Universidade de São Paulo
Instituto de Física

O problema da hierarquia do Modelo Padrão e suas soluções

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Dissertação de mestrado apresentada ao Instituto de Física da Universidade de São Paulo, como requisito parcial para a obtenção do título de Mestre em Ciências.

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São Paulo
2020

FICHA CATALOGRÁFICA
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do Instituto de Física da Universidade de São Paulo

Salla, Gabriel Massoni

O problema da hierarquia do modelo padrão e suas soluções/ The hierarchy problem of the standard model and its solutions. São Paulo, 2020.

Dissertação (Mestrado) – Universidade de São Paulo. Instituto de Física. Depto. de Física Matemática.

Orientador: Prof. Dr. Enrico Bertuzzo
Área de Concentração: Física de partículas elementares e campos

Unitermos: 1.Teoria quântica de campo; 2. Teoria de campos; 3. Física teórica.

USP/IF/SBI-060/2020

University of São Paulo
Physics Institute

The hierarchy problem of the Standard Model and its solutions

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Dissertation submitted to the Physics Institute of the University of São Paulo in partial fulfilment of the requirements for the degree of Master of Science.

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São Paulo
2020

Agradecimentos

A cada palavra e equação escritas fica óbvio para mim que este trabalho é o fruto do apoio de muitos. As próximas palavras são para aqueles que direta ou indiretamente o influenciaram e assim possibilitaram sua realização.

Agradeço à Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) (processo n° 2018/23679-2) pelo auxílio financeiro, sem o qual este projeto não poderia ser desenvolvido.

Agradeço à minha família, que não só apoiou a minha trajetória, mas também sempre suportou de perto todas as minhas chatices e reclamações. Em particular, gostaria de agradecer aos meus pais por todo carinho, pela preocupação, pelos valiosíssimos conselhos e por me ensinarem que o caminho da ciência deve ser trilhado com um coração humilde.

Agradeço aos meus amigos, que fizeram dos meus anos na faculdade os melhores da minha vida. Aos Wiladores, em especial, por me presentearem com lembranças que levarei comigo até meu último dia; são inestimáveis os momentos nos quais ora discutíamos física, ora perdíamos tempo tomando café e falando bobagens. Gostaria ainda de mencionar alguns destes amigos que impactaram diretamente o desenvolvimento deste trabalho. Agradeço à Fernanda, companheira de sala, pelas conversas sobre a vida e a física, que em momento algum falharam em ampliar meus horizontes. Ao Francisco, por sempre se mostrar disponível a esclarecer dúvidas sobre cosmologia e por me ajudar a remover muitos dos preconceitos que tinha com esta área de pesquisa. Ao Martín, cuja dedicação sempre me impressionou, pelas contrutivas discussões sobre os prospectos da pesquisa. À Patrícia, por não somente me apoiar em momentos difícies, como também pelos diversos debates sobre a intersecção da física e da matemática. E ao Pedro, pelas incontáveis discussões que tivemos a respeito dos nossos trabalhos, da física atual e do que nos reserva o futuro no mundo acadêmico.

Agradeço, finalmente, ao prof. Enrico. Há mais de 4 anos estudo sob sua supervisão e creio ser impossível medir em algumas poucas palavras o quanto esta orientação significou para mim. O que posso dizer é que, desde a minha primeira reunião com ele, quando eu ainda estava na graduação, nunca me senti desmotivado. Sua solicitude para as dúvidas mais elementares e sua simpatia ao ensinar são alguns dos motivos que me permitiram sempre seguir em frente. O trabalho que hoje apresento e o pesquisador que amanhã serei são o reflexo de toda sua paciência, dedicação e confiança em mim depositadas. Muito obrigado, professor.

Resumo

Nesta dissertação é estudado o Problema da Hierarquia de física de partículas, em particular tanto a sua formulação como também algumas de suas soluções, nomeadamente Supersimetria, modelos de Higgs Composto, modelos de Higgs Gêmeos e modelos de Relaxamento Cosmológico. Na primeira parte deste trabalho define-se a questão como dois problemas distintos: o Problema da Massa Escalar e o Paradigma da Hierarquia, que estão relacionados respectivamente às inconsistências dentro do contexto de Teorias de Campos Efetivas e ao conceito de naturalidade. São revisadas, a partir de sua formulação teórica baseada em primeiros princípios e com foco em como solucionam o Problema da Hierarquia, as soluções acima mencionadas juntamente com suas implementações mínimas. Além disso, restrições experimentais dos respectivos modelos mínimos são discutidas. Em particular, uma nova abordagem independente de modelo fundamentada em teorias efetivas para os modelos de Relaxamento Cosmológico é desenvolvida. Finalmente, obtém-se uma caracterização de cada solução com base na classificação entre o Problema da Massa Escalar e o Paradigma da Hierarquia.

Palavras-chave: Problema da Hierarquia; Supersimetria; Higgs Composto; Higgs Gêmeos; Relaxamento Cosmológico

Abstract

In this thesis the Hierarchy Problem of particle physics is studied, in particular both its formulation and some of its solutions, namely Supersymmetry, Composite Higgs models, Twin Higgs models and Cosmological Relaxation models. In the first part the Hierarchy Problem is defined as two different problems: the Scalar Mass Problem and the Hierarchy Paradigm, which are related to the inconsistencies within the Effective Field Theory framework and the concept of naturalness, respectively. A review of each of the aforementioned solutions is presented by developing their basic theoretical formulation, focusing on how they solve the Hierarchy Problem, and their minimal implementation. Furthermore, recent experimental bounds of the corresponding minimal models are discussed. On top of this, a novel model-independent approach to Cosmological Relaxation models based on effective theories is developed. Finally, the characterisation of each model in terms of the Scalar Mass Problem and the Hierarchy Paradigm is obtained.

Keywords: Hierarchy Problem; Supersymmetry; Composite Higgs; Twin Higgs; Cosmological Relaxation

Contents

Introduction

At the present time it is clear that the Standard Model (SM) of particle physics is insufficient to describe all fundamental properties of nature, as there are many observations that it fails to address [1, 2, 3]. The most concrete example of an inconsistent prediction of the SM is that of massless neutrinos, whereas we know that they are massive due to their oscillations [3, 4]. Such failures motivate us in searching for new physics beyond the SM (BSM) and show that the SM is nothing but an *Effective Field Theory* (EFT).

With respect to an energy scale Λ, EFT's are quantum field theories that describe the low-energy degrees of freedom (dof) without including explicitly the high-energy dof which lie above Λ [5, 6, 7, 8]. The low- and high-energy regimes of this field theory are usually denoted as Infra-Red (IR) and Ultra-Violet (UV), respectively. The physical principle that allows us proceeding in this manner is the *Wilsonian decoupling principle*, which states that the physics in the far-UV, i.e. above Λ, enters only as small corrections to the theory with only the dof of the IR. In the context of field theory, these small corrections are characterised by effective operators, whose couplings are related to the ones of the UV-theory and can be obtained, after integrating out the heavy modes, by matching both theories at the scale Λ. This scale can be therefore interpreted as the scale at which the predictions of the EFT are not valid anymore. For this reason it is denoted as the cut-off of the EFT. The EFT framework is very fitting to the development of BSM theories and it will be used extensively in this thesis; the relevant concepts and techniques associated with EFT's will be introduced as necessary.

As an EFT, the SM has a corresponding cut-off Λ; however, in spite of the experimental searches, we have yet to find relevant signatures of new physics in the UV. For this reason we *believe* such cut-off to be much larger than the Electro-Weak (EW) scale. As it will be shown, a large cut-off will introduce another problem to the SM, the *Hierarchy Problem* [1, 5, 9, 10]. This problem is much more conceptual compared to other issues of the SM and has a deep connection to the EFT framework. Moreover, it guided the theoretical development of particle physics for the last decades and is still one of the most important motivations for seeking new physics at high-energy experiments.

The objective of this thesis is to once more explore the Hierarchy Problem. In particular, four distinct classes of solutions will be presented*, followed by a discussion of how

*In this thesis other classes of solutions relevant to the study of the Hierarchy Problem, for instance each of them solve the problem.

In order to do so, the thesis is structured in the following manner. In the first chapter the inconsistencies surrounding the Higgs mass will be analysed in detail and a precise definition of the Hierarchy Problem will be given. More precisely, a new classification of the problem into two distinct concepts is proposed. In the upcoming chapters four types of solutions to this problem are studied, of which three, Supersymmetry [11, 12], Composite Higgs Models [13, 14] and Twin Higgs models [15, 16], are more traditional solutions based on the introduction of new symmetries. The fourth type of solution, named Cosmological Relaxation models [17, 18], is much more recent and raises many questions regarding the nature of the Hierarchy Problem. The discussion of each of these solutions include: their motivations, the development of the theoretical framework with emphasis on how they solve the problem, implementation of their minimal realisation, and a brief analysis of the most relevant phenomenology and present experimental bounds. Notably in section 5.2 a novel model-independent approach to Cosmological Relaxation models is explored, with which it is possible to determine general features of this class of models making use only of EFT techniques. A more complete investigation of how these solutions are related to the original formulation of the problem and how they allow us to improve our comprehension of the latter will be left to the conclusion of this thesis.

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models of extra dimensions (Randall-Sundrum), are not discussed due to the lack of space.

1 The facets of the Hierarchy Problem

Let us start by giving a precise formulation of the Hierarchy Problem and the concepts behind it, and explain why it drove most of the theoretical activity over the last 40 years.

This chapter is organised in the following way. In the first section the concepts of Effective Field Theory are used to find the inconsistencies surrounding the Higgs mass and the problem is precisely defined. Section 1.2 is dedicated to the concept of naturalness and how to properly quantify it. A brief outline of what will be studied in this thesis is also given at the end of the chapter.

1.1 Definition of the problem - the Scalar Mass Problem

In the context of renormalization one can introduce an arbitrary hard cut-off Λ as a mean to get finite expressions from loop amplitudes; this is the simplest and more straightforward regularisation method [1, 19]. In Wilsonian analysis the procedure is similar, but with physical interpretation anew, in which Λ acquires a physical meaning: it is the scale below which the theory is defined [8, 20]. If the theory is being probed experimentally at an energy scale $E \ll \Lambda$, then it is possible to calculate the running of the couplings with respect to the energy scale [1, 20, 21]. Such approach to renormalization offers a nice interplay with the concept of Effective Field Theories (EFT), which are just Quantum Field Theories (QFT's) defined only until a maximum energy Λ [8, 21, 22]. When the energy is close to the cut-off, i.e. $E \lesssim \Lambda$, the predictions made by the EFT's are not reliable anymore. The connection between Wilson approach and EFT is made more precise in the following way. A theory described by a Lagrangian $\mathscr{L}$ may be written as [23]

$$\mathscr{L} = \frac{\Lambda^4}{g_*^2}\mathscr{F}, \tag{1.1}$$

where Λ is the cut-off of the theory, g_* is a dimensionless parameter and $\mathscr{F}$ is a dimensionless function of dimensionless combinations of fields, derivatives and Λ. In particular, for a scalar particle ϕ such combination is given by

$$\mathscr{F} = \mathscr{F}\left(\frac{g_\phi \phi}{\Lambda}, \cdots\right), \tag{1.2}$$

with g_ϕ a dimensionless coupling. In general $\mathscr{F}$ can be a very complicated function*, but we can always go to low energy scales and perform a Taylor expansion in $\mathscr{F}$ while respecting the appropriate selection rules of the relevant symmetries. Doing so for the scalar term in Eq. (1.2) will result in the following mass term for ϕ:

$$\mathscr{L} = \frac{\Lambda^4}{g_*^2}\mathscr{F} \xrightarrow{E,|\phi|\ll\Lambda} c\Lambda^2|\phi|^2, \tag{1.3}$$

where c is a dimensionless constant. Note that this constant needs to be small ($|c| \ll 1$) if the scalar belongs to IR particle spectrum, otherwise $m_\phi^2 \sim \Lambda^2$. The same reasoning can be repeated considering more fields, but it does not provide more information than this because we know neither the details of the function $\mathscr{F}$, nor the particular selection rules of the theory. Nevertheless, Eq. (1.3) already gives us precious information at tree-level.

The Standard Model (SM) itself is an EFT from the moment we accept that it cannot be the final theory; there are still many unsolved problems in the SM, for example neutrino oscillations. The SM does not have a mechanism to generate their masses without introducing new operators (a 5-dimensional one in this particular case) or new particles (e.g. right-handed neutrinos) [1, 24, 25]. Besides the neutrinos, other examples of big issues within the SM are the nature of Dark Matter (DM), Dark Energy (DE), the generation of baryon asymmetry and how to couple gravity consistently [2]. It is believed that these problems can be explained by physics at higher energies †. Therefore, above the cut-off new physics *should* enter the scene. If a theory is not defined at all energy scales, physicists must seek generalisations that include physics above Λ. These theories are called UV-completions of the considered EFT.

Let's focus now on some properties of EFT's. Consider an EFT with a coupling g. For energies well below the cut-off ($E \ll \Lambda$), we calculate $\beta(g) = E\frac{\mathrm{d}g}{\mathrm{d}E}$ [20, 21]. When E gets close to Λ, the EFT loses its meaning and we cannot predict anything in the UV (see for instance Figure 1). Hence we will always need to formulate an UV-completion for energies above the cut-off in order to calculate g at higher energies. As a consequence of such fact, measured parameters of a theory have contributions from the EFT but also from the UV.

*Actually, without knowledge of the fundamental theory above Λ it is impossible to determine $\mathscr{F}$.

†More precisely, recent experiments with neutrinoless-double-beta decay indicates that the threshold to new physics lies at TeV scale, which is a concrete motivation to LHC physics [26]. The other problems, however, are much more open and there is a fatal lack of a unique phenomenology.

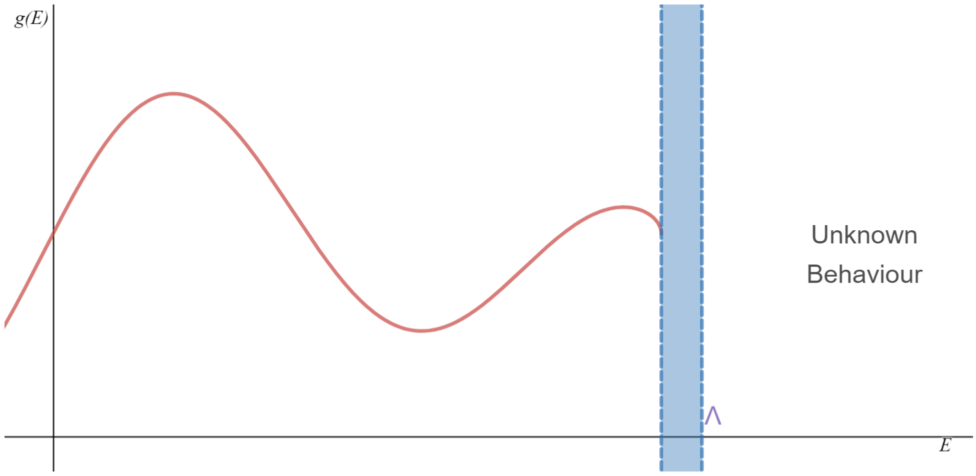


Figure 1: Running of a coupling g in an EFT with cut-off Λ.

Explicitly, the measured coupling g_M, with a given renormalization condition at $E = 0$, can be written as

$$g_M = \int_{-\infty}^{\ln\Lambda} d(\ln\Lambda')\ \beta(g_{\rm EFT}) + \delta g_{\rm UV}. \tag{1.4}$$

On the one hand $g_{\rm EFT}$ is the coupling from the well defined EFT, while on the other hand $\delta g_{\rm UV}$ is the contribution coming from the UV. It is tempting to write it as the integral of $\beta(g_{\rm UV})$ from $\ln\Lambda$ to infinity, but such expression does not make sense because we do not know how the UV-theory works (it is not even restricted to be a field theory!), therefore we write the contributions from the UV *symbolically* as $\delta g_{\rm UV}$.

Our predictive power lies on $\beta(g_{\rm EFT})$, calculated as usual via loop corrections and renormalization methods. Since the β-function is being integrated over the energy, what needs to calculate is essentially the loops themselves. Take as concrete example the corrections to the Higgs mass. There are many corrections to the 2-point function of the Higgs, but in particular the Yukawa couplings with fermions (y), the self quartic coupling (λ) and the quartic couplings g^2 and $(g')^2$ with gauge bosons produce a term proportional to Λ^2 at 1-loop order [1, 9, 16]:

$$\delta M_H^2 = \frac{3\Lambda^2}{8\pi^2}\left(\lambda + \frac{1}{8}(g')^2 + \frac{3}{8}g^2 - y^2\right) + \cdots, \tag{1.5}$$

where we used the cut-off Λ to cut off the internal momenta, and the dots denote subleading terms. The cut-off of the SM is, as far as it is known, above the TeV scale if not the Planck scale itself, whence Eq. (1.5) will produce a very large shift on the mass. Such enormous shift is not observed; the Higgs mass lies at 125 GeV. In renormalization program this is not a problem at all because, as per usual, we get rid of divergent quantities like this through counter-terms, producing finite corrections. Moreover, one could use dimensional regularisation, obtain $\frac{1}{\epsilon}$ divergences and cancel them with counter-terms.

So where is the problem? It starts with Eq. (1.4). Regularisation procedures are required to cancel divergent terms, but in an EFT there are no divergences at all, because all integrals are cut-off by Λ. Then in Eq. (1.4) the huge term in Eq. (1.5) will be present with no counter-term to cancel it*. The fact is: we measure a small mass M_H^2, thus the only way to measure such mass value is to impose that δg_{UV} is just as huge as the contribution in Eq. (1.5). There are two points worth of mention in this last statement. The first is that while working in an EFT one expects that UV corrections are small compared to contributions from energy scales below the cut-off, i.e., that the EFT is a good approximation and heavy physics decouple, as expected by the Wilson decoupling principle [8, 21, 22]. Second, δg_{UV} must not only be of the same order, but must also be set to a particular value in order to give a small measured mass. In other words, we are tuning an unknown parameter of an unknown UV-completion to match the experimental value, i.e. we need to create a certain amount of "fine-tuning" in order to keep light the scalar mass.

A toy-model to help us make more precise the discussion above is given by the following Lagrangian [9, 16]:

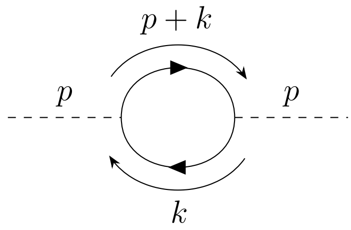
$$\mathscr{L} = \frac{1}{2}\,\partial^\mu\,\phi\,\partial_\mu\,\phi - \frac{m^2}{2}\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi - y\phi\bar{\psi}\psi, \tag{1.6}$$

where ϕ is a real scalar and ψ a Dirac fermion. Suppose that this theory is the UV-completion of a low-energy theory with just the light scalar particle ϕ, i.e. that $m \ll M$. Therefore, for energy scales $E \ll M$, it is reasonable to integrate out the fermion and obtain an EFT for just the scalar particle. The sum of One-Particle-Irreducible (1PI)

*We mean here that no regularisation is needed, which in perturbation theory involves the introduction of equally *divergent* counter-terms. Renormalization is, of course, always essential to express observables in terms of physical quantities.

diagrams at 1-loop is given by [20, 27]

$$i\Sigma(p^2) \;=$$



$$
\begin{aligned}
&= -(-iy)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \mathrm{Tr} \left[\frac{i(\not{k} + \not{p} + M)}{(p+k)^2 - M^2} \frac{i(\not{k} + M)}{k^2 - M^2} \right] \\
&= 4y^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k^2 + \Delta}{(k^2 - \Delta)^2}, \qquad (1.7)
\end{aligned}
$$

where $\Delta = M^2 - p^2 x(1-x)$. Performing dimensional regularisation in Eq. (1.7) we obtain

$$i\Sigma(p^2) = \frac{3iy^2}{4\pi^2} \left[\left(M^2 - \frac{1}{6} p^2 \right) \frac{2}{\epsilon} - M^2 + \frac{1}{6} p^2 + \int_0^1 \mathrm{d}x \; \Delta \ln \frac{4\pi\mu^2 e^{-\gamma_E}}{\Delta} \right], \qquad (1.8)$$

with μ the running scale. Therefore, at 1-loop the corrected inverse propagator reads

$$p^2 - m^2 - \Sigma(p^2) = \left(1 - \frac{y^2}{8\pi^2} \right) p^2 - \left(m^2 - \frac{3y^2 M^2}{4\pi^2} \right), \qquad (1.9)$$

where we have omitted logarithmic terms. The (unormalised) effective Lagrangian becomes then

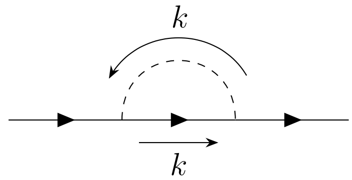
$$\mathscr{L}_{\mathrm{eff}} = \frac{1}{2} \left(1 - \frac{y^2}{8\pi^2} \right) \partial^\mu \phi \, \partial_\mu \phi - \frac{1}{2} \left(m^2 - \frac{3y^2 M^2}{4\pi^2} \right) \phi^2 + \cdots, \qquad (1.10)$$

where the dots denote higher order operators generated by fermionic loops. Note in particular the mass shift, which for the hierarchy chosen, is enormous; in fact it is of the order of the EFT cut-off M. We arrive at a somewhat contradictory situation where, to observe a small effective mass for a scalar, we need to assume a big and precise cancellation between m^2 and M^2. Such cancellation is only possible when $m \sim yM$, in contradiction with our previous assumption if $y \sim \mathcal{O}(1)$. More precisely, the configuration in which we have a heavy fermion and a light scalar can only be achieved if the Yukawa coupling y is of order $\frac{m}{M}$, meaning that the particles interact very weakly. The situation clarified by the example above illustrates the problem with the Higgs mass. More in general, for arbitrary couplings, scalar masses tend to receive large corrections from the UV, and for this very reason it is said that scalar masses are *UV-sensitive*.

This UV-sensitiveness is typical in scalar theories, whereas for fermions or gauge bosons there is a way to maintain masses stable under quantum corrections. Take for example a free massive spin half fermion; there is a global $U(1)_V$ symmetry which transforms the right- and left-handed components of the Dirac spinor in the same way [20]. When the mass is zero, this symmetry is enhanced and becomes a $U(1)_V \times U(1)_A$ which transforms the right- and left-handed components independently [1]. Obviously, the radiactive corrections also respect this bigger symmetry, therefore in the massive case the contributions from loops that breaks $U(1)_A$ must be proportional to the mass itself, so to vanish and restore the symmetry when the mass is taken to zero*. Fermion masses are examples of *technical natural* parameters; when taken to zero the symmetry group is enhanced and as a consequence radiactive corrections must be proportional to this same parameter. If such parameter is small for some reason, then technical naturalness guarantees that it is stable under quantum corrections. Such is not the case for the Higgs mass, because the SM does not have any extra symmetries when $M_H^2 = 0$.

We can illustrate the concept of technical naturalness with the toy-model of Eq. (1.6), considering however that the scalar is much heavier than the fermion, i.e. $M \ll m$. With this hierarchy we may, in an analogous manner, perform the 1-loop match into the low-energy effective theory by integrating out the scalar. Since we are interested only in the behaviour of the mass parameter, it suffices to compute the sum of the 1PI diagrams $\Xi(p)$ with $p = 0$:

$$i\Xi(p=0) =$$



$$\begin{aligned} &= (-iy)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i(\not{k} + M)}{k^2 - M^2} \frac{i}{k^2 - m^2} \\ &= \frac{iy^2 M}{16\pi^2} \left[\frac{2}{\epsilon} + \int_0^1 \ln \frac{4\pi\mu^2 e^{-\gamma_E}}{\Delta} \right], \end{aligned} \tag{1.11}$$

with $\Delta = M^2 + (m^2 - M^2)x$. Eq. (1.11) implies that the mass of the fermion receives contributions that are proportional to M itself, verifying explicitly that M is a technically

*The $U(1)_A$ symmetry is actually not a real symmetry of any QFT, since it is anomalous. The effects of the chiral anomaly are in general non-perturbative, in particular it does not interfere with the computation of the fermion masses, so the argumentation still holds [1, 21] natural parameter. Hence, as opposed to the situation where $m \ll M$, it is possible to construct a low-energy EFT with a light fermion and a heavy scalar for an arbitrary Yukawa coupling.

In summary, the problem we face comes from the clash between the EFT approach and scalar particles. As we saw in Eq. (1.3), the EFT predicts on general grounds that tree-level scalar masses are just as big as the cut-off of the EFT (except in the extreme case $|c| \ll 1$), which in turn implies that the particle cannot belong to the spectrum of IR physics and should be integrated out. If we instead insist on having a light scalar in the spectrum, for which a proper mechanism is needed in order to explain it, the scenario is worsened by taking quantum effects into account: loop corrections to the scalar mass are also of order of the cut-off, therefore UV-effects of the same order, which are uncomputable in the EFT framework, are required to stabilise the mass at its observed value. Note that the issue lies not on the cancellation between EFT- and UV-contributions, but on the fact that the latter must be of the same order of the first. This contradicts the core principle of EFT's, that the low energy theory is a good approximation and that heavy physics decouple. From the EFT point of view $\delta g_{\rm UV}$ should only represent a small correction to the value of g_M at the IR, in other words, $g_{\rm EFT} \gg \delta g_{\rm UV}$. Therefore, the mass of a light scalar is driven to a value of the order of the cut off by quantum corrections. These points are very important, and should be once more emphasised: experimentally we observe a light scalar, while the framework that supports all modern particle physics, the EFT approach, predicts that in general no scalar particle could live at the IR and, if it for some reason belongs to IR spectrum, would not be quantum mechanically stable.

The situation described above fits into the definition of what is a *problem* within physics. We must remember that physics is a predictive, natural science, in the sense that physical theories describe observed phenomena and also predict new ones. The verification of a given theory comes with the observation of the predicted phenomena, from both the qualitative as well quantitative points of view. If the same theory fails to describe some of the observed phenomena, or a predicted phenomenon is not experimentally observed, then it is incompatible with reality; in other words, it has problems. The SM is an extreme example of this, because, while it predicts the wrong value for the neutrino masses, it is extremely successful in every other aspect*. As stressed, the SM embedded into the

*Other smaller issues like b-anomalies or the anomalous $(g-2)_\mu$ are not yet well established, hence framework of EFT is inconsistent with the measured value of the Higgs mass. Since it is a wrong prediction of a well established theory of physics, the situation of the Higgs mass described above is a concrete problem of the SM EFT.

This problem, from now on addressed as the Scalar Mass Problem (SMP), is one of the focuses of this thesis. It is worth remarking that the SMP represents a departure from the usual definition of what is called "Hierarchy Problem" (see section 1.2), which has being constantly criticised in the literature in recent times [26, 28]. In order to solve the SMP, three paths can be followed. First, it could be that the interpretation of the theory and its connection to the experiments is wrong. However, this possibility is excluded, because the SM and EFT are both very successful. Second, it may be that QFT, as a fundamental theory, is not valid anymore above the LHC scale, hence completely new theories to describe UV-physics (that must recover the usual QFT's at low energies) would need to be formulated. Third, it is possible that the theoretical framework is adequate, but that something is missing in the model, for instance new mechanisms (e.g. new symmetries) and/or new particles. This is what beyond the Standard Model (BSM) models are all about and also the approach followed in this thesis.

1.2 Naturalness & Fine Tuning - the Hierarchy Paradigm

The questions raised by the SMP are already enough to motivate us to search for solutions, in particular the ones that extend the SM. However, the history of physics did not follow this path. Instead, particle physicists at the time asked themselves: why is the coefficient c so small? How and why such fine tuned cancellations happen? Note that, although these questions are similar to the ones of the SMP at the quantitative level, they are vastly distinct at the conceptual level, as it will now be clarified.

Naturalness

Consider a physical quantity g with dimension $\dim g$. According to the previous discussion, the only physical parameter at disposal is the cut-off of the theory for which g is defined. Purely on dimensional grounds, one concludes that [9, 16, 10]

$$g = c_g \Lambda^{\dim g}, \tag{1.12}$$

they are not taken into account [3]

with c_g a dimensionless constant. As always, dimensional analysis does not allow us to calculate the coefficient directly, and a theory is needed to compute it. Still, one can always expect what the value of c_g will turn out to be: based on physical intuition one can infer if it is small, large or even of order 1. For the most fundamental theory of nature, where the parameters do not depend on anything else other than Λ, there is no particular reason for c_g to be large or small; it is *natural* for c_g to be of order 1. The Higgs mass is an example of an *unnatural* parameter, because if $\Lambda_{\text{SM}} > 100$ TeV, we must have

$$c_H < 10^{-3}. \tag{1.13}$$

The situation gets even worse if $\Lambda_{\text{SM}} = M_P \sim 10^{19}$ GeV. More in general, scalar masses are unnatural. This statement can be seen directly from Eq. (1.3), where it was deduced that $|c| \ll 1$ if ϕ belongs to IR physics.

Loop corrections are also problematic. Taking the physical Higgs mass to be small, its most relevant 1-loop corrections in Eq. (1.5) introduce Λ^2 divergent terms. As pointed out, in order to stabilise the mass parameter, UV-effects must be of same order. Moreover, they need to cancel the contributions from Eq. (1.5) very precisely; that is why it is said that the UV contribution δg_{UV} is *fine tuned*.

The concept behind this discussion is named *naturalness* and is used as a guiding principle to formulate BSM models. Note, however, that such principle is not a completely physical one and the justification is the following. Ultimately, one is trying to explain why numbers are small and, although they are quantitatively related to physical quantities, theories and models are discarded solely based on them, and not on phenomenological or theoretical inconsistencies. Formulated in another way, unnatural BSM models are just unnatural, and not physically inconsistent. For this reason, if naturalness is used as a guiding principle in order to solve the issues regarding the Higgs mass, one will not be solving the SMP, but something else instead. The latter is the hierarchy between the Electroweak scale and the scale of new physics, which is named as the Hierarchy Paradigm (HPa). The word "paradigm" is chosen instead of "problem", because the question whether a parameter is natural or not is not a problem from the physical point of view, as stressed in the section 1.1. In Table 1 a concise definition of both the SMP and the HPa is given, and note the term "Hierarchy Problem" (HP) will still be used to denote both of them.

Physicists of the last decades have insisted in using naturalness to shape their models* and, although naturalness has some conceptual mishaps, they were not without gain. Many interesting, profound and useful ideas were born from theories that use naturalness as a guiding principle, whence it is worth to insist on it for a while.

Fine Tuning

In order to use naturalness consistently one must first develop a way to quantify how much a parameter is natural or not.

It was stressed that, to obtain the proper value of the Higgs mass, a certain amount of cancellation between radiative corrections and UV contributions was required. From Eq. (1.4), with $g_M = m_h^2 = (125\text{ GeV})^2$ together with the correction in Eq. (1.5) it is possible to compute how precise the cancellation between the contributions must be. Taking $\Lambda = M_P \sim 10^{19}$ GeV one obtains that the cancellation between the two terms must be given within 34 orders of magnitude; that's what one calls "precise cancellation". Such cancellations are also a source of unnaturalness; in the example cited above we have

$$\frac{M_P^2}{(125\text{ GeV})^2} \sim 10^{34},$$

hence the cancellation is given for 10^{-34} parts in 1. The conclusion is that this fact renders the theory unnatural and one is therefore motivated to search for solutions to this problem. However, to make the discussion more precise, we need to quantify how unnatural the theory is, in this manner it will be possible to determine if a solution to the HPa is indeed a solution or not. Such quantification is given by the concept of Fine Tuning (FT), which is, as the name suggests, the precise tuning of the free parameters of the theory in order to obtain the correct measured quantities. Apart from the subjective definition of what is tuned and what not, FT is still a key concept in any EFT since, after all, we expect UV effects to account for only small corrections, i.e. that the heavy physics decouple in the IR regime. All definitions of FT, in a way or another, try to measure the UV-sensitivity of a given EFT with respect to the free parameters of the theory.

One can define the FT as the relative variation of a given observable of the theory with

*The *Swampland Program*, introduced in 2005 [29], must not be forgotten. In this program one tries to determine which low-energy EFT's are consistent with Quantum Gravity (Landscape) and which are not (Swampland). Since it does focus on the consistency of EFT's, it can be classified as an approach to the SMP.

respect to a given parameter. Consider an observable O that depends on a parameter g, which is itself generated at an UV scale Λ. If O is measured at an energy $E \ll \Lambda$, one first solves the RGE of g with the initial condition $g_{\rm UV} = g(\Lambda)$ and then runs it to E, where it will acquire a value $g(E)$. A change δg in the initial condition at the UV results in a change δO in the measured observable:

$$O(g_{\rm UV}) \to O(g_{\rm UV}) + \delta O \simeq O(g_{\rm UV}) + \delta g \left.\frac{\mathrm{d}O}{\mathrm{d}g}\right|_E .$$

The FT Δ is defined by

$$\frac{\delta O}{O} \equiv \Delta \frac{\delta g}{g}, \tag{1.14}$$

which may be rewritten as

$$\Delta = \frac{g}{O}\frac{\mathrm{d}O}{\mathrm{d}g} = \frac{\mathrm{d}\ln O}{\mathrm{d}\ln g}, \tag{1.15}$$

evaluated at energy E. If O depends on more parameters, $\{g_i\}$, then

$$\Delta = \max_i \left|\frac{\mathrm{d}\ln O}{\mathrm{d}\ln g_i}\right|, \tag{1.16}$$

can be taken as the corresponding generalisation. The measure Δ defined above for the FT is known as *Barbieri-Giudici measure* [30]. Though model dependent [31], Eq. (1.16) may be applied to any theory and it represents very well the conceptual idea of FT.

Before proceeding, let us explore the meaning of Δ a little bit more. Consider for instance that the observable O is the coupling g itself; in this case Eq. (1.16) becomes

$$\Delta = \frac{\mathrm{d}\ln g(E)}{\mathrm{d}\ln g(\Lambda)} = \frac{g(\Lambda)}{g(E)}\frac{\mathrm{d}g(E)}{\mathrm{d}g(\Lambda)} \simeq \frac{g(\Lambda)}{g(E)}\frac{\delta g(E)}{\delta g(\Lambda)} \tag{1.17}$$

$$\Rightarrow \frac{\delta g(E)}{g(E)} = \Delta \frac{\delta g(\Lambda)}{g(\Lambda)}. \tag{1.18}$$

With Eq. (1.18) the meaning of the FT becomes extremely clear: a 1% variation on $g(\Lambda)$ results in a $(\Delta \times 1)$% variation on $g(E)$. So what we would expect to be a non-tuned, i.e., natural theory is at maximum a Δ of order 1, otherwise, if $\Delta \gtrsim 1$, then small variations in the UV initial conditions affect severely the measured value, which means that the theory is highly UV-sensitive (see for instance Figure 2).

⋆⋆⋆

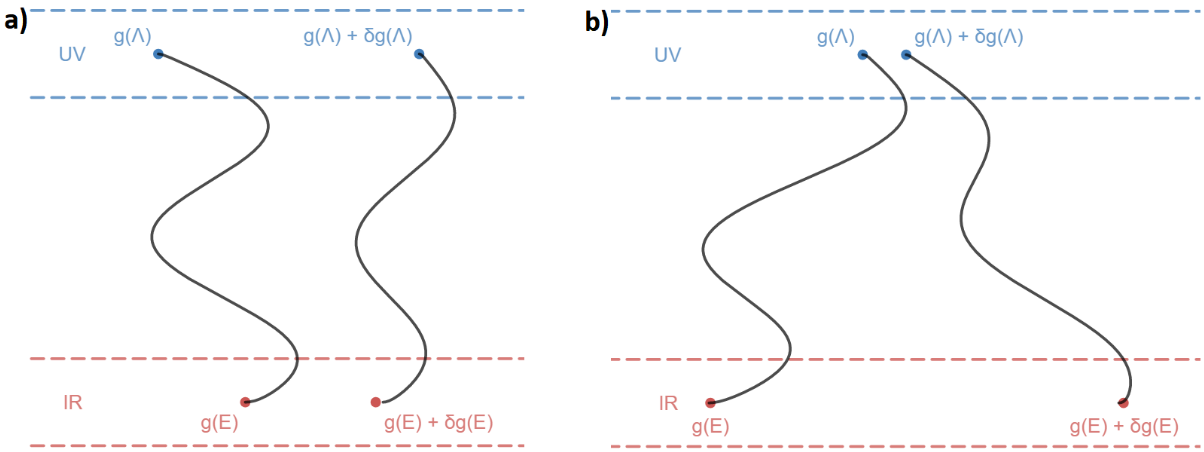


Figure 2: Illustration of the fine tuning through the running of the coupling g. In a) one has $\Delta \lesssim 1$, hence the variation in $g(E)$ is small or of the same order compared to that in the UV, while in b) $\Delta \gg 1$, so a small variation in $g(\Lambda)$ causes a big change in $g(E)$. The expression for $\delta g(E)$ is given by Eq. (1.18).

Since EFT is a very profound and complex subject, there isn't an unique solution to neither the SMP nor the HPa as far as it is known. In the next chapters some of the solutions to the HPa and the SMP are going to be presented. In particular, the focus will be on the ideas and mechanisms behind each model and to use them to build minimal BSM models which are consistent with experimental constrains. They are briefly outlined below.

- The first one is a method based on symmetries. As it will be seen in Chapter 2, there is an unique way to extend the symmetry group of the SM if bosons and fermions are related in some way. The symmetry sought is one that relates bosons and fermions, which can be represented as

$$\delta\phi = \epsilon\psi, \tag{1.19}$$

  where ϕ is a scalar, ψ is a Weyl fermion and ϵ is a spinor that parametrises the transformation. *Supersymmetry* is the name of this theory and it stabilise the Higgs mass due to the appearance of a bosonic loop for every fermionic loop, in a way that the contribution from Eq. (1.5) vanishes exactly.

- The second situation considered is one analogue to the one in Quantum Chromodynamics (QCD) case. In this theory, quarks and gluons form bound states: baryons and mesons. Taking the quarks to be massless, the QCD Lagrangian becomes invariant under a $SU(2)_R \times SU(2)_L$ group, which transforms left- and right-handed components of up and down quarks independently [1, 20, 24]. The spontaneous breaking of such group gives a $SU(2)_V$ and generates three massless Nambu-Goldstone Bosons (NGB's): the pions. The quarks are not massless, hence the group $SU(2)_R \times SU(2)_L$ is explicitly broken by the masses. But as the quarks masses themselves are technical natural parameters, the effects of this explicit break will be small if the masses are small. The up and down are very light [3], therefore the pion masses turn out to be small compared to the QCD scale Λ_{QCD}. In other words, a small mass compared to the energy scale of the theory can be explained through compositness. Models in which the Higgs is taken to be a composite particle in the same way compositness in QCD is understood are known as *Composite Higgs Models*, subject of Chapter 3. In this case the symmetry transformation is a simple shift in the field,

$$\delta\phi = c, \tag{1.20}$$

with c a constant.

- The third class of models to be presented is known as *Neutral Naturalness*. In such models one avoids any new physics charged under the SM gauge group. The motivation for this is precisely the stringent experimental constrains, which comes from experimental particle physics and collider data. One supposes that that the Higgs is a pNGB of an approximate global symmetry that connects the SM with a new, neutral sector [16, 15, 32]. In this case the symmetry transformation of the Higgs is analogous to the one in Eq. (1.20). What is interesting in this type of solution is that, there are some special cases where discrete symmetries become manifest, hence such models have an intriguing interplay between discrete and continuous symmetries. The Higgs mass is computable and is stabilised by the corrections from the neutral sector.

- In Chapter 4 the *Cosmological relaxation* models will be introduced. It is qualitatively different with respect to the mechanisms cited above, due to the fact that it takes into account cosmological considerations. The Higgs mass in this case is driven to the observed experimental value by the cosmological evolution of the universe and it is stabilised by the dynamics of new particles [16, 17]. How such reasoning touches

Hierarchy Problem (HP)	
Scalar Mass Problem (SMP)	**Hierarchy Paradigm (HPa)**
Questions why EFT's with scalar particles contradicts the expectations of decoupling and quantum stability at the low-energy regime	Questions why the ratio between the Higgs mass and the scale of new physics is so small, and how large quantum corrections are cancelled

Table 1: Brief definitions of the Scalar Mass Problem and of the Hierarchy Paradigm, which are the two facets of what it is today named as Hierarchy Problem.

the thin line between the formulation of the SMP and the HPa, and why it is so important to modern physics is going to be explained in detail.

2 Supersymmetry

Supersymmetry (SUSY) is the first solution to the Hierarchy Problem (HP) that will be presented in this thesis. It was first introduced in the 70's, and soon became clear that considering a supersymmetric version of the SM would solve the HP.

This chapter is organised in the following manner. First, the SUSY algebra is introduced from first principles and its most relevant physical properties are pointed out. Then, reducible and irreducible representations of the superalgebra are built, while focusing manly on the superfield representation. In section 2.3, the motivations for SUSY-invariant Lagrangians for both chiral and vector supermultiplets are given and general renormalizable non-abelian supersymmetric gauge theories are constructed. Finally, the Minimal Supersymmetric Standard Model (MSSM) is developed, and particular emphasis on how it solves the HP and on the corresponding FT is given.

2.1 Motivations & SUSY algebra

Supersymmetry was first motivated not from a particular problem within particle physics, such as the Hierarchy Problem, but as an exception to a famous no-go theorem* known as Coleman-Mandula Theorem. Within reasonable assumptions it proves that the S-matrix of a quantum field theory can commute only with the elements of the product between the Poincaré group and internal symmetry groups [34, 35]. Although this theorem is robust, it contains several loopholes. One of them is in the very definition of a symmetry group, which includes the assumption that its elements commute with the S-matrix, in other words, a symmetry is defined based on usual Lie algebras. Hence, the Coleman-Mandula theorem does not necessarily hold if we generalise the Lie algebras to graded Lie algebras [35]. According to the Spin-Statistics theorem [36], the only physical bracket other than the commutator is the anticommutator, so in mathematical terms we would now have a Z_2-graded Lie algebra, or, in simple physical terms, an algebra that mixes commutators and anticommutators.

The relation between the statistics of a given particle and graded Lie algebras is made manifest from the definition of the latter. A graded Lie algebra $\mathfrak{g}$ is a Lie algebra

*No-go Theorems prove that a particular configuration is physically impossible. An example in classical physics is Earnshaw's theorem [33].

supplemented with a gradation, in other words, the algebra $\mathfrak{g}$ can be written as [35]

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n, \tag{2.1}$$

such that $[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m}$. In the equation above, the grade is defined as the integer in the direct sum. The Lie bracket associated with a graded Lie algebra depends on the grade of each element, and can be written compactly as

$$[x, y] = xy - (-1)^{nm} yx, \quad x \in \mathfrak{g}_n, \quad y \in \mathfrak{g}_m. \tag{2.2}$$

From the equation above it is clear that the commutator is the Lie bracket of an algebra with grade zero, while the anticommutator corresponds to an algebra with grade 1, whence a graded Lie algebra that mixes commutators and anticommutators is given by

$$\mathfrak{g} = \bigoplus_{n \in \{0,1\}} \mathfrak{g}_n = \mathfrak{g}_0 \oplus \mathfrak{g}_1. \tag{2.3}$$

The specific grade from Eq. (2.3) is named Z_2-grading and is the adequate mathematical structure for a Lie algebra that relates both commutators and anticommutators.

Admitting the idea of a Z_2-graded Lie algebra, also called superalgebra, new generators, which will play the role of the generators of the symmetry that will extend the Poincaré group non-trivially, must be introduced to the original Lie algebra. These generators are the elements of the algebra $\mathfrak{g}_1$ in Eq. (2.3) and are named Q and $\bar{Q}$ their conjugate. The structure of the superalgebra will be thus fully determined from the moment that all commutators and anticommutators of Q and $\bar{Q}$ with the generators of the internal and Poincaré symmetries are computed.

Before doing so, the Lorentz nature of both Q and $\bar{Q}$ must be made explicit. The whole point of supersymmetry being an exception of the Coleman-Mandula theorem is that these new generators are allowed to have non-trivial Lorentz representations, therefore they are defined to have the following $su(2) \oplus su(2)$ transformations

$$Q \sim (j, j'), \tag{2.4}$$

where j and j' are integers or half-integers with the condition $2(j + j') =$ odd, because otherwise Q and $\bar{Q}$ would be bosonic operators and, by the Coleman-Mandula theorem, would be forced to be Lorentz scalars*. That is to say, Q, and consequently $\bar{Q}$, are Lorentz spinors and can then belong to the algebra $\mathfrak{g}_1$. The determination of j and j' is given in group-theoretical grounds. Take for instance

$$[Q, \bar{Q}]_+ = Q\bar{Q} + \bar{Q}Q.$$

This anticommutator must transform as $(j + j', j + j')$, i.e. as a bosonic operator. Such operator must belong to the Poincaré group [38], but the only generator transforming under such representation is the 4-momentum $P^\mu \sim (\frac{1}{2}, \frac{1}{2})$, so $j = \frac{1}{2}$ and $j' = 0$ is the only possibility. Q and $\bar{Q}$ are therefore Lorentz spinors of the lowest representations, $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ respectively, and so $\bar{Q}$ can be written as $Q^\dagger$ since the representations are mapped into each other by complex conjugation [37].

With the same reasoning we can compute

$$[Q_\alpha, Q_\beta]_+, \quad \alpha, \beta = 1 \text{ or } 2, \tag{2.5}$$

where now α and β denote the indices of the $(\frac{1}{2}, 0)$ representation. The symmetric part in α and β transforms as $(1, 0)$, hence it is zero because there is no such tensor in the Poincaré group [38], while the antisymmetric piece depends on whether there is just one operator Q or more. If there is indeed just one pair of operators $(Q, Q^\dagger)$, the antisymmetric part of Eq. (2.5) is zero, as the anticommutator is symmetric. If not, if there are $\mathcal{N}$ pairs of operators,

$$Q^I \text{ and } (Q^\dagger)^J, \text{ with } I, J = 1, \cdots, \mathcal{N}, \tag{2.6}$$

then Eq. (2.5) will have a contribution from a term proportional to $\epsilon_{\alpha\beta}$ if multiplied by an antisymmetric tensor Z^{IJ}, defined as *central charges*. When $\mathcal{N} > 1$ one usually speeks of *extended* supersymmetries [11, 12, 39].

With similar, though more technical arguments†, one arrives at the full superalgebra

---

*An operator $\mathcal{O}$ with Lorentz representation

$$\mathcal{O} \sim (a, b)$$

is bosonic (fermionic) if the sum $a + b$ is integer (semi-integer) [37].

†The steps presented here are manly based on the original work [40], which also details the argumentation to determine the commutations relations with the Poincaré generators.

for $\mathcal{N} = 1$:

$$[Q_\alpha, Q^\dagger_{\dot\beta}]_+ = 2\sigma^\mu_{\alpha\dot\beta} P_\mu, \tag{2.7a}$$

$$[Q_\alpha, Q_\beta]_+ = [Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}]_+ = [Q^\dagger_{\dot\alpha}, P^\mu]_- = [Q_\alpha, P^\mu]_- = 0, \tag{2.7b}$$

$$[J^{\mu\nu}, Q_\alpha]_- = i(\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta, \tag{2.7c}$$

$$[J^{\mu\nu}, Q^\dagger_{\dot\alpha}]_- = i(\bar\sigma^{\mu\nu})_{\dot\alpha}^{\ \dot\beta} Q^\dagger_{\dot\beta}, \tag{2.7d}$$

where $\dot\alpha$ and $\dot\beta$ are the indices of the $(0, \frac{1}{2})$ representation and take values $\dot{1}$ and $\dot{2}$, σ^μ are the Pauli matrices, $\sigma^{\mu\nu}$ is the spin generator and $J^{\mu\nu}$ the generator of the Lorentz group.

To close the superalgebra one needs in addition the commutation relations with the internal group generators, called here B_n. The Coleman-Mandula theorem states that B_n form a algebra composed of an abelian algebra plus a non-abelian (semi-simple) Lie algebra. The structure of a Z_2-grading forces the commutator between Q's and B's to be proportional to the Q's themselves (see Eq. (2.2)), therefore

$$[Q_\alpha, B_n]_- = q_n Q_\alpha, \tag{2.8a}$$

$$[Q^\dagger_{\dot\alpha}, B_n]_- = -q_n^* Q^\dagger_{\dot\alpha}, \tag{2.8b}$$

The charges q_n defined by the equation above are real, which follows from the generalised Jacobi identities [41]:

$$[O_1, [O_2, O_3]_-]_- + [O_2, [O_3, O_1]_-]_- + [O_3, [O_1, O_2]_-]_- = 0, \tag{2.9a}$$

$$[F_1, [O_2, O_3]_-]_- + [O_2, [O_3, F_1]_-]_- + [O_3, [F_1, O_2]_-]_- = 0, \tag{2.9b}$$

$$[O_1, [F_2, F_3]_+]_- + [F_2, [F_3, O_1]_-]_+ - [F_3, [O_1, F_2]_-]_+ = 0, \tag{2.9c}$$

$$[F_1, [F_2, F_3]_+]_- + [F_2, [F_3, F_1]_+]_- + [F_3, [F_1, F_2]_+]_- = 0, \tag{2.9d}$$

with O_i and F_i bosonic and fermionic operators, respectively. In particular, using that

$[B_n, P^\mu]_- = 0$ and Eq. (2.9c) with $F_2 = Q_\alpha$, $F_3 = Q^\dagger_{\dot\alpha}$ and $O_1 = B_n$ one obtains

$$\begin{aligned} 0 &= [B_n, [Q_\alpha, Q^\dagger_{\dot\alpha}]_+]_- + [Q_\alpha, [Q^\dagger_{\dot\alpha}, B_n]_-]_+ - [Q^\dagger_{\dot\alpha}, [B_n, Q_\alpha]_-]_+ \\ &= 2\sigma^\mu_{\alpha\dot\alpha}[B_n, P^\mu]_- - q_n^*[Q_\alpha, Q^\dagger_{\dot\alpha}]_+ + q_n[Q^\dagger_{\dot\alpha}, Q_\alpha]_+ \\ &= (q_n - q_n^*)[Q_\alpha, Q^\dagger_{\dot\alpha}]_+, \end{aligned} \tag{2.10}$$

showing that $q_n - q_n^* = 0$ must be satisfied. In a extended superalgebra the relations (2.8a) and (2.8b) becomes more complex due to the internal indices I, J, in this case the q_n are promoted to matrices $(q_n)^I_J$, and are hermitian in the internal indices by similar calculation [39, 40]. In the non-extended case one can prove in addition that not all charges q_n are non-vanishing. This follows from the Jacobi identity in Eq. (2.9b) with $F_1 = Q_\alpha$, $O_2 = B_n$ and $O_3 = B_m$,

$$\begin{aligned} 0 &= [Q_\alpha, [B_n, B_m]_-]_- + [B_n, [B_m, Q_\alpha]_-]_- + [B_m, [Q_\alpha, B_n]_-]_- \\ &= if^l_{nm}[Q_\alpha, B_l]_- - q_m[B_n, Q_\alpha]_- + q_n[B_m, Q_\alpha]_- \\ &= \left(if^l_{nm}q_l - q_m q_n + q_m q_n\right) \\ &= if^l_{nm}q_l, \end{aligned} \tag{2.11}$$

where f are the structure constants of the internal algebra. For the non-abelian piece of the internal symmetry, the structure constants cannot be all zero simultaneously, hence $q_l^{\text{non-ab}} = 0$, i.e. in non-extended SUSY the generators Q and $Q^\dagger$ do not affect non-abelian charges. Only the charges associated with the abelian algebra, for which $f = 0$, are allowed to be non-zero. As a result of Eq. (2.11), only one linear combination of the abelian generators has a non-trivial SUSY transformation. This can also be seen from the fact that the entire SUSY algebra is invariant under the abelian transformation

$$Q \to e^{ir}Q, \tag{2.12a}$$

$$Q^\dagger \to e^{-ir}Q^\dagger. \tag{2.12b}$$

The transformation above originates from a $U(1)_R$ group, generated by R, whose commutation relations are

$$[Q_\alpha, R]_- = rQ_\alpha, \tag{2.13a}$$

$$[Q^\dagger_\alpha, R]_- = -rQ^\dagger_\alpha \tag{2.13b}$$

In extended SUSY the $U(1)_R$ group is promoted to a full $U(\mathcal{N})_R$ group [11].

Equations (2.7) and (2.13) contain the most important features of supersymmetric theories. In the following sections they will be used to construct representations, with the final goal being the construction of supersymmetric Lagrangians.

2.2 Representations

With the superalgebra at hands, our task would be to explore its properties and then build representations. Once the SUSY transformations of the fields are determined, the construction of Lagrangians invariant under SUSY becomes straightforward. Our focus will be, however, on the formulation in terms of superfields, a more abstract and practical representation of SUSY. After understanding how to consistently introduce superfields, they will be studied along with the algebraic properties of SUSY.

2.2.1 Superfields

One can study the representations of a field through their transformations properties under the corresponding symmetry group. In the standard case of the Poincaré and internal symmetry groups, this is achieved by constructing the appropriate unitary operators $U(\lambda)$, where λ are the transformation parameters that act on field space. In this latter case, however, the algebra is an usual Lie algebra. For SUSY one must be able to define such operators for Z_2-graded Lie algebras. The most natural way to construct them is to realise that if we contract Q and $Q^\dagger$ in the superalgebra with spinors θ_α and $\theta^\dagger_{\dot\alpha}$, all anticommutators become commutators. The referred contraction of spinors is defined in the usual way with the Levi-Civita tensor [41, 42],

$$\theta Q \equiv \theta^\alpha Q_\alpha = Q^\alpha \theta_\alpha = \theta^\alpha Q^\beta \epsilon_{\alpha\beta}, \tag{2.14a}$$

$$\theta^\dagger Q^\dagger \equiv \theta^\dagger_{\dot\alpha} (Q^\dagger)^{\dot\alpha} = (Q^\dagger)_{\dot\alpha} \theta^{\dagger\,\dot\alpha} = \theta^\dagger_{\dot\alpha} Q^\dagger_{\dot\beta} \epsilon^{\dot\alpha\dot\beta}. \tag{2.14b}$$

For instance, contracting both sides of the anticommutation relation in Eq. (2.7a) with θ^α and $\theta^{\dagger\,\dot\alpha}$ one obtains

$$\begin{aligned}
\theta^\alpha\theta^{\dagger\,\dot\alpha}[Q_\alpha, Q^\dagger_{\dot\alpha}]_+ &= \theta^\alpha\theta^{\dagger\,\dot\alpha}Q_\alpha Q^\dagger_{\dot\alpha} + \theta^\alpha\theta^{\dagger\,\dot\alpha}Q^\dagger_{\dot\alpha}Q_\alpha \\
&= \theta^\alpha\Big(-Q_\alpha\theta^{\dagger\,\dot\alpha}\Big)Q^\dagger_{\dot\alpha} + \theta^\alpha\Big(-Q^\dagger_{\dot\alpha}\theta^{\dagger\,\dot\alpha}\Big)Q_\alpha \\
&= -(\theta^\alpha Q_\alpha)\Big(-Q^\dagger_{\dot\alpha}\theta^{\dagger\,\dot\alpha}\Big) - (\theta^\alpha Q_\alpha)\Big(Q^\dagger_{\dot\alpha}\theta^{\dagger\,\dot\alpha}\Big) \\
&= \theta Q\theta^\dagger Q^\dagger - \theta^\dagger Q^\dagger\theta Q \\
&= [\theta Q, \theta^\dagger Q^\dagger]_-, \qquad (2.15)
\end{aligned}$$

where the anticommuting properties of grassmannian objects was used. The other relation in Eqs. (2.7) and (2.13) can be rewritten analogously.

With the superalgebra written in terms of commutators, U can be written as an exponential of the generators as usual

$$U(\lambda) = U(x,\theta,\theta^\dagger) = \exp\left(ix^\mu P_\mu + i\theta Q + i\theta^\dagger Q^\dagger\right). \qquad (2.16)$$

The unusual part comes from the fact that now U can depend on the spinors θ and $\theta^\dagger$, which, unlike x^μ, have grassmannian nature. The action of such spinor-dependent transformations on the SM fields is unclear, since the fields of usual quantum field theories do not depend on such parameters. The solution to this impasse is simple: one promotes all ordinary fields to *superfields*. A superfield is an object F that depends on $\lambda = (x, \xi, \xi^\dagger)$, where ξ is a spinor and $\xi^\dagger$ its conjugate, and transforms under SUSY as*

$$F(x,\xi,\xi^\dagger) \to U(0,\theta,\theta^\dagger)F(x,\xi,\xi^\dagger). \qquad (2.17)$$

There are still two problems: it is neither known how F is related to the ordinary fields nor how Q_α and $Q^\dagger_{\dot\alpha}$ act on the $(x,\xi,\xi^\dagger)$ space. For the latter, it suffices to calculate

$$U(0,\theta,\theta^\dagger)U(x,\xi,\xi^\dagger) = U(\lambda),$$

*In Eq. (2.17) it is implicit that F is already treated as an ordinary superfield, and not as an operator-valued field. More fundamentally, one would take F to be an operator-valued superfield that transforms as

$$F(x,\xi,\xi^\dagger) \to U(0,\theta,\theta^\dagger)F(x,\xi,\xi^\dagger)U^\dagger(0,\theta,\theta^\dagger),$$

with Q and $Q^\dagger$ abstract operators, and *then* represent both of them as differential operators acting on λ-space, whose action is given exactly by Eq. (2.17) [43].

which can be trivially evaluated by making use of the superalgebra and the Baker-Campbell-Hausdorff formula,

$$\lambda = (x^\mu - i\theta\sigma^\mu\xi^\dagger + i\xi\sigma^\mu\theta^\dagger, \xi + \theta, \xi^\dagger + \theta^\dagger), \tag{2.18}$$

so that the effect of a SUSY transformation on the coordinates is

$$\begin{aligned} x^\mu &\to x^\mu - i\theta\sigma^\mu\xi^\dagger + i\xi\sigma^\mu\theta^\dagger \\ \xi &\to \xi + \theta \\ \xi^\dagger &\to \xi^\dagger + \theta^\dagger, \end{aligned}$$

i.e. just simple translations, which are represented in coordinate space by derivatives

$$Q_\alpha = \frac{\partial}{\partial\,\xi^\alpha} + i\Big(\sigma^\mu\xi^\dagger\Big)_\alpha \partial_\mu, \tag{2.19a}$$

$$Q^\dagger_{\dot\alpha} = \frac{\partial}{\partial\,\xi^{\dot\alpha}} + i(\xi\sigma^\mu)_{\dot\alpha}\,\partial_\mu\,. \tag{2.19b}$$

The differential operators in Eq. (2.19) satisfy the same anticommutation relations (2.7a) and (2.7b), so they are the representation of the SUSY generators in the $(x, \xi, \xi^\dagger)$ space. What is left is to understand how does the superfield is related to the ordinary fields and the spinorial parameters.

Since the Talyor expansion of functions that depend on grassmannian parameters are always finite, the dependency one the spinorial parameters can be introduced as a linear combination of powers of ξ and $\xi^\dagger$. The most general scalar superfield can be written as

$$\begin{aligned} F(x,\xi,\xi^\dagger) \;\; &= \;\; \phi(x) + \xi\psi(x) + \xi^\dagger\chi^\dagger(x) + \xi^2 f(x) + (\xi^\dagger)^2 g(x) + \xi^2\xi^\dagger m^\dagger(x) \\ &\quad + (\xi^\dagger)^2\xi n(x) + \xi\sigma^\mu\xi^\dagger v_\mu(x) + \xi^2(\xi^\dagger)^2 d(x), \end{aligned} \tag{2.20}$$

where the coefficients of this linear combination are none other than the fields themselves, implying that we are building a linear representation of SUSY. Here, the Lorentz structure of the superfield F was specified to be that of a scalar, implying that ϕ, f, g and d are scalars, v_μ is a vector, $\chi^\dagger$, ψ, n and $m^\dagger$ are Weyl spinors. Note how many different kinds of fields appear in F; to relate bosons and fermions is one of the most striking features of

SUSY.

