quad (6.2)$$

where  $\kappa_{ij}^n$  are coefficients that depend on time and  $\{M_i\}$  are a complete set of  $2 \times 2$  matrices; a convenient choice is the set of Pauli matrices  $\{\mathbb{I}_2, \sigma_z, \sigma_+, \sigma_-\}$ . A general recipe to compute the coefficients  $\kappa_{ij}^n$  in Eq. (6.2) is given below in Eq. (6.21). Crucially, as we show, it depends *only* on the matrix  $X$  of the Markovian embedding (4.20).

The memory itself is contained in the dependence of  $\kappa_{ij}^n$  on  $n$ . The dependence on  $i, j$  determines how different elements of  $\theta^r$  affect  $\theta^n$ . For instance, as we will show below, in the case of the BS map [Eq. (4.21)], the only non-zero coefficient will be the one proportional to  $\mathbb{I}_2 \theta \mathbb{I}_2 = \theta$ , which we refer to as  $\kappa_{11}^n$ ; that is, the memory kernel is actually a  $c$ -number,  $\mathcal{K}_n(\theta) = \kappa_{11}^n \theta$ . This implies that the MK is the same for all entries of  $\theta^n$  and each entry  $(\theta^n)_{ij}$  is only affected by the corresponding entry  $(\theta^r)_{ij}$  at past times. Conversely, in the TMS map there will be four non-zero coefficients, corresponding to combinations of  $M_1 = \mathbb{I}_2$  and  $M_2 = \sigma_z$ ; we refer to them as  $\kappa_{11}^n, \kappa_{1,z}^n, \kappa_{z,1}^n$  and  $\kappa_{z,z}^n$ . This means that the memory kernel of  $(\theta^n)_{11}$  will be different from that of  $(\theta^n)_{2,2}$  and so on (each entry will have its own memory kernel). Finally, a memory kernel containing a dependence on  $\sigma_{\pm}$  would imply that  $(\theta^n)_{11}$  would depend on the past values of other entries, such as  $(\theta^r)_{12}$  and  $(\theta^r)_{22}$ .

### 6.2.1 General derivation of the Memory Kernel

We now carry out the derivation of the memory kernel for the Gaussian collisional model. We consider here a more general scenario, which relies only on the structure of the Markovian embedding in Eq. (4.20). We also assume that the system and ancillas are each

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<sup>1</sup>In principle one could also interpret  $G_n$  as being part of the memory kernel. However, we have opted not to do so, since  $G_n$  refers only to the states of the ancillas, while the actual dependence on the state of the system, which is what one expects from a memory kernel to model, is fully contained in  $\mathcal{K}_n$ .

composed of an arbitrary number of modes  $N_S$  and  $N_E$  [Eqs. (4.21) and (4.23) are recovered for  $N_S = N_E = 1$ ]. More specifically, we take the matrices  $X$  and  $Y$  to have the following block structure,

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & Y_{22} \end{pmatrix}, \quad (6.3)$$

where e.g.,  $X_{11}$  and  $X_{22}$  are of size  $2N_S$  and  $2N_E$  respectively. This therefore contemplates both multi-mode systems and ancillas, as well as collisions with longer memory. For instance, if  $E_n$  collides with  $E_{n+1}$  and  $E_{n+2}$ , then we would have  $N_S = 1$  and  $N_E = 2$ .

Our derivation follows the general approach of Nakajima and Zwanzig [10, 11], but adapted to the present context. We begin by noting the following property [115]: the solution of a generic difference equation of the form

$$\psi(n+1) = \alpha\psi(n) + g(n), \quad (6.4)$$

is given by

$$\psi(n) = \alpha^n \psi(0)_+ + \sum_{r=0}^{n-1} \alpha^{n-r-1} g(r). \quad (6.5)$$

This solution holds for arbitrary objects  $\psi$ , provided  $\alpha$  is a linear operator. It therefore holds when  $\psi$  is a vector and  $\alpha$  is a matrix, or when  $\psi$  is a matrix and  $\alpha$  is a superoperator. Thus, for instance, the solution of Eq. (4.20) is

$$\gamma^n = X^n \gamma^0 (X^T)^n + \sum_{r=0}^{n-1} X^{n-r-1} Y (X^T)^{n-r-1}. \quad (6.6)$$

Here the notation  $\gamma^n$ , to denote the time index, becomes a bit ambiguous since  $X^n$  is the matrix  $X$  to the power  $n$ . But there is no room for confusion, since  $X^n$  will be the only quantity where the superscript does not refer to the time.

We now introduce the vectorization operation [116], which transforms a matrix  $A$  into

a vector  $\vec{A} = \text{vec}(A)$  by stacking its columns. For instance,

$$\text{vec} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix}. \quad (6.7)$$

One may verify that, for any three matrices  $A, B, C$ ,

$$\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B). \quad (6.8)$$

With this, the matrix difference equation (4.20) is converted into a vector difference equation

$$\vec{\gamma}^{n+1} = (X \otimes X)\vec{\gamma}^n + \vec{Y}. \quad (6.9)$$

We also introduce projection matrices onto the subspaces of system and ancilla,

$$P_S = \begin{pmatrix} \mathbb{I}_{2N_S} & 0 \\ 0 & 0 \end{pmatrix}, \quad P_E = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_{2N_E} \end{pmatrix}, \quad (6.10)$$

which are of size  $2N_S + 2N_E$ . In the larger space relevant for vectorization there are four possible projections,  $P_S(\dots)P_S$ ,  $P_S(\dots)P_E$  and so on. These operations chop the covariance matrix  $\gamma^n$  in 4 blocks, as in Eq. (4.19). Our interest is in  $P_S(\gamma^n)P_S$ , as it contains the system CM  $\theta^n$ . We therefore also introduce

$$P = P_S \otimes P_S, \quad (6.11)$$

together with its complement  $Q = 1 - P$ . Note, though, that  $Q \neq P_E \otimes P_E$ .

We now multiply Eq. (6.9) by  $P$  and use that  $P + Q = 1$ , together with the fact that  $P\vec{Y} = 0$  [c.f. Eq. (6.3)]. We then get

$$P\vec{\gamma}^{n+1} = P(X \otimes X)P\vec{\gamma}^n + P(X \otimes X)Q\vec{\gamma}^n. \quad (6.12)$$

Similarly, multiplying Eq. (6.9) by  $Q$  we find

$$Q\vec{\gamma}^{n+1} = Q(X \otimes X)Q\vec{\gamma}^n + Q(X \otimes X)P\vec{\gamma}^n + \vec{Y}. \quad (6.13)$$

Now comes the crucial idea of the Nakajima and Zwanzig method [10, 11]. We interpret Eqs. (6.12) and (6.13) as two coupled equations for the variables  $P\vec{\gamma}^n$  and  $Q\vec{\gamma}^n$ . Since our interest is in  $P\vec{\gamma}^n$ , we first solve Eq. (6.13), assuming a given  $P\vec{\gamma}^n$ , and then substitute the result in Eq. (6.12). Eq. (6.13) is of the form (6.4) with  $\alpha = Q(X \otimes X)$  and  $g(n) = Q(X \otimes X)P\vec{\gamma}^n + \vec{Y}$ . Eq. (6.5) then gives

$$Q\vec{\gamma}^n = (Q(X \otimes X))^n Q\vec{\gamma}^0 + \sum_{r=0}^{n-1} (Q(X \otimes X))^{n-r-1} [Q(X \otimes X)P\vec{\gamma}^r + \vec{Y}].$$

Plugging this in Eq. (6.12) we then arrive at

$$P\vec{\gamma}^{n+1} = P(X \otimes X)P\vec{\gamma}^n + \sum_{r=0}^{n-1} \hat{K}_{n-r-1} P\vec{\gamma}^r + \vec{\mathcal{G}}_n, \quad (6.14)$$

where

$$\hat{K}_{n-r-1} = P(X \otimes X)[Q(X \otimes X)]^{n-r-1}Q(X \otimes X), \quad (6.15)$$

is the memory kernel in vectorized form (i.e., as a matrix of size  $(2N_S + 2N_E)^2$ ). The term  $\vec{\mathcal{G}}_n$ , on the other hand, is a function that depends only on the initial state of the ancillas and reads

$$\vec{\mathcal{G}}_n = P(X \otimes X)[Q(X \otimes X)]^n Q\vec{\gamma}^0 + \sum_{r=0}^{n-1} P(X \otimes X)[Q(X \otimes X)]^{n-r-1} \vec{Y}.$$

What is left is to rewrite Eq. (6.14) as an equation for the evolution of the system's CM  $\theta^n$  only. We introduce the  $(2N_S)^2 \times (2N_S + 2N_E)^2$  rectangular matrix  $\pi$  defined such that  $\pi\vec{\gamma}^n = \vec{\theta}^n$ . For instance, in the case  $N_S = N_E = 1$ , the matrix  $\pi$  will be  $4 \times 16$ , of

the form (for more intuition on this matrix, see Appendix B)

$$\pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \end{pmatrix} \quad (6.16)$$

We also notice that  $P = \pi^T \pi$  and  $\pi \pi^T = \mathbb{I}_{(2N_S)^2}$ . Multiplying Eq. (6.14) on the left by  $\pi$  we then get

$$\vec{\theta}^{n+1} = (X_{11} \otimes X_{11}) \vec{\theta}^n + \sum_{r=0}^{n-1} \hat{\mathcal{K}}_{n-r-1} \vec{\theta}^r + \vec{G}_n, \quad (6.17)$$

where we also used the fact that  $\pi(X \otimes X) \pi^T = X_{11} \otimes X_{11}$ . Here  $\vec{G}_n = \pi \vec{\mathcal{G}}_n$  is again a term that depends only on the initial conditions of the ancillas, whereas

$$\hat{\mathcal{K}}_n = \pi \hat{K}_n \pi^T = \pi(X \otimes X) [Q(X \otimes X)]^{n+1} \pi^T,$$

is the memory kernel, now expressed as a matrix of size  $(2N_S)^2 \times (2N_S)^2$  acting on  $\vec{\theta}^r$ . This can also be written more symmetrically, by exploiting the fact that  $Q^2 = Q$ . We can then arrange it as

$$\hat{\mathcal{K}}_n = \pi(X \otimes X) Q [Q(X \otimes X) Q]^n Q(X \otimes X) \pi^T. \quad (6.18)$$

The extra  $Q$ 's outside the square brackets are placed simply to ensure the result also holds for  $n = 0$ . This is the final form of the MK. Crucially, notice how it depends *only* on the matrix  $X$  of the Markovian embedding (4.20).

