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Instituto de Física

Estudos de Laços de Wilson em Teorias 3d Chern-Simons-Matter

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Studies of Wilson Loops on 3d Chern-Simons-Matter Theories

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Abstract

In this work we present some new developments in the study of Wilson loops in 3d Chern-Simons-matter theories.

We begin with a brief introduction of the $\mathcal{N} = 4$ and $\mathcal{N} = 6$ 3d theories and the definition of their BPS Wilson operators. Firstly, a new formulation of the 1/2-BPS Wilson loop is provided, which addresses some of the shortcomings of the original formulation, such as the explicit dependence on the contour parameter and the definition of the operator in terms of the trace of a superconnection, rather than a super-trace. This new formulation also elucidates the moduli space of this operators which is the conifold.

Secondly, we consider wavy deformations of the 1/2-BPS Wilson loop of ABJ(M) and relate them to the Bremsstrahlung function of ABJ(M). We compute these new loops with a deformed contour to leading order in perturbation theory.

Finally, the 1/2-BPS Wilson line of $\mathcal{N} = 4$ is studied as a defect, where we define the *displacement multiplet* in terms of chiral superfields of $\mathfrak{su}(1, 1|2)$ and study 2pt and 4pt correlation functions via a bootstrap approach. The superconformal blocks are derived with the super-Casimir approach, and the CFT data is extracted at strong coupling. We comment on the holographic description of defects in terms of type IIA strings in 10d and M2-branes in the 11d description.

Key-Words: “Wilson Loops”, “ABJ(M)”, “Displacement-Multiplet”, “Superconformal Bootstrap”, “Bremsstrahlung Function”.

Resumo

Nesse trabalho nós apresentamos alguns novos deseonvolvimentos no estudo de laços de Wilson em teorias de Chern-Simons-matter em 3d.

Nós começamos com uma breve introdução das teorias $\mathcal{N} = 4$ e $\mathcal{N} = 6$ em 3d e a definição de seus operadores de Wilson BPS. Primeiramente, uma nova formulação do laço 1/2-BPS é apresentada, tal formulação aborda e resolve alguns dos problemas da formulação original, como a dependência explícita no parâmetro do contorno e a definição do operador em termos do traço de uma superconexão, ao invés de um super-traço. Essa nova formulação também elucida o espaço moduli desses operadores, que é o conifold.

Em seguida, nós consideramos deformações do tipo *wavy* dos operadores de Wilson 1/2-BPS de ABJ(M) e os relacionamos à função de Bremsstrahlung de ABJ(M). Nós computamos esses novos operadores com um contorno deformado em primeira ordem em teoria de perturbação.

Por fim, a linha de Wilson 1/2-BPS de $\mathcal{N} = 4$ é estudada como um defeito, onde definimos o *multipleto de deslocamento* em termos de supercampos quirais de $\mathfrak{su}(1,1|2)$ e estudamos funções de correlação de 2pt e 4pt via o programa de bootstrap. Os blocos superconformes são derivados por meio da técnica de super-Casimir, e a CFT data é extraída a strong coupling. Nós discutimos a descrição holográfica de defeitos em termos de teoria de cordas tipo IIA em 10d e M2-branes na descrição 11d.

Palavras-Chave: “Laços de Wilson”, “ABJ(M)”, “Displacement-Multiplet”, “Bootstrap Superconforme”, “Função Bermsstrahlung”.

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ao vô Sérgio

Contents

Chapter 1

Introduction

Wilson loops are ubiquitous in the study of gauge theories and were first introduced in the study of the quark confinement by Kenneth G. Wilson [1]. They consist of gauge invariant operators supported along a contour, which makes them useful for probing non-local features of field theories, such as the quark/anti-quark potential and the radiation emitted by an accelerated charge. Apart from their importance in calculating physical observables, they are also important in mathematics, being related to the study of knot theory [2].

For any gauge theory, a Wilson operator can be defined in terms of the holonomy of the gauge connection around a contour $\mathcal{C}$,

$$\mathcal{W} = \operatorname{Tr} \mathcal{P} \exp\left(i \oint_{\mathcal{C}} A\right), \quad \text{with} \quad A = A_\mu dx^\mu. \tag{1.1}$$

The gauge fields are in general Lie algebra valued, so that one has to take the trace, and $\mathcal{P}$ stands for a path-ordering.

In the context of supersymmetric fields theories, Wilson operators can be made supersymmetric (BPS), which may render them exactly calculable by a technique called *supersymmetric localization* [3]. As exact results are rare in general quantum systems, the study of BPS loops becomes of great importance for probing non-perturbative aspects of quantum field theories.

The class of supersymmetric operators $\mathcal{O}_{\text{BPS}}$ is defined by the supersymmetry constraint

$$\delta_{\text{SUSY}} \mathcal{O}_{\text{BPS}} = 0, \tag{1.2}$$

for a variation parameterized by some supercharge of the theory. Such operators are often classified in respect to the amount of preserved supersymmetry or the "BPS-ness" of the operator, which is specified by the number of independent supercharges for which (1.2) is satisfied. This way, the so-called $1/N$-BPS operators are the ones that remain invariant under the action of $1/N$ of the total supercharges of the underlying gauge theory.

In this thesis we are interested in the BPS constraints of Wilson operators in the three-dimensional superconformal field theories known as Chern-Simons-matter theories, as well as their solutions. To define BPS Wilson loops in these theories, one needs to generalize the gauge connection of (1.1) to a super-matrix [4], which, apart from containing the gauge fields, accommodates the matter fields which are necessary to satisfy the BPS constraint (1.2).

To illustrate this point, let us consider the example of ABJM theory, which is an $\mathcal{N}=6$ Chern-Simons matter theory. In this theory, a supersymmetry variation[1] transforms the fields of the theory as

$$\begin{aligned}
\delta A_\mu^{(1)} &\sim \left(C_I\psi_{J\beta} + \frac{1}{2}\epsilon_{IJKL}\bar{\psi}_\beta^K\bar{C}^L\right), \\
\delta A_\mu^{(2)} &\sim \left(\psi_{J\beta}C_I + \frac{1}{2}\epsilon_{IJKL}\bar{C}^L\bar{\psi}_\beta^K\right), \\
\delta C_I &\sim \epsilon_{IJKL}\bar{\psi}_\alpha^J, \\
\delta \bar{C}^I &\sim \psi_{J\alpha},
\end{aligned} \tag{1.3}$$

where $A_\mu^{(i)}$ are gauge fields, C^I are complex scalars and ψ_I are complex fermions.[2] With these general transformations, one can engineer a connection which satisfies (1.2) by composing combinations of both gauge and matter fields in a way that their variation amount to a total derivative along the loop. As in these theories the matter fields and gauge fields transform differently in respect to gauge transformations, one needs to introduce the connection as a super-matrix in order to accommodate these different representations.

In [5], the first example of BPS Wilson loop was defined for ABJM theory, this solution is 1/6-BPS and is supported along a straight line or a circle. Such loops are known as *bosonic loops*, due to the bosonic nature of their field content, since in order to preserve 1/6 of the supersymmetry, the super-connection couples to scalar bilinears,[3]

$$\mathcal{L}_{1/6\text{-BPS}} \sim \begin{pmatrix} A^{(1)} + C\bar{C} & 0 \\ 0 & A^{(2)} + \bar{C}C \end{pmatrix}. \tag{1.4}$$

These are the simplest loops in ABJM theory, and they are constructed in a way that the connection is annihilated by the supersymmetry variation.

The 1/2-BPS Wilson loop was first constructed in [6], and it is supported along the same contours. In this case, the super-connection also couples to the fermions of the theory, in addition to the scalar bilinears, and for this reason they are called *fermionic loops*,

$$\mathcal{L}_{1/2\text{-BPS}} \sim \begin{pmatrix} A^{(1)} + C\bar{C} & \bar{\psi} \\ \psi & A^{(2)} + \bar{C}C \end{pmatrix}. \tag{1.5}$$

In this case, we notice that the variation of the super-connection does not vanish, since we cannot ask for supersymmetry variations of the fermions to vanish identically. Rather, it amounts to a gauge transformation, such that the Wilson loop is invariant once the trace is taken.

The original formulation of the fermionic loop has two shortcomings. The first is the necessity to introduce an ad-hoc *twist matrix* in the definition of the loop connection in order to restore gauge invariance of the operator. The second is the

[1] For the complete transformations, see Appendix A.
[2] For more details about the field content of ABJM, see chapter 2.
[3] Bar denotes complex conjugation.

fact that the fermions couple to the operator via grassmann even couplings that are explicit functions of the contour parameter, in other words, the formulation is not manifestly reparametrization invariant. In chapter 3 we review this construction and point out the source of such problems.

In chapter 4 we present a new formulation of the 1/2-BPS Wilson loop which addresses the issues of the original formulation. This is done by means of a gauge transformation, which gets rid of the twist matrix and allows the fermionic couplings to be written in a manifestly reparametrization invariant way. In this new gauge, it is possible to relate the 1/6-BPS solution to the 1/2-BPS by considering deformations of the loop connection.

Such deformations define a new family of 1/6-BPS Wilson loops which are fermionic, and encompasses the 1/2-BPS solution as a special case. This new formulation allows for the identification of the *moduli space* of these loops as the *conifold*, and it was generalized in the context of other CSm theories [7, 8].

One of the physical quantities which concerns this works is the *Bremsstrahlung function*. This is an important observable of gauge theories, since it contains the information of the energy radiated by a quark in the low energy limit. This function can be computed directly from the expectation value of circular Wilson loops [9], via

$$B(\lambda) = \frac{1}{2\pi^2}\lambda\partial_\lambda \log\langle\mathcal{W}_{\text{circle}}\rangle, \tag{1.6}$$

where λ is the 't Hooft coupling of the theory.

From localization [10], we have $\langle\mathcal{W}_{\text{circle}}\rangle$ exactly for the case of 1/2-BPS loops of ABJ(M), which gives us a closed expression for the $B(\lambda)$ function. In the weak coupling regime $\lambda \ll 1$, the Bremsstrahlung function can also be computed through the so-called *wavy line* deformation of the 1/2-BPS loop [11], which provides a pure field theoretical approach to its calculation. In chapter 5 we present our partial findings in the calculation of $B(\lambda)$ via the wavy 1/2-BPS loop of ABJM, which would provide a non-trivial check of this function via Feynman diagrams.

Another important topic in this thesis is the AdS/CFT correspondence, or gauge/gravity duality, which states the dynamical equivalence of certain SCFTs to theories of gravity in an AdS space. Being a weak/strong duality, it maps strongly coupled gauge theories to weakly coupled gravity, so that non-perturbative regimes in the SCFT can be approached via the AdS side.

The most important example of such correspondence for CSm theories is given by the duality between $\mathcal{N} = 6$ CSm theory with gauge group $U(N) \times U(N)$ and Chern-Simons levels k and $-k$, a.k.a ABJM, and M-theory in $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ [12]. The Chern-Simons-levels k are integers which parameterizes the ABJM action and define the 't Hooft coupling of the theory via

$$\lambda = \frac{N}{k}. \tag{1.7}$$

The duality is realized by identifying the free parameters of both sides,

$$\frac{L^3}{l_p^3} = 4\pi\sqrt{2kN}, \tag{1.8}$$

where the M-theory parameters are the AdS radius L and the 11d Planck length l_p. In the special limit of $k^5 \gg N$ the AdS side is reproduced by weakly coupled type IIA strings in $\mathrm{AdS}_4 \times \mathbb{CP}^3$, giving ABJM a description in terms of 10d gravity theory.

In the strong coupling regime of the gauge theory, the degrees of freedom of Wilson loops are encoded in the degrees of freedom of a semi-classical string, such that the expectation value of these operators can be calculated via the minimization of the string action, which defines the proper area of a fundamental string world-sheet. This way,

$$\langle \mathcal{W} \rangle \sim e^{-S_{\mathrm{string},min}}, \tag{1.9}$$

where the string world-sheet is bounded by the Wilson loop contour $\mathcal{C}$.

The identification of Wilson operators with string world-sheets allows us to use the AdS/CFT dictionary to calculate operator insertions on the Wilson line by considering the holographic description of field fluctuations living in the string world-sheet. This provides us with an $\mathrm{AdS}_2/\mathrm{CFT}_1$ instance of the holographic principle, so that the strongly coupled regime of the CFT can be mapped to a weakly interacting theory in the world-sheet.

In this work, we discuss the holographic description of the BPS Wilson lines of CSm theories, where one can map strongly coupled correlation functions on the Wilson line to a weakly interacting field theory in an AdS_2 space. An explicit solution for ABJM 1/2-BPS line is given in [13], and we comment on the still unknown holographic description of 1/2-BPS defects of $\mathcal{N} = 4$ CSm theories in the same spirit.

Another angle from which we are interested in studying BPS Wilson loops is through the defect conformal field theory (dCFT) one. As these operators are supported along straight lines or circles, when inserted in the vacuum, they break the underlying symmetries to the conformal group along the contour.

By introducing a gauge invariant quantity for local operators $\mathcal{O}_i(t_i)$ as

$$\mathcal{W}[\mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n)] \equiv \mathrm{Tr}\,\mathcal{P}[\mathcal{W}_{-\infty,t_1}\mathcal{O}_1(t_1)\mathcal{W}_{t_1,t_2}\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n)\mathcal{W}_{t_n,\infty}], \tag{1.10}$$

where $\mathcal{W}_{t_i,t_j}$ is the untraced Wilson operator integrated from points $x(t_i)$ to $x(t_j)$, we can define the *defect correlation functions* as

$$\langle \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n)\rangle_{\mathcal{W}} \equiv \frac{\langle \mathcal{W}[\mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n)]\rangle}{\langle \mathcal{W} \rangle}, \tag{1.11}$$

defining correlators in a one-dimensional conformal field theory supported by the Wilson contour, which is referred to as the dCFT. In the case of BPS WLs, the defects are also invariant under the set of supercharges annihilating the loop (1.2), thus endowing the dCFT with a supersymmetry structure.

Within supersymmetric dCFT, we are interested in studying correlation functions of protected operators by using the conformal bootstrap approach, which makes use of the one-dimensional superconformal symmetry in a systematic fashion.

The operators we are interested in studying are part of a short multiplet called the *displacement multiplet*, which is fundamental for any dCFT. This multiplet carries this name because it always contains the *displacement operator*, which is present in any dCFT (even in non-supersymmetric ones), since it is defined as the line insertion of momentum generators perpendicular to the defect.[4] In the supersymmetric cases, the displacement multiplet also contains other generators which arise from the insertion of broken charges such as R-symmetry generators and supercharges of the vacuum.

In chapter 6 we introduce the 1/2-BPS Wilson loops of $\mathcal{N} = 4$ as defects. There, the displacement multiplets can be cast into a superfield formalism, which makes the study of correlation functions possible in a suitable superspace. Within this superspace, we can employ the superconformal Casimir equation [14] to derive the conformal blocks of the OPE expansion of such superfields, which allows for the extraction of the conformal block coefficients and *anomalous dimensions* of displacement multiplet operators in the strong coupling regime by employing holographic considerations.

[4]As any line defect inserted in the vacuum of a SCFT defines a breaking of translation invariance in its perpendicular directions, the displacement operator is always present.

Chapter 2

Chern-Simons-matter theories

Introduction

The study of Chern-Simons theories can be motivated in many different ways. One of them was commented in the introductory text, and refers to knot theory [2]. More on the realm of physical interest, this class of theories sees applications ranging from condensed matter theory to quantum gravity. In the study of condensed matter systems, these theories describe the behaviour of topological insulators and the quantum Hall effect. From the quantum gravity aspect, it is important in defining instances of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence.

To define Chern-Simons theory in more precise terms, we introduce its action,

$$S_{\mathrm{CS}} = \frac{k}{4\pi}\int_{\mathcal{M}} d^3x \epsilon^{\mu\nu\rho}\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho + i\frac{2}{3}A_\mu A_\nu A_\rho\right), \tag{2.1}$$

with A_μ a general $U(N)$ gauge field, and $\epsilon^{\mu\nu\rho}$ the totally anti-symmetric tensor. In general the coefficient k is arbitrary, but it needs to be integer-valued for S_{CS} to define a gauge invariant quantum theory.

To see this, we notice that the action (2.1) is invariant under infinitesimal gauge transformations, but it is not generally invariant under large gauge transformations. In this case the action is shifted by

$$S_{\mathrm{CS}} \to S_{\mathrm{CS}} + 2\pi k n, \quad \text{for} \quad n \in \mathbb{Z}. \tag{2.2}$$

Once we consider a quantum theory, where the correlation functions are given by

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \int [\mathcal{D}A_\mu] e^{-iS_{\mathrm{SC}}}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n), \tag{2.3}$$

gauge invariance is restored once we require the coefficient k to be an integer. This coefficient is called the *Chern-Simons level*, and it basically counts loop orders in perturbation theory of Chern-Simons-matter theories. It is interesting to note that (2.1) defines a topological invariant, i.e a quantity that is only sensitive to the topology of the manifold $\mathcal{M}$ in which it is being integrated, which is one of the defining properties of pure Chern-Simons theories.

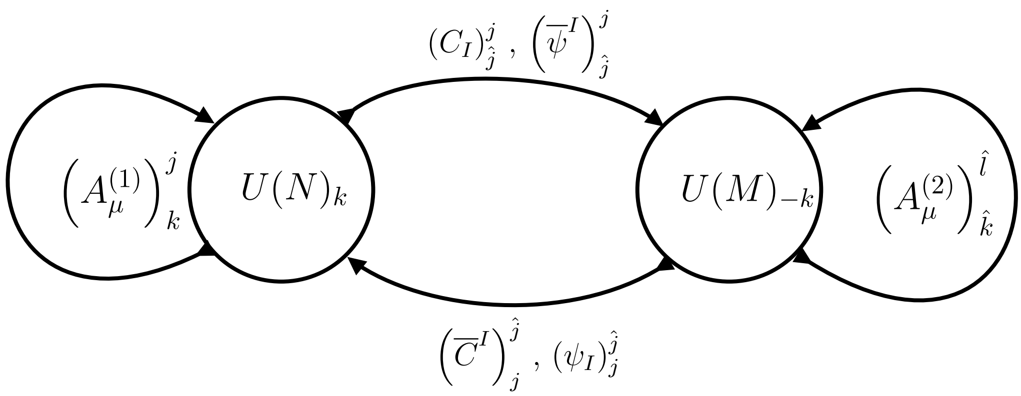
Now that pure Chern-Simons theory is properly introduced, we can focus on the Chern-Simons-matter theories, which, as the name suggests, contain matter fields in addition to gauge fields governed by Chern-Simons terms. This class of theories is defined by supersymmetric addition of matter to the Chern-Simons action. When matter is added in a supersymmetric way, the topological invariance of the pure Chern-Simons action is broken, but a superconformal configuration is defined. Of particular interest to us are the $\mathcal{N} = 4$ and $\mathcal{N} = 6$ theories, which we review in the following.

2.1 ABJ(M) theory

The original construction of ABJM theory, or $\mathcal{N} = 6$ three-dimensional Chern-Simons-matter theory, is given in terms of the low energy limit of M2-branes in M-theory by Aharony, Bergman, Jefferis and Maldacena in 2008 [12].

ABJM contains two gauge fields, $A_\mu^{(1)}$ and $A_\mu^{(2)}$ whose dynamics are given in terms of (2.1), with opposite level, k and $-k$. The matter content consists of four complex scalars $(C_I)_{\hat{j}}^{j}$ and four complex spinors $(\bar{\psi}^I)_{\hat{j}}^{j}$ transforming in the bi-fundamental of the total gauge group $U(N)_k \times U(M)_{-k}$. We denote unhatted indices for $U(N)_k$ and hatted indices for $U(M)_{-k}$. R-symmetries have capital Latin-letters ($I = \{1, 2, 3, 4\}$). The complex-conjugates of the scalars and fermions are denoted $(\bar{C}^I)_j^{\hat{j}}$ and $(\psi_I)_j^{\hat{j}}$ and transform in the anti-bi-fundamental.

This field content of can be cast into the quiver representation:



The quiver contains the two gauge groups as nodes $U(N)_k$ and $U(M)_{-k}$, from the arrows we can read the representation of the fields under the gauge groups. An outward arrow means a fundamental index, and an arrow inward means an antifundamental.

In general, the two gauge groups are parametrized by M, N, defining the ABJ theories. ABJM is a special case of ABJ theories, where $M = N$. This way, ABJ theory has two independent 't Hooft parameters $\lambda = N/k$ and $\hat{\lambda} = M/k$ which collapse to a single one when in the ABJM slice.[1]

ABJM defines a conformal theory in three dimensions with $\mathcal{N} = 6$ supersymmetry, so it enjoys the symmetries of $\mathfrak{osp}(6|4)$. The bosonic subgroup of symmetries

[1]We sometimes refer to the general ABJ theories as ABJ(M).

comprises of the conformal group in 3d, $\mathfrak{so}(1,4)_{\text{conf}}$ and the $\mathfrak{so}(6)_R \cong \mathfrak{su}(4)_R$ R-symmetry,

$$\mathfrak{so}(1,4)_{\text{conf}} \oplus \mathfrak{su}(4)_R \subset \mathfrak{osp}(6|4). \tag{2.4}$$

The fermionic generators are composed of 12 *Poincaré supercharges* represented by antisymmetric $\mathcal{Q}_{IJ\alpha}$, and 12 *superconformal charges* $\mathcal{S}_{IJ\alpha}$, where I and J are R-symmetry indices, and α is a spinor index taking values $\alpha \in \{+,-\}$. A general supersymmerty transformation is the action of a general supercharge

$$Q = \bar{\theta}^{IJ\alpha}\mathcal{Q}_{IJ\alpha} + \bar{\vartheta}^{IJ\alpha}\mathcal{S}_{IJ\alpha}, \tag{2.5}$$

parameterized by 12 independent spinors $\bar{\vartheta}^{IJ}$ and $\bar{\theta}^{IJ}$. The supersymmetry transformations acting on the fields play an important role in defining the BPS loops in this theory, and are explicitly listed in appendix (A.3).

As we are interested in developing a perturbative analysis of the 1/2-BPS Wilson loop in the context of the wavy-line deformations of chapter 5, we write the ABJM action [15],

$$S_{\text{ABJM}} = S_{\text{CS}} + S_{\text{kin}} - V_{\text{Yukawa}} - V_{\text{bos}}, \tag{2.6}$$

with

$$\begin{aligned} S_{\text{CS}} &= -i\frac{k}{4\pi}\int dx^3 \epsilon^{\mu\nu\rho}\text{Tr}\left(A_\mu^{(1)}\partial_\nu A_\rho^{(1)} + i\frac{2}{3}A_\mu^{(1)}A_\nu^{(1)}A_\rho^{(1)} - A_\mu^{(2)}\partial_\nu A_\rho^{(2)} - i\frac{2}{3}A_\mu^{(2)}A_\nu^{(2)}A_\rho^{(2)}\right) \\ S_{\text{kin}} &= \int dx^3 \text{Tr}\left(D_\mu C_I D^\mu \bar{C}^I + i\bar{\psi}\gamma^\mu D_\mu \psi_I\right). \end{aligned} \tag{2.7}$$

The covariant derivatives are defined as

$$\begin{aligned} D_\mu C_I &= \partial_\mu C_I + i(A_\mu^{(1)}C_I - C_I A_\mu^{(2)}), \qquad & D_\mu \bar{C}^I &= \partial_\mu \bar{C}^I - i(\bar{C}^I A_\mu^{(1)} - A_\mu^{(2)}\bar{C}^I), \\ D_\mu \bar{\psi}^I &= \partial_\mu \bar{\psi}^I + i(A_\mu^{(1)}\bar{\psi}^I - \bar{\psi}^I A_\mu^{(2)}), \qquad & D_\mu \psi_I &= \partial_\mu \psi_I - i(\psi_I A_\mu^{(1)} - A_\mu^{(2)}\psi_I). \end{aligned} \tag{2.8}$$

The Yukawa potential reads

$$\begin{aligned} V_{\text{Yukawa}} = -i\frac{2\pi}{k}\int dx^3 \text{Tr}\, \big(&\bar{C}^I C_I \psi_J \bar{\psi}^J - C_I \bar{C}^I \bar{\psi}^J \psi_J \\ &+2C_I \bar{C}^J \bar{\psi}^I \psi_J - 2\bar{C}^I C_J \psi_J \bar{\psi}^I - \epsilon_{IJKL}\bar{C}^I \bar{\psi}^J \bar{C}^K \bar{\psi}^L + \epsilon^{IJKL} C_I \psi_J C_K \psi_L\big), \end{aligned} \tag{2.9}$$

and the bosonic potential is

$$\begin{aligned} V_{\text{bos}} = \frac{4\pi^2}{3k^2}\int dx^3 \text{Tr}\, \big(&C_I \bar{C}^I C_J \bar{C}^J C_K \bar{C}^K + C_I \bar{C}^J C_J \bar{C}^K C_K \bar{C}^I \\ &+4C_I \bar{C}^J C_K \bar{C}^I C_J \bar{C}^K - 6C_I \bar{C}^J C_J \bar{C}^I C_K \bar{C}^K\big). \end{aligned} \tag{2.10}$$

2.2 $\mathcal{N} = 4$ theories

This class of theories was first introduced by Gaiotto and Witten [16], being later generalized in [17] and [18], with the enhancement of the quivers to have an arbitrary number of nodes, and to be either linear or circular. Being an $\mathcal{N}=4$ theory in three dimensions, it enjoys the isometries of $\mathfrak{osp}(4|4)$, whose bosonic subgroups comprise the conformal group and an $\mathfrak{so}(4)_R \cong \mathfrak{su}(2)_A \oplus \mathfrak{su}(2)_B$, so that the operators of these theories are labeled according to the subgroup

$$\mathfrak{so}(1,4)_{\text{conf}} \oplus \mathfrak{su}(2)_A \oplus \mathfrak{su}(2)_B \subset \mathfrak{osp}(4|4). \tag{2.11}$$

In this thesis we are only interested in the linear quiver set-up, in which we have an adjoint $U(N_I)$ gauge field A_I defining the "I$^{\text{th}}$-node" , and adjacent nodes defined by $A_{I\pm1}$, in the adjoint of $U(N_{I\pm1})$, with alternating level $\pm k$. The matter fields comprise a string of alternating hypermultiplet and twisted-hypermultiplets coupling A_I to A_{I+1} and A_I to A_{I-1}, respectively.

The field content of the multiplets are

$$\bullet\text{Hyper: } q^a_I, \psi_{I\dot a}, \bar q_{Ia}, \bar\psi^{\dot a}_I \qquad \bullet\text{ Twisted-Hyper: } \tilde q_{I-1\dot a}, \tilde\psi^a_{I-1}, \bar{\tilde q}^{\dot a}_{I-1}, \bar{\tilde\psi}_{I-1a}, \tag{2.12}$$

with $q, \bar q$ representing scalar fields and $\psi, \bar\psi$ fermionic fields. We define (un-)dotted indices to be fundamental indices of ($\mathfrak{su}(2)_A$) $\mathfrak{su}(2)_B$, namely $a = 1,2$ and $\dot a = \dot 1, \dot 2$.

The linear quiver structure in which we work is given by

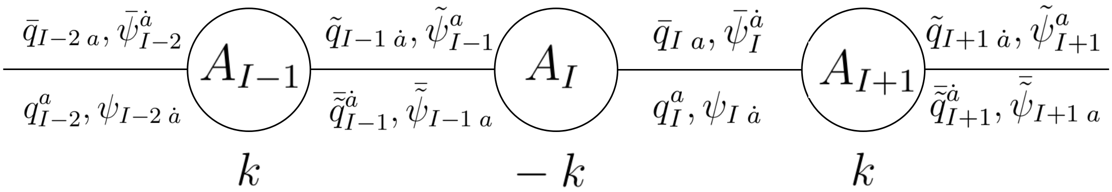


Figure 2.1: Representation the linear quiver. Chern-Simons forms have alternating level k.

As we are interested in Wilson loops that couple the nodes A_I and A_{I+1}, it is useful to define the so-called moment maps, in respect to these nodes. Transforming in the adjoint representation of A_I and A_{I+1} respectively, they are given by the scalar bilinears

$$\begin{aligned} \tilde\mu_I{}^{\dot a}{}_{\dot b} &= \bar{\tilde q}^{\dot a}_{I-1}\tilde q_{I-1\,\dot b} - \frac{1}{2}\delta^{\dot a}_{\dot b}\bar{\tilde q}^{\dot c}_{I-1}\tilde q_{I-1\,\dot c}\,, &\qquad \tilde\mu_{I+1}{}_{\dot b}{}^{\dot a} &= \tilde q_{I+1\,\dot b}\bar{\tilde q}^{\dot a}_{I+1} - \frac{1}{2}\delta^{\dot a}_{\dot b}\tilde q_{I+1\,\dot c}\bar{\tilde q}^{\dot c}_{I+1}\,, \\ \nu_I &= q^a_I \bar q_{I\,a}\,, & \nu_{I+1} &= \bar q_{I\,a} q^a_I. \end{aligned} \tag{2.13}$$

We only list here the moment maps that are relevant for the Wilson loops that will be studied in chapter 6, one can also construct combination of fermionic fields transforming in the adjoint of the nodes [7], which are not moment maps, but currents.

A general superymmetry transformation is parameterized by

$$Q = \theta^{a\dot a\alpha}\mathcal{Q}_{a\dot a\alpha} + \vartheta^{a\dot a\alpha}\mathcal{S}_{a\dot a\alpha}, \tag{2.14}$$

where the spinors carry fundamental indices of $\mathfrak{su}(2)_A \oplus \mathfrak{su}(2)_B$, accounting for four Poincare and four superconformal charges. For more details about preserved superalgebra of these theories see Appendix B.

Chapter 3

Supersymmetric Wilson loops

In this chapter we review the construction of BPS Wilson operators in ABJM theory, outlying important features of the construction of the so-called 1/6-BPS bosonic loops [19] and the 1/2-BPS fermionic loops [6], which are particularly important for the next chapters. By presenting such constructions, we shed light over the supermatrix nature of the connections and its interplay with the BPS condition (1.2).

In this chapter we only present the circular loops, defined by the contour parametrization

$$x^\mu(\tau) = (0, \cos(\tau), \sin(\tau)). \tag{3.1}$$

To search for BPS solutions, we outline that, for circular Wilson loops, the BPS equation holds for a certain linear combination of Poincaré $\mathcal{Q}_{IJ}$ and superconformal $\mathcal{S}_{IJ}$ charges. This is because the action of these charges are related by

$$\mathcal{S}_{IJ} \to x^\mu \gamma_\mu \mathcal{Q}_{IJ}, \tag{3.2}$$

so that a general variation (2.5) can be written as

$$\delta = \left(\bar{\theta}^{IJ} + \bar{\vartheta}^{IJ}(x^\mu \gamma_\mu)\right) \mathcal{Q}_{IJ}. \tag{3.3}$$

By condensing both parameters $\bar{\theta}^{IJ}$ and $\bar{\vartheta}^{IJ}$ into a single spinor $\bar{\Theta}^{IJ}$, the BPS constraint can be expressed as

$$\bar{\Theta}^{IJ} \mathcal{Q}_{IJ} \mathcal{W}_{\text{BPS}} = 0 \qquad \text{for} \qquad \bar{\Theta}^{IJ} = \bar{\theta}^{IJ} \pm (\dot{x}^\mu \gamma_\mu) \bar{\epsilon}^{IJ}, \tag{3.4}$$

with

$$\bar{\vartheta}^{IJ} = \pm i \bar{\epsilon}^{IJ} \sigma^3. \tag{3.5}$$

The spinor $\bar{\Theta}^{IJ}$ is called the Killing spinor, and as we will see, (3.4) is central to defining the correct combinations of charges which annihilate the loops by chosing the appropriate superconformal charges $\bar{\vartheta}^{IJ}$ in terms of the Poincaré $\bar{\theta}^{IJ}$.[1]

[1]Rigorously speaking, the charges are $\mathcal{Q}_{IJ}$ and $\mathcal{S}_{IJ}$, but we refer to the parameters of the SUSY transformations as charges throughout the text.

3.1 1/6-BPS bosonic loops in ABJ(M)

The first constructed BPS operators in ABJM are the 1/6-BPS solutions known as the *bosonic loops*. These loops can be seen as the 1/2-BPS loops of $\mathcal{N} = 2$ [20] containing only bosonic fields in the connection, hence the name.

With the total gauge group being $U(N) \otimes U(M)$, and matter content transforming in the bi-fundamental representation, there are in principle many ways to construct gauge invariant connections, but as the connection is a dimensionless quantity, we are restricted in the combinations that can compose it.

Drawing intuition from the case of $\mathcal{N} = 4$ SYM, where BPS loops couple to scalar fields in the adjoint representation [21], one looks for an analog set-up in ABJM. Although no matter fields transform in the adjoint, the scalar bilinears $C_I \bar{C}^J$ and $\bar{C}^J C_I$ have the correct dimension and transform in the adjoint of $U(N)$ and $U(M)$ respectively, so we can expect that the BPS loops of ABJM contain these combinations.