A remark regarding Eq. (2.20) is in order. For a scalar superfield, one could in principle add the terms

$$\left(\xi\sigma^{\mu}\xi^{\dagger}\right)\left(\xi^{\dagger}\bar{\sigma}^{\nu}\xi\right)h_{\mu\nu}, \quad \xi\sigma^{\mu\nu}\xi F_{\mu\nu}, \tag{2.21}$$

for a symmetric rank 2 tensor $h_{\mu\nu}$ and an antisymmetric $F_{\mu\nu}$, to the superfield in Eq. (2.20). The first term above contains only the information on the trace of $h_{\mu\nu}$, because $\sigma^{(\mu}\bar{\sigma}^{\nu)} \sim \eta^{\mu\nu}$, while the second one is actually zero, since $\xi\sigma^{\mu}\bar{\sigma}^{\nu}\xi = \xi\sigma^{\nu}\bar{\sigma}^{\mu}\xi$. Therefore, these tensor structures cannot be fitted into the scalar superfield, or, more precisely, they do not add any new degrees of freedom to it*.

In conclusion, it was shown how to construct the superfield representation of the superalgebra, that, together with Eq. (2.17), gives the appropriate SUSY transformations of the fields. The expression of F in Eq. (2.20) contains many distinct fields, which is a priori no issue. However, we will see in the following that only two particles are connected by SUSY transformations in $\mathcal{N} = 1$. This does not mean that superfield formulation is inconsistent, but that it is reducible, as it will become clear.

2.2.2 Irreducible representations

Let us turn to the irreducible representations of the SUSY algebra. Our starting point is the Hilbert space of a multi-particle theory: the Fock space. The Fock space is defined as

$$\mathcal{F} = \bigoplus_{n}^{\infty}\left(\bigotimes_{k=0}^{n}\mathcal{H}\right), \tag{2.22}$$

where $\mathcal{H}$ is the Hilbert space spanned by the one-particle states $|p,\sigma\rangle$, with p the on-shell 4-momentum and σ the z-projection of the spin (or helicity for massless particles). It is well known that $\mathcal{H}$ is an irreducible and unitary representation of the Poincaré group [37]. When needed, a "B" or "F" subscript in $\mathcal{F}$ will be written to designate a bosonic or fermionic Fock space, respectively.

In a supersymmetric theory, Q_{α} and $Q^{\dagger}_{\dot{\alpha}}$ may also act on the Fock space in Eq. (2.22),

*The usual definitions are

$$\sigma^{\mu\nu} = \frac{1}{4}\left(\sigma^{\mu}\bar{\sigma}^{\nu} - \bar{\sigma}^{\mu}\sigma^{\nu}\right),$$

while $\bar{\sigma}^{\mu\nu}$ has the bars exchanged. The identities mentioned can be found in [11, 39, 41].

so take for example the action of Q_1 on the one-particle state,

$$Q_1 \left|p, \sigma\right\rangle .$$

It is sufficient* to study the action on standard kets, which are defined as $p_t = (M, 0, 0, 0)$ and as $p_t = (E, 0, 0, E)$ for massive and massless particles, respectively. Consider first the massless case, where the spin is actually the helicity h. Examining the commutation relation (2.7c) with $\mu = 1$ and $\nu = 2$, we obtain the effects of the action of Q_1,

$$[J^{12}, Q_1] = i\left(\sigma^{12}\right)_1^{\ 1} Q_1 + i\left(\sigma^{12}\right)_1^{\ 2} Q_2 \Rightarrow [J^3, Q_1] = \frac{1}{2} Q_1. \tag{2.23}$$

In other words, Q_1 raises the helicity (or the spin for massive case) by half unit. If $|p_l, h\rangle \in \mathcal{F}_B$, i.e. the state-ket of a boson, then

$$Q_1 : \mathcal{F}_B \to \mathcal{F}_F,$$

because the associated helicity is now a half-integer. In the same manner, if $|p_l, h\rangle \in \mathcal{F}_F$, then

$$Q_1 : \mathcal{F}_F \to \mathcal{F}_B.$$

With similar analysis one concludes that Q_1 and $Q^\dagger_{\dot{2}}$ raise and Q_2 and $Q^\dagger_{\dot{1}}$ lower by half unit the helicity. It is worth remarking that this raise and lower of spin, unlike the ones of the raising and lowering spin operators $J^{\pm}$, change the representation from a bosonic (femionic) to a fermionic (bosonic) one.

One cannot raise the helicity indefinitely, as every grassmannian operator is nilpotent, i.e. $(Q^\dagger_{\dot{\alpha}})^2 = Q^2_{\alpha} = 0$, so there is only a finite number of combinations of non-zero operators to apply on $|p_l, h\rangle$. This means that only a finite number of particles (Fock spaces) are connected by SUSY transformations. How many particles depends on the value of $\mathcal{N}$, because it determines the total number of fermionic operators available, as we will now see. Consider the superalgebra in the frame with $P^\mu = p^\mu_l$. Eq. (2.7a) can be written as

$$[Q_\alpha, Q^\dagger_{\dot{\alpha}}]_+ = 2E\left(\sigma^0 + \sigma^3\right)_{\alpha\dot{\alpha}} = 4E\delta_{1\alpha}\delta_{\dot{1}\dot{\alpha}}, \tag{2.24}$$

*This is analogous to the case of the construction of $\mathcal{H}$ itself, where little group representations of the standard vectors are used [37].

or simply

$$[Q_1, Q_1^\dagger]_+ = 4E, \tag{2.25a}$$

$$[Q_2, Q_2^\dagger]_+ = 0, \tag{2.25b}$$

$$[Q_1, Q_2^\dagger]_+ = [Q_2, Q_1^\dagger]_+ = 0. \tag{2.25c}$$

The algebra in this form can be interpreted more easily. The second relation implies that Q_2 and $Q_2^\dagger$ produce zero-norm states, hence do not have any physical action on the states. The first relation straightforwardly implies that Q_1 and $Q_1^\dagger$ behave respectively like annihilation and creation operators. With this interpretation one can build the SUSY representation in the same way the Fock space is built, through the action of creation and annihilation operators on the vacuum $|0\rangle$. One subtlety of this approach regards the vacuum. The vacuum state, here named $|\Omega\rangle$, must be such that it is annihilated by the annihilation operator,

$$Q_1 |\Omega\rangle = 0. \tag{2.26}$$

Note that due to the fact that the theory is SUSY invariant, the state $|0\rangle$ is invariant under SUSY transformations, therefore

$$Q_1^\dagger |0\rangle = 0, \quad Q_1 |0\rangle = 0, \tag{2.27}$$

and for this reason $|0\rangle$ cannot be used to define the vacuum state of the algebra. An appropriate vacuum state that satisfy the equation above can be defined from the fact that Q_1 is nilpotent,

$$|\Omega\rangle \equiv \frac{1}{\sqrt{4E}} Q_1 |p_l, h\rangle . \tag{2.28}$$

In what follows $|\Omega\rangle$ will be used as the vacuum of the algebra.

The analysis proceeds as follows. First, note that this vacuum $|\Omega\rangle$ is not the same as $|0\rangle$, because it is actually a one-particle-state with an associated helicity σ. Second, it follows from CPT invariance that for each state with an helicity h, there must be a state with opposite helicity $-h$ in the particle spectrum. The action of $Q_1^\dagger$ on the vacuum $|\Omega\rangle$ produces a state with helicity $\sigma - \frac{1}{2}$, according to the discussion above. Hence, we must also include a state with helicity $-\sigma$ and another one with helicity $-\sigma + \frac{1}{2}$. One remark regarding the values of sigma is in order. On the one hand there is in

Particle		SUSY Partner	
Helicity	Name	Helicity	Name
0	Scalar	1/2	Scalarino
1/2	Fermion	0	Sfermion
1	Gauge Boson	1/2	Gaugino

Table 2: Supermultiplets for $\mathcal{N} = 1$.

principle no restriction to the values sigma may assume. On the other hand, from the phenomenological point of view, it is not interesting to consider values higher than 1, since elementary particles with spin higher than 1 have not yet been observed. Taking this into consideration we restrict the value of $|\sigma|$ to be only 0, 1/2 or 1.

Returning to the main discussion, there is in total 4 degrees of freedom (dof) which can be brought together into two massless particles. Then, for every massless particle with helicities σ and $-\sigma$ in the spectrum, SUSY invariance requires the inclusion of another massless particle with helicities $-\sigma + 1/2$ and $\sigma - 1/2$. This latter is named the superpartner of the original particle. Moreover, the multiplet composed of a particle and its superpartner is denoted supermultiplet. The supermultiplets for $\mathcal{N} = 1$ are given in Table 2.

In an extended SUSY theory for massless particles the situation is similar. The superalgebra in this case is given by

$$[Q_1^I, (Q_1^\dagger)^J]_+ = 4E\delta^{IJ}, \tag{2.29}$$

$$[Q_2^I, (Q_2^\dagger)^J]_+ = 0, \tag{2.30}$$

and the vacuum with helicity σ is written as

$$|\Omega\rangle_{\mathcal{N}} = \frac{1}{(4E)^{\mathcal{N}/2}} Q_1^1 \cdots Q_1^{\mathcal{N}} |p_l, h\rangle \,, \tag{2.31}$$

such that it is annihilated by all Q_1^I. There are now $\mathcal{N}$ distinct creation operators available. For instance, taking $n < \mathcal{N}$ of these operators and applying to the vacuum in Eq. (2.31), one obtains

$$\frac{1}{(4E)^{n/2}} (Q_1^\dagger)^{I_1} \cdots (Q_1^\dagger)^{I_n} |\Omega\rangle_{\mathcal{N}}, \tag{2.32}$$

that has helicity $\sigma - n/2$ and, due to the fact that the operators can be applied in different orders, is $\binom{\mathcal{N}}{n}$-fold degenerate. If only states with at maximum helicity 1 are being considered, then $\mathcal{N} \leq 4$, wich follows from imposing both $|\sigma| \leq 1$ and $|\sigma - \frac{\mathcal{N}}{2}| \leq 1$. It is important to note that, in extended SUSY, the supermultiplets contains more than two particles (while working in 3+1 dimensions [11, 43]).

For the massive case the procedure is almost identical. For $\mathcal{N} = 1$ one only needs to properly organise the dof to re-obtain Table 2, while in the extend case is a bit more subtle because of internal symmetries between the many SUSY generators and will not be discussed in this thesis [39, 11].

The representations presented above are irreducible as $\mathcal{H}$ itself is irreducible. The next step is to compare these representations with those of superfields. Before proceeding, there are a few algebraic properties of SUSY worth remarking. First, for particles that transform under internal symmetry groups, it follows from the discussion of section 2.1 that in a non-extended superalgebra the elements of a given supermultiplet have all the same non-abelian charges, whereas in the extended case the charges of the superpartner are given by combinations of ${(q_n)^I}_J$ (see Eqs. (2.8a) and (2.8b)). An immediate consequence of this in the context of supersymmetric version of the SM is the prediction of coloured scalars, which, till this moment, were not observed and put therefore strong bounds on such models. Second, from Eq. (2.7b) it is easy to see that the SUSY generators commute with P^2, therefore all particles in a supermultiplet have the same mass. On the one hand this signalises how SUSY solves the Hieararchy Problem, because the Higgs has a fermionic SUSY partner with the same mass and thus, due to the fact that fermion masses are technically natural, the Higgs mass is also technically natural. On the other hand this is very problematic, because none of the SUSY partners have been detected at EW scale. This issue can be solved via SUSY breaking, as we will see in section 2.4.2. Third, all the supermultiplets built above have the same number of bosons and fermions, or, more precisely, they have each the same number of dof for each statistics. This is not a coincidence, as it follows directly from the superalgebra [39, 41]. To prove this, consider the operator F that counts the number of fermions in a given supermultiplet. This operator satisfy

$$[Q_\alpha, F]_+ = 0, \quad [Q^\dagger_{\dot{\alpha}}, F]_+ = 0, \tag{2.33}$$

and in particular

$$[Q_\alpha, (-1)^F]_+ = 0, \quad [Q^\dagger_{\dot{\alpha}}, (-1)^F]_+ = 0. \tag{2.34}$$

Consider the following trace in the supermultiplet space:

$$\mathrm{Tr}\left[(-1)^F [Q_\alpha, Q^\dagger_{\dot\alpha}]_+\right]. \tag{2.35}$$

Using Eq. (2.34) and the cyclic property of the trace, we determine that the trace above is vanishing. From Eq. (2.7a) of the superalgebra together with the fact that all states in a supermultiplet have a well defined, non-zero 4-momentum, one obtains

$$\mathrm{Tr}\left[(-1)^F\right] = 0, \tag{2.36}$$

which proves the initial assertion.

2.2.3 Chiral and vector superfields

In section 2.2.1 it was shown that the superfield representation of the SUSY algebra was reducible, which became even more clear from the construction of the irreducible representations of the superalgebra. In order to connect uniquely both representations, the superfields must be somehow constrained. More precisely, some kind of conditions must be imposed on them such that only the physical dof remain on them.

Superfields are in general complex objects, so a condition one may impose on them is

$$V\left(x, \xi, \xi^\dagger\right) = V^\dagger\left(x, \xi, \xi^\dagger\right). \tag{2.37}$$

Using Eq. (2.20) for a scalar superfield, this condition implies that

$$\begin{aligned} V(x, \xi, \xi^\dagger) \;&=\; C(x) + \xi\eta(x) + \xi^\dagger\eta^\dagger(x) + \xi^2 N(x) + (\xi^\dagger)^2 N(x) \\ &\quad + \xi^2\xi^\dagger\lambda^\dagger(x) + (\xi^\dagger)^2\xi\lambda(x) + \xi\sigma^\mu\xi^\dagger v_\mu(x) + \xi^2(\xi^\dagger)^2 D(x), \end{aligned} \tag{2.38}$$

with C, N, v_μ and D real fields. However, Eq. (2.37) does not seem to be sufficient in diminishing the number of fields to the desired one*. Note that a real superfield is always redundant, in the sense that we may perform the transformation

$$V \to V + F + F^\dagger, \tag{2.39}$$

*These extra fields are required when considering massive fields, which is not the case.

which still is a real superfield for any superfield scalar F. The above transformation induces transformations on the component fields, in particular on the vector field v_μ,

$$v_\mu \to v_\mu + a_\mu + a_\mu^\dagger, \tag{2.40}$$

where a_μ is the vector field from F. Choosing

$$a_\mu + a_\mu^\dagger = \partial_\mu \phi, \tag{2.41}$$

with ϕ a real scalar field, one recovers the usual gauge transformation of a massless vector field. Thus, with the condition that Eq. (2.41) is satisfied, Eq. (2.39) is the generalisation of a gauge transformation in the context of supersymmetric theories. We are then motivated in finding a new superfield that satisfies Eq. (2.41).

From the the differential structure of the SUSY generators in $(x, \xi, \xi^\dagger)$ space, it is implicit that in Eq. (2.17) the action of both operators were defined from the left. If instead $U(0, \theta, \theta^\dagger)$ were to act from the right*, another pair of differential operators would emerge

$$D_\alpha = \frac{\partial}{\partial \xi^\alpha} - i\big(\sigma^\mu \xi^\dagger\big)_\alpha \partial_\mu, \tag{2.42a}$$

$$D_{\dot\alpha}^\dagger = -\frac{\partial}{\partial \xi^{\dagger\,\dot\alpha}} + i(\xi\sigma^\mu)_{\dot\alpha}\,\partial_\mu\,. \tag{2.42b}$$

The operators D_α and $D_{\dot\alpha}^\dagger$ differ from Q_α and $Q_{\dot\alpha}^\dagger$ in some sign and, because of that, the D's anticommute with all Q's and they themselves satisfy very similar anticommutation relations:

$$[D_\alpha, D_{\dot\alpha}^\dagger]_+ = -2\sigma^\mu_{\alpha\,\dot\alpha}\,\partial_\mu, \tag{2.43a}$$

$$[D_\alpha, D_\beta]_+ = 0, \quad [D_{\dot\alpha}^\dagger, D_{\dot\beta}^\dagger]_+ = 0. \tag{2.43b}$$

Since the operators in Eq. (2.42) generate SUSY translations if applied from the right, one may use them to obtain a "differential equation" by applying them to a superfield Φ and requiring generically that

$$D_{\dot\alpha}^\dagger \Phi(x, \xi, \xi^\dagger) = 0. \tag{2.44}$$

*This looks quite silly, but in the context of group theory right and left actions are generally not equivalent. To take the same operators in right-action form and then applying it by the left side is a smart way of obtaining new differential operators with some physical meaning.

A superfield that satisfy Eq. (2.44) is called a chiral superfield. Note in addition that $D^{\dagger}_{\dot\alpha}\Phi$ is still a superfield, which means that $D^{\dagger}_{\dot\alpha}$ and D_α behave like covariant derivatives with respect to SUSY transformations. The "differential equation" in Eq. (2.44) is a very special one because it does not imply in any conditions over the x variable, i.e. Eq. (2.44) does not force the fields to satisfy any differential equation, otherwise such restriction would be in conflict with the classical equations of motion.

The fact that Eq. (2.44) does not imply in any additional differential equation over the x variable can be verified explicitly,

$$\begin{aligned} D^{\dagger}_{\dot\alpha}\Phi &= \chi^{\dagger}_{\dot\alpha} + \theta^{\alpha}\sigma^{\mu}_{\alpha\dot\alpha}\Big(i\,\partial_\mu\,\phi + v_\mu\Big) + i\theta^{\dagger}_{\dot\alpha}g + \theta^2\Big(m^{\dagger}_{\dot\alpha} + \frac{i}{2}\,\partial_\mu\,\psi^{\alpha}\sigma^{\mu}_{\alpha\dot\alpha}\Big) + \\ &+\Big(\theta^{\dagger}_{\dot\alpha}\theta n + i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot\alpha}\,\partial_\mu\,\xi^{\dagger}\theta^{\dagger}\Big) + 2\theta^2\theta^{\dagger}_{\dot\alpha}\Big(d + \frac{1}{4}\Box\phi\Big). \end{aligned}$$

For this expression to be zero, all coefficients of the θ-directions must be zero, which is satisfied by:

$$\begin{aligned} &\chi_\alpha(x) = n_\alpha(x) = 0 \\ &g(x) = 0 \\ &v_\mu(x) = -i\,\partial_\mu\,\phi(x) \\ &m^{\dagger}_{\dot\alpha}(x) = \frac{i}{2}\,\partial_\mu\,\psi^{\alpha}(x)\sigma^{\mu}_{\alpha\dot\alpha} \\ &d(x) = -\frac{1}{4}\Box\phi(x). \end{aligned}$$

Hence the most general chiral superfield is given by

$$\Phi = \phi(x) + \sqrt{2}\xi\psi(x) + \xi^2 f(x) - i\xi\sigma^{\mu}\xi^{\dagger}\,\partial_\mu\,\phi(x) + \frac{i}{\sqrt{2}}\xi^2\,\partial_\mu\,\psi\sigma^{\mu}\xi^{\dagger} - \frac{1}{4}\xi^2(\xi^{\dagger})^2\Box\phi(x), \quad (2.45)$$

where we have re-scaled ψ. For the case of the chiral superfield, it is convenient to introduce a even more compact notation using the *supercoordinates*,

$$y^{\mu} = x^{\mu} - i\xi\sigma^{\mu}\xi^{\dagger}, \quad (2.46)$$

for which the chiral superfield is simply*

$$\Phi(y) = \phi(y) + \sqrt{2}\xi\psi(y) + \xi^2 f(y). \tag{2.47}$$

The previous form of Φ in Eq. (2.45) can be obtained by Taylor expanding in the grassmann variables.

At first sight even the chiral superfield is no good for our purpose, for it has two complex scalars and a Weyl spinor; one scalar more than needed to compose a irreducible supermultiplet. In the next section it will become clear what role does this extra scalar plays and why the superfield in the equation above indeed represents an irreducible supermultiplet. The chiral superfield, defined by Eq. (2.44) and given explicitly in Eq. (2.47), is the first superfield to be introduced and can accommodate the supermultiplets of a scalar field (then ψ is the superpartner) or of a fermion field (with ϕ the superpartner). With the explicit expression of the chiral superfield, it is possible to obtain the infinitesimal SUSY transformations of the ordinary fields. From Eq. (2.17), these are given by

$$(\theta Q + \theta^\dagger Q^\dagger)\Phi = \delta\Phi(x,\xi,\xi^\dagger) = \left[\theta\frac{\partial}{\partial\xi} + \theta^\dagger\frac{\partial}{\partial\xi^\dagger} + i\left(\theta\sigma^\mu\xi^\dagger - \xi\sigma^\mu\theta^\dagger\right)\partial_\mu\right]\Phi(x,\xi,\xi^\dagger).$$

Performing the algebra and matching the coefficients one obtains the infinitesimal SUSY transformations,

$$\delta\phi = \sqrt{2}\theta\psi, \tag{2.48a}$$

$$\delta\psi = i\sqrt{2}\sigma^\mu\theta^\dagger\,\partial_\mu\,\phi + \sqrt{2}\theta f, \tag{2.48b}$$

$$\delta f = i\sqrt{2}\theta^\dagger\bar{\sigma}^\mu\,\partial_\mu\,\psi. \tag{2.48c}$$

Note that the transformation of the field f is nothing but a total derivative.

Having now defined a chiral supermultiplet, let us return to the previous point and

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*The supercoordinates are incredibly practical for chiral superfields because in such coordinates

$$D^\dagger_{\dot\alpha} = -\frac{\partial}{\partial\,\xi^{\dagger\,\dot\alpha}}, \text{ and } D_\alpha = \frac{\partial}{\partial\,\xi^\alpha} - 2i(\sigma^\mu\xi^\dagger)_\alpha\frac{\partial}{\partial\,y^\mu},$$

and therefore the condition (2.44) for a superfield becomes just

$$\frac{\partial}{\partial\xi^\dagger_{\dot\alpha}}\Phi(y) = 0.$$

compute the transformation of the real superfield

$$V \rightarrow V + \Phi + \Phi^\dagger. \tag{2.49}$$

Explicitly, the changes in the fields are:

$$\begin{aligned}
&C \rightarrow C + \phi + \phi^\dagger \\
&\eta \rightarrow \eta + \sqrt{2}\psi \\
&N \rightarrow N + F \\
&v_\mu \rightarrow v_\mu - i\,\partial_\mu \left(\phi - \phi^\dagger\right) \\
&\lambda \rightarrow \lambda - \frac{i}{\sqrt{2}}\bar{\sigma}^\mu\,\partial_\mu\,\psi^\dagger \\
&D \rightarrow D - \frac{1}{4}\Box\left(\phi + \phi^\dagger\right).
\end{aligned}$$

As expected, transformation (2.49) indeed induces the usual gauge transformation, and Eq. (2.49) can be therefore understood as a generalisation of gauge transformation in a SUSY theory. For this very reason the superfield V is called *vector superfield*. The transformations above are not all independent, since they can be reorganised in a way to extract gauge invariant quantities. Let

$$D \Rightarrow D + \frac{1}{4}\Box C, \tag{2.50}$$

$$\lambda \Rightarrow \lambda + \frac{i}{2}\bar{\sigma}^\mu\,\partial_\mu\,\eta^\dagger. \tag{2.51}$$

Redefining D and λ in this way make both gauge invariant under transformation (2.49), allowing us to set C, N and η to zero by performing a particular choice of gauge, known as Wess-Zumino gauge. In this gauge it is clear that the physical fields are v_μ, λ and, a priori, D. V represents the supermultiplet of a vector field, although there is the extra (real) scalar D, which plays the same role of f in the chiral superfield.

In the same manner we can compute the SUSY transformation of the vector field. The infinitesimal transformation is given by

$$\delta V(x,\xi,\xi^\dagger) = \left[\theta\frac{\partial}{\partial\,\xi} + \theta^\dagger\frac{\partial}{\partial\,\xi^\dagger} + i\left(\theta\sigma^\mu\xi^\dagger - \xi\sigma^\mu\theta^\dagger\right)\partial_\mu\right]V(x,\xi,\xi^\dagger), \tag{2.52}$$

which is equivalent to the component fields transformations

$$\delta\lambda_\alpha = i\theta_\alpha D + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}, \tag{2.53a}$$

$$\delta v_\mu = i\Big(\theta\sigma_\mu\lambda^\dagger + \theta^\dagger\bar{\sigma}_\mu\lambda\Big), \tag{2.53b}$$

$$\delta D = \xi^\dagger\bar{\sigma}^\mu\,\partial_\mu\,\lambda - \xi\sigma^\mu\,\partial_\mu\,\lambda^\dagger. \tag{2.53c}$$

Note, again, that the transformation of D is also a total derivative.

2.3 SUSY Lagrangians

A SUSY Lagrangian is nothing more than a Lagrangian which is also invariant under SUSY transformations (up to total derivatives). To build a SUSY Lagrangian, it is enough to gather the fields from a given irreducible supermultiplet and write all possible renormalizable interactions between them, which all together is SUSY invariant. Although straightforward, this procedure is not so simple, because SUSY transformation, albeit not very complicated, involve fields and grassmannian parameters, rendering the algebraic work quite intricate. The superfield formulation built so far is very useful in this regard: Lagrangians built from superfields are manifestly SUSY invariant. In this section it will be shown how to construct SUSY Lagrangians from superfields, while leaving explicit their relation with the irreducible representations, and why they are manifestly SUSY invariant. In particular, the Lagrangians for chiral and vector superfields will be discussed in detail.

2.3.1 Wess-Zumino model

As discussed in section 2.2, the irreducible of a massless chiral supermultiplet consists of a massless complex scalar and a massless spin 1/2 Weyl fermion. The free Lagrangian for such fields is given by

$$\mathscr{L} = \partial_\mu\,\phi^\dagger\,\partial^\mu\,\phi + i\psi^\dagger\bar{\sigma}^\mu\,\partial_\mu\,\psi. \tag{2.54}$$

The objective is to write Eq. (2.54) in terms of superfields. The first step is to note that a kinetic term is written with both the field and its complex conjugate. So the most straightforward term that contains information of both is given by

$$\Phi^\dagger \Phi.$$

This combination is not a chiral superfield, although it is trivially a vector superfield. Nonetheless, it is indeed the desired kinetic term. Writing it explicitly in terms of component fields, one finds

$$\Phi^\dagger \Phi = \cdots + (\xi^\dagger)^2 \xi^2 \Big(\partial_\mu \phi^\dagger \, \partial^\mu \, \phi + i\psi^\dagger \bar{\sigma}^\mu \, \partial_\mu \, \psi + f^\dagger f \Big),$$

where the dots denote terms with other powers of ξ and $\xi^\dagger$. It is thus clear that the term in the equation above is, except for the $f^\dagger f$ term, identical to the Lagrangian in Eq. (2.54). In spite of the presence of this extra term $f^\dagger f$, $\Phi^\dagger \Phi$ is indeed the desired kinetic term for the free Lagrangian of the chiral supermultiplet, and the reason is the following. It must be noted that the irreducible representations are built based on the irreducible representations of the Poincaré group, which is itself build from on-shell states $|p, \sigma\rangle$. Contrary to the representation as Fock spaces, fields in Lagrangians can be off-shell and for this reason it is necessary to have the auxiliary field f in the Lagrangian. More precisely, as it was seen in section 2.2 , a supermultiplet has the same number of dof of fermions and bosons, which is a result that does not depend on whether the particles are on-shell or not, therefore one should have equal dof in both an off- and on-shell Lagrangian. The Lagrangian in Eq. (2.54) clearly has matching dof if the fields are on-shell (2 for a complex scalar and 2 for a Weyl fermion), but not if they are off shell (2 for a complex scalar and 4 for a Weyl fermion). The complex scalar field f (with 2 dof) compensates the two missing bosonic dof off-shell and is relevant only if off-shell, because its EoM are

$$f = f^\dagger = 0. \tag{2.55}$$

The natural introduction of the field f (and of the field D in the case of the vector superfield) is one of the main advantages from a bottom-up approach based on superfields rather than the irreducible representations. In the latter, one would need to introduce the $f^\dagger f$ term by hand and also guess its SUSY transformation [11].

In order to obtain the correct Lagrangian one needs to specifically select the $(\xi^\dagger)^2 \xi^2$

term from $\Phi^\dagger\Phi$. This is easily done using grassmannian calculus [44]. Defining

$$\begin{aligned} d^2\xi &\equiv -\frac{1}{4} d\xi^\alpha d\xi^\beta \epsilon_{\alpha\beta}, \\ d^2\xi^\dagger &\equiv -\frac{1}{4} d\xi^\dagger_{\dot\alpha} d\xi^\dagger_{\dot\beta} \epsilon^{\dot\alpha\dot\beta}, \\ d^4\xi &\equiv d^2\xi d^2\xi^\dagger, \end{aligned}$$

one has

$$\mathscr{L} = \int d^4\xi \; \Phi^\dagger\Phi. \tag{2.56}$$

The Lagrangian above is manifestly SUSY invariant, which can be seen by determining how the Lagrangian transforms under an infinitesimal SUSY transformation. The integration measure in superspace is invariant,

$$d^4\xi \to d^4\xi, \tag{2.57}$$

since SUSY acts as a translation in the superspace (see Eq. (2.18)). The product of chiral superfields $\Phi^\dagger\Phi$ is, as stressed, a vector superfield and its $(\xi^\dagger)^2\xi^2$ component transforms therefore as the $(\xi^\dagger)^2\xi^2$ component of a vector superfield. This transformation is given in Eq. (2.53) and is nothing but a total derivative. Hence, apart from surface terms, the Lagrangian in Eq. (2.56) is SUSY invariant.

The Lagrangian in Eq. (2.56) would be that of a free chiral supermultiplet, i.e. without interactions. In order to build interactions between the fields of the same supermultiplet, one may use the superfields to write down interacting operators. Take for instance

$$\frac{1}{2} M\Phi\Phi. \tag{2.58}$$

This particular quantity is again a chiral superfield (in general, every power of a chiral superfield is a chiral superfield). As a chiral superfield, the ξ^2 component of the operator above transforms like f in Eq. (2.48) and it is thus invariant under infinitesimal SUSY transformation, except for a total derivative. In addition to the operator in Eq. (2.58), one can also consider its complex conjugate, which needs to be integrate over $(\xi^\dagger)^2$ instead of ξ^2. The interacting Lagrangian is therefore given by

$$\mathscr{L}_2 = \frac{1}{2}M \int d^2\xi \ \Phi\Phi + \frac{1}{2}(M)^\dagger \int d^2\xi^\dagger \ \Phi^\dagger\Phi^\dagger$$

which is manifestly SUSY invariant.

Other interactions can be built analogously with one or three fields,

$$\mathscr{L}_{1,3} = \lambda \int d^2\xi \ \Phi + \frac{1}{6}\mathcal{Y} \int d^2\xi \ \Phi^3 + h.c. \tag{2.59}$$

One could in principle add any power of Φ integrated over ξ^2, but not all of these operators are renormalizable. The Lagrangian (2.56) fixes the dimension of f as +2 and Eq. (2.59) introduces a $\phi\phi f$ operator, which is marginal. Therefore, the most general renormalizable Lagrangian of a interacting chiral superfield, known as *interacting Wess-Zumino model*, is

$$\mathscr{L}_{WZ} = \int d^4\xi \ \Phi^\dagger\Phi + \int \Big(d^2\xi \ \lambda\Phi + \frac{1}{2}M\Phi^2 + \frac{1}{6}\mathcal{Y}\Phi^3 + h.c\Big). \tag{2.60}$$

This Lagrangian describes the usual dynamics of a scalar and a fermion plus their SUSY interactions*.

Although very elegant, the Wess-Zumino Lagrangian (2.60) does not leave the interactions between the many particles and SUSY particles explicit. Take, for example, the generalisation of Eq. (2.60) for N chiral superfields†,

$$\mathscr{L} = \int d^4\xi \ \Phi_i^\dagger\Phi_i + \int \Big(d^2\xi \ W(\Phi) + h.c.\Big), \tag{2.61}$$

where W is the *superpotential* defined by

$$W(\Phi) = \frac{1}{2}M_{ij}\Phi_i\Phi_j + \frac{1}{3!}\mathcal{Y}_{ijk}\Phi_i\Phi_j\Phi_k, \tag{2.62}$$

where we take $\lambda_i = 0$ for simplicity. Here the indices i, j and k run from 1 to N and are

*More precisely, one cannot have non-vanishing linear, quadratic and triple operators simultaneously for a single chiral superfield due to $U(1)_R$ invariance [41]. This is not worrisome here, because the purpose of Eq. (2.60) is to motivate the generalisation with more chiral fields below, for which this restriction does not hold.

†Note that the kinetic term is taken to be $\delta_{ij}\Phi_i^\dagger\Phi_j$, which is the simplest choice. Other possibilities for the mixing matrix of the kinetic term are possible, but are out of the scope of this thesis [11, 12]

implicitly summed. The terms quadratic in ξ are

$$\int d^2\xi\ W(\Phi) = M_{ij}\Big(\phi_i f_j - \frac{1}{2}\psi_i\psi_j\Big) + \frac{1}{2}\mathcal{Y}_{ijk}(\phi_i\phi_j f_k - \phi_i\psi_j\psi_k), \tag{2.63}$$

where we have used that the matrices M and $\mathcal{Y}$ are symmetric. Note that the Lagrangian is quadratic on all f_i, so we may integrate them out. The EoM are

$$f_i^\dagger = -M_{ij}\phi_j - \frac{1}{2}\mathcal{Y}_{ijk}\phi_j\phi_k, \tag{2.64a}$$

$$f_i = -M_{ij}^*\phi_j^* - \frac{1}{2}\mathcal{Y}_{ijk}^*\phi_j^*\phi_k^*. \tag{2.64b}$$

Substituting the above equations we obtain the potential V that describes the scalars' interactions, also named f-term,

$$\begin{aligned} V(\phi,\phi^\dagger) &= f_i^\dagger f_i \\ &= \Big|M_{ij}\phi_j + \frac{1}{2}\mathcal{Y}_{ijk}\phi_j\phi_k\Big|^2 \\ &= M_{ij}M_{ik}^*\phi_j\phi_k^* + \frac{1}{2}M_{ij}\mathcal{Y}_{imn}^*\phi_j\phi_m^*\phi_n^* + \\ &\quad + \frac{1}{2}M_{ij}^*\mathcal{Y}_{imn}\phi_j^*\phi_m\phi_n + \frac{1}{4}\mathcal{Y}_{ijk}\mathcal{Y}_{inm}^*\phi_i\phi_j\phi_m^*\phi_n^*, \end{aligned} \tag{2.65}$$

while the interactions with fermions are

$$\mathscr{L}_{\text{int}} = -\frac{1}{2}(M_{ij}\psi_i\psi_j + \mathcal{Y}_{ijk}\phi_i\psi_j\psi_k + h.c.). \tag{2.66}$$

From Eqs. (2.65) and (2.66) the physical meaning of the operators in Eq. (2.62) is obvious: M is the mass parameters of the particles and $\mathcal{Y}$ is the Yukawa coupling. Note that, as a direct consequence of SUSY invariance, the scalar triple and quartic couplings are not independent since they are both given in terms of M and $\mathcal{Y}$.

A remark regarding the interactions is in order. Note that the superpotential $W(\Phi)$ is a function only of Φ, i.e. it does not depend on both Φ and $\Phi^\dagger$, and it is actually a requirement from SUSY. This follows directly by the fact that $\delta\int \mathrm{d}^2\xi\ W$ must be a total derivative, in other words, that the ξ^2 component of W transforms as a field f under SUSY (see Eq. (2.48)). This is only possible if W is a chiral field and, as seen above, products of Φ and $\Phi^\dagger$ are not chiral. From a mathematical point of view this means that SUSY invariance is requiring the superpotetial to be holomorphic in the complex superfield Φ [11, 41, 12].

2.3.2 Gauge theories

The SUSY Lagrangians for vectors superfields will naturally be related to gauge theories, as the first describe massless spin 1 particles. In particular, the vector superfield will carry the same representation under the gauge group of its associated vector field. For $SU(N)$ non-abelian gague theories this implies that the vector superfield carries a index a from the adjoint representation.

Let us focus first on how to write down the kinetic term of a vector supermultiplet. In contrast to the case of the chiral field it is not trivial to build a kinetic term for the vector superfield. The first and foremost difficulty has to do with the strength field tensor $F^a_{\mu\nu}$ itself. The free Lagrangian of a spin 1 particle is as usual

$$-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}, \tag{2.67}$$

so, if we intent to build a Lagrangian which is the square of some superfield, the superfield in question should contain some term proportional to $F^a_{\mu\nu}$, such that its square would be given by

$$i\lambda^{a\dagger}\bar{\sigma}^\mu D_\mu\lambda^a - \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu},$$

where λ^a_α are the respective gauginos. But, as previously noted, a tensor structure like $F^a_{\mu\nu}$ does not enter in a scalar superfield F, whence we cannot construct the appropriate kinetic term from a scalar superfield.

A way to include the $F^a_{\mu\nu}$ in a superfield is to give up on the Lorentz scalar hypothesis, i.e. to allow $F(x,\xi,\xi^\dagger)$ to transform non-trivially under Lorentz transformations. The simplest alternative is to consider a superfield $W_\alpha(x,\xi,\xi^\dagger)$ transforming under (1/2,0), whose first component is a fermion,

$$W_\alpha(x,\xi,\xi^\dagger) = -ig_\alpha(x) + \cdots .$$

To make contact with the vector superfield, we choose g as the gauginos λ^a. To construct the entire W^a_α, we use the most basic property of superfields: the transformation under

SUSY. Note that

$$W^a_\alpha(x,0,0) = -i\lambda^a_\alpha(x),$$

hence, according to Eqs. (2.17) and (2.18),

$$W^a_\alpha(x,0,0) \to U(0,\xi,\xi^\dagger)W^a_\alpha(x,0,0) = W^a_\alpha(x,\xi,\xi^\dagger) \tag{2.68}$$

one obtains the full superfield. Relation (2.68) is in fact a general property of superfields: taking the lowest dimensional component and applying U will result in the full superfield [45]. Notwithstanding, such calculation is a long one, as it starts with Taylor expanding U,

$$\begin{aligned} U(0,\xi,\xi^\dagger)\lambda^a_\alpha &= e^{i\left(\xi Q + \xi^\dagger Q^\dagger\right)}\lambda^a_\alpha \\ &= \lambda^a_\alpha + i\delta\lambda^a_\alpha - \frac{1}{2}\delta^2\lambda^a_\alpha - \frac{i}{3!}\delta^3\lambda^a_\alpha + \frac{1}{4!}\delta^4\lambda^a_\alpha, \end{aligned} \tag{2.69}$$

where δ is the infinitesimal SUSY transformation obtained applying $\xi Q + \xi^\dagger Q^\dagger$ to the superfields. The ξ component of W_α is obviously given by the first term of this expansion, $\delta\lambda^a_\alpha$, which is given by Eq. (2.53)*. Note that this transformation contains the respective field strength tensor $F^a_{\mu\nu}$. This allows us to guess the correct Lagrangian. Since the intentions is to build the kinetic term out of the square of W^a_α, the Yang-Mills Lagrangian shall be given by

$$\mathscr{L}_{YM} = \int \left(d^2\xi \;\; \mathrm{Tr}\, W^2 + h.c.\right) \tag{2.70}$$

where

$$W^2 = W^\alpha W_\alpha = \left(T^b W^{\alpha b}\right)(T^a W^a_\alpha). \tag{2.71}$$

Under SUSY transformation this term transforms as a total derivative because W_α is a chiral superfield (a trivial result from the explicit expression given below). To check if the above reasoning is correct, one may compute the full expression for W^a_α. For such, the transformations of all fields in V^a are needed, which are given in Eq. (2.53). The W^a_α

*Transformations in Eq. (2.53) are for a vector superfield outside the context of gauge theories. To obtain the transformations of component fields in non-abelian gauge theories one just needs to substitute ordinary derivatives by covariant ones and the abelian strength field by the non-abelian one.

superfield is thus given by

$$W_\alpha^a(y) = -i\lambda_\alpha^a(y) + \Big(\delta_\alpha{}^\beta D^a(y) - i(\sigma^{\mu\nu})_\alpha{}^\beta F_{\mu\nu}^a(y)\Big)\theta_\beta + \theta^2\Big(\sigma^\mu D_\mu \lambda^{a\dagger}(y)\Big)_\alpha. \tag{2.72}$$

In component fields Eq. (2.70) can be expanded as

$$\mathscr{L}_{YM} = i(\lambda^a)^\dagger \bar{\sigma}^\mu D_\mu \lambda^a - \frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2}D^a D^a, \tag{2.73}$$

which is indeed the desired and expected result. A remark regarding the field D^a is in order. As stressed, the field D^a plays the same role as f in the chiral superfield, since it contains the missing bosonic dof off-shell. In this case D^a is real, since v_μ^a has 3 dof off-shell.

Having built the SUSY Lagrangians for the chiral and vector superfields, it is already possible to construct Lagrangians that contain interactions between both. For this to be achievable, the relevant chiral superfields must be charged under the non-abelian symmetry in question, in other words, they must carry some non-trivial representation. Since the final goal is to study SUSY versions of the SM, only chiral fields transforming under the fundamental of the gauge group are considered,

$$\Phi = (\Phi_i), \quad i = 1, \cdots, N. \tag{2.74}$$

From the standard knowledge of Quantum Field Theory and our previous results one expects that the Lagrangian that describes the interactions between matter fields and the gauge fields will be given by

$$\begin{aligned} \mathscr{L}_{\text{WZ+YM}} \supset\ & (D_\mu\phi)_i^\dagger (D^\mu\phi)_i + i\psi_i^\dagger(\bar{\sigma}^\mu D_\mu\psi)_i + f_i^* f_i - \\ & -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + i\lambda^{a\dagger}\bar{\sigma}^\mu D_\mu\lambda^a + \frac{1}{2}D^a D^a + \cdots, \end{aligned} \tag{2.75}$$

where the missing terms are the ones taking SUSY interactions into account. In other words, the objective is to determine the theory that couples chiral and vector superfields in a way that is consistent with SUSY invariance.

To this end, let us understand how do superfields behave under non-abelian gauge transformations. An abelian gauge transformation is, according to Eq. (2.49),

$$V \to V' = V + \Lambda + \Lambda^\dagger,$$

with Λ a chiral superfield. In the non-abelian case there is no reason to believe that the non-abelian gauge transformation will retain the same form from Eq. (2.49) due to the non-commutativity. For this reason it is more logical to start with the matter fields, for which the gauge transformations are much more straightforward. The transformation of the chiral superfield Φ is parametrized by Λ^a through exponentiation,

$$\Phi \to \exp\{\Lambda^a T^a\}\Phi, \tag{2.76}$$

which must again be a chiral superfield. Such transformation is almost identical to the non-SUSY case, except for the fact that Λ^a is a chiral superfield. If it wasn't, then an inconsistency would emerge, since a chiral superfield would be transformed into something which is not in general a chiral superfield*. Accepting this transformation as the gauge transformation of a chiral superfield, we note that Eq. (2.76) is not unitary, therefore the Lagrangian (2.56) is not gauge-invariant. In the very same way the covariant derivative is constructed [1], a *compensator* is introduced between $\Phi^\dagger$ and Φ to make Lagrangian (2.56) invariant. This compensator C must depend on V^a so to introduce the field v_μ^a on the Lagrangian. The equation $C(V)$ must satisfy is

$$e^{T^a \Lambda^{a\dagger}} C(V') e^{T^a \Lambda^a} = C(V). \tag{2.77}$$

For the abelian case the equation above is trivially satisfied by†

$$C_{\text{abelian}}(V) = e^{-V}, \tag{2.78}$$

since there are no matrices involved. The general case will maintain the same structure as long as we introduce higher order correction in Eq. (2.49), which, by means of the

*$\Phi(y) \to f(y)\Phi(y)$ is not necessarily chiral for any function f

†Here the coupling constant g is taken to be 1. To re-introduce it, it is enough to make $V^a \to 2gV^a$.

Baker-Campbell-Haussdorff formula, will satisfy Eq. (2.77). The final answer is [41]:

$$C(V) = e^{-V^aT^a}, \tag{2.79}$$

$$V^a \to V^a + \Lambda^a + \Lambda^{a\dagger} + \mathcal{O}(\Lambda V). \tag{2.80}$$

In short, the Wess-Zumino Lagrangian coupled to a non-abelian gauge theory is given by (reintroducing the coupling constant)

$$\mathscr{L} = \mathscr{L}_{YM} + \int d^4\xi\ \Phi^\dagger e^{-2gV^aT^a}\Phi, \tag{2.81}$$

which is exactly the expected answer (2.75) plus new SUSY interactions. The latter can be obtained by expanding* Eq. (2.81):

$$-g\sqrt{2}\Big(\phi_i^\dagger T_{ij}^a \psi_j \lambda^a + \lambda^{a\dagger}\psi_i^\dagger T_{ij}^a \phi_j\Big) + g\phi_i^\dagger T_{ij}^a \phi_j D^a. \tag{2.82}$$

The last term of Eq. (2.82), named D-term, contributes to the scalar potential in Eq. (2.65),

$$V(\phi^\dagger, \phi) = f_i^\dagger f_i + \frac{1}{2}D^aD^a, \tag{2.83}$$

which, after solving the EoM of D^a, becomes

$$V(\phi^\dagger, \phi) = f_i^\dagger f_i + \frac{1}{2}g^2\Big(\phi_i^\dagger T_{ij}^a \phi_j\Big)^2. \tag{2.84}$$

2.4 Minimal Supersymmetric Standard Model

Having understood how to construct Lagrangians for chiral and vector superfields and how to couple them in non-abelian gauge theories, we are in the position to build supersymmetric versions of the SM. In particular, the focus of this section is to build the minimal version of the SM, i.e. the most simple and self consistent theory while taking SUSY into account.

*Using Wess-Zumino gauge, with which $V^aV^b = \frac{1}{2}\xi^2\big(\xi^\dagger\big)^2 A_\mu^a A_\mu^b$ and $V^aV^bV^c = 0$, such that the expansion of the exponential in Eq. (2.81) is given by just three terms.

2.4.1 Particle content

The simplest way to supersymmetrise a theory is to introduce $\mathcal{N} = 1$ SUSY, in other words, to build a theory with all tools developed so far. As a consequence, the SM particle content will be doubled; for every fermion (boson) a boson (fermion) is added to the particle spectrum of the theory.

In the process of building a theory, one must be careful before introducing so many particles to a gauge theory, with one of the reasons being gauge anomalies. Not only gauge anomalies coming from triangle diagrams but also topological anomalies like the Witten anomaly, since the SM gauge group contains an $SU(2)_L$ group [46]. The latter is avoided in the SM for it has only an even number of Weyl fermions: 3 left-handed leptons plus 3 left-handed quarks. The SUSY theory we intent to build has a total of 3 left-handed leptons, 3 left-handed quarks and a higgsino. The latter is the SUSY partner of the Higgs boson, which is a Weyl fermion and a doublet of $SU(2)_L$, because the Higgs is a scalar and itself a doublet of $SU(2)_L$. Therefore, we are left with an odd number of fermions* and the Witten anomaly leaves the theory inconsistent. To cancel such anomaly the addition of a new SUSY Weyl fermion is necessary. This new field is introduced as a new Higgs field, i.e. one needs two Higgs doublets to make the supersymmetric theory consistent†. These two Higgs are called H_u and H_d and their transformations under the symmetry group of the SM are defined as

$$H_u \sim \left(1, 2, \frac{1}{2}\right) \tag{2.85}$$

$$H_d \sim (1, 2, y), \tag{2.86}$$

with y a real number. The hypercharge of the H_d doublet must be such to cancel the remaining gauge anomalies: $U(1)_Y^3$, $SU(2)_L^2 U(1)_Y$ and $U(1)_Y \text{grav}^2$. For example, the anomaly coefficient of the latter is given by[20, 21, 47]

$$U(1)_Y \text{grav}^2 \longrightarrow 3(2Y_{\ell_L} - Y_{e_R} - Y_{\nu_R}) + 3(6Y_{q_L} - 3Y_{u_R} - 3Y_{d_R}) + Y_{\tilde{H}_u} + Y_{\tilde{H}_d} = 0, \tag{2.87}$$

*The wino triplet does not contribute to this counting because it transforms under the adjoint representation, which is real. Only fermions in the chiral representation (1/2,0) contribute to the anomaly [46].

†Another argument of why this new field is Higgs-like will be given below.

Superfield	Boson	Fermion	$SU(3)_c\ \times$	$SU(2)_L$	$\times\ U(1)_Y$
L	$\tilde{\ell}_L$	ℓ_L	1	2	$-\frac{1}{2}$
$\bar{E}$	$\tilde{e}^*_R$	$\bar{e}_R$	1	1	$+1$
Q	$\tilde{q}_L$	q_L	3	2	$+\frac{1}{6}$
$\bar{U}$	$\tilde{u}^*_R$	$\bar{u}_R$	$\bar{3}$	1	$-\frac{2}{3}$
$\bar{D}$	$\tilde{d}^*_R$	$\bar{d}_R$	$\bar{3}$	1	$+\frac{1}{3}$
$\mathcal{H}_u$	H_u	$\tilde{H}_u$	1	2	$+\frac{1}{2}$
$\mathcal{H}_d$	H_d	$\tilde{H}_d$	1	2	$-\frac{1}{2}$
$\mathcal{G}^a$	G^a_μ	$\tilde{G}^a$	Adj	1	0
$\mathcal{W}^a$	W^a_μ	$\tilde{W}^a$	1	Adj	0
$\mathcal{B}$	B_μ	$\tilde{B}$	1	1	0

Table 3: Superfields and corresponding component fields with the appropriate group representations.

where the Y's are the usual hypercharges of the many fermions in the SM and $Y_{\tilde{H}}$ are the hypercharges of the higgsinos. As the first two terms are automatically cancelled in the SM [1, 47], one obtains that

$$Y_{\tilde{H}_u} + Y_{\tilde{H}_d} = 0 \Rightarrow Y_{H_u} = \frac{1}{2}, \quad Y_{H_d} = -\frac{1}{2}. \tag{2.88}$$

It is trivial to see that the other two anomalies are also cancelled by the choice of charges above. In Table 3 all the superfields of the theory with their respective charges are presented. At this point the model is theoretically consistent and the next step is to study the its phenomenological consequences.

The first step is to determine the most general superpotential allowed by the gauge symmetries. The superpotential associated with the Yukawa couplings from the SM is given by [35, 11, 12]

$$W_{\text{Yukawa}} = QY_u\bar{U}\mathcal{H}_u - QY_d\bar{D}\mathcal{H}_d - LY_e\bar{E}\mathcal{H}_d, \tag{2.89}$$

where the Y are matrices in flavour space. The the $SU(2)_L$ indices are implicitly contracted through the Levi-Civita symbol. Here, the necessity for two Higgs superfields is once again manifest. As previously noted, the superpotential must be holomorphic, therefore one cannot use $\mathcal{H}_u$ and $\mathcal{H}_u^*$ simultaneously in expression (2.89) to build gauge invariant operators and two distinct superfields with opposite hypercharges are needed. Using Eq. (2.65) for the scalar sector, the interactions obtained are just quartic interaction between the many squarks and also quartic interactions between two squarks and two Higgs. In particular, the terms

$$V_f \supset (\tilde{u}_R Y_u^* Y_u \tilde{u}_R^* + \tilde{u}_L Y_u^* Y_u \tilde{u}_L^*) H_u^0 H_u^{0*} \tag{2.90}$$

will be the most relevant ones when considering loop corrections. For the interactions with fermions, Eq. (2.66) gives us the SM Yukawa interactions and squark-quark-Higgsino interactions. The superpotential in Eq. (2.89) do not generate the mass terms for neither the Higgs nor Higgsinos, whence it is insufficient to our theory. We must include a superpotential, called *μ-term*,

$$W_{\mu\text{-term}} = \mu \mathcal{H}_u \mathcal{H}_d, \tag{2.91}$$

which is the only possible term with the two Higgs superfields, as the superpotential must be holomorphic and gauge invariant. Note that μ has mass dimension, meaning it could possibly receive large radiative corrections from the UV. This does not happens, because it is supersymmetric, hence can only be renormalized via wave function renormalization [11]. From the f-term of the superpotential $W_{\mu\text{-term}} + W_{\text{Yukawa}}$ comes the mass terms of the Higgs. The scalar potential also receives contribution from the D-term of Eq. (2.84), which for the Higgs it is given by

$$V_{D\text{-term}}(H_u, H_d) = \frac{g^2}{2}\Big(H_u^\dagger \frac{\sigma^a}{2} H_u + H_d^\dagger \frac{\sigma^a}{2} H_d\Big)^2 + \frac{g'^2}{8}\Big(H_u^\dagger H_u - H_d^\dagger H_d\Big)^2, \tag{2.92}$$

where g and g' are the coupling constants of $SU(2)_L$ and $U(1)_Y$, respectively. Together with the mass terms, the potential for H_u and H_d can be written as

$$\begin{aligned} V(H_u, H_d) \;=\; & |\mu|^2\Big(\big|H_u^+\big|^2 + \big|H_u^0\big|^2 + \big|H_d^-\big|^2 + \big|H_d^0\big|^2\Big) + \\ & + \frac{g^2 + g'^2}{8}\Big(\big|H_u^+\big|^2 + \big|H_u^0\big|^2 - \big|H_d^-\big|^2 - \big|H_d^0\big|^2\Big)^2 + \frac{g^2}{2}\big|H_u^+ H_d^{0*} + H_u^0 H_d^{-*}\big|^2. \end{aligned} \tag{2.93}$$

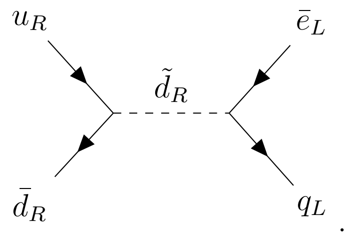
The superpotentials in Eqs. (2.89) and (2.91) already describe the interactions familiar to the SM, but those are not the only ones allowed by SUSY and gauge invariance. An additional superpotential is given by [35, 11]

$$W' = \alpha_i L_i \mathcal{H}_u + \gamma_{ijk} L_i Q_j \bar{D}_k + \delta_{ijk} L_i L_j \bar{E}_k + \beta_{ijk} \bar{U}_i \bar{D}_j \bar{D}_k, \tag{2.94}$$

where i, j and k are family indices and all gauge group indices are properly contracted. It is clear from Eq. (2.94) that W' allows for baryon and lepton number violation at tree-level, and consequently gives contributions to proton decay. To see this consider some of the terms coming from Eq. (2.66),

$$V' \supset \beta_{ijk} u_{Ri} \tilde{d}^*_{Rj} d_{Rk} + \gamma_{ijk} e_{Li} q_{Lj} \tilde{d}^*_{Rk} + h.c. \tag{2.95}$$

These allow for the proton to decay through a squark mediation, given at tree level by the following Feynman diagram



Hence, the decay width can be estimated as [11]

$$\Gamma(p \to \text{lepton} + \text{meson}) \sim |\beta\gamma|^2 \frac{M_p^5}{M_{\tilde{d}}^4}, \tag{2.96}$$

with the proton mass M_p inserted by dimensional analysis. Current bounds set $\Gamma^{-1} > 10^{34}$ years [3], thus, with the squark mass at TeV scale, one obtains

$$|\beta\gamma| < 10^{-26}, \tag{2.97}$$

which renders W' phenomenologically irrelevant. Instead of fine tuning the parameters α, β, γ and δ to match the experimental observations one would rather introduce a theoretical tool to justify why W' is forbidden in our theory. With this reasoning a discrete symmetry, called *R-parity* [11, 39], is introduced, under which usual particles have charge $+1$ and the SUSY partners have charge -1. If R-parity is a good symmetry of nature, only terms with zero or two SUSY particles are allowed in the potential, whence forbidding W'. An important phenomenological consequence of such symmetry is the prediction of a stable SUSY particle, which may be a good candidate for Dark Matter.

2.4.2 SUSY breaking

The Higgs potential in Eq. (2.93) is the prediction of a exact $\mathcal{N} = 1$ supersymmetric SM with two Higgs doublets imposing also R-parity. Note, however, that both quadratic and quartic coefficients are positive, meaning that it cannot break Electroweak (EW) symmetry. Besides, as stressed in previous sections, exact SUSY implies in mass degenerate supermultiplets that are charged under the SM gauge group. From the phenomenological point of view such prediction is already inconsistent with present experiments. Both these issues insinuate that, if nature is somehow supersymmetric, SUSY must be broken.

SUSY spurions

The precise manner in which SUSY breaks is not of interest, since only the low-energy theory is relevant to our discussion. Hence, one can simply parametrise the breaking effects of SUSY by introducing soft operators* to the low-energy effective Lagrangian. The technique of spurions can be used to determine what is the consistent choice of soft operators that break SUSY in the IR. This technique is widely used in the context of EFT's and consists in considering the parameters and couplings that break the given symmetry in the UV as (spurionic) fields [1, 21, 5]. The latter transform under the relevant symmetry such that the UV Lagrangian is manifestly invariant under it. In this way, the low-energy EFT can be written using the selection rules of the symmetry and the breaking effects are properly given in terms of the spurionic fields.

In the case of SUSY theories, the spurion must be charged under SUSY, but shall not break Lorentz invariance [48]. The explicit form of the possible spurions can be determined from the conditions under which SUSY is broken. As discussed in section 2.2, all SUSY generators annihilate the vacuum if SUSY is exact (see Eq. (2.27)), hence, using Eq. (2.7a),

$$\langle 0|P^0|0\rangle = 0, \tag{2.98}$$

in other words, the energy of the vacuum is zero. From the scalar potential of the in-

*Soft operators are operators that break a given symmetry, but that introduce at the maximum logarithmic divergences.

teracting Wess-Zumino model coupled to a non-abelian gauge superfield written in Eq. (2.83), one can interpret the condition above as

$$V\Big(\phi = \phi^\dagger = 0\Big) = 0. \tag{2.99}$$

Eq. (2.99) implies that, in order to SUSY remain unbroken, both f- and D- term must simultaneously vanish. If not, then SUSY is spontaneously broken, as the vacuum is not supersymmetric anymore. We are not interested in understanding how such breaking occurs, i.e. how to write a model with non-vanishing f- or D- terms, but on the impacts of such breaking in the IR. The reasoning above leaves evident that SUSY breaking, without also breaking Lorentz invariance, can only be generated from f- and D-terms, therefore the appropriate spurions must also have this same structure. In what follows only the effects of a f-term spurion will be considered. The D-term spurion, in the context of supersymmetric versions of the SM, is uninteresting, as it could only give non-trivial contributions if there were chiral superfields charged under the adjoint representation of the internal groups [48]. Since there is no such superfield in the (minimal) realisations of the supersymmetric SM, they are henceforth ignored.