To obtain a matrix difference equation for  $\theta^n$  we must “unvec” Eq. (6.17); that is, apply the inverse map of (6.7). Unvecing the first term is trivial since, by Eq. (6.8),

$$\text{unvec}[(X_{11} \otimes X_{11}) \vec{\theta}^n] = X_{11} \theta^n X_{11}^T.$$

The memory kernel (6.18), on the other hand, cannot be unvec'ed as a single product of

$A\theta^n B$ . Instead, it is convenient to express it as

$$\hat{\mathcal{K}}_n = \sum_{ij} \kappa_{ij}^n M_j \otimes M_i, \quad (6.19)$$

where  $\kappa_{ij}^n$  are real coefficients and  $\{M_i\}$  are a set of operators spanning the vector space of  $2N_S$ -dimensional real matrices. Decomposed in this form, the unvecked version of the memory kernel will then be, from (6.8),

$$\mathcal{K}_n(\theta) = \sum_{ij} \kappa_{ij}^n M_i \theta M_j^T. \quad (6.20)$$

Finally, the form of the coefficients  $\kappa_{ij}^n$  can be found if we assume that the  $M_i$  form an orthogonal basis with respect to the Hilbert-Schmidt norm ( $A|B) = \text{tr}(A^T B)$  (which is the case of the Pauli basis, for instance). Multiplying Eq. (6.19) by  $M_j \otimes M_i$  and tracing then yields, by orthogonality,

$$\kappa_{ij}^n = \frac{\text{tr} [(M_j^T \otimes M_i^T) \hat{\mathcal{K}}_n]}{\text{tr}(M_i^T M_i) \text{tr}(M_j^T M_j)}. \quad (6.21)$$

This, together with Eq. (6.18), is all that is required to compute the memory kernel. With all these definitions, one may now finally unvec Eq. (6.17), leading to

$$\theta^{n+1} = X_{11} \theta^n X_{11}^T + \sum_{r=0}^{n-1} \mathcal{K}_{n-r-1}(\theta^r) + G_n, \quad (6.22)$$

where  $G_n = \text{unvec}(\vec{G}_n) = \text{unvec}(\pi \vec{\mathcal{G}}_n)$  is, again, a term depending only on the initial states of the ancillas.

## 6.2.2 Memory Kernel for the BS dynamics

We now illustrate the memory kernel for the two maps considered in Sec. 4.3, starting with the BS dynamics. In general, the structure of the memory kernel will be quite complicated. For the BS dynamics [Eq. (4.21)], however, the only non-zero coefficient in Eq. (6.21) is  $\kappa_{11}^n$ , the term proportional to the identity. In this case the memory kernel is therefore rather simple, as it is just a  $c$ -number multiplying all entries of  $\theta^r$ . A more compact formula for



the MK in this case is given in Appendix B.

Results for the BS dynamics are shown in Fig. 6.3. The upper panel corresponds to  $\lambda_s = 0.5$ , which is similar to Fig. 4.3. As can be seen, for  $\lambda_e > 0$  (Fig. 6.3(a)) the memory kernel's decay is oscillatory, with an exponential envelope. For  $\lambda_e < 0$ , oscillations are also observed, but these are rather different in nature and more asymmetrical with respect to the horizontal axis. When  $\lambda_s = 0.05$  the situation changes (Figs. 6.3(c) and (d)). The dynamics of  $\langle a^\dagger a \rangle$  is still quite similar to that of  $\lambda_s = 0.5$ , shown in Fig. 4.3, except that the time-scales become much longer. But in the MK one sees something entirely different. In particular, one finds that while  $\kappa_{11}^n$  continues to oscillate when  $\lambda_e > 0$ , it now becomes exclusively negative for  $\lambda_e < 0$ . In this case therefore, all past values of  $\theta^r$  tend to contribute negatively to the evolution.

Negative values in the memory kernel are rather important, as they are associated with faster convergence. The reason is that the CM is a positive matrix and the first

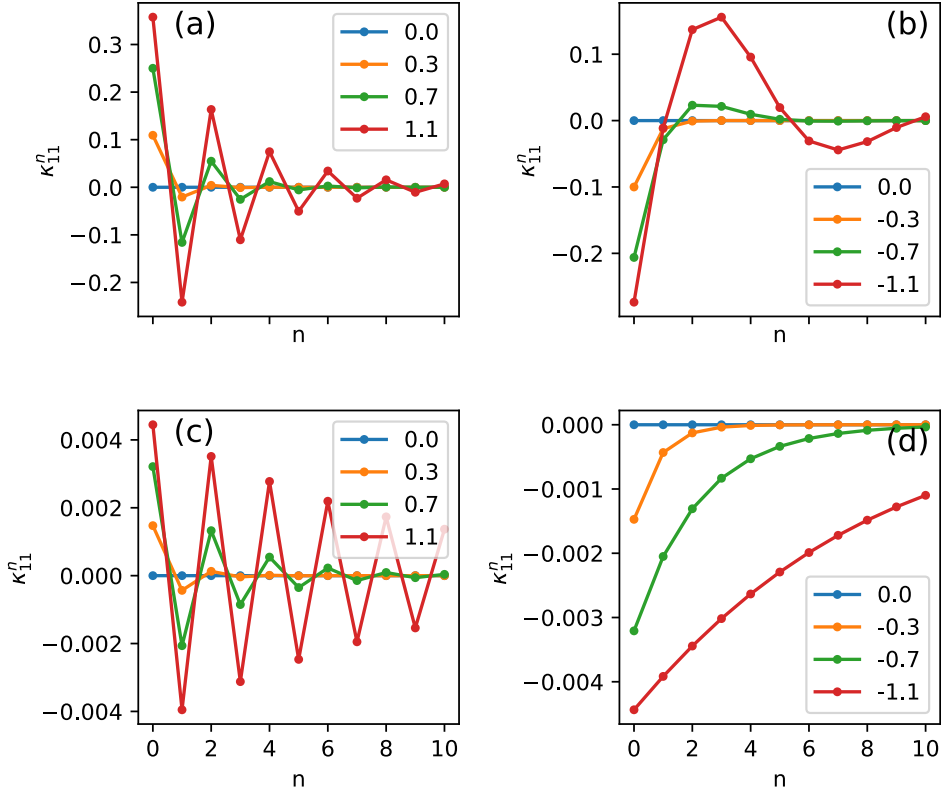


Figure 6.3: The memory Kernel for the BS dynamics, Eq. (4.21). In this case the only non-zero entry in Eq. (6.2) is  $\kappa_{11}^n$ , the term proportional to the identity. The plots are for  $\lambda_s = 0.5$  (upper panel) and  $\lambda_s = 0.05$  (lower panel), with  $\lambda_e > 0$  (left) and  $\lambda_e < 0$  (right).

term in (6.2) is always positive. The negativities observed in Fig. 6.3 therefore represent an accelerated draining of excitations from the system. This sheds light on some of the behaviors previously observed for the number operator (Fig. 4.3) and mutual information (Fig. 6.1).

It is possible to condense a lot of information about the memory kernel by plotting  $\kappa_{11}^n$  in the  $(\lambda_s, \lambda_e)$  plane, for different values of  $n$ . This is shown in Fig. 6.4. Each plot corresponds to a different value of  $n$ , from 0 up to 8. The dependence on the relative signs of  $\lambda_s$  and  $\lambda_e$  is clearly visible, as is the overall damping of the memory with increasing