Following this reasoning, the natural ansatz for a BPS operator contains the scalar bilinears along with the gauge fields, defining the *dressed connections*. Noticing the SUSY variation (A.3) of gauge fields and scalars, one can hope to find a BPS operator fine-tuning an embedding matrix M_J^I which couples the loop to the scalar bilinears.

This way, we can define two independent Wilson loops [19],

$$\mathcal{W}^{(1)}_{\text{bos}} = \operatorname{Tr}\mathcal{P}\exp\left(i\oint \mathcal{A}^{(1)}_{\text{bos}} d\tau\right), \qquad \mathcal{A}^{(1)}_{\text{bos}} = A^{(1)}_\mu \dot{x}^\mu - \frac{2\pi i}{k}|\dot{x}| M_J^I C_I \bar{C}^J, \tag{3.6}$$

and

$$\mathcal{W}^{(2)}_{\text{bos}} = \operatorname{Tr}\mathcal{P}\exp\left(i\oint \mathcal{A}^{(2)}_{\text{bos}} d\tau\right), \qquad \mathcal{A}^{(2)}_{\text{bos}} = A^{(2)}_\mu \dot{x}^\mu - \frac{2\pi i}{k}|\dot{x}| M_J^I \bar{C}^J C_I, \tag{3.7}$$

where the loops are taken over a circular contour $x_\mu(\tau)$, and the matrix M_J^I defines an embedding in the R-symmetry space.

One can inspect [5] that the loops defined above are annihilated by the Killing spinors defined in terms of the Poincaré supercharges

$$\bar{\theta}^{12} \qquad \text{and} \qquad \bar{\theta}^{34}, \tag{3.8}$$

and superconformal charges

$$\bar{\vartheta}^{12} = i\bar{\theta}^{12}\sigma^3 \qquad \text{and} \qquad \bar{\vartheta}^{34} = -i\bar{\theta}^{34}\sigma^3, \tag{3.9}$$

which yields the Killing spinors

$$\bar{\Theta}^{12} = 2\bar{\theta}^{12}\Pi_- \qquad \text{and} \qquad \bar{\Theta}^{34} = 2\bar{\theta}^{34}\Pi_+, \tag{3.10}$$

where we have introduced the *spinor space projectors*

$$\Pi_\pm = \frac{1}{2}\left(1 \pm \frac{\dot{x}^\mu \gamma_\mu}{|\dot{x}|}\right), \tag{3.11}$$

such that

$$\Pi_{\pm}\Pi_{\pm} = \Pi_{\pm}, \tag{3.12}$$
$$\Pi_{\pm}\Pi_{\mp} = 0. \tag{3.13}$$

With the choices of (3.8) and (3.9), one can check that the Killing spinors (3.10) annihilate the connections (3.6), (3.7) for the embedding

$$M^I_J = \mathrm{diag}(-1,-1,1,1). \tag{3.14}$$

To see that the BPS condition (1.2) is satisfied by the connection $\mathcal{A}^{(1)}_{\text{bos}}$ for the given Killing spinors, we can take the variation

$$\delta\mathcal{A}^{(1)}_{\text{bos}} = \delta A^{(1)}_\mu \dot{x}^\mu - \frac{2\pi i}{k}\left(M^I_J \delta C_I \bar{C}^J + M^I_J C_I \delta\bar{C}^J\right), \tag{3.15}$$

so that when the variation is taken in respect to the Killing spinor $\bar{\Theta}^{12}$, we have

$$\delta A^{(1)}_\mu \dot{x}^\mu = \frac{4\pi i}{k}\bar{\Theta}^{12}\left(\gamma_\mu \dot{x}^\mu\right)\left(C_1\psi_2 - \psi_2 C_1 + \epsilon_{12KL}\bar{\psi}^K\bar{C}^L\right), \tag{3.16}$$
$$M^I_J \delta C_I \bar{C}^J = 2M^I_J \bar{\Theta}^{12}\epsilon_{IM12}\bar{\psi}^M\bar{C}^J, \tag{3.17}$$
$$M^I_J C_I \delta\bar{C}^J = 2\bar{\Theta}^{12}\left(M^I_1 C_I\psi_2 - M^I_2 C_I\psi_1\right). \tag{3.18}$$

In the gauge variation (3.16), we write

$$(\dot{x}^\mu\gamma_\mu)_\alpha^{\ \beta} = (2\Pi_+ - 1)_\alpha^{\ \beta}, \tag{3.19}$$

such that

$$\bar{\Theta}^{12}(\dot{x}^\mu\gamma_\mu) = -\bar{\Theta}^{12}. \tag{3.20}$$

This way, plugging expressions (3.16), (3.17), and (3.18) in (3.15) yields

$$\delta_{12}\mathcal{A}^{(1)}_{\text{bos}} = -\bar{\Theta}^{12}\frac{4\pi i}{k}\left(C_1\psi_2 - \psi_2 C_1 + \epsilon_{12KL}\bar{\psi}^K\bar{C}^L + M^I_J\epsilon_{IM12}\bar{\psi}^M\bar{C}^J + M^I_1 C_I\psi_2 - M^I_2 C_I\psi_1\right), \tag{3.21}$$

such that the connection is annihilated by the variation

$$\delta_{12}\mathcal{A}^{(1)}_{\text{bos}} = 0. \tag{3.22}$$

Next, we need to analyze the action of δ_{34}, parameterized by the Killing spinor $\bar{\Theta}^{34}$. This way, the terms of (3.15) read

$$\delta A^{(1)}_\mu \dot{x}^\mu = \frac{4\pi i}{k}\bar{\Theta}^{34}\left(\gamma_\mu \dot{x}^\mu\right)\left(C_3\psi_4 - \psi_4 C_3 + \epsilon_{34KL}\bar{\psi}^K\bar{C}^L\right), \tag{3.23}$$
$$M^I_J \delta C_I \bar{C}^J = 2M^I_J \bar{\Theta}^{34}\epsilon_{IM34}\bar{\psi}^M\bar{C}^J, \tag{3.24}$$
$$M^I_J C_I \delta\bar{C}^J = 2\bar{\Theta}^{34}\left(M^I_3 C_I\psi_4 - M^I_4 C_I\psi_3\right). \tag{3.25}$$

In the variation of the gauge field, we can write

$$(\dot{x}^\mu\gamma_\mu)_\alpha^{\ \beta} = (1 - 2\Pi_-)_\alpha^{\ \beta}, \tag{3.26}$$

so that

$$\bar{\Theta}^{34}(\dot{x}^{\mu}\gamma_{\mu}) = \bar{\Theta}^{34}. \tag{3.27}$$

This way we can plug (3.23), (3.24), and (3.25) in (3.15) to yield

$$\delta_{34}\mathcal{A}^{(1)}_{\text{bos}} = 0. \tag{3.28}$$

Equations (3.22), (3.28) render the connection invariant under the action of the Killing spinors (3.10). As these spinors are parameterized by four independent components of (3.8), we have a total of 4 independent charges annihilating the connection $\mathcal{A}^{(1)}_{\text{bos}}$, so these loops are 1/6-BPS.

We notice that, if the connection is itself invariant under $\bar{\Theta}^{12}$ and $\bar{\Theta}^{34}$, then the whole Wilson loop is also invariant

$$\delta\mathcal{A}^{(1)}_{\text{bos}} = 0 \quad \Longrightarrow \quad \delta\mathcal{W}^{(1)}_{\text{bos}} = 0, \tag{3.29}$$

which renders the bosonic loop $\mathcal{W}^{(1)}_{\text{bos}}$ a 1/6-BPS operator. The invariance of the connection under a supersymmetry transformation is a sufficient condition for the Wilson loop to also be invariant under such transformation, but as we will see in the 1/2-BPS solutions, *it is not a necessary condition.*

The proof of supersymmetry of $\mathcal{A}^{(2)}_{\text{bos}}$ is exactly the same as the one presented for the $\mathcal{A}^{(1)}_{\text{bos}}$ connection, modulo swapping the order of the bilinear of matter fields in the calculation.

In closing, we notice that the configuration of the 1/6-BPS loop relies on the existence of the matrix M^I_J defining its embedding in the R-symmetry space, such that it breaks the R-symmetry group from $\mathfrak{su}(4)_R \to \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. We also notice that this solution is the most supersymmetric solution which can be achieved by requiring that connection $\mathcal{A}$ to be annihilated by the supersymmetric variation,

$$\delta\mathcal{A} = 0. \tag{3.30}$$

Solutions that enjoy more supersymmetry are only attained by considering a relaxed BPS condition, which allows for a supersymmetry variation of the super-connection to amount to a gauge transformation, in a way that the RHS of (3.30) is modified to contain a gauge transformation of a $U(N|M)$ structure, which we introduce in the 1/2-BPS construction.

3.2 1/2-BPS fermionic loops of ABJ(M)

To define the 1/2-BPS operators, one needs to couple the loop to fermionic fields as well as the scalars used in the 1/6-BPS construction. To accommodate the bi-fundamental representations of the fermions one must enhance the loop connection to a super-connection. A superconnection $\mathcal{L}$ of the supergroup $U(N|M)$ has the property of transforming as adjoint of $U(N)$ and $U(M)$ in the diagonals, and as bi-fundamental in the off-diagonals, such that under a gauge transformation of $U_1 \in U(N)$, and $U_2 \in U(M)$, it transforms as

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix} \to \begin{pmatrix} U_1\mathcal{L}_{11}U_1^{\dagger} - iU_1\partial_{\mu}U_1^{\dagger} & U_1\mathcal{L}_{12}U_2^{\dagger}, \\ U_2\mathcal{L}_{21}U_1^{\dagger} & U_2\mathcal{L}_{22}U_2^{\dagger} - iU_2\partial_{\mu}U_2^{\dagger} \end{pmatrix}, \tag{3.31}$$

allowing for the introduction of matter fields in the off-diagonal. In this context, the diagonal entries can naturally accommodate the bosonic dressed connections, since they couple to the scalar bilinears and transform in the adjoint, and the off-diagonal entries are compatible with the coupling to fermionic fields of the theory.

The 1/2-BPS Wilson operators of ABJ(M) were first constructed in [6], and they can be seen as a generalization of the 1/6-BPS of the previous section, now containing bi-fundamental fermions in the off-diagonal entries of the superconnection,

$$\mathcal{L}_{1/2\text{-BPS}} = \begin{pmatrix} \mathcal{A}^{(1)}_{\text{bos}} & -i\sqrt{\frac{2\pi}{k}}|\dot{x}|\eta^{\alpha}_{I}\bar{\psi}^{I}_{\alpha} \\ -i\sqrt{\frac{2\pi}{k}}|\dot{x}|\psi^{\alpha}_{I}\bar{\eta}^{I}_{\alpha} & \mathcal{A}^{(2)}_{\text{bos}} \end{pmatrix}, \tag{3.32}$$

with

$$\begin{aligned} \mathcal{A}^{(1)}_{\text{bos}} &= A^{(1)}_{\mu}\dot{x}^{\mu} - \tfrac{2\pi i}{k}|\dot{x}|M^{I}_{J}C_{I}\bar{C}^{J}\,, \\ \mathcal{A}^{(2)}_{\text{bos}} &= A^{(2)}_{\mu}\dot{x}^{\mu} - \tfrac{2\pi i}{k}|\dot{x}|M^{I}_{J}\bar{C}^{J}C_{I}\,. \end{aligned} \tag{3.33}$$

In contrast to the 1/6-BPS loop, this loop couples to all fields in the matter content of the theory, with the coupling to fermions being achieved via the Grassmann even quantities $\eta^{\alpha}_{I}(\tau)$ and $\bar{\eta}^{I}_{\alpha}(\tau)$, which we refer to as *fermionic couplings*, and to the scalars via a matrix M^{I}_{J},

$$\begin{aligned} &\eta^{\alpha}_{I}(\tau) = \begin{pmatrix} e^{i\tau/2} & -ie^{-i\tau/2} \end{pmatrix}\delta^{1}_{I}\,, \quad \bar{\eta}^{I}_{\alpha}(\tau) = \begin{pmatrix} ie^{-i\tau/2} \\ -e^{i\tau/2} \end{pmatrix}\delta^{I}_{1}\,, \\ &M^{I}_{J} = \text{diag}(-1,1,1,1)\,. \end{aligned} \tag{3.34}$$

To inspect the BPS condition in this loop, we first notice that if we ask for $\mathcal{L}_{1/2\text{-BPS}}$ to be invariant under the action of supercharges, we would have

$$\delta\mathcal{L}_{1/2\text{-BPS}} = \begin{pmatrix} \delta\mathcal{A}^{(1)}_{\text{bos}} & -i\sqrt{\frac{2\pi}{k}}|\dot{x}|\eta^{\alpha}_{I}\delta\bar{\psi}^{I}_{\alpha} \\ -i\sqrt{\frac{2\pi}{k}}|\dot{x}|\delta\psi^{\alpha}_{I}\bar{\eta}^{I}_{\alpha} & \delta\mathcal{A}^{(2)}_{\text{bos}} \end{pmatrix} \overset{(!)}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.35}$$

which can only be acomplished by setting the fermionic couplings to zero, defeating the purpose of enhancing the connection in the first place, and getting back to the set-up of bosonic loops.

As we have said before, connection invariance guarantees loop invariance, but it is not necessary. Instead, we can consider a relaxed condition for the supersymmetry variation of superconnections, which explores the enhanced $\mathfrak{u}(N|M)$ structure of the super-connection.

The idea is to make use of this enhanced gauge symmetry at the level of the super-connection, by considering the case where a supersymmetry variation of the super-connection is equivalent to a $\mathfrak{u}(N|M)$ gauge transformation, so that the traced loop is still an invariant quantity. This way, the BPS condition reads

$$\delta\mathcal{L}_{1/2\text{-BPS}} = \mathfrak{D}_{\tau}\mathcal{G}, \tag{3.36}$$

where operator $\mathfrak{D}_{\tau}$ is the supercovariant derivative of the $\mathfrak{u}(N|M)$ structure, defined by its action on an arbitrary supermatrix $\mathcal{G}$ of $\mathfrak{u}(N|M)$ as

$$\mathfrak{D}_{\tau}\mathcal{G} \equiv \partial_{\tau}\mathcal{G} + i[\mathcal{L}_{1/2\text{-BPS}}, \mathcal{G}]. \tag{3.37}$$

As the action of the supercovariant derivative $\mathfrak{D}_\tau \mathcal{G}$ is a $U(N|M)$ gauge transformation, we are exchanging a supersymmetric transformation of $\mathcal{L}$ for a gauge transformation, which does not contribute once the trace is taken. This way, the BPS condition translates into finding a matrix $\mathcal{G}$ for which (3.36) holds.

Under this BPS constraint, the presented loop is a 1/2-BPS solution preserving all Poincaré charges parameterized by arbitrary constant θ^{IJ}, which amounts to a total of six independent spinors parametrizing the transformations. The preserved superconformal charges have their parameters completely fixed in terms of the Poincaré charges via the identificaction

$$\bar{\vartheta}^{1I} = i\bar{\theta}^{1I}\sigma^3 \quad \text{and} \quad \bar{\vartheta}^{IJ} = -i\bar{\theta}^{IJ}\sigma^3, \quad \text{for } I, J \neq 1. \tag{3.38}$$

This way, we have the six independent Killing spinors written in terms of the projectors,[2]

$$\bar{\Theta}^{1I} = 2\bar{\theta}^{1I}\Pi_- \qquad \text{and} \qquad \bar{\Theta}^{IJ} = 2\bar{\theta}^{IJ}\Pi_+. \tag{3.39}$$

In the light of the constraint (3.37), supersymmetry can be checked by considering the supermatrix

$$\mathcal{G}(\tau) = 2\sqrt{\frac{2\pi}{k}} \begin{pmatrix} 0 & 2\eta(\tau)\bar{\theta}^{1I}C_I \\ -\epsilon_{1IJK}\bar{\theta}^{IJ}\bar{\eta}(\tau)\bar{C}^K & 0 \end{pmatrix}. \tag{3.40}$$

When integrated along the loop, the gauge transformations of the BPS constraint generate an off-set in the superconnection that needs to be compensated in order for the traced operator to be gauge invariant. One can easily recover gauge invariance by the introduction of a *twist matrix* $\mathcal{T}$ as in [4]. Under a finite $\mathfrak{u}(N|M)$ gauge transformation, a Wilson link

$$\mathcal{W}_{1/2\text{-BPS}}(\tau_i, \tau_f) = \mathcal{P}\exp\left(i\int_{\tau_i}^{\tau_f} d\tau \mathcal{L}_{1/2\text{-BPS}}(\tau)\right), \tag{3.41}$$

transforms as

$$\mathcal{W}_{1/2\text{-BPS}}(\tau_i, \tau_f) \to U(\tau_i)\mathcal{W}_{1/2\text{-BPS}}(\tau_i, \tau_f)U^{-1}(\tau_f), \tag{3.42}$$

where

$$U(\tau) = \exp\left(i\mathcal{G}(\tau)\right). \tag{3.43}$$

If we consider the operator defined by the super-trace of a Wilson link compensated by a $\mathcal{T}$ matrix

$$\text{sTr}\,\left(\mathcal{W}_{1/2\text{-BPS}}(\tau_i, \tau_f)\mathcal{T}\right), \tag{3.44}$$

where sTr denotes a *supertrace*,[3] and the ad-hoc $\mathcal{T}$ matrix is such that

$$U(2\pi) = \mathcal{T}U(0)\mathcal{T}^{-1}, \tag{3.45}$$

[2] The killing spinors of the 1/6-BPS solutions are given by $I = 2$ in the first set, and $I, J = 3, 4$ in the second set.

[3] The supertrace of a matrix is defined as $\text{sTr}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{Tr}\,A - \text{Tr}\,D$.

we recover gauge invariance of the circular operator,

$$\begin{aligned}\mathrm{sTr}\,\left(\mathcal{W}_{1/2\text{-BPS}}(0,2\pi)\mathcal{T}\right) \to\ & \mathrm{sTr}\,\left(U(0)\mathcal{W}_{1/2\text{-BPS}}(0,2\pi)U^{-1}(2\pi)\mathcal{T}\right)\\ =\ & \mathrm{sTr}\,\left(U(0)\mathcal{W}_{1/2\text{-BPS}}(0,2\pi)\mathcal{T}U^{-1}(0)\mathcal{T}^{-1}\mathcal{T}\right)\\ =\ & \mathrm{sTr}\,\left(\mathcal{W}_{1/2\text{-BPS}}(0,2\pi)\mathcal{T}\right).\end{aligned} \tag{3.46}$$

We notice that the femionic couplings are anti-periodic in 2π, the scalars are periodic, and the spinors θ^{IJ} are just constants. This way, we have that the matrix $\mathcal{G}(\tau)$ is 2π-anti-periodic, which in turn gives us

$$U(2\pi) = \mathcal{T}U(0)\mathcal{T}^{-1} \qquad \text{for} \qquad \mathcal{T} = -i\sigma^3. \tag{3.47}$$

As consequence we have that the gauge invariant operator is defined by

$$\mathrm{sTr}\,\left(\mathcal{W}(-i\sigma^3)\right) = -i\mathrm{Tr}\,(\mathcal{W}), \tag{3.48}$$

where σ^3 matrix turns the sTr into a Tr.

This is a hint that the defined Wilson operator is not in its most natural form, since the sTr definition is dependent on a non-trivial twist matrix. In fact, this is because the loop is defined in terms of anti-periodic fermionic couplings, an issue that we address in the next chapter.

The expectation value of the 1/6-BPS and 1/2-BPS loops of ABJM can be exactly calculated from the localization technique. In [22], the authors reduced the calculation of the 1/6-BPS loops (3.6), (3.7) to a matrix model, and in [6], it was shown that the same supercharge $\mathcal{Q}$ that is used to localize the 1/6-BPS loop was also shared by the 1/2-BPS loop. In such work, it was also proved that the 1/2-BPS loop is cohomologically equivalent to the 1/6-BPS loop, namely

$$\mathcal{W}_{1/2\text{-BPS}} = \mathcal{W}_{1/6\text{-BPS}} + \mathcal{Q}V, \tag{3.49}$$

which in terms of localization, means that both loops localize to the same matrix model [22], hence, their expectation value is the same.

The solution to the matrix model which describe these loops is given in [10], which allows one to retrieve expansions at weak and strong coupling regimes. The weak coupling regime is given by

$$\left\langle\mathcal{W}_{1/2\text{-BPS}}\right\rangle = 1 + \frac{i\pi}{k}(N-M) - \frac{2\pi^2}{3k^2}\left(N^2 - \frac{5}{2}NM + M^2 - \frac{1}{4}\right) + \mathcal{O}\left(\frac{1}{k^3}\right). \tag{3.50}$$

At the time, there was no field theory computation of the 1/2-BPS loop, and this result provided the first prediction that was later confirmed by the field theory calculations of [23].

Similarly, taking the strong coupling regime of the model yields

$$\left\langle\mathcal{W}_{1/2\text{-BPS}}\right\rangle \sim \exp\left(\pi\sqrt{\frac{M+N}{k}}\right), \tag{3.51}$$

which is in agreement with the prediction from the minimal area of an $AdS_4 \times \mathbb{CP}^3$ string ending on the circular loop at the boundary.

3.3 The Bremsstrahlung function

One of the developments covered by this work is the perturbative calculation of the deformed circular 1/2-BPS Wilson loop in ABJ(M) theory, presented in chapter 5. In our set-up, we calculate a weak coupling expansion of the loop with a deformed contour, which relates to the Bremsstrahlung function. Such function is dependent only on the gauge coupling λ of the theory, and it encodes the information of the energy emitted by a moving quark in the low energy limit via

$$\Delta E = 2\pi B(\lambda) \int dt(\dot{v})^2, \quad \text{for} \quad v \ll 1. \tag{3.52}$$

In this context of deformations of supersymmetric Wilson operators, $B(\lambda)$ was first introduced for $\mathcal{N} = 4$ SYM in [9], where the 1/2-BPS Wilson loop is given by

$$\mathcal{W}_{1/2\text{-BPS}}[\mathcal{C}] = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp\left[\oint_{\mathcal{C}} ds \left(iA_\mu \dot{x}^\mu + |\dot{x}| \vec{\Phi} \cdot \vec{n}\right)\right], \tag{3.53}$$

defined for $\mathcal{C}$ being a circle or straight line. The $\vec{\Phi}$ are scalar fields transforming in the adjoint, and $\vec{n}$ is the scalar coupling such that $|\vec{n}| = 1$.

There is a plethora of equivalent ways of defining the Bremsstrahlung function in terms of Wilson loops, ranging from the introduction of cusps in the Wilson line contour to the language of defect CFTs and the introduction of the so-called *displacement operators*.

As we have a natural intepretation of Wilson lines as heavy charged probes, the most natural way of introducing such function is by defining a cusped Wilson line, where such cusp can be seen as a sudden change of direction of the probe, generating Bremsstrahlung radiation.

Whenever one considers a BPS Wilson line, one is protected from UV divergences because of the supersymmetric nature of the operator, but when a cusp is introduced, the breaking of the supersymmetry generates divergences in the vacuum expectation value of the operator, and regulators are needed. The cusp is defined in both space-time contour x^μ and R-symmetry space, which in the context of $\mathcal{N} = 4$ translates into a discontinuity of the coupling to the scalars by an angle θ.

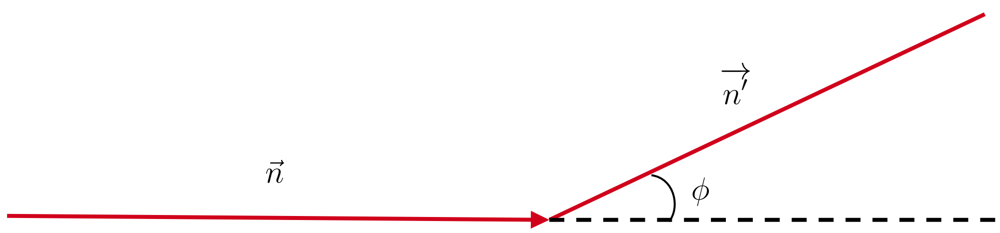


Figure 3.1: The contour of the Wilson line, showing a geometric cusp of angle ϕ. The vectors $\vec{n}$ and $\vec{n'}$ are the S^5 vectors coupling to the scalars of $\mathcal{N} = 4$ SYM. The internal angle is $\cos\theta = \vec{n} \cdot \vec{n'}$.

In this configuration, the expectation value of the loop acquires a log divergence, and can be written as

$$\langle \mathcal{W}_{\text{cusp}} \rangle \sim e^{-\Gamma_{\text{cusp}}(\lambda,\phi,\theta)} \log(L/\epsilon), \tag{3.54}$$

where the function Γ_{cusp} is called the *cusp anomalous dimension*, and ϵ and L are UV and IR cutoffs respectively. To retrieve the $B(\lambda)$ function, we need to take a small limit angle of the cusp anomalous dimension, yielding

$$\Gamma_{\text{cusp}} \sim (\theta^2 - \phi^2)B(\lambda) + o(\phi^4). \tag{3.55}$$

Apart from this rather physical definition of $B(\lambda)$, one can also define it in terms of the expectation value of circular Wilson loops [9]:

$$B(\lambda) = \frac{1}{2\pi^2}\lambda\partial_\lambda \log\langle\mathcal{W}_{\text{circle}}\rangle. \tag{3.56}$$

In the cases that we solve the insertion of the loop exactly (e.g BPS loops of 4d $\mathcal{N} = 4$ SYM and BPS loops of ABJ(M)), this formula provides us with an exact expression for $B(\lambda)$. For example, for the 1/2-BPS of $\mathcal{N} = 4$ SYM we have, in the large N limit,

$$\langle\mathcal{W}^{\mathcal{N}=4}_{\text{circle}}\rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) + \mathcal{O}(1/N^2), \tag{3.57}$$

which can be plugged in (3.56) to collect the exact Bremsstrahlung function

$$B_{\mathcal{N}=4}(\lambda) = \frac{1}{4\pi^2}\frac{\sqrt{\lambda}I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}. \tag{3.58}$$

Unfortunately, the ABJ(M) analogue of this formula is much more complicated then the above [24], but still, it is a known closed expression that can be expanded at any order for any value of the coupling λ.

There is yet another way to define the $B(\lambda)$ function which is central to this work, which is by means of a deformation in the contour of a BPS Wilson loop or line. By introducing a controlled deformation parameter in the contour of the loop, one defines what is called a *wavy* Wilson operator. The appearence of $B(\lambda)$ in this context is formally related to understaning the Wilson operator as a defect in the underlying CFT, and the deformation on the loop as generated by the insertion of displacement operators in the defect CFT [25].

A *wavy* Wilson operator is achieved by considering a small deformation of the contour around a BPS solution. In general, BPS Wilson operators are defined by a circular or straight line contour, so one achieves a wavy operator by deforming the circle or line supporting the operator.

Considering a BPS Wilson line, defined by a straight contour in d-dimensions

$$\mathcal{C} : x^\mu(s) = (s, 0, ..., 0), \tag{3.59}$$

the wavy line is defined by the deformation parameter $\xi^\mu(s)$, which deforms the contour $\mathcal{C}$ into $\mathcal{C}'$ such that,

$$\mathcal{C}' : x^\mu(s) + \xi^i(s) = (s, \xi^i(s)), \tag{3.60}$$

with the requirement of $\xi^i(s)$ to be small in magnitude, which is necessary to avoid divergences coming from emerging cusps and self-intersections of the deformed contour. Notice that the wavy line is in general no longer BPS, since the deformation of the contour violates the BPS constraints. However, the breaking of the line supersymmetry is done in a controlled fashion by tuning the deformation profile.

In [11], it was noted that the quadratic term in ξ follows a universal behaviour due to the superconformal nature of the underlying theory. As the Wilson line is a functional of its contour, the leading order deformation is constrained by rotation and translation invariance to obey

$$I[\xi] = \int ds \int ds^{'} \dot{\xi}^i(s) K(s - s^{'}) \dot{\xi}^i(s^{'}), \tag{3.61}$$

where the kernel has dimensions of $1/\text{distance}^2$, so that

$$K(s - s^{'}) \sim \frac{1}{(s - s^{'})^2}. \tag{3.62}$$

As one can be arbitrarily close to a BPS solution by taking the deformation to vanish, no divergences are expected, so that the kernel must be expressed as a distribution, and the only distribution with the correct properties is given by

$$K(s - s^{'}) = \frac{d}{ds} \frac{P}{s - s^{'}}, \tag{3.63}$$

where P is the principal value distribution, which yields

$$I[\xi] = \int ds \int ds' \frac{(\dot{\xi}(s) - \dot{\xi}(s'))^2}{(s - s')^2}. \tag{3.64}$$

Defining $\mathcal{W}^{(n)}$ as the the loop expansion of order ξ^n in the $\mathcal{W}$ expression, we can relate $B(\lambda)$ to the wavy deformations via [11]

$$\frac{\langle \mathcal{W} \rangle^{(2)}}{\langle \mathcal{W} \rangle^{(0)}} = \pi^2 B(\lambda) I[\xi]. \tag{3.65}$$

This equation provides a way to check $B(\lambda)$ by taking a perturbative calculation in the CFT, which is the developments of chapter 5, where we carry out the calculations in attempt to reproduce the $B(\lambda)$ of ABJ(M) by a pure field theory calculation.

3.3.1 An $\mathcal{N} = 4$ SYM perturbative detour

The main goal of chapter 5 is to calculate the wavy deformation of the 1/2-BPS Wilson loop of ABJM and to relate it to the Bremsstrahlung function via (3.65). Before diving into that calculation, it is instructive to take a perturbative calculation of the deformation of the 1/2-BPS Wilson line of $\mathcal{N} = 4$ SYM at order λ, which is a well understood example, and see how one can achieve a perturbative expansion of (3.65) by solving the expectation value of the LHS perturbatively.

We begin by defining a wavy line $\mathcal{W}_{1/2\text{-BPS}}[\mathcal{C}']$ which is a deformation of the 1/2-BPS straight Wilson line of $\mathcal{N} = 4$,

$$\mathcal{W}_{1/2\text{-BPS}}[\mathcal{C}] = \frac{1}{N} \text{Tr}\, \mathcal{P} \exp \left[g \oint_{\mathcal{C}} ds \left(i A_\mu(x(s)) \dot{x}^\mu(s) + \Phi_i(x(s)) |\dot{x}(s)| \theta^i \right) \right], \tag{3.66}$$

where we have re-scaled the fields $A_\mu \to gA_\mu$ and $\Phi_i \to g\Phi$.

We need to plug (3.66) in expression (3.65) keeping contractions up to order λ and expand the result at order ξ^2 in the deformation. As $\mathcal{W}^{(0)}$ is the undeformed 1/2-BPS Wilson line, its expectation value is given by

$$\langle \mathcal{W}^{(0)} \rangle = 1. \tag{3.67}$$

By calculating $\mathcal{W}^{(2)}$ perturbatively, we can evaluate LHS of (3.65) as a power series in λ, allowing for an identification of the $B(\lambda)$.

To introduce the deformed contour $\mathcal{C}'$, we make $x^\mu \to x^\mu + \xi^\mu$ in (3.66), and to carry out the calculation in order λ, we must expand the loop exponential up to the second order,

$$\langle \mathcal{W}_{1/2\text{-BPS}}[\mathcal{C}'] \rangle = 1 + g \oint ds \langle \mathcal{L}(s) \rangle + \frac{g^2}{2} \oint dsds' \langle \mathcal{L}(s)\mathcal{L}(s') \rangle + \mathcal{O}(\lambda^2). \tag{3.68}$$

As the term in order g contains only 1-point functions, it vanishes, and we are left only with the g^2 term which is of order λ,

$$\frac{1}{N}\frac{g^2}{2}\text{Tr} \oint dsds' \langle (iA_\mu(y)\dot{y}^\mu + \Phi_i(y)|\dot{y}|\theta^i)(iA_\nu(z)\dot{z}^\nu + \Phi_j(z)|\dot{z}|\theta^j) \rangle, \tag{3.69}$$

where we have defined $y \equiv y(s)$ and $z \equiv z(s')$. In the rescaled fields, the propagators are independent of g,

$$\langle A^a_\mu(y) A^b_\nu(z) \rangle = \frac{1}{4\pi^2}\frac{\delta^{ab}\delta_{\mu\nu}}{|y-z|^2},$$
$$\langle \Phi^a_i(y) \Phi^b_j(z) \rangle = \frac{1}{4\pi^2}\frac{\delta^{ab}\delta_{ij}}{|y-z|^2},$$

so that we have[4]

$$\langle \mathcal{W}_{1/2\text{-BPS}}[\mathcal{C}'] \rangle = 1 + \frac{1}{N}\frac{g^2}{2}\frac{1}{4\pi^2}\frac{(N^2-1)}{2}\text{Tr} \oint dsds' \left(\frac{\theta^i\theta^j\delta_{ij}|\dot{y}||\dot{z}| - \dot{y}\cdot\dot{z}}{|y-z|^2} \right). \tag{3.70}$$

Expanding everything in order ξ^2, with the aid of

$$|\dot{y}(s)| = \sqrt{1 + \dot{\xi}(s)\cdot\dot{\xi}(s)},$$
$$|y(s) - z(s')|^2 = (s-s')^2 + (\xi(s) - \xi(s'))^2,$$
$$\dot{y}(s)\cdot\dot{z}(s') = 1 + \dot{\xi}(s)\cdot\dot{\xi}(s'),$$

we recover, in the large N limit, exactly the behaviour (3.64),

$$\langle \mathcal{W}^{(2)}_{1/2\text{-BPS}} \rangle = \frac{g^2 N}{16\pi^2} \oint dsds' \frac{(\dot{\xi}(s) - \dot{\xi}(s'))^2}{(s-s')^2} + \mathcal{O}(\lambda^2). \tag{3.71}$$

[4] We have $\Phi_i = \Phi^a_i T^a$ and $A_\mu = A^a_\mu T^a$, and with the normalization $\text{Tr}\,(T^a T^b) = \delta^{ab}/2$.