The f-term spurion can be thus written as

$$S = \xi^2 f_S, \tag{2.100}$$

where f_S is a constant. It is assumed for simplicity that S is a SM singlet. The next step is to construct operators with the fields of the Table 3 and the spurion S. To study the effects in the IR one must only consider the leading contributions from spurions, which are the ones that generate superrenormalizable operators. Considering M to be a mass scale of the UV*, these are given by

$$\mathscr{L}_{\text{b-term}} = \int d^4\xi \; \frac{S^\dagger S}{M^2} B\mathcal{H}_u\mathcal{H}_d, \tag{2.101a}$$

$$\mathscr{L}_{\text{Higgs masses}} = \int d^4\xi \; \frac{S^\dagger S}{M^2}\Big(M_u^2\mathcal{H}_u^\dagger\mathcal{H}_u + M_d^2\mathcal{H}_d^\dagger\mathcal{H}_d\Big), \tag{2.101b}$$

$$\mathscr{L}_{\text{a-term}} = \int d^2\xi \; \frac{S}{M}\Big(QA_u\bar{U}\mathcal{H}_u - QA_d\bar{D}\mathcal{H}_d - LA_e\bar{E}\mathcal{H}_d\Big), \tag{2.101c}$$

*The physical interpretation of this scale changes as the realisation of SUSY spontaneous breaking in the UV changes [11, 12, 43], but in the IR it is just an input scale that sets the soft scale.

$$\mathscr{L}_{\text{gaugino masses}} = \int d^2\xi \; \frac{S}{M} \, \text{Tr} \left(\tilde{M}_G W^2_{SU(3)} + \tilde{M}_W W^2_{SU(2)} + \tilde{M}_B W^2_{U(1)} \right), \tag{2.101d}$$

$$\mathscr{L}_{\text{sfermion masses}} = \int d^4\xi \; \frac{S^\dagger S}{M^2} \left(L^\dagger C_L L + Q^\dagger C_Q Q + \bar{E}^\dagger C_E \bar{E} + \bar{D}^\dagger C_D \bar{D} + \bar{U}^\dagger C_U \bar{U} \right), \tag{2.101e}$$

where $W^\alpha_{SU(3)}$, $W^\alpha_{SU(2)}$ and $W^\alpha_{U(1)}$ are the spinorial superfields of the respective gauge groups. Furthermore, C_i and A_i are matrices in flavour space and $\tilde{M}_i$, B and M_i^2 are numerical coefficients [11, 49]. For each of the cases in Eq. (2.101) the integration over ξ can be trivially performed. For instance, from Eq. (2.101a) one obtains

$$\begin{aligned} \mathscr{L}_{\text{b-term}} &= \int d^4\xi \; \frac{S^\dagger S}{M^2} B \mathcal{H}_u \mathcal{H}_d \\ &= B \frac{|f_S|^2}{M^2} \int d^4\xi \; \xi^2 (\xi^\dagger)^2 (H_u + \cdots)(H_d + \cdots) \\ &= B \frac{|f_S|^2}{M^2} H_u H_d \\ &\equiv -b H_u H_d, \end{aligned} \tag{2.102}$$

where we associate the soft-SUSY breaking coefficient b with the UV parameters,

$$b = -B \frac{|f_S|^2}{M^2}. \tag{2.103}$$

Performing analogous calculations one obtains all soft-breaking parameters in terms of f_S and M,

$$m_i^2 = -M_i^2 \frac{|f_S|^2}{M^2}, \quad \tilde{M}_i' = -\tilde{M}_i \frac{f_S}{M}, \quad a_i = -A_i \frac{f_S}{M}, \quad \tilde{m}_i = -C_i \frac{|f_S|^2}{M^2}. \tag{2.104}$$

The soft-breaking potential can be therefore be written as

$$\begin{aligned} V_{\text{soft}} &= \frac{1}{2} \left(\tilde{M}_G' \tilde{G} \tilde{G} + \tilde{M}_W' \tilde{W} \tilde{W} + \tilde{M}_B' \tilde{B} \tilde{B} + h.c. \right) + \\ &\quad + \left(\tilde{u}_R^* a_u \tilde{q}_L H_u - \tilde{d}_R^* a_d \tilde{q}_L H_d - \tilde{e}_R^* a_e \tilde{\ell}_L H_d + h.c. \right) + \\ &\quad + \tilde{q}_L^* \tilde{m}_Q^2 \tilde{q}_L + \tilde{u}_R^* \tilde{m}_u^2 \tilde{u}_R + \tilde{d}_R^* \tilde{m}_d^2 \tilde{d}_R + \tilde{\ell}_L^* \tilde{m}_\ell^2 \tilde{\ell}_L + \tilde{e}_R^* \tilde{m}_e^2 \tilde{e}_R + \\ &\quad + b(H_u H_d + h.c.) + m_u^2 |H_u|^2 + m_d^2 |H_d|^2, \end{aligned}$$

which consists of only superrenormalizable operators, as expected. Note that, in particular, besides additional mass term for gauginos, squarks and Higgs, a coupling between the two Higgs and a triple coupling between the squarks and the Higgs are generated.

Electroweak symmetry breaking

With all the allowed soft-breaking parameters, it is now possible to study how Electroweak symmetry breaking (EWSB) occurs. Taking the potential in Eq. (2.105) into account, the full potential for the Higgs fields now reads

$$\begin{aligned} V(H_u, H_d) \;=\;& \frac{g^2+g'^2}{8}\Big(\big|H_u^+\big|^2+\big|H_u^0\big|^2-\big|H_d^-\big|^2-\big|H_d^0\big|^2\Big)^2+\frac{g^2}{2}\big|H_u^+H_d^{0*}+H_u^0H_d^{-*}\big|^2+ \\ &+\big(|\mu|^2+m_u^2\big)\Big(\big|H_u^+\big|^2+\big|H_u^0\big|^2\Big)+\big(|\mu|^2+m_d^2\big)\Big(\big|H_d^-\big|^2+\big|H_d^0\big|^2\Big)+ \\ &+b\Big(H_u^+H_d^- - H_u^0H_d^0+h.c.\Big), \end{aligned} \tag{2.105}$$

with $b > 0^*$. From the equation above it is clear that EWSB is possible, since it contains the unknown soft-breaking parameters m_u^2, m_d^2 and b. The objective is to determine under what conditions these parameters allow for EWSB to take place.

Suppose that at the values

$$\langle H_u\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} v_+ \\ v_u \end{pmatrix}, \quad \langle H_d\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} v_d \\ v_- \end{pmatrix}, \tag{2.106}$$

the potential (2.105) obtains its minimum value. Given the $SU(2)_L$ symmetry we may choose one of these directions to be zero, say, $v_+ = 0$. Moreover, given the $U(1)_Y$ symmetry, v_u can be made real. Note that there is not enough gauge redundancy to set both v_- and v_+ to zero. Notwithstanding, one *can* set $v_- = 0$ by looking at the derivatives of V with respect to the fields. Take for instance the derivative along the charged direction H_u^+ at the minimum:

$$\left.\frac{\partial V}{\partial H_u^+}\right|_{H_i=\langle H_i\rangle} = \frac{v_-}{\sqrt{2}}\left(b+\frac{g^2}{4}v_d^* v_u\right). \tag{2.107}$$

As the above expression is evaluated at the minimum, it must be zero by definition, hence, for arbitrary values of g and b, the only way to satisfy the above equation is to have $v_- = 0$. As a matter of fact, to further study the minimum properties we may set both charged directions in Eq. (2.105) to zero†. Taking the derivatives with respect to

$^*b$ may be taken as positive by doing a global redefinition of the phases of both H_u and H_d.

†Note that this result is consistent with the fact that $U(1)_{\text{EM}}$, the symmetry associated with electromagnetism, is unbroken.

the neutral components at the minimum (2.106), we obtain the following equations for the vacuum expectation values (vevs):

$$\left.\frac{\partial V}{\partial H_u^0}\right|_{H_i=\langle H_i\rangle} = 0 \longrightarrow \left(|\mu|^2 + m_u^2\right)v_u + \frac{g^2+g^{'2}}{8}\left(v_u^2 - |v_d|^2\right)v_u - bv_d = 0, \tag{2.108a}$$

$$\left.\frac{\partial V}{\partial H_d^0}\right|_{H_i=\langle H_i\rangle} = 0 \longrightarrow \left(|\mu|^2 + m_d^2\right)v_d^* - \frac{g^2+g^{'2}}{8}\left(v_u^2 - |v_d|^2\right)v_d^* - bv_u = 0. \tag{2.108b}$$

From the second equation we see that, as v_u is real, v_d must also be real. Hence, the vevs of the Higgs are written as

$$\langle H_u\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_u \end{pmatrix}, \quad \langle H_d\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} v_d \\ 0 \end{pmatrix}, \tag{2.109}$$

with both v_u and v_d real.

It is still unclear whether v_u and v_d are vanishing or not, since this depends on the second derivative of the potential $V(H_u^0, H_d^0)$. For EWSB to take place, the determinant of the Hessian of V at zero must be negative, characterising a local maximum,

$$\det \mathrm{Hess}(V,0) < 0 \Rightarrow b^2 > \left(|\mu|^2 + m_u^2\right)\left(|\mu|^2 + m_d^2\right). \tag{2.110}$$

In spite of the condition set by the equation above, b is not allowed to be arbitrarily large. Another relation between b, μ and the masses m_u^2 and m_d^2 is determined from the requirement that V must be bounded from below regardless the direction in (H_u^0, H_d^0) space [35]. Take for example the real diagonal direction $H_u^0 = H_d^0 = \phi$. The potential in this case becomes

$$V(H_u^0 = H_d^0 = \phi) = \left(2|\mu|^2 + m_u^2 + m_u^2 - 2b\right)\phi^2, \tag{2.111}$$

therefore only if

$$2|\mu|^2 + m_u^2 + m_u^2 > 2b \tag{2.112}$$

will the potential be always bounded from below as $\phi \to \infty$. Note that the above conditions does not allow $m_u^2 = m_d^2$. Furthermore, EWSB will only take place in the narrow region between the inequalities (2.110) and (2.112). From this fact alone it comes as no surprise that the model will be tuned to a certain degree. The quantification of such FT will be studied in section 2.5.

Mass eigenstates

With two Higgs doublets we have a total of 8 dof, from which only the combinations that diagonalise the mass matrix with potential (2.105) are physical observable states. Our next task is to expand the Higgs potential (with both charged and neutral components) around the vevs, compute the mass matrix from the quadratic terms and then diagonalise it. In particular, this procedure shall give us three massless pions, one neutral and two charged, which are the Nambu-Goldstone Bosons (NGB) of EWSB. The remaining 5 mass eigenstates are expected to be massive, of which 2 are charged, 1 is a neutral CP odd scalar and 2 are CP even neutral scalars. The focus will be on the neutral states, as the lightest CP even neutral states is to be identified with the physical Higgs boson h [11, 12].

The diagonalisation of the 8×8 matrix can be simplified by further examining the potential after expanding the fields around their vev

$$H_i^0 \to \frac{v_i}{\sqrt{2}} + H_i^0. \tag{2.113}$$

The potential in Eq. (2.105) is thus given by

$$\begin{aligned} V_2 \;=\;& \Big(|\mu|^2 + m_u^2\Big)\Big((\mathrm{Re}\, H_u^0)^2 + (\mathrm{Im}\, H_u^0)^2 + (\mathrm{Re}\, H_u^+)^2 + (\mathrm{Im}\, H_u^+)^2\Big) + \\ & +\Big(|\mu|^2 + m_d^2\Big)\Big((\mathrm{Re}\, H_d^0)^2 + (\mathrm{Im}\, H_d^0)^2 + (\mathrm{Re}\, H_d^-)^2 + (\mathrm{Im}\, H_d^-)^2\Big) + \\ & +\frac{M_Z^2}{4v^2}\Bigg[(v_u^2 - v_d^2)\Big((\mathrm{Re}\, H_u^0)^2 + (\mathrm{Im}\, H_u^0)^2 - (\mathrm{Re}\, H_d^0)^2 - (\mathrm{Im}\, H_d^0)^2\Big) + \\ & +2v_u^2(\mathrm{Re}\, H_u^0)^2 + 2v_d^2(\mathrm{Re}\, H_d^0)^2 - 4v_u v_d\, \mathrm{Re}\, H_d^0\, \mathrm{Re}\, H_u^0\Bigg] - \\ & -2b\Big(\mathrm{Re}\, H_u^0\, \mathrm{Re}\, H_d^0 - \mathrm{Im}\, H_u^0\, \mathrm{Im}\, H_d^0\Big) + b\Big(H_u^+ H_d^- + h.c.\Big) \\ & +\frac{M_W^2}{v^2}\left(v_u^2\left|H_u^+\right|^2 + v_d^2\left|H_d^-\right|^2\right), \end{aligned} \tag{2.114}$$

where we have defined the following quantities:

$$v^2 = v_u^2 + v_d^2, \tag{2.115a}$$

$$M_Z^2 = \frac{g^2 + g'^2}{4} v^2, \tag{2.115b}$$

$$M_W^2 = \frac{1}{4} g^2 v^2, \tag{2.115c}$$

which are respectively the tree-level vev, the Z boson mass and the W boson mass squared. Another useful definition is the misalignment of the vevs, given by

$$\tan\beta \equiv \frac{v_u}{v_d}. \tag{2.116}$$

Instead of an 8×8 matrix, it is clear from Eq. (2.114) that one needs to diagonalise only three 2×2 matrices, which are given by

$$\begin{pmatrix}\mathrm{Im}\, H_u^0, & \mathrm{Im}\, H_d^0\end{pmatrix}\begin{pmatrix} b\cot\beta & b \\ b & b\tan\beta\end{pmatrix}\begin{pmatrix}\mathrm{Im}\, H_u^0 \\ \mathrm{Im}\, H_d^0\end{pmatrix}, \tag{2.117a}$$

$$\begin{pmatrix}\mathrm{Re}\, H_u^0, & \mathrm{Re}\, H_d^0\end{pmatrix}\begin{pmatrix} b\cot\beta + M_Z^2\sin^2\beta & -b - M_Z^2\sin\beta\cos\beta \\ -b - M_Z^2\sin\beta\cos\beta & b\tan\beta + M_Z^2\cos^2\beta\end{pmatrix}\begin{pmatrix}\mathrm{Re}\, H_u^0 \\ \mathrm{Re}\, H_d^0\end{pmatrix}, \tag{2.117b}$$

$$\begin{pmatrix}H_u^{+*}, & H_d^-\end{pmatrix}\begin{pmatrix} b\cot\beta + M_W^2\cos^2\beta & b + M_W^2\sin\beta\cos\beta \\ b + M_W^2\sin\beta\cos\beta & b\tan\beta + M_W^2\cos^2\beta\end{pmatrix}\begin{pmatrix}H_u^+ \\ H_d^{-*}\end{pmatrix}, \tag{2.117c}$$

where, in order to leave the expressions only in terms of b, β and M_Z^2, we used that

$$\left(|\mu|^2 + m_u^2\right) = b\cot\beta + \frac{M_Z^2}{2}\cos 2\beta, \tag{2.118a}$$

$$\left(|\mu|^2 + m_d^2\right) = b\tan\beta - \frac{M_Z^2}{2}\cos 2\beta, \tag{2.118b}$$

which results from Eqs. (2.108a) and (2.108b). Since the focus is only in the in the lightest neutral component, the charged matrices are not relevant to the discussion anymore, as they do not mix with the neutral states. Even so, we note that two of the three NGB come from the charged components. This clearly follows from the zero determinant and non-vanishing trace, meaning there are two massless eigenstates π^+ and π^-. The other massless pion comes from the imaginary part matrix, as it will now be shown.

The imaginary part matrix in Eq. (2.117a) is a real symmetric matrix, whence it can be diagonalised by the following orthogonal matrix,

$$\begin{pmatrix}\mathrm{Im}\, H_u^0 \\ \mathrm{Im}\, H_d^0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\sin\beta & \cos\beta \\ -\cos\beta & \sin\beta\end{pmatrix}\begin{pmatrix}\pi^0 \\ A^0\end{pmatrix}. \tag{2.119}$$

with mass eigenvalues

$$M_{\pi^0}^2 = 0, \quad M_{A^0}^2 = \frac{2b}{\sin 2\beta}, \tag{2.120}$$

therefore there are indeed three NGB, which which in turn implies in the correct number of massive gauge bosons. The real part matrix in Eq. (2.117b) contains no massless eigenstates and is instead diagonalised by a distinct orthogonal transformation,

$$\begin{pmatrix} \mathrm{Re}\, H_u^0 \\ \mathrm{Re}\, H_d^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} h \\ H^0 \end{pmatrix}, \tag{2.121}$$

with the angle α given by

$$\tan 2\alpha = \frac{M_{A^0}^2 + M_Z^2}{M_{A^0}^2 - M_Z^2} \tan 2\beta. \tag{2.122}$$

The masses of h and H^0 are given by

$$M_{H^0}^2 = \frac{1}{2}\left[M_{A^0}^2 + M_Z^2 + \sqrt{\left(M_{A^0}^2 + M_Z^2\right)^2 - 4M_{A^0}^2 M_Z^2 \cos^2 2\beta}\right], \tag{2.123}$$

$$m_h^2 = \frac{1}{2}\left[M_{A^0}^2 + M_Z^2 - \sqrt{\left(M_{A^0}^2 + M_Z^2\right)^2 - 4M_{A^0}^2 M_Z^2 \cos^2 2\beta}\right]. \tag{2.124}$$

As $m_h^2 < M_{H^0}^2$, the observed Higgs particle is identified with h.

Another way of diagonalising these matrices is by directly identifying the linear combination of H_u and H_d that acquires a vev v and therefore contains all NGB. This particular combination is given by

$$H = \frac{v_u H_u + v_d \tilde{H}_d}{v} = H_u \sin\beta + \tilde{H}_d \cos\beta \tag{2.125}$$

where $\tilde{H}_d$ is defined as usual by

$$\tilde{H}_d = i\sigma^2 H_d^*. \tag{2.126}$$

It is trivial to see that H has vev v,

$$\langle H \rangle = \langle H_u \rangle \sin\beta + \left\langle \tilde{H}_d \right\rangle \cos\beta = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \frac{v_u^2 + v_d^2}{v} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{2.127}$$

and can be therefore identified with the Higgs doublet of the SM. Furthermore, one must also consider the orthogonal direction, which is given by

$$H' = H_u \cos\beta - \tilde{H}_d \sin\beta \tag{2.128}$$

Note that Eqs. (2.125) and (2.128) are equivalent to the following rotation of H_u and H_d,

$$\begin{pmatrix} H \\ H' \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \tilde{H}_d \\ H_u \end{pmatrix}, \tag{2.129}$$

which shows to us that the rotation in Eq. (2.119) has a physical meaning behind it. Either way, the rotation above does not diagonalise the CP even sector, that is composed of $\operatorname{Re} H^0$ and $\operatorname{Re} H'^0$, hence an additional rotation given in Eq. (2.121) is needed.

⋆⋆⋆

The model constructed here is known as the *Minimal Supersymmetric Standard Model* (MSSM). Throughout the discussion it was emphasised that to build a SUSY version of the SM it is not enough to just supersymmetrise the ordinary SM. One also needs to introduce an extra Higgs doublet to cancel anomalies, impose R-parity to avoid proton decay and actually break SUSY softly in order to break EW symmetry spontaneously. All these are important elements of the MSSM and are the minimal requirements for the theory to be consistent. The MSSM is not, though, the unique SUSY version of the SM; there are many non-minimal SUSY extensions of the SM, which won't be discussed in this thesis [35, 38, 39, 41, 11, 43].

2.4.3 Tree-level and 1-loop bounds on m_h^2

A more careful analysis of Eq. (2.124) consists is determining if it is possible to achieve the observed value of the physical Higgs mass, namely $m_h^2 = (125 \text{ GeV})^2$. Take Eq. (2.124),

$$m_h^2(x) = \frac{1}{2}\left[x + M_Z^2 - \sqrt{(x + M_Z^2)^2 - 4xM_Z^2\cos^2 2\beta}\right], \tag{2.130}$$

where $x = M_{A^0}$ is the mass of the neutral CP odd state. In order to study this function, the precise value of x is superfluous, because the function $m_h^2(x)$ is strictly crescent and achieves its maximum value at $x = \infty$,

$$m_h^2(x) \le \lim_{x\to\infty} m_h^2(x) = M_Z^2 \cos^2 2\beta. \tag{2.131}$$

Therefore, if the upper bound $M_Z^2 \cos^2 2\beta$ is lower than 125 GeV, there will be a conflict with the experimental data, which is indeed the case, as $M_Z^2 \simeq (91\text{GeV})^2$ and $|\cos 2\beta| \le 1$. The conclusion is that the tree-level mass (2.124) is incompatible with experiments. Still, one may consider loop contributions and hope that they can be used to push the upper bound (2.131) up to the correct value.

We now proceed to the calculation of 1-loop corrections due to tops and stops. Other contributions coming from gauge bosons, gauginos, leptons, sleptons, and any other quarks or squarks are neglected, as the top Yukawa is by far more relevant then other couplings. The top and stop interact only with the H_u, so only it is affected by those loop corrections. Assuming, for simplicity, that the stops are mass degenerate and that the corresponding a-term in Eq. (2.105) is zero, one can write down the 1-loop Coleman-Weinberg potential, derived in Eq. (A.28) of Appendix A,

$$\Delta V_{\text{eff}}(H_u^0) = \frac{3}{16\pi^2}\left[\left(\tilde{m}_t^2 + M^2\right)^2 \ln\frac{\tilde{m}_t^2 + M^2}{q^2} - M^4 \ln\frac{M^2}{q^2}\right], \tag{2.132}$$

where the first term is the stop contribution, the second term is the top contribution, and

$$M^2 = \left|y_t H_u^0\right|^2$$

with y_t the top Yukawa coupling and q an arbitrary energy scale. Note that, if SUSY were exact, i.e. $\tilde{m}_t^2 = 0$, the contributions of top and stops would cancel exactly, justifying why SUSY could potentially solve the HP. Taking into consideration that the soft-breaking scale is above the EW one, we expand the potential for $\tilde{m}_t^2 \gg M^2$ and obtain*

$$\Delta V_{\text{eff}}(H_u^0) = \frac{3}{16\pi^2} M^4 \left[\ln\frac{\tilde{m}_t^2}{M^2} + \frac{3}{2}\right] + \cdots . \tag{2.133}$$

The dots in the above expression are terms that either do not depend on or are linear in

*The expansion is done for $\tilde{m}_t \gg M$, but this separation cannot be to excessive. The reason for this is clear, as a large-log appears in the effective potential. For a large separation of scales, one must integrate out the heavy modes, match the UV theory with the corresponding EFT and run the couplings accordingly [50].

$|H_u^0|^2$ (while neglecting $\mathcal{O}(M^6)$ terms, which represent higher order contributions). The renormalization condition is chosen such that the effective potential $V_{\text{eff}} = V_{\text{tree}} + \Delta V_{\text{eff}}$ is still minimised by the tree-level vev [35], hence

$$\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}|H_u^0|^2}\left(\frac{v_u}{\sqrt{2}}\right) = 0 \Rightarrow \frac{\mathrm{d}\Delta V_{\text{eff}}}{\mathrm{d}|H_u^0|^2}\left(\frac{v_u}{\sqrt{2}}\right) = 0. \tag{2.134}$$

As consequence, the correction to the mass is proportional to the quartic corrections, thus we do not need to worry with the quadratic terms that fix condition (2.134) but only with the piece proportional to $|H_u^0|^4$. Therefore, the correction to the mass term of H_u is given by

$$\delta m_u^2 = 2v_u^2 \frac{\mathrm{d}^2\Delta V_{\text{eff}}}{\mathrm{d}(|H_u^0|^2)^2} \tag{2.135}$$

evaluated at $v_u/\sqrt{2}$. A straightforward calculation leads to

$$\delta m_u^2 = \frac{3|y_t|^4 v_u^2}{4\pi^2} \ln \frac{\tilde{m}_t^2}{m_t^2}, \tag{2.136}$$

where $m_t = \frac{y_t v_u}{\sqrt{2}}$ is the top mass. Rewriting the above expression in terms only of the top mass, the vev squared and β, one obtains

$$\delta m_u^2 = \frac{3m_t^4}{\pi^2 v^2 \sin^2\beta} \ln \frac{\tilde{m}_t^2}{m_t^2}. \tag{2.137}$$

The radiative correction in Eq. (2.137) shifts the mass term of H_u in potential (2.105),

$$|\mu|^2 + m_u^2 \to |\mu|^2 + m_u^2 + \delta m_u^2.$$

Whence, there will be a shift in the first element of the mass matrix (2.117b),

$$b\cot\beta + M_Z^2 \sin^2\beta \to b\cot\beta + M_Z^2\sin^2\beta + \delta m_u^2,$$

which will in turn produce a new mass eigenvalue for h,

$$\begin{aligned} m_h^2 \;\to\; & \frac{1}{2}\Bigg[M_{A^0}^2 + M_Z^2 + \delta m_u^2 - \Big(M_{A^0}^4 + M_Z^4 + \delta m_u^4 - 2M_{A^0}^2 M_Z^2 \cos^2 4\beta - \\ & -2\delta m_u^2 M_Z^2 \cos^2 2\beta + 2M_{A^0}^2 \delta m_u^2 \cos^2 2\beta\Big)^{1/2}\Bigg]. \end{aligned} \tag{2.138}$$

Just as in the tree-level case, an upper bound for m_h^2 is obtained in the limit $M_{A^0}^2 = \infty$. This limit yields

$$m_h^2 \le M_Z^2 \cos^2 2\beta + \sin^2 \beta \delta m_u^2 = M_Z^2 \cos^2 2\beta + \frac{3m_t^4}{\pi^2 v^2} \ln \frac{\tilde{m}_t^2}{m_t^2} \tag{2.139}$$

$$\Rightarrow m_h^2 \le M_Z^2 + \frac{3m_t^4}{\pi^2 v^2} \ln \frac{\tilde{m}_t^2}{m_t^2}. \tag{2.140}$$

In order to achieve the measured mass for h, the equation

$$(125 \text{ GeV})^2 \le M_Z^2 + \frac{3m_t^4}{\pi^2 v^2} \ln \frac{\tilde{m}_t^2}{m_t^2} \tag{2.141}$$

must be solved for $\tilde{m}_t^2$. Using the experimental values for the Z boson mass and the vev [3] one obtains the following lower bound for the stop mass:

$$\tilde{m}_t^2 \gtrsim (1 \text{ TeV})^2. \tag{2.142}$$

This bound is compatible with present experimental data [51, 52], which are shown in Figure 3.

2.5 Tuning in the MSSM

In section 2.4.3, based on the observed value of the Higgs mass, the stop mass was estimated and it is somewhat compatible with the experimental constrains at 95%CL. Hence, with a stop mass of order TeV the Hierarchy Problem could be potentially solved. However, it will become clear that a certain amount of Fine Tuning (FT) still persists.

The first solution presented in this work is the MSSM, which truly eliminates the quadratic divergences from the Higgs mass, which is clear from the computation of the effective potential in Eq. (2.132), for example. However, this is not enough to judge it free from any tuning, because of the introduction of a new energy scale*: the soft-breaking scale Λ_{soft}. More precisely, since there are new parameters in the theory which are of order Λ_{soft}, new cancellations between them may be needed and raise the FT to dangerous levels.

*If instead we worked not withing the context of EFT, but with a spontaneously broken SUSY, we would have introduced an unification scale at the far UV.

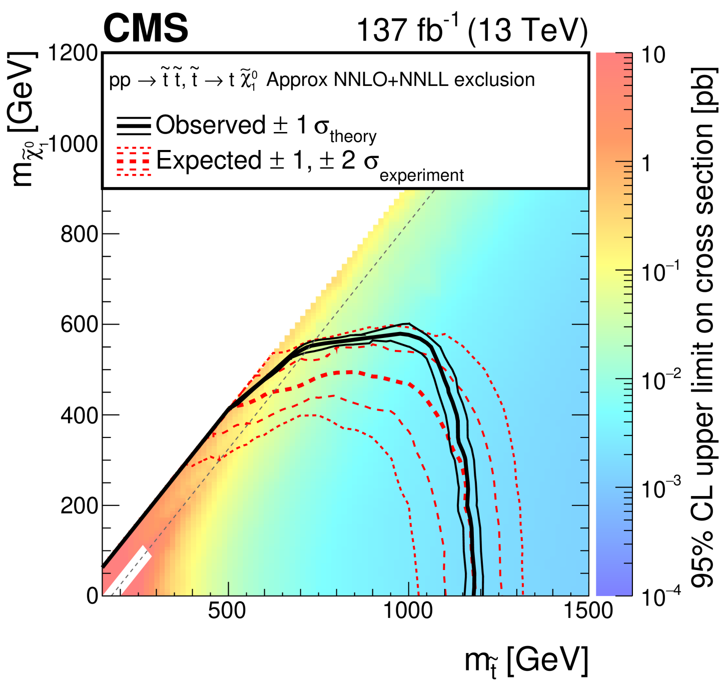


Figure 3: Most recent experimental bounds on the stop mass $\tilde{m}_t$. The plot shows the excluded region at 95% Confidence Level. The process analysed is $pp \to \tilde{t}\bar{\tilde{t}} \to t\bar{t}\tilde{\chi}_1^0\tilde{\chi}_1^0$, where $\tilde{\chi}_1^0$ is the lightest neutralino, with $\sqrt{s} = 13$ TeV and 137 fb^{-1} of integrated luminosity. The thin solid (black) curves show the changes in these limits as the signal cross sections are varied by their theoretical uncertainties. The thick dashed (red) curves present the expected limits. Taken from [52].

For the explicit calculation the mass of the Z boson is the most suitable observable, as it is one of the most precise measurements of the SM. The physical Higgs mass is not used in this case, because the vev and quartic coupling depend implicitly on the SUSY parameters, making the calculations much more complicated. The Z mass, on the other hand, has a very straightforward dependence of the free parameters of the MSSM [30, 53]. Using Eqs. (2.118a) and (2.118b) one obtains

$$M_Z^2\Big|_{\text{tree}} = -2|\mu|^2 + \frac{2(m_d^2 - m_u^2\tan^2\beta)}{\tan^2\beta - 1}, \tag{2.143}$$

at tree-level.

In order to further simplify the discussion, the limit $\tan\beta \gg 1$ is taken, such that

$$M_Z^2\Big|_{\text{tree}} = -2\Big(|\mu|^2 + m_u^2\Big). \tag{2.144}$$

The FT is thus calculated only through the derivatives with respect to m_u^2 and $|\mu|^2$, which are given by

$$\Delta_{|\mu|^2} = \left|\frac{2|\mu|^2}{M_Z^2}\right|, \quad \Delta_{m_u^2} = \left|\frac{2m_u^2}{M_Z^2}\right|. \tag{2.145}$$

The parameter m_u^2 comes from the soft potential in Eq. (2.105), so it is naturally of order of the soft-breaking scale, this latter being given by the value of the stop mass (2.142). As consequence, $\Delta_{m_u^2}$ will not turn out to be very large. On the other hand, the μ parameter is supersymmetric, so there is nothing that forbids it from being as large as unification scale or even the Planck scale. In order to have the FT controlled, however, one needs μ to be of the same order as m_u^2, i.e. the soft-breaking scale. This awkward situation, where a supersymmetric parameter needs to be of the order of soft-breaking scale, is known as *μ-problem*. One way to solve it is to forbid the superpotential in Eq. (2.91) at tree-level and let it be generated by radiative corrections, in this manner it turns out be be automatically of order m_u [35, 11]. Thus, for m_u^2 and $|\mu|^2$ both of order of the soft-breaking scale, the FT on M_Z^2 (or equivalently on the EW scale) at tree-level is

$$\Delta \gtrsim \left(\frac{1\ \text{TeV}}{100\ \text{GeV}}\right)^2 = 100. \tag{2.146}$$

From Eq. (2.144) one can understand the FT in Eq. (2.146) as the amount of precise cancellation between m_u^2 and $|\mu|^2$, which must be of 10^{-2} parts in 1. Though still fine tuned, a FT of order 100 is much more reasonable compared to 10^{34} from before, for this reason the situation represented by Eq. (2.146) is named as the *little Hierarchy Problem*.

If we now include the 1-loop corrections from stop and top loops, Eq. (2.144) becomes

$$M_Z^2\Big|_{\text{1-loop}} = -2\left(|\mu|^2 + m_u^2 + \frac{3m_t^4}{2\pi^2 v^2}\ln\frac{m_{\tilde{t}}^2}{m_t^2}\right), \tag{2.147}$$

where we considered $\tan\beta \to \infty$. Such correction could potentially reduce the FT to an acceptable level, but it turns out that for $m_{\tilde{t}} \sim 1$ TeV the last term in the right-hand side of Eq. (2.147) is at least 100 times smaller then $|\mu|^2 + m_u^2$, so it won't interfere much in the value of Δ [53].

2.6 Conclusions

Throughout this chapter the basic principles of SUSY and the simplest supersymmetric version of the SM, the MSSM, was presented. With it, it became clear how SUSY solves the HP. However, in order to leave the MSSM phenomenologically consistent, one had to introduce new parameters of order TeV or higher in the theory. Those ended up creating more undesired FT, rendering the MSSM unnatural. But note that the situation within the MSSM is much better than before, as the FT was reduced from 10^{34} to 10^2. Also, a small FT is not an essential property of a theory, in the sense that we should not completely discard a theory solely on FT grounds. Notwithstanding, it is still very important that a theory remains untuned, because it translates our physical intuition of what we expect from EFT's and it allows us to have calculable (and hence testable) theories.

3 Composite Higgs Models

In this chapter a different solution to the Hierarchy Problem (HP) will be explored, namely the possibility of considering the Higgs not as an elementary particle but a composite object instead. In the SM, all scalar particles come as bound states of quarks in Quantum Chromo-dynamics (QCD), so the idea of compositness is not an absurdity from this point of view. The theory of hadrons and, in particular, the dynamical mechanism used to obtain it from the one of quarks and gluons contain many of the key concepts for the understanding of Composite Higgs Models (CHM).

This chapter is organised in the following way. In the first section the main motivations for such theories, while quickly explaining how do they work in general, are outlined. Then, in the second section, not only the needed results from group theory, but also some important aspects of $SO(N)$ groups are reviewed. After that, the Minimal Composite Higgs Models (MCHM), i.e. the minimal realisations of compositness compatible with the SM, is explored in depth and the Higgs mass is computed. In the last section the predictions of such models are compared with present experiments and the respective FT is computed.

3.1 Motivations and general idea

As briefly explained in Chapter 1, QCD with just two massless quarks, the up and down quarks, is invariant under a global $SU(2)_L \times SU(2)_R$ which is spontaneously broken to the diagonal group $SU(2)_V$ by the vacuum condensate $\langle \bar{q}q \rangle$. For exactly massless quarks the chiral group $SU(2)_L \times SU(2)_R$ is an exact symmetry and consequently the Nambu-Goldstone Bosons (NGB), the pions, are massless and all other bound states turn out to have well determined masses [1, 21]. This scenario is clearly incompatible with reality, therefore a small and explicit break of the chiral group due to the non-vanishing quark masses must be considered. Performing chiral perturbation theory, one obtains an effective low-energy Lagrangian that describes the strong interactions, which at this energy scale is mediated by the pions themselves [54, 55]. A similar configuration will take place in CHM. In general, one considers a global symmetry group G spontaneously broken to a subgroup $\mathcal{H}$ that gives rise to a new strong dynamics in the SM. Considering the Higgs as a bound state from this strong sector, there are two possibilities: the Higgs is a NGB or it is a resonance. The latter option is not viable, because, if true, we would have already observed many other resonances with similar masses [14]. The first option is more plausible but nonetheless problematic, since the Higgs is massive. To avoid such situation, one needs to break G explicitly, in this manner the Higgs is understood as a pseudo Nabmbu Goldstone Boson (pNGB) [21] and, if the breaking parameter is considerably small, the Higgs has a mass far below the resonances.

That the Higgs mass is stabilised by realising it as a pNGB is the main motivation for CHM. We will see explicitly in section 3.2 that the transformation of the NGB under the global group G is a *shift symmetry*, which is a non-homogeneous symmetry transformation. As a consequence, a mass term for the NGB is forbidden if the symmetry is exact. Introducing an explicit break of this symmetry by allowing a mass term in the Lagrangian renders the mass technically natural, since the symmetry is restored if the mass is taken to zero. More in general, the mass will turn out be to given by the parameters that break the shift symmetry, hence the Higgs mass is protected from receiving large radiative corrections. In this manner the Hierarchy Problem (HP) is solved.

In addition to stabilising the Higgs mass, one can directly compute it, as it will be shown in section 3.3.4. In QCD one may also calculate the pion masses in terms of the quark masses, which is given by the Gell-Mann-Oakes-Renner relation [1, 21]. This relation follows from introducing the mass matrix of the quarks as a spurionic field in the Lagrangian, that after SSB breaks the chiral symmetry and contributes to the pion masses in the low-energy Effective Field Theory (EFT). In CHM, rather than introducing in the Lagrangian a term that explicitly breaks the global group, one *gauges* a subgroup $\mathcal{G} \subset G$. As it will be explained in detail in the following sections, it is not obvious that the gauging of a subgroup breaks explicitly a global symmetry, but we anticipate that this is indeed the case, since loop effects generate a Coleman-Weinberg potential for the Higgs that is not invariant under G. Such method to break the group explicitly is used, because, since all the group structure of the Higgs ought to be determined from G, $\mathcal{H}$ and $G/\mathcal{H}$, the electroweak group $G_{\text{EW}} = SU(2)_L \times U(1)_Y$ must somehow be described by G, implying that a subgroup of G must be gauged such that $G_{\text{EW}} \subseteq \mathcal{H}$. In this picture, the SM becomes the low-energy effective theory of this strong dynamics, which is mediated by the Higgs boson at energies much below than the confinement scale.

Composite Higgs Model as an EFT

Let us make the discussion a bit more precise. The introduction of a new strong sector brings several consequences. For instance, a new energy scale f is dynamically generated from the Dimensional Transmutation (DT) mechanism [23]. The idea behind this mechanism can be understood from the renormalization group equations (RGE) for the gauge coupling of a non-abelian gauge-symmetry at 1-loop,

$$\mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = -C \frac{g^3}{16\pi^2}, \tag{3.1}$$

where $C > 0$ is some constant that depends on the particle content charged under the symmetry [1, 20]. The RGE flow of the equation above implies that the coupling g increases as the energy is lowered, hence in the IR one can take the boundary condition $g(f) \to \infty$ at a scale f and solve the RGE in Eq. (3.1),

$$\frac{4\pi}{g^2(\Lambda)} = -\frac{C}{2\pi} \ln \frac{f}{\Lambda} \Rightarrow f = \Lambda e^{-\frac{8\pi^2}{Cg^2(\Lambda)}} \tag{3.2}$$

where Λ is an energy scale of the UV. So it follows from the RGE in Eq. (3.1) that the scale f is computable from the RGE in terms of the initial conditions in the UV. In this sense f is said to be dynamically generated.

The same scale f in Eq. (3.2) can also be understood, as in QCD, as the scale in which the interactions become too strong and the vacuum confines, which thus characterises the spontaneous breaking of G. Hence, on general grounds, the spontaneous breaking will take place through the vev of a scalar operator Φ, namely $\langle \Phi \rangle$, with magnitude $f^{\dim \Phi}$. As we will see in the next section, the field Φ can be written in terms of its vev as

$$\Phi(x) = f_{U[\Pi]}\left(\tilde{\Phi}(x)\right), \tag{3.3}$$

where $\tilde{\Phi}$ are the radial resonances of the field and $f_{U[\Pi]}$ is a function that depends on the matrix $U[\Pi]$, defined by [1, 20, 21]

$$U[\Pi] \equiv e^{\frac{i\sqrt{2}}{f}\Pi^{\hat{a}}(x)\hat{T}^{\hat{a}}}, \tag{3.4}$$

with $\hat{T}$ the broken generators of the symmetry group and Π the corresponding NGB (in the literature, the matrix $U[\Pi]$ is also called "Goldstone Matrix"). By construction, some of the NGB will arrange themselves in a Higgs doublet of $SU(2)_L$ with 1/2 hypercharge, although the particular details of such procedure obviously depend on the groups G and $\mathcal{H}$ chosen. Therefore, a coset $G/\mathcal{H}$ with dimension less than 4 is incapable of forming the desired complex doublet. Stated in this way, it is trivial to see that a realistic CHM needs at least a coset with dimension 4; this point will be discussed further in section 3.3.

It is therefore of interest to write the EFT of models in which the Higgs is realised as a pNGB. This EFT will describe the low-energy physics below the scale f, where the dof of the new strong sector are already integrated out. In such EFT the relevant dof are the NGB and the light resonances, i.e. the ones with masses below $4\pi f$. As a result, one may use Eq. (1.1) to write the general Lagrangian for a CHM,

$$\mathscr{L}_{\text{CH}} = f^4 \mathscr{F}' \bigg(U[\Pi], \frac{\partial_\mu}{f}, \frac{\tilde{\Phi}}{f^{\dim \tilde{\Phi}}}, \frac{\psi}{f^{\dim \psi}} \bigg). \tag{3.5}$$

In the above equation we have set $\Lambda = 4\pi f$ and $g_* = 4\pi$, which corresponds to a maximally strongly coupled theory, as is QCD, and ψ denote generically the light resonances, that may be bosonic or fermionic. Furthermore, in Eq. (3.5) the SM fields are not yet being considered, that is to say, for the moment all gauge and Yukawas coupling are set to zero. To a great extent, the light resonances play an important phenomenological role in the low-energy regime, since their effects can be used to experimentally probe compositeness [14]. In order to simplify the analysis, however, we will integrate them out and neglect all their leading effects. Therefore, we may write the EFT of the Higgs as

$$\mathscr{L}_{\text{Higgs}} = f^4 \mathscr{F} \bigg(\frac{H}{f}, \frac{\partial_\mu}{f} \bigg), \tag{3.6}$$

where additional NGB other than the Higgs have also been ignored*. The Lagrangian defined in Eq. (3.6) can be expanded in the low-energy regime as a sum of renormalizable and non-renormalizable operators. Since it is known in advance that the Higgs is a complex doublet by construction, the Lagrangian in Eq. (3.6) is given by

$$\mathscr{L}_{\text{Higgs}} \longrightarrow \mathscr{L}^{(\leq 4)}_{\text{Higgs}} + \mathscr{L}^{(6)}_{\text{Higgs}} + \cdots , \tag{3.7}$$

*Note that, despite having integrated $\tilde{\Phi}$ out, the cut off $4\pi f$ was maintained. This is valid in the case considered, for which $g_* = 4\pi$. Otherwise, if $g_* < 4\pi$, one would have to substitute f by $M_{\tilde{\Phi}}$, the mass of the lightest resonance [56].

with

$$\mathscr{L}^{(\leq 4)}_{\text{Higgs}} = |\partial_\mu H|^2 + c_2 f^2 |H|^2 + c_4 |H|^4, \tag{3.8a}$$

$$\mathscr{L}^{(6)}_{\text{Higgs}} = \frac{c_H}{2f^2}\left(\partial_\mu |H|^2\right)^2 + \frac{c_T}{2f^2}\left(H^\dagger \overleftrightarrow{\partial}_\mu H\right)^2 - \frac{c_6}{f^2}|H|^6 + \frac{c_\Box}{2f^2}|\Box H|^2, \tag{3.8b}$$

where the couplings c's are dimensionless. The Lagrangian in Eq. (3.8b) contains all independent dimension 6 operators involving the Higgs and derivatives [56, 57].

The couplings in Eq. (3.8) contains all leading information of the EFT of the Higgs as a pNGB, and can therefore be used to constrain the low-energy model. In particular, much about them can be said by making use of the appropriate selection rules of the theory, which at low-energy are given by the group structure of $\mathcal{H}$ and $G/\mathcal{H}$. For instance, the Higgs, as a NGB, transforms non-linearly under the coset group $G/\mathcal{H}$ and so possesses a shift symmetry (see section 3.2), hence the coefficients c_2, c_4 and c_6 are zero at tree-level and at all orders in perturbation theory if G is exact. Solely based on the shift symmetry, we cannot say much about the coefficients c_H, c_T and $c_\Box$ since they depend on Higgs derivatives and are thus invariant under the shift symmetry. Whether c_H, c_T and $c_\Box$ are zero or not will depend on other factors, for example on the particular selection rules of the unbroken group $\mathcal{H}$, which are yet unspecified. Their respective operators, though, have very distinct phenomenological interpretations, as it will now become clear.

Custodial Symmetry

To study the phenomenology associated with the effective operators in Eq. (3.8b), the interactions with the SM must be first turned on, which implies that the Lagrangian must be complemented with additional operators that include couplings of the Higgs to fermions and gauge bosons. Of particular interest to our present discussion are the interactions between the Higgs and the gauge bosons, which are given by

$$\mathscr{L}^{(6)}_{\text{Higgs-gauge}} = \frac{c_H}{2f^2}\left(\partial_\mu |H|^2\right)^2 + \frac{c_T}{2f^2}\left(H^\dagger \overleftrightarrow{D}_\mu H\right)^2 - \frac{c_6}{f^2}|H|^6 + \frac{c_\Box}{2f^2}\left|D^2 H\right|^2 + \cdots, \tag{3.9}$$

where

$$D_\mu H = \left(\partial_\mu - i g W^a_\mu T^a - g' \frac{1}{2} B_\mu\right) H \tag{3.10}$$

is the appropriate covariant derivative of the Higgs field and the dots denote other dimension 6 operators that involve the Higgs and the gauge bosons*.

*Other operators that are not considered above are relevant to a more precise discussion of the Higgs

Of the operators in Eq. (3.9), the most relevant one to our discussion is

$$O_T = \frac{c_T}{2f^2}\left(H^\dagger \overleftrightarrow{D}_\mu H\right)^2, \tag{3.11}$$

because it gives a tree-level contribution to the masses of gauge bosons. More explicitly, considering that the Higgs acquires a vacuum expectation value (vev)* V, the operator O_T reads

$$O_T\Big|_{H=\langle H\rangle} = -\frac{1}{2}\left(\frac{g^2V^4}{4f^2\cos^2\theta_w}\right)Z_\mu Z_\mu. \tag{3.12}$$

The importance of the equation above comes from the fact that Electroweak precision measurements (EWPM) strongly constrain deviations from the EW sector [1, 58, 59], including the discrepancies from the values of the gauge bosons masses. From a more practical view, such deviations can be parametrised by the ρ-parameter [1, 3] that is defined by

$$\rho \equiv \frac{M_W^2}{M_Z^2\cos^2\theta_w}, \tag{3.13}$$

which is clearly equal to 1 at tree-level in the SM. This parameter has a central role in EWPM, because its deviations from the tree-level value are of order [3]

$$\Delta\rho = \rho_{\text{measured}} - 1 = \mathcal{O}(10^{-3}). \tag{3.14}$$

If an operator like O_T is allowed in the Lagrangian, $\Delta\rho$ would be approximately given by

$$\Delta\rho \simeq c_T\frac{V^4}{f^2v^2} \Rightarrow c_T\frac{V^4}{f^2v^2} \lesssim 10^{-3}. \tag{3.15}$$

For example, if $c_T \sim \mathcal{O}(1)$ and $V \sim v$, we would obtain a severe bound on f:

$$f^2 \gtrsim v^2 \cdot 10^3.$$

With such bound, any interesting emergent phenomena due to the compositness would be washed out and also an enormous FT would be created.

phenomenology [57]. However, the purpose here is just to give a motivation of the types of models that will be studied, for which the operators shown in Eq. (3.9) are sufficient.

*Note that it is not assumed that $V = v$, with v the EW vev of the SM. That the vev of the Higgs is not exactly the same as the one of the SM is to be expected, since the minimisation of the potential of the Lagrangian (3.6) is far less trivial. Nonetheless, it is trivial to see that if $f \gg v$, then $V \simeq v$.

The lesson to be taken from the above discussion is that a custodial symmetry must be present, i.e. a symmetry that protects ρ from large quantum corrections. In the SM there is indeed such a symmetry: the Higgs potential in the unbroken EW phase is invariant under a global $SO(4)_c$ group, where the real components of the Higgs doublet transform in the fundamental of $SO(4)_c$ and the usual $SU(2)_L$ gauge group is a subgroup of it. After EWSB, $SO(4)_c$ breaks down to $SU(2)_c$, also known as the *custodial symmetry group*, which is responsible for protecting ρ (Appendix B is referred to a more detailed discussion of custodial symmetry in the SM). This residual symmetry is only approximate in the SM, because not all interactions present in the SM respect it (e.g. the Yukawa terms). Notwithstanding, even the greatest source of custodial violation, the top Yukawa, only gives a correction of order 10^{-3} to $\Delta\rho$ [1, 3].

From the discussion above we see clearly that an issue with CHM comes from Eq. (3.6), that in general allows for operators that give large corrections to the ρ-parameter, or equivalently, that put strong bounds on f. In order to avoid such scenarios, one needs to be sure that not only $G_{\text{EW}} \subset \mathcal{H}$, but also $SO(4)_c \subseteq \mathcal{H}$. Consequently, the selection rules of the custodial group forbid O_T at tree-level, so that the latter can be only generated radiatively and is therefore suppressed by loop factors, allowing the model to be compatible with EW data*. In what follows, it will always be assumed that $SO(4)_c \subseteq \mathcal{H}$ [1, 14, 56, 60].

At the level of the Lagrangian in Eq. (3.9), custodial symmetry does not forbid neither c_H nor $c_\Box$. We will see in detail that c_H will generally have a non-zero value, while the coefficient $c_\Box$ is not related to selection rules of $\mathcal{H}$ and can only be generated at tree-level by integrating out a massive scalar particle with suitable quantum numbers [56]. The effects of such operator does not affect our analysis and is henceforth neglected.

Yukawa Sector

Up to this point only the properties of gauge bosons were discussed. In particular, as stressed, some of the gauge interactions will explicitly break the symmetry group G, and will thus allow for a mass term to be generated at loop level. In section 3.3.4 these 1-loop effects will be calculated and it will be from this point obvious that the interactions with

*The "T" in O_T refers to the Peskin-Takeuchi parameter T. Along with the other two parameters, S and U, it quantifies the amount of new physics on the EW sector when the new physics is considerably heavy. The T parameter measures specifically the violation of custodial symmetry [58].

gauge bosons are insufficient to trigger spontaneous EWSB, because loops from gauge bosons give only positive contributions to the Higgs mass. For this reason, fermions play a crucial role in the model as they will guarantee that EWSB occurs. More precisely, the leading contribution from the top quark guarantees that the loop induced Higgs mass parameter is negative. Whence, a more precise treatment of the fermion sector is of fundamental importance. This, however, will not be so straightforward and the reason is the following. Take for example the proton and the neutron in QCD, which a doublet of the isospin group [24], whose interactions are mediated by the pions. While the properties of the mediators (pions) are determined exclusively by the group, the fermions (proton and neutron) must be embedded in a representation of the same group in order to determine the appropriate interactions, i.e. the couplings of fermions with the mediators are representation-dependent. The discussion of this topic is for this reason much more subtle, and so it is left to the section 3.3.

3.2 Necessary group theory

In this section some of the technical tools from the group theory needed to construct CHM are reviewed. Also, some of the properties of $SO(N)$ groups, its breaking pattern and, in particular, some representations of $SO(5)$ and $SO(4)$ are discussed.

3.2.1 Callan-Coleman-Wess-Zumino construction

Unlike in the elementary Higgs theory, where the Higgs has a well determined linear transformation under G_{EW}, it is way more difficult to build a Lagrangian for the Higgs field in CHM, since its transformation rule under G is non-linear. As a NGB, the Higgs transforms non-linearly according to the representation of the coset group, $G/\mathcal{H}$. Nevertheless, due to spontaneous symmetry breaking (SSB) pattern, one can still determine the correct structure of the theory of the NBG. Stated in another way, thanks to SSB, we can determine the correct selection rules for the low-energy effective theory. This procedure, known as Callan-Coleman-Wess-Zumino (CCWZ) construction [21], is essential to the construction of CHM.

Let us begin with some definitions. Consider a theory describing some scalar field*

*These fields are assumed to be scalar, but not necessarily elementary. For instance, it could be, as

Φ. Suppose further that this theory is invariant under a symmetry group G, for which Φ transforms according to some representation, namely

$$\Phi \to \Phi' = f_g(\Phi), \quad \forall g \in G. \tag{3.16}$$

Here, $f_g(\cdot)$ is the function that parametrizes the transformation under G, which we assume to be linear*. It should be clear that the notation used in Eq. (3.16) is very general, as it includes not only the usual vector transformation $\Phi \to g\Phi$, but also tensor transformations of different ranks.

If the field Φ acquires a non-vanishing vev, then the symmetry group G is spontaneously broken to a subgroup $\mathcal{H}$. This is equivalent to saying that the vacuum is invariant only by the action of $\mathcal{H}$

$$\langle\Phi\rangle = f_h(\langle\Phi\rangle), \quad \forall h \in \mathcal{H}. \tag{3.17}$$

Therefore, for any infinitesimal parameter ϵ one obtain,

$$\langle\Phi\rangle = f_{e+i\epsilon T}(\langle\Phi\rangle) \simeq f_e(\langle\Phi\rangle) + \epsilon\left[\frac{\partial}{\partial\epsilon} f_{e+i\epsilon T}(\langle\Phi\rangle)\right]_{\epsilon=0} + \mathcal{O}\left(\epsilon^2\right),$$

where $e \in G$ is the identity element and $T \in \mathfrak{h}$, with $\mathfrak{h}$ being the sub-algebra associated with $\mathcal{H}$. So one concludes that

$$\left[\frac{\partial}{\partial\epsilon} f_{e+i\epsilon T}(\langle\Phi\rangle)\right]_{\epsilon=0} = 0, \quad T \in \mathfrak{h}. \tag{3.18}$$

In the same way, the elements outside $\mathcal{H}$ do not leave the vacuum invariant. Given any element of $G\backslash\mathcal{H}$ one obtains

$$\left[\frac{\partial}{\partial\epsilon} f_{e+i\epsilon \hat{T}}(\langle\Phi\rangle)\right]_{\epsilon=0} \neq 0, \quad \hat{T} \in \hat{\mathfrak{g}}, \tag{3.19}$$

where $\hat{\mathfrak{g}} = \mathfrak{g}\backslash\mathfrak{h}$ is the subset of the full algebra $\mathfrak{g}$ that does not leave $\langle\Phi\rangle$ invariant. Due to the fact that $\mathcal{H}$ is a subgroup, the algebra commutation relations may be written as

in QCD, a fermionic pair $\bar{\psi}\psi$.

*If Φ_1 and Φ_2 both follow the transformation rule in Eq. (3.16) and a and b are numbers, then

$$f_g(a\Phi_1 + b\Phi_2) = af_g(\Phi_1) + bf_g(\Phi_2).$$

follows [61, 62]

$$[\mathfrak{h},\mathfrak{h}]_- \subseteq \mathfrak{h}, \quad [\hat{\mathfrak{g}},\mathfrak{h}]_- \subseteq \hat{\mathfrak{g}}, \quad [\hat{\mathfrak{g}},\hat{\mathfrak{g}}]_- \subseteq \mathfrak{g}. \tag{3.20}$$

The development of the theory of SSB relies on the ensuing equation,

$$\Phi(x) = f_U\Big(\tilde{\Phi}(x)\Big), \tag{3.21}$$

where $U = U[\Pi(x)]$ is an element of G that contains solely the information of the NGB, while $\tilde{\Phi}(x)$ contains the other (radial) dof of the field Φ. To prove Eq. (3.21), one needs first to define a bilinear function $(\cdot,\cdot)$ which is G-invariant, i.e. for any fields Φ_1 and Φ_2,

$$(\Phi_1'(x), \Phi_2'(x)) = (f_g(\Phi_1(x)), f_g(\Phi_2(x))) = (\Phi_1(x), \Phi_2(x)), \quad \forall g \in G. \tag{3.22}$$

Consider now the following quantity,

$$S_x(g) \equiv (\Phi(x), f_g(\langle\Phi\rangle)). \tag{3.23}$$

If the group G is assumed to be compact, the quantity above can be minimised [21]. Taking the variation of S with respect to g one obtains

$$\delta S_x(g) = (\Phi(x), \delta f_g(\langle\Phi\rangle)) = (\Phi(x), f_{g+\delta g}(\langle\Phi\rangle) - f_g(\langle\Phi\rangle)). \tag{3.24}$$

The variation of an element g is given by

$$g + \delta g = g\Big(e + i\epsilon_A T^A\Big),$$

where the index A runs through all generators of the group and ϵ_A are infinitesimal parameters. Thus

$$\delta S_x(g) = \epsilon_A\left(\Phi(x), \left[\frac{\partial}{\partial\epsilon_A} f_{g\left(e+i\epsilon_{A'}T^{A'}\right)}(\langle\Phi\rangle)\right]_{\epsilon=0}\right). \tag{3.25}$$

From Eqs. (3.18) and (3.19) it follows that only the action of the unbroken generators is relevant to the above equation. One may rewrite Eq. (3.25) as

$$\delta S_x(g) = \epsilon_{\hat{a}}\left(\Phi(x), \left[\frac{\partial}{\partial\epsilon_{\hat{a}}} f_{g\left(e+i\epsilon_{\hat{b}}\hat{T}^{\hat{b}}\right)}(\langle\Phi\rangle)\right]_{\epsilon=0}\right),$$

where the indices $\hat{a}$ and $\hat{b}$ run only through the broken generators. Now, since the group G is compact, there is an element $U \in G$ such that

$$\delta S_x(U) = 0, \tag{3.26}$$

therefore

$$\left(\Phi(x), \left[\frac{\partial}{\partial \epsilon_{\hat{a}}} f_{U\left(e+i\epsilon_{\hat{b}}\hat{T}^{\hat{b}}\right)}(\langle\Phi\rangle)\right]_{\epsilon=0}\right) = 0. \tag{3.27}$$

Using now the invariance of the bilinear $(\cdot,\cdot)$ under G,

$$\left(f_{U^{-1}}(\Phi(x)), \left[\frac{\partial}{\partial \epsilon_{\hat{a}}} f_{e+i\epsilon_{\hat{b}}\hat{T}^{\hat{b}}}(\langle\Phi\rangle)\right]_{\epsilon=0}\right) = 0. \tag{3.28}$$

Eq. (3.28) shows us that these quantities are orthogonal with respect to the bilinear function.. According to Eq. (3.19), the quantity

$$\left[\frac{\partial}{\partial \epsilon_{\hat{a}}} f_{e+i\epsilon_{\hat{b}}\hat{T}^{\hat{b}}}(\langle\Phi\rangle)\right]_{\epsilon=0}$$

is intrinsically related to the broken dof, i.e. the NGB themselves, hence Eq. (3.28) means that $f_{U^{-1}}(\Phi(x))$ contains no information at all regarding the NGB. Moreover, note that by the definition of $S_x(g)$ in Eq. (3.23) together with Eq. (3.18), one has

$$S_x(g) = S_x(g \cdot h), \quad \forall h \in \mathcal{H}. \tag{3.29}$$

As a consequence, the minimum point U is given up to an element of $\mathcal{H}$. This defines the (right) coset $G/\mathcal{H}$ and allows us to write U as

$$U[\Pi(x)] = e^{\frac{i\sqrt{2}}{f}\hat{T}^{\hat{a}}\Pi^{\hat{a}}(x)}, \tag{3.30}$$

where $\Pi^{\hat{a}}(x)$ are the NGB and we have already inserted the scale f in order to canonically normalise them. In this way, Eq. (3.21) is proved [21, 13].

The transformation rule of the NGB can be deduced from Eq. (3.21). Performing a transformation on Φ one obtains

$$\Phi' = f_g\Big(f_U\big(\tilde{\Phi}\big)\Big) = f_{U'}\big(\tilde{\Phi}'\big), \tag{3.31}$$

where $U' = U[\Pi']$. As $\tilde{\Phi}$ contains no information on the NGB, it can only transform under elements of $\mathcal{H}$, hence

$$\tilde{\Phi} \to \tilde{\Phi}' = f_h\Big(\tilde{\Phi}\Big), \tag{3.32}$$

where, because of Eq. (3.21), $h = h(\Pi, g)$. Therefore, it follows that

$$f_{g\cdot U}\Big(\tilde{\Phi}\Big) = f_{U'\cdot h}\Big(\tilde{\Phi}\Big). \tag{3.33}$$

Since the above equation holds for any $g \in G$, one arrives at

$$U[\Pi'] = g \cdot U[\Pi] \cdot h^{-1}(\Pi, g). \tag{3.34}$$

From Eqs. (3.30) and (3.34), it is obvious that the transformation rule of the NGB is highly non-linear.