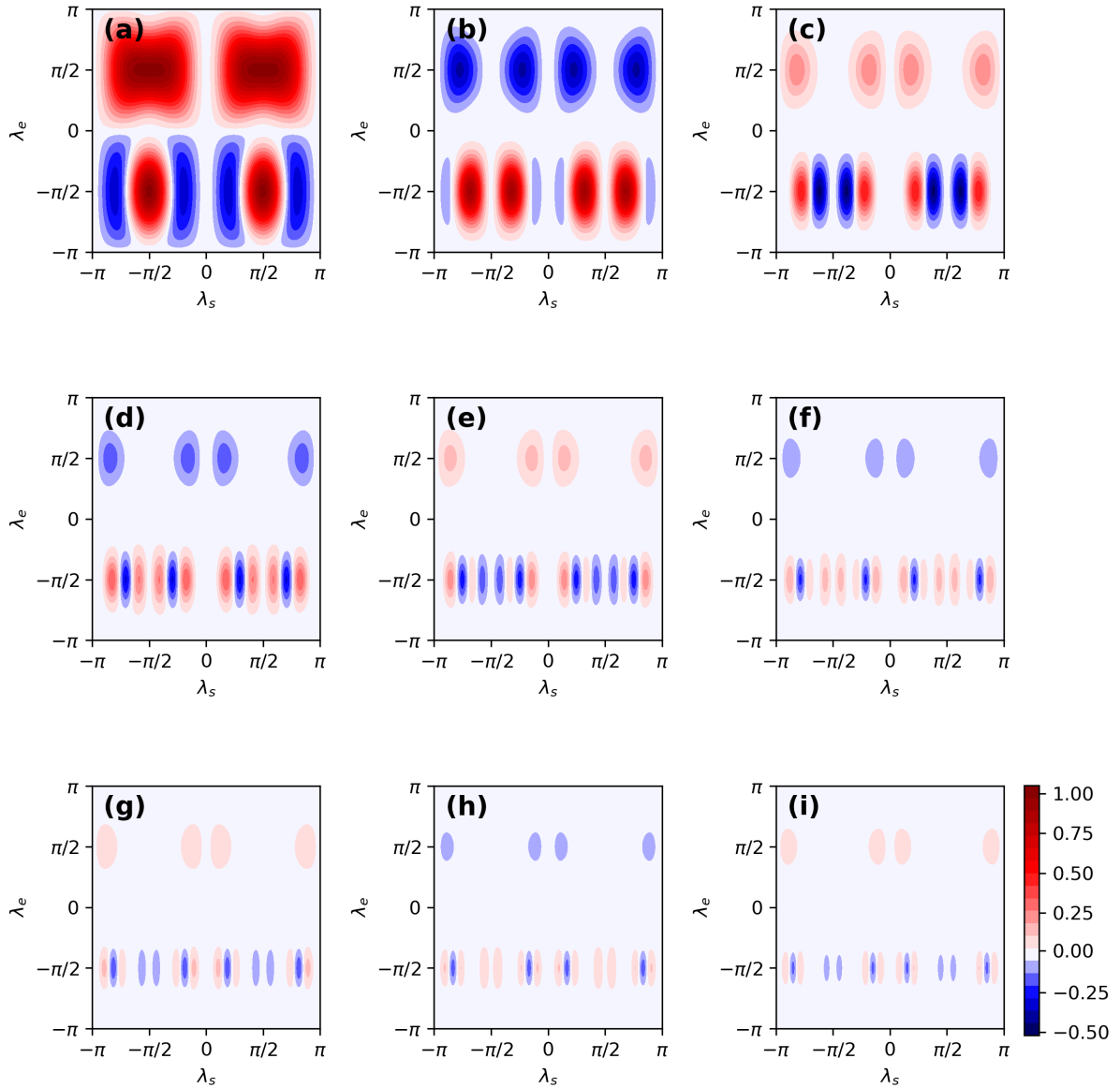


Figure 6.4: Diagrams for the memory kernel of the BS dynamics. Each plot shows  $\kappa_{11}^n$  in the  $(\lambda_s, \lambda_e)$  plane for a different value of  $n$ , from  $n = 0$  to  $n = 8$ .

$n$ . Particularly interesting, this map is able to very clearly pinpoint which regions have negative memory kernels, something which is found to be highly non-trivial.

### 6.2.3 Memory Kernel for the TMS dynamics

Next we turn to the TMS case (4.23). In this case we found the MK is proportional to four terms, namely

$$\mathcal{K}_n(\theta) = \kappa_{11}^n \theta + \kappa_{1z}^n \theta \sigma_z + \kappa_{z1}^n \sigma_z \theta + \kappa_{zz}^n \sigma_z \theta \sigma_z. \quad (6.23)$$

These quantities are plotted in Fig. 6.5 for the stable dynamics ( $\nu_e < \nu_e^{\text{crit}}$ ), with  $\lambda_s = 0.1$ . All four coefficients are found to decay in time in an oscillatory fashion.

The physics of each coefficient, however, is not necessarily transparent. In order to

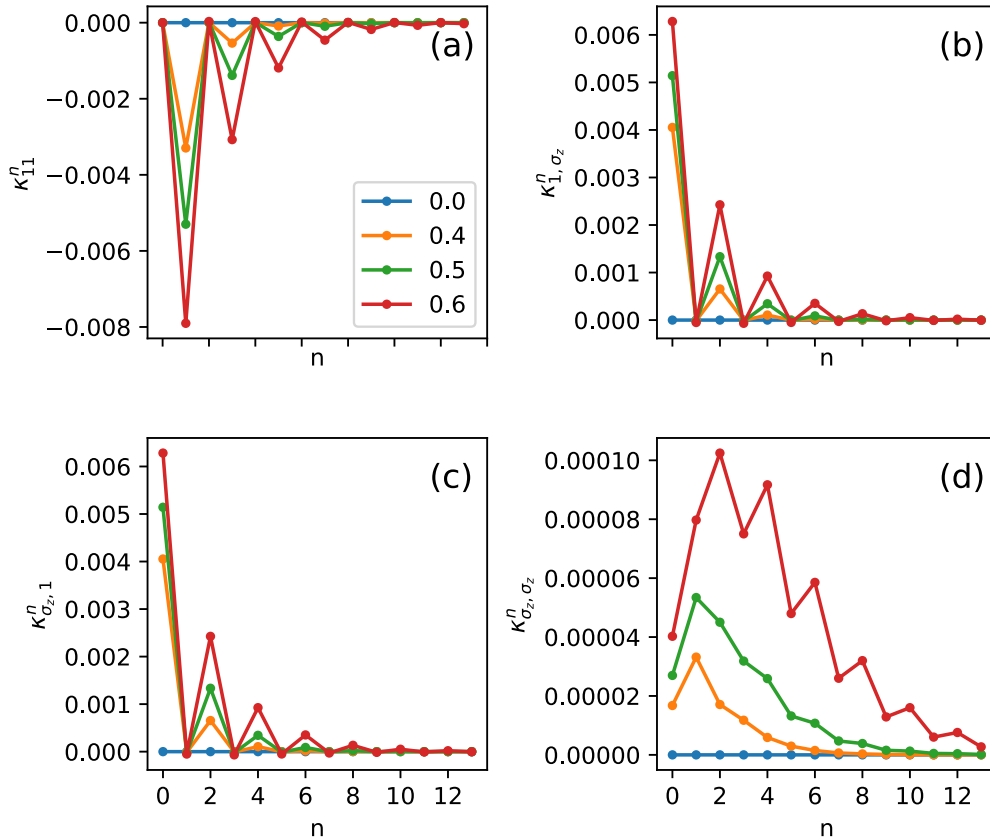


Figure 6.5: The memory Kernel for the (stable) TMS dynamics, Eq. (4.23) with  $\lambda_s = 0.1$  and different values of  $\lambda_e$ . Each curve corresponds to a different entry of Eq. (6.2); namely,  $\kappa_{11}^n$ ,  $\kappa_{1,\sigma_z}^n$ ,  $\kappa_{\sigma_z,1}^n$  and  $\kappa_{\sigma_z,\sigma_z}^n$ .

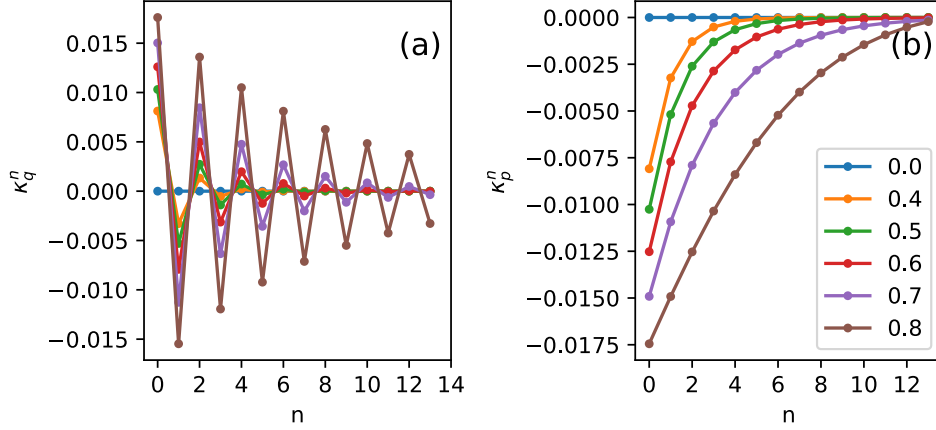


Figure 6.6: The MK for  $\langle Q^2 \rangle$  and  $\langle P^2 \rangle$ , Eq. (6.24), for the TMS dynamics. Other parameters are the same as Fig. 6.5.

gain better intuition, let us focus on the diagonal entries of  $\theta^n$ . In this case one finds that

$$\left( \mathcal{K}_n(\theta) \right)_{11} = (\kappa_{11}^n + \kappa_{1z}^n + \kappa_{z1}^n + \kappa_{zz}^n) \theta_{11}^n := \kappa_q^n \theta_{11}^n, \quad (6.24)$$

$$\left( \mathcal{K}_n(\theta) \right)_{22} = (\kappa_{11}^n - \kappa_{1z}^n - \kappa_{z1}^n + \kappa_{zz}^n) \theta_{22}^n := \kappa_p^n \theta_{22}^n.$$

The coefficients  $\kappa_q^n$  and  $\kappa_p^n$  therefore describe the individual memory kernels of  $\langle Q^2 \rangle$  and  $\langle P^2 \rangle$ , which are different in the TMS dynamics.

These two contributions are shown in Fig. 6.6, for the same parameters as in Fig. 6.5. We also present diagrams in the  $(\lambda_s, \nu_e)$  plane in Figs. 6.7 and 6.8. The plots in Fig. 6.6 reveal an extremely interesting asymmetry between the two quadratures. We see that the memory associated with  $\langle Q^2 \rangle$  is oscillatory, whereas that associated with  $\langle P^2 \rangle$  is always negative and decays monotonically. This asymmetry is a consequence of our choice of two-mode squeezing in the TMS interaction (4.8). Figs. 6.7 and 6.8, however, show that the situation is more intricate. Indeed, for fixed  $(\lambda_s, \nu_e)$ ,  $\kappa_q$  is found to oscillate with  $n$ . But for  $\kappa_p$  this is not necessarily the case.