If one extends this calculation to higher orders in the coupling constant λ, one recovers [11]

$$\langle \mathcal{W}^{(2)}_{1/2\text{-BPS}} \rangle = \left(\frac{\lambda}{16\pi^2} - \frac{\lambda^2}{384\pi^2} \right) \oint ds ds' \frac{(\dot{\xi}(s) - \dot{\xi}(s'))^2}{(s-s')^2} + \mathcal{O}(\lambda^3). \tag{3.72}$$

From the exact $B(\lambda)$, one finds

$$B(\lambda) = \frac{\lambda}{16\pi^2} - \frac{\lambda^2}{384\pi^2} + \frac{\lambda^3}{6144\pi^2} - \frac{\lambda^4}{92160\pi^2}, \tag{3.73}$$

which matches the perturbative calculation.

In chapter 5, we present our approach to recovering the analog procedure for the circular 1/2-BPS Wilson loop of ABJ(M), which is a more intricate problem, since the wavy-line prescription must account for a deformation in the fermionic couplings.

Chapter 4

Wilson loops in ABJ(M)

In chapter 3 we have presented the original formulations of the 1/6-BPS bosonic and the 1/2-BPS fermionic loops of ABJM. The construction of the 1/2-BPS loops relies on the τ-dependent grassman even parameters (3.34) that couple the loop to fermions, and also in the twist matrix $\mathcal{T}$ to recover gauge invariance. These shortcomings are addressed in a new formulation of the BPS operators developed in conjunction with M. Tenser, M. Trépanier and M. Probst. This new formulation was presented in chapter 2 of [26], which is a state-of-the-art review on Wilson loops in CSm theories.

We can trace the necessity of the twist matrix precisely back to the fact that $\eta_I^\alpha(\tau)$ and $\bar{\eta}_\alpha^I(\tau)$ are 2π-anti-periodic. Using a gauge transformation of $\mathfrak{u}(N|M)$, we can define new couplings which are 2π-periodic, avoiding the need for a twist matrix, so that the Wilson loop is naturally gauge invariant in a formulation with the supertrace. In doing so, it is possible to relate the new couplings to the projectors (3.11), allowing for a manifestly reparametrization invariant definition thereof.

With this new formulation in hand, it is possible to construct a family of fermionic 1/6-BPS Wilson loops parameterized by grassman $\mathbb{C}$-numbers which contains the 1/2-BPS loop as a special case. This is a new family of Wilson loops and give a concrete realization of the cohomology of 1/6-BPS Wilson operators in ABJ(M) (3.49). The underlying structure also highlights the role of the breaking of the $\mathfrak{su}(4)$ R-symmetrty of ABJM by these operators and provide an identification of the moduli space of the new loops with a conical singular space, which was not possible before.

4.1 A new formulation of 1/2-BPS loops

The main idea is to propose a $U(1) \times U(1) \subset U(N|M)$ gauge transformation which turns the fermionic couplings into 2π-periodic functions at the same time "untwisting" the loop connection.

Under a general $U(N) \times U(M)$ transformation, the fields of ABJ(M) transform as

$$A_\mu^{(1)} \to U_1 A_\mu^{(1)} U_1^\dagger - iU_1 \partial_\mu U_1^\dagger, \tag{4.1}$$
$$A_\mu^{(2)} \to U_2 A_\mu^{(2)} U_2^\dagger - iU_2 \partial_\mu U_2^\dagger, \tag{4.2}$$
$$\Phi \to U_1 \Phi U_2^\dagger, \tag{4.3}$$
$$\bar{\Phi} \to U_2 \bar{\Phi} U_1^\dagger, \tag{4.4}$$

where Φ is any field in the bi-fundamental representation, and $\bar{\Phi}$ is any field in the anti-bi-fundamental, with U_1 a general element of the $U(N)$ group, and U_2 a general element of $U(M)$.

The proposed gauge transformation is

$$U_1 = e^{-i\Lambda(\tau)}, \tag{4.5}$$
$$U_2 = e^{i\Lambda(\tau)}, \tag{4.6}$$

with

$$\Lambda(\tau) = \frac{(2\tau - \pi)}{8} + \frac{\pi}{2}\Theta(\tau - 2\pi), \tag{4.7}$$

where Θ is the Heaviside Theta function.

Implementing this gauge transformation in the connection $\mathcal{L}_{1/2\text{-BPS}}$ (3.32), we have for the dressed connection $\mathcal{A}_{\text{bos}}^{(1)}$

$$\begin{aligned}\mathcal{A}_{\text{bos}}^{(1)} \to \quad & U_1 A_\mu^{(1)} \dot{x}^\mu U_1^\dagger - iU_1 \partial_\tau U_1^\dagger - \frac{2\pi i}{k} M_J^I U_1 C_I U_2 U_2^\dagger \bar{C}^J U_1^\dagger & (4.8)\\ \to \quad & A_\mu^{(1)} \dot{x}^\mu - \frac{2\pi i}{k} M_J^I C_I \bar{C}^J + \partial_\tau \Lambda(\tau) & (4.9)\\ \to \quad & \mathcal{A}_{\text{bos}}^{(1)} + \partial_\tau \Lambda(\tau). & (4.10)\end{aligned}$$

From the first to second line, we use the fact that the elements (4.6) are diagonal, so they commute with the charged fields.

Analogously, for the second dressed connection $\mathcal{A}_{\text{bos}}^{(2)}$, we have

$$\mathcal{A}_{\text{bos}}^{(2)} \to \quad \mathcal{A}_{\text{bos}}^{(2)} - \partial_\tau \Lambda(\tau). \tag{4.11}$$

Following the transformation laws, the fermionic entries of the superconnection pick up a phase factor coming from the gauge transformation of the fermions,

$$\psi_I^\alpha \to \quad \sqrt{-i} e^{\frac{i\tau}{2}} \psi_I^\alpha \quad \text{and} \quad \bar{\psi}_\alpha^I \to \quad \sqrt{i} e^{-\frac{i\tau}{2}} \bar{\psi}_\alpha^I, \tag{4.12}$$

so that in this new gauge, the 1/2-BPS connection (4.13) reads

$$\mathcal{L}' = \begin{pmatrix} \mathcal{A}_{\text{bos}}^{\prime(1)} & \sqrt{\frac{-4\pi i}{k}} |\dot{x}| \eta_I^{\prime\alpha} \bar{\psi}_\alpha^I \\ \sqrt{\frac{-4\pi i}{k}} |\dot{x}| \psi_I^\alpha \bar{\eta}_\alpha^{\prime I} & \mathcal{A}_{\text{bos}}^{\prime(2)} \end{pmatrix}, \tag{4.13}$$

with the gauge transformed dressed connections

$$\begin{aligned}\mathcal{A}_{\text{bos}}^{\prime(1)} &= A_\mu^{(1)} \dot{x}^\mu - \tfrac{2\pi i}{k} |\dot{x}| M_J^I C_I \bar{C}^J + \tfrac{1}{4} + \tfrac{\pi}{2} \delta(\tau - 2\pi),\\ \mathcal{A}_{\text{bos}}^{\prime(2)} &= A_\mu^{(2)} \dot{x}^\mu - \tfrac{2\pi i}{k} |\dot{x}| M_J^I \bar{C}^J C_I - \tfrac{1}{4} - \tfrac{\pi}{2} \delta(\tau - 2\pi).\end{aligned} \tag{4.14}$$

The 1/4 pieces come from the continuous term in (4.7), whereas the delta distributions arise from the derivative of the Θ function. The original fermionic couplings η_I^α and $\bar{\eta}_\alpha^I$ absorb the τ dependent part of the transformation and an extra $1/\sqrt{2}$ factor to be written as

$$\eta'^{\alpha}_{I}(\tau) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -ie^{-i\tau}\end{pmatrix}\delta_I^1\,, \qquad \bar{\eta}'^{I}_{\alpha}(\tau) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ ie^{i\tau}\end{pmatrix}\delta_1^I. \tag{4.15}$$

With the loop connection cast as (4.13), we can inspect that the contributions from the δ functions in the transformed dressed connections annihilate the twist matrix $\mathcal{T}$ of the gauge invariant quantity

$$\mathcal{W} = \mathrm{sTr}\left(\mathcal{P}\exp\left(i\oint\mathcal{L}(\tau)\right)\mathcal{T}\right). \tag{4.16}$$

To see that, we can separate their contribution from the the connection, since they sit inside a path ordering operator. When integrated over the circular contour, as they are only supported at 2π, we have the contribution

$$\exp\left[\int_{2\pi-\epsilon}^{2\pi+\epsilon} d\tau\frac{i\pi}{2}\begin{pmatrix}\delta(\tau-2\pi) & 0\\ 0 & -\delta(\tau-2\pi)\end{pmatrix}\right] = i\sigma^3 = \mathcal{T}^{-1}, \tag{4.17}$$

which precisely cancels $\mathcal{T}$ from (4.16). This way, the loop can be naturally expressed in terms of the sTr operation.

So far, our new gauge has two important effects: it introduces a constant piece in the dressed connections and it untwists the original loop connection. In this new gauge, we can write the superconnection in a manifestly reparameterization invariant fashion.

To do that, we notice that the $\Pi_\pm$ projectors of (3.11) take the explicit form

$$\Pi_\pm = \frac{1}{2}\begin{pmatrix}1 & \mp ie^{-i\tau}\\ \pm ie^{i\tau} & 1\end{pmatrix}. \tag{4.18}$$

Since the projectors are written in a reparameterization invariant fashion, and we can read the fermionic couplings η' and $\bar{\eta}'$ from its first row and column, we can define the fermionic couplings in a manifestly reparameterization invariant way by defining them as

$$\eta'^{\alpha}_{I} = \sqrt{2}(s\Pi_+)^\alpha\delta_I^1 \quad \text{and} \quad \bar{\eta}'^{I}_{\alpha} = \sqrt{2}(\Pi_+\bar{s})_\alpha\delta_1^I, \tag{4.19}$$

where $s^\alpha = (1,0)$.

Now we can drop the primes in the notation and forget that we have performed a gauge transformation in the first place, so we are able to define the Wilson loop on its own right as

$$\mathcal{W} = (i)\mathrm{sTr}\,\mathcal{P}\exp\left(i\oint\mathcal{L}_{1/2\text{-BPS}}d\tau\right), \tag{4.20}$$

with the superconnection now given by

$$\mathcal{L}_{1/2\text{-BPS}} = \begin{pmatrix}\mathcal{A}^{(1)} & \sqrt{-\frac{4\pi i}{k}}|\dot{x}|\eta_I^\alpha\bar{\psi}_\alpha^I\\ \sqrt{-\frac{4\pi i}{k}}|\dot{x}|\psi_I^\alpha\bar{\eta}_\alpha^I & \mathcal{A}^{(2)}\end{pmatrix}, \tag{4.21}$$

with

$$\begin{aligned} \mathcal{A}^{(1)} &= A^{(1)}_\mu \dot{x}^\mu - \tfrac{2\pi i}{k}|\dot{x}| M_J^I C_I \bar{C}^J - \tfrac{|\dot{x}|}{4|x|}\,, \\ \mathcal{A}^{(2)} &= A^{(2)}_\mu \dot{x}^\mu - \tfrac{2\pi i}{k}|\dot{x}| M_J^I \bar{C}^J C_I + \tfrac{|\dot{x}|}{4|x|}\,, \end{aligned} \tag{4.22}$$

and

$$\eta_I^\alpha = \sqrt{2}(s\Pi_+)^\alpha \delta_I^1, \qquad \bar{\eta}_\alpha^I = \sqrt{2}(\Pi_+ \bar{s})_\alpha \delta_1^I, \qquad M_J^I = \mathrm{diag}(-1,1,1,1), \tag{4.23}$$

thus achieving a formulation which is fully reparameterization invariant, and naturally written in the language of the superconnection via the sTr operation, without the need of an ad-hoc twist matrix.

We notice the particular feature of the $\pm 1/4$ constant pieces in the connection, which are the product of the gauge transformation. These constant pieces are related to a particular feature of the supersymmetry transformation of the fermions which are not expressed in terms of the Killing spinors, but are only written in terms of the superconformal charges. In fact, as shown in Appendix A, they play an essential role in the supersymmetry proof in the new gauge.

As we have simply gauge transformed a 1/2-BPS gauge invariant operator, it is guaranteed that in the operator still is a 1/2-BPS solution to (1.2), nonetheless, an explicit proof of supersymmetry in this new gauge frame is given in Appendix A.

4.2 A new family of 1/6-BPS loops

As discussed in chapter 3, ABJ(M) theories have a great moduli space of BPS Wilson loops, which encompasses the 1/6-BPS bosonic loops as well as the 1/2-BPS fermionic loops.

Within the new formulation of the 1/2-BPS loop, presented last section, it is possible to show that the 1/6-BPS bosonic loops are related by the 1/2-BPS fermionic loops via a deformation of their super-connection. By constructing such a deformation, one defines a family of Wilson loop operators that is fermionic and generically 1/6-BPS containing the 1/2-BPS fermionic loop. The construction is the field realization of the cohomological statement that the 1/2-BPS fermionic loops are related to the 1/6-BPS via a $\mathcal{Q}$-exact term.

Recall that the fermionic 1/2-BPS loops break the R-symmetry of ABJ(M) as $\mathfrak{su}(4) \to \mathfrak{u}(1) \oplus \mathfrak{su}(3)$, while the 1/6-BPS bosonic loops as $\mathfrak{su}(4) \to \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The key ingredient in defining this new family of fermionic Wilson loops is to come up with a mechanism that is capable of controlling the R-symmetry of the loop, interpolating between the two configurations.

In fact, what controls the breaking of R-symmetry is the coupling to the scalar bilinears M_J^I, so that our mechanism needs to be able to control the eigenvalues of this matrix. This can be achieved by considering grassman-odd $\mathbb{C}$-valued parameters α^i, which are charged under an $\mathfrak{su}(2)$ subsector of the original $\mathfrak{su}(4)$ R-symmetry, i.e $i = 1, 2$. With these parameters we can define a deformation which transitions from the 1/6-BPS configuration to the 1/2-BPS one, by engineering a super-matrix coupling to the $C_1, C_2, \bar{C}^1, \bar{C}^2$ scalars.

We begin by noticing that the preserved charges of the 1/6-BPS loop are also preserved by the 1/2-BPS loop,

$$\bar{\theta}^{12} \quad \text{and} \quad \bar{\theta}^{34} = \theta_{12}, \tag{4.24}$$

and the preserved superconformal charges are

$$\bar{\vartheta}^{12} = i\bar{\theta}^{12}\sigma^3 \quad \text{and} \quad \bar{\vartheta}^{34} = -i\bar{\theta}^{34}\sigma^3. \tag{4.25}$$

We introduce the deformation of the bosonic gauge connection

$$\mathcal{L} = \mathcal{L}_{1/6\text{-bos}} + \Delta\mathcal{L}, \tag{4.26}$$

where $\mathcal{L}_{1/6\text{-bos}}$ is the connection

$$\mathcal{L}_{1/6\text{-bos}} = \begin{pmatrix} \mathcal{A}^{(1)}_{\text{bos}} & 0 \\ 0 & \mathcal{A}^{(2)}_{\text{bos}} \end{pmatrix}, \tag{4.27}$$

with the dressed connections given by (3.6) and (3.6). The deformation is defined as

$$\Delta\mathcal{L} = i\sigma^3\delta_+\mathcal{G} - 2\sigma^3\mathcal{G}^2 + \sigma^3\frac{1}{4}, \tag{4.28}$$

where the $\mathcal{G}$ matrix is parameterized by the grassman-odd parameters $\alpha^a \in \mathbb{C}$

$$\mathcal{G} = \sqrt{\frac{2i\pi}{k}} \begin{pmatrix} 0 & \bar{\alpha}^a C_a \\ -\alpha_a \bar{C}^a & 0 \end{pmatrix}. \tag{4.29}$$

We denote lowercase Latin letters for the $1,2$ indices of the $\mathfrak{su}(4)$, which defines an $\mathfrak{su}(2)$ subspace, so that the matrix $\mathcal{G}$ contains the scalars $C_1, C_2, \bar{C}^1, \bar{C}^2$.

We define the variations $\delta\pm$ as being generated from two independent combinations of the four parameters $\bar{\theta}^{12}$ and $\bar{\theta}^{34}$, such that

$$\delta_+ = \bar{\theta}^{12}_+\mathcal{Q}^+_{12} + \bar{\theta}^{34}_-\mathcal{Q}^-_{34} + \bar{\vartheta}^{12}_+\mathcal{S}^+_{12} + \bar{\vartheta}^{34}_-\mathcal{S}^-_{34}, \tag{4.30}$$
$$\delta_- = \bar{\theta}^{12}_-\mathcal{Q}^-_{12} + \bar{\theta}^{34}_+\mathcal{Q}^+_{34} + \bar{\vartheta}^{12}_-\mathcal{S}^-_{12} + \bar{\vartheta}^{34}_+\mathcal{S}^+_{34}. \tag{4.31}$$

The reason to call first variation δ_+ and the second δ_- is that we can use the conjugate notation of charges

$$\bar{\theta}^{34}_- = \theta^+_{12} \quad \text{and} \quad \bar{\theta}^{34}_+ = \theta^-_{12}, \tag{4.32}$$

so that $\delta_\pm$ is parameterized by $\pm$ charges.

Noticing that these superchages amount to 1/6 of the supercharges preserved by ABJ(M), the 1/6-BPS condition can be expressed by

$$\delta_\pm\mathcal{L} = \mathfrak{D}_\tau G, \tag{4.33}$$

where G is some $U(N|M)$ supermatrix.[1]

[1] G does not necessarily need to be $\mathcal{G}$, or related to it at all. But in our construction we'll have $G \sim \mathcal{G}$.

Let's write the explicit form of the deformed connection by evaluating the deformation (4.28) and plugging it back in (4.26). We first evaluate the piece $i\sigma^3\delta_+\mathcal{G}$ of the deformation,

$$i\sigma^3\delta_+\mathcal{G} = i\sqrt{\frac{2i\pi}{k}}\begin{pmatrix} 0 & \bar{\alpha}^a\delta_+ C_a \\ \alpha_a\delta_+\bar{C}^a & 0 \end{pmatrix}. \tag{4.34}$$

To carry out the calculation we need to evaluate the Killing spinors associated to the δ_+ variation, which is sourced by the independent Poincaré parameters in (4.30). Via (3.4), we have

$$\bar{\Theta}^{12}_\alpha = 2(\Pi_+)_\alpha^{\ +}\bar{\theta}^{12}_+, \tag{4.35}$$

$$\bar{\Theta}^{34}_\alpha = 2(\Pi_-)_\alpha^{\ -}\bar{\theta}^{34}_-. \tag{4.36}$$

As we are interested in twisting only the $\mathfrak{su}(2)$ subspace of the R-symmetry, we can recover a notation containing only the index "$a = 1, 2$" by conjugating the R-symmetry indices as

$$\bar{\Theta}^{34}_\alpha = \Theta_{12\,\alpha}, \tag{4.37}$$

so that we now work with the Killing spinors

$$\bar{\Theta}^{12} \quad \text{and} \quad \Theta_{12}. \tag{4.38}$$

With these spinors, we can evaluate (4.34) as

$$i\sigma^3\delta_+\mathcal{G} = (2i)\sqrt{\frac{2i\pi}{k}}\begin{pmatrix} 0 & \bar{\alpha}^a\Theta^\alpha_{ab}\bar{\psi}^b_\alpha \\ \alpha_a\bar{\Theta}^{ab\alpha}\psi_{b\alpha} & 0 \end{pmatrix}. \tag{4.39}$$

This contribution will be responsible for the coupling of fermions, since they are purely off-diagonal.

The piece containing the square $\sim \mathcal{G}^2$, will be responsible for the diagonal pieces, since it involves the square of an off-diagonal matrix. It evaluates to

$$-2\sigma^3\mathcal{G}^2 = \frac{2\pi i}{k}\begin{pmatrix} (2\bar{\alpha}^a\alpha_b)(C_a\bar{C}^b) & 0 \\ 0 & (2\bar{\alpha}^a\alpha_b)(\bar{C}^bC_a) \end{pmatrix}. \tag{4.40}$$

With expressions (4.39), (4.40), we can evaluate the deformed superconnection (4.26) as

$$\mathcal{L} = \begin{pmatrix} \mathcal{A}^{(1)}_{\text{bos}} + \frac{2\pi i}{k}(2\bar{\alpha}^a\alpha_b)(C_a\bar{C}^b) + \frac{1}{4} & (2i)\sqrt{\frac{2i\pi}{k}}\bar{\alpha}^a\Theta^\alpha_{ab}\bar{\psi}^b_\alpha \\ (2i)\sqrt{\frac{2i\pi}{k}}\alpha_a\bar{\Theta}^{ab\alpha}\psi_{b\alpha} & \mathcal{A}^{(2)}_{\text{bos}} + \frac{2\pi i}{k}(2\bar{\alpha}^a\alpha_b)(\bar{C}^bC_a) - \frac{1}{4} \end{pmatrix}. \tag{4.41}$$

This superconnection defines a family of 1/6-BPS solutions parameterized by the grassmann coordinates α^a, and the BPS condition (4.33) can be checked by evaluating the action of the δ_+ and δ_- variations independently,

$$\delta_+\mathcal{L} = 2\mathfrak{D}_\tau\mathcal{G}, \tag{4.42}$$

$$\delta_-\mathcal{L} = 2\mathfrak{D}_\tau(e^{2i\Lambda\sigma^2}\mathcal{G}), \quad \text{with} \quad \Lambda = -\left(\tau + \frac{\pi}{2}\right). \tag{4.43}$$

With the explicit superconnection (4.41), we can tune the parameters α^a such that the connection collapses to the 1/2-BPS solution in the new gauge (4.13). In order to see that we first notice that we can cast (4.41) as

$$\mathcal{L} = \begin{pmatrix} \mathcal{A}^{(1)} & \sqrt{-\frac{4\pi i}{k}}|\dot{x}|\eta_a^\alpha \bar{\psi}_\alpha^a \\ \sqrt{-\frac{4\pi i}{k}}|\dot{x}|\psi_a^\alpha \bar{\eta}_\alpha^a & \mathcal{A}^{(2)} \end{pmatrix}, \tag{4.44}$$

identifying the deformed bosonic connections as

$$\begin{aligned} \mathcal{A}^{(1)} &= \mathcal{A}^{(1)}_{\text{bos}} + \tfrac{2\pi i}{k}|\dot{x}|\Delta M_b^a C_a \bar{C}^b + \tfrac{|\dot{x}|}{4|x|}\,, \\ \mathcal{A}^{(2)} &= \mathcal{A}^{(2)}_{\text{bos}} + \tfrac{2\pi i}{k}|\dot{x}|\Delta M_b^a \bar{C}^a C_b - \tfrac{|\dot{x}|}{4|x|}\,, \end{aligned} \tag{4.45}$$

where the couplings are

$$\eta_b = \sqrt{2}\bar{\alpha}^a \Theta^\alpha_{ab}, \qquad \bar{\eta}^b = \sqrt{2}\bar{\Theta}^{ba}_\alpha \alpha_a, \qquad \Delta M_b^a = 2\bar{\alpha}^a \alpha_b. \tag{4.46}$$

The ΔM piece is crucial to the enhancement of supersymmetry to 1/2-BPS, s this term changes the residual R-symmetry of the loop, effectively enhancing its supersymmetry to 1/2-BPS. The enhancement of supersymmetry can be seen as coming from a particular choice of the family parameters α^a, such that the matrix ΔM has eigenvalues 0 and -2. As it can be directly inspected, this deformation will take

$$M + \Delta M \to \text{diag}(-1, 1, 1, 1), \tag{4.47}$$

which corresponds to changing the R-symmtrey configuration,

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \to \mathfrak{u}(1) \oplus \mathfrak{su}(3). \tag{4.48}$$

Schematically,

$$\underbrace{\mathcal{L}_{\text{bos}} + \Delta\mathcal{L}(\bar{\alpha}^i, \alpha_i)}_{\text{general 1/6-BPS}} \quad \xrightarrow[\text{eigen 0,-2}]{} \quad \mathcal{L}_{\text{1/2-BPS}}. \tag{4.49}$$

There are still interesting questions regarding the moduli space of 1/6-BPS Wilson operators in ABJ(M), namely, if this family encompasses all 1/6-BPS loops, or only a subset of them. For instance other 1/6-BPS loops were already constructed in [27], and it is still to be understood what is the relation of their work with ours.

A nice new feature of the formulation of fermionic loops as the deformation of a bosonic connection is the identification of the moduli space of operators. To identify this space we notice that we have parameterized the super-connection of the 1/6-BPS fermionic family by four independent complex parameters α^i and $\bar{\alpha}^i$, so that the moduli space of connections is $\mathbb{C}^4$. The space of loop operators is a subset of the space of connections, since any two connections related by a gauge transformation define the same loop operator.

Under a constant gauge transformation $U_1 = e^{i\Lambda}$, we have that the connection (4.41) is invariant in its diagonal entries and the fermions acquire a phase

$$\mathcal{L} \to \begin{pmatrix} \mathcal{A}^{(1)}_{\text{bos}} + \frac{2\pi i}{k}(2\bar{\alpha}^a\alpha_b)(C_a\bar{C}^b) + \frac{1}{4} & (2i)\sqrt{\frac{2i\pi}{k}}\bar{\alpha}^a e^{i\Lambda}\Theta^\alpha_{ab}\bar{\psi}^b_\alpha \\ (2i)\sqrt{\frac{2i\pi}{k}}\alpha_a e^{-i\Lambda}\bar{\Theta}^{ab\alpha}\psi_{b\alpha} & \mathcal{A}^{(2)}_{\text{bos}} + \frac{2\pi i}{k}(2\bar{\alpha}^a\alpha_b)(\bar{C}^b C_a) - \frac{1}{4} \end{pmatrix}. \tag{4.50}$$

Defining the new parameters

$$\tilde{\alpha}_a = e^{-i\Lambda}\alpha_a \quad \bar{\tilde{\alpha}}^b = e^{i\Lambda}\bar{\alpha}^b, \tag{4.51}$$

we can write the gauge transformed connection as

$$\mathcal{L} = \begin{pmatrix} \mathcal{A}^{(1)}_{\text{bos}} + \frac{2\pi i}{k}(2\bar{\tilde{\alpha}}^a\tilde{\alpha}_b)(C_a\bar{C}^b) + \frac{1}{4} & (2i)\sqrt{\frac{2i\pi}{k}}\bar{\tilde{\alpha}}^a\Theta^{\alpha}_{ab}\bar{\psi}^b_{\alpha} \\ (2i)\sqrt{\frac{2i\pi}{k}}\tilde{\alpha}_a\bar{\Theta}^{ab\alpha}\psi_{b\alpha} & \mathcal{A}^{(2)}_{\text{bos}} + \frac{2\pi i}{k}(2\bar{\tilde{\alpha}}^a\tilde{\alpha}_b)(\bar{C}^bC_a) - \frac{1}{4} \end{pmatrix}, \tag{4.52}$$

so that this gauge transformation maps

$$\mathcal{L}(\bar{\alpha}^a, \alpha_b) \to \mathcal{L}(\bar{\tilde{\alpha}}^b, \tilde{\alpha}_a), \tag{4.53}$$

which means that a constant gauge transformation reflects as a rescaling of the family parameters such that their product is invariant. Modding out the $\mathbb{C}^*$ action of these gauge transformations, we recover the moduli sapce of loops as $\mathbb{C}^4/\mathbb{C}^*$, which is the space of singular complex matrices known as the *conifold.* For an explicit proof of supersymmetry in this gauge, refer to Appendix A.

Chapter 5

Deformations of the 1/2-BPS WL in ABJM

The study of BPS Wilson loops is highly motivated by the exact results coming from localization, which provides us with weak coupling expansions, such as (3.50), that can be matched by field theory calculations via Feynman diagrams, and strong coupling predictions that can be attained holographically by the calculation of minimal surfaces.

Localization results are given in terms of one-dimensional matrix models, and perturbation theory is developed in terms of standard Feynman diagram calculations. Since these techniques operate under quite distinct frameworks, it is a relevant issue to match calculations coming from these two methods. Such matches are far from trivial in the context of BPS Wilson loops in CSm theories, since both perturbation theory and exact results are filled with intricacies.

One such particularity of Wilson loops in this context arises from the issue of *loop framing*, which can be understood as a regularization procedure for calculating integrals relating to loop expansions in Chern-Simons theory.[1]

To illustrate this point, we can introduce the simplest set-up provided by the expectation value of a Wilson loop around a general contour $\mathcal{C}$ in a pure Chern-Simons theory (2.1),

$$\langle \mathcal{W} \rangle = \langle \mathrm{Tr}\, \mathcal{P} \exp \left(i \oint_{\mathcal{C}} A_\mu dx^\mu \right) \rangle. \tag{5.1}$$

At leading order in the gauge coupling, the only contribution is given by a gauge propagator integrated over the loop contour,

$$\langle \mathcal{W} \rangle |_\lambda \sim \oint_{\mathcal{C}} d\tau_1 d\tau_2 \langle A_\mu(x_1) A_\nu(x_2) \dot{x_1}^\mu \dot{x_2}^\nu \rangle. \tag{5.2}$$

As in general, short distances divergences occur when the insertions on the loop collide $x_1 \to x_2$, a standard regularization procedure is to displace the contour of one of the insertions from $\mathcal{C} \to \mathcal{C}'$, such that

$$x_2^\mu(\tau) = x_1^\mu(\tau) + \delta n^\mu(\tau), \tag{5.3}$$

[1] For a review of loop framing, see chapter 7 of [26]

which when inserted in (5.2) generates a topological invariant known as the *Gauss linking number*

$$\langle \mathcal{W} \rangle|_\lambda \sim \oint_{\mathcal{C}} dx_1^\mu \oint_{\mathcal{C}'} dx_2^\nu \epsilon_{\mu\nu\rho} \frac{(x_1 - x_2)^\rho}{|x_1 - x_2|^3} = \text{Link}(\mathcal{C}, \mathcal{C}'), \tag{5.4}$$

such that this expectation value depends explicitly on the choice of framing (5.3). Thus, in order to define parturbation theory in Chern-Simons theory, one needs the extra data of framing.

Pure field theory calculations in ABJM theory are attained via a standard perturbative set-up called *dimensional regularization with dimensional reduction* (DRED), which operates at *framing zero*. Differently, localization results often operate under a different choice of framing, making the match of observables from the matrix model computations with perturbation non-trivial, since one needs to account for the difference in framing.

Apart from providing non-trivial checks of localization results, the perturbative calculation of BPS operators also provides a clearer understanding of the mechanisms of cancellation of UV divergences of these operators. The first match of the localization results of [10] was given by [23], where the 1/2-BPS loop was calculated at order λ^2 via the DRED scheme, and once the framing factors were identified, it was possible to confirm the localization result (3.50).

In what follows, we want to push forward the perturbative calculations of such operator by considering a λ^2 perturbative calculation in the wavy-line deformation. This way, one could also match the perturbative calculations with a localization prediction for the Bremsstrahlung function [24]

$$B(\lambda)_{\frac{1}{2}} = \frac{\lambda}{8} - \frac{\pi^2 \lambda^3}{48} + \mathcal{O}(\lambda^5). \tag{5.5}$$

Our calculations are mainly based on the same DRED scheme, and our diagrams are the same as the ones in [23], but now we introduce a deformation in the loop contour to follow the wavy-line prescription for the calculation of the Bremmstrahlung function.