As an immediate consequence, it is impossible to simply guess a Lagrangian invariant under Eq. (3.34). Nonetheless, it is still possible to determine the conditions under which the low-energy Lagrangian of Π will turn out to be invariant. Take for instance the 4-divergence of Φ together with Eq. (3.21) and the linearity of f,

$$\begin{aligned}
\partial_\mu \Phi &= \partial_\mu \left(f_U\Big(\tilde{\Phi}(x)\Big)\right) \\
&= f_U\Big(\partial_\mu \tilde{\Phi}(x)\Big) + \partial_\mu U \frac{\partial}{\partial U} f_U\Big(\tilde{\Phi}(x)\Big) \\
&= f_U\left(\partial_\mu \tilde{\Phi}(x) + f_{U^{-1}}\left(\partial_\mu U \frac{\partial}{\partial U} f_U\Big(\tilde{\Phi}(x)\Big)\right)\right).
\end{aligned}$$

Note the last term in the last line of the equation above,

$$f_{U^{-1}}\left(\partial_\mu U \frac{\partial}{\partial U} f_U\Big(\tilde{\Phi}(x)\Big)\right), \tag{3.35}$$

that, irrespective of the explicit form of the representation f, will give rise to terms proportional to

$$U^{-1}\,\partial_\mu U,$$

which, taking Eqs. (3.20) and (3.30) into consideration, can be written as

$$U^{-1}\,\partial_\mu U = i d_\mu^{\hat{a}} \hat{T}^{\hat{a}} + i e_\mu^a T^a, \tag{3.36}$$

with the symbols $d_\mu^{\hat{a}}$ and e_μ^a given by

$$d_\mu^{\hat{a}} = d_{\hat{b}}^{\hat{a}}(\Pi)\,\partial_\mu\,\Pi^{\hat{b}} \quad \text{and} \quad e_\mu^a = e_{\hat{a}}^a(\Pi)\,\partial_\mu\,\Pi^{\hat{a}}, \tag{3.37}$$

where $d_{\hat{a}\hat{b}}(\Pi)$ and $e_{a\hat{a}}(\Pi)$ are numbers*. These two symbols are the fundamental quantities to write our desired Lagrangian for Π. As such, their transformation under G are essential, and can be trivially determined from Eqs. (3.36) and (3.34),

$$\begin{aligned} i\left(d_\mu^{\hat{a}}\right)'\hat{T}^{\hat{a}} + i\left(e_\mu^a\right)'T^a &= U^{-1}[\Pi']\,\partial_\mu\,U[\Pi'] \\ &= h(\Pi,g)\cdot U^{-1}[\Pi]\cdot g^{-1}\,\partial_\mu\left(g\cdot U[\Pi]\cdot h^{-1}(\Pi,g)\right) \\ &= h\cdot U^{-1}\,\partial_\mu\,U\cdot h^{-1} + h\,\partial_\mu\,h^{-1} \\ &= \left(h\hat{T}^{\hat{a}}h^{-1}\right)d_\mu^{\hat{a}} + \left[\left(hT^a h^{-1}\right)e_\mu^a - (\partial_\mu\,h)h^{-1}\right]. \end{aligned}$$

Taking into account the commutation relations (3.20), one obtains

$$h\hat{T}^{\hat{a}}h^{-1} = \mathscr{D}_{\hat{b}}^{\hat{a}}(h)\hat{T}^{\hat{b}}, \tag{3.38a}$$

$$hT^a h^{-1} = \mathscr{E}_b^a(h)T^b, \tag{3.38b}$$

$$(\partial_\mu\,h)h^{-1} = i\mathscr{H}_{a\hat{a}}(h)T^a\,\partial_\mu\,\Pi^{\hat{a}}, \tag{3.38c}$$

where $\mathscr{D}$, $\mathscr{E}$ and $\mathscr{H}$ are all unitary representations of the group. In particular, in the last relation above, the $\partial_\mu\,\Pi^{\hat{a}}$ dependency appears because h depends on x only through Π. With Eq. (3.38), the transformations of $d_\mu^{\hat{a}}$ and e_μ^a are given by,

$$\left(d_\mu^{\hat{a}}\right)' = \mathscr{D}_{\hat{b}}^{\hat{a}}(h)d_\mu^{\hat{b}}, \tag{3.39a}$$

$$\left(e_\mu^a\right)' = \mathscr{E}_b^a(h)e_\mu^b - \mathscr{H}_{a\hat{a}}(h)\,\partial_\mu\,\Pi^{\hat{a}}. \tag{3.39b}$$

Having discussed the transformation of the NGB, it still remains to understand how $\tilde{\Phi}$ enters in the low-energy theory. On the one hand, the group structure of the NGB does not depend on the function f in Eq. (3.16), but only on the representation of the coset. On the other hand, $\tilde{\Phi}$ transforms according to Eq. (3.32), posing much more difficulty in building a Lagrangian for the radial resonances. The main problem comes

*a, b, $\cdots$ denote indices for generators of $\mathfrak{h}$, while $\hat{a}$, $\hat{b}$, $\cdots$ for the ones in $\hat{\mathfrak{g}}$.

with the derivative term, since in Eq. (3.32) the group element h depends on x through Π. Whence, ordinary derivatives applied to $\tilde{\Phi}$ are not suited for the low-energy theory. Instead, one must define a quantity $\tilde{D}_\mu$ such that

$$\left(\tilde{D}_\mu \tilde{\Phi}\right)' = f_h\left(\tilde{D}_\mu \tilde{\Phi}\right), \tag{3.40}$$

in other words, a covariant derivative must be constructed. With this reasoning, the following Ansatz is made: the covariant derivative for $\tilde{\Phi}$ is given by

$$\tilde{D}_\mu \tilde{\Phi} = \partial_\mu \tilde{\Phi} + \left[\frac{\partial}{\partial \epsilon} f_{e+i\epsilon T^a}\left(\tilde{\Phi}\right)\right]_{\epsilon=0} e^a_\mu(\Pi). \tag{3.41}$$

It is now proven that Eq. (3.41) indeed respects Eq. (3.40),

$$\begin{aligned}
\left(\tilde{D}_\mu \tilde{\Phi}\right)' &= \partial_\mu \left[f_h\left(\tilde{\Phi}\right)\right] + \left[\frac{\partial}{\partial \epsilon} f_{e+i\epsilon T^a}\left(f_h\left(\tilde{\Phi}\right)\right)\right]_{\epsilon=0} e^a_\mu(\Pi') \\
&= f_h\left(\partial_\mu \tilde{\Phi}\right) + \left(\partial_\mu h \frac{\partial}{\partial h} f_h\left(\tilde{\Phi}\right)\right) + \\
&\quad + \left[\frac{\partial}{\partial \epsilon} f_{e+i\epsilon T^a}\left(f_h\left(\tilde{\Phi}\right)\right)\right]_{\epsilon=0} \left[\mathscr{E}^a_b(h) e^b_\mu - \mathscr{H}_{a\hat{a}}(h)\, \partial_\mu \Pi^{\hat{a}}\right] \\
&= f_h\left(\partial_\mu \tilde{\Phi}\right) + f_h\left(\left[\frac{\partial}{\partial \epsilon} f_{e+i\epsilon h^{-1} T^a h}\left(\tilde{\Phi}\right)\right]\right)_{\epsilon=0} \mathscr{E}^a_b(h) e^b_\mu + \\
&\quad + \left(\partial_\mu h \frac{\partial}{\partial h} f_h\left(\tilde{\Phi}\right)\right) - \left[\frac{\partial}{\partial \epsilon} f_{e+i\epsilon T^a}\left(f_h\left(\tilde{\Phi}\right)\right)\right]_{\epsilon=0} \mathscr{H}_{a\hat{a}}(h)\, \partial_\mu \Pi^{\hat{a}}.
\end{aligned} \tag{3.42}$$

Note that, since the quantity

$$\left[\frac{\partial}{\partial \epsilon} f_{e+i\epsilon h^{-1} T^a h}\left(\tilde{\Phi}\right)\right]_{\epsilon=0}$$

is linear on $h^{-1} T^a h$, which by Eq. (3.38b) is given by

$$h^{-1} T^a h = \left(\mathscr{E}^{-1}(h)\right)^a_b T^b,$$

one obtains

$$\left[\frac{\partial}{\partial \epsilon} f_{e+i\epsilon h^{-1} T^a h}\left(\tilde{\Phi}\right)\right]_{\epsilon=0} \mathscr{E}^a_b(h) e^b_\mu = \left[\frac{\partial}{\partial \epsilon} f_{e+i\epsilon T^b}\left(\tilde{\Phi}\right)\right]_{\epsilon=0} e^b_\mu. \tag{3.43}$$

Hence, Eq. (3.42) becomes

$$\left(\tilde{D}_\mu \tilde{\Phi}\right)' = f_h\left(\tilde{D}_\mu \tilde{\Phi}\right) + \left(\partial_\mu h \frac{\partial}{\partial h} f_h\left(\tilde{\Phi}\right)\right) - \left[\frac{\partial}{\partial \epsilon} f_{e+i\epsilon T^a}\left(f_h\left(\tilde{\Phi}\right)\right)\right]_{\epsilon=0} \mathscr{H}_{a\hat{a}}(h)\, \partial_\mu \Pi^{\hat{a}}. \quad (3.44)$$

From Eq. (3.38c), it follows that

$$(\partial_\mu h) = i\mathscr{H}_{a\hat{a}}(h) T^a \left(\partial_\mu \Pi^{\hat{a}}\right) h,$$

and together with

$$i\left(T^a h \frac{\partial}{\partial h}\right) f_h\left(\tilde{\Phi}\right) = \left[\frac{\partial}{\partial \epsilon} f_{h+i\epsilon T^a h}\left(\tilde{\Phi}\right)\right]_{\epsilon=0}, \quad (3.45)$$

one concludes that the last two terms on the right-hand side of Eq. (3.44) cancel. This completes the proof that Eq. (3.41) is indeed the desired covariant derivative.

It is now possible to comprehend how to build an adequate low-energy Lagrangian. The low-energy dof, the pions and the radial modes, can be included in the effective Lagrangian only through the quantities $d_\mu^{\hat{a}}$, $\tilde{\Phi}$ and $\tilde{D}_\mu \tilde{\Phi}$, because they have well defined transformations properties under G that are parametrise solely by the elements of low-energy group $\mathcal{H}$. Stated in another way, the G-invariant effective Lagrangian of pions and $\tilde{\Phi}$ can only be written as $\mathcal{H}$-invariant combinations of $d_\mu^{\hat{a}}$, $\tilde{\Phi}$ and $\tilde{D}_\mu \tilde{\Phi}$. This procedure, known as *Callan-Coleman-Wess-Zumino (CCWZ) construction*, allows to write down effective theories that automatically respect the symmetries of the UV physics. Using Eqs. (1.1) and (1.2), the Lagrangian for the corresponding EFT is given by

$$\mathscr{L}_{\text{CCWZ}} = f^4 \mathscr{F}\left(\frac{d_\mu^{\hat{a}}}{f}, \frac{\tilde{\Phi}}{f^{\dim \tilde{\Phi}}}, \frac{\tilde{D}_\mu \tilde{\Phi}}{f^{\dim \tilde{\Phi}+1}}\right), \quad (3.46)$$

which leaves explicit the fact that we are taking the selection rules coming from the SSB into consideration. In particular, if Eq. (3.46) is expanded at the low-energy regime, the first operator involving only the NGB is

$$\mathscr{L}_{\text{CCWZ}} \longrightarrow \frac{f^2}{4} d_\mu^{\hat{a}} d_\mu^{\hat{a}} + \cdots, \quad (3.47)$$

where the coefficient is fixed, because, as it will be seen in the next section, it plays the role of the kinetic term of the NGB, so it must be correctly normalised. Also in next section, Eq. (3.47) will be used to write down the Lagrangian for the MCHM.

3.2.2 $SO(N)$ and its representations

The CCWZ construction derived in the previous section is valid for any compact Lie group of our interest. Still, in order to build a concrete model, one needs to specify G and $\mathcal{H}$. Throughout this section, models with $G = SO(N)$ and $\mathcal{H} = SO(N-1)$ are going to be considered, some of their properties and the relevant quantities to write the correct effective Lagrangian will be derived and calculated.

We begin with the definition of $SO(N)$. The elements of $SO(N)$ by definition leave the N-dimensional euclidian inner product invariant. The matrix representation of this definition is named *fundamental representation*; it consists of $N \times N$, orthogonal, unit determinant matrices. The fundamental representation in some cases can also be understood as the lowest dimensional, non-trivial representation of the group [37, 63]. Higher dimensional representations, i.e. *tensor representations*, are build from the fundamental one. A tensor T of rank n transforms under $SO(N)$ as the tensor product of n fundamentals, namely

$$T \sim \Box \otimes \Box \otimes \cdots \otimes \Box, \tag{3.48}$$

where $\Box$ denotes the fundamental representation. Using the techniques of Young Tableuax one can decompose such tensor representations into smaller, irreducible representations, also called invariant sub-spaces of the given representation [37, 11, 64]. In addition to the fundamental and tensor representations, one can also define the *spinorial representation*. Such representation are motivated from the fact that for $N > 2$ the $SO(N)$ groups are not simply connected. The spinorial representation is defined precisely as the one that is mapped to a simply connected group, also named $\mathrm{Spin}(N)$, which is related to the original $SO(N)$ by a double covering. In more practical terms, the spinorial representation is constructed out of the fundamental one trough the introduction of the gamma matrices that respect the Clifford algebra [64, 65]. The distinct representations of $SO(N)$ will be useful in what follows, in particular in the discussion of the fermion sector.

These concepts, though important in our discussion, are mere definitions. To discuss the actual physics of the composite Higgs, one needs to specify the breaking pattern of the group G, or in other words, one has explicitly chose G and $\mathcal{H}$.. Consider for example the breakin pattern $SO(N) \to SO(N-1)$. The number of NGB associated with it is simply $N-1$, since $SO(N)$ has $\frac{N(N-1)}{2}$ generators. The most interesting point of this breaking pattern is that, with the appropriate choice of N, one is permitted to choose the number of NGB desired. For instance, one just needs to chose $N = 5$ to build a complex doublet with 4 NGB. This is not possible with other groups, for example $SU(N)$, for which one cannot choose the number of NGB so freely. Another interesting feature of this SSB pattern regards the choice of the broken generators. It is quite difficult, if not impossible, to write down explicitly the broken generators. Fortunately, for $SO(N) \to SO(N-1)$ this choice is very simple

$$i\Big(\hat{T}^{\hat{a}}_{\text{fund}}\Big)_{ij} = \frac{1}{\sqrt{2}}\Big[\delta^{\hat{a}}_i \delta^{N-1}_j - \delta^{\hat{a}}_j \delta^{N-1}_i\Big], \quad i,j = 1,\cdots N, \quad \hat{a} = 1,\cdots N-1. \tag{3.49}$$

One may always choose generators in Eq. (3.49) as the broken ones by performing an appropriate rotation on the vacuum $\langle\Phi\rangle$ (the vacua states may not be invariant under general group transformation, but they are all physically equivalent). In Eq. (3.49) it is left explicit that the generators from the fundamental representation, that are $N \times N$, antisymmetric matrices, were used. The generators of the spinorial representation could also be used, but their general expression for $SO(N)$ is much more complex [64, 65].

In order to compute the symbols $d^{\hat{a}}_\mu$ and e^a_μ, the expression for the matrix $U[\Pi]$ is required. Note that the Goldstone matrix can only be explicitly written for a given representation of the generators. The symbols $d^{\hat{a}}_\mu$ and e^a_μ, however, will not depend on the explicit representation of Φ, since it only the representation of the coset is relevant. The Goldstone matrix in the fundamental representation is given by Eq. (3.30),

$$\begin{aligned} U_{\text{fund}}[\Pi] &= \exp\left(\frac{i\sqrt{2}}{f}\Pi^{\hat{a}}\hat{T}^{\hat{a}}_{\text{fund}}\right) \\ &= 1 + \frac{i\sqrt{2}}{f}\Pi^{\hat{a}}\hat{T}^{\hat{a}}_{\text{fund}} + \frac{1}{2}\left[\frac{i\sqrt{2}}{f}\Pi^{\hat{a}}\hat{T}^{\hat{a}}_{\text{fund}}\right]^2 + \frac{1}{6}\left[\frac{i\sqrt{2}}{f}\Pi^{\hat{a}}\hat{T}^{\hat{a}}_{\text{fund}}\right]^3 + \cdots . \end{aligned}$$

With the expression for the broken generators in Eq. (3.49) the Goldstone matrix can be further simplified to

$$U_{\text{fund}}[\Pi] = \begin{pmatrix} \mathbb{1} - \left(1 - \cos\frac{\Pi}{f}\right)\frac{\vec{\Pi}\vec{\Pi}^T}{\Pi^2} & \frac{\vec{\Pi}}{\Pi}\sin\frac{\Pi}{f} \\ -\frac{\vec{\Pi}^T}{\Pi}\sin\frac{\Pi}{f} & \cos\frac{\Pi}{f} \end{pmatrix}, \tag{3.50}$$

where $\Pi = \sqrt{\sum_{\hat{a}}(\Pi^{\hat{a}})^2}$ and $\vec{\Pi} = (\Pi^{\hat{a}})$ is a column vector. Therefore,

$$U^{-1}_{\text{fund}}\,\partial_\mu\, U_{\text{fund}} = i d^{\hat{a}}_\mu \hat{T}^{\hat{a}}_{\text{fund}} + i e^a_\mu T^a_{\text{fund}}, \tag{3.51}$$

where $d^{\hat{a}}_\mu$ is given by

$$d^{\hat{a}}_\mu(\Pi) = \frac{\sqrt{2}\Pi^{\hat{a}}}{\Pi^2}\left(\frac{1}{\Pi}\sin\frac{\Pi}{f} - \frac{1}{f}\right)\vec{\Pi}\cdot\partial_\mu\,\vec{\Pi} - \frac{\sqrt{2}}{\Pi}\sin\frac{\Pi}{f}\,\partial_\mu\,\Pi^{\hat{a}}. \tag{3.52}$$

The e^a_μ symbol can be similarly computed, but serves to no purpose in this thesis, as we won't take resonances into account.

The $SO(4)$ group

As already stressed, the custodial symmetry group $SO(4)_c$ must be a subgroup of $\mathcal{H}$. In what follows some particular properties of this group will be studied.

First, from the commutation relations one can prove the following isomorfism [37, 65, 66]

$$SO(4) \simeq SU(2)_L \times SU(2)_R, \tag{3.53}$$

with $SU(2)_L$ and $SU(2)_R$ independent groups. As a consequence of Eq. (3.53), every representation of $SO(4)$ can be mapped into a representation of the chiral group $SU(2)_L \times SU(2)_R$. For example, a real 4-plet, $\vec{\Pi}$, transforming under the fundamental of $SO(4)$ is equivalent to a pseudo-real 2×2 matrix Σ that transforms under the (2,2) representation of the chiral group*,

$$\Sigma \to \Sigma' = g_L \Sigma g_R^\dagger, \quad (g_L, g_R) \in SU(2)_L \times SU(2)_R. \tag{3.54}$$

This is proved by defining the following pseudo-real matrix

$$\Sigma \equiv \frac{1}{\sqrt{2}}\Big[\sigma^0\Pi^4 + i\sigma^i\Pi^i\Big] = \frac{1}{\sqrt{2}}\sigma^{\hat{a}}\Pi^{\hat{a}}, \quad i = 1,2,3, \quad \hat{a} = 1,2,3,4, \tag{3.55}$$

where $\sigma^{\hat{a}} = (i\vec{\sigma}, \sigma^0)$. From the definition above, one can take the trace of $\Sigma^\dagger\Sigma$, which is

*A pseudo-real 2x2 matrix satisfies

$$\Sigma^* = \sigma_2\Sigma\sigma_2.$$

This conditions follows from the fact that $\vec{\Pi}$ is a real vector with 4 dof, whereas a general complex 2×2 matrix has 8 dof. A pseudo-real matrix instead has 4 dof.

invariant at most by the action of the chiral group, and obtain

$$\mathrm{Tr}\left[\Sigma^\dagger\Sigma\right] = \left|\vec{\Pi}\right|^2. \tag{3.56}$$

Together with the fact that $SO(4)$ and $SU(2)_L \times SU(2)_R$ have the same number of generators, this proves the equivalence of both groups, as the action of both transformation leaves the same quantity invariant.

From the physical point of view, $SO(4)$ is important, because it is the custodial symmetry group of the SM. Mathematically this is explicit from the fact that $G_{\mathrm{EW}} \subset SO(4)$, where in the isomorphism in Eq. (3.53) $SU(2)_L$ is associated with the $SU(2)_L$ of the SM and a subgroup of the $SU(2)_R$ generators corresponds to the hypercharge gauge group. Therefore, the EW group G_{EW} can always be embedded into the custodial $SO(4)_c$.

The $SO(5) \to SO(4)$ breaking pattern

The SSB breaking patterns of interest in this thesis involve only $SO(N)$ groups and, in particular, the ones that have $SO(4)$ as an unbroken subgroup. These considerations motivate us in choosing $\mathcal{H} = SO(4)_c$, such that there is no symmetry other than those already present in the SM in the low-energy regime. This implies, according to the previous discussion, that the strong group in the UV is a $SO(5)$ group. Curiously, the breaking pattern

$$SO(5) \to SO(4), \tag{3.57}$$

produces 4 NGB, which is precisely the number of dof to build the Higgs doublet. From the aforementioned reasons, the CHM based on the breaking pattern in Eq. (3.57) is the *Minimal Composite Higgs Model* (MCHM). Our next objective is to study its implementation.

Before proceeding, some additional properties of the groups $SO(5)$ and $SO(4)$ are needed, in particular regarding the explicit expressions for their generators.

One can determine the explicit expression of the generators of $SO(4)$ in the fundamental representation from the transformation rules of Σ. Under infinitesimal transformation of the $SU(2)_L$ subgroup we have,

$$\delta_L \Pi^{\hat{a}} = -i\epsilon_i \left(T_L^i\right)_{\hat{a}\hat{c}} \Pi^{\hat{c}}, \tag{3.58}$$

$$\delta_L \Sigma = -i\epsilon_i \frac{\sigma^i}{2} \Sigma, \tag{3.59}$$

where, as usual, i =1, 2 or 3 and $\hat{a}$ =1, 2, 3 or 4. The infinitesimal transformation of Σ can also be written in terms of Π by means of Eqs. (3.55) and (3.58),

$$\delta_L \Sigma = \frac{1}{\sqrt{2}} \sigma^{\hat{a}} \delta_L \Pi^{\hat{a}} = -\frac{i}{\sqrt{2}} \epsilon_i \sigma^{\hat{a}} \left(T_L^i\right)_{\hat{a}\hat{b}} \Pi^{\hat{b}}. \tag{3.60}$$

Hence,

$$\epsilon_i \frac{\sigma^i}{2} \Sigma = \frac{1}{\sqrt{2}} \epsilon_i \sigma^{\hat{a}} \left(T_L^i\right)_{\hat{a}\hat{b}} \Pi^{\hat{b}} \tag{3.61}$$

for any infinitesimal parameters ϵ_i. Substituting Eq. (3.55),

$$\frac{1}{2} \sigma^i \sigma^{\hat{b}} = \sigma^{\hat{a}} \left(T_L^i\right)_{\hat{a}\hat{b}}, \tag{3.62}$$

and, inverting the equation above, one obtains the T_L generators

$$\left(T_L^i\right)_{\hat{a}\hat{b}} = \frac{1}{4} \operatorname{Tr}\left[\sigma^{\hat{a}} \sigma^i \sigma^{\hat{b}}\right]. \tag{3.63}$$

In the same manner, by considering a transformation of $SU(2)_R$, one can compute the T_R generators,

$$\left(T_R^i\right)_{\hat{a}\hat{b}} = -\frac{1}{4} \operatorname{Tr}\left[\sigma^{\hat{b}} \sigma^i \sigma^{\hat{a}}\right]. \tag{3.64}$$

The generators in Eqs. (3.63) and (3.64), in the context of the SSB (3.57), are 5×5 matrices,

$$T_L^i \rightarrow \left(\begin{array}{cccc|c} & & & & 0 \\ & T_L^i & & & 0 \\ & & & & 0 \\ & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad T_R^i \rightarrow \left(\begin{array}{cccc|c} & & & & 0 \\ & T_R^i & & & 0 \\ & & & & 0 \\ & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{3.65}$$

In short, Eqs. (3.49), (3.63) and (3.64) are the generators of $SO(5)$ in the fundamental representation.

The same generators in the spinorial representation of $SO(5)$ are constructed out of the Pauli matrices and, for the group in question, are 4×4 matrices. It is not difficult to see that

$$\left(T_L^i\right)_S = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & 0 \end{pmatrix}, \quad \left(T_R^i\right)_S = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad \hat{T}_S^{\hat{a}} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & \sigma_{\hat{a}} \\ \sigma_{\hat{a}}^\dagger & 0 \end{pmatrix} \tag{3.66}$$

satisfy the correct commutation relations. The Goldstone matrix $U_{\text{spin}}[\Pi]$ can be analogously obtained by the exponentiation of $\hat{T}_S$. It is written more easily in terms of Σ of Eq. (3.55),

$$U_{\text{spin}}[\Pi] = \exp\left(\frac{i\sqrt{2}}{f}\Pi^{\hat{a}}\hat{T}_S^{\hat{a}}\right) \tag{3.67}$$

$$= \begin{pmatrix} \mathbb{1}\cdot\cos\frac{|H|}{\sqrt{2}f} & \frac{i\Sigma}{|H|}\sin\frac{|H|}{\sqrt{2}f} \\ \frac{i\Sigma^\dagger}{|H|}\sin\frac{|H|}{\sqrt{2}f} & \mathbb{1}\cdot\cos\frac{|H|}{\sqrt{2}f} \end{pmatrix}, \tag{3.68}$$

with $\mathbb{1}$ the 2×2 identity matrix. One last remark regarding this representation is in order. The spinorial of $SO(4)$ can also be written as the $(2,1)\oplus(1,2)$ of $SU(2)_L \times SU(2)_R$, which follows from the fact that one can always decompose a 4-component spinor into two chiral ones [37].

3.3 Minimal Composite Higgs Model

In the previous sections it was made clear that the minimal realistic model of compositness is the one considering a SSB given by $SO(5) \to SO(4)$. In this sectionits theoretical properties will be explored and how the couplings with gauge bosons change.

3.3.1 General properties

Our first task is to identify the Higgs doublet. Once again, the focus in on the pseudo-real $(2,2)$ representation of the chiral group defined by the Σ matrix in Eq. (3.55), since it is described by 4 real dof, and is given by

$$\Sigma = \frac{1}{\sqrt{2}}\sigma^{\hat{a}}\Pi^{\hat{a}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \Pi_4 + i\Pi_3 & \Pi_2 + i\Pi_1 \\ i\Pi_1 - \Pi_2 & \Pi_4 - i\Pi_3 \end{pmatrix}. \tag{3.69}$$

This matrix can be written in terms of a single doublet as follows. Define the second column vector of the equation above as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \Pi_2 + i\Pi_1 \\ \Pi_4 - i\Pi_3 \end{pmatrix}, \tag{3.70}$$

and note that the first column vector is related to H in the following way:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \Pi_4 + i\Pi_3 \\ \Pi_1 - i\Pi_2 \end{pmatrix} = i\sigma_2 H^* \equiv \tilde{H}, \tag{3.71}$$

which is none other than the charge conjugate doublet. In term of these two doublets the matrix Σ is written as

$$\Sigma = \left(\tilde{H}, H\right), \tag{3.72}$$

or in component form

$$\Sigma_{ij} = \delta_{1j}\tilde{H}_i + \delta_{2j}H_i. \tag{3.73}$$

The transformations properties of H under the chiral group are easily obtained by considering a transformation of $SU(2)_L$,

$$\Sigma_{ij} \to (g_L)_{ik}\Sigma_{kj} = \delta_{1j}(g_L)_{ik}\tilde{H}_k + \delta_{2j}(g_L)_{ik}H_k, \tag{3.74}$$

therefore, in order to respect the transformation law, H and $\tilde{H}$ transform both as a doublet of $SU(2)_L$. One remark regarding the doublets is in order. One should observe that H and $\tilde{H}$ are not distinct doublets, i.e. they carry the same dof. This is a consequence of considering a real 4-plet $\vec{\Pi}$. In the same manner, one could use a complex 4-plet $\vec{\psi}$ to build the matrix Σ. In this case Σ is now complex and possesses two distinct $SU(2)_L$ doublets, ψ_+ and ψ_-, which are related to the original components by

$$\psi_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} i\psi_1 + \psi_2 \\ \psi_4 - i\psi_3 \end{pmatrix}, \quad \psi_- = \frac{1}{\sqrt{2}} \begin{pmatrix} i\psi_1 - \psi_2 \\ \psi_4 + i\psi_3 \end{pmatrix}. \tag{3.75}$$

In short, the pseudo-real $(2, 2)$ representation is equivalent to one complex doublet of $SU(2)_L$, while the complex $(2, 2)$ representation is equivalent to two distinct doublets.

As expected, the action of $SU(2)_L$ of $SO(4)_c$ coincides with the action of the $SU(2)_L$

gauge group of the SM. The action of $SU(2)_R$, though, is not as trivial, since only one combination of its generators will be equivalent to the hypercharge generator. In order to determine this combination, one must study the action of the generators T_R on the doublet H. Consider first the action of T_R^1. This generator is given by Eq. (3.64)

$$T_R^1 = -\frac{i}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) = -\frac{1}{2}\left(\begin{array}{c|c} 0 & \sigma_2 \\ \hline \sigma_2 & 0 \end{array}\right).$$

The variation of $\vec{\Pi}$ associated with the action of this generator is thus given by then

$$\delta_R\vec{\Pi} = -i\epsilon T_R^1\vec{\Pi} = -\frac{\epsilon}{2}\begin{pmatrix} -\Pi_4 \\ +\Pi_3 \\ -\Pi_2 \\ +\Pi_1 \end{pmatrix}. \tag{3.76}$$

Using Eq. (3.70), one can connect the transformation of Π to H,

$$\delta H = \frac{1}{\sqrt{2}}\begin{pmatrix} \delta_R\Pi_2 + i\delta_R\Pi_1 \\ \delta_R\Pi_4 - i\delta_R\Pi_3 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\frac{i\epsilon}{2}\right)\begin{pmatrix} \Pi_4 + i\Pi_3 \\ i\Pi_1 - \Pi_2 \end{pmatrix} = \frac{i\epsilon}{2}\tilde{H}, \tag{3.77}$$

hence, under the transformation of T_R^1, H does not have a proper transformation rule. The same happens for a transformation with T_R^2, i.e. δH is again proportional to $\tilde{H}$. The only generator that induces an appropriate transformation is T_R^3, given by Eq. (3.64),

$$T_R^3 = -\frac{i}{2}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) = \frac{1}{2}\left(\begin{array}{c|c} \sigma_2 & 0 \\ \hline 0 & -\sigma_2 \end{array}\right).$$

The transformation of H in this case is given by

$$\delta_R \vec{\Pi} = -i\epsilon T_R^3 \vec{\Pi} = -\frac{\epsilon}{2}\begin{pmatrix} +\Pi_2 \\ -\Pi_1 \\ -\Pi_4 \\ +\Pi_3 \end{pmatrix} \Longrightarrow \delta H = \frac{1}{\sqrt{2}}\Big(-\frac{i\epsilon}{2}\Big)\begin{pmatrix} \Pi_2 + i\Pi_1 \\ \Pi_4 - i\Pi_3 \end{pmatrix} = -\frac{i\epsilon}{2} H.$$

More precisely, a transformation of T_R^3 leads us to

$$H + \delta H = \Big(1 - i\frac{\epsilon}{2}\Big) H. \tag{3.78}$$

It is now clear that the doublet H has the same transformation properties under G_{EW} as the Higgs boson, provided that we identify T_R^3 as the hypercharge operator. Hence, H has consistent transformation rules and is the physical Higgs field. The same calculations are valid for $\psi_\pm$ of the complex $(2,2)$ in Eq. (3.75), for which they have T_R^3-charge equal to $\pm\frac{1}{2}$.

Having determined the Higgs doublet and the generators of $SO(4)_c$ that corresponds to the generators of G_{EW}, the next step is to write the d symbol in terms of H. Furthermore, the ordinary derivatives in Eq. (3.52) that act on the Higgs field must be promoted to covariant derivatives since G_{EW} is gauged,

$$\partial_\mu H \to D_\mu H = \Big(\partial_\mu - igW_\mu^i \frac{\sigma^i}{2} - ig' \frac{1}{2} B_\mu\Big) H. \tag{3.79}$$

Note that this gauging explicitly breaks $SO(4)$ (see Appendix B). Using in addition that,

$$\Pi^2 = \Pi_1^2 + \Pi_2^2 + \Pi_3^2 + \Pi_4^2 = 2|H|^2, \quad \Big(\partial_\mu \vec{\Pi}\Big)^2 = 2\,\partial_\mu H^\dagger\, \partial_\mu H.$$

we can substitute every Π in Eq. (3.52) by H. As a consequence, the Lagrangian in Eq. (3.47) written in terms of H becomes

$$\frac{f^2}{4} d_\mu^{\hat{a}} d_\mu^{\hat{a}} = \frac{f^2}{2|H|^2} \sin^2 \frac{\sqrt{2}|H|}{f} |D_\mu H|^2 + \frac{f^2}{8|H|^4}\Big(\frac{2|H|^2}{f^2} - \sin^2 \frac{\sqrt{2}|H|}{f}\Big)\Big(\partial_\mu |H|^2\Big)^2, \tag{3.80}$$

where the last term still has an ordinary derivative, because $|H|^2$ is a singlet [13]. Note that we can understand Eq. (3.80) as the "kinetic term" of the Higgs, since in the limit

$f \to \infty$ one obtains

$$\frac{f^2}{4} d_\mu^{\hat{a}} d_\mu^{\hat{a}} \xrightarrow{f\to\infty} |D_\mu H|^2. \tag{3.81}$$

Moreover, it is easy to see that all higher order operators coming from the expansion of the Lagrangian in Eq. (3.47) will vanish in this limit, since they have extra powers of f in the denominator. Therefore, the limit $f \to \infty$ can be understood as the one in which the elementary Higgs theory is recovered.

The next step is to determine what changes does this Lagrangian bring to the observables of the theory. As in the SM, the kinetic term of the Higgs allows us to compute the masses of gauge bosons and their couplings to the Higgs. We proceed to calculate these quantities in the case of MCHM. For this end, the Lagrangian (3.80) is expanded around the Higgs vev in unitary gauge,

$$H \to H = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ V + h \end{pmatrix}, \tag{3.82}$$

with V the vev of the Higgs. The impact on the masses and couplings of gauge bosons come from the term with the covariant derivative in Eq. (3.80), as it is the only term that contains gauge bosons fields. Substituting Eq. (3.82) in Eq. (3.80), one arrives at

$$\begin{aligned} \frac{f^2}{2|H|^2} \sin^2 \frac{\sqrt{2}|H|}{f} |D_\mu H|^2 &= \frac{f^2}{2|H|^2} \sin^2 \frac{\sqrt{2}|H|}{f} \left[H^\dagger \left(g W_\mu^i \frac{\sigma^i}{2} + \frac{g'}{2} B_\mu \right)^2 H + \frac{1}{2} (\partial_\mu h)^2 \right] \\ &= \sin^2 \frac{V+h}{f} \left[\frac{g^2 f^2}{8} (0,1) \begin{pmatrix} \frac{g'}{g} B_\mu + W_\mu^3 & W_\mu^1 - i W_\mu^2 \\ W_\mu^1 + i W_\mu^2 & \frac{g'}{g} B_\mu - W_\mu^3 \end{pmatrix}^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \right. \\ &\quad \left. + \frac{f^2}{2(V+h)^2} (\partial_\mu h)^2 \right] \\ &= \sin^2 \frac{V+h}{f} \left[\frac{g^2 f^2}{4} \left(W_\mu^+ W_\mu^- + \frac{Z_\mu Z_\mu}{2\cos^2\theta_w} \right) + \frac{f^2}{V^2} \left(\frac{\partial_\mu h}{1 + \frac{h}{V}} \right)^2 \right]. \end{aligned}$$

The sine is expanded as

$$\sin^2 \frac{V+h}{f} = \sin^2 \frac{V}{f} + \sin \frac{2V}{f} \frac{h}{f} + \cos \frac{2V}{f} \left(\frac{h}{f} \right)^2 - \frac{2}{3} \sin \frac{2V}{f} \left(\frac{h}{f} \right)^3 + \cdots, \tag{3.83}$$

where the first term in the equation above gives the mass term to the gauge bosons, which are given by

$$M_W^2 = M_Z^2 \cos^2\theta_w = \frac{g^2 f^2}{4}\sin^2\frac{V}{f}. \tag{3.84}$$

However, since both masses are measured with high precision [1, 3], it is very inciting to write them still in terms of the EW vev v,

$$M_W^2 = M_Z^2 \cos^2\theta_w = \frac{g^2 v^2}{4}.$$

This gives us the relation between the three energy scales v, f and V:

$$v^2 = f^2 \sin^2\frac{V}{f}. \tag{3.85}$$

The equation above allows us to identify the relevant, dimensionless quantity that measures the effects of compositness in CHM,

$$\xi = \frac{v^2}{f^2}. \tag{3.86}$$

The parameter ξ defined in Eq. (3.86) is not exclusive to the MCHM, since it is used to characterise all distinct kinds of CHM and es essential for the computation of the FT.

The interaction between h and the gauge bosons in the Lagrangian in Eq. (3.80) can be wrtitten only in terms of ξ,

$$\mathscr{L}_{hWZ} = M_W^2\Big(W_\mu^+ W_\mu^- + \frac{1}{2\cos^2\theta_w} Z_\mu Z_\mu\Big)\times \tag{3.87}$$

$$\times\left(1 + \frac{2h}{v}\sqrt{1-\xi} + \frac{h^2}{v^2}(1-2\xi) - \frac{4h^3}{3v^3}\xi\sqrt{1-\xi} + \cdots\right). \tag{3.88}$$

The first three terms of the expansion in Eq. (3.88) are already present in the SM, while all other higher order couplings with h^n, $n \geq 3$, are absent in the SM. In the limit $f \to \infty$, or equivalently $\xi \to 0$, we recover the SM Lagrangian, for all new couplings are proportional to ξ and hence vanish in this limit.

3.3.2 Coupling to fermions

We now turn the discussion to the fermions. In contrast to gauge bosons, one needs to put much more effort in the model-building for fermions, notably because there is no indication of how do they transform under $SO(5)$ or $SO(4)$. In this section different models corresponding to different embedding of fermions in representations of $SO(5)$ are presented. As previously stressed, this reflects the arbitrariness of the fermion representation in the bottom-up approach we are using.

In the SM, the coupling of the Higgs with the fermions, in particular with the quarks, is given by

$$\mathscr{L}_{\text{Yuk}} = -y_t \bar{q}_L \tilde{H} t_R - y_b \bar{q}_L H b_R + \cdots\ , \tag{3.89}$$

where only the third generation and $\tilde{H} = i\sigma_2 H^*$ is displayed. In CHM one could in principle write the effective Lagrangian for the fermions just as the one in Eq. (3.88) for the Higgs coupled to the gauge bosons was written. With this naive reasoning, the EFT would be given by

$$\mathscr{L}_{\text{Naive}} = f^4 \mathscr{F}\left(\frac{H}{f}, \frac{q_L}{f^{3/2}}, \frac{t_R}{f^{3/2}}, \frac{b_R}{f^{3/2}}, \cdots\right). \tag{3.90}$$

Unfortunately, the above Lagrangian is inconsistent for many reasons. First, the fermions of the SM do not have well determined transformations under G or $\mathcal{H}$, which in turn means thatit is unclear how to build invariant operators from the fields. Second, since the fermions are representation-dependent, the representation of the Goldstone matrix is not arbitrary. It must be such to allows us to construct consistent interactions between the Higgs and the fermions.

In order to write a consistent EFT, one must reformulate the above Lagrangian. One possible way to do so is to promote all fermion fields

$$q_L \to Q_L, \quad t_R \to T_R, \quad b_R \to B_R, \tag{3.91}$$

to fields Q_L, T_R and B_R that have, by definition, a well determined, linear representation under G. Considering an EFT with the promoted fields would leave explicitly that the selection rules of G are being respected. However, this is still not adequate to describe the low-energy interactions between the Higgs and the fermions, because at this energy scale the group G is already broken, so, according to the CCWZ construction, the fields must bear a representation under $\mathcal{H}$, not G. One could avoid this issue and embed the quark fields directly into some representation of the unbroken group, but this would not be the correct path to follow, since the quarks must still be fundamental dof of nature at higher energies and must have, at that energy scale, a well determined representation under G.

Suppose we can embed a quark fields, say q_L, into some representation of G whose transformation is given by the function $f_g(\cdot)$ defined in Eq. (3.16). We now resort to the Goldstone matrix by noticing that the fields

$$\underline{Q}_L \equiv f_{U^{-1}}(Q_L) \tag{3.92}$$

transform only under $\mathcal{H}$, because the transformation of the Goldstone matrix in Eq. (3.34), compensates the undesired transformation of G and leaves only the transformation of $\mathcal{H}$. Therefore, the field in Eq. (3.92) transforms as

$$\underline{Q}_L \longrightarrow f_h\Big(\underline{Q}_L\Big), \quad \forall h \in \mathcal{H}. \tag{3.93}$$

The same is valid for $\underline{T}_R$ and $\underline{B}_R$, which are analogously defined according to Eq. (3.92). In conclusion, the correct EFT for the fermions at low-energy theory is given by

$$\mathscr{L}_{\text{fermions}} = f^4 \mathscr{F}\left(\frac{\underline{Q}_L}{f^{3/2}}, \frac{\underline{T}_R}{f^{3/2}}, \frac{\underline{B}_R}{f^{3/2}}, \cdots\right). \tag{3.94}$$

Expanding the above Lagrangian, the allowed operators are $\mathcal{H}$-invariant combinations of $\underline{Q}_L$, $\underline{T}_R$ and $\underline{B}_R$. Note that, even though Eq. (3.94) is a consistent effective Lagrangian, it is not straightforward to build invariant operators, because Q_L, T_R and B_R are not necessarily embedded into the same representation of G. So in principle one would have to rely on the explicit expression of the fields in order to write down the allowed operators at the low-energy regime, which must give us the SM Yukawa couplings in the limit $f \to \infty$. One can still obtain more information regarding the low-energy operators in the Lagrangian in Eq. (3.94), although a totally general analysis is not possible.

Consider the fundamental representation of $G = SO(N)$, denoted here by $\square^{(N)}$. Taking into account the SSB patter $SO(N) \to SO(N-1)$ and the expression of the broken generators in Eq. (3.49), the fundamental of $SO(N)$ is decomposed after SSB as

$$\square^{(N)} \longrightarrow \square^{(N-1)} \oplus \mathbf{1}^{(N-1)}, \tag{3.95}$$

where $\mathbf{1}^{(N)}$ is a singlet of $SO(N)$. Consider now the rank 2 tensor representation of

$SO(N)$, which is the product of two fundamentals:

$$\square^{(N)} \otimes \square^{(N)}. \tag{3.96}$$

After SSB it is decomposed as

$$\square^{(N)} \otimes \square^{(N)} \longrightarrow \left[\square^{(N-1)} \oplus \mathbf{1}^{(N-1)}\right] \otimes \left[\square^{(N-1)} \oplus \mathbf{1}^{(N-1)}\right] = \cdots \oplus \left[\mathbf{1}^{(N-1)} \otimes \mathbf{1}^{(N-1)}\right]. \tag{3.97}$$

Therefore, from the above equation, one is guaranteed to have at least one singlet of $SO(N-1)$ in the decomposition of $\square^{(N)} \otimes \square^{(N)}$. From Eqs. (3.95) and (3.97) we conclude that every tensor representation of $SO(N)$ contains at least one singlet of $SO(N-1)$ after SSB. This allows us to decompose the fermion fields as

$$\underline{Q}_L = \underset{\wedge}{Q}_L \oplus \cdots , \tag{3.98}$$

where the dots denote non-trivial, representation-dependent terms and $\underset{\wedge}{Q}_L$ is a singlet of $SO(N-1)$. Assuming all promoted fields in Eq. (3.91) are in some tensor representation of $SO(N)$, we may write the fermion Lagrangian at low energies as

$$\mathscr{L}_{\text{fermions}} \supset -y_t f \bar{\underset{\wedge}{Q}}_L \underset{\wedge}{T}_R - y_b f \bar{\underset{\wedge}{Q}}_L \underset{\wedge}{B}_R + h.c. + \cdots , \tag{3.99}$$

where $\underset{\wedge}{T}_R$ and $\underset{\wedge}{B}_R$ are similarly defined. The Lagrangian in Eq. (3.99) may not contain all possible terms, but it already tells us at least how to build an invariant operator in the low-energy regime. Hence, after embedding the fields in a tensor representation, all it is left is to identify the corresponding singlet components and one may readily write down the interacting Lagrangian (3.99). Unfortunately, the same cannot be done for spinorial representations, since their general structure is far more complicated and the decomposition under SSB is not as trivial as it is for the fundamental representation.

Taking all these considerations into account, the next task is to construct explicitly some of the most important representations of $SO(5)$ and $SO(4)$ which are used to build the MCHM. Before proceeding, the methodology of this analysis should be clarified. After choosing a representation of $SO(5)$, it shall be first decomposed into a representation of G_{EW} in order to verify if such representation can indeed accommodate the fermions of the SM. Second, one must explicitly embed the fermions into the promoted fields and then build invariant operators. Note that in the limit $f \to \infty$ one must in addition recover the Yukawa operators of the SM.

The first representation of $SO(5)$ considered is the fundamental one, denoted by $\Box^{(5)}$. Let us see how it decomposes into representations G_{EW}. According to Eq. (3.95) the decomposition of $SO(5)$ into $SO(4)$ is given by

$$\Box^{(5)} \to \Box^{(4)} \oplus \mathbf{1}^{(4)}, \tag{3.100}$$

with $\Box^{(4)}$ the fundamental of $SO(4)$ and $\mathbf{1}^{(4)}$ the respective singlet. As already stressed*, $\Box^{(4)} = (2,2)$ of $SU(2)_L \times SU(2)_R$, so it is further decomposed into

$$\Box^{(4)} \to \mathbf{2} \oplus \mathbf{2},$$

where $\mathbf{2}$ is the fundamental of $SU(2)_L$. With T^3_R as the generator of hypercharge, the decomposition of $\Box^{(4)}$ must come with their respective hypercharges, namely

$$\Box^{(4)} \to \mathbf{2}_{\frac{1}{2}} \oplus \mathbf{2}_{-\frac{1}{2}}. \tag{3.101}$$

At the end, the spontaneous breaking leads us to:

$$\Box^{(5)} \to \Box^{(4)} \oplus \mathbf{1}^{(4)} \to \mathbf{2}_{\frac{1}{2}} \oplus \mathbf{2}_{-\frac{1}{2}} \oplus \mathbf{1}^{(4)}_0. \tag{3.102}$$

The conclusion from the equation above is that none of the quarks can be embedded in the fundamental representation of $SO(5)$, since the wrong hypercharges were obtained. One could try some other representation, but it is easy to see that, due to the decomposition of $SO(4)$ in terms of the chiral group, no representation will have the correct hypercharges. It is possible avoid this problem by adding a $U(1)_X$ group to the group G that has a single generator, named X. If the hypercharge operator is redefined as

$$Y = T^3_R + X, \tag{3.103}$$

one may choose X-charge in a way to obtain the correct hypercharges. In particular, for the fundamental representation considered previously, one can chose the X charge as $2/3$.

*Here (2,2) denote the complex (2,2) representation of the chiral group discussed in section 3.2.2.

In this case the group decomposition in Eq. (3.102) is given by

$$\square^{(5)}_{\frac{2}{3}} \to \square^{(4)}_{\frac{2}{3}} \oplus \mathbf{1}^{(4)}_{\frac{2}{3}} \to \mathbf{2}_{\frac{7}{6}} \oplus \mathbf{2}_{\frac{1}{6}} \oplus \mathbf{1}^{(4)}_{\frac{2}{3}}, \tag{3.104}$$

where in the last decomposition Eq. (3.103) was used. Both q_L and t_R can be embedded into this representation, but another one to embed b_R needs to be built. With the same reasoning, the adequate representation for b_R is

$$\square^{(5)}_{-\frac{1}{3}} \to \square^{(4)}_{-\frac{1}{3}} \oplus \mathbf{1}^{(4)}_{-\frac{1}{3}} \to \mathbf{2}_{\frac{5}{6}} \oplus \mathbf{2}_{\frac{1}{6}} \oplus \mathbf{1}^{(4)}_{-\frac{1}{3}}. \tag{3.105}$$

In short, we embed a generation of quarks in the following representations

$$Q_L \sim \square^{(5)}_{\frac{2}{3}} \quad \text{or} \quad \square^{(5)}_{-\frac{1}{3}}, \tag{3.106a}$$

$$T_R \sim \square^{(5)}_{\frac{2}{3}}, \tag{3.106b}$$

$$B_R \sim \square^{(5)}_{-\frac{1}{3}}. \tag{3.106c}$$

A few remarks are in order. With the addition of a $U(1)_X$ gauge group, the spontaneous breaking pattern considered now is given by

$$SO(5) \times U(1)_X \longrightarrow SO(4) \times U(1)_X, \tag{3.107}$$

which brings no change to the previous discussion of the Higgs, because the new abelian group remains unbroken. Moreover, the X-charge of the Higgs is also set to zero, leaving the covariant derivative the same*.

Let us now begin the explicit embedding. The right-handed top quark t_R is the singlet of $\square^{(5)}_{-\frac{1}{3}}$, therefore one may chose an appropriate basis in which the 5-plet T_R can be written

*The introduction of a new gauged $U(1)_X$ is a subject of many interesting studies in BSM physics [67], but a throughout review of this topic is out of the scope of this thesis.

as

$$T_R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ t_R \end{pmatrix}. \tag{3.108}$$

In order to construct the top Yukawa coupling in the Lagrangian in Eq. (3.99), one needs to embed $\bar{q}_L$ in the $\Box^{(5)}_{-\frac{2}{3}}$ of $SO(5)$. Since it is a doublet of $SU(2)_L$, we set

$$\bar{q}_L = \psi_-, \tag{3.109}$$

where ψ_- denote one of the $SU(2)_L$ doublets of the (2,2) complex representation of the chiral group, which is given in Eq. (3.75).

$$\bar{Q}_L = \left(-i\bar{b}_L, -\bar{b}_L, -i\bar{t}_L, \bar{t}_L, 0\right). \tag{3.110}$$

The computation of the singlet components of $\underline{Q}_L$ and $\underline{T}_R$ is straightforward,

$$\bar{\underline{Q}}_L = \bar{Q}_L U_{\text{fund}}[\Pi] = \left(\cdots, \hat{\bar{Q}}_L\right), \tag{3.111a}$$

$$\underline{T}_R = U^{-1}_{\text{fund}}[\Pi] T_R = \begin{pmatrix} \vdots \\ \hat{T}_R \end{pmatrix}. \tag{3.111b}$$

Using the explicit expression for the Goldstone matrix in the fundamental representation, which are given in Eq. (3.50), one obtains

$$\hat{\bar{Q}}_L = \frac{1}{\Pi} \sin \frac{\Pi}{f} \bar{Q}_L \cdot \vec{\Pi} \quad \text{and} \quad \hat{T}_R = \cos \frac{\Pi}{f} t_R. \tag{3.112}$$

The operator in Eq. (3.94) is thus given by

$$\begin{aligned} y_t f \hat{\bar{Q}}_L \hat{T}_R &= \frac{y_t f}{2\Pi} \sin \frac{2\Pi}{f} \bar{Q}_L \cdot \vec{\Pi}\; t_R \\ &= \frac{y_t f}{2\sqrt{2}|H|} \sin \frac{2\sqrt{2}|H|}{f} \frac{1}{\sqrt{2}} \Big[\bar{b}_L(i\Pi_1 - \Pi_2) + \bar{t}_L(i\Pi_3 + \Pi_4)\Big] t_R \\ &= \frac{y_t f}{2\sqrt{2}|H|} \sin \frac{2\sqrt{2}|H|}{f} \bar{q}_L \tilde{H} t_R. \end{aligned}$$

From the equation above, it is obvious that this term reduces itself to the SM Yukawa coupling in the regime $f \to \infty$. To complete the third generation of quarks, one must still embed the bottom quark into $\square^{(5)}_{-\frac{1}{3}}$. This is done in the exact same way, except that now the left-handed doublet is embedded into the ψ_+ representation of Eq. (3.75). At the end, the effective Lagrangian for the third generation is*:

$$\mathscr{L}_{\text{Yuk}} = -\frac{y_t f}{2\sqrt{2}|H|} \sin \frac{2\sqrt{2}|H|}{f} \bar{q}_L \tilde{H} t_R - \frac{y_b f}{2\sqrt{2}|H|} \sin \frac{2\sqrt{2}|H|}{f} \bar{q}_L H b_R. \tag{3.113}$$

The interacting Lagrangian in Eq. (3.113) is an infinite series of operators, whose couplings will be given by powers of ξ. After EWSB, $\mathscr{L}_{\text{Yuk}}$ becomes

$$\mathscr{L}_{\text{Yuk}} \xrightarrow{\text{EWSB}} \left[\frac{y_t f}{\sqrt{2}}\sqrt{\xi(1-\xi)}\bar{t}t + \frac{y_b f}{\sqrt{2}}\sqrt{\xi(1-\xi)}\bar{b}b\right] \times \left(1 + \frac{h}{v}\frac{1-2\xi}{\sqrt{1-\xi}} - \frac{h^2}{v^2}2\xi + \cdots\right). \tag{3.114}$$

The first term in the expansion above contains the mass terms of the quarks,

$$M_t = \frac{y_t f}{\sqrt{2}}\sqrt{\xi(1-\xi)}, \quad M_b = \frac{y_b f}{\sqrt{2}}\sqrt{\xi(1-\xi)}. \tag{3.115}$$

The second term is the correction to the SM Yukawa coupling, while all other are beyond the SM. As already stressed, the coupling with the fermions depends on their group representation, therefore the dependency of the Yukawa coupling on ξ will change when the representation changes. We define for future purposes the coupling in Eq. (3.114) as

$$k_F^{(\text{fund})}(\xi) \equiv \frac{1-2\xi}{\sqrt{1-\xi}}, \tag{3.116}$$

where "fund" denotes specifically that the fermions are embedded in the fundamental of $SO(5)$.

Let us now turn to spinorial representations, taking for instance the spinorial of $SO(5)$, denoted here by $S^{(5)}$. Its group decomposition after SSB is

$$S^{(5)} \to S^{(4)}, \tag{3.117}$$

with $S^{(4)}$ the spinorial of $SO(4)$. Chosing the X-charge of $S^{(5)}$ as $1/6$ and using the fact

*The other components in Eqs. (3.111) could also be used to build an invariant operator, since they are both 4-plets of $SO(4)$. However, given that $\bar{Q}_L$ and T_R are orthogonal, such operator is redundant.

that the spinorial representation $S^{(4)}$ is equivalent to the $(2,1) \oplus (1,2)$ representation of the chiral group [64, 65], we obtain

$$S^{(5)}_{\frac{1}{6}} \to S^{(4)}_{\frac{1}{6}} \simeq (2,1)_{\frac{1}{6}} \oplus (1,2)_{\frac{1}{6}} \to \mathbf{2}_{\frac{1}{6}} \oplus \mathbf{1}_{\frac{2}{3}} \oplus \mathbf{1}_{-\frac{1}{3}}, \tag{3.118}$$

with $\mathbf{1}$ the singlet of $SU(2)_L$. Hence, one can fit an entire quark generation in $S^{(5)}$. The multiplets for the left-handed quarks, the righ-handed top and bottom are promoted to

$$Q_L = \begin{pmatrix} t_L \\ b_L \\ 0 \\ 0 \end{pmatrix}, \quad T_R = \begin{pmatrix} 0 \\ 0 \\ t_R \\ 0 \end{pmatrix}, \quad B_R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ b_R \end{pmatrix}. \tag{3.119}$$

Given the expression for the Goldstone matrix in the spinorial representation in Eq. (3.68), one can decompose this representation as

$$U^{-1}_{\text{spin}}[\Pi] Q_L = \begin{pmatrix} Q_L^{(1,2)} \\ Q_L^{(2,1)} \end{pmatrix}, \quad U^{-1}_{\text{spin}}[\Pi] T_R = \begin{pmatrix} T_R^{(1,2)} \\ T_R^{(2,1)} \end{pmatrix}, \quad U^{-1}_{\text{spin}}[\Pi] B_R = \begin{pmatrix} B_R^{(1,2)} \\ B_R^{(2,1)} \end{pmatrix}, \tag{3.120}$$

where $(2,1)$ and $(1,2)$ refer to the $SU(2)_R \times SU(2)_L$ representations. From the expressions above it is obvious that there is no $SO(4)$ singlet, as opposed to our analysis of the fundamental representation.

The only independent invariant operators which can be built from the fields in Eq. (3.120) are given by

$$\begin{aligned} \mathscr{L}_{\text{Yuk}} &= y_t f \bar{Q}_L^{(2,1)} \cdot T_R^{(2,1)} + y_b f \bar{Q}_L^{(2,1)} \cdot B_R^{(2,1)} + h.c. \\ &= \frac{f}{\sqrt{2}|H|} \sin \frac{\sqrt{2}|H|}{f} \left[y_t \bar{q}_L \tilde{H} t_R + \bar{q}_L H b_R + h.c. \right]. \end{aligned} \tag{3.121}$$

The masses are now given by

$$M_t = \frac{y_t f \sqrt{\xi}}{\sqrt{2}}, \quad M_b = \frac{y_b f \sqrt{\xi}}{\sqrt{2}}, \tag{3.122}$$

while the Yukawa coupling is

$$k_F^{(\text{spin})}(\xi) = \sqrt{1-\xi}. \tag{3.123}$$

We see that the coupling $k_F(\xi)$ and the expression for the masses change noticeably from one representation to another, although the limit $\xi \to 0$ is the same.

⋆⋆⋆

One could proceed indefinitely and consider many more different representations. For instance, the rank-2, symmetric and traceless tensor representation of $SO(5)$ has some interesting properties: it gives the same coupling strength as in Eq. (3.116), but it allows for more invariant operators in the Lagrangian. In particular, it receives contributions from possible resonances below the scale f [14, 13]. It is also possible to mix different representations, i.e. to embed some fermions into one representation and others into another representation [13]. In short, the possibilities are endless. One thing is certain, though: the more complicated an embedding is, the more model building it requires. In the discussion above, the two simplest MCHM, one with the fundamental representation and the other with the spinorial one, were written down. Any other embedding follows with the exact same analysis.

What could significantly change the discussion is another group structure, in other words, another choice of groups G and $\mathcal{H}$. The next-to minimal composite Higgs model (NMCHM), for example, considers $SO(6) \to SO(5)$; such model gives us the Higgs doublet and an extra singlet as NGB [68, 69]. The options in this direction are also infinite, since one can always consider bigger and bigger groups. With larger groups and extra particles, the model-building possibilities are wider, but at the same time they deviate more and more from the real world, as we haven't had any significant signal of new physics at the TeV scale. Other cosets have been considered and extensively researched; a review on them is given in [70].

3.3.3 Partial Compositness

In the previous section the couplings of fermions to the Higgs were discussed at the level of EFT. In this case the theory is given by the low-energy effective Lagrangian in Eq. (3.94). If one attempts to UV-complete this model, there are essentially two possibilities for the interacting Lagrangian with the fermions. The first one is the analogous of a Yukawa coupling,

$$\mathscr{L}_S = -y_t \bar{Q}_L S_t T_R - y_b \bar{Q}_L S_b B_R + h.c. \tag{3.124}$$

with S_t and S_b scalar operators. These operators have non-trivial expressions given by the fields from the new strong sector, although this does not concern us here. The real issue with Eq. (3.124) is the fact that S_t and S_b have dimension +1, which means that the operators*

$$M_t^2 S_t^2 + M_b^2 S_b^2 \tag{3.125}$$

may be allowed (depending on its transformation rule under G). The presence of these operators could then reintroduce the Hierarchy Problem.