Finally, in Fig. 6.9 we compare the previous result with the case of  $\nu_e$  in the vicinity, and larger than,  $\nu_e^{\text{crit}} = 0.8813$ ; i.e., in the situation where the dynamics diverges. As can be seen, in this case both  $\kappa_q$  and  $\kappa_p$  diverge as well (notice the different scale of the vertical axis). This is therefore contrary to our usual notion of memory: It means that the

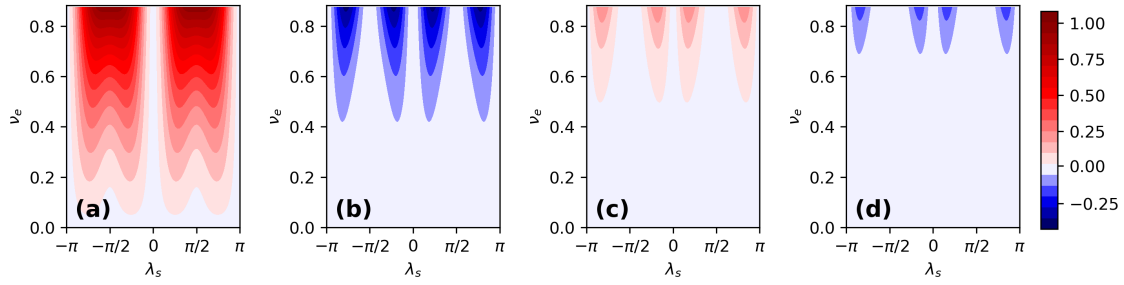


Figure 6.7: Diagrams for the memory kernel coefficient  $\kappa_q^n$  [Eq. (6.24)] of the TMS dynamics, in the  $(\lambda_s, \nu_e)$  plane, for  $n = 0, \dots, 3$ .

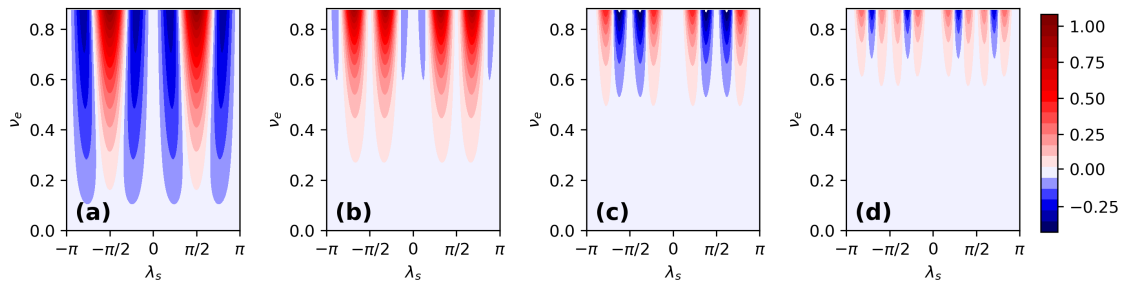


Figure 6.8: Similar to Fig. 6.7, but for  $\kappa_p^n$ .

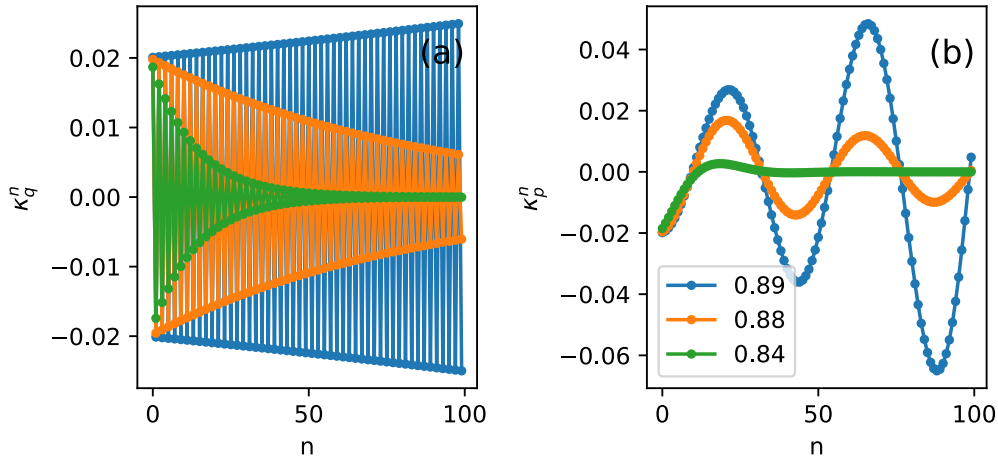


Figure 6.9: Similar to Fig. 6.6, but for values of  $\nu_e$  close to, and larger than,  $\nu_e^{\text{crit}} = 0.8813$ .

system retains a stronger memory from events in the distant past, than those in the recent one. In other words, the relative importance of past events accumulate.

### 6.3 Gaussian CP-Divisibility

Even though the MK explicitly shows the dependence on previous states, this alone does not necessarily imply a non-Markovian dynamic [15]. It is therefore important to contrast the MK with an actual test of non-Markovianity. Here we focus on CP-divisibility of intermediate maps, introduced in Sec. 5.4. The formulation for Gaussian dynamics, at the level of the covariance matrix, can be found in Refs. [107, 108]. Any Gaussian CPTP map, as seen in Sec. 3.4, must have the form  $\theta \rightarrow \mathcal{X}\theta\mathcal{X}^T + \mathcal{Y}$ , where  $\mathcal{X}$  and  $\mathcal{Y}$  are matrices satisfying [67, 117]

$$\mathcal{M}[\mathcal{X}, \mathcal{Y}] := 2\mathcal{Y} + i\Omega - i\mathcal{X}\Omega\mathcal{X}^T \geq 0, \quad (6.25)$$

with  $\Omega = i\sigma_y$  the symplectic form. Here  $\mathcal{M} \geq 0$  means the matrix must be positive semidefinite.

In our case, the evolution of the system's CM, from time 0 to  $n$ , must therefore also be of this form:

$$\theta^n = \mathcal{X}_n\theta^0\mathcal{X}_n^T + \mathcal{Y}_n. \quad (6.26)$$

The matrices  $\mathcal{X}_n$  and  $\mathcal{Y}_n$  can be read from the  $(1, 1)$  block of the general solution (6.6) and are independent of the initial state  $\theta^0$ ; viz.,

$$\mathcal{X}_n = (X^n)_{11}, \quad (6.27)$$

$$\mathcal{Y}_n = (X^n)_{12}\epsilon(X^{nT})_{12} + \sum_{r=0}^{n-1} \left[ X^{n-r-1}Y(X^T)^{n-r-1} \right]_{11}, \quad (6.28)$$

where the subscripts  $i, j$  refer here to specific blocks. This easiness in reading of the corresponding map matrices is another significant advantage of the Markovian embedding representation (4.20).

To probe whether the dynamics is divisible, we consider the map taking the system from  $n$  to  $m > n$ . Assuming that  $\mathcal{X}_n$  and  $\mathcal{Y}_n$  are invertible, which is true in our case, this will have the form [107]

$$\theta^m = \mathcal{X}_{mn}\theta^n\mathcal{X}_{mn}^T + \mathcal{Y}_{mn}, \quad (6.29)$$

where

$$\mathcal{X}_{mn} = \mathcal{X}_m\mathcal{X}_n^{-1}, \quad \mathcal{Y}_{mn} = \mathcal{Y}_m - \mathcal{X}_{mn}\mathcal{Y}_n\mathcal{X}_{mn}^T. \quad (6.30)$$

The dynamics is then considered divisible when the intermediate maps (6.29) are a proper CPTP Gaussian map. That is, when  $\mathcal{M}[\mathcal{X}_{mn}, \mathcal{Y}_{mn}] \geq 0$  [Eq. (6.25)].

The above criteria can be used not only as a dichotomic measure of divisibility, but also as a figure of merit [107]. This is accomplished by defining

$$\mathcal{N}_{mn} = \sum_k \frac{|m_k| - m_k}{2}, \quad \{m_k\} = \text{eigs}\left(\mathcal{M}[\mathcal{X}_{mn}, \mathcal{Y}_{mn}]\right). \quad (6.31)$$

This quantity is always non-negative and the map is divisible iff  $\mathcal{N}_{mn} \equiv 0$  for all  $m, n$ . Otherwise, the magnitude of  $\mathcal{N}_{mn}$  quantifies the extent to which divisibility is broken for that choice of  $m, n$ .

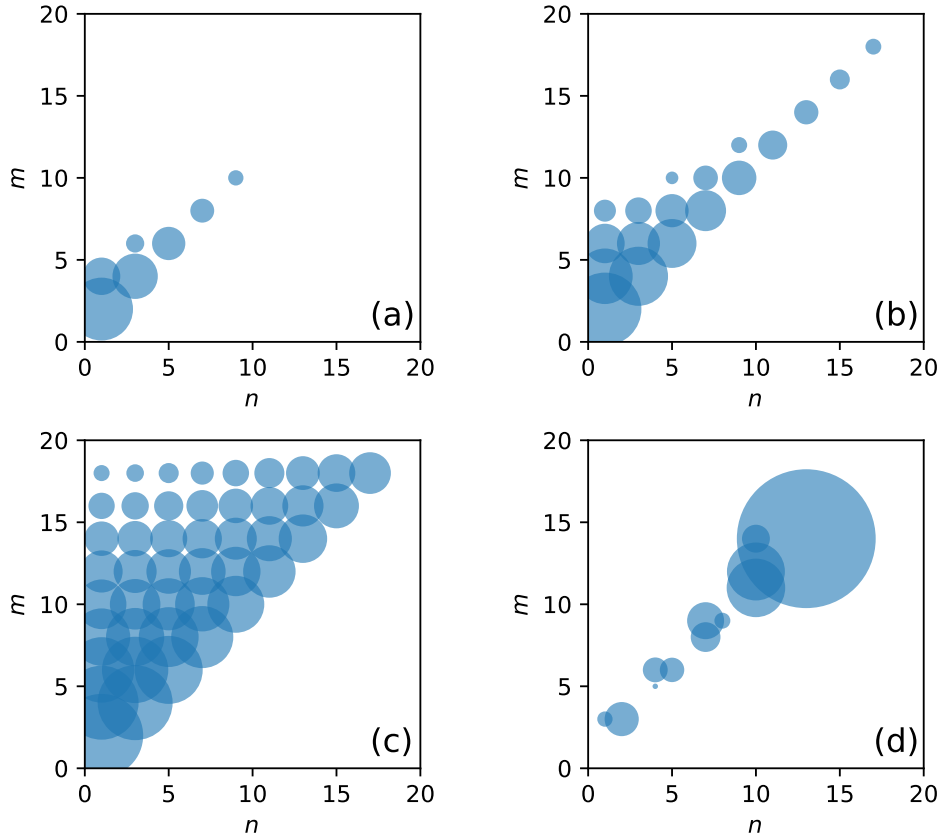


Figure 6.10: Example of the divisibility criteria for the BS dynamics. The plots show  $\mathcal{N}_{mn}$  in the  $(n, m)$  plane, with the size of each point reflecting the magnitude of  $\mathcal{N}_{mn}$ . All curves are for  $\lambda_s = 1.1$  and (a)  $\lambda_e = 0.75$ , (b) 0.9, (c) 1.1 and (d) -0.7.