The observable that we calculate is defined as the insertion of the loop operator (3.2) with the deformed contour (5.7) in the path integral

$$\langle \mathcal{W}[\mathcal{C}, \eta, \bar{\eta}, M] \rangle = \frac{1}{M+N} \int \mathcal{D}[A_\mu^{(1)}, A_\mu^{(2)}, \psi, \bar{\psi}, C, \bar{C}] e^{-S_{\text{ABJM}}} \text{Tr}\, \mathcal{P} \left(e^{-i \oint \mathcal{L}(\tau) d\tau} \right), \tag{5.6}$$

with the action of ABJ(M) (2.6), and we explicitly write the dependence of the operator on the matter couplings and the contour of integration $\mathcal{C}$, which takes the form

$$\mathcal{C}: \quad x^\mu(\tau) = (0, e^{g(\tau)} \cos(\tau), e^{g(\tau)} \sin(\tau)), \tag{5.7}$$

where we can recover the original circular contour for $g(\tau) = 0$. As the loop is periodic, it is convenient to expand $g(\tau)$ in its Fourier components, and parameterize the deformation by its modes b_n

$$g(\tau) = \sum_{n=-\infty}^{\infty} b_n e^{in\tau}. \tag{5.8}$$

For infinitesimal b_n no cusps nor self intersections appear in the contour, which discards extra logarithimic divergences. At order $\mathcal{O}(b_n^2)$ the expectation value of the Wilson operator defines the wavy-circle approximation, such that (5.8) defines a Fourier representation for the ξ parameter of (3.64).

To carry out the perturbative calculations, we need to expand the exponentials in terms of the loop connection. Naturally, the connection encodes the $\mathfrak{u}(N|M)$ structure, such that the diagonal entries transform in the adjoint of the quiver nodes, for that reason, the upper left block is referred to as the $N \times N$ block, and the lower right block is the $M \times M$. The upper off diagonal is the $N \times M$ block and the lower off-diagonal is the $M \times N$.

Since the definition of the operator contains a trace, for each order the non-trivial contributions are always given only by the adjoint blocks. Expanding the loop operator, we have[2]

$$\langle \mathcal{W} \rangle = 1 + \underbrace{(-i)\mathrm{Tr}\left(\oint d\tau \langle \mathcal{L}(\tau) \rangle\right)}_{\mathcal{W}^{(1)}} + \underbrace{(-i)^2\mathrm{Tr}\left(\oint_{\tau_1<\tau_2} d\tau_1 d\tau_2 \langle \mathcal{L}(\tau_1)\mathcal{L}(\tau_2) \rangle\right)}_{\mathcal{W}^{(2)}} + \cdots \tag{5.9}$$

For organization purposes, let us define the Wilson loop expansion as

$$\mathcal{W} = \sum_n \mathcal{W}^{(n)}, \tag{5.10}$$

with $\mathcal{W}^{(n)}$ containing "n" insertions of the connection $\mathcal{L}$. When the trace acts, only the diagonal blocks survive, so that we define $N \times N$ block of $\mathcal{W}^{(n)}$ as $B_N^{(n)}$ and the $M \times M$ block as $B_M^{(n)}$, this way

$$\mathrm{Tr}\,\mathcal{W} = \sum_n \mathrm{Tr}\,[B_N^{(n)} + B_M^{(n)}] \equiv \mathrm{Tr}\,[B_N + B_M]. \tag{5.11}$$

We'll always consider insertions of $\mathcal{W}^{(n)}$ inside a correlator, so we write contractions as operator products and omit the brackets.

5.1 Diagrams

In the calculations that follow, we notice that the sextic scalar terms and Yukawa couplings in (2.6) play no role in our calculations since, in our set-up they would contribute to higher orders in the perturbation parameter λ.

The first contribution to the expansion of the exponential is given by $\mathcal{W}^{(1)}$, which corresponds to the insertion of the connection in the path integral. The trace operation picks up the dressed connections $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, corresponding to the blocks $B_N^{(1)}$ and $B_M^{(1)}$ respectively,

$$\mathrm{Tr}\,\left(\mathcal{W}^1\right) = -i\mathrm{Tr}\,[B_N^{(1)} + B_M^{(1)}] = -i\mathrm{Tr}\oint_{\mathcal{C}'}\left(\mathcal{A}^{(1)} + \mathcal{A}^{(2)}\right). \tag{5.12}$$

[2]The path ordering $\mathcal{P}$ orders the integrals and cancels the $1/n!$ from the exponential.

Each insertion of the dressed connection translates to the insertion of a single gauge field $A^\mu(\tau)$ and a scalar bilinear $\sim C(\tau)\bar{C}(\tau)$. As we are working in the DRED scheme [28, 29], we can consistently discard the tadpoles which arise from the scalar bilinears. As ABJ(M) are fully conformal, the one point function of the gauge field also vanishes, so that

$$\langle \mathcal{W}^{(1)} \rangle = 0. \tag{5.13}$$

5.1.1 Order λ

The term contributing to $\mathcal{O}(\lambda)$ is $\mathcal{W}^{(2)}$, which contains two insertions of the connection $\mathcal{L}(\tau)$,

$$\mathcal{L}(\tau_1)\mathcal{L}(\tau_2) =$$
$$\begin{pmatrix} \mathcal{A}_1^{(1)}\mathcal{A}_2^{(1)} - \frac{2\pi}{k}|\dot{x}_1||\dot{x}_2|(\eta\bar{\psi})_1(\psi\bar{\eta})_2 & -i\sqrt{\frac{2\pi}{k}}\left(|\dot{x}_2|\mathcal{A}_1^{(1)}(\eta\bar{\psi})_2 + |\dot{x}_1|(\eta\bar{\psi})_1\mathcal{A}_2^{(2)}\right) \\ -i\sqrt{\frac{2\pi}{k}}\left(|\dot{x}_1|(\psi\bar{\eta})_1\mathcal{A}_2^{(1)} + |\dot{x}_2|\mathcal{A}_1^{(2)}(\psi\bar{\eta})_2\right) & \mathcal{A}_1^{(2)}\mathcal{A}_2^{(2)} - \frac{2\pi}{k}|\dot{x}_1||\dot{x}_2|(\psi\bar{\eta})_1(\eta\bar{\psi})_2 \end{pmatrix},$$

where numerical subscripts keep track of the curve parameters: $\phi_i = \phi(\tau_i)$. When the trace acts we have the contribution from $\mathcal{W}^{(2)}$,

$$\mathcal{W}^{(2)}|_\lambda \sim \mathrm{Tr}\,[B_N^{(2)} + B_M^{(2)}]|_\lambda. \tag{5.14}$$

Let's focus on $B_N^{(2)}$, since the computation for $B_M^{(2)}$ is completely analogous. Defining $x = x^\mu(\tau_1)$ and $y = y^\mu(\tau_2)$, we have

$$\begin{aligned} B_N &= \left(A_\mu^{(1)}(x)\dot{x}^\mu + \frac{2\pi}{k}|\dot{x}|M_J^I C_I\bar{C}^J(x)\right)\left(A_\nu^{(1)}(y)\dot{y}^\nu + \frac{2\pi}{k}|\dot{y}|M_J^I\bar{C}^J C_I(y)\right) \\ &\quad + \frac{2\pi}{k}|\dot{x}||\dot{y}|(\eta_I^\alpha\bar{\psi}_\alpha^I(x))(\psi_J^\beta(y)\bar{\eta}_\beta^J) \\ &= \underbrace{A_\mu^{(1)}(x)A_\nu^{(1)}(y)\dot{x}^\mu\dot{y}^\nu}_{\mathrm{I}} + \underbrace{\frac{2\pi}{k}A_\mu^{(1)}(x)\dot{x}^\mu|\dot{y}|M_J^I\bar{C}^J C_I(y) + \frac{2\pi}{k}\dot{y}^\nu|\dot{x}|M_J^I C_I\bar{C}^J(x)A_\nu^{(1)}(y)}_{\mathrm{II}} \\ &\quad + \underbrace{\frac{2\pi}{k}|\dot{x}||\dot{y}|(\eta_I^\alpha\bar{\psi}_\alpha^I(x))(\psi_J^\beta(y)\bar{\eta}_\beta^J)}_{\mathrm{III}}. \end{aligned}$$

We have in principle three contributions: a single gluon exchange coming from the gauge propagator in term (I), two scalar tadpoles coming from self contractions of term (II) and a single fermion exchange coming from term (III).

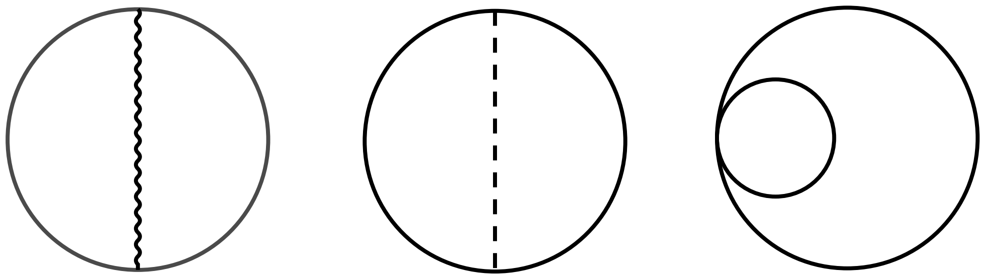
The gluon exchange vanishes due to the antisymmetry of the ϵ tensor, which is contracted with three vectors lying in the plane of the contour. The tadpole can be discarded in our regularization scheme, so that the only non-vanishing contribution comes from the single exchange of a fermion,

$$\begin{aligned} f_{B_M} &= \frac{2\pi}{k}|\dot{x}||\dot{y}|(\eta_I^\alpha\bar{\eta}_\beta^J)\bar{\psi}_\alpha^I(x)\psi_J^\beta(y) \\ &= -\left(\frac{2\pi i}{k}\right)\frac{\Gamma(3/2-\epsilon)}{2\pi^{3/2}}MN|\dot{x}||\dot{y}|(\bar{\eta}\gamma^\mu\eta)\frac{(x-y)_\mu}{((x-y)^2)^{3/2-\epsilon}}. \end{aligned}$$

As the block B_N also contributes with the same single fermion exchange, we have the $\mathcal{O}(\lambda)$ result

$$\mathcal{W}|_{\lambda} = -2MN\frac{\Gamma(3/2-\epsilon)}{2\pi^{3/2}}\left(\frac{2\pi i}{k}\right)\oint_{\tau_1>\tau_2} d\tau_1 d\tau_2 |\dot{x}||\dot{y}|(\bar{\eta}\gamma\eta)^{\mu}\frac{(x-y)_{\mu}}{((x-y)^2)^{3/2-\epsilon}}. \quad (5.15)$$

Diagramatically, the $\mathcal{O}(\lambda)$ contributions are



(single gluon) (single fermion) (scalar tadpole)

Figure 5.1: All contributions at $\mathcal{O}(\lambda)$, the only surviving diagram is the single fermion.

5.1.2 Order λ^2

The calculation at order λ^2 contains contributions of up to four insertions of the connection $\mathcal{L}$, which means we have to expand the loop operators up to fourth order in the exponential to pick up all contributions

$$\mathcal{W}|_{\lambda^2} = \underbrace{1+\mathcal{W}^{(1)}|_{\lambda^2}}_{0} + \mathcal{W}^{(2)}|_{\lambda^2} + \mathcal{W}^{(3)}|_{\lambda^2} + \mathcal{W}^{(4)}|_{\lambda^2}. \quad (5.16)$$

Expanding in terms of $\mathcal{W}^{(n)}$, the non-vanishing contributions are given by

$$\begin{aligned}\mathcal{W}|_{\lambda^2} = \quad & (-i)^2\oint_{\tau_1>\tau_2}\mathcal{L}(\tau_1)\mathcal{L}(\tau_2)|_{\lambda^2} + (-i)^3\oint_{\tau_1>\tau_2>\tau_3}\mathcal{L}(\tau_1)\mathcal{L}(\tau_2)\mathcal{L}(\tau_3)|_{\lambda^2}\\ + \quad & (-i)^4\oint_{\tau_1>\tau_2>\tau_3>\tau_4}\mathcal{L}(\tau_1)\mathcal{L}(\tau_2)\mathcal{L}(\tau_3)\mathcal{L}(\tau_4)|_{\lambda^2}. \end{aligned} \quad (5.17)$$

From the $\mathcal{W}^{(2)}$ term, we have the self-energy corrections of the single fermion exchange and the single gluon exchange, which correspond to diagrams (b) and (d) below. Diagram (c) also comes from this order, by contracting the scalar bilinears. From the $\mathcal{W}^{(3)}$ insertion we have terms $\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3$ in the diagonals, which contracted with the gauge cubic vertex gives rise to diagram (a), and also diagram (f) coming from the contraction of the gauge-fermion vertex. Lastly, from the $\mathcal{W}^{(4)}$ insertion, we have diagram (e) as the double contraction of the fermions.

Having the relevant diagrams in hand, one just has to use the Feynman rules to evaluate the correlators. For simplicity, we separate bosonic and fermionic contributions, spelled out in the following.

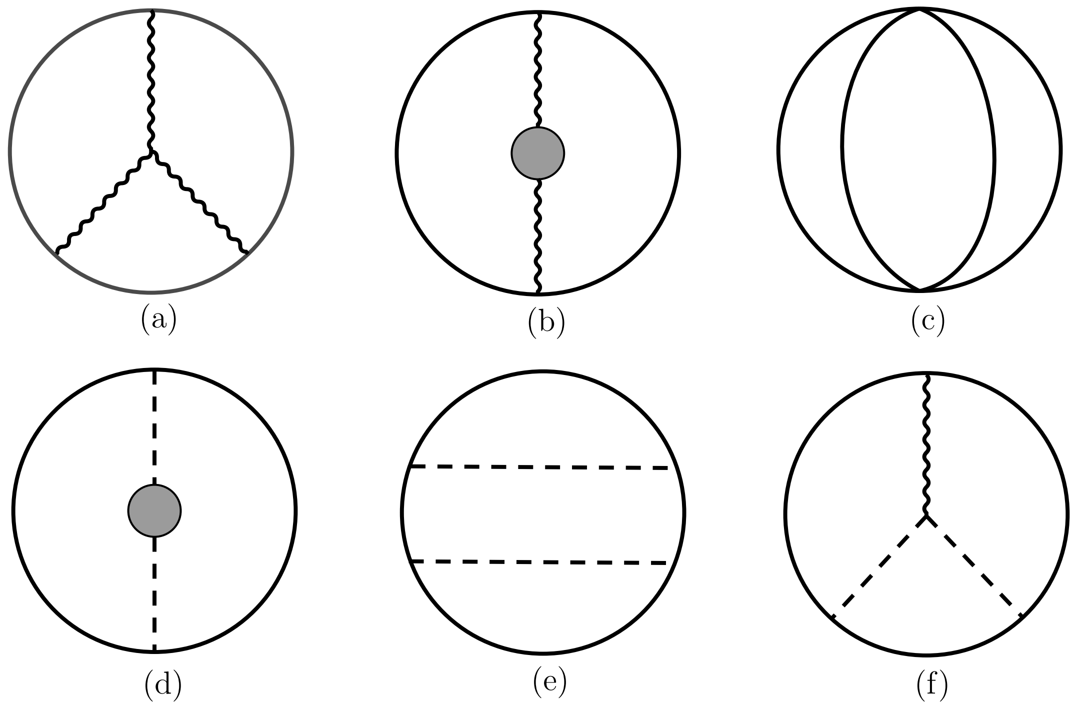


Figure 5.2: Contributions to $\mathcal{O}(\lambda^2)$ diagrams. Diagrams (b), (c) and (d) come from the second order expansion of the exponential in the Wilson loop. Diagram (a) and (f) come from third order, and diagram (e) comes from fourth order.

Bosonic Diagrams

The evaluation of diagram (a) is known from the early studies of Wilson loops in pure Chern-Simons theory [30]. As it is the product of pure gauge interactions we can see it as a pure Chern-Simons contribution. Its value is topologically protected, which means that it does not percieve the small deformations of the wavy loop, so we can recycle a previous known result of the perfect circle. Already accounting for both adjoint blocks, we'll have

$$(\mathrm{a}) = -\frac{M(M^2-1)+N(N^2-1)}{M+N}\frac{\pi^2}{6k^2}. \tag{5.18}$$

As this diagram does not depend on the contour deformation $g(\tau)$, it won't play any role in the $\mathcal{O}(b_n^2)$.

Next we have diagram (b), it is evaluated using the one-loop propagator for the fermions. Considering both contributions of the diagonal blocks, we have

$$(\mathrm{b}) = -\frac{M^2N+N^2M}{M+N}\frac{\Gamma^2(\frac{1}{2}-\epsilon)}{k^2\pi^{1-2\epsilon}}\int d\tau_{1>2}\,\frac{\dot{x}_1\cdot\dot{x}_2}{[(x_1-x_2)^2]^{1-2\epsilon}}. \tag{5.19}$$

Diagram (c) is obtained from two scalar contractions,

$$(\mathrm{c}) = \frac{M^2N+N^2M}{M+N}\frac{\Gamma^2(\frac{1}{2}-\epsilon)}{4k^2\pi^{1-2\epsilon}}\int d\tau_{1>2}\,\frac{|x_1||x_2|}{[(x_1-x_2)^2]^{1-2\epsilon}}\mathrm{Tr}\,(M_J^IM_K^J). \tag{5.20}$$

These are all contributions which are purely bosonic, meaning they are contributions common to the 1/6-BPS bosonic loops. It is interesting to note that with the

1/2-BPS condition $M = \text{diag}(-1, 1, 1, 1)$, we have $\text{Tr}\,(M^I_J M^J_K) = 4$ and diagrams (b) and (c) add up to the same structure as the 1/2-BPS of $\mathcal{N} = 4$ SYM (3.70)

$$(\text{b}) + (\text{c}) = \frac{M^2N + N^2M}{M + N} \frac{\Gamma^2(\frac{1}{2} - \epsilon)}{k^2\pi^{1-2\epsilon}} \int d\tau_{1>2} \frac{|x_1||x_2| - \dot{x}_1 \cdot \dot{x}_2}{[(x_1 - x_2)^2]^{1-2\epsilon}}. \tag{5.21}$$

Fermionic Diagrams

The first fermionic contribution is (d), which is evaluated using the fermion one-loop corrected propagator. The B_N and B_M blocks cancel out due to the grassman even couplings η and $\bar{\eta}$, so we have (d) = 0:

$$(\text{d}) = i\frac{(M - N)}{M + N} \frac{MN}{k^2} \frac{\Gamma(1/2 - \epsilon)}{4\pi^{1-2\epsilon}} \int d\tau_{1>2} \frac{|\dot{x}_1||\dot{x}_2|}{[(x_1 - x_2)^2]^{1-2\epsilon}} \underbrace{\left[\eta^\alpha_I(x_1)\bar{\eta}^I_\alpha(x_2) - \eta^\alpha_I(x_2)\bar{\eta}^I_\alpha(x_1)\right]}_{0}. \tag{5.22}$$

At order $\mathcal{W}^{(3)}$ we have diagram (f) which evaluates to

$$\begin{aligned}(\text{f}) = -\frac{1}{M + N}\left(\frac{2\pi}{k}\right) \int d\tau_{1>2>3} \text{Tr}\Big\{&\eta_{2I}\bar{\eta}^J_3 \Big\langle A^{(2)}_{1\mu}\bar{\psi}^I_2\psi_{3J}\Big\rangle \dot{x}^\mu_1|\dot{x}_2||\dot{x}_3| + \bar{\eta}^I_2\eta_{3J}\Big\langle A^{(2)}_{1\mu}\psi_{2I}\bar{\psi}^J_3\Big\rangle \times \\ &\dot{x}^\mu_1|\dot{x}_2||\dot{x}_3| \; + \; \eta_{3I}\bar{\eta}^J_1\Big\langle \psi_{1I}A^{(1)}_{2\mu}\bar{\psi}^J_3\Big\rangle |\dot{x}_1|\dot{x}^\mu_2|\dot{x}_3| \; + \; \bar{\eta}^I_3\eta_{1J}\Big\langle \bar{\psi}^J_1 A^{(2)}_{2\mu}\psi_{3I}\Big\rangle |\dot{x}_1|\dot{x}^\mu_2|\dot{x}_3| \\ &+ \eta_{1I}\bar{\eta}^J_2\Big\langle \bar{\psi}^I_1\psi_{2J}A^{(1)}_{3\mu}\Big\rangle |\dot{x}_1||\dot{x}_2|\dot{x}^\mu_3 \; + \; \bar{\eta}^I_1\eta_{2J}\Big\langle \psi_{1I}\bar{\psi}^J_2 A^{(2)}_{3\mu}\Big\rangle |\dot{x}_1||\dot{x}_2|\dot{x}^\mu_3\Big\}.\end{aligned} \tag{5.23}$$

And at last, at order $\mathcal{W}^{(4)}$ the only contribution (e) reads

$$\begin{aligned}(\text{e}) = -\frac{1}{M + N}\left(\frac{2\pi}{k}\right)^2 \frac{\Gamma^2\left(\frac{3}{2} - \epsilon\right)}{4\pi^{3-2\epsilon}} \int d\tau_{1>2>3>4}\; &|\dot{x}_1||\dot{x}_2||\dot{x}_3||\dot{x}_4| \times \\ \Big\{[NM^2(\eta_1\gamma^\mu\bar{\eta}_2)(\eta_3\gamma^\nu\bar{\eta}_4) + M^2N(\eta_2\gamma^\mu\bar{\eta}_1)(\eta_4\gamma^\nu\bar{\eta}_3)]&\frac{(x_1 - x_2)_\mu(x_3 - x_4)_\nu}{[(x_1 - x_2)^2(x_3 - x_4)^2]^{\frac{3}{2}-\epsilon}} \\ + [N^2M(\eta_1\gamma^\mu\bar{\eta}_4)(\eta_3\gamma^\nu\bar{\eta}_2) + NM^2(\eta_4\gamma^\mu\bar{\eta}_1)(\eta_2\gamma^\nu\bar{\eta}_3)]&\frac{(x_1 - x_4)_\mu(x_2 - x_3)_\nu}{[(x_1 - x_4)^2(x_2 - x_3)^2]^{\frac{3}{2}-\epsilon}}\Big\}.\end{aligned} \tag{5.24}$$

All these diagrams can also be checked in [23].

5.2 Evaluating the correlators

Taking all the diagrams into account, the goal is to evaluate them at order of b_n^2, and for that we need to regularize all the integrals. We have used two distinct regularization schemes, the DRED scheme and the so-called substitution method [31]. We take advantage of a strategy that involves expanding correlators into an infinite series that can be integrated, then re-sum the result [23].

Since we are working with a circular WL in a superconformal field theory, we expect (3.64) to hold for the second order deformation $g(\tau)$. Expression (3.64) is written for a straight line operator, so we want a circular counterpart of that. In [31], the circular 1/2-BPS WL of $\mathcal{N}=4$ SYM were studied up to $\mathcal{O}(g^4)$ in perturbation, with order $\mathcal{O}(g^2)$ yielding

$$\langle \mathcal{W}^{(\text{SYM})}_{1/2\text{-BPS}} \rangle|_{g^2} = 2I_2(\sqrt{\lambda}) \sum_{n=2} n(n^2-1)|b_n|^2. \tag{5.25}$$

Stripping the λ dependence of this expression, which is the Bremsstrahlung function, we have a Fourier representation of (3.64), where the operator $\mathcal{W}_{1/2\text{-BPS}}$ is the 1/2-BPS circular loop of $\mathcal{N}=4$ SYM, given by (3.53). In what follows we look for a loop deformation of the 1/2-BPS WL of ABJM which reproduces the same universal behaviour on the deformation parameters b_n.

The wavy-line prescription introduces a deformation on the contour of the loop, and as the fermionic couplings of the original formulation are explicitly dependent on the curve parameter, it is not a priori clear if one should also deform the fermionic couplings to reproduce the correct wavy-line set-up. However, if the fermionic couplings are kept fixed, the one is in violation of (3.64), which hints us that the wavy-line deformation must also account for a deformation on the fermionic couplings.

Based on the latitude loops exposed in [4], we were able to come up with an ansatz to correct the fermionic coupings and recover the constraint (3.64) in Fourier space. The deformed fermionic couplings can be written as

$$\eta_I^\alpha(\tau) = e^{g(\tau)/2} \begin{pmatrix} e^{i\tau/2} & -ie^{-i\tau/2} \end{pmatrix} \delta^1_I \,, \qquad \bar{\eta}^I_\alpha(\tau) = e^{g(\tau)/2} \begin{pmatrix} ie^{-i\tau/2} \\ -e^{i\tau/2} \end{pmatrix} \delta^I_1 \,. \tag{5.26}$$

With these couplings, one is in agreement with the expected universal behaviour, so that the prescription of wavy-loops must account for the fermionic deformations in the case of ABJM.

5.2.1 DRED scheme

The DRED scheme follows the standard QFT procedure of defining a continuous dimension to space-time, so that the divergences of the integrals can be condensed to poles in gamma functions, allowing for an analytic expansion which separates the finite parts of the divergent behaviour [29]. We mostly follow the conventions of [23].

As divergences occur when insertions of a correlator collide, i.e when two field insertions approach the same point in the WL contour, we need to introduce a general strategy which we used to regulate them. A general correlator is a path-ordered integration of functions containing poles, which naturally come from the propagators of Wick contractions. For a collision of points τ_1 and τ_2 the divergence can be seen as arising from terms like

$$A = \int_{d\tau_1 < d\tau_2} d\tau_1 d\tau_2 f(\tau_1,\tau_2)(x_1 - x_2)^{-\alpha}, \tag{5.27}$$

where the function $f(\tau_1,\tau_2)$ is free of divergences and consists in Euclidean contractions of x_1, x_2 and derivatives of these. As we are in a circular contour, we can rewrite the dependence on the path x_i as periodic functions, which translates in terms of the correlators being expanded in terms of

$$A = \int_{d\tau_1 < d\tau_2} d\tau_1 d\tau_2 \sum_i f_i(\tau_1, \tau_2) \sin(\tau_1 - \tau_2)^{-\alpha_i}. \tag{5.28}$$

Where again $f_i(\tau_1, \tau_2)$ are a function free of divergences, so that we localize the divergences of the correlators in the sine functions ($\alpha_i > 0$). This makes it possible to use an analytic continuation by expanding the sine as

$$\sin^{-\alpha}(\tau_1 - \tau_2) \to \frac{(2i)}{\Gamma(\alpha)} \sum_{n=0} \frac{\Gamma(n+\alpha)}{n!} (e^{-i(\tau_1 - \tau_2)})^{2n+\alpha}. \tag{5.29}$$

In a sense, we are "hiding" the divergences in the poles of the Gamma functions, allowing for the integration in the contour variables τ_i to be performed. We then re-sum the series and recover the finite result as we impose the integrals to approach $d = 3$, so that the we can recover the finite contribution at each perturbation order.

As an illustration, let us consider the $\mathcal{O}(\lambda)$ contribution which is given by the single fermion exchange (5.15). In this case, the schematic function $f(\tau_1, \tau_2)$ is given by the numerator, and the sine functions come from the expansion of the numerator. Taking (5.15) and expanding it in $\mathcal{O}(g^2)$, we will have the correlator written as a sum of typical terms of the form

$$\mathcal{W}|_{\lambda, g^2} = \sum_{p_1, p_2} \int_{d\tau_1 < d\tau_2} d\tau_1 d\tau_2 \quad C_{p_1, p_2} e^{ip_1\tau_1 + ip_2\tau_2} \sin\left(\frac{\tau_1 - \tau_2}{2}\right)^{-4+2\epsilon}, \tag{5.30}$$

for p_1, p_2 integers, and C_{p_1,p_2} complex-valued constants. For this particular amplitude, we have the sum over 36 pairs of p_1, p_2. With (5.30), we can apply (5.29), which maps the sine functions into gamma functions, allowing for the integration in the τ_1, τ_2 variables. After the integration is done, one just takes the $\epsilon \to 0$ limit, to recover the result for $d = 3$, which yields

$$\mathcal{W}|_{\lambda, g^2} \sim \lambda \sum_{n=-\infty}^{\infty} |b_n|^2 (n^2 - 1)|n|. \tag{5.31}$$

The evaluation of diagrams of $\mathcal{O}(\lambda^2)$ follow similar patterns, although more complicated due to the number of field insertions on the loop, which translates into the number of integrals of τ_i that need to be performed.

5.2.2 Substitution method - A^p

The substitution method relies on the observation that the 1/2-BPS WL is UV finite, so that at each order in $\mathcal{O}(\lambda)$ the contributions must sum up to a finite quantity, or equivalently, the divergences that appear from each diagram must cancel each other.

The A^p method was proposed in [31], here we develop it for the relevant cases, and add some comments regarding the regularization of the divergent integrals.

What we here call the A^p method is a recursive algorithm to calculate integrals of the form

$$A^p_{n_1,n_2} = \frac{1}{4\pi^2}\int d\tau_1 d\tau_2 \, \frac{e^{in_1\tau_1+in_2\tau_2}}{(e^{i\tau_1}-e^{i\tau_2})^p}. \tag{5.32}$$

For $p=0$ this is simply $A^p_{n_1,n_2}=\delta_{0,n_1}\delta_{0,n_2}$. For $p>0$, instead of performing the integrals, we view them as formal objects satisfying the parity condition and recurrence relations, arising from combinations of integrands with factorizable numerators

$$A^p_{n_1,n_2} = (-1)^p A^p_{n_2,n_1}\,, \qquad \sum_{k=0}^{p}\binom{p}{k}(-1)^k A^p_{n_1+p-k,n_2+k} = A^0_{n_1,n_2}\,. \tag{5.33}$$

We will need the values of $p=2$ and $p=4$ in order to regularize the integrals of $\mathcal{O}(\lambda)$ up to order g^2, since the most divergent terms are given by the typical term (5.30) with $\epsilon=0$.

Cases $p=2,4$

For $p=2$ this is solved by

$$A^2_{n_1,n_2} = \frac{1}{4}|n_1-n_2|\delta_{2,n_1+n_2} + C^{(1)}_{n_1+n_2}\,, \tag{5.34}$$

with arbitrary $C^{(1)}_n$.

At quartic order we find

$$A^4_{n_1,n_2} = \frac{1}{96}|n_1-n_2|((n_1-n_2)^2-4)\delta_{4,n_1+n_2} + C^{(2)}_{n_1+n_2}(n_1-n_2)^2 + C^{(3)}_{n_1+n_2}\,. \tag{5.35}$$

Indeed, each of the integrals (for $p>0$) is divergent, but the final expression for the Wilson loop at a given loop order is finite, so the arbitrary C_n do in fact contain these divergences. Here, while computing the contribution of each diagram, we found the same structure of (5.32) and used the specific substitution rules (5.34) and (5.35). We found a final result dependent of the arbitrary C's and set them to zero in the end of the computation. Which hints that setting all C's to zero should be scheme equivalent to DRED.

To see this identification of the DRED scheme with the substitution method, we consider the order λ calculation, where the only contribution is given by the single fermion exchange diagram, and compute it with both prescriptions. In the DRED scheme, we recover the result (5.31), where we it is manifestly divergence free. In the substitution scheme, we have

$$\mathcal{W}|_{\lambda,g^2} \sim \lambda \sum_{n=-\infty}^{\infty} |b_n|^2(n^2-1)|n| + \sum_{n,p} C^{(p)}_n, \tag{5.36}$$

with the sum of constants running through a finite set of integers. This way, we can identify the substitution result with the DRED claculation by taking the remaining constants $C^{(p)}_n$ in the sum to vanish.

5.2.3 Further directions

We are yet to complete the calculation of $\mathcal{W}|_{\lambda^2,g^2}$, the remaining diagrams to be evaluated are (e) and (f) mainly because of technical difficulties related to the procedure of regularization. All other diagrams have already being evaluated and all of them are in agreement with the constraint of the Bremsstrahlung function, namely they evaluate to

$$\text{(b),(c),(d)} = f(\lambda) \sum_n n(n^2-1)|b_n|^2. \tag{5.37}$$

where the sum over the modes is the momentum representation of $I[\xi]$ of (3.64) [31]. Our hope is to perform the remaining calculations, and recover the Bremsstrahlung function of ABJM.

The technical dificulties in the calculation of the remaining diagrams can be traced to the fact that they correspond to more than two field insertions on the Wilson loop. Diagram (e) corresponds to four fermionic insertions on the loop and (f) to two fermionic insertions and a gauge field. In turn, each insertion generates a source of divergence, since the loop integration takes these insertions to collide. This way, instead of having only two integrals to be performed and regularized, as in the solved diagrams, we need a prescription which solves four integrations which is computationally intractable with the current DRED scheme.

Chapter 6

Wilson lines as defects

Apart from being fundamental operators of any gauge theory, Wilson lines are also of importance in the context of space-time defects, since the insertion of BPS extended operators in a general superconformal theory provides a method for defining supersymmetric CFTs inheriting a subgroup of bulk isometries.

Operators of a general d-dimensional superconformal theory are labeled by the quantum numbers of the bosonic subgroup of isometries, which comprises the conformal group $\mathfrak{so}(d+1,1)_{\text{conf}}$ and a space-time commuting R-symmetry group $\mathfrak{so}_R(N)$, which rotates the supercharges of the theory. A familiar example is given by $\mathcal{N}=4$ SYM theory in 4 dimensions, where we have the bosonic subgroup

$$\mathfrak{so}(4,2)_{\text{conf}} \oplus \mathfrak{su}_R(4) \subset \mathfrak{psu}(2,2|4), \tag{6.1}$$

and operators are naturally labeled according to their conformal dimensions Δ, space-time spin j, and $\mathfrak{su}(4)_R$ Dynkin labels $[r_1, r_2, r_3]$.