In order to avoid reintroducing new hierarchies, a different type of interaction, one that does not rely on scalar operators, must be proposed. An alternative is given by the following Lagrangian

$$\mathscr{L}_{\text{PC}} = -y_L \bar{Q}_L F_L - y_{tR} \bar{T}_R F_{tR} - y_{bR} \bar{B}_R F_{bR} + h.c., \tag{3.126}$$

where y are dimensionless coefficients and the new fields F are fermionic field operators of dimension 5/2. The interactions in the Lagrangian (3.126) define the concept of *partial compositness*, which means that the fermions are linearly coupled to the new strong sector. Partial compositness does not reintroduce the HP, because $\bar{F}F$ have dimension higher than 4.

In 4-dimensional field theories it is impossible to build a fundamental 5/2-dimensional operator†, so the only way to write it, while avoiding new hierarchies, is as

$$F \to \frac{f}{4\pi}\Psi, \tag{3.127}$$

where Ψ is a fermion from the strong sector. This allows us to estimate the strength of the Yukawa coupling in Eq. (3.99) in terms of the couplings in Eq. (3.126). Assuming that Ψ is massive, Eq. (3.126) becomes

$$\mathscr{L}_{\text{PC}} \to -f\left[\frac{y_L}{4\pi}\bar{Q}_L\Psi_L + \frac{y_{tR}}{4\pi}\bar{T}_L\Psi_{tR} + \frac{y_{bR}}{4\pi}\bar{B}_L\Psi_{bR} + h.c.\right] - f\bar{\Psi}\Psi. \tag{3.128}$$

---

*The fact that this operator has dimension 1 is not exactly trivial, as it follows from some considerations on the magnitude of the top Yukawa coupling and on how the Wilson coefficient runs [13, 71].

†Many CHM are actually related to 5-dimensional theories through the AdS_5/CFT correspondence. In the 5-dimensional theory one can build the operators F explicitly [72].

Therefore, after integrating Ψ out, the Yukawa couplings are approximately given by [56]

$$y_t \sim \frac{y_L y_{tR}}{4\pi}, \quad y_b \sim \frac{y_L y_{bR}}{4\pi}. \tag{3.129}$$

Other than the estimate in Eq. (3.129), the concept of partial compositness does not provide us with any additional information relevant at low-energies. Still, it is important as it emerges in possible UV-completions of CHM.

3.3.4 Computation of the Higgs mass

It is now feasible to compute the Higgs mass from radiative corrections. To this end, the 1-loop Coleman-Weinberg (CW) potential will be used. We refer the reader to Appendix A for a more detailed discussion on this potential and its explicit derivation. The regularization method employed is the hard cut-off one, because the EFT that describe the Higgs has a well determined cut-off, which is given by the breaking scale f. In the broken phase of the EW symmetry the CW potential of the Higgs is given by Eq. (A.32) of Appendix A

$$V_{\text{CW}}(h) = \pm \sum_i n_i \left[\frac{M_i^2(h) f^2}{32\pi^2} - \frac{M_i^4(h)}{64\pi^2} \left(\ln \frac{f^2}{M_i^2(h)} + \frac{1}{2} \right) \right], \tag{3.130}$$

where the sum is over the particles that couple to the physical Higgs h, n_i is the number of dof of the particle and $M_i(h)$ is its the field dependent mass. Besides, the sign is positive for bosons and negative for fermions.

Let us begin with the gauge bosons. The W boson is a complex, massive vector, so it possesses 6 real dof, while the Z boson has only 3 because it is real. The field dependent masses of the gauge bosons can be computed from their interacting Lagrangian in Eq. (3.88), which are given by

$$M_W^2(h) = M_W^2 \left(1 + \frac{2h}{v} k_V^{\mathbf{3}}(\xi) + \frac{h^2}{v^2} k_V^{\mathbf{4}}(\xi) \right) + \mathcal{O}\left(h^3\right), \tag{3.131}$$

$$M_Z^2(h) = \frac{M_W^2}{c_w^2} \left(1 + \frac{2h}{v} k_V^{\mathbf{3}}(\xi) + \frac{h^2}{v^2} k_V^{\mathbf{4}}(\xi) \right) + \mathcal{O}\left(h^3\right), \tag{3.132}$$

where the standard notation $k_V^{\mathbf{3}}(\xi)$ and $k_V^{\mathbf{4}}(\xi)$ to denote respectively the triple and quartic gauge bosons couplings with the Higgs is being used.

The largest contribution from the fermion sector comes from the top quarks, as it has largest coupling to the Higgs. Hence, only the contribution from the top quark to the CW potential will be considered, which in the broken phase of EW symmetry is a coloured Dirac fermion with has 12 dof. As noted in section 3.3.2, the interactions of the top with the Higgs depends on the representation of $SO(5)$ chosen. We can, however, write the field dependent mass as

$$M_t(h) = M_t\left(1 + \frac{h}{v}k_F(\xi)\right) + \mathcal{O}\left(h^2\right), \tag{3.133}$$

with $k_F(\xi)$ the representation-dependent coupling.

Using Eqs. (3.131), (3.132) and (3.133) in Eq, (3.130), one obtains the CW potential for h

$$\begin{aligned} V_{\text{CW}}(h) &= \frac{f^2 M_W^2}{32\pi^2}\left(6 + \frac{3}{c_w^2}\right)\left(1 + \frac{2h}{v}k_V^{\mathbf{3}} + \frac{h^2}{v^2}k_V^{\mathbf{4}}\right) - \\ &- \frac{M_W^4}{64\pi^2}\left(1 + \frac{2h}{v}k_V^{\mathbf{3}} + \frac{h^2}{v^2}k_V^{\mathbf{4}}\right)^2\left[6\left(\ln\frac{f^2}{M_W^2(h)} + \frac{1}{2}\right) + \frac{3}{c_w^2}\left(\ln\frac{f^2}{M_Z^2(h)} + \frac{1}{2}\right)\right] - \\ &- \frac{12 f^2 M_t^2}{32\pi^2}\left(1 + \frac{h}{v}k_F\right)^2 + \frac{24 M_t^4}{64\pi^2}\left(1 + \frac{h}{v}k_F\right)^2\left(\ln\frac{f^2}{M_t^2(h)} + \frac{1}{2}\right) \end{aligned} \tag{3.134}$$

where only the interactions already present in the SM are being taken into account, because all other operators generated by the composite nature of the Higgs are suppressed by powers of ξ, hence they are negligible in a first approximation. The Higgs mass parameter is given by the second derivative of the potential, which is given by

$$\begin{aligned} M_h^2 &= \left.\frac{\mathrm{d}^2 V_{\text{CW}}}{\mathrm{d}h^2}\right|_{h=0} \\ &= \frac{f^2}{16\pi^2 v^2}\left[M_W^2\left(6 + \frac{3}{c_w^2}\right)k_V^{\mathbf{4}} - 12 M_t^2 k_F^2\right] - \\ &- \frac{3M_W^4}{8\pi^2 v^2}\left[2\left(k_V^{\mathbf{3}}\right)^2 + k_V^{\mathbf{4}}\right]\left(\ln\frac{f^2}{M_W^2} + \frac{1}{2c_w^2}\ln\frac{f^2 c_w^2}{M_W^2}\right) - \frac{3M_W^4\left(k_V^{\mathbf{3}}\right)^2}{4v^2\pi^2}\left(1 + \frac{1}{2c_w^2}\right) + \\ &+ \frac{9M_t^4 k_F^2}{2\pi^2}\left(\ln\frac{f^2}{M_t^2} - \frac{2}{3}\right), \end{aligned} \tag{3.135}$$

Since the breaking scale f is considerably larger than the EW scale, i.e. $f \gg v$, the largest contribution to the mass in Eq. (3.136) comes from the term proportional to f^{2*}. The

*It is worth stressing again that the hierarchy between f and v cannot be too large, otherwise additional other terms proportional to $\ln f^2$ are therefore sub-leading. Inspecting Eq. (3.136) it is clear that the dominant term, due to the top quark, makes M_h^2 a negative parameter. This drives spontaneous EWSB. In CHM spontaneous EW is thus achieved through radiative corrections unlike what happens in the SM.

The Higgs acquires a vev $\frac{V}{\sqrt{2}}$ after EWSB, which is approximately given by

$$V^2 \simeq -\frac{M_h^2}{\lambda}, \tag{3.136}$$

where M_h^2 is given in Eq. (3.136) and λ is the quartic coupling that is also generated by quantum effects. The latter is given by the fourth derivative of the CW potential (3.134)

$$\begin{aligned} \lambda &= \left.\frac{\mathrm{d}^4 V_{\mathrm{CW}}}{\mathrm{d}h^4}\right|_{h=0} \\ &= \frac{1}{32\pi^2}\left\{\frac{288 M_t^4 k_F^4}{v^4}\left(\ln\frac{f^2}{M_t^2}-\frac{11}{3}\right)-\right. \\ &- \frac{24 M_W^4}{v^4}\left[4\left(k_V^{\mathbf{3}}\right)^4 - 12\left(k_V^{\mathbf{3}}\right)^2 k_V^{\mathbf{4}} - 3\left(k_V^{\mathbf{4}}\right)^2\right]\left(1+\frac{1}{2c_w^4}\right) - \\ &- \left.\frac{72 M_W^4 (k_V^{\mathbf{4}})^2}{v^4}\left[\ln\frac{f^2}{M_W^2}+\frac{1}{2c_w^4}\ln\frac{c_w^2 f^2}{M_W^2}\right]\right\}. \end{aligned} \tag{3.137}$$

We note in the equation above that in order to have $\lambda > 0$, we need $f \gtrsim 1.1$ TeV. Combining the Higgs mass parameter of Eq. (3.136) with the quartic coupling of Eq. (3.137) one gets the physical Higgs mass. Neglecting the contributions from the gauge bosons, it reads

$$\begin{aligned} m_h^2 &= 2V^2\lambda \\ &= 2v^2\left(\frac{\sin^{-1}\sqrt{\xi}}{\sqrt{\xi}}\right)^2\lambda \\ &\simeq \frac{18 M_t^4}{\pi^2 v^2}\left(\frac{\sin^{-1}\sqrt{\xi}}{\sqrt{\xi}}\right)^2 k_F^4(\xi)\left[\ln\frac{f^2}{M_t^2}-\frac{11}{3}\right], \end{aligned} \tag{3.138}$$

where in the second line Eq. (3.85) was used to write V in terms of v and ξ. This equation makes clear that the model dependent coupling to fermions affects the physical Higgs mass only by the $\mathcal{O}(1)$ parameter k_F.

We stress that Eq. (3.134) is the Higgs potential that can be computed inside the

effects (e.g. the running of the couplings) must be taken into account due to the presence of large logs.

EFT (the "IR contribution"). There may in principle be another "UV-contribution" which is dominated by physics at the cut-off scale Λ. For instance, an additional contribution to the $|H|^2$ term in the potential may come from $\mathcal{H}$-breaking couplings between H and composite resonances. These contributions are uncomputable in the EFT, and introduce an additional source of model dependency in the Higgs mass.

3.4 Experimental results & Fine tuning

Since experiments have not yet reached the energy scale at which the composite structure of the Higgs becomes manifest, our only choice is to work with the effective theory of the MCHM at lower energies. Stated in another way, we are not in position to detect new fundamental particles or resonances yet, so the only option is to measure their residual effects on the EFT, in particular the deviations of the couplings of fermions and gauge bosons with respect to their SM value. From the discussion of sections 3.3.1 and 3.3.2, these couplings are now functions of ξ, therefore, if ξ is different from zero, then certainly one will be able to detect some deviations from the SM values.

For gauge bosons, the most important coupling is the one between one Higgs and two gauge bosons. In the MCHM its strength is controlled by (see Eq. (3.88))

$$k_V^{\mathbf{3}}(\xi) = \sqrt{1-\xi}. \tag{3.139}$$

The importance of the coupling $k_V^{\mathbf{3}}(\xi)$ is due to experimental considerations: the principal decay channels of the Higgs, $H \to \bar{b}b$, $H \to WW^*$, $H \to \tau^+\tau^-$, $H \to ZZ^*$, $H \to \gamma Z$ and $H \to \gamma\gamma$, involve the triple coupling in Eq. (3.139). In particular, the latter two have no tree-level contribution, hence involve $\bar{t}t$ and gauge bosons loops, which in turn allow us to probe the values of both $k_V^{\mathbf{3}}$ and k_F [3]. The quartic coupling in the MCHM, given in terms of ξ by

$$k_V^{\mathbf{4}}(\xi) = 1 - 2\xi, \tag{3.140}$$

is still not experimentally relevant, as its measurement are much more challenging [3, 14]. But, at the same time, it is the one who gives the leading contribution to the Higgs mass (see Eq. (3.138)).

In the limit $\xi \to 0$, the couplings of gauge bosons and fermions return to their SM value, because $k_V^{\mathbf{3}}(0) = k_V^{\mathbf{4}}(0) = k_F(0) = 1$, even if the explicit expressions for $k_F(\xi)$ are

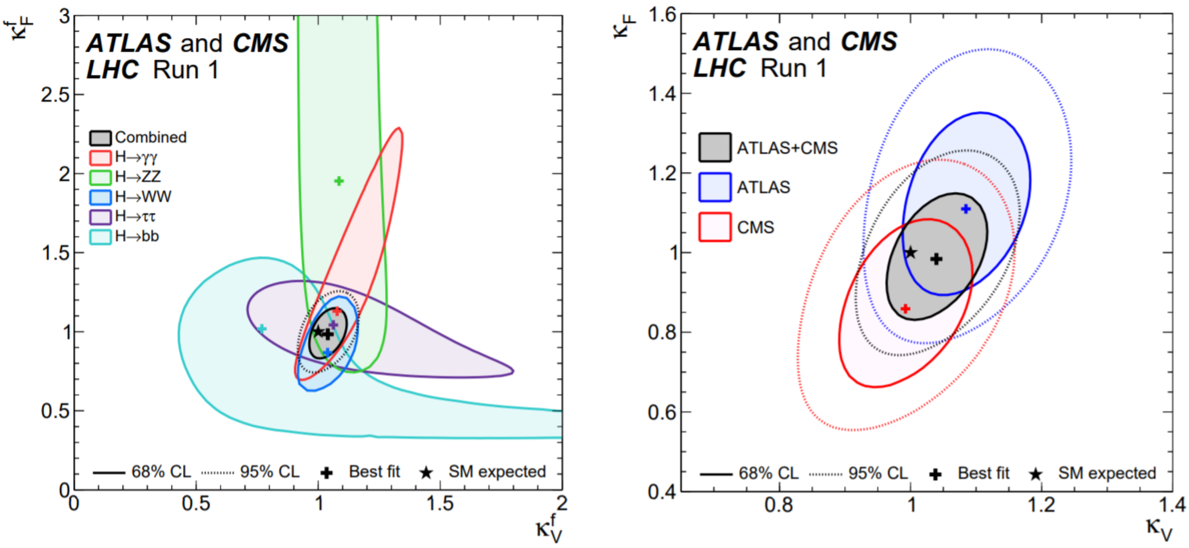


Figure 4: Most likely contours in $(k_V^{\mathbf{3}}, k_F)$ plane. On the left one has the constrains from the individual decay channels of the Higgs, while on the right the combined data from both ATLAS and CMS. Figure taken from [3].

different for distinct representations. Through the measurement of distinct decay channels of the Higgs, one can trace the most likely contour in $(k_F, k_V^{\mathbf{3}})$ plane. The corresponding plot is given in Figure 4. With this analysis one finds out that

$$k_V^{\mathbf{3}} = 1.04(5), \quad k_F = 0.98(11), \tag{3.141}$$

which at 68% CL is compatible with the SM values [3]. Using such data, one can completely exclude the possibility of $\xi > 0.01$ at 1σ, which in turn means that a certain amount of FT is being created. The FT is given by the tuning in the Higgs mass, which was computed in Eq. (3.138),

$$m_h^2 \simeq \frac{3M_t^4}{\pi^2 v^2}\left(\frac{\sin^{-1}\sqrt{\xi}}{\sqrt{\xi}}\right)^2 k_F^4(\xi)\left[\ln\frac{f^2}{M_t^2} - \frac{11}{3}\right]$$

with only the leading order term from the top quark. To achieve $m_h = 125$ GeV, one needs $f \simeq 6$ TeV, for which the associated FT is at least of order 10.

3.5 Conclusions

In this chapter a new kind of solution to the Hierarchy Problem was discussed, one that relies on ideas present on the SM itself, in particular in QCD. In contrast to SUSY theories, which relies on cancellations within the supermultiplets, the CHM exhibit a completely different way of approaching the HP, for it introduces the Higgs as a pNGB of a new strong dynamics. This class of models are interesting for many reasons. From one point of view, their model-building possibilities are diversified, and from another standpoint they have connections with much more exquisite, 5-dimensional theories, which offer possible UV-completions for CHM. Besides, the phenomenology is very rich, in particular after the introduction of resonances, and many of its predictions can be detected at colliders. Just like SUSY, CHM reduce the unnaturalness problem by a lot, but remains tuned nonetheless.

4 Neutral Naturalness

Neutral Naturalness is another class of models that solves the HP, which, in contrast to SUSY and CHM, are much less constrained by EW precision data and collider searches. The minimal model of Neutral Naturalness, the Twin Higgs model (TH), introduces an invisible copy of the SM which is related to the original one via a Z_2 symmetry, and realises the Higgs as a pNGB of a broken global symmetry. These two ingredients combined eliminate the UV-sensitivity of the Higgs mass and explain why it is so small.

This chapter is structured as follows. In the first section the motivations are discussed and in the second section the explicit implementation of the model is shown. In the third section some comments on the cosmological problems associated with such models are made. In the final section an explicit break of the Z_2 symmetry is introduced and the corresponding FT of the TH model is computed.

4.1 Motivations & Twin Worlds

Both solutions presented so far, supersymmetry (SUSY) and composite Higgs models (CHM), predict new particles charged under the SM gauge group. In particular, SUSY had coulored squarks, while CHM predicted a tower of bound states from the new strong sector, which are detectable at the LHC. We saw in both cases that this implies in stringent experimental constrains, which in turn raise the fine tuning (FT) to dangerous levels. It would be therefore interesting if one could build a model in which the new particles are completely neutral under the SM gauge group, in particular under $SU(3)_c$, given the very strong bounds on coloured particles. In this manner, one would be able to loosen the bounds from collider physics, for instance from the EW precision measurements (EWPM).

In addition to a new, neutral sector, we still need a mechanism to stabilise (and possibly predict) the Higgs mass parameter in order to solve the HP. We have learned from CHM that the Higgs mass is protected from dangerous radiative corrections if understood as a pNGB, which in that case was interpreted as a bound state from a new strong sector. In the Neutral Naturalness scenario the Higgs is still considered as a pNGB, but without an underlying strong dynamics, because this was precisely what made the CHM so sensitive to experimental constrains. For this purpose, we consider the simplest potential that triggers spontaneous symmetry breaking (SSB), which is given by [15]

$$V(\mathscr{H}) = -M^2|\mathscr{H}|^2 + \lambda|\mathscr{H}|^4, \tag{4.1}$$

where $\mathscr{H}$ is a complex scalar field that contains information on the Higgs doublet. By taking $M^2 > 0$, this potential will lead to the SSB of the yet unspecified global symmetry group G, which will break down to a subgroup $\mathcal{H}$ and as a result will give us the Higgs doublet as a NGB. Moreover, the interactions of the NGB Higgs with the other NGB and the unbroken modes must be the only ones that connect our SM to the new neutral sector, else we would be once again constrained by precision and collider data.

To summarise, Neutral Naturalness theories have two main ingredients: the Higgs as pNGB and new particles neutral under the SM, both of them interacting only via the potential in Eq. (4.1). The tools to study the physics of a NGB Higgs were already developed in Chapter 3, so it remains only to model the neutral sector in a way that the Hierarchy Problem (HP) is solved. It turns out that such modelling is not straightforward [15]. To see this, consider the quadratic corrections to the Higgs mass given in Eq. (1.5),

$$\delta m^2 = \frac{3\Lambda^2}{8\pi^2}\left(\lambda + \frac{1}{8}(g')^2 + \frac{3}{8}g^2 - y^2\right) \equiv \frac{g_{\rm SM}^2\Lambda^2}{64\pi^2}. \tag{4.2}$$

This particular correction is actually the coefficient of the quadratic term of the Coleman-Weinberg (CW) potential

$$V_{\rm CW}^{\rm SM}(H) \supset \frac{g_{\rm SM}^2\Lambda^2}{64\pi^2}|H|^2, \tag{4.3}$$

computed with the SM particle content. Turning to the CW potential generated by the potential in Eq. (4.1), it will depend only on $|\mathscr{H}|^2$, if the symmetry group G is exact. In this way the NGB embedded in $\mathscr{H}$ will not acquire a potential, even after quantum corrections are included. This suggests that in order to protect the Higgs mass from UV-sensitiveness, the CW potential of the theory must be of the form

$$V_{\rm CW}(\mathscr{H}) \supset \frac{g^2\Lambda^2}{64\pi^2}|\mathscr{H}|^2, \tag{4.4}$$

for some dimensionless coupling g. But, if the theory is to generate correctly the SM interactions, the corrections in Eq. (4.3) are inevitable. Therefore, one must add a contribution from the neutral sector to Eq. (4.3) such that it results in Eq. (4.4). On general grounds, the field $\mathscr{H}$ contains other independent, scalar degrees of freedom (dof), which interact with the Higgs and the neutral sector. The neutral dof of $\mathscr{H}$, denoted here by H', will receive quantum corrections from the interactions in the potential (4.1) and also from the interactions with other neutral particles. These can be written as

$$V_{\text{CW}}^{\text{neutral}}(H') \supset \frac{g_N^2 \Lambda^2}{64\pi^2}|H'|^2, \tag{4.5}$$

where g_N^2 is some dimensionless coupling. Since it is assumed that H and H' are independent, or more precisely, that they come from independent sectors of our theory which interact only via Eq. (4.1), the field $\mathscr{H}$ can be brought into the following form by an appropriate choice of basis

$$\mathscr{H} = \begin{pmatrix} H \\ H' \end{pmatrix}. \tag{4.6}$$

Demanding that

$$\frac{g^2 \Lambda^2}{64\pi^2}|\mathscr{H}|^2 = \frac{g_{\text{SM}}^2 \Lambda^2}{64\pi^2}|H|^2 + \frac{g_N^2 \Lambda^2}{64\pi^2}|H'|^2, \tag{4.7}$$

one obtains that the dimensionless coefficients must be the same,

$$g^2 = g_{\text{SM}}^2 = g_N^2. \tag{4.8}$$

The equation above implies that the couplings of the neutral sector must be tightly related to the ones of the SM. One cannot, however, draw any conclusions for the individual couplings, as it is unknown how interactions in the neutral sector take place.

Since there is no experimental evidence on neither the gauge structure, nor the particle content of the neutral sector, one may choose them arbitrarily, as long this choice respect both Eqs. (4.4) and (4.8). Thus, the most straightforward choice is to suppose that the neutral sector is identical to the SM, in the sense that every particle from the SM has now a *twin* partner, that have the same statistics and the same quantum numbers. These quantum numbers, however, are not the ones from the usual SM gauge group, but from the *twin* gauge group $SU(3)'_c \times SU(2)'_L \times U(1)'_Y$. To choose the structure of the SM as the one of the neutral sector does not seem to be the simplest one, since the SM is a very complex model. Nevertheless, there are two important advantages with such choice. First, the interactions of the SM are very well understood, both from the theoretical, as well from the phenomenological point of view. Second, the introduction of an identical copy of the SM together with Eq. (4.8) leaves the symmetry that interchanges the usual SM with the twin SM, and vice-versa, manifest. In particular, this discrete symmetry is a Z_2 symmetry, whose action is given by [16, 15]

$$Z_2 : \text{ SM} \longleftrightarrow \text{twin SM}, \tag{4.9}$$

where fields, gauge groups and couplings are interchanged. Eq. (4.9) is crucial to the construction of Neutral Naturalness theories, because it is this discrete symmetry that enforces Eq. (4.4) and therefore protects the Higgs mass.

Note that in this scenario the neutral scalar H' turns out to be a complex doublet of the twin EW group with 1/2 twin hypercharge. This implies that the field $\mathscr{H}$, that triggers SSB of the global symmetry group in potential (4.1), is a complex bi-doublet of $SU(2)_L \times SU(2)'_L$ with 4 complex dof. The potential in Eq. (4.1) is thus invariant under a global $SU(4)$ group, which breaks down to a $SU(3)$ group after SSB. We see, however, that the potential (4.1) is invariant under a larger $SO(8)$ group, since $\mathscr{H}$ has 8 real dof. In this case the SSB is given by $SO(8) \to SO(7)$, which has the same number of NGB as $SU(4) \to SU(3)$. As we will see in the next sections, the two descriptions of the SSB pattern are not equivalent due to their distinct group structures, but their cosets are isomorphic, therefore the NGB Higgs may be described by either $SO(8)/SO(7)$ or $SU(4)/SU(3)$. In addition, we observe that the field H in Eq. (4.6), which was used to motivate the introduction of a neutral sector, is not the Higgs field, or more precisely, it is not the field that acquire the EW vacuum expectation value (vev). To not generate confusion, the doublets H and H' will now be denoted respectively as $H^{(1)}$ and $H^{(2)}$, while the Higgs, the one that breaks EW symmetry, will still be denoted as H.

The idea of understanding this neutral sector as a copy of the SM is not restricted to a *single* twin SM. Instead, one may create $N-1$ neutral copies of it. In this case it is straightforward to see that one can write $\mathscr{H}$ as

$$\mathscr{H} = \begin{pmatrix} H^{(1)} \\ H^{(2)} \\ \vdots \\ H^{(N)} \end{pmatrix}, \tag{4.10}$$

where $H^{(1)}$ is the complex doublet of the EW group of the SM, while all other $H^{(i)}$ are associated with independent, neutral copies of the SM. In analogy to the case of a single neutral sector, one needs all individual couplings to be the same in every sector. In other words, only by implementing a Z_N discrete symmetry that interchanges these sectors can one guarantee that the quadratic term of the CW potential depends only on $|\mathscr{H}|^2$ and, thus, protects the Higgs mass parameter from quadratic divergences. The complex scalar field $\mathscr{H}$ now has $4N$ real dof, therefore the SSB pattern is given by $SO(4N) \longrightarrow SO(4N-1)$, which in general contains many NGB besides the Higgs.

To emphasise, Neutral Naturalness models consist in introducing neutral sectors that have the same particle content and gauge structure of the SM. What connects our SM and these neutral copies of the SM is the interaction potential given in Eq. (4.1), which contains interactions between the EW doublets of each sector*. This potential is invariant under a larger global symmetry, whose SSB gives us the physical Higgs as a NGB. The mass of the Higgs boson is protected from quadratic divergences by implementing a Z_N discrete symmetry that interchanges the sectors; this symmetry guarantees that the quadratic term of the CW potential depends only on $|\mathscr{H}|^2$.

The focus in this thesis is the construction of the minimal realisation of this idea, namely the case where $N = 2$.

4.2 Twin Higgs Model

In this section the minimal model of Neutral Naturalness, the Twin Higgs (TH) model [15], is worked out, in which one invisible copy of the SM is taken into consideration. The focus is on the implementation of this model, in particular on how to couple it to gauge bosons and fermions. Also, at the end of this section the physical mass of the Higgs from radiative corrections is computed.

*The potential in Eq. (4.1) does not contain the only interactions between the sectors, because the Z_N symmetry allows the following renormalizable, gauge invariant operators

$$\mathscr{L}_{\text{mix}} = \frac{\epsilon_{ij}}{2} B^{(i)}_{\mu\nu} B^{(j)}_{\mu\nu},$$

which characterises a kinetic-mixing between the hypercharge gauge bosons. This kind of coupling will be relevant in section 4.3 when the cosmological impacts of TH are discussed.

4.2.1 Implementation

The interacting potential between the SM and the neutral sector is given by Eq. (4.1)

$$V(\mathscr{H}) = -M^2|\mathscr{H}|^2 + \lambda|\mathscr{H}|^4, \tag{4.11}$$

with

$$\mathscr{H} = \begin{pmatrix} H^{(1)} \\ H^{(2)} \end{pmatrix}, \tag{4.12}$$

where the supscript (1) refers to the usual SM, while (2) refers to the twin SM. We stress again that the global symmetry $SO(8)$ will be spontaneously broken to $SO(7)$. Just as in the case of the composite Higgs, this global symmetry contains all group structure of the Higgs, since it lives in its coset group. This implies that the EW groups, both the SM one as well the twin one, are contained in $SO(8)$; this latter, therefore, must be partially gauged. From a group theoretical point of view this is consistent, since

$$SO(8) \supset SO(4)^{(1)} \times SO(4)^{(2)}, \tag{4.13}$$

and each $SO(4)$ is isomorphic to a $SU(2)_L \times SU(2)_R$ group. The product of both EW groups, namely

$$SU(2)_L^{(1)} \times U(1)_Y^{(1)} \times SU(2)_L^{(2)} \times U(1)_Y^{(2)}, \tag{4.14}$$

is indeed a subgroup of $SO(8)$, where one identifies the third generator of $SU(2)_R$ as the hypercharge operator. Note that if one instead considered $SU(4)$ as the global group, one could still decompose it in $SU(2)_L^{(1)} \times SU(2)_L^{(2)}$, but wouldn't be able to define the hypercharges directly; only by adding two independent $U(1)$ groups would the hypercharge groups be consistently introduced*. From now on we will always assume that the global symmetry group is $SO(8)$.

The kinetic term for $\mathscr{H}$ is

$$\mathscr{L}_{\text{kin}} = |D_\mu \mathscr{H}|^2, \tag{4.15}$$

*Choosing another subgroup of $SU(4)$, in particular $SU(2)\times SU(2)\times U(1)$, wouldn't solve the problem, because the decomposition of the fundamental representation of $SU(4)$ under this subgroup implies that the resulting two $SU(2)$ doublets, i.e. $H^{(1)}$ and $H^{(2)}$, have opposite hypercharges. Hence, the Z_2 symmetry would be explicitly broken. Besides, this $U(1)$ acts on both the SM and the twin SM, which is yet another reason to avoid this choice [63].

with the covariant derivative given by

$$D_\mu \mathscr{H} = \begin{pmatrix} D_\mu H^{(1)} \\ D_\mu H^{(2)} \end{pmatrix}, \tag{4.16}$$

where

$$D_\mu H^{(i)} = \partial_\mu H^{(i)} - igW_\mu^{a(i)} \frac{\sigma^a}{2} H^{(i)} - i\frac{g'}{2} B_\mu^{(i)} H^{(i)} \tag{4.17}$$

and $W_\mu^{a(i)}$, $B_\mu^{(i)}$ are the gauge bosons of $SU(2)_L^{(i)} \times U(1)_Y^{(i)}$. Note that the gauge couplings are the same, as a consequence of the discrete Z_2 symmetry. The partial gauging of a global group naturally leads to an explicit break of the group via radiative corrections, and allows the Higgs to acquire a non-vanishing mass. Such quantum corrections, however, do not reintroduce quadratic divergences in the Higgs mass due to the Z_2 symmetry. This will be shown explicitly in section 4.2.3, but let us anticipate that the CW potential generated at 1-loop, already given in Eq. (3.130), in the case of consideration is approximately

$$V_{\text{CW}}(H^{(i)}) \simeq \frac{g_{\text{SM}}^2 \Lambda^2}{64\pi^2} \left| H^{(i)} \right|^2 + \kappa(\Lambda) \left| H^{(i)} \right|^4, \tag{4.18}$$

where κ is a function that depends logarithmically on the cut-off Λ and g_{SM}^2 is a SM-like coupling. It is trivial to see that the quadratic term of the sum of both potentials is accidentally $SO(8)$ symmetric,

$$V_{\text{CW}}(H^{(1)}) + V_{\text{CW}}(H^{(2)}) \simeq \frac{g_{\text{SM}}^2 \Lambda^2}{64\pi^2} |\mathscr{H}|^2 + \kappa(\Lambda) \left(\left| H^{(1)} \right|^4 + \left| H^{(2)} \right|^4 \right), \tag{4.19}$$

and as a consequence introduce only logarithmically divergent terms to the Higgs potential. The quartic term, which in the case of the above potential, is not $SO(8)$ symmetric, will allow the Higgs to acquire a mass. Thus, we observe that the Z_2 parity is of central importance to protect the Higgs mass, even when $SO(8)$ is explicitly broken.

In order to compute the Higgs mass from the radiative corrections of the gauge boson and fermion loops, one needs to know the details of the SSB, for instance how to parametrise the fields $H^{(1)}$ and $H^{(2)}$ in terms of H. First of all, we define the breaking scale of the $SO(8)$ group, which is given by the minimisation of the potential (4.1):

$$\left. \frac{\mathrm{d}V}{\mathrm{d}|\mathscr{H}|^2} \right|_{\mathscr{H} = \langle \mathscr{H} \rangle} = 0 \Rightarrow |\langle \mathscr{H} \rangle|^2 = \frac{M^2}{2\lambda} \equiv \frac{f^2}{2}. \tag{4.20}$$

This vacuum state is conveniently aligned with the invisible sector, such that twin QED is unbroken,

$$\langle \mathscr{H} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ f \end{pmatrix} \Longleftrightarrow \left\langle H^{(1)} \right\rangle = 0, \quad \left\langle H^{(2)} \right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ f \end{pmatrix}. \tag{4.21}$$

The vacuum $\langle \mathscr{H} \rangle$ breaks $SO(8)$ down to $SO(7)$, meaning that there are a total of 7 NGB, of which only 4 are needed to build the Higgs doublet. What is the physical meaning of the extra NGB? Since the twin EW group is a subgroup of the larger $SO(8)$ and is gauged, they ought to be eaten-up by the twin gauge bosons. Note that one can do this, because, given the covariant derivatives in Eq. (4.17), it is always possible to go to the unitary gauge and let the twin gauge bosons absorb the NGBs. Indeed, one just needs to know which generators are associated with the dof of the Higgs field. To this end, the broken generators in the $SU(4)$ representation of the SSB are listed, as it is easier to understand what is happening in the case in which $H^{(1)}$ and $H^{(2)}$ are complex doublets rather than real 4-plets. The generators in this case are hermitian and traceless, so a possible basis for the broken generators (in the fundamental representation) is given by

$$\begin{aligned} \left\{ \hat{T}^{\hat{a}} \right\} = \Bigg\{ & \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 0 & 0 \\ \hline 0 & 0 & & \\ 1 & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} & & 0 & -i \\ & & 0 & 0 \\ \hline 0 & 0 & & \\ i & 0 & & \end{array}\right), \left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & 1 \\ \hline 0 & 0 & & \\ 0 & 1 & & \end{array}\right), \left(\begin{array}{cc|cc} & & 0 & 0 \\ & & 0 & -i \\ \hline 0 & 0 & & \\ 0 & i & & \end{array}\right), \\ & \left(\begin{array}{cc|cc} 0 & 0 & & \\ 0 & 0 & & \\ \hline & & 0 & 1 \\ & & 1 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & & \\ 0 & 0 & & \\ \hline & & 0 & -i \\ & & i & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & & \\ 0 & 0 & & \\ \hline & & 1 & 0 \\ & & 0 & -1 \end{array}\right) \Bigg\}, \end{aligned} \tag{4.22}$$

where a normalization factor of $1/\sqrt{2}$ is understood. One clearly sees that the last three generators, in the basis given in Eq. (4.22), act only on the twin sector, and therefore one can use the $SU(2)_L^{(2)} \times U(1)_Y^{(2)}$ gauge redundancy to eliminate them; this choice defines the unitary gauge in the twin sector. In this gauge, the Goldstone matrix contains only the Higgs boson and is written as

$$U[H] = \exp\left[\frac{2iH^{\hat{a}}}{f}\hat{T}^{\hat{a}}\right] = \exp\left[\frac{\sqrt{2}i}{f}\left(\begin{array}{ccc|c} & & & H_1 \\ & & & H_2 \\ & & & 0 \\ \hline H_1^* & H_2^* & 0 & 0 \end{array}\right)\right], \tag{4.23}$$

with H_1 and H_2 the complex components of the Higgs doublet H, such that

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}. \tag{4.24}$$

We are interested in writing the non-linear sigma model for the Goldstone Higgs H, so we must substitute the field $\mathscr{H}$ by [16, 73]

$$\mathscr{H}(x) = U[H(x)]\left(\langle\mathscr{H}\rangle + \frac{\sigma(x)}{\sqrt{2}}\right) = \frac{f}{\sqrt{2}}\left(1 + \frac{\sigma(x)}{f}\right)\begin{pmatrix} \frac{iH}{|H|}\sin\frac{\sqrt{2}|H|}{f} \\ 0 \\ \cos\frac{\sqrt{2}|H|}{f} \end{pmatrix}, \tag{4.25}$$

with $|H| = \sqrt{H^\dagger H}$ and $\sigma(x)$ the radial resonance. With the non-linear parametrization above one obtains

$$H^{(1)} = \frac{f}{\sqrt{2}}\left(1 + \frac{\sigma}{f}\right)\frac{iH}{|H|}\sin\frac{|H|\sqrt{2}}{f} \Rightarrow \left|H^{(1)}\right|^2 = \frac{f^2}{2}\left(1 + \frac{\sigma}{f}\right)^2 \sin^2\frac{\sqrt{2}|H|}{f}, \tag{4.26a}$$

$$H^{(2)} = \frac{f}{\sqrt{2}}\left(1 + \frac{\sigma}{f}\right)\begin{pmatrix} 0 \\ \cos\frac{\sqrt{2}|H|}{f} \end{pmatrix} \Rightarrow \left|H^{(2)}\right|^2 = \frac{f^2}{2}\left(1 + \frac{\sigma}{f}\right)^2 \cos^2\frac{\sqrt{2}|H|}{f}. \tag{4.26b}$$

If there were no radiative corrections, or in other words, if the $SO(8)$ symmetry were to remain exact, one would obtain a constant potential for H, because

$$|\mathscr{H}|^2 = \left|H^{(1)}\right|^2 + \left|H^{(2)}\right|^2 = \frac{f^2}{2}\left(1 + \frac{\sigma}{f}\right)^2. \tag{4.27}$$

In this way the vev of H is zero, hence

$$\left|\left\langle H^{(1)}\right\rangle\right| = 0, \quad \left|\left\langle H^{(2)}\right\rangle\right| = \frac{f}{\sqrt{2}}, \tag{4.28}$$

which is equivalent to Eq. (4.21). As estimated in Eq. (4.19), quantum corrections, however, do break $SO(8)$ explicitly and allows a non-constant potential $V(\sigma, H)$ to be generated. For instance, considering the corrections at 1-loop in the CW potential (4.19) one obtains the following potential

$$\begin{aligned} V(\sigma, H) &\simeq \cdots + \kappa(\Lambda)\left(\left|H^{(1)}\right|^4 + \left|H^{(2)}\right|^4\right) \\ &\simeq \cdots + \frac{\kappa(\Lambda) f^4}{4}\left(1+\frac{\sigma}{f}\right)^4\left(\sin^4\frac{\sqrt{2}|H|}{f} + \cos^4\frac{\sqrt{2}|H|}{f}\right) \\ &\simeq \cdots + \frac{\kappa(\Lambda) f^4}{4}\left(1 - \frac{1}{2}\sin^2\frac{2\sqrt{2}|H|}{f}\right), \end{aligned} \tag{4.29}$$

where the dots denote $SO(8)$ symmetric terms and interacting terms between H and σ. The minimisation of the above potential is straightforward

$$\left.\frac{\mathrm{d}V(\sigma, H)}{\mathrm{d}|H|}\right|_{H=\langle H\rangle} = 0 \Rightarrow \sin\frac{4\sqrt{2}|\langle H\rangle|}{f} = 0. \tag{4.30}$$

To determine the value of $\langle H\rangle$ that satisfy Eq. (4.30), one needs to know the sign of the function $\kappa(\Lambda)$. From the analysis of the second derivative of $V(\sigma, H)$ with respect to $|H|$ we learn that

$$\text{if} \;\; \kappa(\Lambda) > 0 \Rightarrow \langle H\rangle \neq 0, \tag{4.31a}$$

$$\text{if} \;\; \kappa(\Lambda) \leq 0 \Rightarrow \langle H\rangle = 0. \tag{4.31b}$$

In section 4.2.3 the function $\kappa(\Lambda)$ will be computed explicitly, but we anticipate that the condition (4.31a) is satisfied only by including the contribution from the fermion sector. Hence, assuming that $\langle H\rangle \neq 0$, one obtains from Eq. (4.30)

$$\frac{\sqrt{2}|\langle H\rangle|}{f} = \frac{\pi}{4}. \tag{4.32}$$

Substituting Eq. (4.32) into Eq. (4.26) and taking the Higgs vev to be aligned with the unbroken direction of the EW group, we obtain the vevs of $H^{(1)}$ and $H^{(2)}$

$$\left\langle H^{(1)}\right\rangle = \left\langle H^{(2)}\right\rangle = \frac{f}{2}\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.33}$$

which is manifestly Z_2 invariant.

A remark on the choice of the vev (4.21) is in order. First, note that the potential in Eq. (4.29) is proportional to the breaking scale f, hence f is the mass scale of the scalar sector of the model, as both the SM as well the neutral sectors are described by $V(\sigma, H)$. It will be shown below that, due to the Z_2 symmetry, the scale f is equal to the EW vev v, which means that the overall mass scale of the theory is the EW scale. As we shall see shortly, this implies that the TH model is incompatible with the observed phenomenology. In order to make the model compatible with experimental data, we will see in section 4.4 how to modify the potential $V(\mathscr{H})$ such that the twin sector and the SM acquire mass scales proportional to f and v, respectively, and how to create an hierarchy $f > v$ that leaves the neutral sector heavier than the SM. For this to work, Eq. (4.21) must hold, therefore it is of utmost importance to keep the vev of $\mathscr{H}$ aligned with the twin sector. It is clear, however, that this choice apparently breaks Z_2 spontaneously. In principle this is not so problematic, since the the theory is still Z_2 symmetric, although at low energies the mechanism that protects the Higgs mass would be certainly concealed. What is interesting is that, due to $SO(8)$ being explicitly broken by quantum effects, the Z_2 symmetry is restored and the vev becomes Z_2 invariant; in other words, there is an intriguing interplay between the continuous and discrete symmetries.

Before analysing the modification in the EW sector, it is important to discuss first the scalar sector. The Higgs field H acquires a vev and can be thus written as

$$H(x) = \begin{pmatrix} 0 \\ \frac{V}{\sqrt{2}} + \frac{1}{\sqrt{2}}\rho(x) \end{pmatrix} \tag{4.34}$$

in unitary gauge, with $|\langle H \rangle| = \frac{V}{\sqrt{2}}$. Here, $\rho(x)$ is the radial resonance of the Higgs doublet. From Eq. (4.26) it is clear that the radial resonances σ and ρ will interact and mix [74]. Their interactions are given by the potential in Eq. (4.1), that together with the 1-loop corrections in Eq. (4.18) reads

$$\begin{aligned} V(\sigma,\rho) &= -\left(M^2 + \delta m^2\right)\frac{f^2}{2}\left(1+\frac{\sigma}{f}\right)^2 + \frac{\lambda f^4}{4}\left(1+\frac{\sigma}{f}\right)^4 \\ &\quad + \frac{\kappa(\Lambda) f^4}{4}\left(1+\frac{\sigma}{f}\right)^4\left[1-\frac{1}{2}\sin^2\left(\frac{2V}{f}+\frac{2\rho}{f}\right)\right], \end{aligned} \tag{4.35}$$

where δm^2 is quadratic piece of the 1-loop contribution, given in Eq. (4.2). The last term in the potential in Eq. (4.35) contains the terms that mix both resonances and is proportional to function $\kappa(\Lambda)$. The quadratic term in the potential above gives us the mass matrix of the scalar sector. This latter is given by

$$V_{\text{mass}}(\sigma,\rho) = \frac{1}{2}(\sigma\ \rho)\mathcal{M}\begin{pmatrix}\sigma\\ \rho\end{pmatrix}, \tag{4.36}$$

with

$$\mathcal{M} = \begin{pmatrix} 2\lambda f^2 - \delta m^2 + f^2\kappa(\Lambda)\left(1-\frac{1}{2}\sin^2\frac{2V}{f}\right) & -\frac{3\kappa(\Lambda)f^2}{2}\sin\frac{4V}{f} \\ -\frac{3\kappa(\Lambda)f^2}{2}\sin\frac{4V}{f} & -\kappa(\Lambda)f^2\cos\frac{4V}{f}\end{pmatrix}. \tag{4.37}$$

When the Z_2 symmetry is exact, Eq. (4.32) imposes that $\frac{V}{f} = \frac{\pi}{4}$, hence the off-diagonal terms in Eq. (4.37) vanish. In this case there is no mixing and the lightest state can be identified with the neutral, radial excitation of the Higgs field h. Since $\kappa(\Lambda)$ is loop-suppressed, we expect that $\lambda > \kappa(\Lambda)$, therefore the ρ-resonance is the neutral Higgs h. If the discrete Z_2 group is not exact, there will be a mixing and h will be a linear combination of the radial resonances; will be discussed in section 4.4. In section 4.2.3 the mass of h will be computed in detail, but from Eq. (4.37) there is already an estimate to it,

$$m_h^2 \sim \kappa(\Lambda)f^2. \tag{4.38}$$

Let us now discuss the modifications in the EW sector. The gauge bosons of both sectors become massive when the EW doubles acquire vevs. The kinetic term of the Lagrangian in this case is given by

$$\begin{aligned}|D_\mu\langle\mathscr{H}\rangle|^2 &= \left|D_\mu\left\langle H^{(1)}\right\rangle\right|^2 + \left|D_\mu\left\langle H^{(2)}\right\rangle\right|^2 \\ &= M_{W(1)}^2 W_\mu^{(1)+}W_\mu^{(1)-} + M_{W(2)}^2 W_\mu^{(2)+}W_\mu^{(2)-} \\ &\quad + \frac{M_{Z(1)}^2}{2}Z_\mu^{(1)}Z_\mu^{(1)} + \frac{M_{Z(2)}^2}{2}Z_\mu^{(2)}Z_\mu^{(2)},\end{aligned}$$

where $M_{W(i)}$ and $M_{Z(i)}$ are the masses of the $W^{(i)}$ and $Z^{(i)}$ bosons, respectively. It is trivial to see that the masses of the W's and the Z's are given by

$$M_{W(1)}^2 = M_{W(2)}^2 = \frac{g^2 f^2}{4}, \tag{4.39a}$$

$$M_{Z(1)}^2 = M_{Z(2)}^2 = \frac{g^2 f^2}{4\cos^2\theta_w}, \tag{4.39b}$$

where g is the $SU(2)_L$ gauge coupling and θ_w is the weak angle. Note that the masses of the respective gauge bosons of both sectors are equal, which is a consequence of the manifest Z_2 symmetry of the theory. In order to obtain the correct experimental value for the masses, one needs to impose

$$f = v, \tag{4.40}$$

with v being the EW vev.

With Eq. (4.40) we see that the expression for the masses are unchanged with respect to the SM. On the other hand the couplings between H and the gauge bosons are modified, as it will now become clear. After SSB, spontaneous EW symmetry breaking (EWSB) is also triggered, hence the Higgs also acquire a vev. In the unitary gauge together with Eq. (4.32) one has

$$H(x) = \begin{pmatrix} 0 \\ \frac{\pi f}{4\sqrt{2}} + \frac{1}{\sqrt{2}} h(x) \end{pmatrix}, \tag{4.41}$$

where h is the neutral, CP-even component of the Higgs field. Substituting the above equation into Eq. (4.25), we obtain for the twin sector

$$\begin{aligned} \left|D_\mu H^{(2)}\right|^2 &= \frac{f^2}{2}\left|D_\mu \begin{pmatrix} 0 \\ \left(1+\frac{\sigma}{f}\right)\cos\left(\frac{\pi}{4}+\frac{1}{f}h\right) \end{pmatrix}\right|^2 \\ &= \frac{1}{2}\left(1+\frac{\sigma}{f}\right)^2 (\partial_\mu h)^2 \sin^2\left(\frac{\pi}{4}+\frac{1}{f}h\right) + \frac{1}{2}(\partial_\mu \sigma)^2 \cos^2\left(\frac{\pi}{4}+\frac{1}{f}h\right) + \\ &+ \frac{g^2 f^2}{2}\left[W_\mu^{(2)+} W_\mu^{(2)-} + \frac{1}{2\cos\theta_w} Z_\mu^{(2)} Z_\mu^{(2)}\right]\cos^2\left(\frac{\pi}{4}+\frac{1}{f}h\right). \end{aligned} \tag{4.42}$$

In the same manner for the SM sector

$$\begin{aligned} \left|D_\mu H^{(1)}\right|^2 &= \frac{1}{2}\left(1+\frac{\sigma}{f}\right)^2 (\partial_\mu h)^2 \cos^2\left(\frac{\pi}{4}+\frac{1}{f}h\right) + \frac{1}{2}(\partial_\mu \sigma)^2 \sin^2\left(\frac{\pi}{4}+\frac{1}{f}h\right) + \\ &+ \frac{g^2 f^2}{4}\left[W_\mu^{(1)+} W_\mu^{(1)-} + \frac{1}{2\cos\theta_w} Z_\mu^{(1)} Z_\mu^{(1)}\right]\sin^2\left(\frac{\pi}{4}+\frac{1}{f}h\right). \end{aligned} \tag{4.43}$$

We see that the sum of Eqs. (4.42) and (4.43) gives us a correct normalised kinetic term for h and its coupling to gauge bosons of both sector. In particular, note that, since

$$\cos^2\left(\frac{\pi}{4}+\frac{1}{f}h\right) = 1 - \sin^2\left(\frac{\pi}{4}+\frac{1}{f}h\right),$$

the couplings of h with the SM and the twin sector have opposite signs. This implies that the quadratic piece of the 1-loop radiative corrections to the mass of h are exactly cancelled.

⋆⋆⋆

The implementation of the TH model above is obviously not consistent with reality. The source of inconsistency has its roots in Eqs. (4.32) and (4.40), since the new physics, which should become manifest at the breaking scale f, is exactly at the EW scale. Such situation is discarded by many reasons. First, the Taylor expansion of the trigonometric functions in Eqs. (4.42) and (4.43), which determine the coupling of h with the gauge bosons, does not reproduce the interactions of the SM. Take Eq. (4.43) for example:

$$\begin{aligned}\sin^2\left(\frac{\pi}{4}+\frac{1}{f}h\right) &= \left[\frac{1}{\sqrt{2}}\left(\sin\frac{h}{f}+\cos\frac{h}{f}\right)\right]^2 \\ &= \frac{1}{2}\left[1+\sin\frac{2h}{f}\right] \\ &= \frac{1}{2}\left[1+\frac{2h}{f}-\frac{4}{3}\left(\frac{h}{f}\right)^3+\cdots\right].\end{aligned}$$

It is clear from the above expansion that neither Eq. (4.42) nor Eq. (4.43) contain the quartic coupling with gauge bosons, but predict only odd-power interactions instead.

As stressed previously in section 3.4, the quartic coupling is still not well understood experimentally, so it is difficult to reject the above formulation solely based on its absence. A second, more concrete point is the following. Consider the total decay width of the Higgs, that in the TH model is given by [27]

$$\Gamma_{\text{total}} = \Gamma(h \to \text{SM}) + \Gamma(h \to \text{Neutral}), \tag{4.44}$$

where "Neutral" refers to the particles within the neutral sector. From Eqs. (4.40), (4.42) and (4.43) one can relate the decay widths from both sectors*

$$\Gamma(h \to \text{SM}) = \Gamma(h \to \text{Neutral}). \tag{4.45}$$

*The fermion sector has yet to be discussed, but it will be seen in section 4.2.2, that, due to Z_2 invariance, the relation in (4.45) is valid.

The neutral sector is invisible to us, as none of its particles are charged under gauge group of the SM, as a consequence the invisible branching ratio is given by

$$\mathrm{Br}(h \to \mathrm{inv}) \gtrsim 50\%. \tag{4.46}$$

According to the Particle Data Group (PDG) [3], the branching ratio for invisible decays of the Higgs is

$$\mathrm{Br}(h \to \mathrm{inv}) < 24\%, \tag{4.47}$$

at 95% CL. Therefore the prediction in Eq. (4.46) is clearly excluded by data.

Both problems have their origin in the exact Z_2 symmetry of the TH model. For instance, Eq. (4.45) is a direct consequence of the manifest Z_2 symmetry of the model, while Eq. (4.40), that states that the SSB scale of the global $SO(8)$ is located at the EW scale, follows directly from the invariance under Z_2 of the vev of $\mathscr{H}$. In addition, the vev of the Higgs doublet was calculated in Eq. (4.32) from the minimisation of the Z_2-symmetric potential (4.29), which in turn implied that there were no even-power couplings to the gauge bosons. One may see that, if the Z_2 group were to be broken, both problems could be avoided. By introducing a source of Z_2-breaking one could create a hierarchy between v and f, such that the twin sector becomes heavier in comparison to the SM, and also modify the minimisation of the potential (4.1) in order to change the vev f of $\mathscr{H}$. In the next few sections we are going to discuss aspects of the Z_2 symmetric theory like the introduction of fermions and the computation of the Higgs mass, before discussing in section 4.4 how to break Z_2 consistently.

4.2.2 Fermion sector

Our next objective is to introduce the fermions in the theory. It was seen in the previous discussion that there is an interesting interplay between the discrete Z_2 and the continuous $SO(8)$ symmetries. For instance, the latter is an exact symmetry of the scalar sector and it is broken in the gauge bosons sector by the partial gauging of the group, but, due to the Z_2 symmetry, it generates a $SO(8)$-symmetric quadratic term under radiative corrections. Since the symmetry is broken only by the small gauge couplings, $SO(8)$ is not just an accidental symmetry, but a physical one instead, on which we can rely to study the physical properties of the theory.

Such considerations take into account only the scalar and gauge boson sectors. The fermion sector requires a more careful discussion, because we do not know what role does the $SO(8)$ symmetry play in it. More precisely, as long as the Higgs mass parameter is protected from quadratic divergences, one is free to chose whether $SO(8)$ is a symmetry of the fermion sector or not.

Following this line of reasoning, there are two different approaches to the fermion sector. The first one is to treat the global symmetry group $SO(8)$ as an accidental symmetry, which means that one does not need to build a $SO(8)$ invariant Lagrangian for the fermions, but we must use the Z_2 symmetry to make the quadratic term of the CW potential (accidentally) $SO(8)$-symmetric. The second possibility is to treat $SO(8)$ as a physical symmetry, in other words, to build our theory according to its selection rules and embed all our fields into appropriate representations [15]. The latter approach is similar to the one studied in section 3.3.2 for a CHM, where an EFT was built according to the selections rules of $SO(5)$ and $SO(4)$, but requires much more effort as we will see.

First case: $SO(8)$ as an accidental symmetry

Let us first not impose the $SO(8)$ symmetry and just built an interacting Lagrangian that reproduces the SM terms and that protects the Higgs mass. Considering just the quarks, the Yukawa operators in the SM are given by

$$\mathscr{L}_{(1)} = -y_t \bar{q}_L^{(1)} \tilde{H}^{(1)} t_R^{(1)} - y_b \bar{q}_L^{(1)} H^{(1)} b_R^{(1)} + h.c., \tag{4.48}$$

where as usual $\tilde{H}^{(1)} = i\sigma_2 H^{(1)*}$. Imposing the Z_2 symmetry one must add the following Lagrangian to account for the interactions in the twin sector:

$$\mathscr{L}_{(2)} = -y_t \bar{q}_L^{(2)} \tilde{H}^{(2)} t_R^{(2)} - y_b \bar{q}_L^{(2)} H^{(2)} b_R^{(2)} + h.c. \tag{4.49}$$

The Yukawa Lagrangian is thus given by the sum of the two terms,

$$\mathscr{L}_{\text{Yuk}} = \mathscr{L}_{(1)} + \mathscr{L}_{(2)} \,. \tag{4.50}$$

Note that the Lagrangian in the above equation is not $SO(8)$ invariant, as the quarks from both sectors do not mix through any symmetry transformation. The corresponding CW potential will be computed in detail in section 4.2.3, but we anticipate that its quadratic term is accidentally $SO(8)$ symmetric and contributes only logarithmcally to the Higgs mass parameter.

It is also interesting to see how this cancellation works in the broken phase of the theory. Take the lowest order expansion of the EW doublets from Eq. (4.25),

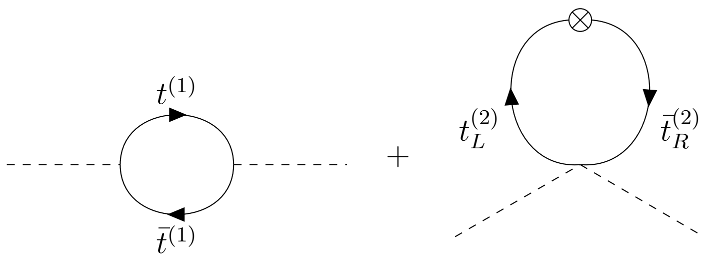
$$H^{(1)} \simeq iH, \tag{4.51a}$$

$$H^{(2)} \simeq \begin{pmatrix} 0 \\ \frac{f}{\sqrt{2}} - \frac{1}{\sqrt{2}f}|H|^2 \end{pmatrix}. \tag{4.51b}$$

The interactions with the top and twin top quarks, for example, becomes

$$\mathscr{L}_{\text{top}} = iy_t\bar{q}_L^{(1)} H t_R^{(1)} + y_t \bar{t}_L^{(2)} t_R^{(2)} \frac{1}{\sqrt{2}f}|H|^2 - y_t \bar{t}_L^{(2)} t_R^{(2)} \frac{f}{\sqrt{2}} + h.c., \tag{4.52}$$

hence the two relevant 1-loop diagrams that contribute to the Higgs mass are



. (4.53)

The diagram on the left is just the usual fermion loop of the SM, while the diagram on the right is the contribution from the neutral sector. The insertion $\otimes$ on the twin quark loop corresponds to the third term in the Lagrangian (4.52), which comes from the zeroth order expansion of the cosine. From the above diagrams we note that there are other types of diagrams that can cancel quadratic divergences besides the squark loop in SUSY and, in contrast, do not require the addition of particles charged under the SM [16, 73].

Second case: $SO(8)$ as a physical symmetry

The Lagrangian in Eq. (4.50) already describes the fermion sector consistently, but treats $SO(8)$ as an accidental symmetry. Stated in another way, the $SO(8)$ group has no physical content; it is just an accidental symmetry of the scalar sector. However, the construction of the fermion sector is always redundant from our bottom-up approach, since one needs to choose the representation of the fermions under the relevant groups. Hence, if there are no other reasons against, one might consider the $SO(8)$ symmetry as a physical symmetry and use $SO(8)$ invariance to build the interacting Lagrangian.

Following this line of reasoning, the first step is to promote the fermion fields,

$$q_L^{(1)}, q_L^{(2)} \to Q_L, \quad t_R^{(1)}, t_R^{(2)} \to T_R, \quad b_R^{(1)}, b_R^{(2)} \to B_R \tag{4.54}$$

where the fields Q_L, T_R and B_R are embedded into some representation of $SO(8)$. One can build an $SO(8)$ invariant Yukawa term with the promoted quark fields (4.54) and $\mathscr{H}$. Unlike what happens in CHM, where one could chose any representation that could fit the quarks representations, there is not much liberty in the TH model regarding the choice of the representation. This is a consequence of the fact that the promoted quark fields couple linearly to $\mathscr{H}$, which transforms under the fundamental representation of $SO(8)$. Hence, Q_L must transform under the fundamental representation of $SO(8)$, while T_R and B_R must be singlets of it. In this way the Yukawa Lagrangian is $SO(8)$ invariant. In principle we could also embed Q_L, T_R and B_R into larger representations of $SO(8)$. For example, we could have Q_L transforming in the fundamental and T_R/B_R in the symmetric rank-2 tensorial representation*. In the following it is chosen that Q_L transforms under the fundamental representation while T_R/B_R are singlets, as the discussion with tensorial representations is analogous.