### 6.3.1 BS dynamics

We begin our investigation of  $\mathcal{N}_{mn}$  by focusing on the BS dynamics [Eq. (4.21)]. An example of the behaviour of (6.31) is shown in Fig. 6.10, where we plot  $\mathcal{N}_{mn}$  in the  $(n, m)$  plane, with fixed  $\lambda_s = 1.1$  and different values of  $\lambda_e$ . The magnitude of  $\mathcal{N}_{mn}$  is represented by the size of each point. These diagrams are interpreted as follows. We start with Fig. 6.10(a). In this case we see that, for  $n = 1$ ,  $\mathcal{N}_{mn}$  is non-zero only for  $m = 2$  and  $m = 4$ , being smaller in the latter. For  $n = 2$  the map is always divisible. And for  $n = 3$ , it is not divisible only for  $m = 4$  and 6. These irregularities are a consequence of the oscillatory character of the parameters appearing, e.g., in Eq. (4.21). Still concerning Fig. 6.10(a), we see notwithstanding that as  $n$  gets large, the map tends to be Markovian for all  $m$ . As we increase  $\lambda_e$ , however, as in Figs. 6.10(b) and (c), we see that overall the regions where  $\mathcal{N}_{mn} > 0$  tend to increase. They increase both as a function of  $n$ , as well as a function of  $m$  for fixed  $n$ .

When  $\lambda_e < 0$ , however, strange things happen [Fig. 6.10(d)]. In this case we find that there can be highly irregular values of  $(n, m)$  which yield non-zero  $\mathcal{N}_{mn}$  which, in fact, can reach significantly large values. For instance, the largest value plotted in Fig. 6.10(d) is for  $n = 13$ ,  $m = 14$  and has the value  $\mathcal{N} \sim 69.7$ . For  $n = 16$ ,  $m = 17$ , however, one finds  $\mathcal{N} \sim 10309$  (not shown). This is to be contrasted with Fig. 6.10(a), whose largest value is  $\mathcal{N} = 3.42$ . We present these results simply to emphasize that  $\mathcal{N}_{mn}$  can oscillate violently. The reason is due to the term  $\mathcal{X}_n^{-1}$  in Eq. (6.30), which can blow up for certain values of  $\lambda_s$ ,  $\lambda_e$  and  $n$ .

Next we turn to the divisibility of a single collision; that is, with  $m = n + 1$ . Plots of  $\mathcal{N}_{n+1,n}$  in the  $(\lambda_s, \lambda_e)$  plane are shown in Fig. 6.11. The overall behaviour is found to alternate with even and odd  $n$ . For  $n$  even, the map is always divisible for  $\lambda_e > 0$  and potentially non-divisible within certain regions of  $\lambda_e < 0$ . Conversely, for  $n$  odd, one finds that divisibility breaks down in significant portions of the  $(\lambda_s, \lambda_e)$  plane. An additional illustration of the complex dependence of  $\mathcal{N}_{n+1,n}$  on  $\lambda_s, \lambda_e, n$  is provided in Fig. 6.12, where we plot  $\mathcal{N}_{n+1,n}$  as a function of  $n$  for selected values of  $\lambda_s$  and  $\lambda_e$ . From this figure, both the even/odd behavior, as well as the dramatic variations in the  $(\lambda_s, \lambda_e)$  plane can be more clearly appreciated.

The behavior of  $\mathcal{N}_{n+1,n}$  in Fig. 6.11 is exacerbated close to the special points  $\lambda_{s(e)} =$



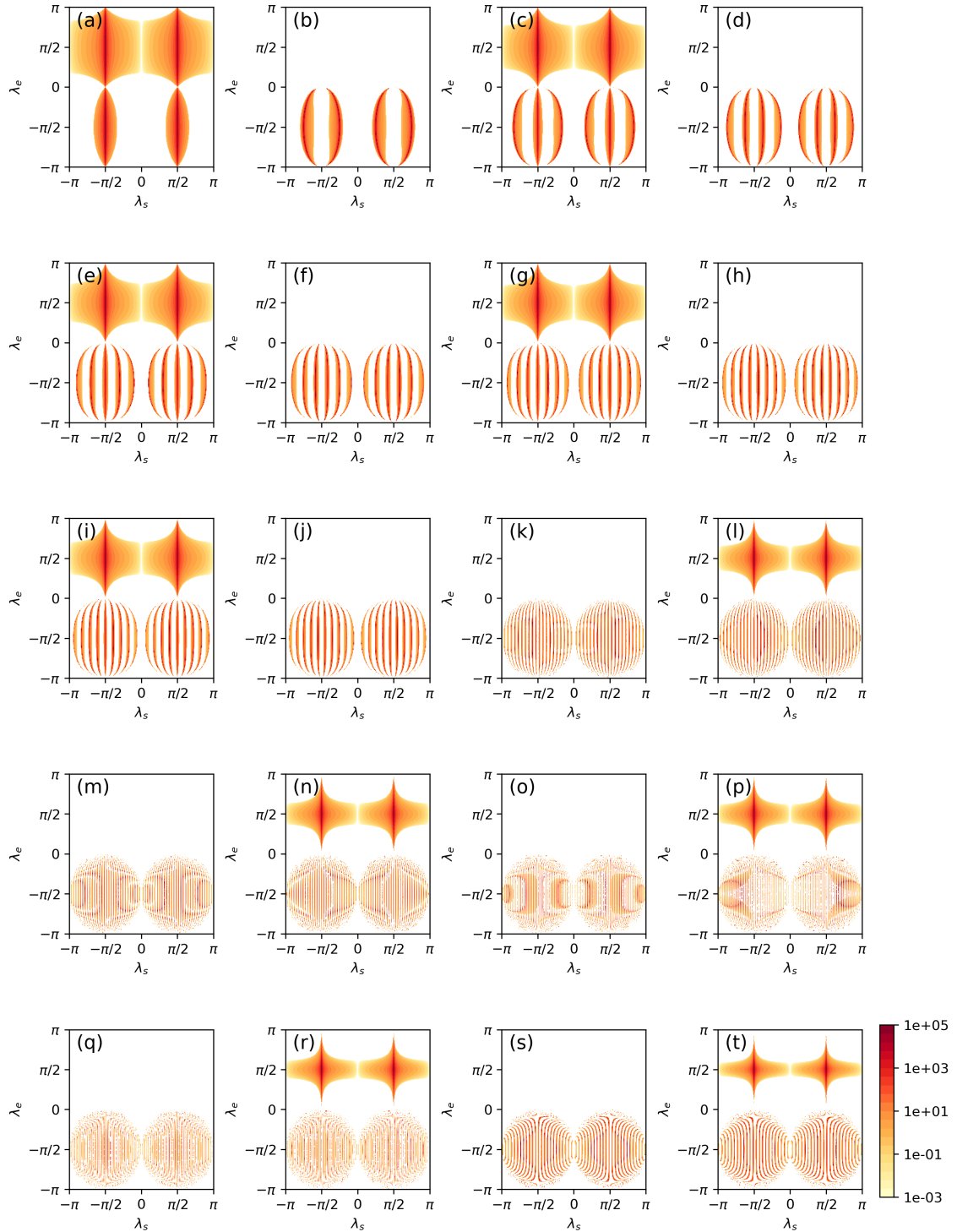


Figure 6.11: CP-divisibility measure  $\mathcal{N}_{n+1,n}$  [Eq. (6.31)] in the  $(\lambda_s, \lambda_e)$  plane, for the BS dynamics. Each plot corresponds to a different values of  $n$ : in the first 2 lines,  $n$  ranges from 1 to 8 in steps of 1. In the remaining lines,  $n = 9, 10, 20, 21, 30, 31, 40, 41, 50, 51$  and 100, 101.

$\pi/2$ . For instance, in the vicinity of  $\lambda_s = \pi/2$ , the dynamics is non-divisible even for infinitesimally small  $\lambda_e$ . This occurs because  $\lambda_s = \pi/2$  corresponds to the full SWAP, where the CM of the system is completely transferred to the ancilla. As a consequence, when the next ancilla arrives to interact with the system, it will always contain a significant amount of information about it. We therefore expect that in the limit  $n \rightarrow \infty$  the diagrams in Fig. 6.11 should converge to narrow lines going through these special points (although, unfortunately, we cannot actually verify this since the simulation cost becomes prohibitive for extremely large  $n$ ).