In what follows, we are particularly interested in superconformal theories in three-dimensional space-time, where the $\mathcal{N}$-extended supersymmetric theory enjoys the isometries of $\mathfrak{osp}(\mathcal{N}|4)$, and operators are labeled in respect to the subgroup

$$\mathfrak{so}(1,4)_{\text{conf}} \oplus \mathfrak{so}(\mathcal{N})_R \subset \mathfrak{osp}(\mathcal{N}|4). \tag{6.2}$$

From the Lagrangean point of view, these theories are described by Chern-Simons terms couple to matter fields, which have been reviewed in chapter 2 for the case of $\mathcal{N}=6$ and $\mathcal{N}=4$.

When considering the insertion of BPS Wilson operators in the vacuum of these theories, we have a breaking of the vacuum symmetries, i.e a breaking of the symmetries of the underlying theory (bulk theory), and a particular subgroup of the underlying superconformal isometries is preserved, giving rise to a one-dimensional conformal theory supported in the Wilson operator contour, which is known as the *defect conformal field theory* (dCFT).

In general, the bulk symmetries are broken to the conformal group in one dimension $\mathfrak{su}(1,1)_{\text{conf}}$, a $\mathfrak{u}(1)_{J_0}$ "twist" mixing rotations and R-symmetries, residual R-symmetries and the supercharges preserved by the defect:

$$\mathfrak{osp}(\mathcal{N}|4) \Longrightarrow \mathfrak{su}(1,1)_{\text{conf}} \oplus \mathfrak{so}(\mathcal{N}^*)_R \oplus \mathfrak{u}(1)_{J_0} \oplus \text{supercharges}, \quad \text{with} \quad \mathcal{N}^* < \mathcal{N}. \tag{6.3}$$

In the dCFT, operators are also labeled in respect to its bosonic subgroup, which is bound to be a subset of the bulk bosonic isometries. In the non-supersymmetric case, a Wilson line breaks all fermionic generators, and only a one-dimensional conformal group is preserved, defining an ordinary one-dimensional CFT. In a supersymmetric case, a BPS line preserves some amount of the bulk supersymmetry, so that the dCFT corresponds to a superconformal one-dimensional theory, which is the setting of interest to us.

The reason we are interested in BPS dCFTs is that correlation functions of these defects can be studied via the superconformal bootstrap approach, since they define superconformal field theories in one dimension. This approach relies only on the symmetry group preserved by the Wilson lines, and in principle requires no definition of the Wilson line in terms of the fields of the bulk theory, providing a purely group theoretic approach to the problem, thus achieving non-perturbative set-up to the study of the defects. When a definition of the Wilson line in terms of Lagrangean fields is existent, one can map operators of the dCFT to insertion of bulk fields in the line, and explore perturbative dynamics on the dCFT by evaluating Feynman diagrams.

From the AdS/CFT perspective, a Wilson line is the boundary of a world-sheet of a propagating string in the AdS side (1.9). The preserved dCFT isometries are then reproduced by isometries on this world-sheet, so that field fluctuations of the dCFT can be mapped to an effective field theory living in the world-sheet, in an AdS_2/CFT_1 instance of the holographic principle. The gravity description of the system makes it possible to study correlation functions of gauge invariant operators on the line by means of Witten diagrams [32].

The 1/2-BPS line of ABJ(M) was recently studied in this context in [13], and we we present our studies of the 1/2-BPS lines of the $\mathcal{N} = 4$ Chern-Simons-matter theories in the same spirit, providing a weak coupling description of the fundamental operators of the theory, as well as a superspace representation of their correlation functions.

The $\mathfrak{psu}(1,1|2)$ symmetry of the defect is used to constrain the correlation function of chiral superfields, where we derive the superblocks associated to the correlation functions via the super-covariant approach [14], and calculate a strong coupling correction to the free theory based on a "minimal solution" explored in [13]. In closing, we discuss the holographic description of this set-up in terms of low energy M-branes in an orbifold setting.

Generalities

The presence of the 1/2-BPS Wilson line in the 3d Chern-Simons-matter theories breaks the $\mathfrak{osp}(\mathcal{N}|4)$ symmetry to $\mathfrak{su}(1,1|\frac{1}{2}\mathcal{N})$ [33]. We are interested in the study of correlation functions of operators that live in the dCFT, namely, of operators charged under the $\mathfrak{su}(1,1|\frac{1}{2}\mathcal{N})$ group and supported along the line.

Given local operators $\mathcal{O}_i(t_i)$ of the bulk theory, i.e. operators transforming according to a representation of $\mathfrak{osp}(\mathcal{N}|4)$ inserted in the line at the point t_i and transforming in a representation of $U(N|N)$,[1] the defect correlation function is de-

[1] In order to define gauge invariant operators, one needs to respect the embedding of the su-

fined as

$$\langle \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n)\rangle_{\mathcal{W}} \equiv \frac{\langle \mathcal{W}[\mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n)]\rangle}{\langle\mathcal{W}\rangle}, \tag{6.4}$$

with

$$\mathcal{W}[\mathcal{O}_2(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n)] \equiv \operatorname{Tr}\mathcal{P}[\mathcal{W}_{t_i,t_1}\mathcal{O}(t_1)\mathcal{W}_{t_1,t_2}\mathcal{O}(t_2)\cdots\mathcal{O}(t_n)\mathcal{W}_{t_n,t_f}]. \tag{6.5}$$

With $\mathcal{W}_{t_i,t_j}$ being the gauge compensator between t_i and t_f.[2] With these definitions we can induce operators in the dCFT by inserting bulk operators in the line.

Given an infinitesimal variation of the Wilson line, one can rewrite it as an operator insertion of the variation of the Wilson connection as

$$\frac{\langle\delta\mathcal{W}\cdots\rangle}{\langle\mathcal{W}\rangle} = -i\int dt\langle\delta\mathcal{L}(t)\cdots\rangle_{\mathcal{W}}. \tag{6.6}$$

Once we consider that the variation in (6.6) can be generated by any element of the algebra $\mathfrak{osp}(\mathcal{N}|4)$, we naturally have that the elements in the preserved sub-algebra $\mathfrak{su}(1,1|\frac{1}{2}\mathcal{N})$ decouple from the ones which are broken, since symmetries of the line will yield zero for RHS of (6.6). The broken generators will give rise to operators living in the defect, with well defined quantum numbers under the preserved $\mathfrak{su}(1,1|\frac{1}{2}\mathcal{N})$ symmetry.

Given a broken generator "G" in $\mathfrak{osp}(\mathcal{N}|4)$, it defines an operator supported on the line via its action on the Wilson line operator

$$[G,\mathcal{W}] \equiv i\delta_G\mathcal{W} = \int dt\mathcal{W}[\mathbb{G}(t)]. \tag{6.7}$$

Equation (6.7) can be understood as a kind of "pullback" for operators, since it yields an operator $\mathbb{G}(t)$ on the line, transforming in a representation of the subalgebra $\mathfrak{su}(1,1|\frac{1}{2}\mathcal{N}) \subset \mathfrak{osp}(\mathcal{N}|4)$.

The breaking of bulk translations by a line defect implies in the breaking of Ward identites associated to the directions perpendicular to the Wilson line. In particular for a line placed at the x^1 axis, the conservation of the energy momentum tensor is schematically modified to

$$\partial^\mu T_{\mu m}(x) = \delta^2(x_\perp)\mathbb{D}_m(x), \tag{6.8}$$

where "m" stands for the x^1-orthogonal directions. The operator $\mathbb{D}$ is called the *displacement operator*, and it carries the information about the linear response of the Wilson operator to deformations of the Wilson line contour.

This operator can be seen as arising from the insertion of a linear combination of the broken translation generators $G_\pm \sim P_2 \pm iP_3$ in (6.7).[3] In particular, the two

perconnection in $U(N_1|N_2)$ for the "ψ_1-line". We take $N_I = N$ working with a single 't Hooft parameter $\lambda = N/k$.

[2]It is implicit the limit $t_i \to -\infty$ and $t_f \to \infty$ in (6.5).

[3]Assuming the line is placed at the x_1 axis.

point function of the displacement operator is fixed by the dCFT superconformal symmetry and bound to satisfy

$$\langle \mathbb{D}_m(t_1)\mathbb{D}_n(t_2)\rangle = \frac{\delta_{mn}C_\Phi}{|t_{12}|^4}, \tag{6.9}$$

reflecting the fact that this operator is a conformal primary of protected conformal dimension $\Delta = 2$.

In the supersymmetric setting, the displacement operator defines the top component of a chiral multiplet of $\mathfrak{su}(1,1|\frac{1}{2}\mathcal{N})$, and their correlation functions can be studied in superspace, allowing us to extract any correlation function of any operator inside the multiplet by expanding the correlation function of the chiral superfields. In particular, all 2pt correlation functions are immediately fixed by the superconformal two point functions of the chiral fields, and all 4pt functions are given once the 4pt of the superprimary of the multiplet is known. As shown in [9], the constant normalizing the 2pt function (6.9) is related to the bremsstrahlung function as

$$C_\Phi(\lambda) = 12B(\lambda). \tag{6.10}$$

We concentrate on the analysis of the $\mathfrak{osp}(\mathcal{N}|4)$, but in principle it can be reproduced for any BPS defect in different dimensions. The goal is to derive the components of the displacement multiplet, to cast it as (anti-)chiral superfield, and to use the superconformal invariants of this space to constrain the correlation functions.

Once the correlation functions are constrained, we develop the superconformal boostrap approach in terms of the super-Casimir differential equation [14], which makes it possible to express correlators in a convariant basis of the underlying dCFT symmetry, i.e the supercoformal blocks.

Given the superconformal blocks, we perturb the correlators around the free theory and calculate the anomalous dimensions of the exchanged operators in the s-channel OPE of the relevant correlation functions.

The program

Let us organize the steps which we take to study these defects, outlying the important parts that will be discussed in more detail as we carry out the calculations in the next sessions. We are ultimately interested in the systematic study of dCFT correlators of operators sitting in the short multiplet by the name of displacement multiplet. This particular short multiplet can be cast as a chiral superfield, which makes it possible to use the full power of superconformal symmetry to constrain all 4pt funtions of this multiplet in terms of the 4pt function of the superprimary of the multiplet. By using a supersymmetric generalization of the Casimir equation, we derive the superblocks of the s-channel OPE of the said chiral fields, and by the means of a holographic ansatz, it is possible to calculate the anomalous dimensions of the operators flowing in the OPE at the strong coupling limit.

One begins by inspecting the symmetries of the 1/2-BPS defect, which generates the symmetry group of the dCFT, ultimately dictating the dynamics of the chiral fields. To do that, we can divide the generators of the bulk theory into two disjoint sets, the *symmetries of the defect* S, and the *broken generators* G, such that

$$[S, \mathcal{W}] = 0, \quad S \in \mathfrak{osp}(\mathcal{N}|4), \tag{6.11}$$

and

$$[G, \mathcal{W}] \sim \mathbb{G} \neq 0, \quad G \in \mathfrak{osp}(\mathcal{N}|4), \tag{6.12}$$

so that the broken generators spawn the operators of the displacement multiplet via (6.7). This gives rise to both the symmetry group $\mathfrak{su}(1,1|\frac{1}{2}\mathcal{N})$ of the dCFT and the field content of the multiplet that we are interested in studying.

After we identified the operators of the multiplet and the symmetry group, we classify these operators in respect to the quantum numbers of the line, generally consisting in the conformal dimension Δ, the j_0 twist charge, and the Dynkin labels of the preserved R-symmetry of the defects. At this stage, it is possible that we miss some of the operators in the displacement multiplet. This is because not all of them are generated by the direct insertion of the broken generators in the line.

To understand this, we note that the line operators defined by (6.7) are invariant under the addition of total derivatives, which may spawn extra operators. On the other hand, we also notice that the operators come from the generators of the superalgebra $\mathfrak{osp}(\mathcal{N}|4)$, which must satisfy the super-Jacobi consistency conditions. In fact, the imposition of the super-Jacobi identities fixes all freedom in choosing the derivative terms in (6.7), spawning extra operators in the multiplet that ensure consistency of the algebra at the same time completing the displacement multiplet.

Once the multiplet is completed, we can verify that it is a 1/2-BPS multiplet of the dCFT algebra, meaning it is annihilated by 1/2 of the preserved supercharges of the defect. Being a 1/2-BPS multiplet, it can be cast as a chiral superfield in a suitable superspace.

The superspace structure naturally arises from the $\mathfrak{su}(1,1|\frac{1}{2}\mathcal{N})$, so we have the space-time coordinate t of the line, and $\mathcal{N}/2$ complex Grassman coordinates θ_a, and their derivatives

$$\partial^a = \frac{\partial}{\partial \theta_a} \qquad \bar{\partial}^a = \frac{\partial}{\partial \bar{\theta}^a}, \tag{6.13}$$

with $a = 1, \cdots, \frac{1}{2}\mathcal{N}$ defining the superspace coordinates (t, θ_a). The superspace is also endowed with the covariant derivatives

$$D^a = \partial^a + \bar{\theta}^a \partial_t, \qquad \bar{D}_a = \bar{\partial}_a + \theta_a \partial_t. \tag{6.14}$$

We also introduce a chiral(anti-) coordinates, such that $\bar{D}_a y = 0$ and $D^a \bar{y} = 0$,

$$y = t + \theta_a \bar{\theta}^a, \quad \bar{y} = t - \theta_a \bar{\theta}^a, \tag{6.15}$$

which is the natural coordinate system of the chiral fields. This way, we may define the superconformal chiral fields by the chirality conditions

$$\bar{D}_a \Phi = 0 \qquad D^a \bar{\Phi} = 0. \tag{6.16}$$

The perk of working in superspace is that we can constrain the correlators of the whole multiplet by studying the correlators of the correspinding chiral fields. Recalling that we are in a superconformal setting, apart from supersymmetry, correlators are constrained by the 1d conformal group. The Ward identities of the dilatation, translation and special conformal transformations constrain the 2pt functions of any operator to satisfy

$$\langle \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{|t_{12}|^{2\Delta_1}}, \qquad t_{ij} \equiv t_i - t_j, \tag{6.17}$$

where the operator $\mathcal{O}_i$ has dimension Δ_i. Supersymmetry further constrains the 2pt functions of the chiral fields by the Ward identities associated to the preserved supercharges, so that the ordinary distance t_{ij} must be enhanced to the supersymmetric distance $\langle i\bar{j}\rangle$,

$$\langle i\bar{j}\rangle = y_i - \bar{y}_j - 2\theta_{ai}\bar{\theta}^a_j, \tag{6.18}$$

so that the superconformal 2pt of the chiral fields is written as

$$\langle \Phi(y_1,\theta_1)\bar{\Phi}(y_2,\theta_2)\rangle = \frac{C_\Phi}{\langle 1\bar{2}\rangle^{2\Delta_\Phi}}. \tag{6.19}$$

As each component of the chiral fields corresponds to an operator of the displacement multiplet, the grassmann expansion of (6.19) yields all two point functions thereof.

Our main object of study consists in j_0-neutral 4pt functions of the chiral superfields. As we are in a one-dimensional theory, insertions of operators in a correlator come with a specific ordering[4] so that we have in principle two distinct correlators, which are constrained by superconformal symmetry as

$$\langle \Phi(y_1,\theta_1)\bar{\Phi}(\bar{y}_2,\bar{\theta}_2)\Phi(y_3,\theta_3)\bar{\Phi}(\bar{y}_4,\bar{\theta}_4)\rangle = \frac{C_\Phi^2}{\langle 1\bar{2}\rangle^{2\Delta_\Phi}\langle 3\bar{4}\rangle^{2\Delta_\Phi}}\, f(\mathcal{Z})\,, \tag{6.20}$$

$$\langle \Phi(y_1,\theta_1)\bar{\Phi}(\bar{y}_2,\bar{\theta}_2)\bar{\Phi}(\bar{y}_3,\bar{\theta}_3)\Phi(y_4,\theta_4)\rangle = -\frac{C_\Phi^2}{\langle 1\bar{2}\rangle^{2\Delta_\Phi}\langle 4\bar{3}\rangle^{2\Delta_\Phi}}\, h(\mathcal{X})\,, \tag{6.21}$$

where

$$\mathcal{Z} = \frac{\langle 1\bar{2}\rangle\langle 3\bar{4}\rangle}{\langle 1\bar{4}\rangle\langle 3\bar{2}\rangle} \qquad\qquad \mathcal{X} = -\frac{\langle 1\bar{2}\rangle\langle 4\bar{3}\rangle}{\langle 1\bar{3}\rangle\langle 2\bar{4}\rangle} \tag{6.22}$$

are the supersymmetric generalization of the usual conformal cross ratios z and χ,

$$\mathcal{Z}|_{\theta\to 0} \equiv z = \frac{t_{12}t_{34}}{t_{14}t_{23}}, \qquad \mathcal{X}|_{\theta\to 0} \equiv \chi = \frac{t_{12}t_{34}}{t_{13}t_{24}}. \tag{6.23}$$

The correlator (6.20) is called the chiral-anti-chiral, and the correlator (6.21) is called the chiral-chiral. In more than one dimension they would be related by crossing, but in one dimension one is only allowed to take the OPE of neighboring operators, so they need to be considered independently.

By definition, the constructed chiral fields gives us the operators of the multiplet via a grassmann expansion,

$$\Phi(y,\theta) = \Phi_0(y) + \theta_a\Phi_1^a + \cdots \tag{6.24}$$

$$\bar{\Phi}(\bar{y},\theta) = \bar{\Phi}_0(\bar{y}) + \bar{\theta}^a\bar{\Phi}_{1a} + \cdots \tag{6.25}$$

[4] We take the ordering $t_1 < t_2 < t_3 < t_4$.

with Φ_i being operators of the multiplet. The θ_a free terms Φ_0, $\bar{\Phi}_0$ correspond to the super-primaries of the multiplet and the fields contracted with the Grassmann coordinates are their descendants, which are achieved from the super-primaries by action of the preserved supercharges.

An expansion of (6.21) and (6.20) generates all 4pt functions of the operators in the displacement multiplet, and as we will show in detail in the next sessions, the 4pt functions of descendents are written in terms of the 4pt function of the super-primary of the multiplet, so that by solving for the super-primary yields solutions to all other correlators. To see this, notice that the 4pt functions of the descendants are generated by Grassmann expanding the LHS of (6.20) and (6.21). On the other hand, the expansion of the RHS generates expressions containing derivatives of $f(\mathcal{Z})$ and $h(\mathcal{X})$, which are known once these functions are known.

This way, the natural object of study is given by the 4pt functions of the super-primaries, which is given by taking the fermionic coordinates $\theta_a \to 0$ in (6.20) and (6.21),

$$\langle \Phi_0(t_1)\bar{\Phi}_0(t_2)\Phi_0(t_3)\bar{\Phi}_0(t_4)\rangle = \frac{C_\Phi^2}{t_{12}^{2\Delta_{\Phi_0}} t_{34}^{2\Delta_{\Phi_0}}} f(z)\,, \tag{6.26}$$

$$\langle \Phi_0(t_1)\bar{\Phi}_0(t_2)\bar{\Phi}_0(t_3)\Phi_0(t_4)\rangle = \frac{C_\Phi^2}{t_{12}^{2\Delta_{\Phi_0}} t_{34}^{2\Delta_{\Phi_0}}} h(\chi)\,. \tag{6.27}$$

So far superconformal symmetry had two main effects in our set-up. Firstly, it constrains all 4pt correlators of displacement multiplet operators as a function of the 4pt function of the super-primary operator, and secondly, it constrains the 4pt functions of super-primaries to follow (6.26) and (6.27), so that all information is encoded into the unknown functions f and h.

By considering an s-channel ((12)-(34)) OPE expansion of (6.26) and (6.27) we can write the correlators in terms of the conformal blocks of $\mathfrak{su}(1,1|\frac{1}{2}\mathcal{N})$, following the traditional bootstrap approach.[5] In this sense, we can write

$$\langle \Phi_0(t_1)\bar{\Phi}_0(t_2)\Phi_0(t_3)\bar{\Phi}_0(t_4)\rangle = \frac{C_\Phi^2}{t_{12}^{2\Delta_{\Phi_0}} t_{34}^{2\Delta_{\Phi_0}}} \sum_{\Delta=0}^{\infty} c_\Delta G_\Delta(z)\,, \tag{6.28}$$

$$\langle \Phi_0(t_1)\bar{\Phi}_0(t_2)\bar{\Phi}_0(t_3)\Phi_0(t_4)\rangle = \frac{C_\Phi^2}{t_{12}^{2\Delta_{\Phi_0}} t_{34}^{2\Delta_{\Phi_0}}} \sum_{\Delta=0}^{\infty} \tilde{c}_\Delta \tilde{G}_\Delta(\chi)\,, \tag{6.29}$$

with the super-primaries Φ_0 with dimension Δ_{Φ_0}, and G_Δ, $\tilde{G}_\Delta$ are the conformal block of dimension Δ associated to the exchange of an operator of dimension Δ in the OPE. This way we have

$$f(z) = \sum_{\Delta=0}^{\infty} c_\Delta G_\Delta(z)\,, \tag{6.30}$$

$$h(\chi) = \sum_{\Delta=0}^{\infty} \tilde{c}_\Delta \tilde{G}_\Delta(\chi)\,. \tag{6.31}$$

[5] We derive these conformal blocks by the super-Casimir approach [14], with more details in the case of the $\mathcal{N}=4$ defect.

This is in principle only a rewriting of the correlators in terms of the conformal blocks, achieving a conformal partial wave decomposition. The upside of this, is that we can consider a perturbations of their LHS and conformal symmetry bounds the correlator to be expressed in terms of the conformal blocks by means of the constants c_Δ and $\tilde{c}_\Delta$.

As our defects have well defined representations in terms of the Lagrangean fields of the theory, we can construct the weak coupling description of the super-primaries of (6.28) and (6.29), which allows for a perturbative calculation of the LHS. In our perturbation scheme, we first consider a Wick contraction at tree level, allowing for the collection of the coefficients c_Δ around the free theory regime. Evaluating the correlators at the free theory limit, we define the $f^{(0)}(z)$ and $h^{(0)}(\chi)$ functions as the solution to (6.26) and (6.27), which defines our leading order results of the conformal block expansions

$$f^{(0)}(z) = \sum_{\Delta=0}^{\infty} c_\Delta^{(0)} G_\Delta(z) \; = \; 1 + \sum_{n=0}^{\infty} c_n^{(0)} G_{\Delta_n^{(0)}}(z), \tag{6.32}$$

$$h^{(0)}(\chi) = \sum_{\Delta=0}^{\infty} \tilde{c}_\Delta^{(0)} \tilde{G}_\Delta(\chi) \; = \; 1 + \sum_{n=0}^{\infty} \tilde{c}_n^{(0)} \tilde{G}_{\Delta_n^{(0)}}(\chi), \tag{6.33}$$

where we take the identity out of the sums, and parameterize them with integers n, such that the n^{th} exchanged operator has classical dimension $\Delta_n^{(0)}$.

Next up, we use the free theory solution as a saddle point, and look for a strong coupling correction to the correlators coming from holographic considerations. The strong coupling limit of the dCFT can be understood as a weakly interacting effective field theory in the string world-sheet associated to the Wilson line. The Wilson line defines a straight line contour, which renders the string world-sheet as an AdS_2 slice of the bulk, so that the dual description of our dCFT is given in terms of propagating fields in an AdS_2 space. This way, the correlation functions of the dCFT can be mapped to correlation functions of fields in the AdS_2 space that can be calculated via Witten diagrams [32].

The perturbation scheme that we consider introduces a correction to the coefficients c_Δ and to the dimensions Δ of the exchanged operators in the free theory expansion. The perturbation parameter is ϵ, such that we have

$$c_n = c_n^{(0)} + \epsilon c_n^{(1)} \tag{6.34}$$

$$\Delta_n = \Delta_n^{(0)} + \epsilon \gamma_n, \tag{6.35}$$

with γ_n defining the *anomalous dimension* of the n^{th} exchanged operator.

To understand the perturbation parameter ϵ, one must turn to the holographic calculation of the correlators, where one has the identification of ϵ with the inverse of the tension parameter of the world-sheet description. In the string theory side, the AdS_2 is the world-sheet which is described as the minimal solution to a string action with tension T, such that

$$\epsilon = \frac{1}{4\pi T}. \tag{6.36}$$

In the small ϵ regime, the correlators can be calculated via Witten diagrams [13, 32, 34], and in general, a correlator is expressed in terms of the so-called D-functions [35, 36].

The calculation of this correction in [37] inspired an ansatz for the bootstrap by noticing that the general D-functions carry logarithmic and power-like divergences for $\chi \to 0$ and $\chi \to 1$, as this general behaviour needs to be reproduced by the correlator in the dCFT side.

To implement the ansatz in the correlators, we first define the "hat" of any function f to be $\hat{f}$, with

$$\hat{f}(\chi) = \frac{f(\frac{\chi}{\chi-1})}{\chi^{\Delta_{\Phi_0}}}, \tag{6.37}$$

such that crossing symmetry is defined as

$$\hat{f}(\chi) = \hat{f}(1-\chi). \tag{6.38}$$

In this new frame, the ϵ order correction to the correlators can be attained by the general ansatz

$$\hat{f}(\chi) = \hat{f}^{(0)}(\chi) + \epsilon \hat{f}^{(1)}(\chi), \tag{6.39}$$

with

$$\hat{f}^{(1)}(\chi) = r(\chi)\log(1-\chi) + r(1-\chi)\log(\chi) + q(\chi), \tag{6.40}$$

where $r(\chi)$ and $q(\chi)$ are rational functions with poles reproducing the holographic behaviour of the D-functions. Furthermore, we constrain

$$q(\chi) = q(1-\chi), \tag{6.41}$$

to guarantee crossing symmetry of the correlator.

Up until now, the functions which parameterize the ansatz are completely arbitrary. From the holographic computations, it is expected that they have poles at physical values of $\chi \to 0$ and $\chi \to 1$, so we can expand the functions in a Laurent series

$$r(\chi) = \sum_{m=-M_1}^{M_2} r_m \chi^m, \quad \text{and} \quad q(\chi) = \sum_{l=-L_1}^{L_2} q_l \chi^l (1-\chi)^l, \tag{6.42}$$

for $M_2 \geq -M_1$ and $L_2 \geq -L_1$, explicitly accounting for the expected divergences.[6] Notice that $q(\chi)$ is written in terms of crossing symmetric monomials, ensuring the crossing symmetry of the series. With (6.42) the ansatz (6.40) can be understood as being parameterized by an infinite number of coefficients r_m and q_l.

By using (6.40) as an ansatz for the $\mathcal{O}(\epsilon)$, we have in principle infinite possibilities, since the functions $q(\chi)$ and $r(\chi)$ are free. For each choice of functions it is possible to calculate the anomalous dimensions γ_n of the operators, which in general grow with a power-like behaviour in n. In order to construct a *minimal solution*, we consider the ansatz which produces the mildest behaviour in the anomalous dimensions, as done in [13], which is achieved by considering the functions as[7]

$$r(\chi) = \frac{r_{-2}}{\chi^2} + \frac{r_{-1}}{\chi} \qquad \text{and} \qquad q(\chi) = \frac{q_{-1}}{\chi(1-\chi)}, \tag{6.43}$$

defining our *minimal ansatz* in terms of three free parameters, r_{-1}, r_{-2} and q_{-1}.

[6] $M_1, M_1, L_1, L_2 \in \mathbb{Z}$.

[7] For a more complete discussion, see [34]

6.1 1/2-BPS line defect in $\mathcal{N} = 4$ theories

The Wilson line we are interested was defined in [38] and supported along a linear quiver. These are the 1/2-BPS solutions defined along a straight line, and are known as the ψ_1 solutions.

6.1.1 The 1/2-BPS Wilson line

We are interested in studying the 1/2-BPS operator which preserves $\mathfrak{su}(2)_A$ and breaks $\mathfrak{su}(2)_B$. Such operator was called the 'ψ_1-loop' in [38], it is coupled to just two adjacent nodes of the quiver and is defined as follows. The connection of the ψ_1 WL is

$$\mathcal{W} = \mathrm{sTr}\,\mathcal{P}\exp\left(i\int_{-\infty}^{\infty}\mathcal{L}\,dt\right), \qquad \mathcal{L} = \begin{pmatrix} \mathcal{A}_I & -i\bar{\alpha}\psi_{I\dot{1}-} \\ i\alpha\bar{\psi}^{\dot{1}}_{I+} & \mathcal{A}_{I+1} - \frac{1}{2} \end{pmatrix}, \tag{6.44}$$

where α and $\bar{\alpha}$ are not complex conjugate to each other and must satisfy $\alpha\bar{\alpha} = 2i/k$. The bosonic connections read

$$\mathcal{A}_I = A_{t,I} + \frac{i}{k}\left(\nu_I - \tilde{\mu}_I{}^{\dot{1}}{}_{\dot{1}} + \tilde{\mu}_I{}^{\dot{2}}{}_{\dot{2}}\right), \qquad \mathcal{A}_{I+1} = A_{t,I+1} + \frac{i}{k}\left(\nu_{I+1} - \tilde{\mu}_{I+1}{}^{\dot{1}}{}_{\dot{1}} + \tilde{\mu}_{I+1}{}^{\dot{2}}{}_{\dot{2}}\right), \tag{6.45}$$

where we have defined the moment maps as (2.13). The line is supported at $x^2 = x^3 = 0$, and preserves the Poincaré charges are parameterized by

$$\theta^+_{a\dot{1}}, \qquad \theta^-_{a\dot{2}}, \tag{6.46}$$

along with the superconformal ones

$$\vartheta^+_{a\dot{1}}, \qquad \vartheta^-_{a\dot{2}}. \tag{6.47}$$

The 3d $\mathcal{N} = 4$ theories we are considering have the symmetry group of $\mathfrak{osp}(4|4)$, whose bosonic part is $\mathfrak{so}(1,4)_{\mathrm{conf}} \oplus \mathfrak{so}(4)_R$, which comprehends the conformal group in 3d and the rotations of supercharges. The "ψ_1-lines" preserve an $\mathfrak{su}(2)_A$ subgroup of R-symmetries by construction, so that we can understand the breaking of bulk R-symmetry by the presence of such defect as

$$\mathfrak{so}(4)_R \cong \mathfrak{su}(2)_A \oplus \mathfrak{su}(2)_B \Longrightarrow \mathfrak{su}(2)_A. \tag{6.48}$$

In order to understand the preserved isometries of the 3d conformal algebra which survive the presence of the defect, we need to account for the presence of a twist, which mixes the broken R-symmetry generator and a rotation in a non-trivial fashion.

The presence of the line keeps the invariance under translations and special conformal transformations along itself, and also of dilatations, which accounts for a conformal group in one dimension

$$\mathfrak{so}(1,4)_{\mathrm{conf}} \Longrightarrow \{P_1, K_1, D\} \cong \mathfrak{su}(1,1)_{\mathrm{conf}}. \tag{6.49}$$

We now have to look for a twist symmetry of the loop, and to do that, we need to take a closer look at the breaking of R-symmetry in the line connection. In the adjoint of $U(N_1)$, we have

$$\mathcal{A}_I = A_{t,1} + \frac{i}{k}(\underbrace{-\bar{\tilde{q}}^{\dot{1}}_{I-1}\tilde{q}_{\dot{1}\;I-1} + \bar{\tilde{q}}^{\dot{2}}_{I-1}\tilde{q}_{\dot{2}\;I-1}}_{\text{broken }\mathfrak{su}(2)_B} + \underbrace{q^a_I\bar{q}_{a\;I}}_{\text{manifest }\mathfrak{su}(2)_A}). \tag{6.50}$$

Although the $\mathfrak{su}(2)_B$ is broken, we still have singlets of the diagonal, since in the $\mathfrak{su}(2)_B$ space, we would have $\mathbb{M}^{\dot{b}}_{\dot{a}}\bar{\tilde{q}}^{\dot{a}}\tilde{q}_{\dot{b}}$ with

$$\mathbb{M}^{\dot{b}}_{\dot{a}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.51}$$

Representing the generators of $\mathfrak{su}(2)_B$ as $\bar{R}^{\ \dot{b}}_{\dot{a}}$, we would have an invariance of $\mathcal{A}_{(I)}$ by $\bar{R}^{\ \dot{1}}_{\dot{1}}$ and $\bar{R}^{\ \dot{2}}_{\dot{2}}$. Notice that the fermions coupling to the line are $\psi^{+}_{\dot{1}}$ and $\bar{\psi}^{\dot{1}}_{+}$, so that they are eigenvectors of $M^{23} = -\mathcal{M}^{\ +}_{+}$ rotations as well,

$$[\bar{R}^{\ \dot{1}}_{\dot{1}}, \psi^{+}_{\dot{1}}] = \frac{1}{2}\psi^{+}_{\dot{1}}, \qquad [\mathcal{M}^{\ +}_{+}, \psi^{+}_{\dot{1}}] = -\frac{1}{2}\psi^{+}_{\dot{1}}. \tag{6.52}$$

From this, we can easily define a twist annihilating the fermions

$$J_0 = \left(\mathcal{M}^{\ +}_{+} + \bar{R}^{\dot{1}}_{\dot{1}}\right). \tag{6.53}$$

This generator can be seen to annihilate the loop connection, since the diagonal entries are scalars and invariant under $\bar{R}^{\dot{1}}_{\dot{1}}$. This is a charge preserved by $\mathcal{W}$ which commutes with the other generators, namely it is a non-trivial central ideal[8] of the isometries preserved by the defect.