To check if such representations can fit the respective quarks from both sectors, one must decompose them into representations of the EW groups. For instance, the fundamental of $SO(8)$, denoted here by $\square^{(8)}$, is decomposed as [63, 75]

$$\begin{array}{ccccc} SO(8) \supset & SO(4)^{(1)} \times SO(4)^{(2)} & \supset & SU(2)_L^{(1)} \times U(1)_Y^{(1)} \times SU(2)_L^{(2)} \times U(1)_Y^{(2)} \\ \square^{(8)} \to & \left[\square^{(4)} \otimes \mathbf{1}^{(4)}\right] \oplus \left[\mathbf{1}^{(4)} \otimes \square^{(4)}\right] & \to & \left(\mathbf{2}_{\frac{1}{2}} \oplus \mathbf{2}_{-\frac{1}{2}}\right)^{(1)} \oplus \left(\mathbf{2}_{\frac{1}{2}} \oplus \mathbf{2}_{-\frac{1}{2}}\right)^{(2)} \end{array}$$

where $\square^{(4)}$ is the fundamental of $SO(4)$, $\mathbf{1}^{(4)}$ is a singlet of $SO(4)$, $\mathbf{2}$ is a doublet of $SU(2)_L$ with its respective hypercharge. One clearly sees from the above decomposition that the fundamental of $SO(8)$ alone cannot describe the quark fields, due to the wrong hypercharge of the doublets. Just as in CHM, one needs to introduce another local $U(1)_X$

*The right handed fields cannot transform under the fundamental of $SO(8)$, because there are no singlets of $SU(2)_L^{(1)} \times SU(2)_L^{(2)}$ in it.

which will contribute to the hypercharge as in Eq. (3.103). Therefore, the adequate representation of the promoted quarks under $SO(8) \times U(1)_X$ are

$$Q_L \sim \square^{(8)}_{\frac{2}{3}} \quad \text{or} \quad \square^{(8)}_{-\frac{1}{3}}, \tag{4.55a}$$

$$T_R \sim \mathbf{1}^{(8)}_{\frac{2}{3}}, \tag{4.55b}$$

$$B_R \sim \mathbf{1}^{(8)}_{-\frac{1}{3}}, \tag{4.55c}$$

with $\mathbf{1}^{(8)}$ a singlet of $SO(8)$. Choosing the X charge of $\mathscr{H}$ as zero, one can write the Yukawa Lagrangian as

$$\mathscr{L}_{\text{Yuk}} = -y_T \bar{Q}_L \mathscr{H} T_R - y_B \bar{Q}_L \mathscr{H} B_R + h.c., \tag{4.56}$$

which is manifestly $SO(8) \times U(1)_X$ invariant.

There is still a subtlety regarding the colour groups [15]. In the Lagrangian (4.56), a general transformation of $SO(8)$ mixes the quarks from both sectors and consequently their colour states. As a result the group $SU(3)^{(1)}_c \times SU(3)^{(2)}_c$ is explicitly broken, since the colours states are not diagonal anymore. This problem is solved by introducing an additional group G_c that contains both colour groups as a subgroup. The choice of this group is not very important to us*, so we generically state that all promoted quarks in Eq. (4.54) bear some representation R of G_c, which is decomposed as

$$R \to (\mathbf{3} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{3}) \tag{4.57}$$

under $SU(3) \times SU(3)$, with $\mathbf{3}$ the fundamental of $SU(3)$.

In the end, following this approach one concludes that the symmetry group that consistently implements the $SO(8)$ symmetry in the Yukawa sector is $G_c \times SO(8) \times U(1)_X$. The most interesting point of this analysis is the fact that the symmetry group does not distinguish what belongs to the SM or the twin SM; in this sense both sectors are unified.

*A possible option for G_c is $SU(9)$, which has well defined decomposition into $SU(3) \times SU(3)$ [63, 75].

4.2.3 Computation of the Higgs mass

The next step is to compute the Higgs mass at 1-loop level. The CW potential with cut-off regularization si used (given in Eq. (3.130)) and computed in the unbroken phase of the global symmetry. The contributions to the CW potential are split in two pieces: the ones from the SM and the ones from the neutral sector. Both of them are related by the Z_2 symmetry, hence contribute in the same way.

Let us start with the contributions from the gauge bosons. In the unbroken phase none of the gauge bosons have mass, so their field dependent masses are

$$M_W^2(H^{(i)}) = \frac{1}{2}g^2\left|H^{(i)}\right|^2, \quad M_B^2(H^{(i)}) = \frac{1}{2}(g')^2\left|H^{(i)}\right|^2, \tag{4.58}$$

for the $W^{a(i)}$ and $B^{(i)}$ bosons, respectively. The sum of the potentials of both sectors is thus

$$\begin{aligned} V_{\text{CW, vector}} &= \frac{\Lambda^2}{64\pi^2}\Big(9g^2 + 3(g')^2\Big)|\mathscr{H}|^2 - \qquad (4.59)\\ &-\frac{\left|H^{(1)}\right|^4}{256\pi^2}\left[9g^4\left(\ln\frac{2\Lambda^2}{g^2|H^{(1)}|^2} + \frac{1}{2}\right) + 3(g')^4\left(\ln\frac{2\Lambda^2}{(g')^2|H^{(1)}|^2} + \frac{1}{2}\right)\right] - \\ &-\frac{\left|H^{(2)}\right|^4}{256\pi^2}\left[9g^4\left(\ln\frac{2\Lambda^2}{g^2|H^{(2)}|^2} + \frac{1}{2}\right) + 3(g')^4\left(\ln\frac{2\Lambda^2}{(g')^2|H^{(2)}|^2} + \frac{1}{2}\right)\right], \end{aligned}$$

Note that the first term in the equation above protects the Higgs mass from Λ^2 due to the $SO(8)$ symmetry, while the terms that depend logarithmically do not.

The corrections from the interactions in Eq. (4.1) of the EW doublets with themselves introduce further terms to the CW potential. Notwithstanding, they are all $SO(8)$ symmetric, because the field dependent masses are manifestly $SO(8)$ symmetric,

$$M^2_{H^{(i)}}(\mathscr{H}) = -M^2 + 2\lambda|\mathscr{H}|^2, \tag{4.60}$$

hence the CW potential $V_{\text{CW, scalar}}$ generated from the interactions of the scalar sector do not contribute to the Higgs mass.

Let us calculate the corrections from the fermion sector. In section 4.2.2 two distinct approaches were developed: one that did not enforced the $SO(8)$ symmetry on the Yukawa Lagrangian and another that did. The latter approach forced us to promote the fermion fields and the symmetry group of the model, which added much model dependency to our theory, for instance in the colour group G_c and in the interactions of the gauge boson associated with $U(1)_X$. Another remark on this approach is the fact that the Yukawa Lagrangian in Eq. (4.56) is $SO(8)$ symmetric, therefore does not directly contribute to the Higgs mass through the CW potential. If the contributions from the gauge boson sector could alone achieve spontaneous EWSB, this wouldn't be an issue, but it will be seen below that the potential in Eq. (4.59) is insufficient to trigger spontaneous EWSB; a contribution from the fermions sector, which has the correct sign, is needed. This problem could be avoided by adding $SO(8)$-breaking operators to the Lagrangian in Eq. (4.56), but this, again, would be a source of model dependency. Given the aforementioned reasons, we abandon the approach of the $SO(8)$ symmetric Yukawa Lagrangian and proceed from now on with Lagrangian (4.50), which just assumes the Z_2 symmetry. Considering only the top quark, whose field dependent mass is

$$M_t^2\left(H^{(i)}\right) = |y_t|^2 \left|H^{(i)}\right|^2, \tag{4.61}$$

we obtain the CW potential

$$\begin{aligned} V_{\text{CW, top}} &= -\frac{3|y_t|^2\Lambda^2}{8\pi^2}|\mathscr{H}|^2 + \\ &+\frac{3|y_t|^4}{16\pi^2}\left[\left|H^{(1)}\right|^4\left(\ln\frac{\Lambda^2}{|y_t|^2|H^{(1)}|^2}+\frac{1}{2}\right)+\left|H^{(2)}\right|^4\left(\ln\frac{\Lambda^2}{|y_t|^2|H^{(2)}|^2}+\frac{1}{2}\right)\right]. \end{aligned} \tag{4.62}$$

The full CW potential is given by

$$V_{\text{CW}} = V_{\text{CW, scalar}} + V_{\text{CW, top}} + V_{\text{CW, vector}}. \tag{4.63}$$

The Higgs mass parameter M_H^2 is computed from the second derivative with respect to the Higgs field at $H = 0$. This calculation leads us to

$$M_H^2 = -\frac{3f^2}{16\pi^2}\left[|y_t|^4\left(\ln\frac{2\Lambda^2}{|y_t|^2 f^2}-1\right)-\frac{3g^4}{16}\left(\ln\frac{4\Lambda^2}{g^2 f^2}-1\right)-\frac{(g')^4}{16}\left(\ln\frac{4\Lambda^2}{(g')^2 f^2}-1\right)\right], \tag{4.64}$$

which is clearly negative due to the top Yukawa being much larger than the gauge couplings. Hence, spontaneous EWSB is triggered through quantum effects. To compute the physical Higgs mass, i.e. the mass of the radial excitation h, we use the quartic coupling, given by the fourth derivative of the CW potential

$$\begin{aligned}\lambda &= \frac{\mathrm{d}^4 V_{\mathrm{CM}}}{\mathrm{d}h^4} \\ &= \frac{3}{\pi^2}\left[|y_t|^4\left(\ln\frac{2\Lambda^2}{|y_t|^2 f^2}-\frac{5}{4}\right)-\frac{3g^4}{16}\left(\ln\frac{4\Lambda^2}{g^2 f^2}-\frac{5}{4}\right)-\frac{(g')^4}{16}\left(\ln\frac{4\Lambda^2}{(g')^2 f^2}-\frac{5}{4}\right)\right].\end{aligned} \tag{4.65}$$

A remark regarding the approximate expression for the CW potential used in section 4.2.1 is in order. In Eq. (4.19) the approximation that $\Lambda \gg f$ was used, which corresponds to neglect the fluctuations around the vev and to substitute the fields by their vev in the logarithms in Eqs. (4.59) and (4.62). With such approximation one obtains the same results as in Eqs. (4.64) and (4.65) for the Higgs mass parameter and quartic coupling, respectively, but without the constant coefficients. In short, for $\Lambda \gg f$ one has

$$\lambda \simeq \kappa(\Lambda) = \frac{3}{\pi^2}\left[|y_t|^4 \ln\frac{2\Lambda^2}{|y_t|^2 f^2}-\frac{3g^4}{16}\ln\frac{4\Lambda^2}{g^2 f^2}-\frac{(g')^4}{16}\ln\frac{4\Lambda^2}{(g')^2 f^2}\right], \tag{4.66}$$

where the function $\kappa(\Lambda)$ is the same from Eq. (4.19).

Neglecting the contributions from the gauge bosons and also considering $\Lambda \gg f$, the physical mass of the Higgs m_h^2 is

$$\begin{aligned} m_h^2 &\simeq 4|\langle H\rangle|^2\lambda \\ &\simeq \frac{3f^2|y_t|^4}{8}\ln\frac{\Lambda^2}{M_t^2}, \end{aligned} \tag{4.67}$$

where we have used Eq. (4.32). To achieve $m_h = 125$ GeV from Eq. (4.67), still using $f = v$, one needs $\Lambda \simeq 2$ TeV.

4.3 Cosmological constrains

In section 4.2.1 we stressed that the TH model is not realistic, since it is not consistent with present data from experimental particle physics. This lead us to the conclusion that the Z_2 discrete symmetry must be somehow broken. In this section it will be seen that the TH model, as it is, is also incompatible with cosmological data.It will become clear that, in order to alleviate this tension, there need to be a source of Z_2-breaking.

Due to the exact Z_2 symmetry, the TH model predicted, in particular, that the twin particles have the same mass as their SM counterparts. As it was shown, this implied in several inconsistencies with EW data. Such predictions impacts not only particle physics, but also other areas of physics, for instance cosmology. The evolution of the universe is tightly constrained by particle physics because in the early universe, when the temperature was high and neutral matter had yet to be created, the subatomic interactions between the particles dictated how the universe evolved [76, 77]. Conversely, a beyond the SM (BSM) model can be confronted with cosmological data in order to see if it predicts the correct evolution of the universe. Many of the relevant parameters obtained from cosmological observations have a much higher precision than their counterparts in particle physics, hence can constrain BSM models much better. However, by using cosmological data, one is assuming the universe to be described by the standard model of cosmology ΛCDM, which is characterised by 6 parameters [78, 77]. Appendix C is referred to a brief introduction to the ΛCDM and some of the most basic concepts of cosmology. What we need to keep in mind is that, although cosmology offers precise data, they are qualitatively different from the ones of particle physics, in the sense that both assume distinct theoretical and experimental backgrounds. With this in mind, we proceed to understand how cosmology constrains the TH model.

One of the most important quantities in cosmology relevant to particle physics is the *effective number of neutrinos*, denoted by $N_{\rm eff}$ [77, 79, 80]. This parameter represents the amount of degrees of freedom (dof) that, together with the photon, contributes to the energy density associated with radiation. At later times, after the radiation domination epoch, the latter can be written as

$$\begin{aligned} \rho_{\rm rad} &= \rho_\gamma + \rho_{\rm rest} \\ &= \frac{\pi^2 T_\gamma^4}{30}\left[2 + \sum_i n_i \left(\frac{T_{{\rm dec},i}}{T_\gamma}\right)^4\right], \end{aligned} \tag{4.68}$$

where the sum is over all particles that contribute to radiation density, n_i is the number of dof (times $\frac{7}{8}$ for fermions), T_γ is the photon temperature and $T_{{\rm dec},i}$ is the decoupling temperature of the ith-particle (in other words, the temperature at which the particle and the photons ceased to be in thermal equilibrium). The expression above is conventionally rewritten as

$$\rho_{\rm rad} = \rho_\gamma\left[1 + \frac{7}{8}\left(\frac{T_\nu}{T_\gamma}\right)^4 N_{\rm eff}\right], \tag{4.69}$$

with T_ν the decoupling temperature of the neutrinos, for which $\frac{T_\nu}{T_\gamma} \simeq \left(\frac{4}{11}\right)^{\frac{1}{3}}$, and*

$$N_{\text{eff}} = \frac{8}{7}\left(\frac{T_\gamma}{T_\nu}\right)^4 \sum_i \frac{n_i}{2}\left(\frac{T_{\text{dec},i}}{T_\gamma}\right)^4. \tag{4.70}$$

One the one hand, in the SM, only the three neutrinos contribute significantly to ρ_{rad} and gives us

$$N_{\text{eff, SM}} = 3, \tag{4.71}$$

where the tiny effects from neutrino oscillations and non-equilibrium dynamics are suppressed [81]. On the other hand, in TH model, not only the SM neutrinos, but also the twin photon and twin neutrinos contribute to the radiation energy density. In this case the effective number of neutrinos reads

$$N_{\text{eff, TH}} = N_{\text{eff, SM}} + \frac{8}{7}\left(\frac{T_{\gamma^{(2)}}}{T_{\nu^{(1)}}}\right)^4 + 3\left(\frac{T_{\nu^{(2)}}}{T_{\nu^{(1)}}}\right)^4. \tag{4.72}$$

The decoupling temperatures is defined as the one for which the scattering rate is of the same order as the Hubble parameter. Hence, T_{dec} is a function of the coupling constants and the masses of the particles involved. Due to the Z_2 symmetry, the twin sector is identical to the SM, so the decoupling temperatures are also the same. Therefore,

$$N_{\text{eff, TH}} - N_{\text{eff, SM}} = \frac{8}{7}\left(\frac{11}{4}\right)^{\frac{4}{3}} + 3 \simeq 7. \tag{4.73}$$

The most recent experiments from the PLANCK collaboration gives the value of N_{eff} [82],

$$N_{\text{eff}} = 2.99(17), \tag{4.74}$$

in this way Eq. (4.73) is clearly incompatible with Eq. (4.74)†.

Another remark regarding T_{dec} is in order. In Eq. (4.73) the interactions between the two sectors was not taken into account, because these interactions, which are mediated by the Higgs, are highly suppressed. Hence, the decoupling temperature of the twin particles

*One must observe that Eq. (4.69) is misleading, because it gives the idea that only neutrinos contribute to ρ_{rad}, but it is clear from Eq. (4.70) that other particles can contribute to the radiation energy density.

†The recent tension regarding the Hubble parameter can be reflected to a $\mathcal{O}(1)$ uncertainty in N_{eff}. Even so, it cannot explain Eq. (4.73) [79, 83]

shown in Eq. (4.72) depends mostly on the interactions of the neutral sector. There is, however, another operator allowed by Z_2 symmetry, namely [32]

$$\mathscr{L}_{\text{mix}} = \frac{\epsilon}{2} B^{(1)}_{\mu\nu} B^{(2)}_{\mu\nu}, \tag{4.75}$$

which is a kinetic mixing between the $U(1)_Y$ gauge bosons. Considering the Lagrangian in Eq. (4.75), one sees that the SM and the twin SM can interact more at tree-level, this in turn implies that both sectors stay longer in thermal equilibrium and therefore raise not only $T_{\gamma^{(2)}}$ and $T_{\nu^{(2)}}$, but also $T_{\gamma^{(1)}}$ and $T_{\nu^{(1)}}$.

Once more we experience the disastrous phenomenological consequences of an exact Z_2 symmetry. To reduce the contribution from the neutral sector in Eq. (4.73) one must lower the decoupling temperatures $T_{\gamma^{(2)}}$ and $T_{\nu^{(2)}}$, and also avoid that the SM and the twin SM remain too long in thermal equilibrium. This can be done in three ways. First, one can create an hierarchy between both sectors, such that $f > v$ [16, 15, 73, 79]. In this way all massive particles from the twin sector become heavier than the SM ones and as a consequence contribute less to radiation energy density. Moreover, SM particles do not annihilate so often into twin particles, hence the twin sector is not re-heated. Second, one may reduce the couplings that connect the SM with the twin sector, in particular the ϵ coupling in Eq. (4.75). With a small kinetic mixing, the twin particles interact less with the SM and decouple from it much earlier. Third, in order to specifically reduce $T_{\gamma^{(2)}}$, which contributes the most in Eq. (4.73), one can break twin QED and let the twin photon become massive [15]. From the three methods aforecited, the first is the simplest and easily implemented (see section 4.4), while the third is the most radical and model-dependent one. Note that both of them break Z_2 explicitly. The second method does not break Z_2, but, since there is no theoretical justification to impose that ϵ is very small or even zero, it is a source of tuning.

⋆⋆⋆

It is clear that cosmological constrains are not satisfied by TH with an exact Z_2 symmetry. The issue with N_{eff} is the most prominent, but not the only one. Other problems are, for example, if the twin matter can be interpreted as Dark Matter (DM) [32]. Such questions are much more subtle and are not the focus of this thesis. What we wish to stress in this section is, that cosmology introduces many difficulties to the construction of a phenomenologically acceptable TH model and offers the most stringent experimental bounds.

4.4 Z_2 breaking & Fine-Tuning

In this section it is discussed how to introduce an explicit break of the Z_2 symmetry, motivated by the discussion in sections 4.2 and 4.3. Moreover, the fine-tuning (FT) associated with this breaking is computed.

Our main goal is to raise the masses of the twin particles in a way the Higgs mass is still protected from quadratic divergences. The masses of all particles come from the Higgs mechanism, therefore an hierarchy between the vevs v and f must be created. This implies that there must be a source of Z_2-breaking in Eq. (4.1), which will modify the minimisation of the potential. The new potential of the scalar sector is

$$V\left(H^{(1)}, H^{(2)}\right) = -M^2\left(\left|H^{(1)}\right|^2 + \left|H^{(2)}\right|^2\right) + \lambda\left(\left|H^{(1)}\right|^2 + \left|H^{(2)}\right|^2\right)^2 + \Delta M^2\left|H^{(1)}\right|^2. \quad (4.76)$$

The potential above contains an extra $\left|H^{(1)}\right|^2$ operator compared to potential (4.1), which is the only non-trivial, soft operator allowed* and obviously breaks the Z_2 symmetry of the potential.

Note that the potential (4.76) also breaks $SO(8)$ explicitly, so the minimisation must be done separately for $\left|H^{(1)}\right|^2$ and $\left|H^{(2)}\right|$. However, ΔM^2 is taken to be small enough to still consider the global SSB as a good approximation. The twin EW doublet $H^{(2)}$ acquires the following vev,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}|H^{(2)}|^2}\right|_{H^{(2)}=\left\langle H^{(2)}\right\rangle} = 0 \Rightarrow \left\langle H^{(2)}\right\rangle = \frac{f}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}, \quad (4.77)$$

where $\left\langle H^{(2)}\right\rangle$ is aligned with the unbroken twin QED direction and $f^2 = \frac{M^2}{\lambda}$. The vev above triggers $SO(8) \to SO(7)$ and the fields can be thus parametrized non-linearly in terms of H, according to Eq. (4.25). For a cut-off $\Lambda \gg f$, the potential in Eq. (4.76)

*Another possible operator is $\left|H^{(2)}\right|^2$, but its effects are equivalent to the ones of $\left|H^{(1)}\right|^2$.

becomes

$$V(\sigma, H) = \cdots + \frac{f^4\kappa(\Lambda)}{4}\left[1 - \frac{1}{2}\sin^2\frac{2\sqrt{2}|H|}{f}\right] + \frac{\Delta M^2 f^2}{2}\sin^2\frac{\sqrt{2}|H|}{f}, \tag{4.78}$$

where the dots denote constant ($SO(8)$ symmetric) terms and interacting terms between H and σ, and $\kappa(\Lambda)$ is the same function as in Eq. (4.66), that introduces 1-loop quantum corrections from fermions and gauge bosons. One can then minimise with respect to H,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}|H|}\right|_{H=\langle H\rangle} = 0 \Rightarrow \sin^2\frac{\sqrt{2}|\langle H\rangle|}{f} = \frac{1}{2}\left(1 - \frac{\Delta M^2}{f^2\kappa(\Lambda)}\right). \tag{4.79}$$

Note that the expression above reduces to Eq. (4.32) in the case $\Delta M^2 = 0$. In order to obtain an hierarchy between the Higgs vev and the SSB scale, i.e. $\frac{f}{\sqrt{2}} \gg |\langle H\rangle|$, the right-hand side of Eq. (4.79) must be much smaller than 1. Therefore, one can Taylor expand Eq. (4.79) and obtain

$$|\langle H\rangle|^2 = \frac{f^2}{4}\left(1 - \frac{\Delta M^2}{f^2\kappa(\Lambda)}\right). \tag{4.80}$$

Before computing the physical mass of the Higgs, one needs to diagonalize the mass matrix of the scalar sector in order to obtain the correct linear combination of radial resonances, which is identified with the neutral Higgs h. The mass matrix obtained in Eq. (4.37) is modified in two aspects. First, Eq. (4.32) is not valid anymore, hence there is a non-vanishing mix between σ and ρ. Second, one needs to add the contribution from the soft-breaking operator. Naming $\frac{V}{\sqrt{2}} = |\langle H\rangle|$, this yields the new mass matrix

$$\mathcal{M}' = \begin{pmatrix} 2\lambda f^2 + \Delta M^2\sin^2\frac{V}{f} - \delta m^2 + f^2\kappa(\Lambda)\left(1 - \frac{1}{2}\sin^2\frac{2V}{f}\right) & \Delta M^2\sin\frac{2V}{f} - \frac{3\kappa(\Lambda)f^2}{2}\sin\frac{4V}{f} \\ \Delta M^2\sin\frac{2V}{f} - \frac{3\kappa(\Lambda)f^2}{2}\sin\frac{4V}{f} & \Delta M^2\cos\frac{2V}{f} - \kappa(\Lambda)f^2\cos\frac{4V}{f} \end{pmatrix}. \tag{4.81}$$

Since now there is a clear hierarchy between the scales, $f \gg V$, the off-diagonal elements are much smaller then the diagonal ones*. Hence, in a first approximation, σ and ρ are the eigenstates of $\mathcal{M}'$, where ρ is associated with h.

*A matrix $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ has eigenvalues $\lambda_{1,2} = \frac{1}{2}\left[a + d \mp \sqrt{(a-d)^2 + 4b^2}\right]$. In the case of the matrix

Taking such considerations into account, the physical mass of the Higgs is already computed in Eq. (4.81). It reads

$$m_h^2 \simeq \Delta M^2 \cos\frac{2V}{f} - \kappa(\Lambda) f^2 \cos\frac{4V}{f}. \tag{4.82}$$

Using Eq. (4.80) and imposing that $|\langle H\rangle| = \frac{v}{\sqrt{2}}$, we obtain

$$m_h^2 \simeq 4|\langle H\rangle|^2 \kappa(\Lambda). \tag{4.83}$$

To obtain the correct value for m_h^2 we must impose that the quartic coupling $\kappa(\Lambda)$ to be around 0.13, as in the SM [3]. To obtain such values we must create an amount of FT.

The quartic coupling introduces FT, because we must maintain an hierarchy $\Lambda \gg f$. However, since $\kappa(\Lambda)$ depends logarithmcally on Λ and f, the associated FT is order unity, hence negligible. The main source of FT comes from the physical mass m_h^2. The FT of the mass with respect to the breaking parameter ΔM^2 is given by Eq. (1.16)

$$\Delta = \left|\frac{\Delta M^2}{m_h^2}\frac{\partial m_h^2}{\partial \Delta M^2}\right| = \frac{\Delta M^2}{m_h^2}. \tag{4.84}$$

For $f = 1$ TeV, one needs $\Delta M^2 \simeq (330 \text{ GeV})^2$ to achieve the correct value of the Higgs mass. Hence, $\Delta \simeq 7$, which is acceptable. However, if $f = 5$ TeV, then $\Delta M^2 \simeq (1700 \text{ GeV})^2$, therefore $\Delta \simeq 200$.

⋆⋆⋆

The potential in Eq. (4.76) represents the most straightforward way to introduce the Z_2-breaking in the TH model. With it, one can raise the value of f and consequently raise the masses of the twin particles at the cost of an order 100 FT. This method of breaking the Z_2 symmetry does not spoil the mechanism that protects the Higgs mass, since it is only introduced via super-renormalizable operators. Other ways of implementing the Z_2-breaking, for instance by letting the twin photon to become massive or to create an hierarchy between the couplings of both sectors, require much model building and will certainly re-introduce the quadratic divergences to the Higgs mass.

(4.81) we have $a \gg b, d$, hence

$$\lambda_1 \simeq \frac{d}{2} \ll \lambda_2 \simeq \frac{1}{2}(2a+d).$$

4.5 Conclusions

The TH model is the first solution to the HP that does not rely on particles charged under the SM to stabilise the Higgs mass. As a consequence, the constrains from EW precision measurements are less stringent; with an appropriate amount of Z_2-breaking, the mass scale of the twin sector is such that all constrains from EW precision measurements are satisfied, at the cost of a FT $\Delta \sim \mathcal{O}(10^2)$ (while neglecting the kinetic-mixing between the abelian gauge bosons). This solution is particularly interesting, because one may introduce an arbitrary number of neutral sectors, offering rich model-building prospects. Moreover, the phenomenology of such models is not complicated, since the gauge structure and particle content of such sectors are identical to the ones of the SM. The main difficulty lies on the cosmological bounds. It is not straightforward to correctly describe the evolution of the universe with such models, without introducing an unreasonable amount of FT. To summarise, Neutral Naturalness models offer an original way of solving the HP, but, in order to be phenomenologically consistent, must be tuned.

5 Cosmological Relaxation Models

5.1 Motivations

In Chapter 4 the idea of Neutral Naturalness was developed, whose purpose was to solve the Hierarchy Problem (HP) while evading most of the experimental constrains from particle and collider data. It was found out, however, that essentially every implementation of this idea is in disagreement with cosmological data. Although the theoretical and experimental frameworks of cosmology are distinct from the ones of particle physics, the early universe is described by the dynamics of the elementary particles and therefore the latter affects the cosmological history as a whole. As a consequence, one cannot ignore cosmology when formulating Beyond the Standard Model (BSM) theories. In the discussion of the Twin Higgs (TH) model cosmological constrains (section 4.3) were introduced after formulating the model. From our experience, it did not turn out to be a good strategy to impose cosmological considerations *after* the model was formulated, as one needed to introduce an unreasonable amount of Fine Tuning to satisfy them. Instead, a more compelling approach to this embarrassing situation would be to take cosmology seriously from the start, in other words, one should use cosmology as a guiding principle and not just as a source of experimental constrains. Note that we are in any way proposing to completely fuse cosmology with particle physics, but to bring them somewhat closer. In light of this, how should one proceed? One should first note that the most important objective of cosmology is to describe the evolution of the universe and justify why it is what we observe today. Hence, one must somehow include this evolution into our model-building and, if the HP is to be solved, the Higgs shall be tightly related to it.

With such motivation, let us determine what does the cosmological evolution of the early universe, when elementary particles were the relevant degrees of freedom (dof), add to their description. One particularly interesting effect is that of temperature. At primordial times the universe was an almost uniform hot soup of particles, i.e. it was a thermodynamical system. Hence, the early universe possessed thermal energy and the equilibrium temperature T at a given time t thus determined the energy scale of the particles in this thermal bath. A non-zero temperature has significant impact on Quantum Field Theories (QFT's) since the temperature enters in the expression for the generating functional, which in this case is most often called partition function. Therefore, in addition the usual quantum corrections, there are *thermal corrections* [84]. For a scalar field, for instance the Higgs, the 1-loop corrected potential can be written as

$$V_{\text{eff}}(H,T) = V_0(H) + V_1(H,T), \tag{5.1}$$

where V_0 is the tree-level potential,

$$V_0(H) = -M_H^2|H|^2 + \lambda|H|^4, \tag{5.2}$$

and V_1 is the 1-loop contribution from thermal effects, which must satisfy the condition

$$V_1(H,T) \xrightarrow{T\to 0} 0, \tag{5.3}$$

since there is no thermal bath when $T = 0$. For $T \neq 0$, one can Taylor expand V_1 and obtain the following expression for the effective potential

$$V_{\text{eff}}(H,T) = M_{H,\text{eff}}^2(T)|H|^2 - rT|H|^3 + \lambda_{\text{eff}}(T)|H|^4, \tag{5.4}$$

where $r > 0$. The new coefficients λ_{eff} and $M_{H,\text{eff}}^2$ depend on the temperature and on the relevant dof (e.g. the masses of the particles that couple to the Higgs). The effective mass may be put into the form

$$M_{H,\text{eff}}^2(T) = -M_H^2 + cT^2, \tag{5.5}$$

with $c > 0$. In our expanding universe the temperature depends on time and decreases as the universe expands (see Appendix C, Eq. (C.11)), hence $M_{H,\text{eff}}^2$ is a dynamical variable, in the sense that it depends on time. Moreover, its derivative with respect to time is proportional to the Hubble parameter H_b,

$$\dot{M}_{H,\text{eff}}^2 = -2cT^2 H_b(t), \tag{5.6}$$

leaving explicit that the dynamics of $M_{H,\text{eff}}^2$ depends on the evolution and particle content of the universe and on the temperature.

Temperature indeed plays a significant role in the description of the Higgs in the early universe. Eqs. (5.5) and (5.6), however, do not offer a solution to the HP and the reason is simple. Temperature effects, how important they were at the very beginning of the universe, become less important as time passes. In particular, the temperature today is so small compared to the Electroweak (EW) scale that

$$M_{H,\text{eff}}^2 \simeq -M_H^2, \quad \dot{M}_{H,\text{eff}}^2 \simeq 0. \tag{5.7}$$

In conclusion, as far as today's experiments are concerned, temperature effects are negligible and they cannot possibly solve the HP.

We see clearly from the discussion above that, in order to solve the HP, the cosmological evolution must induce some change in the Higgs mass parameter at zero temperature. This naturally implies that M_H^2 must be somehow dynamical. In particular, one should expect from this dynamics the exact opposite of thermal corrections, i.e. the dynamical effects on M_H^2 shall be more relevant in the IR scale rather than in the UV, because the Higgs mass must be stabilised at its experimental value in the IR and not in the UV. Models that implement this idea are known as *Cosmological Relaxation* (CR) models [17].

To summarise, CR models make the Higgs mass parameter dynamical, i.e. it varies with time as the universe itself expands. As stressed above, the dynamics we refer to is not the one coming from thermal effects, since they cannot solve the HP and are irrelevant in the IR. Instead, CR models introduce a new mechanism that will render the zero-temperature mass M_H^2 dynamical and stable at the EW scale.

5.2 General EFT approach

5.2.1 Back-reaction mechanism

The objective is to write an EFT that implements the idea behind CR models and that will allow us to make general statements on such models. To this end, the Lagrangian from Eq. (1.1) is written down,

$$\mathscr{L} = \frac{\Lambda^4}{g_*^2} \mathscr{F}\left(\frac{D_\mu}{\Lambda}, \frac{H}{\Lambda}, \frac{\phi}{\Lambda^{\dim\phi}}\right), \tag{5.8}$$

where Λ is the cut-off of the EFT, g_* is a dimensionless coupling, D_μ is the appropriate covariant derivative and H is the Higgs field. The other parameter ϕ in the function $\mathscr{F}$ is defined as all other relevant dof of the theory, regardless if they are from the SM or from BSM, that connect the Higgs to the evolution of the universe. In other words, ϕ will be responsible for the dynamics of M_H^2.

The next step is to introduce some of the concepts of cosmology in Eq. (5.8). The expansion of the universe may be separated in two pieces: the homogeneous and isotropic expansion, and its perturbations. The first is characterised by the classical Einstein's equations and the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, given respectively by Eqs. (C.1) and (C.3) in Appendix C, while the latter are described by the perturbed Einstein's equations and the fluctuations of the classical metric and matter fields [77, 85]. Since ϕ is the bridge between the Higgs and the cosmic evolution, it can be decomposed as

$$\phi(x) = \bar{\phi}(t) + \delta\phi(x), \tag{5.9}$$

where $\bar{\phi}(t)$ represents the classical, non-perturbed evolution of the universe and $\delta\phi(x)$ is the quantum fluctuation, which is tightly related to the anisotropies and inhomogeneities of the cosmological evolution. As it is usual in cosmology, the analysis will be first focused on the classical evolution and than later on the quantum effects.

The classical behaviour of the CR mechanism will be described by the classical mode $\bar{\phi}$ in the tree-level Lagrangian. Eq. (5.8) is rewritten as

$$\mathscr{L}_0 = \frac{\Lambda^4}{g_*^2}\mathscr{F}_0\left(\frac{D_\mu}{\Lambda}, \frac{H}{\Lambda}, \frac{\bar{\phi}}{\Lambda^{\dim\phi}}\right), \tag{5.10}$$

where the subscript 0 means that this is the tree-level Lagrangian. To study the phenomenology of Eq. (5.10) one would usually expand it in the low energy regime as series of operators. In the present case, however, one must be more careful for two reasons. First, one neither know if there are any additional global selection rules nor the Lorentz and gauge structure of ϕ, which makes it impossible to appropriately write down operators without assumptions on ϕ. Second, although the cosmological relaxation becomes more important at the IR, this mechanism will first take place at the far UV (at a primordial era of the universe), where one is not allowed to expand the Lagrangian. On the one hand, the first point is a real issue and will depend on our model-building. It will be discussed in more detail in section 5.2.3. On the other hand, the latter issue is not so problematic, because one can simply go to the low-energy regime where the expansion is valid. The subtlety lies in the initial conditions of $\bar{\phi}$; as the explicit expression of the UV Lagrangian is unknown, one cannot give a precise value for its initial condition at the UV, hence, for the mechanism to properly work, it must not depend heavily on the initial conditions of $\bar{\phi}$ *. In order to understand the mechanism behind the CR models, we will proceed without expanding the Lagrangian in Eq. (5.10).

Let us now discuss how M_H^2 and $\bar{\phi}$ interact. By definition, $\bar{\phi}$ must render M_H^2 dynamical, however a definition for the mass parameter is lacking, due to the insistence in not writing Eq. (5.10) in terms of operators. An alternative definition for the tree-level mass parameter in terms of $\mathscr{F}_0$ is proposed,

$$M_H^2 \equiv -\frac{\Lambda^4}{g_*^2}\left(\frac{\partial \mathscr{F}_0}{\partial |H|^2}\right)_{H=0}, \tag{5.11}$$

which recovers the usual definition when one Taylor-expands $\mathscr{F}_0$. Note that the equation above defines an effective mass, in the sense that it may depend on the fields that couple to $|H|^2$. Another remark to Eq. (5.11) is that it is assumed that $|H|^2$ is a gauge invariant operator at all energy scales below the cut-off Λ (which could not be true at the UV if, for instance, the Higgs were part of a multiplet of a larger gauge group). If M_H^2 is to vary with time, its derivative with respect to time,

$$\begin{aligned} \dot{M}_H^2 &= -\frac{\Lambda^4}{g_*^2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{F}_0}{\partial |H|^2}\right)_{H=0} \\ &= -\frac{\Lambda^4}{g_*^2}\left[\dot{\bar{\phi}}\left(\frac{\partial^2 \mathscr{F}_0}{\partial \bar{\phi}\partial |H|^2}\right)_{H=0} + \dot{\bar{\phi}}^\dagger\left(\frac{\partial^2 \mathscr{F}_0}{\partial \bar{\phi}^\dagger\partial |H|^2}\right)_{H=0} + \cdots\right], \end{aligned} \tag{5.12}$$

where the dots denote terms that depend on higher derivatives of $\bar{\phi}$, must be non-zero. From Eq. (5.12) one sees that this is satisfied under the following conditions

$$\dot{\bar{\phi}} \neq 0 \quad \text{and} \quad \left(\frac{\partial^2 \mathscr{F}_0}{\partial \bar{\phi}\partial |H|^2}\right)_{H=0} \neq 0 \Rightarrow \dot{M}_H^2 \neq 0, \tag{5.13}$$

where it is assumed that the contributions from higher derivatives are negligible. In section 5.2.3 it will become clear how important these contributions are, but we anticipate that they are very suppressed and therefore Eq. (5.13) is a very good approximation. The first condition of Eq. (5.13), $\dot{\bar{\phi}} \neq 0$, is trivially satisfied if the Equations of Motion (EoM) of $\bar{\phi}$ have non-constant solutions. The second condition on the second derivative of $\mathscr{F}_0$ is not

*Note that this discussion is very similar, if not identical to the motivation of the Fine Tuning (FT) measure in section 1.2. In some sense, it is required from the start that the model is free of FT coming from the initial conditions of $\bar{\phi}$.

so straightforward, but, if one analyses the low-energy regime, it means that a non-trivial interaction between $\bar{\phi}$ and $|H|^2$ must take place. At this point it is impossible to give an explicit expression for such interaction at the IR, because, as stressed, the Lorentz structure of $\bar{\phi}$ is still unknown.

Let us discuss in more detail the first condition in Eq. (5.13). From Eq. (5.12) it is clear that M_H^2 is directly proportional to the time-derivative of $\bar{\phi}$. In general, the EoM of $\bar{\phi}$ will depend on its potential V_0, that is defined by

$$V_0(\bar{\phi}) \equiv -\frac{\Lambda^4}{g_*^2}\mathscr{F}_0\Big|_{H=0,\ \partial_\mu=0}, \tag{5.14}$$

and on the interactions with the Higgs*. Since it is assumed that $\bar{\phi}$ interacts only via $|H|^2$, or that at least it is the most relevant interaction with the Higgs field, the Higgs will contribute to the EoM with terms proportional to its vacuum expectation value (vev). From this perspective, it is obvious that the vev of the Higgs plays a crucial role in the evolution of $\bar{\phi}$, which of course depends on the value and the sign of M_H^2. It is clear from this viewpoint that one can separate the analysis in two parts: the case where at the initial time $t_0 \sim \Lambda^{-1}$ the Higgs mass parameter was positive, and the case where it was negative. In the first situation the Higgs has a vanishing vev and therefore does not contribute to the EoM of $\bar{\phi}$. Even so, M_H^2 must evolve to its SM value, which is negative. In order to guarantee that M_H^2 will be dragged to the correct value today, one needs a non-trivial potential $V_0(\bar{\phi})$ to generate the dynamics of $\bar{\phi}$. The second case, in which the initial value of M_H^2 is already negative, is much more intricate. In this situation the vev is non-vanishing and would affect the EoM considerably. However, one must not forget the thermal effects of Eq. (5.5); if the initial value of M_H^2 is as large as the temperature, the correction from the temperature could cancel it and reduce the value of the physical vev. For this reason it is quite complicated to make statements on the dynamics of $\bar{\phi}$ in a model-independent way in this situation. To proceed with the discussion we suppose, without proof, that even in this case one needs a potential V_0 at some point to control the evolution of $\bar{\phi}$.

To summarise, it is crucial to have a non-trivial potential. However, this is not enough to successfully describe the CR mechanism, because in addition of having a dynamical

*If ϕ is a BSM field, one needs to consider the possibility of it interacting with the other field from the SM. We will return to this point in section 5.2.3.

mass parameter, one needs M_H^2 to converge to the SM value $M_{H,\text{SM}}^2$ as time passes. More precisely, for a general potential V_0, the field $\bar{\phi}$ will continue to evolve irrespective to the value of M_H^2, such that one cannot guarantee that $\dot{M}_H^2 \to 0$ as $M_H^2 \to M_{H,\text{SM}}^2$. The solution to this problem is conceptually simple: $\bar{\phi}$ must be aware of the value M_H^2 in a way that

$$\dot{\bar{\phi}} \to 0 \quad \text{as} \quad M_H^2 \to M_{H,\text{SM}}^2.$$

In other words, not only $\bar{\phi}$ affects M_H^2 but also M_H^2 must affect $\bar{\phi}$; only under such circumstances can one assure that the system is able to converge to the SM scenario.

Before proceeding with the discussion, let us be a bit more precise with what is meant with *converge*. Until now were trying to describe the dynamical evolution of $\bar{\phi}$ and M_H^2, i.e. a parametrisation of the pair $(\bar{\phi}, M_H^2)$ with respect to time. As it was seen, given a generic initial condition $(\bar{\phi}_0, M_{H,0}^2)$ at t_0, the EoM of $\bar{\phi}$ together with Eq. (5.12) dictate how it evolves with time. In particular, in order to observe the correct EW scale today, this pair must converge to

$$(\bar{\phi}, M_H^2) \xrightarrow{t \to t_f} (\bar{\phi}_\text{eq}, M_{H,\text{SM}}^2), \tag{5.15}$$

with t_f the final time and $\bar{\phi}_\text{eq} \equiv \bar{\phi}(t_f)$. To solve the Hierarchy Problem one needs the pair to be stable at the equilibrium point $(\bar{\phi}_\text{eq}, M_{H,\text{SM}}^2)$, this implies that

$$\frac{\partial V_0}{\partial \bar{\phi}}(\bar{\phi}_\text{eq}) = 0, \quad \frac{\partial^2 V_0}{\partial \bar{\phi}^\dagger \partial \bar{\phi}}(\bar{\phi}_\text{eq}) > 0. \tag{5.16}$$

From Eq. (5.12), the stability conditions of Eq. (5.16) are sufficient to also stabilise M_H^2 at the classical level, under the condition that the contributions from higher derivatives of $\bar{\phi}$ are suppressed. So, Eq. (5.15) is being referred when the system is said to *converge*, in which the pair $(\bar{\phi}, M_H^2)$ converges to the equilibrium point $(\bar{\phi}_\text{eq}, M_{H,\text{SM}}^2)$.

One remark regarding the initial conditions is in order. We stress again that the initial condition of $(\bar{\phi}, M_H^2)$ is not well determined, since it lies at the UV, where one has no details of the function $\mathscr{F}_0$. For this reason, the pair must converge to $(\bar{\phi}_\text{eq}, M_{H,\text{SM}}^2)$ for a wide variety of initial conditions*. Adding this condition to the stability one, one finds

*It must and can not be totally independent from the initial conditions, because, as pointed out, the explicit mechanisms for positive and negative $M_{H,0}^2$ are completely distinct. A review on many of the CR models with distinct initial conditions for the Higgs mass parameter is given in [18].

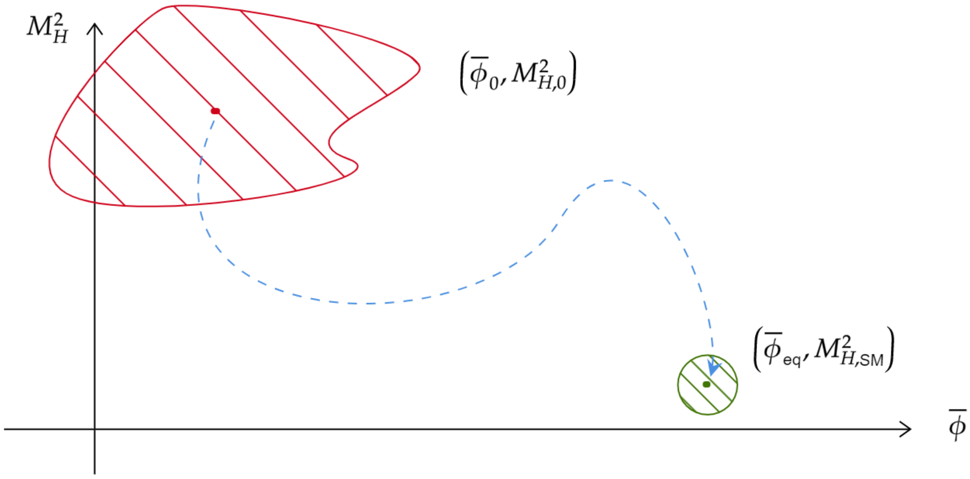


Figure 5: Illustration of the evolution of the initial condition $(\bar{\phi}_0, M_{H,0}^2)$ to the attractor point. For a wide range of initial conditions (red region), the system converges to a arbitrarily close neighbourhood of $(\bar{\phi}_{\rm eq}, M_{H,\rm SM}^2)$ (green region).

out that $(\bar{\phi}_{\rm eq}, M_{H,\rm SM}^2)$ must be an *attractor point* of the system. Stated in another way, for a generic initial condition at t_0, $(\bar{\phi}, M_H^2)$ will be arbitrarily close to the $(\bar{\phi}_{\rm eq}, M_{H,\rm SM}^2)$ after sufficient time and from then on will remain in its neighbourhood. See for instance Figure 5.

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In this section we have constructed what is called a *back-reaction mechanism*, i.e. a dynamical system in which the relevant dof depends on time such that their evolution equations are tightly connected. From Eqs. (5.11) and (5.12) it was determined how $\bar{\phi}$ affects M_H^2 and, in particular, how it renders the latter dynamic (see Figure 6). As it was seen, for this system to converge, one concluded that a back-reaction from M_H^2 is necessary, in other words, the Higgs mass parameter must somehow affect the dynamics of $\bar{\phi}$. However, there is no clear way of how this could possibly take place and it is simple understand why. The other solutions presented so far (Supersymmetry, Composite Higgs and Neutral Naturalness) rely on symmetries to solve the HP, which implies that the respective EFT's have some additional selection rules. As a consequence, for a given particle content the models can be uniquely determined. In the case of Cosmological Relaxation this is not true, since one relies on dynamics rather than symmetries, therefore there is no way to uniquely determine how the back-reaction mechanism works only from the general EFT approach.

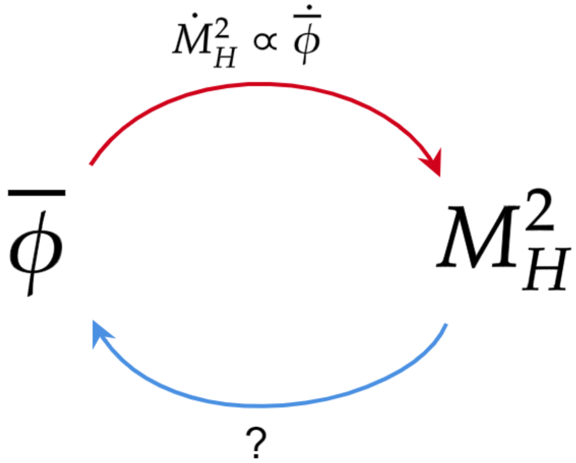

Figure 6: The back-reaction mechanism of $\bar{\phi}$ and M_H^2. The red arrow is given by Eqs. (5.11) and (5.12), whereas it is impossible to determine the blue arrow in a model independent way.

5.2.2 Quantum effects

The next step is to discuss the properties of quantum corrections to the back-reaction mechanism, in particular we will consider 1-loop corrections. The Lagrangian in Eq. (1.1) at 1-loop can be written as

$$\mathscr{L}_1 = \frac{\Lambda^4}{g_*^2}\left[\mathscr{F}_0\left(\frac{D_\mu}{\Lambda}, \frac{H}{\Lambda}, \frac{\phi}{\Lambda^{\dim\phi}}\right) + \frac{g_*^2}{16\pi^2}\mathscr{F}_1\left(\frac{D_\mu}{\Lambda}, \frac{H}{\Lambda}, \frac{\phi}{\Lambda^{\dim\phi}}\right)\right], \tag{5.17}$$

where $\mathscr{F}_0$ is the same tree-level function of Eq. (5.10), $\mathscr{F}_1$ is the function generated at 1-loop and ϕ is given by Eq. (5.9). Eq. (5.17) leaves manifest the difficulty of describing quantum effects in CR models only from the EFT approach: $\mathscr{F}_1$ is yet another unknown function and there is no indication of how to describe the quantum fluctuation $\delta\phi$. Indeed, it is not obvious how the quantum fluctuations of ϕ evolve during the cosmological evolution and what impact can they have today*.

Although there is not much to say about the evolution of the quantum fluctuations, one can analyse how they behave near the attractor point. Suppose that the classical mode is already stabilised, in this case the field ϕ is given by

$$\phi(x) = \bar{\phi}_{\text{eq}} + \delta\phi(x). \tag{5.18}$$

*In general they are not irrelevant. Take the example from the standard cosmology, where the quantum modes of the fields have evolved to the non-linear, large structures we observe today [77].

To study the dynamics of $\delta\phi$ in the neighbourhood of the attractor point, we shall perturb the potential around $\bar{\phi}_{\rm eq}$. Using Eq. (5.16), the tree-level potential can be Taylor-expanded as

$$V_0\left(\delta\phi + \bar{\phi}_{\rm eq}\right) - V_0\left(\bar{\phi}_{\rm eq}\right) \simeq \frac{1}{2}\delta\phi^\dagger \left.\frac{\partial^2 V_0}{\partial\phi^\dagger\partial\phi}\right|_{\bar{\phi}_{\rm eq}} \delta\phi. \tag{5.19}$$

The 1-loop correction to the potential is defined as

$$V_1(\phi) \equiv -\frac{\Lambda^4}{16\pi^2}\left.\mathscr{F}_1\right|_{H=0,\ \partial_\mu=0}, \tag{5.20}$$

which may be expanded as

$$V_1\left(\delta\phi + \bar{\phi}_{\rm eq}\right) - V_1\left(\bar{\phi}_{\rm eq}\right) \simeq \delta\phi^\dagger \left.\frac{\partial V_1}{\partial\phi^\dagger}\right|_{\bar{\phi}_{\rm eq}} + \left.\frac{\partial V_1}{\partial\phi}\right|_{\bar{\phi}_{\rm eq}} \delta\phi. \tag{5.21}$$

In the equation above higher order terms are neglected, because they do not contribute at 1-loop level. In conclusion, apart from interactions with the Higgs, the potential for $\delta\phi$ near $\bar{\phi}_{\rm eq}$ is quadratic and if one neglects the cosmic evolution, the resulting EoM are those of an harmonic oscillator with a constant source $\frac{\partial V_1}{\partial\phi}(\bar{\phi}_{\rm eq})$. If one takes into account the FLRW metric, then a damping term appears in the EoM (see for instance Eq. (C.27) for the case of a scalar field). Therefore, the quantum fluctuations of ϕ will eventually die out as time passes, which means that the field ϕ becomes asymptotically static. This, however, does not exclude phenomenological signals at accessible energies and the reason is the following. Depending on the Lorentz and gauge structure of ϕ, there could be a neutral, scalar component of ϕ that mixes with the Higgs [86, 87]. If the mixing term is considerably large, this could make ϕ observable at the IR. But, as it was stressed, this depends on the explicit choice of ϕ, so this possibility is not pursued in this section.

One last remark on quantum fluctuations regards the quantum tunnelling. Even if one develops a way to drive the classical mode $\bar{\phi}$ to the equilibrium point $\bar{\phi}_{\rm eq}$, there is no guarantee that quantum fluctuations in this vicinity are not able to tunnel to another minimum of the potential and in this manner destabilise the Higgs mass parameter at an observable rate. In order to not spoil the classical stabilisation, the life-time of the vacuum defined by $\bar{\phi}_{\rm eq}$ should be at least larger than the age of the universe [17]. Due to the many theoretical complications, the study of this phenomenon lies outside of the scope of this thesis. However, this issue will again be mentioned in section 5.2.3.

5.2.3 Minimal Model?

From the EFT approach one could determine the existence of a back-reaction mechanism at the classical level and how quantum fluctuations of ϕ behave near the equilibrium point. The Lorentz and gauge structure of ϕ, however, remain unknown. In this section it will be discussed which would be the most appropriate Lorentz representation for ϕ and how it could guide us in the model-building of CR models.

Before going into the details of the possible Lorentz structures that ϕ may possess, one needs first to determine if ϕ could be a field from the SM or if it must be a BSM field. To this end, let us suppose ϕ is a field from the SM and expand the tree-level Lagrangian (5.10) in the low-energy regime $E \ll \Lambda$,

$$\mathscr{L}_0 \supset \left[g_0\Lambda^2 + \sum_i \frac{g_i}{\Lambda^{\dim \mathcal{O}_i - 2}}\mathcal{O}_i\right]|H|^2, \tag{5.22}$$

where $\{g_i\}$ are coupling constants and $\{\mathcal{O}_i\}$ are all possible gauge-invariant* operators with SM fields, except the Higgs. Note that $\dim \mathcal{O}_i \geq 4$, therefore all operators $\mathcal{O}_i|H|^2$ turn out to be non-renormalizable and their impact in the IR is suppressed by powers of the cut-off. It is clear that irrelevant operators play a role similar to that of temperature, they become less important as the energy scale decreases. Hence, their effects are irrelevant in the IR and cannot be the underlying dof behind the CR mechanism. For the effects of some operator $\mathcal{O}$ in Eq. (5.22) to be relevant at the IR, it must have at most dimension 2. Since one cannot built such operator in the SM, it is thus necessary for ϕ to be a BSM field.

We list below all possible Lorentz-invariant operators with dimension ≤ 2 for each Lorentz structure that ϕ can have†. In this list the selection rules from global symmetry groups are not taken into account.

- Fermions

  Fermions are complex fields, hence can carry representations under the gauge group, which leads to some model-building dependence. Nevertheless, irrespective to the gauge-representation, if ϕ is a fermion field ψ, then the lowest dimensional Lorentz

*Here gauge invariance refers to invariance under the SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$, since we are already in the low-energy regime.

†In order to keep the discussion more simple, it is excluded from this listing the spin-2 and spin-3/2 cases, since they are intrinsically connected to gravity and Supersymmetry, respectively [21, 35]

invariant operator is $\bar{\psi}\psi$. This operator has dimension 3 and therefore ϕ cannot be a fermion.

- Vectors

  If ϕ is a vector field X_μ, it can be either real or complex. In both cases it can bear some representation under gauge groups, hence there is model-dependency. The possible operators in this case are

  $X_\mu^\dagger X_\mu$: This term violates unitarity in the UV and requires the model to be UV-completed.

  $\partial_\mu X_\mu$: This could only be considered if X_μ is an uncharged matter field.

- Complex Scalar

  If ϕ is a complex scalar field Φ, it can be charged under the gauge groups. The possible operators are

  Φ and $\Phi^\dagger$: Allowed if Φ is a singlet.

  $|\Phi|^2$: Always allowed.

- Real Scalar

  If ϕ is a real field, it cannot transform under complex representations of any group, but only under real ones (adjoint representations for instance). The possible operators are

  ϕ: Allowed if ϕ is a singlet.

  ϕ^2: Always allowed.

From our analysis, it is clear that the only excluded situation is the case where ϕ is a fermion. The other possibilities are allowed, but have distinct levels of model-dependency. The worst choice would be the vector one, since, in addition to the dependence on the representations under the gauge groups, there is also the problem with unitarity. In comparison to the complex scalar, the real scalar has much less model-dependency, so it is the minimal choice. From now on we will work only with real scalar fields.

One remark is in order. Although marginal operators have significant effects, relevant operators will dominate at the IR, therefore, for the CR mechanism to properly work, one must have relevant operators coupled to the Higgs mass parameter. It is worth noting that marginal operators could have larger impact if the anomalous dimension of the fields were large, but this would require for a dark-sector to generate this anomalous dimension. Neglecting the contributions from marginal and irrelevant operators and considering that ϕ is a single field, the Higgs mass parameter in the low-energy regime $E \ll \Lambda$ is given by

$$\mathscr{L} \supset -M_H^2(\phi)|H|^2, \quad M_H^2(\phi) = g_0\Lambda^2 - g_1\Lambda\phi, \tag{5.23}$$

where it is supposed that ϕ is a gauge invariant operator. The minus sing in the second term is chosen for future convenience. From the equation above it is clear that the Higgs mass is driven to $M_{H,\mathrm{SM}}^2$ by the cancellation between the two terms, which is achieved when ϕ obtains the value

$$\bar{\phi}_{\mathrm{eq}} = -\frac{M_{H,\mathrm{SM}}^2 - g_0\Lambda^2}{g_1\Lambda}. \tag{5.24}$$

The first problem encountered by considering a real scalar field is again the Scalar Mass Problem and the Hierarchy Paradigm and one needs an additional mechanism to stabilise the mass of ϕ. We are already familiar with many mechanisms that were discussed in the past chapters, so one may implement one of them to stabilise the mass of ϕ. Neither Supersymmetry nor Neutral Naturalness are good choices, because both models introduce several particles to the particle spectrum; as it was seen in section 4.4, too many additional particles render the model incompatible with cosmological data. The most adequate mechanism is to realise ϕ a pseudo Nambu-Goldstone-Boson (pNGB) of an approximate, spontaneously broken global symmetry. In this scenario ϕ has a shift-symmetry that forbids a mass term at tree-level, which could be generated by quantum corrections if the global symmetry is somehow broken. The coupling $\phi|H|^2$ from Eq. (5.23) is an interaction that breaks the shift-symmetry explicitly and it can generate a potential $V_1(\phi)$ at 1-loop.

It is worth remarking that the relation between the 1-loop induced potential V_1 and V_0 remains unclear. More precisely, it is unknown which of them is more relevant to the dynamics of ϕ. At this point in the discussion we may only state that, due to the lack of symmetries in the EFT, a tree-level potential V_0 is not forbidden at the scale Λ. In what follows it is assumed that V_0 dominates the dynamics of ϕ, but it will become clear in section 5.3.1, in a specific model, under what conditions this is true and why it is necessary.