We may also study similar diagrams for collisions that are more broadly spaced in time. In Fig. 6.13 we present results for  $\mathcal{N}_{m,1}$  for different values of  $m$ . This therefore describes the long-term memory of the map, concerning the first collision. Two features stand out from this figure. First, as one would expect, the overall region in the  $(\lambda_s, \lambda_e)$  plane where the map is CP-divisible tends to shrink with increasing  $m$ . However, the regions around  $\lambda_s = \pm\pi/2$  tend to be remarkably persistent, remaining highly non-divisible

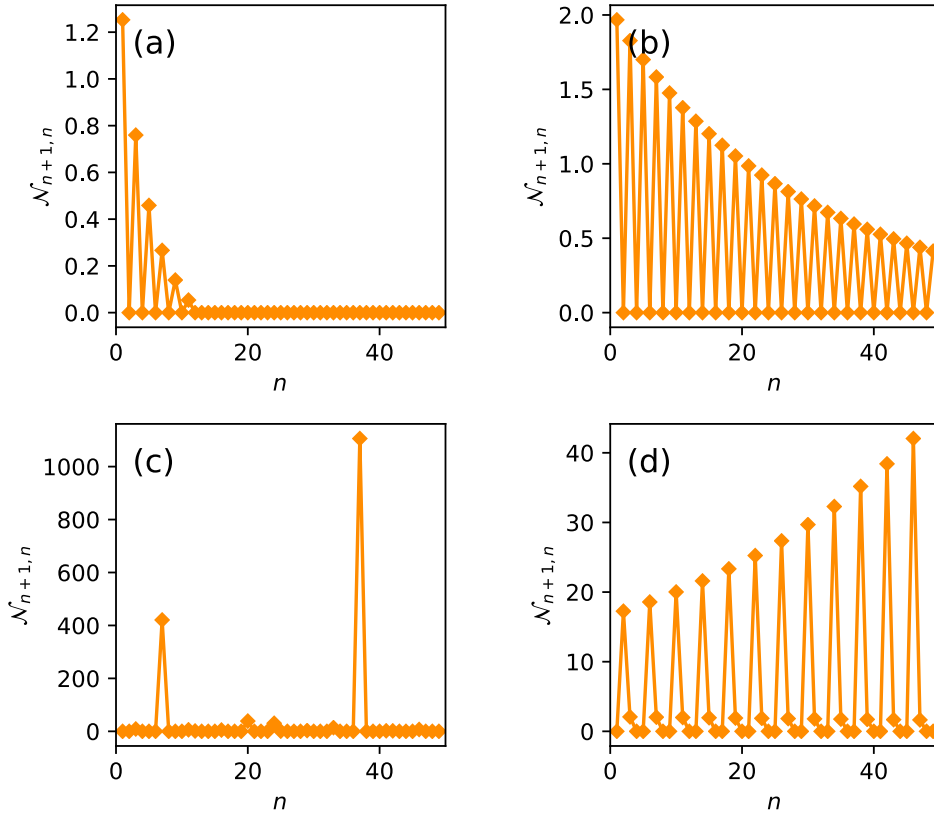


Figure 6.12: CP divisibility measure  $\mathcal{N}_{n+1,n}$  as a function of  $n$ , for the BS dynamics with  $\lambda_s = 0.8$  and  $\lambda_e = 0.9, 1.3, -0.5, -0.8$ . Complements Fig. (6.11).

even for large  $m$ .

The results in Figs. 6.11 and 6.13 refer to divisibility for specific times  $(n, m)$ . We can also combine all data and ask, for which regions in the  $(\lambda_s, \lambda_e)$  plane, the BS dynamics is divisible for all  $(n, m)$ . This is shown in Fig. 6.14. As expected, for most choices of parameters, the map will not be CP-divisible for some  $(n, m)$ . Notwithstanding, there are regions where the map is always divisible. These regions tend to be concentrated close to  $\lambda_e = 0$  (or  $\lambda_e = \pi$ , which is equivalent). And they exist even for large values of  $\lambda_s$ .

A direct comparison with the memory kernel, Sec. 6.2, is not generally possible since

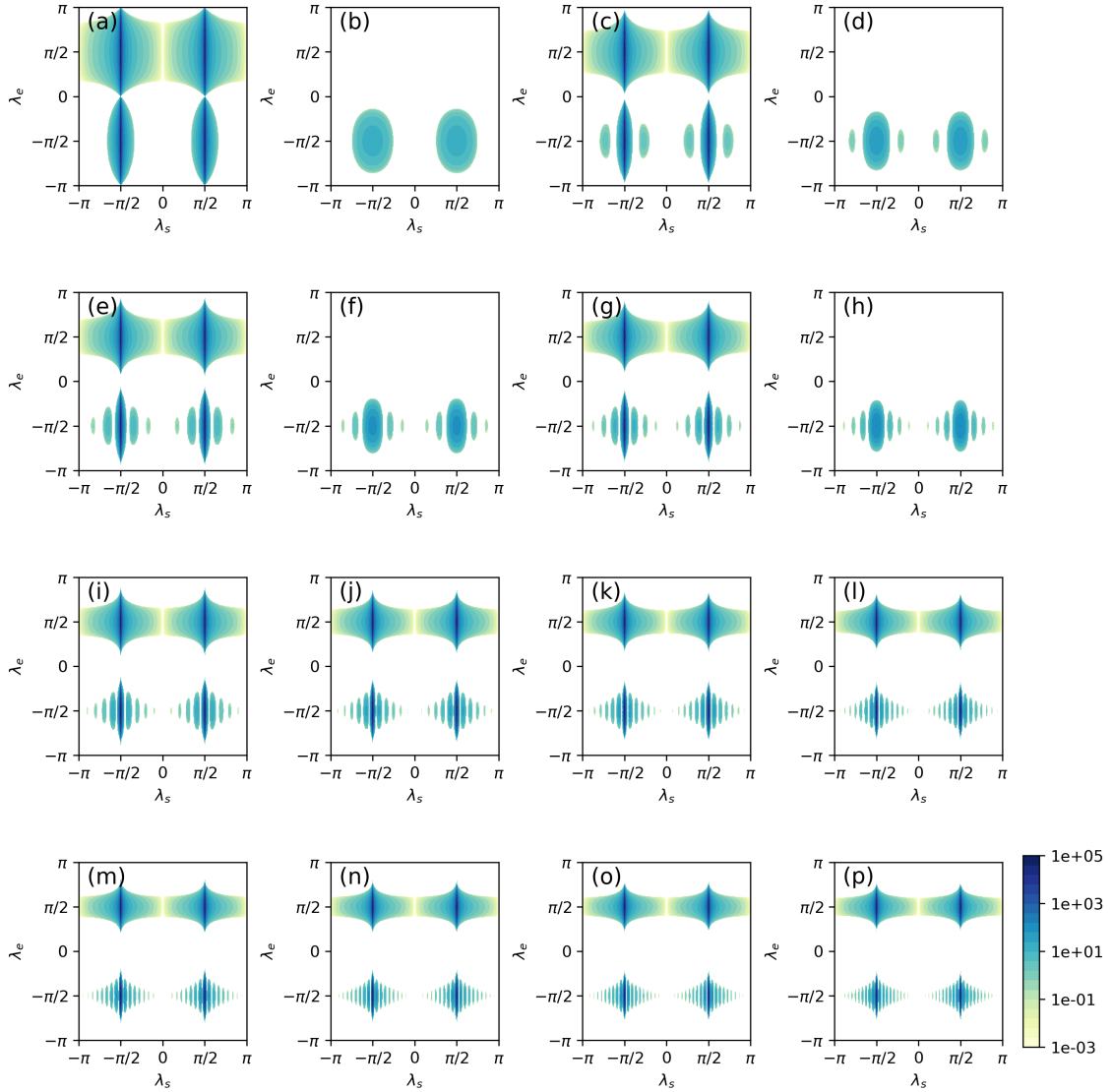


Figure 6.13: CP-divisibility measure,  $\mathcal{N}_{m,1}$  [Eq. (6.31)] in the  $(\lambda_s, \lambda_e)$  plane, for the BS dynamics. Each plot corresponds to a different values of  $m$ : in the first 2 lines,  $m$  ranges from 2 to 9 in steps of 1. In the 3rd and 4th, from  $m = 10$  to 24 in steps of 2. lines,

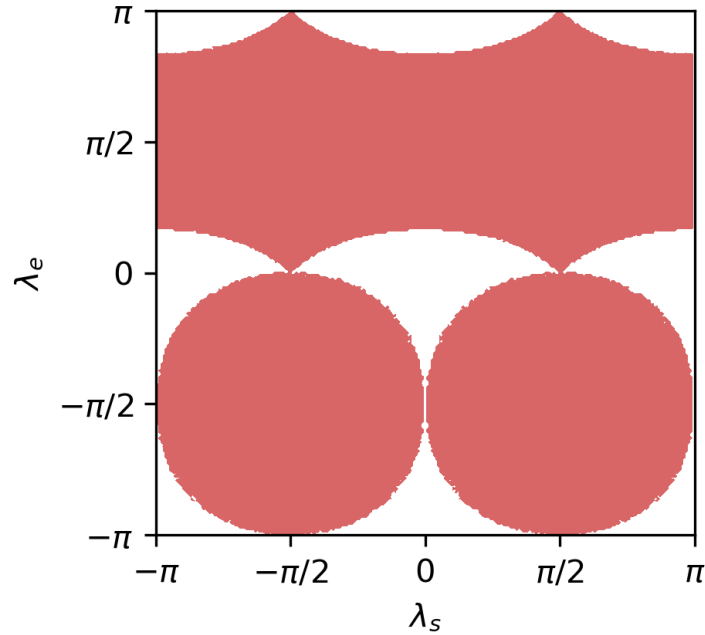


Figure 6.14: Regions in the  $(\lambda_s, \lambda_e)$  plane where the BS dynamics is *not* CP-divisible for at least one choice of  $(n, m)$ .

both refer to different physical aspects of the problem. But if we focus on  $\mathcal{N}_{n+1,n}$ , then some comparison is possible. Recall that the MK describes how the dynamics from  $n \rightarrow n + 1$  is affected by previous times. Thus, regions where the memory kernel is large tend to be accompanied by regions where  $\mathcal{N}_{n+1,n} > 0$ . This is indeed the case, as can be seen by comparing Fig. 6.11 with 6.4.