This $\mathfrak{u}(1)_{j_0}$ charge along with the conformal group on the line $\mathfrak{su}(1,1)_{\text{conf}}$ and the residual $\mathfrak{su}(2)_A$ R-symmetry makes the bosonic part of the symmetry group of the 1/2 BPS line $\mathfrak{su}(1,1)_{\text{conf}} \oplus \mathfrak{su}(2)_A \oplus \mathfrak{u}(1)_{j_0}$. The fermionic symmetries of the defect are given by the preserved supercharges, which are

$$Q^{1\dot{1}}_{+}, Q^{2\dot{1}}_{+}, Q^{1\dot{2}}_{-}, Q^{2\dot{2}}_{-} \quad \text{and} \quad S^{1\dot{1}}_{+}, S^{2\dot{1}}_{+}, S^{1\dot{2}}_{-}, S^{2\dot{2}}_{-}. \tag{6.54}$$

Along with the bosonic ones, they form the group preserved by the line $\mathfrak{su}(1,1|2)$, where the projection of the algebra is over the central ideal j_0. This way, the highest weight states of the algebra are labeled as $[\Delta, j_0, h]$, where we diagonalize in respect to the dilatation, the j_0 charge and the $\mathfrak{su}(2)_A$ Dynking label.

6.1.2 Displacement multiplet

As inspected last section, the presence of the 1/2-BPS Wilson line breaks the $\mathfrak{osp}(4|4)$ symmetry of the theory to $\mathfrak{su}(1,1|2)$, so that the broken generators when inserted in the line give rise to the displacement multiplet transforming in a representation of the defect symmetry group. With the symmetries in hand, we can analyze the displacement multiplet, labeling the operators accordingly.

[8] The appearance of such operator was first discussed in [33] for the $\mathcal{N}_1 = 4$ case.

In order to calculate the labels associated to the j_0 charge of a general line operator $\mathbb{G}$, one needs to evaluate

$$[J_0, \mathbb{G}] = j_0 \mathbb{G}. \tag{6.55}$$

The conformal dimension $\Delta_{\mathbb{G}}$ associated to an operator $\mathbb{G}$ coming from the insertion of a broken generator G of dimenison Δ_G is always increased by one, since one needs to account for the dimension of the differential dt in the definition of the line operator, so that [9]

$$\Delta_{\mathbb{G}} = \Delta_G + 1. \tag{6.56}$$

Supercharges. A generic Poincaré supersymmetry transformation is parametrized by $\theta^{\alpha}_{a\dot{b}}$, so that the variation reads

$$\delta \equiv \theta^{\alpha}_{a\dot{b}} \mathcal{Q}^{a\dot{b}}_{\alpha}, \tag{6.57}$$

where $\mathcal{Q}^{a\dot{b}}_{\alpha}$ are the Poincaré supercharges. With these definitions and the preserved parameters of the "ψ_1-loop" (6.46), we have the set of conserved charges

$$\mathcal{Q}^{a\dot{1}}_{+}, \mathcal{Q}^{a\dot{2}}_{-}, \tag{6.58}$$

in turn we have the broken supercharges as

$$\mathcal{Q}^{a\dot{1}}_{-}, \mathcal{Q}^{a\dot{2}}_{+}, \tag{6.59}$$

which are fundamental vectors of $\mathfrak{su}(2)_A$ with $h=1$. As the fermionic generators have dimension $1/2$, when inserted in the line, they generate operators with $\Delta = 3/2$. To complete its lables, we need only to claculate j_0.

The broken supercharges give rise to the line operators

$$[\mathcal{Q}^{a\dot{1}}_{-}, \mathcal{W}] = \int dt\, \mathcal{W}[\mathbb{\Lambda}^{a\dot{1}}_{-}(t)], \qquad [\mathcal{Q}^{a\dot{2}}_{+}, \mathcal{W}] = \int dt\, \mathcal{W}[\mathbb{\Lambda}^{a\dot{2}}_{+}(t)], \tag{6.60}$$

which are fermions in the **2** of $\mathfrak{su}(2)_A$. The j_0-charges are given by inserting J_0, $\mathcal{Q}$ and $\mathcal{W}$ into a Jacobi identity. For the "$\dot{1}, -$" charges we have

$$[J_0, [\mathcal{Q}^{a\dot{1}}_{-}, \mathcal{W}]] + [\mathcal{Q}^{a\dot{1}}_{-}, \underbrace{[\mathcal{W}, J_0]}_{0}] + [\mathcal{W}, [J_0, \mathcal{Q}^{a\dot{1}}_{-}]] = 0. \tag{6.61}$$

We can evaluate the action of J_0 in a supercharge as

$$[J_0, \mathcal{Q}^{a\dot{1}}_{-}] = -\mathcal{Q}^{a\dot{1}}_{-}. \tag{6.62}$$

Plugging it back in the Jacobi identity (6.61), it gives $j_0 = -1$, completing the labels

$$\mathbb{\Lambda}^{a\dot{1}}_{-} : \left[\frac{3}{2}, -1, 1\right]. \tag{6.63}$$

[9] In what follows, we only consider labels of operators on the line, and drop indices of Δ.

The $\dot{2},+$ case is calculated exactly in the same way, yielding

$$[J_0, \mathcal{Q}_+^{a\dot{2}}] = \mathcal{Q}_+^{a\dot{2}}, \tag{6.64}$$

with resulting labels given by

$$\mathbb{\Lambda}_+^{a\dot{2}} : \left[\frac{3}{2}, 1, 1\right]. \tag{6.65}$$

In summary, we have derived the representation of the fermionic operators of the displacement multiplet, and the final result is

$$\mathbb{\Lambda}_-^{a\dot{1}} : \left[\frac{3}{2}, -1, 1\right], \qquad \mathbb{\Lambda}_+^{a\dot{2}} : \left[\frac{3}{2}, 1, 1\right]. \tag{6.66}$$

R-charges. The 1/2-BPS solution breaks the $\mathfrak{su}(2)_B$ of $\mathfrak{osp}(4|4)$, which is generated by $\bar{R}_{\dot{a}}^{\ \dot{b}}$, and as we have seen, the generator $\bar{R}_{\dot{1}}^{\ \dot{1}}$ conspires with the rotation $\mathcal{M}_+^{\ +}$ to give the J_0 symmetry of the defect. This leaves us with two broken R-symmetry generators $\bar{R}_{\dot{1}}^{\ \dot{2}}$, $\bar{R}_{\dot{2}}^{\ \dot{1}}$. For each of the broken charges we define the operators

$$[\bar{R}_{\dot{a}}^{\ \dot{b}}, \mathcal{W}] = \int dt\ \mathcal{W}[\bar{\mathbb{R}}_{\dot{a}}^{\ \dot{b}}(t)]. \tag{6.67}$$

The task now reduces to finding the representation of $\bar{\mathbb{R}}_{\dot{a}}^{\ \dot{b}}$. As R-symmetry generators are dimensionless, when inserted in the line they yield operators of $\Delta = 1$. They are also singlets of $\mathfrak{su}(2)_A$, so $h = 0$.

When acting on the R-symmetry subspace, the J_0 operator has only contributions from $\bar{\mathbb{R}}_{\dot{1}}^{\ \dot{1}}$, since R-symmetries commute with space-time rotations. We can see that the two operators have their degeneracy lifted by the j_0-charge[10]

$$[J_0, \bar{\mathbb{R}}_{\dot{1}}^{\ \dot{2}}] = \bar{\mathbb{R}}_{\dot{1}}^{\ \dot{2}}, \qquad [J_0, \bar{\mathbb{R}}_{\dot{2}}^{\ \dot{1}}] = -\bar{\mathbb{R}}_{\dot{2}}^{\ \dot{1}}. \tag{6.68}$$

which gives us

$$\bar{\mathbb{R}}_{\dot{1}}^{\ \dot{2}} : [1, 1, 0], \qquad \bar{\mathbb{R}}_{\dot{2}}^{\ \dot{1}} : [1, -1, 0]. \tag{6.69}$$

Translations The broken translations are given by P_{++} and P_{--}. Their insertion in the line gives rise to the displacement operators

$$[P_{++}, \mathcal{W}] = \int dt\ \mathcal{W}[\mathbb{D}(t)], \qquad [P_{--}, \mathcal{W}] = \int dt\ \mathcal{W}[\bar{\mathbb{D}}(t)]. \tag{6.70}$$

As P_{--} and P_{++} are momentum generators, they have unit dimension, so that the displacement operators have canonical dimension $\Delta = 2$. Being scalars of the R-symmetry, we have $h = 0$.

We can calculate their j_0 label by acting with J_0. The relevant Jacobi identity is

$$[J_0, [\mathbf{P}, \mathcal{W}]] + [\mathbf{P}, \underbrace{[\mathcal{W}, J_0]}_{0}] + [\mathcal{W}, [J_0, \mathbf{P}]] = 0. \tag{6.71}$$

[10]The $\bar{\mathbb{R}}_{\dot{2}}^{\ \dot{2}}$ is not present in the displacement multiplet, since $\bar{\mathbb{R}}_{\dot{2}}^{\ \dot{2}} = -\bar{\mathbb{R}}_{\dot{1}}^{\ \dot{1}}$ due to the trace condition $\bar{\mathbb{R}}_{\dot{a}}^{\ \dot{a}} = 0$ for $\mathfrak{su}(2)$.

Where $\mathbf{P}$ is either P_{--} or P_{++}. As the displacement operator is a singlet in R-symmetry space, the only contribution of the J_0 charge is $\mathcal{M}_+^{\ +}$, so that

$$[J_0, \mathbb{D}] = \mathbb{D}, \qquad\qquad [J_0, \bar{\mathbb{D}}] = -\bar{\mathbb{D}}, \tag{6.72}$$

so that

$$\mathbb{D} : [2, 1, 0]\,, \qquad \bar{\mathbb{D}} : [2, -1, 0]\,. \tag{6.73}$$

With this, we have classified all operators arising from the insertion of broken generators. The next step is to act with the preserved supercharges on them and close the algebra using super-Jacobi identities, ensuring consistency of the algebra and completing the multiplet. The definition of defect operators corresponding are summarized in Table (6.1).

Operator	$[\Delta, j_0, h]$
$\bar{\mathbb{R}}_{\dot{1}}^{\ \dot{2}}$	$[1, 1, 0]$
$\bar{\mathbb{R}}_{\dot{2}}^{\ \dot{1}}$	$[1, -1, 0]$
$\mathbb{A}_+^{a\dot{2}}$	$\left[\frac{3}{2}, 1, 1\right]$
$\mathbb{A}_-^{a\dot{1}}$	$\left[\frac{3}{2}, -1, 1\right]$
$\mathbb{D}$	$[2, 1, 0]$
$\bar{\mathbb{D}}$	$[2, -1, 0]$

Table 6.1: Dynkin labels of the displacement multiplet operators.

6.1.3 Consistency conditions

We are now concerned with the compatibility of the developed displacement multiplet of 1/2 BPS defects in $\mathcal{N}=4$ theories with the $\mathfrak{osp}(4|4)$ superalgebra. Note that the prescription of generating operators in the displacement multiplet by the insertion of $\mathfrak{osp}(4|4)$ broken charges in the Wilson operator is defined up to total derivatives.

At each node of the diagram (6.1), we can in principle add a term which is $\partial_\tau \mathcal{O}$, since these operators are always supported along the line. Most importantly, the freedom to add total derivatives in the multiplet is necessary to ensure that the action of the supercharges is compatible with the bulk superalgebra. Here we investigate whether (6.1) is consistent with the $\mathfrak{osp}(4|4)$ algebra in its current form (without total derivative terms), or if it is necessary to include total derivative terms.

The starting point is to study the action of conserved charges $\mathcal{Q}_+^{a\dot{1}}$ and $\mathcal{Q}_-^{a\dot{2}}$. We inspect the action of $\mathcal{Q}_-^{a\dot{2}}$ in the $\mathbb{D}$ multiplet structure via super-Jacobi identities, since the naïve action of these charges on the operators yields trivial zeros which are incompatible with the supergroup structure.

Displacement operators

Let us start by checking the consistency of the action of preserved charges on the displacement operator itself. In order to do that, we consider the super-Jacobi identity of the preserved charges and $\mathbb{D}$,

$$\{\mathcal{Q}_-^{a\dot{2}},[\mathcal{Q}_+^{b\dot{1}},\mathbb{D}]\}-\{\mathcal{Q}_+^{b\dot{1}},[\mathbb{D},\mathcal{Q}_-^{a\dot{2}}]\}+[\mathbb{D},\{\mathcal{Q}_-^{a\dot{2}},\mathcal{Q}_+^{b\dot{1}}\}]\overset{(!)}{=}0. \tag{6.74}$$

Starting with the third term, we have

$$\{\mathcal{Q}_-^{a\dot{2}},\mathcal{Q}_+^{b\dot{1}}\}=4\epsilon^{ab}\epsilon^{\dot{2}\dot{1}}P_{-+}, \tag{6.75}$$

Noticing that $P_{+-}=P_1=P$.[11] This term generates the derivative term via

$$[\mathbb{D},\{\mathcal{Q}_-^{a\dot{2}},\mathcal{Q}_+^{b\dot{1}}\}]=4\epsilon^{ab}[P,\mathbb{D}]=-4\epsilon^{ab}\partial_\tau\mathbb{D}. \tag{6.76}$$

With the result (6.83) we can rewrite it as

$$[\mathbb{D},\{\mathcal{Q}_-^{a\dot{2}},\mathcal{Q}_+^{b\dot{1}}\}]=\{\mathcal{Q}_+^{b\dot{1}},\partial_\tau\Lambda_+^{a\dot{2}}\}. \tag{6.77}$$

The first term vanishes since $\mathbb{D}$ is the top component of the multiplet, $[\mathcal{Q}_+^{b\dot{1}},\mathbb{D}]=0$. Plugging (6.77) into the super-Jacobi identity gives us

$$\{\mathcal{Q}_+^{b\dot{1}},[\mathcal{Q}_-^{a\dot{2}},\mathbb{D}]\}+\{\mathcal{Q}_+^{b\dot{1}},\partial_\tau\mathbb{\Lambda}_+^{a\dot{2}}\}\overset{(!)}{=}0,\qquad \{\mathcal{Q}_+^{b\dot{1}},[\mathcal{Q}_-^{a\dot{2}},\mathbb{D}]+\partial_\tau\mathbb{\Lambda}_+^{a\dot{2}}\}\overset{(!)}{=}0, \tag{6.78}$$

so

$$[\mathcal{Q}_-^{a\dot{2}},\mathbb{D}]=-\partial_\tau\mathbb{\Lambda}_+^{a\dot{2}}. \tag{6.79}$$

Thus fixing the action of the charges on the $\mathbb{D}$ operator.

Fermionic operators

We now turn our attention to the identities of the fermions $\mathbb{\Lambda}_+^{a\dot{2}}$ in the multiplet. We notice that we have to study two distinct set-up of identities. The first one concerns the insertion of a preserved charge Q, a broken charge $\mathbf{Q}$ and the the $\mathcal{W}$ operator, schematically

$$\{Q,[\mathbf{Q},\mathcal{W}]\}-\{\mathbf{Q},[\mathcal{W},Q]\}+[\mathcal{W},\{Q,\mathbf{Q}\}]\overset{(!)}{=}0, \tag{6.80}$$

and the second is the insertion of three fermionic charges, consisting of a fermionic line operator $\mathbb{\Lambda}$, and two preserved Poncaré charges Q, schematically

$$[Q^{(1)},\{Q^{(2)},\mathbb{\Lambda}\}]+[Q^{(2)},\{\mathbb{\Lambda},Q^{(1)}\}]+[\mathbb{\Lambda},\{Q^{(2)},Q^{(1)}\}]\overset{(!)}{=}0. \tag{6.81}$$

Let us start with the first set of Jacobi identites. We have the preserved supercharges (6.58), so we first study their action on the fermionic operator $\mathbb{\Lambda}_+^{a\dot{2}}$, which gives rise to the displacement operator $\mathbb{D}$. First we observe that $\mathbb{\Lambda}_+^{a\dot{2}}$ is annihilated by the $\mathcal{Q}_-^{a\dot{2}}$ charges, so we only have to evaluate the action of $\mathcal{Q}_+^{a\dot{1}}$ on the fermionic operator. Considering a Jacobi identity of the type (6.80), we first notice that the middle term always vanishes, since by definition Q is preserved by the defect, and we are left with

$$\{Q,\mathbb{\Lambda}\}=[\{Q,\mathbf{Q}\},\mathcal{W}], \tag{6.82}$$

[11] The P action is defined as $-\partial_\tau$ in accordance with (6.104).

where we defined $[\mathbf{Q},\mathcal{W}]=\mathbb{A}$. If we plug the charges $\mathcal{Q}_{+}^{a\dot{1}}$, $\mathcal{Q}_{+}^{b\dot{2}}$ as the preserved and broken one, respectively, we collect

$$\{\mathcal{Q}_{+}^{b\dot{1}},\mathbb{A}_{+}^{a\dot{2}}\}=4\epsilon^{ba}\mathbb{D}. \tag{6.83}$$

Next, we have to consider the second type of identities (6.81),

$$[\mathcal{Q}_{-}^{c\dot{2}},\{\mathcal{Q}_{+}^{b\dot{1}},\mathbb{A}_{+}^{a\dot{2}}\}]+[\mathcal{Q}_{+}^{b\dot{1}},\{\mathbb{A}_{+}^{a\dot{2}},\mathcal{Q}_{-}^{c\dot{2}}\}]+[\mathbb{A}_{+}^{a\dot{2}},\{\mathcal{Q}_{-}^{c\dot{2}},\mathcal{Q}_{+}^{b\dot{1}}\}]\overset{(!)}{=}0. \tag{6.84}$$

The first term reads

$$[\mathcal{Q}_{-}^{c\dot{2}},\{\mathcal{Q}_{+}^{b\dot{1}},\mathbb{A}_{+}^{a\dot{2}}\}]=4\epsilon^{ab}[\mathcal{Q}_{-}^{c\dot{2}},\mathbb{D}]. \tag{6.85}$$

With the result (6.79), we reach

$$[\mathcal{Q}_{-}^{c\dot{2}},\{\mathcal{Q}_{+}^{b\dot{1}},\mathbb{A}_{+}^{a\dot{2}}\}]=-4\epsilon^{ba}\partial_{\tau}\mathbb{A}_{+}^{c\dot{2}}. \tag{6.86}$$

The third term generates a derivative again

$$[\mathbb{A}_{+}^{a\dot{2}},\{\mathcal{Q}_{-}^{c\dot{2}},\mathcal{Q}_{+}^{b\dot{1}}\}]=-4\epsilon^{cb}[\mathbb{A}_{+}^{a\dot{2}},P]=-4\epsilon^{cb}\partial_{\tau}\mathbb{A}_{+}^{a\dot{2}}. \tag{6.87}$$

Of course, we do not know a priori what the second term evaluates to, but we can see that it needs to generate epsilon tensors and derivatives of the fermionic operator. We notice that the final expression contains a $\dot{2}$ index, while the supercharge of the second term is $\mathcal{Q}_{+}^{b\dot{1}}$. So that if

$$\{\mathbb{A}_{+}^{\bar{a}\dot{2}},\mathcal{Q}_{-}^{c\dot{2}}\}\sim\bar{\mathbb{R}}_{\dot{1}}^{\dot{2}} \tag{6.88}$$

we can see that the $\mathfrak{su}(2)$ index will rotate to $\dot{2}$ as we wish. Furthermore, anti-commutators of supercharges naturally contain the $\mathfrak{su}(2)$ invariant, and we need to fit it in a derivative, so the natural ansatz is

$$\{\mathbb{A}_{+}^{a\dot{2}},\mathcal{Q}_{-}^{c\dot{2}}\}=4\epsilon^{ca}\partial_{\tau}\bar{\mathbb{R}}_{\dot{1}}^{\dot{2}}. \tag{6.89}$$

This way, the second term in (6.84) evaluates to

$$[\mathcal{Q}_{+}^{b\dot{1}},\{\mathbb{A}_{+}^{a\dot{2}},\mathcal{Q}_{-}^{c\dot{2}}\}]=-4\epsilon^{ac}\partial_{\tau}\mathbb{A}_{+}^{b\dot{2}}, \tag{6.90}$$

which means that (6.84) evaluates to

$$\epsilon^{ba}\partial_{\tau}\mathbb{A}_{+}^{c\dot{2}}+\epsilon^{cb}\partial_{\tau}\mathbb{A}_{+}^{a\dot{2}}+\epsilon^{ac}\partial_{\tau}\mathbb{A}_{+}^{b\dot{2}}=0, \tag{6.91}$$

as desired.

R-symmetry operators

Lastly, we need to perform the consistency check on the $\bar{\mathbb{R}}_{\dot{1}}^{\dot{2}}$ operator. Its Jacobi identity reads

$$\{\mathcal{Q}_{-}^{a\dot{2}},[\mathcal{Q}_{+}^{b\dot{1}},\bar{\mathbb{R}}_{\dot{1}}^{\dot{2}}]\}-\{\mathcal{Q}_{+}^{b\dot{1}},[\bar{\mathbb{R}}_{\dot{1}}^{\dot{2}},\mathcal{Q}_{-}^{a\dot{2}}]\}+[\bar{\mathbb{R}}_{\dot{1}}^{\dot{2}},\{\mathcal{Q}_{-}^{a\dot{2}},\mathcal{Q}_{+}^{b\dot{1}}\}]\overset{(!)}{=}0. \tag{6.92}$$

The first term reads

$$\{\mathcal{Q}_{-}^{a\dot{2}},[\mathcal{Q}_{+}^{b\dot{1}},\bar{\mathbb{R}}_{\dot{1}}^{\dot{2}}]\}=-\epsilon^{cb}\partial_\tau\bar{\mathbb{R}}_{\dot{1}}^{\dot{2}}, \tag{6.93}$$

while the third term is

$$[\bar{\mathbb{R}}_{\dot{1}}^{\dot{2}},\{\mathcal{Q}_{-}^{a\dot{2}},\mathcal{Q}_{+}^{b\dot{1}}\}]=\epsilon^{cb}\partial_\tau\bar{\mathbb{R}}_{\dot{1}}^{\dot{2}}. \tag{6.94}$$

which means that one can consistently choose

$$[\mathcal{Q}_{-}^{c\dot{2}},\bar{\mathbb{R}}_{\dot{1}}^{\dot{2}}]=0, \tag{6.95}$$

so that the Jacobi identity holds. This is different than in the ABJ(M) case, where one needs to introduce an extra operator $\mathbb{F}$ which satisfies (6.95). In fact, as $\bar{\mathbb{R}}_{\dot{1}}^{\dot{2}}$ is the super-primary of this multiplet. Defining

$$\bar{\mathbb{R}}_{\dot{1}}^{\dot{2}}\equiv\mathbb{R},\qquad \mathbb{\Lambda}_{+}^{a\dot{2}}\equiv\mathbb{\Lambda}^a,\qquad \mathbb{D}, \tag{6.96}$$

$Q^a\equiv\mathcal{Q}_{+}^{a\dot{1}}$ and $\bar{Q}^a\equiv\mathcal{Q}_{-}^{a\dot{2}}$ charges, we can write a summary of supersymmetry transformations as

$$\begin{aligned}
&[Q^b,\mathbb{R}]=\mathbb{\Lambda}^b, && [\bar{Q}_b,\mathbb{R}]=0,\\
&\{Q^b,\mathbb{\Lambda}^a\}=4\epsilon^{ba}\mathbb{D}, && \{\bar{Q}_b,\mathbb{\Lambda}^a\}=4\delta_b^a\partial_\tau\mathbb{R},\\
&[Q^b,\mathbb{D}]=0, && [\bar{Q}_b,\mathbb{D}]=-\partial_\tau\mathbb{\Lambda}_b.
\end{aligned} \tag{6.97}$$

As the preserved fermionic charges act on the states $|\Delta,j_0,h\rangle$, one recovers another state $|\Delta',j_0',h'\rangle$ in the same multiplet (with the same Casimir numbers). In fact, we acted with the preserved charges (6.58) in the states generated by the operators in Table (6.1) and recover other operators of the multiplet. The result can be visualized as in Figure 6.1. As we can see, the multiplet naturally decouples into actions of the Q^a and $\bar{Q}_a$ charges, with any mixing between them annihilating the states, thus defining (anti-)chiral superfields.

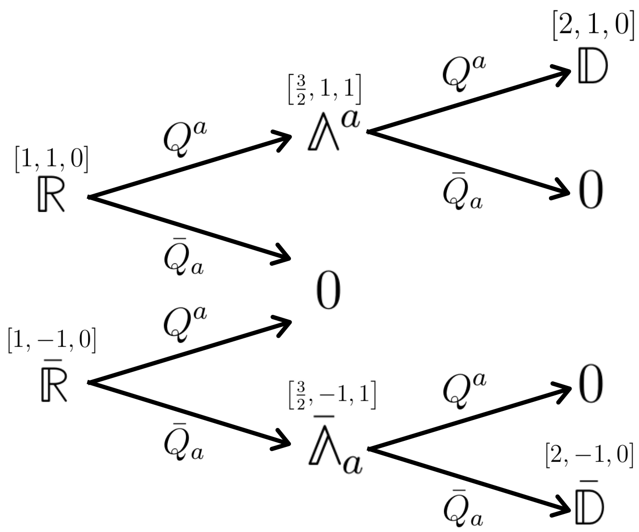


Figure 6.1: Representation of 1/2 BPS multiplet. Zeros are enhanced to total derivative operators along the line defect.

6.1.4 Chiral correlators

As the displacement multiplet is naturally described by a chiral superfield, we wish to use a superspace representation of the preserved $\mathfrak{psu}(1,1|2)$ defect algebra to constrain the correlation functions of operators in the displacement multiplet. In the present case, we are interested in recasting the displacement multiplet dropping the $\mathfrak{u}(1)_{j_0}$ charge of the chiral superfield, since it is projected out of the algebra, where $\mathbb{R}$ is the super-primary operator, which is an $\mathfrak{su}(2)$ scalar with mass dimension of $\Delta=1$.

Accounting for the $\mathfrak{su}(2)$ structure of $\mathfrak{psu}(1,1|2)$, we introduce two sets θ_a, with $a=1,2$. Using this superspace, we can cast the displacement multiplet and its conjugate as a chiral and anti-chiral fields, where the superprimary is given by the operators $\mathbb{R}$, and $\bar{\mathbb{R}}$:

$$\Phi(y,\theta)=\mathbb{R}(y)+\theta_a\mathbb{A}^a(y)-\frac{1}{2}\theta_a\theta_b\epsilon^{ab}\mathbb{D}(y), \tag{6.98}$$

$$\bar{\Phi}(\bar{y},\bar{\theta})=\bar{\mathbb{R}}(\bar{y})+\bar{\theta}^a\bar{\mathbb{A}}_a(\bar{y})+\frac{1}{2}\bar{\theta}^a\bar{\theta}^b\epsilon_{ab}\bar{\mathbb{D}}(\bar{y}). \tag{6.99}$$

2-point functions

In order to read off the 2-point functions, we need to expand (6.19) in the Grassmann parameters, where now we have $\mathfrak{su}(2)$ products $\theta_i\cdot\bar{\theta}_j\equiv T_{ij}$ to expand. For any T_{ij}, we have $T_{ij}^3=0$, so that our expansions truncate at second order

$$\frac{1}{(y_{12}-2T_{12})^2}=\frac{1}{y_{12}^2}+\frac{2(2T_{12})}{y_{12}^3}+\frac{3(2T_{12})^2}{y_{12}^4}, \tag{6.100}$$

yielding

$$\langle\Phi(t_1,\theta_1)\bar{\Phi}(t_2,\theta_2)\rangle=C_\Phi\left(\frac{1}{t_{12}^2}+\frac{2(2\theta_1\cdot\bar{\theta}_2)}{t_{12}^3}+\frac{3(2\theta_1\cdot\bar{\theta}_2)^2}{t_{12}^4}\right). \tag{6.101}$$

From this expansion we easily collect

$$\langle\mathbb{R}(t_1)\bar{\mathbb{R}}(t_2)\rangle=\frac{C_\Phi}{t_{12}^2},\qquad\langle\mathbb{A}^a(t_1)\bar{\mathbb{A}}_b(t_2)\rangle=-\frac{4C_\Phi\delta^a_b}{t_{12}^3},\qquad\langle\mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\rangle=\frac{24C_\Phi}{t_{12}^4}. \tag{6.102}$$

4-point functions

As before, we expand expression (6.26) in the Grassmann parameters and match with the superfield expansion in terms of the operators in the multiplet, collecting

$$\begin{aligned}
\langle\mathbb{R}(t_1)\bar{\mathbb{R}}(t_2)\mathbb{R}(t_3)\bar{\mathbb{R}}(t_4)\rangle&=\frac{C_\Phi^2}{t_{12}^2t_{34}^2}f,\\
\langle\mathbb{A}^{a_1}(t_1)\bar{\mathbb{A}}_{a_2}(t_2)\mathbb{A}^{a_3}(t_3)\bar{\mathbb{A}}_{a_4}(t_4)\rangle&=\frac{C_\Phi^2}{t_{12}^3t_{12}^3}\left[\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}z\left(zf''-3f'\right)+4\delta^{a_1}_{a_4}\delta^{a_3}_{a_2}f\left(f'+zf''\right)\right],\\
\langle\mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\mathbb{D}(t_3)\bar{\mathbb{D}}(t_4)\rangle&=\frac{16C_\Phi^2}{t_{12}^4t_{34}^4}\left[36f+z\left(4z^2-2z-32\right)f'+2z^2\left(7z^2+z+7\right)f''\right.
\end{aligned}$$

$$+z^3(z-1)(8z+4)f^{(3)}+z^4(z-1)^2f^{(4)}\big], \quad (6.103)$$

where all f and its derivatives are functions of z. All 4pt functions are defined in terms of the same function $f(z)$, so that if one solves the correlators of primaries, the rest of the multiplet is easily solved by evaluating the derivatives.

6.1.5 Bootstrapping the supercorrelators

So far we have exploited the symmetry constraints on two and four point functions of operators in the displacement multiplet, rewriting them in terms of an unknown function $f(z)$. In this section, we develop an in-depth leading and next-to-leading order computation of the chiral-anti-chiral correlator, as the chiral-chiral is obtained similarly.

Conformal blocks

We are interested in calculating the function $f(z)$ which parameterizes all the 4-pt functions of operators in the displacement multiplet of the 1/2-BPS defect of $\mathcal{N}=4$ theories (6.103). One fundamental ingredient to the program is the conformal blocks of the $\mathfrak{psu}(1,1|2)$ conformal group. In order to calculate the blocks for the chiral-antichiral 4pt function, we need to employ a conformal partial wave analysis, where the s-channel is explored and the Casimir operator labels the irreducible representations exchenged in the OPE.

A differential representation for the generators is given by

$$\begin{aligned}
P &= -\partial_t, \qquad\qquad D = -t\partial_t - \tfrac{1}{2}\theta_a\partial^a - \tfrac{1}{2}\bar\theta^a\bar\partial_a - \Delta,\\
K &= -t^2\partial_t - (t+\theta\bar\theta)\theta_a\partial^a - (t-\theta\bar\theta)\bar\theta^a\bar\partial_a - (\theta\bar\theta)^2\partial_t - 2\,t\,\Delta + j_0\,\theta\bar\theta,\\
Q^a &= \sqrt{2}\left(\partial^a - \bar\theta^a\partial_t\right),\\
\bar Q_a &= -\sqrt{2}\left(\bar\partial_a - \theta_a\partial_t\right),\\
S^a &= -i\sqrt{2}\left[(t+\theta\bar\theta)\partial^a - (t-\theta\bar\theta)\bar\theta^a\partial_t - 2\bar\theta^a\bar\theta^b\bar\partial_b - (2\Delta+j_0)\bar\theta^a\right],\\
\bar S_a &= -i\sqrt{2}\left[(t-\theta\bar\theta)\bar\partial_a - (t+\theta\bar\theta)\theta_a\partial_t - 2\theta_a\theta_b\partial^b - (2\Delta-j_0)\theta_a\right],\\
R_a{}^b &= \theta_a\partial^b - \bar\theta^b\bar\partial_a - \tfrac{1}{2}\,\delta_a^b\,(\theta_c\partial^c - \bar\theta^c\bar\partial_c).
\end{aligned} \tag{6.104}$$

Note that the twist operator J_0 is absent since it is a central extension of the algebra, i.e it annihilates all generators and is itself a Casimir. In fact, it means that it is a quantum number that is unchanged by the action of any element of the $\mathfrak{su}(1,1|2)$. In this respect, by projecting out this generator we work in $\mathfrak{psu}(1,1|2)$.