Let us discuss V_0 in more detail. It is already expected that V_0 breaks the continuous shift-symmetry of the NGB ϕ, because else it would be a constant potential. Therefore, one may write V_0 as a sum of two distinct contributions:

$$V_0(\phi) = V_{0,\text{break}}(\phi) + V_{0,\text{inv}}(\phi). \tag{5.25}$$

The first contribution $V_{0,\text{break}}$ in the equation above is by definition a potential that breaks the shift-symmetry completely, i.e. it does not exist any real number a for which $V_{0,\text{break}}(\phi + a) = V_{0,\text{break}}(\phi)$ is satisfied. Since it breaks the shift-symmetry explicitly, it may be written as a power series,

$$V_{0,\text{break}}(\phi) = c_0 + c_1\Lambda^3\phi + \frac{1}{2}c_2\Lambda^2\phi^2 + \cdots\ , \tag{5.26}$$

where the c's are independent couplings. In particular, the coefficient $c_2\Lambda^2$ that represents the mass of ϕ must be small and stabilised in order to avoid the SMP and the HP. However, it is impossible to state that c_2 is stable under quantum effects if all the couplings g_1 and $\{c_n\}$ are independent, because in this case radiative corrections depend on all these couplings. In short, due to the many independent sources of explicit symmetry breaking, c_2 is not quantum mechanically stable. This issue can be solved if all the couplings that break the shift-symmetry are assumed to depend solely on g_1. More precisely, if

$$c_n = c_n(g_1) \text{ such that } c_n(0) = 0, \quad \forall n, \tag{5.27}$$

then the explicit breaking of the shift-symmetry is controlled only by g_1 and one can thus stabilise the mass parameter of ϕ *. For a sufficiently small g_1 and considering the low-energy regime we can write $V_{0,\text{break}}$ as

$$V_{0,\text{break}} \simeq r_0\Lambda^4 - r_1 g_1^\alpha \Lambda^3\phi, \tag{5.28}$$

where r_1 is a number, $r_0 > 0$ and $\alpha > 0$. To simplify our discussion, $\alpha = 1$ is chosen. The expansion above is valid, because the linear term is the most relevant, non-constant one in the IR regime and will therefore dictate the dynamics of ϕ. It is interesting to note

*Note, however, that due to the unknown origin of the coupling in Eq. (5.23), we cannot guarantee that g_1 is small. One needs to assume that it is small for the technical naturalness argument to work.

that the source of shift-symmetry breaking is given only by the coupling g_1, which could be an indication that the origin of $V_{0,\mathrm{break}}$ may be the same as of the interaction $\phi|H|^2$. Unfortunately, the underlying physics of the interaction in Eq. (5.23) is also unknown, hence there is not much to say about it.

The second contribution in Eq. (5.25) consists of terms that may be invariant under a discrete sub-group of the continuous translation symmetry. For instance, assume that for a real a the potential $V_{0,\mathrm{inv}}$ satisfies

$$V_{0,\mathrm{inv}}(\phi + na) = V_{0,\mathrm{inv}}(\phi), \tag{5.29}$$

for any integer n. Note that, though this potential restores a discrete sub-group of the shift-symmetry, it is not forbidden since $V_{0,\mathrm{break}}$ and the interaction (5.23) break the symmetry completely, hence g_1 remains technically natural. Although the origin of $V_{0,\mathrm{inv}}$ in the UV is also uncertain, it must have a connection to the SSB of the global symmetry group that has ϕ as its NGB, because the potential restores part of the shift-symmetry associated with the same global symmetry. Moreover, it is most likely that this potential is non-perturbative, as it encodes global properties of ϕ.

One remark regarding the potentials in Eq. (5.25) is in order. Note that $V_{0,\mathrm{inv}}$ is a periodic function of period a, therefore is has sites of typical size a. Then, if $g_1 = 0$, i.e. if shift-symmetry wasn't broken, the values of the field ϕ would be restricted to the ones in the interval $[0, a]$. In this manner it wouldn't be possible for $\bar{\phi}$ to evolve to the value in Eq. (5.24), which certainly lies outside of this interval. However, $g_1 \neq 0$ and so ϕ can assume values outside $[0, a]$. Even so, it is impossible for $V_{0,\mathrm{inv}}$ to drive $\bar{\phi}$ to $\bar{\phi}_{\mathrm{eq}}$ due to its periodicity, whence one needs a non-vanishing $V_{0,\mathrm{break}}$ in order to generate the required dynamics.

From the aforementioned properties of both potentials, one can qualitatively understand how the evolution of $\bar{\phi}$ will proceed and, in particular, how it will be classically stabilised. The classical mode will be stabilised by $V_{0,\mathrm{inv}}$ at the site centred at the equilibrium point $\bar{\phi}_{\mathrm{eq}}$ and it will arrive at this particular site through the slope of the potential $V_{0,\mathrm{break}}$. As for how this mechanism take place in detail, one needs a concrete model to make quantitative statements. One possible drawback to this scenario is regarding the quantum fluctuations mentioned in section 5.2.2. It is well known that a quantum field can tunnel classical barriers, for instance those of the periodic potential $V_{0,\mathrm{inv}}$, which means

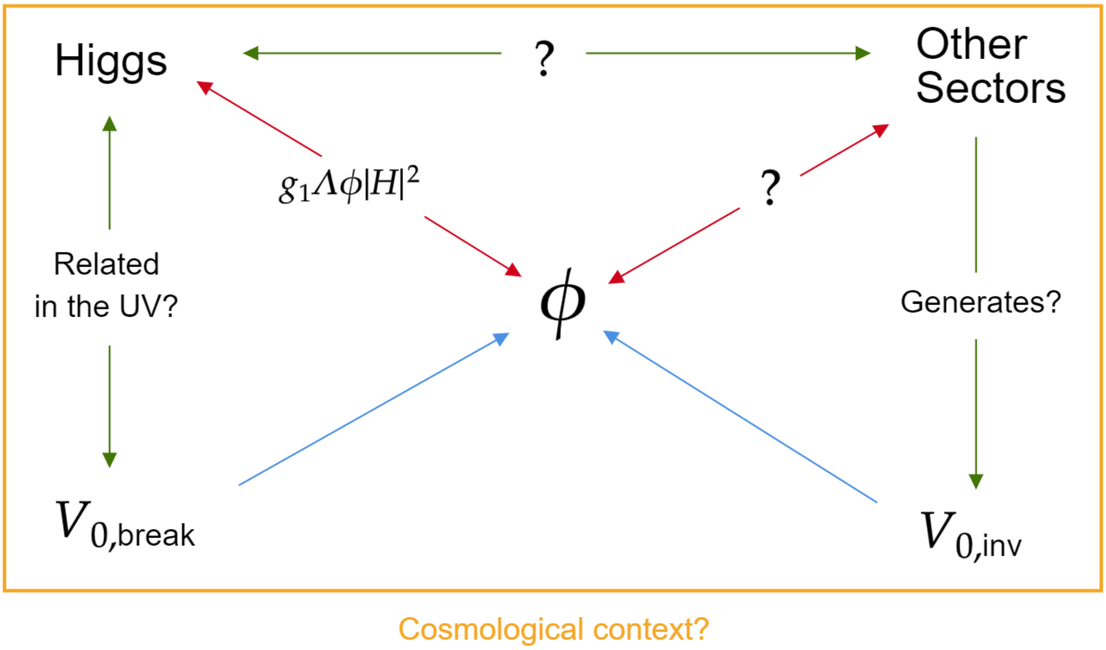

Figure 7: Diagrammatic representation of the many ingredients of the CR model. $V_{0,\text{break}}$ and $V_{0,\text{inv}}$ are the tree-level potentials that describe the dynamics of the classical mode $\bar{\phi}$, given by Eqs. (5.25), (5.28) and (5.29). The field ϕ interacts with the Higgs through the coupling in Eq. (5.23), which may be related to the potential $V_{0,\text{break}}$ in the UV. Here, "Other Sectors" denote all other dof besides the Higgs that may be relevant to the CR model. In particular, it may contain some reminiscent dof of the SSB of the global symmetry that could have generated $V_{0,\text{inv}}$. However, there is no indication of how they interact with ϕ and with the Higgs. In addition, everything is inside of the yet-undetermined cosmological context.

that, while the equilibrium configuration $\bar{\phi}_{\text{eq}}$ is classically stable, it may not be so at the quantum level. As a consequence, the field ϕ can spread throughout many different minima of the potential and would thus create distinct vacua configurations in the universe [17, 18]. But again, it is difficult to make precise statements, because tunnelling effects depend on numerous factors, e.g. the height of the barriers and the distance between adjacent barriers, so this point won't be discussed anymore.

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In this section it was outlined how the model building will proceed, where the options with the least amount of model-dependency have been chosen. In this way the motivations for a minimal model were given. Notwithstanding, it is not adequate to call it a minimal model, because most of our choices were not uniquely determined due to the lack of symmetries.

Before turning to the discussion of a specific model,the conclusions and choices made along this section are emphasised. First, it was determined that ϕ cannot be a SM particle due to the relevance of its effects at the IR. After that it was investigated which would be the most adequate Lorentz structure for ϕ, and the conclusion was that a real scalar has the least amount of model-dependency. Then it was discussed how to avoid the HP for ϕ. ϕ was chosen to be realised as a pNGB of a spontaneously broken approximate global symmetry, because, out of the mechanisms studied so far, it is the most fitting mechanism while taking cosmological considerations into account. The potential that is responsible for the evolution of ϕ will have a tree- and 1-loop-level potential, however the tree-level contribution V_0, which is not forbidden by any symmetries at the UV, is in general more relevant and necessary. This potential can be split into two pieces: $V_{0,\text{break}}$, that breaks the shift-symmetry completely, and $V_{0,\text{inv}}$, that is invariant under a discrete sub-group of the shift-symmetry. In order to stabilise the mass of ϕ consistently, the couplings in $V_{0,\text{break}}$ must all depend on g_1, such that it is the only source of explicit symmetry breaking. Furthermore, its most relevant contribution at the IR is the linear term in ϕ. It was conjectured that the coupling $\phi|H|^2$ and $V_{0,\text{break}}$ may have the same origin at the UV. The other contribution $V_{0,\text{inv}}$ is periodic and with it, apart from issues of quantum tunnelling, we can expect $\bar{\phi}$ to be successfully stabilised. It is presumed that $V_{0,\text{inv}}$ is the result of a non-perturbative effect of the SSB of the global symmetry. See Figure 7 for a diagrammatic representation of our discussion.

5.3 $L + N$ Relaxion model

Having explored the general properties of CR models from the EFT point of view, we now proceed to study a particular realisation of it, the $L + N$ *relaxion model* [17, 18]. It will first be shown how to implement this model in terms of our previous discussion. Then, the theoretical conditions and bounds for which the model is consistent with the CR mechanism will be presented. Finally, some of the experimental bounds will be studied.

5.3.1 Implementation

The correct implementation of the CR model at low-energy demands the specification of the particle content and their interactions with ϕ, the choice of a cosmological context and the identification of the back-reaction mechanism. We proceed to study them one by one.

Particle content and back-reaction

It has been argued in section 5.2.3 that one needs both potentials $V_{0,\text{inv}}$ and $V_{0,\text{break}}$ in order to stabilise the classical mode $\bar{\phi}$. In addition, there are indications that $V_{0,\text{inv}}$ may be generated from non-perturbative effects of other dof, besides the Higgs, that are relevant to the CR mechanism (the "Other Sectors" in Figure 7). In what follows, this hint is pursued as way to make the model more minimal and elegant*.

With this line of reasoning, we chose to introduce a new strongly-interacting sector with an underlying non-abelian gauge group G_s, whose precise structure does not concern us here. This choice is motivated from the fact that such theories have well studied non-perturbative effects and can indeed generate periodic potentials [1, 21]. More precisely, we are interested in the impact of the following term,

$$\mathscr{L}_\theta = \theta \frac{c_s^2}{32\pi^2} C^a_{\mu\nu} \tilde{C}^a_{\mu\nu}, \tag{5.30}$$

where $C^a_{\mu\nu}$ is the strength field tensor of the gauge group G_s, with a the index of the adjoint representation, $\tilde{C}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} C^{a\alpha\beta}$ is the dual of $C^a_{\mu\nu}$, c_s is the gauge coupling and θ is the coefficient of the operator. The operator (5.30) is a marginal operator that breaks CP in the strong gauge sector and accounts only for boundary terms, since it can be rewritten as a total derivative. As a consequence, this operator does not affect the theory at the level of perturbation theory.

It can, however, have non-perturbative effects [1, 21, 88]. For instance, in a free theory with a massless Dirac fermion ψ charged under this gauge group, the Lagrangian is invariant under the transformations of the following global group,

$$U(1)_V \times U(1)_A, \tag{5.31}$$

where $U(1)_V$ transforms left- and right-handed fermions in the same way, while $U(1)_A$ transforms them with opposite phases. For this reason they are called *vector* and *axial* symmetries, respectively. In analogy with QCD, one expects that at a scale Λ_{SSB} the gauge

*It is not attempt to do the same with $V_{0,\text{break}}$, because it would require us to study the UV-completions of the model, which is out of the scope of this thesis.

interaction becomes strong and that the fermion condenses into the vacuum expectation value $\langle\bar{\psi}\psi\rangle$, which breaks the global symmetry group spontaneously. The SSB pattern is given by

$$U(1)_V \times U(1)_A \to U(1)_V, \tag{5.32}$$

since the fermion condensate is invariant only under $U(1)_V$. The Goldstone matrix U, that in this case is just a phase, transforms under $U(1)_A$ in the same way $\langle\bar{\psi}\psi\rangle$ does. The term in Eq. (5.30) may have significant impact on the SSB, as it depends explicitly on the gauge fields that are strongly interacting. Notwithstanding, due to the anomalous nature of the $U(1)_A$ symmetry, one can perform an axial rotation on the fermion that eliminates the operator (5.30) from the Lagrangian. On the one hand, if the fermion is exactly massless, then the Lagrangian is invariant under $U(1)_A$ and the θ-parameter can be completely removed from it. Therefore, the parameter θ is not physical in this case. On the other hand, if the fermion has a non-vanishing mass M, $U(1)_A$ is explicitly broken. As a consequence, the axial rotation that eliminates the term in Eq. (5.30) renders the mass term complex and θ-dependent. In this latter scenario, one may promote the θ-dependent mass $M(\theta)$ to a spurionic field that transforms in the same way as $U^\dagger$ under $U(1)_A$. Whence, after SSB, the lowest order mass-dependent operator one may write in the Lagrangian is given by*.

$$\Lambda^3_{\text{SSB}}\Big(MU + U^\dagger M^\dagger\Big). \tag{5.33}$$

Taking the vacuum expectation value of U in the equation above leads us to the following potential,

$$V(\bar{\theta}) \sim \Lambda^3_{\text{SSB}}|M|\cos\bar{\theta}, \tag{5.34}$$

where $\bar{\theta}$ is a function of θ and the vacuum expectation value of the pNGB.

The potential in Eq. (5.34) is known as *$\bar{\theta}$-potential* and is a direct, non-perturbative effect of the operator in Eq. (5.30). In particular, it is a periodic potential, as required by our previous discussion, however it is constant as $\bar{\theta}$ is just function of constant parameters. In order to fit the mechanism above in the CR mechanism, one needs $\bar{\theta}$ to be a dynamical

*Note that due to the fact that the axial symmetry is anomalous, i.e. it was never a symmetry of the generating functional, its selection rules must not b satisfied to build the EFT after SSB. This point does not interfere with our discussion, but it is of particular importance in the determination of the mass of the pNGB, which is not protected by shift-symmetry. In this regard, the angular mode of the SSB of an axial symmetry is not a pNGB in the usual sense.

quantity, in other words, a field. This is easily done by promoting $\bar{\theta}$ to a field [1, 21, 88],

$$\bar{\theta} \to \bar{\theta} + \frac{a(x)}{f}, \tag{5.35}$$

where a is a real scalar and f is an energy scale.The potential of Eq. (5.34) can be re-obtained by following the same procedure, but with the axial rotation parameter given by $\theta + \frac{a(x)}{f}$ rather than θ. In this manner we obtain a potential for a, which is given by*

$$V(\bar{\theta}, a) \sim \Lambda^3_{\text{SSB}} |M| \cos\left(\bar{\theta} + \frac{a(x)}{f}\right). \tag{5.36}$$

This peculiar field a is called *axion* [88, 89] and is characterised by the interaction

$$\mathscr{L}_{aC\tilde{C}} = \frac{c_s^2}{32\pi^2} \frac{a}{f} C^a_{\mu\nu} \tilde{C}^a_{\mu\nu}, \tag{5.37}$$

that results in the potential in Eq. (5.36) given an appropriate particle content. The axion is indeed an appropriate choice for the field ϕ, as it may be understood as a pNGB from either a spontaneously broken axial $U(1)_{PQ}$, which is usually denoted as Peccei-Quinn symmetry and broken at the scale f [89], or from a Clockwork mechanism [90, 48], for example. From now on it is assumed that ϕ is an axion and, since it responsible for the CR mechanism, it is called *relaxion*.

From this discussion one anticipates how the relaxion, together with the strong gauge sector, naturally introduces a back-reaction of the Higgs mass parameter. In the example above, which is inspired from the mechanism that takes place in QCD, the potential $V(\bar{\theta})$ is proportional to the mass $|M|$ of the fermion. If this fermion acquires its mass through the Higgs mechanism, just like the quarks do, the amplitude of the potential is thus proportional to the vev of the Higgs. Therefore, as M_H^2 varies, so does the potential in Eq. (5.36), which in turn affects the dynamics of the relaxion. In other words, this is a concrete back-reaction from M_H^2. We can be even more precise by noting that the amplitude of the potential is large if the vev is large, implying that it becomes more difficult for the classical mode $\bar{\phi}$ to overcome the barriers as the vev grows. With such reasoning it is obvious that the model is guaranteed to stabilise the relaxion if its evolution

*Due to the space-time dependence of the axial rotation parameter, the kinetic term of the fermion will generate couplings proportional to $\partial_\mu a$. Such interactions, that are relevant for the phenomenology of a, are not discussed here.

makes v grow. In section 5.3.2, how this back-reaction affects the evolution of $\bar{\phi}$ will be studied in detail.

Our example was based on the QCD scenario and it would be compelling if one could take take G_s as the QCD colour group, because then the model would be even more minimal. However, this is not possible due to the physical $\bar{\theta}$-parameter of QCD, $\bar{\theta}_{\text{QCD}}$ [17, 18]. Through the measurement of the electric dipole moment (EDM) of the neutron one can infer an upper bound for $\bar{\theta}_{\text{QCD}}$, which is given by $|\bar{\theta}_{\text{QCD}}| \lesssim 10^{-10}$ [3, 88]. In the presence of an axion, this parameter is dynamical and is stabilised near the experimental value. This is achieved by minimising the potential in Eq. (5.36), for which the axion acquires a vev $\langle a \rangle \simeq -\bar{\theta} f$. In CR models such is not the case, as the relaxion must be stabilised at the equilibrium configuration in Eq. (5.24) and as a consequence the relaxion will in general drive $\bar{\theta}_{\text{QCD}}$ to a value excluded by the EDM of the neutron. For this reason one cannot chose G_s as the QCD colour group, but need to introduce a new, hidden strong sector instead.

Although it is impossible to use the framework of QCD in the CR model, the back-reaction of the periodic potential, i.e. that the height of the potential (5.34) is proportional to the vev of the Higgs, is certainly interesting and could be used to stabilise $\bar{\phi}$ correctly. In order to retain this property, the fermions charged under the new strong gauge group G_s must somehow interact with the Higgs. The most straightforward and minimal way to do so is to introduce four left-handed Weyl fermions [17, 18, 91]: two $SU(2)_L$ doublets L and L^c that transform under the fundamental and anti-fundamental of G_s with $-1/2$ and $+1/2$ hypercharges, respectively, and two singlets under the SM gauge group, N and N^c, that transform respectively only under the fundamental and anti-fundamental of G_s. The most general renormalizable, interacting Lagrangian is thus given by

$$\mathscr{L}_{L+N} \supset \frac{c_s^2}{32\pi^2} \frac{\phi}{f} C^a_{\mu\nu} \tilde{C}^a_{\mu\nu} - M_L L L^c - M_N N N^c - y H L N^c - \tilde{y} \tilde{H} L^c N + h.c., \tag{5.38}$$

where M_L and M_N are Dirac masses, y and $\tilde{y}$ are the Yukawa couplings and we are using the Weyl notation to contract the $SU(2)_L$ and the spinorial indices. In the Lagrangian above it was already included the coupling between the relaxion and the gauge bosons, which will generate the periodic potential of the relaxion. In the next section it will be seen in detail how the condensation takes place and how the periodic potential depends on the Higgs vev.

Cosmological context

Some assumptions regarding the cosmological era in which the CR will take place still need to be made. This choice is crucial to the model for two reasons. First, new particles and interactions leave imprints on high precision measurements of the cosmological evolution, for instance the Big Bang Nucleo-synthesis (BBN), the Cosmic Microwave Background (CMB) and Large structure formation [77, 92, 93]. For this reason more primordial eras of the cosmic evolution are preferred. Second, the EoM of the relaxion, which is a Klein-Gordon equation analogous to the one in Eq. (C.27) in Appendix C, depends explicitly on the Hubble parameter H_b. In particular, it controls the damping of $\bar{\phi}$, hence one must have H_b under control if the evolution of $\bar{\phi}$ is to be successfully stopped.

A reasonable choice for the cosmological epoch is inflation, an era before radiation domination, in which the universe had an accelerated expansion (Appendix C is referred for more details). One characteristic of this epoch is that the Hubble parameter is approximately a constant, that implies in a constant damping term in the EoM. However, even with this damping, it is difficult to avoid the relaxion from gaining a lot of kinetic energy as it evolves due to the potential in Eq. (5.28), which could make the stabilisation difficult. To avoid such situation and have the dynamical evolution of the relaxion under control, we assume that it is in a slow-roll regime, i.e. it satisfies the conditions of Eqs. (C.25) and (C.28). Stated in another way, the Hubble friction in the slow-roll regime will allow for the relaxion to be stabilised by its potential V_0. In what follows it is assumed that the CR mechanism takes place during inflation and that the relaxion slow-rolls, but the inflationary sector is left unspecified and it is assumed that there is no relevant interaction between the relaxion and the inflaton.

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At this point one can precisely see how the CR mechanism in the $L+N$ model works. At the beginning of inflation, when the relaxion has a generic small initial value, say $\phi_0 \simeq 0$, the relaxion starts its evolution. We have argued that the vev v must grow as the relaxion evolves, hence the initial mass of the Higgs shall be positive (i.e. the coefficient g_0 in Eq. (5.23) is positive) and will decrease as $\bar{\phi}$ approaches $\bar{\phi}_{\text{eq}}$. Choosing without loss of generality that $\bar{\phi}_{\text{eq}} > 0$, this implies that $g_1 > 0$ and that there is a critical value

$\bar{\phi}_{\rm crit}$ for which $M_H^2 = 0$. From this point onward the vev becomes non-vanishing and the barrier of the potential grows until $\bar{\phi}$ cannot overcome it.

Note that for this argument to work, one needs the potential $V_{0,\rm break}$ to induce the correct evolution, in other words, it must drive $\bar{\phi}$ to large, positive values. Before the critical point $\bar{\phi}_{\rm crit}$ the EoM of $\bar{\phi}$ is approximately given by

$$3H_b\dot{\bar{\phi}} - r_1 g_1 \Lambda^3 = 0, \tag{5.39}$$

where the slow-roll condition (C.28) and the potential in Eq. (5.28) are used. The equation above is trivially solved for a constant Hubble parameter and $\bar{\phi}$ grows if the coefficient r_1 of the potential (5.28) is positive. One remark regarding quantum corrections is in order. We stress again that the relaxion-Higgs interaction in Eq. (5.23) generates a quantum potential at 1-loop. In particular, using Eq. (3.130) for the 1-loop Coleman-Weinberg potential one obtains the following potential:

$$\Delta V_1(\phi) = \frac{g_1\Lambda^3}{16\pi^2}\phi + \cdots\ , \tag{5.40}$$

where the dots denote logarithmic terms. Note that this potential has the opposite sign in comparison to $V_{0,\rm break}$ in Eq. (5.28), therefore, by the EoM, it will drive $\bar{\phi}$ to negative values. In order to avoid the quantum contribution ΔV_1 to dominate the EoM, one needs

$$r_1 > \frac{1}{16\pi^2}. \tag{5.41}$$

With these properties it is thus straightforward to determine the conditions under which the classical mode of the relaxion stops evolving; one must simply solve the EoM after the Higgs acquires a vev and compute $\bar{\phi}_{\rm eq}$ in terms of the parameters of the model. In section 5.3.2 the stopping conditions of the relaxion will be studied in more depth.

5.3.2 Theoretical constrains

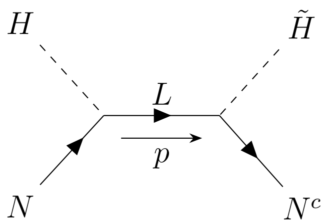
In this section the $L + N$ model outlined in section 5.3.1 is studied in more detail. More precisely, we will study the conditions under which the model works properly. All conditions derived in this section are summarised in Table 4 [17, 18, 91, 94].

Strong Sector

Let us first discuss the Lagrangian (5.38) in more depth, in particular let us see how the potential depends on the vev of the Higgs and what kind of constrains does it introduce.

Before proceeding, one must determine the hierarchy between the energy and mass scales involved. As it will be seen below, in order to successfully form a Higgs-dependent potential, one needs the mass of at least one of the fermions to be below the condensation scale $4\pi F$. This condition allows for the respective fermion to form a condensate. In order to simplify the discussion, it is assumed that only N and N^c condenses, i.e. $M_N < 4\pi F < M_L$. Here, the mixing between N (N^c) and the electrically neutral component of L (L^c) is ignored, which is small if the mixing given by the Yukawa couplings y and $\tilde{y}$ in Eq. (5.38) is not very large. In addition, it is assumed that $v > F$, in such way that electroweak symmetry breaking (EWSB) happens before the condensation, because it will avoid some phenomenological complications (for the opposite situation $v < F$ see [95]). Furthermore, for the Lagrangian in Eq. (5.38) to be valid at energies below the cut-off Λ, the SSB of the symmetry that generated the relaxion must have broken, hence $f > \Lambda$.

Given a sufficient separation between M_L and $4\pi F$, one may integrate L and L^c out for energies $4\pi F < E \ll M_L$. The leading effect of this integration is given by the diagram*



$$= y\tilde{y}^* \frac{iM_L}{p^2 - M_L^2} \xrightarrow{p\to 0} -\frac{iy\tilde{y}^*}{M_L}, \qquad (5.42)$$

which results in the effective Lagrangian

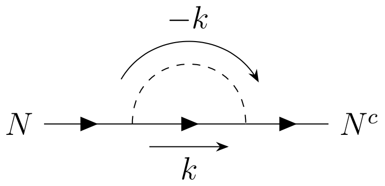
$$\mathscr{L}_{L+N} \to \mathscr{L}_{\text{eff}} = \frac{c_s^2}{32\pi^2}\frac{\phi}{f}C^a_{\mu\nu}\tilde{C}^a_{\mu\nu} - \left[M_N + \delta M_N + \frac{y\tilde{y}}{M_L}|H|^2\right]NN^c + h.c. \qquad (5.43)$$

In the effective Lagrangian above the 1-loop radiative correction δM_N from closing the

*Here the Feynman rules for Weyl fermions [11, 42] are used.

Higgs loop in the diagram (5.42),

$$\delta M_N =$$



$$\simeq \frac{y\tilde{y}}{16\pi^2} M_L \ln \frac{\Lambda}{M_L}, \tag{5.44}$$

for $\Lambda \gg M_L$, are already included, which can be relevant to our considerations as it is proportional to M_L.

At some point the Higgs will get a vev, hence the effective Lagrangian can be written as

$$\mathscr{L}_{\text{eff}} = \frac{c_s^2}{32\pi^2}\frac{\phi}{f} C^a_{\mu\nu}\tilde{C}^a_{\mu\nu} - \left[M_N + \delta M_N + \frac{y\tilde{y}}{2M_L}(v+h)^2\right] N N^c + h.c., \tag{5.45}$$

with h the radial mode of the Higgs doublet. Lowering the energy, the strong group G_s will condense below the scale $4\pi F$. Following the same procedure used to obtain the potential (5.36), one arrives at the expression for the periodic potential of the relaxion:

$$V_{0,\text{inv}}(\phi) \simeq (4\pi F)^3 \left[M_N + \delta M_N + \frac{y\tilde{y}}{2M_L}(v+h)^2\right] \cos\frac{\phi}{f}, \tag{5.46}$$

where contributions from higher order operators are neglected. At this point the assumption made regarding the EW scale and the condensation scale, i.e $F < v$, is very relevant, because it implies that the radial mode h is classical and it is given by its EoM.

We may separate the potential $V_{0,\text{inv}}$ in two pieces:

$$V_{0,\text{inv}} = \Lambda_b^4 \cos\frac{\phi}{f} + \mu_b^4 \frac{h}{v}\left(\frac{h}{v} + 2\right) \cos\frac{\phi}{f}, \tag{5.47}$$

where it was defined

$$\Lambda_b^4(\phi) \equiv (4\pi F)^3 \left[M_N + \delta M_N + \frac{y\tilde{y}}{2M_L} v^2(\phi)\right], \tag{5.48}$$

and

$$\mu_b^4(\phi) \equiv \frac{y\tilde{y}}{2M_L}(4\pi F)^3 v^2(\phi). \tag{5.49}$$

On the one hand, the term proportional to μ_b^4 is h-dependent and induces, in particular, a mix between ϕ and h. The phenomenology of this scenario will be explored in section 5.3.3. On the other hand, the quantity in Eq. (5.48) is independent of h and represents the vev-dependent barrier that will stop the relaxion. In addition, for the vev to dominate the barrier, one needs that

$$M_N + \delta M_N < \frac{y\tilde{y}}{2M_L} v^2(\phi) \tag{5.50}$$

during the final stages of the evolution. This implies that at the equilibrium configuration one may approximately write

$$\Lambda_b^4\left(\bar{\phi}_{\rm eq}\right) \simeq \mu_b^4\left(\bar{\phi}_{\rm eq}\right). \tag{5.51}$$

Inflation

Let us now study some of the constrains that emerge from cosmological considerations. First, it is assumed that the CR mechanism takes place during inflation, in a way that the relaxion is not the inflaton. This implies that the energy density of the relaxion cannot dominate over the energy-density of the inflaton, else inflation would not be driven by the latter. We thus impose that

$$\rho_{\rm inflaton} > \rho_\phi. \tag{5.52}$$

Using the expression of the energy-density of the inflaton in the slow-roll regime in Eq. (C.24) and the expression for the zero-point energy of the relaxion in Eq. (5.28), the condition above can be rewritten as

$$3M_P^2 H_b^2 > r_0 \Lambda^2 \Rightarrow H_b > \sqrt{\frac{r_0}{3}} \frac{\Lambda^2}{M_P}, \tag{5.53}$$

where $M_P = (8\pi G)^{-1/2}$ is the reduced Planck mass and H_b is the Hubble parameter.

Second, in section 5.3.1 we have argued that the relaxion should be in a slow-roll regime, which requires Eqs. (C.25) and (C.28) to be satisfied. Before the critical point $\bar{\phi}_{\rm crit}$, for which $M_H^2 = 0$, the slow-roll velocity of $\bar{\phi}$ is approximately given by

$$\dot{\bar{\phi}}_{\rm SR} = \frac{r_1 g_1 \Lambda^3}{3H_b}, \tag{5.54}$$

which is obtained from the EoM using the slow-roll condition (C.28). Eq. (C.25) thus implies

$$\frac{1}{2}\left(\frac{r_1 g_1 \Lambda^3}{3H_b}\right)^2 < r_0 \Lambda^4 \Rightarrow r_1^2 g_1^2 \Lambda^2 < 6H_b^2. \tag{5.55}$$

From the second slow-roll condition in Eq. (C.28) together with Eq. (5.28) we obtain

$$\frac{r_1 g_1 M_P}{\sqrt{6} r_0 \Lambda} < 1, \tag{5.56}$$

which indicates that the coupling g_1 must be quite small if $r_0 \sim r_1 \sim \mathcal{O}(1)$.

Third, inflation should take place at an energy scale below the cut-off, such that it is reasonable to expand the effective Lagrangian (5.8) in a series of operators. Therefore

$$H_b < \Lambda. \tag{5.57}$$

Fourth, we assume that the classical mode is dominant during the relaxion excursion. This means that the overdensity δ_ϕ of the relaxion energy density is small [85] and whence

$$\delta_\phi = \frac{H_b^2}{2\pi \dot{\phi}_{\rm SR}} < 1 \Rightarrow H_b < \frac{(r_1 g_1)^{\frac{1}{3}}}{\sqrt{3}} \Lambda. \tag{5.58}$$

One remark regarding the equation above is in order. Depending on the shape of the potential V_0, Eq. (5.58) does not necessarily hold. The behaviour of the quantum fluctuation $\delta\phi$ in those cases is as important as the classical mode and impacts the evolution of the latter since they can exchange energy. Such scenarios are not studied in this thesis, which are carefully considered in [18], and the focus on the Hubble friction as the source of energy dissipation.

Fifth, in order for the mechanism to take place during inflation, the barrier in Eq. (5.29) must have already formed. This puts a constraint on the scale of inflation,

$$H_b < 4\pi F. \tag{5.59}$$

The sixth condition regards the period of reheating [85, 96]. After the end of inflation, the temperature of the universe is typically much smaller than the initial temperature of the radiation domination era. For this reason, an additional period by the end of inflation, named reheating, is necessary to bring the universe to the correct temperature. A problem for the stabilisation mechanism may arise from this epoch, since the barrier of the periodic potential vanishes if the reheating temperature $T_{\rm RH}$ by the end of the reheating is larger than the confinement scale. In this case, due to the lack of $V_{0,\rm inv}$, the relaxion would not be stabilised anymore and would therefore be free to keep evolving. In order to avoid these problems, it is imposed that the reheating temperature is smaller compared to the confinement scale,

$$T_{\rm RH} < 4\pi F, \tag{5.60}$$

in such a way that the barrier of the potential does not vanish after the reheating.

Lastly, one must compute the total number of e-folds necessary to slow down the relaxion during the inflationary period and allow it to completely scan the Higgs mass parameter. For this calculation it is assumed that the CR mechanism lasts during the entire inflationary period, such that one may write the total number of e-folds in Eq. (C.32) as

$$N_e = \int_{\bar{\phi}_0}^{\bar{\phi}_{\rm eq}} {\rm d}\bar{\phi}\frac{H_b}{\dot{\bar{\phi}}}. \tag{5.61}$$

Note that this equation holds even if the relaxion is not the inflaton, as we are assuming. Since in the entire excursion of the relaxion both H_b and $\dot{\bar{\phi}} \simeq \dot{\bar{\phi}}_{\rm SR}$ remain approximately constant, N_e is given by

$$\begin{aligned} N_e &\simeq \left(\bar{\phi}_{\rm eq} - \bar{\phi}_0\right)\frac{\dot{\bar{\phi}}_{\rm SR}}{H_b} \\ &\simeq \frac{g_0\Lambda}{g_1}H_b\frac{3H_b}{r_1 g_1\Lambda^3} \\ &\simeq \frac{3g_0 H_b^2}{r_1 g_1^2\Lambda^2}. \end{aligned} \tag{5.62}$$

Stopping condition

We now proceed to discuss how the relaxion stops its evolution. As stressed, the relaxion is expected to stop due to the back-reaction of the periodic potential in Eq. (5.47), whose barrier grows with the Higgs vev. At the critical point $\bar{\phi}_{\rm crit}$ the total energy density of the relaxion is given by

$$E_{\rm crit} = \frac{1}{2}\dot{\bar{\phi}}_{\rm SR}^2 - g_0 r_1\Lambda^4, \tag{5.63}$$

where the zero-point energy $r_0\Lambda^4$ is suppressed. Note that the first contribution in $E_{\rm crit}$ is purely kinetic energy, whereas the second contribution comes from the potential energy

Sector	Explanation	Condition
Strong Sector	N and N^c condense	$M_N < 4\pi F < M_L$
	Condensation below EW scale	$v > F$
	Broken global symmetry	$f > \Lambda$
	vev dominates the barrier	$v^2 > 2\frac{M_L M_N}{y\tilde{y}} + \frac{M_L^2}{16\pi^2}\ln\frac{\Lambda}{M_L}$
Inflation	Inflaton energy density dominates	$H_b > \sqrt{\frac{r_0}{3}}\frac{\Lambda^2}{M_P}$
	Classical > Quantum	$H_b < \frac{(r_1 g_1)^{1/3}}{\sqrt{3}}\Lambda$
	Number of e-folds	$N_e \simeq \frac{3g_0 H_b^2}{r_1 g_1^2 \Lambda^2}$
	Inflation below the cut-off	$H_b < \Lambda$
	Barrier forms	$H_b < 4\pi F$
	Reheating	$T_{\rm RH} < 4\pi F$
	Slow-roll	$r_1^2 g_1^2 \Lambda^2 < 6H_b^2$
	Slow-roll	$\frac{r_1 g_1 M_P}{\sqrt{6 r_0}\Lambda} < 1$
Stopping Condition	Height of the barrier necessary to stop the relaxion	$\Lambda_b^4\left(\bar{\phi}_{\rm eq}\right) = r_1 g_1 \Lambda^3 f$

Table 4: Conditions discussed in section 5.3.2 imposed in order for the CR mechanism in the $L + N$ model to be consistent.

of $V_{0,\rm break}$. From energy conservation, if there were no dissipation, the relaxion would stop in a configuration for which the potential energy density equals to $E_{\rm crit}$, which exists since the barrier of $V_{0,\rm inv}$ grows as $\bar{\phi}$ grows. If one adds the Hubble friction, that dissipates the kinetic energy of the relaxion, the relaxion will stop before this configuration. The exact expression for the equilibrium configuration $\bar{\phi}_{\rm eq}$ is given by the asymptotic behaviour of the solution of the EoM of $\bar{\phi}$,

$$\ddot{\bar{\phi}} + 3H_b\dot{\bar{\phi}} + V_0' = 0, \tag{5.64}$$

with V_0 given by Eqs. (5.25), (5.28) and (5.47).

In general Eq. (5.64) cannot be solved analytically [18]. However, in the case that H_b^{-1} is much smaller than the time Δt spent by the relaxion to cross a wiggle of the periodic potential, in other words, when the Hubble friction is very strong, the EoM can be written approximately as

$$\dot{\bar{\phi}} = -\frac{V_0'}{3H_b}, \tag{5.65}$$

and implies that the relaxion will stop immediately as $V_0' = 0$. From Eqs. (5.28) and

(5.47) one obtains that this condition is first satisfied when [17]

$$\Lambda_b^4\left(\bar{\phi}_{\text{eq}}\right) = r_1 g_1 \Lambda^3 f. \tag{5.66}$$

One can also study the opposite situation, i.e. $H_b \Delta t \ll 1$, in which the Hubble friction is small. As a consequence, the relaxion oscillates quickly through the barriers and loses very little kinetic energy, hence the averaged velocity of $\bar{\phi}$ is still described by the slow-roll velocity (5.54). After a careful analysis of the EoM [18], it is possible to show that the barrier of the periodic potential at the stopping configuration is approximately given by

$$\Lambda_b^4 \lesssim \frac{1}{2}\dot{\bar{\phi}}_{\text{SR}}^2, \tag{5.67}$$

which agrees with our previous argumentation based on energy conservation*.

⋆⋆⋆

In this section some constrains on the $L + N$ model that assure the mechanism to properly work were studied. They are summarised in Table 4. In addition to those, one can use them to compute an upper bound for the cut-off Λ in terms of only the height of the barrier Λ_b at the equilibrium point and of the number of e-folds N_e [18, 91, 94].

*Note that the potential $V_{0,\text{break}}$ varies very little from the critical point to the equilibrium one, whence the kinetic energy is almost entirely transferred to the periodic potential.

Starting from Eq. (5.62) for the number of e-folds one obtains

$$
\begin{aligned}
N_e &\simeq \frac{g_0 H_b^2}{r_1 g_1^2 \Lambda^2} \\
&= \frac{g_0}{r_1 g_1^2 \Lambda^2} H_b^5 \frac{1}{H_b^3} \\
&> \frac{g_0}{r_1 g_1^2 \Lambda^2} \left(\sqrt{\frac{r_0}{3}} \frac{\Lambda^2}{M_P} \right)^5 \left(\frac{3^{3/2}}{r_1 g_1 \Lambda^3} \right) \\
&= \left(\frac{g_0 r_0^{5/2}}{3 r_1^2} \right) \frac{\Lambda^5}{g_1^3 M_P^5} \\
&= \left(\frac{g_0 r_0^{5/2}}{3 r_1^2} \right) \frac{\Lambda^5}{M_P^5} \left(\frac{r_1^3 \Lambda^9 f^3}{\Lambda_b^{12}} \right) \\
&> \left(\frac{g_0 r_0^{5/2}}{3 r_1^2} \right) \frac{\Lambda^5}{M_P^5} \left(\frac{r_1^3 \Lambda^9 H_b^3}{\Lambda_b^{12}} \right) \\
&> \left(\frac{g_0 r_0^{5/2}}{3 r_1^2} \right) \frac{\Lambda^5}{M_P^5} \left(\frac{r_1^3}{\Lambda_b^{12}} \right) \left(\frac{r_0}{2} \right)^{3/2} \frac{\Lambda^{15}}{M_P^3} \\
&= \left(\frac{g_0 r_0^4 r_1}{3^{5/2} M_P^8} \right) \frac{\Lambda^{20}}{\Lambda_b^{12}},
\end{aligned} \tag{5.68}
$$

where Eqs. (5.53), (5.57), (5.58) and (5.66) were used. Using that the reduced Planck mass is $M_P \simeq 10^{18}$ GeV, one has the following bound for the cut-off,

$$
\Lambda \lesssim \left(\frac{3^{5/2}}{g_0 r_1 r_0^4} \right)^{1/20} \left(\frac{\Lambda_b}{10^3 \text{ GeV}} \right)^{3/5} \left(\frac{N_e}{10^{26}} \right)^{1/20} 10^{10} \text{ GeV}, \tag{5.69}
$$

where the value 10^{26} for the number of e-folds is the maximum allowed with reasonable fine tuning. Note that the cut-off can be as high as 10^{10} GeV.

5.3.3 Experimental constrains

Having derived the theoretical constrains of the $L + N$ model, we proceed to study some of its the phenomenological signatures. Our brief analysis will be focused on general properties of relaxion models and on some aspects of the phenomenology of the $L + N$ model.

Higgs-Relaxion mixing

We begin with a more general analysis of relaxion models. Independently on the choice of the additional particle content and interactions, all relaxion models have the linear coupling with the Higgs, given by Eq. (5.23), as a consequence there is a mixing between the Higgs and the relaxion [86, 87], so one needs to diagonalise the corresponding mass-matrix to obtain the mass eigenstates. Before writing down this mass-matrix, let us understand what are the impacts of this mixing at the phenomenological level. Suppose that the the diagonalisation is achieved by the rotation

$$\begin{pmatrix} \phi \\ h \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \tilde{\phi} \\ \tilde{h} \end{pmatrix}, \tag{5.70}$$

with $\tilde{\phi}$ and $\tilde{h}$ the respective mass eigenstates. This implies that the all the interactions of the Higgs in the SM will be now proportional to $\cos\theta$, for instance the coupling to the top k_F and the triple-gauge boson coupling $k_V^{\mathbf{3}}$, which were first introduced in section 3.4 in the context of Composite Higgs Models. They are now given by

$$k_F \to k_F \cos\theta, \quad k_V^{\mathbf{3}} \to k_V^{\mathbf{3}} \cos\theta. \tag{5.71}$$

From the experimental values in Eq. (3.141), extracted from Figure 4, one concludes that this angle is very small and even compatible with zero, implying that the mixing is very suppressed. To determine how does this fact impact the model, one needs to compute θ in terms of the parameters of the model.

The relevant terms in the Lagrangian are given by Eqs. (5.23) and (5.47),

$$\mathscr{L}_{\text{mix}} = g_1 \Lambda \phi \frac{1}{2}(v+h)^2 - \Lambda_b^4 \cos\frac{\phi}{f} - \frac{1}{2} m_h^2 h^2 - \frac{1}{2} m_\phi^2 \phi^2, \tag{5.72}$$

where m_h^2 and m_ϕ^2 are the mass parameters of h and ϕ, respectively. In order to obtain the eigenstates today, one expands the relaxion around its equilibrium point, $\phi \to \bar{\phi}_{\text{eq}} + \phi$, in this manner the Lagrangian is given by

$$\begin{aligned} \mathscr{L}_{\text{mass}} &= -\frac{1}{2}\begin{pmatrix} \phi & h \end{pmatrix} \mathcal{M} \begin{pmatrix} \phi \\ h \end{pmatrix} \\ &= -\frac{1}{2}\begin{pmatrix} \phi & h \end{pmatrix} \begin{pmatrix} m_\phi^2 - \frac{\Lambda_b^4}{f^2}\cos\frac{\bar{\phi}_{\text{eq}}}{f} & -g_1 \Lambda v \\ -g_1 \Lambda v & m_h^2 \end{pmatrix} \begin{pmatrix} \phi \\ h \end{pmatrix}. \end{aligned} \tag{5.73}$$

The rotation in Eq. (5.70) diagonalises the mass-matrix above,

$$\mathscr{L}_{\text{mass}} = -\frac{1}{2}\begin{pmatrix}\tilde{\phi} & \tilde{h}\end{pmatrix}\begin{pmatrix}\tilde{m}_\phi^2 & 0 \\ 0 & \tilde{m}_h^2\end{pmatrix}\begin{pmatrix}\tilde{\phi} \\ \tilde{h}\end{pmatrix}, \tag{5.74}$$

with $\tilde{m}_\phi^2$ and $\tilde{m}_h^2$ the mass eigenvalues. Therefore the mixing angle can be written as

$$\tan 2\theta = \pm\frac{1}{\sqrt{\left(\frac{\tilde{m}_h^2-\tilde{m}_\phi^2}{2\mathcal{M}_{h\phi}}\right)^2 - 1}}, \tag{5.75}$$

where $\mathcal{M}_{h\phi}$ is the off-diagonal term of the mass matrix $\mathcal{M}$.

The fact that θ is very small implies that the denominator of Eq. (5.75) is quite large,

$$\frac{\tilde{m}_h^2 - \tilde{m}_\phi^2}{2\mathcal{M}_{h\phi}} \gg 1. \tag{5.76}$$

To expand the above equation in terms of the parameters of the model, one makes the assumption that the physical mass of the relaxion $\tilde{m}_\phi^2$ is much smaller than $\tilde{m}_h^2$. This is reasonable from the fact that axion-like particles (ALP's) tend to have small masses, but this depends on what kind of UV-completion generates the ALP, which in the particular case of the relaxion is not determined [3, 88]. Using Eqs. (5.51) and (5.66), and taking into account that $\tilde{m}_h^2 \sim v^2$, one obtains

$$\frac{\tilde{m}_h^2 - \tilde{m}_\phi^2}{2\mathcal{M}_{h\phi}} \simeq \frac{\tilde{m}_h^2}{2g_1\Lambda v} \simeq \frac{v}{g_1\Lambda}. \tag{5.77}$$

From Eq. (5.76) above is rewritten as

$$\frac{1}{g_1}\frac{v}{\Lambda} \gg 1. \tag{5.78}$$

Therefore, one obtains the following estimate for the coupling g_1,

$$g_1 \ll \frac{v}{\Lambda} \ll 1, \tag{5.79}$$

where the expectation that $v \ll \Lambda$ is being considered. Eq. (5.79) is an indication that g_1 is *extremely* small.

One remark regarding the bound (5.79) is in order. As stressed, every relaxion model will have at least a mixing with the Higgs from the linear interaction in Eq. (5.23). Nevertheless, additional sources of mixing appear from the particular characteristics of each model. Take for instance the $L + N$ model developed so far. This model has a Higgs-dependent barrier, which modifies non-trivially the mass-matrix $\mathcal{M}$. In particular, the off-diagonal term becomes

$$\mathcal{M}_{h\phi} \to \mathcal{M}_{h\phi} = -g_1 \Lambda v - 4\frac{\mu_b^4}{vf} \sin\frac{\bar{\phi}_{\text{eq}}}{f}, \tag{5.80}$$

due to the Higgs-dependent barrier in Eq. (5.47). At the equilibrium point, using Eqs. (5.66) and (5.51), and supposing that

$$\sin\frac{\bar{\phi}_{\text{eq}}}{f} \sim \mathcal{O}(1), \tag{5.81}$$

which reflects our ignorance on the precise value of $\bar{\phi}_{\text{eq}}$, Eq. (5.80) can be written as

$$\mathcal{M}_{h\phi} = -g_1 \Lambda v - 4\frac{\mu_b^4}{vf} \sin\frac{\bar{\phi}_{\text{eq}}}{f} \simeq -g_1 \Lambda v - 4 g_1 r_1 \frac{\Lambda^3}{v} \simeq -g_1 \frac{\Lambda^3}{v}\left(1 + \frac{v^2}{\Lambda^2}\right) \simeq -g_1 \frac{\Lambda^3}{v}, \tag{5.82}$$

where in the second equality it was used that r_1 is also a $\mathcal{O}(1)$ parameter (see Eq. (5.41)) and in the last step that $v \ll \Lambda$. From the result above we obtain the following bound

$$g_1 \ll \left(\frac{v}{\Lambda}\right)^3 \ll 1, \tag{5.83}$$

which is clearly distinct from the one obtained in Eq. (5.79), though they both imply in a small g_1.

This example shows to us how intricate it is to make general, model-independent predictions in relaxion models. Although the mixing between the Higgs and the relaxion is indeed a general characteristic of relaxion models, it is clear that each particular model will have a distinct value for the mixing, since they may have additional contributions to the mass-matrix. Even worse, these can suppress the contribution from the linear interaction (5.23), as it is clear from the derivations of the estimates to the coupling g_1 in Eqs. (5.79) and (5.83), making the model-independent signals more concealed. In order to understand more clearly how difficult it is to make general statements regarding relaxion models, let us consider the $L + N$ model once again. We have seen in section 5.3.2 that this model predicts the condensation of the fermions N and N^c, that at low energies manifests itself as an electrically neutral meson* field $\tilde{\eta}$. This field enters in the low-energy EFT after the condensation of the new strong group through the Goldstone matrix $U = \exp\left(-i\tilde{\eta}/\sqrt{2}f_{\tilde{\eta}}\right)$, with $f_{\tilde{\eta}}$ the corresponding decay constant. Expanding the potential (5.47) while taking into account $\tilde{\eta}$ results in a mixing of it between both the Higgs and the relaxion. In other words, the $L+N$ models predicts not only additional terms to $\mathcal{M}_{h\phi}$, but also that the mass-matrix is a 3×3 matrix that mixes $\tilde{\eta}$, ϕ and h, which thus implies in a more complicated diagonalisation. In this case all off-diagonal terms are still all proportional to the coupling g_1, therefore one expects similar results to hold, i.e. a very small g_1.

We emphasise once more that the fact that the Higgs mixes with the relaxion is general and independent of which particular model is being considered. *How* it mixes, however, is model-dependent. We can only expect that the experimental bounds on k_F and $k_V^{\mathbf{3}}$ force some of the mixing angles to be very small, which in turn could imply in a bound for g_1, as in the case of Eqs. (5.79) and (5.83).

The mixing between the Higgs and the relaxion would be a general phenomenological consequence from the point of view of collider physics. It remains to analyse the phenomenology from the cosmological perspective. Unfortunately, the relaxion itself does not leave any relevant imprint on the cosmological history of the universe, because the relaxion mechanism takes place at primordial times of the universe (be it during inflation or not), and there are no direct experimental probes of those epochs. Furthermore, since $g_1 \ll 1$, the relaxion is very weakly interacting, hence it will not have any relevant impact on more recent cosmological events like the Cosmic Microwave Background (CMB) or Big-Bang Nucleosynthesis (BBN). We see that, with respect to the other solutions to the Hierarchy Paradigm, the CR models are almost invisible to experimental searches of cosmology.

$L+N$ phenomenology

As the model-dependent experimental signs are more pronounced as the model independent ones, one could in principle study the phenomenology of the $L+N$ model in more depth. A complete review of those, however, is out of the scope of this thesis. In

*This meson is the analogue of η' of QCD, that is associated with the anomalous axial symmetry of the fermions. As noted in section 5.3.2, this meson will have a mass of the order of the condensation scale $4\pi F$ since it is not protected by shift-symmetry.

spite of that, it is necessary to point out some of the relevant signatures in order to have a more complete understanding of the model. Based on the analysis done in [91], we will comment on three phenomenological aspects of the $L + N$ model.

The first and most straightforward consequence of having fermions charged under the SM gauge group is their impact on the EW precision measurements (EWPM), which were already mentioned in previous chapters [1, 58]. The EWPM can be split in two parts: oblique and non-oblique corrections. "Oblique" refers to corrections to the propagators of gauge bosons and can be parametrised by the extended Peskin-Takeuchi set of parameters $STUVWX$*. The constrains on the parameters of the new strong sector, namely M_L, M_N, y and $\tilde{y}$, are obtained by making use of the experimental values of the accessible EW observables and by minimising the corresponding χ^2 function. Some of the results obtained in [91] are plotted in Figure 8a), which shows the excluded region in the $y - M_L$ plane in the case that $y = \tilde{y}$ and for distinct values of M_N.

It is clear from Figure 8a) that the oblique effects already constrain some of the parameter space, but there are also relevant non-oblique signatures in the $L + N$ model. The most important one regards the decay channels of the Higgs, because, due to the Yukawa couplings in Eq. (5.38), the Higgs can decay into the fermions from the new strong sector. Moreover, the fermions will hadronize into the unstable meson $\tilde{\eta}$, which will at some point decay to SM particles. As a consequence, the branching ratios of the Higgs, in particular the invisible one, are modified and further constrains on the parameter space are obtained. The main results of [91] are shown in Figure 8b).

In addition to the EWPM, one can obtain some interesting bounds from cosmology. We stressed that the relaxion itself is almost invisible to cosmological searches, but the fermions from the new strong sector interact with the SM and can therefore leave some traces during the cosmological evolution. For instance, after the temperature of the universe is lower than the condensation scale $4\pi F$ the strong group will hadronize and $\tilde{\eta}$ mesons will be produced. They will eventually decay into either hadrons from the SM or radiatively into photons and leptons, which implies that the distributions of protons, neutrons, electrons and photons may be modified. One of the cosmological events that may be disrupted by the decays of the $\tilde{\eta}$ meson is Big-Bang Nucleo-synthesis (BBN), the

*The Peskin-Takeuchi parameters STU can unequivocally measure the deviations of the EW sector caused by new physics only if this latter is much heavier than the EW scale. If it is not far or even below the EW scale, the additional three parameters VWX are needed [58, 97, 59].

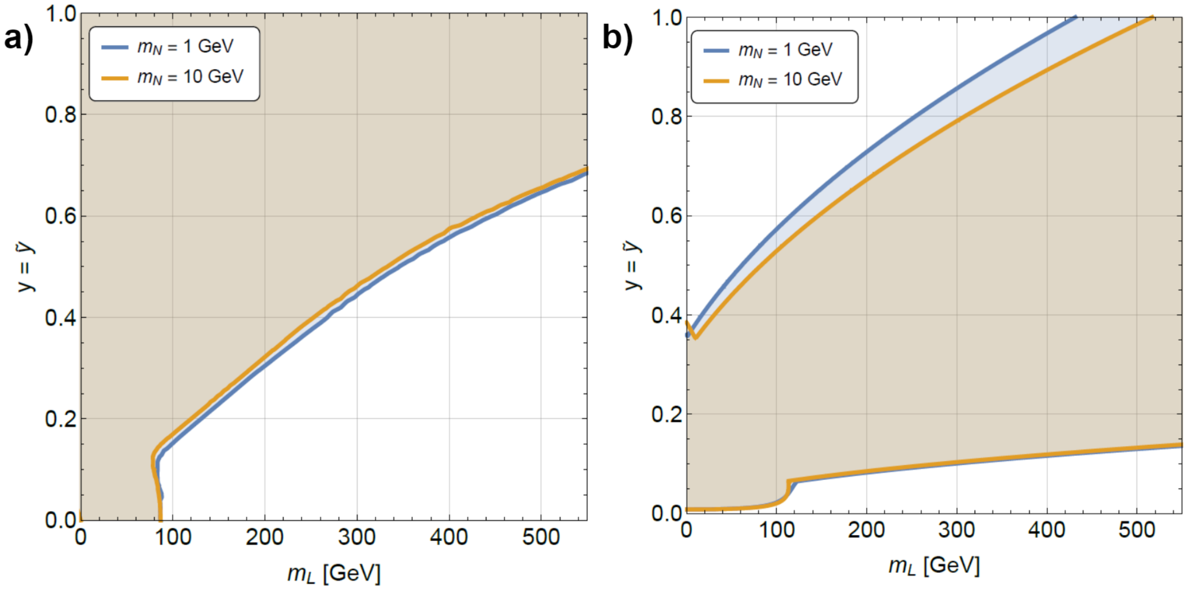


Figure 8: Excluded regions (coloured) at 95% CL from the bounds on **a)** oblique EWPM and **b)** Higgs decays, for two distinct values of M_N. In the plots above it is considered that $y = \tilde{y}$ and that the strong group is given by $G_s = SU(3)$. Both plots were taken from [91].

epoch in which light elements were formed and that took place at $T \lesssim$ MeV [77, 85]. In this manner, depending on the value of the condensation scale and on the life-time of $\tilde{\eta}$, some regions of the parameter space are excluded by taking the precise data from the BBN into account [3, 98]. The results of [91] are shown in Figure 9. The excluded region in this case depends heavily on how $\tilde{\eta}$ decays. If it decays into hadrons of the SM, which can alter the ratio of protons and neutrons, then its life-time cannot exceed 10^{-1} s. If it decays radiatively into photons and leptons, the upper bound of the life-time is thus 10^4 s [98].

5.4 Conclusions

Cosmological Relaxation models offer a novel way of approaching the Hierarchy Paradigm. With respect to other solutions, CR models take the evolution of the universe into account from the beginning, which allows us to make contact with cosmological phenomena more consistently. Furthermore, they do not rely on new symmetries, neither local nor global ones, but on dynamics instead. As it was seen, this makes the discussion in terms of EFT rather difficult, because there are no selection rules available to unequivocally determine the properties of the mechanism at low energies. Notwithstanding, some of the most

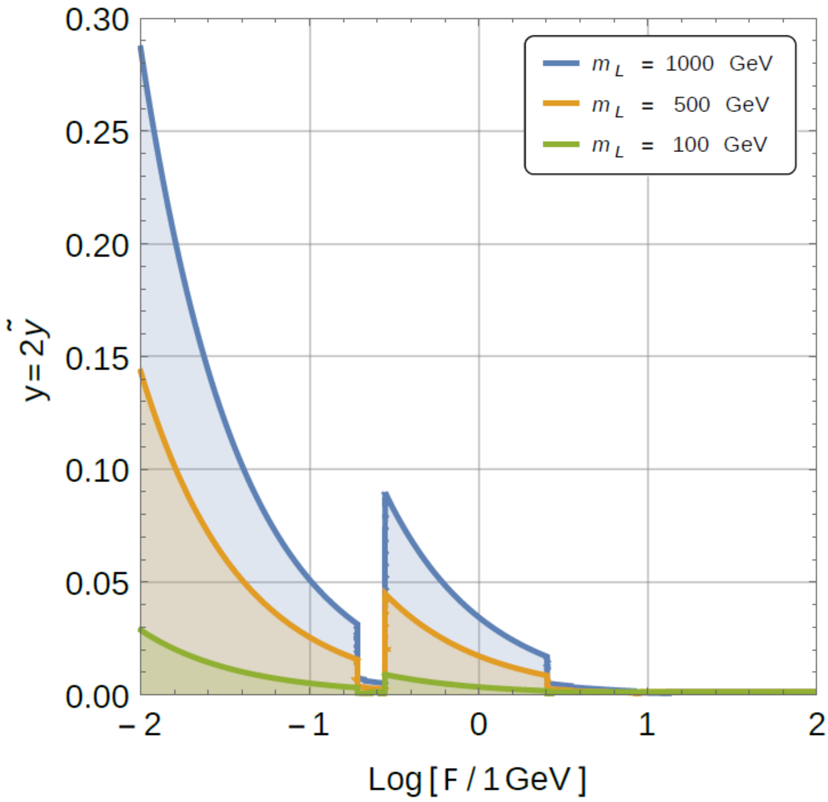


Figure 9: Excluded region (coloured) from the bounds imposed by BBN, in term of the Yukawa couplings and the condensation scale F, for distinct values of M_L and for $G_s = SU(3)$. Taken from [91]

important features of the CR mechanism, for instance the back-reaction of the Higgs mass parameter, are encoded in the EFT formulation and, as it was shown, the relaxion models have a solid motivation based on this framework. Altogether the CR model, and more in particular the relaxion models, stabilises the Higgs mass parameters due to the coupled dynamical evolution of M_H^2 and ϕ during primordial epochs of the universe, and therefore disregards the notion that new physics should be around TeV scale.