### 6.3.2 TMS dynamics

The situation for the TMS dynamics is dramatically different. Diagrams for  $\mathcal{N}_{n+1,n}$  in the  $(\lambda_s, \nu_e)$  plane are shown in Fig. 6.15 for different values of  $n$ . In contrast to the BS maps, now most of parameter space is non-divisible. Moreover, the region where it is non-divisible increases for longer times. And finally, what is perhaps the least intuitive, the regions where the map is non-divisible are denser for *small*, instead of large,  $\nu_e$  (although the values of  $\mathcal{N}_{n+1,n}$  are correspondingly smaller). This is a consequence of the fact that the TMS dynamics spontaneously creates excitations in the system, which implies that for large  $\nu_e$  a substantial amount of noise is introduced, making the map more likely to be divisible. If  $\nu_e = 0$  the map is, of course, divisible by construction. However, the

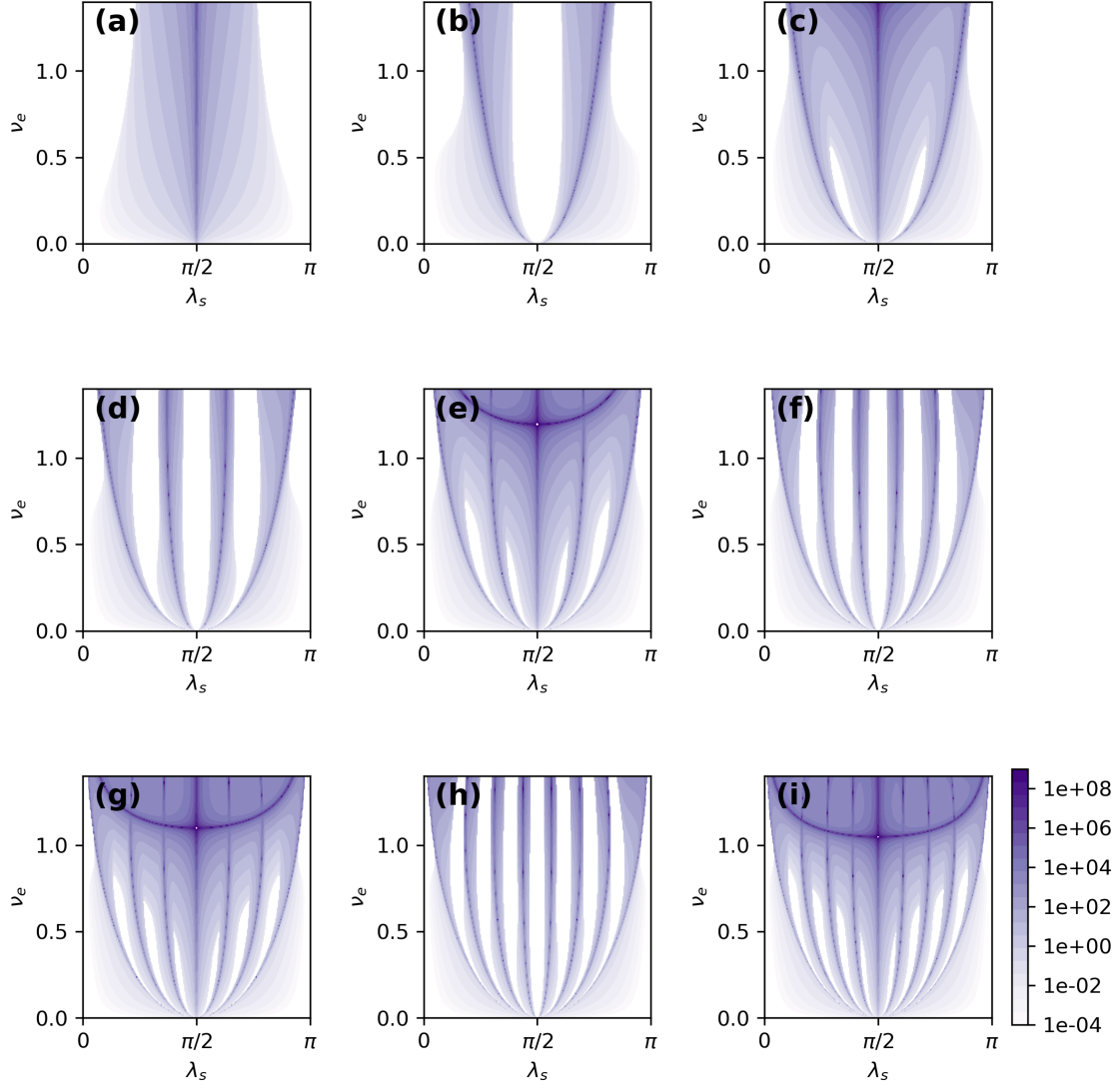


Figure 6.15: CP-divisibility measure,  $\mathcal{N}_{n+1,n}$  [Eq. (6.31)] in the  $(\lambda_s, \nu_e)$  plane, for the TMS dynamics. Each plot corresponds to a different values of  $n$ , from 1 to 9 in steps of 1.

results in Fig. 6.15 show that for arbitrarily small, but non-zero  $\nu_e$ , the map is already non-divisible, albeit with a small  $\mathcal{N}_{n+1,n}$ . As with the BS dynamics, one could also combine all these diagrams to ask whether there are regions in the  $(\lambda_s, \nu_e)$  where the map is always divisible, for all  $(n, m)$ .

The answer to this question is, in this case, negative: for the TMS dynamics the dynamics is never divisible, except for the trivial line  $\nu_e = 0$ . This represents a major difference in comparison with the BS dynamics and, once again, it is ultimately a property of the entangling nature of the two-mode squeezing interaction (4.8).

# Chapter 7

## Conclusions

In this dissertation we presented a robust framework for studying non-Markovianity in collisional models from multiple perspectives. We showed that the dynamics can be cast in terms of a Markovian embedding of the covariance matrix [Eq. (4.20)], which yields closed expressions for the mutual information [Eq. (5.1)], the memory kernel [Eq. (6.1)] and the divisibility monotone [Eq. (6.31)]. We analyzed in detail two types of interactions, a beam-splitter implementing a partial SWAP, and a two-mode squeezing which entangles the ancillas and, at the same time, feeds excitations into the system. By analyzing several memory quantifiers for these two representative scenarios, this helped to shed light on the intricate mechanism behind memory effects in the quantum domain. Results of this work were reported in the preprint [94].

Essential to this work was the use of two main ingredients described in Chapter 4. First, collisional models, which allow us to introduce non-Markovianity in a fully controllable way. By means of the Markovian embedding (Sec. 4.2), the full non-Markovian dynamics is captured in the map  $\Phi(\cdot)$  [Eq. (4.3)], and the dynamics it generates [Eq. (4.4)], with  $\varrho^n$  containing all the relevant information characterizing the non-Markovianity of the evolution. And second, continuous-variable Gaussian operations, which replace the complicated dynamics of the density matrix with a much simpler map for the covariance matrix (Sec. 4.3). The global evolution at the level of the density matrix [Eq. (4.2)] is converted into an equation for the global covariance matrices [Eq. (4.10)], and the unitaries [Eqs. (4.6)-(4.8)] are replaced by symplectic matrices [Eqs. (4.11)-(4.15)].

The Markovian embedding then allowed us to compute the memory effects in the

dynamics of  $\gamma^n$ . To begin with, the mutual information [Eq. (5.1)] is easily computed from the symplectic eigenvalues of  $\gamma^n$ . Next, the time-non-local dynamics defining the memory kernel [Eq. (5.2)] can be rewritten at the level of the system covariance matrix [Eq. (6.2)]. The memory kernel depends only on the matrix  $X$  and can be computed using [Eq. (6.18)]. One can also write a Kraus decomposition of the MK with coefficients  $\kappa_{ij}^n$  [Eq. (6.21)]. Finally, the intermediate map taking the system from time  $n$  to time  $m$  is given by [Eqs. (6.29) and (6.30)]. A monotone of CP-divisibility is obtained [Eq. (6.31)] and depends only on the  $X$  and  $Y$ . All methods described are implemented in the associated python, which provides an efficient and simple way of simulating a broad range of scenarios [76].

In particular, we have focused on two types of maps. The system-ancilla interaction was always fixed to be of beam-splitter-type (partial SWAP). But the ancilla-ancilla interaction could be either beam-splitter or a two-mode squeezing. The behaviour of the two are dramatically different. For the former, we have found that the combination of the two beam-splitter interactions lead to strong resonance effects that cause most quantities to oscillate in time and also depend sensibly on the relative signs of the interaction strengths (c.f. Figs. 6.4 or 6.11). For the BS dynamics, there is also a non-negligible portion of parameter space in which the dynamics is always Markovian (Fig. 6.14). Conversely, in the TMS dynamics excitations are constantly being generated in the system. As a consequence, the dynamics is only stable for certain values of the interaction strength (Fig. 4.4). If the interaction is too strong, the occupations in the system diverge (never reach a steady-state). Interestingly, this is also reflected in the memory kernel, which acquires infinitely long memory (Fig. 6.9). The TMS dynamics is also always non-Markovian (never CP-divisible; Fig. 6.15), unless the ancilla-ancilla interaction is strictly zero. This reflects the entangling nature of the two-mode squeezing. The magnitude of the non-Markovianity, of course, is small for weak interactions. This is clearly seen, for instance, in the memory kernel, Fig. 6.7.

Our framework can be readily extended to a broad range of scenarios. We begin by mentioning problems which are straightforward extensions of our results. Throughout the dissertation, we have focused on ancillas initially prepared in the vacuum state. Studying different initial preparations would be interesting since the memory kernel does not

depend on this, but CP-divisibility does. It would be particularly interesting to study the introduction of single-mode squeezing in the ancillas.

Another natural extension would be to consider different types of interactions, as in Refs [61, 65]. In particular, one thing that we have not explored are interactions that lead to “non-diagonal” memory kernels. As discussed below [Eq. (6.2)], a MK involving the identity or  $\sigma_z$  is always diagonal, meaning that each entry of  $\theta^n$  is only affected by the same entry at past times. A memory kernel involving  $\sigma_{\pm}$ , however, would imply, for instance, that  $\langle Q^2 \rangle_n$  could be affected by past values of  $\langle P^2 \rangle_n$ . This could, in principle, generate a plethora of interesting effects. Another possibility would be the inclusion of stochastic SWAPs, as in Refs. [58, 59].