The quadratic Casimir operator is given by

$$C^{(2)} = D^2 - \frac{1}{2}\{K,P\} + \frac{i}{8}\left[\bar S_a, Q^a\right] - \frac{i}{8}\left[S^a, \bar Q_a\right] - \frac{1}{2}R_a{}^bR_b{}^a, \tag{6.105}$$

which yields the eigenvalue

$$c_2 = \Delta(\Delta+1), \tag{6.106}$$

when acting in a highest weight state $[\Delta,0]$.

By acting with the quadratic Casimir (6.105) in the insertions $1,2$ of (6.20),[12] we generate a differential equation for the blocks G_Δ. We define the differential operator

$$\mathcal{D}_{1,2}=-\frac{1}{2}C^{(2)}_{1,2}, \tag{6.107}$$

where $C^{(2)}_{1,2}$ is given by (6.105) with every operator acting on the points t_1 and t_2, as in

$$C^{(2)}_{1,2}=\{D_1+D_2,D_1+D_2\}-\frac{1}{2}\{K_1+K_2,P_1+P_2\}+\cdots \tag{6.108}$$

Then the differential equation is[13]

$$\mathcal{D}_{1,2}\left(\frac{f(\mathcal{Z})}{\langle 1\bar{2}\rangle\langle 3\bar{4}\rangle}\right)=c_2\left(\frac{f(\mathcal{Z})}{\langle 1\bar{2}\rangle\langle 3\bar{4}\rangle}\right). \tag{6.109}$$

Collecting the bosonic term, which accounts for the 4pt function of primaries, we have the differential equation

$$z\left((2-z)f'(z)-(z-1)zf''(z)\right)=\Delta(\Delta+1)f(z), \tag{6.110}$$

whose solution gives us the blocks

$$G_\Delta(z)=(-z)^\Delta{}_2F_1(\Delta,\Delta,2\Delta+2;z). \tag{6.111}$$

Leading Order

We start by developing the weak coupling description of the super-primary operators in terms of the Lagrangean fields. Our goal is to perturb the theory around the free limit, where all interactions vanish.

Recalling the definition of $\mathbb{R}$ and $\bar{\mathbb{R}}$ from (6.96), we are looking for an object with the same $\mathfrak{su}(2)$ R-symmetry index structure, and quantum numbers $[1,1,0]$ and $[1,-1,0]$. Luckily, the quiver structure of the theory is constraining enough to arrive at the ansatz

$$\mathbb{R}(t)=\frac{2i}{k}\begin{pmatrix}(\tilde{\mu}_I)^{\dot{2}}{}_{\dot{1}} & 0\\ 0 & (\tilde{\mu}_{I+1})^{\dot{2}}{}_{\dot{1}}\end{pmatrix}\quad\text{and}\quad\bar{\mathbb{R}}(t)=-\frac{2i}{k}\begin{pmatrix}(\tilde{\mu}_I)^{\dot{1}}{}_{\dot{2}} & 0\\ 0 & (\tilde{\mu}_{I+1})^{\dot{1}}{}_{\dot{2}}\end{pmatrix}. \tag{6.112}$$

Which naturally satisfies the complex structure, because the moment maps are mapped to each other by complex conjugation.

Ignoring overall factors, which can be absorbed into the C_Φ constant, one evaluates the 4pt function[14]

$$\langle\mathbb{R}(t_1)\bar{\mathbb{R}}(t_2)\mathbb{R}(t_3)\bar{\mathbb{R}}(t_4)\rangle_{\mathcal{W}}=\frac{C_\Phi\lambda^4}{t_{12}^2t_{34}^2}\left(1-\left(1+\frac{1}{N^2}\right)z+z^2\right), \tag{6.113}$$

[12] Recalling the superspace structure, the particle numbers $1,2$ are given by the coordinates $t_1,\theta_1,t_2,\theta_2$.

[13] Computations are carried out in the conformal frame: $t_3\to 0$, $t_4\to\infty$.

[14] We take $N_I=N$ and the 't Hooft coupling $\lambda=N/k$.

where we have performed Wick contractions following a free theory of $\mathfrak{su}(2)$ charged bosons

$$\langle \bar{\tilde{q}}_{\dot{a}}(t_1)_i^{\hat{j}} \tilde{q}^{\dot{b}}(t_2)_{\hat{l}}^{k} \rangle = \delta_{\dot{b}}^{\dot{a}} \delta_i^k \delta_{\hat{l}}^{\hat{j}} \frac{1}{|t_{12}|}. \tag{6.114}$$

From (6.113), one can read off the leading order piece of $f(z)$ as

$$f^{(0)}(z) = 1 - \left(1 + \frac{1}{N^2}\right) z + z^2. \tag{6.115}$$

Given the leading order $f^{(0)}(z)$, we now need to look for its conformal block expansion in the s-channel OPE of (6.113). We now have to expand (6.115), which yields the conformal block expansion

$$f^{(0)}(z) = 1 + \sum_{n=0}^{\infty} c_n^{(0)} \, G_{n+1}(z). \tag{6.116}$$

In order to extract the coefficients c_n that render (6.116) true, one power expands the equation in z around the origin. This procedure reorganizes the blocks and generates a set of recursive equations for the coefficients. In the large N limit, we have

$$c_n^{(0)} = \frac{\sqrt{\pi} 4^{-n-1} (n+2) \left((n+1)^2 + n\right) \Gamma(n+1)}{\Gamma\left(n + \frac{3}{2}\right)}, \tag{6.117}$$

which tells us that the primary itself flows through the s-channel OPE, since we need a block of $\Delta = 1$ to reproduce the linear term $\sim z$ in $f^{(0)}(z)$. This set-up is dissimilar to the $\mathcal{N}=6$ counterpart, where the OPE was in the double-particle limit. In the next session we provide the first order perturbation of this result, accounting for the anomalous dimensions of the exchanged operators.

Next-to-Leading Order

Following the perturbation scheme, we can relate the ansatz function to the conformal block expansion (6.30), which can now be transformed into a "hat" equation in the χ variable as

$$\hat{f}(\chi) = \frac{1}{\chi^2} + \sum_{\Delta=1}^{\infty} c_\Delta \hat{G}_\Delta(\chi), \tag{6.118}$$

where we have

$$\hat{G}_\Delta(\chi) = \chi^{-2} G_\Delta\left(\frac{\chi}{\chi - 1}\right). \tag{6.119}$$

Expanding both sides of (6.118) in $\mathcal{O}(\epsilon)$ yields

$$\hat{f}^{(1)}(\chi) = \sum_{n=0}^{\infty} \left(c_n^{(1)} \hat{G}_{\Delta_n^{(0)}}(\chi) + c_n^{(0)} \gamma_n^{(1)} \partial_\Delta \hat{G}_\Delta(\chi)|_{\Delta=\Delta_n^{(0)}} \right), \tag{6.120}$$

allowing for the extraction of the anomalous dimensions and corrections to the conformal block expansion.

Our minimal ansatz is a priori a three-parameter family of functions, as the constants q_{-1}, r_{-1}, r_{-2} are free. We can always absorb one of them in the definition of the perturbation parameter ϵ. Matching the expansion around $z\to 0$ of (6.120) allows us to reach a solution for any value of these parameters, which means that the conformal block expansion does not constrain the space of solutions for the given ansatz as in the $\mathcal{N}=6$ case, where we had a regularity condition.

Instead, we reach a recursive set of equations which are solved for $c_n^{(1)}$ and γ_n as a function of the free parameters. We have an analytic function for the anomalous dimensions

$$\gamma_n = \frac{n(n+3)}{n(n+3)+1} r_{-2} + r_{-1}, \tag{6.121}$$

and the first corrections read

$$c_n^{(1)} = q_{-1} - r_{-2}, \frac{1}{4}\left(q_{-1} + r_{-2} - 4r_{-1}\right), \frac{1}{90}\left(6q_{-1} - 56r_{-2} + 25r_{-1}\right), \cdots \tag{6.122}$$

The anomalous dimensions can be checked by applying the orthogonality relation[15]

$$-\frac{1}{2i\pi}\oint \frac{1}{(1-z)^2} G_{1+n}(z) G_{-2-m}(z) = \delta_{n,m}. \tag{6.123}$$

Chiral-Chiral correlator

In the same spirit of bootstrapping the chiral-anti-chiral correlator (6.26) with a minimal ansatz (6.43), one can also follow the same procedure for the chiral-chiral correlator (6.27). The partial wave expansion of the correlator

$$h(\chi) = 1 + \sum_{\Delta>0} \tilde{c}_\Delta \tilde{G}_\Delta(\chi). \tag{6.124}$$

We start by deriving the superconformal blocks which account for the exchange of operators in the s-channel of (6.21), which is given by the same super-Casimir insertion (6.105). This again generates a differential equation for the superblocks $\tilde{G}_\Delta(\chi)$, which yields the solution

$$\tilde{G}_\Delta(\chi) = \chi^\Delta {}_2F_1(\Delta, \Delta, 2+2\Delta, -\chi). \tag{6.125}$$

The leading order coefficients are again given by the Wick contractions with the super-primaries (6.95), which gives us $h^{(0)}(\chi)$ as

$$h^{(0)}(\chi) = 1 + \left(1 + \frac{1}{N^2}\right)\chi + \chi^2. \tag{6.126}$$

Again, expanding (6.124) in terms of the operators in s-channel, we have

$$1 + \left(1 + \frac{1}{N^2}\right)\chi + \chi^2 = 1 + \sum_{n=0}^{\infty} \tilde{c}_n^{(0)} \tilde{G}_{1+n}(\chi), \tag{6.127}$$

which is solved by for $\tilde{c}_n^{(0)} = c_n^{(0)}$, relating the coefficients of the chiral-chiral and chiral-antichiral in a trivial way.

[15] Notice that the Casimir eigenvalue equation always defines a Sturm-Liouville problem, allowing for the definition of orthogonality relations among the blocks.

The next-to-leading order, is given by matching (6.124) in first order in the perturbation parameter ϵ. In this order the minimal ansatz generates the function $h^{(1)}(\chi)$ as[16]

$$h^{(1)}(\chi)=\frac{\chi}{1-\chi}\left(-q\left(\frac{\chi}{1-\chi}\right)-r\left(\frac{1}{1-\chi}\right)\log(\chi)+\left(r\left(\frac{1}{1-\chi}\right)+r\left(\frac{\chi}{\chi-1}\right)\right)\log(1-\chi)\right). \tag{6.128}$$

Plugging it in the expansion we have

$$h^{(1)}(\chi)=\sum_{n=0}^{\infty}\left((\tilde{c}_n^{(0)}+\epsilon\tilde{c}_n^{(1)})\tilde{G}_{\gamma_n^{(1)}+n+1}(\chi)\right)|_{\mathcal{O}(\epsilon)}, \tag{6.129}$$

which can be solved for the coefficient corrections $c_n^{(1)}$ and the anomalous dimensions $\gamma_n^{(1)}$. Contrary to the chiral-antichiral correlator, the solution to (6.129) constrains one of the parameters of the ansatz, namely, the non-trivial solutions are spanned by

$$r_{-2}=-q_{-1}. \tag{6.130}$$

The anomalous dimensions are given by

$$\tilde{\gamma}_n^{(1)}=-\frac{(n^2+n-3)\,q_{-1}+r_{-1}}{n^2+n-1}, \tag{6.131}$$

and the first coefficient corrections

$$\tilde{c}_n^{(1)}=\frac{1}{2}\left(2r_{-1}-3q_{-1}\right),\frac{1}{24}\left(-28q_{-1}-15r_{-1}\right),\frac{1}{900}\left(-5179q_{-1}-526r_{-1}\right),\cdots \tag{6.132}$$

Some comments about the degrees of freedom of the ansatz are in order. In the ABJM case, the solution presented no free parameters, which could be seen as consequence of two features. The first one is the fact that the conformal block expansion is completely regular for $\chi\to 0$ (and by crossing $\chi\to 1$), which posits a set of constraints in the coefficients r_m and q_l, fixing all the coefficients r_m in terms of the coefficients q_l, apart from r_0 and r_{-1}. The second important feature is the selection rules for the mixing of $\mathbb{F}\bar{\mathbb{F}}$, which only allows long multiplets of dimension $\Delta_{\text{long}}>3$. In turn it generates a constraint in the conformal block expansion which allows one to fix the remaining degrees of freedom.

The $\mathcal{N}=4$ case is different, since there is no known constraint in the dimension of exchanged long multiplets, defining a less constrained set-up. The block expansion of the chiral-chiral correlator yields one constraint, which fixes one of the parameters (6.130). Once the ansatz is fixed, we can extract the CFT data by a small χ limit of the conformal block expansion. This procedure yields a recursive set of equations for the anomalous dimension and first order correction to the coefficients $\gamma_n^{(1)}$ and $c_n^{(1)}$ as functions of the parameters q_{-1},r_{-1}. As we can always absorb one of the free parameters in the definition of ϵ, we are left with one free parameter.

[16] Notice we are not working with the hat function $\hat{h}(\chi)$, but with $h(\chi)$.

Chapter 7

Outlook

In this thesis we have presented three distinct projects regarding the study of BPS Wilson loops in CSm theories. Let us briefly summarize the main results and comment on possible ramifications of this work.

A new formulation of BPS loops: In chapter 4 we have discovered a new gauge for the formulation of BPS loops in ABJM theory which is written in the natural language of the supertrace of a connection. In this new gauge we pay the price of having a constant piece $\pm 1/4$ in the connections, but gain the ability of identifying the moduli space and to write the operators in a manifeslty reparameterization invariant way.

In this new formulation, we have constructed a systematic framework for the definition of 1/6-BPS fermionic loops from the 1/6-BPS bosonic loops via the prescription of deforming the loop connections [26]. The deformation introduces free parameters which endow the moduli space of connections with a manifold structure, which makes it possible to identify the moduli space of Wilson loops by modding out connections that are related by gauge transformations, constituting a novel feature in the study of Wilson loops in CSm theories.

This prescription was first generalized in [8], where these techniques are applied in the context of BPS Wilson loops of arbitrary theories with $\mathcal{N} \geq 2$ in S^3. There it was possible to identify the moduli space of loops as generalized cones in complex space, which are essentially the generalization of the conifold to higher dimensions. It was also pointed out that such spaces may have interesting physical features, so that it would be pertinent to study it from a geometrical point of view.

In [7] the same techniques were applied in the context of $\mathcal{N} = 4$ CSm theories, where it yields a two-parameter family of loops by the deformation of bosonic loops. Apart from defining new loops, these deformations can be seen as a general organizational principle for structuring relations between loops with different BPS ranks, which is highly desirable since CSm theories often carry a rich moduli space of loops.

Deformations of the 1/2-BPS loop in ABJM: The second study concerns the wavy-line deformation of the 1/2-BPS Wilson loop of ABJM theory and the calculation of the Bremmstrahlung function. The wavy-line prescription specifies a deformation of the Wilson operator contour, and as the original matter couplings to the operator were contour dependent, it was unclear whether one should consider a deformation of these couplings as well.

Carrying out the perturbative calculations, we have checked that indeed one must correct the fermionic couplings of the loop in order to satisfy a universal constraint of Wilson lines in superconformal field theories [11]. As a historical note, the corrections to the fermionic couplings (5.26) inspired the idea to look for a gauge transformation which ultimately lead to the new gauge of chapter 4.

Apart from the developments regarding the wavy-line prescription, we also applied a novel regularization method, which we call the substitution method, firstly defined on [31], to calculate the divergent integrals from the diagrams. This method is more computationally efficient than the standard DRED scheme, since it relies on substitution rules rather than explicit integration over the loop contour, and it could in principle be used to regularize any loop integral.

We have managed to calculate the wavy-line deformation at order λ both in the DRED scheme and the new regularization method, with perfect agreement. The computation at order λ^2 is still incomplete, due to the difficulties in treating diagrams (e) and (f) which contain more than two loop insertions. The next step in the direction of finishing these calculations would consist in a generalization of the substitution method for the regularization of integrals with three and four sources of divergence.

The 1/2-BPS defect of $\mathcal{N} = 4$ Chern-Simons-matter theories: In this study, we carried out the description of the 1/2-BPS Wilson lines known as the ψ_1 solutions as dCFTs, where we have identified the symmetry group of the defect as the subgroup of the bulk symmetry $\mathfrak{su}(1,1|2) \subset \mathfrak{osp}(4|4)$, as well as the complete displacement multiplet.

By casting the displacement multiplet as a chiral superfield, we have successfully employed a superconformal bootstrap program to the study of 4pt functions of operators of that multiplet, where we have derived the relevant superconformal blocks via the techniques of [14], and by expanding the OPEs in this basis, we achieved a conformal partial wave expansion allowing for the extraction of the anomalous dimensions of the exchanged operators in the dCFT in the strong coupling limit via a holographic ansatz parameterized by three unknowns.

By considering the chiral-chiral and chiral-anti-chiral 4pt functions, we managed to fix one of the parameters, and unlike the ABJM case [13], there is no known constraints on the anomalous dimensions from pure representation theory of the exchanged operators, which means the $\mathcal{N} = 4$ case is less constrained than the $\mathcal{N} = 6$. In turn, we cannot fix any more parameters of the minimal ansatz, so that we end up with a one parameter family of solutions once we absorb one the two remaining parameters in the definition of the perturbation parameter ϵ.

We also note that the program we have used to study this defect in chapter 6 can be applied to a wide range of defects in different dimensions. For instance in [33], we have a complete classification of maximally supersymmetric defects in theories of varying supersymmetry in $3 \leq d \leq 6$ from the group theory point of view, which could be used as a guide to similar constructions.

Another future direction which requires attention is the holographic description of the defects. A good starting point is to consider the well known holographic description of the 1/2-BPS Wilson line of ABJM and try to reason towards a holographic description of the studied 1/2-BPS line of $\mathcal{N}=4$.

The original formulation of ABJM theory with level k is given in terms of 11-dimensional supergravity in $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$, which gives rise to a 10-dimensional description in terms of Type IIA strings in $\mathrm{AdS}_4 \times \mathbb{CP}^3$ by considering the limit of large k [12]. Within the string theory description, the Wilson line is reproduced by the string world-sheet which is bounded by the Wilson line contour via (1.9), so that insertions on the dCFT associated to the 1/2-BPS defect can be studied by sourcing fields in the string world-sheet.

In [13], the authors managed to calculate the 4pt functions of displacement multiplet operators in this set-up, by explicitly constructing the string world-sheet solution and expanding the Nambu-Goto action as a weakly interacting effective field theory which reproduces the dynamics of the dCFT in the strong coupling limit.

The string world-sheet which is given by an $\mathrm{AdS}_2 \subset \mathrm{AdS}_4$ and localized in the $\mathbb{CP}^3$, such that in the boundary of the AdS_4 it reproduces the Wilson line contour. This way, we have the string action

$$S_B = \frac{1}{2} T \int d^2\sigma \sqrt{h} h^{\mu\nu} \left[\frac{1}{z^2} \left(\partial_\mu x^r \partial_\nu x_r + \partial_\mu z \partial_\nu z \right) + 4 G^{\mathbb{CP}^3}_{MN} \partial_{[\mu} Y^M \partial_{\nu]} Y^N \right], \tag{7.1}$$

with $x^r = (x^0, x^1, x^2)$ parameterizing the Euclidean boundary of AdS_4, z the AdS radial coordinate, and Y^N general coordinates of the $\mathbb{CP}^3$ space.

By expanding (7.1) around the AdS_2 world-sheet geometry, one generates an effective field theory in the coordinates transverse to the world-sheet, which are in one-to-one correspondence with the operators of the displacement multiplet. This way, the correlators of the multiplet can be calculated holographically by insertions of the fluctuations with corresponding representations.

The holographic set up for the $\mathcal{N}=4$ case is quite different from the ABJM case, since there is no known description of the 1/2-BPS defects in terms of 10-dimensional gravity. Naively, we would expect that the description of the defect could be attained by considering orbifolds over an 11-dimensional description in terms of M2-branes, which could generate a 10-dimensional description via the $k \to \infty$ compactification.

As a general naive prescription, one could follow an analog M2-brane construction of the original ABJM formulation [12], where the 10-dimensional description is achieved by considering a $\mathbb{Z}_k$ orbifold of S^7. In this spirit, the description of M2-branes which are $\mathfrak{su}(1,1|2)$ invariant are given in terms of the $\mathbb{Z}_k$ orbifold followed by an extra $\mathbb{Z}_p \otimes \mathbb{Z}_q$ orbifold of S^7, for integers p and q [38, 39].

These extra orbifolds, which are necessary to reproduce the correct R-symmetry breaking of the S^7 to the $\mathfrak{su}(2)_A \times \mathfrak{su}(2)_B$ of the $\mathcal{N}=4$ CSm theories, end up complicating the process of taking the large k limit to reproduce a 10-dimensional description, since the coordinates of the compactified manifold do not reproduce the R-symmetry manifestly, as in the ABJM case. In turn, it makes the process of identifying the operators of the displacement multiplet with transverse fluctuations non-trivial, which calls for a more in-depth analysis of the holographic construction.

Appendix A

Useful notations and definitions in ABJ(M)

A.1 Supersymmetry conventions

We work in three-dimensional Euclidean space $g_{\mu\nu} = \mathrm{diag}(1,1,1)$, with the gamma-matrices chosen as $(\gamma^\mu) = \{-\sigma^3, \sigma^1, \sigma^2\}$. The spinor indices are lowered and raised as $(\gamma^\mu)^\alpha{}_\beta = \epsilon^{\alpha\rho}(\gamma^\mu)_\rho{}^\delta \epsilon_{\beta\delta}$, where

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{A.1}$$

Omitted spinor indices $\alpha = \pm$ follow the convention

$$\phi\psi = \phi^\alpha \psi_\alpha = -\phi_\alpha \psi^\alpha. \tag{A.2}$$

The supersymmetry transformations are the ones written in [4]:

$$\begin{aligned}
\delta A_\mu &= \frac{4\pi i}{k} \bar{\Theta}^{IJ\alpha} (\gamma_\mu)_\alpha{}^\beta \left(C_I \psi_{J\beta} + \frac{1}{2} \epsilon_{IJKL} \bar{\psi}^K_\beta \bar{C}^L \right), \\
\delta \hat{A}_\mu &= \frac{4\pi i}{k} \bar{\Theta}^{IJ\alpha} (\gamma_\mu)_\alpha{}^\beta \left(\psi_{J\beta} C_I + \frac{1}{2} \epsilon_{IJKL} \bar{C}^L \bar{\psi}^K_\beta \right), \\
\delta \bar{\psi}^I_\beta &= -2i \bar{\Theta}^{IJ\alpha} (\gamma^\mu)_{\alpha\beta} D_\mu C_J - \frac{4\pi i}{k} \bar{\Theta}^{IJ}_\beta (C_J \bar{C}^K C_K - C_K \bar{C}^K C_J) \\
&\quad - \frac{8\pi i}{k} \bar{\Theta}^{JK}_\beta C_J \bar{C}^I C_K - 2i \bar{\epsilon}^{IJ}_\beta C_J \, , \\
\delta \psi_I^\beta &= -i \bar{\Theta}^{KL\alpha} \epsilon_{IJKL} (\gamma^\mu)_\alpha{}^\beta D_\mu \bar{C}^J + \frac{2\pi i}{k} \bar{\Theta}^{KL\beta} \epsilon_{IJKL} (\bar{C}^J C_M \bar{C}^M \\
&\quad - \bar{C}^M C_M \bar{C}^J) + \frac{4\pi i}{k} \bar{\Theta}^{KL\beta} \epsilon_{KLMN} \bar{C}^M C_I \bar{C}^N - i \bar{\epsilon}^{KL\beta} \epsilon_{IJKL} \bar{C}^J \, , \\
\delta C_I &= \bar{\Theta}^{KL^\alpha} \epsilon_{IJKL} \bar{\psi}^J_\alpha \, , \\
\delta \bar{C}^I &= 2 \bar{\Theta}^{IJ\alpha} \psi_{J\alpha} \, ,
\end{aligned} \tag{A.3}$$

where $\Theta^{IJ} = \bar{\theta}^{IJ} - (\dot{x}^\mu \gamma_\mu) \bar{\epsilon}^{IJ}$ is a conformal Killing spinor, with superconformal parameter

$$\bar{\vartheta}^{IJ} = \pm i \bar{\epsilon}^{IJ} \sigma^3. \tag{A.4}$$

The Poincaré ($\bar{\theta}^{IJ}$) and superconformal parameters ($\bar{\vartheta}^{IJ}$) satisfy the relations

$$\bar{\theta}^{IJ} = -\bar{\theta}^{JI}\,, \qquad \theta_{IJ} = \frac{1}{2}\epsilon_{IJKL}\bar{\theta}^{KL}\,, \tag{A.5}$$

$$\bar{\vartheta}^{IJ} = -\bar{\vartheta}^{JI}\,, \qquad \vartheta_{IJ} = \frac{1}{2}\epsilon_{IJKL}\bar{\vartheta}^{KL}\,. \tag{A.6}$$

In Euclidean space, there is no reality condition, i.e., $\bar{\theta} \neq \theta^\dagger$ and $\bar{\vartheta} \neq \vartheta^\dagger$.

A.2 Feynmann rules

From [23] we have the following Feynman rules of ABJM. We only list the terms which contribute to $\mathcal{O}(\lambda^2)$ in the calculation of the 1/2-BPS loop of section 5.1.

- Tree-level vector propagators are:

$$\left\langle A_\mu^{a(1)}(x) A_\nu^{b(1)}(y)\right\rangle^{(0)} = \delta^{ab}\left(\frac{2\pi i}{k}\right)\frac{\Gamma\left(\frac{3}{2}-\epsilon\right)}{2\pi^{\frac{3}{2}-\epsilon}}\epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{[(x-y)^2]^{\frac{3}{2}-\epsilon}}\,, \tag{A.7}$$

$$\left\langle A_\mu^{a(2)}(x) A_\nu^{b(2)}(y)\right\rangle^{(0)} = -\delta^{ab}\left(\frac{2\pi i}{k}\right)\frac{\Gamma\left(\frac{3}{2}-\epsilon\right)}{2\pi^{\frac{3}{2}-\epsilon}}\epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{[(x-y)^2]^{\frac{3}{2}-\epsilon}}\,; \tag{A.8}$$

- One-loop vector propagators:

$$\left\langle A_\mu^{a(1)}(x) A_\nu^{b(1)}(y)\right\rangle^{(1)} = \delta^{ab}\left(\frac{2\pi}{k}\right)^2 N_2 \frac{\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{4\pi^{3-2\epsilon}}\left[\frac{\delta_{\mu\nu}}{[(x-y)^2]^{1-2\epsilon}} - \partial_\mu\partial_\nu\frac{(x-y)^{2\epsilon}}{4\epsilon(1+2\epsilon)}\right], \tag{A.9}$$

$$\left\langle A_\mu^{a(2)}(x) A_\nu^{b(2)}(y)\right\rangle^{(1)} = \delta^{ab}\left(\frac{2\pi}{k}\right)^2 N_1 \frac{\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{4\pi^{3-2\epsilon}}\left[\frac{\delta_{\mu\nu}}{[(x-y)^2]^{1-2\epsilon}} - \partial_\mu\partial_\nu\frac{(x-y)^{2\epsilon}}{4\epsilon(1+2\epsilon)}\right]; \tag{A.10}$$

- Scalar propagator:

$$\left\langle (C_I)_i^{\hat{j}}(x)(\bar{C}^J)(y)_{\hat{k}}^{l}\right\rangle^{(0)} = \delta_I^J\delta_i^l\delta_{\hat{k}}^{\hat{j}}\frac{\Gamma\left(\frac{1}{2}-\epsilon\right)}{4\pi^{\frac{3}{2}-\epsilon}}\frac{1}{[(x-y)^2]^{\frac{1}{2}-\epsilon}}\,; \tag{A.11}$$

- Tree-level fermion propagator:

$$\left\langle (\psi_I^\alpha)_{\hat{i}}^{j}(x)(\bar{\psi}_\beta^J)_k^{\hat{l}}(y)\right\rangle^{(0)} = -i\delta_I^J\delta_{\hat{i}}^{\hat{l}}\delta_k^j\frac{\Gamma\left(\frac{3}{2}-\epsilon\right)}{2\pi^{\frac{3}{2}-\epsilon}}\frac{(\gamma^\mu)^\alpha{}_\beta(x-y)_\mu}{[(x-y)^2]^{\frac{3}{2}-\epsilon}}\,; \tag{A.12}$$

- One-loop fermion propagator:

$$\left\langle (\psi_I^\alpha)_{\hat{i}}^{j}(x)(\bar{\psi}_\beta^J)_k^{\hat{l}}(y)\right\rangle^{(1)} = -i\delta_I^J\delta_{\hat{i}}^{\hat{l}}\delta_k^j\left(\frac{2\pi}{k}\right)(N_2-N_1)\frac{\Gamma^2\left(\frac{1}{2}-\epsilon\right)}{16\pi^{3-2\epsilon}}\frac{1}{[(x-y)^2]^{1-2\epsilon}}\,. \tag{A.13}$$

The interactions are given by

- Gauge cubic vertex

$$-\frac{ik}{12\pi}\epsilon^{\mu\nu\rho}\int d^3x\ f_1^{abc}A_\mu^{a(1)}A_\nu^{b(1)}A_\rho^{c(1)}\,,\quad \frac{ik}{12\pi}\epsilon^{\mu\nu\rho}\int d^3x\ f_2^{abc}A_\mu^{a(2)}A_\nu^{b(2)}A_\rho^{c(2)}\,, \tag{A.14}$$

where f_i^{abc} is the structure constant of $U(N_i)$;

- Gauge-fermion cubic vertex

$$-\int d^3x\ \mathrm{Tr}\left(\bar\psi^I\gamma^\mu\psi_i A_\mu^{(1)} - \bar\psi^I\gamma^\mu A_\mu^{(2)}\psi_I\right). \tag{A.15}$$

We work with hermitian generators of $U(N_1)$ and $U(N_2)$. The colour convetions are

$$\mathrm{Tr}\,(T^aT^b) = \delta^{ab},\quad \mathrm{Tr}\,(\hat T^{\hat a}\hat T^{\hat b}) = \delta^{\hat a\hat b}, \tag{A.16}$$

and

$$\sum_{a=1}^{N_1^2}(T^a)_{ij}(T^a)_{kl} = \delta_{ij}\delta_{kl},\quad \sum_{\hat a=1}^{N_2^2}(\hat T^{\hat a})_{ij}(\hat T^{\hat a})_{kl} = \delta_{\hat i\hat j}\delta_{\hat k\hat l}. \tag{A.17}$$

A.3 Proof of supersymmetry of the 1/2-BPS loop of ABJM

In order to prove supersymmetry of the 1/2-BPS loop (4.20), we need to act with 1/2 of the supercharges of ABJ(M) and prove that the constraint of supersymmetry is respected. For self-consistency and clarity, we repeat the definitions of the operator and supercharges that are used in the variation,

$$\mathcal{W} = i\ \mathrm{sTr}\,\mathcal{P}\exp\left(i\oint\mathcal{L}_{1/2\text{-BPS}}d\tau\right), \tag{A.18}$$

with the superconnection now given by

$$\mathcal{L}_{1/2\text{-BPS}} = \begin{pmatrix} \mathcal{A}^{(1)} & \sqrt{-\frac{4\pi i}{k}}|\dot x|\eta_I^\alpha\bar\psi_\alpha^I \\ \sqrt{-\frac{4\pi i}{k}}|\dot x|\psi_I^\alpha\bar\eta_\alpha^I & \mathcal{A}^{(2)} \end{pmatrix}, \tag{A.19}$$

with

$$\begin{aligned} \mathcal{A}^{(1)} &= A_\mu^{(1)}\dot x^\mu - \tfrac{2\pi i}{k}|\dot x|M_J^I C_I\bar C^J - \tfrac{|\dot x|}{4|x|}\,,\\ \mathcal{A}^{(2)} &= A_\mu^{(2)}\dot x^\mu - \tfrac{2\pi i}{k}|\dot x|M_J^I \bar C^J C_I + \tfrac{|\dot x|}{4|x|}\,, \end{aligned} \tag{A.20}$$

and

$$\eta_I^\alpha = \sqrt2(s\Pi_+)^\alpha\delta_I^1,\qquad \bar\eta_\alpha^I = \sqrt2(\Pi_+\bar s)_\alpha\delta_1^I,\qquad M_J^I = \mathrm{diag}(-1,1,1,1). \tag{A.21}$$