We have studied in addition the $L + N$ model, that is a specific realisation of the CR mechanism, together with some of its phenomenological consequences. From the theoretical point of view, interesting features of this model are its simplicity and the explicit representation of the back-reaction through the Higgs-dependent barrier. From the more phenomenological view, the $L + N$ model offers a rich variety of experimental signals, which spans from EW precision measurements to deviations from Big-Bang Nucleosynthesis, making the model testable in many scenarios.

Conclusions

In this thesis the Hierarchy Problem of the SM of particle physics was studied along with some of its solutions. Particular emphasis to the formulation of the question, as there can only be answers to well defined questions, was given. After presenting four distinct solutions to the problem, they will be now analysed from the point of view of the classification made in Chapter 1, in terms of the Scalar Mass Problem (SMP) and the Hierarchy Paradigm (HPa), taking into account whereas they are symmetry-based or dynamical solutions.

We begin our discussion with symmetry-based solutions. Models that propose to solve the Hierarchy Problem by introducing new symmetries are very similar, because they all introduce new selection rules in the IR that protect the Higgs mass parameter at tree- and quantum-level. Exact symmetries, however, render the phenomenology incompatible with experiments, so all the new symmetries must be broken, implying that the low-energy expansion of the corresponding EFT must contain operators that explicitly break them. These operators parametrise how the effects of new physics generate the mass term for the Higgs and are characterised by a new energy scale f, that is the scale at which new physics is expected to be observed. As a consequence, the Higgs acquires a non-vanishing mass, which depends on the scale f and on the relevant couplings, and is in general much smaller than the Planck mass M_P. With such reasoning, the question being addressed is obviously the SMP, since one relies on the techniques and predictions of the EFT in order to stabilise the Higgs mass parameter in the IR. Note that till this point the fine tuning (FT) is non-existing, as one is entirely within the EFT framework.

The fact that the mass of the Higgs is computable and given in terms of f, allows us to predict the value of the latter or, at the least, to have an order of magnitude estimate. For instance, from the explicit expression for the Higgs mass in the Minimal Supersymmetric SM (MSSM) one expected that the stop mass, proxy for the soft-breaking scale of SUSY, would be around 1 TeV. However, for all symmetry-based solutions studied in this thesis, the respective experimental signatures were not observed. Hence, the scale f is larger than the value computed from the EFT, meaning that the prediction of the latter is at odd with observations. This situation is usually circumvented assuming a certain amount of FT, in other words, one assumes that there are unknown UV effects that contribute to the stabilisation of the Higgs mass. Concretely, we assume that there is an additional contribution $m_h^2\big|_{\rm UV}$ to the physical Higgs mass m_h^2,

$$m_h^2 = m_h^2\Big|_{\rm EFT} - m_h^2\Big|_{\rm UV},$$

where $m_h^2\big|_{\rm EFT}$ is the EFT prediction, for instance Eq. (2.139) in the case of the MSSM, and the cancellation between both terms is such to obtain the observed value for the physical mass. As stressed in Chapter 1, such UV effects do not respect the decoupling of heavy modes, which is a core principle of the EFT framework, and must be introduced by hand. Therefore, from the moment that it is assumed that there is a certain amount of FT being created, such that the Higgs mass parameter is stabilised at its experimental value, we are abandoning the EFT approach and asking ourselves why the Higgs mass is smaller than what we expected it to be, i.e. the HPa is being addressed.

In essence, the symmetry-based solutions studied in this thesis have a connection with both SMP and HPa. On the one hand in their theoretical formulation, these models properly address the SMP and make predictions consistent with the principles of EFT. On the other hand, the disagreement with experiments and the introduction of FT deviate the model from an approach entirely within the EFT framework, introducing in this manner the HPa.

Let us now turn to the solutions that are based on dynamics. From our EFT analysis of the CR models, which implement the dynamics through the cosmic evolution, one concludes that such solutions are completely different with respect to symmetry-based ones. In particular, the focus is no longer how to generate a mass term for the Higgs given a particular broken symmetry, but in how the corresponding EFT evolves from the far-UV to the IR and allows the Higgs to be dynamically stabilised. In Chapter 5 we described how the pair $(M_H^2, \bar{\phi})$ evolved classically, and showed that it depends on the evolution equations and on the initial conditions. The latter could be a potential source of FT, in the sense that the evolution could only work by considering very specific initial conditions. However, from the EFT approach it was clear that the whole mechanism depends only mildly on them (apart from the sign of M_H^2), which followed directly from the fact that the explicit form of the theory at the UV is unknown. Hence, from this side the models do not suffer from FT problems. The same is not true for the coupled evolution equations, since they are only partially described by the general EFT approach.

From general arguments it was determined how the dynamics of ϕ affected M_H^2, but the necessary back-reaction is completely model-dependent. Therefore, there may be some FT associated with the new degrees of freedom (dof) that generate the back-reaction, which usually manifests itself through the stopping conditions of ϕ. Even so, this will depend on the experimental constraints of each model and, at least for the $L + N$ model, the allowed parameter space is still quite large.

What is clear from this discussion is that CR models are much more associated with the SMP rather than the HPa, as the core of the models are built upon the EFT framework and the FT is only introduced, if it is, from model-dependent factors. Given that CR models address more intensely the SMP, one must deepen the understanding of EFT description of dynamical-based approaches to the Hierarchy Problem. In what follows the relation between CR models and EFT's will be discussed in more detail, in particular regarding the decoupling principle.

It is not trivial that the Wilsonian decoupling principle is respected in the case of EFT's with an associated dynamic. After all, note that in CR models the stabilisation of the Higgs mass requires very large field values in the IR (in some cases even larger than the Planck mass), which in principle is a contradiction to the decoupling of heavy modes. As pointed out recently in [28, 99], this is not the case, and the reason lies precisely on the dynamical nature of the models: The fact that the CR mechanism rely on very heavy modes to stabilise the Higgs mass parameter in the IR is not a violation of the Wilsonian decoupling, because there is sufficient time to observe them. More precisely, at a given energy scale E, the EFT predicts that the probability for the production of quanta with energies much larger than E is very suppressed. However, if one waits long enough, or if has sufficient samples, one can overcome these suppressed probabilities and indeed observe physical states with energies above the present threshold. One subtlety of this argument lies on the requirement that the relevant fields are tightly connected to the evolution, such that the production of quanta with a given energy depends on time. This is true for the case of CR models, since the evolution of the universe defines the energy scale of the thermal bath and therefore changes the respective dispersion relations at each instant.

One remark is in order. Although the arguments above explains the apparent contradiction of CR models with a very clear physical reasoning, it does not elucidate what is their underlying physics. In other words, it is not straightforward to determine which fundamental dof render the dynamical evolution of the EFT consistent with the decoupling principle. As stressed in [28], one possible realisation in the context of CR models uses the graviton, the particle that mediates gravitational interactions, because it is precisely the field that induces the cosmological evolution. Nevertheless, the details of this idea are not explicitly stated and it is possible that this is not the only feasible alternative. A more precise analysis of this situation could be subject of future investigation.

⋆⋆⋆

It is clear from our discussion that the Hierarchy Problem is, even after decades of research, still a rich subject with many theoretical as well experimental prospects. Not only its solutions keep guiding us to the search for new physics beyond the standard model, but also with each new solution the comprehension of the problem itself is improved. In this manner, the conceptual and phenomenological difficulties behind the Hierarchy Problem are slowly unravelled, and so the path to the future of particle physics is paved.

A Effective actions and potentials

In this appendix we review the concepts of effective actions and the techniques of effective potential. This is a review based on [1, 8, 20, 21, 100, 101].

The method of effective action in Quantum Field Theories is an alternative approach to calculate quantum contributions. Instead of computing loop diagrams for each amplitude of interest, all loop effects are resumed in the Lagrangian (or more precisely in the action), such that all vertices produced by this new action, the so called effective action, already contains the renormalized quantum effects.

This appendix is structured in the following way. In the first section we develop the basic intuition behind the concept of effective action. Then, we show explicitly how to compute the effective action and the corresponding effective potential. In particular, we perform these calculations using two distinct regularisation methods.

A.1 The effective action

To compute any given n-point function in a Quantum Field Theory it is enough to take derivatives of the generating functional $Z[J]$ with respect to the external current J. Concretely, let $Z[J]$ be given by

$$Z[J] = \int \mathcal{D}\phi \; e^{iS[\phi]+i\int \mathrm{d}^4x \;\; \phi(x)J(x)}, \tag{A.1}$$

where $\phi(x)$ are some set of fields, whose precise structure does not concern us here, and $S[\phi]$ is the classical action. To simplify the following discussion, we will use the connected generating functional, defined from Eq. (A.1) as

$$Z[J] = e^{iW[J]}, \tag{A.2}$$

which generates only connected amplitudes. Eq. (A.1) can be rewritten as

$$iW[J] = \int \left[\mathcal{D}\phi\right]_c \; e^{iS[\phi]+i\int \mathrm{d}^4x \;\; \phi(x)J(x)}, \tag{A.3}$$

where the subscript c in the integration measure leaves explicit that we are summing only in the connected diagrams. As usual, the connected n-point function $G_c^{(n)}$ can be written as the following functional derivative:

$$G_c^{(n)}(x_1, \cdots, x_n) = \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)}\Big|_{J=0}. \tag{A.4}$$

To take into account for quantum effects, Eq. (A.4) must be supplemented with a renormalization procedure, which involves the redefinition of the parameters and many loop integrals for each n-point function of interest. This necessity steams from the fact that the action $S[\phi]$ is fully classical, in the sense that it does not contain any information regarding the quantum nature of the theory and depends therefore only on the bare couplings and fields. From this reasoning we may put forward the question if there exists a modified action $\Gamma[\phi]$ that contains all renormalized quantum effects and will therefore allows us to compute any $G_c^{(n)}$ with only tree-level diagrams.

It is indeed possible to obtain such an action $\Gamma[\phi]$. Based on the previous discussion, we may define it formally and indirectly as

$$\begin{aligned} iW[J] &= \int [\mathcal{D}\phi]_c \; e^{iS[\phi]+i\int \mathrm{d}^4x \;\; \phi(x)J(x)} \\ &= \int_{\text{tree}} [\mathcal{D}\phi]_c \; e^{i\Gamma[\phi]+i\int \mathrm{d}^4x \;\; \phi(x)J(x)}, \end{aligned} \tag{A.5}$$

where *tree* denotes that the integral is only over the tree-level diagrams*. Obviously, Eq. (A.5) permits us to compute Γ directly in terms of the classical action S. Naively we may expect the following equation to hold,

$$i\Gamma \sim \int_{\text{loops}} [\mathcal{D}\phi]_c \; e^{iS[\phi]},$$

where *loops* denotes that we are summing only over the loop diagrams. To make sense of the equation above, we need to formulate it more precisely. First, note that Γ is a functional of ϕ and contains information of all amplitudes with an arbitrary number of external states. Therefore, the argument of S inside the integral must be shifted by some $\tilde{\phi}$ in order to insert $\tilde{\phi}$ in each external line of each n-point function computed from

*The fact that Eq. (A.5) is restricted to tree-level diagrams means that the integral is selecting only the field configurations ϕ_J that satisfy the equations of motion, which are now given by

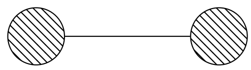
$$\frac{\delta\Gamma[\phi_J]}{\delta\phi_J(x)} = -J(x).$$

From the equation above one can relate $\Gamma[\phi_J]$ and $W[J]$ through a Legendre transformation [21]

S. Hence, the action Γ in the naive equation above becomes a function of this external configuration $\tilde{\phi}$ and may be rewritten as

$$i\Gamma[\tilde{\phi}] \sim \int_{\text{loops}} [\mathcal{D}\phi]_c \, e^{iS[\phi+\tilde{\phi}]}. \tag{A.6}$$

However, the equation above is still insufficient to describe $\Gamma[\phi]$, because we need to better specify the integration domain. To understand how to perform the integral, take for instance a loop diagram, with an arbitrary number of external states, that cannot be separated into two disconnected diagrams by removing an internal line. Diagrams of this type are called 1 Particle Irreducible (1PI) and contribute to the integral in Eq. (A.6). Consider now a second loop diagram which is not 1PI, because in the absence of an internal state it is simply the product of two disconnected 1PI loop diagrams. This latter can be represented as



(A.7)

where each blob is a 1PI loop diagram. We note that diagrams such as the one in Eq. (A.7) do not contribute to the integral in Eq. (A.6) and the reason is the following. As stressed, 1PI diagrams are ones that need to be taken into account to compute $\Gamma[\phi]$, and so the vertices produced by it contain information on all 1PI diagrams. The diagram in Eq. (A.7) is nothing but a tree-level composition of 1PI diagrams, therefore it does not belong to the integration in Eq. (A.6), but to the one in Eq. (A.5). In conclusion, only 1PI must be considered and hence

$$i\Gamma[\tilde{\phi}] = \int_{\text{1PI}} [\mathcal{D}\phi]_c \, e^{iS[\phi+\tilde{\phi}]} \tag{A.8}$$

is the correct expression* for the effective action $\Gamma[\phi]$. In terms of exponential the equation above is given by

$$e^{i\Gamma[\tilde{\phi}]} = \int_{\text{1PI}} \mathcal{D}\phi \, e^{iS[\phi+\tilde{\phi}]}, \tag{A.9}$$

in which connected and disconnected diagrams are considered.

*In the discussion above only perturbation theory is considered. If other effects steaming from non-perturbative physics are relevant, these must be added to the computation of $\Gamma[\phi]$ [1].

A.2 Coleman-Weinberg potential

Next we show how to compute the effective action $\Gamma[\phi]$ explicitly. Let us first consider the action of a real scalar field given by

$$S[\phi] = \int \mathrm{d}^4x \left[\frac{1}{2}(\partial\,\phi)^2 - V(\phi)\right], \tag{A.10}$$

with $V(\phi)$ the corresponding potential. To compute $\Gamma[\phi]$ in Eq. (A.9) we must first shift the field ϕ by some $\tilde{\phi}$,

$$\begin{aligned} S[\phi+\tilde{\phi}] &= \int \mathrm{d}^4x \left[\frac{1}{2}\Big(\partial\,\phi + \partial\,\tilde{\phi}\Big)^2 - V(\phi+\tilde{\phi})\right] \\ &= \int \mathrm{d}^4x \left[\frac{1}{2}(\partial\,\phi)^2 + \frac{1}{2}(\partial\,\tilde{\phi})^2 + \partial\,\phi\,\partial\,\tilde{\phi} - V(\tilde{\phi}) - \phi V'(\tilde{\phi}) - \frac{1}{2}\phi^2 V''(\tilde{\phi}) + \cdots\right], \end{aligned} \tag{A.11}$$

where the dots denote higher order terms in the expansion of the potential. From the above expansion we may compute $\Gamma[\tilde{\phi}]$ perturbatively, such that it may be written as

$$\Gamma[\tilde{\phi}] = \Gamma^{(0)}[\tilde{\phi}] + \Gamma^{(1)}[\tilde{\phi}] + \cdots\,, \tag{A.12}$$

where $\Gamma^{(n)}[\tilde{\phi}]$ denotes the nth loop contribution. It is thus obvious that

$$\Gamma^{(0)}[\tilde{\phi}] = \int \mathrm{d}^4x \left[\frac{1}{2}\Big(\partial\,\tilde{\phi}\Big)^2 - V(\tilde{\phi})\right]. \tag{A.13}$$

The 1-loop contribution involves terms only up to ϕ^2, but since the integration is restricted to 1PI diagrams, linear terms do not contribute. Hence,

$$e^{i\Gamma^{(1)}[\tilde{\phi}]} = \int_{1\mathrm{PI}} \mathcal{D}\phi\ \exp\Big\{-i\int \mathrm{d}^4x\ \frac{1}{2}\phi\Big(\Box + V''(\tilde{\phi})\Big)\phi\Big\} \propto \frac{1}{\sqrt{\det\Big(\Box + V''(\tilde{\phi})\Big)}} \tag{A.14}$$

$$\Rightarrow\ i\Gamma^{(1)}[\tilde{\phi}] = -\frac{1}{2}\,\mathrm{tr}\ln\Big(\Box + V''(\tilde{\phi})\Big), \tag{A.15}$$

where the trace is over space-time indices as well any other internal index.

Before computing the 1-loop effective action in Eq. (A.15), some remarks are in order. First, the example above is very simple, because it contains only a single real scalar field that self-interacts via its potential. More generally, ϕ interacts with other fields, and thence receives more radiative corrections. In such situations the integration in Eq. (A.14) must be performed for each field, or more precisely, for each dof that interacts with ϕ. This implies that the effective action $i\Gamma^{(1)}$ will be proportional to the total number of dof of the corresponding field. Second, for bosons, i.e. scalars and vectors, the integration procedure is very similar to the one in Eqs. (A.14) and (A.15), with the difference of taking all dof into account. The case of fermions is a bit different and must be briefly discussed. Note that for some operator $\mathcal{O}(\tilde{\phi})$

$$e^{i\Gamma^{(1)}[\tilde{\phi}]} = \int_{\text{1PI}} \mathcal{D}\bar{\psi}\mathcal{D}\psi \;\exp\Big\{i\int \mathrm{d}^4x\; \bar{\psi}\Big(i\partial\!\!\!/ - \mathcal{O}(\tilde{\phi})\Big)\psi\Big\} \propto \det\Big(i\partial\!\!\!/ - \mathcal{O}(\tilde{\phi})\Big) \tag{A.16}$$

$$\Rightarrow\; i\Gamma^{(1)}[\tilde{\phi}] = \operatorname{tr}\ln\Big(i\partial\!\!\!/ - \mathcal{O}(\tilde{\phi})\Big). \tag{A.17}$$

Using the properties of the trace we obtain that

$$\operatorname{tr}\ln\Big(i\partial\!\!\!/ - \mathcal{O}(\tilde{\phi})\Big) = \operatorname{tr}\ln\Big(-i\partial\!\!\!/ - \mathcal{O}(\tilde{\phi})\Big), \tag{A.18}$$

therefore

$$\begin{aligned} i\Gamma^{(1)}[\tilde{\phi}] &= \operatorname{tr}\ln\Big(i\partial\!\!\!/ - \mathcal{O}(\tilde{\phi})\Big) \\ &= \frac{1}{2}\Big[\operatorname{tr}\ln\Big(i\partial\!\!\!/ - \mathcal{O}(\tilde{\phi})\Big) + \operatorname{tr}\ln\Big(-i\partial\!\!\!/ - \mathcal{O}(\tilde{\phi})\Big)\Big] \\ &= \frac{1}{2}\operatorname{tr}\ln\Big[\Big(i\partial\!\!\!/ - \mathcal{O}(\tilde{\phi})\Big)\Big(-i\partial\!\!\!/ - \mathcal{O}(\tilde{\phi})\Big)\Big] \\ &= \frac{1}{2}\operatorname{tr}\ln\Big[\Box + \mathcal{O}(\tilde{\phi})^2 + i\partial\!\!\!/\mathcal{O}(\tilde{\phi})\Big], \end{aligned} \tag{A.19}$$

which is in the same form as Eq. (A.15) and can be therefore computed in an analogous manner.

A.2.1 Dimensional regularisation

Let us compute

$$iD[\tilde{\phi}] = \frac{1}{2}\operatorname{tr}\ln\left[\Box + U\right], \tag{A.20}$$

where U is an operator that can depend on $\tilde{\phi}(x)$ (and therefore on the space-time coordinates). Note that the equation above may represent both Eq. (A.15) and (A.19), expect for some sign.

The trace may be taken over either the space-time-coordinates x or over the momentum p. Some care must be taken, however. On the one hand note that the operator U

is a local operator in x, so its action is well defined on the $|x\rangle$ basis. On the other the D'Alembert operator can be trivially represented in the momentum basis $|p\rangle$ by

$$\langle x| \left(-P^2\right) |p\rangle = -(i\,\partial)^2 \langle x|p\rangle = -(i\,\partial)^2 e^{-ixp}, \tag{A.21}$$

with P the momentum operator. In this manner $D[\tilde{\phi}]$ becomes

$$\begin{aligned} iD[\tilde{\phi}] &= \frac{1}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \langle p|\ln\left[-P^2+U(X)\right]|p\rangle \\ &= \frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \langle p|x\rangle\,\langle x|\ln\left[-P^2+U(X)\right]|p\rangle \\ &= \frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4}\ e^{ipx}\ln\left[-(i\,\partial)^2+U(x)\right]e^{-ipx}. \end{aligned} \tag{A.22}$$

In order to cancel the exponentials in the equation above we must commute one of them with the differential operator $\mathrm{tr}\ln\left[-(i\,\partial)^2+U(x)\right]$. Note that this operator depends also on x through $U(x)$, whence it is incorrect to simply substitute $i\,\partial$ by p, because $i\,\partial$ and $U(x)$ do not commute. The action of the exponential is, in this case, to shift $i\,\partial$ by p,

$$\begin{aligned} iD[\tilde{\phi}] &= \frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \ln\left[-(i\,\partial+p)^2+U(x)\right] \\ &= \frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \ln\left[-(i\,\partial-p)^2+U(x)\right] \\ &= \frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \ln\left[-p^2+U(x)+(2ip\cdot\partial+\Box)\right], \end{aligned} \tag{A.23}$$

where in the second line we made a change of variable in p.

The expression in Eq. (A.23) contains two distinct contributions to the 1-loop effective action. The first, that comes from $-p^2+U(x)$, does not contain any derivative of $\tilde{\phi}$ and will therefore only contribute to the its potential (in the case of the scalar field). The second contribution comes from the terms with derivatives $2ip\cdot\partial+\Box$, which accounts for corrections to the kinetic term and derivative interactions. The contribution from derivatives were recently studied in [101], where the two derivative term was explicitly computed.

In this thesis we are only interested in the 1-loop effects to the potential, hence we will set $\partial = 0$ from now on. Then, we must compute

$$iD[\tilde{\phi}] = \frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \ln\left[-p^2 + U(x)\right]. \tag{A.24}$$

In this section we will use dimensional regularisation (dim reg) to evaluate Eq. (A.24). Due to the logarithm we cannot compute D directly with the techniques of dim reg. A way out is to represent the logarithm as

$$\ln x = -\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\ \frac{1}{x^\alpha}\right)_{\alpha=0}, \tag{A.25}$$

in this manner the integrand becomes a rational polynomial that can be regularised with dim reg. Therefore,

$$\begin{aligned} iD[\tilde{\phi}] &= \frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \ln\left[-p^2 + U(x)\right] \\ &= -\frac{1}{2}\int \mathrm{d}^4x \left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\ \int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \frac{1}{[-p^2+U(x)]^\alpha}\right)_{\alpha=0} \\ &= -\frac{i}{2}\int \mathrm{d}^4x\ \frac{\mu^{4-d}}{(4\pi)^{d/2}} U(x)^{d/2}\Gamma\left(-\frac{d}{2}\right), \end{aligned} \tag{A.26}$$

where $\Gamma(x)$ is the gamma function, $d = 4 - \epsilon$ and μ is the running scale. Performing the expansion in ϵ and using $\overline{MS}$ scheme we obtain

$$iD[\tilde{\phi}] = -\frac{i}{64\pi^2}\int \mathrm{d}^4x\ U^4\left[\ln\frac{\mu^2}{U^2} + \frac{3}{2}\right]. \tag{A.27}$$

As stressed, Eq. (A.27) is not precisely the potential, because we must sum over all dof that interact with ϕ. We can thus write the 1-loop effective action as

$$\Gamma^{(1)}[\tilde{\phi}] = -\int \mathrm{d}^4x\ V_{\mathrm{CW}}(\tilde{\phi}) = -\int \mathrm{d}^4x\ \sum_i \mp\frac{n_i}{64\pi^2}U_i(\tilde{\phi})^4\left[\ln\frac{\mu^2}{U_i(\tilde{\phi})^2} + \frac{3}{2}\right], \tag{A.28}$$

where the sum is over the particles that interact with ϕ, n_i is the corresponding dof of these particles and the sign is negative for bosons and positive for fermions. This effective action defines V_{CW}, known as the 1-loop Coleman-Weinberg potential.

A.2.2 Cut-off regularisation

We now evaluate Eq. (A.24) with cut-off regularisation. First, we must Wick rotate the integral by setting $p_E^0 = -ip^0$,

$$iD[\tilde{\phi}] = \frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \ln\Big[-p^2 + U(x)\Big] = \frac{i}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p_E}{(2\pi)^4}\ \ln\Big[p_E^2 + U(x)\Big]. \quad \text{(A.29)}$$

With a momentum cut-off set by $p_E^2 = \Lambda^2$, the integral over the momenta is given by

$$\begin{aligned}\int \frac{\mathrm{d}^4p_E}{(2\pi)^4}\ \ln\Big[p_E^2 + U(x)\Big] &= \frac{1}{16\pi^4}\int \mathrm{d}\Omega_4 \int_0^{\Lambda^2} \frac{\mathrm{d}p_E^2}{2}\ p_E^2 \ln\Big[p_E^2 + U(x)\Big] \quad \text{(A.30)}\\ &= \frac{1}{16\pi^2}\int_0^{\Lambda^2} \mathrm{d}p_E^2\ p_E^2 \ln\Big[p_E^2 + U(x)\Big]\\ &= \frac{1}{64\pi^2}\Big[\Big(2U^2 - \Lambda^2\Big)\Lambda^2 - 2\Big(U + \Lambda^2\Big)\ln\Big(U + \Lambda^2\Big) + 2U^2 \ln U\Big].\end{aligned}$$

Expanding the equation above for $\Lambda^2 \gg U$ and neglecting constant term we obtain

$$iD[\tilde{\phi}] = \frac{i}{2}\int \mathrm{d}^4x \left[\frac{U(\tilde{\phi})\Lambda^2}{32\pi^2} - \frac{U(\tilde{\phi})^2}{64\pi^2}\left(\ln\frac{\Lambda^2}{U(\tilde{\phi})} + \frac{1}{2}\right)\right]. \quad \text{(A.31)}$$

The Coleman-Weinberg potential is therefore given by

$$V_{\mathrm{CW}}(\tilde{\phi}) = \sum_i \pm\frac{n_i}{2}\left[\frac{U_i(\tilde{\phi})\Lambda^2}{32\pi^2} - \frac{U_i(\tilde{\phi})^2}{64\pi^2}\left(\ln\frac{\Lambda^2}{U_i(\tilde{\phi})} + \frac{1}{2}\right)\right], \quad \text{(A.32)}$$

where the sum is over the particles that interact with ϕ, n_i is the corresponding dof of these particles and the sign is positive for bosons and negative for fermions.

B Custodial Symmetry in the SM

In this appendix we review the concept of custodial symmetry in the SM.

In the SM custodial symmetry refers to an approximate, non-abelian symmetry in the electroweak (EW) sector. This symmetry is of great significance, because it has profound impact on observables of the EW sector of the SM. The custodial symmetry is already manifest at the low-energy theory, the Fermi theory, where the gauge bosons were integrated out. To see this we must first write the low-energy theory in terms of the parameters of the EW theory.

At low energies the Fermi-Lagrangian receives two distinct contributions from the EW sector: charged currents (CC) interactions and neutral currents (NC) interactions. In the EW theory these are given by the following interacting Lagrangian

$$\mathscr{L} = M_W^2 W_\mu^+ W_\mu^- + \frac{1}{2} M_Z^2 Z_\mu Z_\mu + \frac{g}{\sqrt{2}}\Big(W_\mu^+ J_\mu^- + W_\mu^- J_\mu^+\Big) + g_Z Z_\mu J_\mu^0, \tag{B.1}$$

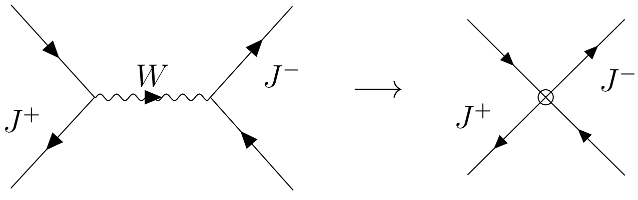
where g is the $SU(2)_L$ gauge coupling and $g_Z^2 = g^2 + g'^2$, with g' the hypercharge coupling. In the SM the currents are explicitly given by [25]

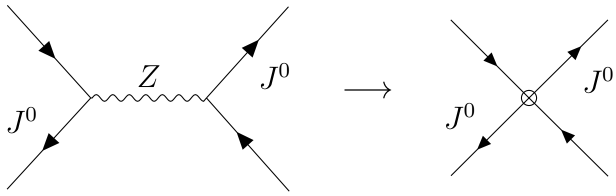
$$J_\mu^+ = \bar{e}_L \gamma^\mu \nu_L + \bar{d}_L \gamma^\mu V_{\text{CKM}}^\dagger u_L, \tag{B.2a}$$

$$J_\mu^- = \bar{\nu}_L \gamma^\mu e_L + \bar{u}_L \gamma^\mu V_{\text{CKM}} d_L, \tag{B.2b}$$

$$J_\mu^0 = \sum_f \bar{f} \gamma^\mu \Big(T^3 P_L - Q \sin^2\theta_w\Big) f, \tag{B.2c}$$

where the sum in the neutral current is over all fermions and a sum over all generations in the charged currents is understood. When the typical energy of the experiment is much smaller than M_W, the interaction between the currents is diagrammatically represented by





The current-current interactions on the right-hand side of the above diagrams are obtained by integrating the massive gauge bosons out and its coupling strength by matching with the UV-theory. This is done explicitly by considering the Equations of Motion (EoM) of the gauge bosons*

$$W_\mu^+ = -\frac{g}{M_W^2\sqrt{2}} J_\mu^+, \tag{B.3a}$$

$$W_\mu^- = -\frac{g}{M_W^2\sqrt{2}} J_\mu^-, \tag{B.3b}$$

$$Z_\mu = -\frac{g_Z}{M_Z^2} J_\mu^0. \tag{B.3c}$$

Inserting the EoM in Eq. (B.3) back into Lagrangian (B.2) we obtain

$$\begin{aligned} \mathscr{L}_{\text{Fermi}} &= M_W^2\left(-\frac{g}{M_W^2\sqrt{2}}J_\mu^+\right)\left(-\frac{g}{M_W^2\sqrt{2}}J_\mu^-\right) + \frac{1}{2}M_Z^2\left(-\frac{g_Z}{M_Z^2}J_\mu^0\right)^2 + \\ &\quad +\frac{g}{\sqrt{2}}\left[-\frac{g}{M_W^2\sqrt{2}}J_\mu^+J_\mu^- - \frac{g}{M_W^2\sqrt{2}}J_\mu^+J_\mu^-\right] - \frac{g_Z^2}{M_Z^2}J_\mu^0J_\mu^0 \\ &= -\frac{g^2}{2M_W^2}J_\mu^+J_\mu^- - \frac{g_Z^2}{2M_Z^2}J_\mu^0J_\mu^0 \\ &\equiv -\frac{G_F^{\text{CC}}}{\sqrt{2}}J_\mu^+J_\mu^- - \frac{G_F^{\text{NC}}}{\sqrt{2}}J_\mu^0J_\mu^0 \\ &= -\frac{G_F^{\text{CC}}}{\sqrt{2}}\left[J_\mu^+J_\mu^- + \frac{G_F^{\text{NC}}}{G_F^{\text{CC}}}\left(J_\mu^0\right)^2\right], \end{aligned} \tag{B.4}$$

where G_F^{CC} and G_F^{NC} denote the couplings of CC and NC, respectively, which are defined in term of the gauge couplings and masses of the mediators as

$$G_F^{\text{CC}} = \frac{g^2}{\sqrt{2}M_W^2}, \quad G_F^{\text{NC}} = \frac{g_Z^2}{\sqrt{2}M_Z^2}. \tag{B.5}$$

The ρ parameter is the coefficient of the NC interaction, namely

$$\rho \equiv \frac{G_F^{\text{NC}}}{G_F^{\text{CC}}}. \tag{B.6}$$

*We are considering the currents to be conserved, hence $\partial_\mu W_\mu^\pm = 0$ and $\partial_\mu Z_\mu = 0$.

In the SM at tree-level this parameter is exactly 1, since

$$\rho_{\text{tree}} = \frac{g_Z^2}{g^2}\frac{M_W^2}{M_Z^2} = \frac{M_W^2}{M_Z^2 \cos_w^\theta} = 1. \tag{B.7}$$

Under radiative corrections the ρ parameter receives only very small corrections due to the fact that it is protected by an $SO(3)$ symmetry. This symmetry is made manifest in Lagrangian (B.4) by rewriting the charged currents as

$$J_\mu^\pm = J_\mu^1 \pm i J_\mu^2. \tag{B.8}$$

The Lagrangian in this case is

$$\mathscr{L}_{\text{Fermi}} = -\frac{G_F^{\text{CC}}}{\sqrt{2}}\left[\left(J_\mu^1\right)^2 + \left(J_\mu^2\right)^2 + \left(J_\mu^0\right)^2 + \Delta\rho\left(J_\mu^0\right)^2\right], \tag{B.9}$$

where we have defined

$$\Delta\rho = \frac{G_F^{\text{NC}}}{G_F^{\text{CC}}} - 1, \tag{B.10}$$

which is the deviation of the ρ parameter with respect to the tree-level value. If $\Delta\rho = 0$, the Lagrangian (B.9) is invariant under the so called custodial $SO(3)$ symmetry, under which the currents $\left(J_\mu^1, J_\mu^2, J_\mu^0\right)$ transform as a triplet. It is obvious then that $\Delta\rho$ is technically natural, hence it is stable under quantum corrections.

From the above discussion it is clear that the origin of custodial symmetry rests in the Higgs sector, because the ρ parameter is directly connected with the masses of the gauge bosons. The Higgs as a doublet of $SU(2)_L$ can be written in terms of 4 real components:

$$H = \begin{pmatrix} H_2 + iH_1 \\ H_4 - iH_3 \end{pmatrix}, \tag{B.11}$$

with H_1, H_2, H_3 and H_4 real. The potential written in terms of those is

$$\begin{aligned} V(H) &= -m^2|H|^2 + \lambda|H|^4 \\ &= -m^2\left(H_1^2 + H_2^2 + H_3^2 + H_4^2\right) + \lambda\left(H_1^2 + H_2^2 + H_3^2 + H_4^2\right)^2. \end{aligned} \tag{B.12}$$

Therefore it is clear that $V(H)$ has a $SO(4)$ global symmetry, under which (H_1, H_2, H_3, H_4) is a 4-plet. After electroweak symmetry breaking (EWSB) the $SO(4)$ is broken down to

$SO(3)$, which is exactly the same we have encountered in the Fermi Lagrangian. This follows from the fact that the devoured directions (H_1, H_2, H_3) form a triplet of $SO(3)$, meaning that the gauge bosons, and as a consequence the respective currents, will transform under $SO(3)$.

The $SO(4)$ may be written as

$$SO(4) \simeq SU(2)_L \times SU(2)_R,$$

with the $SU(2)_L$ being the same that of the SM. The fundamental of $SO(4)$ is then equivalent to the $(2,2)$ representation of the chiral group (see sections 3.2.2 and 3.3.1) and every doublet H of $SU(2)_L$ can be brought into the matrix form

$$\Sigma = (\tilde{H}, H), \tag{B.13}$$

where $\tilde{H} = i\sigma^2 H^*$ and

$$\Sigma \to U_L \Sigma U_R^\dagger \tag{B.14}$$

under $SU(2)_L \times SU(2)_R$, with $U_L \in SU(2)_L$ and $U_R \in SU(2)_R$. After EWSB Σ acquires a vev given by

$$\Sigma \to \langle\Sigma\rangle = \left(\left\langle\tilde{H}\right\rangle, \langle H\rangle\right) = \frac{v}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{B.15}$$

which means that the chiral group breaks down to the diagonal group. Note that the Lagrangian of the Higgs may be rewritten in terms of Σ, leaving the custodial symmetry manifest. This Lagrangian is given by

$$\begin{aligned} \mathscr{L}_{\text{Higgs}} &= |\partial_\mu H|^2 + m^2|H|^2 - \lambda|H|^4 \\ &= \frac{1}{2}\operatorname{Tr}\left[\partial_\mu \Sigma^\dagger \partial_\mu \Sigma\right] + m^2 \det\Sigma - \lambda(\det\Sigma)^2, \end{aligned} \tag{B.16}$$

where we have not yet coupled the gauge bosons. It is trivial to see that the Lagrangian (B.16) is invariant under $SU(2)_L \times SU(2)_R$.

In the SM we have two sources of custodial violation: the gauge boson sector and the Yukawa sector. Custodial symmetry is broken in the gauge sector because only the third generator of $SU(2)_R$ is gauged, as it is the one that generates hypercharge transformations (see subsection 3.3.1). The covariant derivative of Σ is in this case given by

$$D_\mu \Sigma = \partial_\mu \Sigma - i\frac{g}{2} W_\mu^a \sigma^a \Sigma + i\frac{g'}{2} \Sigma \sigma^3 B_\mu. \tag{B.17}$$

Under a global transformation of $SU(2)_L$ the gauge bosons transform as

$$W_\mu^a \frac{\sigma^a}{2} \to U_L W_\mu^a \frac{\sigma^a}{2} U_L^\dagger, \tag{B.18}$$

which leaves the Lagrangian invariant. On the other hand, under an arbitrary global transformation of $SU(2)_R$

$$\Sigma \sigma^3 \to U_L \Sigma U_R^\dagger \sigma^3 \neq U_L \Sigma \sigma^3 U_R^\dagger. \tag{B.19}$$

Hence, the Lagrangian is not invariant and the custodial symmetry is therefore broken by the hypercharge coupling. Radiative corrections to $\Delta\rho$ from the gauge sector are proportional to the hypercharge coupling g', which is very small.

In the Yukawa sector the situation is different. Consider the Yukawa terms of the quark sector:

$$\mathscr{L}_{\text{Yuk}} = -\bar{q}_L \tilde{H} Y_u u_R - \bar{q}_L H Y_d d_R + h.c., \tag{B.20}$$

with Y_u and Y_d are matrices in flavour space. The above Lagrangian can be rewritten in terms of Σ; we obtain

$$\mathscr{L}_{\text{Yuk}} = -\bar{q}_L \Sigma \begin{pmatrix} Y_u & 0 \\ 0 & Y_d \end{pmatrix} q_R + h.c., \tag{B.21}$$

where we have defined

$$q_R \equiv \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \tag{B.22}$$

which is a doublet of $SU(2)_R$. If $Y_u = Y_d$, then the Yukawa sector is manifestly invariant under the chiral symmetry group. This also means that custodial symmetry is exact in the Yukawa sector only if each generation is mass degenerate, since the masses are proportional to the corresponding Yukawas. We know, however, that this is not true, as the up-type and down-type quarks of all generations have distinct couplings. Quantum corrections from the fermion sector to the ρ parameter will thus depend explicitly on the mass splitting of the fermions from each generation, and will vanish if the generations are mass degenerate. The most important contributions come from the third generation, in particular from the top quark, for it has the largest coupling. Nevertheless, such corrections are still very small [1].

C Cosmology

In this appendix we review some of the most relevant concepts of cosmology, in particular the standard Big-Bang model and Inflation.

C.1 The Standard Model - ΛCDM model

C.1.1 Fundamental Principles

Cosmology is understood as the science that describes the evolution of the universe and explains why it is what we observe today. The Standard Model of modern cosmology, also known as ΛCDM model, has three fundamental hypothesis that allows us to make qualitative and quantitative predictions [77, 78, 85]:

1) The universe can be described by Einstein's General Relativity.

Historically, General Relativity (GR) was not conceived as a tool to explain the evolution of universe, but as a generalisation of Special Relativity (SR), which in particular gives a description of gravitational phenomena consistent with relativity principles. The fact that it is possible to use GR in cosmology can be seen from Einstein's equations [102, 103],

$$G_{\mu\nu}(g_{\mu\nu}) \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{C.1}$$

where $R_{\mu\nu}$ and R are the Ricci tensor and scalar, respectively, that are functions of the metric tensor $g_{\mu\nu}(x)$, G is Newton's constant and $T_{\mu\nu}$ is the energy-momentum tensor. The left-hand side of Eq. (C.1) represents the geometry of space-time, since it depends only on quantities that depends of the metric tensor, while the right-hand side represents the matter content of such space-time, i.e. what it is inside of it. With such interpretation it is intuitive that Eq. (C.1) can describe the evolution of the universe, which is just a big space-time configuration, and its particle content.

To solve the Einstein's equations means to compute the metric tensor $g_{\mu\nu}(x)$, in other words to determine the geometric structure of the space-time as a function of its matter content. In the context of cosmology we give particular attention to the time dependency of $g_{\mu\nu}$, as we want to understand how the universe evolves with time. Unfortunately, unless some simplifications are imposed, it is in general impossible to solve Eq. (C.1) due to its high non-linearity.

One remark is in order. The Einstein's equations in Eq. (C.1) are the Equations of Motion (EoM) of the action [85, 102],

$$S = S_{\text{matter}} + \frac{1}{8\pi G}\int \mathrm{d}^4x\ \sqrt{-g}R, \tag{C.2}$$

where $g = \det g_{\mu\nu}$, with respect to $g_{\mu\nu}(x)$. The second term in the equation above, also known as the Einstein-Hilbert action, describes the left-hand side of the Eq. (C.2) and is the result of a summation of an infinite series of actions of a *classical* massless spin 2 field. For this reason Eq. (C.1) should be understood as classical EoM.

2) The universe is approximately homogeneous and isotropic.

At small distance scales, i.e. sub-galactic and galactic scales, the universe is neither homogeneous nor isotropic, since there are many distinct structures (for example, stars and planets) distributed in a non-uniform way in the sky. We should note that all observations today are made by detecting light signals, hence what we are observing is how the universe *was* at the time those light signals were emitted. In this way small scale observations represent the universe at later times of its evolution. However, as we probe larger and larger distances, the universe becomes more homogeneous and isotropic, which implies that the universe at very earlier times was approximately homogeneous and isotropic [77, 85, 92]. Assuming space isotropy and space homogeneity it is possible to determine the form of the metric tensor,

$$g_{\mu\nu}(t) = \begin{pmatrix} -1 & & & \\ & a^2(t) & & \\ & & a^2(t) & \\ & & & a^2(t) \end{pmatrix} \tag{C.3}$$

which is known as the Friedmann-Lemaître-Robertson-Walker (FLRW) metric*. In Eq. (C.3) $a(t)$ is the scale-factor and must be determined through the Einstein's equations. The scale-factor represents the growth of the universe, therefore it quantifies how the measure of distance changes as time passes [77]. We adopt the convention that the scale-factor today is 1.

*Here we use the standard convention of cosmology, that the Minkovski metric is given by $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Moreover, we assume the curvature is zero.

Another interesting and useful way to represent the scale-factor is using the concept of red-shift [77, 78, 85]. If the universe is expanding, the wave-length of a light signal grows, so the ratio of the wave-lengths of a photon at a later time t_1 with respect to an earlier time t_0 is given by

$$\frac{\lambda(t_1)}{\lambda(t_0)} > 1. \tag{C.4}$$

Since the wave-length is a length scale, the ratio above is equal to the ratio of the respective scale-factors. We can thus define

$$\frac{\lambda(t_1)}{\lambda(t_0)} = \frac{a(t_1)}{a(t_0)} \equiv 1 + z, \tag{C.5}$$

where z is the *red-shift*. Hence, smaller red-shifts represent later periods of the cosmological evolution, while larger red-shifts represent earlier ones.

3) Matter content of the universe

In order to determine the scale-factor $a(t)$ from Eq. (C.1) we need to specify the energy-momentum tensor $T_{\mu\nu}$. In the standard model of cosmology there are three distinct contributions to the content of the energy momentum tensor [77, 92]. The first is *radiation*, in other words relativistic degrees of freedom (dof). The particles that contribute mostly to radiation throughout the evolution of the universe are the photon and the neutrinos, but, due to their mass, neutrinos eventually become non-relativistic and cease to be radiation. The second kind of contribution is *matter*, i.e. non-relativistic dof. It can be further split into two: Cold Dark Matter (CDM) and baryonic matter. "Baryon" is how cosmologists name stable, visible matter, which is composed of protons, neutrons and electrons. CDM is another kind of matter that does not interact with the photon, but whose existence is inferred through measurements of the Cosmic Microwave Background (CMB), of bullet clusters and of the rotation velocity of spiral galaxies, for example. The third contribution to the energy-momentum tensor is what we call *Dark Energy* (DE). The theoretical and phenomenological nature of DE is to this day unknown and its existence is postulated to fit experimental data that probe the total energy density of the universe. In particular, 70% of the energy of the universe is approximately composed of DE, while 27% is given by CDM and only 3% is baryonic matter [3].

With the hypothesis of isotropy and homogeneity of space-time and supposing that there are no interactions between the particles, we can simplify the explicit form of the energy-momentum tensor $T_{\mu\nu}$. As a consequence of these symmetries, the energy-momentum tensor has the form of a perfect-fluid [102, 77, 104],

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix}, \tag{C.6}$$

where ρ is the total energy density and P is the pressure.

⋆⋆⋆

With the three hypothesis above we can characterise the ΛCDM model of cosmology by identifying the evolution equations and the free parameters of the model. Without perturbations and interactions, the evolution equations are simply the 00- and ii-components of Eq. (C.1) [77, 85, 92],

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho, \tag{C.7a}$$

$$\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi G P, \tag{C.7b}$$

where the dot denotes derivative with respect to the cosmic time t. Eq. (C.7), also known as the Friedmann equations, defines the Hubble parameter,

$$H_b(t) \equiv \frac{\dot{a}(t)}{a(t)}. \tag{C.8}$$

Another way to write Eq. (C.7a) is with the definition of the critical density [92] ρ_{crit},

$$\rho_{\text{crit}} \equiv \frac{3H_{b0}^2}{8\pi G}, \tag{C.9}$$

with H_{b0} the Hubble parameter today, whence

$$\frac{H_b^2}{H_{b0}^2} = \frac{\rho}{\rho_{\text{crit}}}. \tag{C.10}$$

In order to solve the equation above completely we need to know how the energy density depends on time. From simple dimensional analysis we can see that* [92]

$$\rho_{\text{radiation}} \propto \frac{1}{a^4(t)}, \quad \rho_{\text{matter}} \propto \frac{1}{a^3(t)}, \tag{C.12}$$

therefore we may define,

$$\frac{\rho_{\text{radiation}}}{\rho_{\text{crit}}} \equiv \frac{\Omega_r}{a^4(t)}, \quad \frac{\rho_{\text{matter}}}{\rho_{\text{crit}}} \equiv \frac{\Omega_{\text{cdm}} + \Omega_{\text{baryons}}}{a^3(t)}, \tag{C.13}$$

where the Ω's represent the ratio of the respective energy densities to the critical density today. The case of DE is more complicated and model-dependent, but observations lead us to considering [3, 85]

$$\frac{\rho_{\text{DE}}}{\rho_{\text{crit}}} \sim \Omega_\Lambda. \tag{C.14}$$

With such dynamics, DE can be understood as a cosmological constant, i.e. an additional constant term -2Λ in the Einstein-Hilbert action in Eq. (C.2). Neglecting curvature effects, the Friedmann equation (C.7a) can be written as

$$\frac{H_b^2}{H_{b0}^2} = \frac{\rho_{\text{radiation}} + \rho_{\text{matter}} + \rho_{\text{DE}}}{\rho_{\text{crit}}} = \frac{\Omega_r}{a^4} + \frac{\Omega_{\text{cdm}} + \Omega_{\text{baryons}}}{a^3} + \Omega_\Lambda. \tag{C.15}$$

Note that by considering the equation above today we have the constrain

$$\Omega_r + \Omega_{\text{cdm}} + \Omega_{\text{baryons}} + \Omega_\Lambda = 1. \tag{C.16}$$

In summary, together with the constrain of Eq. (C.16), we have a total of 4 independent input parameters.

If we took curvature into account, in other words, if we had considered that the universe is not necessarily flat, there would be an additional contribution to the energy density due to the energy contained in the curvature. This contribution gives an additional factor of $\frac{\Omega_K}{a^2}$ to the right-hand side of Eq. (C.15) and is an extra input parameter of the model [85, 92]. Another assumption we have made is regarding the neutrinos. The energy

*The energy of a photon in equilibrium in a thermal bath with temperature T has energy $E = k_B T$, where k_B is Boltzmann's constant. From Eq. (C.12) we can conclude that temperature depends on the scale factor as [77]

$$T(t) \propto \frac{1}{a(t)}. \tag{C.11}$$

density of neutrinos is not taken as a free parameter, because in the ΛCDM it is usually fixed by setting the sum of their masses to a fixed values [3, 77, 85].

From Eq. (C.15) it is obvious that the universe has distinct phases of evolution, each one characterised by a different energy density. For example, at very primordial times the scale-factor was very small, hence the radiation contribution, due to the factor of a^{-4} in the denominator, was dominant. This era is named *radiation domination era.* After the radiation domination, matter dominated and at later times DE started dominating. Each of these eras have distinct properties and phenomena which are necessary for a precise description of the universe, for instance Big-Bang Nucleo-synthesis (BBN), the photon decoupling and the formation of large structures [77, 78, 85, 92, 93]. The discussion of these topics require a more careful analysis and a thorough review of perturbation theory, so we will not develop them in details.

C.1.2 Perturbations

Since the universe is not completely homogeneous and isotropic, we need to consider perturbations to Eq. (C.15) and the interactions between the particles. Such perturbations mark a departure from the classical regime given by the Einstein's equations and introduce fluctuations whose origin is not classical.

There are two types of perturbations: perturbations of the metric tensor and perturbations of the distribution functions of the particles. The first represent fluctuations of the metric around the FLRW metric and can be classified into scalar, vector and tensor perturbations. However, only the scalar perturbations, that couples to matter and radiation, and tensor ones, that generate gravitational waves, are relevant to the description of the cosmic evolution [85, 93]. Perturbations of the distribution functions on the other hand represent non-equilibrium aspects of the thermal bath and affect the components of the energy-momentum tensor [77, 93]. Both types of perturbations are connected by the perturbed Einstein's equations,

$$G_{\mu\nu}(g_{\mu\nu} + \delta g_{\mu\nu}) = 8\pi G(T_{\mu\nu} + \delta T_{\mu\nu}). \tag{C.17}$$

In addition to the aforementioned perturbations, we must also take interactions into account. The interactions between the particles, in particular between the ones of the Standard Model of particle physics, have non-trivial impacts on the abundance of each particle species and on the formation of structures. More precisely, the number density n of a given particle species, that in equilibrium scales as a^{-3}, is given by the Boltzmann equation [77]

$$a^{-3}\frac{\mathrm{d}(a^3 n)}{\mathrm{d}t} = C, \tag{C.18}$$

where C is the collision integral, which is a functional of all the distribution functions of the particles that interact with the corresponding particle. Moreover, C depends on the squared amplitude of such interactions, that are given by the techniques and theories of particle physics.

We can clearly see that there are many parameters and functions that need to be introduced to describe perturbations and interactions in an appropriate manner, hence we could expect that the set of free parameters of the model quickly grows. Fortunately, these new parameters are not all independent, since they are all coupled to each other by the differential equations from Eqs. (C.17) and (C.18). Even more, the initial conditions of all of them depend on only one perturbation parameter from the metric tensor. However, there is a priory no reason for such perturbations to exist, since we have no mechanism to generate them. It is thus necessary to complement our model with such mechanism [93, 105].

C.2 Inflation

C.2.1 Problems with the ΛCDM

In addition to the lack of a mechanism to generate the initial conditions of the perturbations, the ΛCDM model described in section B.1 has many other conceptual problems. These issues are used to motivate the modern theory of inflation.

One of the problems is called *horizon problem* [85]. "Horizon" is used to name the largest 4-dimensional surface of the observable universe which is causally connected by a light-like signal*. At the beginning of the evolution, the scale-factor was very small, therefore the horizon was small and essentially everything was in causal contact. As the

*This is quantified by the conformal time [77],

$$\eta = \int_0^t \frac{\mathrm{d}t'}{a(t')},$$

that measures the distance a light-like signal could have travelled from the initial time $t = 0$ to t. If the distance between two events is greater than η, they were never in causal contact.

universe expands, the horizon grows and some patches of the universe cease to be in causal contact. If we track different regions of the sky at a given red-shift back to primordial times, we will determine the typical angular separation for which the events are causally connected at this red-shift. This defines the mean correlation angle at some red-shift z, so we should expect to measure such angle in experiments. One particularly interesting red-shift is $z \sim 1000$, when photons decoupled from matter, and has a correlation angle of order 1 degrees [77]. The CMB, that is the measure of the temperature of the entire sky at decoupling, shows that all regions in the sky seems to be correlated, as every corner of the sky has approximately the same temperature, with fluctuations of the order of 10^{-5} parts in 1 [3, 77]. In other words, the temperature of the sky is almost isotropic at that time. In an analogous way we could expect that the universe would not be homogeneous, because only small volumes of it were causally connected in the past. Measurements of the large structures of the universe, however, show that the universe was still very homogeneous at $z \lesssim 1$, contradicting our expectation [85].

Another very important problem is the *flatness problem* [77, 85, 93]. The universe today is very flat, such that in the standard ΛCDM curvature is neglected. More precisely, the energy density of curvature today Ω_{K0} is smaller than 0.01. We can compute how much curvature energy density we must have had at primordial times in order to observe Ω_{K0} using the Friedmann equation in a radiation dominated era. At red-shifts of the order of the Planck time Ω_K is given by

$$|\Omega_K| \lesssim 10^{-60}. \tag{C.19}$$

The question is how can we guarantee that our universe was extremely flat at the beginning.

To solve the above-mentioned problems we conjecture that there was another era before radiation domination, named inflation [93, 105], in which the cosmic expansion was very quick and accelerated. In this scenario the horizon grows immensely and could, in principle, causally connect a larger region of the universe. In addition, due to the rapid expansion, any curvature fluctuations are washed away. In what follows we will develop a concrete model of inflation.

C.2.2 Single field inflationary model

One of the hypothesis of inflation is that $\ddot{a} > 0$ during inflation. To see how to satisfy such constrain, we can combine the Friedmann equations in Eq. (C.7) such that we eliminate the factor H_b^2. Doing this we obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P). \tag{C.20}$$

In order to have an accelerated universe,

$$\rho + 3P < 0 \Rightarrow P < 0, \tag{C.21}$$

where the we have used that $\rho > 0$ always. The condition in Eq. (C.21) is not satisfied by neither matter ($P = 0$) nor radiation ($P = \rho/3$), therefore we must introduce another kind of d.o.f. [77, 105]. The simplest model that satisfies the equation above is the one of a single real scalar field, which is given by the action [93, 105, 104]

$$S[\phi] = \int \mathrm{d}^4x\sqrt{-g}\,\mathscr{L} = \int \mathrm{d}^4x\sqrt{-g}\Big[\frac{1}{2}g_{\mu\nu}\,\partial^\mu\,\phi\,\partial^\nu\,\phi - V(\phi)\Big]. \tag{C.22}$$

Since ϕ is the particle responsible for driving inflation, we name it *inflaton*. The energy density and the pressure are components of the energy-momentum tensor,

$$\begin{aligned} T_{\mu\nu} &= \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\,\mathscr{L} \\ &= 2\frac{\delta\,\mathscr{L}}{\delta g^{\mu\nu}} - g_{\mu\nu}\,\mathscr{L} \\ &= \partial_\mu\,\phi\,\partial_\nu\,\phi - g_{\mu\nu}\Big[\frac{1}{2}\,\partial^\alpha\,\phi\,\partial_\alpha\,\phi - V(\phi)\Big]. \end{aligned} \tag{C.23}$$

Whence, using the non-perturbed FLRW metric of Eq. (C.3), we obtain

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{C.24}$$

Condition (C.21) thus implies that

$$V(\phi) \gg \frac{1}{2}\dot{\phi}^2. \tag{C.25}$$

Before continuing the discussion, we should remark that ϕ is a field in the quantum mechanical sense, i.e. it has a classical, dominant behaviour and quantum fluctuations around the classical configuration. Therefore it is natural to separate the field in two parts [85],

$$\phi(x) = \phi(t) + \delta\phi(x), \tag{C.26}$$

where the classical mode $\phi(t)$ depends only on time due to isotropy and homogeneity. The origin of cosmological perturbations are the quantum fluctuation $\delta\phi$, whose relation with the other perturbation parameters can be given at the end of inflation [77].

The condition in Eq. (C.25) is known as *slow-roll* condition, because it states that the field will remains approximately constant during inflation (it will slowly change) [77, 85, 93]. Moreover, from the EoM of ϕ, that are given by the Klein-Gordon equation

$$\ddot{\phi} + 3H_b\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0, \tag{C.27}$$

we are lead to the additional condition

$$\ddot{\phi} \ll 3H_b\dot{\phi}. \tag{C.28}$$

The equation above is also known as a slow-roll condition, as it constrains the acceleration of ϕ.

What determines the dynamics of the inflaton is solely its potential, hence it would be convenient to write the slow-roll conditions (C.25) and (C.28) in terms of V and its derivatives. Up to the second derivative of the potential, the slow-roll conditions can be parametrised by two parameters, namely [77, 85, 93]

$$\epsilon = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{H_b}, \quad \delta = \frac{\ddot{\phi}}{H_b\dot{\phi}}. \tag{C.29}$$

Using the Friedmann equations (C.7) and the slow-roll conditions (C.25) and (C.28) we may rewrite the parameters above only in terms of the potential,

$$\epsilon = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{H_b} \simeq \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2, \tag{C.30}$$

$$\delta = \frac{\ddot{\phi}}{H_b \dot{\phi}} \simeq \epsilon - \frac{1}{8\pi G} \frac{V''}{V} \simeq -\epsilon + \frac{\dot{\epsilon}}{2H_b \epsilon}. \tag{C.31}$$

Note that $\epsilon, |\delta| < 1$.

Every deviation from the slow-roll regime can be measured by these two parameters. In particular, they are used to define the spectral indices of scalar and tensor perturbation modes at the end of inflation [3, 77, 85]. Without going into the details, the spectral indices are the powers with which such perturbations grow in momentum space. However, these indices describe only the momentum dependency, so we need two additional parameters to measure their amplitude. In the ΛCDM the tensor perturbations are neglected and only the scalar perturbations coming from the quantum fluctuations of ϕ are taken into account [3, 93]. Hence, we add only two input parameters to our model. In short, neglecting curvature and tensor perturbations, and fixing the sum of the neutrino masses, the ΛCDM model has total of 6 free parameters: H_{b0}, Ω_r, Ω_{cdm}, Ω_{baryon}, the scalar spectral index and the scalar perturbation amplitude. Nevertheless, due to many technical and practical reasons, the ΛCDM model is not written in terms of these specific parameters, but this does not affect our discussion [3, 93].

One last remark is in order. As every other epoch of the universe, inflation lasts for some time. The duration of the inflationary period is measured by the number of *e-folds* N [77],

$$N = \ln\left(\frac{a_f}{a_i}\right), \tag{C.32}$$

with a_f and a_i the scale-factors at the end and beginning of inflation, respectively. The equation above can be rewritten in terms of the slow-roll parameters,

$$N = -\frac{1}{\sqrt{4\pi G}} \int_{\phi_i}^{\phi_f} \frac{\mathrm{d}\phi}{\sqrt{\epsilon}}, \tag{C.33}$$

where ϕ_f is the field configuration of ϕ at the end of inflation and ϕ_i is the initial one. Most of the experimental data can be explained with $N \simeq 40 - 60$ [3, 77, 85].

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