In Ref. [66], the authors asked, within a collisional model context, whether it was possible to pinpoint the backflow of information to either changes in the environmental states, or the buildup of correlations between system and environment. This question is relevant since these are the two main ingredients entering in the 2nd law of thermodynamics in the quantum domain [118, 119]. That is to say, they are the two quantities measuring the degree of irreversibility of a process. We believe that this question could be directly addressed within our framework. Whether the Markovian embedding suffices for this end, however, is not clear at the moment. For instance, the mutual information studied in Sec. 5.2 is only between  $S$  and  $E_{n+1}$  and thus correspond to only a part of the full system-environment correlations. Notwithstanding, even if the embedding does not suffice, the fact that the approach deals only with covariance matrices still allows one to study numerically dynamics involving large numbers of ancillas.

Concerning less trivial extensions, throughout this work we have assumed that the Markov memory length is 1. That is, each ancilla  $E_n$  only propagates information to its nearest neighbor. The extension to arbitrary memory length, as studied in Refs. [61, 65], would be quite interesting. And it is also amenable to our framework, provided one extends the Markovian embedding to have longer memory.

Finally, we mention that the basic ideas set up could also serve as a starting point for exploring the Gaussian formulation of process tensors [111–113], which provide an alternative, and much broader, way of characterizing non-Markovianity. In fact, this could perhaps also be used as a way to bridge process tensors and the memory kernel.



# Appendix A

## Stability Theory

We are interested in studying the fixed point stability of the Markovian embedding equation (4.20), i.e. solutions that satisfy  $\gamma^{n+1} = \gamma^n$ . To this end, we use the vectorized form (6.9) and label the vectorized fixed point solution as  $\vec{\gamma}^*$ :

$$\vec{\gamma}^* = X \otimes X \vec{\gamma}^* + \vec{Y}. \quad (\text{A.1})$$

As long as  $\det(\mathbb{I} - X \otimes X) \neq 0$  a fixed point solution can be readily found as

$$\vec{\gamma}^* = (\mathbb{I} - X \otimes X)^{-1} \vec{Y}. \quad (\text{A.2})$$

The stability of  $\vec{\gamma}^*$  will be associated to the eigenvalues of the  $X \otimes X$  matrix. Or, what is equivalent, the eigenvalues of  $X$ . If their modulus are below 1, the fixed point will be a globally asymptotic state (GAS) and all trajectories will converge to  $\gamma^*$  for large enough  $n$ . Otherwise, it may diverge.

The eigenvalues of the matrix  $X$  for the BS channel, Eq. (4.21), read

$$\frac{1}{2} \left( -wx + x \pm \sqrt{(w+1)^2 x^2 + 4wy^2} \right). \quad (\text{A.3})$$

Using the  $(\lambda_s, \lambda_e)$  parametrization, one finds that the only values not satisfying the GAS conditions are  $\lambda_e = \pm\pi/2$  or  $\lambda_s = 0, \pi$ , which represent, respectively, the case where no particle flow to the ancillas and when the system does not interact at all. Excluding those

points, the fixed point is a GAS given by:

$$\gamma_{BS}^* = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}. \quad (\text{A.4})$$

That is, the map tends to homogenize the system to the same initial state of the ancillas. This, of course, is what is expected of a beam-splitter/partial SWAP dynamics. It is notwithstanding interesting that it remains true even in the case of ancilla-ancilla interactions and non-Markovian dynamics.

Similarly, the eigenvalues of  $X$  for the TMS case, Eq. (4.23), read

$$\begin{aligned} & \frac{1}{2} \left( (1 + \tilde{w})x \pm \sqrt{(\tilde{w} - 1)^2 x^2 - 4\tilde{w}y^2} \right), \\ & \frac{1}{2} \left( (1 - \tilde{w})x \pm \sqrt{(\tilde{w} + 1)^2 x^2 + 4\tilde{w}y^2} \right). \end{aligned} \quad (\text{A.5})$$

These eigenvalues only fulfill the GAS requirements in the interval where  $\nu_e \in [0, \sinh^{-1}(1)]$ . This therefore defines the critical value  $\nu_e^{\text{crit}} = \sinh^{-1}(1)$ , after which the dynamics diverges. Inside this interval, the fixed point is a GAS given by

$$\gamma_{TMS}^* = \begin{pmatrix} \left( \frac{2 \sinh^2(\nu_e)}{1 - \sinh^2(\nu_e)} + 1 \right) \epsilon & 0 \\ 0 & \left( \frac{2 \sinh^2(\nu_e)}{1 - \sinh^2(\nu_e)} + 1 \right) \epsilon \end{pmatrix}. \quad (\text{A.6})$$

Thus, we see that system and ancilla once again tend to homogenize. However, the ancilla initial state  $\epsilon$  is now amplified by a factor which is always larger than unity and diverges when  $\nu_e = \nu_e^{\text{crit}}$ . We also call attention to the fact that  $\gamma_{TMS}^*$  is a product state, so that no correlations survive in the long-time limit.

# Appendix B

## Memory Kernel for the BS dynamics

In this appendix we discuss how to obtain a more compact expression for the memory kernel (6.2), in the case of the BS dynamics. This case is simpler because the only non-zero coefficient is  $\kappa_{11}^n$ , which is proportional to the identity map. That is to say, in this case the MK is actually just a  $c$ -number, instead of a superoperator.

To accomplish this, we exploit in more detail the tensor structure of the matrices used in Sec. 6.2 (now all specialized to  $N_S = N_E = 1$ ). We begin by noting that the matrix  $X$  of the BS dynamics, Eq. (4.21), can also be written as

$$X = \chi \otimes \mathbb{I}, \quad \chi = \begin{pmatrix} x & y \\ yw & -xw \end{pmatrix}, \quad (\text{B.1})$$

where  $\chi$  is now a simple  $2 \times 2$  matrix and, in this appendix,  $\mathbb{I}$  will always refer to the identity of dimension 2. Similarly, the projection operator  $P_S$  in Eq. (6.10) can be written as

$$P_S = p_s \otimes \mathbb{I}, \quad p_s = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (\text{B.2})$$

Thus, the matrix  $P$  in Eq. (6.11) becomes

$$P = p_s \otimes \mathbb{I} \otimes p_s \otimes \mathbb{I}. \quad (\text{B.3})$$

This type of tensor structure is simply a consequence of the vectorization procedure, Eq. (6.8).

The matrix  $p_s$  can be further decomposed as

$$p_s = |0\rangle\langle 0|, \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (\text{B.4})$$

Dirac's notation is introduced here just for clarity; the state  $|0\rangle$  is completely unrelated to the actual Hilbert space basis of the system. The advantage of this decomposition is that it allows us to write the isometry  $\pi$ , in Eq. (6.16), as

$$\pi = \langle 0| \otimes \mathbb{I} \otimes \langle 0| \otimes \mathbb{I}. \quad (\text{B.5})$$

This now clearly shows that  $\pi$  contracts slots 1 and 3, while acting trivially on 2 and 4.

At this point, it is convenient to simplify the notation and introduce indices 1, 2, 3, 4, to refer to which slot of the tensor product the operators act. Thus, for instance, we will henceforth write

$$X \otimes X = \chi \otimes \mathbb{I} \otimes \chi \otimes \mathbb{I} := \chi_1 \chi_3, \quad (\text{B.6})$$

meaning  $\chi_1$  acts on slot 1 and  $\chi_3$  on slot 3. Similarly,  $P = p_s^1 p_s^3$  and, therefore,  $Q = 1 - p_s^1 p_s^3 := Q_{13}$  is a matrix acting only on slots 1 and 3 (we emphasize that  $Q_{13}$  cannot be written as a simple product of an operator acting on 1 and another acting on 3). Notice how the special structure appearing in Eq. (B.6) is unique of the BS dynamics. For other types of dynamics,  $X \otimes X$  would in general act non-trivially on all four slots. Due to this simplification, the quantity appearing inside  $\pi(\dots)\pi^T$  in Eq. (6.18) will be an operator acting only on slots 1 and 3.

Next we turn to Eq. (6.21), describing the coefficients  $\kappa_{ij}^n$ . The contraction  $\pi(\dots)\pi^T$  eliminates slots 1 and 3, so that  $(M_j^T \otimes M_i^T)$  is effectively multiplying matrices from slots 2 and 4. Thus, one may equivalently write

$$(M_j^T \otimes M_i^T)\pi(\dots)\pi^T = \pi \left[ (\mathbb{I} \otimes M_j^T \otimes \mathbb{I} \otimes M_i^T) \dots \right] \pi^T,$$

where  $(\dots)$  refers to all terms inside  $\pi(\dots)\pi^T$  in Eq. (6.18). But from the arguments above, these quantities act only on slots 1 and 3. Combining this with the fact that  $\text{tr}(A \otimes B) = \text{tr}(A) \text{tr}(B)$  explains why, in the BS case, the only non-trivial coefficient

will be  $\kappa_{11}^n$ , corresponding to  $M_i = M_j = \mathbb{I}$ . This coefficient may then be written as

$$\kappa_{11}^n = \text{tr}_{13} \left\{ \pi_{13} \left[ \chi_1 \chi_3 (Q_{13} \chi_1 \chi_3 Q_{13})^n \chi_1 \chi_3 \right] \pi_{13}^T \right\},$$

where the remaining trace is now only over slots 1 and 3. Finally, we use Eq. (B.5) to express  $\pi$  in terms of  $\langle 0|$ . This allows us to write

$$\kappa_{11}^n = \langle 00| \bar{\chi} (\bar{Q} \bar{\chi} \bar{Q})^n \bar{\chi} |00\rangle, \quad (\text{B.7})$$

where  $|00\rangle = |0\rangle \otimes |0\rangle$ ,  $\bar{\chi} = \chi \otimes \chi$  and  $\bar{Q} = \mathbb{I}_4 - p_s \otimes p_s$  are all objects of dimension 4. Eq. (B.7) therefore provides a compact representation of the memory Kernel for the BS dynamics. It is expressed solely in terms of  $|0\rangle$ ,  $\chi$  and  $p_s$ , [Eqs. (B.1) and (B.4)]. And it requires exponentiating only operators of dimension 4, in comparison with (6.18) which would have dimension 16.

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