We want to take the variation with respect to supersymmetry transformations parameterized by

$$\Theta^{1I} = \bar\theta^{1I} + \bar\vartheta^{1I}(x^\mu\gamma_\mu) = \bar\theta^{1I}(1-\dot x^\mu\gamma_\mu), \tag{A.22}$$

$$\Theta^{IJ} = \bar\theta^{IJ} + \bar\vartheta^{IJ}(x^\mu\gamma_\mu) = \bar\theta^{IJ}(1+\dot x^\mu\gamma_\mu),\quad I,J\neq 1, \tag{A.23}$$

which means we have chosen

$$\bar{\vartheta}^{1I\alpha} = i\bar{\theta}^{1I\beta}(\sigma^3)_\beta{}^\alpha, \quad \bar{\vartheta}^{IJ\alpha} = -i\bar{\theta}^{IJ\beta}(\sigma^3)_\beta{}^\alpha, \quad I, J \neq 1. \tag{A.24}$$

We want to show that the variation of the superconnection reduces to a supercovariant derivative acting on a supermatrix $G \in \mathfrak{u}(N|M)$,

$$\delta_{susy}\mathcal{L}(\tau) = \mathfrak{D}_\tau G \equiv \partial_\tau G + i[\mathcal{L}, G]. \tag{A.25}$$

The supermatrix G has to be anti-diagonal, since the susy transformations of the bosonic fields do not contain any derivatives. Therefore we have

$$G = \begin{pmatrix} 0 & g_1 \\ \bar{g}_2 & 0 \end{pmatrix}, \tag{A.26}$$

so

$$\mathfrak{D}_\tau G = \begin{pmatrix} i\sqrt{-\frac{4\pi i}{k}}|\dot{x}|(\eta_I^\alpha \bar{\psi}_\alpha^I \bar{g}_2 - g_1 \psi_I^\alpha \bar{\eta}_\alpha^I) & \mathcal{D}_\tau g_1 \\ \mathcal{D}_\tau \bar{g}_2 & i\sqrt{-\frac{4\pi i}{k}}|\dot{x}|(\psi_I^\alpha \bar{\eta}_\alpha^I g_1 - \bar{g}_2 \eta_I^\alpha \bar{\psi}_\alpha^I) \end{pmatrix}, \tag{A.27}$$

where we define the dressed covariant derivatives,

$$\mathcal{D}_\tau g_1 = \partial_\tau g_1 + i(\mathcal{A}^{(1)} g_1 - g_1 \mathcal{A}^{(2)}), \tag{A.28}$$

$$\mathcal{D}_\tau \bar{g}_2 = \partial_\tau \bar{g}_2 - i(\bar{g}_2 \mathcal{A}^{(1)} - \mathcal{A}^{(2)} \bar{g}_2). \tag{A.29}$$

This way, the 1/2-BPS constraint (A.25) is satisfied once we find g_1 and $\bar{g}_2$ that satisfy the following conditions

$$\text{(A) } \delta\mathcal{A}^{(1)} = i\sqrt{-\frac{4\pi i}{k}}|\dot{x}|(\eta_I^\alpha \bar{\psi}_\alpha^I \bar{g}_2 - g_1 \psi_I^\alpha \bar{\eta}_\alpha^I). \qquad \text{(B) } \mathcal{D}_\tau g_1 = \sqrt{-\frac{4\pi i}{k}}|\dot{x}|\eta_I^\alpha \delta\bar{\psi}_\alpha^I.$$

$$\text{(C) } \mathcal{D}_\tau \bar{g}_2 = \sqrt{-\frac{4\pi i}{k}}|\dot{x}|\delta\psi_I^\alpha \bar{\eta}_\alpha^I. \qquad \text{(D) } \delta\mathcal{A}^{(2)} = i\sqrt{-\frac{4\pi i}{k}}|\dot{x}|(\psi_I^\alpha \bar{\eta}_\alpha^I g_1 - \bar{g}_2 \eta_I^\alpha \bar{\psi}_\alpha^I). \tag{A.30}$$

To calculate the LHS of (A.25), we apply the supersymmetry variation on the connection, which yields

$$\delta\mathcal{A}^{(1)} = \frac{8\pi i}{k} C_I \psi_1 (1 + \dot{x}^\mu \gamma_\mu) \bar{\theta}^{1I} + \frac{8\pi i}{k} \theta_{1I} (1 + \dot{x}^\mu \gamma_\mu) \bar{\psi}^1 \bar{C}^I, \tag{A.31}$$

$$\delta\mathcal{A}^{(2)} = \frac{8\pi i}{k} \psi_1 C_I (1 + \dot{x}^\mu \gamma_\mu) \bar{\theta}^{1I} + \frac{8\pi i}{k} \theta_{1I} (1 + \dot{x}^\mu \gamma_\mu) \bar{C}^I \bar{\psi}^1, \tag{A.32}$$

$$\begin{aligned} \delta\bar{\psi}_\alpha^1 &= 2i\gamma^\nu (1 + \dot{x}^\mu \gamma_\mu) \bar{\theta}^{1I} D_\nu C_I - \frac{4\pi i}{k} (1 + \dot{x}^\mu \gamma_\mu) \bar{\theta}^{1I} \left[C_I M_R{}^S \bar{C}^R C_S - M_R{}^S C_S \bar{C}^R C_I \right] \\ &- \frac{8\pi i}{k} (1 - \dot{x}^\mu \gamma_\mu) \bar{\theta}^{IJ} C_I \bar{C}^1 C_J - 2(\sigma_3) \bar{\theta}^{1I} C_I, \end{aligned} \tag{A.33}$$

$$\begin{aligned} \delta\psi_1^\alpha &= -2i\theta_{1I} (1 + \dot{x}^\mu \gamma_\mu) \gamma^\nu D_\nu \bar{C}^I + \frac{4\pi i}{k} \theta_{1I} (1 + \dot{x}^\mu \gamma_\mu) \left[\bar{C}^I M_R{}^S C_S \bar{C}^R - M_R{}^S \bar{C}^R C_S \bar{C}^I \right] \\ &+ \frac{8\pi i}{k} \theta_{IJ} (1 - \dot{x}^\mu \gamma_\mu) \bar{C}^I C_1 \bar{C}^J - 2\theta_{1I} (\sigma_3) \bar{C}^I \Big\}. \end{aligned} \tag{A.34}$$

The problem of finding the correct g_1 and $\bar{g}_2$ that satisfy the BPS condition reduces to the calculation of

$$\delta(\eta_I^\alpha \bar{\psi}_\alpha^I) = 4i\ \mathcal{D}_\tau(\eta_1^\alpha C_I)\ \bar{\theta}_\alpha^{1I}, \tag{A.35}$$

$$\delta(\psi_I^\alpha \bar{\eta}_\alpha^I) = -4i\ \theta_{1I}^\alpha\ \mathcal{D}_\tau(\bar{C}^I \bar{\eta}_\alpha^1). \tag{A.36}$$

Once we have (A.35) and (A.36), it is easy to see that

$$g_1 = \sqrt{-\frac{4\pi i}{k}4i(\eta_1^\alpha C_I)\bar{\theta}_\alpha^{1I}} \qquad \text{and} \qquad \bar{g}_2 = -\sqrt{\frac{-4\pi i}{k}4i\theta_{1I}^\alpha(\bar{C}^I\bar{\eta}_\alpha^1)} \tag{A.37}$$

automatically satisfy conditions (B) and (C). Using that $\bar{\eta}_\alpha^1\eta_1^\beta = \frac{1}{2}(1+\dot{x}^\mu\gamma_\mu)_\alpha{}^\beta$, we see that they also satify (A) and (D).

In order to derive (A.35) and (A.36), we can write

$$\begin{aligned}
&\delta(\eta_1^\beta\bar{\psi}_\beta^1) = \\
&\eta_1^\beta[\underbrace{-2i\bar{\Theta}^{1L\alpha}(\gamma^\mu)_{\alpha\beta}D_\mu C_L}_{\text{(I)}}\underbrace{-\frac{4\pi i}{k}\bar{\Theta}_\beta^{1L}(C_L\bar{C}^M C_M - C_M\bar{C}^M C_L)}_{\text{(II)}} \\
&\quad\underbrace{-\frac{8\pi i}{k}\bar{\Theta}_\beta^{IJ}C_I\bar{C}^1C_J}_{\text{(III)}}\underbrace{-2i(\bar{\epsilon}_\beta^{1L}C_L)}_{\text{(IV)}}].
\end{aligned} \tag{A.38}$$

Focusing on terms (II) and (III), we have

$$\begin{aligned}
\text{(II)+(III)} &= -\frac{4\pi i}{k}\bar{\Theta}_\beta^{1L}(C_L\bar{C}^M C_M - C_M\bar{C}^M C_L) - \frac{8\pi i}{k}\bar{\Theta}_\beta^{IJ}C_I\bar{C}^1C_J \\
&= -\frac{4\pi i}{k}\bar{\Theta}_\beta^{1L}(C_L\bar{C}^M C_M - C_M\bar{C}^M C_L + 2C_1\bar{C}^1C_L - 2C_L\bar{C}^1C_1) \\
&\quad \underbrace{-\frac{8\pi i}{k}(\bar{\Theta}_\beta^{IJ}C_I\bar{C}^1C_J,(I,J)\neq 1),}_{\text{III.b}}
\end{aligned} \tag{A.39}$$

where we have separated indices 1 from term (III). The term (III.b) vanishes when contracted with η,

$$\begin{aligned}
\eta_1^\beta\bar{\Theta}_\beta^{IJ}C_I\bar{C}^1C_J &= \sqrt{2}(s\Pi_+)^\beta\bar{\theta}^{IJ\alpha}(2\Pi_+)_{\alpha\beta}(C_I\bar{C}^1C_J) \\
&= 2-\sqrt{2}(s\Pi_+)^\beta(\Pi_+)_\beta^\alpha\bar{\theta}_\alpha^{IJ}(C_I\bar{C}^1C_J) \\
&= 2\sqrt{2}\,\underbrace{(s\Pi_+)^\beta(\Pi_-)_\beta^\alpha}_{0}\,\bar{\theta}_\alpha^{IJ}(C_I\bar{C}^1C_J),
\end{aligned}$$

where we have used the projector property

$$(\Pi_\pm)_\alpha^\beta = -(\Pi_\mp)_\alpha^\beta. \tag{A.40}$$

So we have that (II)+(III) evaluates to

$$\eta_1^\beta((\text{II})+(\text{III})) = -\frac{4\pi i}{k}\eta_1^\beta\bar{\Theta}_\beta^{1L}(C_L M_S^R\bar{C}^S C_R - M_S^R C_R\bar{C}^S C_L). \tag{A.41}$$

Focusing on the contraction of η with the Killing spinor, we can write

$$\begin{aligned}\eta_1^\beta \bar{\Theta}_\beta^{1L} &= \eta_1^\beta \bar{\theta}^{1I\rho}(1-\dot{x}^\mu\gamma_\mu)_{\rho\beta} \\ &= -\eta_{\beta 1}\bar{\theta}^{1I\rho}(1-\dot{x}^\mu\gamma_\mu)_\rho^{\ \beta} \\ &= \eta_{\beta 1}\bar{\theta}^{1I\rho}(1+\dot{x}^\mu\gamma_\mu)_\rho^{\ \beta},\end{aligned} \tag{A.42}$$

rewriting the coupling in terms of the projectors, it is easy to see that we have

$$\eta_1^\beta \bar{\Theta}_\beta^{1L} = 2\eta_1^\beta \bar{\theta}_\beta^{1I}. \tag{A.43}$$

which means (A.41) can be written as

$$\eta_1^\beta((\mathrm{II})+(\mathrm{III})) = -\frac{8\pi i}{k}\eta_1^\beta\bar{\theta}_\beta^{1L}(C_L M_S^R \bar{C}^S C_R - M_S^R C_R \bar{C}^S C_L). \tag{A.44}$$

We still have to evaluate the contributions of terms (I) and (IV) in (A.38). Getting back to term (I), we have

$$\begin{aligned}\eta_1^\beta(\mathrm{I}) &= -2i\eta_1^\beta\bar{\Theta}^{1L\alpha}(\gamma^\mu)_{\alpha\beta}D_\mu C_L \\ &= 2i\eta_{1\beta}\bar{\Theta}^{1L\alpha}(\gamma^\mu)_\alpha^{\ \beta}D_\mu C_L \\ &= 2i\eta_{1\beta}\bar{\theta}^{1L\rho}(1-\dot{x}^\nu\gamma_\nu)_\rho^{\ \alpha}(\gamma^\mu)_\alpha^{\ ;\beta}D_\mu C_L.\end{aligned} \tag{A.45}$$

Using

$$(1-\dot{x}^\nu\gamma_\nu)_\rho^{\ \alpha}(\gamma^\mu)_\alpha^{\ \beta} = -2\dot{x}^\mu\delta_\rho^\beta, \tag{A.46}$$

we reach

$$\eta_1^\beta(\mathrm{I}) = -4i\eta_{1\beta}\bar{\theta}^{1L\beta}\dot{x}^\mu D_\mu C_L \tag{A.47}$$

$$= 4i\eta_1^\beta\bar{\theta}_\beta^{1L}\dot{x}^\mu D_\mu C_L. \tag{A.48}$$

Adding up the terms (I), (II), and (III), contracted with the coupling η, we have

$$\begin{aligned}\eta_1^\beta((\mathrm{I})+(\mathrm{II})+(\mathrm{III})) &= 4i\eta_1^\beta\bar{\theta}_\beta^{1L}\dot{x}^\mu D_\mu C_L - \frac{8\pi i}{k}\eta_1^\beta\bar{\theta}_\beta^{1L}(C_L M_S^R\bar{C}^S C_R - M_S^R C_R\bar{C}^S C_L) \\ &= 4i\eta_1^\beta\bar{\theta}_\beta^{1L}\left(\dot{x}^\mu D_\mu C_L - \frac{2\pi}{k}(C_L M_S^R\bar{C}^S C_R - M_S^R C_R\bar{C}^S C_L)\right) \\ &= 4i\eta_1^\beta\bar{\theta}_\beta^{1L}\left(\dot{x}^\mu D_\mu C_L + iA_\mu^{(1)}\dot{x}^\mu C_L - iC_L A_\mu^{(2)}\dot{x}^\mu - \frac{2\pi}{k}(C_L M_S^R\bar{C}^S C_R - M_S^R C_R\bar{C}^S C_L)\right) \\ &= 4i\eta_1^\beta\bar{\theta}_\beta^{1L}\left(\partial_\tau C_L + i(A_\mu^{(1)}\dot{x}^\mu - \frac{2\pi i}{k}M_S^R C_R\bar{C}^S)C_L - iC_L(A_\mu^{(2)}\dot{x}^\mu - \frac{2\pi i}{k}M_S^R\bar{C}^S C_R)\right) \\ &= 4i\eta_1^\beta\bar{\theta}_\beta^{1L}\left(\partial_\tau C_L + i\mathcal{A}^{(1)}C_L - iC_L\mathcal{A}^{(2)} - \frac{iC_L}{2}\right) \\ &= 2\eta_1^\beta\bar{\theta}_\beta^{1L}C_L + 4i\eta_1^\beta\bar{\theta}_\beta^{1L}\mathcal{D}_\tau C_L.\end{aligned} \tag{A.49}$$

We can put η inside the derivative at the expense of an extra term

$$4i\eta_1^\beta\bar{\theta}_\beta^{1L}\mathcal{D}_\tau C_L = 4i\bar{\theta}_\beta^{1L}\mathcal{D}_\tau(\eta_1^\beta C_L) - 4i\bar{\theta}_\beta^{1L}(\partial_\tau\eta_1^\beta)C_L. \tag{A.50}$$

Finally, we have the pieces (I-III) evaluating to

$$(\mathrm{I})+(\mathrm{II})+(\mathrm{III}) = 4i\bar\theta^{1L}_\beta \mathcal{D}_\tau(\eta_1^\beta C_L) - 4i\bar\theta^{1L}_\beta(\partial_\tau\eta_1^\beta)C_L + 2\eta_1^\beta\bar\theta^{1L}_\beta C_L. \tag{A.51}$$

Now we focus on the last contribution, the term (IV). We have

$$\begin{aligned} -2i\eta_1^\beta\bar\epsilon^{1L}_\beta C_L &= -2i\eta_1^\beta(i\bar\theta^{1L\alpha}(\sigma^3)_{\alpha\beta})C_L \\ &= 2\eta_1^\beta(\bar\theta^{1L\alpha}(\sigma^3)_{\alpha\beta})C_L \\ &= -2\eta_1^\beta((\sigma^3)_\beta^{\ \alpha})\bar\theta^{1L}_\alpha C_L \end{aligned} \tag{A.52}$$

Adding up contributions all contributions, namely (A.51) and (A.52), we reach

$$\delta(\eta_1^\beta\bar\psi^1_\beta) = 4i\bar\theta^{1L}_\beta\mathcal{D}_\tau(\eta_1^\beta C_L) - 4i\bar\theta^{1L}_\beta(\partial_\tau\eta_1^\beta)C_L + 2\eta_1^\beta\bar\theta^{1L}_\beta C_L - 2\eta_1^\beta((\sigma^3)_\beta^{\ \alpha})\bar\theta^{1L}_\alpha C_L \tag{A.53}$$

Notice that the last three terms in (A.53) cancel each other

$$\begin{aligned} -4i\bar\theta^{1L}_\beta(\partial_\tau\eta_1^\beta)C_L + 2\eta_1^\beta\bar\theta^{1L}_\beta C_L - 2\eta_1^\beta((\sigma^3)_\beta^{\ \alpha})\bar\theta^{1L}_\alpha C_L &= 2(\eta_1(1-\sigma^3) - 2i\partial_\tau\eta_1)\bar\theta^{1L}C_L \\ &= 0. \end{aligned} \tag{A.54}$$

So we finally have

$$\delta(\eta_1^\beta\bar\psi^1_\beta) = 4i\bar\theta^{1L}_\beta\mathcal{D}_\tau(\eta_1^\beta C_L). \tag{A.55}$$

The derivation of (A.34) is analogous to the derivation of (A.33), so one can follow the same strategy. Reading the superconformal transformations to ψ, we can write

$$\begin{aligned} &\delta(\psi_1^\beta\bar\eta^1_\beta) = \\ &[\underbrace{-2i\Theta^\alpha_{1L}(\gamma^\mu)_\alpha^{\ \beta}D_\mu\bar C^L}_{(\mathrm{I})}\underbrace{+\frac{4\pi i}{k}\Theta^\beta_{1L}(\bar C^L C_M\bar C^M - \bar C^M C_M\bar C^L)}_{(\mathrm{II})} \\ &\quad\underbrace{+\frac{8\pi i}{k}\Theta^\beta_{IJ}\bar C^I C_1\bar C^J}_{(\mathrm{III})}\underbrace{-2i(\epsilon^\beta_{1L}\bar C^L)}_{(\mathrm{IV})}]\bar\eta^\beta_1. \end{aligned}$$

We also have, by the reality condition of Θ_{IJ}, that the supercharges satisfy

$$\begin{cases} \Theta_{1I} = \theta_{1I}(1+\dot x^\mu\gamma_\mu) \\ \Theta_{IJ} = \theta_{IJ}(1-\dot x^\mu\gamma_\mu), \text{ for I,J} \neq 1 \end{cases}$$

. Following the former analysis yields

$$\delta(\psi_1^\beta\bar\eta^1_\beta) = -4i\theta^\beta_{1I}\mathcal{D}_\tau(\bar\eta^1_\beta\bar C^I) + \underbrace{4i\theta^\beta_{1I}\partial_\tau(\bar\eta^1_\beta)\bar C^I + 2\theta^\beta_{1I}\bar\eta^1_\beta\bar C^I - 2\theta^\alpha_{1I}(\sigma^3)_\alpha^{\ \beta}\bar C^I\bar\eta^1_\beta}_{0}. \tag{A.56}$$

The derivation of (A.31) and (A.32) are the same, except with ψ and C switching places, they consist in using the same tricks as in deriving the former relations.

Appendix B

Defect symmetries

In chapter 6 we study the 1/2-BPS Wilson line of $\mathcal{N}=4$ CSm theories. In this context of defects, the 1/2-BPS line defines an $\mathfrak{su}(1,1|2)$ dCFT, which can be seen as the reduction of the bulk $\mathfrak{osp}(4|4)$ to the BPS line. In this appendix, we explicitly construct the $\mathfrak{su}(1,1|2)$ from the original bulk symmetries of $\mathcal{N}=4$ CSm theories, and specify all conventions used in the calculations of that chapter.

B.1 $\mathfrak{osp}(4|4)$

We use the conventions and definitions of [40]. The 3d $\mathcal{N}=4$ superconformal algebra is $\mathfrak{osp}(4|4)$, and its bosonic sub-algebra $\mathfrak{so}(1,4)\oplus\mathfrak{su}(2)_A\oplus\mathfrak{su}(2)_B$ consist of conformal and R-symmetry transformations. We work in three-dimensional Euclidean space $\mathbb{R}^3$, with isometry generators P_μ, $\mathcal{M}_{\mu\nu}$, D, K_μ for translations, rotations and dilatations, and special conformal transformations. The $\mathfrak{su}(2)_A$ and $\mathfrak{su}(2)_B$ R-symmetry are $R_a{}^b$ and $\bar{R}_{\dot{a}}{}^{\dot{b}}$, respectively. The bosonic sector reads

$$[M_\alpha{}^\beta, P_{\gamma\delta}] = \delta_\gamma{}^\beta P_{\alpha\delta} + \delta_\delta{}^\beta P_{\alpha\gamma} - \delta_\alpha{}^\beta P_{\gamma\delta}\,, \tag{B.1}$$
$$[M_\alpha{}^\beta, K^{\gamma\delta}] = -\delta_\alpha{}^\gamma K^{\beta\delta} - \delta_\alpha{}^\delta K^{\beta\gamma} + \delta_\alpha{}^\beta K^{\gamma\delta}\,, \tag{B.2}$$
$$[M_\alpha{}^\beta, M_\gamma{}^\delta] = -\delta_\alpha{}^\delta M_\gamma{}^\beta + \delta_\gamma{}^\beta M_\alpha{}^\delta\,, \quad [D, P_{\alpha\beta}] = P_{\alpha\beta}\,, \quad [D, K^{\alpha\beta}] = -K^{\alpha\beta}\,, \tag{B.3}$$
$$[P_{\alpha\beta}, K^{\gamma\delta}] = 4\delta_{(\alpha}{}^{(\gamma} M_{\beta)}{}^{\delta)} + 4\delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} D\,, \tag{B.4}$$
$$[R_a{}^b, R_c{}^d] = -\delta_a{}^d R_c{}^b + \delta_c{}^b R_a{}^d\,, \quad [\bar{R}_{\dot{a}}{}^{\dot{b}}, \bar{R}_{\dot{c}}{}^{\dot{d}}] = -\delta_{\dot{a}}{}^{\dot{d}} \bar{R}_{\dot{c}}{}^{\dot{b}} + \delta_{\dot{c}}{}^{\dot{b}} \bar{R}_{\dot{a}}{}^{\dot{d}}\,, \tag{B.5}$$

where we have the spinor embedding

$$P_{\alpha\beta} \equiv (\gamma^\mu)_{\alpha\beta} P_\mu\,, \quad K^{\alpha\beta} \equiv (\gamma^\mu)^{\alpha\beta} K_\mu\,, \quad M_\alpha{}^\beta \equiv \frac{i}{2}(\gamma^\mu\gamma^\nu)_\alpha{}^\beta \mathcal{M}_{\mu\nu}\,. \tag{B.6}$$

The algebra (B.1)–(B.5) is represented on a dimension Δ scalar primary operator $\mathcal{O}_{a\dot{a}}(x)$ in the $(\mathbf{2},\mathbf{2})$ irrep of $\mathfrak{su}(2)_A\oplus\mathfrak{su}(2)_B$ as

$$\begin{aligned}
&[P_\mu, \mathcal{O}_{a\dot{a}}(x)] = i\partial_\mu \mathcal{O}_{a\dot{a}}(x)\,, \quad [K_\mu, \mathcal{O}_{a\dot{a}}(x)] = i(x^2\partial_\mu - 2x_\mu(x\cdot\partial) - 2\Delta x_\mu)\mathcal{O}_{a\dot{a}}(x)\,,\\
&[\mathcal{M}^{\mu\nu}, \mathcal{O}_{a\dot{a}}(x)] = i(x^\mu\partial^\nu - x^\nu\partial^\mu)\mathcal{O}(x)\,, \quad [D, \mathcal{O}_{a\dot{a}}(x)] = (x\cdot\partial + \Delta)\mathcal{O}_{a\dot{a}}(x)\,,\\
&[R_a{}^b, \mathcal{O}_{c\dot{c}}(x)] = \delta_c{}^b \mathcal{O}_{a\dot{c}} - \frac{1}{2}\delta_a{}^b \mathcal{O}_{c\dot{c}}\,, \quad [\bar{R}_{\dot{a}}{}^{\dot{b}}, \mathcal{O}_{c\dot{c}}(x)] = \delta_{\dot{c}}{}^{\dot{b}} \mathcal{O}_{c\dot{a}} - \frac{1}{2}\delta_{\dot{a}}{}^{\dot{b}} \mathcal{O}_{c\dot{c}}.
\end{aligned} \tag{B.7}$$

The fermionic sector of the algebra reads:

$$\{\mathcal{Q}_{\alpha a\dot{a}}, \mathcal{Q}_{\beta b\dot{b}}\} = 4\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}P_{\alpha\beta}\,, \qquad \{\mathcal{S}^{\alpha}{}_{a\dot{a}}, \mathcal{S}^{\beta}{}_{b\dot{b}}\} = 4\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}K^{\alpha\beta}\,, \tag{B.8}$$

$$[K^{\alpha\beta}, \mathcal{Q}_{\gamma a\dot{a}}] = i\left(\delta_{\gamma}{}^{\alpha}\mathcal{S}^{\beta}{}_{a\dot{a}} + \delta_{\gamma}{}^{\beta}\mathcal{S}^{\alpha}{}_{a\dot{a}}\right)\,, \qquad [P_{\alpha\beta}, \mathcal{S}^{\gamma}{}_{a\dot{a}}] = -i\left(\delta_{\alpha}{}^{\gamma}\mathcal{Q}_{\beta a\dot{a}} + \delta_{\beta}{}^{\gamma}\mathcal{Q}_{\alpha a\dot{a}}\right)\,, \tag{B.9}$$

$$[M_{\alpha}{}^{\beta}, \mathcal{Q}_{\gamma a\dot{a}}] = \delta_{\gamma}{}^{\beta}\mathcal{Q}_{\alpha a\dot{a}} - \frac{1}{2}\delta_{\alpha}{}^{\beta}\mathcal{Q}_{\gamma a\dot{a}}\,, \qquad [M_{\alpha}{}^{\beta}, \mathcal{S}^{\gamma}{}_{a\dot{a}}] = -\delta_{\alpha}{}^{\gamma}\mathcal{S}^{\beta}{}_{a\dot{a}} + \frac{1}{2}\delta_{\alpha}{}^{\beta}\mathcal{S}^{\gamma}{}_{a\dot{a}}\,, \tag{B.10}$$

$$[D, \mathcal{Q}_{\alpha a\dot{a}}] = \frac{1}{2}\mathcal{Q}_{\alpha a\dot{a}}\,, \qquad [D, \mathcal{S}^{\alpha}{}_{a\dot{a}}] = -\frac{1}{2}\mathcal{S}^{\alpha}{}_{a\dot{a}}\,, \tag{B.11}$$

$$[R_{a}{}^{b}, \mathcal{Q}_{\alpha c\dot{c}}] = \delta_{c}{}^{b}\mathcal{Q}_{\alpha a\dot{c}} - \frac{1}{2}\delta_{a}{}^{b}\mathcal{Q}_{\alpha c\dot{c}}\,, \qquad [R_{a}{}^{b}, \mathcal{S}^{\alpha}{}_{c\dot{c}}] = \delta_{c}{}^{b}\mathcal{S}^{\alpha}{}_{a\dot{c}} - \frac{1}{2}\delta_{a}{}^{b}\mathcal{S}^{\alpha}{}_{c\dot{c}}\,, \tag{B.12}$$

$$[\bar{R}_{\dot{a}}{}^{\dot{b}}, \mathcal{Q}_{\alpha c\dot{c}}] = \delta_{\dot{c}}{}^{\dot{b}}\mathcal{Q}_{\alpha c\dot{a}} - \frac{1}{2}\delta_{\dot{a}}{}^{\dot{b}}\mathcal{Q}_{\alpha c\dot{c}}\,, \qquad [\bar{R}_{\dot{a}}{}^{\dot{b}}, \mathcal{S}^{\alpha}{}_{c\dot{c}}] = \delta_{\dot{c}}{}^{\dot{b}}\mathcal{S}^{\alpha}{}_{c\dot{a}} - \frac{1}{2}\delta_{\dot{a}}{}^{\dot{b}}\mathcal{S}^{\alpha}{}_{c\dot{c}}\,, \tag{B.13}$$

and also

$$\{\mathcal{Q}_{\alpha a\dot{a}}, \mathcal{S}^{\beta}{}_{b\dot{b}}\} = 4i\left[\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}\left(M_{\alpha}{}^{\beta} + \delta_{\alpha}{}^{\beta}D\right) + \delta_{\alpha}{}^{\beta}\left(\varepsilon_{\dot{a}\dot{b}}R_{ab} + \varepsilon_{ab}\bar{R}_{\dot{a}\dot{b}}\right)\right]\,. \tag{B.14}$$

All other (anti)commutators vanishing.

B.2 $\mathfrak{su}(1,1|2)$ sub-algebra

Inside the $\mathfrak{osp}(4|4)$ it is possible to identify the $\mathfrak{su}(1,1|2)$ sub-algebra preserved by the 1/2 BPS Wilson line. The $\mathfrak{su}(1,1)$ generators are those of the one-dimensional conformal group, i.e. $\{D, P \equiv P_1, K \equiv K_1\}$, satisfying

$$[P,K] = -2D \qquad [D,P] = P \qquad [D,K] = -K \tag{B.15}$$

The SU(2) generators $R_{a}{}^{b}$ are traceless, i.e. $R_{a}^{\ a} = 0$. The $\mathfrak{u}(1)$ twist is given by

$$J_0 = \left(\mathcal{M}_{+}^{\ +} + \bar{R}_{\dot{1}}^{\ \dot{1}}\right)\,. \tag{B.16}$$

The twist is actually a central extension of the algebra, i.e it commutes with all other generators.

The fermionic generators are given by a reorganization of the preserved supercharges $Q_{+}^{1\dot{1}}, Q_{+}^{2\dot{1}}, Q_{-}^{1\dot{2}}, Q_{-}^{2\dot{2}}$, together with the corresponding superconformal charges. Our notation is

$$Q^a = \mathcal{Q}_{+}^{a\dot{1}} \qquad S^a = \mathcal{S}_{+}^{a\dot{1}} \qquad \bar{Q}_a = \epsilon_{ab}\mathcal{Q}_{-}^{b\dot{2}} \qquad \bar{S}_a = \epsilon_{ab}\mathcal{S}_{-}^{b\dot{2}} \tag{B.17}$$

so that,

$$\{Q^a, \bar{Q}_b\} = -4\delta_b^a P, \qquad \{S^a, \bar{S}_b\} = -4\delta_b^a K, \tag{B.18}$$

$$\{Q^a, \bar{S}_b\} = -4i\left[\delta_b^a\left(D + J_0\right) + R_{\ b}^{a}\right], \quad \{\bar{Q}_a, S^b\} = 4i\left[\delta_a^b\left(D - J_0\right) - R_{a}^{\ b}\right]. \tag{B.19}$$

Finally, the non-vanishing mixed commutators are

$$[D, Q^a] = \frac{1}{2}Q^a, \qquad [D, \bar{Q}_a] = \frac{1}{2}\bar{Q}_a, \qquad [K, Q^a] = iS^a, \quad [K, \bar{Q}_a] = -i\bar{S}_a, \tag{B.20}$$

$$[D, S^a] = -\frac{1}{2}S^a, \qquad [D, \bar{S}_a] = -\frac{1}{2}\bar{S}_a, \qquad [P, S^a] = iQ^a, \quad [P, \bar{S}_a] = -i\bar{Q}_a, \tag{B.21}$$

$$[R_a{}^b, Q^c] = -\delta^c_a Q^b + \tfrac{1}{2}\delta^b_a Q^c, \quad [R_a{}^b, \bar{Q}_c] = \delta^b_c \bar{Q}_a - \tfrac{1}{2}\delta^b_a \bar{Q}_c, \quad [J_0, Q^a] = 0, \qquad [J_0, \bar{Q}^a] = 0, \tag{B.22}$$

$$[R_a{}^b, S^c] = -\delta^c_a S^b + \tfrac{1}{2}\delta^b_a S^c, \quad [R_a{}^b, \bar{S}_c] = \delta^b_c \bar{S}_a - \tfrac{1}{2}\delta^b_a \bar{S}_c, \quad [J_0, S^a] = 0, \qquad [J_0, \bar{S}^a] = 0. \tag{B.23}$$

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